ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

# Extremal Selections of Multifunctions Generating a Continuous Flow

Thesis submitted for the degree of
"Magister Philosophiæ"

CANDIDATE
Graziano Crasta

SUPERVISOR
Prof. Alberto Bressan

October 1993

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

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# 1 - Introduction

Let $F : [0,T] \times I\!R^n \mapsto 2^{I\!R^n}$ be a continuous multifunction with compact, not necessarily convex values. If $F$ is Lipschitz continuous, it was shown in [4] that there exists a measurable selection $f$ of $F$ such that, for every $x_0$, the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \qquad x(0) = x_0$$

has a unique Caratheodory solution, depending continuously on $x_0$.

In this paper, we prove that the above selection $f$ can be chosen so that $f(t,x) \in extF(t,x)$ for all $t, x$. More generally, the result remains valid if $F$ satisfies the following Lipschitz Selection Property:

(LSP) *For every $t, x$, every $y \in \overline{co}F(t,x)$ and $\varepsilon > 0$, there exists a Lipschitz selection $\phi$ of $\overline{co}F$, defined on a neighborhood of $(t,x)$, with $|\phi(t,x) - y| < \varepsilon$.*

We remark that, by [7,9], every Lipschitz multifunction with compact values satisfies (LSP). Another interesting class, for which (LSP) holds, consists of those continuous multifunctions $F$ whose values are compact and have convex closure with nonempty interior. Indeed, for any given $t, x, y, \varepsilon$, choosing $y' \in int\,\overline{co}F(t,x)$ with $|y' - y| < \varepsilon$, the constant function $\phi \equiv y'$ is a local selection from $\overline{co}F$ satisfying the requirements.

In the following, $\Omega \subseteq I\!R^n$ is an open set, $\overline{B}(0,M)$ is the closed ball centered at the origin with radius $M$, $\overline{B}(D;\ MT)$ is the closed neighborhood of radius $MT$ around the set $D$, while $\mathcal{AC}$ the Sobolev space of all absolutely continuous functions $u : [0,T] \mapsto I\!R^n$, with norm $\|u\|_{\mathcal{AC}} = \int_0^T \left(|u(t)| + |\dot{u}(t)|\right)\, dt$.

**Theorem 1.** *Let $F : [0,T] \times \Omega \mapsto 2^{I\!R^n}$ be a bounded continuous multifunction with compact values, satisfying (LSP). Assume that $F(t,x) \subseteq \overline{B}(0,M)$ for all $t, x$ and let $D$ be a compact set such that $\overline{B}(D;\ MT) \subset \Omega$. Then there exists a measurable function $f$, with*

$$f(t,x) \in extF(t,x) \qquad \forall t, x, \tag{1.1}$$

*such that, for every $(t_0, x_0) \in [0,T] \times D$, the Cauchy problem*

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0 \tag{1.2}$$

*has a unique Caratheodory solution $x(\cdot) = x(\cdot, t_0, x_0)$ on $[0,T]$, depending continuously on $t_0, x_0$ in the norm of $\mathcal{AC}$.*

*Moreover, if* $\varepsilon_0 > 0$ *and a Lipschitz continuous selection* $f_0$ *of* $\overline{co}F$ *are given, then one can construct* $f$ *with the following additional property. Denoting by* $y(\cdot, t_0, x_0)$ *the unique solution of*

$$\dot{y}(t) = f_0(t, y(t)), \qquad y(t_0) = x_0, \tag{1.3}$$

*for every* $(t_0, x_0) \in [0,T] \times D$ *one has*

$$\left|y(t, t_0, x_0) - x(t, t_0, x_0)\right| \leq \varepsilon_0 \qquad \forall t \in [0,T]. \tag{1.4}$$

The proof of the above theorem, given in section 3, starts with the construction of a sequence $f_n$ of selections from $\overline{co}F$, which are piecewise Lipschitz continuous in the $(t,x)$-space. For every $u : [0,T] \mapsto I\!R^n$ in a class of Lipschitz continuous functions, we then show that the composed maps $t \mapsto f_n(t, u(t))$ form a Cauchy sequence in $\mathcal{L}^1\big([0,T];\ I\!R^n\big)$, converging pointwise almost everywhere to a map of the form $f(\cdot, u(\cdot))$, taking values within the extreme points of $F$. This convergence is obtained through an argument which is considerably different from previous works. Indeed, it relies on a careful use of the likelihood functional introduced in [3], interpreted here as a measure of "oscillatory non-convergence" of a set of derivatives.

Among various corollaries, Theorem 1 yields an extension, valid for the wider class of multifunctions with the property (LSP), of the following results, proved in [5], [4] and [6], respectively.

(i) Existence of selections from the solution set of a differential inclusion, depending continuously on the initial data.

(ii) Existence of selections from a multifunction, which generate a continuous flow.

(iii) Contractibility of the solution sets of $\dot{x} \in F(t,x)$ and $\dot{x} \in extF(t,x)$.

These consequences, together with an application to bang-bang feedback controls, are described in section 4.

## 2 - Preliminaries

As customary, $\bar{A}$ and $\overline{co}\,A$ denote here the closure and the closed convex hull of $A$ respectively, while $A\backslash B$ indicates a set–theoretic difference. The Lebesgue measure of a set $J \subset I\!R$ is $m(J)$. The characteristic function of a set $A$ is written as $\chi_A$.

In the following, $\mathcal{K}_n$ denotes the family of all nonempty compact convex subsets of $I\!R^n$, endowed with Hausdorff metric. A key technical tool used in our proofs will be the function $h : I\!R^n \times \mathcal{K}_n \mapsto I\!R \cup \{-\infty\}$, defined by

$$h(y,K) \doteq \sup\left\{ \left(\int_0^1 |w(\xi) - y|^2 \, d\xi\right)^{\frac{1}{2}}; \quad w : [0,1] \mapsto K, \quad \int_0^1 w(\xi)\, d\xi = y \right\} \tag{2.1}$$

with the understanding that $h(y,K) = -\infty$ if $y \notin K$. Observe that $h^2(y,K)$ can be interpreted as the maximum variance among all random variables supported inside $K$, whose mean value is $y$. The following results were proved in [3]:

**Lemma 1.** *The map $(y,K) \mapsto h(y,K)$ is upper semicontinuous in both variables; for each fixed $K \in \mathcal{K}_n$ the function $y \mapsto h(y,K)$ is strictly concave down on $K$. Moreover, one has*

$$h(y,K) = 0 \qquad \text{if and only if} \qquad y \in extK, \tag{2.2}$$

$$h^2(y,K) \le r^2(K) - \left|y - c(K)\right|^2, \tag{2.3}$$

*where $c(K)$ and $r(K)$ denote the Chebyschev center and the Chebyschev radius of $K$, respectively.*

For the basic theory of multifunctions and differential inclusions we refer to [1]. As in [2], given a map $g : [0,T] \times \Omega \mapsto I\!R^n$, we say that $g$ is directionally continuous along the directions of the cone $\Gamma^N = \{(s,y);\ |y| \le Ns\}$ if

$$g(t,x) = \lim_{k\to\infty} g(t_k, x_k)$$

for every $(t,x)$ and every sequence $(t_k, x_k)$ in the domain of $g$ such that $t_k \to t$ and $|x_k - x| \le N(t_k - t)$ for every $k$. Equivalently, $g$ is $\Gamma^N$-continuous iff it is continuous w.r.t. the topology generated by the family of all conical neighborhoods

$$\Gamma^N_{(t,x,\varepsilon)} \doteq \{(s,y)\ ; \quad \hat{t} \le s \le \hat{t} + \varepsilon,\ |y - \hat{x}| \le N(s - t)\}. \tag{2.4}$$

A set of the form (2.4) will be called an $N$–cone.

Under the assumptions on $\Omega$, $D$ made in Theorem 1, consider the set of Lipschitzean functions

$$Y \doteq \{u : [0,T] \mapsto \overline{B}(D, MT); \qquad |u(t) - u(s)| \leq M|t-s| \quad \forall t, s\}.$$

The Picard operator of a map $g : [0,T] \times \Omega \mapsto \mathbb{R}^n$ is defined as

$$\mathcal{P}^g(u)(t) \doteq \int_0^t g(s, u(s))\, ds \qquad\qquad u \in Y.$$

The distance between two Picard operators will be measured by

$$\left\|\mathcal{P}^f - \mathcal{P}^g\right\| \;=\; \sup\left\{\left|\int_0^t [f(s,u(s)) - g(s,u(s))]\, ds\right| \;;\quad t \in [0,T],\;\; u \in Y\right\}. \tag{2.5}$$

The next Lemma will be useful in order to prove the uniqueness of solutions of the Cauchy problems (1.2).

**Lemma 2.** *Let $f$ be a measurable map from $[0,T] \times \Omega$ into $\overline{B}(0,M)$, with $\mathcal{P}^f$ continuous on $Y$. Let $D$ be compact, with $\overline{B}(D,\ MT) \subset \Omega$, and assume that the Cauchy problem*

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \qquad t \in [0,T] \tag{2.6}$$

*has a unique solution, for each $(t_0, x_0) \in [0,T] \times D$.*

*Then, for every $\epsilon > 0$, there exists $\delta > 0$ with the following property. If $g : [0,T] \times \Omega \to \overline{B}(0,M)$ satisfies $\left\|\mathcal{P}^g - \mathcal{P}^f\right\| \leq \delta$, then for every $(t_0, x_0) \in [0,T] \times D$, any solution of the Cauchy problem*

$$\dot{y}(t) = g(t, y(t)) \qquad y(t_0) = x_0 \qquad t \in [0,T] \tag{2.7}$$

*has distance $< \varepsilon$ from the corresponding solution of (2.6). In particular, the solution set of (2.7) has diameter $\leq 2\varepsilon$ in $C^0\big([0,T]; \mathbb{R}^n\big)$.*

**Proof.** If the conclusion fails, then there exist sequences of times $t_\nu$, $t'_\nu$, maps $g_\nu$ with $\left\|\mathcal{P}^{g_\nu} - \mathcal{P}^f\right\| \to 0$, and couples of solutions $x_\nu, y_\nu : [0,T] \mapsto \overline{B}(D; MT)$ of

$$\dot{x}_\nu(t) = f(t, x_\nu(t)), \qquad \dot{y}_\nu(t) = g_\nu(t, y_\nu(t)) \qquad t \in [0,T], \tag{2.8}$$

with

$$x_\nu(t_\nu) = y_\nu(t_\nu) \in D, \qquad \left|x_\nu(t'_\nu) - y_\nu(t'_\nu)\right| \geq \varepsilon \qquad \forall \nu. \tag{2.9}$$

By taking subsequences, we can assume that $t_\nu \to t_0$, $t'_\nu \to \tau$, $x_\nu(t_0) \to x_0$, while $x_\nu \to x$ and $y_\nu \to y$ uniformly on $[0, T]$. From (2.8) it follows

$$\begin{aligned}\left| y(t) - x_0 - \int_{t_0}^{t} f(s, y(s))\, ds \right| &\le \left| y(t) - y_\nu(t) \right| + \left| x_0 - y_\nu(t_0) \right| \\ &+ \left| \int_{t_0}^{t} \left[ f(s, y(s)) - f(s, y_\nu(s)) \right] ds \right| + \left| \int_{t_0}^{t} \left[ f(s, y_\nu(s)) - g_\nu(s, y_\nu(s)) \right] ds \right| . \end{aligned} \tag{2.10}$$

As $\nu \to \infty$, the right hand side of (2.10) tends to zero, showing that $y(\cdot)$ is a solution of (2.6). By the continuity of $\mathcal{P}^f$, $x(\cdot)$ is also a solution of (2.6), distinct from $y(\cdot)$ because

$$|x(\tau) - y(\tau)| = \lim_{\nu \to \infty} |x_\nu(\tau) - y_\nu(\tau)| = \lim_{\nu \to \infty} \left| x_\nu(t'_\nu) - y_\nu(t'_\nu) \right| \ge \varepsilon .$$

This contradicts the uniqueness assumption, proving the lemma.

## 3 - Proof of the main theorem

Observing that $extF(t,x) = ext\overline{co}F(t,x)$ for every compact set $F(t,x)$, it is clearly not restrictive to prove Theorem 1 under the additional assumption that all values of $F$ are convex. Moreover, the bounds on $F$ and $D$ imply that no solution of the Cauchy problem

$$\dot{x}(t) \in F(t, x(t)), \qquad x(t_0) = x_0, \qquad t \in [0, T],$$

with $x_0 \in D$, can escape from the set $\overline{B}(D,\ MT)$. Therefore, it suffices to construct the selection $f$ on the compact set $\Omega^\dagger \doteq [0,T] \times \overline{B}(D,\ MT)$. Finally, since every convex valued multifunction satisfying (LSP) admits a globally defined Lipschitz selection, it suffices to prove the second part of the theorem, with $f_0$ and $\varepsilon_0 > 0$ assigned.

We shall define a sequence of directionally continuous selections of $F$, converging a.e. to a selection from $extF$. The basic step of our constructive procedure will be provided by the next lemma.

**Lemma 3.** *Fix any $\varepsilon > 0$. Let $S$ be a compact subset of $[0,T] \times \Omega$ and let $\phi : S \to I\!R^n$ be a continuous selection of $F$ such that*

$$h(\phi(t,x), F(t,x)) < \eta \qquad \forall (t,x) \in S, \tag{3.1}$$

*with $h$ as in (2.1). Then there exists a piecewise Lipschitz selection $g : S \to I\!R^n$ of $F$ with the following properties:*

*(i) There exists a finite covering $\{\Gamma_i\}_{i=1\dots,\nu}$, consisting of $\Gamma^{M+1}$-cones, such that, if we define the pairwise disjoint sets $\Delta^i \doteq \Gamma_i \setminus \bigcup_{\ell<i} \Gamma_\ell$, then on each $\Delta^i$ the following holds:*

*(a) there exist Lipschitzean selections $\psi_j^i : \overline{\Delta^i} \mapsto I\!R^n$, $j = 0, \dots, n$, such that*

$$g\big|_{\Delta^i} = \sum_{j=0}^{n} \psi_j^i \, \chi_{A_j^i}. \tag{3.2}$$

*where each $A_j^i$ is a finite union of strips of the form $\big([t', t'') \times I\!R^n\big) \cap \Delta^i$.*

*(b) For every $j = 0, \dots, n$ there exists an affine map $\varphi_j^i(\cdot) = \langle a_j^i, \cdot \rangle + b_j^i$ such that*

$$\varphi_j^i(\psi_j^i(t,x)) \le \varepsilon, \qquad \varphi_j^i(z) \ge h(z, F(t,x)), \qquad \forall (t,x) \in \overline{\Delta^i}, \quad z \in F(t,x). \tag{3.3}$$

*(ii) For every $u \in Y$ and every interval $[\tau, \tau']$ such that $(s, u(s)) \in S$ for $\tau \le s < \tau'$, the following estimates hold:*

$$\left| \int_\tau^{\tau'} \left[\phi(s, u(s)) - g(s, u(s))\right] ds \right| \le \varepsilon, \tag{3.4}$$

$$\int_\tau^{\tau'} \left|\phi(s, u(s)) - g(s, u(s))\right| ds \; \le \; \varepsilon + \eta(\tau' - \tau). \tag{3.5}$$

**Remark 1.** Thinking of $h(y, K)$ as a measure for the distance of $y$ from the extreme points of $K$, the above lemma can be interpreted as follows. Given any selection $\phi$ of $F$, one can find a $\Gamma^{M+1}$-continuous selection $g$ whose values lie close to the extreme points of $F$ and whose Picard operator $\mathcal{P}^g$, by (3.4), is close to $\mathcal{P}^\phi$. Moreover, if the values of $\phi$ are near the extreme points of $F$, i.e. if $\eta$ in (3.1) is small, then $g$ can be chosen close to $\phi$. The estimate (3.5) will be a direct consequence of the definition (2.1) of $h$ and of Hölder's inequality.

**Remark 2.** Since $h$ is only upper semicontinuous, the two assumptions $y_\nu \to y$ and $h(y_\nu, K) \to 0$ do not necessarily imply $h(y, K) = 0$. As a consequence, the a.e. limit of a convergent sequence of approximately extremal selections $f_\nu$ of $F$ need not take values inside $extF$. To overcome this difficulty, the estimates in (3.3) provide upper bounds for $h$ in terms of the affine maps $\varphi_j^i$. Since each $\varphi_j^i$ is continuous, limits of the form $\varphi_j^i(y_\nu) \to \varphi_j^i(y)$ will be straightforward.

**Proof of Lemma 3.** For every $(t,x) \in S$ there exist values $y_j(t,x) \in F(t,x)$ and coefficients $\theta_j(t,x) \geq 0$, with

$$\phi(t,x) = \sum_{j=0}^{n} \theta_j(t,x) y_j(t,x), \qquad \sum_{j=0}^{n} \theta_j(t,x) = 1,$$

$$h\big(y_j(t,x), F(t,x)\big) < \varepsilon/2.$$

By the concavity and the upper semicontinuity of $h$, for every $j = 0, \ldots, n$ there exists an affine function $\varphi_j^{(t,x)}(\cdot) = \langle a_j^{(t,x)}, \cdot \rangle + b_j^{(t,x)}$ such that

$$\varphi_j^{(t,x)}(y_j(t,x)) < h\big(y_j(t,x), F(t,x)\big) + \frac{\varepsilon}{2} < \varepsilon,$$

$$\varphi_j^{(t,x)}(z) > h\big(z, F(t,x)\big) \qquad \forall z \in F(t,x).$$

By (LSP) and the continuity of each $\varphi_j^{(t,x)}$, there exists a neighborhood $\mathcal{U}$ of $(t,x)$ together with Lipschitzean selections $\psi_j^{(t,x)} : \mathcal{U} \mapsto I\!R^n$, such that, for every $j$ and every $(s,y) \in \mathcal{U}$,

$$\left| \psi_j^{(t,x)}(s,y) - y_j(t,x) \right| < \frac{\varepsilon}{4T}, \tag{3.6}$$

$$\varphi_j^{(t,x)}\big(\psi_j^{(t,x)}(s,y)\big) < \varepsilon. \tag{3.7}$$

Using again the upper semicontinuity of $h$, we can find a neighborhood $\mathcal{U}'$ of $(t,x)$ such that

$$\varphi_j^{(t,x)}(z) \geq h\big(z, F(s,y)\big) \qquad \forall z \in F(s,y), \quad (s,y) \in \mathcal{U}', \quad j = 0, \ldots, n. \tag{3.8}$$

Choose a neighborhood $\Gamma_{t,x}$ of $(t,x)$, contained in $\mathcal{U} \cap \mathcal{U}'$, such that, for every point $(s,y)$ in the closure $\overline{\Gamma}_{t,x}$, one has

$$|\phi(s,y) - \phi(t,x)| < \frac{\varepsilon}{4T}, \tag{3.9}$$

It is not restrictive to assume that $\Gamma_{t,x}$ is a $(M+1)$-cone, i.e. it has the form (2.4) with $N = M+1$. By the compactness of $S$ we can extract a finite subcovering $\{\Gamma^i;\ 1 \leq i \leq \nu\}$, with $\Gamma_i \doteq \Gamma_{t_i,x_i}$. Define $\Delta^i \doteq \Gamma_i \setminus \bigcup_{j<i} \Gamma_j$ and set $\theta_j^i = \theta_j(t_i,x_i)$, $y_j^i = y_j(t_i,x_i)$, $\psi_j^i = \psi_j^{(t_i,x_i)}$, $\varphi_j^i = \varphi_j{}^{(t_i,x_i)}$. Choose an integer $N$ such that

$$N > \frac{8M\nu^2 T}{\varepsilon} \tag{3.10}$$

and divide $[0,T]$ into $N$ equal subintervals $J_1,\ldots,J_N$, with

$$J_k = \big[t_{k-1},\ t_k\big), \qquad\qquad t_k = \frac{kT}{N}. \tag{3.11}$$

For each $i,k$ such that $\big(J_k \times I\!R^n\big) \cap \Delta^i \neq \emptyset$, we then split $J_k$ into $n+1$ subintervals $J^i_{k,0},\ldots,J^i_{k,n}$ with lengths proportional to $\theta^i_0,\ldots,\theta^i_n$, by setting

$$J^i_{k,j} = \big[t_{k,j-1},\ t_{k,j}\big), \qquad t_{k,j} = \frac{T}{N}\cdot\Big(k+\sum_{\ell=0}^{j}\theta^i_\ell\Big), \qquad t_{k,-1} = \frac{Tk}{N}.$$

For any point $(t,x)\in\overline{\Delta^i}$ we now set

$$\begin{cases} g^i(t,x) \doteq \psi^i_j(t,x) \\ \bar g^i(t,x) = y^i_j \end{cases} \qquad \text{if } \ t\in\bigcup_{k=1}^{N} J^i_{k,j}. \tag{3.12}$$

The piecewise Lipschitz selection $g$ and a piecewise constant approximation $\bar g$ of $g$ can now be defined as

$$g = \sum_{i=1}^{\nu} g^i \chi_{\Delta^i}, \qquad\qquad \bar g = \sum_{i=1}^{\nu} \bar g^i \chi_{\Delta^i}. \tag{3.13}$$

By construction, recalling (3.7) and (3.8), the conditions (a), (b) in (i) clearly hold.

It remains to show that the estimates in (ii) hold as well. Let $\tau,\ \tau'\in[0,T]$ and $u\in Y$ be such that $\big(t,u(t)\big)\in S$ for every $t\in[\tau,\tau']$, and define

$$E^i = \big\{t\in I\ ;\quad (t,u(t))\in\Delta^i\big\}, \qquad\qquad i=1,\ldots,\nu.$$

From our previous definition $\Delta^i \doteq \Gamma_i\setminus\bigcup_{j<i}\Gamma_j$, where each $\Gamma_j$ is a $(M+1)$-cone, it follows that every $E^i$ is the union of at most $i$ disjoint intervals. We can thus write

$$E^i = \Big(\bigcup_{J_k\subset E^i} J_k\Big)\cup \hat E^i,$$

with $J_k$ given by (3.11) and

$$m(\hat E^i) \le \frac{2iT}{N} \le \frac{2\nu T}{N}. \tag{3.14}$$

Since

$$\phi(t_i,x_i) = \sum_{j=0}^{n}\theta^i_j y^i_j, \tag{3.15}$$

the definition of $\bar{g}$ at (3.12), (3.13) implies

$$\int_{J_k} [\phi(t_i, x_i) - \bar{g}(s, u(s))] \; ds = m(J_k) \cdot \left[ \phi(t_i, x_i) - \sum_{j=0}^{n} \theta_j^i y_j^i \right] = 0.$$

Therefore, from (3.9) and (3.6) it follows

$$\begin{aligned} \left| \int_{J_k} [\phi(s, u(s)) - g(s, u(s))] \; ds \right| &\leq \left| \int_{J_k} [\phi(s, u(s)) - \phi(t_i, x_i)] \; ds \right| \\ &+ \left| \int_{J_k} [\phi(t_i, x_i) - \bar{g}(s, u(s))] \; ds \right| + \left| \int_{J_k} [\bar{g}(s, u(s)) - g(s, u(s))] \; ds \right| \\ &\leq m(J_k) \cdot \left[ \frac{\varepsilon}{4T} + 0 + \frac{\varepsilon}{4T} \right] = m(J_k) \cdot \frac{\varepsilon}{2T}. \end{aligned}$$

The choice of $N$ at (3.10) and the bound (3.14) thus imply

$$\left| \int_{\tau}^{\tau'} [\phi(s, u(s)) - g(s, u(s))] \; ds \right| \leq 2M \cdot m\Big( \bigcup_{i=1}^{\nu} \hat{E}^i \Big) + (\tau' - \tau) \frac{\varepsilon}{2T} \leq 2M\nu \cdot \frac{2\nu T}{N} + \frac{\varepsilon}{2} \leq \varepsilon,$$

proving (3.4).

We next consider (3.5). For a fixed $i \in \{1, \ldots, \nu\}$, let $E^i$ be as before and define

$$\xi_{-1} = 0, \qquad \xi_j = \sum_{\ell=0}^{j} \theta_\ell^i, \qquad w^i(\xi) = \sum_{j=0}^{n} y_j^i \chi_{[\xi_{j-1}, \; \xi_j]}.$$

Recalling (3.15), the definition of $h$ at (2.1) and Hölder's inequality together imply

$$\begin{aligned} h\big(\phi(t_i, x_i), \; F(t_i, x_i)\big) &\geq \Big( \int_0^1 \big|\phi(t_i, x_i) - w^i(\xi)\big|^2 \; d\xi \Big)^{\frac{1}{2}} \\ &\geq \int_0^1 \big|\phi(t_i, x_i) - w^i(\xi)\big| \; d\xi = \sum_{j=0}^{n} \theta_j^i \big|\phi(t_i, x_i) - y_j^i\big|. \end{aligned}$$

Using this inequality we obtain

$$\begin{aligned} \int_{J_k} \big|\phi(t_i, x_i) - \bar{g}(s, u(s))\big| \; ds &= m(J_k) \cdot \sum_{j=0}^{n} \theta_j^i \big|\phi(t_i, x_i) - y_j^i\big| \\ &\leq m(J_k) \cdot h\big(\phi(t_i, x_i), F(t_i, x_i)\big) \leq \eta \cdot m(J_k), \end{aligned}$$

and therefore, by (3.9) and (3.6),

$$
\begin{aligned}
&\int_{J_k} \big|\phi(s,u(s)) - g(s,u(s))\big|\; ds \\
&\qquad \le \int_{J_k} \big|\phi(s,u(s)) - \phi(t_i,x_i)\big|\; ds + \int_{J_k} \big|\bar g(s,u(s)) - g(s,u(s))\big|\; ds \\
&\qquad\quad + \int_{J_k} \big|\phi(t_i,x_i) - \bar g(s,u(s))\big| \\
&\qquad \le m(J_k)\cdot\left[\frac{\varepsilon}{4T} + \frac{\varepsilon}{4T} + \eta\right] = m(J_k)\cdot\left(\frac{\varepsilon}{2T} + \eta\right).
\end{aligned}
$$

Using again (3.14) and (3.10), we conclude

$$
\int_\tau^{\tau'} \big|\phi(s,u(s)) - g(s,u(s))\big|\; ds \le (\tau'-\tau)\left(\frac{\varepsilon}{2T}+\eta\right) + 2M\nu\cdot\frac{2\nu T}{N} \le \varepsilon + (\tau'-\tau)\eta.
$$

Q.E.D.

Using Lemma 3, given any continuous selection $\tilde f$ of $F$ on $\Omega^\dagger$, and any sequence $(\varepsilon_k)_{k\ge1}$ of strictly positive numbers, we can generate a sequence $(f_k)_{k\ge1}$ of selections from $F$ as follows.

To construct $f_1$, we apply the lemma with $S = \Omega^\dagger$, $\phi = f_0$, $\varepsilon = \varepsilon_1$. This yields a partition $\{A_1^i;\ \ i = 1,\ldots,\nu_1\}$ of $\Omega^\dagger$ and a piecewise Lipschitz selection $f_1$ of $F$ of the form

$$
f_1 = \sum_{i=1}^{\nu_1} f_1^i \chi_{A_1^i}.
$$

In general, at the beginning of the $k$-th step we are given a partition of $\Omega^\dagger$, say $\{A_k^i;\ \ i=1,\ldots,\nu_k\}$, and a selection

$$
f_k = \sum_{i=1}^{\nu_k} f_k^i \chi_{A_k^i},
$$

where each $f_k^i$ is Lipschitz continuous and satisfies

$$
h\big(f_k(t,x), F(t,x)\big) \le \varepsilon_k \qquad\qquad \forall (t,x)\in \overline{A_k^i}.
$$

We then apply Lemma 3 separately to each $A_k^i$, choosing $S = \overline{A_k^i}$, $\varepsilon = \varepsilon_k$, $\phi = f_k^i$. This yields a partition $\{A_{k+1}^i;\ \ i = 1,\ldots,\nu_{k+1}\}$ of $\Omega^\dagger$ and functions of the form

$$
f_{k+1} = \sum_{i=1}^{\nu_{k+1}} f_{k+1}^i \chi_{A_{k+1}^i}, \qquad\qquad \varphi_{k+1}^i(\cdot) = \langle a_{k+1}^i, \cdot\rangle + b_{k+1}^i,
$$

where each $f^i_{k+1} : \overline{A^i_{k+1}} \mapsto I\!R^n$ is a Lipschitz continuous selection from $F$, satisfying the following estimates:

$$\varphi^i_{k+1}(z) > h(z, F(t,x)) \qquad \forall (t,x) \in A^i_{k+1}, \tag{3.16}$$

$$\varphi^i_{k+1}\big(f^i_{k+1}(t,x)\big) \leq \varepsilon_{k+1} \qquad \forall (t,x) \in A^i_{k+1}, \tag{3.17}$$

$$\left| \int_\tau^{\tau'} \big[f_{k+1}(s,u(s)) - f_k(s,u(s))\big]\, ds \right| \leq \varepsilon_{k+1}, \tag{3.18}$$

$$\int_\tau^{\tau'} \big|f_{k+1}(s,u(s)) - f_k(s,u(s))\big|\, ds \leq \varepsilon_{k+1} + \varepsilon_k(\tau' - \tau), \tag{3.19}$$

for every $u \in Y$ and every $\tau, \tau'$, as long as the values $(s, u(s))$ remain inside a single set $A^i_k$, for $s \in [\tau, \tau')$.

Observe that, according to Lemma 3, each $A^i_k$ is closed-open in the finer topology generated by all $(M+1)$–cones. Therefore, each $f_k$ is $\Gamma^{M+1}$–continuous. By Theorem 2 in [2], the substitution operator $\mathcal{S}^{f_k} : u(\cdot) \mapsto f_k(\cdot, u(\cdot))$ is continuous from the set $Y$ defined at (2.5) into $\mathcal{L}^1\big([0,T];\ I\!R^n\big)$. The Picard map $\mathcal{P}^{f_k}$ is thus continuous as well.

Furthermore, there exists an integer $N_k$ with the following property. Given any $u \in Y$, there exists a finite partition of $[0,T]$ with nodes $0 = \tau_0 < \tau_1 < \cdots < \tau_{n(u)} = T$, with $n(u) \leq N_k$, such that, as $t$ ranges in any $[\tau_{\ell-1},\ \tau_\ell)$, the point $(t, u(t))$ remains inside one single set $A^i_k$. Otherwise stated, the number of times in which the curve $t \mapsto (t, u(t))$ crosses a boundary between two distinct sets $A^i_k$, $A^j_k$ is smaller that $N_k$, for every $u \in Y$. The construction of the $A^i_k$ in terms of $(M+1)$–cones implies that all these crossings are transversal. Since the restriction of $f_k$ to each $A^i_k$ is Lipschitz continuous, it is clear that every Cauchy problem

$$\dot{x}(t) = f_k(t, x(t)), \qquad x(t_0) = x_0$$

has a unique solution, depending continuously on the initial data $(t_0, x_0) \in [0,T] \times D$.

From (3.18), (3.19) and the property of $N_k$ it follows

$$\begin{aligned} \left| \int_0^t \big[f_{k+1}(s,u(s)) - f_k(s,u(s))\big]\, ds \right| &\leq \sum_{\ell=1}^{L} \left| \int_{\tau_{\ell-1}}^{\tau_\ell} \big[f_{k+1}(s,u(s)) - f_k(s,u(s))\big]\, ds \right| \\ &\leq N_k \varepsilon_{k+1}, \end{aligned} \tag{3.20}$$

where $0 = \tau_0 < \tau_1 < \cdots < \tau_L = t$ are the times at which the map $s \to (s, u(s))$ crosses a boundary between two distinct sets $A_k^i$, $A_k^j$. Since (3.20) holds for every $t \in [0, T]$, we conclude

$$\left\| \mathcal{P}^{f_{k+1}} - \mathcal{P}^{f_k} \right\| \leq N_k \varepsilon_{k+1}. \tag{3.21}$$

Similarly, for every $u \in Y$ one has

$$\begin{aligned} \left\| f_{k+1}(\cdot, u(\cdot)) - f_k(\cdot, u(\cdot)) \right\|_{\mathcal{L}^1([0,T];\, \mathbb{R}^n)} &\leq \sum_{\ell=1}^{n(u)} \int_{\tau_{\ell-1}}^{\tau_\ell} \left| f_{k+1}(s, u(s)) - f_k(s, u(s)) \right| \, ds \\ &\leq \sum_{\ell=1}^{n(u)} \left[ \varepsilon_{k+1} + \varepsilon_k (\tau_\ell - \tau_{\ell-1}) \right] \;\leq\; N_k \varepsilon_{k+1} + \varepsilon_k T. \end{aligned} \tag{3.22}$$

Now consider the functions $\varphi_k : \mathbb{R}^n \times \Omega^\dagger \to \mathbb{R}$, with

$$\varphi_k(y, t, x) \doteq \langle a_k^i, y \rangle + b_k^i \qquad \text{if} \qquad (t, x) \in A_k^i. \tag{3.23}$$

From (3.16), (3.17) it follows

$$\varphi_k(y, t, x) \geq h(y, F(t, x)) \qquad \forall (t, x) \in \Omega^\dagger, \quad y \in F(t, x), \tag{3.24}$$

$$\varphi_k\big(f_k(t, x), t, x\big) \leq \varepsilon_k \qquad \forall (t, x) \in \Omega^\dagger. \tag{3.25}$$

For every $u \in Y$, (3.18) and the linearity of $\varphi_k$ w.r.t. $y$ imply

$$\begin{aligned} &\left| \int_0^T \left[ \varphi_k\big(f_{k+1}(s, u(s)), s, u(s)\big) - \varphi_k\big(f_k(s, u(s)), s, u(s)\big) \right] ds \right| \\ &\qquad \leq \sum_{\ell=1}^{n(u)} \max\left\{ |a_k^1|, \ldots, |a_k^{\nu_k}| \right\} \cdot \left| \int_{\tau_{\ell-1}}^{\tau_\ell} \left[ f_{k+1}(s, u(s)) - f_k(s, u(s)) \right] ds \right| \\ &\qquad \leq N_k \cdot \max\left\{ |a_k^1|, \ldots, |a_k^{\nu_k}| \right\} \cdot \varepsilon_{k+1}. \end{aligned} \tag{3.26}$$

Moreover, for every $\ell \geq k$, from (3.19) it follows

$$\begin{aligned} &\int_0^T \left| \varphi_k\big(f_{\ell+1}(s, u(s)), s, u(s)\big) - \varphi_k\big(f_\ell(s, u(s)), s, u(s)\big) \right| ds \\ &\qquad \leq \max\left\{ |a_k^1|, \ldots, |a_k^{\nu_k}| \right\} \cdot \int_0^T \left| f_{\ell+1}(s, u(s)) - f_\ell(s, u(s)) \right| ds \\ &\qquad \leq \max\left\{ |a_k^1|, \ldots, |a_k^{\nu_k}| \right\} \cdot \big( N_\ell \varepsilon_{\ell+1} + \varepsilon_\ell T \big). \end{aligned} \tag{3.27}$$

Observe that all of the above estimates hold regardless of the choice of the $\varepsilon_k$. We now introduce an inductive procedure for choosing the constants $\varepsilon_k$, which will yield the convergence of the sequence $f_k$ to a function $f$ with the desired properties.

Given $f_0$ and $\varepsilon_0$, by Lemma 2 there exists $\delta_0 > 0$ such that, if $g : \Omega^\dagger \mapsto \overline{B}(0, M)$ and $\|\mathcal{P}^g - \mathcal{P}^{f_0}\| \leq \delta_0$, then, for each $(t_0, x_0) \in [0,T] \times D$, every solution of (2.7) remains $\varepsilon_0$-close to the unique solution of (1.3). We then choose $\varepsilon_1 = \delta_0/2$.

By induction on $k$, assume that the functions $f_1, \ldots, f_k$ have been constructed, together with the linear functions $\varphi_\ell^i(\cdot) = \langle a_\ell^i, \cdot \rangle + b_\ell^i$ and the integers $N_\ell$, $\ell = 1, \ldots, k$. Let the values $\delta_0, \delta_1, \ldots, \delta_k > 0$ be inductively chosen, satisfying

$$\delta_\ell \leq \frac{\delta_{\ell-1}}{2} \qquad \ell = 1, \ldots, k, \tag{3.28}$$

and such that $\|\mathcal{P}^g - \mathcal{P}^{f_\ell}\| \leq \delta_\ell$ implies that for every $(t_0, x_0) \in [0,T] \times D$ the solution set of (2.7) has diameter $\leq 2^{-\ell}$, for $\ell = 1, \ldots, k$. This is possible again because of Lemma 2. For $k \geq 1$ we then choose

$$\varepsilon_{k+1} \doteq \min \left\{ \frac{\delta_k}{2N_k}, \ \frac{2^{-k}}{N_k}, \ \frac{2^{-k}}{N_k \cdot \max\left\{|a_\ell^i|; \ 1 \leq \ell \leq k, \ 1 \leq i \leq \nu_\ell \right\}} \right\}. \tag{3.29}$$

Using (3.28), (3.29) in (3.21), with $N_0 \doteq 1$, we now obtain

$$\sum_{k=p}^{\infty} \|\mathcal{P}^{f_{k+1}} - \mathcal{P}^{f_k}\| \leq \sum_{k=p}^{\infty} N_k \cdot \frac{\delta_k}{2N_k} \leq \sum_{k=p}^{\infty} \frac{2^{p-k}\delta_p}{2} \leq \delta_p \tag{3.30}$$

for every $p \geq 0$. From (3.22) and (3.29) we further obtain

$$\sum_{k=1}^{\infty} \|f_{k+1}(\cdot, u(\cdot)) - f_k(\cdot, u(\cdot))\|_{\mathcal{L}^1} \leq \sum_{k=1}^{\infty} \left( N_k \cdot \frac{2^{-k}}{N_k} + \frac{2^{1-k}T}{N_k} \right) \leq \sum_{k=1}^{\infty} (2^{-k} + 2^{1-k}T) \leq 1 + 2T. \tag{3.31}$$

Define

$$f(t,x) \doteq \lim_{k \to \infty} f_k(t,x) \tag{3.32}$$

for all $(t,x) \in \Omega^\dagger$ at which the sequence $f_k$ converges. By (3.31), for every $u \in Y$ the sequence $f_k(\cdot, u(\cdot))$ converges in $\mathcal{L}^1([0,T]; \mathbb{R}^n)$ and a.e. on $[0,T]$. In particular, considering the constant functions $u \equiv x \in \overline{B}(D, MT)$, by Fubini's theorem we conclude that $f$ is defined a.e. on $\Omega^\dagger$. Moreover, the substitution operators $\mathcal{S}^{f_k} : u(\cdot) \mapsto f_k(\cdot, u(\cdot))$ converge to the operator $\mathcal{S}^f : u(\cdot) \mapsto f(\cdot, u(\cdot))$ uniformly on $Y$. Since each $\mathcal{S}^{f_k}$ is continuous, $\mathcal{S}^f$ is also continuous. Clearly, the Picard map $\mathcal{P}^f$ is continuous as well. By (3.30) we have

$$\|\mathcal{P}^f - \mathcal{P}^{f_k}\| \leq \sum_{k=p}^{\infty} \|\mathcal{P}^{f_{k+1}} - \mathcal{P}^{f_k}\| \leq \delta_p \qquad \forall p \geq 1.$$

Recalling the property of $\delta_p$, this implies that, for every $p$, the solution set of (2.7) has diameter $\leq 2^{-p}$. Since $p$ is arbitrary, for every $(t_0, x_0) \in [0, T] \times D$ the Cauchy problem can have at most one solution. On the other hand, the existence of such a solution is guaranteed by Schauder's theorem. The continuous dependence of this solution on the initial data $t_0, x_0$, in the norm of $\mathcal{AC}$, is now an immediate consequence of uniqueness and of the continuity of the operators $\mathcal{S}^f$, $\mathcal{P}^f$. Furthermore, for $p = 0$, (3.30) yields $\|\mathcal{P}^f - \mathcal{P}^{f_0}\| \leq \delta_0$. The choice of $\delta_0$ thus implies (1.4).

It now remains to prove (1.1). Since every set $F(t, x)$ is closed, it is clear that $f(t, x) \in F(t, x)$. For every $u \in Y$ and $k \geq 1$, by (3.24)–(3.27) the choices of $\varepsilon_k$ at (3.29) yield

$$\begin{aligned}
\int_0^T h\big(f(s,u(s)),\ F(s,u(s))\big)\ ds &\leq \int_0^T \varphi_k\big(f(s,u(s)),s,u(s)\big)\ ds \\
&\leq \int_0^T \varphi_k\big(f_k(s,u(s)),s,u(s)\big)\ ds \\
&\quad + \left| \int_0^T \big[\varphi_k\big(f_{k+1}(s,u(s)),s,u(s)\big) - \varphi_k\big(f_k(s,u(s)),s,u(s)\big)\big]\ ds \right| \\
&\quad + \sum_{\ell=k+1}^{\infty} \int_0^T \big|\varphi_k\big(f_{\ell+1}((s,u(s)),s,u(s)\big) - \varphi_k\big(f_\ell(s,u(s)),s,u(s)\big)\big|\ ds \\
&\leq 2^{1-k}T + 2^{-k} + \sum_{\ell=k+1}^{\infty} (2^{-\ell} + 2^{1-\ell}T).
\end{aligned} \tag{3.33}$$

Observing that the right hand side of (3.33) approaches zero as $k \to \infty$, we conclude that

$$\int_0^T h\big(f(t,u(t)),\ F(t,u(t))\big)\ dt = 0.$$

By (2.2), given any $u \in Y$, this implies $f(t,u(t)) \in extF(t,u(t))$ for almost every $t \in [0,T]$. By possibly redefining $f$ on a set of measure zero, this yields (1.1).

# 4 - Applications

Throughout this section we make the following assumptions.

(H) $F : [0,T] \times \Omega \mapsto \overline{B}(0,M)$ is a bounded continuous multifunction with compact values satisfying (LSP), while $D$ is a compact set such that $\overline{B}(D,\ MT) \subset \Omega$.

An immediate consequence of Theorem 1 is

**Corollary 1.** *Let the hypotheses (H) hold. Then there exists a continuous map* $(t_0, x_0) \mapsto x(\cdot, t_0, x_0)$ *from* $[0,T] \times D$ *into* $\mathcal{AC}$, *such that*

$$\begin{cases} \dot{x}(t,t_0,x_0) \in extF\big(t, x(t,t_0,x_0)\big) & \forall t \in [0,T], \\ x(t_0,t_0,x_0) = x_0 & \forall t_0, x_0. \end{cases}$$

Another consequence of Theorem 1 is the contractibility of the sets of solutions of certain differential inclusions. We recall here that a metric space $X$ is contractible if there exist a point $\tilde{u} \in X$ and a continuous mapping $\Phi : X \times [0,1] \to X$ such that:

$$\Phi(v,0) = \tilde{u}, \qquad \Phi(v,1) = v, \qquad \forall v \in X.$$

The map $\Phi$ is then called a *null homotopy* of $X$.

**Corollary 2.** *Let the assumptions (H) hold. Then, for any* $\bar{x} \in D$, *the sets* $\mathcal{M}$, $\mathcal{M}^{ext}$ *of solutions of*

$$x(0) = \bar{x}, \qquad \dot{x}(t) \in F(t, x(t)) \qquad t \in [0,T],$$

$$x(0) = \bar{x}, \qquad \dot{x} \in extF(t, x(t)) \qquad t \in [0,T],$$

*are both contractible in* $\mathcal{AC}$.

**Proof.** Let $f$ be a selection from $extF$ with the properties stated in Theorem 1. As usual, we denote by $x(\cdot, t_0, x_0)$ the unique solution of the Cauchy problem (1.2). Define the null homotopy $\Phi : \mathcal{M} \times [0,1] \to \mathcal{M}$ by setting

$$\Phi(v,\lambda)(t) \doteq \begin{cases} v(t) & \text{if} \quad t \in [0, \lambda T], \\ x(t, \lambda T, v(\lambda T)) & \text{if} \quad t \in [\lambda T, T]. \end{cases}$$

By Theorem 1, $\Phi$ is continuous. Moreover, setting $\tilde{u}(\cdot) \doteq u(\cdot, 0, \bar{x})$, we obtain

$$\Phi(v,0) = \tilde{u}, \qquad \Phi(v,1) = v, \qquad \Phi(v,\lambda) \in \mathcal{M} \qquad \forall v \in \mathcal{M},$$

proving that $\mathcal{M}$ is contractible. We now observe that, if $v \in \mathcal{M}^{ext}$, then $\Phi(v, \lambda) \in \mathcal{M}^{ext}$ for every $\lambda$. Therefore, $\mathcal{M}^{ext}$ is contractible as well.

Our last application is concerned with feedback controls. Let $\Omega \subseteq \mathbb{R}^n$ be open, $U \subset \mathbb{R}^m$ compact, and let $g : [0,T] \times \Omega \times U \to \mathbb{R}^n$ be a continuous function. By a well known theorem of Filippov [8], the solutions of the control system

$$\dot{x} = g(t, x, u), \qquad u \in U, \tag{4.1}$$

correspond to the trajectories of the differential inclusion

$$\dot{x} \in F(t,x) \doteq \{g(t,x,\omega) \quad \omega \in U\}. \tag{4.2}$$

In connection with (4.1), one can consider the "relaxed" system

$$\dot{x} = g^{\#}(t, x, u^{\#}), \qquad u^{\#} \in U^{\#}, \tag{4.3}$$

whose trajectories are precisely those of the differential inclusion

$$\dot{x} \in F^{\#}(t,x) \doteq \overline{co}F(t,x).$$

The control system (4.3) is obtained defining the compact set

$$U^{\#} \doteq U \times \cdots \times U \times \Delta_n = U^{n+1} \times \Delta_n,$$

where

$$\Delta_n \doteq \left\{ \theta = (\theta_0, \ldots, \theta_n) \ ; \quad \sum_{i=0}^{n} \theta_i = 1, \quad \theta_i \geq 0 \quad \forall i \right\}$$

is the standard simplex in $\mathbb{R}^{n+1}$, and setting

$$g^{\#}(t,x,u^{\#}) = g^{\#}(t,x,(u_0,\ldots,u_n,(\theta_0,\ldots,\theta_n))) \doteq \sum_{i=0}^{n} \theta_i f(t,x,u_i).$$

Generalized controls of the form $u^{\#} = (u_0, \ldots, u_n, \theta)$ taking values in the set $U^{n+1} \times \Delta_n$ are called *chattering controls.*

**Corollary 3.** *Consider the control system (4.1), with $g : [0,T] \times \Omega \times U \mapsto \overline{B}(0,M)$ Lipschitz continuous. Let $D$ be a compact set with $\overline{B}(D;\ MT) \subset \Omega$. Let $u^{\#}(t,x) \in U^{\#}$ be a chattering feedback control such that the mapping*

$$(t,x) \mapsto g^{\#}(t,x,u^{\#}(t,x)) \doteq f_0(t,x)$$

*is Lipschitz continuous.*

*Then, for every* $\varepsilon_0 > 0$ *there exists a measurable feedback control* $\bar{u} = \bar{u}(t,x)$ *with the following properties:*

*(a) For every* $(t,x)$*, one has* $g(t,x,\bar{u}(t,x)) \in extF(t,x)$*, with* $F$ *as in (4.2).*

*(b) for every* $(t_0,x_0) \in [0,T] \times D$*, the Cauchy problem*

$$\dot{x}(t) = g\big(t, x(t), \bar{u}(t,x(t))\big), \qquad x(t_0) = x_0$$

*has a unique solution* $x(\cdot, t_0, x_0)$*,*

*(c) if* $y(\cdot, t_0, x_0)$ *denotes the (unique) solution of the Cauchy problem*

$$\dot{y} = f_0(t, y(t)), \qquad y(t_0) = x_0,$$

*then for every* $(t_0, x_0)$ *one has*

$$\big|x(t,t_0,x_0) - y(t,t_0,x_0)\big| < \varepsilon_0, \qquad \forall t \in [0,T].$$

**Proof.** The Lipschitz continuity of $g$ implies that the multifunction $F$ in (4.2) is Lipschitz continuous in the Hausdorff metric, hence it satisfies (LSP). We can thus apply Theorem 1, and obtain a suitable selection $f$ of $extF$, in connection with $f_0$, $\varepsilon_0$. For every $(t,x)$, the set

$$W(t,x) \doteq \big\{\omega \in U\ ;\quad g(t,x,\omega) = f(t,x)\big\} \subset I\!R^m$$

is a compact nonempty subset of $U$. Let $\bar{u}(t,x) \in W(t,x)$ be the lexicographic selection. Then the feedback control $\bar{u}$ is measurable, and it is trivial to check that $\bar{u}$ satisfies all required properties.

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