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COVARIANT AND GAUGE - INVARIANT DESCRIPTION OF COSMOLOGICAL DENSITY INHOMOGENEITIES

Thesis submitted for the degree of
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Thesis presented by

Marco Bruni

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We asked a taxi driver in Rome how
should we spell Great Attractor in Italian
and he replied: "You mean Grande Attrattore?
Like Sofia Loren?"

Avishai Dekel (1988) [6]

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INTRODUCTION

A major problem in present-day cosmology is that of understanding the formation of the structures seen at each length-scale on which the universe is observed. It is a common belief that the main role in this formation is played by gravity, and the most widely accepted theory of gravitation is general relativity. On the other hand, the universe shows a great degree of isotropy in the Cosmic Microwave Background. In the framework of Big Bang cosmologies, this is interpreted as the remnant of an earlier hot and very homogeneous stage in the history of our universe, supporting the idea that Friedmann-Lemaître-Robertson-Walker (FLRW from now on) models give a reasonably good description of it. The picture which we have of the formation of the structures, is then that they originated from small fluctuations in an almost-FLRW universe. Therefore, a basic step towards the explanation of such processes is the formulation of a relativistic theory of perturbations of FLRW models:

> "This is a key part of the story, since the rules which came out of this study define the basic problems of galaxy formation."
>
> M. S. Longair [36]

This task was fulfilled by Lipshits in 1946 [34], and in some respects this is the end of the story. Indeed, as Lipshits himself pointed out [34,35], his approach suffers from gauge problems, and much of the succeeding literature has been devoted either to working-out results in one gauge or another, or to trying to eliminate these problems. In the usual approach, one starts with an exact isotropic and homogeneous FLRW space-time $\bar{S}$, and then perturbs it to obtain a physical universe S. Then the perturbation in each quantity is the difference between the value which it has at given point in the physical space-time S and the value at the *corresponding point* in the background $\bar{S}$; by considering all points, the perturbation field is determined. For example, the energy density perturbation is

$$\delta\mu \equiv \mu - \bar{\mu} \ .$$

However, what we can observe is μ, as well as other quantities in the real universe S, and *not* $\bar{\mu}$ (or other quantities in $\bar{S}$). It follows that $\delta\mu$ is ill-defined, because there are many ways of splitting μ into a background part $\bar{\mu}$ and a perturbation $\delta\mu$. We cannot recover $\bar{S}$ by observing S, unless some fitting procedure is specified [15,16]. In other words, the correspondence between $\bar{S}$ and S is not uniquely defined, because it can be changed by a *gauge-transformation*:

> "a gauge transformation ... changes the point in the background space-time corresponding to a point in the physical space-time. Thus even if a quantity is a scalar under coordinate transformations, the value of the *perturbation* in the quantity will *not* be invariant under gauge transformations if the quantity is non-zero and position dependent in the background."

Consequently,

> "if the gauge condition imposed to simplify the metric leaves a residual gauge freedom, the perturbation equations will have spurious gauge mode solutions which can be completely annulled by a gauge transformation and have no physical reality".
>
> J. M. Bardeen [1]

The resulting problem is that the quantity $\delta\mu$, calculated in perturbation calculations, is completely dependent on the gauge chosen, and unless this gauge is fully specified the modes found for this quantity maybe spurious (due to residual gauge freedom); while if it is fully specified, its relation to what we really want to know (the spatial variation of density in the universe) is complicated and difficult to interpret.

A successful attempt to avoid this kind of problem was made by Bardeen in 1980. Indeed, he introduced a set of gauge-invariant quantities describing the perturbation in the matter and the geometry and determined evolution equations for them [1]. Bardeen's formalism is a theory of some complexity, which can now be regarded as the standard theory of density perturbations. However, Bardeen's variables are related to density perturbations, but are not those perturbations themselves (they include metric tensor Fourier components and other quantities in cunning combinations). Thus they do not have a simple physical interpretation, *unless some gauge is specified.* Indeed:

> "What the gauge-invariant formalism does do, is take advantage of the homogeneity of the background space-time to make all quantities explicitly independent of the choice of spatial coordinates. Any spatially gauge-invariant quantity can be made formally gauge-invariant under changes in the hypersurfaces conditions, *but its physical interpretation will refer to one particular hypersurface condition* unless it is a perturbation in something which is zero or time-independent in the background."
>
> J. M. Bardeen [1]

The aim of this thesis is to present a simple alternative approach to cosmological inhomogeneities in almost-FLRW universes. This approach, introduced in two recent papers (Ellis and Bruni [17]; Ellis, Hwang and Bruni [19]; respectively EB and EHB from now on), is both fully covariant and gauge invariant; thus it avoids the usual problems. The key step for us is contained in the last sentence of the above quotation: the variables which we introduce *are not* perturbations in some non-zero quantities in the background. Instead we use a set of *spatial gradients in the physical universe* to describe inhomogeneity, the most significant being the *comoving fractional density gradient* $\mathcal{D}_a$. Thus

[1] Unpublished. I took this quotation from a lecture by Efstathiou at an SUSSP Summer School ("The Early Universe", Edinbourgh (1989)). He cited it while arguing against the use of Bardeen's formalism, since he is a supporter of the synchronous-gauge.

these variables are not *defined* by comparison with the background, as it is for $\delta\mu$, and they have not an unperturbed part. Since these quantities are zero in the background (an exactly spatially homogeneous and isotropic FLRW), they are gauge - invariant both in their definitions and in their physical interpretation. We do not need any assumption of "smallness" in defining our variables: actually they represent spatial variations in a universe which may be as inhomogeneous as we like. Therefore, in the framework of the covariant fluid approach to cosmology [11], we derive exact fully non - linear equations for our variables. When linearized, these equations represent the evolution of these quantities in an almost - FLRW universe. Thus we proceed in a direction opposite to that of the usual approach, since we directly obtain equations in the physical almost - FLRW universe, while the background, an exact FLRW, appears as the zero - order solutions to these equations. We believe that this is conceptually more satisfactory, and we hope that it will make it possible to develop a higher order gauge - invariant analysis of density inhomogeneities. We obtain standard results, equivalent to those of Bardeen [29], but we believe that our derivation is simpler and that its physical interpretation is more transparent.

Chapter 1 is mainly devoted to the discussion of the gauge problem which arises in the usual approach to cosmological density perturbations. Following a very brief sketch of the latter, section 1.1 defines what a gauge transformation is and shows how it changes a generic tensorial quantity; also, definitions of gauge invariance and gauge freedom are given. Section 1.2 illustrates a mapping view of the gauge problem, discussing the various features of the latter. Next the possible solutions to the gauge problem are considered, i.e., the choice of a particular gauge and the definition of gauge - invariant variables.

In the first section of chapter 2 the covariant fluid approach to cosmology is briefly reviewed. In such an approximation the matter content of the universe is described as a continuous fluid; this description of matter is complementary to the particle distribution function representation [9], and we can regard the

4 - velocity of the fluid element as the average velocity of the particles in that volume.

Section 2.2 deals with the definition of gauge - invariant quantities which fit naturally with the covariant fluid description adopted here and which describe the inhomogeneity of the real universe S. One of these variables is the density gradient X_a: this has been considered previously by Hawking (1966) [26] and Olson (1976) [43], but they did not notice that the density fluctuation problem in an almost - FLRW can be treated in terms of this variable only. A most significant quantity for describing the time evolution of density fluctuations (see section 2.2.2) is the *comoving fractional density gradient* $\mathcal{D}_a$ introduced in EB [17], because it incorporates the relative evolution of density between neighbouring comoving volumes of fluid. Thus we identify $\mathcal{D}_a$ as the new variable in terms of which the perturbation problem in cosmology can be treated in a simple and gauge - invariant way. Other relevant quantities which we introduce are the expansion gradient Z_a and the *comoving expansion gradient* $\mathcal{Z}_a$. The physical interpretation of all of our variables is straightforward and gauge - independent.

The aim of chapter 3 is to derive a linear system of two first - order equations from which $\mathcal{D}_a$ can be determined in the perfect fluid assumption. However, we shall see that for a general space - time S the various density gradients which we have introduced ($\mathcal{D}_a$ and related quantities) are coupled with the expansion gradients Z_a or $\mathcal{Z}_a$, as well as with other fluid - variables, through a set of exact non - linear first - order equations.

Section ?? considers a general space - time S, and well - known [26,11] exact non - linear equations for the various standard variables of the fluid flow approximation are presented. They are not directly relevant for our discussion of inhomogeneity in an almost - FLRW universe, but they are introduced in order to give a complete picture of the general non - linear case. In section 3.1.3, we derive the exact equations of motion (presented in EB [17]) for our set of gauge - invariant variables.

In section 3.2 we consider an almost - FLRW universe. By this we mean a universe model in which all of the gauge - invariant quantities are first -

order with respect to the energy density μ, the pressure p and the expansion Θ ("zero - order" quantities, i.e., non vanishing in the background FLRW). We linearize the equations previously derived, obtaining pairs of two coupled first - order linear equations. In particular we derive a pair of equations coupling $\mathcal{D}_a$ with $\mathcal{Z}_a$ (EB [17]). Also, an evolution equation for the *comoving curvature gradient* $\mathcal{C}_a$ is derived in the linear approximation (EHB [19]). An equivalent system of equations coupling $\mathcal{D}_a$ with $\mathcal{C}_a$ is then obtained.

Section 3.3.1 deals with the mapping which we have chosen from the background exact FLRW model (S) to the realistic almost - FLRW universe $(\bar{S})$, and section 3.3.2 discusses a technical point arising in the case of rotating universe models.

In chapter 4 we apply the above mentioned linear equations to some specific cases. In particular we restrict our analysis to adiabatic (isentropic) perturbations. We show explicitly how the linear systems of two first - order equations derived in section 3.2 decouple from the other evolution equations, by virtue of the assumed equation of state. We consider systems for $\mathcal{D}_a$ and $\mathcal{Z}_a$, as well as for $\mathcal{D}_a$ and $\mathcal{C}_a$. Also, we introduce a "Bardeen - like" variable Φ_a, and derive a linear equation for it (section 4.2.1).

In section 4.3 we derive second - order linear equations for $\mathcal{D}_a$ and Φ_a and analyse some of their properties (EHB [19]). We consider a Jeans instability for matter against gravity, giving a correction to a previous result by Jackson [30]. A first integral is obtained in the long - wavelength limit, corresponding to an analogous well - known [2,38] first integral. Also, we comment on the isocurvature and scalar modes.

The aim of section 4.4 is to obtain explicit solutions of the second - order linear equations. We consider the zero - pressure fluid (dust), and the pure radiation case, for which explicit solutions are obtained in the long - wavelength limit. Since our equations and variables are gauge - invariant, we do not obtain the usual decaying gauge - mode. Also, we briefly consider the case of a mixture of perfect fluids when those perfect fluids all have the same 4 - velocity.

Finally, section 4.5 considers the evolution of density on neighbouring world-lines. It is shown how we can define scalar quantities closely related to our gradients, and we derive exact and linear evolution equations for them.

Chapter 1

THE GAUGE PROBLEM

Cosmological observations are very intriguing: on one hand, looking at galaxy distribution, we can say that

> "large structures are a common feature of all surveys large enough to contain them."
>
> M. J. Geller and J. P. Huchra [24]

On the other hand, we observe a Cosmic Microwave Background (CMB) which is

> "within the limits of our observations, isotropic, unpolarized, and free from seasonal variation."
>
> A. A. Penzias and R. W. Wilson [45]

In the framework of Big Bang cosmologies, CMB is interpreted as the remnant of an earlier hot and very homogeneous stage in the history of our universe, supporting the idea that FLRW models give a reasonably good description of it. More precisely, the high degree of isotropy in the CMB puts severe limits on the inhomogeneity of the matter distribution at early epochs, therefore it is a common belief that the structures which we see originated through the action of gravity from small inhomogeneities in an almost - FLRW universe. A basic step towards the understanding of such processes is then the formulation of a relativistic theory of linear perturbations of the expanding, isotropic and homogeneous Friedmann - Lemaître - Robertson - Walker models (FLRW from now on).

Such a theory

> "springs into existence virtually full-grown with the work of Lifshits (1946)."
>
> W. H. Press and E. T. Vishniac [46]

Actually, Lifshits [34] was interested in the dynamical stability of FLRW models with respect to perturbations. The theory which he developed has become standard and is presented in many text-books on cosmology (see for example Börner [4], and Peebles [44], and Weinberg [58]). However, as we shall see, such a theory suffers from gauge problems, as Lifshits himself [34,35] and other authors [48] have pointed out. The usual approach is then to fix a particular gauge, and work within that; however, gauge ambiguities can remain citebi:previ.

Another, different approach to the problem was pioneered by Hawking in 1966 [26]. His approach is fully covariant, but nevertheless gauge-affected [11,43].

Finally, one can formulate a gauge-invariant theory of cosmological perturbations, avoiding the gauge problems. This has been accomplished by Bardeen in 1980 [1], who introduced a set of gauge-invariant quantities describing perturbations in the matter and in the geometry. As we have mentioned in the introduction, the problem with Bardeen's formalism is that the physical interpretation of his variables depends on the choice of one particular hypersurface condition, i.e. Bardeen's variables acquire a clear physical meaning only in some particular gauge.

In this chapter, we shall first very briefly introduce the usual approach to perturbations in cosmology (section 1.1). A discussion of the gauge problem that arises in this approach follows in section 1.2.

FLRW models are standard in cosmology, and we refer to standard text-books [4,58] in the subject for an introduction to them. A characterization of FLRW models in the covariant fluid approach adopted in this thesis is given in section 2.2.1.

1.1 Some Features of the Usual Approach to Perturbations

We outline here only those features of the usual approach to perturbations in cosmology which are relevant for the subsequent discussion of the gauge problem. See the above quoted references for a detailed exposition.

We consider an idealised universe model $\bar{S}$ (usually taken to be a FLRW universe). Each quantity in this model will be indicated with an overbar, e.g. the energy density will be denoted by $\bar{\mu}$ and the pressure by $\bar{p}$. We perturb this model to obtain a "realistic" or "lumpy" universe S, where the physical quantities will be denoted by the same symbols as in $\bar{S}$ but without overbars (e.g. the energy density is μ and the pressure is p). The perturbation in each quantity is then the difference between the value which it has at a given point in the physical space-time S and the value at *the corresponding point* in the background $\bar{S}$. Considering all points, the perturbation field is determined. For example, the metric perturbation is

$$\delta g_{ab} = g_{ab} - \bar{g}_{ab} \; , \tag{1.1}$$

while for the perturbation in the energy momentum tensor we have

$$\delta T_{ab} = T_{ab} - \bar{T}_{ab} \; . \tag{1.2}$$

Two assumptions are implicit in writing the above equations: one is obvious, while the other is obscure (see the discussion in the next section). The first is that the unperturbed metric is a solution of the Einstein equations with the unperturbed energy momentum tensor as the source term (we shall call this the "zero-order" solution). The second is that the perturbations are "small". Following these two assumptions one substitutes g_{ab} and T_{ab} in the Einstein equations, subtracts the zero-order solution, neglects higher-order terms, and obtains *linear* equations for the metric perturbation (1.1) with the energy momentum perturbation as the source term (1.2). Also, one carries out the same procedure with the energy momentum conservation equations to obtain equations of motion for the matter.

1.1.1 Gauge Transformations, Gauge Invariance, Gauge Freedom

The procedure outlined above suffers from *gauge problems.* These follow from the *gauge transformations*, i.e. , infinitesimal coordinate transformations such that

$$\bar{x}^a \;\rightarrow\; x^a = \bar{x}^a - \varepsilon^a(x) \;, \tag{1.3}$$

where $\varepsilon^a(x)$ is an arbitrary infinitesimal vector field. This induces a change in any tensor $\mathcal{T}$ such that, *at the same coordinate point*

$$\mathcal{T}(x) = \bar{\mathcal{T}}(x) + L_\varepsilon \bar{\mathcal{T}}(x) \;, \tag{1.4}$$

where $L_\varepsilon \bar{\mathcal{T}}$ is the Lie derivative of $\mathcal{T}$ along ε [50]. For scalars and vectors we have[1]

$$L_\varepsilon f \equiv f_{;a}\varepsilon^a \;, \tag{1.5}$$

$$L_\varepsilon V_a \equiv V^b \varepsilon_{b;a} + V_{a;b}\varepsilon^b \;, \tag{1.6}$$

and analogous expressions hold for tensors of any rank (cf. Weinberg [58]). It is straightforward to verify that $\delta g_{ag} = L_\varepsilon \bar{g}_{ab}$ is a solution of the linearized Einstein equations with source term $\delta T_{ab} = L_\varepsilon \bar{T}_{ab}$, therefore, *because of the linearity* of the equations, we can always find other solutions of the form $\delta g_{ab} + L_\varepsilon \bar{g}_{ab}$ for any given solution δg_{ab}. Thus the linearized Einstein equations are said to be *gauge - invariant* with respect to the transformation (1.3), and the freedom which we have in choosing coordinates is said to be the *gauge freedom.* Again quoting Bardeen [1], it clearly follows from (1.4) and (1.5) that

> " even if a quantity is a scalar under coordinate transformations, the value of the *perturbation* in the quantity will *not* be invariant under gauge transformations if the quantity is non - zero and position dependent in the background."

[1] We omit the overbar here, since these *definitions* of Lie derivatives have nothing to do with the background. However, in the present context, it is clear from (1.4) that we are considering Lie derivatives of the background quantities.

It also follows from (1.4) and (1.6) that the perturbation in a vectorial quantity is gauge-invariant only if the unperturbed quantity vanishes in the background. In the usual approach, the gauge freedom is then used to *fix the gauge*, i.e., one can choose a particular form for ε^a to impose some restriction, usually on the metric perturbations. For example, ε^a can be chosen so that

$$\delta g_{\mu 0} = \delta g_{00} = 0 \ , \tag{1.7}$$

fixing the synchronous gauge. However, problems still remain.

1.2 The Gauge Problem

It is very easy to be misled by the "obvious" way of investigating density perturbations: following the procedure sketched above the energy density perturbation is

$$\delta\mu \equiv \mu - \bar{\mu}. \tag{1.8}$$

However this approach obscures the real situation. It suggests that there is something very special about the way the original model $\bar{S}$ is related to the lumpy model, whereas in reality this is not so. Suppose we consider the lumpy universe model S, not knowing how the model $\bar{S}$ was used to make the construction; can we uniquely recover $\bar{S}$ from S ? Without further restriction, the answer is No; for without a specific prescription for approximating the lumpy model by the smooth one, the quantities in the background model $\bar{S}$ are not uniquely determined from the lumpy model S (in equation (1.1), the only restriction relating the two models is that δg_{ab} is "small" in some suitable sense; it is far from obvious how one can extract $\bar{g}_{ab}$ from g_{ab} in a unique way). In fact the definition of the background model in S is equivalent to defining a map Φ from $\bar{S}$ to S, mapping the density in $\bar{S}$ into a background density $\bar{\mu}$ in S (for notational convenience, we use the same symbol for quantities in $\bar{S}$ and their images in S, e.g. the image $\Phi(\bar{\mu})$ in S of $\bar{\mu}$ in $\bar{S}$ is simply denoted by $\bar{\mu}$). The perturbations defined are completely dependent on how that map is chosen (Figure 1.1). This is the gauge freedom in defining the perturbation.

As delineated in section 1.1.1, the situation is usually expressed in terms

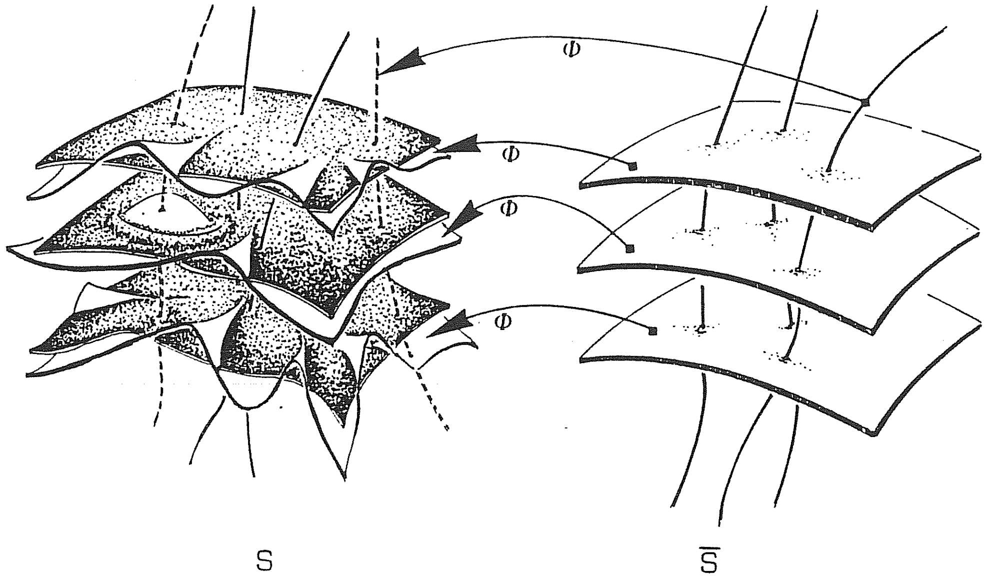

Figure 1.1: The perturbed density $\delta\mu$ is defined by a mapping Φ of an idealised world model $\bar{S}$ into a more accurate world model S; for Φ maps surfaces $\{\bar{\mu} = const\}$ from $\bar{S}$ into S, where they can be compared with the actual surfaces $\{\mu = const\}$.

of the coordinate choice in S, it being understood that the coordinates in S correspond to coordinates chosen in $\bar{S}$, so that a choice of coordinates determines a map from $\bar{S}$ into S; thus the gauge freedom is represented as a freedom of coordinate choice in S (see equation 1.3). However, we want here to specifically consider the map Φ from $\bar{S}$ into S, noting that we have coordinate freedom both in $\bar{S}$ and in S which we can usefully adapt to the chosen map Φ.

Thus the actual situation is that what we are given to study is the real (lumpy) universe S (this is all we can measure), and we define the perturbed quantities and their evolution by the way we specify a mapping Φ of the (fictitious) idealised space-time $\bar{S}$ into S. The determination of the best way to make this correspondence can be called the "Fitting problem" for cosmology [15,16]; there are various ways to do this, so the answer is not unique. Once we completely specify the map Φ, there is no arbitrariness in $\delta\mu$; insofar as Φ is unspecified, that quantity is arbitrary.

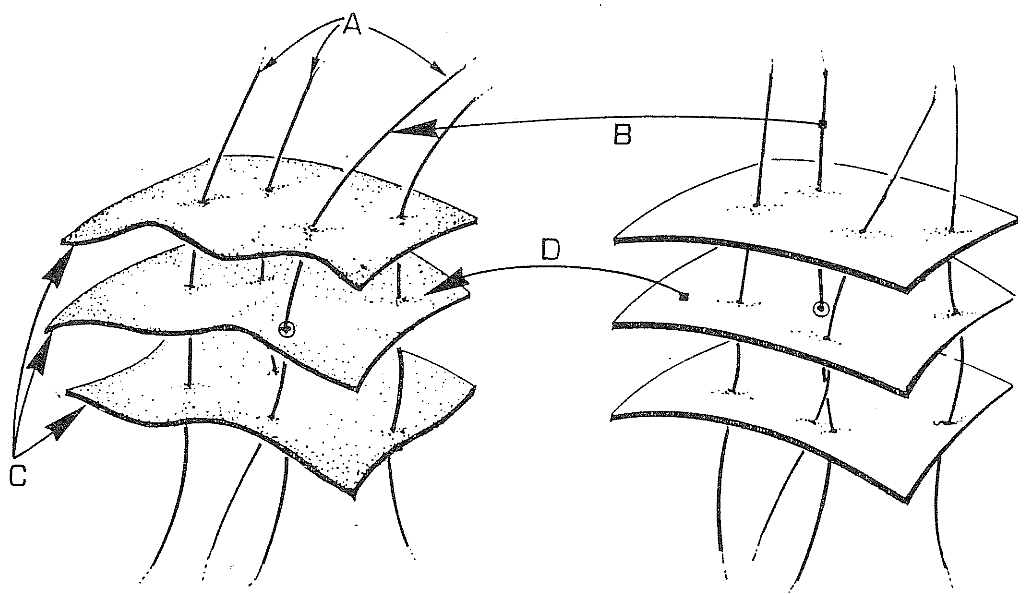

Figure 1.2: The map Φ has four aspects: (A) choice of a family of time lines in each space-time; (B) choice of a particular correspondence of time lines in the family in $\bar{S}$ to particular time lines in the family in S; (C) choice of a family of spacelike surfaces in each space-time; (D) choice of a particular correspondence of surfaces from the family in $\bar{S}$ to surfaces in the family in S.

1.2.1 Gauge Specification

It is convenient to think of this map as having four aspects (Figure 1.2):

(A) We define a family of world lines $\bar{\gamma}$ in $\bar{S}$ and a corresponding family of world lines $\bar{\gamma}$ in S. This determines the world lines in each space-time along which we will compare the evolution of density fluctuations. There is an obvious choice in $\bar{S}$, namely the fundamental flow lines; this will often be the best choice in S also, but others (e.g. normals to a chosen set of surfaces) may be convenient.

(B) We define a specific correspondence between individual world lines $\bar{\gamma}_i$ in $\bar{S}$ and individual world lines $\bar{\gamma}_i$ in S. This specifies which specific observer's observations we shall compare with which. In the case where $\bar{S}$ is an FLRW universe, this choice does not matter because of the spatial homogeneity of those models.

(C) We define a family of spacelike surfaces $\bar{\Sigma}$ in $\bar{S}$ and a corresponding family $\bar{\Sigma}$ in S; these are the "time surfaces" in each space-time. There is an obvious choice in $\bar{S}$, namely the surfaces of homogeneity $\{\bar{t} = const\}$; this means the image of these surfaces in S (that is, the surfaces $\{\bar{t} = const\}$ in S) *are* the idealised surfaces of constant density $\{\bar{\mu} = const\}$ we use to define the density perturbations. There is a variety of choice for the surfaces $\bar{\Sigma}$ in S, as discussed in depth by Bardeen [1].

(D) We define a correspondence between particular surfaces $\bar{\Sigma}_i$ in the family $\bar{\Sigma}$ in $\bar{S}$ and particular surfaces $\bar{\Sigma}_i$ in the family $\bar{\Sigma}$ in S, and so assign particular time values $\bar{t}$ to each event q in S. This is crucial: this specifies which specific point q in S corresponds to a point $\bar{q}$ in $\bar{S}$, and completes the specification of the map Φ. In particular, the time evolution of a density perturbation $\delta\mu$ is now defined, because this choice, by assigning particular values $\bar{\mu}$ to each surface $\bar{\Sigma}_i$ in S (the "unperturbed value" of the density) defines $\delta\mu$ via equation (1.8).

If we follow the normal convention, we understand (C) to *define* the coordinate surfaces $\{t = const\}$ in S (taking them as the same as the surfaces $\{\bar{t} = const\}$); and (D) to assign particular values to t at each event q in S by this map: $t_q = \bar{t}_q$. However this choice is not forced on us. Note that in general neither t nor $\bar{t}$ will measure proper time along the world lines in S.

1.2.2 The Arbitrariness of $\delta\mu$

The problem is that the definition of $\delta\mu$ depends both on the choice of the surfaces $\bar{\Sigma}$ in S and on the allocation of density values to these surfaces. We can for example choose $t = \bar{t}$ and then set the dependence of $\delta\mu$ on the spatial coordinates to zero through the gauge freedom (C), by choosing the surfaces $\bar{\Sigma}$ as surfaces of constant density μ in S; because these surfaces are regarded as surfaces of constant reference density, we will then have $\delta\mu$ constant on these surfaces (they will be spacelike if the universe S is sufficiently like a

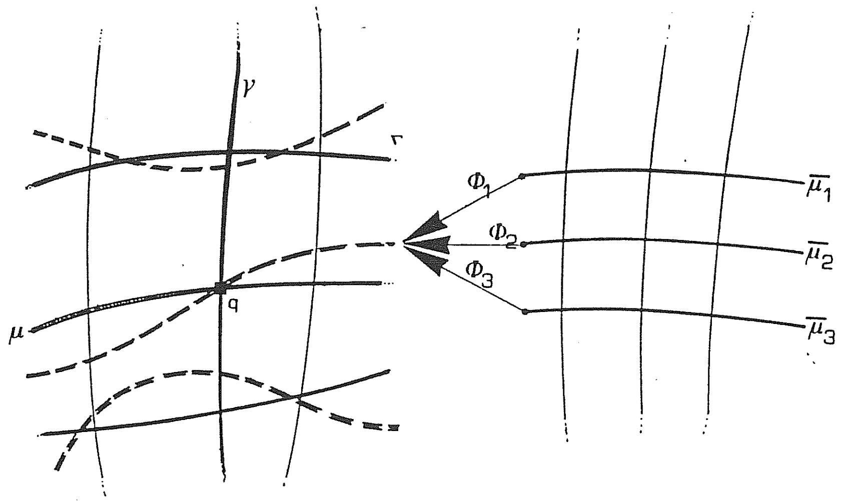

Figure 1.3: By varying the assignation (D) of particular surfaces in $\bar{S}$ to surfaces in S, we can give the density perturbation $\delta\mu = \mu - \bar{\mu}$ at the event q in S (where the world line γ intersects the surface $\{\bar{\mu} = const\}$) any value we like.

FLRW universe), and as they are also surfaces of constant $\bar{t}$, we will find $\delta\mu = \delta\mu(t)$. In many ways this is an obvious choice for the time surfaces (the constant density surfaces are covariantly defined in S, and correspond precisely to the surfaces of homogeneity in the idealised model $\bar{S}$, which are also surfaces of constant density).

Furthermore, given a choice of the family of surfaces $\bar{\Sigma}$ in S, we can still assign any value we like to $\delta\mu$ at a particular event through the gauge freedom (D), by changing the assignation of values $\bar{\mu}$ to the surfaces $\bar{\Sigma}$. Thus in particular, given any choice whatever of the time surfaces, we can set $\delta\mu$ to zero at an event q at $t = t_0$ on any world line γ, by choosing $\bar{\mu}_q = \mu_q$; this is a possible assignation of a values of the "ideal" density $\bar{\mu}$ to the event q where $t = t_0$ intersects γ (Figure 1.3).

How this propagates along the chosen time lines then depends on the gauge choice and the fluid equation of state. We can choose a gauge where $\delta\mu$ vanishes at every point of γ by assigning the mapping of densities to

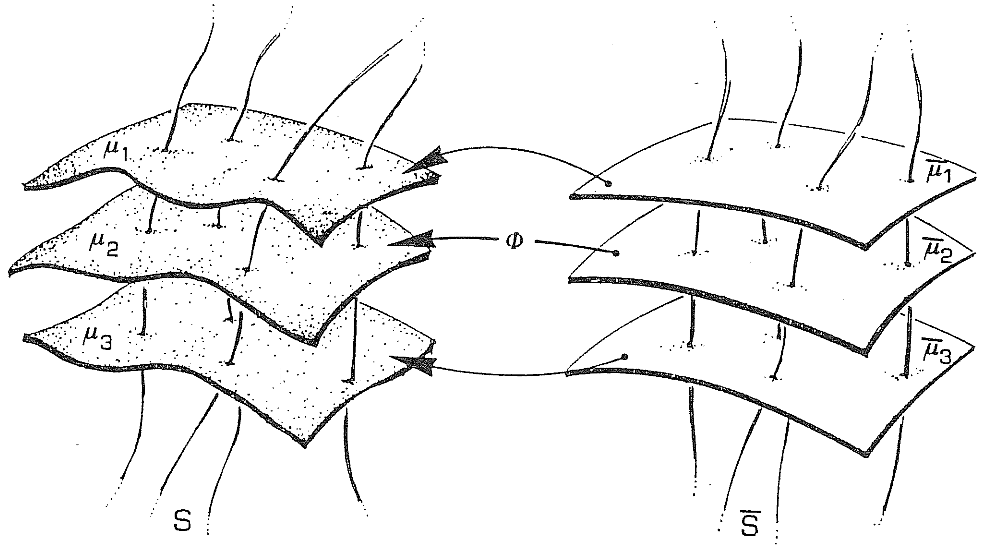

Figure 1.4: By choosing Φ so that the surfaces $\{\bar{\mu} = const\}$ in S are the same as the surfaces $\{\mu = const\}$, and then choosing the correspondence (D) to assign the same numerical values to $\bar{\mu}$ on each surface as μ has on it, we obtain a *zero density-perturbation* gauge. Note that the proper time τ between any two of these surfaces in S will vary spatially, in general; the physical density variation is coded in this spatial variation of $dt/d\tau$.

satisfy the condition $\mu(t) = \bar{\mu}(t)$ on γ. This choice is obtained in Bardeen's formalism [1] by choosing the arbitrary function $T(\tau)$ (his notation, see his equation (3.1)) to be given (in terms of his variables) by

$$T = -\frac{\delta}{3(1+w)(\dot{S}/S)}$$

on γ, where the right hand side will only depend on the conformal time τ along any chosen world-line γ. Then his equation (3.7) shows $\tilde{\delta} = 0$, i.e. the energy density perturbation vanishes along γ in the new gauge.

If we combine these two choices, we will have chosen a gauge where $\delta\mu = 0$ identically; we map the FLRW model into the lumpy universes by mapping surfaces of constant density $\bar{\mu}$ into surfaces of constant density μ with the same numerical values (Figure 1.4).

We might call this the *zero density-perturbation gauge.* This possibility will not of course mean that there are no spatial variations of density; in this gauge, inhomogeneities will be represented by the fact that the proper time separating a surface of coordinate time $\bar{t}_1$ from a surface of coordinate time $\bar{t}_2$, measured along the normals to these surfaces, varies spatially (corresponding to the normals to these surfaces being non-geodesic).

The basic problem, then, is this arbitrariness in definition of $\delta\mu$, because $\delta\mu$ (a) is not gauge invariant: it can be assigned any value we like at any event by appropriate gauge choice; and (b) is not observable even in principle, unless the gauge is fully specified by an *observationally based* procedure (as otherwise $\bar{\mu}$ is not an observable quantity).

As a result, if we are to use $\delta\mu$ in a satisfactory way to describe density perturbations, we must either leave some gauge freedom, and keep full track of the consequences of *all* this freedom; or find a satisfactory, unique way of making the gauge choices (A)-(D) discussed above. The alternative is to look for gauge-invariant quantities that code the information we want.

1.2.3 Fixing a Gauge

One way of approaching the problem is to choose a satisfactory specific gauge (specifying completely (A)-(D) above). We mention four possibilities [2].

In each case we choose the corresponding world lines in S and $\bar{S}$ to be the fundamental flow lines. The issue then is the choice of time surfaces, and then a specific correspondence between these surfaces.

Proper Time Gauges

One possibility is to define clearly equivalent proper times in the two models, and use this to completely specify both time functions and so fix the gauge. The obvious choice (cf. Olson [43]) is to choose proper time along the fluid flow lines from the big bang in both models. This is conceptually a clean solution to the problem, provided we can start at the big bang and follow the

---

[2] We omit two of Bardeen's option [1]: (a) We do not consider surfaces of simultaneity determined by radar, because such surfaces in $\bar{S}$ do not coincide with the surfaces $\{\bar{t} = const\}$ there [14]. (b) We do not consider zero shear surfaces because they are not invariantly defined, and cannot exist in most space-times; furthermore in general such surfaces in $\bar{S}$ do not correspond to the surfaces $\{\bar{t} = const\}$ [40]

evolution of each model from then on.

The problem, as pointed out by Bardeen [1], who refers to this as a *synchronous gauge*, is that the definition is non-local. If we observe the universe *today*, this proposal means we cannot define $\delta\mu$ directly from these observations but have to do so by integrating the field equations all the way back to the big bang and then deducing from this integration what $\delta\mu$ is today. Apart from issues of practicality, this is clearly an unsatisfactory procedure.

Flow - Orthogonal Hypersurfaces

A second possibility is to choose the surfaces of constant time as surfaces orthogonal to the fluid flow. However this choice (called *comoving hypersurfaces* by Bardeen) is only possible if the fluid vorticity is zero, so it is not a generic strategy. Furthermore it is not clear how to assign specific values of time or density uniquely to these surfaces (unless the acceleration is zero, proper time measured along one flow line from the big bang to a given surface will be different from that time measured along another world line).

Equivalent Scalars

A third possibility is to identify equivalent scalars in S and $\bar{S}$, that define spacelike surfaces in $\bar{S}$. The obvious choices are the energy density μ (leading to the "zero density-perturbations" discussed above, with $\bar{\mu}_q = \mu_q$)) or the fluid expansion Θ (giving Bardeen' s *uniform-Hubble-constant hypersurfaces*, with $\bar{\Theta}_q = \Theta_q$). The problem is that then the information on spatial density fluctuations is coded in a way that is hard to unravel.

Spatial Averaging

A fourth approach is to define the ideal density $\bar{\mu}$ in the lumpy model S as a suitable *average density* in S: $\bar{\mu} =< \mu >$, where $< . >$ denotes some suitable spatial average (cf. Lyth and Mukherjee [38]). This is equivalent to specifying a fitting procedure of the fictitious model to the real universe based on this averaging. This is indeed a reasonable thing to do [15,16], and may be expected to lead to integral conditions such as the Traschen integral constraints [53,54,52], as discussed by Ellis and Jaklitsch [18].

This procedure may well give us the physical information we want. However one will then have to take seriously the problems associated with averaging in general relativity, for example the degree to which averaging commutes with the Einstein field equations [13,?]. It also demands investigation of how this average depends on the choice of space-sections over which the average is taken.

The results obtained for the evolution of $\delta\mu/\mu$ from the various gauge choices are different (see Bardeen's paper [1] for an extensive discussion; and see also Goode [25]). In each of the last three cases considered, we have to concern ourselves with the relation between coordinate time and proper time along the fluid flow lines. In the first three cases, clearly the definitions are such that they have the correct correspondence limit: if S *is* a FLRW model, they define as surfaces $\{\bar{\mu} = const\}$ the surfaces $\{\mu = const\}$ in those universes. However the fourth approach is the most fundamental: it tackles the major issue, on what scale is the real universe approximated by the FLRW model [13]. From the viewpoint of the present paper, the averaging implied is a sophisticated way of comparing evolution along neighbouring world lines in the real fluid. In the next section we shall see there are simpler and more direct ways of making this comparison.

1.2.4 Gauge Invariant Variables

The fundamental requirement for a gauge invariant quantity is that it be invariant under the choice of the mapping Φ. The simplest case is a scalar $\bar{f}$ that is constant in the unperturbed space-time $\bar{S}$ ($\bar{f} = const$), or any tensor $\bar{f}^{ab}{}_{cd}$ that vanishes in $\bar{S}$: $\bar{f}^{ab}{}_{cd} = 0$. The reason is that in each case the mapped quantity $\bar{f}$ in S will also be constant, so *the choice of correspondence Φ does not matter; they will all define the same perturbation $\delta f = f - \bar{f}$.*

The only other possibility for gauge invariant quantities is a tensor that is a constant linear combination of products of Kronecker deltas (Stewart and Walker [51], Lemma 2.2)

What are the simple covariantly defined gauge invariant quantities in a FLRW universe? We can easily determine them by writing down a list of all the simple covariantly defined quantities in a general fluid flow, and then seeing which ones vanish in a FLRW universe model (the other two options in the Stewart and Walker lemma are not useful in our context, as the only invariantly defined constant in the FLRW universes is the cosmological constant, and no tensors that are constant products of Kronecker deltas occur naturally).

To carry this out, it is convenient to use the general formalism developed by Schücking, Ehlers, and Trümper. We turn to this in the next chapter.

Chapter 2

GAUGE - INVARIANT VARIABLES

In the first section of this chapter we briefly review the covariant fluid approach to cosmology. In such an approximation the matter content of the universe is described as a continuous fluid; this can be thought to be divided into small[1] volume elements. At each point of the space - time we can assign a 4 - velocity vector u^a representing the velocity of the volume element of fluid surrounding that point. This description of matter is complementary to the particle distribution function representation (see e.g. Ehlers (1971)[9]), and we can regard the 4 - velocity of the fluid element as the average velocity of the particles in that volume.

The second part of the present chapter deals with the definition of gauge - invariant quantities that naturally fit in the covariant fluid description adopted here and which describe the inhomogeneity of the real universe. In particular we identify the *comoving fractional density gradient* as the new variable (introduced in EB [17]) in terms of which the perturbation problem in cosmology can be treated in a simple and gauge - invariant way.

The physical interpretation of these variables is straightforward; moreover they are not referred to as perturbations[2], since in our approach we proceed

[1]From the mathematical point of view "small" means arbitrarily small, but from the physical point of view it means much smaller than the scale of interest.

[2]There is nothing dealing with smallness in their definition.

in a direction opposite to that of the usual treatment of perturbations in cosmology. We start by considering a generic universe, filled in with a general-relativistic fluid as inhomogeneous as we like; this fluid is described by variables obeying exact non-linear equations as we shall see in the next chapter. Then we restrict ourselves to an almost FLRW model for which these equations can be linearized; at this point the gauge-invariance of those variables becomes relevant, for in studying an almost FLRW universe we have to compare it with an exact FLRW.

2.1 The Covariant Fluid Approach

We consider now a completely general perfect-fluid flow in a curved spacetime. To characterize this fluid we introduce the covariant approach to general relativity as is presented for example in the papers of Hawking [26] and Ellis [11,12]. The presentation I give here is an attempt to satisfy a requirement of self-consistency of this thesis, avoiding details irrelevant to the content of the following chapters. The unsatisfied reader can refer to the papers quoted above, and references therein.

In the following we denote by μ the energy density of the fluid and by p the pressure. We assume a signature $(-,+,+,+)$, $c=1$, and $\kappa \equiv 8\pi G$, where G is the Newton's gravitational constant. Also, we denote 4-dimensional indices by Latin letters and $T^{a\ldots b}{}_{c\ldots(d\ldots e)\ldots f}$, $T^{a\ldots b}{}_{c\ldots[d\ldots e]\ldots f}$ is the standard notation for symmetrization and skew-symmetrization of $T^{a\ldots b}{}_{c\ldots d\ldots e\ldots f}$ with respect to the indices $e\ldots f$.

2.1.1 Kinematical Quantities

In the context of cosmology, there will always be a preferred family of world-lines (the *fundamental world lines*) representing the motion of observers in the universe ("fundamental observers") which are at rest with respect to our volume element of fluid. We will often refer to the flow lines as "fluid flow lines", since we will use the fluid approximation.

Let the normalized 4-velocity vector tangent to these world lines be

$$u^a = \frac{dx^a}{d\tau} \quad \Rightarrow \quad u^a u_a = -1 \ , \tag{2.1}$$

where τ is proper time along the fluid flow lines: at any point of the space-time u^a is the 4-velocity of the volume-element of fluid surrounding that point. Then we can define the projection tensor into the tangent 3-spaces orthogonal to u^a (the rest-space of an observer moving with 4-velocity u^a) as

$$h_{ab} \equiv g_{ab} + u_a u_b \quad \Rightarrow \quad h^a{}_b h^b{}_c = h^a{}_c, \quad h_a{}^b u_b = 0 \ . \tag{2.2}$$

It must be noted that the 3-planes defined at each point by h_{ab} do not in general mesh together to form 3-surfaces in the space-time (see section 2.1.5).

The time derivative of any tensor $T^{a...b}{}_{c...d}$ along the fluid flow lines is simply the covariant derivative along u^a

$$\dot{T}^{a...b}{}_{c...d} \equiv T^{a...b}{}_{c...d;e} u^e \ . \tag{2.3}$$

It is important to note that, because of (2.1), this is the derivative with respect to *proper time* defined along these lines: in other words $\dot{T}^{a...b}{}_{c...d}$ is the rate of change of $T^{a...b}{}_{c...d}$ as measured by a fundamental observer.

The 4-acceleration is then defined as

$$a^a \equiv \dot{u}^a = u^a{}_{;b} u^b \ , \tag{2.4}$$

and from the second of (2.1) it follows that $a^a u_a = 0$.

A relevant quantity in the fluid-flow picture is the *connecting vector* η^a, joining any two given flow lines at all time (see section ??). It can be show that η^a is *Lie dragged* [50] along u^a, i.e., its Lie derivative along the fluid flow lines vanishes. This implies

$$\dot{\eta}^a = \eta_{a;b} u^b = u_{a;b} \eta^b \ , \tag{2.5}$$

so that a significant quantity in our approach is the covariant derivative of the 4-velocity. Hence It is convenient to split $u_{a;b}$, for which we will need to define new variables. This we take up next.

The *expansion scalar (volume expansion)* Θ is the trace of $u_{a;b}$

$$\Theta \equiv u^a{}_{;a} \ , \qquad (2.6)$$

which represents the isotropic part of the expansion of the fluid. For instance, the action of Θ alone during a small time interval on a sphere of fluid changes the latter in a larger (smaller) sphere with the same orientation (see Fig. 2.1 (a)).

The *shear tensor* is the spatial trace-free symmetric part of $u_{a;b}$

$$\sigma_{ab} \equiv h_a{}^c h_b{}^d u_{(c;d)} - \frac{1}{3}\Theta h_{ab} \quad \Rightarrow \quad \sigma_{ab}u^b = 0 \ , \qquad (2.7)$$

Its action distorts the sphere leaving unchanged its volume and the directions of the shear principal axis (see Fig. 2.1 (b)). The shear magnitude is

$$\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma ab \geq 0 \ , \quad \sigma = 0 \quad \Leftrightarrow \quad \sigma_{ab} = 0 \ . \qquad (2.8)$$

The *vorticity tensor* ω_{ab} is the skew-symmetric spatial part of $u_{a;b}$

$$\omega_{ab} \equiv h_a{}^c h_b{}^d u_{[c;d]} \quad \Rightarrow \quad \omega_{ab}u^b = 0 \ , \qquad (2.9)$$

with magnitude

$$\omega^2 \equiv \frac{1}{2}\omega_{ab}\omega^{ab} \geq 0 \ . \qquad (2.10)$$

Since ω_{ab} is skew-symmetric, all the information contained in it can be put in a vector, the *vorticity vector*

$$\omega^a \equiv \frac{1}{2}\eta^{abcd}u_b\omega_{cd} \quad \Leftrightarrow \quad \omega_{ab} = \eta_{abcd}\omega^c u^d \ , \qquad (2.11)$$

$$\omega_a u^a = 0 \ , \quad \omega = 0 \quad \Leftrightarrow \quad \omega^a = 0 \quad \Leftrightarrow \quad \omega_{ab} = 0 \ ,$$

where η^{abcd} is the totally skew-symmetric tensor:

$$\eta^{abcd} = \eta^{[abcd]} \ , \quad \eta^{1234} = (-g)^{-\frac{1}{2}} \ , \quad g \equiv det(g_{ab}) \ . \qquad (2.12)$$

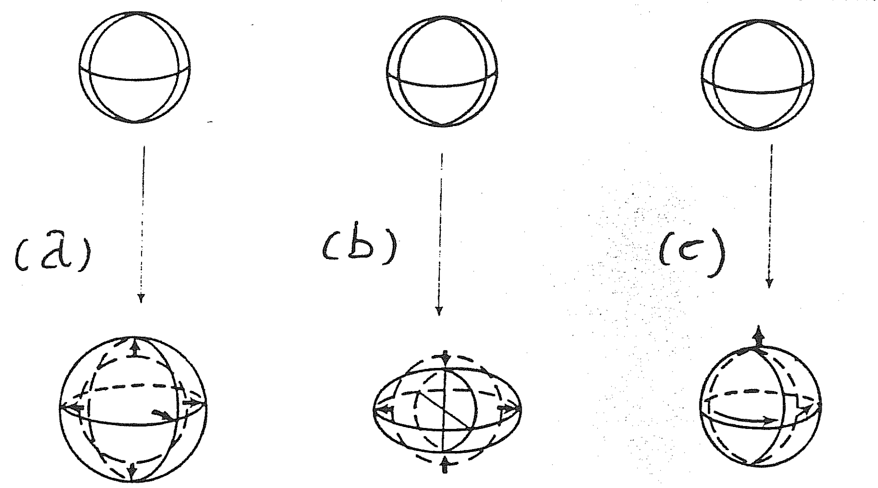

Figure 2.1: (a) The action of the expansion Θ alone on a sphere of fluid during a small time interval: the sphere change volume, but its shape and orientation is unchanged. (b) The shear modifies the shape of the sphere, leaving unchanged its volume and orientation. (c) The vorticity vector rotates the sphere around the axis defined by its direction, leaving its volume and shape unmodified.

The action of ω^a alone rotates the sphere, leaving its shape and volume unchanged (see Fig. 2.1 (c)).

With the definitions given above, the first covariant derivative of the 4-velocity vector is completely determined

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\Theta h_{ab} - a_a u_b \ . \tag{2.13}$$

It is convenient to define a representative *length scale* $S(\tau)$ by the relation

$$\dot{S}/S = \frac{1}{3}\Theta \quad \Leftrightarrow \quad \Theta = \frac{1}{S^3}\frac{d(S^3)}{d\tau} \ , \tag{2.14}$$

which determine S up to a constant factor along each world line. Hence, the volume of any fluid element varies as S^3 along the flow lines (this quantity is the generalization to arbitrary anisotropic flows of the Robertson-Walker scale parameter), so that S represents the *average* distance behaviour of the fluid. This can be understood if we refer to the definition of Θ and σ_{ab}

and to Fig. 2.1: in general a sphere of fluid will expand in an anisotropic way during each small time interval, but if we *average* the expansion along different directions over this time interval the shear effect will cancel out and the resulting effect is described by Θ (S) alone.

It is clear from the above definition of S that Θ is just proportional to the familiar Hubble parameter $H(\tau)$ when we consider a FLRW model.

2.1.2 Matter: Conservation Equations

If T^{ab} is the energy momentum stress tensor, the covariant form of the energy momentum conservation equations is

$$T^{ab}{}_{;b} = 0 \ . \tag{2.15}$$

From now on we will assume a one component perfect fluid unless otherwise specified; in this case T^{ab} takes the well-known form

$$T^{ab} = \mu u_a u_b + p h_{ab} = (\mu + p) u_a u_b + p g_{ab} \ , \tag{2.16}$$

and μ and p will be constrained through an equation of state. In section 3.1.1 we will separate equation (2.15) in its time and space components.

2.1.3 Geometry

Riemann, Ricci and Weyl Tensors

The Riemann tensor R_{abcd} is the tensor which describes the curvature of the space-time. It is defined by the commutation relation satisfied by the covariant derivatives of any arbitrary 4-vector (Ricci identity). In particular for the 4-velocity vector Ricci's identity is

$$u_{a;d;c} - u_{a;c;d} = R_{abcd} u^b \ . \tag{2.17}$$

The Riemann tensor satisfy the symmetry properties

$$R_{[ab][cd]} = R_{abcd} = R_{cdab}, \quad R_{a[bcd]} = 0 \ , \tag{2.18}$$

giving 20 independent components. The Riemann tensor can be decomposed into its "trace", i.e., the Ricci tensor (10 independent components)

$$R_{ab} \equiv R^{c}{}_{acb} \ , \tag{2.19}$$

and its "trace-free" part, the Weyl tensor C_{abcd} (the remaining 10 components)

$$C^{ab}{}_{cd} \equiv R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{1}{3}Rg^{[a}{}_{[c}g^{b]}{}_{d]} \quad \Rightarrow \quad C^{ab}{}_{ad} = 0 \ , \tag{2.20}$$

where $R = R^{a}{}_{a}$ is the Ricci scalar. We can further spit the Weyl tensor into its "electric" and "magnetic" parts[3], respectively defined by

$$E_{ac} \equiv C_{abcd}u^{b}u^{d} \ , \quad H_{ac} \equiv \frac{1}{2}\eta_{ab}{}^{gh}C_{ghcd}u^{b}u^{d} \ ; \tag{2.21}$$

$$E_{ab} = E_{(ab)} \ , \quad H_{ab} = H_{(ab)} \ , \quad E^{a}{}_{a} = H^{a}{}_{a} = 0 \ , \quad E_{ab}u^{b} = H_{ab}u^{b} = 0 \ .$$

Then the Weyl tensor can be written as

$$C_{abcd} = (\eta_{abpq}\eta_{cdrs} + g_{abpq}g_{cdrs})u^{p}u^{r}E^{qs} - (\eta_{abpq}g_{cdrs} + g_{abpq}\eta_{cdrs})u^{p}u^{r}H^{qs} \ , \tag{2.22}$$

$$g_{abcd} \equiv g_{ac}g_{bd} - g_{ad}g_{bc} \ .$$

It is interesting to note that the physical interpretation of the gravitational field E_{ab} is clarified by its Newtonian counterpart[4]: E_{ab} represents the tidal force, inducing shear in the fluid flow lines (see equation (3.6); H_{ab} has no Newtonian counterpart.

Bianchi Identities

The Riemann tensor satisfies the Bianchi identities

$$R_{ab[cd;e]} = 0 \ ; \tag{2.23}$$

[3]The reason for this terminology is that E_{ab} and H_{ab} satisfy a "Maxwellian form" of the Bianchi's identity (see section 3.1.2).

[4]The Newtonian analogue of the general-relativistic fluid approximation is developed in detail in Ellis (1971) [11]. The extention of the covariant fluid analysis of cosmological density inhomogeneities to its Newtonian analogue is given in Ellis (1989) [20].

in 4 dimensions these are equivalent to

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6} g^{c[a} R^{;b]} \ . \tag{2.24}$$

Written in this form, they are differential equations relating the components of the Ricci and Weyl tensor. Contraction of (2.24) implies

$$G^{ab}{}_{;b} = 0 \ , \quad G_{ab} \equiv R_{ab} - \frac{1}{2} R \, g_{ab} \ , \tag{2.25}$$

where G_{ab} is the Einstein tensor.

2.1.4 Einstein Equations

Up to now we have not related geometry with the matter contents of the space-time. This relation is established by the Einstein equations

$$G_{ab} + \Lambda g_{ab} = \kappa T_{ab} \ , \tag{2.26}$$

where we include the cosmological constant term Λ for generality. The cosmological constant problem is a controversial one, but not at all old-fashioned, see e.g. the recent review by Weinberg (1989) [59].

The Einstein equations tell us that the Ricci tensor R_{ab} is determined directly from the matter energy-momentum-stress tensor at each point. If we substitute (2.26) in the Bianchi identities (2.24), we see that the Weyl tensor represents the "free gravitational field", determined non-locally by matter and suitable boundary conditions.

2.1.5 Intrinsic Curvature

When the fluid vorticity vanishes (and only then) there exists a family of 3-surfaces $\Sigma_\perp$ everywhere orthogonal to the fluid flow vector u^a. Indeed it is possible to show that

$$\begin{aligned} \omega = 0 \quad &\Leftrightarrow \quad u_{[a} u_{b;c]} = 0 \\ &\Leftrightarrow \quad \exists \ \ f, g : \ u_a = f g_{;a} \ . \end{aligned} \tag{2.27}$$

In other words, for the surfaces $\Sigma_\perp$ to exist, it must be possible to write u_a as a 4-gradient. Then the surfaces $\Sigma_\perp \equiv \{g = constant\}$ are instantaneous surfaces of simultaneity for all the fundamental observers, i.e., the surfaces $\Sigma_\perp$ define a cosmic time. However this can be locally normalized to measure proper time along each flow line only if $a_a = 0$.

Since for $\omega = 0$ these surfaces exist, one can define an intrinsic curvature tensor for them from the Ricci identity in $\Sigma_\perp$. For any vector V^a in $\Sigma_\perp$: $V^a u_a = 0$, we have

$$ {}^{(3)}\nabla_c {}^{(3)}\nabla_b V_a - {}^{(3)}\nabla_b {}^{(3)}\nabla_c V_a = V_d {}^{(3)}R^d{}_{abc} \ , \tag{2.28} $$

where ${}^{(3)}\nabla_a$ is the 3-covariant derivative defined in $\Sigma_\perp$ as the total projection of the 4-covariant one[5]. Finally, the Ricci tensor for these 3-spaces can be written as

$$ \begin{aligned} {}^{(3)}R_{ab} = h_a{}^f h_b{}^g \left(-S^{-3}(S^3\sigma_{fg})^{\cdot} + a_{(f;g)}\right) + a_a a_b + \\ + \frac{1}{3} h_{ab}(-\frac{2}{3}\Theta^2 + 2\sigma^2 + 2\kappa\mu + 2\Lambda - A) \end{aligned} \tag{2.29} $$

for a general perfect fluid.

Now, in a general fluid flow, we can define the quantity

$$ \mathcal{K} = 2(-\frac{1}{3}\Theta^2 + \sigma^2 + \kappa\mu + \Lambda) \ . \tag{2.30} $$

Then, when $\omega = 0$, this quantity acquires a special significance: it is the Ricci scalar ${}^{(3)}R$ of the 3-dimensional spaces $\Sigma_\perp$; that is, $\omega = 0 \Rightarrow {}^{(3)}R = \mathcal{K}$. A brief discussion about the meaning of this variable when $\omega \neq 0$ is postponed to section 3.3.2.

When $\omega \neq 0$, we can of course define a 3-covariant derivative at each point, by using the projection tensor h_{ab} (as we did up to now). However we want to stress here that, if we compute the commutator of these derivatives, the tensor that rests defined in this way will not have al the usual curvature tensor symmetries.

[5]We already used it in the case of function, i.e., ${}^{(3)}\nabla_a f = h_a{}^b f_{.b}$. We introduce this (standard) notation to consider 3-covariant derivatives of tensors, since it is compact; e.g. ${}^{(3)}\nabla_a V_b = h_a{}^c h_b{}^d V_{d;c}$; ${}^{(3)}\nabla_c {}^{(3)}\nabla_b V_a = h_c{}^d h_b{}^e h_a{}^f (h_e{}^m h_f{}^r V_{r;m})_{;d}$; and so on.

2.2 Gauge Invariant Quantities

2.2.1 Characterization of FLRW Universes

We can describe a FLRW universe using the quantities defined in the previous section. If we assume that the matter and radiation content of the universe can be depicted as a perfect-fluid with equation of state $p = p(\mu)$, then a FLRW model is a space-time characterised by the conditions [8,11]:

$$\sigma_{ab} = \omega_{ab} = a^a = 0 \ . \tag{2.31}$$

These imply

$$\mu = \mu(t), \quad p = p(t) \ , \quad \Theta = \Theta(t) \ , \tag{2.32}$$

where t is the cosmic time defined (up to a constant) by the FLRW fluid flow vector: $u_a = -t_{,a}$. Thus

$$h_a{}^b\mu_{;b} = h_a{}^b p_{;b} = h_a{}^b\Theta_{;b} = 0 \ , \tag{2.33}$$

and these models are spatially homogeneous and isotropic since there are no directions defined in the 3-space orthogonal to u_a. Further conditions satisfied by FLRW models are

$$E_{ab} = 0 \ , \quad H_{ab} = 0 \ , \tag{2.34}$$

i.e., the Weyl tensor vanishes. Thus these space-times are conformally flat (their metric can be written as $g = \Omega\eta$, where η is the metric of flat space).

2.2.2 Gauge-Invariant Variables

We now consider an almost FLRW universe, i.e, a universe described by equations that differ from that of an exact FLRW only by linear terms (this statement will be clarified in the next chapter). We come back to consider a general fluid, represented by a general equation of state.

From the above characterization of a FLRW model plus the Stewart and Walker Lemma discussed in the previous chapter [51], the basic gauge invariant quantities for an almost-FLRW universe are as follows:

(1) The vorticity ω_{ab}, shear σ_{ab}, and acceleration a^a;
(2) The electric and magnetic parts E_{ab}, H_{ab} of the Weyl tensor C_{abcd};
(3) The matter tensor components

$$q_a \equiv -h_a{}^c T_{cd} u^d \ , \quad \pi_{ab} \equiv h_a{}^c h_b{}^d T_{cd} - \frac{1}{3}(h^{cd}T_{cd})h_{ab} \ . \tag{2.35}$$

These components of the energy - momentum - stress tensor vanish identically in the perfect fluid case; however, we may need to consider non-zero components of these tensors in some physically significant situations (cf. section ??).

These are the simplest covariantly defined quantities which vanish in FLRW models, and so are gauge invariant. The problem is that the list so far does not contain quantities characterizing the variation of the zero-order variables (the energy density μ, pressure p, and fluid expansion Θ), which are in general non-zero in expanding FLRW models, and so are not gauge invariant. However we can find associated gauge invariant quantities, namely the *orthogonal spatial gradients* of these variables. We define

$$X_a \equiv \kappa h_a{}^b \mu_{,b} \ , \quad Y_a \equiv \kappa h_a{}^b p_{,b} \ , \quad Z_a \equiv h_a{}^b \Theta_{,b} \ , \tag{2.36}$$

(we include the gravitational constant κ in these definitions for later convenience) where each quantity is gauge invariant, since they all vanish in the FLRW universes (2.2;2.32).

We can easily come up with further gauge invariant quantities by finding more complex invariantly defined quantities that vanish in the FLRW universe models, e.g., the gradients of the squared magnitudes of the shear and vorticity, $(\omega^2)_{,a}$ and $(\sigma^2)_{,a}$, the scalar products of the shear with the Weyl tensor components, $\sigma^{ab}E_{ab}$ and $\sigma^{ab}H_{ab}$, and so on. These will not be significant to us in considering linearization around the FLRW universes, for they will be of second or higher order. However there are two other gauge-invariant quantities that will be important subsequently, namely the divergence of the acceleration, and its spatial gradient:

$$A \equiv a^c{}_{;c} \ , \quad A_a \equiv h_a{}^b A_{,b} \ . \tag{2.37}$$

In the case of vanishing vorticity, the Ricci scalar ${}^{(3)}R$ of the orthogonal 3-spaces is gauge invariant if and only if the homogeneous space-sections in $\bar{S}$ are flat, i.e. if that idealised universe is at the critical density. However its spatial gradient is always gauge-invariant. Thus for a general fluid flow, it is interesting to define from $\mathcal{K}$ (see (2.30)) the gauge invariant quantity

$$\mathcal{K}_a \equiv h_a{}^b \mathcal{K}_{,b} = -\frac{4}{3}\Theta Z_a + 2X_a + 2(\sigma^2)_{,c} h^c{}_a \ , \tag{2.38}$$

the equivalence following from the definitions (2.36). Since for $\omega = 0$ we have $\mathcal{K} = {}^{(3)}R$, $\mathcal{K}_a$ is an *intrinsic curvature gradient.* Then *isocurvature fluctuations* can be defined as the zero-vorticity perturbations for which $\mathcal{K}_a = 0$.

2.2.3 The Key Variables

The point of this discussion is that instead of concentrating on $\delta\mu$, with the arbitrariness that implies, we can find three simple gauge invariant quantities that will give us the information we need to discuss the time evolution of density perturbations, without the complexity of the Bardeen [1] analysis.

The first is *the spatial projection of the energy density gradient*, i.e. the vector $X_a \equiv \kappa h_a{}^b \mu_{,b}$. This vanishes in the FLRW universes, and so is a gauge invariant quantity; it is covariantly defined in the real universe. This variable naturally arises in the covariant fluid approach (Hawking (1966) [26]; Olson (1976) [43]). However it was not recognized as the gauge-invariant variable in terms of which the whole perturbation problem can be formulated.

X_a is measurable in the sense that (a) it can be determined from virial theorem estimates (indeed, dynamical mass estimates determine precisely spatial density gradients), and (b) the contribution to it from luminous matter can be found by observing gradients in the numbers of observed sources and estimating the mass to light ratio (Kristian and Sachs [33], equation (39)). It describes the density inhomogeneities which we wish to investigate, for if there is an overdensity which is a viable proto - galaxy, this will be evidenced by a non-zero value of X_a (the magnitude of X_a directly indicating how rapid the spatial variation of density is). Thus X_a seems to encapsulate much of the information we want.

However, we normally will wish to compare the density gradient with the existing density, to characterize it significance. Thus we can define the second quantity, the *fractional density gradient*

$$\mathcal{X}_a \equiv \frac{X_a}{\kappa\mu} = h_a{}^b \left(\frac{\mu_{,a}}{\mu}\right) , \tag{2.39}$$

which is also gauge-invariant, and represents the relative importance of the density gradient. While both are observable in principle, it is a moot point whether X_a or $\mathcal{X}_a$ is more easily observable in practice.

Both these vectors can be used to determine the spatial variation of the energy density μ. One important point should be noticed. In the case where $\omega = 0$, they will characterize the distribution of the density μ in the 3-spaces $\Sigma_\perp$ orthogonal to the fluid flow (which might naturally be chosen as the surfaces $\{t = const\}$). However when $\omega \neq 0$, no such orthogonal 3-surfaces exist. These vectors still characterize the gradient of μ orthogonal to u^a, but cannot be immediately integrated to give the distribution of density in the surfaces $\{t = const\}$ for a suitable set of coordinates [55,7] because these surfaces cannot be everywhere orthogonal to the fluid flow lines. Even if $\omega = 0$, the time t such that the surfaces $\{t = const\}$ are orthogonal to the fluid flow will not measure proper time τ along the fluid flow lines unless the acceleration is zero also, that is, unless there are no pressure gradients.

There remains a problem with $\mathcal{X}_a$: it is not dimensionless. This is essentially related to the fact that when we consider the time evolution of the fluid, both X_a and $\mathcal{X}_a$ represent the change in density to a *fixed distance*, whereas in the context of considering the growth of proto - galaxy-galaxy fluctuations we want to consider density variations at a *fixed comoving scale*. In other words, when we are interested in the time evolution of density inhomogeneities we have to define a quantity for which the overall damping effect of the general expansion cancels out. Thus the third quantity of interest is the *comoving fractional density gradient* obtained by multiplying (2.39) by the scale factor $S(\tau)$:

$$\mathcal{D}_a \equiv S\mathcal{X}_a \ , \tag{2.40}$$

which is gauge-invariant and dimensionless. We must remember here that S is defined only up to a constant by (2.14), so $\mathcal{D}_a$ is similarly defined up to a constant along each flow line; this reflects the fact that it represents the density variation to *any* neighbouring comoving region. The time variation of this quantity precisely reflects the relative growth of density in neighbouring fluid comoving volumes, and this is what we wish to investigate. Because S represents the averaged volume behaviour, $\mathcal{D}_a$ represents the *average* behaviour of comoving density fluctuations, rather than the growth of a specific fluctuation; this is represented by Δ, a quantity that will be defined in section 4.5. We concentrate here on $X_a, \mathcal{X}_a$, and $\mathcal{D}_a$ because they are space-time fields, whereas Δ is not.

The vector $\mathcal{D}_a$ can be separated into a direction e_a and magnitude $\mathcal{D}$ where

$$\mathcal{D}_a = \mathcal{D}e_a, \ e_a e^a = 1, \ e_a u^a = 0 \ \ \Rightarrow \ \ \mathcal{D} = (\mathcal{D}^a\mathcal{D}_a)^{1/2}. \tag{2.41}$$

The magnitude $\mathcal{D}$ is the gauge-invariant variable that most closely corresponds to the intention of the usual $(\delta\mu/\mu)$ in representing the fractional density increase in a comoving density fluctuation. The crucial difference from the usual definition is that $\mathcal{D}$ represents a (real) spatial fluctuation, rather than a (fictitious) time fluctuation.

The vectors X_a, $\mathcal{X}_a$ are closely related to the vectors Y_a and Z_a defined above, equation (2.36); indeed they are dynamically dependent on each other, as will be shown in the following chapter. But we have also defined the comoving density gradient $\mathcal{D}_a$, therefore we find useful to define the *comoving* *expansion gradient*

$$\mathcal{Z}_a \equiv S Z_a \tag{2.42}$$

as the natural companion variable of $\mathcal{D}_a$. All these variables are gauge invariant, and directly determinable (at any desired scale) from a description of the real (lumpy) universe at that scale. Thus our further analysis will concentrate on these quantities.

The gauge-invariant variables introduced in this section are only useful if we can determine a self-consistent set of dynamic equations for those quantities. This is what we consider in the next chapter.

Chapter 3

DYNAMIC EQUATIONS

In the second part of the previous chapter we introduced a set of variables describing the inhomogeneity of a general space - time S: since they vanish in an exact FLRW ($\bar{S}$), they are gauge - invariant. The density gradient X_a has been considered previously as a gauge invariant variable by Hawking (1966) [26] and Olson (1976) [43], but they did not notice that the density fluctuation problem in an almost FLRW can be treated in terms of this variable only. A most significant quantity for describing the time evolution of density fluctuations (see section 2.2.2) is the *comoving fractional density gradient* $\mathcal{D}_a$ introduced in EB [17], because it incorporates the relative growth of density in neighbouring comoving volumes of fluid. The final aim of this chapter is therefore to derive a linear system of two first - order equations [1] from which $\mathcal{D}_a$ can be determined.

However, we shall see that for a general space - time S the density gradients $X_a, \mathcal{X}_a$ or $\mathcal{D}_a$ are coupled with the expansion gradients Z_a or $\mathcal{Z}_a$, as well as with the shear σ_{ab} and the vorticity ω_{ab}, through a set of exact non - linear first - order equations. We shall assume a perfect fluid throughout this chapter, unless otherwise specified.

In the first section we consider a general space - time S, and we present

[1] To avoid confusion we will use "first - order equation" (second - order) to refer to the order of the derivatives with respect to time, and "linear equation" when we refer to the degree of approximation.

the evolution equation for μ, p and Θ, as well as for σ_{ab} and ω_{ab}. We also introduce the "Maxwellian form" of the Bianchi identities (2.24) for the Weyl tensor components E_{ab}, H_{ab} (this will clarify the reason for their names). All these equations are well known and can be found in Hawking (1966) [26] and Ellis (1971) [11] as well as in other classic papers. They are not directly relevant to our discussion of the inhomogeneity in an almost FLRW universe, but we present them here in order to have a complete picture of the general non-linear case. Finally, we derive the equations of motion presented in EB [17] for the gauge-invariant variables $a_a, X_a, \mathcal{X}_a, \mathcal{D}_a, Z_a, \mathcal{Z}_a$ (first-order in an almost FLRW). All the equations in this first section are exact and non-linear (apart from the energy and momentum conservation equations).

In the second section we consider an almost FLRW universe. By this we mean a universe model in which all the gauge-invariant quantities are first-order with respect to the energy density μ, the pressure p and the expansion Θ. we shall refer to these latter quantities as "zero-order", since they are non vanishing in an exact FLRW model. We linearize the equations derived in section 1, and we shall see how they decouple. Equivalent systems of two first-order linear equations coupling any of the density gradients with the expansion gradients are obtained. In particular we derive a pair of equations coupling $\mathcal{D}_a$ with $\mathcal{Z}_a$ (presented in EB [17]). Also, an evolution equation for the *comoving curvature gradient* $\mathcal{C}_a$ is derived in the linear approximation (EHB [19]). An equivalent system of equations coupling $\mathcal{D}_a$ with $\mathcal{C}_a$ is then obtained.

The final section of this chapter deals with the mapping we have chosen from the background exact FLRW model (S) to the realistic almost FLRW universe ($\bar{S}$). Indeed, since we examine the evolution of each quantity along the fluid flow lines, we map fluid flow lines in $\bar{S}$ to fluid flow lines in S. We shall explain how this specific choice does not affect the gauge-invariance of our variables (and equations). Finally, we shall briefly discuss the meaning of the quantity $\mathcal{K}$ (defined by (2.30)) in the case of non-vanishing vorticity.

3.1 Exact Equations

3.1.1 Zero - Order Quantities

Conservation Equations

The tensorial energy - momentum conservation equation (2.15) can be separated into time (energy conservation) and space (momentum conservation) components. For a perfect fluid the energy - momentum tensor has the form (2.16). Inserting this in equation (2.15) gives

$$T^{ab}{}_{;b} = [\dot{\mu} + (\mu + p)\Theta]u^a + (\mu + p)a^a + Y^a = 0 \ . \tag{3.1}$$

The time component is obtained by projecting the above equation along u_a. Since $u^a u_a = -1, a^a u_a = Y^a u_a = 0$ we obtain the *energy conservation equation*

$$\dot{\mu} + (\mu + p)\Theta = 0 \, . \tag{3.2}$$

In the same way, using the projection tensor h_{ab}, we obtain the space component of (3.1), the *momentum conservation equation*

$$\kappa(\mu + p)a_a + Y_a = 0 \, . \tag{3.3}$$

The time-evolution of p is determined by (3.2) when we specify an equation of state determining p from μ.

Raychauduri Equation

The third zero - order variable is Θ, whose evolution along the fluid flow lines is given by the *Raychaudhuri equation*:

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - A + \frac{1}{2}\kappa(\mu + 3p) - \Lambda = 0 \ , \tag{3.4}$$

where A is defined by (2.37). This equation was derived by Raychauduri (1955) [47] in the case of dust and generalized by Ehlers (1961) [8] to the case of non - vanishing pressure. It is also the *trace* of the Ricci identity (2.17) (actually it is obtained by contracting the spatial part of (2.17) projected along

u^d). The Raychauduri equation is the fundamental equation of gravitational attraction that establishes that in general relativity $(\mu + 3p)$ is the active gravitational mass of the fluid[2]. For $\mu + 3p > 0$ we have from (3.4) a volume contraction; we also see from (3.4) that Λ contributes as a constant repulsive force, and a similar repulsive role is played by the acceleration divergence A and by the vorticity. On the other end the shear term tends to shrink the volume.

A further useful quantity we defined in section 2.1 is $\mathcal{K}$ (see (2.30)): its time derivative along the fluid flow lines obeys the equation

$$(\mathcal{K} - 2\sigma^2)^{\cdot} = \frac{2}{3}\Theta(6\sigma^2 - \mathcal{K} - 4\omega^2 - 2A)\,. \tag{3.5}$$

This is immediately obtained by differentiating (2.30) and using the Raychauduri equation (3.4) and the energy conservation equation (3.2); when the vorticity vanishes, this is an equation for ${}^{(3)}R$.

3.1.2 Shear, Vorticity, and Maxwell - like Equations

By the perfect fluid assumption, the matter tensor components (2.35) q_a and π_{ab} vanish in the general space - time S, so their "propagation equations" are $q_a = 0$, $\pi_{ab} = 0$. The equations for shear and vorticity as well as for the Weyl tensor components E_{ab} and H_{ab} are given below.

Shear and Vorticity Equations

Propagation equations for σ_{ab} and ω_{ab} are also obtained via the Ricci identity (2.17). The *symmetric trace - free part* of the tensor $((PIPPO)_{ab})$ obtained by taking the spatial part of (2.17) projected along u^d is the *shear evolution equation*:

$$h_a{}^f h_b{}^g(\sigma_{fg})^{\cdot} - h_a{}^f h_b{}^g a_{(f;g)} - a_a a_b + \omega_a\omega_b + \sigma_{af}\sigma^f{}_b +$$

[2]This is the role played by the mass density *only* in the Newtonian theory: the p term in the active gravitational mass is responsible for the *regeneration of pressure* bringing to gravitational collapse in general relativity.

$$+\frac{2}{3}\Theta\sigma_{ab}+\frac{1}{3}h_{ab}(A-\omega^2-2\sigma^2)+E_{ab}=0\,. \tag{3.6}$$

The *vorticity evolution equation*

$$h^a{}_b(\omega^b)^{\cdot}+\frac{2}{3}\Theta\omega^a-\sigma^a{}_b\omega^b-\frac{1}{2}\eta^{abcd}u_b a_{c;d}=0 \tag{3.7}$$

is then the *skew symmetric part* of $(PIPPO)_{ab}$.

Contraint Equations

Three further equations follow from the Ricci identity (2.17).

They can be regarded as constraint equations, since they do not involve time derivatives:

$$h^e{}_b(\omega^{bc}{}_{;c}-\sigma^{bc}{}_{;c}+\frac{2}{3}Z^b)+(\omega^e{}_b+\sigma^e{}_b)a^b=0\,, \tag{3.8}$$

$$\omega^a{}_{;a}=2\omega^a a_a\,, \tag{3.9}$$

$$H_{ad}=2a_{(a}\omega_{d)}-h_a{}^e h_d{}^g\left(\omega_{(e}{}^{b;c}+\sigma_{(e}{}^{b;c}\right)\eta_{g)fbc}u^f\,. \tag{3.10}$$

Maxwell - like Bianchi Identities

We have seen in section 2.1 that the Bianchi identities (2.23) obeyed by the Riemann tensor can be cast in the form (2.24), relating the Weyl tensor components to the Ricci tensor components. As the Weyl tensor can be decomposed into components E_{ab} and H_{ab} (see eq. (2.22)) and the Ricci tensor is related to the energy - momentum tensor via the Einstein equations (2.26), one can substitute (2.22) and (2.26) in (2.24), obtaining four tensorial equations for E_{ab} and H_{ab} that are rather similar to the Maxwell equations. In the perfect fluid case these *Maxwell - like Bianchi identities* are:

divE :

$$h^t{}_a E^{as}{}_{;d}h_s{}^d-\eta^{tbpq}u_b\sigma_p{}^d H_{qd}+3H^t{}_s\omega^s=\frac{1}{3}X^t\,, \tag{3.11}$$

divH :

$$h^t{}_a H^{as}{}_{;d}h_s{}^d+\eta^{tbpq}u_b\sigma_p{}^d E_{qd}-3E^t{}_s\omega^s=(\mu+p)\omega^t\,, \tag{3.12}$$

$\dot{E}$:

$$h_a{}^m h_c{}^t (E^{ac})^{\cdot} + h_a{}^{(m}\eta^{t)rsd} u_r H^a{}_{s;d} - 2H_q{}^{(t}\eta^{m)bpq} u_b a_p + \Theta E^{mt} + \\ + h^{mt}(\sigma^{ab}E_{ab}) - 3E_s{}^{(m}\sigma^{t)s} - E_s{}^{(m}\omega^{t)s} = -\frac{1}{2}(\mu + p)\sigma^{tm} , \quad (3.13)$$

$\dot{H}$:

$$h^{ma}h^{tc}(H_{ac})^{\cdot} - h_a{}^{(m}\eta^{t)rsd} u_r E^a{}_{s;d} + 2E_q{}^{(t}\eta^{m)bpq} u_b a_p + \\ + h^{mt}(\sigma^{ab}H_{ab}) + \Theta H^{mt} - 3H_s{}^{(m}\sigma^{t)s} - H_s{}^{(m}\omega^{t)s} = 0 . \quad (3.14)$$

The form of these equations is the reason for calling E_{ab} and H_{ab} the electric and magnetic component of the Weyl tensor.

3.1.3 Exact Evolution Equations for Gauge - Invariant Variables

We pass now to sketch the derivations of the equations presented in EB [17] for the new gauge - invariant variables defined in section 2.2.2.

Acceleration Propagation Equation

In order to derive an evolution equation for the acceleration we express a^a using the momentum conservation equation (3.2):

$$a^a = -\frac{Y^a}{\kappa(\mu + p)} . \quad (3.15)$$

Differentiating with respect to the proper time and projecting orthogonally to u^a we have

$$h_a{}^c(a_c)^{\cdot} = -\frac{h_a{}^c(Y_c)^{\cdot}}{\kappa(\mu + p)} + a_a\left(1 + \frac{dp}{d\mu}\right)\Theta , \quad (3.16)$$

where we substituted for $\dot{\mu}$ from the energy conservation equation (3.2) and we used $\dot{p} = \frac{dp}{d\mu}\dot{\mu}$, with $dp/d\mu$ taken along the fluid flow lines. Now with the same tricks and after some algebra we can write

$$h_a{}^c(Y_c)^{\cdot} = -\kappa(\mu + p)\left(\frac{dp}{d\mu}\Theta\right)_{,b} h^b{}_a - \\ - \kappa\Theta\left(1 + \frac{dp}{d\mu}\right)Y_a - \kappa h_a{}^b p_{,c} u^c{}_{;b} - \kappa a_a \frac{dp}{d\mu}\Theta(\mu + p) .$$

Using (2.13) to express $u^c{}_{;b}$ and substituting in (3.16) we finally obtain

$$h_a{}^c(a_c)^{\cdot} = a_a\Theta\left(\frac{dp}{d\mu} - \frac{1}{3}\right) + h_a{}^b\left(\frac{dp}{d\mu}\Theta\right)_{,b} - a_c(\omega^c{}_a + \sigma^c{}_a) \ . \tag{3.17}$$

Density Gradient Equation

To obtain a propagation equation for the spatial gradients of the energy density we could proceed from its definition, $X_a = \kappa h_a{}^b\mu_{,b}$, differentiating with respect to the proper time and projecting orthogonally to u^a [3].

However it is more interesting to proceed directly from (3.2), because the propagation equation we derive for X_a is the *spatial variation of the energy conservation equation.* Taking the spatial gradient of (3.2) and using the definitions (2.36), we obtain

$$h_a{}^b(\dot{\mu})_{;b} + Z_a(\mu + p) + \Theta X_a + \Theta Y_a = 0 \ . \tag{3.18}$$

Now we can write

$$\begin{aligned} h_a{}^b(\dot{\mu})_{;b} &= h_a{}^b(\mu_{;c}u^c)_{;b} = \\ &= h_a{}^b\mu_{;b;c}u^c + h_a{}^b\mu_{;c}(\sigma^c{}_{;b} + \omega^c{}_{;b} + \frac{1}{3}\Theta h^c{}_b) \ , \end{aligned}$$

where we used $\mu_{;c;b} = \mu_{;b;c}$ (μ is a scalar) and we substituted for $u^a{}_{;b}$ from (2.13). It is useful to express the first term in the last part of the previous equality as

$$\begin{aligned} h_a{}^b\mu_{;b;c}u^c &= h_a{}^d(h_d{}^b\mu_{;b})_{;c}u^c - h_a{}^d(h_d{}^b)_{;c}u^c\mu_{;b} = \\ &= h_a{}^b(X_b)^{\cdot} - Y_a\Theta \ , \end{aligned}$$

where in the last step we again used (3.2) and (3.15).

Substituting in (3.18) and using the definitions (2.36), we finally obtain

$$h_a{}^b(X_b)^{\cdot} + X_b(\sigma^b{}_a + \omega^b{}_a) + \frac{4}{3}\Theta X_a + Z_a(\mu + p) = 0. \tag{3.19}$$

[3] in this case a key step would be $(\mu_{,c})^{\cdot} = (\dot{\mu})_{,c} - \mu_{,b}u^b{}_{;c}$, followed by the substitution for $u^a{}_{;b}$ from (2.13), a^a from (3.15) and $\dot{\mu}$ from the energy conservation equation (3.2).

We can cast this equation in the following form:

$$S^{-4}h_c{}^a(S^4X_a)^{\cdot} = -\kappa(\mu+p)Z_c - (\omega^a{}_c + \sigma^a{}_c)X_a \ , \qquad (3.20)$$

showing that the time variation of X_a is determined by the source term Z_a and by the non-linear term coupling X_a with the shear and vorticity.

Expansion Gradient Equation

We now want to derive an evolution equation for the spatial gradient of the expansion. As for X_a, we could start from the definition $Z_a = h_a{}^b\Theta_{;b}$ and spatially-project its time derivative. However we proceed from (3.4), because the equation we obtain for Z_a is the *spatial variation of the Raychauduri equation.* Taking the spatial gradient of (3.4) we have

$$h_a{}^b(\dot{\Theta})_{,b} + \frac{2}{3}\Theta Z_a + \frac{1}{2}X_a + \frac{3}{2}Y_a + h_a{}^b\left[2(\sigma^2)_{,b} - 2(\omega^2)_{,b} - A_a\right] = 0 \ , \qquad (3.21)$$

and the first term of this equation can be reexpressed as

$$\begin{aligned} & h_a{}^b(\Theta_{,c}u^c)_{;b} = \\ & \quad h_a{}^d(h_d{}^b\Theta_{,b})_{;c}u^c - h_a{}^d\Theta_{,b}(h_d{}^b)_{;c}u^c + Z_b(\sigma^b{}_a + \omega^b{}_a) + \frac{1}{3}\Theta Z_a = \\ & \quad h_a{}^b(Z_b)^{\cdot} - \dot{\Theta}a_a + Z_b(\sigma^b{}_a + \omega^b{}_a) + \frac{1}{3}\Theta Z_a \ . \end{aligned}$$

As before we used (2.13) to express $u^a{}_{;b}$, and the definitions (2.36). If now we substitute the last expression in (3.21), using (3.4) to express $\dot{\Theta}$, we obtain finally

$$h_a{}^b(Z_b)^{\cdot} + \Theta Z_a - a_a\mathcal{R} + h_a{}^b\left(\frac{1}{2}X_b + 2(\sigma^2)_{,b} - 2(\omega^2)_{,b} - A_b\right) + Z_b(\sigma^b{}_a + \omega^b{}_a) = 0 \ . \qquad (3.22)$$

We defined

$$\begin{aligned} \mathcal{R} &\equiv -\frac{1}{3}\Theta^2 - 2\sigma^2 + 2\omega^2 + A + \kappa\mu + \Lambda = \\ &= \frac{1}{2}\mathcal{K} + A - 3\sigma^2 + 2\omega^2 \ , \end{aligned} \qquad (3.23)$$

where $\mathcal{K}$ (defined by (2.30)) is the Ricci curvature ${}^{(3)}R$ of the surfaces orthogonal to the fluid flow when $\omega = 0$ (see section 2.1.5).

Equation (3.22) can be put in the form

$$S^{-3}h_a{}^b(S^3 Z_b)^{\cdot} = a_a\mathcal{R} + h_a{}^b\left[-\frac{1}{2}X_b - 2(\sigma^2)_{,b} + 2(\omega^2)_{,b} + A_b\right] - Z_b(\sigma^b{}_a + \omega^b{}_a)\ , \tag{3.24}$$

with $\mathcal{R}a_a$, X_a, A_a, $h_a{}^b(\sigma^2)_{,b}$, and $h_a{}^b(\omega^2)_{,b}$ acting as source terms, while the last non-linear term couples Z_a with the shear and vorticity.

Pressure and Curvature Gradients

We could derive the equation for the evolution of the pressure gradient Y_a proceeding from its definition, but this would not be an independent equation. Indeed, when the equation of state of the fluid is known, the evolution of Y_a will follow from that for X_a (see next chapter).

We could also derive a propagation equation for $\mathcal{K}_a$, either by taking the spatial gradient of (3.5) or by taking the spatial projection of the time derivative of (2.38). However, the resulting equation would be rather cumbersome, involving also the time derivative of σ. Therefore, we postpone the derivation of an equation for $\mathcal{K}_a$ to the next section, where we consider the linear approximation.

Equations for $\mathcal{X}_a$, $\mathcal{D}_a$ and $\mathcal{Z}_a$

The evolution equations for the fractional density gradient $\mathcal{X}_a$ and the comoving fractional density gradient $\mathcal{D}_a$ could be derived starting from their definitions (2.39;2.40), spatially projecting their derivatives with respect to the proper time. Also, we could derive such equations for $\mathcal{X}_a$ and $\mathcal{D}_a$ taking the spatial gradient of, respectively, $(3.2)/\mu$ and $S\times$ $(3.2)/\mu$, where (3.2) is the energy conservation equation.

However, both $\mathcal{X}_a$ and $\mathcal{D}_a$ are simply related to X_a. Therefore, it is more convenient to use this interdependence to express the relation between the time derivative of either $\mathcal{X}_a$ or $\mathcal{D}_a$ and the time derivative of X_a, since we have already derived the evolution equation for the latter.

These relations are:

$$\mathcal{X}_a = \frac{X_a}{\kappa\mu} \quad \Rightarrow \quad (\mathcal{X}_a)^{\cdot} = \frac{(X_a)^{\cdot}}{\kappa\mu} + \Theta\left(1 + \frac{p}{\mu}\right)\mathcal{X}_a \ , \tag{3.25}$$

and

$$\mathcal{D}_a = \frac{SX_a}{\kappa\mu} = S\mathcal{X}_a \quad \Rightarrow \quad (\mathcal{D}_a)^{\cdot} = S(\mathcal{X}_a)^{\cdot} + \frac{1}{3}\Theta\mathcal{D}_a \ . \tag{3.26}$$

Accordingly, the propagation equations for $\mathcal{X}_a$ and $\mathcal{D}_a$ follow from (3.19). They are

$$h_c{}^a(\mathcal{X}_a)^{\cdot} = \mathcal{X}_c\Theta\left(\frac{p}{\mu} - \frac{1}{3}\right) - Z_c\left(1 + \frac{p}{\mu}\right) - \mathcal{X}_a(\omega^a{}_c + \sigma^a{}_c) \ , \tag{3.27}$$

and

$$h_c{}^a(\mathcal{D}_a)^{\cdot} = \frac{p}{\mu}\Theta\mathcal{D}_c - \left(1 + \frac{p}{\mu}\right)\mathcal{Z}_a - \mathcal{D}_a(\omega^a{}_c + \sigma^a{}_c) \ , \tag{3.28}$$

where we have used $\mathcal{Z}_a = SZ_a$. The equation for this latter variables can be derived from that for Z_a (3.22) in the same way, and we write it here for completeness:

$$\begin{aligned} S^{-2}h_a{}^b(S^2\mathcal{Z}_b)^{\cdot} = & -\frac{1}{2}\kappa\mathcal{D}_a - \mathcal{Z}_b(\sigma^b{}_a + \omega^b{}_a) + \\ & + S\left\{a_a\mathcal{R} + A_a + h_a{}^b\left[2(\omega^2)_{,b} - 2(\sigma^2)_{,b}\right]\right\} \ . \end{aligned} \tag{3.29}$$

3.1.4 Some Remarks on Exact Equations

It should be emphasized that, given the perfect fluid assumption, the equations presented in this section are *exact propagation equations*, valid in any fluid flow whatever.

We see from (3.20; 3.27; 3.28) and from (3.24; 3.29) that the density gradients $X_a, \mathcal{X}_a$ and $\mathcal{D}_a$ are coupled with the expansion gradients Z_a and $\mathcal{Z}_a$. For example, a significant feature follows immediately from (3.20): provided $(\mu + p) \neq 0$, $Z_a \neq 0 \Rightarrow X_a \neq 0$. The converse result ($X_a \neq 0 \Rightarrow Z_a \neq 0$) will hold in general as well (if $X_a \neq 0$, $Z_a = 0$ then the right hand side of (3.24) must be zero; this is unlikely to remain true even if it is true at some initial time). Indeed, the equations for $X_a, \mathcal{X}_a, \mathcal{D}_a$, Z_a and $\mathcal{Z}_a$ contain non-linear terms coupling these quantities with shear, vorticity, acceleration, as well as the acceleration divergence A and its gradient A_a. Therefore, to consider a closed non-linear system of equations, one should take into account the evolution of all these quantities. Also, Maxwell-like equations should be included, because E_{ab} appear as a source term in the shear equation (3.6), and H_{ab} is coupled with E_{ab} in the $\dot{E}$ equation (3.13). Finally, the constraint equations (3.8;3.9;3.10) must be satisfied.

It is not surprising that to consider the full non-linear equation for the density gradient involve to take into account so many other quantities. After all, the full non-linear system is equivalent to the complete content of Einstein equations. We only chosen new variables, more suitable for the study of the growth in time of spatial density inhomogeneities.

An interesting point is then to adopt some reasonable hypothesis [4], to see if the problem to determine this growth can be solved.

The first step on this line (at which this thesis stand) is the linear approximation. Indeed, the two equations for X_a ($\mathcal{X}_a$ or $\mathcal{D}_a$) and Z_a ($\mathcal{Z}_a$) decouple from the others for an almost FLRW universe. We consider this case in the next section.

3.2 Linearisation about Robertson-Walker Universes

We now specialize the equations given in the previous section to the situation where the universe is almost FLRW. We do so by treating the quantities μ, p, and Θ as zero-order ($\mathcal{O}(0)$), because they do not vanish in an exact FLRW. We also assume that all the following quantities, and their derivatives, are first-order ($\mathcal{O}(1)$):
the shear (2.7), the vorticity (2.9), the acceleration (2.4), the electric and

[4] Physically motivated, and corresponding to some mathematical assumption.

magnetic part of the Weyl tensor (2.21), the gradients of density, pressure and expansion (2.36) (2.39) (2.40) (2.42), the divergence of acceleration and its gradient (2.37), the curvature gradient (2.38).

Moreover, since we have assumed a perfect fluid, the energy momentum tensor components (2.35) vanish.

To proceed to linearize, in each equation we drop the higher order terms relative to the lower order ones, keeping only the lowest two orders[5]. Note this does not mean we can always drop the 2nd order terms (in some equations the largest term is 1st order); and also that although we treat the pressure p as zeroth order, it may vanish; but we must allow for those cases where it is large.

The basic equations resulting from this process are given in Hawking's pioneering paper [26] (see his equations (13) to (19)): they are the linearization of equations (3.6;3.7;3.4; 3.11; 3.12; 3.13; 3.14) presented in section 3.1.2. However we consider here only those equations that are relevant for the time evolution of the gauge-invariant spatial density gradients, because the growth of density fluctuations in an almost FLRW universe is what we wish to investigate.

On carrying out this procedure, we still remain with a non yet fully linearized term, the gradient of the acceleration divergence[6]. This term will be treated separately, and we shall show how it reduces to the 3-Laplatian of the pressure gradient. After that, we will have linearised the covariant equations for our gauge-invariant variables about an as yet unspecified FLRW universe model; as the linearised equations hold for all choices of background FLRW models, they are gauge-invariant.

[5] As usual, the product of $\mathcal{O}(0) \times \mathcal{O}(0)$ is $\mathcal{O}(0)$, the product $\mathcal{O}(0) \times \mathcal{O}(1)$ is $\mathcal{O}(1)$, the product $\mathcal{O}(1) \times \mathcal{O}(1)$ is $\mathcal{O}(2)$, etc.

[6] Athough we assumed A_a to be first-order, it is not so if we express it as a function of pressure and density gradients, but see below.

3.2.1 Propagation Equations

Zero - Order Variables

The first equations that are relevant for determining the density fluctuation behaviour along the flow lines are those giving the evolution of zero - order quantities, i.e. , the energy and momentum conservation equations (3.2) and (3.3), which are unaffected by the linearisation procedure, and the linearised Raychaudhuri equation. This is

$$\dot{\Theta} + \frac{1}{3}\Theta^2 - A + \frac{1}{2}\kappa(\mu + 3p) - \Lambda = 0 \ , \tag{3.30}$$

where we have dropped the σ^2, ω^2 terms in (3.4).

First - Order Variables

The linearised equations for propagation of X_a and Z_a follow from (3.20;3.24) in the same way. They are

$$S^{-4}h_c{}^a(S^4X_a)^{\cdot} = -\kappa(\mu + p)Z_c \ , \tag{3.31}$$

and

$$S^{-3}h_c{}^a(S^3Z_a)^{\cdot} = -\frac{1}{2}X_c + \frac{1}{2}\mathcal{K}a_c + A_c \ , \tag{3.32}$$

where now

$$\frac{1}{2}\mathcal{K} = -\frac{1}{3}\Theta^2 + \kappa\mu + \Lambda = \mathcal{R}, \quad \dot{\mathcal{K}} = -\frac{2}{3}\Theta(\mathcal{K} + 2A) \ . \tag{3.33}$$

Note that linearizing $\mathcal{R}$ from (3.23) also gives a term A; however we drop this term because in (3.32) $\mathcal{R} = \frac{1}{2}\mathcal{K}$ is multiplied by the first - order quantity a_c, so A only gives a second - order contribution to (3.32).

The equation for $\mathcal{X}_a$ follows directly (or can be obtained by linearising (3.27), by dropping the last bracket). Similarly, the linearised equation for $\mathcal{D}_a$ and $\mathcal{Z}_a$ follow directly from (3.28) and (3.29); they are:

$$h_a{}^c(\mathcal{D}_c)^{\cdot} = \frac{p}{\mu}\Theta\mathcal{D}_a - \left(\frac{p}{\mu} + 1\right)\mathcal{Z}_a \ , \tag{3.34}$$

$$h_c{}^a(\mathcal{Z}_a)^{\cdot} = -\frac{2}{3}\Theta\mathcal{Z}_c - \frac{1}{2}\kappa\mu\mathcal{D}_c + S\left(\frac{1}{2}\mathcal{K}a_c + A_c\right) \ . \tag{3.35}$$

These are the linearised equations determining the propagation of the gradients along the fluid flow lines, and they are the basic perturbation equations. Indeed, we can already see from (3.31;3.32) and (3.34;3.35) that in the linear approximation the evolution equations for the density gradients and the expansion gradients decouple from that of the other quantities. Their development depends however on the equation of state of the fluid, as we shall see in the next chapter. Actually, the acceleration a_c term in the equations for Z_a and $\mathcal{Z}_a$ is just proportional to the pressure gradient Y_c, and this will be reduced to $\mathcal{D}_a$ via the equation of state. This also suggests that the gradient of acceleration divergence, A_c, can be reduced to Y_c in some way. Thus a last step remains toward the full linearization of the pairs of equations for the density and expansion gradients, namely the linearization of the A_c term.

Before to turn to this, we derive now an evolution equation for the curvature gradient $\mathcal{K}_a$. This is an easy task in the linear approximation, in which we obtain from (2.38)

$$\mathcal{C}_a = -\frac{4}{3}\Theta \mathcal{Z}_a + 2\kappa\mu\mathcal{D}_a \ , \tag{3.36}$$

where we defined the *comoving curvature gradient*

$$\mathcal{C}_a \equiv S\mathcal{K}_a \tag{3.37}$$

as the natural companion variable of $\mathcal{D}_a$ and $\mathcal{Z}_a$. A propagation equation for it, with $\mathcal{D}_a$ as source term, can be obtained by taking the spatial projection of the proper time derivative of (3.36), using (3.2;3.4) and (3.34;3.35), and substituting back for $\mathcal{Z}_a$ via (3.36). In doing that we obtain the following alternative pair of equation

$$S^{-2}h_a{}^b(S^2\mathcal{C}_b)^{\cdot} = \mathcal{K}\Theta^{-1}\left(\frac{1}{2}\mathcal{C}_a - \kappa\mu\mathcal{D}_a\right) - \frac{4}{3}\Theta S\left(\frac{1}{2}\mathcal{K}a_a + A_a\right) \ , \tag{3.38}$$

$$h_a{}^b(\mathcal{D}_a)^{\cdot} = \left(\frac{p}{\mu}\Theta - \frac{3}{2}\kappa(\mu+p)\Theta^{-1}\right)\mathcal{D}_a + \frac{3}{4}\Theta^{-1}\left(\frac{p}{\mu}+1\right)\mathcal{C}_a \ , \tag{3.39}$$

showing that $\mathcal{D}_a$ can be also determined through its coupling with the curvature gradient. In practice this is because (2.38) reduce to a constraint between $\mathcal{K}_a, \mathcal{D}_a$ and $\mathcal{Z}_a$, equation (3.36), in the linear approximation.

Linearizing the A_c Term

We want now to fully linearize the $\frac{1}{2}\mathcal{K}a_c + A_c$ term appearing in the rhs of the equations for $Z_a, \mathcal{Z}_a$ and $\mathcal{K}_a$.

The acceleration, given by (3.15), is determined via the momentum conservation (3.3), it is just proportional to the pressure gradient Y_a and it is a first-order quantity. Therefore we need to determine $\mathcal{K}$ only at zero-order in this term, i.e., we can neglect the A term in the second of (3.33), because the difference it makes in determining $\mathcal{K}$ is first-order. Accordingly, we have

$$\dot{\mathcal{K}} = -\frac{2}{3}\Theta + \mathcal{O}(1) \quad \Rightarrow \quad \mathcal{K} = \frac{6k}{S^2}, \quad \dot{k} = 0 \ , \tag{3.40}$$

where k corresponds to the curvature constant for background FLRW model underlying in our zero-order assumption. The outcome we obtain in this way is

$$\frac{1}{2}\mathcal{K}a_a = -\frac{1}{\kappa(\mu + p)}\frac{3k}{S^2}Y_a \ . \tag{3.41}$$

To linearize the A_a term it is first of all convenient toshow that A can be written as

$$A = -\frac{h_a{}^b Y^a{}_{;b}}{\kappa(\mu + p)} + O(2) \ .$$

Indeed, from the definition of A we have

$$A \equiv a^a{}_{;a} = a^a{}_{;b}(h^b{}_a - u^b u_a) = h_a{}^b a^a{}_{;b} - \ddot{u}^a u_a \ ,$$

and using

$$a^a u_a = 0 \Rightarrow \ddot{u}^a u_a = -a^a a_a \ ,$$

we see that this is a second-order term. Thus, using (3.15), we obtain at first-order

$$A = h_a{}^b a^a_{;b} = -h_a{}^b\left(\frac{Y^a}{(\mu + p)}\right)_{;b} = -\frac{h_a{}^b Y^a{}_{;b}}{\kappa(\mu + p)} + O(2) \ ,$$

or

$$A = \frac{{}^{(3)}\nabla_a Y^a}{\kappa(\mu + p)} \ , \tag{3.42}$$

where ${}^{(3)}\nabla_a$ is the 3 - covariant derivative defined in section 2.1.5. Now, using the definition of A_a and the above result we can write

$$A_a \equiv h_a{}^b A_{,b} = {}^{(3)}\nabla_a A = -{}^{(3)}\nabla_a \left(\frac{{}^{(3)}\nabla_b Y^b}{\kappa(\mu + p)} \right) ,$$

that is, at first - order,

$$A_a = -\frac{{}^{(3)}\nabla_a \, {}^{(3)}\nabla_b \, {}^{(3)}\nabla^b p}{\kappa(\mu + p)} . \tag{3.43}$$

But ${}^{(3)}\nabla_a \, {}^{(3)}\nabla^a \, {}^{(3)}\nabla_b p = {}^{(3)}\nabla_a \, {}^{(3)}\nabla_b \, {}^{(3)}\nabla^a p = \frac{1}{\kappa\mu} {}^{(3)}\nabla_a \, {}^{(3)}\nabla_b Y^a$, and using the commutation rule of the ${}^{(3)}\nabla$'s, namely the 3 - Ricci identity (2.28), we have

$$ {}^{(3)}\nabla_a \, {}^{(3)}\nabla_b Y^b = {}^{(3)}\nabla_b \, {}^{(3)}\nabla_a Y^b - \frac{1}{3} Y_a {}^{(3)}R , \tag{3.44}$$

where we have substituted for the 3 - dimensional Ricci tensor ${}^{(3)}R^b{}_a$ from (2.30), using ${}^{(3)}R^b{}_a = \frac{1}{3} h_a{}^{b(3)}R$ at the zero - order needed in the previous equation. But at the zero - order we have ${}^{(3)}R = \mathcal{K} = 6k/S^2$, therefore we finally obtain

$$A_a = \frac{1}{\kappa(\mu + p)} \left(\frac{2k}{S^2} Y_a - {}^{(3)}\nabla^2 Y_a \right) , \tag{3.45}$$

on using the notation ${}^{(3)}\nabla^2 V_b \equiv {}^{(3)}\nabla_a \, {}^{(3)}\nabla^a V_b$ for the 3 - dimensional Laplatian. If we sum the above result with the previous one, equation (3.41), we obtain the required first - order expression

$$\frac{1}{2}\mathcal{K} a_a + A_a = -\frac{1}{\kappa(\mu + p)} \left(\frac{k}{S^2} Y_a + {}^{(3)}\nabla^2 Y_a \right) . \tag{3.46}$$

Thus we have explicitly shown that the term $\frac{1}{2}\mathcal{K} a_a + A_a$ can be reexpressed as a function of Y_a only. Then it remains only one step to close the systems of linear equations for the density and expansion gradients we have obtained in this section. Namely, we have to consider an equation of state that will enable us to express Y_a in terms of X_a. This will be the task of the next chapter.

As we said in 2.1.5, the quantity we calculate as a curvature tensor, using the usual definition from commutation of second derivatives (2.28), will not have all the usual curvature tensor symmetries when $\omega \neq 0$. Nevertheless the zero - order equations, representing the curvature of the 3 - spaces orthogonal to the fluid flow in the background model, will agree with the linearized equations up to the required accuracy.

3.2.2 Constraint Equations

While the constraint equations are not needed to determine the propagation of interesting quantities along the flow lines, they must of course be satisfied at some initial time on each world line. This gives interesting information about what is and is not possible.

Specifically, the linearised momentum constraint equation we obtain from (3.8) ((10) in Hawking [26]) is

$$h_a{}^b(\omega_b{}^c{}_{;c} - \sigma_b{}^c{}_{;c}) = -\frac{2}{3}Z_a. \tag{3.47}$$

This shows that if Θ varies spatially, i.e. $Z_a \neq 0$, then either the shear or the vorticity must also be non - zero. Conversely only restricted shear and vorticity perturbations will be compatible with Z_a remaining zero.

Similarly, the linearised "div E" Maxwell - like Bianchi identities (3.11) ((13) in Hawking [26]) is

$$E^{ab}{}_{;b} = \frac{1}{3}h^{ab}\kappa\mu_{,b} = \frac{1}{3}X^a \tag{3.48}$$

showing that the electric part E_{ab} of the Weyl tensor must be non - zero if there is a non - zero density gradient (i.e. if $X_a \neq 0$).

These results give a warning that consistent solutions to the field equations may demand inclusion of non - zero gauge - independent variables not initially anticipated.

Finally, linearization of (3.9) simply reduces to show that vorticity divergence vanish at first - order:

$$\omega^a{}_{;a} = 0 + \mathcal{O}(2) \ , \tag{3.49}$$

while from (3.10) we obtain

$$H_{ad} = -h_a{}^e h_d{}^g \left(\omega_{(e}{}^{b;c} + \sigma_{(e}{}^{b;c}\right) \eta_{g)fbc} u^f \ , \qquad (3.50)$$

determining H_{ab} at first - order.

3.3 The Implied "Gauge"

3.3.1 Natural Map

Before turning to specific equations of state, we briefly consider the gauge issue relative to the formulation here. Our equations are gauge - invariant, so we can choose any map Φ we like from $\bar{S}$ to S when using this formalism (just as we can use any coordinates we like in $\bar{S}$ and in S, because the formalism is covariant). However there is a natural map Φ from an idealised FLRW model $\bar{S}$ to the realistic model S associated with our formalism, which is the obvious one to choose unless there is some good reason to use a different correspondence. We consider here this map, naturally implied by the analysis (see Fig. 3.1).

namely to examine the propagation of each quantity along the fluid flow lines. Because of the perfect fluid form (11), these are uniquely defined provided $(\mu + p) \neq 0$, which we almost always assume. The naturally associated map Φ from $\bar{S}$ to S maps fluid flow lines to fluid flow lines. This means we compare observations made by fundamental observers in the two universes.

(B) Because of the spatial homogeneity of the FLRW models it does not matter which specific flow line in $\bar{S}$ is mapped into which one in S.

(C) The implied time coordinate t in S is proper time along the fluid flow lines; it is the time-coordinate of normalised comoving coordinates (t, y^ν) (see e.g. Ehlers [7], Ellis [11], Treciokas and Ellis [55]), characterised by $\dot{t} \equiv t_{,a}u^a = 1$, $\dot{y}^\nu \equiv (y^\nu)_{,a}u^a = 0$. It is arbitrary by choice of some initial surface Σ_0, i.e the freedom in t is

$$t \rightarrow t' = t + f(y^\nu) \qquad (3.51)$$

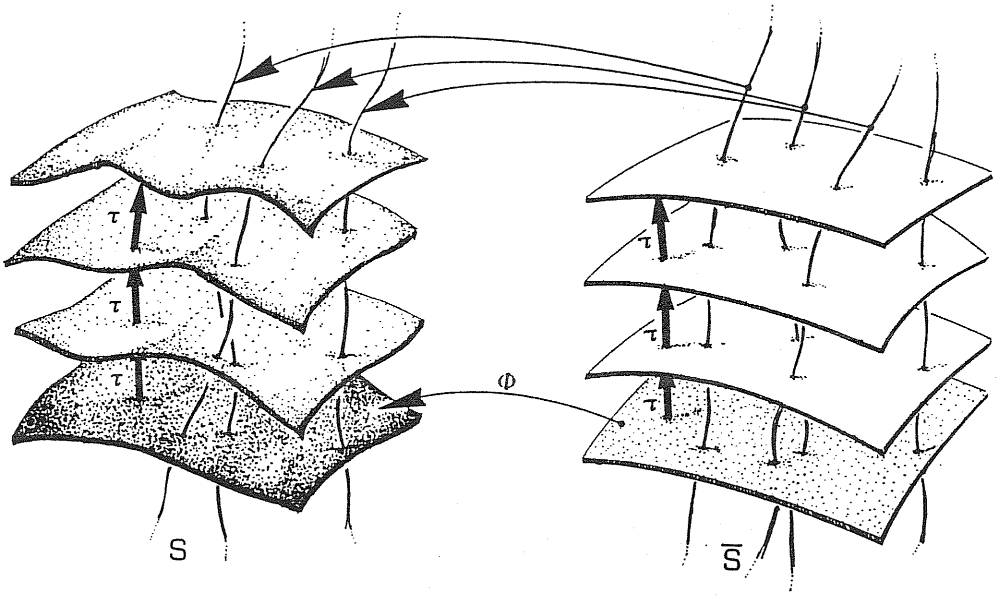

Figure 3.1: In the *proper time gauge* (the time coordinate denotes proper time along the fluid flow lines), we have freedom to choose an initial time surface $\Sigma_0 : \{t = t_0\}$ arbitrarily; then the other time surfaces are determined by measuring proper time from it along the fluid flow lines. This has the advantage of corresponding to time measurements made by fundamental observers.

where f is an arbitrary function of the "spatial coordinates" y^ν. Thus we compare evolution in the universes $\bar{S}$ and S with respect to proper time measured by the fundamental observers in each model (the standard time $\bar{t}$ in $\bar{S}$ is proper time along the fluid flow lines, without the freedom (3.51) because we take $\bar{t} = 0$ at the big bang $\bar{S} = 0$). The objections raised to this choice by Bardeen [1] do not apply here, for the variables X_a, Y_a, and Z_a will be small in any space-time region where the universe S is "near" some FLRW universe $\bar{S}$, irrespective of how the time coordinate t is chosen, and the *definition* of our variables is independent of the time choice; thus the "non-locality" issue discussed previously (see section 1.2.2) does not affect the physical interpretation of our variables.

(D) The specific map of times from the idealised model $\bar{S}$ to the realistic model S will be represented by a choice of constants of integration in the solutions to the zero - order propagation equations (3.2) and (3.30), which then determine the solutions to the propagation equations (3.31;3.32), (3.34;3.35) or (3.38;3.39) for the gauge - independent variables; in effect, the zero - order solutions are arbitrary by independent constants along each world line. This choice corresponds to the gauge freedom above, and may be thought of as choosing specific initial conditions for the perturbed universe at an initial time t_0.

In the present approach, the definition of the perturbation quantities is independent of the gauge chosen; however we have to choose a specific gauge to obtain detailed specific solutions of the equations (just as we have to choose specific coordinates to write down a specific detailed solution to covariant equations). This freedom should be left to the end (being represented by the integration constants that naturally arise). Variation of these constants then corresponds to variation of the gauge, and also enables us to explore the effects of different initial conditions on the evolution of the gauge-independent variables (or equivalently, to explore their evolution in families of differing FLRW models instead of only one model). The essential problem in the non - gauge invariant approach - that the *definition* of $\delta\mu$ depends on this choice - does not arise with the variables proposed here.

3.3.2 Meaning of $\mathcal{K}$ when $\omega \neq 0$

When $\omega \neq 0$, there are no surfaces orthogonal to the family of fluid flow lines, but we can find normalised comoving coordinates $\{t, y^\nu\}$ satisfying (3) (see Ehlers [7] Treciokas and Ellis [55]). Using such coordinates, the surfaces $\{t = const\}$ can be set orthogonal to a particular chosen world line γ and almost orthogonal to neighbouring world lines, by the remaining gauge freedom (43) (e.g. if we choose an initial surface $\{t = t_0\}$ to be generated by orthogonal geodesics emanating from γ). Then $\mathcal{K}$, given by (12), will be nearly the Ricci-scalar of these 3-spaces on and near γ. Note however these surfaces do not directly correspond to the FLRW surfaces $\{t = const\}$ when there are spatial density gradients, because if $X_a \neq 0$ the surfaces $\{\mu = const\}$ do not lie orthogonal to the world-lines; similarly if $Z_a \neq 0$ the surfaces $\{\Theta = const\}$ do not lie orthogonal to the world lines.

More generally, if u^a is not too different from the normals n^a to a family of surfaces, then $\mathcal{K}$ will be not too different from the Ricci scalar of those 3-spaces. The meaning of "not too different" can be made precise by either using (a) a formalism equivalent to the ADM [7] lapse and shift formalism (cf. Bardeen [1] section VI), (b) the tilted flow vector formalism of King and Ellis (1973), or (c) adapted comoving coordinates mentioned above.

[7] Acronym for: Arnowitt, Deser, and Misner; they developed a particular formalism in 1962. See for example: Wald (1984) [57].

Chapter 4

SIMPLE APPLICATIONS

In this chapter we shall apply the linear equations obtained in the previous chapter to some specific cases. In particular we shall restrict our analysis to a perfect fluid which can be described by a simple equation of state. In this approximation, the final aim of this chapter is to derive a second-order linear equation for $\mathcal{D}_a$ (or some related variable), and analyse its properties and simple solutions.

In section 1 we briefly comments on the different kinds of fluid that we can treat within our approach (or some generalization of it), and we discuss the different equations of state that arise for them.

In the second section we restrict ourselves to the case of fluids which have an equation of state for which the energy density is the only independent thermodynamic variable. In this case we can speak of adiabatic (isentropic) perturbations. Then we explicitly show how the linear systems of two first-order equations derived in section 3.2 decouple from the other evolution equations, by virtue of the assumed equation of state. We consider systems for $\mathcal{D}_a$ and $\mathcal{Z}_a$, as well as for SX_a and $\mathcal{C}_a$. Also, we introduce a "Bardeen-like" variable Φ_a, and we derive a linear equation for it.

In section 3 we derive second-order linear equations for $\mathcal{D}_a$ and Φ_a and we analyse some of their properties. We use a harmonic expansion for our variables. This is a standard technique, which enables us to derive ordinary differential equations for the harmonic components of the various variables. We consider the Jeans instability for matter against gravity, giving a correction to a previous result by Jackson [30]. A first integral is obtained in the long - wavelength limit, corresponding to an analogous well - known [2,38] first integral. Also, we comment on the isocurvature and scalar modes.

The aim of the fourth section is to obtain explicit solutions of the second - order linear equations derived in the previous section. We consider the zero - pressure fluid (dust), and the pure radiation case, for which explicit solutions are obtained in the long - wavelength limit. Since our equations and variables are gauge - invariant, we do not obtain the usual decaying gauge - mode. Also, we briefly consider the case of a mixture of perfect fluids when those perfect fluids all have the same 4 - velocity.

Finally, in the last section we consider the evolution of density on neighbouring world - lines. We show how we can define scalar quantities closely related to our gradients, and we derive exact and linear evolution equations for them.

4.1 Specific Matter Descriptions

In section 3.2 we have obtained linear evolution equations valid in an almost FLRW universe. These are systems of two first - order equations coupling, for example, the comoving fractional density gradient $\mathcal{D}_a$ with the comoving expansion gradient $\mathcal{Z}_a$ (or $\mathcal{D}_a$ with the comoving curvature gradient $\mathcal{C}_a$). However, the equations for $\mathcal{Z}_a$ (3.34) and $\mathcal{C}_a$ (3.38) contain an additional term, which has been shown to be a function of the pressure gradient Y_a (see equation 3.46). Therefore, we still need an equation of state, describing the physics of the situation, in order to close our system of first - order equations.

The intent of this section is to briefly comment on general equations of state, before to apply our equations to specific cases.

4.1.1 Fluid Equations of State

In general we may wish to study perturbations with a scalar field, fermionic matter, or other matter sources; or using a kinetic theory description. We here concern ourselves with situations where a simple or multifluid description is appropriate. Three rather different cases arise.

Imperfect Fluids

Imperfect fluids will have non-zero energy flux vectors q_a and/or anisotropic pressures π_{ab} (these are the components of the matter tensor (2.35)). These could occur due to dissipative processes, when suitable equations of state will determine the quantities q_a and π_{ab} (see e.g. Ehlers [8]); the approach to density fluctuations introduced in EB [17] and presented in this thesis has been generalized to this case by Hwang and Vishniac (1989) [29].

However, this description is also appropriate for *multi-component perfect fluids with different 4-velocities.* In the latter case it would be natural to describe the situation relative to the 4-velocity of the dominant component; the effective stress tensor of any other perfect fluid, moving relative to this 4-velocity, will be that of an imperfect fluid [31]. This would be the situation for example in *isothermal perturbations* where the surfaces of constant matter density are different from the surfaces of constant radiation density, for in general their 4-velocities will differ, leading to such phenomena as radiation drag.

The methods introduced in EB [17] can be adapted to this case, but the resulting equations are rather more complex than those presented here.

Non-Barotropic Perfect Fluids

Non-barotropic perfect fluid occur when there are two essential thermodynamic variables, so that the matter tensor still has the form (2.16) and all the equations in chapter 3 hold, but $p \neq p(\mu)$. The importance of this is that then in general $a_{[a}X_{b]} \neq 0$, so that a_a and X_c are not parallel in (3.31), (3.32), implying Z_a (and so X_a) will not be Fermi-propagated[1] along the fluid flow-lines (they will rotate relative to a local inertial rest frame).

A particular case of interest is that of *multi-component perfect fluids with the same 4-velocity*, e.g. baryonic matter plus radiation that is isotropic about that matter. This might be expected to be the case in *isentropic perturbations*[2], where both the matter and radiation are significant but the surfaces of constant matter density are the same as the surfaces of constant radiation density (the baryon to photon ratio is constant), then in general their 4-velocities must coincide, else this condition will not be maintained. We can then represent the equation of state in terms of the simple relativistic *γ-law equation of state*

$$p = (\gamma - 1)\mu, \tag{4.1}$$

where $\gamma = \gamma(S)$ takes a simple form when the fluid components are non-interacting (cf. Madsen and Ellis [39]).

Barotropic Perfect Fluids

Barotropic perfect fluids are perfect fluids where p and μ are functionally dependent: $p = p(\mu)$. Then there will be a well-defined speed of sound $v_s = (dp/d\mu)^{1/2}$ limiting communication by fluid processes, and from (3.3), a_a and X_b are necessarily functionally dependent and parallel. Equations introduced in chapter 3 apply.

The simplest situation is when v_s is constant (cf. Olson [43]); then the relativistic γ-law description (4.1) may be used where now γ is constant. The important cases are $\gamma = 1$ (dust), $\frac{4}{3}$ (radiation), or 0 (false vacuum). The third case is very briefly considered below. The other cases will be discussed later, in section 4.4.

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[1] That is, their Fermi derivative is non-zero. See Hawking and Ellis (1973) [27], pages 80, 81.

[2] Also referred to as *adiabatic*.

4.1.2 False Vacuum

The "false vacuum" equation of state occurs if the stress tensor is Lorentz invariant, i.e. if $T_{ab} \propto g_{ab}$. This will be a good representation of the stress tensor of a scalar field ϕ when $\dot{\phi}$ is nearly zero (e.g. it underlies the concept of exponential inflation in the early universe.)

The false vacuum is equivalent to a perfect fluid for which $\mu + p = 0$; we see directly from (3.31) that then $S^4 X_a$ is constant along the fluid flow lines (which are not uniquely defined, in this case). Thus spatial density gradients die away as S^{-4}, independent of their wavelength; relative gradients $\mathcal{X}_a$ also die away as S^{-4}; but comoving fractional density gradient $\mathcal{D}_a$ die away as S^{-3}.

4.2 Relevant Systems of First - Order Equations

4.2.1 Adiabatic Perturbations

As we said above, for a general perfect fluid the pressure p and energy density μ are related by a suitable equation of state with two independent thermodynamic variables. For example, if s is the entropy then we can express it in the form

$$p = p(\mu, s) \ , \tag{4.2}$$

but as always in thermodynamics many other representations are possible for the equation of state. From the perfect fluid assumption it follow that the entropy is constant along each flow line (see equation (3.13) in Ellis (1971) [11]), thus *there is only one independent thermodynamic variable along each world line.* However, for the fluid in consideration, the entropy can vary spatially. In this case, it follows from (4.2) that that

$$\frac{1}{\kappa} Y_a = \left(\frac{\partial p}{\partial \mu}\right)_{(s)} {}^{(3)}\nabla_a \mu + \left(\frac{\partial p}{\partial s}\right)_{(\mu)} {}^{(3)}\nabla_a s \ , \tag{4.3}$$

where, as usual in thermodynamic, the subscripts means that the derivative with respect to one variable is computed while the other is fixed. The assumption we now make is that we can ignore the second term (pressure variations caused by spatial entropy variations) relative to the first (pressure variations caused by energy density variations). This is at any effect equivalent to assume vanishing of the spatial entropy variation, that is, we can use an equation of state of the form

$$p = p(\mu) \tag{4.4}$$

for our practical purpose. The perturbations we are going to consider are usually referred to as *adiabatic* or *isentropic*.

We also ignore spatial variations in the scale function S (which would at most cause 2nd order variations in the propagation equations). Then (ignoring terms due to the spatial variation of $dp/d\mu$, which will again cause second order variations) we find that we can express the term (3.46) as

$$S(\frac{1}{2}\mathcal{K}\dot{u}_a + A_a) = -\frac{1}{(1+p/\mu)}\left(\frac{dp}{d\mu}\right)\left(\frac{k}{S^2}\mathcal{D}_a + {}^{(3)}\nabla^2\mathcal{D}_a\right) . \tag{4.5}$$

With this result we can now explicitly close the system of equations obtained in section 3.2.

Before to turn to this, we follow Bardeen [1] by defining

$$w = p/\mu, \;\; c_s^2 = dp/d\mu \;\Rightarrow\; \left(\frac{p}{\mu}\right)^{\cdot} \equiv \dot{w} = -(1+w)(c_s^2 - w)\Theta \; , \tag{4.6}$$

for easy comparison with the literature.

4.2.2 The System for $\mathcal{D}_a, \mathcal{Z}_a$

We will focus here on the gauge invariant variables $\mathcal{D}_a$ and $\mathcal{Z}_a$; with the definitions (4.6) and the result (4.5) the equations (3.34; 3.35) become

$$h_a{}^c(\mathcal{D}_c)^{\cdot} = w\Theta\mathcal{D}_a - (w+1)\mathcal{Z}_a \; , \tag{4.7}$$

$$h_c{}^a(\mathcal{Z}_a)^{\cdot} = -\frac{2}{3}\Theta\mathcal{Z}_c - \frac{1}{2}\kappa\mu\mathcal{D}_c - \frac{c_s^2}{(1+w)}\left(\frac{k}{S^2}\mathcal{D}_a + {}^{(3)}\nabla^2\mathcal{D}_a\right) \; , \tag{4.8}$$

where k, arising from (3.40; 3.41), is the usual curvature constant for the background FLRW model. The above system of first - order linear evolution equations is now a close system for $\mathcal{D}_a$ and $\mathcal{Z}_a$ only. Before to consider two alternative descriptions of the growth of density inhomogeneities we comment on some general properties of the equations above.

(1) Inhomogeneity on a world line γ is indicated by at least one of $\mathcal{D}_a$, $\mathcal{Z}_a$ being non-zero. Because the equation governing its evolution are homogeneous, inhomogeneity cannot arise spontaneously: if both $\mathcal{D}_a$ and $\mathcal{Z}_a$ are zero at any event p on γ, then they are both zero at all events on γ; if either is non-zero at any event on γ, they are both non-zero at almost all events on γ (one or the other may be zero at exceptional events).

(2) In general, $\mathcal{D}_a$ and $\mathcal{Z}_a$ are not parallel. However if they are parallel at one event p on γ, they are parallel at all events on γ; and if either vanishes at any event q on γ, they are parallel at all events on γ where they are non-zero. In these cases, the vector equations reduce to scalar equations, giving the rate of change of the relevant magnitude along γ (see section 4.3.7).

4.2.3 A System involving $\mathcal{C}_a$

In an analogous way we can obtain from (3.38; 3.39) a close system involving $\mathcal{C}_a$. However, since we want explicitly show here the close correspondence between our work and that of Lyth and Mukherjee [38], we shall couple $\mathcal{C}_a$ with the comoving density gradient SX_a. For this pair of variables we obtain

$$h_a{}^b(SX_b)^{\cdot} = -(SX_a)\left[\Theta + \frac{3}{2}\kappa\mu(1+w)\Theta^{-1}\right] + \frac{3}{4}\Theta^{-1}\kappa\mu(1+w)\frac{(S^2\mathcal{C}_a)}{S^2}\,, \quad (4.9)$$

$$\begin{aligned} S^{-2}h_a{}^b(S^2\mathcal{C}_b)^{\cdot} = {} & \frac{6k}{S^2}\Theta^{-1}\left(\frac{1}{2}\mathcal{C}_a - SX_a\right) + \\ & + \frac{4}{3}\Theta\frac{c_s^2}{\kappa\mu(1+w)}\left(\frac{k}{S^2}(SX_a) + {}^{(3)}\nabla^2(SX_a)\right)\,, \end{aligned} \quad (4.10)$$

which (remembering the Hubble parameter H is given by $H = \frac{1}{3}\Theta$) closely correspond to the system of equations (24; 28) of Lyth and Mukherjee [38], except that we did not yet harmonically analyzed our equations (we shall turn to this in section 4.3.3), and our system is for a general curvature background[3].

Note that, because of the linear constraint (3.36), the comments at the end of the previous section apply also to the pair $SX_a, \mathcal{C}_a$.

4.2.4 Equation for a "Bardeen - like" Variable

We want now introduce a new vector formally corresponding to the Bardeen [1] potential Φ_H (up to a constant). This enable us to an easy comparison. We can thus define

$$\Phi_a \equiv \kappa\mu S^2\mathcal{D}_a \;=\; S^3X_a\ , \tag{4.11}$$

where we have not required a harmonic analysis to make this definition. A point however must be stressed: our variable Φ_a is a vector, while the Bardeen Φ_H is not. Moreover the correspondence is purely formal, in the sense that while our Φ_a is just proportional to $\mathcal{D}_a$, Φ_H is a potential for the Bardeen's variable ϵ_m[4] (this is a gauge - invariant quantity for the density perturbation in the Bardeen approach). A more extended analysis of the relation between our and Bardeen approach can be found in Hwang and Vishniac (1989) [29].

The rate of variation of the Bardeen - like variable (4.11) follows directly from the equations above; it is given by

$$\dot{\Phi}_{\perp a} + \frac{1}{3}\Theta\Phi_a + \frac{3}{2}(w+1)\frac{\kappa\mu}{\Theta}\Phi_a = \frac{3}{4}(1+w)\frac{\kappa\mu}{\Theta}(S^2\mathcal{C}_a) \tag{4.12}$$

where we have written it in terms of the comoving curvature gradient $\mathcal{C}_a$ and we use the subscript $\perp$ to denote projection orthogonal to u^a. The equation coupling with (4.12) is just (4.10), with a trivial substitution $S\dot{X}_a \;\rightarrow\; \Phi_a$ from (4.11).

[3]Actually, the correspondence between equation (4.10) and equation (28) of [38] became clear if we remember (3.40); then $\delta k \rightarrow \frac{S^2\mathcal{C}_a}{6}$.

[4]Note however that, if we armonically analyze Φ_H and ϵ_m, their components become proportional.

4.3 Second - Order Equations

The system of equations given above can now be used to obtain second order equations for $\mathcal{D}_a$ and Φ_a.

4.3.1 A Second - Order Equation for $\mathcal{D}_a$

Now differentiation of (4.7) and projection orthogonal to u^a gives a second-order equation for $\mathcal{D}_a$ (we use (3.30, 4.8, 4.6) and (4.5) in the process). As before we use the subscript $\perp$ to denote projection orthogonal to u^a, i.e. we write $h_a{}^c(\mathcal{D}_c)^{\cdot\cdot} \equiv \ddot{\mathcal{D}}_{\perp a}$, $h_a{}^c(\mathcal{D}_c)^{\cdot} \equiv \dot{\mathcal{D}}_{\perp a}$. We find

$$\begin{aligned}\ddot{\mathcal{D}}_{\perp a} &+ \left(\frac{2}{3} - 2w + c_s^2\right)\Theta\dot{\mathcal{D}}_{\perp a} - \\ &- \left[\left(\frac{1}{2} + 4w - \frac{3}{2}w^2 - 3c_s^2\right)\kappa\mu + (c_s^2 - w)\frac{12k}{S^2} + (5w - 3c_s^2)\Lambda\right]\mathcal{D}_a + \\ &+ c_s^2\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) = 0\ . \qquad (4.13)\end{aligned}$$

This equation is the basic result of this chapter; the rest of the discussion examines its properties and special cases. It is a second order equation determining the evolution of the gauge-invariant density variation variable $\mathcal{D}_a$ along the fluid flow lines, equivalent to the central equation of Bardeen's paper [1] (see Hwang and Vishniac [29]). It has the form of a wave equation with extra terms due to the expansion of the universe, gravity, the spatial curvature, and the cosmological constant. We bracket the last two terms together because when we make a harmonic decomposition corresponding to that made by Bardeen (see section 4.3.3), these terms together give the harmonic eigenvalues n^2.

The form of the equation (4.13) allows for a variation of $w = p/\mu$ with time. However if $w = const$, then from (4.6), $c_s^2 = w$, and the equation simplifies to

$$\begin{aligned}\ddot{\mathcal{D}}_{\perp a} &+ \left(\frac{2}{3} - w\right)\Theta\dot{\mathcal{D}}_{\perp a} - \\ &- \left(\frac{(1-w)(1+3w)}{2}\kappa\mu + 2w\Lambda\right)\mathcal{D}_a + w\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) = 0 \qquad (4.14)\end{aligned}$$

This reduce to an ordinary differential equation for the particular case of vanishing pressure matter (dust), $w = 0$, for which thus we have a separate evolution of perturbations along each flow line. The matter source term in (4.14) vanishes if $w = 1$ (the case of "stiff matter" $\Leftrightarrow p = \mu$) or $w = -1/3$ (the case $p = -\mu/3$, corresponding to matter with no active gravitational mass). Between these two limits ("ordinary matter"), the matter term is positive and tends to cause the density gradient to increase ("gravitational aggregation"); outside these limits, the term is negative and tends to cause the density gradient to decrease ("gravitational smoothing"). A positive Λ-term tends to cause gravitational aggregation if w is positive (but smoothing if w is negative). Also the sign of the damping term (giving the adiabatic decay of inhomogeneities) is positive if $2/3 > w$ (that is, $2\mu > 3p$) but negative otherwise (they adiabatically grow rather than decay in this case).

While this form is expressed in terms of $\kappa\mu$, it is convenient for many applications to substitute for μ from (3.33) and (3.40), that is, at zero order[5]

$$\kappa\mu = \frac{1}{3}\Theta^2 + \frac{3k}{S^2} - \Lambda \ . \tag{4.15}$$

We do so and drop Λ (which can be represented by setting $w = -1$) to obtain

$$\ddot{\mathcal{D}}_{\perp a} + \left(\frac{2}{3} - w\right)\Theta\dot{\mathcal{D}}_{\perp a} - \\ - \frac{(1-w)(1+3w)}{2}\left(\frac{1}{3}\Theta^2 + \frac{3k}{S^2}\right)\mathcal{D}_a + w\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) = 0 \tag{4.16}$$

a form convenient for most applications.

4.3.2 A Second - Order Equation for Φ_a

We can directly find the second order equation for Φ_a (4.11) from the equations above, (4.12; 4.10), obtaining

$$\ddot{\Phi}_{\perp a} + \dot{\Phi}_{\perp a}\Theta\left(\frac{4}{3} + c_s^2\right) + \left\{(c_s^2 - w)\kappa\mu - \frac{2k}{S^2}(1 + 3c_s^2) + \Lambda(1 + c_s^2)\right\}\Phi_a + \\ + c_s^2\left(\frac{2k}{S^2}\Phi_a - {}^{(3)}\nabla^2\Phi_a\right) = 0 \ , \tag{4.17}$$

[5]We neglected an A term in (3.33). Note that this is just the Friedmann equation, valid along each flow line.

which simplifies in the case $c_s^2 = w$, $\Lambda = 0$ to the form

$$\ddot{\Phi}_{\perp a} + \dot{\Phi}_{\perp a}\Theta\left(\frac{4}{3} + w\right) - \frac{2k}{S^2}(1+3w)\Phi_a + w\left(\frac{2k}{S^2}\Phi_a - {}^{(3)}\nabla^2\Phi_a\right) = 0 \ . \quad (4.18)$$

4.3.3 Harmonic Decomposition

It is standard [1,26,34,35,38,43] to decompose the variables harmonically, thus effectively separating out the time and space variations; this conveniently represents the idea of a comoving wavelength for the matter inhomogeneities. In our case we do so by writing $\mathcal{D}_a$ in terms of harmonics $Q_a^{(n)}$ from which the background expansion has been factored out. It must however stressed that the harmonic analysis is straightforward only for finite (compact) 3-spaces [4] (see also D'Eath [5], quoted in Bardeen [1]); then, what is usually done is to consider finite comoving volumes with periodic boundary conditions [38], and to use a discrete harmonic analysis there. We only briefly touch these problems here, and we do not give an extensive introduction to the different kinds of harmonics (see for example: Börner [4], Bardeen [1], or Kodama and Sasaki [32]).

We start with the defining equations for the scalar harmonics

$$(Q^{(n)})_{;c}u^c = 0 \ , \quad {}^{(3)}\nabla^2 Q^{(n)} = -\frac{n^2}{S^2}Q^{(n)} \ , \quad (4.19)$$

corresponding to Bardeen's scalar Helmholtz equation (2.7), but expressed covariantly following Hawking [26]. From these quantities we define the gradient [6] harmonics (cf. [1], equations (2.8), (2.10); we do not divide by the wave number, however, so our equations are valid even if $n = 0$)

$$Q_a^{(n)} \equiv S\ {}^{(3)}\nabla_a Q^{(n)} \Rightarrow \quad Q_a^{(n)}u^a = 0 \ , \quad (Q_a^{(n)})_{;c}u^c \simeq 0 \ ,$$

$${}^{(3)}\nabla^2 Q_a^{(n)} = -\frac{(n^2 - 2k)}{S^2}Q_a^{(n)} \ , \quad (4.20)$$

[6] We call these "gradient harmonics" to distinguish them from the intrinsically vectorial harmonics with vanishing divergence [1,4,32]. Any generic vector in the 3-space can be completely decomposed with these two kinds of harmonics.

where the factor S ensure that these gradient harmonics are approximately covariantly constant along the fluid flow lines in the almost-FLRW case. Then we can write μ in terms of these harmonics as

$$\mu = \sum_n \mu^{(n)} Q^{(n)}, \quad {}^{(3)}\nabla \mu^{(n)} \simeq 0, \tag{4.21}$$

$\mu^{(n)}$ being the nth. harmonic component of μ (approximately constant in the directions orthogonal to u^a; they cannot be *exactly* constant in all these directions if $\omega \neq 0$, for if they were they would define surfaces orthogonal to the fluid flow lines, see section 2.1.5). As usual the harmonics are orthogonal to each other in a suitable measure (the details depending on whether $k = +1$, 0, or -1), so the coefficients $\mu^{(n)}$ can be determined by suitable weighted integrals of μ. However we have to worry about the convergence of these integrals; this may require consideration of finite boxes in the universe, or subtraction of a time-varying function from μ before doing the harmonic analysis. In the latter case it may be preferable to use an alternative representation:

$$\mu = \mu_0 (1 + \sum_n \delta^{(n)} Q^{(n)}), \quad {}^{(3)}\nabla \mu_0 \simeq 0, \quad {}^{(3)}\nabla \delta^{(n)} \simeq 0, \tag{4.22}$$

where μ_0 is a solution of the zeroth-order equations and $\delta^{(n)}$ are the nth. fractional harmonic component of μ (again, these functions are approximately constant in the directions orthogonal to u^a). Now suitable choice of the background solution (e.g. such that the Traschen integral constraints [54] are satisfied) will make all the harmonic coefficients small and ensure these integrals (specifically, that for $n = 0$) converge. In this case there is a gauge arbitrariness in defining the harmonics, that will affect the higher order terms but not the linear calculations of this chapter (because we do not use the absolute values of these coefficients, but rather their spatial gradients).

In either case it then follows from the definition of $\mathcal{D}_a$ that

$$\mathcal{D}_a = \sum_n \mathcal{D}^{(n)} Q_a^{(n)}, \quad {}^{(3)}\nabla_b \mathcal{D}^{(n)} \simeq 0 \tag{4.23}$$

where $\mathcal{D}^{(n)}$ is the harmonic component of $\mathcal{D}_a$ corresponding to the comoving wave-number n, containing the time- variation of that component; to first order, $\mathcal{D}^{(n)} \equiv (\mu^{(n)}/\mu) \equiv \delta^{(n)}$. Putting this decomposition in the linearised equations (4.13), (4.14) or (4.16), the harmonics decouple. Thus for example we obtain from (4.16) the nth harmonic equation

$$\ddot{\mathcal{D}}^{(n)} + (\frac{2}{3} - w)\Theta\dot{\mathcal{D}}^{(n)} - \left\{\frac{(1-w)(1+3w)}{2}\left(\frac{1}{3}\Theta^2 + \frac{3k}{S^2}\right) - w\frac{n^2}{S^2}\right\}\mathcal{D}^{(n)} = 0 \tag{4.24}$$

(valid for each $n \geq 0$), showing how the growth of the inhomogeneity depends on the comoving wavelength. Clearly we can similarly harmonically analyse the other equations, e.g. the second order equation (4.18) for the Bardeen variable.

4.3.4 Jeans Instability

To determine explicitly the solutions of the second-order equations we have obtained, we have to substitute for μ, Θ and S from the zero-order equations.

Speed of Sound

We can examine solutions in the case where the divergence term is the dominant term, by examining the case where Θ, $\kappa\mu$, k/S^2 and Λ can be neglected. We see then directly from (4.13) that c_s introduced above *is* the speed of sound (and that imaginary values of c_s, that is, negative values of $dp/d\mu$, lead to exponential growth or decay rather than oscillations).

Instability Criterion

The Jeans' criterion is that gravitational collapse will tend to occur if the combination of the matter term and the divergence term in (4.13) or (4.14) is positive; that is, if

$$\frac{1}{2}(1-w)(1+3w)\kappa\mu\mathcal{D}_a > w\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) \tag{4.25}$$

when $c_s^2 = w$ (we include the curvature term also, because it comes from the divergence term A_a). Using the harmonic decomposition, this can be expressed in terms of an equivalent scale: from (4.24), gravitational collapse tends to occur for a mode $\mathcal{D}^{(n)}$ if

$$\frac{1}{2}(1-w)(1+3w)\kappa\mu > w\frac{n^2}{S^2}, \tag{4.26}$$

that is

$$\left\{(1-w)\left(\frac{1}{w}+3\right)\frac{\kappa\mu}{2}\right\}^{1/2} > \frac{n}{S}. \tag{4.27}$$

In terms of wavelengths, the Jeans' length is defined by

$$\lambda_J \equiv \frac{2\pi S}{n_J} = c_s\sqrt{\frac{\pi}{G\mu}\frac{1}{(1-w)(1+3w)}} \tag{4.28}$$

where we have expressed the result in terms of the usual gravitational constant G. Thus gravitational collapse will occur for small n (wavelengths longer than λ_J), but not for sufficiently large n (wavelengths less than λ_J), for the pressure gradients are then large enough to resist the collapse and lead to oscillations instead (cf. Jackson [30], but his answer appears to be in error; we here present a corrected version of his result).

4.3.5 A First Integral in the Long Wavelength Limit

Suppose we can ignore A and so A_a; then the Raychauduri equation (3.30) becomes

$$3\ddot{S}/S = -\frac{1}{2}\kappa\mu(1+w) + \Lambda\ , \tag{4.29}$$

and provided $\dot{S} \neq 0$ we can multiply by $S\dot{S}$ and integrate: we find

$$3(\dot{S})^2 - (\kappa\mu + \Lambda)S^2 = -3k\ , \quad \dot{k} = 0\ , \tag{4.30}$$

which is just the Friedmann equation which governs the time evolution of FLRW universe models; it is the same as equation (4.15). When $\omega = 0$, $\mathcal{K} = {}^{(3)}R$ and k, constant on each world line γ, characterises the 3 - space curvature of the 3-surfaces $\Sigma_\perp$ where they intersect γ (when $\omega \neq 0$ this is approximately true, see section 2.1.5).Thus in this case *there is a separate FLRW evolution along each world line* [37,38]; these evolutions will differ only in their energies and starting times [26]. Note the difference from (3.33), (3.40) here: in general we are able to use (3.40) to determine $\mathcal{K}$ as far as the propagation equation for $\mathcal{D}_a$ is concerned; but this ignores first order corrections to this equation, which we must take into account if we use it to determine $\Theta(t)$ or $\mu(t)$; the separate world lines evolve separately in general, and (3.40) does not describe their evolution accurately. However in the case considered now, we *can* use (3.40) for these purposes, that equation being the same as (4.30) under these conditions, and giving the independent evolution of $S(t)$ along each world line. The evolution of the spatial variation of density will be then governed by the equations above, where now we drop the divergence terms, that is from (3.45),

$$A = 0 \quad \Rightarrow \quad A_a = 0 \quad \Leftrightarrow \quad c_s^2 \left(\frac{2k}{S^2} \mathcal{D}_a - {}^{(3)}\nabla^2 \mathcal{D}_a \right) = 0 \; ; \tag{4.31}$$

then the second-order propagation equations become ordinary differential equations along the fluid flow lines, easily solved for particular equations of state (e.g. see section 4 below).

There is a first integral in this case if additionally

$$\Lambda = 0, \quad k = 0, \tag{4.32}$$

which has been used extensively in analysing perturbations during inflation (cf. [2,21]). It is obtained in the following way: define Φ_a by (4.11). Under the restrictions (4.31; 4.32), from (4.12) and remembering that now by (4.15) $\frac{1}{3}\Theta^2 = \kappa\mu$,

$$\dot{\Phi}_{\perp a} + \left[\frac{1}{3} + \frac{1}{2}(w+1) \right] \Theta \Phi_a = \frac{1}{4}(1+w)\Theta(S^2 C_a) \tag{4.33}$$

while (4.10) shows

$$S^2 C_a = C_a, \quad (C_a)^{\cdot} = 0. \tag{4.34}$$

Combining these results, we find

$$\Phi_a + (\frac{2}{\Theta}) \frac{\dot{\Phi}_a + \frac{1}{3}\Theta\Phi_a}{1+w} = \frac{C_a}{2}, \tag{4.35}$$

a first integral of the equations (cf Bardeen et al. [2], Lyth [37], Lyth and Mukherjee [38]).

If additionally $\{w = const\} \Leftrightarrow c_s^2 = w$, the second- order equation (4.18) reduces to

$$\ddot{\Phi}_{\perp a} + \dot{\Phi}_{\perp a}\Theta\left(\frac{4}{3} + w\right) = 0 \tag{4.36}$$

which can be directly integrated to give

$$\dot{\Phi}_{\perp a} = \frac{\phi_a}{S^{(4+3w)}}, \quad \dot{\phi}_{\perp a} = 0 \tag{4.37}$$

which can be put in (4.35) to give Φ_a explicitly in terms of two constants. In the particular case where $\phi_a = 0$ (no decaying mode), then we have the integral

$$\Phi_a = \Phi_{0a} = \frac{C_a}{2\left(1 + \frac{2}{3(1+w)}\right)} \tag{4.38}$$

showing how this constant relates to the spatial variation in the curvature of the perturbed model. A generalisation of this argument by Mukhanov [41] and Vishniac [56] applies if w is not constant. If now the equation of state varies dramatically over a short time interval during a phase transition, although w varies C_a stays constant at the transition (see (4.34) and [21]), so Φ_a varies greatly; enabling us to follow the evolution of Φ_a through the different stages of an inflationary universe [2,21]. We obtain the isocurvature case (see section 4.3.6 below) when $C_a = 0$.

4.3.6 Isocurvature Modes

We look now at the implications of imposing geometrical restrictions on the fluctuations. Specifically, suppose the isocurvature condition following from (3.36) is satisfied at an initial time: under what circumstances will it remain satisfied? From (4.10) we find that isocurvature inhomogeneities, that is perturbations with $\mathcal{D}_a \neq 0$ such that

$$\mathcal{C}_a = 0 \quad \Leftrightarrow \quad \mathcal{Z}_a = \frac{3}{2}\frac{\kappa\mu}{\Theta}\mathcal{D}_a\ , \tag{4.39}$$

are possible only if

$$\frac{3k}{S^2}\left(-\frac{3}{2}(1+w)\kappa\mu + c_s^2\Theta^2\right)\mathcal{D}_a - c_s^2\Theta^2\left(\frac{2k}{S^2}\mathcal{D}_a - {}^3\nabla^2\mathcal{D}_a\right) = 0\ . \tag{4.40}$$

In the pure dust case, when $p = 0 \Rightarrow c_s^2 = 0$, this reduces to

$$k\mathcal{D}_a = 0 \; , \tag{4.41}$$

showing that in this case, isocurvature perturbations with non-zero density gradients are only possible if $k = 0$, i.e., if the unperturbed universe has flat spatial sections.

Equation (4.40) shows that this is also the case for non - vanishing pressure fluids. In this case we have that such solutions are possible if $k = 0$ and the divergence term vanishes, i.e. (4.31) is satisfied, whatever the value of Λ. When $c_s^2 = w$ and $\Lambda = 0$ this corresponds to the particular case $C_a = 0, \phi_a \neq 0$ of the integrals discussed above, that is, *the isocurvature condition is precisely equivalent to existence of decaying mode alone.* Although we have not obtained a formal proof, it seems likely these are the only such solutions, that is, (4.40) can only remain true at all times for ordinary matter (more specifically, matter such that $w \neq \frac{2}{3}$) if $k = 0$ and (4.31) holds.

4.3.7 Scalar Modes

In general, $\mathcal{D}_a$ and $\mathcal{Z}_a$ are not parallel. Even if they are parallel at one event p on a world-line γ, the divergence term A_a in (4.8) will in general mean that they will not stay parallel; thus they will be essentially vector rather than scalar solutions. However each harmonic mode is effectively a scalar solution (as it is an eigenmode). Also, when the divergence term may be ignored, that is (4.31) is satisfied, then there is scalar solution, arising from initial data for which $\mathcal{D}_a$ and $\dot{\mathcal{D}}_a$ are parallel (so $\mathcal{D}_a$ and $\mathcal{Z}_a$ are parallel to each other all along the world line). For example (4.14) has a scalar mode obeying

$$\ddot{\mathcal{D}} + (\frac{2}{3} - w)\Theta\dot{\mathcal{D}} - \left(\frac{(1-w)(1+3w)}{2}\kappa\mu + 2w\Lambda\right)\mathcal{D} = 0 \tag{4.42}$$

where $\mathcal{D}$ is the magnitude defined in (??). We can always find such "scalar" solutions (take initial data at p on γ with $\dot{\mathcal{D}}_a$ parallel to $\mathcal{D}_a$, and (4.31) satisfied); they will indicate the extreme behaviour of the vector solutions. Thus we may use the scalar equations to investigate the maximum rates at which density inhomogeneities can grow. Note that the scalar equation

(4.42) is just the harmonic equation (4.24) for $n = 0$. As we obtain the same equations in both cases, $\mathcal{D}^{(0)}$ varies precisely as $\mathcal{D}$ along the flow lines; that is the $n = 0$ harmonic equation *is* the scalar mode.

4.4 Specific Fluid Solutions

The Zero - Order Equations

We can integrate the zero - order equation along each flow line. The energy conservation equation (3.2) shows

$$\mu = \frac{M_1}{\mathcal{S}^{3(1+w)}} \; , \quad \dot{M}_1 = 0 \; , \tag{4.43}$$

and the Raychaudhuri equation (3.30) can be integrated to give the Friedmann equation (4.30), as we already shown in section 4.3.5.

4.4.1 Pressure - free Matter

This is the case of "pure gravity", often called "dust"; $p = 0, \mu > 0$ so no kinetic or pressure effects are taken into account. Thus it is not very physical, but enables us to see how gravity alone functions. It is usually taken to be a reasonable approximation to the equation of state of the universe at late times.

Pressure free matter must move geodesically: from the momentum equation (3.3), we have

$$p = 0 \Rightarrow a^a = 0 \Rightarrow A = A_a = 0 \tag{4.44}$$

exactly, at any length - scale (there are no pressure gradients to deviate the motion from free- fall). This enables us to omit the projection tensors in the perturbation evolution equations. Moreover the Raychauduri equation (3.30) reduces to the Friedmann equation (4.30) not only at *zero* - order, but also at *first* - order.

The Perturbation Equations

In this case the second-order equation (4.13) reduces to

$$\ddot{\mathcal{D}}_a + \frac{2}{3}\Theta\dot{\mathcal{D}}_a - \frac{1}{2}\left(\frac{1}{3}\Theta + \frac{3k}{S^2}\right)\mathcal{D}_a = 0 \; . \tag{4.45}$$

Since (4.44) holds exactly, we can obtain solutions of the above equation directly, without harmonic decomposition. Thus in this case we can integrate (4.45) directly along characteristic, that is, fluid flow lines *are* characteristic for the "pure gravity problem" at any wavelength. This is in accord with the analysis of Ehlers et al. [10]. To determine the solutions explicitly, we have to substitute for $S(t)$ from the zero-order equations (or change to the conformal time variable $\eta = \int dt/S(t)$ and give S in terms of that time along each world line). Before looking at two simple cases, we comment on some general properties of these equations.

In the dust case equation (4.42) reduces to

$$\ddot{\mathcal{D}} + \frac{2}{3}\Theta\dot{\mathcal{D}} - \frac{1}{2}\kappa\mu\mathcal{D} = 0 \; , \tag{4.46}$$

but this holds at any length - scale.

In the case considered here (vanishing pressure), because the evolution along each world line is independent, the evolution of each of X_a, $\mathcal{X}_a$ and $\mathcal{D}_a$ is unaffected by the wavelength of the density fluctuations ((4.45) and (4.46) hold independent of wavelength). Furthermore, the evolution is unaffected by particles horizons and/or Hubble radius; they are irrelevant to this evolution, whether we consider large or small scale inhomogeneity, because the individual world lines evolve independently.

Equation (4.46) is the standard equation for zero-pressure density perturbation growth relative to proper time along the flow lines in an expanding universe, obtained by E. Lifshitz [34] in his pioneering study of the instability of FLRW models. It can also be obtained from Newtonian theory [3]. We have here obtained it as an equation describing scalar modes of the vector equation (4.45).

The Einstein Static Universe

As a first example, we consider a universe that is static at some event p on a world line γ: $\ddot{S} = \dot{S} = 0$ at p. Then from (4.29), (4.30) $k = +1$, $\frac{1}{2}\kappa\mu = S^{-2} = \Lambda$ at p. Equation (55) becomes

$$(\mathcal{D}_a)^{\cdot\cdot} = \frac{1}{2}\kappa\mu\mathcal{D}_a \tag{4.47}$$

at p, independent of the wavelength of the fluctuation, showing the gravitational instability to inhomogeneity; any non-zero initial inhomogeneity in a static situation will grow. This supplements the usual proof of instability to homogeneous (FLRW) modes, which follows direct from the Raychaudhuri equation (4.29).

The Einstein-de Sitter Universe

For comparison with the standard case, we consider the simplest expanding solution, the Einstein-de Sitter universe with $k = 0 = \Lambda$. Then the zero order solution is

$$S(t) = a(t - t_*)^{2/3},\ a = (3\kappa M_1/4)^{1/3},\ \Theta = 2/(t - t_*) \tag{4.48}$$

where t is proper time along the world lines; a, M_1 and t_* are constants. From equations (4.45) and the related equations one can obtain for the other density gradients, we find as follows: in a parallel propagated orthonormal frame along a world line, the spatial density gradients X_a have power-law solutions

$$X_a = a_{+a}(t - t_*)^{-2} + a_{-a}(t - t_*)^{-11/3} \tag{4.49}$$

where the a_{ia} are constant along each world line; that is there are only decaying modes. Correspondingly, the fractional spatial density gradients $\mathcal{X}_a$ have power law solutions

$$\mathcal{X}_a = b_{+a} + b_{-a}(t - t_*)^{-5/3} \tag{4.50}$$

where the b_{ia} are constant on each world line. Again there is no growing mode. Finally the comoving fractional density gradients $\mathcal{D}_a$ has power law solutions

$$\mathcal{D}_a = c_{+a}(t - t_*)^{2/3} + c_{-a}(t - t_*)^{-1} \tag{4.51}$$

where the c_{ia} are constant on each world line, giving the expected modes with powers of 2/3 and −1. From (2.41) it follows that the magnitude $\mathcal{D}$ goes as

$$\mathcal{D} = (\mathcal{D}_a\mathcal{D}^a)^{1/2} = (c_{+a}c_+{}^a(t - t_*)^{4/3} + \\ + 2c_{+a}c_-{}^a(t - t_*)^{-1/3} + c_{-a}c_-{}^a(t - t_*)^{-2})^{1/2} \tag{4.52}$$

showing that there is also an extra mode in this magnitude in the general case, that is if c_{+a} and $c_-{}^a$ are not parallel. Note however this is the magnitude of the scaled energy density gradient $\mathcal{D}_a$, which does not necessarily directly relate the density change between neighbouring world - lines (it gives the density variation in the instantaneous direction of maximum density change, but particles in that direction at one time will not necessarily remain in that direction at other times). The relative density change Δ between two comoving fluid elements will not show this extra mode, because it will be governed by equation (4.78), identical to equation (4.46) for the scalar modes of the vector equation. The growth of this quantity will thus show only the 2/3 and −1 modes, agreeing with the standard results for growth of $\delta\mu/\mu$ in terms of proper time along the flow lines [34,26,43].

It is quite clear in our analysis that these are physically well - defined modes of growth and decay of a density inhomogeneity; whereas, because of the remaining gauge freedom (choice of the initial surface from which to measure proper time), the situation is much more ambivalent if we use the usual variables. Because the evolution along each world line γ_i individually is like a FLRW model F_i, it is clear that the "best fit" FLRW model along γ_i *is* F_i (irrespective of the world model $\bar{S}$ we first thought of). If we define the map Φ to assign the reference density $\bar{\mu}$ correspondingly, we will have chosen the zero density perturbation gauge (see section 1.2.1). Suppose we more conventionally choose a time coordinate which measures proper time along the world lines in S. Then Olson shows (see page 329 of his paper [43]) that the decaying mode of $\delta\mu/\mu$ can be eliminated by the remaining gauge freedom, while this is not true for the growing mode. However, when the decaying mode has been eliminated from $\delta\mu/\mu$, it will still be evident in other quantities. The gauge-invariant approach avoids this kind of problem.

As the equation (4.46) is a standard, we will not discuss its properties further here; the solutions for $k = +1$ and $k = -1$ may be found, for example, in Weinberg's book [58], see p.573 on.

4.4.2 Radiation

In the case of pure radiation, $\gamma = 4/3$, $w = 1/3 = c_s^2$. Then we find from (4.16)

$$\ddot{\mathcal{D}}_{\perp a} + \frac{1}{3}\Theta\dot{\mathcal{D}}_{\perp a} - \frac{2}{3}\left(\frac{1}{3}\Theta^2 + \frac{3k}{S^2}\right)\mathcal{D}_a + \frac{1}{3}\left(\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\right) = 0 \qquad (4.53)$$

and from (4.42) the scalar form

$$\ddot{\mathcal{D}} + \frac{1}{3}\Theta\dot{\mathcal{D}} - \frac{2}{3}\left(\frac{1}{3}\Theta^2 + \frac{3k}{S^2}\right)\mathcal{D} = 0, \qquad (4.54)$$

valid when we can ignore the divergence term (that is, in the low-frequency limit). When $k = 0$ then $S(t) \propto t^{1/2}$ and we obtain in the long-wavelength limit

$$\mathcal{D}_a = t\ d_{+a} + t^{-1/2}d_{-a}, \quad \dot{d}_{ia} = 0 \qquad (4.55)$$

(where t is proper time along the flow lines). The corresponding standard result in the synchronous and comoving proper time gauges is different, being modes proportional to t and to $t^{1/2}$ (cf. e.g. [43,1]); however we obtain the same growth law as derived in the comoving time orthogonal gauge and equivalent gauges (cf. e.g. [49,1]). As our variables are gauge-independent and covariantly defined, we believe they show the latter gauges represent the physics more accurately than any other. Note that we obtain no fictitious modes (proportional to t^{-1}) as happens e.g. in Olson's paper, because we are using gauge-independent variables.

The Jeans length criterion (4.27) is now

$$\{2\kappa\mu\}^{1/2} > n/S \quad \Leftrightarrow \quad \lambda < \lambda_J = \sqrt{\frac{1}{4}\frac{\pi}{G\mu}} \tag{4.56}$$

as usual. Because our equations reduce effectively to the Bardeen equations, their further properties (e.g. solutions when $k \neq 0$) are essentially dealt with in his paper, so we will not discuss them further here.

4.4.3 A mixture of simple fluids

As the last of simple applications, we consider a multi - component fluid (matter plus radiation plus a cosmological constant). A non-interacting mixture of matter and radiation with the *same* 4-velocity is like a perfect fluid, that is (??) applies where the total energy density μ is now given by

$$\mu = \mu_1 + \mu_2 + \mu_3 \equiv M_1/S^3 + M_2/S^4 + M_3, \ \dot{M}_i = 0 \tag{4.57}$$

and the total pressure p by

$$p = p_2 + p_3 \equiv (1/3)M_2/S^4 - M_3 \tag{4.58}$$

where M_1 represents the amount of matter present, M_2 the amount of radiation present and M_3 a cosmological constant.

The relativistic γ-law equation of state

$$p = (\gamma - 1)\mu \tag{4.59}$$

can still be used in this case; it is related to w and c_s^2 (see (4.60) by

$$w = \gamma - 1, \quad c_s^2 = (\gamma - 1) + \mu\frac{d\gamma}{d\mu}. \tag{4.60}$$

The quantity γ is a constant for a simple fluid; in the present case we have an effective $\gamma(S)$ of the form

$$\gamma = \frac{M_1/S^3 + (4/3)M_2/S^4}{M_1/S^3 + M_2/S^4 + M_3} \tag{4.61}$$

(Madsen and Ellis [39]). When Λ vanishes ($M_3 = 0$), γ smoothly decreases from 4/3 to 1 as the universe expands, and there is a smooth transition from the radiation dominated to the matter dominated behaviour.

We can use the equations in the rest of this chapter in this situation, with μ representing the total energy density and p the total pressure; at most stages $\dot{\gamma}$ will be small and can be neglected, so we can use (4.14) rather than (4.13). The Jeans' length will be given by (4.28), where w is given by (4.60) and (4.61). Because of the possible independent spatial variation of M_1 and M_2, the isocurvature behaviour will be richer than in the simple fluid case, but (4.10) remains valid and we find as before that perturbations such that $\mathcal{K}_a = 0$ are consistent with the evolution equations if $k = 0$ and $c_s^2\{\frac{2k}{S^2}\mathcal{D}_a - {}^{(3)}\nabla^2\mathcal{D}_a\} = 0$; and it seems likely that these are the only isocurvature solutions.

If the fluids interact significantly, we can no longer describe the situation by the simple equation of state (4.61); nevertheless, just as in the case of a dissipative "perfect fluid" (i.e. a fluid with stress-tensor (??) but non-zero bulk viscosity) we can still use the equations in this chapter for the evolution of density gradients, provided we add suitable equations of state describing the interactions and dissipative processes occurring.

However the situation for multi-component fluids with different 4-velocities is more complex; generalisations of the equations given here are needed for that case.

4.5 Evolution on Neighbouring World - Lines

4.5.1 Relative Position Vector

As we said in section 2.1.1, a relevant quantity in the fluid flow approximation is the *relative position vector* $\eta^a_\perp = h^a{}_b\eta^b$ linking any two world lines O, G, where the connecting vector η^a obeys the Lie Derivative relation [8,11]

$\eta^a{}_{;b}u^b = u^a{}_{;b}\eta^b$. The time variation of $\eta^a_\perp$ is given by [8,11]

$$h_{ab}(\frac{\eta^b_\perp}{S})^{\cdot} = (\sigma_{ab} + \omega_{ab})\frac{\eta^b_\perp}{S} \ . \tag{4.62}$$

We see from this equation that $\frac{\eta^a_\perp}{S} = \eta^a_{\perp 0}$ is constant in an exact FLRW (where $\sigma = \omega = 0$). Since $\eta^a_\perp$ connects two generic observers, at a distance as large as we wish, it will be a zero-order quantity in an almost FLRW space-time; however we see from (4.62) that the correction to its variation with $S(t)$ are first-order in an almost FLRW.

4.5.2 Scalar Quantities

Relative Density Variation

With the above defined relative position vector we can build up scalar quantities related with our gauge-invariant spatial gradients. For example, in considering galaxy formation, what we really wish to examine is the relative density growth in two neighbouring comoving volumes. Using $\eta^a_\perp$ we can define the following scalar

$$\Delta \equiv \left(\frac{\mu_{,a}}{\mu}\right)\eta^a_\perp = \mathcal{X}_a\eta^a_\perp \ , \tag{4.63}$$

which express the relative difference in density between O and G. Note that we did not assumed any "smallness" in defining Δ. However, in an almost FLRW our gradients are first-order, therefore we need $\eta^a_\perp$ only at order zero in (4.63), that is, $\Delta \simeq \mathcal{X}_a S(t)\eta^a_{\perp 0}$. Now, since $\mathcal{D}_a = S(t)\mathcal{X}_a$, we see that

$$\Delta \simeq \mathcal{D}_a\eta^a_{\perp 0} \ , \tag{4.64}$$

that is, the time variation of the density difference between two neighbouring comoving volumes is determined by $\mathcal{D}_a$. With this, the density $\tilde{\mu}$ on G is related to the density μ on O by

$$\tilde{\mu} = \mu(1 + \Delta + O(\Delta^2)) \ . \tag{4.65}$$

Note that Δ is a gauge-invariant scalar, because it is made by the scalar product with a gauge-invariant gradients.

Other Scalars

In the some way we can build up scalar quantities from any gradient we wish. Let f any quantity, for which we can define the two related spatial gradients

$$f_a = h_a{}^b f_{,b} \ , \quad \mathcal{F}_a \equiv S f_a \ , \tag{4.66}$$

as we did when we defined X_a and $\mathcal{X}_a$. The corresponding scalar is [7]

$$\Delta_{(f)} \equiv f_{,a}\eta^a_\perp = f_a \eta^a_\perp \ , \tag{4.67}$$

and at first - order

$$\Delta_{(f)} \simeq \mathcal{F}_a \eta^a_{\perp 0} \ , \tag{4.68}$$

in complete analogy with (4.64).

4.5.3 Propagation Equations for Scalars

Equation for a Generic Scalar

One can obtain an exact first - order propagation equation for $\Delta_{(f)}$ from the equation for f_a, just using the definition (4.67) and differentiating with respect to time. With a few of algebra we obtain

$$(\Delta f)^{\cdot} = (f_a \eta^a_\perp)^{\cdot} = \frac{\eta^a_\perp}{S}\left[h_a{}^b (\mathcal{F}_b)^{\cdot}\right] + \mathcal{F}_a \left[h^a{}_b \left(\frac{\eta^b_\perp}{S}\right)^{\cdot}\right] \ , \tag{4.69}$$

expressing the relation between the exact equations for $\Delta_{(f)}$ and $\mathcal{F}_a$ (we have substitute f_a with $\mathcal{F}_a$).

It is immediate to see from (4.69) that, at first - order, the equation for $\Delta_{(f)}$ is obtained by multiplying the constant vector $\eta^a_{\perp 0}$ with the linear equation satisfied by $\mathcal{F}_a$, that is

$$(\Delta f)^{\cdot} = \eta^a_{\perp 0}\left[h_a{}^b (\mathcal{F}_b)^{\cdot}\right]_L + \mathcal{O}(2) \ , \tag{4.70}$$

where the subscript L means we are considering the linear equations for $\mathcal{F}_a$.

[7] The subscript (f) *not* a vectorial index, but obviously refer to the quantity f of which $\Delta_{(f)}$ is the spatial scalar variation.

Equation for Δ

For Δ, defined by (4.63), the exact equation we obtain from (3.27) and (4.62) is:

$$\dot{\Delta} = \frac{p}{\mu}\Theta\Delta - (1 + \frac{p}{\mu})\Xi \,, \tag{4.71}$$

where

$$\Xi \equiv \Theta_{,a}\eta^a_\perp = Z_a\eta^a_\perp \tag{4.72}$$

$$\Xi \simeq \mathcal{Z}_a\eta^a_{\perp 0} \,, \tag{4.73}$$

is the scalar related to $\mathcal{Z}_a$, and the expansion $\tilde{\Theta}$ on G is related to the expansion Θ on O by $\tilde{\Theta} = \Theta + \Xi + O(\Xi^2)$. From (3.24) and (4.62), the exact first-order propagation equation for Ξ is,

$$\dot{\Xi} = -\frac{2}{3}\Theta\Xi - \frac{1}{2}\kappa\mu\Delta + \{\mathcal{R}\dot{u}_c + A_c - 2\sigma^2{}_{,c} + 2\omega^2{}_{,c}\}\eta^c_\perp \,. \tag{4.74}$$

4.5.4 The Case of Zero Pressure

We conclude this section deriving the equations satisfied by Δ in the case of vanishing pressure. In the case of *dust*, we find from (4.71) and (4.74) the simple exact relations

$$\dot{\Delta} = -\Xi \,, \tag{4.75}$$

$$\dot{\Xi} = -\frac{2}{3}\Theta\Xi - \frac{1}{2}\kappa\mu\Delta + \{-2\sigma^2{}_{,c} + 2\omega^2{}_{,c}\}\eta^c_\perp \tag{4.76}$$

leading to the completely general *exact second order equation*

$$\ddot{\Delta} = -\frac{2}{3}\Theta\dot{\Delta} + \frac{1}{2}\kappa\mu\Delta + \{2\sigma^2{}_{,c} - 2\omega^2{}_{,c}\}\eta^c_\perp. \tag{4.77}$$

The last equation is linearised in the almost-FLRW context by dropping the last bracket, to give

$$\ddot{\Delta} + \frac{2}{3}\Theta\dot{\Delta} - \frac{1}{2}\kappa\mu\Delta = 0, \tag{4.78}$$

which is essentially the well-known equation (4.46) for zero pressure density perturbations (obtained in the standard literature by other means).

CONCLUSIONS

This thesis has presented the results obtained in two recent papers [17,19]. We introduced there new gauge - invariant variables characterizing density inhomogeneities in cosmology, and derived evolution equations for them. These variables have a straightforward physical gauge - independent interpretation (they represent the *spatial* variation of energy density in the real universe), unlike Bardeen's variables [1].

However, in this thesis, I tried also to give a more comprehensive exposition of the subject including a summary of the theoretical framework within which the present approach to cosmological density inhomogeneities naturally fits. Following an introduction to the gauge problems affecting the standard approach to density perturbations in cosmology (chapter 1), we summarized the covariant fluid flow approach to cosmology [11] (section 2.1) and the standard equations describing the fluid motion in this approach (see section ??).

We have found a set of covariantly defined gauge-invariant quantities that characterise spatial density variation in almost Robertson - Walker universes. In particular, we have identified the quantity $\mathcal{D}_a$, the *comoving fractional density gradient* and its magnitude $\mathcal{D}$, defined by equations (2.40)-(2.41), as the covariant and gauge invariant quantities which embody most closely the intention of the usual (gauge-dependent) definition $\delta\mu/\mu$. These, and other closely related variables, are presented in section 2.2.2.

We have obtained exact (fully non-linear) evolution equations for these quantities (section 3.1.3), showing how these latter are coupled with many other fluid variables in the general case. Then we outlined a linearization procedure in almost Robertson - Walker universes, and applied it to our equations, obtaining linear first - order propagation equations for our variables (section 3.2.1). In the perfect fluid assumption, we obtained second - order linear equations for $\mathcal{D}_a$ and Φ_a, a vectorial variable corresponding to Bardeen's scalar potential Φ_H (section 4.3). These equations are equivalent to the general Bardeen gauge invariant equations (see [29]). However, our basic definitions and equations are valid independently of any harmonic analysis, although they can be harmonically decomposed if desiderated.

We analysed the properties of these equations, reproducing many standard results in a unified and transparent way. We have deduced: (1) the speed of sound in a barotropic relativistic fluid, (2) the Jeans criterion for gravitational instability, correcting a previous result by Jackson [30]; (3) the long-wavelength limit of those equations, and corresponding first integrals [2,38] that exist for $\Lambda = k = 0$; (4) restrictions on the nature of isocurvature inhomogeneities; (5) evolution of the density contrast between neighbouring world lines (see sections 4.3.4; 4.3.5; 4.3.6; 4.5).

Section 4.4 presented solutions to our equations in the simplest cases. We first examined the case of pressure - free matter, finding the standard $+\frac{2}{3}, -1$ modes in the Einstein - de Sitter background.

In the case of pure radiation with $k = 0$, we derived solutions in the long - wavelength limit, obtaining different growth rates (relative to proper time along the fluid flow lines) from those given by standard analyses using the synchronous and comoving proper time gauges; our results agree with those obtained in the comoving time - orthogonal gauge.

Also, we have briefly considered the case of a mixture of perfect fluids when those perfect fluids all have the same 4-velocity vector u^a.

Our equations are exact and non - linear. When linearized, a comparison with the usual approach to cosmological density perturbations shows that we obtain equations equivalent to the standard ones but in a much more transparent way, because in the standard approach the *definition* of the density fluctuation $\delta\mu$ depends on the gauge chosen. In our case we need a specific gauge to write down the solutions to the equations, but the definitions of the fundamental quantities are gauge-invariant. The key difference is that the standard approach compares two evolutions (the actual one, and a fictitious background one) along a world line, whereas our variables specifically reflect the spatial density variation in the fluid (they compare evolutions along neighbouring world-lines in the actual universe).

In a recent paper, the formalism presented here has been extended by Hwang and Vishniac (1989) [29] to include the case of imperfect fluids, while a Newtonian version has been formulated by Ellis (1989) [20]. Also, aspects of the effects of averaging on the effective field equations (cf. Ellis [13], Futamase [22]) have been considered within the framework of our approach (Futamase (1989) [23]).

Because we have obtained fully non-linear equations describing the evolution of our variables, we can hope to extend our analysis to look at non-linear effects. In particular, when vorticity vanishes, the constraint equation (??) suggests that we could consider a second-order effect of shear in the evolution of the density gradients. Also, the perturbations of a scalar field could be investigated using our approach.

It is interesting to compare our approach with that of Bardeen. In Bardeen's approach, a central role is played by his variables Φ_H and ϵ_m, related by equation 4.3 in his paper [1], which show that Φ_H is a potential for ϵ_m. However, these variables do not directly represent the density contrast, unless some gauge is fixed. It can be shown [42] that these quantities correspond to the Weyl tensor components E_{ab}. This is not mysterious, since E_{ab} is a potential for our variable X_a in the linear approximation (see equation (3.48), 13 in Hawking [26]). We have introduced a vectorial variable, Φ_a, which formally corresponds to Bardeen' Φ_H. Also, Hwang and Vishniac have shown that our variable $\mathcal{D}_a$ is related to Bardeen's ϵ_m when harmonic analysis is used. However, the relations between the two formalisms require further investigation.

Finally, we note that we the constraint equations remain to be considered, and that it remain to be verified that their solutions are preserved along the fluid flow lines by the propagation equations. A full analysis of almost-FLRW universe models must of course examine these issues.

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