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G-CONVERGENCE OF QUASI-LINEAR ORDINARY DIFFERENTIAL OPERATORS OF MONOTONE TYPE

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Academic Year 1989-90

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"MAGISTER PHILOSOPHIAE" THESIS

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CONTENTS

CHAPTER 1

INTRODUCTION

In this thesis, our purpose is to deal with a particular notion of convergence for differential operators, said G-convergence. The importance of this notion is connected with a problem which arises in a natural way in the study of perturbations of differential equations. To introduce this problem, let us begin with the model case, first studied by E. De Giorgi and S. Spagnolo, of a second order linear partial differential equation. Let us assume that Ω is an open bounded subset of R^n, $f \in L^2(\Omega)$, $a^h_{i,j} : \Omega \to \mathrm{R}$ are measurable functions such that

$$(1.0) \qquad \begin{aligned} &a^h_{i,j}(x) = a^h_{j,i}(x) \quad \text{a.e. in } \Omega \\ &\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a^h_{i,j}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \end{aligned}$$

for a.e. $x \in \Omega$, for every $\xi \in \mathrm{R}^n$, with $0 < \lambda \leq \Lambda < +\infty$, $h \in \mathrm{N}$, $i,j = 1,\ldots,n$, and $(u_h)_{h\in\mathrm{N}}$ is the sequence of weak solutions of

$$(1.1)_h \qquad \begin{aligned} &-D_j(a^h_{i,j}D_iu_h) = f \\ &u_h \in W^{1,2}_0(\Omega). \end{aligned}$$

Assumptions (1.0) yield, by simple calculations, that

$$\begin{aligned} &\|u_h\|_{W^{1,2}_0} \leq C(\lambda,\Lambda,n)\|f\|_{L^2} \\ &\|a_hDu_h\|_{(L^2)^n} \leq C(\lambda,\Lambda,n)\|f\|_{L^2}. \end{aligned}$$

These inequalities and the reflexivity of the Sobolev space $W^{1,2}_0(\Omega)$ and the space $(L^2(\Omega))^n$, imply that there exists a subsequence $h_k \to +\infty$ and two functions $u \in W^{1,2}_0(\Omega)$ and $g \in (L^2(\Omega))^n$ such that $u_{h_k} \rightharpoonup u$ weakly in $W^{1,2}_0(\Omega)$ and $a_{h_k}Du_{h_k} \rightharpoonup g$ weakly in $(L^2(\Omega))^n$.

The question naturally arises if it is possible to characterize u as a weak solution of a suitable problem, that is if there exists a matrix $a_{i,j}$ of the same type as $a^h_{i,j}$, such that u satisfies

$$(1.1) \qquad \begin{aligned} & -D_j(a_{i,j}D_i u) = f \\ & u \in W_0^{1,2}(\Omega). \end{aligned}$$

A positive answer to this problem was given by E. De Giorgi and S. Spagnolo (see [15], [26], [27], [28]), with the introduction of the theory of the G-convergence. This notion was then extended to the non-symmetric case by F. Murat and L. Tartar, who stated also the convergence of the momenta, that means $a_{h_k} Du_{h_k} \rightharpoonup aDu$ weakly in $(L^2(\Omega))^n$, (see [19], [29], [30]). The quasi-linear case with linear principal part was studied by L. Boccardo, T. Gallouet and F. Murat (see [5], [6], [7]).

Under proper hypotheses of continuity and monotonicity, the same question arises in the case of partial differential operators with quasi-linear principal part, in a general Sobolev space $W^{1,p}$ with $p > 1$. This case was considered by F. Murat and L. Tartar (see [20]) and by V. Chiadò-Piat, G. Dal Maso and A. Defranceschi (see [12]). The notion of G-convergence was extended by L.C. Piccinini to the case of ordinary differential equations (see [23], [24], [25]).

In this thesis, we are interested in studying the properties of the G-convergence in the special case of non linear systems of ordinary differential equations with Dirichlet boundary conditions; our aim is, in particular, to apply some typical tools of partial differential equations to this case.

More specifically, if (a, b) is an open bounded interval of R, $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_h, a : I \times \mathrm{R}^n \to \mathrm{R}^n$ are measurable on $I \times \mathrm{R}^n$, strictly monotone in R^n for almost every $x \in I$ and satisfy suitable continuity and strict monotonicity conditions for every $h \in \mathrm{N}$, we consider, for $h \in \mathrm{N}$, the family of boundary value problems

$$(1.2)_h \qquad \begin{aligned} & u_h \in (W^{1,p}(I))^n \\ & -(a_h(x, u_h'))' = f_h \quad \text{a.e. in } I \\ & u_h(a) = \alpha \quad u_h(b) = \beta \end{aligned}$$

and the boundary value problem

$$(1.2) \qquad \begin{aligned} & u \in (W^{1,p}(I))^n \\ & -(a(x,u'))' = f \quad \text{a.e. in } I \\ & u(a) = \alpha \quad u(b) = \beta \end{aligned}$$

for arbitrary $\alpha, \beta \in \mathbb{R}^n$ and $(f_h)_{h\in\mathbb{N}}, f \in (W^{-1,q}(I))^n$. Then we say that $a_h \xrightarrow{G} a$ (a_h G-converges to a), if and only if for every sequence $(f_h)_{h\in\mathbb{N}}$ converging stronlgy to f in $(W^{-1,q}(I))^n$, the corresponding sequence $(u_h)_{h\in\mathbb{N}} \in (W^{1,p}(I))^n$ of solutions of the systems $(1.1)_h$ satisfies

$$u_h \rightharpoonup u \quad \text{weakly in } (W^{1,p}(I))^n$$

$$a_h(x, u_h'(x)) \to a(x, u'(x)) \quad \text{strongly in } (L^q(I))^n,$$

where u is the solution of (1.2).

Following the paper of Chiadò Piat-Dal Maso-Defranceschi (see [12] remark 3.9), the definition of G-convergence can be extended, with some suitable modification, to the case in which we drop out the hypotheses of continuity and strict monotonicity, but in this case we cannot deal anymore with single-valued functions only. In fact, it is possible to obtain non trivial examples, in which a sequence of single-valued functions G-converges to a really multivalued map (see, for instance, [13] ex. 4.7). Therefore, in order to achieve a compactness result with respect to G-convergence, it is natural to consider the general case, where $a_h(x,\cdot)$ and $a(x,\cdot)$ are maximal monotone multivalued maps, a case which is not included in the previous works of Piccinini.

The main result of this thesis is a compactness theorem (see theorem 6.12), which states that from any sequences $(a_h)_{h\in\mathbb{N}}$ of maps belonging to the class described before, we can extract a subsequence $(a_{h_k})_{k\in\mathbb{N}}$ that G-converges to a map a belonging to the same class. This is a consequence of an extension theorem for sections of maximal monotone maps (see theorem 4.3), applied to the sequence $(a_h^{-1})_{h\in\mathbb{N}}$ of the inverse maps with respect to the second variable.

This theorem gives an alternative and simpler proof, with respect to the one in [12], of the compactness of the G-convergence in the case of one independent variable.

Moreover, our method gives a characterization of the G-limit for ordinary differential operators in terms of a suitable convergence of the inverse maps. In particular, in the single-valued case (see corollary 6.8), we obtain that $a_h \xrightarrow{G} a$ if and only if $a_h^{-1}(\cdot,\eta) \rightharpoonup a^{-1}(\cdot,\eta)$ weakly in $L^p(I)$, and when $n = 1$ this is an extension to the quasi-linear case of the result obtained by S. Spagnolo in [28] in the linear case.

Finally, we observe that in the case of equations, i.e. n=1, it is possible to characterize the G-convergence by means of the weak-L^p convergence of the primitives associated to the inverse maps a_h^{-1}. This provides a new proof of a result which could also be obtained by means of a theorem by P. Marcellini and C. Sbordone (see [18]), taking into account the connection between G-convergence and Γ-convergence proved by A. Defranceschi in [14].

However, the main conclusion in this case is the characterization of the G-convergence by means of weak L^p lower and upper limits (see definition 6.5), which can be compared to the notion of weak L^p limit of multivalued maps given by Z. Arstein in [2]. Since our results are obtained by dealing with the operators associated with the inverse maps a_h^{-1}, the main difficulties encountered in the work were: 1) to establish conditions on the inverse maps $(a_h^{-1})_{h\in N}$, which are equivalent to the G-convergence of $(a_h)_{h\in N}$; 2) to guarantee that a monotone function of a suitable class has only one maximal monotone extension.

This thesis is organized as follows: in chapter 2 we recall some definitions and known results about maximal monotone operators and convex functions, without giving proof; in chapter 3 we consider measurable functions and measurable multivalued maps, giving the proof of three properties very useful for our thesis; in chapter 4 we state an extension theorem on monotone operators, which results to be a fundamental tool in the development of chapter 6; in chapter 5 we briefly study the existence problem for differential inclusions with Dirichlet boundary conditions; chapter 6 contains the main results on the G-convergence for non linear systems of ordinary differential equations and finally in chapter 7 we consider the G-convergence for the scalar case.

The results of chapter 4, 5, 6 and 7 are contained in [1].

CHAPTER 2

MAXIMAL MONOTONE MAPS AND CONVEX FUNCTIONS

In this chapter we will recall some definitions and some known results about maximal monotone operators and convex functions, which can be found in the classical licterature. For this reason we give only the references, even if these results will be very useful for the sequel.

Let X and Y two sets, we denote by $\mathcal{P}(Y)$ the collection of the subsets of Y. A multivalued function $F : X \to Y$ is a map that to every $x \in X$ associates a subset $Fx \subseteq Y$. If for every $x \in X$, the set Fx is a single point of Y, we say that the map F is single valued.

The sets

$$D(F) = \{x \in X \ : \ Fx \neq \emptyset\},$$

$$G(F) = \{[x,y] \in X \times Y \ : \ y \in Fx\}$$

$$\text{and} \quad R(F) = \bigcup_{x \in X} Fx$$

are called the *domain*, the *graph* and the *range* of F, respectively.

In this thesis work, we refer to the particular case in which the set X is a topological vector space and Y is its topological dual, denoted by X^*. Finally, the duality pairing between X and X^* is indicated by $< \, , \, >$.

We begin with the definition and the main properties of maximal monotone operators.

Def. 2.1

We say that a set $A \subseteq X \times X^*$ is *monotone* if and only if

$$< y - \eta, x - \xi > \geq 0 \quad \forall [x,y], [\xi, \eta] \in A.$$

The set A is *maximal monotone* if it is monotone and it is not contained in any other monotone set, i.e. the following property holds:

if $[x, y] \in X \times X^*$ is such that

$$< y - \eta, x - \xi > \geq 0 \quad \forall [\xi, \eta] \in A$$

then $[x, y] \in A$.

Def. 2.2

A multivalued operator $F : X \to X^*$ is said *maximal monotone* if such is its graph.

Theorem 2.3 (See, for instance, [22] chap. III, section 2.3)

Let $F : X \to X^*$ and $F^{-1} : X^* \to X$ be two multivalued operators such that

$$y \in Fx \quad \Longleftrightarrow \quad x \in F^{-1}y.$$

Then F is maximal monotone if and only if such is F^{-1}.

Def. 2.4

A multivalued operator $F : X \to X^*$ is *coercive* if

$$\lim_{\substack{\|x\| \to +\infty \\ x^* \in Fx}} \frac{< x^*, x >}{\|x\|} = +\infty.$$

Theorem 2.5 (See, for instance, [22] chap. III, theorem 2.10)

Let X be a reflexive Banach space and let $F : X \to X^*$ be a maximal monotone multivalued operator. Then, if F is coercive, $R(F) = X^*$.

Theorem 2.6 (See, for instance, [22] chap. III, theorem 4.2)

Let X be a reflexive Banach space and $F : X \to X^*$ be a maximal monotone multivalued function. Suppose that for every $x \in X$ there exists a neighborhood U of x such that

$$\bigcup_{y \in U \cap D(F)} Fy$$

is a bounded set, i.e. F is locally bounded in X. Then F^{-1} is surjective.

Def 2.7

Let X be a reflexive Banach space. We say that a set $A \subseteq X \times X^*$ is *demiclosed*, if for every sequence $([x_n, y_n])_{n \in \mathbb{N}} \subset A$ with $x_n \to x$ in the strong topology of X and $y_n \rightharpoonup y$ in the weak topology of X^*, it follows that $[x, y] \in A$.

Def. 2.8

Let $F : X \to X^*$ be a multivalued operator. We say that F is *upper semicontinuous* in X if for every $x \in D(F)$ and for every neighborhood V of Fx there exists a neighborhood U of x such that for every $y \in U \cap D(F)$ it follows that $Fy \subset V$.

Theorem 2.9 (See, for instance, [22] chap. III, theorem 2.3)

Let X be a reflexive Banach space and $F : X \to X^*$ be a multivalued monotone operator. Suppose that for each $x \in X$, Fx is a non empty closed convex subset of X^*.

If, moreover, F is also upper semicontinuous with respect to the strong topology of X and the weak topology of X^*, then F is maximal monotone.

Now we want to consider convex functions and some of their properties.

Def. 2.10

The function $V : X \to \overline{\mathbf{R}} = \mathbf{R} \cup \{\pm\infty\}$ is said *convex* if for every $\lambda \in [0,1]$ and for every pair $x, y \in X$ we have

$$V(\lambda x + (1-\lambda)y) \leq \lambda V(x) + (1-\lambda)V(y)$$

whenever the right-hand side is defined.

V is said proper if $V(x) < +\infty$ for every $x \in X$ and there exists $\tilde{x} \in X$ such that $V(\tilde{x}) > -\infty$.

Next theorem points out the connection between convexity and continuity.

Theorem 2.11 (See, for instance, [16] chap. I, corollary 2.3)

Every proper convex function on a space of finite dimension is continuous on the interior of its effective domain, where the effective domain is the subset of all points in which the function takes a finite value.

We want to introduce now a concept that generalizes the notion of gradient for those convex functions which are not differentiable.

Def. 2.12

The *subdifferential* of the convex function $V : X \to \overline{\mathbf{R}}$ is the multivalued operator

$$\partial V : X \to X^*$$

defined as follows:

$$x^* \in \partial V(x) \iff V(y) \geq V(x) + < x^*, y - x > \quad \forall y \in X.$$

Next theorem shows that the subdifferential of a convex function has the fundamental property of being maximal monotone.

Theorem 2.13 (See, for instance, [22] chap. III, proposition 2.13)

Let X be a reflexive Banach space. Let $V : X \to \overline{\mathbf{R}}$ be a convex proper lower semicontinuos function (i.e. $V(x) \leq \liminf_{x_n \to x} V(x_n)$). Then $\partial V : X \to X^*$ is a maximal monotone mappings.

CHAPTER 3

MEASURABLE MULTIVALUED MAPS

In this chapter, we want to give some notions about measurable functions and measurable multivalued maps.

Def. 3.1

Let $(X, \mathcal{A})$, $(Y, \mathcal{B})$ be two measurable spaces and let $u : X \to Y$. We say that u is measurable with respect to $\mathcal{A}$ and $\mathcal{B}$ if, for every subset $B \in \mathcal{B}$, the inverse image $u^{-1}(B) \in \mathcal{A}$.

We observe that, when $Y = \mathbf{R}^n$ and $\mathcal{B}$ is the σ-field of all Borel subset of $\mathbf{R}^n$, then u is measurable with respect to $\mathcal{A}$ and the σ-field of Borel subsets if and only if $u^{-1}(B) \in \mathcal{A}$ for every open subset $B \subseteq \mathbf{R}^n$, or equivalently if and only if $u^{-1}(C) \in \mathcal{A}$ for every closed subset $C \subseteq \mathbf{R}^n$. This equivalence, as we shall see later, does not remain true in the multivalued case.

Let $(X, \mathcal{A}, \mu)$ be a measurable space; next theorem shows the space of all measurable functions defined on X with values in $\overline{\mathbf{R}}$ forms a complete lattice with respect to the order relation given by $f \leq g$ μ-a.e. in X. We want to point out that, when we deal with a non countable family, this problem is not trivial.

Theorem 3.2 (See, for instance, [21] chap. II, proposition 4.1)

Let $(X, \mathcal{A}, \mu)$ be a measurable space with finite measure. Then, for every (countable or not) family $(f_j)_{j\in J}$ of measurable functions defined on X with values in $\overline{\mathbb{R}}$, there exist two measurable functions F^s and F^i such that , for every measurable function g defined on X:

$$f_j \le g \quad \text{a.e. in } X \quad \forall j \in J \quad \Leftrightarrow \quad F^s \le g \quad \text{a.e. in } X$$
$$f_j \ge g \quad \text{a.e. in } X \quad \forall j \in J \quad \Leftrightarrow \quad F^i \ge g \quad \text{a.e. in } X.$$

Proof. If the family $(f_j)_{j\in J}$ is countable, we set $F^s = \operatorname{Sup}_{j\in J} f_j$ and $F^i = \inf_{j\in J} f_j$.

On the contrary, let h be a strictly increasing continuous function from $[-\infty, +\infty]$ to $(-\infty, +\infty)$, for instance $h = \text{arctg}$. The upper bound σ of

$$\int_X h(\operatorname{Sup}_{\tilde{J}} f_j) d\mu$$

is finite when $\tilde{J}$ varies in the countable subsets of J, moreover σ is obtained by a countable subset of index J_0, in fact it is enough to set $J_0 = \cup\{J_n : n \in \mathbb{N}\}$, where J_n is such that

$$\int_X h(\operatorname{Sup}_{J_n} f_j) d\mu + \frac{1}{n} \ge \sigma.$$

Let us define $F^s = \operatorname{Sup}_{j\in J_o} f_j$. Hence, for every measurable function g such that $f_j \le g$ a.e. in X for every $j \in J$, we have that $F^s \le g$ a.e. in X. To prove the inverse inequality, it is enough to show that $f_j \le F^s$ a.e. in X for every $j \in J$. By construction of J_0, it follows that for all $j \in J$

$$\int_X h(\operatorname{Sup}(f_j, F^s)) d\mu = \int_X h(F^s) d\mu = \sigma;$$

therefore we obtain that $h(\operatorname{Sup}(f_j, F^s)) = h(F^s)$ a.e. in X and hence $\operatorname{Sup}(f_j, F^s) = F^s$ a.e. in X and for every $j \in J$. In a similar way we prove the existence of F^i, and this concludes the proof.■

We want to consider now multivalued maps.

We recall that by $\mathcal{B}(\mathbf{R}^n)$ we denote the σ-field of all Borel subsets of $\mathbf{R}^n$.

Let $(X, \mathcal{A})$ be a measurable space and let $F : X \to \mathbf{R}^n$ be a multivalued function from the space X to the family of non empty closed subsets of the space $\mathbf{R}^n$. For every $B \subseteq \mathbf{R}^n$ the inverse image of B under F is denoted by

$$F^{-1}(B) = \{x \in X \ : \ Fx \cap B \neq \emptyset\}.$$

Let consider the following measurability conditions:

1) for each Borel set $B \subseteq \mathbf{R}^n$, $F^{-1}(B) \in \mathcal{A}$;

2) for each closed set $C \subseteq \mathbf{R}^n$, $F^{-1}(C) \in \mathcal{A}$;

3) for each open set $U \subseteq \mathbf{R}^n$, $F^{-1}(U) \in \mathcal{A}$;

4) there exists a sequence σ_h of measurable selections such that $Fx = \mathrm{cl}\{\sigma_h(x) : h \in \mathbf{N}\}$ for almost every x (a selection of F is a map $\sigma : X \to \mathbf{R}^n$ such that $\sigma(x) \in Fx$ for almost every x);

5) $G(F) \in \mathcal{A} \otimes \mathcal{B}(\mathbf{R}^n)$.

<u>**Def. 3.3**</u>

We say that a multivalued function $F : X \to \mathbf{R}^n$ is *measurable* (with respect to $\mathcal{A}$ and $\mathcal{B}(\mathbf{R}^n)$) if condition 2) is verified.

This definition of measurability is linked to the other conditions on F listed above by the following theorem.

Theorem 3.4 (See, for instance, [11] chap. III, section 2)

Let $(X, \mathcal{A})$ be a measurable space. Let $F : X \to \mathbf{R}^n$ be a multivalued function with non empty closed values. Then the following conditions holds:

i) 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Leftrightarrow$ 4) $\Rightarrow$ 5);

ii) if there exists a complete σ-finite measure μ defined on $\mathcal{A}$, then all conditions 1)-5) are equivalent.

Theorem 3.5 (See, for instance, [11] chap. III, theorem 23)

Let $(X, \mathcal{A}, \mu)$ be a measurable space, where μ is a complete σ-finite measure defined on $\mathcal{A}$. If G belongs to $\mathcal{A} \otimes \mathcal{B}(\mathbf{R}^n)$, then the projection $\mathrm{pr}_x G$ belongs to $\mathcal{A}$.

It is interesting to note that the equivalence between conditions 2) and 5) is true for certain multivalued maps, even if the measure space is not complete. This result was found by Chiadò-Piat, Dal Maso, Defranceschi in [12] and it will be very useful in the sequel; therefore we prefer to state it, giving also the proof.

Theorem 3.6 (See [12], theorem 1.3)

Let $(X, \mathcal{A}, \mu)$ be a measurable space, where μ is a complete σ-finite measure. Let $F : X \to \mathbf{R}^n \times \mathbf{R}^m$ be a multivalued function with non empty closed values. Let $H : X \times \mathbf{R}^n \to \mathbf{R}^m$ be the multivalued function defined by

$$H(x, \xi) = \{\eta \in \mathbf{R}^m : [\xi, \eta] \in Fx\}.$$

Then the following conditions are equivalent:

i) F is measurable with respect to $\mathcal{A}$ and $\mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m)$;

ii) $G(F) \in \mathcal{A} \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m)$;

iii) H is measurable with respect to $\mathcal{A}\otimes\mathcal{B}(\mathbf{R}^n)$ and $\mathcal{B}(\mathbf{R}^m)$;

iv) $G(H)\in\mathcal{A}\otimes\mathcal{B}(\mathbf{R}^n)\otimes\mathcal{B}(\mathbf{R}^m)$.

Proof. By theorem 3.4 we have that i) is equivalent to ii) and iii) implies iv). Since $G(F)=G(H)$, we obtain easily that ii) is equivalent to iv). To conclude the proof of the theorem we shall show that ii) implies iii). To this aim it is enough to prove that ii) yields $H^{-1}(C)\in\mathcal{A}\otimes\mathcal{B}(\mathrm{R}^n$ for every compact subset $C\subseteq\mathbf{R}^m$. Let us fix a compact subset $C\subseteq\mathbf{R}^m$. By taking into account the definition of H, we have that

$$(3.1)\qquad H^{-1}(C)=\{[x,\xi]\in X\times\mathbf{R}^n:\exists\eta\in\mathbf{R}^m:[\xi,\eta]\in Fx\cap(\mathbf{R}^n\times C)\}.$$

Let B denotes the set of all $x\in X$ such that $Fx\cap(\mathbf{R}^n\times C)$ is not empty. By ii) and the projection theorem 3.5 it follows that $B\in\mathcal{A}$. If Φ is the multivalued function from X to $\mathbf{R}^n\times\mathbf{R}^m$ defined by $\Phi x=Fx\cap(\mathbf{R}^n\times C)$, then $D(\Phi)=B$ and (3.1) becomes

$$(3.2)\qquad H^{-1}(C)=\{[x,\xi]\in X\times\mathbf{R}^n:\exists\eta\in\mathbf{R}^m:[\xi,\eta]\in\Phi x\}.$$

Since $G(\Phi)=G(F)\cap(X\times\mathbf{R}^n\times C)\in\mathcal{A}\otimes\mathcal{B}(\mathbf{R}^n)\otimes\mathcal{B}(\mathbf{R}^m)$, by theorem 3.4 there exists a sequence $[\varphi_h,g_h]$ of measurable functions from B to $\mathbf{R}^n\times\mathbf{R}^m$ such that

$$(3.3)\qquad \Phi x=\mathrm{cl}\{[\varphi_h(x),g_h(x)]:h\in\mathbf{N}\}$$

for every $x\in B$. By taking (3.3) into account let us define the set

$$(3.4)\qquad M=\{[x,\xi]\in X\times\mathbf{R}^n:x\in B,\xi\in\mathrm{cl}\{\varphi_h(x):h\in\mathbf{N}\}\}.$$

We shall prove that $M=H^{-1}(C)$. The inclusion $H^{-1}(C)\subseteq M$ follows easily from (3.2), (3.3), (3.4). To prove that $M\subseteq H^{-1}(C)$, let us fix $[x,\xi]\in M$. By definition there exists a subsequence $(\varphi_{\sigma(h)})$ of (φ_h) such that $(\varphi_{\sigma(h)}(x))$ converges to ξ. Moreover, the corresponding sequence $(g_{\sigma(h)}(x))$ belongs to the compact set C. Hence, by passing, if necessary, to a subsequence we may assume that $(g_{\sigma(h)}(x))$ converges to some $\eta\in\mathbf{R}^m$. By

(3.3) we have $[\xi,\eta] \in \Phi x$, hence $[\xi,\eta] \in H^{-1}(C)$, which concludes the proof of the equality $M = H^{-1}(C)$. Since $M = \{[x,\xi] \in X \times \mathbf{R}^n : x \in B,\ \inf_{h\in\mathbf{N}} |\xi - \varphi_h(x)| = 0\}$, we have that $M \in \mathcal{A} \otimes \mathcal{B}(\mathbf{R}^n)$ and the proof of the theorem is accomplished.■

Since it will be useful for the sequel, we observe that when $n = m$ the roles of ξ and η can be interchanged.

To conclude this chapter, we would like to pass from the previous general situation to the particular case of the space L^q and of the Sobolev space $W^{1,p}$.

From now on, we denote by p a fixed real number, $1 < p < +\infty$, and by q its dual exponent, $\frac{1}{p} + \frac{1}{q} = 1$; moreover $I = (a,b)$ is a fixed bounded open interval of $\mathbf{R}$, and by $|I|$ we indicate the Lebesgue measure of I.

<u>Def. 3.7</u>

By $W^{1,p}(I)$ we denote the Sobolev space of all functions $u \in L^p(I)$ whose distributional derivative u' belongs to $L^p(I)$, with the norm

$$\|u\|_{W^{1,p}} := \|u\|_{L^p} + \|u'\|_{L^p}.$$

The Sobolev space $W_0^{1,p}(I)$ is the closure of $C_c^\infty(I)$ in the topology of $W^{1,p}(I)$. We denote by $W^{-1,q}(I)$ the dual space of $W_0^{1,p}(I)$. Instead of the norm of $W^{1,p}(I)$, thanks to the Poincaré inequality, we can introduce in $W_0^{1,p}(I)$ the equivalent norm:

$$\|u\|_{W_0^{1,p}} = \|u'\|_{L^p}$$

and in $W^{-1,q}(I)$ we consider the dual norm:

$$\|F\|_{W^{-1,q}} = \operatorname*{Sup}_{\substack{u\in W_0^{1,p}(I) \\ \|u\|_{W_0^{1,p}}=1}} < F, u > .$$

For further properties of the Sobolev spaces, we refer to [9].

Since in the sequel we will often deal with vector valued functions, we observe that the space $L^q(I, \mathbb{R}^n) = \{f : I \to \mathbb{R}^n \ : \ \int_I |f(x)|^q < +\infty\}$ is equivalent to the space $(L^q(I))^n$ and in the same way $W^{1,p}(I, \mathbb{R}^n)$ is equivalent to $(W^{1,p}(I))^n$. To simplify the notation, we will always write L^q instead of $L^q(I)$ or $(L^q(I))^n$ and $W^{1,p}$ instead of $W^{1,p}(I)$ or $(W^{1,p}(I))^n$, when no confusion is possible.

<u>Lemma 3.8</u>

Let $(g_h)_{h\in\mathbb{N}} \subseteq L^q$ $(q > 1)$ be a sequence such that

i) $g_h \rightharpoonup g$ weakly in L^q,

ii) $g'_h \to g'$ strongly in $W^{-1,q}$.

Then $g_h \to g$ strongly in L^q.

Proof. Without loss of generality, we may assume that g and g' are equal to zero. So we have to prove that $\|g_h\|_{L^q} \to 0$. By the theory of Sobolev spaces, we can find a function $f_h \in L^q$ such that

$$g'_h = f'_h \quad \text{and}$$

$$\|g'_h\|_{W^{-1,q}} = \|f_h\|_{L^q} .$$

Since we are in the one-dimensional case, we can write

$$g_h = f_h + c_h$$

for a suitable choice of the constants c_h.

By *ii*), we have that $f_h \to 0$ strongly in L^q and by *i*) we have that $g_h \rightharpoonup 0$ weakly in L^q, hence the sequence of constants $(c_h)_{h\in\mathbf{N}}$ converges to 0 in $\mathbf{R}$ and so $g_h \to 0$ strongly in L^q. ∎

CHAPTER 4

AN EXTENSION THEOREM FOR MONOTONE MAPPINGS

In this chapter, we present a construction which permits to extend, in a unique way, a monotone map defined on a dense subset of a reflexive Banach space to a maximal monotone operator. In other words, this theorem says that a maximal monotone operator is completely known by its behaviour on a section defined on a dense subset.

Moreover, this explicit construction permits to preserve some of the properties of the initial monotone map. These results can be found in [1].

Proposition 4.1(See, for instance, [10], proposition 7.2)

Let X be a reflexive Banach space, A a bounded subset of X, x_0 a point in the weak closure of A. Then there exists an infinite sequence $(x_k)_{k\in \mathbb{N}}$ in A converging weakly to x_0 in X.

Proof. We prove first that there exists a countable subset A_0 of A such that x_0 lies in the weak closure of A_0. For each integer n, let B^n be the product of n copies of the unit ball in X^*. Then B^n is compact in the product of the weak topologies. For each element $[w_1, \ldots, w_n]$ of B^n and each positive integer m, we may find an element x of A such that

$$|(w_j, x - x_0)| < m^{-1} \quad (1 \leq j \leq n).$$

Moreover, this inequality holds on a weak neighborhood of the point $[w_1, \ldots, w_n]$ in B^n for the same choice of x. Since B^n is compact, we may find a finite set $F_{n,m}$ of points

x such that for each $[w_1, \ldots, w_n]$ at least one point x of $F_{n,m}$ serves to verify the above inequality. Set $A_0 = \cup_{n,m} F_{n,m}$, the union being taken over all pairs of positive integers n and m. Then A_0 is countable, and it follows by the above prescription that x_0 lies in the weak closure of A_0.

Let X_0 be the separable closed subspace of X spanned by the points of A_0. Then x_0 lies in the closure of $A \cap X_0$, and X_0 is reflexive since it is a closed subspace of a reflexive space. To show that x_0 is the weak limit of a sequence from A_0 in X, it suffices to do the same in X_0. Since X_0 is separable and reflexive, however, the weak topology on X_0 is metrizable and each limit point of A_0 in the weak topology on X_0 is the weak limit of an infinite sequence from A_0. ■

<u>Corollary 4.2</u>

Let X be a Banach space, Y a reflexive Banach space, $M \subseteq X \times Y$ a subset with the following property: for every $\overline{x} \in X$, there exists $\epsilon > 0$ and $C > 0$ such that, if $|x - \overline{x}| < \epsilon$ and $[x, y] \in M$, then $|y| < C$.

Assume that $[x_0, y_0]$ belongs to $\overline{M}^{s \times w}$, the closure of M in the product of the strong topology of X and the weak topology of Y, then there exists a sequence $([x_k, y_k])_{k \in \mathrm{N}}$ belonging to M such that $x_k \to x$ strongly in X and $y_k \rightharpoonup y$ weakly in Y.

Proof. Choose ϵ corresponding to x_0, as prescribed by the property of M and for every x in X, let $M(x) = \{y \in Y \ : \ [x, y] \in M\}$. For every $\eta > 0$, we define $M(x_0, \eta)$ as the union of all sets $M(x)$ with $|x - x_0| < \eta$.

Since $[x_0, y_0] \in \overline{M}^{s \times w}$, it follows that for every ball $B_{\frac{\epsilon}{k}}$ of center x_0 and radius $\frac{\epsilon}{k}$ and every neighborhood V of y_0 in the weak topology of Y, the set $M \cap (B_{\frac{\epsilon}{k}} \times V)$ is non empty; let $[x, y] \in M \cap (B_{\frac{\epsilon}{k}} \times V)$, hence $|x - x_0| < \frac{\epsilon}{k}$ and $y \in M(x) \cap V$, and then $y \in M(x_0, \frac{\epsilon}{k}) \cap V$. This allows us to say that for every $k \in \mathrm{N}$ and for every neighborhood V of y_0 in the weak topology of Y, the set $M(x_0, \frac{\epsilon}{k}) \cap V \neq \emptyset$, i.e. for every $k \in \mathrm{N}$, y_0 belongs to the closure of $M(x_0, \frac{\epsilon}{k})$ in the weak topology of Y.

Let k fixed and recall that $M(x_0, \frac{\epsilon}{k})$ is bounded in Y, then by proposition 4.1, we can find a sequence $(y_n^k)_{n\in N} \subseteq M(x_0, \frac{\epsilon}{k})$ such that $(y_n^k)_{n\in N}$ converges to y_0 weakly in Y; correspondingly we find a sequence $(x_n^k)_{n\in N} \subseteq B_{\frac{\epsilon}{k}}$ such that $[x_n^k, y_n^k] \in M$. Repeat this argument for every k and denote by Y_0 the separable closed subset of Y spanned by $(y_n^k)_{n,k\in N}$.

Since $(x_n^k)_{n,k\in N} \subseteq B_\epsilon$, then $(y_n^k)_{n,k\in N}$ is contained in a bounded subset of Y_0, which is reflexive and separable, hence the weak topology is metrizable on its bounded subsets (see, for instance, [9] chap. III, corollary 24 and theorem 25'); in particular, we can find for every $k \in N$ an index $n_k \in N$ such that $d(y_{n_k}^k, y_0) < \frac{\epsilon}{k}$, where d is the distance which gives the metric on the bounded subset of Y_0 containing $(y_n^k)_{n,k\in N}$. Correspondingly, the subsequence $(x_{n_k}^k)_{k\in N}$ is such that $|x_{n_k}^k - x_0| < \frac{\epsilon}{k}$ and $[x_{n_k}^k, y_{n_k}^k] \in M$; hence $x_{n_k}^k \to x_0$ strongly in X and $y_{n_k}^k \rightharpoonup y_0$ weakly in Y, which concludes the proof.■

<u>Theorem 4.3</u>

Let X be a reflexive Banach space and let $Y \subseteq X$ be a dense subset of X.

Let $\gamma : Y \to X^*$ be a monotone and locally bounded single valued operator, that is

$$i) \ < \gamma(y_1) - \gamma(y_2), y_1 - y_2 > \geq 0 \qquad \forall y_1, y_2 \in Y$$
$$ii) \ \forall x_0 \in X \ \ \exists \ \epsilon > 0, \ \exists C_\epsilon > 0 \ \ s.t. \ \ \|\gamma(y)\|_{X^*} \leq C_\epsilon \ \ \forall y \in Y \cap B(x_0, \epsilon).$$

Let Γ be the operator defined by

$$[x, f] \in G(\Gamma) \subseteq X \times X^* \iff < \gamma(y) - f, y - x > \geq 0 \ \ \forall y \in Y.$$

Then Γ is the unique maximal monotone extension of γ and $D(\Gamma) = X$.

Moreover, if $\tilde{\Gamma}$ is the map defined by

$$G(\tilde{\Gamma}) = \overline{G(\gamma)}^{s \times w}$$

then $\Gamma(x) = \overline{\text{co}}\tilde{\Gamma}(x)$ for every $x \in X$ where, for every $x \in X$, $\overline{\text{co}}\tilde{\Gamma}(x)$ denotes the closure of the convex envelope of $\tilde{\Gamma}(x)$.

Proof. Let $[x_i, f_i] \in G(\Gamma)$ $i = 1,2$ and let $x = \frac{x_1+x_2}{2}$.

Hence $x_1 = x + \frac{x_1-x_2}{2}$ $x_2 = x - \frac{x_1-x_2}{2}$, and by the definition of Γ

$$< f_1 - \gamma(y), x + \frac{x_1 - x_2}{2} - y >\geq 0 \quad \forall y \in Y$$
$$< f_2 - \gamma(y), x - \frac{x_1 - x_2}{2} - y >\geq 0 \quad \forall y \in Y.$$

By adding we have

$$\frac{1}{2} < f_1 - f_2, x_1 - x_2 > -2 < \gamma(y), x - y > + < f_1 + f_2, x - y >\geq 0 \quad \forall y \in Y.$$

By the density, we can find $y_n \to x$ and taking in account the local boundedness of γ, we obtain

$$< f_1 - f_2, x_1 - x_2 >\geq 0 \quad \forall [x_1, f_1], [x_2, f_2] \in G(\Gamma).$$

Therefore the operator Γ is monotone; the maximality and the uniqueness are obvious by the characterization of its graph.

Finally we have to prove that $D(\Gamma) = X$. Let $x \in X$ and let $y_n \to x$ in X; as γ is monotone, we have that

$$(4.1) \qquad < \gamma(y_n) - \gamma(y), y_n - y >\geq 0 \qquad \forall y \in Y.$$

By the reflexivity of the space and the local boundedness of γ, we know that $\exists\ (y_{n_k})_{k\in\mathbf{N}}$ such that $\gamma(y_{n_k}) \rightharpoonup z$. Passing to the limit in (4.1) we obtain that

$$< z - \gamma(y), x - y >\geq 0 \qquad \forall y \in Y$$

and hence $[x, z] \in G(\Gamma)$; in particular $x \in D(\Gamma)$ or, for the arbitrariety of x, $D(\Gamma) = X$. We observe also that, since $D(\Gamma) = X$ and Γ is monotone, then it is locally bounded.

In order to prove the second part of the lemma, we begin observing that, if $[x, f] \in G(\tilde{\Gamma})$, applying corollary 4.2 with $M = G(\gamma) \subseteq X \times X^*$, it follows that there exists a sequence $([y_n, \gamma(y_n)])_{n\in\mathrm{N}}$ such that $y_n \to x$ strongly in X and $\gamma(y_n) \rightharpoonup f$ weakly in X^*, that is

$$\tilde{\Gamma}x = \{f \in X^* \ : \ \exists (y_n)_{n\in\mathrm{N}} \subseteq X \text{ s.t. } y_n \to x \text{ and } \gamma(y_n) \rightharpoonup f\}.$$

Clearly $D(\tilde{\Gamma}) = X$, $\tilde{\Gamma}(x) \subseteq \Gamma(x)$ for every $x \in X$, $\tilde{\Gamma}$ is locally bounded and $G(\tilde{\Gamma})$ is demiclosed, hence it is also upper semicontinuous from the strong topology of X to the weak topology of X^*. Let us take now $\overline{\text{co}}\tilde{\Gamma}(x)$, we will prove that $G(\overline{\text{co}}\tilde{\Gamma})$ is demiclosed. By contradiction, let $([x_n, f_n])_{n\in\mathrm{N}} \subseteq G(\overline{\text{co}}\tilde{\Gamma})$ with $x_n \to x$ and $f_n \rightharpoonup f$, but $f \notin \overline{\text{co}}\tilde{\Gamma}(x)$. By the Hahn-Banach theorem (see, for instance, [9] chap. I), there exist $z \in X$ and $\delta \in \mathrm{R}$ such that

$$\text{(4.2)} \qquad \begin{aligned} &< f, z > +\delta > 0 \\ &< x^*, z > +\delta < 0 \quad \forall x^* \in \overline{\text{co}}\tilde{\Gamma}(x). \end{aligned}$$

Since the half space $\{x^* \in X^* \ : < x^*, z > +\delta < 0\}$ is a convex neighborhood of $\tilde{\Gamma}(x)$ in the weak topology of X^* and $\tilde{\Gamma}$ is upper semicontinuous, we have that for n large enough, if $y^* \in \tilde{\Gamma}(x_n)$, then $\beta y^* + \delta < 0$. Passing to the closed convex envelope of $\tilde{\Gamma}(x_n)$, this inequality is still true, hence in particular $< f_n, z > +\delta \leq 0$; passing to the limit, we obtain $\beta f + \delta \leq 0$ which contradicts (4.2). This implies that $[x, f] \in G(\overline{\text{co}}\tilde{\Gamma})$, or $\overline{\text{co}}\tilde{\Gamma}$ is demiclosed. It is clear that $\overline{\text{co}}\tilde{\Gamma}(x) \subseteq \Gamma(x)$ for every $x \in X$, thus this map is monotone and locally bounded and these properties, jointly with the demiclosure of its graph, imply its upper semicontinuity. Moreover, we observe that $\overline{\text{co}}\tilde{\Gamma}(x)$ is closed and convex and, since $D(\overline{\text{co}}\tilde{\Gamma}) = X$, thus, by theorem 2.9, it is maximal monotone, which implies $\overline{\text{co}}\tilde{\Gamma}(x) = \Gamma(x)$ for all $x \in X$ and this concludes the proof.∎

<u>Remark 4.4</u>

Let γ, Γ, Y be as in the previous theorem and let $\mathcal{C} : X \to X^*$ be a multivalued map with closed convex values and demiclosed graph. Assume that $\gamma(y) \in \mathcal{C}(y)$ for every $y \in D(\gamma) = Y$; then the preceding theorem assures that $\Gamma(x) \subseteq \mathcal{C}(x)$ for every $x \in X$. In fact Γ is constructed passing throw the demiclosure of $G(\gamma)$ and the closure of the convex envelope of $\tilde{\Gamma}(x)$, and these operations preserve the inclusion into the multivalued map $\mathcal{C}$.

CHAPTER 5

THE DIRICHLET PROBLEM FOR DIFFERENTIAL INCLUSIONS

This chapter is devoted to a brief study of the Dirichlet problem for second order systems of ordinary differential inclusions. In particular, we want to consider the existence of solutions for this kind of problems. It is known, by a general theory, that under suitable coercivness and boundedness hypotheses, the Dirichlet problem admits at least one solution. Really this theory takes into account generic partial differential operators and, for this reason, it makes use of very complicated tools. On the other hand, in our particular case, the scalar nature of the indipendent variable x, permits us to overcome many difficulties and to develop an indipendent and simpler existence theory, that we consider useful to present here.

For the sequel, $m_1, m_2 \in L^1(I)$ will be two non-negative functions and $c_1 > 0, c_2 > 0$ two constants. By $\mathcal{L}(I)$ we denote the σ-field of all Lebesgue measurable subsets of I. The Euclidean norm and the scalar product in $\mathbf{R}^n$ are denoted by $|\cdot|$ and $(\cdot,\cdot)$, respectively.

<u>Def. 5.1</u>

By M_I^p we denote the class of all multivalued functions $a : I \times \mathbf{R}^n \to \mathbf{R}^n$ with closed values which satisfy the following conditions:

(i) for a.e. $x \in I$ the multivalued function $a(x,\cdot) : \mathbf{R}^n \to \mathbf{R}^n$ is maximal monotone;

(ii) a is measurable with respect to $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n)$ and $\mathcal{B}(\mathbf{R}^n)$, i.e.

$$a^-(C) = \{[x,\xi] \in I \times \mathbf{R}^n : a(x,\xi) \cap C \neq \emptyset\} \in \mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n)$$

for every closed set $C \subseteq \mathbf{R}^n$;

(iii) the estimates

$$|\eta|^q \leq m_1(x) + c_1(\eta, \xi), \tag{5.1}$$

$$|\xi|^p \leq m_2(x) + c_2(\eta, \xi), \tag{5.2}$$

hold for a.e. $x \in I$, for every $\xi \in \mathbf{R}^n$ and for every $\eta \in a(x, \xi)$.

Proposition 5.2

If $a \in M_I^p$, then $D(a(x,\cdot)) = \mathbf{R}^n$ for a.e. $x \in I$. Moreover the multivalued map $b : I \times \mathbf{R}^n \to \mathbf{R}^n$, defined by

$$\xi \in b(x, \eta) \iff \eta \in a(x, \xi). \tag{5.3}$$

belongs to the class M_I^q, corresponding to m_2, m_1 and c_2, c_1.

With a little abuse of notation, the map b, that is the inverse map of a with respect to ξ, well be denoted by a^{-1}.

Proof. 1) By (5.1), for a.e. $x \in I$ the maximal monotone operator $a(x, \cdot)$ is locally bounded, hence by theorem 2.6 $a^{-1}(x, \cdot)$ is surjective. This implies that $a(x, \xi) \neq \emptyset$ for every $\xi \in \mathbf{R}^n$ or $D(a(x,\cdot)) = \mathrm{R}^n$ for a.e. $x \in I$.

Let, now, $N \subset I$ with $|N| = 0$ be the set where (5.1) and (5.2) hold for every $x \in I \setminus N$, and fix $x \in I \setminus N$. By (5.2), it results that a is coercive, hence applying theorem 2.5, we have that for every $\eta \in \mathbf{R}^n$, there exists $\xi \in \mathbf{R}^n$ s.t. $\eta \in a(x, \xi)$. Let call $b(x, \cdot)$ the map that associates to every $\eta \in \mathbf{R}^n$ the values $\xi \in \mathbf{R}^n$ for which $\eta \in a(x, \xi)$. We have to show that $b \in M_I^q$.

Since for x fixed in $I \setminus N$, $b(x,\cdot)$ is the inverse map of $a(x,\cdot)$ and $a(x,\cdot)$ is maximal monotone, by theorem 2.3 also $b(x,\cdot)$ is maximal monotone. Clearly the estimates (5.1) and (5.2) hold for b, with the interchanged constants c_2, c_1 and the interchanged functions m_2, m_1. It remains to prove the measurability of b with respect to $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n)$ and $\mathcal{B}(\mathbf{R}^n)$, but this is a direct consequence of theorem 3.6.■

Def. 5.3

To every $a \in M_I^p$ we associate the operator A : $W^{1,p} \to L^q$ defined by

$$\mathrm{A}u = \{g \in L^q : g(x) \in a(x, u'(x)) \text{ for a.e. } x \in I\}$$

and to every $b = a^{-1} \in M_I^q$ we associate the operator

B : $L^q \to L^p$ defined by

$$\mathrm{B}g = \{v \in L^p : v(x) \in b(x, g(x)) \text{ for a.e. } x \in I\}.$$

As usual, we identify the multivalued maps A and B with their graphs.

Moreover, adapting an argument of Brezis (see [8] ex. II.3.3), it is possible to prove that the properties of b imply the maximal monotonicity of B, as the next proposition shows. Therefore, B has closed and convex image and demiclosed graph.

Proposition 5.4

Let $b \in M_I^q$ and B as in definition 5.3, then B is maximal monotone.

Proof. We begin by showing that

$$R(\lambda B + J) = L^p \tag{5.4}$$

for every $\lambda > 0$, where $J : L^q \to L^p$ is the multivalued map defined by

$$J(g) = |g|^{q-2} g.$$

More precisely we want to show that for every $v \in L^p$ the differential inclusion

$$(\lambda B + J)(g) \ni v \tag{5.5}$$

has, for every $\lambda > 0$, a solution $g \in L^q$.

By the properties of b, there exists a subset N contained in I with $|N| = 0$, such that for every $x \in I \setminus N$, $b(x, \cdot)$ is maximal monotone. Let fix $x \in I \setminus N$, then for every $\lambda > 0$ the inclusion

$$\lambda b(x, \eta) + |\eta|^{q-2}\eta \ni v(x) \tag{5.6}$$

has at least one solution $\eta_x \in \mathbf{R}^n$, because the left-hand side of (5.6) is maximal monotone (see, for instance, [22] chap. III, theorem 3.6) and by (5.1) it is also coercive, then it is possible to apply theorem 2.5. Moreover, by the selection theorem of Aumann and Von Neumann (see, for instance, [11], theorem III.22), we can find a measurable function g such that

$$\lambda b(x, g(x)) + |g(x)|^{q-2} g(x) \ni v(x) \quad \forall x \in I \setminus N \tag{5.7}$$

that is (5.7) holds a.e. in I. It remains to prove that $g \in L^q$. We observe that (5.7) implies that there exists $f(x) \in b(x, g(x))$ a.e. in I such that

$$\lambda f(x) + |g(x)|^{q-2} g(x) = v(x) \quad \text{a.e. } x \in I$$

and then by (5.1)

$$
\begin{aligned}
2\|g\|_{L^q}^q &\le \|m_1\|_{L^1} + \|g\|_{L^q}^q + \lambda \int_I f(x)g(x)dx = \\
&= \|m_1\|_{L^1} + \int_I (\lambda f + Jg)(x)dx \le \|m_1\|_{L^1} + \|\lambda f + Jg\|_{L^p}\|g\|_{L^q}^q \\
&\Rightarrow \|g\|_{L^q}^{q-1} \le \frac{1}{2}\left[(\|m_1\|_{L^1} + 1) + \|\lambda f + Jg\|_{L^p}\right] < +\infty
\end{aligned}
$$

since $\|\lambda f + Jg\|_{L^p} = \|v\|_{L^p}$ and this proves (5.4). Now, by (5.4) we obtain immediately that B is maximal monotone, in fact if C is a monotone operator containing B and $v \in Cg$, then

$$\lambda v + Jg \in \lambda Cg + Jg. \tag{5.8}$$

On the other hand, since $R(\lambda B + J) = L^p$, there exists $h \in L^q$ such that

$$\lambda v + Jg \in \lambda Bh + Jh. \tag{5.9}$$

As $B \subseteq C$, we have

$$\lambda v + Jg \in \lambda Ch + Jh. \tag{5.10}$$

By taking (5.8) and (5.10) into account, the strict monotonicity of the operator $\lambda C + J$ yelds $g = h$ a.e. in I. Thus (5.9) becomes

$$\lambda v + Jg \in \lambda Bg + Jg$$

that is $v \in Bg$ or $C \subseteq B$, which gives the maximality of B and concludes the proof. ∎

We can now prove the existence theorem, which can be found in [1] remark 3.4.

Theorem 5.5

For every $a \in M_I^p$ and $f \in W^{-1,q}$, the boundary value problem of the type:

$$
(5.11) \qquad \begin{aligned}
& u \in W_o^{1,p} \quad g \in L^q \\
& -g' = f \\
& g(x) \in a(x, u'(x)) \quad \text{a.e. in } I
\end{aligned}
$$

has at least one solution $u \in W_o^{1,p}$.

We observe that in the case of a strictly monotone and single-valued map a, (5.11) reduces to the problem (1.2) mentioned in the introduction.

Proof. We would like to show that (5.11) always admits at least one solution, which is in general not unique. We recall that given $f \in W^{-1,q}$, we can fix $\gamma \in L^q$ such that $\gamma'(x) = f(x)$ a.e. in I and so we will write $g(x) = c + \gamma(x)$ a.e. in I. Then we have to determine the constant $c \in \mathbf{R}^n$, so that there exists $u \in W_o^{1,p}$ with

$$
u'(x) \in b(x, c + \gamma(x)) \quad \text{a.e. in } I.
$$

To this aim, it is enough to find a measurable selector $\psi(x) \in b(x, c + \gamma(x))$ a.e. in I such that $\int_I \psi(x)\, dx = 0$; hence let us define the map $\mathcal{I} : \mathrm{R}^n \to \mathrm{R}^n$ as follows:

$$
\mathcal{I}(c) = \left\{ \int_I \psi(x)\, dx \ : \ \psi \in L^p,\ \psi(x) \in b(x, c + \gamma(x)) \text{ a.e. in } I \right\}.
$$

By the properties of B defined in def. 5.3, $\mathcal{I}$ locally takes values in a compact subset of R^n and has closed graph, hence the next lemma 5.6 yelds that it is upper semicontinuous. Moreover, $\mathcal{I}$ is monotone and has non empty closed convex values for every $c \in \mathrm{R}^n$, hence $\mathcal{I}$ is maximal monotone by theorem 2.9. Finally, $\mathcal{I}$ is coercive and so it is onto by theorem 2.5. Then $0 \in R(\mathcal{I})$, or equivalently there exist a constant $c \in \mathbf{R}^n$ and a function

$\psi(x) \in b(x, c+\gamma(x))$ a.e. in I such that

$$\int_I \psi(x)\, dx = 0.$$

Clearly, if we set

$$u(x) = \int_a^x \psi(t)\, dt,$$

we obtain that u satisfies our problem and belongs to $W_0^{1,p}$. ∎

For semplicity, we have considered homogeneus boundary conditions, but it is obvious that the same proof can be adapted to any other boundary conditions.

<u>Lemma 5.6</u> (See, for instance [4] chap. I, corollary 1.1)

Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a locally bounded multivalued map with closed graph and closed values. Then F is upper semicontinuous in $\mathbf{R}^n$.

Proof. Given $x \in \mathbf{R}^n$, let V be any open neighborhood of Fx; since Fx is bounded and closed, it is also compact, hence $d(Fx, V^c) > 0$. Set $\epsilon = \frac{1}{2} d(Fx, V^c)$, then $B(Fx, \epsilon) = \{z \in \mathbf{R}^n : d(z, Fx) < \epsilon\} \subseteq V$. We would like to show that it is possible to find $\delta > 0$ such that $Fy \subseteq B(Fx, \epsilon)$ for every $y \in B(x, \delta)$; this means, in particular, that $Fy \subseteq V$ for every $y \in B(x, \delta)$. By the arbitrariety of x in $\mathbf{R}^n$ we obtain the upper semicontinuity of F in $\mathbf{R}^n$. To this aim we proceed by contradiction. Suppose on the contrary that, given $x \in \mathbf{R}^n$, there exists $\epsilon > 0$ such that $Fy \not\subseteq B(Fx, \epsilon)$. For every $n \in \mathbf{N}$, let $\delta_n = \frac{1}{n}$, therefore there exists $y_n \in B(x, \frac{1}{n})$ and $z_n \in Fy_n$ such that $z_n \notin B(Fx, \epsilon)$. Since $y_n \to x$ when $n \to +\infty$, by hypothesis we have that $(z_n)_{n \in \mathbf{N}}$ is a bounded sequence in $\mathbf{R}^n$, then there exists a subsequence $(z_{n_k})_{k \in \mathbf{N}}$ and a point $z \in \mathbf{R}^n$ such that $z_{n_k} \to z$, moreover $d(z_{n_k}, Fx) \geq \epsilon$ and passing to the limit, this implies that $d(z, Fx) \geq \epsilon$ or $z \notin Fx$.

On the other hand, (y_{n_k}, z_{n_k}) belongs to the graph of F, which is closed; passing to the limit, we obtain that $(x, z) \in G(F)$ that implies $z \in Fx$, which is a contraddiction. This concludes the proof.∎

CHAPTER 6

G-CONVERGENCE FOR MAXIMAL MONOTONE OPERATORS

As we pointed out in the introduction, this chapter, jointly with the next one, contains the main results of this thesis, that is the fundamental theorems on the G-convergence for systems of ordinary differential equations with boundary conditions. In particular we give a characterization of the G-convergence by means of the inverse maps and we state a G-compactness theorem, whose proof is completely indipendent from the one given in the work of Chiadò-Piat, Dal Maso, Defranceschi in [12]. In our case, in fact, we take strongly into account that the problem is defined on a subset of the one-dimensional space $\mathbf{R}$ and hence it does not arise any problem in passing from the derivatives to the primitives of a vector function.

After an introductive part, the theorems of the remaining of the chapter can be found in [1] section III.

We begin considering a convergence that was formulated in abstract terms by Kuratowski (see [17] section 29), as a general concept of set convergence in an arbitrary topological space (X, τ).

<u>Def. 6.1</u>

Let $(\mathrm{E}_h)_{h\in\mathrm{N}}$ be a sequence of subsets of X. We define the *sequential lower limit* and the *sequential upper limit* of $(\mathrm{E}_h)_{h\in\mathbf{N}}$ by

$$1) K_{seq}(\tau)\liminf_{h\to\infty}\mathrm{E}_h = \{x\in X : \exists x_h \xrightarrow{\tau} x, \exists k\in\mathbf{N}, \forall h\geq k : x_h\in\mathrm{E}_h\}$$
$$2) K_{seq}(\tau)\limsup_{h\to\infty}\mathrm{E}_h = \{x\in X : \exists h_k\to+\infty, \exists x_k \xrightarrow{\tau} x, \forall k\in\mathbf{N} : x_k\in\mathrm{E}_{h_k}\}.$$

Then we say that the sequence $(\mathrm{E}_h)_{h\in\mathbf{N}}$ $K_{seq}(\tau)$- *converges to a set* E *in* X if

$$K_{seq}(\tau)\liminf_{h\to\infty}\mathrm{E}_h = K_{seq}(\tau)\limsup_{h\to\infty}\mathrm{E}_h = \mathrm{E}$$

and in this case we write $K_{seq}(\tau)\lim_{h\to\infty}\mathrm{E}_h = \mathrm{E}$.

We denote by w the weak topology of L^p, by s the strong topology of L^q and by $\tilde{w}$ the weak topology of $W^{1,p}$.

Def. 6.2

Let $(a_h)_{h\in\mathbf{N}}$ be a sequence belonging to M_I^p. We say that $(a_h)_{h\in\mathbf{N}}$ G-*converges* to $a\in M_I^p$ and write $a_h \xrightarrow{G} a$ if

$$K_{seq}(\tilde{w}\times s)\limsup_{h\to\infty}\mathrm{A}_h \subseteq \mathrm{A},$$

where A_h and A are the operators associated to a_h and a by definition 5.3, and as usual A_h and A are identified with their graphs.

Remark 6.3

It is worthwhile to point out that the G-convergence satisfies the following properties:

i) if $(a_n)_{n\in\mathbf{N}}\subseteq M_I^p$ and $a_n = a_0$ for every $n\in\mathbf{N}$, then $(a_n)_{n\in\mathbf{N}}$ G-converges to a_0;

ii) if $(a_n)_{n\in\mathbf{N}} \subseteq M_I^p$ and $(a_n)_{n\in\mathbf{N}}$ G-converges to a, then every subsequence $(a_{n_k})_{k\in\mathbf{N}}$ of $(a_n)_{n\in\mathbf{N}}$ G-converges to a;

iii) if $(a_n)_{n\in\mathbf{N}} \subseteq M_I^p$ and if every subsequence $(a_{n_k})_{k\in\mathbf{N}}$ of $(a_n)_{n\in\mathbf{N}}$ contains a further subsequence which G-converges to a, then $(a_n)_{n\in\mathbf{N}}$ G-converges to a;

iv) if $(a_n)_{n\in\mathbf{N}} \subseteq M_I^p$ G-converges, then its limit is unique.

The properties i), ii) and iii) are a direct consequence of definition 6.2; for the uniqueness of the G-limit we refer to theorem 6.7.

Remark 6.4

We observe that by the definition, it follows immediately that the inclusion

$$K_{seq}(\tilde{w} \times s)\limsup_{h\to\infty} \mathrm{A}_h \subseteq \mathrm{A}$$

is equivalent to the following condition:

$$\begin{aligned}
&\text{if } h_k \to +\infty \\
&\quad g_k \in a_{h_k}(x, u_k'(x)) \text{ a.e. in } I \\
&\quad u_k \rightharpoonup u \quad \text{weakly in } W^{1,p} \\
&\quad g_k \to g \quad \text{strongly in } L^q \\
&\text{then } g(x) \in a(x, u'(x)) \text{ a.e. in } I.
\end{aligned}$$

When $n = 1$, this condition is equivalent to the one given in the paper of Chiadò-Piat, Dal Maso, Defranceschi (see [12], remark 3.9): in fact, in this case, the topology σ introduced there for the space L^q coincides, by lemma 3.8, with the strong topology of L^q.

Theorem 6.5

Let $(a_h)_{h\in\mathbf{N}} \in M_I^p$ and $(b_h)_{h\in\mathbf{N}} \in M_I^q$, $b_h = a_h^{-1}$ and let B_h be the operator associated to b_h by definition 5.3. Then the following conditions are equivalent:

$$1)\ a_h \xrightarrow{G} a$$
$$2)\ K_{seq}(s \times w) \limsup_{h\to\infty} \mathrm{B}_h \subseteq \mathrm{B}$$

where B is the operator associated to $b = a^{-1} \in M_I^q$.

Proof. As a consequence of 1) we have that if

$$h_k \to +\infty$$
$$g_k(x) \in a_{h_k}(x, u_k'(x)) \quad \text{a.e. in } I$$
$$u_k \rightharpoonup u \quad \text{weakly in } W^{1,p}$$
$$g_k \to g \quad \text{strongly in } L^q$$

then $g(x) \in a(x, u'(x))$ for a.e. $x \in I$.

Let now $[g, v] \in K_{seq}(s \times w) \limsup_{h\to\infty} \mathrm{B}_h$, then there exists $[g_k, v_k] \in \mathrm{B}_{h_k}$ such that $[g_k, v_k] \overset{(s\times w)}{\to} [g, v]$. In particular, if we set

$$u_k(x) = \int_a^x v_k(t)dt$$
$$u(x) = \int_a^x v(t)dt$$

we obtain that $u_k \rightharpoonup u$ weakly in $W^{1,p}$. Since $v_k(x) \in b_{h_k}(x, g_k(x))$ a.e. in I, we have $g_k(x) \in a_{h_k}(x, v_k(x))$ a.e. in I and, by the G-convergence of the sequence $(a_h)_{h\in\mathbf{N}}$, it follows that $g(x) \in a(x, v(x))$ a.e. in I, or $v(x) \in b(x, g(x))$ a.e. in I, that is $[g, v] \in B$.

Viceversa, let us assume 2) and let

$$h_k \to +\infty$$
$$u_k \rightharpoonup u \quad \text{weakly in } W^{1,p}$$
$$g_k \to g \quad \text{strongly in } L^q$$
$$g_k(x) \in a_{h_k}(x, u_k'(x)) \quad \text{a.e. in } I,$$

we would like to conclude that $g(x) \in a(x, u'(x))$ a.e. in I.

Set $v_k = u_k'$, hence $v_h \rightharpoonup v$ weakly in L^p and $v = u'$. As $g_k(x) \in a_{h_k}(x, u_k'(x))$, we obtain $v_k(x) \in b_{h_k}(x, g_k(x))$ a.e. in I, or equivalently $[g_k, v_k] \in \mathrm{B}_{h_k}$; since we have also $[g_k, v_k] \stackrel{(s\times w)}{\to} [g, v]$, by 2) it follows that $[g, v] \in \mathrm{B}$, which is equivalent to $v(x) \in b(x, g(x))$ a.e. in I, or $g(x) \in a(x, v(x))$ a.e in I.■

For every $\eta \in \mathbf{R}^n$ and every $b \in M_I^q$, we set

$$\mathrm{B}^\eta := \{v \in L^p : v(x) \in b(x, \eta) \text{ for a.e. } x \in I\}.$$

Theorem 6.6

Let D be a dense subset of $\mathbf{R}^n$. If B_h and B are as in the previous theorem, then the following conditions are equivalent:

$$1) K_{seq}(s \times w) \limsup_{h\to\infty} \mathrm{B}_h \subseteq \mathrm{B}$$
$$2) K_{seq}(w) \limsup_{h\to\infty} \mathrm{B}_h^\eta \subseteq \mathrm{B}^\eta \quad \forall \eta \in D.$$

Proof. 1) implies 2) is trivial.

Conversely, assume 2) and let $[\tilde{g},\tilde{v}] \in K_{seq}(s\times w)\limsup_{h\to\infty} \mathrm{B}_h$, hence there exist a subsequence $(B_{h_k})_{k\in\mathrm{N}}$ of $(B_h)_{h\in\mathrm{N}}$ and a sequence $[\tilde{g}_k,\tilde{v}_k] \in B_{h_k}$ such that $[\tilde{g}_k,\tilde{v}_k] \overset{(s\times w)}{\to} [\tilde{g},\tilde{v}]$. Fix, now, a countable dense subset D_0 of D. By the measurable selection theorem due to Aumann and Von Neumann (see, for instance, [11], theorem III.22) and by the boundedness condition (5.2), for every $\eta \in D_0$ and for every $k \in \mathrm{N}$ there exists a function $v_k^\eta \in L^p$ with $v_k^\eta(x) \in b_{h_k}(x,\eta)$ a.e. in I. Moreover the inequality (5.2) implies, thanks to the reflexivity of the space L^p, that for every η fixed in D_0, there exists a subsequence of $(v_k^\eta)_{k\in\mathrm{N}}$ which converges weakly in L^p. By a diagonal method, we can construct a further subsequence, that, for simplicity, we still denote by $(v_k^\eta)_{k\in\mathrm{N}}$ and a function v^η, such that $v_k^\eta \rightharpoonup v^\eta$ weakly in L^p for every $\eta \in D_0$; by 2) $v^\eta(x) \in b(x,\eta)$ a.e. in I, and by (5.2) $v^\eta \in L^p$. As a consequence of the monotonicity of b_h, for a.e. $x \in I$ we have that

$$(v_k^\eta(x) - \tilde{v}_k(x), \eta - \tilde{g}_k(x)) \geq 0 \quad \forall \eta \in D_0.$$

Passing to the limit, we obtain for a.e. $x \in I$

$$(v^\eta(x) - \tilde{v}(x), \eta - \tilde{g}(x)) \geq 0 \quad \forall \eta \in D_0. \tag{6.1}$$

Let I' be the set of all points $x \in I$ such that (6.1) holds, $v^\eta(x) \in b(x,\eta)$ and $b(x,\cdot)$ is maximal monotone and locally bounded. Clearly, $|I \setminus I'| = 0$. For a given $x \in I'$, we can apply theorem 4.3 with $X = \mathbf{R}^n$, $Y = D_0$, $\gamma(\eta) = v^\eta(x)$, and we obtain that $\tilde{v}(x) \in b(x,\tilde{g}(x))$. This gives $[\tilde{g},\tilde{v}] \in \mathrm{B}$ and concludes the proof.∎

<u>Theorem 6.7</u>

Let $(a_h)_{h\in\mathrm{N}} \subseteq M_I^p$. Then, if $(a_h)_{h\in\mathrm{N}}$ G-converges, its limit is unique.

Proof. By contradiction, assume that $a_h \overset{G}{\to} a$ and $a_h \overset{G}{\to} \tilde{a}$, with a and $\tilde{a} \in M_I^p$ and $a \neq \tilde{a}$; then, by theorem 6.5 and 6.6, $K_{seq}(w)\limsup_{h\to\infty} \mathrm{B}_h^\eta \subseteq \mathrm{B}^\eta$ and $K_{seq}(w)\limsup_{h\to\infty} \mathrm{B}_h^\eta \subseteq \tilde{\mathrm{B}}^\eta$ for every $\eta \in \mathbf{R}^n$ with $\mathrm{B}^\eta \neq \tilde{\mathrm{B}}^\eta$ for at least one $\eta \in \mathbf{R}^n$.

As we did in the preceding theorem, if D_0 is a countable dense subset of $\mathbf{R}^n$, we can construct a sequence $(v_k^\eta)_{k\in\mathbf{N}}$ and a function v^η, such that $v_k^\eta(x) \in b_{h_k}(x,\eta)$ a.e. in I and $v_k^\eta \rightharpoonup v^\eta$ weakly in L^p for all $\eta \in D_0$. We observe that $v^\eta \in \mathrm{B}^\eta \cap \bar{\mathrm{B}}^\eta$ for all $\eta \in D_0$ by the condition on $K_{seq}(w)\limsup\limits_{h\to\infty}\mathrm{B}_h^\eta$; hence, applying theorem 4.3 with $X = \mathbf{R}^n$, $Y = D_0$ and $\gamma(\eta) = v^\eta(x)$ for x fixed a.e. in I, it follows that γ has one and only one maximal monotone extension, hence $\mathrm{B}^\eta = \bar{\mathrm{B}}^\eta$ for every $\eta \in \mathbf{R}^n$. This implies that $a = \bar{a}$ and concludes the proof.■

Theorems 6.5 and 6.6 give a characterization of the G-convergence of the sequence $(a_h)_{h\in\mathbf{N}}$ in terms of a suitable convergence of the inverse maps b_h. In particular, when the map $b = a^{-1} \in M_I^q$ is single-valued, these results give a very simple characterization, as the following corollary points out.

<u>Corollary 6.8</u>

Let D be a dense subset of $\mathbf{R}^n$, $(a_h)_{h\in\mathbf{N}}$ and $a \in M_I^p$, $(b_h)_{h\in\mathbf{N}} = (a_h^{-1})_{h\in\mathbf{N}}$ and $b = a^{-1} \in M_I^q$. Moreover, assume that b is single-valued. Then the following conditions are equivalent:

1) $a_h \xrightarrow{G} a$

2) for every $\eta \in D$, there exists a sequence $(v_h^\eta)_{h\in\mathbf{N}}$, such that

$$v_h^\eta(x) \in b_h(x,\eta) \quad \text{a.e. in } I$$
$$\text{and} \quad v_h^\eta \rightharpoonup b(\cdot,\eta) \quad \text{weakly in } L^p.$$

Proof. Assume that 1) holds, then by theorems 6.5 and 6.6, it follows that

$$K_{seq}(w)\limsup_{h\to\infty} \mathrm{B}_h^\eta \subseteq \mathrm{B}^\eta \quad \text{for every } \eta \in D,$$

where now B^η contains only the function $b(x,\eta)$. For every $\eta \in D$, as we have already proved in the preceding theorems, the growth condition (5.2) implies that

$$K_{seq}(w)\limsup_{h\to\infty} B_h^\eta \neq \emptyset;$$

hence

$$K_{seq}(w)\limsup_{h\to\infty} B_h^\eta = B^\eta. \tag{6.2}$$

Equality (6.2) and again the growth condition (5.2) imply that, for every $\eta \in D$, if $v_h^\eta \in B_h^\eta$, every subsequence $(v_k^\eta)_{k\in\mathbf{N}}$ of $(v_h^\eta)_{h\in\mathbf{N}}$ has a further subsequence which weakly converges to $b(\cdot,\eta)$; hence all the sequence $(v_h^\eta)_{h\in\mathbf{N}}$ weakly converges to $b(\cdot,\eta)$ and 2) is proved.

Conversely, assume that 2) holds. By theorem 6.5 and 6.6, it is enough to prove that $K_{seq}(w)\limsup_{h\to\infty} B_h^{\bar\eta} \subseteq B^{\bar\eta}$ for every $\bar\eta \in D$, where D is a dense subset of $\mathbf{R}^n$ and $B^{\bar\eta}$ contains only the function $b(x,\bar\eta)$. Let $\bar v \in K_{seq}(w)\limsup_{h\to\infty} B_h^{\bar\eta}$; then there exist $h_k \to +\infty$ and a sequence $(\bar v_k)_{k\in\mathbf{N}}$ such that $\bar v_k(x) \in b_{h_k}(x,\bar\eta)$ a.e. in I and $\bar v_k \rightharpoonup \bar v$ weakly in L^p. By the monotonicity of $b_{h_k}(x,\cdot)$ we have

$$< \bar v_k(x) - v_{h_k}^\eta(x), \bar\eta - \eta > \geq 0 \quad \forall \eta \in D \text{ and a.e. in } I.$$

Passing to the limit, we obtain

$$< \bar v(x) - b(x,\eta), \bar\eta - \eta > \geq 0 \quad \forall \eta \in D \text{ and a.e. in } I$$

and by theorem 4.3, this implies that $\bar v(x) = b(x,\bar\eta)$, but this is true for a.e. $x \in I$, and hence the theorem is proved.■

If also b_h is single-valued for every $h \in \mathbf{N}$, the result obtained takes the form

$$a_h \xrightarrow{G} a \iff b_h(\cdot,\eta) \overset{w-L^p}{\rightharpoonup} b(\cdot,\eta) \quad \forall \eta \in D.$$

Remark 6.9

In general it is not possible to say that, if a_h G-converges to a, for every $\eta \in \mathbf{R}^n$

$$K_{seq}(w) \limsup_{h\to\infty} \mathrm{B}_h^\eta = \mathrm{B}^\eta$$

nor that

$$K_{seq}(w) \limsup_{h\to\infty} \mathrm{B}_h^{\eta_h} = \mathrm{B}^\eta$$

for a suitable sequence $\eta_h \to \eta$, as the following counterexample shows.

Let $a < c < b$ and $I = (a, c] \cup (c, b)$. We set

$$b_h(x,\eta) = b_h^1(\eta) = \begin{cases} 1 & \text{if } \eta \in [0, +\infty) \\ h\eta + 1 & \text{if } \eta \in (\frac{-2}{h}, 0) \\ -1 & \text{if } \eta \in (-\infty, \frac{-2}{h}] \end{cases}$$

when $x \in (a, c]$ and

$$b_h(x,\eta) = b_h^2(\eta) = \begin{cases} 1 & \text{if } \eta \in [\frac{2}{h}, +\infty) \\ h\eta - 1 & \text{if } \eta \in (0, \frac{2}{h}) \\ -1 & \text{if } \eta \in (-\infty, 0] \end{cases}$$

when $x \in (c, b)$.

Clearly if

$$b(x,\eta) = b(\eta) = \begin{cases} 1 & \text{if } \eta \in (0, +\infty) \\ [-1, 1] & \text{if } \eta = 0 \\ -1 & \text{if } \eta \in (-\infty, 0) \end{cases}$$

we have that

$$K_{seq}(w) \limsup_{h\to\infty} \mathrm{B}_h^{\eta_h} \subseteq \mathrm{B}^\eta.$$

By contradiction, assume that $K_{seq}(w)\limsup_{h\to\infty}\mathrm{B}_h^{\eta_h} = \mathrm{B}^\eta$ and take $0 \in \mathrm{B}^0$. Then there exists a subsequence $b_{h_k}(x, \eta_k)$ such that $b_{h_k}(\cdot, \eta_k) \rightharpoonup 0$ weakly in L^p when $\eta_k \to 0$. Without loss of generality, we can suppose that $\eta_k \geq 0$ for all $k \in \mathrm{N}$, but in this case, when $x \in (a, c]$, $b_{h_k}(x, \eta_k) = b_{h_k}^1(\eta_k) \geq 1$ and hence $b_{h_k}^1(\eta_k)$ does not converge to zero on $(a, c]$. As a consequence, it follows that

$$K_{seq}(w) \limsup_{h\to\infty} \mathrm{B}_h^{\eta_h} \subsetneqq \mathrm{B}^0$$

for any $\eta_k \to 0$.

We now introduce an auxiliary notion of convergence, called τ-convergence, on the family of maps belonging to M_I^q, that has two important properties: the first one is that this convergence implies the G-convergence (see lemma 6.11) and the second one is that this convergence is compact (see theorem 6.12).

We would like to point out that the τ-convergence does not satisfy condition iii) of remark 6.3. Since this property is very useful in many concrete cases, we preferred, as a definition of G-convergence, the one given in def. 6.2, by means of the notion of set convergence formulated by Kuratowski, instead of the simpler definition which could be given in terms of the τ-convergence.

Def. 6.10

Let $(b_h)_{h\in\mathbb{N}}, b \in M_I^q$; we say that the sequence $(b_h)_{h\in\mathbb{N}}$ τ- converges to b, and write $b_h \xrightarrow{\tau} b$ if and only if for every $\eta \in \mathbb{Q}^n$ there exist v_h^η, $v^\eta \in L^p$ such that

$$v_h^\eta \rightharpoonup v^\eta \quad \text{weakly in } L^p,$$
$$v_h^\eta(x) \in b_h(x,\eta) \quad \text{a.e. in } I,$$
$$v^\eta(x) \in b(x,\eta) \quad \text{a.e. in } I.$$

Lemma 6.11

If $(b_h)_{h\in\mathbb{N}}, b \in M_I^q$ and $(b_h)_{h\in\mathbb{N}}$ τ-converges to b, then $(a_h)_{h\in\mathbb{N}}$ G-converges to a.

Proof. By theorems 6.5 and 6.6, it is enough to prove that the condition

$$(6.3) \qquad K_{seq}(w) \limsup_{h\to\infty} \mathrm{B}_h^{\bar{\eta}} \subseteq \mathrm{B}^{\bar{\eta}} \quad \forall \bar{\eta} \in \mathbf{R}^n$$

is satisfied; hence let $\bar{\eta} \in \mathbf{R}^n$ and $v \in K_{seq}(w) \limsup_{h\to\infty} \mathrm{B}_h^{\bar{\eta}}$. Then there exists a subsequence $v_k(x) \in b_{h_k}(x, \bar{\eta})$ a.e. in I such that $v_k \rightharpoonup v$ weakly in L^p. By the hypothesis, for every $\eta \in \mathbf{Q}^n$ there exist $v_h^\eta(x) \in b_h(x, \eta), v^\eta(x) \in b(x, \eta)$ a.e. in I such that $v_h^\eta \rightharpoonup v^\eta$ weakly in L^p. By the monotonicity of $b_{h_k}(x, \cdot)$, it follows that, for a.e. $x \in I$,

$$< v_k(x) - v_{h_k}^\eta(x), \bar{\eta} - \eta > \geq 0 \quad \forall \eta \in \mathbf{Q}^n$$

and passing to the limit, we obtain that for a.e. $x \in I$

$$< v(x) - v^\eta(x), \bar{\eta} - \eta > \geq 0 \quad \forall \eta \in \mathbf{Q}^n.$$

In order to prove that $v(x) \in b(x, \bar{\eta})$ it is enough to apply theorem 4.3 as in the proof of theorem 6.6. Then we conclude that $v \in \mathrm{B}^{\bar{\eta}}$, or (6.3) holds.■

We can now state the main result of this thesis work; in fact, as it was pointed out at the beginning of the chapter, we prove a compactness theorem for the τ-convergence, which implies the compactness of the G-convergence for systems of ordinary differential equations. Roughly speaking, the compactness of the G-convergence is a consequence of the next result and of the relation between the G-convergence and the τ-convergence, obtained in the previous theorems and lemmas of this chapter.

<u>Theorem 6.12</u>

Given a sequence $(b_h)_{h\in\mathbf{N}} \in M_I^q$, there exists a subsequence $(b_{h_k})_{k\in\mathbf{N}}$ which τ-converges to $b \in M_I^q$.

Proof. For every $\eta \in \mathbf{Q}^n$, the measurable selection theorem (see, for instance, [11], theorem III.22) and the boundedness condition (5.2) assure the existence of a measurable selection $v_h^\eta(x) \in b_h(x,\eta)$ a.e. in I, with $v_h^\eta \in L^p$ for every $h \in \mathrm{N}$. Moreover, by the boundedness conditions (5.2) on b_h, we have that

$$|v_h^\eta(x)| \leq C\left(m(x) + |\eta|^{q-1}\right) \quad \forall \eta \in \mathbf{Q}^n, \text{ for a.e. } x \in I \text{ and } \forall h \in \mathrm{N} \tag{6.4}$$

with $m \in L^p$. Thanks to the reflexivity of the space L^p, for every η fixed in $\mathbf{Q}^n$, there exists a subsequence of $(v_h^\eta)_{h\in\mathrm{N}}$ which weakly converges in L^p. By a diagonal method, we can construct a further subsequence $(v_k^\eta)_{k\in\mathrm{N}}$ and a function $v^\eta \in L^p$ such that v_k^η converges to v^η weakly in L^p for every $\eta \in \mathbf{Q}^n$. Clearly v^η still satisfies (6.4), hence there exists $N \subseteq I$ with $|N| = 0$, such that for every $x \in I \setminus N$, the map $\eta \to v^\eta(x)$ is monotone, locally bounded on $\mathbf{Q}^n$ and satisfies conditions (5.1), (5.2). Let, for fixed $x \in I$, $\gamma_x : \mathbf{Q}^n \to \mathbf{R}^n$ be the function defined by

$$\gamma_x(\eta) = \begin{cases} v^\eta(x) & \text{if } x \in I \setminus N \\ |\eta|^{q-2}\eta & \text{if } x \in N \end{cases}$$

By theorem 4.3, there exists a unique maximal monotone extension of γ_x on all of $\mathbf{R}^n$; let call Γ_x this extension, and define the operator $b : I \times \mathrm{R}^n \to \mathrm{R}^n$ such that $b(x,\cdot) = \Gamma_x$ for every $x \in I$. By construction, it follows that $b_{h_k} \xrightarrow{T} b$. It remains to prove that $b \in M_I^q$.

It is clear that $\gamma_x(\eta)$ is measurable in x for every η fixed in $\mathbf{Q}^n$. We would like to show that b is measurable with respect to $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n)$ and $\mathcal{B}(\mathbf{R}^n)$. Recall that

$$G(b) = \bigcap_{\eta_q \in \mathbf{Q}^n} C_q$$

$$\text{where} \quad C_q := \bigcap_{\eta_q \in \mathbf{Q}^n} \{[x,\xi,\eta] \in I \times \mathbf{R}^n \times \mathbf{R}^n \ : \ (\gamma_x(\eta_q) - \xi, \eta_q - \eta) \geq 0\}.$$

Since every C_q is measurable and we have a countable intersection, it follows that $G(b) \in \mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^n)$. Let now

$$Fx = \{[\xi,\eta] : \xi \in b(x,\eta)\}$$

then $G(F) = G(b) \in \mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^n)$. Applying theorem 3.6, we obtain the measurability of F from $\mathcal{L}(I)$ to $\mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^n)$. Set

$$H(x,\eta) = \{\xi : [\xi,\eta] \in Fx\} = \\ = \{\xi : \xi \in b(x,\eta)\} = b(x,\eta)$$

applying again theorem 3.6 we obtain that H and hence b is measurable from $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R}^n)$ to $\mathcal{B}(\mathbf{R}^n)$. To conclude the proof, we have to show that the boundedness conditions (5.1) and (5.2) are satisfied, or explicitely:

$$|\eta|^q \leq m_1(x) + c_1(\eta,\xi)$$
$$|\xi|^p \leq m_2(x) + c_2(\eta,\xi)$$

hold for a.e. $x \in I$, for every $\eta \in \mathbf{R}^n$ and for every $\xi \in b(x,\eta)$. To obtain this, let define for a.e. $x \in X$ fixed, the multivalued map $C_x : \mathbf{R}^n \to \mathbf{R}^n$ as follows:

$$C_x(\eta) := \{\xi \in \mathbf{R}^n \ : \ (5.1) \text{ and } (5.2) \text{ hold } \}.$$

Clearly C_x is has closed convex values and demiclosed graph and $\gamma_x(\eta) \in C_x(\eta)$ for every $\eta \in \mathbf{Q}^n$; hence, by remark 4.4, for every $\eta \in \mathbf{R}^n$ $b(x,\eta) \subseteq C_x(\eta)$ and this concludes the proof. ∎

CHAPTER 7

G-CONVERGENCE IN THE SCALAR CASE

In the scalar case, i.e. when $n = 1$, for every $x \in I$ and every $\eta \in \mathbf{R}$, the maximal monotone map $b_h(x,\eta)$ is an interval of $\mathbf{R}$; if we denote by $\phi_h^-(x,\eta)$ and $\phi_h^+(x,\eta)$, respectively, the lower and upper bounds of this interval, then we have the following representation:

$$b_h(x,\eta) = [\phi_h^-(x,\eta), \phi_h^+(x,\eta)] \tag{7.1}$$

with the functions ϕ_h^+ and ϕ_h^- monotone with respect to η.

The $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurability of $\phi_h^{\pm}(\cdot,\cdot)$ is an easy consequence of the $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurability of $b_h(\cdot,\cdot)$, in fact

$$\begin{aligned}\mathcal{L}(I) \otimes \mathcal{B}(\mathrm{R}) \ni & b_h^-([a,+\infty)) = \{[x,\eta] \in I \times \mathrm{R} \ : \ b(x,\eta) \cap [a,+\infty) \neq \emptyset\} = \\ =& \{[x,\eta] \in I \times \mathrm{R} \ : \phi_h^+(x,\eta) \geq a\} = (\phi_h^+)^{-1}([a,+\infty)).\end{aligned}$$

By (5.2) $\phi_h(x,\cdot), \phi_h^+(x,\cdot) \in L^q$.

Moreover, if $b_h \in M_I^q$, it is always possible to write

$$V_h(x,\eta) = \int_0^\eta \phi_h^-(x,\tau)d\tau \tag{7.2}$$

so that it results:

$$b_h(x,\eta) = \partial_\eta V_h(x,\eta) \tag{7.3}$$

with $V_h : I \times \mathrm{R} \to \mathrm{R}$ convex with respect to η. Hence in this case, it is possible to obtain the same results as in the previous chapter, using the convergence of the primitives $V_h(x,\eta)$; in fact, it follows that $V_h(x,\eta)$ satisfies the following conditions:

(i) V_h is measurable with respect to $\mathcal{L}(I)\otimes\mathcal{B}(\mathbf{R})$ and $\mathcal{B}(\mathbf{R})$

(ii) the estimate

$$c_3|\eta|^q - m_3(x)|\eta| \le V_h(x,\eta) \le c_4|\eta|^q + m_4(x)|\eta| \tag{7.4}$$

holds for a.e. $x\in I$ and for every $\eta\in\mathbf{R}$, where the functions $m_3, m_4 \in L^p$ and the constants $0 < c_3 \le c_4 < +\infty$ depend only on the functions and the constants which appear in the definition of M_I^q. Also the converse is true, that is if V_h is convex with respect to η and satisfies (i) and (ii), then $b_h \in M_I^q$. For the measurability see [3], theorem 2.3; for the estimates it is enough to do some simple calculations and finally the maximal monotonicity of b_h is a consequence of theorem 2.13.

In particular in this case, the compactness of the G-convergence is a direct consequence of the compactness of $V_h(x,\eta)$ in the weak topology of L^p, as the next theorem shows (see [1] section 4).

<u>Theorem 7.1</u>

Let $(a_h)_{h\in\mathbf{N}}$, $a\in M_I^p$ and let $(b_h)_{h\in\mathbf{N}}$, $b\in M_I^q$ be the corresponding inverse maps; let, for a.e. $x\in I$, $b_h(x,\eta)=\partial_\eta V_h(x,\eta)$ and $b(x,\eta)=\partial_\eta V(x,\eta)$, with $(V_h)_{h\in\mathbf{N}}$ and V convex with respect to η and satisfying (i) and (ii), and finally let D_0 be a countable dense subset of R.

Then the following conditions are equivalent

1) $a_h \xrightarrow{G} a$,

2) $V_h(\cdot,\eta) \rightharpoonup V(\cdot,\eta) \quad \forall\eta\in D_0$ weakly in L^p.

Moreover condition 2) is compact.

Proof. By inequality (7.4) we have, for a.e. $x \in I$, at the same time, the continuity of the primitives in the second variable (by theorem 2.11) and their boundedness in L^p for every $\eta \in \mathrm{R}$ fixed. Hence, by a diagonal method, it is possible to construct a subsequence of $(V_h)_{h\in\mathbf{N}}$, which converges in the sense of 2) to a function V of the same type, defined on $I \times D_0$. Since V is locally uniformly continuous in η for a.e. $x \in I$, it is possible to extend this convergence to every $\eta \in \mathrm{R}$, in fact we can well define for every $\eta \in \mathbf{R}$ and for a.e. $x \in I$ the function

$$V(x,\eta) = \lim_{\eta_h \to \eta} V(x,\eta_h)$$

where $(\eta_h)_{h\in\mathbf{R}} \subseteq D_0$ and $\eta_h \to \eta$ in $\mathbf{N}$. It is clear that the definition is indipendent of the sequence $(\eta_h)_{h\in\mathbf{N}}$. To show that $V_h(\cdot,\eta) \rightharpoonup V(\cdot,\eta)$ weakly in L^p for every $\eta \in \mathbf{R}$, it is enough to consider the weak convergence on D_0 and the lipschitz continuity of $V_h(x,\cdot)$, uniform with respect to $h \in \mathbf{N}$, due to (7.2) and (5.2). This argument gives the compactness of the primitives for this notion of convergence.

Now it is possible to follow two different ways.

The first one takes into account the equivalence between G-convergence and Γ-convergence proved by A. Defranceschi in [14] and the relation between Γ-convergence and the convergence expressed in 2) due to P. Marcellini and C. Sbordone in [18], as we mentioned in the introduction.

The second one is a direct proof. Assume that 2) holds, then adapting the proof of lemma 3.1 in [18] and taking (7.2) and (5.2) into account, we obtain that $V_h(\cdot, g_h(\cdot)) \rightharpoonup V(\cdot, g(\cdot))$ weakly in L^1, for every $(g_h)_{h\in\mathbf{N}}$ converging strongly to g in L^q. Suppose that

$$\begin{aligned}
&h_k \to +\infty \\
&g_k(x) \in a_{h_k}(x, u'_k(x)) \quad \text{a.e. in } I \\
&g_k \to g \quad \text{strongly in } L^q \\
&u_k \rightharpoonup u \quad \text{weakly in } W^{1,p},
\end{aligned}$$

we would like to conclude that $g(x) \in a(x, u'(x))$ a.e. in I or $u'(x) \in b(x, g(x))$ a.e. in I. Since $g_k(x) \in a_{h_k}(x, u'_k(x))$ a.e. in I, we have $u'_k(x) \in b_{h_k}(x, g_k(x))$ a.e. in I, or

$u'_k(x) \in \partial_\eta V_{h_k}(x, g_k(x))$ a.e. in I. This implies that for every $\eta \in \mathbf{R}$

$$V_{h_k}(x,\eta) - V_{h_k}(x, g_k(x)) \geq u'_k(x)(\eta - g_k(x)) \quad \text{for a.e. } x \text{ in } I$$

and passing to the limit, it follows that

$$V(x,\eta) - V(x, g(x)) \geq u'(x)(\eta - g(x)) \quad \text{for a.e. } x \text{ in } I$$

or $u'(x) \in \partial_\eta V(x, g(x)) = b(x, g(x))$ a.e. in I, and this proves 1).

The converse is a consequence of this step, the compactness of condition 2) and the uniqueness of the G-limit, proved in theorem 6.7. In fact, let $a_h \overset{G}{\to} a$; by the compactness, we have that there exist a subsequence $h_k \to +\infty$ and a function $\tilde{V}$ such that $V_{h_k}(\cdot,\eta) \rightharpoonup \tilde{V}(\cdot,\eta)$ for every $\eta \in D_0$, with $\tilde{V}$ satisfying the same properties of V_h. Since we proved that 2) implies 1), it follows that $a_{h_k} \overset{G}{\to} \tilde{a}$, where $\tilde{a}^{-1}(x,\eta) = \partial_\eta \tilde{V}(x,\eta)$ a.e. in I and for every $\eta \in \mathbf{R}$; by the uniqueness of the G-limit, we obtain that $\tilde{a} \equiv a$ or equivalently $\tilde{V} \equiv V$ (we agree to assume that every primitive function is zero when $\eta = 0$), and this concludes the proof. ■

By (7.1) it is clear that the multivalued map b_h has two special selections, i.e. ϕ_h^+ and ϕ_h^-; hence we can particularize the characterization of the G-convergence in terms of the convergence of ϕ_h^+ and ϕ_h^-. For this purpose, we have to develop some preliminary tools (see [1], appendix).

Theorem 7.2

Let (f_n) be a sequence of functions contained in $L^p(I)$ and (f_n) equidominated, i.e. there exists $m \in L^1(I)$ such that

$$|f_n(x)|^p \leq m(x) \quad \text{for a.e. } x \text{ in } I.$$

Set

$$G^i = \{g \in L^p(I) \ : \ \int_B g(x)dx \le \liminf_{n\to+\infty} \int_B f_n(x)dx \quad \forall B \in \mathcal{B}(I)\}.$$

Then the following conditions hold:

1) $g_1, g_2 \in G^i \implies g_1 \vee g_2 \in G^i$

2) $g_h \uparrow g \ \ g_h \in G^i \implies g \in G^i$.

Moreover, there exists a unique $g^i \in G^i$ with the following properties:

(i) $g(x) \le g^i(x) \quad \forall g \in G^i$

(ii) if $h \in L^p$ and $g(x) \le h(x) \quad \forall g \in G^i$ then $g^i(x) \le h(x)$ a.e. in I.

Proof. 1) Assume that $g_1 \ge g_2$ on A_1 and $g_1 < g_2$ on A_2, then $I = A_1 \cup A_2$ and

$$\begin{aligned}\int_B g_1 \vee g_2 dx &= \int_{B\cap A_1} g_1 dx + \int_{B\cap A_2} g_2 dx \le \\ &\le \liminf_{n\to+\infty} \int_{B\cap A_1} f_n dx + \liminf_{n\to+\infty} \int_{B\cap A_2} f_n dx \le \\ &\le \liminf_{n\to+\infty} \int_B f_n dx.\end{aligned}$$

2) It is a consequence of the monotone convergence theorem.

Since G^i is a collection of measurable functions, by theorem 3.2 there exists a function g^i which satisfies the properties (i) and (ii) and such that $g^i = \operatorname{Sup}_{i\in J_0} g_i$ for a suitable choice of a countable set J_0. Now we have to prove that $g^i \in G^i$, therefore set

$$\tilde{g}_1 = g_1 \quad \tilde{g}_2 = \tilde{g}_1 \vee g_2 \ \ldots\ldots\ldots \ \tilde{g}_h = \tilde{g}_{h-1} \vee g_h$$

then $\tilde{g}_h \le \tilde{g}_{h+1}$ and $\tilde{g}_h \ge g_h$ a.e. in I and hence

$$\operatorname{Sup}_{h\in\mathbb{N}} \tilde{g}_h \ge \operatorname{Sup}_{h\in\mathbb{N}} g_h = g^i.$$

Since by 1) $\tilde{g}_h \in G^i$, it follows that $g^i \geq \operatorname{Sup}_{h\in\mathbf{N}} \tilde{g}_h$ and so $\operatorname{Sup}_{h\in\mathbf{N}} \tilde{g}_h = g^i$. We conclude the proof noting that $\operatorname{Sup}_{h\in\mathbf{N}} \tilde{g}_h = \lim_{h\to+\infty} \tilde{g}_h$ and by 2) $\lim_{h\to+\infty} \tilde{g}_h \in G^i$.■

Theorem 7.3

In the hypothesis of theorem 7.2, let us consider the set

$$\mathcal{H} = \{h \in L^p \ : \ \exists n_k \uparrow +\infty \text{ s.t. } f_{n_k} \rightharpoonup h \text{ weakly in } L^p\}.$$

Then $\mathcal{H}$ is not empty and $\inf \ \mathcal{H} = \max \ G^i$.

Proof. Since $(f_n)_{n\in\mathbf{N}}$ is equidominated in L^p, it is weakly precompact in L^p, hence $\mathcal{H}$ is not empty. By theorem 3.2, we have that there exists $\inf \mathcal{H}$; it remains only to prove the equality $\inf \ \mathcal{H} = \max \ G^i$. Fixed $B \in \mathcal{B}$, there exists a subsequence $(f_k)_{k\in\mathbf{N}}$ of $(f_n)_{n\in\mathbf{N}}$ such that

$$\liminf_{n\to+\infty} \int_B f_n dx = \lim_{k\to+\infty} \int_B f_k dx.$$

Since $(f_k)_{k\in\mathbf{N}}$ is equidominated, it is possible to extract a further subsequence $(f_h)_{h\in\mathbf{N}}$ weakly convergent to h such that:

$$\lim_{k\to+\infty} \int_B f_k dx = \lim_{h\to+\infty} \int_B f_h dx = \int_B h \ dx \geq \int_B \inf \mathcal{H} \ dx.$$

This implies that

$$\liminf_{n\to+\infty} \int_B f_n dx \geq \int_B \inf \mathcal{H} \ dx$$

and this relation holds for every $B \in \mathcal{B}(I)$, thus $\inf \mathcal{H} \in G^i$.

Moreover let $g \in G^i$ and $h \in \mathcal{H}$. By definition, there exists a subsequence $(f_k)_{k\in\mathbf{N}}$ of $(f_h)_{h\in\mathbf{N}}$ weakly convergent to h, then

$$\int_B g \ dx \leq \liminf_{n\to+\infty} \int_B f_n \ dx \leq \liminf_{k\to+\infty} \int_B f_k dx = \int_B h \ dx.$$

This inequality holds for every $B \in \mathcal{B}(I)$, then for a.e. $x \in I$ it follows that $g(x) \leq h(x)$. By the arbitrariety of h in $\mathcal{H}$ and g in G^i and since $\inf \mathcal{H} \in G^i$, we obtain that

$$\max G^i \leq \inf \mathcal{H} \leq \max G^i.$$

This conclude the proof.■

Remark 7.4

By duality we can consider the set

$$G^s = \{g \in L^p(I) \; : \; \int_B g(x)dx \geq \limsup_{n \to +\infty} \int_B f_n(x)dx \quad \forall B \in \mathcal{B}(I)\}$$

and we obtain that min $G^s = \text{Sup } \mathcal{H}$.

We can now give the following definition.

Def. 7.5

Let $(\phi_h)_{h \in \mathbf{N}}$ be a sequence of functions belonging to L^p, which is equidominated. We say that the L^p function g^i is the *weak L^p lower limit* of the sequence if and only if g^i is the maximal function such that

$$\int_B g^i(x)dx \leq \liminf_{h \to +\infty} \int_B \phi_h(x)dx \quad \forall B \in \mathcal{B}(I)$$

The *weak L^p upper limit* g^s is defined in a dual way.

As we proved

$$g^i = \inf\ \mathcal{H} \quad \text{and} \quad g^s = \text{Sup}\ \mathcal{H} \tag{7.5}$$

where the infimum and the supremum are taken in the lattice structure of L^p.

We can now state the main theorem of this chapter (see [1] section 4), which characterizes the G-convergence of maximal monotone operators, in the scalar case, in terms of a suitable convergence of their inverse maps.

<u>Theorem 7.6</u>

Let $(a_h)_{h\in\mathbf{N}} \in M_I^p$ be a sequence of maximal monotone operators and let $(b_h)_{h\in\mathbf{N}} \in M_I^q$ be the sequence of their inverse maps, as in proposition 5.2. Then the following conditions are equivalent:

$$\begin{aligned} &1)\ a_h \xrightarrow{G} a \\ &2)\ \phi^-(x,\eta) \le g^i_\eta(x) \le g^s_\eta(x) \le \phi^+(x,\eta) \\ &\quad \text{for a.e. } x \in I \text{ and } \forall \eta \in \mathrm{R} \end{aligned}$$

where $b(x,\eta) = [\phi^-(x,\eta), \phi^+(x,\eta)]$ for a.e. $x \in I$ and for every $\eta \in \mathbf{R}$, g^i_η is the *weak L^p lower limit* of the sequence $(\phi^-_h(\cdot,\eta))_{h\in\mathbf{N}}$ and g^s_η is the *weak L^p upper limit* of the sequence $(\phi^+_h(\cdot,\eta))_{h\in\mathbf{N}}$.

Proof. Assume that 1) holds. We recall that, since $\phi^-_h(x,\eta) \in b_h(x,\eta)$ then $\eta \in a_h(x, \phi^-_h(x,\eta))$. By the boundedness condition (5.2), fixed η in $\mathbf{R}$, $\phi^-_{h_{k(\eta)}}(\cdot,\eta) \rightharpoonup h^\eta(\cdot)$ weakly in L^p; noting that $\phi^-_{h_{k(\eta)}}$ and h^η are derivatives of $W^{1,p}$ functions, it follows by the condition (5.2) and the G-convergence that h^η is a solution of the inclusion $\eta \in a(x, h^\eta(x))$ a.e. in I or equivalently $h^\eta(x) \in b(x,\eta)$ a.e. in I. In particular we obtain that $h^\eta(x) \ge \phi^-(x,\eta)$ for a.e. x in I. Since this is true for every weakly-L^p convergent subsequence of $(\phi_h^-(\cdot,\eta))_{h\in\mathbf{N}}$, by (7.5) it follows that

$$g_\eta^i(x) \geq \phi^-(x,\eta)$$

for a.e. x in I. In the same way it is possible to prove the opposite inequality for g_η^s and ϕ^+ and the first part of the theorem is proved.

Conversely, let us suppose that 2) holds. We would like to show that if

$$\begin{aligned}
&h_k \to +\infty \\
&u_k \rightharpoonup u \quad \text{weakly in } L^p \\
&g_k \to g \quad \text{strongly in } L^q \\
&g_k(x) \in a_{h_k}(x, u_k'(x)) \quad \text{a.e. in } I \ \text{ and}
\end{aligned}$$

then

$$g(x) \in a(x, u'(x)) \quad \text{a.e. in } I.$$

This is equivalent to show that

$$u'(x) \in b(x, g(x)) \quad \text{a.e in } I. \tag{7.6}$$

As in the proof of theorem 6.6, we can construct a subsequence $(v_k^\eta)_{k\in\mathbf{N}}$ of $(\phi_{h_k}^-(\cdot,\eta))_{k\in\mathbf{N}}$, which weakly converges to a function $h^\eta \in L^p$, for every $\eta \in \mathbf{Q}$. The properties of the *weak L^p lower* and *upper limits* mentioned above assure that $g_\eta^i(x) \leq h^\eta(x) \leq g_\eta^s(x)$ a.e. in I, and hence by 2) $h^\eta(x) \in b(x,\eta)$ a.e. in I.

By the monotonicity, for a.e. $x \in I$, we have

$$(u_k'(x) - v_k^\eta(x), g_k(x) - \eta) \geq 0 \quad \forall \eta \in \mathbf{Q}.$$

Passing to the weak limit, we obtain that, for a.e. $x \in I$

$$(u'(x) - h^\eta(x), g(x) - \eta) \geq 0 \quad \forall \eta \in \mathbf{Q}.$$

We can now consider, for a.e. $x \in I$, the map $\gamma_x : \mathbf{Q} \to \mathbf{R}$ defined by $\gamma_x(\eta) = h^\eta(x) \in b(x, \eta)$. By theorem 4.3 γ_x has, as unique maximal monotone extension, the operator $\Gamma_x(\cdot) = b(x, \cdot)$, then it follows that $u'(x) \in b(x, g(x))$, and this holds for a.e. $x \in I$, hence (7.6) holds and the proof is complete. ∎

If $\phi_h^-(\cdot, \eta) \rightharpoonup g_\eta^i(\cdot)$ weakly in L^p and $\phi_h^+(\cdot, \eta) \rightharpoonup g_\eta^s(\cdot)$ weakly in L^p, then we say that the multivalued operator $\tilde{b}(x, \eta) := [g_\eta^i(x), g_\eta^s(x)]$ is the convex weak limit of the subsequence $(b_h(x, \eta))_{h \in \mathbf{N}} = \left([\phi_h^-(x, \eta), \phi_h^+(x, \eta)]\right)_{h \in \mathbf{N}}$, and the definition agrees with the one introduced by Arstein in [2], theorem 4.1. Therefore, in this case, the statement of the previous theorem 7.6 takes the following form:

If $\phi_h^-(\cdot, \eta) \rightharpoonup g_\eta^i(\cdot)$ weakly in L^p and $\phi_h^+(\cdot, \eta) \rightharpoonup g_\eta^s(\cdot)$ weakly in L^p, then $a_h \xrightarrow{G} a$ if and only if for every $\eta \in \mathbf{R}$ the convex weak limit in the sense of Arstein of the sequence $(b_h(\cdot, \eta))_{h \in \mathbf{N}}$ is contained in $b(\cdot, \eta)$.

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