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METHODS OF CONFORMAL SYMMETRY IN TWO DIMENSIONAL STATISTICAL MECHANICS

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METHODS OF CONFORMAL SYMMETRY IN TWO DIMENSIONAL STATISTICAL MECHANICS

Introduction

This work is intended as a general survey of methods and results of conformal symmetry is two-dimensions applied to the study of statistical mechanics models. A renewed, strong impulse in this subject has been given by Belavin, Polyakov and Zamolodchikov with their paper [1.1] in 1984, but early results trace back to old ideas of Polyakov [I.2] and Migdal [I.3] in 1970, about conformal invariance in phase transtions physics and its consequences in operator product expansions ("bootstrap equations").

Since then, and along with renormalization group approach to critical phenomena, conformal symmetry has provided many insights both in statistical mechanics and in quantum field theory, constituting a deep link between them (see for example [I.7]). An interplay with string theories has been present since the early days of dual models, but the new interest in superstring theories has inspired some major results, such as the extension to the superconformal symmetry (see [I.16]) and the manifestation of supersymmetry in a system realized in nature, the tricritical Ising model (see [I.17]). Besides that, a line of development risen from [I.1] and based on the concept of modular invariance has produced a classification of two-dimensional critical phenomena using Dynkin diagrams (see [I.5] for a complete review).

We will review here the fundamental concepts of conformal symmetry in two dimensions in their modern formulation after [I.1], along with methods developed from them for studying two-dimensional statistical systems. Such methods will concern the computation of correlation functions, the study of the operator algebra and the operator content of a given model, the determination of physically interesting functions and quantities , such as partition functions, conformal anomaly and so on. The two-dimensional Ising model will be used as a "gymnasium" in order to show applications of general results to a concrete model: even if it has some simplifying features, it is perfectly suitable to illustrate many aspects of the computations in detail. We hope to provide a wide understanding of the role played by conformal symmetry principles in two-dimensional physics.

This thesis is organized as follows:

- Chapter I is devoted to a complete review of conformal invariance in two-dimensions in the formulation of Belavin, Polyakov and Zamolodchikov. Major results, such as conformal Ward identities, differential equation for correlators, Kac formula and fusion rules for operator algebra are given, and are exhibited in the case of critical Ising model.
- Chapter II can be considered as a natural continuation of the first one, as it deals with a method for computing correlation functions as given by differential equation of chapter I: Coulomb gas representation of correlations, as developed by Dotsenko and Fateev (see [II.1]) , is explained in detail and a simple computation is given .
- Chapter III is devoted to the study of conformal symmetry in a restricted geometry, with an eye to applications in lattice statistical mechanics. In this context, modular invariance can be profitably used in order to determine the operator content of a statistical model.

A particular emphasis is given to the interpretation of the conformal anomaly as a finite size scaling correction to the free energy : in quantum field theory language, one can say that the central charge is related to the Casimir effect (see [III.5]) . As an application , the computation of the central charge for the Ising model is presented. Possibly, one can go further : this characteristic feature of finite geometry may provide a way to determine the conformal anomaly of models for which it is not known, but whose free energy is available. Work in this direction is in progress ([III.19]) .

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Chapter I : Conformal Invariance in Two Dimensions

Introduction

This chapter will be devoted to a survey of the formulation of conformal invariance in two dimensional physics as given by Belavin, Polyakov and Zamolodchikov (see [I.1]). We shall furthermore consider the Ising model as an example of application of their ideas to two dimensional statistical models.

Conformal invariance has been playing a central role in statistical mechanics since Polyakov introduced it early in 1970 (see [I.2]) and showed the main consequences it has in studying phase transitions of statistical models and also operator algebras in quantum field theories (see [I.3], [I.15]).

In short, the idea is that the scale invariance of critical phenomena may be generalized to be a local symmetry just as in the case of local gauge symmetries in quantum field theory (see [I.2] and [I.4]). This may be explained using renormalization group concepts. In fact, considering a lattice system and uniformly rescaling the lattice spacing by a factor λ, correlation functions of operators φ_i get transformed as follows

$$< \varphi_1(r_1) \dots \varphi_N(r_N) > = \prod_{i=1}^{N} \lambda^{-x_i} < \varphi_1(r'_1) \dots \varphi_N(r'_N) >$$

x_i being the scale dimensions of φ_i and $r' = \lambda \cdot r$. Assuming rotational invariance of the fixed point Hamiltonian, which is valid for isotropic models, this relation has to be true also if the lattice is rotated through some fixed angle. Therefore due to locality of renormalization group transformations, it holds also if we consider a non uniform rescaling $\lambda = \lambda(r)$ as long as this mapping corresponds locally to a dilatation and a rotation, i.e. as long as $\lambda(r)$ is a conformal mapping. In two dimensions conformal symmetry becomes an infinite symmetry, any analytic mapping $\lambda(r)$ being a conformal one: as we will see in the following, this fact has important consequences that may be enforced in a wide class of two dimensional models, either exactly solvable or not.

I.1 Conformal transformations

Let us start with the definition of a conformal transformation: this is a coordinate transformation

$$\xi_\alpha \rightarrow \eta_\alpha(\xi) \qquad \alpha = 1,\dots,D \tag{I.1}$$

such that the metric tensor gets modified in the following way:

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = \frac{\partial\xi^\gamma}{\partial\eta^\alpha}\frac{\partial\xi^\delta}{\partial\eta^\beta}\, g_{\gamma\delta} = \rho(\xi)\, g_{\alpha\beta} \tag{I.2}$$

$\rho(\xi)$ being some function of ξ^α.

For dimension D>2 the conformal group of transformation (I.1) is finite dimensional and its generators are given by translations, rotations, dilatations and special conformal transformations (see e.g. [I.2], [I.3] and[I.4]). In the case D=2 transformations (I.1) form a much bigger group: that is the group of conformal analytical transformations of a complex variable. This richer structure allows to get more information about two dimensional systems: as we will see in the following, the existence of conformal Ward identities or the possibility of getting correlation functions from some differential equation are due to the infinite dimensions of the conformal group in D=2.

For these reasons from now on we shall concentrate on the two-dimensional case, exploiting its special features in order to study conformal properties of statistical systems.

I.2 Conformal Ward Identities and the Stress Energy Tensor

One of the more important ingredient in a conformal theory is the stress energy tensor that we now introduce. Let us consider a two-dimensional conformal theory (i.e., a theory which is symmetric under (I.1)) involving some local fields $\varphi_i(\xi)$ and then perform an infinitesimal conformal transformation around the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$ $(\mu,\nu = 1,2)$:

$$\xi_\mu \to \xi_\mu - \varepsilon_\mu(\xi)$$

$$\delta g_{\mu\nu} = g'_{\mu\nu} - \delta_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu \tag{I.3}$$

Having in mind a functional formulation of our theory, we can write, for the variation under (I.3) of a generic correlation function $\langle X\rangle = \langle \varphi_1(\xi_1)...(\varphi_N)(\xi_N)\rangle$,

$$\delta_\varepsilon \langle X\rangle = 0 = \langle \delta_\varepsilon X\rangle - \langle X \delta_\varepsilon A\rangle \tag{I.4}$$

where A is the action appearing in the definition of the partition function (see [I.5]).

Relation (I.4) has the form of a Ward identity, and in fact it is called "conformal Ward identity". The stress energy tensor is introduced precisely at this point: it appears in the variation of the action with respect to the metric (see [I.6] or other standard textbooks). We have

$$\delta_\varepsilon A = \int d^2\xi \, T^{\mu\nu}(\xi) \partial_\mu \varepsilon_\nu(\xi) \tag{I.5}$$

where $T^{\mu\nu}$ denotes the stress-energy tensor. Using this last relation and denoting with $\delta_\varepsilon \varphi_i$ the variation of the fields under (I.3), we can get a more explicit form of the conformal Ward identity (I.4):

$$\sum_{k=1}^{N} \langle \varphi_1(\xi_1) \dots \delta_\varepsilon \varphi_k(\xi_k) \dots \varphi_N(\xi_N) \rangle = \int d^2\xi \, \partial_\mu \varepsilon_\nu \langle T^{\mu\nu}(\xi) \, \varphi_1(\xi_1) \dots \varphi_N(\xi_N) \rangle \tag{I.6}$$

I.3 Properties of the Stress Energy Tensor $T^{\mu\nu}$

It is well known that in a general conformal theory the stress energy tensor has to be conserved, symmetric and traceless ([I.1],[I.7]). In two dimensions these properties are better stated with the aid of complex coordinates. Therefore we put

$$\begin{cases} z = \xi^1 + i\xi^2 \\ \bar{z} = \xi^1 - i\xi^2 \end{cases} \tag{I.7}$$

and we treat them as independent coordinates. In term of these new variables the conformal group of transformations (I.1) is given by all analytic or antianalytic functions:

$$z \to \zeta(z) \qquad \bar{z} \to \bar{\zeta}(\bar{z}) \tag{I.8}$$

The stress energy tensor has only two independent components which are given by

$$\begin{cases} T_{zz} \equiv T = T_{11} - T_{22} + 2iT_{12} \\ T_{\bar{z}\bar{z}} \equiv \bar{T} = T_{11} - T_{22} - 2iT_{12} \end{cases} \tag{I.9}$$

and which can be shown to be analytic or antianalytic functions of z and $\bar{z}$ respectively:

$$\begin{cases} T = T(z) \\ \bar{T} = \bar{T}(\bar{z}) \end{cases} \tag{I.10}$$

A remarkable feature of this relation, as well as (I.8), is the complete separation between z and $\bar{z}$ dependence: even though we consider $\mathbb{C}^2$ by taking z and $\bar{z}$ as independent, the conformal group $\mathcal{G}$ of transformations (I.8) is completely factorizable into a direct product

$$\mathcal{G} = \Gamma \otimes \bar{\Gamma} \tag{I.11}$$

where $\Gamma(\bar{\Gamma})$ is given by all analytic substitution of $z(\bar{z})$. This observation allows us to concentrate on the properties of group Γ in what follows: the same results will be also valid for $\bar{\Gamma}$.

In this complex language the conformal Ward identity (I.6) transforms into

$$< \delta_\varepsilon X > = \oint_C dz\, \varepsilon(z) < T(z) X > \qquad (I.12)$$

where C is a circle surrounding all the points z_k, $k = 1,\ldots,N$, in which the fields φ_k present in X are evaluated (see also [I.9]).

We consider now conformal properties of T under infinitesimal analytic transformations

$$z \to z + \varepsilon(z) \qquad (I.13)$$

deferring the study of those of the fields φ_i to the next sections. Thanks to a theorem due to Lüscher and Mack (see [I.8] and references therein) one can show that the stress energy tensor transforms as follows:

$$\delta_\varepsilon T(z) = \varepsilon(z)\, T'(z) + 2\varepsilon'(z) T(z) + \frac{c}{12}\, \varepsilon''(z) \qquad (I.14)$$

where a prime denotes a z-derivation.

At this stage the constant c appearing in (I.14) is treated as a parameter of the theory, since it cannot be determined from first principles: the only constraint we can give on it is that

$$c > 0 \qquad (I.15)$$

under very general requirements, such as positivity and uniqueness of the vacuum state, or reality of stress energy tensor.

In order to make closer contact with quantum field theory, which is also the framework used to prove Lüscher-Mack theorem, we can introduce σ and τ coordinates by

$$\begin{cases} z = \exp(\tau + i\sigma) \\ \bar{z} = \exp(\tau - i\sigma) \qquad \tau, \sigma \in \mathbb{R}\, , \quad 0 < \sigma \leq \pi \end{cases} \qquad (I.16)$$

Considering a "radial ordering" $\mathcal{R}$ in the euclidean time τ (see [I.7]), correlation functions may be rewritten as (*)

$$< X > \;=\; < 0 \mid \mathcal{R}[\, \varphi_{j_1}(\sigma_1,\tau_1) \ldots \varphi_{j_n}(\sigma_n,\tau_n) \,] \mid 0 > \qquad \text{(I.17)}$$

and if the generator T_ε is given by

$$T_\varepsilon \;=\; \oint_{\log|z|=\tau} \varepsilon(z) T(z)\, dz$$

then we may express the variation $\delta_\varepsilon \varphi_k$ as a commutator

$$\delta_\varepsilon \varphi_k(\sigma,\tau) = [\, T_\varepsilon , \varphi_k(\sigma,\tau)] \qquad \text{(I.18)}$$

Therefore, from (I.14) we can get

$$[T_\varepsilon , T(z)] \;=\; \varepsilon(z)\, T'(z) \;+\; 2\varepsilon'(z)\, T(z) \;+\; \frac{c}{12}\varepsilon'''(z) \qquad \text{(I.19)}$$

It is very useful, at this point, to introduce operators L_n , $n \in \mathbb{Z}$, as moments of T(z) in its Laurent expansion:

$$T(z) \;=\; \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}} \qquad \text{(I.20)}$$

(and the corresponding L_n for T(z)).

It is clear that expansion (I.20) has to be understood as a formal one (**), but the important point is that these L_m turn out to be the generators of a Virasoro algebra, as we will see in a moment.

I.4 Conformal Covariance and Virasoro Algebra

Simple substitution of Laurent series (I.20) into commutation relation (I.19), giving the transformation properties of T, shows that the operators L_n, $n \in \mathbb{Z}$, satisfy the following algebra:

(*) see next section for the definition of the vacuum state

(**) it is only known that (I.20) has convergent matrix elements for $|z| = 1$, see [I.8]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\delta_{n+m,0}(n^3-n) \qquad (I.21)$$

This is a Virasoro algebra, and is the so-called "central extension" of the algebra of the differential operators

$$\ell_n = z^{n+1}\frac{d}{dz} \quad , \quad n \in \mathbb{Z} \qquad (I.22)$$

Thanks to relation (I.21) we see that the parameter c entering the transformation law of stress energy tensor is the central charge of the Virasoro algebra.

Algebra (I.21) admits the set $\{L_{-1}, L_0, L_{+1}\}$ as a subalgebra (note that for $n=\pm1,0$ the central charge term is absent): each operator generates some elementary transformation, such as translations (L_{-1}), dilations (L_0) and special conformal transformations (L_1). Using variables σ,τ introduced above we see that the operator

$$H = L_0 + \bar{L}_0 \qquad (I.23)$$

generates "time" shifts, and so it can be taken as a hamiltonian. Therefore the $|0>$ state appearing in (I.17) can be taken as the ground state of H. Requiring analyticity of T(z) as z goes to zero or to infinity, one can show that

$$L_n|0>=0 \qquad n\geq -1$$
$$L_n|0>=|\text{ new states}> \quad n\leq -2 \qquad (I.24)$$

and

$$<0|L_n=0 \quad n<1 \qquad (I.25)$$

so that the vacuum is left invariant by $L_{\pm1}$ and L_0 as expected.

Before considering conformal properties of local fields $\varphi_j(\xi)$ we quote a last result concering the stress energy tensor T. In fact its Laurent expansion, along with Virasoro algebra and relations (I.24), (I.25) allows us to compute, at least in principle, any correlation function of the form

$$<T(\xi_1)...T(\xi_N)>$$

In the case of a two points function, simple algebraic manipulations yield the following result

$$< T(z_1)\, T(z_2) > \quad = \quad \frac{c}{2(z_1 - z_2)^4} \tag{I.26}$$

which shows again that $c > 0$.

From the point of view of computation of c this is also a practical result: once we know the explicit form of stress energy tensor, in order to find c it will be sufficient to perform an Operator Product Expansion (O.P.E.) of T(z) with itself and look for the most singular part.

I.5 Primary Fields and Conformal Families

Studying the conformal properties of local fields it turns out that the most important class of fields is given by those which have the simplest possible transformation law. The finite and infinitesimal versions of this law are (*)

$$\phi_{h,\bar{h}}(z,\bar{z}) \quad \rightarrow \quad \left(\frac{dz'}{dz}\right)^{h} \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{\bar{h}} \phi_{h,\bar{h}}(z',\bar{z}') \tag{I.27}$$

and

$$\delta_{\varepsilon}\phi_h(z) \quad = \quad \varepsilon(z)\frac{\partial}{\partial z}\phi_h(z) \; + \; h\varepsilon'(z)\phi_h(z) \tag{I.28}$$

Fields satisfying (I.27) and (I.28) are called primary fields, $x \equiv h+\bar{h}$ is the anomalous scale dimensions of the primary field $\phi_{h,\bar{h}}$ and $s \equiv h-\bar{h}$ is its spin (as can be seen considering a dilatation or a rotation in (I.27)). Obviously the class of primary fields does not exhaust all possible fields that can appear in a given conformal theory: for instance, if one considers the O.P.E. of the stress energy tensor with a primary field φ_k, there already show up fields having a more complicate transformation law than (I.27), (I.28). However it can be proved that the algebra of the operator product expansion quoted above is given by the so called conformal family of the primary field φ_h (see [I.1]).

A conformal family $[\varphi_h]$ of a primary field φ_h is made up by φ_h itself (which

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(*) In the first relation we have restored for a moment both z and $\bar{z}$ dependence plays the role of an "ancestor") and by an infinite set of "secondary" or "descendant" fields which are given by

$$\varphi_h^{(-k_1,\dots,-k_N)}(z) = L_{-k_1}(z)\dots\dots L_{-k_N}(z)\,\varphi_h(z) \tag{I.29}$$

with

$$L_{-k}(z) = \oint d\zeta\, \frac{T(\zeta)}{(\zeta - z)^{k+1}} \tag{I.30}$$

and whose dimensions are

$$\Delta_h^{(-k_1,\dots\dots,-k_N)} = h + k_1 + \dots\dots + k_N \tag{I.31}$$

thus forming an integer spaced series.

As we will see in the following, conformal families will have a central part in the classification of representations of Virasoro algebra. For the time being, we limit ourselves to mention some important consequences of (I.27) and conformal Ward identity, concerning correlation functions of stress energy tensor with local fields. Let us consider for the moment a string of primary fields $X = \varphi_h(z_1)\dots\,\varphi_h(z_N)$: if we insert transformation law (I.28) into relation (I.12) and consider Cauchy representation for ε and ε', then, eliminating an integration over z, we get

$$< T(z)\,\varphi_{h_1}(z_1)\dots\varphi_{h_N}(z_N) > = \sum_{k=1}^{N} \left\{ \frac{h_k}{(z-z_k)^2} + \frac{1}{z-z_k}\frac{\partial}{\partial z_k} \right\} <\varphi_{h_1}(z_1)\dots\varphi_{h_N}(z_N)> \tag{I.32}$$

which relates a correlation function of T and φ_h with that of φ_h only, through the application of a differential operator; this is only the simplest example of such identities, and it can be generalized. In fact one can consider any correlation function of the form

$$< T(\zeta_1)\,\dots\dots\, T(\zeta_N)\,\varphi_{h_1}(z_1)\,\dots\dots\,\varphi_{h_M}(z_M) > \tag{I.33}$$

and repeatedly apply (I.32), and also any correlation function of secondary fields

$$< \varphi_{h_1}^{(k_1)}(z_1)\,\dots\dots\,\varphi_{h_N}^{(k_N)}(z_N) > \tag{I.34}$$

since they are nothing else than coefficients of an operator product expansion of $T(z)$ and a primary field φ_h . In that manner we get rather cumbersome formulas (see [I.1]), but the main feature to point out is that every correlator (I.33), (I.34) may be expressed in terms of a correlator of primary fields only, through the action of some combination of differential operators, like the one appearing in (I.32). This is a first instance of how conformal invariance severely restricts the possible form of a correlation function. However, it is not the only one: when dealing with the so called minimal conformal theories we will be able to have differential equations for every correlator.

I.6 Representations of Virasoro Algebras and Degenerate Conformal Families

First of all, let us give the transformation law (I.28) of a primary field in terms of a commutator with operators L_m: this will be useful in the following for the study of representations of Virasoro algebra. Inserting the Laurent expansion (I.20) into eq.(I.18), and expanding also the l.h.s. of (I.28) we can easily get

$$[\, L_m \,,\, \varphi_h(z)\,] \;=\; z^{m+1}\frac{\partial}{\partial z}\,\varphi_h(z) \;+\; h(m+1)z^m\varphi_h(z) \qquad \text{(I.35)}$$

Using relations like this, we can also see, from their definition (I.29), that descendant fields in a conformal family have variations $\delta_\varepsilon\varphi_h^{\{k\}}$ which may be expressed as combinations of field in the same conformal family (*). This fact suggests that conformal families correspond to various representations of the conformal algebra.

To show that, we resort to a highest weight vector (H.W.V.) construction of the representation space. Let us define primary states as

$$|\, h > = \varphi_h(0)\,|\, 0> \qquad \text{(I.36)}$$

where φ_h is a primary field.

(*) Recall that a conformal family contains also all the derivatives of its ancestor

States (I.36) are in fact highest weight vectors for the operator L_o : commutation relations (I.35) allow us to write

$$L_m |h> = 0 \quad \text{if} \quad m > 0$$

$$L_o |h> = h |h> \qquad (I.37)$$

On the other hand, application of L_m, m<0 gives new state ("descendants" of |h>), which are in one to one correspondence with fields in the conformal family $[\varphi_h]$ of φ_h: we have

$$L_{-k_1} \ldots\ldots L_{-k_m} | h> \quad = \quad \varphi_h^{\{-k_1,\ldots, -k_m\}}(0) \, |0> \quad , \quad k_i > 0 \qquad (I.38)$$

and these are eigenstates of L_0 relatively to the eigenvalues

$$\Delta_h^{(-k_1,\ldots\ldots,-k_N)} \quad = \quad h + k_1 + \ldots\ldots + k_N \qquad (I.39)$$

Therefore we argue that representations of the Virasoro algebra (I.21) correspond to some spaces $V_{(h,c)}$ (*), each of them being isomorphic to a conformal family $[\varphi_h]$: those spaces are called Verma modules.

There exist cases in which the structure of representation space becomes much simpler, being present a reduced number of states: they correspond to degenerate conformal families. It happens that for particular values of the conformal weight h, the space $V_{(h,c)}$ contains a special kind of vectors $|\xi>$: these are vectors such that

$$L_n |\xi> = 0 \quad n>0, \qquad |\xi> \in V_{(h,c)} \qquad (I.40)$$

$$L_o |\xi> = (h+M) \, |\xi> \qquad M > 0 \qquad (I.41)$$

and they are called "null vectors". A conformal family containing fields which give rise to such vectors will be called a degenerate conformal family. Let us

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(*) We keep also the suffix "c" in $V_{(h,c)}$ since the representations of the algebra (I.21) will also depend on the value of the central charge.

examine in more detail eqs.(I.40) and (I.41). The last one simply says that $|\xi>$ is not a highest weight vector for $V_{(h,c)}$: it belongs to the same level of

$\varphi_n^{\{-k \dots, -k\}}(0)|0>$, with $\Sigma \; k_i = M$. On the contrary the first one is the

characteristic condition of a highest weight vector, so that $|\xi>$ seems to be a H.W.V. without actually being that: it may be considered as a H.W.V. of "its own" Verma modulus $V_{(h+M,c)}$. Its hallmark is that (I.40) holds for it with each n>0, and not only starting from a certain N onward, like in the case of "normal" secondary states (I.38).

We shall now try to determine what are the conditions under which the phenomenon described above can take place.

From equation (I.41) we see that $|\xi>$ may be written in the form

$$|\xi> \quad = \quad L_{-k_1} \dots\dots L_{-k_N} |h> \qquad k_i > 0$$

or their linear combination. Therefore, from eq.(I.40) we get

$$\| \xi \|^2 \quad = \quad <\xi|\xi> \quad = \quad <h| L_{k_1} \dots\dots L_{k_N} |\xi> \quad = \quad 0 \qquad \text{(I.42)}$$

where we used the fact that $L_n^+ = L_{-n}$. Thus we come to the following important point: the existence of a null vector in a Verma modulus $V_{(h,c)}$ is related to the degeneracy of the metric in that space. Since a given modulus $V_{(h,c)}$ is the direct

sum of submodules $V^{(n)}_{(h,c)}$ of level n,

$$V_{(h,c)} \quad = \quad \bigoplus_{n=0}^{\infty} V^{(n)}_{(h,c)} \qquad \text{(I.43)}$$

each one with a basis

$$L_{-k_1} \dots\dots L_{-k_\nu} |h> \quad , \quad k_\nu \geq \dots\dots \geq k_1 \geq 1 \quad , \quad \sum_{i=1}^{\nu} k_i \; = \; n \qquad \text{(I.44)}$$

and such that

$$L_0 V^{(n)}_{(h,c)} = (h+n) V^{(n)}_{(h,c)} \qquad \text{(I.45)}$$

the study of the degeneracy of the metric in $V_{(h,c)}$ may be reduced to one in the submodules $V^{(n)}_{(h,c)}$ (see [I.8]). An analysis of this kind yields some constraint on the possible values of h and c. In fact, a simple application of algebra (I.21) gives

$$0 \leq \| L_{-n} |h\rangle \|^2 = \langle h| L_n L_{-n} |h\rangle = \langle h| [L_n, L_{-n}] |h\rangle =$$

$$= 2nh + \frac{c}{12} n (n^2 - 1) \qquad \forall n \geq 0 \qquad \text{(I.46)}$$

which for n=1 and $n \to \infty$ implies

$$h \geq 0$$
$$c \geq 0 \qquad \text{(I.47)}$$

As a typical (and easy) example, we may study the metric in the submodulus at level n=2: here a basis is given by

$$L_{-2}|h\rangle, L^2_{-1}|h\rangle \qquad \text{(I.48)}$$

and we can form the following matrix of inner product

$$M = \begin{pmatrix} \langle h | L_2 L_{-2} |h\rangle & \langle h| L_2 L^2_{-1} | h\rangle \\ \langle h | L^2_1 L_{-2} | h\rangle & \langle h | L^2_1 L^2_{-1} | h\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h(2h+1) \end{pmatrix} \qquad \text{(I.49)}$$

In order to be non degenerate, the matrix M has to satisfy the following condition:

$$0 \leq \det M = 2h(16h^2 - 2h(5-c) + c) \qquad \text{(I.50)}$$

with h and c positive.

This is a constraint on h and c, and we see that there may be null vectors in $V^{(n)}_{(h,c)}$ only for the following values of h

$$h = \frac{1}{16} [\, 5 - c \pm \sqrt{(c-1)(c-25)} \,] \qquad \text{(I.51)}$$

to which it corresponds

$$|\xi> \;=\; (L_{-2} + \frac{3}{2(2h+1)} L_{-1}^2)\;|h> \qquad (I.52)$$

(cfr. [I.1]). This ends the analysis for the n=2 case, which shows that there are special values of h that makes the representation $V_{(h,c)}$ a degenerate one. In fact, for h as given in (I.51), there exist null vectors in $V_{(h,c)}$ which can be considered, as one can see from (I.40) , (I.41) , the highest weight vectors of the submodules $V_{(h+M,c)}$ of $V_{(h,c)}$ (in the case of (I.52), $V_{(h+2,c)}$). Thus $V_{(h,c)}$ turns out to be reducible, but the fact that null vectors have actually zero norm (see (I.42)) allows us to recover irreducible representation by consistently putting equal to zero null vectors and all their descendants (I.38):

$$|\xi> = 0 \qquad (I.53)$$

Correspondingly the Verma modulus $V_{(h,c)}$ will contain much less states than a "normal" one, having eliminated all those belonging to $V_{(h+M,c)}$.

The study of the degeneracy of the metric in the general case, i.e. in any submodulus $V^{(n)}_{(h,c)}$, giving all the special values of h, has been completed by Kac (see [I.10]).

In the following section we shall comment on his result and study the consequences of eq.(I.53) in a conformal invariant theory.

I.7 Operator Algebras and Minimal Theories

As we already mentioned, the complete analysis of the metric in a given Verma modulus has been treated by Kac, who was also able to find a general formula for the determinants of the matrices of inner products, like the one in (I.49) (see [I.11],[I.12]). By setting them equal to zero he found also all the values of h corresponding to a degenerate conformal family: they are labelled by some positive integers p, q and are given by

$$h_{(p,q)} = h_0 + \frac{1}{4}(\alpha_+ p + \alpha_- q)^2 \qquad (I.54)$$

where

$$h_0 = \frac{1}{24}(c-1)$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \qquad (I.55)$$

Since we do not want to have negative or complex conformal dimensions $h_{(p,q)}$ in (I.54), we have to limit the possible values of c to the interval

$$0 < c \leq 1 \qquad (I.56)$$

Considering this domain, Friedan, Qiu and Shenker ([I.11],[I.12]) showed that unitarity requirements constrain much further the central charge: in fact it has to be quantized within the interval (I.56) according to the following formula

$$c = 1 - \frac{6}{m(m+1)} \qquad m \geq 2 \qquad (I.57)$$

and in consequence of that the dimensions given in (I.54) have to be parametrized as

$$h_{(p,q)} = \frac{[p(m+1) - qm]^2}{4m(m+1)} \quad , \quad 1 \leq p \leq m\text{-}1 \quad , \quad 1 \leq q \leq p \qquad \text{(I.58)}$$

Formulas (I.57) and (I.58) determine finally all the value of c and h which can give rise to degenerate representations. Degenerate primary fields, i.e. primary fields corresponding to degenerate conformal families, will be accordingly denoted as $\varphi_{(p,q)}$, their conformal weight being $h_{(p,q)}$. According to (I.53), considering one of these fields we will find some state $|\xi>$ that has to be put equal to zero and is generated by a descendant $\varphi^{\{-k_i\}}_{(p,q)}$ of $\varphi_{(p,q)}$, as we can see in (I.38). In order to be consistent, we then have to put equal to zero also the field corresponding to $|\xi>$ (hereafter called a "null field"), and this have important consequences on the correlation functions containing degenerate primary fields.

As we have seen in sect. I.5, any correlation function of secondary fields may be expressed in terms of one of primary fields only, through the action of a differential operator. Consider now a null field $\varphi^{\{-k_i\}}_{(p,q)}$ and one of its correlation functions with other primary fields φ_i: having put $\varphi^{\{-k_i\}}_{(p,q)}$ equal to zero will then yield a differential equation for the correlator of the degenerate primary field $\varphi_{(p,q)}$ with φ_i. We can take the case of (I.52) as an example. To this null vector it corresponds the null field

$$\xi = \varphi_h^{\{-2\}} + \frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} \varphi_h \qquad \text{(I.59)}$$

(see [I.1]), where, having taken the minus sign in (I.51), the degenerate field φ_h is given by $\varphi_{(1,2)}$. Given the correlator $<\xi(z)\varphi_1(z_1)>$ with another primary field φ_1, we have the following differential equation for $<\varphi_{(1,2)}(z)\,\varphi_1(z_1)>$:

$$\left[\frac{3}{2(2h+1)} \frac{\partial^2}{\partial z^2} - \frac{h_1}{(z-z_1)^2} - \frac{1}{z-z_1} \frac{\partial}{\partial z_1} \right] < \varphi_{(1,2)}(z)\,\varphi_1(z_1) > \; = 0 \qquad \text{(I.60)}$$

where we have used the fact that

$$< \varphi^{\{-2\}}_{(1,2)}(z)\,\varphi_1(z_1) > \; = \left[-\frac{h_1}{(z-z_1)^2} - \frac{1}{z-z_1} \frac{\partial}{\partial z_1} \right] < \varphi_{(1,2)}(z)\,\varphi_1(z_1) >$$

Equations like (I.60) are extremely useful when computing actual correlation functions and this aspect makes the case of degenerate conformal families a very interesting one. Anyway, this is not their only benefit. In fact these differential equations constrain also the operator algebra of degenerate primary fields. If we consider the O.P.E. of a degenerate field with another primary field φ_k, it turns out that operators in the expansion cannot have arbitrary conformal weights: these are determined by the differential equation concerning the degenerate field we are dealing with.

For these reasons it is worthwhile to examine the so called "minimal models" in which all primary fields are degenerate. Actually, it comes out that the parametrization of c and $h_{(p,q)}$ given in (I.57) and (I.58) corresponds to such models, in which every degenerate representation has an infinite number of null vectors, in different levels. Putting equal to zero each of them, the infinite equations we get give rise to an operator algebra involving a finite number of degenerate conformal families, whose weights are given by (I.58).

This results may be summarized in the so called "fusion rules" for degenerate fields, which determine the operator algebra:

$$\psi_{(p_1,q_1)}\,\varphi_{(p_2,q_2)} = \sum_{k=|p_1-p_2|+1}^{p_1+p_2-1}\ \sum_{j=|q_1-q_2|+1}^{q_1+q_2-1} [\,\phi_{(k,j)}\,] \tag{I.61}$$

where k(j) is even if p_1+p_2, (q_1+q_1) is odd and viceversa.

Eq. (I.61) is written in a very concise form: the notation $[\Phi_{(p,q)}]$ in the r.h.s. shows which conformal family appears in the sum, rather than the precise form of the expansion and the coefficients. A remarkable feature to point out is that those minimal theories correspond to actual two-dimensional models. In the next section we shall see the Ising models as a simple example, but is by now very well known (see [I.11],) that the series (I.57), (I.58) gives a lot of statistical models, which can then be studied with the methods of conformal symmetry.

I.8 Ising Model : an Example

In this section the Ising model will be treated as an example to which we can apply the results found in the previous paragraphs. Commenting on it, we shall also try to get some new insight into the structure of conformal theories.

It is very well known (see [I.13], [I.14]) that the Ising model has also a fermion formulation: at the critical point in the continuum limit it is described by the lagrangian of a free Majorana fermion with zero mass

$$\mathcal{L} = \frac{1}{2}\psi\,\partial_{\bar{z}}\psi + \frac{1}{2}\bar{\psi}\,\partial_z\bar{\psi} \tag{I.62}$$

where we have used complex variables. Integrating over z,z we consider the action

$$S = \int dz\, d\bar{z}\, \mathcal{L}(\psi,\bar{\psi}) \tag{I.63}$$

and then we perform a conformal transformation

$$z \rightarrow \zeta(z) \tag{I.64}$$

Since we have

$$\begin{cases} d\zeta = \frac{d\zeta}{dz}\, dz \\ \frac{\partial}{\partial\zeta} = \frac{dz}{d\zeta}\frac{\partial}{\partial z} \end{cases} \tag{I.65}$$

and complex conjugate relations, the transformed action is

$$S' = \frac{1}{2}\int dz\, d\bar{z}\ \left(\psi' \frac{d\zeta}{dz}\partial_{\bar{z}}\psi' + \bar{\psi}'\frac{d\bar{\zeta}}{d\bar{z}}\partial_z\bar{\psi}' \right)$$

and is equal to the old one if and only if

$$\begin{cases} \psi'(\zeta)\left(\frac{d\zeta}{dz}\right)^{1/2} = \psi(z) \\ \bar{\psi}'(\bar{\zeta})\left(\frac{d\bar{\zeta}}{d\bar{z}}\right)^{1/2} = \bar{\psi}(\bar{z}) \end{cases} \tag{I.66}$$

from which we see the conformal weights of primary fields ψ and ψ:

$$\begin{aligned} \psi &\rightarrow h = 1/2\,; & \bar{h} = 0 \\ \bar{\psi} &\rightarrow h = 0\,; & \bar{h} = 1/2 \end{aligned} \tag{I.67}$$

Consistently with the value of spin ($h-\bar{h} = \pm\, 1/2$). As a consequence of conformal symmetry we have that the two point function of primary fields must be of the following form:

$$< \varphi_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\, \varphi_{h_2,\bar{h}_2}(z_2,\bar{z}_2) > \ = \ \delta_{h_1,h_2}\, \delta_{\bar{h}_1,\bar{h}_2}\, (z_1 - z_2)^{-2h_1}\, (\bar{z}_1 - \bar{z}_2)^{-2\bar{h}_2} \tag{I.68}$$

and therefore we get for ψ and ψ

$$< \psi(z)\,\psi(z') > \ = \ \frac{1}{z - z'} \qquad\qquad < \bar{\psi}(\bar{z})\,\bar{\psi}(\bar{z}') > \ = \ \frac{1}{\bar{z} - \bar{z}'}$$

$$< \psi(z)\,\bar{\psi}(\bar{z}') > \ = \ < \bar{\psi}(\bar{z})\,\psi(z') > \ = \ 0 \tag{I.69}$$

Let us consider now the stress tensor as given by the Noether theorem

$$T(z) = \frac{\delta\mathcal{L}}{\delta(\partial_{\bar{z}}\psi)}\partial_z\psi = -\frac{1}{2} :\psi(z)\,\partial_z\,\psi(z): \tag{I.70}$$

and taken in normal ordered form. Performing the operator product expansion with the aid of Wick's theorem, we find that

$$<T(z)T(z')> = (1/4)[<\psi(z)\partial_{z'}\psi(z')> < \partial_z\psi(z)\,\psi(z')> - <\psi(z)\,\psi(z')> < \partial_z\psi(z)\partial_{z'}\psi(z')>] \tag{I.71}$$

and, thanks to (I.69)

$$< T(z)\, T(z') > \quad = \quad \frac{1}{4(z-z')^4} \tag{I.72}$$

Comparing this result with (I.26), we deduce that for the Ising model

$$c_{Ising} = 1/2 \tag{I.73}$$

We can then start studying the operator algebra for this model: inserting (I.73) into (I.57) we see that it corresponds to m=3 in the series of Friedan, Qiu and Shenker. Hence, the allowed conformal weights are given by (I.58)

$$h_{(p,q)} \quad = \quad \frac{(4p-3q)^2 - 1}{48} \qquad \text{with} \qquad 1 \leq p \leq 2 \;\; , \;\; 1 \leq q \leq p \tag{I.74}$$

and may be plotted in a p,q diagram to form the so-called "conformal grid":

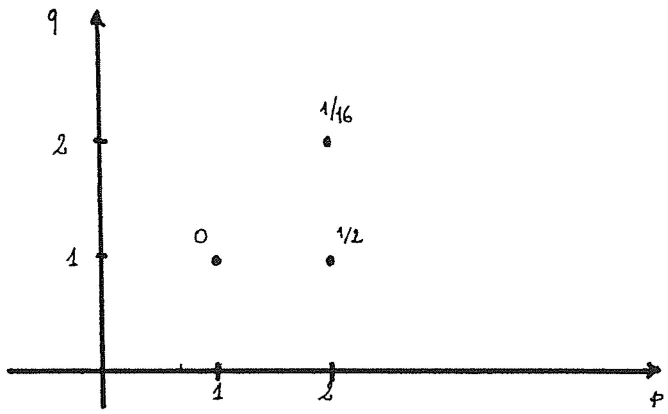

Fig. 1: Conformal grid for the Ising model: the conformal weight is shown associated with each point

These diagrams are a synthetic way to exhibit all the conformal weights allowed by (I.58) in a given model: a degenerate primary operator will correspond to each of them, but this correspondence is not properly biunique.

In fact, the values shown in fig. 1 are also reproduced by operators with indices (p,q) given by (2,3) for h = 0, (1.3) for h = 1/2 and (1,2) for h=1/16, so that we have a doubling of the operators of given conformal weight. However, as pointed out in [I.9], the possible ambiguities arising from this fact are only apparent: on the contrary it can be profitably used in determining the operator algebra (see [I.1]). With this remark in mind we can identify the operators appearing in our model: in order to do that we have to take into account also the $\bar{z}$ dependence, neglected till now. This produces conformal weights $\bar{h}_{(p,q)}$ given by the same formula (I.74), and therefore conformal fields will be denoted as $\varphi_{(h,\bar{h})}$ rather than $\varphi_{(p,q)}$. Comparing scale dimensions $x=h+\bar{h}$ of each field with known results about Ising model, we see that the operator algebra involves

$$\varphi_{(0,0)} = I$$

$$\varphi_{(1/2,1/2)} = \varepsilon$$

$$\varphi_{(1/16,1/16)} = \sigma \qquad \text{(I.75)}$$

where I is the identity operator, ε the energy density and σ the magnetisation. Now, a simple application of (I.61) will give the operator product relations: the operator algebra for the Ising model is then summarized as follows

$$\begin{array}{lll} I\cdot\varepsilon = [\varepsilon] & \quad & \varepsilon\cdot\varepsilon = [I] \\ I\cdot\sigma = [\sigma] & & \varepsilon\cdot\sigma = [\sigma] \\ I\cdot I = [I] & & \sigma\cdot\sigma = [I]+[\varepsilon] \end{array} \qquad \text{(I.76)}$$

This ends our analysis of Ising model: we have seen how conformal symmetry may be used to determine the operator content of a given model.

In the next chapter we shall dwell further on this aspect and also we shall consider some application to the study of correlation functions.

* * *

Chapter II : "Coulomb Gas" Approach to Conformal Symmetry

Introduction

The formulation of conformal invariance as given by Belavin, Polyakov and Zamolodchikiov in their work [I.1] has been a major breakthrough in the study of two-dimensional statistical systems. Not only it recovers known results, but it also provides a unified framework for various classes of statistical models, which allow new computations. One of the most important results concerns correlation functions, namely the possibility of getting them from some differential equation: this and the study of the operator algebra of a given model yield also new relations for critical indices (see e.g. [II.2]).

In this chapter we shall review a general technique for getting, with conformal invariance methods, correlation functions based on an integral representation for them. The integral representation stems from the study of a Coulomb gas model in two-dimensions, and is therefore called " Coulomb gas representation".

We shall firstly illustrate some features of Coulomb gas model in two-dimensions: besides being another example of applications of conformal symmetry to an actual model, its importance arises from the fact that many statistical models in two-dimensions may be led back to it, so that it is a good guide for their study (see e.g. [II.3] for a review on this topic). Since Dotsenko and Fateev have been the first to stress the importance of Coulomb gas representation for correlation functions in the context of conformal symmetry, the main reference throughout this chapter will be to their original work, [II.1].

II.1 Coulomb Gas Model as a Free Field Theory

In this section we shall define this model as a free field theory in two dimensions.

Let us consider a neutral scalar field without mass , φ , described by a partition function

$$Z = \int \mathcal{D}\varphi \;\; \exp(-A[\varphi]) \tag{II.1}$$

with the following action

$$A[\varphi] \quad = \quad \frac{1}{4} \int \; dz\, d\bar{z} \;\; \partial_z \varphi \; \partial_{\bar{z}} \varphi \tag{II.2}$$

where we have employed, as usual, complex variables z , $\bar{z}$. As one can see from the equation of motion for φ , this is not a primary field of our model : in fact its two points function reads as follows

$$< \varphi(z)\, \varphi(z') > \quad \sim \quad \log \frac{1}{| z - z' |} \tag{II.3}$$

Comparison with (I.68) shows that our basic field is not primary. On the contrary , we may consider exponentials of it as primary fields of the Coulomb gas model. Let us define a "vertex operator" $V_\alpha (z,\bar{z})$ as

$$V_\alpha (z,\bar{z}) \quad = \quad : \exp [i\alpha\, \varphi (z,\bar{z})] : \tag{II.4}$$

where the parameter α is usually called the charge of the vertex V_α .

For the correlator of two vertices we then get the following result

$$< V_\alpha(z)\; V_{-\alpha}(z') > \quad \sim \quad \frac{1}{| z - z' |^{4\alpha^2}} \tag{II.5}$$

which shows that V_α is a primary fields with weights

$$h_\alpha \;\; = \;\; \bar{h}_\alpha \;\; = \;\; \alpha^2 \tag{II.6}$$

This , in turn , determines also the transformations properties of vertex operators under a conformal change of coordinates $z \rightarrow f(z)$: we have

$$V_\alpha(z,\bar{z}) \quad \rightarrow \quad (\frac{df}{dz})^{\alpha^2} \, (\frac{d\bar{f}}{d\bar{z}})^{\alpha^2} \; V_\alpha(f(z),\bar{f}(\bar{z})) \tag{II.7}$$

Let us comment shortly on (II.5) . The combination $\pm\,\alpha$ for the charges in this formula is not an accident : actually in order to get a result different from zero in any correlation function , vertex fields V_α appearing in it have to satisfy the following neutrality condition :

$$\sum_i \alpha_i = 0 \tag{II.8}$$

We now examine what happens if (II.8) is not true : put the model in a box of size R and then evaluate $< V_\alpha \, V_{\alpha'} >$, $\alpha + \alpha' \neq 0$, by usual functional methods.

Considering the exact expression for the two point function (cfr. [II.1])

$$< \varphi(z)\, \varphi(z') > \quad = \quad 4 \log \frac{R}{| z - z' |} \tag{II.9}$$

it turns out that

$$< V_\alpha(z) \; V_{\alpha'}(z') > \quad = \quad \frac{\int \mathcal{D}[\varphi] \; e^{i\alpha\,\varphi(z)} \; e^{i\alpha'\varphi(z')} \; e^{-A[\varphi]}}{Z} \quad =$$

$$\sim \quad \frac{R^{-2(\alpha+\alpha')}}{| z - z' |^{-4\alpha\alpha'}} \tag{II.10}$$

Therefore the correlator vanishes in the limit $R \to \infty$, unless condition (II.8) holds , giving $\alpha + \alpha' = 0$.

Having determined what are the primary fields of our model and their conformal properties, let us proceed to consider the stress energy tensor and the computation of the central charge. As usual, in what follows we shall neglect the $\bar{z}$ dependence in order to keep the notation as simple as possible.

An application of the Noether theorem gives the relevant component of the stress energy tensor

$$T(z) \equiv T_{zz} = - \frac{\delta \mathcal{L}}{\delta(\partial_{\bar{z}}\varphi)} \partial_z\varphi = -\frac{1}{4} :\partial_z\varphi \; \partial_z\varphi: \tag{II.11}$$

which is an analytic function of z .

From the Wick theorem we then get

$$< T(z)\, T(z') > \quad = \quad \frac{1}{16} < :(\partial_z \varphi)^2: \; :(\partial_{z'}\varphi)^2: > \quad =$$

$$= \quad \frac{1}{2(z - z')^4} \qquad \text{(II.12)}$$

and this yields finally

$$c_{\text{Coulomb}} = 1 \qquad \text{(II.13)}$$

Formula (II.13) shows that the Coulomb gas model may be obtained from the series (I.57) as the limiting case $m = \infty$. Therefore , if we want to employ it in the study of other models of the series of Friedan, Qiu and Shenker , we have to find a way to shift the value of the central charge away from that of (II.13) .

We will see in the following section that this is possible by modifying , in a suitable way , the boundary conditions at infinity of the field involved in the model.

II.2 Modified Coulomb Gas Model : a Charge at Infinity

In this section we shall define a Coulomb gas model with a conformal anomaly different from one . In order to do that we have to introduce a charge $-2\alpha_0$ placed at infinity : as we'll see this results in a change of boundary conditions of $\varphi(z)$.

A first consequence of this is that neutrality condition in the form of (II.8) is no longer valid , but it has to be replaced by the following relation

$$\sum_i \alpha_i \quad = \quad 2\alpha_0 \qquad \text{(II.14)}$$

and only those fields satisfying (II.14) can have correlation functions different from zero . As an example we can consider the two points function given in (II.10) : it becomes

$$< V_\alpha(z)\; V_{2\alpha_0-\alpha}(z') > \quad \sim \quad \frac{1}{(z - z')^{2\alpha(\alpha-2\alpha_0)}} \qquad \text{(II.15)}$$

from which we see that conformal weights of the two fields V_α and $V_{2\alpha_0-\alpha}$ are the same

$$h_\alpha = h_{2\alpha_0-\alpha} = \alpha^2 - 2\alpha\alpha_0 \tag{II.16}$$

in such a way that V_α and $V_{2\alpha_0-\alpha}$ can be considered as a sort of conjugate fields.

By virtue of (II.16) we see that also transformation law (II.5) gets modified : since V_α is a primary field in our model , (II.16) implies that performing a conformal transformation $z \to f(z)$ we must have

$$V_\alpha(z) \to (\frac{df}{dz})^{\alpha^2-2\alpha\alpha_0} V_\alpha(f(z)) \tag{II.17}$$

Note that in order to get this result we have to change also the transformation properties of the scalar field φ . This is not a primary field , and we can recover (II.7) simply supposing that

$$\varphi(z) \to \varphi(f(z)) \tag{II.18}$$

under $z \to f(z)$.

The case of (II.17) is slightly less simple : we have to require that

$$\varphi(z) \to \varphi(f(z)) + 2i\alpha_0 \log[f'(z)] \sim$$

$$\sim j(z) + \varepsilon(z)\partial_z\varphi(z) + 2i\alpha_0\varepsilon'(z) \tag{II.19}$$

in order to reproduce it. In fact , using (II.19) and (II.7), we have , under $z \to f(z)$,

$$V_\alpha(z) = :e^{i\alpha\varphi(z)}: \to (\frac{df}{dz})^{\alpha^2} :e^{i\alpha[\varphi(f(z)) + 2i\alpha_0 \log f'(z)]}: =$$

$$= (\frac{df}{dz})^{\alpha^2-2\alpha\alpha_0} :e^{i\alpha\varphi(f(z))}: \tag{II.20}$$

which finally yields (II.17) .

The main consequence of (II.19) is that it modifies the boundary conditions at infinity of the field φ with the appearance of $2i\alpha_0\log[f'(z)]$, and this fact give rise to a sort of improvement of the stress energy tensor , in which it appears a new term.

In order to see how it can happen, and to determine this new contribution, we have to refer back to the very definition of the stress energy tensor in a lagrangian field theory (see [II.14]). Actually, if we consider the variation of our action (II.2) under the infinitesimal version of (II.19), we find that it is given by two contributions :

$$\delta A = I_1 + 2i\alpha_0 I_2 \tag{II.21}$$

with

$$I_1 = \int d\zeta\, d\bar{\zeta}\, \left[\frac{\delta \mathcal{L}}{\delta(\partial_\zeta \varphi)} \partial_\zeta(\varepsilon \partial_\zeta \varphi) + \frac{\delta \mathcal{L}}{\delta(\partial_{\bar{\zeta}} \varphi)} \partial_{\bar{\zeta}}(\varepsilon \partial_\zeta \varphi)\right] \tag{II.22}$$

$$I_2 = \int d\zeta\, d\bar{\zeta}\, \left[\frac{\delta \mathcal{L}}{\delta(\partial_\zeta \varphi)} \partial_\zeta \varepsilon' + \frac{\delta \mathcal{L}}{\delta(\partial_{\bar{\zeta}} \varphi)} \partial_{\bar{\zeta}} \varepsilon'\right] \tag{II.23}$$

Clearly the term given by (II.22) concerns the usual relation (II.18), while the second one is related to the introduction of the charge at infinity : if it was zero, then I_1 would yield the conservation of a stress energy tensor as given by (II.11). Now by evaluating the functional derivatives of the lagrangian in (II.2) and performing some integrations by parts under suitable hypoteses on the function ε (which is always possible, since ε is arbitrary) we get for I_2

$$I_2 = \frac{1}{2} \int d\zeta\, d\bar{\zeta}\, [\partial_{\bar{\zeta}}(\partial_\zeta^2 \varphi)]\, \varepsilon \tag{II.24}$$

Combining this with the usual result for I_1, and imposing

$$\delta A = 0 \tag{II.25}$$

according to the conformal symmetry of the theory, we find that the stress energy tensor gets modified as follows :

$$T(z) = (-1/4) :\partial_z\varphi\, \partial_z\varphi: + i\alpha_0\, \partial_z{}^2\varphi \tag{II.26}$$

in consequence of the presence of the charge at infinity.

Having worked out the form of the stress energy tensor for the modified Coulomb gas model, we are ready to compute the value of the central charge for it. As usual we are interested in the short distance expansion of the product $T(z) . T(z')$, whose most divergent part is given by the terms

$$(1/16) :\partial_z\varphi\ \partial_z\varphi : \ :\partial_{z'}\varphi\ \partial_{z'}\varphi: \tag{II.27}$$

and

$$-\alpha_0^2 :\partial_z^2\varphi : \ :\partial_{z'}^2\varphi : \tag{II.28}$$

As in the previous computation , (II.27) yields the old result (II.13) , while the term (II.28) shifts the central charge away from one . Contracting fields in (II.28) with Wick theorem and performing derivatives , we end up with the following contribution to $< T(z)\, T(z') >$:

$$\frac{-12\alpha_0^2}{(z-z')^4} \tag{II.29}$$

Therefore the total conformal anomaly for the Coulomb gas model with a charge at infinity is given by

$$c_{\alpha_0 \neq 0} = 1 - 24\alpha_0^2 \tag{II.30}$$

This is the last result for this section : in the next one we shall see , while introducing integral representations for conformal correlators , how to recover the series of conformal weights (I.58) with this parametrization of the conformal anomaly.

II.3 Screening Operators and Integral Representation for Correlation Functions.

In this section we shall show how the Coulomb gas model can be used for the computation of the correlation functions : as a result we find that they can be expressed as integrals of complex variables .

In order to keep contact with actual statistical models we work with a charge $-2\alpha_0$ at infinity and we make some requirements on the general properties of operators V_α of the Coulomb model , so that they can be identified with physical operators of the statistical system at hand each time.

Let us start by considering a four-points correlator of some operator φ :

$$< \varphi(z_1)\ \varphi(z_2)\ \varphi(z_3)\ \varphi(z_4) > \qquad \text{(II.31)}$$

We want to represent it as a correlation function of four vertex fields

$$< V_{\alpha_1}\ V_{\alpha_2}\ V_{\alpha_3}\ V_{\alpha_4} > \qquad \text{(II.32)}$$

which have to be suitably chosen according to the following requirements :

i) the correlator (II.31) must be non vanishing ;

ii) all the vertex fields must have the same conformal weight in order to represent the operator φ with each of them.

Since we are considering a charge $-2\alpha_0$ at infinity , condition i) implies simply that charges in (II.32) have to satisfy the neutrality condition (II.14) , while the second one may be enforced only by using either V_α or $V_{2\alpha_0-\alpha}$ in (II.32) . In so doing we can easily see that any combination of V_α and $V_{2\alpha_0-\alpha}$ in (II.32) gives rise to a vanishing result.

A way to solve this problem could be constituted by the insertion in (II.32) of an operator that do not spoil the conformal properties of the correlation function and is able to cancel the exceeding charge carried by V_α and $V_{2\alpha_0-\alpha}$, so that we can recover the neutrality condition (II.14) . We will call such an operator a "screening operator ".

One of the properties of a screening operator is that it must have a zero conformal weight , in order to be a conformal invariant object : therefore it has to be proportional to the identity operator and in our Coulomb gas model this is represented by $V_{2\alpha_0}$ and $V_{\alpha=0}$. Unfortunately these two verices are not suitable for our aims , so that we are left with the possibility of considering an integral operator of the following form :

$$Q = \oint_C dz\ \ J(z) \qquad \text{(II.33)}$$

with C being some curve in the complex plane .

Clearly the operator J(z) in (II.33) must have $h = 1$, if we we want Q to be conformal invariant.

Since the only primary fields we have at hand are the usual vertex fields V_α , we have to impose the conformal invariance condition on their conformal weights in order to see if an operator like Q actually exists or not. Indeed the equation

$$h_\alpha = \alpha^2 - 2\alpha\alpha_0 = 1 \tag{II.34}$$

(cfr. (II.16)) has two solutions

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1} \tag{II.32}$$

so that we can build up two screening operators

$$Q_\pm = \oint_C dz \; V_{\alpha_\pm}(z) \tag{II.36}$$

with all the required properties.

Thus operators $Q_\pm$ can be inserted in any correlation function (II.32) without affecting its conformal behaviour but altering the balance of the charges.

Actually there is one more condition that vertices V_α have to fulfill if we want to represent correlators (II.31) with (II.32) : this is the quantization of their charge , which yields in turn a parametrization of the conformal weights analogous to that found in (I.54) when computing Kac determinants . We'll see it as follows.

Let us consider the four-points correlator

$$< V_\alpha V_\alpha V_\alpha V_{2\alpha_0 - \alpha} > \tag{II.37}$$

which has a charge surplus given by

$$\sum_i \alpha_i - 2\alpha_0 = 2\alpha \tag{II.38}$$

We can easily realize that this can be cancelled by the insertion of a certain number of Q_+ and Q_- operators if and only if the charge α is quantized according to

$$\alpha_{(p,q)} = \frac{[(1-p)\alpha_+ + (1-q)\alpha_-]}{2} \tag{II.39}$$

with p and q being some positive integers.

From this we learn that we can use only vertices $V_{\alpha_{(p,q)}}$ for our purposes : the magic is that these vertex fields have exactly the same conformal weights as those of primary fields in a degenerate theory as examined in the first chapter.

In fact , evaluating $h_{\alpha_{(p,q)}}$ for $V_{\alpha_{(p,q)}}$ we find

$$h_{\alpha_{(p,q)}} = \alpha^2_{(p,q)} - 2\alpha_{(p,q)}\alpha_0 = \frac{[(p\alpha_+ + q\alpha_-)^2 - (\alpha_+ + \alpha_-)^2]}{4} \tag{II.40}$$

which , after short manipulations and taking into account (II.30) , yields precisely (I.54) .

We can therefore conclude that the Coulomb gas model may be used very profitably to study correlation functions of primary fields in minimal models , by reperesenting each field with a vertex of appropriate conformal weight (II.40) . Since we have to insert screening operator $Q_\pm$ in order to get a non vanishing result , we will express correlators as integrals of complex variables : for example , referring back to the notation of the first chapter , we can have

$$< \varphi_{(p,q)}(z_1)\, \varphi_{(p,q)}(z_2)\, \varphi_{(p,q)}(z_3)\, \varphi_{(p,q)}(z_4) > \sim$$

$$\sim \oint_{C_1} du_1 \ldots \oint_{C_n} du_n \oint_{S_1} dv_1 \ldots \oint_{S_n} dv_n < V_{\alpha_{(p,q)}}(z_1) V_{\alpha_{(p,q)}}(z_2) V_{\alpha_{(p,q)}}(z_3) V_{\alpha_{(p,q)}}(z_4) \times$$

$$\times V_{\alpha_-}(u_1) \ldots V_{\alpha_+}(v_n) > \tag{II.41}$$

On the other hand , correlations functions of primary fields satisfy certain differential equations , so that what we are doing with (II.41) is just to give the integral representation of the solution of an equation like the one given in (I.60).The usefulness of relations like (II.41) lies in the fact that correlators of vertex operators are indeed computable with standard methods as those giving (II.5) , (II.10) or (II.15).

Let us consider a particular correlation function as an example of what we have said till now : look at

$$< \varphi_{(p,q)}(z_1)\ \varphi_{(1,2)}(z_2)\ \varphi_{(1,2)}(z_3)\ \varphi_{(p,q)}(z_4) > \quad \equiv \quad < \phi > \tag{II.42}$$

According to (II.41) this is given by

$$< \phi > \quad = \oint_C dv\ < V_{\alpha_{(p,q)}}(z_1)\ V_{\alpha_{(1,2)}}(z_2)\ V_{\alpha_{(1,2)}}(z_3)\ V_{\alpha_{(p,q)}}(z_4)\ V_{\alpha_-}(v) > \tag{II.43}$$

The multipoint vertex correlator reads as

$$< V_{\alpha_1}(z_1)\ V_{\alpha_2}(z_2)\ V_{\alpha_3}(z_3)\ V_{\alpha_4}(z_4)\ V_{\alpha_5}(z_5) > \ \sim \ \prod_{i<j} (z_i - z_j)^{2\alpha_i\alpha_j} \tag{II.44}$$

(cfr. [II.1]) , and then we get

$$< \phi > \ \sim \ z^{2\alpha_{(1,2)}\alpha_{(p,q)}} (1 - z)^{2\alpha^2_{(1,2)}} \oint_C dv\ v^{2\alpha_+\alpha_{(p,q)}} (v - 1)^{2\alpha_+\alpha_{(1,2)}} (v - z)^{2\alpha_+\alpha_{(1,2)}} \tag{II.45}$$

after having performed a projective transformation $z_1 \to 0$, $z_2 \to z$, $z_3 \to 1$, $z_4 \to \infty$.

There are two independent ways to choose the path C on which perform the integration (either from 0 to z or from 1 to ∞), and this correspond to consider one of the two independent solutions of the hypergeometric equation , both expressed in integral form.

The integrals we obtain are

$$I_1 = \int_1^\infty dv\ v^a (v - 1)^b (v - z)^c = \frac{\Gamma(b+1)\,\Gamma(-a-b-c-1)}{\Gamma(-a-c)}\ F(-c, -a-b-c-1; -a-c; z) \tag{II.46}$$

and

$$I_2 = \int_0^z dv\, v^a (1-v)^b (z-v)^c = z^{1+a+c} \frac{\Gamma(a+1)\Gamma(c+1)}{\Gamma(a+c+2)} F(-b,a+1;a+c+2;z) \qquad \text{(II.47)}$$

with F being the hypergeometric function.

Thus $< \phi >$ may be given a a combination of hypergeometric functions, and therefore it has all their singularity problems.

In the next section we shall consider them, and in particular we shall study the single valuedness of conformal correlators : as a result we shall see that monodromy properties of correlation functions will help us in their actual computation.

II.4 Monodromy Properties of Conformal Correlators

In chapter one we have seen that conformal symmetry allows us to get correlation functions of primary fields involved in a statistical model as solutions of certain complex differential equations . We have then to expect that such functions will be affected by some polydromy problems , as it happens , in general , in complex differential equations' theory (see [II.5] or any other standard textbook) . The Coulomb gas approach provides a straight and neat way to handle those problems : we shall treat them in a particular example and their study will allow us to find the explicit form of a correlator , up to some constant.

Let us consider the following correlation function

$$< \varphi_{(p_1,q_1)}(0)\ \varphi_{(1,2)}(z)\ \varphi_{(p_3,q_3)}(1)\ \varphi_{(p_4,q_4)}(\infty) > \qquad \text{(II.48)}$$

which , according to what we have found in the last section , is represented by the integral

$$\int_C dt \quad < V_{\alpha_1}(0)\ V_{\alpha_2}(z)\ V_{\alpha_3}(1)\ V_{\alpha_4}(\infty)\ V_{\alpha_+}(t) > \quad =$$

$$= z^{2\alpha_1\alpha_2} (1-z)^{2\alpha_2\alpha_3} \int_C dt\ \ t^a (t-1)^b (t-z)^c \qquad \text{(II.49)}$$

where $\alpha_i = \alpha_{(p_i,q_i)}$ are given by (II.39) , $\alpha_2 = \alpha_{(1,2)}$, and , according to (II.44) ,

$$a = 2\alpha_1\alpha_+ \quad , \quad b = 2\alpha_3\alpha_+ \quad , \quad c = 2\alpha_2\alpha_+ \tag{II.50}$$

As before there are two independent choices for the path C , given in fig. 1 , and the integrals we have to consider are exactly those in (II.46) , (II.47) (with a , b , and c as in (II.50)) . The correlator (II.48) is then expressed as a linear combination of the two integrals , but if we want to consider a physical correlation function we have to restore also the $\bar{z}$ dependence , neglected till now. By the factorization property of the conformal group outlined in the first chapter , this simply amounts to multiply the results obtained with their complex conjugate. Therefore the physical correlator we are interested in may be written as

$$G(z,\bar{z}) = \sum_{i,j=1}^{2} X_{ij} \; I_i(z) \; \overline{I_j(z)} \tag{II.51}$$

with X_{ij} being some coefficients.

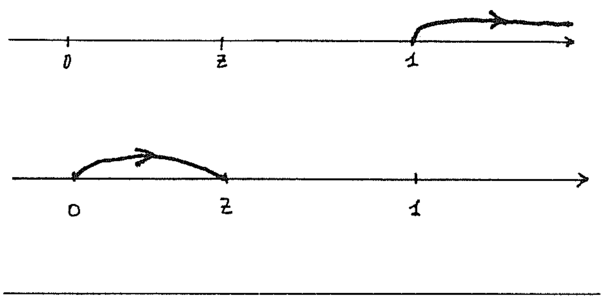

fig. 1 : independent paths of integration in (II.49)

A characteristic feature of any physical function is that it has to be single valued and we'll make it sure by working out appropriate conditions on (II.51).

We know that I_1 and I_2 have singularities in 0 , 1 and ∞ : if we perform an analytic continuation of them along some path C_0 or C_1 surrounding 0 or 1 , they will in general change their values . The new where $\alpha_i = \alpha_{(p,q)}$ are given by (II.39) , $\alpha_2 = \alpha_{(1,2)}$, and , according to (II.44) ,

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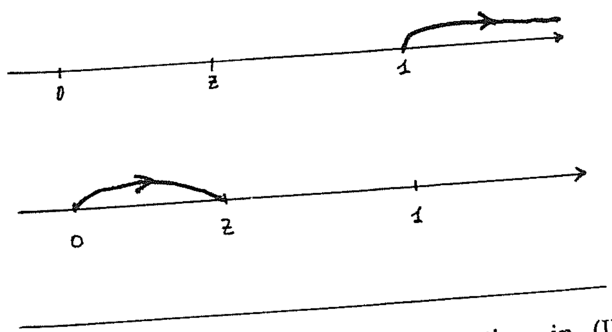

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A characteristic feature of any physical function is that it has to be single valued and we'll make it sure by working out appropriate conditions on (II.51).

We know that I_1 and I_2 have singularities in 0 , 1 and ∞ : if we perform an analytic continuation of them along some path C_0 or C_1 surrounding 0 or 1 , they will in general change their values . The new functions we get , however , are still solutions of the same differential equation and so they are a linear combination of the old ones . In short, we may summarize the situation as follows :

i) if we continue along C_0 , then $I_i(z)$ is transformed into

$$\tilde{I}_i(z) = \sum_{j=1}^{2} g^0_{ij} \, I_j(z) \qquad \text{(II.52)}$$

ii) if we continue along C_1 then $I_i(z)$ becomes

$$\tilde{\tilde{I}}_i = \sum_{j=1}^{2} g^1_{ij} \, I_j(z) \qquad \text{(II.53)}$$

(see fig. 2).

Matrices g^0 , g^1 in (II.52) , (II.53) are called " monodromy matrices " and generate the monodromy group .

In our case we can determine g^0 by direct inspection of (II.46) and (II.47) : continuing around 0 we see that I_1 remains invariant , while I_2 acquires a phase factor $e^{2i\pi(1+a+c)}$. Therefore g^0 is given by :

$$g^0 = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\pi(1+a+c)} \end{pmatrix} \qquad \text{(II.54)}$$

This diagonal form forces $G(z,\bar{z})$ in (II.51 to be diagonal too , if we want to mantain it single valued :

$$G(z,\bar{z}) = X_1 I_1(z) \overline{I_1(z)} + X_2 I_2(z) \overline{I_2(z)} \qquad \text{(II.55)}$$

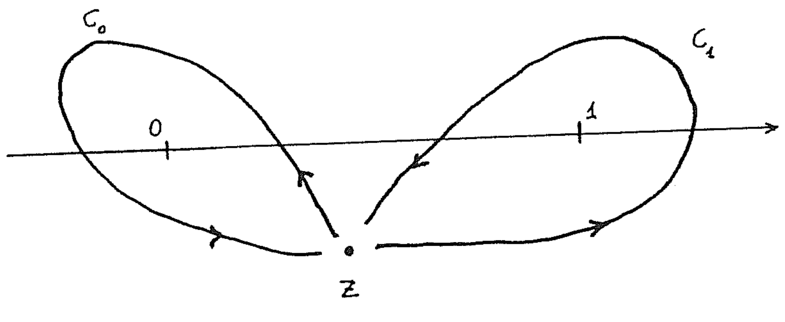

fig. 2 : continuation paths around 0 and 1 .

For what concerns g^1, we have to resort to the following expansion of $I_i(z)$ in terms of $I_i(1-z)$ (cfr. [II.1])

$$I_i(z) = \sum_{k=1}^{2} \beta_{ik} I_k(1-z) \qquad \text{(II.56)}$$

which has the advantage to turn also g^1 into a diagonal form. From (II.55) we then get

$$G(z,\bar{z}) = \sum_{i,j,k=1}^{2} X_i \beta_{ik} \beta_{ij} I_k(1-z) \overline{I_j(1-z)} \qquad \text{(II.57)}$$

so that if we want G to be invariant also under g^1 we have to require that

$$\sum_{i=1}^{2} X_i \beta_{ik} \beta_{ij} = 0 \qquad \text{if} \quad k \neq j \qquad \text{(II.58)}$$

This relation, in turn, allows us to compute the ratio X_1/X_2 :

$$\frac{X_1}{X_2} = -\frac{\beta_{22}\,\beta_{21}}{\beta_{11}\,\beta_{12}} \qquad \text{(II.59)}$$

Thus we are left with the problem of determining the coefficients β_{ik} and this is done by determining in which way the path of $I_i(z)$ may be deformed into that of $I_k(1-z)$. Hence, these appear with some sine factors in front, and the final result is

$$\beta = \begin{pmatrix} \beta_{11} & \beta_{21} \\ \beta_{12} & \beta_{22} \end{pmatrix} = \frac{1}{\sin[\pi(b+c)]} \begin{pmatrix} \sin(\pi a) & -\sin(\pi c) \\ -\sin[\pi(a+b+c)] & -\sin(\pi b) \end{pmatrix} \qquad \text{(II.60)}$$

with a, b and c given in (II.51). Therefore the ratio (II.59) is

$$\frac{X_1}{X_2} = \frac{\sin[\pi(a+b+c)]\ \sin(\pi b)}{\sin(\pi a)\ \sin(\pi c)} \qquad \text{(II.61)}$$

and finally (II.55) gives

$$G(z,\bar{z}) \sim \sin(\pi b) \sin[\pi (a+b+c)] \; |I_1(z)|^2 + \sin(\pi a) \sin(\pi c) \; | I_2(z)|^2 \qquad (II.62)$$

up to some constant.

This is the announced result and ends our analysis of the Coulomb gas approach to conformal symmetry . Of course methods and results outlined here may be generalized in order to consider more complicated situations , as is done in [II.1] and [II.6] . Furthermore , applications to actual two dimensional statistical models lead to interesting results concerning critical indexes , as in the case of O(n) and Potts models (see [II.1] and references quoted therein) . These features go far beyond the aims of the present work , so we have limited ourselves to describe in general the Coulomb gas model as a useful and profitable way to employ conformal symmetry concepts in actual computations.

In the next chapter we shall survey some features more closely related to lattice statistical mechanics , and in particular we will describe a method for evaluating the conformal anomaly in a statistical system.

* * *

Chapter III : Statistical Mechanics in a Finite Geometry

Introduction and Summary

As we already pointed out in the first chapter, conformal invariance is a richer symmetry in two-dimensions than in any other case, and therefore allows for considerable improvements in the study of two-dimensional systems.

One such improvement is constituted by the possibility of relating the behaviour of a statistical system in a finite or semi-infinite geometry (like, e.g., a torus, a strip or the half plane) to that of the same system in the infinite plane through a conformal transformation of coordinates (see [III.1], [III.2], [I.4]).

In particular , this correspondence is very useful when considering finite size scaling in two dimensions, allowing, for example, the computation of surface cirtical exponents for a wide class of models (see [I.4] and [II.2]).

This chapter will be entirely devoted to the study of statistical systems in a restricted geometry , which allows us to enforce modular invariance for their partition function and hence to determine the model operator content. Besides that, one can stress, as a main consequence of conformal symmetry in a finite configuration, the interpretation of the central charge of a statistical model as a finite size scaling correction to its free energy , thus relating, in quantum field theory context, the conformal anomaly to the Casimir effect ([III.5]) .

We will start by studying conformal invariant theories in the upper half complex plane : this is intended as an introductory exercise , since we will show which modifications arise in the conformal symmetry machinery of the first chapter as a consequence of finite geometry . In particular we'll find that the factorization of the theory into analytic and antianalytic parts is no longer valid, and hence we'll be forced to slightly modify conformal Ward identities. In so doing we change also , in a very natural way, differential equations for correlators of primary fields : for example , in the case of a scalar primary field φ , it turns out that the correlation function $< \varphi(z_1,\bar{z}_1) \ldots\ldots \varphi(z_N,\bar{z}_N) >$ in the half plane satisfies the same equation as $< \varphi(z_1,\bar{z}_1) \ldots\ldots \varphi(z_{2N},\bar{z}_{2N}) >$ in the infinite plane considered as a function of z_i only. As we will see in the case of Ising model this leads to the correct surface exponents.

As a subsequent step we'll illustrate in detail the concept of modular invariance for the partition function of a statistical system put on a torus : this idea makes finite geometry so interesting , because it allows to study the number of primary operators of the related conformal algebra. In short , the situation is as follows : if we consider a model put on a parallelogram of sides l , l' with periodic boundary conditions, its partition function $Z(l,l')$ can be decomposed into a sum containing characters , $\chi(\delta)$, of Virasoro algebra , the ratio $\delta = l/l'$, and the number of primary conformal operators of the theory $\mathcal{N}(p,q;\bar{p},\bar{q})$, where $p,q,\bar{p},\bar{q}$ are integers parametrizing c in (I.57) . Enforcing modular invariance on Z implies

$$Z(l,l') = Z(l',l)$$

and this gives a non trivial constraint on the number $\mathcal{N}$, which may be interpreted as an eigenvalue of a certain matrix. This eigenvalue problem is not solved in the general case : available solutions concern very simple models and give their operator content. As an example , we will consider the Ising model, finding again its operators I , ε and σ .

In the last two sections we shall be concerned with statistical models on an infinite strip of width L : rewriting conformal Ward identities for the partition function in this geometry, one can derive the conformal anomaly as a correction of order $1/L$ to the free energy in the limit $L \to \infty$. Thus , one has an alternative way to compute the central charge for a statistical model : instead of evaluating a short distance expansion for a stress energy tensor , one has to know the expression of the free energy of the model in the strip, and then study the limit $L \to \infty$. This is explicitly done still in the case of Ising model, for which we know the partition function and the free energy on the strip , for example from its grassmann variable formulation ([III.3]) , but this method may be very well suitable for any two dimensional model whose free energy is known , and in particular for those whose conformal anomaly has not yet been computed. Work in this direction is in progress ([III.19]) .

Let us then consider conformal symmetry in a half plane.

III.1 Correlation Functions in a Semi-infinite Geometry

The results of chapter I concerning conformal invariant systems in the complex plane may be easily generalized to the case of the half plane. Let us consider a theory whose operators $\varphi(z,\bar{z})$ live in the upper half plane Im $z\geq0$: in order to keep the geometry, conformal transformations have then to be limited to the real analytic ones, i.e. to those satisfying

$$\overline{f(z)} = f(\bar{z}) \qquad \text{(III.1)}$$

Consequently the factorization between analytic and antianalytic components of the stress energy tensor is no longer valid: on the contrary we have to extend the definition of T also on the lower half plane, and we do that by posing

$$T(z) = \bar{T}(z) \qquad \text{for} \quad \text{Im} z < 0 \qquad \text{(III.2)}$$

With these restrictions, and relabeling the complex conjugate $\bar{z}$ with z' , we may write down the conformal Ward identities for the half plane: in the case of a multipoint function of a scalar primary field $\varphi(z,z')$ we have

$$\frac{1}{2\pi i}\oint_C dz\ \varepsilon(z) < T(z)\,\varphi(z_1,z_1')\,....> + \frac{1}{2\pi i}\oint_{\bar{C}} dz\,\varepsilon(z)\ <T(z)\varphi(z_1,z_1')\,.....> =$$

$$= \sum_i (\varepsilon'(z_i)\,h + \varepsilon(z_i)\frac{\partial}{\partial z_i} + \varepsilon'(z_i')\,h + \varepsilon(z_i')\frac{\partial}{\partial z_i'}\ ) < \varphi(z_1,z_1')\,.....>$$

(III.3)

having considered an infinitesimal transformation $z \to z+\varepsilon(z)$ and integrations contour C and $\bar{C}$ as in fig. 1. It is easily seen that the parts of C and $\bar{C}$ neighbouring the real axis give zero contribution if $T = \bar{T}$ for Imz=0, and indeed this is the appropriate boundary condition on the stress energy tensor in our half plane configuration. Therefore, integrals in (III.3) can be joined together and evaluated on a larger circle surrounding z_i and z'_i: the net result is that

$$< T(z)\, \varphi\, (z_1, z_1') \ldots > \quad = \quad \{ \sum_i [\frac{h}{(z - z_i')^2} + \frac{1}{z - z_i'} \frac{\partial}{\partial z_i}] +$$

$$+ \sum_i [\frac{h}{(z - z_i')^2} + \frac{1}{z - z_i'} \frac{\partial}{\partial z_i'}] \} < \varphi(z_1, z_1') \ldots > \tag{III.4}$$

having resorted to Cauchy representation for ε and ε' as usual. The rest of the analysis goes through in the same way as in the case of the infinite plane, so that comparing (III.4) with already known results (see (I.32)) we are left with the fact that the correlator $<\varphi(z_1,\bar{z}_1) \ldots \varphi(z_N,\bar{z}_N)>$ as a function of z_i, z_i satisfies, in the half plane , the same differential equation as that of $<\varphi(z_1,\bar{z}_1) \ldots \varphi(z_{2N},\bar{z}_{2N})>$ in the infinite plane, considered as a function of z_i only. Let us consider a four points function in the infinite plane

$$G^{(4)} = < \varphi(z_1)\, \varphi(z_2)\, \varphi(z_3)\, \varphi(z_4) > \tag{III.5}$$

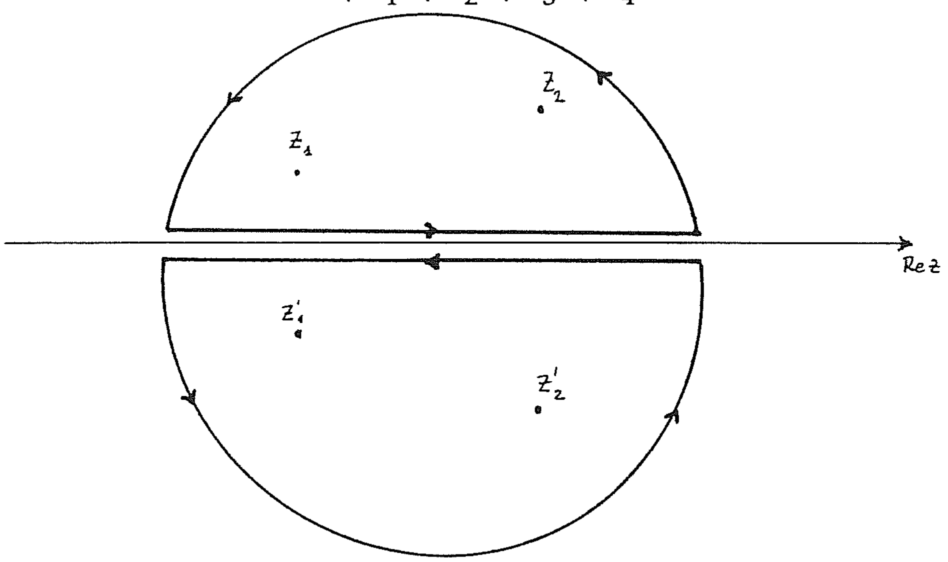

fig. 1 : integration contours for (III.3)

According to the early results of Polyakov (see [I.2]) , (III.5) is constrained to be of the form

$$G^{(4)} = \left(\frac{z_{13}\, z_{24}}{z_{12}\, z_{23}\, z_{34}\, z_{14}}\right)^{2h} F(\xi) \tag{III.6}$$

having put

$$z_{ij} = z_i - z_j \tag{III.7}$$

and

$$\xi = \frac{z_{12}\, z_{34}}{z_{13}\, z_{24}} \tag{III.8}$$

If the field φ is degenerate at some level, then $G^{(4)}$ has to satisfy a differential equation which, thanks to (III.6), is a differential equation for $F(\xi)$ with regular singular points at $\xi = 0,1,\infty$. We can determine the behaviour of F in a neighbourhood of $\xi = 0$ (*) by resorting to a short distance expansion of the field φ, which reads as follows

$$\varphi(z_1)\, \varphi(z_2) \sim \sum c_k\, (z_1 - z_2)^{-2h + h_k}\, \varphi_k(z_1) \tag{III.9}$$

Substituting it in (III.5) we have

$$G^{(4)} \sim \sum c_k\, (z_1 - z_2)^{-2h + h_k}\, \varphi_k(z_1) \tag{III.10}$$

and we see that the allowed exponents in

$$F(\xi) \sim \xi^{\alpha} \qquad \xi \to 0 \tag{III.11}$$

are $\alpha = h_k$ (III.12)

h_k being the scaling dimension of an operator appearing in (III.9). If we now

----

(*) similar considerations may be carried on also for $\xi = 1,\infty$.

consider the two point function in the half plane,

$$G_s^{(2)} \quad = \quad < \varphi(z_1,\bar{z}_1) \;\; \varphi(z_2,\bar{z}_2) > \tag{III.13}$$

this has to be of the form

$$G_s^{(2)} \quad = \quad [\frac{(z_1 - \bar{z}_1) (z_2 - \bar{z}_2)}{(z_1 - z_2) (\bar{z}_1 - \bar{z}_2) (z_1 - \bar{z}_2) (\bar{z}_1 - z_2)}]^{2h} F_s(\xi) \tag{III.14}$$

where now

$$\xi = \frac{(z_1 - z_2) (\bar{z}_1 - \bar{z}_2)}{(z_1 - \bar{z}_1) (z_2 - \bar{z}_2)} \tag{III.15}$$

Thanks to the above considerations, for $F_s(\xi)$ we have the same results as for $F(\xi)$. Let us take as an example the two points spin correlation function in Ising model: in this case the operator algebra is (see sect. I.8)

$$\sigma\sigma \sim [\mathbb{I}] + [\varepsilon] \tag{III.16}$$

while

$$h_\sigma = 1/16$$
$$h_\varepsilon = 1/2 \tag{III.17}$$
$$h_{\mathbb{I}} = 0$$

From this we get

$$G_\sigma^{(2)} \quad \equiv \quad < \sigma(z_1,\bar{z}_2) \;\; \sigma(z_2,\bar{z}_2) > \quad =$$

$$= \quad [\frac{(z_1 - \bar{z}_1) (z_2 - \bar{z}_2)}{(z_1- z_2) (\bar{z}_1 - \bar{z}_2) (z_1 - \bar{z}_2) (\bar{z}_1 - z_2)}]^{\frac{1}{8}} \quad F_\sigma(\xi) \tag{III.18}$$

where

$$\xi \;=\; - \frac{| z_1 - z_2 |^2}{4y_1 \, y_2} < 0 \tag{III.19}$$

in this case (we have put $z_i = x_i - iy_i$) .

Let $\xi \to -\infty$ with y_i fixed, thus giving $|x_1 - x_2| \to \infty$: assuming $F_\sigma(\xi) \sim \xi^{-\alpha}$ from (III.18) we get

$$G^{(2)}_\sigma(z_1, \bar{z}_1, z_2, \bar{z}_2) \underset{\xi \to \infty}{\sim} \frac{1}{|x_1 - x_2|^{1/2}} \xi^{-\alpha} \qquad \text{(III.20)}$$

and since in this limit we have $\xi \sim - |x_1 - x_2|^2$, the final result is

$$G^{(2)}_\sigma(z_1, \bar{z}_1, z_2, \bar{z}_2) \underset{|x_1 - x_2| \to \infty}{\sim} |x_1 - x_2|^{-1/2 - 2\alpha} \qquad \text{(III.21)}$$

The allowed exponents a in (III.21) may be determined form the algebra (III.16). An argument similar to that leading to (III.12) shows that we can have only

$$\alpha_k = -4h_\sigma + h_k \qquad \text{(III.22)}$$

where k refers either to the identity $\mathbb{I}$ or to the energy operator ε. Therefore we have the two solutions

$$\alpha_{\mathbb{I}} = -1/4 \qquad \text{(III.23)}$$

$$\alpha_\varepsilon = 1/4$$

Since the two point function $G^{(2)}$ has to vanish in the limit we are considering , $\alpha_{\mathbb{I}}$ is ruled out and we are left with

$$G^{(2)}_\sigma \underset{|x_1 - x_2| \to \infty}{\sim} |x_1 - x_2|^{-1} \qquad \text{(III.24)}$$

that thus determines the critical index $\eta_{\parallel}$ for the Ising model (see [III.2], [I.4])

$$\eta_{\parallel} = 1 \qquad \text{(III.25)}$$

which agrees with known results (see [I.4]).

This example shows how conformal invariance, recast in a finite geometry context, may be used to compute surface critical exponents.

In the next sections we shall consider a restricted geometry and show how it can be used in order to determine the operator content of a conformal theory.

III.2 Unitary Conformal Theories in a Restricted Geometry

In this section we shall be concerned with a statistical system put on a rectangle of sides ℓ and ℓ' with periodic or toroidal boundary conditions. It is known (see [III.1]) that the partition function for such a system can be given in terms of a transfer matrix $\hat{H}$ as follows

$$Z(\ell, \ell') = \mathrm{Tr}\,[\exp(-\ell' \hat{H})] \tag{III.26}$$

This result can be extended to the case of a parallelogram by considering a complex ratio ℓ / ℓ' (see fig. 2): in both cases enforcing the requirement of modular invariance

$$Z(\ell,\ell') = Z(\ell',\ell) \tag{III.27}$$

we obtain useful constraints on the transfer matrix H, which allow to determine its eigenvalues and their degeneracy, and therefore the operator content of the theory we are considering.

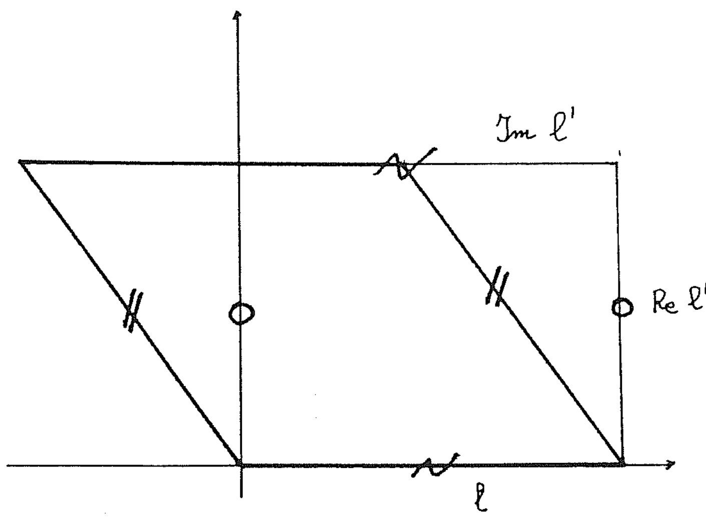

fig. 2 : rectangle and parallelogram with toroidal boundary conditions

For the moment we will study the case of ℓ and ℓ' both real and , as a first consequence of modular invariance, we shall show that a unitary theory with a finite number of primary operators must necessarily have

$$c < 1 \qquad \text{(III.28)}$$

and therefore fall into the classification (I.57) of Friedan, Qiu and Shenker.

Let us consider the theory given by the partition function (III.26) in the limit $\ell\ ,\ \ell' \rightarrow \infty$ with $\delta \equiv \ell'/\ell$ fixed. We have to resort to a complete orthonormal set of eigenstates of $\hat{H}$ with energy E_n in order to evaluate the trace in $Z(\ell,\ell')$, and we obtain

$$Z(\ell,\ell') = e^{-E_0\ell'} \sum_n e^{-(E_n - E_0)\ell'} \qquad \text{(III.29)}$$

The particular way in which we have rewritten the partition function permits to employ the relation between the ground state energy E_0 and the central charge c (see [III.1], [III.5], [III.6]) which holds in the limit under consideration and for our toroidal boundary conditions

$$E_0 \sim f\ell - \frac{\pi c}{6\ell}\ ,\ \ell,\ell' \rightarrow \infty,\ \delta \text{ fixed} \qquad \text{(III.30)}$$

f being the bulk free energy per unit area. As a consequence of that , the partition function becomes

$$Z(\ell,\ell') = e^{(-fA + \pi c\delta/6)} \sum_n e^{[-(E_n - E_0)\ell']} \qquad \text{(III.31)}$$

A being the area of our rectangle.

Let us examine the expression we have obtained so far: in the sum only those energy gaps going like ℓ^{-1} can contribute in our limit. The same hypotheses leading to (III.30) allow also to show the following expression for the energy gaps

$$E_n - E_0 = \frac{2\pi x_n}{\ell} \qquad \text{(III.32)}$$

where x_n are the anomalous scale dimensions of conformal fields corresponding to E_n-eigenstates of $\hat{H}$ (see [III.1] and [III.2]). By virtue of (III.32) we have that all independent conformal scaling operators will contribute to the sum in (III.31) and therefore it can be recast according to the various conformal families, summing over the ancestor and descendants secondary fields of level $(N,\bar{N})$ in each of them. At a given level $(N,\bar{N})$ in a family, the operators have the form (I.29), and hence their number is $P(N)\cdot P(\bar{N})$, if P(n) gives the number of partitions of n into positive numbers.

This observation allows us to give an upper bound on the partition function (III.31). In fact, since any term in (III.31) is positive in a unitary theory, and in general not all the $P(N)\cdot P(\bar{N})$ secondary operators are really independent (recall the degenerate cases), the contribution from one conformal family of scale dimension x is certainly less than or equal to

$$e^{(-2\pi x_{N,\bar{N}}\,\delta)} \sum_{N,\bar{N}} P(N)\,P(\bar{N})\; e^{[-2\pi(N+\bar{N})\delta]} \qquad \text{(III.33)}$$

$x_{N,\bar{N}}$ being given by

$$x_{N,\bar{N}} \;=\; x + N + \bar{N} \qquad \text{(III.34)}$$

(cfr. (I.31)). At this point, it can be easily shown (see [III.1]) that the sum in (III.33) is equal to $[1/f(\delta)]^2$ where

$$f(\delta) \;=\; \prod_{n=1}^{\infty} (1 - e^{-2\pi n\delta}) \qquad \text{(III.35)}$$

(f^{-1} is called the "generating function" for P(N)) and this finally yields the desired upper bound on $Z(\ell,\ell')$:

$$Z(\ell,\ell') \;\leq\; e^{(-fA + \pi c\delta/6)}\,[f(\delta)]^{-2} \sum_{\substack{\text{primary}\\ \text{operators}}} e^{(-2\pi x_p\,\delta)} \qquad \text{(III.36)}$$

x_p being the anomalous scale dimensions of the primary fields in the theory. This bound , along with the requirement of modular invariance of Z which we now impose, permit to prove that if the number of primary operators in the theory is finite, then c has to be strictly less than one. According to (III.27) it must be

$$Z(\delta) = Z(\delta^{-1}) \tag{III.37}$$

substituting δ with δ^{-1} in (III.31) we can easily get

$$Z(\delta^{-1}) = e^{(-fA + \pi c/\delta 6)} \sum_n e^{(-2\pi x_n/\delta)} \tag{III.38}$$

let us consider now the limit $\delta \to 0$: then only the identity operator with $X_n = 0$ will contribute in the sum, giving

$$Z(\delta^{-1}) \underset{\delta \to 0}{\sim} e^{(-fA + \pi c/6\delta)} \tag{III.39}$$

In order to enforce condition (III.36), we have to study also its limit for $\delta \to 0$: thanks to an inversion relation for $f(\delta)$ (see [III.1]) which relates its δ dependence to δ^{-1} one ,

$$f(\delta) = \delta^{-1/2} e^{\pi(\delta - \delta^{1})/12} f(\delta^{-1}) \tag{III.40}$$

and since

$$f(\delta^{-1}) \underset{\delta \to 0}{\sim} 1 \tag{III.41}$$

the l.h.s. of (III.36) takes the following form in the limit $\delta \to 0$

$$\delta e^{(-fA + \pi/6\delta)} \sum_{\substack{\text{primary} \\ \text{operators}}} 1 \tag{III.42}$$

Therefore, if $\mathcal{N}$ is the number of primary operators in the theory, (III.39) and (III.42) yield

$$e^{(-fA + \pi c/6\delta)} \leq \mathcal{N} \delta e^{(-fA + \pi/6\delta)} \tag{III.43}$$

which holds in the limit $\delta \to 0$. Now simple algebraic manipulations lead to

$$\exp[\pi(c-1)/6\delta] \le \mathcal{N}\cdot\delta \qquad \text{(III.44)}$$

in the limit $\delta \to 0$: since an $\mathcal{N}$ finite makes ~~r.h.s.~~ vanish in the limit , also l.h.s. has to go to zero, and this can happen for $\delta \to 0$ if and only if

$$c < 1 \qquad \text{(III.45)}$$

which is the desired result. This implies that for a theory with a finite number of primary operators, c must be given by (I.57).

In the next section we shall dwell further on the implications of modular invariance of the partition function in a restricted geometry: in particular we will see how it can be used for determining the content of primary operators in a given theory.

III.3 Operator Content of a Conformal Theory

Throughout this section we shall consider a unitary theory with a finite number of primary operators (which therefore satisfy (III.45)) put on a parallelogram with toroidal boundary conditions (see fig. 2). The new shape may be obtained by considering ℓ' as a complex number, while the partition function is written with the aid of the trasfer matrix $\hat{H}$ and the momentum operator $\hat{K}$ (see [II.1]):

$$Z(\ell,\ell') = \mathrm{Tr}\,[\, e^{-\hat{H}\,\mathrm{Re}\,\ell'}\, e^{-i\hat{K}\mathrm{Im}\ell'}\,] \qquad \text{(III.46)}$$

As before, taking a complete orthonormal set of eigenstates of $\hat{H}$ and $\hat{K}$ we obtain

$$Z(\ell,\ell') = e^{(-f A + \pi c\,\mathrm{Re}\delta/6)} \sum_n e^{[-E_n \mathrm{Re}\ell' - i k_n \mathrm{Im}\ell']} \qquad \text{(III.47)}$$

Examining the sum in (III.47) we see that a conformal operator of scale dimensions (h,h) will give a term $\exp[-2\pi(h\delta+h\delta^*)]$ and therefore the contribution of each conformal family will be of the form

$$\chi(\delta)\,.\,\bar{\chi}(\delta^*) \tag{III.48}$$

where

$$\chi(\delta) = \sum_N d(N)\, e^{[-2\pi(h+N)\delta]} \tag{III.49}$$

The quantity $\chi(\delta)$ is called a character, and since it gives the contribution of a conformal family it pertains to a certain representation of the Virasoro algebra: therefore it may be denoted by $\chi_{p,q}(\delta)$ resorting to the notation of Chapter I; the number d(N) simply gives the possible degeneracy of an operator at level N. Characters $\chi_{p,q}(\delta)$ are actually given by the following formula (see [III.1], [III.7])

$$\chi_{p,q}(\delta) = [f(\delta)]^{-1}\, g_{p,q}(\delta) \tag{II.50}$$

where f(δ) is given in (III.35) and $g_{p,q}(\delta)$ is

$$g_{p,q}(\delta) = \sum_{k=-\infty}^{+\infty} [\exp\{-\frac{2\pi\delta}{4m(m+1)}[(2m(m+1)k + (m+1)p - mq)^2 - 1]\} + - \{q \to -q\}] \tag{III.51}$$

Therefore, if $\mathcal{N}(p,q:\bar{p},\bar{q})$ is the number of primary operators with conformal weights $h_{p,q}$, $\bar{h}_{\bar{p},\bar{q}}$ (cfr. I.58), for the partition function in (III.47) it turns out

$$Z(\ell,\ell') = e^{(-fA + \pi c \operatorname{Re}\delta/6)} \sum_{p,q;\bar{p},\bar{q}} \mathcal{N}(p,q;\bar{p},\bar{q})\, \chi_{p,q}(\delta)\, \bar{\chi}_{\bar{p},\bar{q}}(\delta^*) \tag{III.52}$$

where the indices p,q run in the domain

$$1 \le q \le p \le m-1 \tag{III.53}$$

m being the integer which parametrizes c in (I.57).

The form (III.52) for $Z(\ell,\ell')$ is very well suitable for enforcing modular invariance requirements: these will translate in some constraints on the numbers $\mathcal{N}(p,q;\bar{p},\bar{q})$ which will be given in the form of sum rules. An application of Poisson's formula to $g_{p,q}(\delta)$ brings δ to the denominator of the exponential in (III.51), and further employing the inversion relation (III.40) we get the dependence of $Z(\ell,\ell')$ upon δ^{-1} as follows

$$Z(\ell,\ell') = e^{(-fA+\pi c\,\mathrm{Re}\,\delta^{-1}/6)} \frac{2\,|f(\delta^{-1})|^{-2}}{m(m+1)} \sum_{p,q;\bar{p},\bar{q}} \mathcal{N}(p,q;\bar{p},\bar{q}) \cdot$$
$$\cdot \Big[\sum_{r=-\infty}^{+\infty} e^{\left(-\frac{\pi(r^2-1)}{2\delta m(m+1)}\right)} \sin\Big(\frac{r\pi p}{m}\Big) \sin\Big(\frac{r\pi q}{m+1}\Big)\Big] \cdot$$
$$\cdot \Big[\sum_{\bar{r}=-\infty}^{+\infty} e^{\left(-\frac{\pi(\bar{r}^2-1)}{2\delta^* m(m+1)}\right)} \sin\Big(\frac{\bar{r}\pi\bar{p}}{m}\Big) \sin\Big(\frac{\bar{r}\pi\bar{q}}{m+1}\Big)\Big] \qquad \text{(III.54)}$$

On the other hand we can work out the expression of $Z(\ell,\ell') \equiv Z(\delta^{-1})$ just in the same way as that of $Z(\ell,\ell') \equiv Z(\delta)$, obtaining

$$Z(\delta^{-1}) = e^{(-fA+\pi c\,\mathrm{Re}\,\delta^{-1}/6)} \sum_{p,q;\bar{p},\bar{q}} \mathcal{N}(p,q;\bar{p},\bar{q}) \cdot$$
$$\cdot \Big[\sum_{k=-\infty}^{+\infty} \Big\{ \exp\Big(-\frac{\pi}{2\delta m(m+1)}\big[(2m(m+1)k + (m+1)p - mq)^2\big]\Big) - (q \to -q)\Big\}\Big] \cdot$$
$$\cdot \Big[\sum_{\bar{k}=-\infty}^{+\infty} \{\delta \to \delta^*,\ k \to \bar{k},\ p \to \bar{p},\ q \to \bar{q}\}\Big]$$

(III.55)

Modular invariance then requires that (III.54) and (III.55) have to be equal, and comparing powers of $\exp(1/\delta)$ and $\exp(1/\delta^*)$ we get conditions that have to be fullfilled by numbers $\mathcal{N}(p,q;p,q)$. For example, the leading terms corresponding to $r = \bar{r} = \pm 1$ in (III.54) and to $k = \bar{k} = 0$, $p = q = \bar{p} = \bar{q} = 1$ in (III.55) yield the following equation:

$$\frac{8}{m(m+1)} \sum_{p,q;\bar{p},\bar{q}} \mathcal{N}(p,q;\bar{p},\bar{q}) \sin\frac{\pi p}{m} \sin\frac{\pi q}{m+1} \sin\frac{\pi \bar{p}}{m} \sin\frac{\pi \bar{q}}{m+1} =$$

$$= \mathcal{N}(1,1;1,1) = 1 \qquad \text{(III.56)}$$

since the identity operator can appear just once in a given theory. Actually the number of constraints we get for $\mathcal{N}(p,q;p,q)$ is not infinite, as it may seem at first glance, and this is due to the particular structure of the sums in (III.54). In fact, thanks to the presence of sine factors, r and r such that

$$r,\bar{r} = \begin{cases} 0 & \text{mod } m \\ 0 & \text{mod } m+1 \end{cases} \qquad \text{(III.57)}$$

will not contribute to it, and this allows to reparametrize the sums by putting (see [III.1])

$$r^2 = (2m(m+1)k'+(m+1)p'-mq')^2 \qquad \text{(III.58)}$$

with $1 \leq |q'| \leq p' \leq m-1$. If we substitute (III.58) into (III.54), we express it in the same form of (III.55): consequently the sums over k in (III.55) and those in (III.54) may be eliminated and we end up with a finite number of conditions on $\mathcal{N}(p,q;\bar{p},\bar{q})$:

$$\sum_{p,q;\bar{p},\bar{q}} \mathcal{N}(p,q;\bar{p},\bar{q}) \, (-1)^{(p+q)(p'+q') + (\bar{p}+\bar{q})(\bar{p}'+\bar{q}')} \, .$$

$$\cdot \sin\left(\frac{\pi p p'}{m}\right) \sin\left(\frac{\pi q q'}{m+1}\right) \sin\left(\frac{\pi \bar{p}\bar{p}'}{m}\right) \sin\left(\frac{\pi \bar{q}\bar{q}'}{m+1}\right) =$$

$$= \frac{m(m+1)}{8} \mathcal{N}(p',q';\bar{p}',\bar{q}') \qquad \text{(III.59)}$$

The set of equations thus obtained, indexed by integers $(p',q';\bar{p}',\bar{q}')$ in the interval (III.53), may be interpreted as an eigenvalue equation in the form

$$M\mathcal{N} = \mathcal{N} \qquad \text{(III.60)}$$

where M may be thought of as a direct product of two matrices, has a total number of $[(\frac{1}{2})m(m-1) \times (\frac{1}{2})m(m-1)]^2$ elements corresponding to the possible values of p and q's in (III.53), and is given by

$$M^{(m)}_{\substack{p,q\,;\,p',q' \\ \bar{p},\bar{q}\,;\,\bar{p}',\bar{q}'}} = \frac{8}{m(m+1)} \, (-1)^{(p+q)\,(p'+q') \; + \; (\bar{p}+\bar{q})\,(\bar{p}'+\bar{q}')} \; .$$

$$\cdot \; \sin(\frac{\pi p p'}{m}) \; \sin(\frac{\pi q q'}{m+1}) \; \sin(\frac{\pi \bar{p}\bar{p}'}{m}) \; \sin(\frac{\pi \bar{q}\bar{q}'}{m+1}) \qquad \text{(III.61)}$$

Solving equation (III.60) is a very difficult task, and yet a general solution for arbitrary values of m is not at our disposal (see [III.1]). The best one can do is to study for the lowest m values, being helped by some simplifying observations pertaining the case under consideration. In the next section, as an example, we shall work out in any detail the solution for the simplest case m=3, corresponding , once again , to the Ising model.

III.4 Operator Content of Ising Model: an Application

First of all, let us make some general requirements which hold for any value of m, starting from the fact that the eigenvectors we have to find must be constituted by non negative integers. Furthermore, the particular boundary conditions we have considered from the beginning (i.e. periodicity) imply that there may be only integer spin operators, the others being excluded: we shall comment again on the influence of boundary conditions at the end of this section, and for the moment we put

$$\mathcal{N}(p,q;\bar{p},\bar{q}) \quad = \quad 0 \qquad \text{unless} \quad h_{p,q} - \bar{h}_{\bar{p},\bar{q}} \in \mathbb{Z} \qquad \text{(III.62)}$$

Finally, reality of partition functions yields

$$\mathcal{N}(p,q;\bar{p},\bar{q}) \quad = \quad \mathcal{N}(\bar{p},\bar{q};p,q) \qquad \text{(III.63)}$$

With these remarks in mind, let us now consider the case of an Ising model, i.e. m=3 . Conformal weights are given as usual by Kac formula (I.58)

$$h^{m=3}_{p,q} = \frac{(4p - 3q)^2 - 1}{48} \qquad \text{(III.64)}$$

with $p = 1, q = 1$ (III.65)

and $p = 2, q = 1,2$ (III.66)

(and respective formulas for $\bar{h}_{\bar{p},\bar{q}}$), and therefore the possible values are

$$\begin{cases} h_{1,1} = \bar{h}_{1,1} = 0 \\ h_{2,1} = \bar{h}_{2,1} = ½ \\ h_{2,2} = \bar{h}_{2,2} = 1/16 \end{cases} \qquad \text{(III.67)}$$

from which we see that the only integer spin operators that can be present in the theory are the scalar operators.

Substituting m=3 into (III.61) we get the matrix we have to study in our case: this is

$$M^{(3)}_{\substack{p,q\,;\,p',q' \\ \bar{p},\bar{q}\,;\,\bar{p}',\bar{q}'}} = \frac{2}{3}(-1)^{(p+q)(p'+q') + (\bar{p}+\bar{q})(\bar{p}'+\bar{q}')} \cdot$$

$$\cdot \sin(\frac{\pi p'p}{3}) \sin(\frac{\pi qq'}{4}) \sin(\frac{\pi \bar{p}\bar{p}'}{3}) \sin(\frac{\pi \bar{q}\bar{q}'}{4}) \qquad \text{(III.68)}$$

Enforcing the constraint (III.62), we have to consider only those values of indexes which give rise to scalar operators, and hence we have to put

$$p = \bar{p} \ , \ q = \bar{q} \qquad \text{(III.69)}$$

This condition reduces $M^{(3)}$ to a simple 3x3 matrix as follows

$$M^{(3)} = \begin{pmatrix} M^{(3)}_{\substack{1,1\,;\,1,1 \\ 1,1\,;\,1,1}} & M^{(3)}_{\substack{1,1\,;\,2,1 \\ 1,1\,;\,2,1}} & M^{(3)}_{\substack{1,1\,;\,2,2 \\ 1,1\,;\,2,2}} \\ M^{(3)}_{\substack{2,1\,;\,1,1 \\ 2,1\,;\,1,1}} & M^{(3)}_{\substack{2,1\,;\,2,1 \\ 2,1\,;\,2,1}} & M^{(3)}_{\substack{2,1\,;\,2,2 \\ 2,1\,;\,2,2}} \\ M^{(3)}_{\substack{2,2\,;\,1,1 \\ 2,2\,;\,1,1}} & M^{(3)}_{\substack{2,2\,;\,2,1 \\ 2,2\,;\,2,1}} & M^{(3)}_{\substack{2,2\,;\,2,2 \\ 2,2\,;\,2,2}} \end{pmatrix} \qquad \text{(III.70)}$$

and also gives at once $\mathcal{N}(p,q;p,q)$ in the form

$$\mathcal{N}(p,q;\bar{p},\bar{q}) \quad = \quad \mathcal{N}(p,q;p,q) \qquad \text{(III.71)}$$

consistently with (III.63).

In order to determine $\mathcal{N}$ we relabel the multi-index (p,q;p,q) in a way suggested by the form of $M^{(3)}$ itself (see III.70): let us put

$$(p,q;p,q) = \alpha \qquad \alpha = 1,2,3 \qquad \text{(III.72)}$$

so that

$$(1,1;1,1) = 1 \qquad (2,1;2,1) = 2 \qquad (2,2;2,2) = 3 \qquad \text{(III.73)}$$

This yields $\mathcal{N}$ as a three-component vector

$$\mathcal{N} = (N_1\,, N_2\,, N_3\,) \qquad \text{(III.74)}$$

and also relabels the matrix $M^{(2)}$; for example

$$M^{(3)}_{\substack{2,1;\;2,1 \\ 2,1;\,2,1}} = M^{(3)}_{2,2}$$

The various components of $M^{(3)}$ in (III.70) are computed by simply replacing the corresponding values of p,q, p',q' into (III.68) , and the result is

$$M^{(3)} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix} \qquad \text{(III.75)}$$

Therefore the eigenvalue equation (III.60) is , in the case of Ising model,

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \qquad \text{(III.76)}$$

and this yields the following linear system

$$\begin{cases} 4N_1 = N_1 + N_2 + 2N_3 \\ 4N_2 = N_1 + N_2 + 2N_3 \\ 4N_3 = 2(N_1 + N_2) \end{cases} \qquad \text{(III.77)}$$

which finally gives $\mathcal{N}$ as

$$\mathcal{N} = N_1 \ (1,1,1) \qquad \text{(III.78)}$$

In order to determine N_1 we have to know the operator content for some index in the Kac table (I.58) : since $\alpha=1$ in (III.73) corresponds to the identity operator we have

$$N_1 = 1 \qquad \text{(III.79)}$$

and hence

$$\mathcal{N} = (1,1,1) \qquad \text{(III.80)}$$

Taking into account condition (III.71) and the relabeling (III.72) we can easily conclude that for the Ising model

$$\mathcal{N}(p,q;\bar{p},\bar{q}) = \delta_{p,\bar{p}} \ \delta_{q,\bar{q}} \qquad \text{(III.81)}$$

in agreement with (III.63) and (III.62).

This result is consistent with our previous analysis of chapter I, in fact it simply means that the allowed operators are the identity, the energy density and the magnetization which are all scalar operators (cfr.(I.75)).

For what concerns higher values of m, as we already mentioned, a general solution does not exist. Nonetheless it may be shown (see [III.1]) that (II.81) is a solution of (III.60) for any m, and Cardy (see [III.1]) has been able to solve (III.60) up to m = 5 thus determining the operator content of the tricritical Ising model and 3-state Potts model. As a final observation on (III.81) we note that the eigenvalue equation (III.60) is a much more fruitful constraint than the requirement of the closure of the operator algebra: in the case of Ising model the algebra actually closes by considering just $\mathbb{I}$ and ε (see (I.76)) but from (III.81) we see that the theory, to be consistent, has to involve also the magnetization σ.

The whole analysis we have exposed here has been based on the assumption of periodicity at the boundaries of the parallelogram in fig. 2. Actually other boundary conditions may be considered, and they obviously affect the operator content of the theory. Cardy [III.8] has analysed completely all the various cases of boundary conditions, pertaining the symmetries of the statistical models considered, such as antiperiodic for Z_2 models, cyclic and twisted for 3-state Potts model and so on. Rewriting the partition function with these new conditions (see also [III.9]) it is easy to work out the corresponding sum rules for $\mathcal{N}(p,q;\bar{p},\bar{q})$ in the form of (III.60) and then determine the operator content. A few simplifications may occur in some cases, and it can be seen that in a given sector each model can use only a subset of the allowed dimension in Kac formula (I.58) (for the complete list of results refer to [III.8]).

In the next section we shall concentrate once again on the determination of the conformal anomaly: this time we shall obtain it from finite size geometry considerations, and this will enable us to work out a relation valid also for models in which it is not possible to define a stress energy tensor.

III.5 Conformal Anomaly as a Finite Size Effect

In this section we shall consider as usual a statistical system in a restricted geometry, namely an infinite strip in the complex plane, and we will give a relation between the free energy of the system and its conformal anomaly , resorting to some finite size scaling results.

Let us consider the strip of fig. 3, with width $L\pi$, in which we impose periodic boundary conditions on the edges at $\pm L\pi / 2$.

This makes it a cylinder, and the coordinate transformation taking z in the plane to w in the cylinder is given by

$$w = -\frac{iL}{2} \log z \tag{III.82}$$

or

$$z = \exp\left(\frac{2iw}{L}\right) \tag{III.83}$$

Note that limiting (III.82) to the principal branch of the logarythm, they give the proper transformation for the strip in fig. 3 with cylindrical boundary conditions.

For what concerns the stress energy tensor, its expression in the strip may be worked out with the aid of the finite version of (I.14): $T_s(w)$ in the strip is given in terms of T(z) in the plane by

$$T_s(w) \quad = \quad T(z)\left(\frac{dz}{dw}\right)^2 \quad + \quad \frac{c}{12}\{z,w\} \tag{III.84}$$

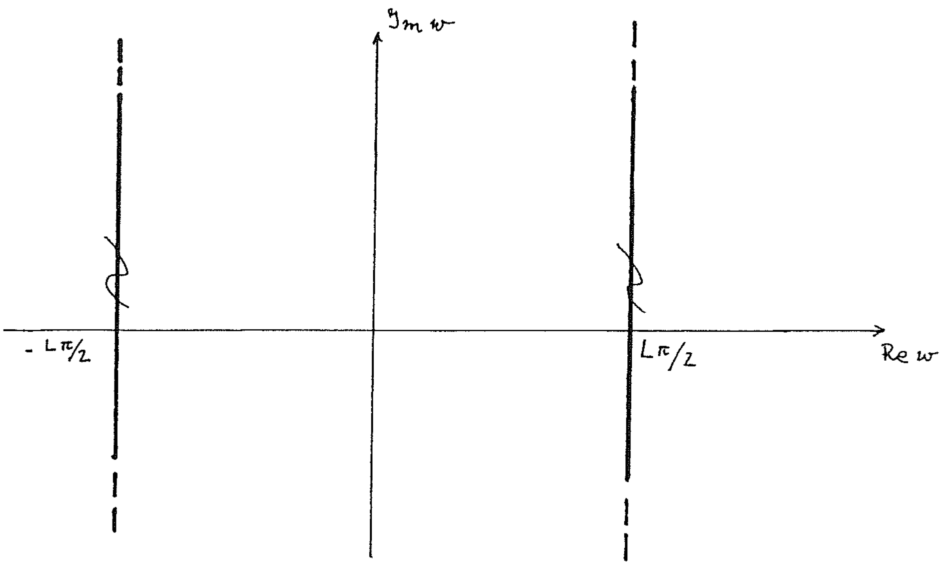

Fig. 3: Infinite strip of width $L\pi$ with periodic boundary conditions

where

$$\{z,w\} = \frac{z'''(w)}{z'(w)} - \frac{3}{2}\left(\frac{z''(w)}{z'(w)}\right)^2 \tag{III.85}$$

After a little algebra we obtain

$$T_s(w) = -\frac{4}{L^2}\left(T(z) - \frac{c}{24}\right) \tag{III.86}$$

Since we can have (see [III.5])

$$< T(z) > = 0 \tag{III.87}$$

in the plane, it turns out that

$$< T_s(w) > = \frac{c}{6L^2} \tag{III.88}$$

Therefore even if we normalize T in order to make it vanish in the plane, it is surely non zero in the strip (cfr.[III.5]).

Let us now examine which form is taken by conformal Ward identities in our geometry. Instead of considering them in the form (I.5), where they are written for a correlator of some field, we study the Ward identity for the partition function itself, that in the strip reads as

$$\delta_\varepsilon \log Z - \int d^2w <T_s^{\mu\nu}(w)> \partial_\mu \varepsilon_\nu(w) = 0 \tag{III.89}$$

ε being an infinitesimal transformation and Z the partition function of our system. The integral (III.89) is considered as given by

$$\int_s d^2w = \int_{-L\pi/2}^{L\pi/2} dw_1 \int_{-M/2}^{M/2} dw_2 \tag{III.90}$$

and afterwards M will be let go to infinity.

In the case of an infinitesimal dilatation of our strip

$$L \rightarrow (1+\Delta)\, L \tag{III.91}$$

with $\delta L = \Delta \cdot L$, it turns out that

$$\partial_\mu \varepsilon_\nu = \Delta\, \delta_{\mu,1}\, \delta_{\nu,1} \tag{III.92}$$

and

$$\delta_{\varepsilon} \log Z = \frac{\delta \log Z}{\delta L} \Delta.L \qquad \text{(III.93)}$$

Substituting (III.90), (III.92) and (III.93) into the conformal Ward identity (III.89) we obtain

$$\frac{\delta \log Z}{\delta L} \Delta.L - \int_{-L\pi/2}^{L\pi/2} dw_1 \int_{-M/2}^{M/2} dw_2 \quad < T_s^{11}(w) > \Delta = 0 \qquad \text{(III.94)}$$

From (I.9) and (III.88) it is easily seen that

$$< T_s^{11}(w) > = \frac{1}{2} (< T_s(w) > + < \overline{T}_s(\overline{w}) >) = < T_s(w) > = \frac{c}{6L^2} \qquad \text{(III.95)}$$

and plugging it in (III.94) we have

$$\frac{\delta \log Z}{\delta L} \Delta L - \int_{-L\pi/2}^{L\pi/2} dw_1 \int_{-M/2}^{M/2} dw_2 \frac{c}{6L^2} \Delta = 0 \qquad \text{(III.96)}$$

Carrying out the integrations, this finally yields

$$\frac{\delta \log Z}{\delta L} \Delta L - \frac{c\pi M}{6L^2} \Delta L = 0 \qquad \text{(III.97)}$$

which gives

$$\frac{\delta \log Z}{\delta L} = \frac{c\pi M}{6L^2} \qquad \text{(III.98)}$$

Let us now define the free energy per unit length of the strip as

$$F(L) = -\lim_{m \to \infty} (\frac{1}{M}) \log Z \qquad \text{(III.99)}$$

and let us combine (III.99) with the result (III.98): we get

$$\frac{\delta F(L)}{\delta L} = - \frac{c\pi}{6L^2} \qquad \text{(III.100)}$$

In order to determine F(L) we have to integrate this expression between L and ∞ , thus obtaining

$$F(\infty) - F(L) = \frac{c\pi}{6\lambda}\Big|_L^\infty = -\frac{c\pi}{6L} \qquad \text{(III.101)}$$

The finite size scaling hypothesis (see [III.5], [II.11]) gives the following form for $F(\infty)$ in the case of periodic boundary conditions:

$$F(\infty) = f_v\ L \qquad \text{(III.102)}$$

f_v being the free energy per volume. Therefore the final result is

$$F(L) = f_v L + \frac{c\pi}{6L} \qquad \text{(III.103)}$$

This is the desired equation, which gives the conformal anomaly as a finite size effect in a statistical system. Relation (III.103) has been worked out here in the assumption of periodic boundary conditions on the edges of the strip, but our considerations hold true also in the case of other type of conditions, leading to a slightly different result (see [III.5]).

Furthermore, the correspondence between the free energy and the ground state of the hamiltonian corresponding to the infinitesimal transfer matrix of the system (see, e.g. [III.10]), allows us to interpret the central charge as related to the magnitude of a Casimir effect in a (1+1)-dimensional quantum field theory (see [III.5]).

The relation we have found may be checked either analytically or numerically in a wide class of models, such as Ising or gaussian models. Exploiting their correspondence with a Gaussian model (III.103) has been proven valid numerically also in the case of six and eight vertex models.

A recent line of development arising from it concerns Bethe ansatz computations of central charge in statistical models (see, for example, [III.12], [III.13]). We will consider, in the next section, a very detailed check of (III.103) in the case of the Ising model: in order to do that, we will introduce a Grassmann variable formulation of it (see [III.3], [III.4]) which appears to be very useful and profitable, since it can be generalized to any free fermion model.

III.6 Grassmann Formulation of the Ising Model and Central Charge Computations.

This section is devoted to a survey of the Grassmann variables techniques introduced and developed in statistical mechanics by Samuel (see [III.3],[III.4] and [III.14]).

This formulation establishes very deep links between quantum field theory and statistical mechanics, since it is able to treat a statistical system as a fermion field theory, and, besides that, it can be applied either to exactly solved or unsolved statistical models in two and theree dimensions (see [III.14]).

Applications of anticommuting variables in statistical mechanics trace back to the graphical representation of two-dimensional statistical models solvable with Pfaffian methods. A typical example is the closed packed dimer problems (see [III.15]) which in turn can be seen to be equivalent to a free fermion vertex model (see [III.16]). Consequently many models which have a graphical representation can be treated with the aid of Grassmann variables. This is also the case of Ising model in two dimensions: the Grassmann formulation we shall illustrate for it in the sequel, allows us to study it as a particular case of a free fermion eight vertex model.

Let us consider an Ising model in a two-dimensional lattice with horizontal and vertical couplings J_h and J_v ; its graphical formulation can be constructed by drawing lines surrounding regions with constant spin. The partition function can be then written in term of the partition function of closed non overlapping polygons, which can at most intersect, and one gets

$$Z_{\text{Ising}}(J_h, J_v) \quad = \quad g \; Z_{\text{closed polygons}}(z_h, z_v) \qquad \text{(III.104)}$$

with

$$g \;=\; \exp\left[N(\beta J_v + \beta J_h)\right] \qquad \text{(III.105)}$$

N being the number of sites in the lattice and $\beta = K_B T$. Parameters z_h and z_v are Boltzmann weights for horizontal and vertical Bloch walls, which are given by

$$z_h = \exp(-2\beta J_h) \quad z_v = \exp(-2\beta J_v) \qquad \text{(III.106)}$$

in the case of the Ising model.

In order to give the explicit expression of $Z_{\text{closed polygons}}$ we resort to two sets of Grassmann variables which can be associated to a site (μ , ν) of the lattice: considering the possibility of two kinds of sites (x and o sites) these sets are given by

$$\eta^{h^0}_{\mu\nu} \ , \ \eta^{h^\times}_{\mu\nu} \tag{III.107}$$

and

$$\eta^{v^0}_{\mu\nu} \ , \ \eta^{v^\times}_{\mu\nu} \tag{III.108}$$

where h and v stand for horizontal and vertical.

We have then

$$Z_{c.p.}(z_h, z_v) \quad = \quad (-1)^N \int d\eta^0 \, d\eta^\times \ \exp(-A) \tag{III.109}$$

with

$$A = A_{\text{Bloch wall}} + A_{\text{corner}} + A_{\text{monomer}} \tag{III.110}$$

Each term in the action creates the corresponding part of the polygon, which can be considered as a fermion trajectory: it turns out that (cfr.[III.3])

$$A_{\text{Bloch wall}} \quad = \sum_{\mu,\nu} (z_h \eta^{h^\times}_{\mu\nu} \eta^{h^\circ}_{\mu+1,\nu} + z_v \eta^{v^\times}_{\mu\nu} \eta^{v^\circ}_{\mu,\nu+1}) \tag{III.111}$$

$$A_{\text{corner}} = \sum_{\mu , \nu} (a_1 \eta^{h^\times}_{\mu\nu} \eta^{v^\circ}_{\mu\nu} + a_2 \eta^{v^\times}_{\mu\nu} \eta^{h^\circ}_{\mu\nu} + a_3 \eta^{v^\times}_{\mu\nu} \eta^{h^\times}_{\mu\nu} + a_4 \eta^{v^\circ}_{\mu\nu} \eta^{h^\circ}_{\mu\nu}) \tag{III.112}$$

$$A_{\text{monomer}} = \sum_{\mu,\nu} (b_h \eta^{h^\circ}_{\mu\nu} \eta^{h^\times}_{\mu\nu} + b_v \eta^{v^\circ}_{\mu\nu} \eta^{v^\times}_{\mu\nu}) \tag{III.113}$$

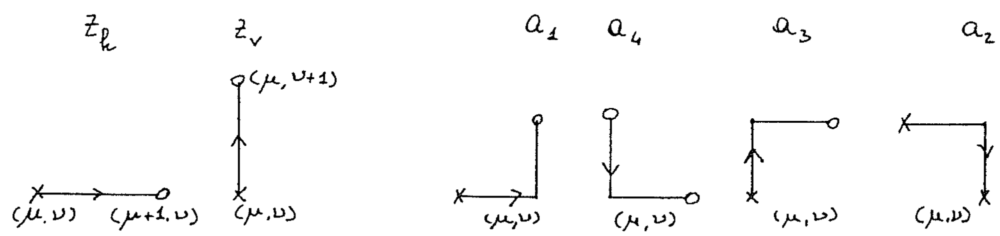

fig. 4 : Bloch walls and corners with their weights. The arrows represent the order of η variables

The first creates a vertical or horizontal unit of Bloch wall from an x site to an o one, while the second gives corners necessary to build up polygons (see fig. 4); the third one, finally, fills all free sites with o−x pairs ("monomers"): since the square of an anticommuting variables vanishes, the polygons so drawn cannot overlap.

The partition function (III.109) is actually quite general, and in fact it represents a free fermion eight vertex model, that we now define (see [III.17]).

Let us consider a square lattice, and on each one of its edges put an arrow which can point toward a certain site, but with the following constraint: the only allowed configurations are those in which there is an even number of arrows flowing into or out of each site. This permits a maximum of eight individual vertex configurations, given in Fig. 5, each one with an energy ε_i and a Boltzmann weight

$$\omega_i = \exp(-\varepsilon_i/k_BT) \qquad i = 1, \dots, 8 \qquad \text{(III.114)}$$

Like the Ising model, also the eight vertex model admits a graphical representation which can be constructed by drawing an heavy line on an edge of the lattice if the corresponding arrow points down or to the left (see fig. 5).

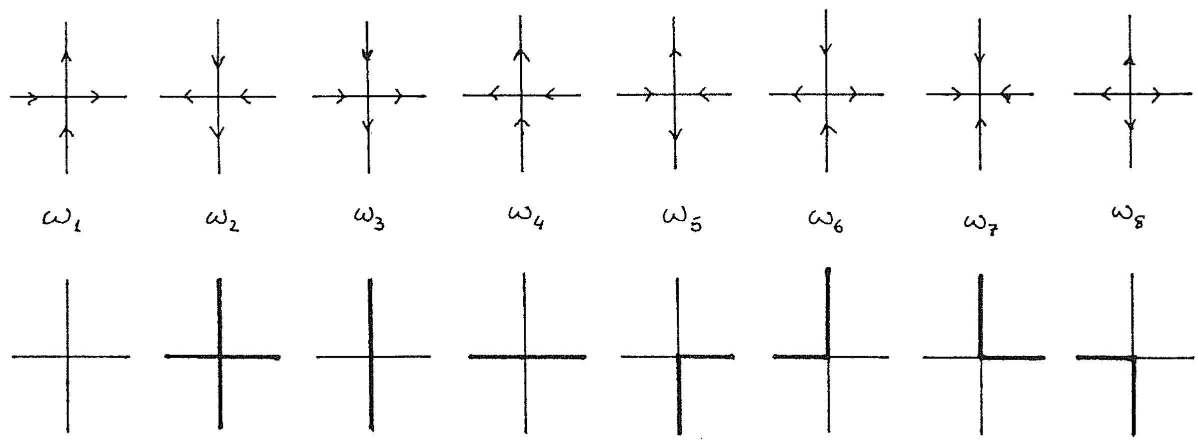

fig. 5 : The eight allowed vertex configurations, with their line representation and their weights

Then to each line configuration we can assign a Boltzmann weight (III.114), and referring to the meaning of the various terms of the action in (III.109) it can be given (see [III.3]) a complete correspondence between (III.114) and the weights a_i, b_i of (III.112) and (III.113), which is depicted in fig.6 .

From it we can easily see that the Grassmann variables formulation of the model given in (III.109) is constructed in such a way that it enforces automatically the free fermion condition. This is a further constraint on the model, namely on its vertex weights (III.114), and reads as follows

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8 \qquad \text{(III.115)}$$

Under the assumption (III.115) (which is readily verified by weights a_i, b_i in fig. 6) the eight vertex model has been firstly solved in 1970 by Fan and Wu (see [III.16]).

We shall now explicitly compute the partition function and the free energy from the general formula (III.109): the result can recover the Ising model by imposing (III.106) and setting all $a_i = b_i = -1$, and then it can be used to determine the conformal anomaly in this case as a finite size effect.

Let us consider, for definiteness, a square lattice with 2N+1 rows and 2M+1 columns, so that we have (2M+1)×(2N+1) sites, and first of all let us go to the momentum space via a Fourier transform

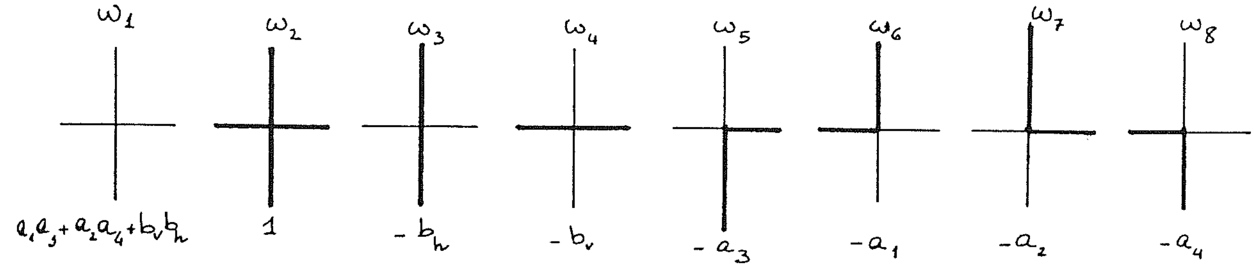

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fig. 6: Correspondence between ω_i and a_i, b_i. The minus sign is due to the presence of $(-1)^N$ in (III.109). Note that substituting the proper weights a_i, b_i, (III.115) is readily verified.

$$\eta^{r}_{\mu\nu} = \sum_{s,t} (2M+1)^{-1/2} (2N+1)^{-1/2} \exp\left(\frac{2\pi i \mu s}{2M+1} + \frac{2\pi \nu t}{2N+1}\right) \xi^{r}_{st} \qquad \text{(III.116)}$$

where s and t go from -M to M and from -N to N respectively and ξ^{r}_{st} is a set of anticommuting variables equivalent to that of η's.

Equation (III.116) involves periodic boundary conditions, and the transformation it represents has a jacobian equal to one , so that the functional integral (III.109) remains unaffected: one has to think of (μ,ν) as space variables, and of (t,s) as momenta, in the following way

$$(\mu,\nu) \leftrightarrow (x,y)$$
$$\left(\frac{2\pi s}{2M+1}, \frac{2\pi t}{2N+1}\right) \leftrightarrow (p_x, p_y) \qquad \text{(III.117)}$$

In consequence of (III.116) the free fermion action (III.110) transforms in

$$A = \sum_{s,t} [\, z_h \xi^{h^\times}_{st} \xi^{h^\circ}_{st} \exp\left(\frac{2\pi i s}{2M+1}\right) + z_v \xi^{v^\times}_{st} \xi^{v^\circ}_{st} \exp\left(\frac{2\pi i t}{2N+1}\right) + a_1 \xi^{h^\times}_{st} \xi^{v^\circ}_{st} +$$
$$+ a_3 \xi^{v^\times}_{st} \xi^{h^\circ}_{st} + a_2 \xi^{v^\times}_{st} \xi^{h^\times}_{-s,-t} + a_4 \xi^{v^\circ}_{st} \xi^{h^\circ}_{-s,-t} + b_h \xi^{h^\circ}_{st} \xi^{h^\times}_{st} + b_v \xi^{v^\circ}_{st} \xi^{v^\times}_{st} \,] \qquad \text{(III.118)}$$

Employing the usual rules of integration of Grassmann variables (see, e.g. [III.3]) we get for the free fermion partition function

$$Z_{\text{f.f.}} = \left[\prod_{s,t} L(s,t) \right]^{1/2} \qquad \text{(III.119)}$$

where L(s,t) is given by

$$L(s,t) = h_s h_{-s} v_t v_{-t} - a_1 a_3 (h_s v_t + h_{-s} v_{-t}) - a_2 a_4 (h_s v_{-t} + h_{-s} v_t) +$$
$$+ (a_1 a_3 + a_2 a_4)^2 \qquad \text{(III.120)}$$

with

$$h_s = b_h - z_h \exp\left(\frac{2\pi i s}{2M+1}\right) \qquad \text{(III.121)}$$

$$v_t = b_v - z_v \exp\left(\frac{2\pi i t}{2N+1}\right) \qquad \text{(III.122)}$$

Therefore for the free energy, which is the quantity we have to compute in order to determine the central charge, we obtain

$$-\beta F = \frac{1}{2} \sum_{s,t} \log L(s,t) \qquad \text{(III.123)}$$

in a finite square lattice with (2N+1) (2M+1) sites. Since our considerations have been quite general till now, result (III.123) holds true for any free fermion eight vertex model: we can specialize it for the case of Ising model by posing

$$a_i = b_h = b_v = -1 \qquad i = 1,\dots,4 \qquad \text{(III.124)}$$

In order to compute the conformal anomaly for the Ising model we have also to enforce an isotropy constraint on it, so to make sure that conformal symmetry can take place at criticality: this additional condition is simply

$$J_h = J_v \equiv J \qquad \text{(III.125)}$$

which gives

$$z_h = z_v = \exp(-2\beta J) \qquad \text{(III.126)}$$

If we substitute (III.124) and (III.126) in (III.120) we get the free energy for the Ising model:

$$-\beta F_{Ising} = \frac{1}{2} \sum_{s,t} \log L_{Ising}(s,t) \qquad \text{(III.127)}$$

where

$$L_{Ising}(s,t) = h^I_s h^I_{-s} v^I_t v^I_{-t} - (h^I_s v^I_t + h^I_{-s} v^I_{-t}) - (h^I_s v^I_{-t} + h^I_{-s} v^I_t) + 4 \qquad \text{(III.128)}$$

with

$$h_s^I = -[\,1 + \exp(-2\beta J)\exp(\frac{2\pi i s}{2M+1})\,] \tag{III.129}$$

$$v_t^I = -[\,1 + \exp(-2\beta J)\exp(\frac{2\pi i t}{2N+1})\,] \tag{III.130}$$

According to the analysis of the previous section, if we consider our statistical system on an infinite strip we can get its central charge as a finite size scaling contribution to the free energy: since we assume periodic boundary condition, we will refer to formula (III.103). We can fit (III.127) to the case of an infinite strip by performing a sort of thermodynamic limit in one direction, by letting, say, N go to infinity: this transforms one of the two sums in (III.127) into an integral , according to the usual relation

$$\sum_{t=-N}^{N} L(t\Delta t)\,.\,\Delta t \quad \sim \quad \int_{-\pi}^{\pi} dt\; L(t) \tag{III.131}$$

where now

$$\Delta t = \frac{2\pi}{2N+1} \tag{III.132}$$

Thus we can easily obtain the free energy per unit length for the Ising model in an infinite strip

$$-\beta f_{Ising} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{dp}{2\pi}\sum_{s=-M}^{M}\log L_{Ising}(s,p) \tag{III.133}$$

Easy algebraic manipulations yield the form of L_{Ising} as follows

$$L_{Ising}(s,p) = 4\,e^{-4\beta J}\,[\,\cosh^2(2\beta J) - \sinh(2\beta J)\,(\cos\frac{2\pi s}{2M+1} + \cos p)\,] \tag{III.134}$$

and substituting it into (II.133) we get

$$-\beta f_{Ising} = (2M+1)\log(2e^{-2\beta J}) +$$

$$+ \frac{1}{2}\int_0^{2\pi} \frac{dp}{2\pi} \sum_{s=-M}^{M} \log\left[\cosh^2(2\beta J) - \sinh(2\beta J)\left(\cos\frac{2\pi s}{2M+1} - \cos p\right)\right]$$

(III.135)

As we are interested in corrections of order 1/M for large M (cfr. (III.103)) to the free energy, we can neglect the first term in (III.135). The integral over p can be easily evaluated with the aid of the tables (see [III.18]): in particular we use the integration formula

$$\int_0^{2\pi} \frac{dp}{2\pi} \log\left[A - B\cos(p) + C\sin(p)\right] = \log\frac{1}{2} + \log\left[A + \sqrt{A^2 - B^2 - C^2}\right]$$

(III.136)

which holds provided $A^2 > B^2 + C^2$, and where we can pose

$$A = \cosh^2(2\beta J) - \sinh(2\beta J)\cos\frac{2\pi s}{2M+1}$$

$$B = \sinh(2\beta J) \qquad \text{(III.137)}$$

$$C = 0$$

Neglecting once again the first term in (III.136), which gives a contribution proportional to (2M+1), we are left with

$$-\beta f_{Ising} = \sum_{s=1}^{M} \log\left[\cosh^2(2\beta J) - \sinh(2\beta J)\cos\frac{2\pi s}{2M+1} + \right.$$

$$\left. + \sqrt{\left(\cosh^2(2\beta J) - \sinh(2\beta J)\cos\frac{2\pi s}{2M+1}\right)^2 - \sinh^2(2\beta J)}\right]$$

(III.138)

where we have exploited the simmetry of the summand under $s \rightarrow -s$ and omitted the irrelevant term corresponding to s=0. We now impose criticality condition for the Ising model, in order to make it conformal invariant: at the self dual point we have (see [III.17])

$$\cosh^2(2\beta J) = 2 \tag{III.139}$$

and choosing $\sinh 2\beta J = -1$ it turns out

$$-\beta f_{\text{critical Ising}} = \sum_{s=1}^{M} \log \left[2 + \cos\frac{2\pi s}{2M+1} + \sqrt{\left(2 + \cos\frac{2\pi s}{2M+1}\right)^2 - 1}\ \right] \tag{III.140}$$

The problem of determining the conformal anomaly is now reduced to that of studying the contribution O(1/M) to the sum (III.140) in the limit of great M. This is done by noticing ([III.19]) that in a sum involving a function of the cosine, the coefficient of the wanted term is given by

$$\sum_{s=1}^{M} f\left(\cos\frac{2\pi s}{2M+1}\right) \underset{\substack{M\to\infty \\ O(1/M)}}{\sim} -\frac{x}{24}\frac{d}{dx} f\left(\cos\frac{x}{2}\right)\Big|_{x=2\pi} \tag{III.141}$$

Let us then consider

$$\mathcal{F}(x) = -\frac{x}{24}\frac{d}{dx} \log \left[2 + \cos\frac{x}{2} + \sqrt{\left(2 + \cos\frac{x}{2}\right)^2 - 1}\ \right] \tag{III.142}$$

which after a short manipulation gives

$$\mathcal{F}(x) = \frac{x}{48} \frac{1}{2 + \cos\frac{x}{2} + \sqrt{\left(2 + \cos\frac{x}{2}\right)^2 - 1}} \left\{ \sin\frac{x}{2} + \frac{\sin\frac{x}{2}\left(2 + \cos\frac{x}{2}\right)}{\sqrt{\left(2 + \cos\frac{x}{2}\right)^2 - 1}} \right\} \tag{III.143}$$

Examining its limit for $x \to 2\pi$ we have two terms

$$\mathcal{F}(x) = \mathcal{F}_1(x) + \mathcal{F}_2(x) \tag{III.144}$$

the first of which vanishes

$$\mathcal{F}_1(x) = \frac{x \sin\frac{x}{2}}{48\left(2 + \cos\frac{x}{2} + \sqrt{\left(2 + \cos\frac{x}{2}\right)^2 - 1}\ \right)} \to 0 \quad \text{as } x \to 2\pi \tag{III.145}$$

For the second, expanding around $x = 2\pi$, we find

$$\mathcal{F}_2(x) \underset{x \to 2\pi}{\simeq} - \frac{\pi}{48} \frac{x-2\pi}{1/2(x-2\pi)} = - \frac{\pi}{24} \tag{III.145}$$

and this is the coefficient of $1/M$ in the sum (III.140) as $M \to \infty$. Since in this limit the length of the strip is $L = 2M$ we get from (III.140) and (III.145)

$$+ \beta f_{\text{crit. Ising}} = + \frac{\pi}{24} \cdot \frac{\pi}{M} = + \frac{\pi}{12} \cdot \frac{1}{L} \tag{III.146}$$

which, compared with (III.103) after having restored the Boltzmann factors, finally yields

$$C_{\text{Ising}} = 1/2 \tag{II.147}$$

in agreement with (I.73).

The method outlined here in order to evaluate (III.147) , based on a finite size scaling interpretation of the central charge of a statistical system , has not to be thought of simply as another way to derive already known results.

Its interest lies in the fact that it may provide a relatively simple, analytic procedure to compute the conformal anomaly for the 6 and 8 vertex models, which may be an alternative to a rather cumbersome application of the Bethe ansatz (see [III.12], [III.13]). Besides it can be further generalized to recover the cases of 32 and 128 vertex models, whose free energy has been already worked out (see [III.20], [III.21] and [III.22]), and also a closed packed dimer problem, for which a clear graphical solution (see [III.3] and [III.15]) can be given .Work in this direction in in progress.

These comments end our analysis of statistical systems in a restricted geometry, and also our survey of conformal symmetry implications in statistical mechanics, in which we have reviewed various methods for studying critical systems, and computing their physically interesting quantities.

Since the publication of the fundamental work of Belavin , Polyakov and

Zamolodchikov [I.1] , the general interest in these subjects has continuously increased. Various lines of development, such as the classification of critical systems according to modular invariance, the extension to the supersymmetric case or the study of parafermionic systems, have sharpened the unification of two branches of physics which did already find major instances of synthesis: quantum field theory and the study of critical phenomena.

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