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of

"MAGISTER PHILOSOPHIAE"

CONSTRUCTION OF THE BRST INVARIANT STRING GROUND STATE AND QUANTUM PROPERTIES OF THE RELATED WAVE FUNCTION

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CONTENTS

INTRODUCTION

String theory has in recent years been object of a growing interest. The basic motivations are well known [1]:

1) The theory seems to be, in all its formulations, consistent with the requirements imposed by quantum mechanics, relativity, causality and so on.

2) String theory contains the gravity in the low energy limit, i.e. the spectrum exhibits a massless spin-two excitation mode that interacts at low energies obeying the laws of general covariance.

3) There exists a certain agreement between many predictions of string theories and those of the standard model.

Despite of these advantages, string theories are not free of problems. First of all they are conceived from a first quantization point of view. The main shortcoming of this approach lies in the fact that it is intrinsecally perturbative [1,2]. The partition function and the mean values are written as a sum over topologies, i.e. Riemann Surfaces of genus g, as: $Z = \sum_g Z_g$ $(g = 0,1,...)$. Each Z_g has to be considered as the contribution to the partition function coming from a g-loop approximation. So it is difficult to investigate the nonperturbative features of the theory.

Another still unsolved question is the relation between the geometrical (via path integral) and the algebraic (via operator algebra) formulations of string theory.

In fact there exists no global definition for surfaces of genus g>1 of the vacuum state invariant in the string Hilbert space under conformal transformations [3].

Nevertheless it seems useful to extend also for these surfaces the operator formalism which was successfully developed for the case of the sphere and of the torus [3,4,5].

Apart from string field theory [2,6], many new recipes have recently been found to cope with these problems [7,8], which remain within the framework of the first quantized strings. In particular we refer to [7], in which there is also defined a sort of vacuum state for Riemann Surfaces with genus g>1. This is based on the usual vacuum $|0>$ of [3] derived in the simple case of the sphere.

In this thesis we provide two methods to compute the vacuum |0> following the lines depicted in [9]. The first method is based on a saddlepoint evaluation [10] (see section 3.1) and the second on the definition of the path integral [11] (see Appendix B). We have recognized that in both cases we get equal results apart from a total derivative term added to the Lagrangian in the former approach (section 3.2).

After that we study the quantum properties of the vacuum |0> and in particular its invariance under the BRST operator (section 3.2).

A brief discussion is carried out on the differences between the BRST charge of [3] and that coming from the canonical BRST quantization of the string in the complex formalism (section 2.2).

A short introduction to BRST formalism is contained in section 2.1.

A study of conformal field theory and of the properties of the vacuum under conformal transformations are contained in section 1.3 and Appendix C respectively.

We stress the operatorial quantization of the string theory finding all constraints (see section 1.1 and 1.3) at the classical level and then quantizing along the lines of [12] (section 1.3). The future aim is to extend this formalism to surfaces of genus g>1. To this purpose we have included part of our preliminary investigations on higher genus surfaces in section 1.2 and Appendix A. In this appendix the complex formalism and the notations we used throughout this thesis are also explained.

CHAPTER ONE

1.1-Introduction: the Classical String Lagrangian

Let's start with the usual Polyakov's Lagrangian [13] for an Euclidean theory in two dimensions:

$$L = \frac{1}{2}\sqrt{g}\, g^{a\,b}\, \partial_a X^{\mu} \partial_b X_{\mu} \qquad (1.1.1)$$

where $X^{\mu} : M \dashrightarrow R^d$ is a map from a 2-dim manifold M (the so called worldsheet) to R^d (the target space) and $g_{a\,b}$ is a metric over $M(g_{ab} = g_{ba})$.
We list here the invariances of the string action based on (1.1.1) [14]:

1)Worldsheet reparametrization invariances, constituted by the group Diff(M) of diffeomorphisms acting on the coordinates of M.
2)Weyl invariances Conf(M): this is not a pointwise transformation group and it is given by the transformations of the type: $g_{a\ b} \dashrightarrow e^{\phi(x)} g_{a\ b}$ (x are coordinates on M).
The lagrangian (1.1.1) describes a gauge theory. We furnish now a sketchy computation of the related constraints to provide a better understanding of what will follow [15].

I)First remark that the matrix $\tilde{g}^{ab} = \sqrt{g}\, g^{ab}$ appearing in (1.1.1) has only two degrees of freedom,the others being fixed by the conditions:

$$\begin{cases} \tilde{g}_{ab} = \tilde{g}_{ba} \\ \det \tilde{g}_{ab} = 1 \end{cases} \qquad (1.1.2)\,.$$

II)So $\tilde{g}^{ab}$ can be parametrized in the following way:

$$\tilde{g}^{ab} = \begin{bmatrix} \dfrac{1}{\lambda_1} & -\dfrac{\lambda_2}{\lambda_1} \\ -\dfrac{\lambda_2}{\lambda_1} & \dfrac{\lambda_2^2}{\lambda_1} + \lambda_1 \end{bmatrix} \qquad (1.1.3).$$

This obeys clearly (1.1.2) and displays immediately the two degrees of freedom λ_1, λ_2.

III)Let σ and τ be the parameters of M and denote $\dot{A}(\sigma,\tau) = \partial_\tau A(\sigma,\tau)$, $A'(\sigma,\tau) = \partial_\sigma A(\sigma,\tau)$.

Now substituting (1.1.3) in (1.1.1) we have:

$$L = \frac{1}{2}\left[\frac{1}{\lambda_1}\dot{X}^2 - 2\frac{\lambda_2}{\lambda_1}\dot{X}X' + \left(\frac{\lambda_2^2}{\lambda_1} + \lambda_1\right) X'^2 \right] \qquad (1.1.4).$$

IV)The momentum associated to the coordinate X is:

$$p_\tau = \frac{\partial L}{\partial(\dot{X})} = \frac{1}{\lambda_1}(\dot{X} - \lambda_2 X') \qquad (1.1.5).$$

It is straightforward to see that using (1.1.5) properly in (1.1.4) one obtains:

$$L = \dot{X} p_\tau - \frac{\lambda_1}{2}(p_\tau^2 + X'^2) - \lambda_2 p_\tau X' \qquad (1.1.6)$$

This form of the Lagrangian shows immediately that:

1) the theory exhibites two constraint (first class) [15] equal to those of the Nambu-Goto Lagrangian:

$$\begin{cases} p_\tau^2 + X'^2 = 0 \\ p_\tau X' = 0 \end{cases} \qquad (1.1.7)$$

2)The Hamiltonian related to (1.1.6) vanishes apart from the contribution of the Lagrange multipliers.

3)The theory is something different from the standard ones because the Lagrangian contains the constraints [15]. This means, for example, that apart (1.1.7) we have four additional (trivial) constraints given by:

$$p_{g_{ab}} = \frac{\partial L}{\partial(\partial_\tau g_{ab})} \qquad (1.1.8)$$

4)The constraints (1.1.7) are the generators of the reparametrization invariance, so that the gauge group is Diff(M), whose elements are the diffeomorphisms connected with the identity.

1.2-Covariant Quantization

Let's go on with the first quantization of the simple bosonic string action:

$$S = \int_M d^2\xi \; g^{ab} \partial_a X \partial_b X \qquad (1.2.1)$$

(From now on target space indices will be omitted.) As in the case of standard gauge theories the quantization can be performed starting with the path integral [16] :

$$Z = \int \mathcal{D} g \mathcal{D} X \; e^{-S(g,X)} \qquad (1.2.2)$$

A measure $\mathcal{D}$ g is induced by the metric [17]:

$$(\delta_1 g, \delta_2 g) = \int_M d^2 z \sqrt{g} \; (g^{ab} g^{rs} + c\, g^{ar} g^{bs}) \, \delta g_{ar} \, \delta g_{bs} \qquad (1.2.3)$$

where c is a constant.

This measure is determined using the relation $\int \mathcal{D} g \; e^{-\|\delta g\|} = 1$.

Eqn. (1.2.3) provides the most general metric for δg, taking into account that the variations δg can be decomposed in a traceless and a trace part, which are reciprocally orthogonal. Analogously the measure for X is given by:

$$(\delta_1 X, \delta_2 X) = \int_M d^2 z \; \delta_1 X \, \delta_2 X \qquad (1.2.4)$$

At a first sight (1.2.2) seems to describe a free field theory because the action is at most quadratic in the fields. Nevertheless the string can split into two parts [18], as is pictorially shown in fig.1.1

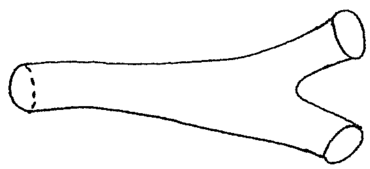


fig. 1.1

Note that the interaction point is not definite

It is easy to see by reparametrization invariance that this is the only allowed vertex. Then we can express string interactions as in fig. 1.2.:

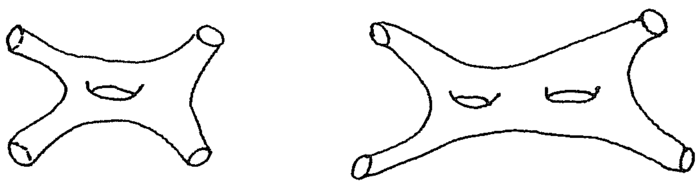


fig 1.2

We see that the interactions between strings introduce manifolds with different topologies, characterized by the number of handles of the manifold, called the genus g. So the partition function in (1.2.2) is effectively a sum of the kind:

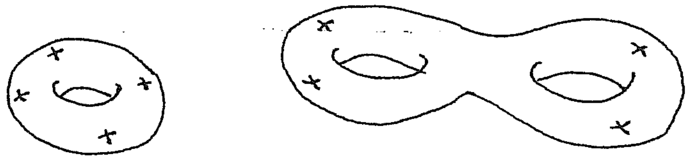
$$Z = \sum_{\text{topologies}} \int D\, g\ D\, X\ e^{-S(g,X)} = \sum_{g} Z_g \qquad (1.2.5)$$

Exploiting again reparametrization invariance on graphs like those in fig 1.2 and considering the external legs coming from infinity, these graphs become Riemann Surfaces in which the external legs appear as points.(see fig 1.3).

The problem of the gauge fixing was until now ignored. We quote here just the most widespreaded methods [3]:

1)Light cone gauge: one uses the Dirac procedure to convert the I^{st} class constraints (1.1.7) to second class ones eliminating some components of X's and p's.

fig. 1.3
The external legs in fig 1.2 are now the points indicated by crosses



This lift the covariance in the target space but has the advantage that with this choice one can retain a positive definite Hilbert space.

2) Conformal gauge: the Polyakov's approach allows us to fix some components of the Lagrange multipliers g^{ab}.

For example in (1.1.6) one can choose [15] :

$$\lambda_1 = 1 , \lambda_2 = 0 \qquad (1.2.6).$$

From the first paragraph it is easy to see that this amounts to put :

$$p_\tau = \dot{X} \qquad (1.2.7)$$

and so the constraints (1.1.7) become:

$$\begin{cases} \dot{X}^2 + X'^2 = 0 \\ \dot{X} X' = 0 \end{cases} \qquad (1.2.8).$$

These are the generators of conformal transformations. So this second gauge is covariant in the target space but it is not able to eliminate completely the gauge freedom. Despite of this, we will use afterwards the conformal gauge.

We discuss here some points of the quantization in this gauge for surfaces of genus $g \geq 2$ [14,16,17,19,20]. The method we use is the generalized Faddeev-Popov mechanism. The trick consists in trading the functional integration over the metric g in each term of the sum (1.2.5) for an integration in the gauge parameters, which can then be factorized as an infinite constant.[14,16].

A problem is to find a good parametrization of the metric space *M*. For this purpose we choose a particular metric g [14]. Using the uniformization theorem for surfaces of genus $g \geq 2$, we choose $\hat{g}_{ab}$ in such a way that the related curvature tensor $R_{\hat{g}}$ is constant over all M, i.e. $R_{\hat{g}} \geq 1$.

From A-44 (see appendix A) we see how g behaves under a small variation of $\hat{g}$ [19,21]:

$$\delta g_{ab} = \delta \phi\, g_{ab} + (P_1 v)_{ab} + \delta y^A \varphi^A{}_{ab} \qquad (1.2.9)$$

These variations correspond to an orthogonal decomposition of the tangent space of metrics $T_g(M)$:

$$T_g(M) = \mathrm{Conf}(M) \oplus \mathrm{Diff}_0(M) \oplus (\mathrm{Teichmuller}) \qquad (1.2.10)$$

We notice that in this case φ^A is not varying under a Weyl transformation, i.e. $\varphi^A|_g = \varphi^A|_{g'}, g' = e^{\phi} g$. But we need that also the y^A variables don't depend on ϕ. In fact when we do the change of variables $\delta g \to (\delta\Phi, v, y)$:

$$\begin{bmatrix} \delta g_{mn}(\delta \phi) \\ \delta g_{mn}(P_1) \\ \delta g_{mn}(y^A) \end{bmatrix} = \begin{bmatrix} 1 & \nabla^c dv_c & * \\ 0 & P_1 & * \\ 0 & 0 & \varphi^A \end{bmatrix} \begin{bmatrix} \delta \phi \\ \delta v \\ \delta y^A \end{bmatrix} \qquad (1.2.11)$$

One has to check that the parameters Φ, v, y^A form an integrable system [22]. If this is the case, the derivatives $\delta/\delta\Phi$, $\delta/\delta v$, $\partial/\partial y^A$ defined below must commute:

$$\frac{\delta}{\delta\phi}=\int_M d^2 z\ \delta\phi(z)\frac{\delta}{\delta\phi(z)} \quad ; \quad \frac{\delta}{\delta v}=\int_M d^2 z\ \delta v(z)\frac{\delta}{\delta v(z)} \quad ; \quad \frac{d}{dy^A}=\frac{\partial}{\partial y^A} \qquad (1.2.12)$$

But if we compute the commutator between two of these derivatives, for example:

$$\left[\frac{\delta}{\delta\phi},\partial_A\right] g_{a\,b} \qquad (1.2.12a)$$

his does not vanish.

Explicitly,since $\delta/\delta\Phi\ g_{a\,b} = g_{a\,b}\delta\phi$ and $\partial/\partial y^A\ g_{a\,b} = \varphi^A{}_{a\,b}$ we have:

$$\frac{\delta}{\delta\phi}\varphi^A_{ab} = \partial_A g_{ab}\ \delta\phi = \varphi^A_{ab}\ \delta\phi \qquad (1.2.13)$$

This leads to a nonsense: $\varphi^A{}_{a\,b}$ does not depend on Φ because these tensors are related to transformations which are orthogonal to conformal transformations as explained before.

Accordingly $\partial/\partial\phi\ (\varphi^A{}_{ab}) = 0$ and we get $0 = \varphi^A{}_{a\,b}\ \delta\Phi \neq 0$! This implies that the y^A variables involve a dependence on ϕ.

The recipe that solves this problem is the use of a nonorthogonal decomposition [22], adopting a new basis $\{\tau^A\}$ on Teichmuller space which transforms non trivially under the conformal group.

Our situation is the following: in (1.2.9) we have the decomposition:

$$\delta g_{ab} = g_{ab}\ \delta\phi + (P_1\ v)_{ab} + \delta y^A\ \varphi_{A\ ab} \qquad (1.2.9)$$

where φ_{Aab} does not depend on ϕ and dy^A depends on ϕ. We rewrite (1.2.9) allowing a non orthogonal decomposition:

$$\delta g_{ab} = g_{ab}\,\delta\phi + (P_1 v)_{ab} + \delta\rho_A\,\tau^{A\,a}_{\ \ b}\,g_{ab} \qquad (1.2.9a).$$

This should be the correct decomposition usually accepted, with $\delta\rho_A$ and δy^B related between themselves by a matrix depending on the Weyl parameter ϕ. We will verify in a while that it is so.

Eqn.(1.2.11) should be replaced by:

$$\begin{bmatrix} \delta g_{mn}(\delta\phi) \\ \delta g_{mn}(P_1) \\ \delta g_{mn}(y^A) \end{bmatrix} = \begin{bmatrix} 1 & \nabla^c dv_c & * \\ 0 & P_1 & * \\ 0 & 0 & \Lambda_g \end{bmatrix} \begin{bmatrix} \delta\phi \\ \delta v \\ \delta y^A \end{bmatrix} \qquad (1.2.14)$$

where Λ_g is an orthogonal projection from the φ^A's to the new τ^A's [14].(1.2.12a)

If we perform this change of variables we get the measure of the path integral in terms of $\delta\Phi$, v, δy^A.

$$D\,g = D\,f\,D\,v\;\; d^{3g-3}\,y^A\, d^{3g-3}\,y^A\,|\det P_1|\det\Lambda_g \qquad (1.2.14a)$$

We want now give an explicit form to det Λg [19]. For this proposal we have to construct the basis $\{\tau^A\}$. We recall the notations of appendix A:

$\{\tau^{Az}{}_{\bar z}\}$ are the Beltrami differentials.

$\{\varphi^A{}_{zz}\}$ is their dual orthonormal basis, the holomorphic quadratic differentials.

Since we have:

$$\int_M dz\wedge d\bar{z}\, \tau^{Az}_{\bar{z}}\, \varphi_{B\,zz} = \delta^{A}_{B} \qquad (1.2.15)$$

$$\int_M dz\wedge d\bar{z}\varphi^{A}_{zz}\, \bar{\varphi}_{B\,\bar{z}\bar{z}}\, g^{z\bar{z}} = N^{A}_{B} \qquad (1.2.16)$$

($N_{A\,B}$ is not diagonal in general) we infer that $g^{z\bar{z}}\,\bar{\varphi}_{B\,\bar{z}\bar{z}}$ lies in the dual space of φ. So it can be expressed as linear combinations of the τ's in this way:

$$\bar{\varphi}_{A\,\bar{z}\bar{z}} = A_{AB}\, g_{\bar{z}z}\, \tau^{B\,z}_{\bar{z}} \qquad (1.2.17)$$

where A is a constant matrix. The expression of A is obtained inserting the definition of τ in terms of $\bar{\varphi}$ given by 1.2.17:

$$(A^{-1})^{AC} \int_M dz\wedge d\bar{z}\, g^{z\bar{z}}\, \bar{\varphi}_{C\,\bar{z}\bar{z}}\, \varphi_{B\,zz} = \delta^{A}_{B} \qquad (1.2.18)$$

This implies [19]:

$$A_{AB} = \left\{ \int_M dz\wedge d\bar{z}\, g^{z\bar{z}}\, \bar{\varphi}_{A\,\bar{z}\bar{z}}\, \varphi_{B\,zz} \right\} \qquad (1.2.19)$$

Using this result in eqn (1.2.17) we get the wanted relations between τ's and φ's:

$$t^{Az}_{\bar z} = g^{z\bar z} \left\{ \int_M dz \wedge dz\, g^{z\bar z}\, \bar\varphi_{A\bar z\bar z}\, \varphi_{B\,zz} \right\}^{-1} \bar\varphi_{B\bar z\bar z} \qquad (1.2.20)$$

Here we see that the φ's have an explicit dependence on the conformal parameter Φ trough the metric tensor $g^{z\bar z}$. In fact if put eqn. (1.2.9a) in terms of the τ^A, using (1.2.17), we get:

$$\delta g_{ab} = g_{ab}\, \delta\phi + (P_1\, v)_{ab} + \delta y^A\, (A)_{AB}\, g_{cb}\, \tau^{Bc}_{a}$$

Eqn. 1.2.9a is obtained from the subtitution $dy_A\, (A)_{AB} = \delta\rho_B$. Here it is shown an explicit dependence of y^A from $(A)_{AB}$ which contains the conformal parameter. Now we can check that the commutator (1.2.12a) is effectively 0. We have computed it explicitly in the new basis and verified that this is true.

At this point we derive the Jacobian det Λ_g [19]. A measure $Dg^{\perp}$ is defined requiring:

$$\int Dg^{\perp} e^{-\| dg^{\perp}\|^2} = 1 \qquad (1.2.21)$$

where:

$$\|\delta g^{\perp}\|^2 = \int_M dz \wedge d\bar z\, g_{z\bar z}\, dy_A\, \tau^{A\,z}_{\bar z}\, d\bar y_B\, \bar\tau^{B\,\bar z}_{\ z} \qquad (1.2.22)$$

as one sees applying the ortogonality condition (1.2.10) and using the metric decomposition A-44 in the definition of $\|\delta g\|^2$. In terms of the φ's $|\delta g^{\perp}\|^2$ becomes:

$$||\delta g^{\perp}|| = (A^{-1})^{AB}\, dy_A dy_B \tag{1.2.23}$$

If we perform the change of variables $\delta g^{\perp}$ -->y^A in eqn. (1.2.21) we have the condition:

$$\int \prod_{i=1}^{3g-3} dy \wedge d\bar{y}\ \det \Lambda_g\, e^{-\{ y_A \bar{y}_B (A^{-1})^{AB} \}} = 1 \tag{1.2.23a}$$

Here we have formally indicated the Jacobian of the change of variables with $\det(\Lambda_g)$.

After the integration of the gaussian integral we find:

$$\det \Lambda_g = \det(A^{AB})^{-1} = \det\left[\int_M dz \wedge d\bar{z}\, g^{z\bar{z}}\, \bar{\varphi}_{A\, \bar{z}\bar{z}}\, \varphi_{B\, zz} \right]^{-1} \tag{1.2.24}$$

Now using the relation above together with the eqn.(1.2.14) for Dg it is possible to express the partition function Z_g as [19]:

$$Z_g = \int D f D v\ d^{3g-3} y_A\, d^{3g-3} \bar{y}_B \frac{|\det (P_1)|\, e^{-S}}{\det\left[\int dz \wedge d\bar{z}\, g^{z\bar{z}}\, \bar{\varphi}_A\, \varphi_B \right]} \quad (g > 1) \tag{1.2.25}$$

As we see the determinants define in a certain sense a measure and in fact they are positive numbers.

Moreover they don't depend on v^z, because they are computed at the point $g=\hat{g}$ and the effect of the trace part of the diffeomorphisms,which gives rise to a conformal anomaly,can be adsorbed in the integration over Φ [19,23].

Hence the Dv^z integration can be performed and its contribution is an infinite constant which we easily factor out in the normalization constant. Also the integration in DX can be done explicitly since the action S is quadratic in the X fields and gives:

$$(\det D')^{-\frac{D}{2}} \int d^D X_0 \qquad (1.2.26)$$

Here det Δ' is the determinant of the Laplacian without the zero modes, which are the constant fields X_0 and yield the $\int d^D X_0$ term [16].
As a result we get from (1.2.25):

$$Z_{g>1} = \int D\phi \; d^{26} X_0 \, d^{3g-3} y \;\; d^{3g-3} \bar{y} \; \frac{|\det(P_1)|}{\det[\,...\,]} \; (\det \Delta')^{-\frac{D}{2}} \qquad (1.2.27)$$

In D=26 dimensions also the integration over Φ is harmless because there is a cancellation of the conformal anomalies coming from det Δ' and $|\det P_1|$: in this case the $D\Phi$ integration can be performed [13,14,24]. Here we just mention the result one would get in the
case g=0 and for a complex parametrization of M, because it will be useful afterwards.

$$Z_g = \int D X\, Db\, Dc \; e^{-\int_M d^2 z \left(\frac{1}{2} \partial_z X \partial_{\bar{z}} X + b \bar{\partial} c + \text{c.c} \right)} \qquad (1.2.28)$$

We have used here the expressions of $|\det P_1|$ in terms of the ghost fields. X,b,c are free fields. Eqn. (1.2.28) will be derived further in the framework of BRST quantization.

1.3-OPERATORIAL FORMULATION OF STRING THEORY

We discuss here some topics of the operatorial formulation of the string theory [3,22,25,26,27,28]. Such a formalism has been developed thoroughly only in the case of surfaces of genus g=0 and g=1, namely the sphere S^2 and the torus T^2.
We restrict us to the simple case of the sphere, for which we introduce the following atlas $\{U_\alpha, \Phi_\alpha\}$, ($\alpha = 1,2$):

$$U_1 = S^2/\{s\}\ ,\ U_2 = S^2/\{n\} \qquad (1.3.1)$$

$$\phi_1 :\ U_1 \dashrightarrow C_1 + \{ (\sigma,\tau) : -\infty < \tau \leq 0 ; 0 \leq \sigma \leq 2\pi \}$$

$$\phi_2 :\ U_2 \dashrightarrow C_2 + \{ (\sigma,\tau) : 0 \leq \tau < +\infty ; 0 \leq \sigma \leq 2\pi \} \qquad (1.3.2)$$

{n} and {s} are two arbitrary but opposite points of the sphere and C_α are semiinfinite cylinders. Since S^2 allows a complex structure, we define on C_α a complex coordinate system $w,\overline{w}$ as:

$$w = \tau + i\sigma$$

$$\overline{w} = \tau - i\sigma \qquad (1.3.3).$$

Further we can go on the complex plane $\mathbb{C}$ projecting C_1 and C_2 on the unitary disc D centered in the origin by means of the conformal transformations $z = e^{-w}$ and $\tilde{z}=e^{-w}$ respectively.
On $\mathbb{C}$, the overlapping region $U_1 \cap U_2$ is simply the disc D/{0} and the transition function between z and z is the conformal map:

$$z = \frac{1}{\tilde{z}} \qquad z,\tilde{z} \in D/\{0\} \qquad (1.3.4).$$

The remaining part of this section is devoted to show,in a very intuitive manner, that the quantum string theory on the sphere is equivalent to a conformal field theory.

We will organize the job as follows:

1) Conformal field theory briefly reviewed.
2) Quantization of the string theory in the operatorial formalism.
3) The stress-energy tensor for the string theory case.

1) <u>Conformal field theory briefly reviewed:</u> a conformal field theory is a theory invariant under conformal transformations (see Appendix A).

Infinitesimally they look like:

$$z \dashrightarrow z + v(z)$$
$$\bar{z} \dashrightarrow \bar{z} + \bar{v}(\bar{z}) \qquad (1.3.5).$$

Notice that here we have chosen the disc D containing the image of the point {s} of the sphere: this does not imply any loss of generality.

A simple basis for $v,\bar{v}$ is [26]:

$$v(z) = \varepsilon\ z^{n+!}$$
$$\bar{v}(\bar{z}) = \bar{\varepsilon}\ \bar{z}^{n+1} \qquad -\infty < n < +\infty\ ,\ \varepsilon << 1 \qquad (1.3.6).$$

These transformations have as generators:

$$L_n = - z^{n+1} \frac{d}{dz}$$

$$L_n = - z^{n+1} \frac{d}{dz} \qquad (1.3.7)$$

We see that not all the L_m's are well defined over the whole sphere. As a matter of fact,their extension in the local coordinate system $\tilde{z},\tilde{\bar{z}}$, which contains the image of the point {n}, is:

$$L_n = z^{1-n} \frac{d}{dz}$$

$$\tilde{\bar{L}}_n = \tilde{\bar{z}}^{1-n} \frac{d}{d\tilde{\bar{z}}} \qquad (1.3.8).$$

From eqns. (1.3.7) and (1.3.8) it is easy to see that the regularity in the origin requires that the only allowed values of n are 0 and ± 1 [26]. $L_0, L_{\pm 1}$ generate the transformation group $SL(2,\mathbb{C})$, i.e. the group of automorphisms of the complexified sphere. They are related to the Conformal Killing Vectors in the sense that they are global solutions of the equation:

$$\partial_z c^z = 0 \qquad (1.3.9)$$

The other L_n's, with $n \neq 0, \pm 1$, correspond on the contrary to the local solutions of (1.3.9).

The commutation relations of the L_n's give rise to a Lie algebra of the kind [25]:

$$[\, L_m , L_n \,] = (\, n - m \,)\, L_{n+m} \qquad (1.3.10).$$

Unfortunately this classical Virasoro algebra does't match with the quantum string theory on the sphere.

Due to the operational nature of the L_n's, they need a prescription in their definition.

So the "true" algebra, to which we will always refer from now on, is actually the Virasoro algebra:

$$[L_m , L_n] = (n - m) L_{n+m} + \frac{c}{12} (m^3 - m) \delta_{n+m,0} \qquad (1.3.11).$$

The additional term appearing in eqn.(1.3.11) spoils the conformal invariance and can be thought as a quantum mechanical breakdown of that symmetry [25]. In any case we notice that the subalgebra SL(2,$\mathbb{C}$) generated by $L_{0,\pm 1}$ is unaltered by the anomalous term: the automorphisms of the sphere still remain a global symmetry.
At this point we introduce the conformal fields: they are tensors

$$\phi^{h,\bar{h}}_{\underbrace{z..z}_{h \text{ times}} ; \underbrace{\bar{z}..\bar{z}}_{\bar{h} \text{ times}}}$$

whose transformation rules under a general conformal variation w=w(z) is of the form:

$$\phi^{h,\bar{h}}_{\underbrace{w..w}_{h \text{ times}} ; \underbrace{\bar{w}..\bar{w}}_{\bar{h} \text{ times}}} \qquad \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} = \phi^{h,\bar{h}}_{z..z ; \bar{z}..\bar{z}} \qquad (1.3.12).$$

From this equation, it should be clear that they transform homogeneously under the conformal group, i.e. [26]:

$$\begin{aligned} [L_m (z) , \phi^{h,\bar{h}} (z,\bar{z})] &= z^m [z \frac{d}{dz} + (m + 1)] h \, \phi^{h,\bar{h}} (z,\bar{z}) \\ [\bar{L}_m (\bar{z}) , \phi^{h,\bar{h}} (z,\bar{z})] &= \bar{z}^m [\bar{z} \frac{d}{d\bar{z}} + (m + 1)] \bar{h} \, \phi^{h,\bar{h}} (z,\bar{z}) \end{aligned} \qquad (1.3.13).$$

Remark: of course in a field theory there are also fields with non homogeneous transformation rules.

An example is provided in string theory by the X's, whose propagator is given by [3]:

$$< X(z), X(w) > = \ln | z - w |^2$$

We don't face us with this problem, because [27] showed that all information about a conformal quantum field theory is contained completely in the correlation functions of the conformal fields only.

It is clear in the context of string theory how this is possible. All the states in the Hilbert space generated by the inhomogeneous fields acting on the vacuum are spurious states [3]. They don't contribute to the physical amplitudes.

This discussion should be more clear when we will introduce the BRST formalism.

Now we come back to eqn. (1.3.13).

All the local transformations appearing there can be summarized expressing them in terms of the so called stress-energy tensor T_{ab} [27]:

$$T_{zz}(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n(z_1)}{(z - z_1)^{n+2}}$$

$$\bar{T}_{\bar{z}\bar{z}}(\bar{z}) = \sum_{n=-\infty}^{+\infty} \frac{\bar{L}_n(\bar{z}_1)}{(\bar{z} - \bar{z}_1)} \quad ; \; T_{z\bar{z}} = 0 \qquad (1.3.14).$$

Some remarks about this equation are in order:

1) We have defined T at the point z instead of the point zero as is done in [3], because the origin is just the singular point of the L_n's.

2) (1.3.14) has to be understood as a formal Laurent expansion, which allows us to rederive the generators of Virasoro algebra by means of a

Cauchy-Riemann integral:

$$L_n(z) = \oint_{C_z} \frac{d\xi}{2\pi i} (\xi - z)^{n+1}\, T(\xi) \qquad (1.3.15).$$

Here C_z is an arbitrary path surrounding the point z.

3)The tensorial properties of T are somewhat hidden in (1.3.14), but from (1.3.15) we see that T must be of the form T_{zz}, since the L_m's have a vectorial nature (see eqn. 1.3.7).

4)The fact that the conformal transformation z -->f(z) and the anticonformal transformations $\bar{z}$ --> $\bar{f}(\bar{z})$ are independent, involves that the trace part of the stress-energy tensor vanishes. In real coordinates, indeed, the trace $T^a{}_a$ is the $T_{z\bar{z}}$ component in complex coordinates, which is zero, as eqn. (1.3.14) shows.

Finally we can use the expression of commutators as complex contour integrals [3,23] to rederive the transformation rules of conformal fields in terms of T:

$$[\, L_m(z)\ ,\ \phi^{(h,\bar{h})}(z,\bar{z})\,] = \int_{C_z} \frac{d\xi}{2\pi i} (\xi - z)^{m+1}\, T(\xi)\, \phi^{(h,\bar{h})}(z,\bar{z}) \qquad (1.3.16).$$

We see from this equation (and from the previous discussion) that a conformal field theory in two dimensions is completely determined by its traceless stress-energy tensor [3,27,29].

2)<u>Quantization of the string theory in the operatorial formalism.</u>

We quantize now the string action:

$$S = \int d^2z \, (\frac{1}{2} \partial_z X \partial_{\bar{z}} X + (b\bar{\partial}c \; + c.c.)) \qquad (1.3.17)$$

(see eqn. 1.2.28). We use an operatorial approach along the lines of [10].

I) The Lagrangian in (1.3.17) has the following expression in real coordinates:

$$L = \eta^{ab} \partial_a X \partial_b X + b_{ab} \eta^{ac} \partial_c c^b \qquad (1.3.18).$$

The conjugate momenta of coordinates X, c and b are computed below:

$$p_X = \partial_\tau X \qquad (1.3.19)$$

$$p_{c^\tau} = \frac{\partial L}{\partial(\partial_\tau c_\tau)} = b_{\tau\tau}$$

$$p_{c^\sigma} = \frac{\partial L}{\partial(\partial_\tau c^\sigma)} = b_{\tau\sigma}$$

$$p_{b_{\tau\tau}} = \frac{\partial L}{\partial(\partial_\tau b_{\tau\tau})} = 0 \quad ; p_{b_{\sigma\tau}} = \frac{\partial L}{\partial(\partial_\tau b_{\sigma\tau})} = 0 \qquad (1.3.20).$$

The four eqns. of (1.3.20) are actually constraints; a further investigation tells us that they are second class constraints [10].

II) We build the commutation relations of ouer field in terms of the Poisson Brackets { , }:

$$\{X, p_X\} = \delta(\sigma-\sigma') \qquad (1.3.21)$$

$$\{c^\sigma, p_{c^{\sigma'}}\} = \delta(\sigma-\sigma') \qquad \{c^\tau, p_{c^\tau}\} = \delta(\sigma-\sigma')$$

$$\{b_{\tau\tau}, p_{b_{\tau\tau}}\} = \delta(\sigma-\sigma') \qquad \{b_{\tau\sigma}, p_{b_{\tau\sigma}}\} = \delta(\sigma-\sigma') \qquad (1.3.22).$$

Here it was used the fact that $b_{\sigma\tau}$ and $b_{\sigma\sigma}$ are not additional degrees of freedom since b_{ab} is a traceless antisymmetric tensor.

III) We eliminate the second class constraints introducing the Dirac Brackets $\{,\}_{DB}$ [30]:

$$\{A,B\}_{DB} = \{A,B\} - \{A,\varphi_\alpha\} C^{-1}_{\alpha\beta} \{\varphi_\beta,B\} \qquad (1.3.23)$$

Where :

A,B are any two observables of σ,τ;

φ_α are the constraints, namely:

$$\begin{cases} \varphi_1 = p_{b_{\tau\tau}} \\ \varphi_2 = p_{c^\tau} - \frac{1}{2} b_{\tau\tau} \\ \varphi_3 = p_{c^\sigma} - \frac{1}{2} b_{\sigma\tau} \\ \varphi_4 = p_{b_{\sigma\tau}} \end{cases} \qquad (1.3.24)$$

$C_{\alpha\beta}{}^{-1}$ is the inverse of the matrix :

$$C_{\alpha\beta} = \{\varphi_\alpha,\varphi_\beta\} \qquad (1.3.25)$$

and the summation over α and β includes also the integration over the σ variables. In terms of the Dirac Brackets we obtain that the only non vanishing commutators are the following two [31]:

$$\begin{aligned} \{c^\tau, b_{\tau\tau}\}_{DB} &= \delta(\sigma+\sigma') \\ \{c^\sigma, b_{\tau\sigma}\}_{DB} &= \delta(\sigma+\sigma') \end{aligned} \qquad (1.3.26).$$

We see that $b_{\tau\tau}$ can be considered the conjugate momenta to c^{σ} and $b_{\tau\sigma}$ the conjugate momenta to c^{σ}.

We turn now to complex coordinate: in this case the conjugate variables are b_{zz} and c^{z}, since we have the following commutations relations:

$$\{c^{z}, b_{zz}\}_{DB} = \{c^{\tau} + ic^{\sigma}, \frac{(b_{\tau\tau} - i\, b_{\sigma\tau})}{2}\} = \delta(\sigma+\sigma') \qquad (1.3.27)$$

$$\{c^{z}, b_{zz}\}_{DB} = \delta(\sigma+\sigma') \qquad (1.3.27a).$$

IV) We exploite the equations of motion coming from the complex action (1.3.17) [12], which are:

$$\partial_{\bar z}\, b_{zz} = \partial_{\bar z}\, c^{z} = 0$$

$$\partial_{z}\, \partial_{\bar z}\, X = 0 \qquad (1.3.28)$$

On the disc the most general solutions of eqns. (1.3.28) are expressed as:

$$\begin{cases} X = \sum_{n=-\infty}^{+\infty} (X_n z^n + \bar X_n \bar z^n) \qquad P = \sum_{n=-\infty}^{+\infty} n X_n z^n + n \bar X_n \bar z^n \\ b_{zz} = \sum_{n=-\infty}^{+\infty} b_n z^n \\ c^{z} = \sum_{n=-\infty}^{+\infty} c_n z^n \end{cases} \qquad (1.3.29).$$

We quantize at this point the theory transforming the coefficients X_n, $\bar X_n$, b_n and c_n in operators acting on a certain Hilbert space. The Dirac quantization recipe ({,}-->1/i[,]) tells us that the commutation relations become:

$$[\hat X_n, \hat P_m]_- = i n\, \delta_{n,m}$$

$$[\hat c_m, \hat b_m]_+ = i\, \delta_{n+m,0} \qquad (1.3.30)$$

where the ^ symbol denotes operators. If we represent $\hat b_m$ as $\delta/\delta c_{-m}$ we have the usual commutation relations:

$$[\hat{X}_n, \hat{P}_m]_- = i n \delta_{n,m}$$

$$[\hat{c}_n, \frac{\delta}{\delta c_m}]_+ = i \delta_{n,m} \quad (1.3.31).$$

Remark: our commutation relations differ from those of [3] because of an extra i factor.

This depends on the definition of BRST transformations one is adopting.

V) Until now we have constructed a theory of operator fields without specifying in which space of states they are acting. We refer to [29] for the demostration that the string theory on the sphere can be effectively represented in an Hilbert space of states.

Here we limit ourselves to the explicit contruction of the vacuum |0>, which will be fundamental in the forthcoming computations.

To accomplish this purpose one computes the amplitude:

$$\langle \underline{X}, \underline{c}; \tau = 0 | 0; \tau = -T \rangle$$

between a state at time -T with $X_n = c_n = 0$ for each integer n and a state at time 0 with classical configuration $\underline{X}$ and $\underline{c}$. This amplitude can be rewritten in terms of the time independent wave functions $\Phi_n(X,c)$, which are eigenstates of the energy:

$$\langle \underline{X}, \underline{c}; 0 | 0; -T \rangle = \sum_{n=0}^{\infty} \phi_n(\underline{X}, \underline{c}) \phi_n^*(0) e^{-E_n T} \quad (1.3.32).$$

Here we understand why we have excluded the b's.

Since we want to express all wave functions in a coordinate representation and the b's are the momenta conjugate to the coordinates c (see eqn. (1.3.30)), they have no right to appear.

Now if we perform the T --> +∞ limit, we see that only the lowest energy contribution survives in (1.3.3), i.e. that of the ground state which gives:

$$\langle \underline{X}, \underline{c}; 0 | 0; -T \rangle \underset{T \to \infty}{\sim} \phi_0(\underline{X}, \underline{c}) \phi_0^*(0) e^{-E_0 T} \quad (1.3.33).$$

From this equation we single out the vawe functions of the vacuum $\Phi_0(\underline{X},\underline{c})$ at the time $\tau = 0$.

In a first quantized theory $\Phi_0(\underline{X},\underline{c})$ has to be understood as a state in the Hilbert space [9]. The other states can be reached easily acting on $\Phi_0(\underline{X},\underline{c})$ with the creation and annihilation operators.

The only problem is the explicit computation of the amplitude in eqn. (1.3.33), which is equivalent to the evaluation of a path integral.

In fact:

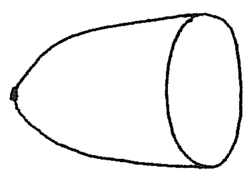
$$\langle \underline{X}, \underline{c}; 0 | 0; -T \rangle = \int_{\substack{c|_{\partial M} = \underline{c} \\ X|_{\partial M} = \underline{X}}} \mathcal{D}X \mathcal{D}b \mathcal{D}c \, e^{-\int_M \left(\frac{1}{2} \partial_z X \partial_{\bar{z}} X + b \bar{\partial} c + c.c. \right)} \qquad (1.3.34)$$

as one can see from the point particle analogy.

In the above expression M is a cylinder (see fig. 1.4) and possesses consequently a boundary ∂M corresponding to the circle S^1 parametrized as $\tau = 0$ and $0 \le \sigma \le 2\pi$.

1.4a: point particle case

Fig 1.4b: string case



3) <u>The stress-energy tensor in the string theory case.</u>

We derive now the stress-energy tensor:

$$T^{ab} = -\frac{4\pi}{\sqrt{g}} \frac{\delta S}{\delta g^{ab}} \qquad (1.3.35)$$

for thκe bosonic string and we show that it is effectively traceless.

A)GHOST SECTOR:the starting point s the action in (1.3.17) expressed in a covariant form:

$$S^{gh} = - \int \nabla^{z} b_{zz} c^{z} \sqrt{g}\, d^2 z + c.c. \tag{1.3.36}$$

The standard action $S_{gh} = \int b\bar{\partial}c + c.c.$ is recovered remembering the expressions of the covariant derivative $\nabla_z : \Upsilon^n \to \Upsilon^{n-1}$ [see A-2] in the conformal gauge $g^{ab} \to e^{\Phi}\eta^{ab}$:

$$\nabla_z = \partial_z + n\, \partial_z \phi \quad ; \quad \nabla^{z} = g^{z\bar{z}} \partial_{\bar{z}} \tag{1.3.37}$$

If we compute δS^{gh} using the formula of [3,23].

$$\delta \nabla^{z} = \frac{1}{2} \delta g^{zz} \nabla_z + \frac{1}{2} n \nabla_z \left(\delta g^{zz}\right) \tag{1.3.38}$$

with $n=1$,
we get as a result:

$$\delta S^{gh} = - \int d^2 z \sqrt{g} \left[\frac{1}{2} \nabla^{z} b_{zz} c^{z} \delta g^{zz} + \delta g^{zz} b_{zz} \nabla_z c^{z}\right] \tag{1.3.39}$$

We see that the variations in $\delta g^{z\bar{z}}$ are not appearing. This happens because the classical action does't depend on the conformal factor ϕ.
In fact $\sqrt{g}\, g^{z\bar{z}}$ is independent from ϕ in the bosonic action and also $\sqrt{g}\, \nabla^{z}$ is independent on ϕ because $\sqrt{g}\, \nabla^{z} = g_{z\bar{z}}\, g^{z\bar{z}} \partial_{\bar{z}} = \partial_{\bar{z}}$.
From the definition of the stress energy tensor T^{ab} in eqn. (1.3.35) this entails that T^{ab} is traceless and so the ghost sector is invariant under conformal transformations.
As a matter of fact we get from (1.3.35):

$$\begin{aligned} T_{zz}(z) &= \frac{1}{2}\left(\partial_z b_{zz} c^{z}\right) + b_{zz} \partial_z c^{z} \\ \bar{T}_{\bar{z}\bar{z}}(\bar{z}) &= \frac{1}{2}\left(\partial_{\bar{z}} b_{\bar{z}\bar{z}} c^{\bar{z}}\right) + b_{\bar{z}\bar{z}} \partial_{\bar{z}} c^{\bar{z}} \\ T_{z\bar{z}} &= T_{\bar{z}z} = 0 \end{aligned} \tag{1.3.40}$$

Normal ordering of operators is understood whenever necessary.

The related generators of the Virasoro algebra are now easily computed:

$$L^{gh}_n(0) = \oint_{C_{z=0}} \frac{dz}{2\pi i} z^{n+1} T(z) = \sum_m \frac{b_{n+m} c_{-m}}{2} [m-n] \qquad (1.3.41).$$

B)BOSONIC PART: we start from $S^X = \int d^2z\sqrt{g}\, g^{\alpha\beta}\partial_\alpha X\, \partial_\beta X.(\alpha,\beta=z,\bar{z})$.
Following an analogous procedure as before we find:

$$\delta S^X = \int d^2z \sqrt{g}\, \frac{\delta g^{zz}}{2} \partial_z X \partial_z X + c.c. \qquad (1.3.42)$$

and

$$T^X_{\bar{z}\bar{z}}(\bar{z}) = \frac{\partial_{\bar{z}} X \partial_{\bar{z}} X}{2} \quad ; \quad T^X_{zz}(z) = \frac{\partial_z X \partial_z X}{2} \qquad (1.3.43).$$

If we rewrite eqn. (1.3.43) in real coordinates τ,σ we obtain:

$$-\bar{T}(\bar{z}) = -\dot{X}^2 + X'^2 - 2i\dot{X}X'$$

$$-T(z) = -\dot{X}^2 + X'^2 + 2i\dot{X}X' \qquad (1.3.44).$$

In this form it is transparent that the stress-energy tensor generates the gauge group of conformal transformations, since it is a linear combination of the first class constraints (1.2.8).

Finally the computation of L_n^X gives:

$$\tilde{L}^X_n = \sum_{l=-\infty}^{+\infty} \frac{\alpha_l \alpha_{n-l}}{2} \qquad (1.3.45)$$

where we used for the X's the following Laurent expansion (compare with that of 1.3.29):

$$X^\mu(z) = q_0 - i p_0 \ln z + i \sum_{n\neq 0} \frac{1}{n} \alpha_n z^{-n}$$

$$p_0 = \alpha_0 \qquad (1.3.46).$$

Eqn. (1.3.45) shows that for n=0 we have:

$$L_0^X = \sum_{l=-\infty}^{+\infty} \frac{\alpha_l \alpha_{-l}}{2} \tag{1.3.47}$$

L_0 is not well defined since the α_1, α_{-1} don't commute. The normal ordering prescription gives us:

$$:L_0: = L_0 - \frac{(D-2)}{24} \tag{1.3.48}$$

In the case D=26 this result agrees with [25].

CHAPTER TWO

2.1 Introduction: BRST Quantization

The BRST quantization is a way to quantize the gauge theories which generalizes the Faddev-Popov mechanism.([redacted]).
As is well known, BRST invariance turns out to be crucial in string theories [22,32]. It is not only an useful tool to prove the unitarity in the conformal gauge, but also provides a method to select automatically the physical vertex operators [3].
Moreover, if it is valid at the quantum level, it implies the absence of anomalies [22]. Here we give some general notions about BRST quantization, following the approach of Kugo and Ojima [33,34].

I)The starting point is the construction of a global BRST symmetry from the gauge symmetry. The associated charge must satisfy the following properties:

nihilpotency:

$$Q_B^2=0 \tag{2.1.1}$$

hermiticity:

$$Q_B^+ = Q_B \tag{2.1.2}$$

These conditions assure [33] the unitarity and hermiticity of the theory.

Example: QED with scalars.

$$\mathcal{L} = \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + D_\mu \varphi_i D^\mu \varphi_i$$

The infinitesimal gauge transformations are written in terms of a parameter $\varepsilon(x)$ in this way:

$$\delta_\varepsilon A_\mu(x) = D_\mu \varepsilon(x) \qquad D_\mu \text{ covariant derivative}$$

$$\delta_\varepsilon \varphi_i(x) = i g \varepsilon^a(x) T_{a\,ij}\, \varphi_j(x) \qquad T^a \text{ generators of the gauge group.} \tag{2.1.3}$$

The corresponding BRST transformations δ_B are builted introducing the ghosts c^a such that:

$$\varepsilon^a = \lambda\, c^a \tag{2.1.4}$$

Here λ is a Grassman constant which is purely immaginary to meet condition (2.1.2).

Now we define

$$\delta_B A_\mu(x) = \lambda D_\mu c(x)$$

$$\delta_B \varphi_i(x) = i g \lambda c^c(x) T_{ij\,c}\, \varphi_j(x) \tag{2.1.5}$$

$$\delta_B c^a(x) = -\lambda g f^a{}_{bc}\, c^b(x) c^c(x) \tag{2.1.6}$$

The δ_B variation of c^a is such that $\delta_B{}^2$ vanishes when applied to the fields.

Since $\delta_B * = \{Q_B, *\}_\pm$, ($*$ an arbitrary operator), this entails that $Q_B{}^2 = 0$.

Finally we have to introduce the antighosts to guarantie the conservation of the fermion number. Their transformation rules under δ_B are:

$$\delta_B b^a(x) = i \lambda B^a(x)$$

$$\delta_B B^a(x) = 0 \tag{2.1.7}$$

where B^a are a set of bosonic fields which play the role of Lagrange multipliers.

Notice that also for the B's and the b's the $\delta_B{}^2 = 0$ property is valid.

One verifies that (2.1.5) defines a global symmetry at the classical level.

So the related BRST current J^B, derived form the Noether theorem:

$$J^B_\mu = D^\nu c \frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} + (c \cdot g T \varphi)_i \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi_i)} - \frac{1}{2} g (c \times c) \frac{\partial \mathcal{L}}{\partial(\partial^\mu c)} + i B \frac{\partial \mathcal{L}}{\partial(\partial^\mu b)} \tag{2.1.8}$$

is conserved:

$$\partial_\mu J^{\mu B} = 0 \qquad (2.1.9).$$

From (2.1.9) we can compute the charge Q_B:

$$Q_B = \int d^3x \, J_0^B \qquad (2.1.10).$$

Because of the conservation of the ghost number, there is another symmetry of the action connected with the following transformations:

$$c \dashrightarrow c\, e^{\theta} \qquad b \dashrightarrow b\, e^{-\theta} \qquad (2.1.11).$$

This invariance leads to an additional conserved current J^c and to a charge Q_c, called the ghost number operator.

II) The second step is to provide a condition to discard the unphisical states, i.e. those with negative norm. Accordingly a physical state must obey the following relation:

$$Q_B |phys> = 0 \qquad (2.1.12).$$

In string theory it is easy to see that this equation,together with (2.1.1), is equivalent to impose:

$$L_m|0> = 0 \qquad n \geq -1 \qquad (2.1.13)$$

which is the usual gauge fixing.

The proof in a more general case is not trivial and we refer to [33] for this.

From eqn. (2.1.12) one can split the space of the states into three pieces:

A) <u>Unphysical vectors</u> {|unphys>} which don't satisfy condition (2.1.12).

B) <u>Physical vectors with strictly positive norm</u>, denoted from now on as {|phys>}.

C) <u>Physical vectors which are annihilated trivially by Q_B</u>.

These are of the form

$|\chi\rangle = Q_B\,|unphys\rangle$. (2.1.13a)

The $|\chi\rangle$ vectors have these two properties:

i) They have vanishing norms.

ii) They decouple completely from the states $|phys\rangle$ of B).

As a a matter of fact:

$\langle\chi|phys\rangle = \langle unphys|Q_B{}^{+}|phys\rangle = \langle unphys|Q_B|phys\rangle = 0$ (2.1.14)

This means that physics does not change under the transformations:

$|phys\rangle \dashrightarrow |phys\rangle + |\chi\rangle$ (2.1.15)

Of course we should check that also the mean values of physical observables $\hat{O}_{phys}$ are unaffected by (2.1.15). In general this purpose is achievied only if $\hat{O}_{phys}$ is invariant under the BRST symmetry.

In formulas:

$[Q_B, \hat{O}_{phys}]_{\pm}\quad Q_B$ (2.1.16).

If eqn. (2.1.16) is verified, the zero normed states $|\chi\rangle$ decouple completely from the physical quantities:

$\langle phys_1 + \chi_1|\hat{O}_{phys}|phys_2 + \chi_2\rangle = \langle phys_1|\hat{O}_{phys}|phys_2\rangle$ (2.1.17).

In correspondence with the categories A),B) and C) of states, we have three distinct kinds of operators:

a) Unphysical operators $\hat{C}$, not obeying (2.1.16).

They send physical states $|phys\rangle$ into unphysical ones:

$$\hat{C} : \{|phys\rangle\} \rightarrow \{|unphys\rangle\} \qquad (2.1.18)$$

b) Physical operators $\hat{B}$, which satisfy (2.1.16) trivially, and are such that:

$$\hat{B} : \{|phys\rangle\} \rightarrow \{|\chi\rangle\} \qquad (2.1.19).$$

All these $\hat{B}$'s, called also null operators, are written as follows:

$$\hat{B} = [\hat{C}, Q_B]_{\pm} \qquad (2.1.20)$$

with $\hat{C}$ unphysical.

c) Finally we have the true physical operators $\hat{A}$, which transform physical states into unphysical ones :

$$\hat{A} : \{|phys>\} \rightarrow \{|phys>\} \qquad (2.1.21).$$

III) The last step is to add a gauge fixing term to the original Lagrangian L_0, in order to break the dangerous gauge invariance.

Since we don't want that the physical expectation values change, it is natural to choose an operator of the $\hat{B}$ type,which is given by:

$$\mathcal{L}_{GF+FP} = -i\delta_B(\bar{c}_a F^a) = \{Q_B, \bar{c}_a F^a\} \qquad (2.1.22)$$

F^a is a gauge fixing, which in this BRST formalism can be also dependent on the ghosts [33,35].

Finally, starting from the total Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{GF+FP} \qquad (2.1.22a)$$

the path integral formulation is builted.

Remarks:

1) Q_B is an operator of ghost number 1 and acts on the space V^n of states with ghost number n as follows [18]:

$$Q_B : V^n \rightarrow V^{n+1} \qquad (2.1.23)$$

Since $Q_B{}^2 = 0$, the BRS charge defines a cohomology operator.

The physical states $|phys> \in V^n$, which obey the condition (2.1.12), are collected in equivalence classes of a cohomology group. We will denote it as $H^n(G,R)$, where G is the group of the original gauge invariance and R is the representation on which the ghosts live. The charge Q_B can be expressed as (see [3]):

$$Q_B = c^i G_i - \frac{1}{2} f_{ij}^{\ k} c^i c^j b_k \qquad (2.1.24)$$

in terms of the generators G_i belonging to the representation R

($[G_i,G_j]=if^k{}_{ij}G^k$). The states $|X\rangle$ of (2.1.13a) are the exact forms of H^n, i.e. the trivial solutions of eqn. (2.1.12). The fact that two states belong to the same equivalence class if their difference is a $|X\rangle$ vector, is simply a restatement of the independence of physical quantities under the transformations (2.1.15).

2) Since the Lagrangian $L_0 + L_{GF+FP}$ does not depend on the constant modes c_0 and b_0 of c and b, we have a degeneration of the vacuum $|0\rangle_{gh}$ in the ghost sector [36].

Note that the bosonic and fermionic sectors are independent so that:

$$|0\rangle = |0\rangle_X \otimes |0\rangle_{gh}$$

because of this degeneration, we can define a ground state $|+\rangle_{gh}$ annihilated by c_0:

$$c_0|+\rangle_{gh} = 0 \qquad (2.1.25)$$

but then the anticommutation relation (see (1.3.31)) $[c_0,b_0]_+ = 1$, tells us that also $|-\rangle_{gh} = b_0\,|+\rangle_{gh}$, which is annihilated by b_0, is a good ground state [36]. Nevertheless b and c don't enter simmetrically in the theory: the b's are rank two tensors ans the c's are vectors. Indeed, if we denote as Q_{c+} the ghost number of $|+\rangle_{gh}$ and as Q_{c-} that of $|-\rangle_{gh}$, we see that their differ because:

$$Q_{c+} = Q_{c-} + 1 \qquad (2.1.26).$$

The knowledge of the absolute values of Q_{c+} and Q_{c-} needs a prescription for the ghost number operator like normal ordering. The explicit computation gives us [36]:

$$\hat{Q}_c = \frac{1}{2}\left(c_0 b_0 - b_0 c_0\right) + \sum_{n=1}^{\infty}\left(c_{-m} b_m - b_{-m} c_m\right) \qquad (2.1.27).$$

From this equation we see that $Q_{c\pm} = \pm\, 1/2$.

So in string theory, due to the singular character of the charge Q_c, we have states with half integer ghost number. It is possible to show that the true vacuum is the state $|-\rangle_{gh}$, with ghost number -1/2, otherwise the condition

(2.1.11) is not equivalent to the standard gauge fixing (2.1.13) [36].

The vacuum $|0\rangle_{gh}$ which we will compute later satisfies effectively this condition.

3) The BRST quantization has been developed in many different formulations. The following approach, due to Fradkin and Vilkovisky [35], will turn useful afterwards.

Hamiltonian approach of Fradkin and Vilkovisky [37]:

i)We start from the generators of the symmetry gauge group G_i, whose commutation relations between themselves and with the Hamiltonian H_0 are:

$$\{G_i, G_j\}_\pm = G_k f^k_{ij} \qquad (2.1.28)$$

$$\{H_0, G_i\} = G_j V^j_i \qquad (2.1.29).$$

(2.1.19) displays the invariance of the Hamiltonian under the symmetry group.

ii) The BRS charge is constructed directly as in (2.1.24):

$$Q_B = c^i G_i + \frac{1}{2}(-1)^{n_i} b_i U^i_{jk} c^j c^k \qquad (2.1.30)$$

where c^i, b_i are variables with opposite statistic to that of the constraint G_i satisfying the following commutation relations:

$$\{c^i, b_j\}_\pm = \delta^i_j \qquad (2.1.31)$$

and

$$n^i = \begin{cases} 0 & \text{if } G_i \text{ is bosonic} \\ 1 & \text{if } G_i \text{ is fermionic} \end{cases}$$

The condition which selects the physical states is as before:

$$Q_B |phys\rangle = 0 \qquad (2.1.32).$$

iii) The gauge fixing term added to the Hamiltonian is again a null operator:

$$H = H_0 + H_{GF+FP} = H_0 + b_i V^i_j c^j + \{F, Q\}_\pm \qquad (2.1.33)$$

Here F is a general gauge fixing.

From (2.1.33) one can quantize the theory.

2.2 BRST Quantization of the string.

We employ the previous formalism to the actual case of the string:

$$S = \int d^2z \sqrt{g} \left(g^{zz} \partial X \partial X + g^{z\bar{z}} \partial X \bar{\partial} X + c.c. \right) \qquad (2.2.1)$$

This action is invariant under the riparametrization group, which in complex coordinates is composed by the transformations of the type:

$$\begin{aligned} z &\to z + v(z, \bar{z}) \\ \bar{z} &\to \bar{z} + \bar{v}(z, \bar{z}) \end{aligned} \qquad (2.2.2).$$

These transformations look like the conformal ones, but they mix z with z.

The gauge acts on the fields X and the Lagrange multipliers g in this way:

$$\delta X = v^z \partial_z X + \bar{v}^{\bar{z}} \partial_{\bar{z}} X$$

$$\delta g_{zz} = \nabla_z v_z \qquad \delta g_{\bar{z}\bar{z}} = \nabla_{\bar{z}} v_{\bar{z}}$$

$$\delta g_{z\bar{z}} = \left(\nabla_z v^z + \nabla_{\bar{z}} v^{\bar{z}} \right) g_{z\bar{z}} \qquad (2.2.3).$$

Accordingly with the BRST quantization procedure, we introduce the ghosts c^z such that:

$$\begin{cases} v^z \to \lambda c^z \\ v^{\bar{z}} \to \bar{\lambda} c^{\bar{z}} = -\lambda c^{\bar{z}} \end{cases} \qquad (2.2.4).$$

With these substitutions eqn. (2.2.3) gets:

$$\delta_B X = \lambda \left(c^z \partial_z X - c^{\bar{z}} \partial_{\bar{z}} X \right)$$

$$\delta_B g_{zz} = \lambda \nabla_z c_z \qquad \delta_B g_{\bar{z}\bar{z}} = -\nabla_{\bar{z}} c_{\bar{z}}$$

$$\delta_B g_{z\bar{z}} = \lambda(\nabla_z c^z - \nabla_{\bar{z}} c^{\bar{z}})\, g_{z\bar{z}}$$

$$\delta_B g = c^z \partial_z g - c^{\bar{z}} \partial_{\bar{z}} g \qquad (2.2.5).$$

The variation $\delta_B c^z$ is determined by the requirement $\delta_B(X,g,c) = 0$. This entails, after a little computation, that:

$$\delta_B c^z = \lambda\, c^z \partial_z c^z + \lambda\, c^{\bar{z}} \partial_{\bar{z}} c^z$$

$$\delta_B c^{\bar{z}} = -\lambda\, c^z \partial_z c^{\bar{z}} - \lambda\, c^{\bar{z}} \partial_{\bar{z}} c^{\bar{z}} \qquad (2.2.6).$$

The set of BRS fields is completed by the antighosts $b_{zz}, b_{\bar{z}\bar{z}}$ and the related Lagrange multipliers $B_{zz}, B_{\bar{z}\bar{z}}$, with nihilpotent transformation rules:

$$\delta_B b_{zz} = i B_{zz} \qquad \delta_B b_{\bar{z}\bar{z}} = i B_{\bar{z}\bar{z}}$$

$$\delta_B B_{zz} = \delta_B B_{\bar{z}\bar{z}} = 0 \qquad (2.2.7).$$

Remark: since no use of the equations of motion was made to assure the condition $\delta_B{}^2 = 0$ throughout eqns. (2.2.5), (2.2.6), (2.2.7), the BRST invariance is valid off shell, apart anomalies.

We fix now the gauge freedom emploing the conformal gauge prescription:

$$g_{zz} = g_{\bar{z}\bar{z}} = 0$$

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2} e^{\phi} \qquad (2.2.8).$$

The related gauge fixing Lagrangian is, according to (2.1.22):

$$\mathcal{L}_{GF+FP} = \sqrt{g}\, \delta_B \left(b_{zz}\, g^{zz} + b_{\bar{z}\bar{z}}\, g^{\bar{z}\bar{z}} \right) \qquad (2.2.9)$$

whose explicit computation yields:

$$\mathcal{L}_{GF+FP} = \sqrt{g}\, \left(i B_{zz}\, g^{zz} + b_{zz} \nabla^z c^z + i B_{\bar{z}\bar{z}}\, g^{\bar{z}\bar{z}} + b_{\bar{z}\bar{z}} \nabla^{\bar{z}} c^{\bar{z}} \right) \qquad (2.2.10).$$

From eqn. (2.2.10) the Lagrange multiplier role of B_{zz} and $B_{\bar{z}\bar{z}}$ is evident.

We turn now to the path integral formulation writing the partition function:

$$Z=\int \mathcal{D}g_{zz}\,\mathcal{D}g_{z\bar z}\,\mathcal{D}c^{z}\,\mathcal{D}b_{\bar z\bar z}\,\mathcal{D}B_{\bar z\bar z}\,\mathcal{D}(c.c.)\, e^{-\int_H d^2z\,(\mathcal{L}_0+\mathcal{L}_{GF+FP})}=\int\mathcal{D}(\ldots)\mathcal{D}(\overline{\ldots})\times$$

$$\times\, e^{-\int_H d^2z\sqrt{g}\,[g^{zz}\partial X\partial X+g^{z\bar z}\partial X\bar\partial X+iB_{zz}g^{zz}+b_{zz}\nabla^{z}c^{z}+c.c.]} \qquad (2.2.11).$$

The usual form of Z is recovered after an integration over the Lagrange multipliers and further over the component of the metric tensor g_{zz} and $g_{\bar z\bar z}$:

$$Z=\int\mathcal{D}g_{zz}\,\mathcal{D}g_{z\bar z}\,\mathcal{D}c^{z}\,\mathcal{D}b_{zz}\,\mathcal{D}(c.c.)\;\delta(g_{zz})\,\delta(g_{\bar z\bar z})\;\times$$

$$\times\, e^{-\int_H d^2z\sqrt{g}\,[g^{z\bar z}\partial X\bar\partial X+g^{zz}\partial X\partial X+b_{zz}\nabla^{z}_{c}c^{z}+c.c.]}=\int\mathcal{D}g_{z\bar z}\,\mathcal{D}b\,\mathcal{D}c\; e^{-\int_H d^2z\,[\frac{1}{2}\partial X\bar\partial X+b\bar\partial c+c.c.]} \qquad (2.2.12).$$

The final result is the partition function (1.2.28) one gets with the Faddeev-Popov method.

The action appearing in (2.2.12) still retains some trace of the initial BRS symmetry. The related current $j_z{}^B, j_{\bar z}{}^B$ can be computed directly from the current $J_z{}^B, J_{\bar z}{}^B$ connected with the BRST transformations:

$$J^{B}_{z}=\frac{\partial\mathcal{L}}{\partial(\partial_{\bar z}g^{z\bar z})}\,\delta_B g^{z\bar z}+\frac{\partial\mathcal{L}}{\partial(\partial_{\bar z}b_{zz})}\,\delta_B b_{zz}+\frac{\partial\mathcal{L}}{\partial(\partial_{\bar z}X)}\,\delta_B X+\frac{\partial\mathcal{L}}{\partial(\partial_{\bar z}c^{z})}\,\delta_B c^{z}+$$

$$+\frac{\partial\mathcal{L}}{\partial(\partial_{\bar z}g^{\bar z\bar z})}\,\delta_B g^{\bar z\bar z}+\frac{\partial\mathcal{L}}{\partial(\partial B_{zz})}\,\delta_B B_{zz}+c.c. \qquad (2.2.13).$$

Eqn. (2.2.13) is just an application of the Noether theorem.

Now $j_z{}^B$ is achievied simply taking into account the constraints (2.2.8) and the fact that after their imposition g^{zz} is not more appearing in the Lagrangian (see eqn. 2..2.12).

We obtain:

$$j^{B}_{\bar z}=(c^{z}\partial_z X-c^{\bar z}\partial_{\bar z}X)\bar\partial X-(c^{z}\partial_z c^{\bar z}+c^{\bar z}\partial_{\bar z}c^{\bar z})\,b_{\bar z\bar z}$$

$$j^{B}_{z}=(c^{z}\partial_z X-c^{\bar z}\partial_{\bar z}X)\partial X+(c^{z}\partial_z c^{z}+c^{\bar z}\partial_{\bar z}c^{z})\,b_{zz} \qquad (2.2.14).$$

It is easy to see that the related charge Q_F is still nihilpotent, but this time only on-shell, i.e. after the use of the equations of motion (1.3.28) [22].

Hence the effective current leading to $Q_F{}^2 = 0$ is instead :

$$T^B_z = c^z \partial_z X(z) \partial_z X(z) + c^z \partial_z c^z(z)\, b_{zz}(z) \qquad (2.2.15)$$

We recognize here, apart an ininfluent total derivative, the current of [3], which generates the conformal group transformations:

$$\begin{aligned} &[Q_F, X^\mu]_- = c^z \partial_z X^\mu \\ &[Q_F, c^z]_+ = c^z \partial_z c^z \\ &[Q_F, b_{zz}]_+ = T^X_{zz} + T_{zz}^{gh} \end{aligned} \qquad (2.2.16).$$

<u>Comment:</u>

We can provide only an hint of what has happened resting on an analysis performed with the Fradkin-Vilkovisky method.

We started with a charge Q_B, conserved off shell, related with the reparametrization group.

The choice of gauge fixing has restricted us to the conformal group, which is, on a chart, a subgroup of the initial one. In this way, the reparametrization current J^B merges in the conformal current j^B, but in this procedure we have reduced the invariance of the theory.

In particular we have discarded so many generators that the Lie algebra of the residual group is not able to close without the aid of the equations of motion.

This explains intuitively why the operator Q_F is now nihilpotent on-shell.

At this point a remark is in order: when we supposed that $Q_B{}^2 = 0$ holds off shell, we were neglecting the anomalies: in fact, even if in our simply case we don't have to face the anomalies coming from the moduli space [32], we still encounter the conformal anomaly.

Until now we have tacitly assumed to be in D=26 dimensions to avoid problems [36]. Nevertheless, to find anomalies, one has to verify the Ward Identities, so the question arises, if we have to use the full BRST symmetry of (2.1.5),(2.1,6),(2.1.7) or the reduced symmetry of (2.2.16). This point was recognized by Mansfield [22]. The generator of the W. Identities he proposes:

$$\Omega_{WI} = \int d^2z \sqrt{g} \left(\xi^\mu \hat{c}^z \partial_z \hat{X}^\mu - \gamma_z \hat{c}^z \partial_z \hat{c}^z + 2\beta^{zz} \left(T^X_{zz}(\hat{X}) + T^{gh}_{zz}(\hat{b}, \hat{c}) + c.c. \right) \right)$$

$$\hat{X}^\mu = \frac{1}{\sqrt{g}} \frac{\delta}{\delta \xi^\mu}, \quad \hat{c}^z = \frac{1}{\sqrt{g}} \frac{\delta}{\delta \gamma^z} \quad \hat{b}_{zz} = \frac{1}{\sqrt{g}} \frac{\delta}{\delta \beta^{zz}} \tag{2.2.17}$$

acting on the functional $Z(\xi^\mu, \beta, \gamma)$ with sources ξ^μ, γ^z, β^{zz} is based on the full BRST symmetry.

Remark: in the approach of [3] one uses an operational formalisms, where explicit use of the equations of motion (1.3.28) is made, so the charge appearing there is Q_F: this is not possible in the path integral formulation.

CHAPTER THREE

In this chapter we give an explicit form to the ground state |0> along the lines explained in section 1.3.

We provide two methods of computation.

In the first, one extracts the boundary contribution to the amplitude (1.3.4), fixing suitable boundary conditions [38].

In the other we use the definition of path integral a la Feynman and Hibbs [11] discretizing the time τ axis in many intervals and then performing the limit in which the amplitude of these intervals goes to zero.

As we will see, the two vacua look very different: this is because in the first approach one is lead to add a boundary term in the usual ghost action,which can be considered as a canonical transformation of the dynamical variables b and c.

So the fields appearing in one vacuum are actually different from the fields appearing in the other.

We have found the operator which relates the two vacua and we have verified that, apart from this transformation, they coincide.

After that we use the BRST formalism developed in chapter two in order to see that our vacuum is effectively annihilated by the BRST charge of [3].

Since we are in an operatorial formulation of the theory, it is not necessary to employ the full BRST charge Q_B as explained at the end of the last chapter.

Note that to comply with the standard notation, throughout this chapter we use the symbol Q_B for the charge Q_F.

3.1-Computation of the vacuum wave function: Saddle-Point Evaluation.

The amplitude < $\underline{X}$,$\underline{c}$;0 | 0; -T> of (1.3.34) which we want to compute is basically a determinant for the Laplacian operator $\partial_z \partial_{\bar{z}}$ and for the P_1 operator $g^{z\bar{z}}\partial_{\bar{z}}$ (see appendix A). Since they are operators defined on a manifold with boundary, (in our explicit case the semiinfinite cylinder of section 1.3), we need to fix their boundary conditions to determine their eigenvalues uniquely.

We start with the simple case of the Laplacian. Since we use a sort of saddlepoint evaluation to compute determinants, we rewrite the X fields as:

$$X = X_{cl} + \delta X \qquad (3.1.1).$$

where X_{cl} is a field obeying the classical equations of motion and δX is its quantum fluctuation.

Now the classical action, in terms of the shifted fields (3.1.1) looks like:

$$S(X_{cl} + dX) = S(X_{cl}) + \left[i \int_{\partial M} d\bar{z}\, (\partial_{\bar{z}} X_{cl})\, \delta X \; - i \int_{\partial M} dz (\partial_z X_{cl}) \delta X \; - i \int_{\partial M} dz\, (\partial_z \delta X)\, \delta X \right] +$$

$$+ i \int_M d\bar{z} dz\, \delta X (\partial_z \partial_{\bar{z}}\, \delta X) \qquad (3.1.2).$$

Notice that here we have employed the equations of motion:

$$\partial_z \partial_{\bar{z}} X_{cl} = 0 \qquad (3.1.3).$$

Of course, if we want to derive the correct determinant, we have to require that the linear part in δX vanishes. This result can be achieved if we impose:

$$\delta X_{|\partial M} = 0 \qquad (3.1.4).$$

With eqn. 3.14 we have choosen the so called Dirichlet boundary conditions for the fields X.

Actually O. Alvarez in [24] claims that these boundary conditions are too strong because one loses the reparametrization invariance on the boundary. So he proposes more sophisticated Modified Dirichlet Boundary Conditions. Nevertheless one shows [24] that in the case of the Laplacian acting on scalar fields X, the choice (3.1.4) does not lead to pathologies.

In terms of the shifted fields, we obtain from the X integration in (1.3.34):

$$< X_B, b_B, c_B ; 0 \,|\, 0 ; -T > = e^{-S_{cl}(X_{cl})} \int Db\, Dc \; (\det \Delta)^{-\frac{d}{2}} e^{-\int_M d^2z\, (b\bar{\partial}c + c.c)} \qquad (3.1.5).$$

where: $X_B = X_{|\partial M}$, $b_B = b_{|\partial M}$, $c_B = c_{|\partial M}$ (3.1.5a),

X_B, b_B, c_B are classical fields,

$$S_{cl}(X_{cl}) = \int_M dz\, d\bar{z}\; \partial_z X_{cl}\, \partial_{\bar{z}} X_{cl} \qquad (3.1.6)$$

and Δ is the Laplacian without the zero modes X_0.

As we see, we were able to separate the contribution of the classical field configurations at the boundary from the Δ determinant, which is a pure number.

The same procedure can be developed in the case of the ghosts, but we have to use more care in the choice of the boundary conditions. This is because the operator $P_1 = \bar{\partial}$ is not selfadjoint like the Laplacian.

We determine the boundary conditions for P_1 and P_1^+ requiring the following properties [24]:

i) P is a differential operator: we want the freedom of integrate by parts without worry with surface terms.

ii) $\det(P_1^+P_1) = |\det(P_1)|$ defines a functional measure: accordingly the operator P^+P has to be definite positive. Here we are neglecting the zero modes.

Let's start with the property i).

First of all we have to introduce an inner product in the space of tensors.

In general, for two tensors $\psi_{z..z}$ and $\phi_{z..z}$ in $\mathcal{T}^n$ this is provided by:

$$< \psi , \phi > = \int_M d^2 z \sqrt{g}\ (g_{z\bar{z}})^n (\psi) * \phi \qquad (3.1.7)$$

$$(\psi_{z..z})^* = \psi_{\bar{z}..\bar{z}} \qquad (3.1.8).$$

Secondly, we proceed shifting the fields b and c:

$$\begin{aligned} b &= b_{cl} + \delta b \\ c &= c_{cl} + \delta c \end{aligned} \qquad (3.1.9).$$

Now we can compute the following difference for the quantum fluctuations δb and δc:

$$< b , P_1\, \delta c > - < P_1^+\, \delta b, \delta c > = - \mathrm{Im}\, \{ \int_{\partial M} dz\ \delta b_{zz}\ \delta c^z \} \qquad (3.1.10).$$

If we want to integrate freely by parts, the RHS has to vanish. But δc is a vector representing the infinitesimal reparametrizations: so [23] it is a tangent vector on M.

On the boundary, in complex coordinates, this entails:

$$\mathrm{Re}\ \delta c^{z}|_{\partial M} = 0 \qquad (3.1.11).$$

Accordingly in (3.1.9) we need to require that:

$$\mathrm{Im}\ \delta b_{zz}|_{\partial M} = 0 \qquad (3.1.12).$$

Analogously we verify the positivity, i.e. property ii).
We find for $P_1{}^{+}P_1$ that:

$$< \delta c\ , P_1^{+} P_1\ \delta c > - < P_1\ \delta c\ ,, P_1\ \delta c > = \mathrm{Im} \left\{ \int_{\partial M} dz \sqrt{g}\ \delta\ c^{z}\ (P_1\ \delta c)_{zz} \right\} \qquad (3.1.13).$$

This equation, taking into account (3.1.11), means that tha positivity of $P_1{}^{+} P_1$ is assured if:

$$\mathrm{Im}\ (P\delta c)_{zz}|_{\partial M} \equiv \mathrm{Im}\ (\nabla_z\ \delta c_z|_{\partial M}) = 0 \qquad (3.1.14).$$

We have still to see that also $P_1 P_1{}^{+}$ has strictly positive eigenvalues.
In a similar way as before, we find the additional condition:

$$\mathrm{Re}\ (\nabla^{z}\ \delta b_{zz})|_{\partial M} = 0 \qquad (3.1.15).$$

To summarize we collect our results scattered in formulas (3.1.11), (3.1.12), (3.1.14) and (3.1.15):

$\mathrm{Re}\ \delta c^{z}|_{\partial M} = 0$ by definition of c^{z}

$$\text{Im}\, \delta b_{zz|\partial M} = 0 \qquad \text{requiring the freedom of integrate by parts}$$

$$\text{Im}\, \nabla^z \delta c_{z\, |\partial M} = 0 \qquad \text{positivity of } P_1^+ P_1$$

$$\text{Re}\, (\nabla^z \delta b_{zz})_{|\partial M} = 0 \qquad \text{positivity of } P_1 P_1^+ \qquad (3.1.16).$$

In real coordinates we recover the results of [24]:

$$n^b \delta c_{b\, |\partial M} = 0$$
$$n_b t_a \delta b_{ba\, |\partial M} = 0$$
$$n_b t_a (P_1 \delta c)^{ab}{}_{|\partial M} = 0$$
$$n^a (P_1^+ \delta b)_{a\, |\partial M} = 0 \qquad (3.1.17)$$

where n^a and t^a are respectively the normal and tangent vectors of the boundary ∂M.

One may wonder of the fact that we have give conditions also on the derivatives of the fields because our problem is of the first order. Nevertheless one should recognize that these conditions are more appropriate in this case than the simpler Dirichlet ones:

$$\text{Re}\, \delta c_{\,|\partial M} = \text{Im}\, \delta c_{\,|\partial M} = 0$$

$$\text{Re}\, \delta b_{\,|\partial M} = \text{Im}\, \delta b_{|\partial M} = 0 \qquad (3.1.18).$$

At this point we compute the ghost action:

$$S_{gh}(b,c) = \int_M d^2z\, (b\bar{\partial} c + \text{c.c}) \qquad (3.1.19)$$

in terms of the shifted fields:

$$S_{gh}[b_{cl} + \delta b, c_{cl} + \delta c] = S_{gh}[b_{cl}, c_{cl}] + i \int_M d^2z\, \delta b_{zz} \bar{\partial}\, \delta c^z + i \int_{\partial M} dz\, b_{zz_{cl}}\, \delta c^z \qquad (3.1.20).$$

We see that the action (3.1.19) does not lead to the determinant of $\partial_{\bar{z}}$ because we have an additional non vanishing boundary term. In order to eliminate this linear term in the quantum fluctuation we are forced to add an extra piece to the action (3.1.19).

We express it in real coordinates ξ^a:

$$S'_{gh}[b,c] = S_{gh}[b,c] - \int_{\partial M} dS^a\, b_{ac}\, t^c\, t_b\, c^b \qquad (3.1.21).$$

Remark: if we parametrize the boundary with a local parameter λ so that the coordinates get $\xi^a(\lambda)$ on ∂M, we can write the infinitesimal surface element $d\, S^a$ as:

$$dS^a = ds\, n^a \qquad (3.1.22)$$

with:

$$ds = \sqrt{g_{ab} \frac{d\xi^a}{d\lambda} \frac{d\xi^b}{d\lambda}}\, d\lambda$$

the element of arc lenght.

The additional term to the Lagrangian in (3.1.21) is only a total derivative which defines a canonical transformation of the ghost fields: moreover it suppresses exactly the boundary term of (3.1.20) yielding just the correct determinant.

Unfortunately this cancellation is more subtle in complex coordinates because it happens only when we consider the contribution to the action of b and c together with their complex conjugates.

This was the main reason to the use of real coordinates in (3.1.21).

in any case , in terms of z and $\bar{z}$, the new action is given by:

$$S'_{gh}(b,c) = \int_M d^2z\, b_{zz}\, \partial_{\bar{z}} c^z - \int_{\partial M} dz\, (b_{zz} - b_{\bar{z}\bar{z}})\,(c^z - c^{\bar{z}}) \qquad (3.1.23).$$

Since we are sure that now we will obtain the correct determinant, we integrate over b and c in (3.1.5), so that the amplitude $<X_B, b_B, c_B ; 0| 0 ; -T>$ gets:

$$<X_B, b_B, c_B ; 0 | 0 ; -T> \underset{T\to\infty}{=} N\, e^{-S_{cl}(X_{cl}) - S_{gh}(b_{cl}, c_{cl})} \qquad (3.1.24).$$

In N we have factored all determinants: all the dependence on X_B, b_B, c_B is confined in the exponential term which has to be proportional to the vacuum wave function $\phi_0(X_B, b_B, c_B)$. In particular we see from (3.1.23) that the ghost contribution to $\phi_0(X_B, b_B, c_B)$ is determined only by the imaginary parts of b_{cl} and c_{cl}.

The appearence of the b fields with respect to the original formula (1.3.34) is not amazing because we have mixed the b and c fields with a canonical transformation.

Finally we find explicitly the form of the wave function $\phi_0(X_B, b_B, c_B)$ in terms of the boundary data X_B, b_B, c_B.

We have to compute:

$$S_{cl}(X_{cl}) = \int_M d^2z\, \partial_z X_{cl}\, \partial_{\bar{z}} X_{cl} \qquad (3.1.25)$$

and

$$S_{gh} = \int_M d^2z\, b_{zz}\, \partial_{\bar z}\, c^z - \int_{\partial M} dz\, (b_{zz} - b_{\bar z\bar z})\, (c^z - c^{\bar z}) + \text{c.c.} \qquad (3.1.26)$$

with the conditions (3.1.5a).

We develope the classical fields on the semiinfinite cylinder in Fourier series ($z=\tau+i\sigma$):

$$X_{cl} = \sum_{n=0}^{\infty} (X_n e^{nz} + \bar X_n e^{n\bar z}) \qquad (3.1.27)$$

$$c^z = \sum_{n=0}^{\infty} c_n e^{nz} \qquad (3.1.28)$$

$$c^{\bar z} = \sum_{n=0}^{\infty} \bar c_n e^{nz} \qquad (3.1.29)$$

$$b_{zz} = \sum_{n=0}^{\infty} b_n e^{nz} \qquad (3.1.30)$$

$$b_{\bar z\bar z} = \sum_{n=0}^{\infty} \bar b_n e^{n\bar z} \qquad (3.1.31).$$

One can show that these are the most general solutions regular at $\tau \to -\infty$.

Substituting (3.1.27 .... 3.1.31) in (3.1.25) and (3.1.26) we get from (3.1.24), apart from a normalization constant:

$$\phi_0 = e^{-\sum_{n\geq 1}^{\infty} \pi n X_n \bar X_n}\; e^{-\sum_{n\geq 1}^{\infty} (b_n \bar c_n + \bar b_n c_n)} \qquad (3.1.32).$$

Equivalently this expression can be restated through the imaginary parts of b and c on the boundary [10]:

$$\mathrm{Im}\, b_{zz_{cl}} = \sum_{n=-\infty}^{n=+\infty} \theta_n e^{-in\sigma} \quad (3.1.33)$$

$$\mathrm{Im}\, c^z_{cl} = \sum_{n=-\infty}^{n=+\infty} \hat{\theta} e^{-in\sigma} \quad (3.1.34).$$

Using (3.1.33) and (3.1.34) we rewrite ϕ_0 as:

$$\phi_0 (X, \theta, \hat{\theta}) = e^{-\sum_{n\geq 1}^{\infty} \pi n X_n \bar{X}_n} \; e^{-\sum_{n\geq 1}^{\infty} (\hat{\theta}_{-n} \theta_n - \hat{\theta}_n \theta_{-n})} \quad (3.1.35)$$

Notice that in (3.2.32) and (3.2.35) there is also the contribution of the conjugate fields $\bar{b}$ and $\bar{c}$.

3.2-An Alternative Method to Compute the Vacuum Wave Function.-Properties of the Wave Functions.

We provide here an expression of the Wave function ϕ_0 derived from an alternative approach to that of section 3.1.

Actually, since the computation is quite lenghty, we have confined it in the appendix B and here we state just the result:

$$\phi_0^k = b_0 \prod_{n>0}^{\infty} c_n \quad (3.2.1)$$

We use the index k to distinguish this new ϕ_0 from the previous one.

We will strive for a relation between the vacuum of (3.2.1) and that of (3.1.35).

To do this we introduce the momenta conjugate to the variables θ and $\hat{\theta}$ of (3.1.33) and (3.1.34). From the commutation relations (1.3.26) we see that these momenta are given by:

$$c^{\sigma} \equiv \sum_{n=-\infty}^{+\infty} \hat{\pi} \, e^{-in\sigma} \tag{3.2.2}$$

$$[\hat{\theta}_n, \hat{\pi}_m]_+ = i\delta_{n+m,0} \tag{3.2.3}$$

and

$$b_{\tau\tau} \equiv \sum_{n=-\infty}^{n=+\infty} p_n \, e^{-in\sigma} \tag{3.2.4}$$

$$[\theta_n, \pi_m] = i\,\delta_{n+m,0} \tag{3.2.5}$$

c^S and $b_{\tau\tau}$ are proportional to Re c^Z and Re $b_{\tau\tau}$ respectively.

Now, since the boundary term added to the action (3.1.2) resembles a Fourier Transform acting on the Wave Function, we seek for the inverse of this transformation $\mathfrak{F}$.

This is of the form:

$$\mathfrak{F}[\phi_0] = \int d\hat{\theta}_{-n} \, d\hat{\theta}_n \, e^{i(\hat{\theta}_n \hat{\pi}_{-n} + \hat{\theta}_{-n}\hat{\pi}_n)} A(\theta, \hat{\theta}) \tag{3.2.6.}$$

As a matter of fact, if we choose as $A(\theta, \hat{\theta})$ our wave function ϕ_{0gh} we get:

$$\mathfrak{F}[\phi_0] = \int d\hat{\theta}_{-n} \, d\hat{\theta}_n \, e^{-(\hat{\theta}_{-n}\theta_n - \hat{\theta}_n\theta_{-n})} \, e^{i(\hat{\theta}_n \hat{\pi}_{-n} + \hat{\theta}_{-n}\hat{\pi}_n)} =$$

$$= \prod_{n=1}^{\infty} (\hat{\pi}_{-n} - i\theta_{-n})(\hat{\pi}_n + i\theta_n) \tag{3.2.7.}$$

But from the definitions of $\hat{\pi}$ and $\hat{\theta}$ we recognize that (3.2.7) can be rewritten in terms of c and $\bar{c}$:

$$\mathfrak{F}[\phi_0] = \prod_{n=1}^{\infty} \bar{c}_{-n} c_n \qquad (3.2.8).$$

This is equal to the Fourier transform of (3.2.1) in the constant mode b_0 remembering that here we have included also the contribution of the complex conjugate ghost c. In fact we can show easily that (3.2.8) has the right ghost number - 1/2.

The bosonic part, instead, did not present shortcomings.

From now on we will consider as our vacuum wave function that of (3.2.1):

$$|0> = |0>_X \otimes |0>_{gh} = e^{-\sum_{n \geq 1} \pi n X_n X_{-n}} \quad b_0 \prod_{m \geq 0} c_m \otimes (c.c.) \qquad (3.2.9)$$

Notice that in (3.2.9) we have put $X_n = X_{-n}$.

Now we will investigate on the BRST properties of the vacuum state.

We list the creation and annihilation operators:

A) Bosonic sector:

destruction operators are:

$$\alpha_n = -i\left[\frac{1}{\pi}\frac{\partial}{\partial X_{-n}} + n X_n\right] \qquad (n>0) \qquad (3.2.10)$$

$$\tilde{\alpha}_n = -i\left[\frac{1}{\pi}\frac{\partial}{\partial X_n} + n X_{-n}\right] \qquad (n>0) \qquad (3.2.11).$$

The costant modes α_0, $\tilde{\alpha}_0$ are considered later.

The creation operators are instead $\alpha_{-n}, \tilde{\alpha}_{-n}$,(n>0).

The commutation relations are:

$$[\alpha_n , \alpha_m]_- = n\, \delta_{n+m,0} \qquad (3.2.12)$$

$$[\tilde{\alpha}_n , \tilde{\alpha}_m]_- = n\, \delta_{n+m,0} \qquad (3.2.13).$$

B) Ghost sector: (see eqns. B-29 andB-30).

destruction operators:

$$c_n , b_n \equiv \frac{\delta}{\delta c_{-n}} \qquad n>0 \qquad (3.2.14)$$

creation operators:

$$c_{-n} , b_{-n} \equiv \frac{\delta}{\delta c_n} \qquad n>0 \qquad (3.2.15).$$

Moreover b_0 is a destruction operator and c_0 is the related creation operator.

From (3.2.10) and (3.2.11) we see that the constant modes appears in a wrong way in the bosonic vacuum $| 0 >_X$.

In fact, applying α_0 and $\tilde{\alpha}_0$ to $| 0 >_X$we obtain:

$$\alpha_0 \Psi_0 = \tilde{\alpha}_0 \Psi_0 = 0 \qquad (3.2.16)$$

Moreover:

$$P_0 \Psi_0 = \frac{\delta}{i \delta X_0} \Psi_0 = 0 \qquad (3.2.17).$$

So our ground state has zero momentum: but we know that the true ground state of the bosonic string is a state with tachyonic momentum. We give the correct quantum numbers to the vacuum defining a new vacuum in this manner:

$$|0> = e^{i\,kX_0}\,|0>_X \otimes |0>_{gh} \qquad (3.2.18)$$

with $k^2=8$.

We check now that this vacuum is effectively annihilated by the BRST charge as the physicality condition (2.1.12) requires.

First we construct Q_B from the current related to transformations (2.2.16):

$$j^B{}_z = c^z\partial_z X(z)\,\partial_z X + c^z\partial_z\, c^z\, b_{zz} \qquad (2.1.16).$$

In two dimensions eqn. (2.1.10) yields for the classical charge:

$$Q^B_{cl} = \int_{C_0} \frac{dz}{2\pi i} j^B_z \qquad (3.2.19)$$

The explicit integration of (3.2.19) gives, after a little reshuffling of terms:

$$Q^B_{cl} = \sum_{n=-\infty}^{+\infty} c_{-n}\, L^X_n + \sum_{l,m} c_l\, c_m\, (1\text{-}m)\, b_{-n-1} \qquad (3.2.20)$$

Now we turn to the quantistic theory; operators are regularized with normal ordering.

$$:Q^B: = \sum_{n=-\infty}^{+\infty} c_{-n} :L^X_n: + \sum_{l,m} :c_l\, c_m\, (1-m)\, b_{-n-1}: \qquad (3.2.21).$$

After a straightforward calculation we see that the only divergent term in (3.2.21) is L_0^X.

Using eqn (1.3.48) we obtain:

$$Q^B = \sum_{n=-\infty}^{+\infty} c_{-n} :L_n^X: + \sum_{l,m} (1 - m) : c_l c_n b_{-m-l} : + \frac{(D-2)}{12} c_0 \qquad (3.2.22)$$

We apply this BRST charge to the vacuum of (3.2.18) in D = 26 dimensions.

The term

$$\sum_{l,m} (1-m) : c_l c_m b_{-m-l} :$$

vanishes automatically acting on $| 0 >$: this can be shown in a straightforward way checking all possible cases l=m=0, l=0 but $m \neq 0$ and so on.

So the remaining task consists in to realize that also:

$$\sum_{n=-\infty}^{+\infty} (c_n : L_n^X : + 2c_0) | 0 > = 0 \qquad (3.2.23).$$

It is easy to see that this is true provided we use the tachyonic vacuum of (3.2.18).

In conclusion the vacuum of (3.1.18) has to be the correct vacuum, since its ghost number, its tachyonic quantum numbers and finally the physicality condition are all verified.

This ends our analysis.

We should remark that in this discussion we have not exhausted all possible methods of vacuum state computation. For example there is the approach of the geometric quantization [8] and that of the holomorphic representation. In any case they are all based on the principles stated at the end of section 1.3

APPENDIX A

A.1 Complex and Almost Complex Structures

Let M be a real differentiable manifold with its tangent space T(M).
We suppose that M satisfies the following two properties [11,39,40,41]:
1) M is even dimensional, with dimension 2m for each $x \in M$.
2) $H^{2m}(M,R) \neq 0$, i.e. M is orientable.
So it can be defined on $T_x(M)$ a $2m \times 2m$ matrix J such that $J^2 = -1$, which is the analogous of the complex number $\sqrt{-1}$ in the simple complex analysis.
In order to preserve the operation of multiplication of vectors by J, the group of symmetries $GL(m,\mathbb{R})$ acting on $T_x(M)$ should be restricted to matrices which commute with J.
This leads to the group of symmetries $GL(m,\mathbb{C})$, which is a subgroup of $GL(m,\mathbb{R})$.
If it is possible to do this for each point $x \in M$ then we say that M has an almost complex structure.
Remark:the properties 1) and 2) are only necessary.
The necessity of M to be 2m dimensional is explained by consistency:

$$0 \leq (\mathrm{Det}\, J)^2 = \mathrm{Det}(J^2) = \det(-\mathbb{1}) = (-1)^n \qquad \text{A-1}$$

and this implies n>0.
The necessity of the orientability is more difficult and we skip its demonstration.
In the case of a Riemann Surface, for which m = 1, we have:

$$J^a{}_b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{A-2}$$

This tensor can be diagonalized only using complex coordinates z and $\bar{z}$.

In fact let:

$$z = x + iy \qquad \text{A-3}$$
$$\bar{z} = x - iy$$

where x,y are real coordinates on M, then:

$$J^{z}{}_{\bar{z}} = J^{\bar{z}}{}_{z} = 0 \qquad \text{A-4}$$
$$J^{z}{}_{z} = -J^{\bar{z}}{}_{\bar{z}} = -i$$

This is called the canonical form of J.

We go now one step beyond the almost complex structure, extending the complex coordinates z and $\bar{z}$ on an entire open subset U of M and not only in a point x

This implies that we have to put the tensor J in the canonical form A-4 on all U, solving in this set the following Beltrami equations in the z variable:

$$J^{a}{}_{b} \frac{dz}{d\xi^{a}} = i \frac{dz}{d\xi^{a}} \qquad \text{A-5}$$

where $\xi^a = (x,y)$ are the real coordinates of M.

When it is possible to find solutions of eqn. A-5 for each open set U_α of a given covering of M, then we say that M has a <u>complex structure.</u>

A similar problem arises in the general relativity theory: one want to have a flat metric $g_{\mu\nu} = \delta_{\mu\nu}$ in an entire neighborhood of a point p.

The necessary and sufficient condition in order to get this result is the vanishing of the curvature tensor R_{ijkl} in this neighborhood.

In our case the role of R is played by the Nijenhuis tensor [18]:

$$N^{i}_{jk} = 2\left(J^{h}_{j}\,\partial_h J^{i}_{k} - J^{h}_{k}\,\partial_h J^{i}_{j} - J^{i}_{h}\,\partial_j J^{h}_{k} + J^{i}_{h}\,\partial_k J^{k}_{i}\right) \qquad \text{A-6}$$

In fact one can state the following theorem [39]:

<u>Theorem:</u> an almost complex structure J corresponds to a complex structure iff:

$$N^{i}_{jk} = 0$$

The necessity of this condition is easily verified:
if $J^a{}_b = i\delta^a{}_b$ the partial derivatives of J in A-6 are vanishing giving N = 0.

Remarks:

1) A complex structure on a real differential manifold determines a complex manifold [41].

In fact we have now coordinates z_α on each patch U_α of M.

In $U_\alpha \cap U_\beta$, since the transition functions must preserve the canonical form of J, it is easy to proof that the only possible transformation functions are holomorphic, i.e. of the kind:

$$z_\alpha = z_\alpha(z_\beta) \qquad \text{A-7}$$

For example the component $J^z{}_z$ transforms under A-7 as:

$$J'^{z_\alpha}{}_{z_\alpha} = J^{z_\beta}{}_{z_\beta} \frac{dz_\beta}{dz_\alpha} \frac{dz_\alpha}{dz_\beta} = J^{z_\beta}{}_{z_\beta}$$

2) We can extend this discussion to the case of a Riemannian manifold M, in which, by definition, a metric tensor is defined.

In this case the transition functions on $T_x(M)$ are elements of $SO(2n,\mathbb{R})$ and the complex group G such that $G \subset SO(2n,\mathbb{R})$ and preserves the complex structure, is U(n).

A-2. The Complex Formalism for String Theory

We fix the notations which were used in this thesis [16,24].

We know that, after imposing the conformal gauge, the action of the bosonic string still remains invariant under the conformal group, previously called holomorphic transformation group.:

$$z' = f(z) \qquad \text{A-8}$$

We have also seen that these trasformations allow a complex structure on M.

At this point we can introduce a metric in order to rise and lower indices.

In string theory the metric ds^2 can be written locally as [16]:

$$ds^2 = \frac{1}{2} e^{\phi} (dz \otimes d\bar{z} + d\bar{z} \otimes dz) = e^{\phi} (dx^2 + dy^2) \qquad \text{A-9}$$

Strictly speaking, ϕ is not a scalar: in fact its transformations underA-8 are:

$$\phi \rightarrow \phi + \ln \left| \left(\frac{\partial z}{\partial z'} \right) \right|^2 \qquad \text{A-10}$$

From A-10 we verify that only the subgroup SO(2) ≈ U(1) of GL(1,ℂ) mantains the metric invariant.

In matrix form the metric A-9 becomes:

$$\| g_{ab} \| = \begin{pmatrix} 0 & g_{\bar{z}z} \\ g_{z\bar{z}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} e^{\phi} \\ \frac{1}{2} e^{\phi} & 0 \end{pmatrix} \qquad \text{A-11}$$

Acting with this metric all tensors can be expressed with n^+ z upper indices and n^- z lower indices without z indices.

Ex.:

$$g^{z\bar{z}} T^{zz}{}_{\bar{z}} = T^{zzz}$$

Their transformation rules under conformal group are:

$$T' = \left(\frac{\partial z'}{\partial z} \right)^{n_+ - n_-} T \qquad \text{A-12}$$

We see that this transformation is characterized only by the difference $n=n^+ - n^-$. Following Alvarez [24] we call $\mathcal{T}^n$ the space of tensors with $n>0$ (upper unpaired indices) and $\mathcal{T}^{-n}$ the space of tensors with $n<0$ (unpaired lower indices).
The covariant derivatives acting on these spaces are:

$$\nabla^z_n : \mathcal{T}^n \to \mathcal{T}^{n+1} \quad ; \quad \nabla^z_n T \equiv g^{z\bar{z}} \partial_{\bar{z}} T \qquad \text{A-13a}$$

$$\nabla_z^n : \mathcal{T}^n \to \mathcal{T}^{n-1} \quad ; \quad \nabla_z^n T \equiv (g^{z\bar{z}})^n \partial_z \left[(g_{z\bar{z}})^n T \right] \qquad \text{A-13b}$$

They are expressed in terms of the partial derivatives:

$$\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad \text{A-14}$$

Now we are in the position to compute the operator P_1, which sends vectors v^a in traceless symmetric tensors of rank two h_{ab}.
The components of v in the complex coordinates are [24]:

$$\begin{cases} v^z = v^1 + i v^2 \\ v^{\bar{z}} = v^1 - i v^2 \end{cases} \qquad \text{A-15}$$

while h_{ab} becomes:

$$\begin{cases} h_{zz} = h^{11} - i h^{12} \\ h_{\bar{z}\bar{z}} = h^{11} - i h^{12} \\ h_{z\bar{z}} = h_{\bar{z}z} = 0 \end{cases} \qquad \text{A-16}$$

v^z is in $\mathcal{T}^{-1}$ and $h_{zz} = (g_{zz})^2 h^{zz}$ is in $\mathcal{T}^{+2}$ according with our conventions.
So v^a belongs to $\mathcal{T}^1 \quad \mathcal{T}^{-1} = S^1$ and h_{ab} is decomposed in $\mathcal{T}^2 \quad \mathcal{T}^{-2} = S^2$.
Then the operator P_1 going from S^1 to S^2 is, according to A-13:

$$P_1 = \begin{pmatrix} \nabla_1^{z} & 0 \\ 0 & \nabla_z^{-1} \end{pmatrix} \qquad \text{A-17}$$

The adjoint of P_1, we computed explicitly elsewhere, is:

$$P_1^{+} = \begin{pmatrix} -\nabla_z^{2} & 0 \\ 0 & -\nabla_{-2}^{z} \end{pmatrix} \qquad \text{A-18}$$

A-3: Deformations of the Complex Structure-Moduli

The purpose of this section is to find the relation between deformation in the complex structure of a manifold and the moduli.

We change the complex structure $J^a{}_b$ on M with a small perturbation $\tau^a{}_b$:

$$J^a{}_b \rightarrow J'^a{}_b = J^a{}_b + \tau^a{}_b \quad (a,b = z,\bar{z}) \qquad \text{A-19}$$

$J'^a{}_b$ defines a new complex structure only if it obeys the conditions:

$$J'^a{}_b \, J'^b{}_c = -\delta^a{}_c \qquad \text{A-20}$$

$$N^a{}_{bc}(J') = 0 \qquad \text{A-21}$$

Since τ is supposed to be small, we retain only first order terms in τ in the successive calculations.

The conditions A-20 then become:

$$J^a{}_b \tau^b{}_c + \tau^a{}_b J^b{}_c = 0 \qquad \text{A-22}$$

Remembering that:

$$J^a{}_b = \begin{Vmatrix} i\delta^z{}_z & 0 \\ 0 & -i\delta^{\bar z}{}_{\bar z} \end{Vmatrix} \qquad \text{A-23}$$

it is straightforward to find from A-22 that:

$$\tau^z{}_z = \tau^{\bar z}{}_{\bar z} = 0 \qquad \text{A-24}$$

On the contrary no constraint to $\tau^z{}_{\bar z}$ comes out exploiting condition A-20.

Now we check A-21.

Since $J^a{}_b$ is constant, its derivatives don't appear, and we obtain:

$$J^d{}_b \partial_d \tau^a{}_c - J^d{}_c \partial_d \tau^a{}_b - J^a{}_d \partial_b \tau^d{}_c + J^a{}_d \partial_c \tau^d{}_b = 0 \qquad \text{A-25}$$

Also this expression at a first sight vanishes automatically.

But A-25 is a differential equation on a manifold and we have to solve it globally; eqn. A-25 is formulated instead over a local patch of M.

In order ot understand better this point, we solve A-21 in the more general case of a complex amnifold with several variables z_i, $\bar z_j$ (i,j=1,..,n).

Later we will restrict ourselves to the case n=1 of a Riemann Surface.

Repeating the same procedure as before we find that all components of $N^a{}_{bc}$ vanish except:

$$N^{z_i}{}_{\bar z_j \bar z_k} = \partial_{\bar z_j} \tau^{z_i}{}_{\bar z_k} - \partial_{\bar z_k} \tau^{z_i}{}_{\bar z_j} \qquad \text{A-26}$$

So we get a condition on τ :

$$\bar{\partial}_{\bar{z}_j} \tau^{z_i}{}_{\bar{z}_k} - \bar{\partial}_{\bar{z}_k} \tau^{z_i}{}_{\bar{z}_j} = 0 \qquad \text{A-27}$$

We can think now at $\tau^{z_i}{}_{\bar{z}_k}$ as one (0,1) form living in the holomorphic tangent space $\mathcal{T}^1$.

The action of the external derivative $\bar{\partial}$ on $\tau^{z_i}{}_{\bar{z}_k}$, when computed in local coordinates, takes exactly the form A-26.

So eqn. A-27 can be rewritten as:

$$\bar{\partial} \tau^{z_k} = 0 \qquad \text{A-28}$$

The nontrivial solutions of A-27 or equivalently A-28 lie in the first Dolbeault cohomology group of $\mathcal{T}^1$: $H(\mathcal{T}^1)$.

Of course in the case of a Riemann Surface A-27 and A-28 are automatically 0: on a Riemann Surface the whole space of (0,1) forms consists in closed forms, but this does not imply that the are all trivial.

In fact, as we will show in the next section, $H^1(\mathcal{T}^1)$ is different from 0 on a Riemann Surface and its inequivalent classes define a vector space of dimension 3g-3.

We indicate a basis on this space as $\{\tau^{Az}{}_{\bar{z}}\}$ (A=1,..,3g-3).

Each τ^A defines one inequivalent complex structure of M; all the complex structures can be reached by the deformations:

$$\tau(y) = y^A \tau^A \qquad \text{A-29}$$

The coordinates y^A are called moduli and build the so called Teichmuller space.

In principle also the diffeomorphisms change the complex structure since they mix z with z.

An infinitesimal diffeomorphism z + δz can be written as:

$$z' = z + \varepsilon_z v^z(z,\bar{z})$$

$$\bar{z}' = \bar{z} + \bar{\varepsilon}_{\bar{z}} \bar{v}^{\bar{z}}(z,\bar{z}) \qquad \text{A-30}$$

where v^z is the z component of a vector.

Being a tensor, the complex structure $J^a{}_b$ transforms as a tensor under A-30:

$$J^a{}_b = \frac{dx^a}{dx'^{a'}} \frac{dx'^{b'}}{dx^b} J'^{a'}{}_{b'} \qquad (a,b,a',b' = z,\bar{z})$$

A-31

Explicitly:

$$J'^{z'}{}_{z'} = J^z{}_z \quad ; \quad J'^{\bar{z}'}{}_{\bar{z}'} = J^{\bar{z}}{}_{\bar{z}}$$

A-32

and:

$$J'^{z'}{}_{\bar{z}'} = J^z{}_{\bar{z}} + 2i\,\partial_{\bar{z}} v^z$$

A-33

So for a diffeomorphism:

$$J^z{}_{\bar{z}} \propto \partial_{\bar{z}} v^z$$

A-34

From a cohomological point of view, this is a trivial solution of A-27 corresponding to an exact form: in other words diffeomorphisms transform complex structures in other different but inequivalent complex structures.

We explain now the connection between this rather abstract formalism and the metric g_{ab}.

We remark that, given a metric g_{ab}, the complex structure $J^a{}_b$ compatible with it is:

$$J^a{}_b = \sqrt{g}\,\varepsilon_{ac}\, g^{cb}$$

A-35

with $\varepsilon_{12} = -\varepsilon_{21} = -1$, $\varepsilon_{22} = \varepsilon_{11} = 0$.

So if we change J this has in principle an influence also on g_{ab}.

In order to find this correlation, we start with

$$J^a{}_b = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

on M.

According to A-35 the local metric is:

$$ds^2 = e^{\phi}\, dz\, d\bar{z} \qquad \text{A-36}$$

Now we transform infinitesimally J (see A-19):

$$J'^{z}{}_{z} = J^{z}{}_{z}$$

$$J'^{\bar{z}}{}_{\bar{z}} = J^{\bar{z}}{}_{\bar{z}} \qquad \text{A-37}$$

$$J'^{z}{}_{\bar{z}} = \tau^{z}{}_{\bar{z}}$$

We want to compare this variation of J with the one given by a variation of the metric g_{ab}.

In whatever way the metric tensor changes, we can express it as

$$\text{A-38} \qquad ds'^2 = e^{\phi}\, |dz + \mu^{z}{}_{\bar{z}}\, d\bar{z}|^2$$

where $\mu^{z}{}_{z}$ is called a Beltrami differential.

In fact A-38 represent an arbitrary metric on M.In matrix form:

$$g'^{ab} = e^{\phi} \begin{pmatrix} 2\bar{\mu} & 1+\mu\bar{\mu} \\ 1+\mu\bar{\mu} & 2\mu \end{pmatrix} \qquad \text{A-39}$$

Now we are able to compute the structure $J' = \sqrt{g'}\, \varepsilon_{ac}\, g'^{cb}$ compatible with g'.

Remembering that:

$$\varepsilon_{zz} = \varepsilon_{\bar{z}\bar{z}} = 0$$

$$\varepsilon_{z\bar{z}} = -\varepsilon_{\bar{z}z} = i \qquad \text{A-40}$$

and that, for small μ, $\sqrt{g} \approx 1$, we have:

$$J'^{z}_{\ z} = i \qquad J'^{\bar z}_{\ \bar z} = -i$$

$$J'^{z}_{\ \bar z} = i2\mu^{z}_{\ \bar z} \qquad \text{A-41}$$

Matching this result with the expression of J' obtained after a direct change of the complex structure, we see that, apart from a negligible constant, $\mu^{z}_{\ \bar z}$ coincides with the deformations τ: a change of the complex structure is equivalent to change the metric.

In formulas:

$$\mu^{z}_{\ \bar z} \approx y^A \tau^A \qquad \text{A-42}$$

Analogous results are found in the case of transformations by diffeomorphisms:

$$\mu^{z}_{\ \bar z} = \partial_{\bar z} v^z \qquad \text{A-43}$$

(A-43 is true apart of a constant whic is ininfluent because it can be incorporated in the definition of v^z).

At this point we find what is the transformation of the metric under the actions of the conformal group, reparametrization group and moduli.

From A-39, A-42 and A-43 we get:

$$\delta(e^{\phi}\, dz d\bar z) =$$

$$\delta\phi\, e^{\phi}\, dz d\bar z + e^{\phi}\, \bar\partial v^z\, d\bar z^2 + e^{\phi}\, \delta y^A \tau^A\, d\bar z^2 + c.c. \qquad \text{A-44}$$

This is the central result of this section.

A-4:Determination of the Dimension of Teichmuller Space.

Let's introduce the dual basis of τ^A, i.e. the tensors φ^A such that:

$$\int_M \tau^A \varphi_B \, dz \wedge d\bar{z} = \delta^A{}_B \qquad \text{A-45}$$

We want to compute the number of the φ^A's [42].

They have two important properties:

1) Since the scalar product A-43 has to give complex numbers, we want all tensorial indices saturated.

It is easy to prove that φ_B must have the tensorial structure φ_{Bzz}.

2) One can also check that $\varphi^A{}_{zz}$ is holomorphic if we want the the $\mu^z{}_{\bar{z}}$'s orthogonal to diffeomorphisms.

In fact, following our previous discussion, two deformations $\mu_1{}^z{}_z$ and $\mu_2{}^z{}_z$ are equivalent only if:

$$\mu_1{}^z{}_{\bar{z}} = \mu_2{}^z{}_{\bar{z}} + \partial_{\bar{z}} v^z \qquad \text{A-46}$$

Equivalence implies that the components of $\mu_1{}^z{}_{\bar{z}}$ and $\mu_2{}^z{}_{\bar{z}}$ with respect to a vector basis are the same. i.e.

$$\int \mu_1 \varphi^A \, dz \wedge d\bar{z} = \int \mu_2 \varphi^A \, dz \wedge d\bar{z}$$

So if we write:

$$\mu_1{}^z{}_{\bar{z}} = \sum_A y^A \mu^{A\,z}{}_{\bar{z}}$$

$$\mu_2{}^z{}_{\bar{z}} = \sum_A y'^A \mu^{A\,z}{}_{\bar{z}} + \bar{\partial} v^z \qquad \text{A-47}$$

μ_1 and μ_2 are equivalent only if $y^{A'} = y^A$ and

$$\int_M dz \wedge d\bar{z} \, \partial_{\bar{z}} v^z \varphi^A{}_{zz} = 0 \qquad \text{A-48}$$

Integrating by parts we have:

$$-\int_M dz\wedge d\bar{z}\, v^z\, \partial_{\bar{z}}\, \varphi^A_{zz} = 0 \qquad \text{A-49}$$

for each v^z and this implies that $\varphi^A{}_{zz}$ is indeed holomorphic.

We define at this point the canonical line bundle K, as the space of the tensors transforming under a conformal variation as

$$c_z = c_w \frac{dz}{dw}$$

C_z is called a section of K.

From the above discussion we can conclude that the $\varphi^A{}_{zz}$ are global holomorphic sections of the line bundle K^2 to compute their number we have to find the solutions of the equation:

$$\partial_{\bar{z}}\, \varphi_{zz} = 0$$

This is equivalent to compute the dimension $h^0(K^2)$ of the cohomology group $H^0(K^2)$.

We need for this task the following ingredients:

1) The Riemann-Roch theorem for an arbitrary line bundle L:

$$h^0(M,L) - h^1(M,L) = \deg L - g + 1 \qquad \text{A-50}$$

where deg L is defined below, eqn. A-51.

2) The values of $h^0(L)$ and $h^1(L)$ listed in table A-1: these are obtained using the Riemann-Roch theorem, the Serre duality and some reasonable considerations.

As an example we derive h^0 and h^1 in the case deg L = 0.

Given one global section s of L, deg L consists in the sum:

$$\deg L = \sum_i n_i - \sum_j m_j \qquad \text{A-51}$$

TABLE A-1

deg(L)	$h^0(L)$	$h^1(L)$
<0	0	g-1-deg(L)
0	$\begin{cases} 1 \\ 0 \end{cases}$	$\begin{cases} g & \text{if } L \cong 0 \\ g-1 & \text{if } L \ncong 0 \end{cases}$
2g-2	$\begin{cases} g \\ g-1 \end{cases}$	$\begin{cases} 1 & \text{if } L \cong K \\ 0 & \text{if } L \ncong K \end{cases}$
>2g-2	deg(L) + 1 -g	0

i runs over the zeroes and j over the poles of s and n_i (n_j) is the order of the i-th (j-th) zero (pole).

All global sections of L must obey the bound A-51.

So one holomorphic section cannot have zeroes because deg L = 0 by hypothesis.

On a compact manifold this means that either there is only a constant section or there are not.

In the first case we have $h^0(L) = 1$, because there exists the constant section: this implies that L is trivial, i.e. its transition functions can be put equal to one everywhere.

in the second case we have $h^0(L) = 0$.

When h^0 is known, also h^1 is known from the Riemann-Roch theorem.

The other results of table A-1 were obtained in a similar way.

3) The last ingredient we need is deg(K).

We find that for K:

$$h^0(K) = h^1(K^{-1} \otimes K) = h^1(0) = g \qquad \text{A-52}$$

$$h^1(K) = h^0(K^{-1} \otimes K) = h^0(0) = 1 \qquad \text{A-53}$$

The first equality in A-52, A-53 is obtained from Serre duality.

Then we have remembered that $h^1(0)$ is the number of harmonic differential.

$h^0(0)$ was already computed above.

After a substitution of A-52, A-53 in Riemann-Roch theorem we get:

$$\deg L = 2g - 2 \qquad \text{A-54}$$

At this point we have all the tools to compute the number of holo quadratic differentials $\varphi^A{}_{zz}$.

The derivation is trivial and we give only the final results in table A-2.

TABLE A-2

dimensions of moduli space

genus	dim. of moduli space	# of conformal killing vectors
g>1	3g-3	0
g=1	1	1
g=0	0	3

APPENDIX B

Computation of the Vacuum Wave Function.

We want to rederive the ground state wave function of (3.1.35) with an alternative method which do not imply the addition of surface terms in the Lagrangian.
We deal here only with the ghost contribution since the bosonic sector do not suffer from the problem.
So we have to compute the amplitude:

$$\langle c_B ; 0 | 0 ; -T \rangle = \int_{\substack{c_B = c(f) \\ 0 = c(i)}} \mathcal{D}b\,\mathcal{D}c\; e^{\int b \bar{\partial} c} \times (c.c.)$$

where (c.c.) is the term due to the presence of the complex conjugate fields in the action.
Notice also that we have fixed only the boundary value of c: in fact in this case b and c cannot appear contemporarily in the final result (see section 1.3).
Of course c(f) is a classical field.
Now we expand b and c on the semiinfinite cylinder in the following way:

$$b_{zz}(\sigma, \tau) = \sum_{n=-\infty}^{+\infty} b_m(\tau)\, e^{im\sigma} \qquad \text{B-2}$$

$$c^{z}(\sigma, \tau) = \sum_{n=-\infty}^{+\infty} c_m(\tau)\, e^{im\sigma} \qquad \text{B-3.}$$

Eqns. B-2 and B-3 define a change of variables for the path integral in B-1 whose Jacobian determinant is equal to one:

$$< c_B ; 0 | 0_i ; -T > = \int \prod_{m=-\infty}^{+\infty} db_m \prod_{n=-\infty}^{+\infty} dc_n \times$$

$$\times \exp\left\{ -\sum_{\mu,\nu=-\infty}^{+\infty} \int d^2 z \, e^{i\mu\sigma} b_\mu(\tau) (\partial_\tau + i \partial_\sigma) c_\nu(\tau) e^{i\nu\sigma} \right\} + (c.c.) \qquad \text{B-4.}$$

Some care must be used for the measure in B-4 because of the anticommuting nature of the ghosts.

Our choice agrees with the meaning of B-4 as determinant of $\bar{\partial}$.

After a little of work integrating in the σ variable, we obtain from B-4:

$$< \; > = \prod_m \int db_m \, dc_{-m} \times$$

$$\times \exp\left\{ -2\pi \int_{-T}^{0} d\tau \left[b_m(\tau) \partial_\tau c_{-m}(\tau) + b_m c_{-m} \, m \right] \right\} \qquad \text{B-5.}$$

We have omitted here the (c.c.) terms which can be treated in an analogous manner. We rewrite the action appearing in B-5 in a more compact form, working on a single term of the infinite product $\prod_m$ and eliminating the index m:

$$< \; >_m = \int dp \, dq \, \exp\left\{ -2\pi \int_{-T}^{0} d\tau \left[p\dot{q} + m p q \right] \right\} \qquad \text{B-6}$$

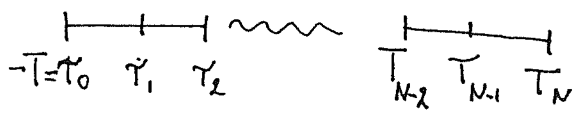
where we have used the following substitution:

$$b_m = p \; , \quad c_{-m} = q \qquad \text{B-7}$$

p,q anticommuting variables.

Now we discretize the [-T, 0] interval dividing it into N intervals (fig B-1).

Fig B.1



The infinitesimal element is given by:

$$\Delta\tau = \frac{(\tau_0 - \tau_N)}{N+1} = \frac{T}{N+1} \qquad \text{B-8}$$

and c(i)=c(t_0) ; c(f) = c(t_N) B-8a.

From the definition of path integral (see Feynman and Hibbs) we get:

$$\langle\ \rangle_m = \lim_{N\to\infty} \int \frac{dp_{N+1}}{2\pi} \prod_{i=1}^{N} \frac{dp_i}{2\pi}\, dq_i \times$$

$$\times \exp\left\{-2\pi \sum_{j=1}^{N+1} \left[p_j q_j \left(1+m\frac{\Delta\tau}{2}\right) - p_j q_{j-1}\left(1-\frac{\Delta\tau m}{2}\right)\right]\right\} \qquad \text{B-9.}$$

The integration in dp_i is very easy to perform but firstly we have to pass with the differentials dp_i over the dq_i's.

Doing this carefully we obtain:

$$\langle\ \rangle_m = \lim_{N\to\infty} \int \prod_{j=1}^{N} dq_i \prod_{j=1}^{N} \frac{dp_i}{2\pi} \frac{dp_{N+1}}{2\pi} (-1)^{\xi} \times$$

B-10

$$\times \exp\left\{-2\pi \sum_{j=1}^{N+1} \left[p_j q_i \left(1+\frac{m\Delta\tau}{2}\right) - p_j q_{j-1}\left(1-\frac{m\Delta\tau}{2}\right)\right]\right\}$$

with

$$\xi = \begin{cases} \frac{N}{2} & \text{when } N \text{ is even} \\ \frac{N+1}{2} & \text{when } N \text{ is odd} \end{cases} \qquad \text{B-11.}$$

Remark: it is clear that at the end of computations this dependence on $(-1)^{\xi}$ must disappear because otherwise the limit N --> +∞ does not exist.

Integrating over p_i's and p_{N+1} eqn. B-11 becomes:

$$\langle\ \rangle_m =$$

$$= \lim_{N\to\infty} (-1)^{\xi} (-1)^{N} \int \prod_{i=1}^{N} dq_i \prod_{j=1}^{N+1} \left\{q_j\left(1+\frac{m\Delta\tau}{2}\right) + q_{j-1}\left(1-\frac{m\Delta\tau}{2}\right)\right\} \qquad \text{B-12.}$$

We put now eqn. B-12 in a more suitable form for the dq_i integration:

$$\langle\ \rangle_m = \lim_{N \to \infty} (-1)^{\chi+\zeta+N} \int dq\,(q_1 a - q_0 b)\, dq_2\,(q_2 a - q_1 b) \cdots\cdots$$

$$\cdots\, dq_{N-1}\,(q_{N-1} a - q_{N-2} b)\, dq_N\,(q_N a - b q_{N-1})(a q_{N+1} - b q_N) \qquad \text{B-13}$$

where:

$$a = \left(1 + \frac{\Delta\tau m}{2}\right)$$

$$b = \left(1 - \frac{\Delta\tau m}{2}\right) \qquad \text{B-14}$$

and

$$\chi = \begin{cases} \frac{N}{2} & \text{when } N \text{ is even} \\ \frac{(N-1)}{2} & \text{when } N \text{ is odd} \end{cases}$$

At this point we verify that:

$$(-1)^{N+\zeta+\chi} = 1 \qquad \forall N \qquad \text{B-15,}$$

allowing the limit N --> ∞.

The integration in the variables q_i can be done in an iterative way and the result is:

$$\langle\ \rangle_m = \lim_{N \to \infty} \left(a^{N+1} q_{N+1} - b^{N+1} q_0\right) \qquad \text{B-16.}$$

Since:

$$a = 1 + \frac{\Delta\tau m}{2} \approx \exp\left(\frac{m\,\Delta\tau}{2}\right) \qquad \text{B-17}$$

and analogously:

$$b \simeq \exp\left(-m\frac{\Delta\tau}{2}\right) \qquad \text{B-17a}$$

we have:

$$\begin{cases} a^2 \approx \left(1+\frac{\Delta\tau m}{2}\right)\exp\left(\frac{m\Delta\tau}{2}\right) \simeq \exp\left(\frac{2m\Delta\tau}{2}\right) \\ b^2 \approx \exp\left(-\frac{2m\Delta\tau}{2}\right) \end{cases}$$

$$\vdots$$

$$\begin{cases} a^{N+1} \approx \exp\left((N+1)\, m\frac{\Delta\tau}{2}\right) \\ b^{N+1} \approx \exp\left(-\frac{mT}{2}\right) \end{cases} \qquad \text{B-18}$$

and $< >_m$ becomes:

$$\langle\ \rangle_m = \left[e^{mT/2} c_{-m}(f) - e^{-mT/2} c_{-m}(i)\right] \qquad \text{B-19.}$$

Putting this result in eqn. B-5 we obtain the total amplitude B-1:

$$\langle\ \rangle = \prod_{m=-\infty}^{+\infty}\left[e^{mT/2} c_{-m}(f) - e^{-mT/2} c_{-m}(i)\right] \times$$

$$\times \prod_{n=-\infty}^{+\infty}\left[e^{-nT/2} c^*_{-m}(f) - e^{nT/2} c^*_{-m}(i)\right] \qquad \text{B-20.}$$

In B-20 we have also included the contribution of the (c.c.) term in c*(f) and c*(i). The dominant contribution is provided by:

$$\langle\ \rangle \approx \left(\prod_{m<0} c_m(i)\right) c_0(f) \prod_{m>0} c_m(f)\, e^{\sum_{k\geq 1} kT} \times (c.c.) \qquad \text{B-21}$$

where

$$\sum_{k=1}^{\infty} k = -\frac{1}{12} \qquad \text{B-22}$$

using the zeta function regularization.

Comparing this amplitude with 1.3.33 we recognize that:

$$\phi_0^* = \prod_{m<0} c_m(i) \qquad \text{B-23}$$

$$\phi_0 = c_0(f) \prod_{m>0} c_m(f) \qquad \text{B-24}$$

$$E_0 = \frac{1}{12} \qquad \text{B-25.}$$

B-25 is the energy of the vacuum state.

Finally we find the creation and annihilation operators for ϕ_0.

according to the commutation relations of (1.3.30) they are given by:

annihilation operators

$$c_m, b_m \equiv \frac{\delta}{\delta c_{-m}} \qquad m > 0 \qquad \text{B-26.}$$

creation operators

$$c_{-m}, b_{-m} = \frac{\delta}{\delta c_n} \qquad n > 0 \qquad \text{B-27.}$$

acting with these operators we can find all the excited states which span theHilbert space of string theory

Moreover we can effectively verify that the vacuum defined from ϕ_0 has ghost number 1/2.

In fact the ghost number operator of Q_c applied on the vacuum gives

$$\hat{Q}_c |0\rangle_{gh} = \left[\frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{m=1}^{\infty} (c_{-m} b_m - b_{-m} c_m)\right]\left(c_0 \prod_{n>0} c_m\right) \qquad \text{B-28.}$$

The only part of Q_c which is not vanishing is:

$$\hat{Q}_c |0\rangle_{gh} = \frac{1}{2} c_0 b_0 \left(c_0 \prod_{m>0} c_m \right) = \frac{1}{2} |0\rangle_{gh} \qquad \text{B-29.}$$

But this is the wrong vacuum as we said in section 2.1.

The true vacuum is recovered putting by hand the right constant modes:

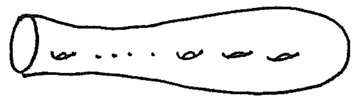
$$|0\rangle_{gh} = b_0 \prod_{m>0} c_m \qquad \text{B-30}$$

APPENDIX C

Properties of the Vacuum under Conformal Transformations.

We verify the crucial invariance of the vacuum |0> under the conformal killing vectors of (1.3.9).

We see this problem for the general case of a manifold with boundary M (fig. C.1).

fig. C.1 $M =$ 

The "vacuum" on M is:

$$|0\rangle = \int_{c|_{\partial M} = c_B} \mathcal{D}b\, \mathcal{D}c\; e^{-\int_M b \bar{\partial} c}$$

We want that |0> is invariant under transformations of the type:

$$c \longrightarrow c' = c + \psi$$

with ψ holomorphic global section over M.

As a matter of fact let's compute:

$$|0\rangle_{c+\psi} = \int_{c|_{\partial M} = c_B + \psi_B} \mathcal{D}b\, \mathcal{D}(c+\psi)\; e^{-\int_M b \bar{\partial}(c+\psi)}$$

Now we know that:

$$\mathcal{D}(c+\psi) = \mathcal{D}c$$

$$\bar{\partial}(c+\psi) = \bar{\partial} c$$

The only problem is the boundary ∂M.

But we know that in general in a path integral the expansion of c in series will be of the form:

$$c = \sum_{n=-\infty}^{+\infty} c_n z^n + \sum_{n=-\infty}^{+\infty} \bar{c}_n \bar{z}^n + \sum_{n,m} c_{n,m} (z^n \bar{z}^m) + \cdots$$

In any case we see that the holomorphic part (i.e. the first term in the expansion) never contributes to the path integral because of the presence of the differential operator $\partial_{\bar{z}}$ in the Lagrangian.

So we can perform the shift c --> c + ψ as we want if ψ is a globally defined vector.

In this case it is possible to write:

$$\int_{H \; c_{|\partial H} = c_B + \psi_B} b \bar{\partial} c = \int_{H \; c_{|\partial H} = c_B} b \bar{\partial} c$$

because in the second member ∂ (c-ψ) = ∂c since ψ_b is conformal.

So we have:

$$|0\rangle_{c+\psi} = \int_{c_{|\partial H} = c_B} \mathcal{D}b \mathcal{D}c \, e^{-\int_H b \bar{\partial} c} = |0\rangle_c$$

In particular all this discussion is valid when ψ is a conformal killing vector.

The only problem is that such conformal killing vectors does't exists for surfaces of genus g>0.

Finally we check that our vacuum computed in appendix B is invariant under global conformal tranformations.

$$|0\rangle_c = \prod_{n<0} c_n$$

From our discussion it seems that:

$$\prod_{n<0} (c_n + \psi_n) = \prod_{n<0} c_n$$

but this is not possible.

How can we explain this?

The fact is that ψ_n is holomorphic and so on M it can be expressed in a Laurent expansion whose coefficients for n<0 are vanishing.

But this implies simply that ψ_n does not enter on $\prod_{n<0} c_n$.

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