ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Thesis submitted for the degree of "Magister Philosophiae"

CONTINUATION THEOREMS FOR THE EXISTENCE OF PERIODIC SOLUTIONS TO FIRST ORDER DIFFERENTIAL SYSTEMS

Candidate:
Anna CAPIETTO

Supervisor:
Prof. Fabio ZANOLIN

Academic Year 1987/88

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Strada Costiera 11

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Contents

CHAPTER ZERO

INTRODUCTION AND NOTATION

We deal with the periodic boundary value problem (BVP)

(1.1) $\dot{x} = F(t,x)$
(1.2) $x(0) = x(\omega)$,

where $F : [0,\omega]\times C \to \mathbf{R}^m$ is a continuous function, $C \subset \mathbf{R}^m$ and $\omega>0$.
Our aim is to prove the existence of a solution $x(\cdot)$ to (1.1)-(1.2) such that, for all $t \in [0,\omega]$, $x(t)$ belongs to a given subset of C.

The periodic boundary value problem plays a central role in the qualitative theory of ODEs for its significance in the physical sciences. Biology, mathematical economy and hydrodynamics provide models for situations in which the set C is not the whole space.

We use topological methods. Generally speaking, solutions to (1.1)-(1.2) are obtained as fixed points of operators defined in function spaces. Following a classical procedure, one first performs (conditions leading to) a-priori bounds for the solutions of (1.1)-(1.2) (or, more precisely, for the solutions of certain related equations) - the so-called "transversality conditions"; secondly, the Brouwer degree of a suitable autonomous map is required to be nonzero. If $C = \mathbf{R}^m$, two main approaches have been used.
The first one goes back to the Leray-Schauder theorem. Indeed, one writes

$$F(t,x) := f(t,x;1),$$

where $f = f(t,x;\lambda) : [0,\omega]\times\mathbf{R}^m\times[0,1]\to\mathbf{R}^m$, and imbeds (1.1) in a family of parametrized equations

$$\dot{x} = \lambda f(t,x;\lambda),$$

$\lambda \in (0,1)$; then, one requires some suitable a-priori bounds for the solutions of *these* parametrized problems. Subsequently, one considers the averaged map

$$\overline{f}_0(x) := \frac{1}{\omega}\int_0^{\omega} f(s,x;0)ds.$$

and asks that its Brouwer degree (relatively to some open subset of $\mathbf{R}^m$) is nonzero. The existence of at least one solution to (1.1)-(1.2) is proved by imbedding the given problem in an abstract functional-analytic framework via the Liapunov-Schmidt reduction. Along these lines, J. Mawhin has obtained theorem 3.1. We point out that in this result a more refined type of degree is used - the "coincidence degree" - and, furthermore, a condition more general than a-priori bounds - the "bound set condition" - is introduced.
On the other hand, M.A. Krasnosel'skii developed a method (see theorem 3.2) which consists of the search of fixed points of the translation operator (Poincaré-Andronov map) $\pi_\omega : x(0) \longmapsto x(\omega)$. Besides the "$\omega$-irreversibility condition" (which concerns the solutions of equation (1.1) with initial value on the boundary of a given subset of $\mathbf{R}^m$), it is required that the Brouwer degree of the map $F_0(x) := F(0,x)$ is nonzero.
Along this direction, R. Srzednicki has recently given a contribution to the periodic BVP in the framework of Ważewski's method (see th. 3.3 and cor. 3.1).

In recent years, some results were obtained in the case when the underlying space does not have a linear structure. For instance, in [4,19,20], [12,18,22] the case in which the set C is a regular manifold, a convex set or a conical shell, respectively, are studied.

Our work fits into the above framework. Instead of the Brouwer degree, we use the fixed point index, first for maps defined in subsets of $\mathbf{R}^m$ and secondly in a more abstract situation (which we explain below).

Our first result (theorem 4.1) is a continuation theorem for the existence of periodic solutions lying in convex sets. More precisely, let C be a closed convex subset of $\mathbf{R}^m$ and let $G \subset C$ be a bounded set, open relatively to C; we prove the existence of a solution $x(\cdot)$ to (1.1)-(1.2) such that x(t) belongs to the closure of G relatively to C for all t. This result is in the lines of the first method sketched above, i.e. the Liapunov-Schmidt reduction and Mawhin's continuation theorem 3.1, which we generalize to the case of convex sets. The proof is carried out using the concept of invariance (cf. Nagumo's theorem, cone conditions) and the properties of the fixed point index. In the corollaries and in the applications we use the concept of "block" (cf. [9]) and Srzednicki's theorem 2.1.

This first part of our work shows (as was also pointed out in [38]) that with some simple changes in the proof of theorem 4.1 it is possible to obtain a generalization of Krasnosel'skii's theorem too.

As a further step, we produce our main result, i.e. a continuation theorem for the periodic BVP in flow-invariant ENRs (theorem 4.2). We recall that a metric space X is an Absolute Neighbourhood Retract (ANR) if and only if X is homeomorphic to a subset Y of a Banach space B and Y is a neighbourhood retract of B. In the particular case $B = \mathbf{R}^m$, the set X is called an Euclidean Neighbourhood Retract (ENR). Our result is unifying, both regarding the methods we use and with respect to the properties we require for the set C. Indeed, on the one hand we imbed our problem in a functional-analytic framework (which, however, is not the same as in [38]) and we study, again, the family of parametrized equations $\dot{x} = \lambda f(t,x;\lambda)$, $\lambda \in (0,1)$; on the other hand, we use the properties of the translation operator. Furthermore, since regular manifolds, closed convex sets and conical shells are ENRs, our result contains all the situations mentioned above. Nevertheless, we point out that, in the particular case of convex sets, theorem 4.1 is slightly more general than theorem 4.2; in any case, the two proofs are completely different.
In the functional-analytic framework in which we imbed (1.1)-(1.2), which is inspired by the study of the Poincaré map π_ω, we use the definition and properties of the concept of "process". We refer the reader to the proof of theorem 4.2, since details are too long to be repeated here. As in our previous work, we find solutions to (1.1)-(1.2) as fixed points of an operator defined in $\mathcal{Z} := \{x:[0,\omega]\to C,\ x \text{ continuous}\}$. A crucial result (see [31]) ensures that $\mathcal{Z}$ is an ANR if and only if C is an ANR. In this situation, we can use the fixed point index theory for ANRs as introduced by A. Granas in [23]; more precisely, through this concept we define (as in [20,49]) the "index of rest points". This very notion, which plays the role of the Brouwer degree, illustrates the meaning of the word "unifying" that we used above.

Chapter 1 is devoted to the fixed point index for ANRs. After some algebraic topology preliminaries (essentially, the definition of Lefschetz number), we give, following [23], the axioms of the fixed point index. In section 1.6 we study the Euler-Poincaré characteristic of an ENR, which turns out to be quite an useful tool for the computation of the index of rest points.

Chapter 2 contains the definition and some properties of the index of rest points.

Chapter 3 is a survey of a number of theorems on the existence of periodic solutions to (1.1)-(1.2) that we often recall in our results.

Chapter 4 contains our main results (theorem 4.1 and theorem 4.2) and some corollaries, which extend theorems quoted in Chapter 3. We point out that, since the proof of theorem 4.2 is based on processes, we need the uniqueness for the solutions of the Cauchy problems which we consider. Hence, we assume for simplicity that all the vector fields we deal with are locally lipschitzian in the space variable. This regularity assumption was not required in [20,38]. However, we point out that in such cases a "continuous" version of our theorems can be obtained by means of a standard perturbation argument based on Weierstrass-Stone and Ascoli-Arzelà theorems. We also notice that, without loss of generality, we can assume (if it is convenient) $F : [0,\omega]\times A\rightarrow \mathbf{R}^m$, with A any open set such that $C\subset A\subset \mathbf{R}^m$ (see remark 4.5).

Chapter 5 contains an extension to the case of ENRs of the Krasnosel'skii method of guiding functions; besides, we compute an analogous of the "index of nondegeneracy" for a potential function (see [33, p.84]).

Chapter 6 is devoted to the applications. In section 6.1 we consider equations arising from biological models (e.g. the Lotka-Volterra system) and we prove the existence of solutions lying in the convex set $C := \mathbf{R}^m_+$ (i.e. non-negative solutions). In section 6.2 we study a situation in which it is natural to work in a domain with holes. This situation occurs, for example, in hydrodynamic applications.

The m-dimensional real euclidean space $\mathbf{R}^m$ is endowed with the usual inner product $(\cdot|\cdot)$, norm $|\cdot| = (\cdot|\cdot)^{1/2}$ and distance $d(\cdot,\cdot)$. We denote by $\{e_i,\ i = 1,..,m\}$ its canonical basis and we define $\hat{x}_i := (x_1,..,x_{i-1},0,x_{i+1},..x_m)$. $\mathbf{R}_+$ (resp. $\mathbf{R}^+$) is the set of nonnegative (resp. positive) reals. Given any metric space Y, we denote by $B(x,R)$ (resp. $B[x,R]$) the open (resp. closed) ball of center $x\in Y$ and radius $R>0$. For $A\subset B\subset Y$, by $\mathrm{int}_B A$, $\mathrm{fr}_B A$, $\mathrm{cl}_B A$ we mean, respectively, the interior, boundary and closure of the set A relatively to B; cardA is the cardinality of the set A. We omit the subscript whenever no confusion occurs and we also set $S(x,R) := \mathrm{fr}B(x,R)$.

For a closed convex set $K\subset \mathbf{R}^m$ we denote by $\mathcal{N}(u,K)$ the set of outer normals to K at $u\in \mathrm{fr}K$. We recall that $\eta\in \mathcal{N}(u,K)$ if and only if $\eta\in \mathbf{R}^m\setminus\{0\}$ and $K\subset\{x\in \mathbf{R}^m : (x - u\,|\eta)\leq 0\}$. If X is a normed space, $|\cdot|_X$ denotes its norm. For $V : \mathbf{R}^m\rightarrow \mathbf{R}$ and $-\infty\leq a<b\leq+\infty$ we define $[a<V<b] := \{x\in \mathbf{R}^m: a<V(x)<b\}$ and $[V=a] := \{x\in \mathbf{R}^m: V(x) = a\}$.

Let $\omega>0$ be a fixed constant. For a continuous vector function $x(\cdot) : [0,\omega]\rightarrow \mathbf{R}^m$, we set $|x|_\infty := \sup\{|x(t)|,\ t\in[0,\omega]\}$, $|x|_1 := \int_0^\omega |x(s)|ds$ (L_1-norm) and $\overline{x} := \frac{1}{\omega}\int_0^\omega x(s)ds$. Furthermore, for a function $y = y(t;\lambda) : [0,\omega]\times[0,1]\rightarrow \mathbf{R}^m$, we write $\overline{y}_\lambda := \frac{1}{\omega}\int_0^\omega y(s;\lambda)ds$.

Finally, we denote the Brouwer degree by d_B.

CHAPTER 1

THE FIXED POINT INDEX AND RELATED TOPICS

1.1. This chapter is devoted to the fixed point index, which is one of the most important concepts in this thesis; it is widely used expecially in Chapter 4, where we prove our main results.

We introduce in an axiomatic way the fixed point index for a rather general class of spaces and maps. To this aim, we follow [23,24,40].

1.2. First, we must point out that some knowledge of algebraic topology is needed. More precisely, we use (singular) homology theory. We refer to the Eilenberg-Steenrod axioms as stated, for example, in [51]. For brevity, we do not recall explicitly the axioms and main properties of homology theory. (See also [5,13]).

1.3. Now, we introduce the class of spaces for which we define the fixed point index.

DEFINITION 1.1. *A metric space* X *is an ABSOLUTE NEIGHBOURHOOD RETRACT (ANR) if and only if for every metric space* Y, *for every closed subset* M *of* Y *and for every continuous map* $f: M \to X$ *there exists a continuous extension* $\tilde{f}$ *of* f *which is defined in an open set containing* M.

REMARK 1.1. We point out that, equivalently, X is an ANR if and only if X is homeomorphic to a subset Y of a Banach space B and Y is a neighbourhood retract of B.

DEFINITION 1.2. *If* X *is an ANR and the Banach space* B *in the above remark is* $\mathbf{R}^m$, *then* X *is called an EUCLIDEAN NEIGHBOURHOOD RETRACT (ENR)*.

Now we recall some elementary properties and examples of ANRs (for a general treatment of ANRs, see [31]).

(a) (Dugundji) [14]. If X is a closed convex subset of a normed linear space, then X is an ANR (indeed, X is a retract of the space).

(b) If X is a closed subset of a normed linear space Y and if there exists a family $\{C_j\}_{j\in J}$ of closed convex subsets of Y such that $X = \bigcup_{j\in J} C_j$ and $\{C_j\}_{j\in J}$ is a locally finite covering of X, then X is an ANR.

(c) A retract of an ANR is an ANR.

(d) Every open subset of an ANR is an ANR.

This latter fact implies that any ANR is locally ANR. The converse of proposition (d) is also true; namely:

(e) (Hanner) [28]. If X is a metric space and every $x \in X$ is contained in an open neighbourhood N_x which is an ANR, then X is an ANR.

In particular, a metrizable Banach manifold is an ANR (this fact will be extensively used in the sequel).
Finally, we recall:

(f) (West) [53]. Every compact ANR is homotopically equivalent to a compact polyhedron (see also Section 1.6 below).

(g) ([31]). Let X be a compact metrizable space and Y a metrizable space. Let d be a distance which defines the topology of Y. Consider the function space $\Omega = \{f : X \to Y, \text{continuous}\}$, endowed with the distance d^*, where $d^*(f,g) := \sup_{x\in X} d(f(x), g(x))$. Then, Ω with the d^*-topology is an ANR if and only if Y is an ANR.

1.4. Before writing the axioms for the fixed point index, we introduce the notions of generalized trace and Lefschetz number as given by J. Leray (see [35]). Indeed, the Lefschetz fixed point theory is an useful preliminary to the fixed point index theory. Moreover, the fixed point index theory contains, as a corollary, the Lefschetz fixed point theorem for compact ANRs (see [23, p. 222]).

In what follows, all the vector spaces we consider are over the field of rational numbers $\mathbf{Q}$.
We recall that a graded vector space $E = \{E_q\}_{q=0}^{\infty}$ is of finite type if:

(i) $\dim E_q < \infty$ for each q ;

(ii) $E_q = 0$ for all except finitely many q.

In the sequel, we denote by $H_q(X)$ the q-th dimensional (singular) homology group, with coefficients in the field $\mathbb{Q}$ of rational numbers, of a space X.
We also recall that, if $f:X \to X$ is a continuous function, then the induced linear map:

$$f_* := \{f_q\}_{q=0}^{\infty}, \quad f_q : H_q(X) \to H_q(X)$$

is defined.
If the graded vector space $H(X) = \{H_q(X)\}_{q=0}^{\infty}$ is of finite type, then we can consider the ordinary trace $tr(f_q)$ of the linear map f_q, for each q, and give the following:

DEFINITION 1.3. *The Lefschetz number of the map* f *is given by the formula* :

$$\lambda(f) := \sum_{q=0}^{\infty} (-1)^q \, tr(f_q).$$

Since $\{H_q(X)\}_{q=0}^{\infty}$ is of finite type, the above sum is finite and the definition is meaningful.
Indeed, we can give the definition of Lefschetz number also in a more general situation. More precisely, if $\{H_q(X)\}_{q=0}^{\infty}$ is not of finite type, then we can define, for each q,

$$N(f_q) := \bigcup_{n \geq 1} \mathrm{Ker}\,(f_q)^n$$

and

$$\tilde{H}_q(X) := \frac{H_q(X)}{N(f_q)} .$$

Since $f_q(N(f_q)) \subset N(f_q)$, then we can consider the induced endomorphism:

$$\tilde{f}_q : \tilde{H}_q(X) \to \tilde{H}_q(X).$$

Now, if $\dim \tilde{H}_q(X) < +\infty$ for each q, we can give the following definition:

DEFINITION 1.4. *The Lefschetz number of the map* f *is given by the formula* :

$$\Lambda(f) := \sum_{q=0}^{\infty} (-1)^q \operatorname{tr}(\tilde{f}_q).$$

We remark that if $\{H_q(X)\}_{q=0}^{\infty}$ is of finite type, then $\operatorname{tr}(f_q) = \operatorname{tr}(\tilde{f}_q)$, and the two definitions of Lefschetz number coincide.

EXAMPLE 1.1. If X is a compact ANR, then $\{H_q(X)\}_{q=0}^{\infty}$ is of finite type and $\Lambda(f)$ is defined for all continuous $f:X \to X$.

1.5. THE FIXED POINT INDEX.

Let X be an ANR and let W be an open subset of X . Let $f : W \to X$ be a continuous function. We begin with the following

DEFINITION 1.5. *The triple* (X,W,f) *is called ADMISSIBLE if the set* $S = \{x \in W: f(x) = x\}$ *is compact (possibly empty) and there exists an open neighbourhood* V *of* S *such that* $\mathrm{cl}V \subset W$ *and* $f_{|\mathrm{cl}V}$ *is compact.*

REMARK 1.2. Frequently, the axioms of the fixed point index are given in a less abstract framework, i.e. one assumes that W is a (bounded) open subset of X and $f : \mathrm{cl}W \to X$ is a compact map such that $f(x) \neq x$ for all $x \in \mathrm{fr}W$. Indeed, if this is true, then the triple (X,W,f) is admissible.

THE AXIOMS.

To any given admissible triple (X,W,f) we associate an integer

$$i_X(f,W)$$

called the fixed point index of f on W (relatively to X) satisfying the following properties:

I. EXCISION

Let W' be an open subset of W with $S \subset W'$ and let $f' = f_{|W'} : W' \to X$. Then,

$$i_X(f,W) = i_X(f',W') .$$

(Note that the triple (X,f',W') is admissible.)

II. ADDITIVITY

Assume that $W = \bigcup_{i=1}^{n} W_i$ and let $f_i := f_{|W_i}$, $S_i := S \cap W_i$. If $S_i \cap S_j = \emptyset$, $i \neq j$, then

$$i_X(f,W) = \sum_{i=1}^{n} i_X(f_i,W_i) .$$

III. FIXED POINT PROPERTY

If $i_X(f,W) \neq 0$, then $S \neq \emptyset$, i.e. the map f has a fixed point.

IV. HOMOTOPY

Let $H : W \times [0,1] \to X$ be a continuous homotopy, and let $H_t : W \to X$ be defined by $H_t(x) := H(t,x)$. Assume that $S := \bigcup_{t \in [0,1]} \{x \in W : H_t(x) = x\}$ is compact and there is an open neighbourhood V of S such that $clV \subset W$ and $H_{|clV \times [0,1]}$ is a compact mapping. Then,

$$i_X(f,H_0) = i_X(f,H_1) = \text{constant with respect to } t .$$

V. MULTIPLICATIVITY

If the triples (W_1,X_1,f_1), (W_2,X_2,f_2) are admissible, then

$$i_{X_1 \times X_2}(f_1 \times f_2, W_1 \times W_2) = i_{X_1}(f_1,W_1) \cdot i_{X_2}(f_2,W_2) .$$

VI. COMMUTATIVITY

Let U_1, U_2 be open subsets of X_1, X_2, respectively; assume that $f_1 : U_1 \to X_2$, $f_2 : U_2 \to X_1$ are continuous maps and that the map f_1 is compact in a neighbourhood of $\{x \in U_1 : f_2 f_1(x) = x\}$ (or the map f_2 is compact in a neighbourhood of $\{x \in U_2 : f_2(x) = x\}$). Consider the composite maps:

$$f_2 f_1 : f_1^{-1}(U_2) \to X_1$$

$$f_1 f_2 : f_2^{-1}(U_1) \to X_2 .$$

If one of the triples

$$(f_1^{-1}(U_2), X_1, f_2 f_1) \ , \ (f_2^{-1}(U_1), X_2, f_1 f_2)$$

is admissible, then so is the other and, in this case,

$$i_{X_1}(f_2 f_1, f_1^{-1}(U_2)) = i_{X_2}(f_1 f_2, f_2^{-1}(U_1)) .$$

VII. NORMALIZATION

If $W = X$ and the map f is compact, then the Lefschetz number of f is defined and

$$i_X(f,W) = \Lambda(f) .$$

REMARK 1.3. We point out that the axioms given above are not independent. For instance, axiom III (fixed point property) is an easy consequence of the additivity axiom.

Now, we recall a useful property of the fixed point index which follows from the commutativity axiom:

PROPOSITION 1.1 (Contraction property of the fixed point index). *Let* W *be an open subset of* X *and* $f: W \to X$ *a continuous map for which the index* $i_X(f,W)$ *is defined. If a metric ANR* Y *is a subset of* X *such that the inclusion* $j : Y \subset X$ *is continuous and* $f(W) \subset Y$, *then*

$$i_X(f,W) = i_Y(f_{|W \cap Y}, W \cap Y).$$

REMARK 1.4. First of all, we point out that if we assume that X is a compact ANR and we denote by Id the identity map, then $i_X(Id,X) = \Lambda(Id)$; this number, which depends only on the set X itself, has many important topological properties, which we recall in section 1.6. Furthermore, we remark that the normalization axiom is, essentially, the Lefschetz fixed point theorem.

It seems interesting to recall also the "weak" form of the normalization axiom (and to compare it with the analogous property of the Brouwer degree):

VII BIS. "WEAK" NORMALIZATION

Assume that W = X; if the triple (W,X,f) is admissible and f(x) = p for all x, then

$$i_x(f,W) = \begin{cases} 1 & \text{if } p \in W \\ 0 & \text{if } p \notin W \end{cases} .$$

The proof of the existence of the fixed point index for ANRs is omitted for brevity. See [23, th.7.1, th.10.1] for details.

1.6. THE EULER-POINCARE' CHARACTERISTIC.

The Euler-Poincaré characteristic of a set is another important tool in this thesis. Although it is defined in algebraic topology, in recent years it has turned out to be very useful from the point of view of analysis too. Accordingly, after the abstract definition, we briefly give an intuitive explanation of this important concept; as a consequence, we outline the way in which it can be viewed, and used, by analysts. Let $C \subset \mathbf{R}^m$ be a compact ENR (indeed, it is sufficient to consider a set which has the homotopy type of a polyhedron).

DEFINITION 1.6. *The Lefschetz number of the identity map* Id_C *is called the EULER-POINCARE' CHARACTERISTIC of the set* C *, and it is denoted by* $\chi(C)$.

From the definition of Lefschetz number, it can be seen that

$$\chi(C) = \sum_{q=0}^{\infty} (-1)^q \dim H_q(C) ,$$

where $b_q := (-1)^q \dim H_q(C)$ is called the q-th Betti number of C.
From the definition and from the homotopy axiom of singular homology it follows that the Euler-Poincaré characteristic is a homotopy invariant i.e. if C_1, C_2 are compact ENRs and if they have the same homotopy type, then $\chi(C_1) = \chi(C_2)$.
Actually, if C is a nonempty pathwise connected space, then

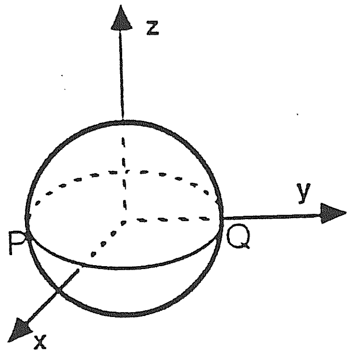
$$H_0(C) \simeq Q ,$$

so that $b_0(C) = 1$ (see [51]).
Indeed, $b_0(C)$ is the number of connected components of the set itself. In the same way, one may view the Betti numbers b_q, $q \geq 1$, as the "measure" of a form of higher-dimensional connectivity.

Another important property of χ is the following:
if the set C is a CW-complex [13], then

$$\chi(C) = \sum_{q=0}^{\infty} (-1)^q \alpha_q ,$$

where α_q is the number of "q-cells" which partition the set C. In particular, $\chi(C)$ is independent on the CW-decomposition of the set C.
For example, the m-dimensional sphere S(0,1) is a CW-complex, and for m = 2 a possible decomposition is the following:



$$S(0,1) = \{P\} \cup \{Q\} \cup S_1 \cup S_2 \cup S_3 \cup S_4 \text{ ,where}$$
$$S_1 = \{(x,y,z) \in S(0,1) : z = 0, x > 0\},$$
$$S_2 = \{(x,y,z) \in S(0,1) : z = 0, x < 0\},$$
$$S_3 = \{(x,y,z) \in S(0,1) : z > 0\},$$
$$S_4 = \{(x,y,z) \in S(0,1) : z < 0\},$$

so that $\chi(S(0,1)) = 2$ (for m = 2).
Thus, the computation of the Euler-Poincaré characteristic is, roughly speaking, a generalization of Euler's polyhedron formula, which asserts that (for m = 2) $\alpha_0 - \alpha_1 + \alpha_2 = 2$ for every decomposition of S(0,1) into disjoint celles (α_i = number of i-cells).
Obviously, we are interested in calculating $\chi(C)$ in more general situations; to this end, we recall that (see [13, p.105]), under rather general hypotheses,

$$\chi(C_1) + \chi(C_2) - \chi(C_1 \cap C_2) = \chi(C_1 \cup C_2) \ .$$

The following useful formulas can be proved by the axioms and properties of singular homology:

$$\chi(\{P\}) = 1, \quad \chi(B[0,1]) = 1, \quad \chi(S(0,1)) = 1 + (-1)^m \ ,$$

$$\chi(S_h) = 2 - 2h$$

where S_h denotes an orientable surface of genus h (see [13, p.106]).
Furthermore, we recall that if M is a compact manifold and if the dimension of M is odd, then $\chi(M) = 0$ (see [51]).

We now outline the way in which the Euler-Poincaré characteristic, a purely algebraic topology object, can be used in order to apply topological methods in the search of periodic solutions to differential systems.
Roughly speaking, in many cases the Euler-Poincaré characteristic of certain subsets of $\mathbf{R}^m$ turns out to be equal to the "index of rest points" which we define in Chapter 2. In other words, in this thesis (expecially in the applications) the Euler-Poincaré characteristic "plays the role" of the topological degree and/or the fixed point index, i.e. the hypothesis $\chi(C) \neq 0$ in some cases implies (the existence of fixed points of suitable operators and consequently) the existence of periodic solutions of (1.1)-(1.2).
In some sense, we establish a link between the topological nature of a manifold (or, in general, of a compact ENR) and the possible kinds of singularities of a vector field on such a set.

Actually, this link is in the classical Poincaré-Hopf theorem; indeed, if we denote by i_x the "index of isolated singularities" x of a given vector field on a (sufficiently regular) manifold M , then

$$\sum_x i_x = \chi(M) \ .$$

Let T(M) denote the tangent bundle of M. The famous Poincaré-Hopf theorem states that if $\chi(M) \neq 0$ then any smooth vector field $v : M \to T(M)$ must vanish somewhere.
In the case m = 2, for example, since $\chi(S(0,1)) = 2$, a vector field defined on S(0,1) has at least one singular point.

The results we present in this thesis include the Poincaré-Hopf theorem as a particular case.

Before ending this section we observe that, among others, H. Groemer, V.A. Efremovic and Yu.B. Rudjak [15,25] have given a characterization of the Euler-Poincaré characteristic for compact polyhedra by means of the following axioms:

1. ADDITIVITY

$$\chi(C_1 \cup C_2) = \chi(C_1) + \chi(C_2) - \chi(C_1 \cap C_2)\,.$$

2. NORMALIZATION

$$\chi(\emptyset) = 0\ ,\ \chi(\{P\}) = 1.$$

Indeed, we point out that since compact ANRs are homotopically equivalent to compact polyhedra (see [53]) then we have a characterization of the Euler-Poincaré characteristic for compact ANRs by adding to the additivity and normalization axioms the following property:

3. HOMOTOPY EQUIVALENCE
   If C_1 and C_2 have the same homotopy type, then

$$\chi(C_1) = \chi(C_2).$$

In this way, the Euler-Poincaré characteristic is determined by a number of axioms, just as in the case of the Brouwer degree and the fixed point index, independently of its construction through singular homology.

CHAPTER 2

THE INDEX OF REST POINTS

In this chapter we introduce another important concept for this thesis, the index of rest points, by means of the fixed point index (see Chapter 1). In our results (Theorems 4.1, 4.2 and Corollaries) the "index of rest points" plays the role of the Brouwer degree in classical theorems for the periodic BVP (see [33,34,38]).

Let X be an ENR, and let π be a dynamical system in X. Assume that there are no rest points of π in $\mathrm{fr}U$ and U is relatively compact. Then, we know (see Chapter 1) that the fixed point index $i_X(\pi_t,U)$ (where $\pi_t : x \longmapsto \pi(t,x)$) has a constant value for $0 < t < \varepsilon$, provided that ε is sufficiently small. In this situation, we can give the following:

DEFINITION 2.1. (see [49]). *The INDEX OF REST POINTS of the dynamical system π in the set U is given by the formula:*

$$I(\pi,U) := \lim_{\varepsilon\to 0^+} i_X(\pi_t,U) .$$

In what follows, we often deal with a dynamical system π which is induced by an (autonomous) differential system of the type

$$(2.1) \qquad \dot{x} = f(x) ,$$

where $f : \mathbf{R}^m \to \mathbf{R}^m$ is locally lipschitzian. More precisely, since the aim of this thesis is to study the (non autonomous) periodic BVP

$$\begin{cases} \dot{x} = F(t,x) \\ x(0) = x(\omega) \end{cases} ,$$

it will be quite natural (see Section 4.3) to deal with the (autonomous) vector fields

$$\bar{F}(x) := \frac{1}{\omega}\int_0^\omega F(s,x)ds$$

or

$$F_0(x) := F(0,x) .$$

REMARK 2.1. We point out that in Section 4.3 we use the "index of rest points" in a situation which is slightly more general than the one of Definition 2.1. Roughly speaking, we will consider a "flow invariant" ENR $C\subset\mathbf{R}^m$ and a bounded set $G\subset C$, open relatively to C.

Now, we recall (following [49]) some properties of the index of rest points.

PROPOSITION 2.1 [49, prop.4.3, th.5.1, th.6.1].

(i) *Assume that* U *and* W *are open,* $U\subset W$, clW *is compact and there are no rest points in* (clW)\U*; then:*

$$I(\pi,W) = I(\pi,U).$$

(ii) *Assume that* $U_1,U_2,\ldots,U_r$ *are open subsets of* X *such that* $U_i\cap U_j = \emptyset$ *for* $i\neq j$ *and there are no rest points in* $X\setminus(\bigcup_{i=1}^{r} U_i)$*; then,*

$$I(\pi,X) = \sum_{i=1}^{r} I(\pi,U_i).$$

(iii) *Assume that* X *is compact; then,*

$$I(\pi,X) = \chi(X).$$

(iv) *Assume that there are no rest points in the set* clU*; then,*

$$I(\pi,U) = 0.$$

(v) *Assume that* π *is generated by the equation* $\dot{x} = Ax$, *where* A *is a real nonsingular matrix; let* k *denote the number of its eigenvalues having positive real parts; then, for any open set* U, $0\in U$,

$$I(\pi,U) = (-1)^k.$$

(vi) *Assume that* π *is a dynamical system in* $\mathbf{R}^m$ *generated by a function* $f : \mathbf{R}^m \to \mathbf{R}^m$. *Suppose that* f *is locally lipschitzian. Let* $U \subset \mathbf{R}^m$ *be an open bounded set and suppose that* $f(x) \neq 0$ for $x \in \mathrm{fr}U$; *then,*

(2.2) $$I(\pi,U) = (-1)^m d_B(f,U,0).$$

(vii) *Assume that there is a compact subset* K *of* X *such that, for every* $x \in X$, $\pi([0,\infty[,x) \cap K \neq \varnothing$; *then,*

- X *is of finite type;*
- K *has a rest point provided that* $\chi(X) \neq 0$;
- *If there are no rest points in* frK, *then*

$$I(\pi,\mathrm{int}K) = \chi(X).$$

A homotopy property for the index of rest points has been proved in the particular case of tangent vector fields by Furi and Pera in [20]; for the general case, we refer the reader to lemma 5.1 below.

The actual computation of the index of rest points can be performed, in some particular cases, in a rather straightforward manner. Before recalling Srzednicki's result in this direction we need the following

DEFINITION 2.2. *A subset* S *of* X *is called a SECTION of the flow* π *if there exists* $\delta>0$ *such that* $\pi_{|(-\delta,\delta)\times S}$ *is an homeomorphism with an open range.*

Let B be a compact subset of X and let S^+, S^- be sections such that

$$\mathrm{cl}S^+ \cap \mathrm{cl}S^- = \varnothing$$

and

$$\pi((-\delta,\delta),S^+) \cap \pi((-\delta,\delta),S^-) = \varnothing.$$

DEFINITION 2.3. *The set B is called a BLOCK if the following conditions are satisfied:*

(1) $(\mathrm{cl}S^{\pm}\setminus S^{\pm})\cap B=\varnothing$;

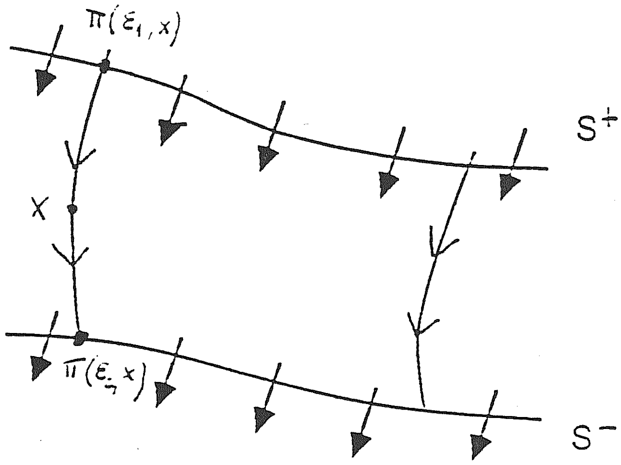
(2) $\pi((-\delta,\delta),S^{+})\cap B=\pi([0,\delta),S^{+}\cap B)$,

$\pi((-\delta,\delta),S^{-})\cap B=\pi((-\delta,0],S^{-}\cap B)$;

(3) *for each* $x\in \mathrm{fr}B\setminus(S^{+}\cup S^{-})$ *there are* $\varepsilon_1,\varepsilon_2\in\mathbf{R}$, $\varepsilon_1<0<\varepsilon_2$, *such that* $\pi(\varepsilon_1,x)\in S^{+}$, $\pi(\varepsilon_2,x)\in S^{-}$ *and*

$$\pi([\varepsilon_1,\varepsilon_2],x)\subset \mathrm{fr}B.$$

The picture below gives an intuitive explanation of the above definition.



REMARK 2.2. A slightly different definition of block can also be found in the literature; namely, roughly speaking, one may require that each point of frB is (in Wazewski's terminology) either a "strict ingress point" or a "strict egress point", i.e. "sliding" points are not allowed. Nevertheless, we point out that, although the two definitions are not equivalent, the existence of a block according to definition 2.3 is equivalent to the existence of a block of this "second type" (cf. definition 3.1 and corollary 3.1). In what follows, we denote the set of "egress points" by:

$$b^{-}:=\{x\in B:\exists\ (t_n)>0,\ t_n\to 0\ \text{s.t.}\ \pi(t_n,x)\notin B\}.$$

Now, we can state Srzednicki's result, which we will often recall in the next chapters. Namely, we have:

THEOREM 2.1 [49, theorem 4.4]. *Let* B *be a block and* b^- *the set of "egress points". Assume that* B, b^- *are* ENRs. *Then,*

$$I(\pi,B) = \chi(B) - \chi(b^-).$$

A concept analogous to the index of rest points has been introduced by Furi and Pera in [20] for flows on manifolds satisfying suitable assumptions. More precisely, for a vector field $f(\cdot)$ as in (2.1) they define $\chi(f)$, the "Euler characteristic of the vector field f "; the properties of this characteristic are analogous to those of the fixed point index (i.e. the solution, excision, additivity, homotopy, normalization properties hold). Indeed, if π is the dynamical system induced by (2.1), then we have:

$$I(\pi,U) = \chi(-f)\ .$$

Finally, it is worth mentioning the following

PROPOSITION 2.2 [20]. *Let* M *be an m-dimensional manifold as in* [20] *and let* f *be a smooth tangent vector field with a compact set of zeros. Then,*

$$\chi(-f) = (-1)^m \chi(f). \tag{2.3}$$

REMARK 2.3. Under the assumptions of (v) in proposition 2.1 and proposition 2.2, we have that (2.2) and (2.3) imply

$$\chi(f) = d_B(f,U,0).$$

For further discussion about the computation of the index of rest points, see Remark 4.4.

CHAPTER 3

THE PROBLEM OF THE EXISTENCE OF PERIODIC SOLUTIONS TO FIRST ORDER DIFFERENTIAL SYSTEMS

3.1. In this chapter we recall the theorems that constitute the main literature for the results we prove in Chapter 4. While quoting such results, we will make the reader acquainted of the framework in which we work and of the techniques we use. For brevity, we omit all the proofs. Let $C \subset \mathbf{R}^m$ be a closed ENR, for example a closed convex set with nonempty interior. The main purpose of our work is to prove the existence of a solution $x(\cdot)$ to the periodic BVP:

$$\begin{cases} \dot{x} = F(t,x) & (3.1) \\ x(0) = x(\omega) & (3.2) \end{cases}$$

such that, for all $t \in [0,\omega]$, $x(t)$ belongs to a certain subset of C.
We also recall that whenever $F: \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^m$ is ω-periodic in the first variable then any solution of (3.1)-(3.2) is the restriction of a classical $\mathcal{C}^1$ ω-periodic solution of (3.1), defined on the whole real line.
Among the topological methods, two main approaches, which we now discuss, have been used when $C = \mathbf{R}^m$.

3.2. The first approach has its origin in the Leray-Schauder theorem. Indeed, (3.1)–(3.2) is transformed into an equivalent coincidence equation in function spaces

$$Lx = Nx, \tag{3.3}$$

with L a linear (not necessarily invertible) operator and N a (nonlinear) Nemitzky operator ([8,21]). Then, under rather general hypotheses, (3.3) may be replaced by an equivalent fixed-point problem:

$$x = Mx. \tag{3.4}$$

This procedure is usually accomplished by the Liapunov-Schmidt reduction or its generalizations. More precisely, one writes

$$F(t,x) := f(t,x;1),$$

where $f = f(t,x;\lambda) : [0,\omega]\times\mathbf{R}^m\times[0,1] \rightarrow \mathbf{R}^m$; then, one imbeds (3.1) in the family of parametrized equations

$$(3.1_\lambda) \qquad \dot{x} = \lambda f(t,x;\lambda)\ ,\ \lambda\in(0,1).$$

In this direction, the most important result is due to J. Mawhin. Namely, we have:

THEOREM 3.1 [38]. *Let* $f : [0,\omega]\times\mathbf{R}^m\times[0,1] \rightarrow \mathbf{R}^m$ *be continuous and let* $G\subset\mathbf{R}^m$ *be an open bounded set. Assume:*

(a1) *("bound set condition")*
for any $x(\cdot)$ *solution of* (3.1_λ) *with* $x(0) = x(\omega)$ *and* $x(t)\in \mathrm{cl}G$ *for all* t, *it follows that* $x(t)\in G$ *for all* t;

(a2) $\bar{f}_0(z) \neq 0$ *for* $z\in \mathrm{fr}G$, *with*

$$\bar{f}_0(z) := \frac{1}{\omega}\int_0^\omega f(s;z;0)ds\ ;$$

(a3) $d_B(\bar{f}_0,G,0) \neq 0$.

Then, there is a solution $x(\cdot)$ *of* (3.1)-(3.2) *such that* $x(t)\in \mathrm{cl}G$ *for all* t.

Some analogous results have been obtained in recent years in the case when the underlying space C does not have a linear structure. More precisely, under the further requirement that the solution remains in a cone (or, more generally, in a convex set) a natural assumption is that the operator M in (3.4) maps the set C onto itself. In this line, R.E. Gaines and J. Santanilla have proved the existence of solutions to (3.1)-(3.2) lying in a convex set (see [22,45,46]).

In Section 4.2 we prove a continuation theorem for periodic solutions in convex sets which generalizes the results quoted above. The proof is performed in the functional-analytic framework of Mawhin's coincidence degree.

3.3. A second point of view was developed by M.A. Krasnosel'skii (see [33,34]); it consists of the search of fixed points of the translation operator (Poincaré-Andronov map) $\pi_\omega : x(0) \longmapsto x(\omega)$.
In this setting he proved, for the case $C = \mathbf{R}^m$, the following:

THEOREM 3.2 [33]. *Let* $F : [0,\omega]\times\mathbf{R}^m \to \mathbf{R}^m$ *be continuous and such that uniqueness and global existence for the solutions of the associated Cauchy problems is guaranteed. Assume:*

(b1) *(" ω-irreversibility condition")*
there is no solution $x(\cdot)$ *of* $\dot{x} = F(t,x)$ *such that* $x(0) = x(k) \in \mathrm{fr}G$ *for some* $0 < k < \omega$;

(b2) $F(0,z) \neq 0$ *for* $z \in \mathrm{fr}G$;

(b3) $d_B(F(0,\cdot),G,0) \neq 0$.

Then, there is a solution $x(\cdot)$ *of* (3.1)-(3.2)*such that* $x(0) \in \mathrm{cl}G$.

Along the lines of Krasnosel'skii's theorem, the situation in which the set C is a convex set or a conical shell has been studied, for example, by K. Deimling, M.L.C. Fernandes and F. Zanolin. See [12,18,22] for the details.

Besides the Krasnosel'skii theorem, further developments were achieved by R. Srzednicki along the lines of Ważewski's method (see [52]). Namely, we have:

THEOREM 3.3. [50, Theorem 1] *Let* p *be an* ω*-periodic process on a metric space* X *. Let* P, P^- *be subsets of* X *such that:*

(c1) P *and* P^- *are compact ANRs and* $P^- \subset P$*;*

(c2) *(Ważewski's condition)*

$$P^{-}\times R = \{(x,t)\in P\times R : \exists(\varepsilon_n) > 0 , \varepsilon_n \to 0 , p(\varepsilon_n,x,t) \notin P\} ;$$

(c3) $\chi(P) - \chi(P^{-}) \neq 0$
(χ *denotes the Euler-Poincaré characteristic).*

Then, there exists $x_0\in P$ *such that* $p(t,x_0,0)\in P$ *and*

$$p(t,x_0,0) = p(t + \omega,x_0,0)$$

for any $t \geq 0$.

Before stating an important corollary of Theorem 3.3, we recall the following:

DEFINITION 3.1 [50]. *Consider the Cauchy problem:*

(3.1) $$\dot{x} = F(t,x)$$

(3.5) $$x(t_0) = x_0$$

where $F : R\times\Omega \to \mathbf{R}^m$ *is continuous,* Ω *is an open subset of* $\mathbf{R}^m$ *and* $x_0\in \Omega$. *Let* p,q *be nonnegative integers,* $p + q > 0$. *Let* L^k , $k = 1, ..., p + q$, *be* $p + q$ *functions of class* C^1 *in* Ω . *We define:*

$$B : = cl_{\Omega}\{x\in \Omega : L^k(x) < 0 , k = 1,...,p + q\},$$
$$\pi^j : = \{x\in B : L^j(x) = 0\}, j = 1,..., p + q .$$

The set B *defined above is called a BLOCK of type* (p,q) *if the following conditions are satisfied:*

(*1) $$(\nabla L^k(x) \mid F(t,x)) > 0, \quad (t,x)\in \mathbf{R}\times\pi^k, \quad k = 1,..,p;$$

(*2) $$(\nabla L^k(x) \mid F(t,x)) < 0, \quad (t,x)\in \mathbf{R}\times\pi^k, \quad k = p + 1,..,p + q.$$

We also denote (in Ważewski's terminology) by

$$b^- := \bigcup_{k=1}^{p} \pi^k .$$

the set of "egress points".

REMARK 3.1. As we noticed in Remark 2.2, the above definition is not equivalent to Definition 2.3; we point out that the former corresponds to the definition of block of "second type" we mentioned in Remark 2.2. Accordingly, given a block as in definition 3.1 we can always obtain a block as in Definition 2.3.
For a complete discussion about blocks, see [9].

Now, we can state the following:

COROLLARY 3.1 [50, th.2] . *Assume that* Ω *is an open subset of* R^m *and* $f : R\times\Omega \rightarrow R^m$ *is of class* C^1 *and* ω*-periodic in* t . *Let* B *be a block of type* (p,q), *according to the above definition. If* B *and* b^- *are compact ANRs and* $\chi(B) - \chi(b^-) \neq 0$, *then there exists a point* $x_0 \in \text{int}B$ *such that the solution* $x(t)$ *of the Cauchy problem* (3.1)-(3.5) *is* ω*-periodic in* t . *Moreover,* $x(t) \in \text{int}B$ *for each* $t \in R$.

Finally, we consider the case in which the set C is a manifold, possibly with boundary. We denote by T(C) the tangent bundle of C.This situation has been studied by M. Furi and M.P. Pera in [19,20]; in [20] they use the Euler characteristic χ of a vector field. This object is equal, up to a minus sign, to the index of rest points of the dynamical system induced by the given vector field (see Chapter 2). In [20], Furi and Pera prove a bifurcation theorem which implies the following:

THEOREM 3.4. [20, th. 2.4] *Let* $F : R\times C \rightarrow T(C)$ *be a* ω*-periodic continuous vector field and let* $\bar{F}$ *be the averaged vector field associated to* F. *Assume that:*

(d1) *the set* $\{p \in C : \bar{F}(p) = 0\}$ *is a compact subset of* intC ;

(d2) *the Euler characteristic* $\chi(\bar{F})$ *is nonzero ;*

(d3) *all the possible* ω*-periodic solutions* $x(\cdot)$ *of the parametrized equation:*

$$\dot{x} = \lambda F(t,x) \quad , \quad \lambda \in (0,1] \ ,$$

lie in a compact subset of int$C$.

Then, (3.1)-(3.2) *has a solution.*

Theorem 4.2 extends the quoted results; moreover, we give a proof which, in some sense, unifies the two approaches we have sketched in this chapter. We will return to this remark in Section 4.3.

3.4. A final preliminary to our results is needed.
We have already mentioned that we want to prove the existence of solutions lying in some subset C of $\mathbf{R}^m$. To this aim, a key hypothesis is the flow-invariance of the set C itself, so we end this chapter by recalling some well-known facts about invariance.

Let $C \subset \mathbf{R}^m$ be a closed set and $F : J \times C \to \mathbf{R}^m$ be a continuous function, where $J \subset \mathbf{R}$ is a nondegenerate interval with interior I .
We denote by

$$T(z;C) := \{v \in \mathbf{R}^m : \liminf_{h \to 0^+} d(z + hv, C)/h = 0\}$$

the (Bouligand) tangent cone to C at z .
Recall that, according to a classical theorem of M. Nagumo ([39]), for each $(t_0,x_0) \in I \times C$ the Cauchy problem

$$\begin{cases} \dot{x} = F(t,x) \\ x(t_0) = x_0 \end{cases}$$

has a solution $x(\cdot) = \mathrm{dom}\, x(\cdot) \to C$ defined on a right maximal neighbourhood of t_0 if and only if

(3.6) $\qquad F(t,z) \in T(z;C)$ for all $t \in I$, $z \in \mathrm{fr}C$.

Equivalently, if $F^* : J \times \mathbf{R}^m \to \mathbf{R}^m$ is any continuous extension of f, then (3.6) ensures that the set C is (weakly) positively invariant with respect to the equation $\dot{x} = F^*(t,x)$, i.e. for each $(t_0,x_0) \in I \times C$ there is at least a solution $x(\cdot)$ of $\dot{x} = F^*(t,x)$ with $x(t_0) = x_0$ and such that $x(t) \in C$ in its right maximal interval of existence.

Accordingly, since we are interested in solutions lying in the set C, there will be no loss of generality if we assume $F(t,\cdot)$ defined on the whole space $\mathbf{R}^m$ whenever (3.6) is assumed.

CHAPTER 4

THE MAIN RESULTS

4.1. In this chapter we present our main results.We will examine, essentially, two cases. First (Section 4.2), we deal with the case in which the set C is a closed convex subset of $\mathbf{R}^m$ with nonempty interior. The study of this situation is contained in [6].
Secondly (Section 4.3), we prove the main result of this thesis, i.e. a continuation theorem for the periodic BVP in flow-invariant ENRs. This theorem and other related results are contained in [7].
We point out that in the case of convex sets the former result (Theorem 4.1 below) is slightly more general than the latter (Theorem 4.2). Moreover, we notice that the two proofs are completely different.

4.2. Throughout this section, we suppose that $C \subset \mathbf{R}^m$ is a closed convex set with $\mathrm{int}C \neq \emptyset$, and denote by

$$r : \mathbf{R}^m \to C$$

its canonical projection ($r(x)$ is such that $|r(x) - x| = \mathrm{dist}(x,C)$). It is well-known that r is non-expansive. For brevity, we denote by $\mathcal{N}(u)$ the set $\mathcal{N}(u,C)$. Let $\emptyset \neq G \subset C$ be a bounded set which is open relatively to C and let $f = f(t,x;\lambda) : [0,\omega]\times\mathbf{R}^m\times[0,1] \to \mathbf{R}^m$ be a continuous function.
We want to find solutions to the differential system

$$(4.1) \qquad \dot{x} = f(t,x;1)$$

verifying the boundary condition

$$(4.1') \qquad x(0) = x(\omega) \ .$$

Such functions will be called ω-periodic for brevity.

We will further want to prove that at least one solution of (4.1)-(4.1') takes values in the set $cl_C G$ for all values of t.
We now state our first result.

THEOREM 4.1. *Suppose that the following conditions are satisfied :*

(e1) *for each* $u \in frC \cap G$, *there is* $\eta_u \in \mathcal{N}(u)$ *such that*

$$(f(t,u;\lambda) \mid \eta_u) \leq 0$$

for all $t \in [0,\omega]$ *and* $\lambda \in [0,1]$;

(e2) *for any* $x(\cdot)$, *ω-periodic solution of*

$$(4.1_\lambda) \qquad \dot{x} = \lambda f(t,x;\lambda) , \qquad \lambda \in (0,1),$$

such that $x(t) \in cl_C G$, *for all* $t \in [0,\omega]$, *it follows that* $x(t) \in G$ *for all* $t \in [0,\omega]$;

(e3) *the fixed point index* $i_C(r(I + \bar{f}_0), G)$ *is defined and*

$$i_C(r(I + \bar{f}_0), G) \neq 0 ,$$

where $\bar{f}_0(z) := \frac{1}{\omega}\int_0^\omega f(s,z;0)\, ds \quad (z \in \mathbf{R}^m) \ , \ I := Id_{\mathbf{R}^m}$.

Then the equation $\dot{x} = f(t,x;1)$ *has at least one ω-periodic solution* $x(\cdot)$ *such that* $x(t) \in cl_C G$ *for all* $t \in [0,\omega]$.

REMARK 4.1. We observe that (e1) is equivalent to the (apparently more restrictive) condition:

(e1') for each $u \in frC \cap G$ and for each $\eta \in \mathcal{N}(u)$, $(f(t,u;\lambda) \mid \eta) \leq 0$ for all $t \in [0,\omega]$ and $\lambda \in [0,1]$.

Indeed, obviously, (e1') implies (e1). On the other hand, assume that (e1) holds and let $u \in frC \cap G$. Take $\varepsilon > 0$ such that $K := C \cap B[u,2\varepsilon] \subset C \cap G$.
Observe that K is a compact convex set with $intK \neq \emptyset$. By Urysohn's lemma, there is a continuous function $\phi_\varepsilon: \mathbf{R}^m \to \mathbf{R}_+$ such that $\phi_\varepsilon(x) = 1$ for $x \in B[u,\varepsilon]$ and $\phi_\varepsilon(x) = 0$ for $x \notin B(u,2\varepsilon)$. Define

$$f_\varepsilon(t,x;\lambda) := \phi_\varepsilon(x)\, f(t,x;\lambda) \ .$$

It can be easily checked that, by (e1), for each $v \in frK$ there is $\eta_v \in \mathcal{N}(v,K)$ such that $(f_\varepsilon(t,v;\lambda) \mid \eta_v) \leq 0$, for all t and λ. Note that $\mathcal{N}(v,K) = \mathcal{N}(v)$ for every $v \in frC \cap B(u,2\varepsilon)$. Then we can apply a flow-invariance result for convex sets (see, for instance, [17, cor.3]) and get that K is weakly positively invariant for the equation $\dot{x} = f_\varepsilon(t,x;\lambda)$. By the Nagumo theorem (see [39,55]) we know that $(f_\varepsilon(t,v;\lambda) \mid \eta) \leq 0$ for every $\eta \in \mathcal{N}(v,K)$. Hence, for $v = u$ we obtain $(f(t,u;\lambda) \mid \eta) \leq 0$ for each $\eta \in \mathcal{N}(u)$. Therefore, (e1') is proved. ∎

In the sequel, the following result will be used.

LEMMA 4.1. *Let* $u \in frC$ *and* $z \neq 0$ *be such that*

(4.2) $\quad (z \mid \eta) \leq 0$ *for all* $\eta \in \mathcal{N}(u)$.

Then

(4.3) $\quad (r(u + z) - u \mid z) > 0$.

Proof. Indeed, for any $y \in C$ we have $r(y+z) \in B[y+z,|z|]$, and so $(r(y+z) - y \mid z) \geq 0$. Moreover, $(r(y+z) - y \mid z) = 0$ if and only if $y \in frC$ and $r(y+z) = y$, so that $z \in \mathcal{N}(y)$. As, by (4.2), $z \notin \mathcal{N}(u)$, then (4.3) follows. ∎

Now we are in position to prove our main result.

Proof of Theorem 4.1. First of all, we observe that, since $intC \neq \emptyset$, we have

$(p - u \mid \eta) < 0$ for any $p \in \mathrm{int}C$, $u \in \mathrm{fr}C$ and $\eta \in \mathcal{N}(u)$.
Choose a point $p \in \mathrm{int}C$ and consider the functions

$$f_n(t,x;\lambda) := f(t,x;\lambda) + n^{-1}(p - x) , \qquad (n \in \mathbf{N}).$$

We are going to prove that, for $n \in \mathbf{N}$ large enough, the equation

$$(4.4) \qquad \dot{x} = f_n(t,x;1)$$

has a ω-periodic solution $x_n(\cdot)$ and $x_n(t) \in \mathrm{cl}_C G$ for all t. Then, as f_n converges to f uniformly on compact sets, there will be a subsequence of (x_n) converging uniformly to a ω - periodic solution $x(\cdot)$ of (4.1) and so $x(t) \in \mathrm{cl}_C G$ for all t.
To this end, it is convenient to introduce an abstract framework (following J. Mawhin [38]) and use a continuation theorem for the coincidence equation $Lx = Nx$.
Let $Z := \mathcal{C}[0,\omega]$ with the sup-norm and let $X := \{z \in Z : z(0) = z(\omega)\}$. Define $L : x \mapsto \dot{x}$, a linear Fredholm mapping of index zero, with $\mathrm{dom}L \subset X$, $\mathrm{dom}L = \{x \in X : x \in \mathcal{C}^1[0,\omega]\}$. The linear projectors $Q : Z \to Z$, $Qz := (1/\omega)\int_0^\omega z(s)ds$, and $P = Q_{\mid X} : X \to X$ are considered too.
We denote by $K_{P,Q} : Z \to \mathrm{Ker}P \cap \mathrm{dom}L$ the generalized inverse of L (see [38, p.7]). Finally, let N, N_n $(n \in \mathbf{N})$ be the Nemitzky operators from $X \times [0,1]$ to Z induced by f, f_n respectively. Then, equations (4.1) and (4.4), with the periodic boundary conditions, are equivalent, respectively, to the fixed point equations

$$(4.5) \qquad x = Px + K_{P,Q}N(x;1) + JQN(x;1) ,$$

$$(4.6) \qquad x = Px + K_{P,Q}N_n(x;1) + JQN_n(x;1) ,$$

where $J = \mathrm{Id}_{\mathbf{R}^m}$ (see [21,38]).
Now we introduce the set

$$\Omega := \{x \in X : x(t) \in G \cap \mathrm{int}C \text{ for all } t \in [0,\omega]\} .$$

It is clear that Ω is an open bounded subset of X.
Moreover, $\mathrm{cl}\Omega \subset \{x \in X : x(t) \in \mathrm{cl}_C G$ for all $t \in [0,\omega]\}$, so that

$$\mathrm{fr}\Omega \subset S_1 \cup S_2$$

with

$$S_1 = \{x \in X : \forall t \ \ x(t) \in G \ \text{ and } \ \exists t_0 \text{ s.t. } x(t_0) \in \mathrm{fr}C\} ,$$
$$S_2 = \{x \in X : \ \forall t \ \ x(t) \in \mathrm{cl}_C G \ \text{ and } \ \exists t_0 \text{ s.t. } x(t_0) \in \mathrm{fr}_C G\} .$$

Observe that S_2 is closed and $S_1 \cap S_2 = \emptyset$.
It is a standard fact to check that N and N_n are L-compact on $\mathrm{cl}\Omega\times[0,1]$ (see[38, p.12]).
We want to prove that for n sufficiently large

(4.7) $$x \neq Px + \lambda K_{P,Q}N_n(x;\lambda) + JQN_n(x;\lambda)$$

holds for each $(x,\lambda) \in \mathrm{fr}\Omega\times[0,1]$.
Without loss of generality, we can assume that (e2) holds for $\lambda \in (0,1]$. Then, (e2) and (e3) imply that

(4.8) $$x \neq Px + \lambda K_{P,Q}N(x;\lambda) + JQN(x;\lambda)$$

for all $\lambda \in [0,1]$ and $x \in S_2$. In fact, (e3) ensures that $QN(x;0) = \bar{f}_0(x) \neq 0$ for $x \in S_2 \cap \mathrm{Ker}L = \mathrm{fr}_C G$. Now we claim that there is n_1 such that for every $n \geq n_1$, (4.7) holds for all $\lambda \in [0,1]$ and $x \in S_2$. Indeed, it is sufficient to observe that the sequence of operators (N_n) converges to N uniformly on $\mathrm{cl}\Omega\times[0,1]$ and that

$$\inf \{|x - Px - \lambda K_{P,Q}N(x;\lambda) - JQN(x;\lambda)|_X : x \in S_2 , \lambda \in [0,1]\} > 0$$

(recall that $K_{P,Q}N$ and QN are compact on $S_2\times [0,1]$ and S_2 is closed).
Furthermore, we note that, for every $n \in \mathbb{N}$, (4.7) holds for all $\lambda \in [0,1]$ and $x \in S_1$; this claim follows at once arguing like in [38, p.74-75] since, for every n,

(4.9) $$(f_n(t,u;\lambda) \mid \eta_u) < 0$$

for all $t \in [0,\omega]$, $u \in \mathrm{fr}C \cap G$, $\lambda \in [0,1]$.
Therefore, we have proved that, for n sufficiently large, (4.7) holds on $\mathrm{fr}\Omega\times[0,1]$ and so, equivalently,

(4.10) $$Lx \neq \lambda N_n(x;\lambda) \ , \qquad \text{on} \ \ (\mathrm{dom}L \cap \mathrm{fr}\Omega)\times[0, 1] \ ,$$

(4.11) $$QN_n(x;0) \neq 0 \ , \qquad \text{on} \ \ (\mathrm{Ker}L \cap \mathrm{fr}\Omega) \ .$$

Now we want to use the Mawhin continuation theorem ([21, Theorem 4.1]), and so we need only to prove that

(4.12) $$d_B(JQN_n(\cdot,0)_{|\mathrm{Ker}\,L}\,, \Omega\cap\mathrm{Ker}\,L\,, 0) = d_B(\,\bar{f}_{n,0}, G\cap\mathrm{int}C, 0) \neq 0 \ ,$$

where $\bar{f}_{n,0}(z) := (1/\omega)\int_0^{\omega} f_n(s,z;0)ds.$

Setting for simplicity $G' := G\cap\mathrm{int}C$, we have $\bar{f}_{n,0}(w) \neq 0$ for $w\in \mathrm{fr}G'$ and

$$(-1)^m d_B(\,\bar{f}_{n,0}, G', 0) = d_B(-\,\bar{f}_{n,0}, G', 0) \ =$$

$$= i_{R^m}(I + \bar{f}_{n,0}, G') \ = \ i_{R^m}((I + \bar{f}_{n,0})\, r \ , G') \ .$$

Since $r : R^m \setminus C \rightarrow \mathrm{fr}C$ and $G'\sqsubset\mathrm{int}C$, then

$$i_{R^m}((I + \bar{f}_{n,0})\, r \ , G') = i_{R^m}((I + \ \bar{f}_{n,0})\, r \ , r^{-1}(G')).$$

Furthermore we can write, using the commutativity property of the index,

$$i_{R^m}((I +\bar{f}_{n,0})\, r \ , \ r^{-1} (G')) = i_C(r \ (I +\bar{f}_{n,0}) \ , \ G').$$

We point out that for every $n \geq n_1$

(4.13) $$u \neq r(u +\bar{f}_{n,0}(u))$$

holds for all $u\in G\cap\mathrm{fr}C$. In fact, by (4.9) and Remark 4.1, we have that $(\bar{f}_{n,0}(u) \mid \eta) \leq 0$ for all $\eta\in \mathcal{N}(u)$. We know that $\bar{f}_{n,0}(u) \neq 0$ by (4.11); then, by Lemma 4.1, $r(u +\bar{f}_{n,0}(u))-u\neq 0$.

We also note that (4.13) holds for all $u \in fr_C G \cap frC$, provided that n is sufficiently large ($n \geq n_2$). Indeed, by (e3), $\inf\{|r(u+\bar{f}_0(u)) - u|: u \in fr_C G\} > 0$ and so the same is true replacing $\bar{f}_0$ with $\bar{f}_n$, for n large enough.

Hence, we can use the excision property of the index and write, for $n \geq n_1 + n_2$,

$$i_C(r(I + \bar{f}_{n,0}), G') = i_C(r(I + \bar{f}_{n,0}), G).$$

Finally, $i_C(r(I + \bar{f}_{n,0}), G) = i_C\big(r(I + \bar{f}_0), G\big)$ for n sufficiently large (as $\bar{f}_{n,0} \to \bar{f}_0$ uniformly on $cl_C G$) and (4.12) follows from (e3).

The validity of (4.10)-(4.11)-(4.12) provides, for almost all $n \in \mathbf{N}$, the existence of a solution $x_n(\cdot) \in domL \cap cl\Omega$ such that

$$\begin{aligned} x_n &= Px_n + K_{P,Q}N_n(x_n;1) + JQN_n(x_n;1) \\ &= Px_n + K_{P,Q}N(x_n;1) + JQN(x_n;1) + K_{P,Q}[N_n(x_n;1) - N(x_n;1)] + \\ &\quad + JQ[N_n(x_n;1) - N(x_n;1)]. \end{aligned}$$

By the previously listed properties of the operators, it is clear that the sequence (x_n) is relatively compact and so, passing to the limit on an appropriate subsequence, we get $x_n \to x^*$ with $x^* \in domL \cap cl\Omega$ and

$$x^* = Px^* + K_{P,Q}N(x^*;1) + JQN(x^*;1),$$

that is

$$Lx^* = N(x^*;1).$$

The proof is therefore complete. ∎

REMARK 4.2 . We observe that the assumption

(4.14) $\qquad \bar{f}_0(z) \neq 0 \quad \text{for} \quad z \in fr_C G$

is not sufficient to ensure that $i_C(r(I+\bar{f}_0), G)$ is defined. Indeed, from Lemma 4.1, it is clear that the index is defined if and only if (4.14) and

$$\bar{f}_0(z) \notin \mathcal{N}(z) \quad \text{for } z \in fr_C G \cap frC$$

hold. Moreover, we note that in the hypothesis (e3) the function $\bar{f}_0$ may be substituted by any map f^* with $f^*(z) = \phi(z)\cdot\bar{f}_0(z)$ and $\phi : \mathbf{R}^m \to \mathbf{R}^+$ a continuous function. In this case, we have $i_C(r(I+\bar{f}_0), G) = i_C(r(I+f^*), G)$.

We note that (e2) easily follows provided that we find a-priori bounds for the ω-periodic solutions of (4.1_λ) lying in C. In this case, take $G = B(0,R) \cap C$ with R large enough.

We point out that our result makes sense in the case when $G \not\subset intC$. In fact, if $G \subset intC$ condition (e1) is vacuously satisfied and (e3) reduces to

$$d_B(\bar{f}_0, G, 0) = (-1)^m \, i_C(r(I+\bar{f}_0), G) \neq 0,$$

so that in this case theorem 4.1 reduces to Mawhin's theorem 3.1.

On the other hand, if C is compact and G = C then condition (e2) is vacuously satisfied, whence (e3) holds with $i_C = 1$, since $r(I+\bar{f}_0) : C \to C$ and C is contractible. In this case theorem 4.1 reduces to a classical result of existence of periodic solutions in convex sets (see [26],[33, th.3.2],[36, cor.2.1]).

In many concrete situations, the actual computation of the fixed point index may be performed using the operator of translation along orbits. Namely, assume that the autonomous differential equation

$$\dot{x} = \bar{f}_0(x)$$

induces a local flow π in $\mathbf{R}^m$ (e.g. $\bar{f}_0$ locally lipschitzian) and C is flow-invariant with respect to this flow, i.e. $(\bar{f}_0(u) \mid \eta) \le 0$ for all $u \in frC$, $\eta \in \mathcal{N}(u)$. Then, for $\varepsilon > 0$ sufficiently small, we have

$$(4.15) \qquad i_C(r(I+\bar{f}_0), G) = i_C(\pi_\varepsilon, G),$$

where $\pi_\varepsilon : w \longmapsto \pi(\varepsilon, w)$.

The proof of this claim follows a standard procedure (see [33,49]), showing that the continuous homotopy

$$h(t,x) := \begin{cases} r\big(x + t^{-1}(\pi(t\varepsilon,x) - x)\big) & 0 < t \leq 1 \\ r(x + \varepsilon \bar{f}_0(x)) & t = 0 \end{cases}$$

is fixed point free on $fr_C G$ (use Lemma 4.1). From (4.15), it is also clear that

$$\lim_{\varepsilon \to 0^+} i_C(\pi_\varepsilon, G) = i_C(r(I + \bar{f}_0), G).$$

Therefore, we can use Srzednicki's theorem 2.1. If $B = cl_C G$ is a block (according to C. Conley [9]) with b^- the set of "egress points" of B then

$$i_C((I + \bar{f}_0), G) = \chi(B) - \chi(b^-),$$

provided that B, b^- are ENRs. (As before, we denote by χ the Euler characteristic). For analogous results, see also theorem 3.4 and [19].

In some applications it is convenient to deal with a set G which is defined as intersection of sublevel sets of suitable Liapunov-like functions. Such possibility has been widely developed in [21,38], dealing with the so-called "bound sets".

In the light of our result and as an example in this direction, we investigate the case in which the set G is the part of the convex set C lying between two level surfaces of a functional V. More precisely, the following situation is considered.

Let $V : \mathbf{R}^m \to \mathbf{R}$ be a continuous function and suppose that for $-\infty \leq a < b < +\infty$, the set $G := \{x \in C : a < V(x) < b\}$ is defined. It is clear that G is open relatively to C. We suppose that G is bounded.

It can be seen that

$$cl_C G \subset [a \leq V \leq b]$$

and

$$fr_C G \subset [V = a] \cup [V = b] .$$

Suppose that there are closed (possibly empty) sets $H, K \subset C$, with $H \cap K = \emptyset$, such that $fr_C G = H \cup K$ and let V be of class C^1 in a neighbourhood of $fr_C G$, with

(4.16) $\nabla V(u) \neq 0$ for $u \in fr_C G$.

Furthermore, we require

(4.17) $(\nabla V(u) \mid \eta) \leq 0$, for each $u \in H \cap frC$ and $\eta \in \mathcal{N}(u)$,

(4.18) $(\nabla V(u) \mid \eta) \geq 0$, for each $u \in K \cap frC$ and $\eta \in \mathcal{N}(u)$

Finally, let $\psi : \mathbf{R}^m \to \mathbf{R}$ be any continuous function such that

(4.19) $\psi(x) = 1$ for $x \in H$, $\psi(x) = -1$ for $x \in K$.

Now we have

COROLLARY 4.1. *Suppose that* $F : [0,\omega] \times \mathbf{R}^m \to \mathbf{R}^m$ *is a continuous function which satisfies the following conditions :*

(f1) *for each* $u \in frC \cap G$, *there is* $\eta \in \mathcal{N}(u)$ *such that*

$$(F(t,u) \mid \eta_u) \leq 0, \quad \textit{for all } t \in [0,\omega] ;$$

(f2) $$(F(t,u) \mid \nabla V(u)) \geq 0 \quad \textit{for all } (t,u) \in [0,\omega] \times H ,$$

$$(F(t,u) \mid \nabla V(u)) \leq 0 \quad \textit{for all } (t,u) \in [0,\omega] \times K ;$$

(f3) $$i_C(r(I + \psi \nabla V), G) \neq 0 .$$

Then the equation $\dot{x} = F(t,x)$ *has at least one* ω*-periodic solution* $x(\cdot)$ *such that* $x(t) \in cl_C G$ *for all* t.

Proof. We define $f(t,x;\lambda) := (1-\lambda)h(x) + \lambda F(t,x)$, $\lambda \in [0,1]$, where $h(x) := r(x + \psi(x)\nabla V(x)) - x$.
We show that (e1), (e2), (e3) of theorem 4.1 are fulfilled with respect to the function f.
In order to verify (e1), it will be sufficient (by (f1)) to show that

$$(4.20) \qquad (h(u) \mid \eta_u) \leq 0 \quad \text{for all } u \in frC \cap G$$

holds. But, as $r(u + \psi(u)\nabla V(u)) \in C$, then, by a general property of convex sets, $\big(r(u + \psi(u)\nabla V(u)) - u \mid \eta\big) \leq 0$ for any $\eta \in \mathcal{N}(u)$ and (4.20) is achieved.
To prove (e2), we observe that if $x(\cdot)$ is a ω-periodic solution of (4.1_λ) with $x(t) \in cl_C G$ for all t and $x(t_0) \in fr_C G$, then (by a standard argument) the function $v(t) := V(x(t))$ verifies $v(0) = v(\omega)$ and has a local maximum (or minimum) for $t = t_0$. Furthermore, v is differentiable at t_0. Hence, by chain rule, we have $(f(t_0,x(t_0);\lambda) \mid \nabla V(x(t_0))) = 0$. Then, it is clear that (e2) will follow from

$$(4.21) \qquad (f(t,u;\lambda) \mid \nabla V(u)) \neq 0 \quad \text{for all } u \in fr_C G \;,\; \lambda \in (0,1)\,.$$

Let $u \in H$ and observe that, by the properties of r,

$$(h(u) \mid \nabla V(u)) = \big(r(u + \psi(u)\nabla V(u)) - u \mid \nabla V(u)\big) =$$

$$= \psi(u)\big(r(u + \psi(u)\nabla V(u)) - u \mid \psi(u)\nabla V(u)\big) \geq 0.$$

If $(h(u) \mid \nabla V(u)) = 0$ then, arguing as in the proof of lemma 4.1, $u \in H \cap frC$ and $\psi(u)\nabla V(u) \in \mathcal{N}(u)$. Then, by (4.17), $(\nabla V(u) \mid \psi(u)\nabla V(u)) \leq 0$ and so $\psi(u) \leq 0$ contradicting the definition of the function ψ. Thus, we have proved that

$$(4.22) \qquad (h(u) \mid \nabla V(u)) > 0 \;, \quad \text{for } u \in H.$$

In the same manner, one shows that

(4.23) $(h(u) \mid \nabla V(u)) < 0$, for $u \in K$.

Finally, (4.22), (4.23) and (f2) imply (4.21) and (e2) is verified.

At last, we must check (e3). We observe that, as $f(t,x;0) = h(x)$, then $\bar{f}_0(z) = h(z) = r(z + \psi(z)\nabla V(z)) - z$ and so $r(I + \bar{f}_0) = r(I + r(I + \psi\nabla V) - I) = r(I + \psi\nabla V)$. Now, it is sufficient to verify that the index is well defined, that is $h(z) \neq 0$ for all $z \in fr_C G$. In view of the preceding remarks (see lemma 4.1 and remark 4.2), this property follows since $\nabla V(u) \neq 0$ for all $u \in fr_C G$ and $\psi(u)\nabla V(u) \notin \mathcal{N}(u)$ for $u \in fr_C G \cap frC$ by (4.16), (4.17) and (4.18).
Then, theorem 4.1 applies and the result is achieved. ∎

REMARK 4.3. In recent years, several results have been obtained concerning the computation of the topological degree or the fixed point index associated to nonlinear maps of gradient type (see [3,11,30,33,44]). Here we just present a simple example in which the index is found in the situation examined in corollary 4.1.

LEMMA 4.2. *Suppose further that* $[V \leq b]$ *is bounded and let the sets* $[V > b]$, $[V < a]$ *(resp.,* $[V < b]$, $[V > a]$*) be contractible. Moreover, assume that*

$$V\big(r(x + \psi(x)\nabla V(x))\big) > b \quad (< b) \textit{ for } x \in [V = b] \quad ,$$

$$V\big(r(x + \psi(x)\nabla V(x))\big) < a \quad (> a) \textit{ for } x \in [V = a] \quad .$$

Then, $i_C(r(I + \psi\nabla V), G) = -1$ *(= 1, respectively).*

The proof is omitted since it is a straightforward application of contractibility of sets and homotopy invariance and additivity of the index. ∎

A situation similar to that described in corollary 4.1 and lemma 4.2 has been examined in [50] using the Ważewski approach [52].
For a further result on the computation of the index for gradient maps, in the spirit of Krasnosel'skii's approach [33, lemma 6.5], see lemma 5.2.

From corollary 4.1 and lemma 4.2 it is possible to obtain and improve various results from [22,45,46]. For instance, we have :

COROLLARY 4.2. *Let* $g : [0,\omega]\times R^m \to R^m$ *be continuous and suppose there are constants* $0 < a < b$ *such that for all* $t\in[0,\omega]$:

(4.24) $\quad (g(t,x) \mid x) \leq 0 \quad (\geq 0) \quad$ *for* $|x| = a\,,\ x \geq 0$,

(4.25) $\quad (g(t,x) \mid x) \geq 0 \quad (\leq 0) \quad$ *for* $|x| = b\,,\ x \geq 0$.

Finally, assume that for each $i = 1,\dots,m$ *and* $t\in[0,\omega]$:

(4.26) $\quad g_i(t,\hat{x}_i) \geq 0 \quad$ *for* $a \leq |\hat{x}_i| \leq b\ ,\ \hat{x}_i \geq 0$.

Then, equation $\dot{x} = g(t,x)$ *has at least one* ω*-periodic solution with values in the set* $\{x\in R^m_+ : a \leq |x| \leq b\}$.

Proof. The result can be easily obtained from corollary 4.1 by setting : $F = g$, $C = R^m_+$, and $V(x)=|x|$. Observe that $u\in frC$ if and only if $u = \hat{u}_i$ for some i and, in this case, $\eta\in\mathcal{N}(u)$ if and only if $\eta = \sum_{i\in J} \mu_i e_i$, where $\mu_i \geq 0$, $\sum_{i\in J} \mu_i > 0$, and $J = \{j : u = \hat{u}_j\}$. (f1) and (f2) follow from (4.26) and (4.24), (4.25). In particular, $H = [V = b]$ and $K = [V = a]$ (resp., $H =[V = a]$ and $K = [V = b]$). Then, using lemma 4.2, we find $i_C(r(I + \psi\nabla V)\ ,\ G) = -1$ (=1, respectively).
The proof is complete. ∎

For the last step in the proof (computation of the index) we can use, alternatively, theorem 2.1 by Srzednicki. Indeed, it is sufficient to observe that $\chi(cl_C G) = 1$ and $\chi((cl_C G)^-) = 2$ (= 0, respectively), where the Euler characteristic is computed for the flow induced by $\dot{x} = \psi(x)\nabla V(x)$, with ψ smooth enough.
Corollary 4.2 improves [22, th.3.1], where, instead of (4.25), $g(t,x) \geq -x$ was assumed. On the same line, it is easy to find improvements of other results (e.g. [45, th.4.1], [46, th.3.2]).

A careful reading of the proof of theorem 4.1 shows that variants of this result may be obtained by suitably changing the definition of the open bounded set $\Omega \subset X := \{x : [0,\omega] \to \mathbf{R}^m$, continuous and verifying (4.1')$\}$. In particular, it is possible to produce various results analogous to the classical theorems of M.A. Krasnosel'skii [33, Chap.6]. For brevity, we do not give the proofs of these theorems, but we refer the reader to corollary 4.5 below, where we extend - with a different technique - Krasnosel'skii's theorem to ENRs.

We end this section by observing that it is possible to obtain an (equivalent) variant of theorem 4.1 by reversing the inequality in condition (e1), and replacing $\overline{f}_0$ with $-\overline{f}_0$ in (e3). Indeed, this can be achieved with the standard change of variable $t \to \omega - t$ which transforms equation (4.1) into $y' = -f(s,y;1)$ where $s = \omega - t$, $y(s) = x(\omega - t)$.
All the corollaries can be modified accordingly.
We also point out that the results may be extended to the second order differential systems

$$-\ddot{x} = F(t,x,\dot{x})$$

and to some classes of differential-delay equations .

4.3. Throughout this section, we suppose that $C \subset \mathbf{R}^m$ is an ENR.
As usual, we deal with the periodic boundary value problem

(4.27) $\qquad \dot{x} = F(t,x)$,

(4.28) $\qquad x(0) = x(\omega)$, $(\omega > 0)$,

where

$$F(t,x) := f(t,x;1)$$

and $f = f(t,x;\lambda) := [0,\omega] \times \mathbf{R}^m \times [0,1] \to \mathbf{R}^m$ is a continuous function which is locally lipschitzian in x. Once for all, we point out that such assumption is not strictly necessary in our proofs, but it avoids the requirement of the uniqueness of the solutions to all the Cauchy problems which will be considered henceforth.
In what follows, we denote by X the complete metric space of the continuous functions $x(\cdot):[0,\omega] \to C$ endowed with the distance d^*, $d^*(x_1,x_2) := |x_1 - x_2|_\infty$. From [31, p.186], we know

that (X,d^*) is a (metric) ANR. We want to prove the existence of solutions to (4.27)-(4.28) belonging to certain subsets of X.
To this end, we produce a continuation theorem (on the line of [37,38]) involving the averaged system

(4.29) $$\dot{x} = \bar{f}_0(x)$$

Observe that the map $\bar{f}_0$ is locally lipschitzian; accordingly, (4.29) induces a local dynamical system $\bar{\pi}$ with phase space $\mathbf{R}^m$. We also note that if the set C is positively invariant for $\dot{x} = f(t,x;0)$, then the same property is true for $\bar{\pi}$ (see Lemma 4.3 below).
We further remark that if $G \subset C$ is a bounded set, open relatively to C, such that

(4.30) $$\bar{f}_0(x) \neq 0 \quad \text{for all } x \in \mathrm{fr}_C G$$

holds, then there is $\varepsilon_0 > 0$ such that the map $\bar{\pi}_\varepsilon : x \longmapsto \pi(\varepsilon,x)$ is fixed point free on $\mathrm{fr}_C G$, for all $0 < \varepsilon \leq \varepsilon_0$. Therefore, whenever C is positively invariant for $\bar{\pi}$ and (4.30) holds, the fixed point index $i_C(\pi_\varepsilon,G)$ is defined and it is constant with respect to ε, for all $0 < \varepsilon \leq \varepsilon_0$. In this situation, according to Chapter 2, the index of rest points

(4.31) $$I(\bar{\pi},G) := \lim_{\varepsilon \to 0^+} i_C(\bar{\pi}_\varepsilon , G)$$

is well defined.

REMARK 4.4. Concerning the computation of the index $I(\bar{\pi},G)$, we consider some special cases (see also Chapter 2).

(i) If G=C (C compact), then $I(\bar{\pi},G)=\chi(C)$, (see [20,49]).

(ii) If $\mathrm{cl}G \subset \mathrm{int}C$, then $I(\bar{\pi},G) = (-1)^m d_B(\bar{f}_0,G,0)$ (see [33,49]).

(iii) If C is a closed convex set with nonempty interior, then $I(\bar{\pi} , G) = i_C(r(I+\bar{f}_0),G)$, $r : \mathbf{R}^m \to C$ being the canonical projection (see section 4.2 above and [6]).

(iv) More generally, if $B = clG$ is a block (according to C. Conley [9]), with b^- the set of "egress points" of B, then $I(\bar{\pi},G) = \chi(B) - \chi(b^-)$, provided that B and b^- are ENRs (see theorem 2.1).

(v) Finally, if $\bar{\pi}$ is dissipative, i.e. there is a compact set $\mathcal{K} \subset C$ such that for each $x \in C$ there is $t_x \geq 0$ with $\bar{\pi}(t,x) \in \mathcal{K}$ for all $t \geq t_x$, then C is of finite type and $I(\bar{\pi},G) = \chi(C)$ for every $G \supset \mathcal{K}$ (see proposition 2.1).

Now we are in position to state the main result of this thesis. In what follows points of C are identified with constant functions.

THEOREM 4.2. *Assume*

(g1) C *is positively invariant for* $\dot{x} = f(t,x;\lambda)$, $\lambda \in [0,1]$.

Let $\Omega \subset X$ *be an open bounded set such that the following conditions are satisfied:*

(g2) *there is no* $x(\cdot) \in fr_X\Omega$, *with* $x(0) = x(\omega)$, *such that*

$$(4.27_\lambda) \qquad \dot{x} = \lambda f(t,x;\lambda), \ \lambda \in (0,1);$$

(g3) $\bar{f}_0(z) \neq 0$ *for all* $z \in C \cap fr_X\Omega$;

(g4) $I(\bar{\pi},\Omega \cap C) \neq 0$.

Then, (4.27)-(4.28) *has at least one solution* $x(\cdot) \in cl_X\Omega$.

Observe that in the particular case $C = \mathbf{R}^m$ assumption (g1) is trivially verified, while condition (g4) is equivalent to $d_B(\bar{f}_0,\Omega \cap \mathbf{R}^m,0) \neq 0$, so that we obtain theorem 3.1. Actually, in [37] the local lipschitzianity of f is not supposed; however, in the special case $C = \mathbf{R}^m$ we can relax such regularity assumption on f using a standard perturbation argument.
The following result is crucial for the proof.

LEMMA 4.3. *Assume* (g1). *Then, for each* $\alpha,\beta \geq 0$ *and* $0 \leq \lambda_i \leq 1$, i=1,2, C *is flow-invariant for* $\dot{x} = \alpha f(t,x;\lambda_1) + \beta \bar{f}_{\lambda_2}(x)$.

Proof. At first, we observe that the function $\alpha f(t,x;\lambda_1) + \beta \bar{f}_{\lambda_2}(x)$ is locally lipschitzian in x, so that the uniqueness for the solutions of the associated Cauchy problems is guaranteed. Recall that, by the characterization of flow-invariant sets in terms of tangent cones (see [2,10]), (g1) implies that $f(t,z;\lambda) \in T(z;C)$, for all $t \in [0,\omega]$, $z \in \mathrm{fr}C$ and $\lambda \in [0,1]$, where T(z;C) is a suitable tangent cone to C at z. Without loss of generality (see [41, Th.3.9]), we can assume that T(z;C) is closed and convex (for instance, the Clarke tangent cone can be chosen). Then, by the mean value theorem [2, p.21], $\bar{f}_\lambda(z) \in T(z;C)$ for all $z \in \mathrm{fr}C$ and $\lambda \in [0,1]$. Finally, the convexity and the cone property of T(z;C) imply that $\alpha f(t,z;\lambda_1) + \beta \bar{f}_{\lambda_2}(z) \in T(z;C)$.
The proof is complete. ∎

Proof of Theorem 4.2. At first, we prove our result under the supplementary assumption that there is a constant $A > 0$ such that

$$(4.32) \qquad |f(t,x;\lambda)| \leq A$$

for all $t \in [0,\omega]$, $x \in \mathbf{R}^m$, $\lambda \in [0,1]$. The general situation will be examined at the end of the proof.
Without loss of generality, we also suppose that (g2) holds with $\lambda \in (0,1]$ in (4.27_λ) (otherwise, the result is already proved for $x \in \mathrm{fr}_X\Omega$).
We begin with some technical preliminaries.
Let $\varepsilon \in (0,\omega)$ be arbitrarily small but fixed. We define the following functions:

$$\theta(\lambda) := (\lambda\omega - \varepsilon) / (\omega - \varepsilon) , \qquad \varepsilon/\omega \leq \lambda \leq 1;$$

$$\phi(\theta,t) := [\varepsilon + \theta(\omega - \varepsilon)]\,(t/\omega) , \qquad 0 \leq \theta \leq 1, \quad 0 \leq t \leq \theta;$$

$$g(s,y;\theta) := \begin{cases} f\big(s\omega/\phi(\theta,\omega),y;\lambda(\theta)\big) , & 0 \leq \theta \leq 1 , \ 0 \leq s \leq \phi(\theta,\omega) , \ y \in \mathbf{R}^m , \\ f(\omega,y;\lambda(\theta)) , & 0 \leq \theta \leq 1 , \ s > \phi(\theta,\omega) , \ y \in \mathbf{R}^m . \end{cases}$$

Observe that $g:\mathbf{R}_+\times\mathbf{R}^m\times[0,1]\to\mathbf{R}^m$ is continuous and such that uniqueness and global existence for the associated Cauchy problems are guaranteed. Accordingly, if we denote by $u(\sigma,z,\cdot;\theta)$ the solution of

$$(4.33) \qquad \begin{aligned} \dot{y} &= g(s,y;\theta) \\ y(\sigma) &= z \end{aligned}$$

then a one-parameter family of processes is defined. Using (g1), it can be easily checked that, for each $\theta\in[0,1]$, the set C is positively invariant for the corresponding process u. We further note that, since

$$\lambda = \lambda(\theta) = \phi(\theta,\omega)/\omega,$$

then the function y(s) is a solution of (4.33) for $s\in[0,\phi(\theta,\omega)]$ if and only if the function

$$(4.34) \qquad x(t) := y(\phi(\theta,\omega)t/\omega) = y(\phi(\theta,t))$$

is a solution of (4.27_λ) with $t\in[0,\omega]$.

The existence of solutions to (4.27)-(4.28) will be achieved producing a fixed point for a suitable operator defined on X. We will carry out this programme using the properties of the fixed point index for compact operators in metric ANRs (see [23]); more precisely, some admissible homotopies will be constructed.

As a first step, we introduce a nonlinear operator M defined on $X\times[0,1]$ as follows:

$$M(x,\theta) := u(0,x(\omega),\phi(\theta,\cdot);\theta) \quad , \quad \theta\in[0,1].$$

By the flow-invariance of C, $M : X\times[0,1]\to X$; moreover, by the Ascoli-Arzelà theorem, M is compact on $cl_X\Omega\times[0,1]$. Using the definition of u and (4.34), it is immediately seen that x is a fixed point of $M(\cdot,\theta)$ for some $\theta\in[0,1]$ if and only if x is a solution of (4.27_λ) with $\lambda\in[\varepsilon/\omega,1]$ and $x(0) = x(\omega)$. In particular, (4.27)-(4.28) is solvable if and only if $M(\cdot,1)$ has a fixed point. Hence, this claim and assumption (g2) imply that $M(x,\theta) \neq x$ for $x\in fr_X\Omega$ and $\theta\in[0,1]$. Therefore, M is an admissible homotopy and so

$$(4.35) \qquad i_X(M(\cdot,1),\Omega) = i_X(M(\cdot,0),\Omega).$$

Secondly, we denote by $v(\sigma,z,\cdot;\mu)$ the solution of

$$\dot{y} = (1 - \mu)\,\overline{f}_0(y) + \mu g(s,y;0)$$
$$y(\sigma) = z,$$

with $\mu\in[0,1]$.
As before, a one-parameter family of processes is defined. By Lemma 4.3 we have that, for each $\mu\in[0,1]$, the set C is flow-invariant for the corresponding process v as well. Now, we consider another nonlinear operator $N(x,\mu)$, defined on $X\times[0,1]$ as follows:

$$N(x,\mu) := v(0,x(\omega),\phi(0,\cdot);\mu).$$

Arguing as before, $N : X\times[0,1]\to X$ and it is compact on $cl_X\Omega\times[0,1]$. Moreover,

$$N(x,1) = M(x,0).$$

We want to prove that N is an admissible homotopy. To this end, we observe that x is a fixed point of $N(\cdot,\mu)$ if and only if $x(\cdot)$ is a solution of

$$(4.36) \qquad \dot{x} = (\varepsilon/\omega)[(1 - \mu)\,\overline{f}_0(x) + \mu f(t,x;\varepsilon/\omega)]$$

with $x(0) = x(\omega)$.
We claim that there is $\varepsilon_0 > 0$ (small enough) such that $N(x,\mu) \neq x$ for all $x\in fr_X\Omega$ and $\mu\in[0,1]$, provided that $\varepsilon\in(0,\varepsilon_0]$. (Recall that the function g and, consequently, the operator N depend on the constant ε chosen at the beginning of the proof). In fact, assume the contrary, i.e. that for each $n\in N$ there are $\varepsilon_n\in[0,\omega]$ with $\lim \varepsilon_n = 0$, $\mu_n\in[0,1]$ and $x_n\in fr_X\Omega$ such that $N(x_n,\mu_n) = x_n$. Then, from (4.36) and (4.32) we have:

$$(4.37) \qquad |\dot{x}_n|_\infty \le (\varepsilon_n/\omega)A\ .$$

Moreover, as Ω is bounded there is a constant $R > 0$, independent of n, such that $|x_n|_\infty \le R$. By the Ascoli-Arzelà theorem, we get that there is $x^*\in fr_X\Omega$ such that (up to a subsequence) $x_n(\cdot)\to x^*(\cdot)$ in the d^*-metric.

Clearly, $x^*(t) = \text{constant} = x^* \in C \cap fr_X\Omega$ (use (4.37)). We can also assume (passing, possibly, to a further subsequence) that $\lim \mu_n = \mu^* \in [0,1]$. Taking the mean value of (4.36) and dividing by (ε_n/ω), we obtain, for each n,

$$0 = \Big[(1-\mu_n)\frac{1}{\omega}\int_0^\omega \bar{f}_0(x_n(t))dt \; + \; \mu_n\frac{1}{\omega}\int_0^\omega f(t,x_n(t)\,;\,\varepsilon_n/\omega)dt\Big]\,.$$

Passing to the limit as $n \to +\infty$, we get

$$0 = (1-\mu^*)\;\bar{f}_0(x^*) + \mu^*\frac{1}{\omega}\int_0^\omega f(t,x^*;0)dt = \;\bar{f}_0(x^*)\,,\text{ with } \; x^* \in fr_X\Omega\,.$$

Thus, a contradiction with (g3) is reached. Hence, the claim is proved and we can write:

$$i_X(M(\cdot,0),\Omega) = i_X(N(\cdot,1),\Omega) = i_X(N(\cdot,0),\Omega). \tag{4.38}$$

Finally, we define a third homotopy. Let $\bar{\pi}: \mathbf{R}\times\mathbf{R}^m \to \mathbf{R}^m$ be the dynamical system induced by $\dot{x}=\bar{f}_0(x)$ and observe that, with the notation introduced along the proof,

$$\bar{\pi}(t,z) = v(0,z,t;0).$$

By Lemma 4.3, C is positively invariant with respect to $\bar{\pi}$. A nonlinear operator H is defined on $X\times[0,1]$ as follows:

$$H(x,\beta) := \bar{\pi}((1-\beta)\varepsilon + \beta\phi(0,\cdot),x(\omega)).$$

As before, $H : X\times[0,1]\to X$ and it is compact on $cl_X\Omega\times[0,1]$. Moreover,

$$N(x,0) = H(x,1).$$

In this case, $x\in X$ is a fixed point of $H(\cdot,\beta)$ if and only if $x(t) \equiv y((1-\beta)\varepsilon + \beta\varepsilon(t/\omega))$, with $y:[0,\varepsilon]\to C$ an ε-periodic solution of

$$\dot{y} = \bar{f}_0(y)$$
$$y(0) = x(\omega).$$

We claim that there is $\varepsilon_1 \in (0,\varepsilon_0]$ such that $H(x,\beta) \neq x$ for all $x \in fr_X\Omega$ and $\beta \in [0,1]$, provided that $\varepsilon \in (0,\varepsilon_1]$. Assume the contrary; then for each $n \in \mathbb{N}$ there are $\varepsilon_n \in [0,\omega]$ with $\lim \varepsilon_n = 0$, $\beta_n \in [0,1]$ and $x_n \in fr_X\Omega$ such that $H(x_n,\beta_n) = x_n$. We consider the auxiliary functions $z_n \in X$ defined by $z_n(t) := y_n(\varepsilon_n t/\omega)$, where $\dot{y}_n = \bar{f}_0(y_n)$ and $y_n(0) = y_n(\varepsilon_n) = x_n(\omega)$. For such z_n we have:

(4.39) $$\dot{z}_n = (\varepsilon_n/\omega)\bar{f}_0(z_n),$$
$$z_n(0) = z_n(\omega) = x_n(\omega)$$

and

(4.40) $$x_n(t) = z_n((1 - \beta_n)\omega + \beta_n t).$$

Arguing as in the preceding claim, we easily get $|\dot{z}_n|_\infty \leq (\varepsilon_n/\omega)A$ and $|z_n(0)| \leq R$ (with $R > 0$ a suitable constant independent of n). Again, the Ascoli-Arzelà theorem implies that (passing, possibly, to subsequences) $z_n(\cdot) \to z^*(\cdot)$ in the d*-metric, with $z^*(\cdot) \equiv z^* =$ constant and $\lim\beta_n = \beta^* \in [0,1]$. Furthermore by (4.40) $x_n(\cdot) \to z^*$ in the d*-metric, with $z^* \in C \cap fr_X\Omega$. Taking the mean value of (4.39) and dividing by (ε_n/ω) we get

$$\frac{1}{\omega}\int_0^\omega \bar{f}_0(z_n(t))dt = 0\,;$$

hence, passing to the limit as $n \to +\infty$, we have $\bar{f}_0(z^*) = 0$, with $z^* \in C \cap fr_X\Omega$ and a contradiction with (g3) is reached.
Therefore, the claim is proved and we can write:

(4.41) $$i_X(N(\cdot,0),\Omega) = i_X(H(\cdot,1),\Omega) = i_X(H(\cdot,0),\Omega).$$

By definition of H, we have:

$$H(x,0) = \bar{\pi}(\varepsilon, x(\omega)) = \bar{\pi}_\varepsilon(x(\omega)),$$

where $\bar{\pi}_\varepsilon$ is the ε-Poincaré map ($0 < \varepsilon \le \varepsilon_1$). Hence, since $H(\cdot,0):X \to C$, by the contraction property of the fixed point index (see proposition 1.1) we have:

$$(4.42) \qquad i_X(H(\cdot,0),\Omega) = i_C(H(\cdot,0),\Omega\cap C) = i_C(\bar{\pi}_\varepsilon,\Omega\cap C).$$

(Observe that, as a consequence of the last claim, $\bar{\pi}_\varepsilon$ is fixed point free on $fr_C(\Omega\cap C)\subset C\cap fr_X\Omega$).
In conclusion, we have proved that, via (4.35), (4.38), (4.41) and (4.42), the integer $i_C(M(\cdot,1),\Omega) = i_C(\bar{\pi}_\varepsilon,\Omega\cap C)$ is constant with respect to ε, for $\varepsilon > 0$ small enough.
Then,

$$i_X(M(\cdot,1),\Omega) = \lim_{\varepsilon\to 0^+} i_C(\bar{\pi}_\varepsilon,\Omega\cap C) = I(\bar{\pi},\Omega\cap C)\ .$$

Assumption (g4) provides (see Chapter 1) the existence of a fixed point $x\in\Omega$ of $M(\cdot,1)$. Therefore, the conclusion is established.
In the case when (4.32) is not satisfied, the proof can be repeated for the equation

$$(4.43) \qquad \dot{x} = f(t,x;1)\cdot\rho(|x|),$$

where $\rho:R_+\to[0,1]$ is lipschitzian and such that $\rho(x) = 1$ for $|x| \le R$, $\rho(x) = 0$ for $|x| \ge 2R$ and $cl_X\Omega\subset B(0,R)$.

Of course, the local flow $\bar{\pi}$ induced by (4.29) coincides with the flow induced by $\dot{x} = \bar{f}_0(x)\rho(|x|)$ in a neighbourhood of $\Omega\cap C$ and, moreover, any solution of (4.43)-(4.28) such that $x\in cl_X\Omega$ is also a solution of (4.27)-(4.28).
The proof is complete. ∎

REMARK 4.5. As we mentioned in Chapter 3, the flow-invariance condition (g1) may be stated in an equivalent geometrical manner using tangent cones. Indeed, (g1) holds if and only if (h1) $\qquad f(t,z;\lambda)\in T(z;C)$, *for all* $t\in[0,\omega]$, $z\in \mathrm{fr}C$, $\lambda\in[0,1]$

is satisfied (see [2,10]). Hence, if $f(t,x;\lambda)$ is defined only for $x\in C$, then (h1) ensures that all the processes considered in the proof of Theorem 4.2 are defined.
A standard situation in which the function $f(t,\cdot;\lambda)$ is defined just on the set C occurs, for example, when C is a regular manifold; in this case, $f(t,z;\lambda)\in A\cap -A$, where $A = T(z;C)$, whenever f is a tangent vector field and so (h1) holds. Accordingly, our result is general enough to be applied to the setting considered in [4,19,20].
If C is a convex set like in section 4.2, then (h1) reduces to $(f(t,z;\lambda)|\eta) \leq 0$ for each $\eta\in \mathcal{N}(z,C)$.

REMARK 4.6. It is possible to obtain a variant of Theorem 4.2 assuming, besides (g2) and (g3), the following conditions which replace (g1) and (g4):

(g_1^-) $\qquad$ C *is negatively invariant for* $\dot{x}=f(t,x;\lambda)$, $\lambda\in[0,1]$;

(g_4^-) $\qquad \lim_{\varepsilon\to 0^-} i_C(\pi_\varepsilon,\Omega\cap C) \neq 0$.

This can be accomplished by the standard change of variables $t\longmapsto \omega-t$ which transforms equation (4.27) into $\dot{x}=-f(s,x;1)$, where $s=\omega-t$.
Observe that if, furthermore, the critical set

$$Z = \{z\in C: \bar{f}_0(z) = 0\}$$

is compact and $\Omega\cap C\supset Z$, then $\lim_{\varepsilon\to 0^-} i_C(\pi_\varepsilon,\Omega\cap C)$ is exactly $\chi(\bar{f}_0)$, the "characteristic of the vector field" $\bar{f}_0$ defined in [20] (see section 3.3 above). It is also clear that (g_1^-) is equivalent to

(h_1^-) $\qquad f(t,z;\lambda)\in -T(z;C)$, *for all* $t\in[0,\omega]$, $z\in \mathrm{fr}C$, $\lambda\in[0,1]$.

so that the situation considered in [20] fits our hypotheses (see also Remark 4.8).

Now, we present, as immediate corollaries of Theorem 4.2, an extension to ENRs of two classical results of existence of solutions for the periodic problem (4.27)-(4.28). Namely, only a (suitable) different choice of the set $\Omega\subset X$ is needed.
In what follows, $G\subset C$ denotes a bounded set which is open relatively to C. Observe that $cl_C G = clG$.
In the lines of theorem 3.1 we can prove the following:

COROLLARY 4.3. *Assume* (g1) *and suppose that the following conditions are satisfied:*

(h2) *for any* $x(\cdot)$, *solution of* (4.27_λ)-(4.28) *such that* $x(t)\in clG$ *for all* $t\in[0,\omega]$, *it follows that* $x(t)\in G$ *for all* $t\in[0,\omega]$;

(h3) $\bar{f}_0(z)\neq 0$ *for all* $z\in fr_C G$;

(h4) $I(\bar{\pi},G)\neq 0$.

Then, (4.27)-(4.28) *has at least one solution* $x(\cdot)$ *such that* $x(t)\in clG$, *for all* $t\in[0,\omega]$.

Proof. In the setting of Theorem 4.2 we define:

$$\Omega=\{x\in X\colon x(t)\in G,\ \forall t\in[0,\omega]\}.$$

It can be checked that Ω is bounded and open relatively to X.
Furthermore, the following facts hold true:

$$\Omega\cap G=G;$$
$$cl_X\Omega\subset\{x\in X\colon x(t)\in clG,\ \forall t\in[0,\omega]\};$$
$$fr_X\Omega\subset\{x\in X\colon x(t)\in clG,\ \forall t \text{ and } \exists t_0 \text{ with } x(t_0)\in fr_C G\}.$$

Hence, (h2) and (h4) imply (g2) and (g4), respectively. Finally, (g3) follows from (h3) since

$$\mathrm{fr}_C G \subset C \cap \mathrm{fr}_X \Omega.$$

Therefore, Theorem 4.2 applies and the proof is complete. ∎

REMARK 4.7. Hypothesis (h2) is a transversality condition at boundary points as considered in theorem 3.1. However, in (h2) not all the boundary is concerned, but only points of $\mathrm{fr}_C G$ are taken into account. This advantage is balanced by a weak boundary condition which is implicitly required in (g1). As we already observed in the previous section, the flow-invariance assumption (g1) is equivalent to the cone condition (h1). However, since in Corollary 4.3 we study solutions lying in $\mathrm{cl}_C G$, we realize that it is possible to obtain a slight improvement of Corollary 4.3 by relaxing (h1). Namely, we have:

COROLLARY 4.3'. *Besides* (h2), (h3), (h4), *assume*

(h_1') $$f(t,z;\lambda) \in T(z;C)$$

for all $t \in [0,\omega]$, $z \in \mathrm{fr}C \cap \mathrm{cl}G$, $\lambda \in [0,1]$.

Then, the same conclusion of Corollary 4.3 *holds.*

The proof of this result can be achieved via a standard perturbation argument based on the Ascoli-Arzelà theorem (see [18] for an analogous situation).
In the particular case when C is a closed convex set with nonempty interior, Corollary 4.3' can be seen as a consequence of theorem 4.1.

A simple application of Corollary 4.3 is based on the fact that assumption (h2) is fulfilled whenever a-priori bounds for the solutions of (4.27_λ)-(4.28) can be produced. Accordingly, we have (recall Remark 4.6):

PROPOSITION 4.1. *Assume that, for all* $t \in [0,\omega]$, $z \in \mathrm{fr}C$ *and* $\lambda \in [0,1]$,

$$f(t,z;\lambda)\in T(z;C) \quad (\textit{respectively}, f(t,z;\lambda)\in -T(z;C)).$$

Suppose that there is a compact set $K\subset C$ *containing all the solutions of* (4.27_λ)-(4.28) *and such that* $\{z\in C: \bar{f}_0(z) = 0\}\subset K$. *Let* $G\subset C$ *be a bounded set, open relatively to* C, *such that* $K\subset G$ *and suppose that*

$$\lim_{\varepsilon\to 0^+} i_C(\bar{\pi}_\varepsilon,G) \neq 0 \quad (\textit{respectively}, \lim_{\varepsilon\to 0^-} i_C(\bar{\pi}_\varepsilon,G) \neq 0) .$$

Then, (4.27)-(4.28) *has at least one solution with values in* K.

Observe that, by the excision property of the fixed point index, the limits $\lim_{\varepsilon\to 0^\pm} i_C(\bar{\pi}_\varepsilon,G) \neq 0$ are independent of the choice of $G\supset K$. The above proposition clearly contains Theorem 3.4 by Furi and Pera; in fact, according to the notations introduced in [20], $\lim_{\varepsilon\to 0^-} i_C(\bar{\pi}_\varepsilon,G) = \chi(\bar{f}_0)$.
We note that there is no loss of generality, in our setting, if we take $K = B[0,R_0]\cap C$ and $G = B(0,R)\cap C$ for any $R > R_0$. In this way, we obtain a generalization to arbitrary ENRs of an useful principle due to Mawhin [37, Th.4].
Finally, we remark that Proposition 4.1 is suitable for C non-compact. Indeed, if C is compact then we can choose $K = G = C$ and $f(t,x;\lambda) \equiv F(t,x)$. Accordingly, Proposition 4.1 recovers a classical result on the existence of periodic orbits in compact positively (negatively) invariant ENRs with non-zero Euler characteristic (cf. Poincaré-Hopf theorem).

Secondly, by means of another choice of the set Ω in Theorem 4.2, we prove two corollaries of our main result which are in the lines of the well-known Krasnosel'skii theorem [33, Th.6.1].

COROLLARY 4.4. *Besides* (g1), (h3) *and* (h4), *assume further*

(h_2') *there is no solution of* (4.27_λ)-(4.28) *with* $x(0)\in fr_C G$;

(h5) $|f(t,x;\lambda)| \leq A|x| + B$, *for all* $t\in [0,\omega]$, $x\in C$, $\lambda\in [0,1]$.

Then, (4.27)-(4.28) *has at least one solution* $x(\cdot)$ *such that* $x(0)\in cl_C G$.

Proof. First of all, we note that (h5) ensures the global existence for all the Cauchy problems associated to (4.27_λ) with initial values in C.
Then, there is a constant $R > 0$, independent of λ, such that $|x|_\infty < R$ for every $x(\cdot)$ solution of (4.27_λ) with $x(0)\in clG$.
In this situation, the appropriate definition of the set Ω (in order to apply Theorem 4.2) is the following:

$$\Omega := \{x\in X: x(0)\in G, |x|_\infty < R\}.$$

Obviously, Ω is bounded and open relatively to X. Observe that $\Omega\cap C = G$ and $fr_X\Omega \subset \{x\in X: x(0)\in fr_C G, |x|_\infty \leq R\}\cup\{x\in X: x(0)\in clG, |x|_\infty = R\}$. Then, by the choice of R, it is immediately seen that (h_2') implies (g2). Since $fr_C G \subset C\cap fr_X\Omega$, arguing like in the proof of the previous corollary, from (h3) and (h4) we obtain (g3) and (g4), respectively. Then we can apply Theorem 4.2 and the proof is complete. ∎

As a consequence of Corollary 4.4 we immediately get an extension of Krasnosel'skii's theorem to arbitrary ENRs. Precisely, we consider the equation:

(4.44) $\dot{x} = g(t,x)$

with $g : [0,\omega]\times\mathbf{R}^m\to\mathbf{R}^m$ continuous, locally lipschitzian in x and such that the (forward) global existence for the solutions of the associated Cauchy problems with initial values in C is guaranteed. Then, we have:

COROLLARY 4.5. *Suppose that the following conditions are satisfied:*

(k1) C *is positively invariant for equation* (4.44);

(k2) *there is no solution* $x(\cdot)$ *of* (4.44) *such that* $x(0) = x(k)\in fr_C G$, *for some* $0 < k < \omega$;

(k3) $g(0,z) \neq 0$ *for* $z \in fr_C G$.

Let π^0 *be the (local) flow induced by* $\dot{x} = g(0,x)$ *and assume*:

(k4) $I(\pi^0,G) \neq 0$.

Then, (4.44)-(4.28) *has at least one solution* $x(\cdot)$ *with* $x(0) \in clG$.

According to Krasnosel'skii's terminology, assumption (k2) means that the points of $fr_C G$ are points of "ω-irreversibility".

Proof. By the global existence, there is a constant $R > 0$ such that $|x|_\infty < R$ for every $x(\cdot)$ solution of (4.44) with $x(0) \in clG$. Let $\rho: R^m \to [0,1]$ be a locally lipschitzian function such that $\rho(x) = 1$ for $|x| \leq R$ and $\rho(x) = 0$ for $|x| \geq 2R$.
Now we define, for $\lambda \in [0,1]$, $f(t,x;\lambda) := \rho(x) g(\lambda t, x)$ and observe that (by the choice of R, $\rho(\cdot)$) $x \in X$ is a solution of (4.27_λ)-(4.28) with $x(0) \in fr_C G$ if and only if $y(t) := x(t/\lambda)$ is a solution of $\dot{y} = g(t,y)$ with $y(0) = y(\lambda\omega) \in fr_C G$. Then, (h_2') follows from (k2). We also remark that (k1) implies (g1) and (k3), (k4) are nothing but (h3), (h4) respectively. Finally, (h5) is fulfilled with $A = 0$ and $B = \sup\{|g(t,x)|, t \in [0,\omega], x \in C, |x| \leq R\}$. Then, Corollary 4.4 applies and the result is achieved. ∎

An analogous result was obtained by R. Srzednicki (see theorem 3.3 and corollary 3.1) where $g: R \times R^m \to R^m$ is ω-periodic in the t-variable.
Instead of (k2), in corollary 3.1 it is assumed that there exists a set $B \subset R^m$ which is a block (according to definition 3.1) with respect to the vector field $g(t,\cdot)$, *for each* t. However, we point out that corollary 4.5 is not contained in corollary 3.1, as the following example shows.

EXAMPLE 4.1. Let $\phi: [0,\omega] \to R^+$ be a function of class C^1 such that the following properties are satisfied:

(4.45) $\dot{\phi}(0) / \phi(0) = \dot{\phi}(\omega) / \phi(\omega)$

and

(4.46) $\dot{\phi}(\omega/2)\, \dot{\phi}(0) < 0.$

Let $\psi: R \rightarrow R$ be continuous, ω-periodic and such that

$$\psi(t) = \dot{\phi}(t) / \phi(t)$$

for each $t \in [0,\omega]$.
Now, consider the Cauchy problem

(4.47) $\dot{x} = \psi(t)x$

(4.48) $x(0) = x_0.$

We claim that if $\phi(0) \neq \phi(\omega)$, then (4.47) has a unique ω-periodic solution which is identically zero.
Indeed, the solution of (4.47)-(4.48) is given by

$$x(t) = (x_0 / \phi(0))\phi(t),$$

and it is easily seen that, if $x_0 \neq 0$, then

$$x(\omega) = (x_0 / \phi(0))\, \phi(\omega) \neq x_0;$$

thus, the claim is proved.

Now, if we take $C := R$, $G := B(0,R)$, $R>0$ and $g(t,x) := \psi(t)x$, then we can prove that all the hypotheses of corollary 4.5 are satisfied. Indeed, if we further assume $\phi(t) < \phi(0)$ for all t, then (k2) is straightforward; moreover, $g(0,z) \neq 0$ for each $z \in fr_C G$ and $I(\pi^0, G)$ is defined and different from zero. Thus, corollary 4.5 is applicable.
On the other hand, we have, by (4.45) and (4.46), that there is $t^* \in [0,\omega]$ such that $\dot{\phi}(t^*)/\phi(t^*) = 0.$

Thus, (*1) or (*2) in definition 3.1 of block with respect to the vector field $x \longmapsto \psi(t)x$ may be violated, and so the assumptions of corollary 3.1 are not satisfied.
We finally note that the set clG *is* a block with respect to $g(0,\cdot)$. (See also Remark 6.1 and Remark 6.2).

REMARK 4.8. Straightforward variants of Corollaries 4.4 and 4.5 may be easily obtained following Remark 4.5 and Remark 4.6. In particular, the case of C negatively invariant may be treated as well.
The above result clearly generalizes [33, Th.6.1]; in the special case in which C is a manifold and g is a tangent vector field Corollary 4.5 reduces to theorem 3.4.
Finally, we point out that when we deal with arbitrary closed ENRs, it is not possible to extend the results of this section to second order differential systems

$$-\ddot{x} = F(t,x,\dot{x})$$

in a straightforward manner.

CHAPTER 5

FURTHER RESULTS.

Let $C \subset \mathbf{R}^m$ be a closed ENR.
We consider the equation

(5.1) $$\dot{x} = g(t,x),$$

with $g : [0,\omega] \times \mathbf{R}^m \to \mathbf{R}^m$ continuous, locally lipschitzian in x and such that the global existence for the solutions of the associated Cauchy problems, with initial values in C, is guaranteed. Our aim is to obtain a consequence of Corollary 4.5 in which the transversality condition (k2) follows by means of some explicit geometrical hypotheses on the vector field g. More precisely, we examine an extension to ENRs of the concept of guiding function (see[33,34]).
Now, we introduce the concept of guiding function relatively to the set C.

DEFINITION 5.1. *Let* $\Phi:\mathbf{R}^m \to \mathbf{R}^m$ *be a continuously differentiable function with* $\nabla\Phi$ *locally lipschitzian on* C. *We say that* Φ *is a guiding function for the equation* (5.1) *relatively to* C *if there is* $R_0 > 0$ *such that* $B(0,R_0) \cap C \neq \emptyset$ *(to avoid trivialities) and*

(5.2) $$(\nabla\Phi(x) \mid g(t,x)) > 0$$

for all $t \in [0,\omega]$, $x \in C$ *and* $|x| \geq R_0$.

In particular, it follows that $\{x \in C : \nabla\Phi(x) = 0\} \subset B(0,R_0) \cap C$.
We confine ourselves to guiding functions satisfying the additional condition:

(Φ1) C *is positively invariant for*

(5.3) $$\dot{x} = \nabla\Phi(x).$$

Then, if we denote by π^Φ the (local) flow induced by (5.3), we have that, by (5.2) and (ϕ1), the index of rest points $I(\pi^\Phi, B(0,R_0) \cap C)$ is defined for any $R \geq R_0$ (see Chapter 2) and it is constant with respect to $R \geq R_0$ by the excision property. Hence, the integer

$$(5.4) \qquad J_C(\Phi,\infty) := \lim_{R\to+\infty} I(\pi^{\phi},B(0,R)\cap C)$$

is well defined.

REMARK 5.1. Up to now, we have just followed, verbatim, the corresponding definition of guiding function in $\mathbf{R}^m$ given by Krasnosel'skii ([33, § 6.3]), modulo the natural modifications due to the more general setting. Now, we explain the meaning of (5.4) in some particular cases. If $C = \mathbf{R}^m$, then (Φ1) is vacuously satisfied and

$$(5.5) \qquad J_C(\Phi,\infty) = (-1)^m \gamma(\Phi,\infty),$$

where γ is the "index of non-degeneracy" of Φ, according to [33, p.84]).
If C is a (regular) manifold, it turns out that

$$J_C(\Phi,\infty) = \chi(-\nabla\Phi),$$

where $\chi(-\nabla\Phi)$ is the characteristic of the (tangent) vector field $-\nabla\Phi$, according to [20, p.325].
If C is compact, then

$$J_C(\Phi,\infty) = I(\pi^{\Phi},C) = \chi(C),$$

where $\chi(C)$ is the Euler-Poincaré characteristic of C.

For the proof of the next theorem, we need a preliminary result relating homotopic fields with the indexes of the corresponding flows.
Let $h = h(x;\lambda):\mathbf{R}^m\times[0,1]\to\mathbf{R}^m$ be continuous and such that, for each $\lambda\in[0,1]$, the solutions for the Cauchy problems

$$(5.6) \qquad \dot{x} = h(x;\lambda)$$

$$(5.7) \qquad x(0) = z$$

are unique. We denote by π^{λ} the local flow induced by (5.6).
Then we have:

LEMMA 5.1. *Let* G *be a bounded subset of* $\mathbf{R}^m$, *open relatively to* C. *Assume that, for each* $\lambda \in [0,1]$, C *is positively invariant with respect to equation* (5.6). *If*

(L1) $\qquad h(x;\lambda) \neq 0$

holds for all $x \in fr_C G$ *and* $\lambda \in [0,1]$, *then*

(5.8) $\qquad I(\pi^0, G) = I(\pi^1, G).$

Particular cases of this result have been already examined in [20]. For the reader's convenience, we give the complete proof in the general situation.

Proof. We set

$$\eta := \inf\{|h(z;\lambda)| : z \in fr_C G,\ \lambda \in [0,1]\};$$

by (L1), $\eta > 0$. We define $x(t,z;\lambda)$ to be the solution of (5.6)-(5.7) and observe that, according to the notations previously introduced:

$$x(\varepsilon,z;\lambda) = \pi_\varepsilon^\lambda(z).$$

First of all, we note that there is $K > 0$ such that $x(\cdot)$ is defined on $[0,K]$, for each $z \in fr_C G$ and $\lambda \in [0,1]$. Then, the set

$$\mathcal{B} = \{x(t,z;\lambda) : t \in [0,K],\ z \in fr_C G,\ \lambda \in [0,1]\}$$

is a compact subset of C.
Finally, let $M > 0$ be such that

$$|h(w;\lambda)| \leq M$$

for each $w \in \mathcal{B}$ and $\lambda \in [0,1]$.

Fix ε_0 such that $0 < \varepsilon_0 < K$. Then, for any $\varepsilon \in (0,\varepsilon_0]$, we have:

$$x(\varepsilon,z;\lambda) - z = \varepsilon \int_0^1 h(x(\theta\varepsilon,z;\lambda);\lambda)\, d\theta$$

$$= \varepsilon \int_0^1 \left[h(x(\theta\varepsilon,z;\lambda);\lambda) - h(z;\lambda)\right] d\theta + \varepsilon h(z;\lambda) .$$

Since

$$|x(\theta\varepsilon,z;\lambda) - z| \le \varepsilon_0 M$$

for every $\theta \in [0,1]$, $\varepsilon \in (0,\varepsilon_0]$, $z \in fr_C G$ and $\lambda \in [0,1]$, by the uniform continuity of h on $\mathcal{B}\times[0,1]$ we have:

$$|h(x(\theta\varepsilon,z;\lambda);\lambda) - h(z;\lambda)| < \eta/2$$

for ε_0 small enough. Hence, we obtain:

$$(1/\varepsilon)|x(\varepsilon,z;\lambda) - z| \ge |h(z;\lambda)| - \eta/2 \ge \eta/2$$

for all $z \in fr_C G$, $\lambda \in [0,1]$, $\varepsilon \in (0,\varepsilon_0]$.

Then, we have proved that

$$i_C(x(\varepsilon,\cdot;\lambda),G) = \text{constant}$$

for all $\lambda \in [0,1]$, $\varepsilon \in (0,\varepsilon_0]$.

Therefore, (5.8) follows immediately. ∎

Now we are in position to state the main result of this Chapter. As before, we denote by X the complete metric space of the continuous functions $x(\cdot) : [0,\omega] \to C$ endowed with the distance d^*, $d^*(x_1,x_2) := |x_1 - x_2|_\infty$.

THEOREM 5.1. *Let Φ be a guiding function for equation* (5.1) *relatively to* C *and suppose that* C *is positively invariant for* (5.1) *and* (5.3).

Then, there is a solution $x(\cdot) \in X$ *to* (5.1)-(3.2) *(i.e. an* ω*-periodic solution), provided that*

(Φ2) $\qquad J_C(\Phi,\infty) \neq 0.$

Proof. We apply Corollary 4.5 with respect to the set $G = B(0,R) \cap C$, where $R > R^*$, and $R^* := \sup\{x(t): t, t_0 \in [0,\omega], \dot{x} = g(t,x), x(t_0) \leq R_0\}$. Then, (k2) and (k3) follow from the definition of guiding function, arguing like in [34, p.48]. Finally, we observe that (5.2) implies that the function $h(x;\lambda) := (1-\lambda)g(0,x) + \lambda\nabla\Phi(x)$ satisfies (L1) of lemma 5.1, so that

$$I(\bar{\pi},G) = I(\pi^0,G) = I(\pi^1,G) = J_C(\Phi,\infty)$$

Then, (Φ2) implies (k4) and the proof is complete. ∎

Clearly, Theorem 5.1 is an extension of Krasnosel'skii's result [33, Th.6.5], [34, Th.13.1] to the case of a flow-invariant ENR. In [33,34], various criteria are proposed in order to evaluate $\gamma(\Phi,\infty)$ for $C = \mathbf{R}^m$. In particular, it is proved that

$$\gamma(\Phi,\infty) = (-1)^m \text{, for } \lim_{|x| \to +\infty} \Phi(x) = -\infty$$

and

$$\gamma(\Phi,\infty) = 1 \text{, for } \lim_{|x| \to +\infty} \Phi(x) = +\infty.$$

On the same line and combining arguments from [33,49], we can prove an analogous result for $J_C(\Phi,\infty)$.

LEMMA 5.2. *Let* Φ *be a guiding function relatively to* C, *verifying* (Φ1) *and*

(Φ 3) $\qquad \lim_{\substack{|x| \to +\infty \\ x \in C}} \Phi(x) = -\infty.$

Then, C *is of finite type and* $J_C(\Phi,\infty) = \chi(C)$.

Essentially, this result follows from (vii) in Proposition 4.1. However, according to our hypotheses, equation (5.3) does not induce a dynamical system as required in [49] but just a (local) semi-flow on C. Thus, we give the details of the proof for the reader's convenience.

Proof. Obviously, if C is bounded then (Φ3) is vacuously satisfied and Remark 5.1 immediately gives the result. Hence, we consider the case of C unbounded.
First, we observe that

$$(5.9) \qquad \nabla\Phi(x) \neq 0$$

for all $x \in C$, $|x| \geq R_0$. Now, we fix $c^* \in \mathbf{R}$ such that

$$c^* \leq \inf\{\Phi(x) : x \in C, |x| \leq R_0\}.$$

Then, for any $c \leq c^*$, we consider the sets:

$$\begin{aligned} K_c &:= \{x \in C : \Phi(x) \geq c\}, \\ L_c &:= \{x \in C : \Phi(x) = c\}, \\ M_c &:= \{x \in C : \Phi(x) \leq c\}. \end{aligned}$$

By (Φ1), (Φ3) and (5.9), it follows that for every $c \leq c^*$, we have: K_c is compact and flow-invariant for (5.3), $L_c = \mathrm{fr}_C K_c = \mathrm{fr}_C M_c$ and each $x \in L_c$ is a strict egress point for M_c (according to Ważewski [52]). Let $x \in M_{c^*}$; we want to show that there is $t_x \geq 0$ such that $\pi^\Phi(t_x,x) \in L_{c^*}$. Indeed, let us assume $\Phi(x) = c < c^*$; then, there is $\eta > 0$ such that $|\nabla\Phi(y)| \geq \eta$ for every $y \in K_c \backslash K_{c^*}$. Following [33, Lemma 6.5], the function $\phi(t) := \Phi(\pi^\Phi(t,x))$ is such that $\phi(t) \geq c$ for all $t \geq 0$ and $\dot{\phi}(t) \geq \eta^2$ for all $t \geq 0$ such that $\pi^\Phi(t,x) \in K_c \backslash K_{c^*}$. Then, arguing by contradiction, it can be seen that the solution of (5.3) with initial value x meets L_c at a time t_x, with $t_x \leq (c^* - c)/\eta^2$. Note that such t_x is unique. If $x \in L_{c^*}$, the claim follows with $t_x = 0$.
By Ważewski's Lemma, we know that the map $x \longmapsto t_x$, $x \in M_{c^*}$, is continuous (see [9,52]) and so K_{c^*} is a strong deformation retract of C via the homotopy

$$(x,\lambda) \longmapsto \pi^\Phi(\lambda t_x, x), \quad x \in M_{c^*}$$

$$(x,\lambda) \longmapsto x, \qquad\qquad x \in K_{c^*}.$$

Then, K_{c^*} is a compact ENR and C and K_{c^*} have the same homotopy type. Accordingly, C is of finite type and

$$\chi(C) = \chi(K_{c^*}) = \chi(K_c)$$

for every $c \leq c^*$.
Now, it is clear that $\pi^\Phi(t,x) \neq x$ for every $x \in M_{c^*}$ and $t > 0$ (see also the proof of Theorem 5.1); hence, for any $\varepsilon > 0$, $i_C(\pi^\phi_\varepsilon, B(0,R) \cap C)$ is defined whenever $B(0,R) \supset K_{c^*}$. Fix such an R and let $c \leq c^*$ be such that $K_c \supset B[0,R] \cap C$. Then, using the excision and contraction properties of the fixed point index, we can write:

$$i_C(\pi^\Phi_\varepsilon, B(0,R) \cap C) = i_C(\pi^\Phi_\varepsilon, \mathrm{int}_C K_c) = i_{K_C}(\pi^\Phi_\varepsilon, K_c) .$$

On the other hand, π^ϕ_ε is homotopic to the identity Id on K_c (moving the points along the semi-orbits); consequently,

$$i_{K_C}(\pi^\Phi_\varepsilon, K_c) = i_{K_C}(\mathrm{Id}, K_c) = \chi(K_c) .$$

From the above inequalities, letting $\varepsilon \to 0^+$, $R \to +\infty$ and recalling the definition of $J_C(\Phi,\infty)$, we have the conclusion. ∎

REMARK 5.2. A simple application of Lemma 5.2 can be performed when C is convex and flow-invariant with respect to (5.3). Indeed, in such a case $\chi(C) = 1$ and so (Φ2) holds. We notice that, even in this simple situation, the validity of (Φ2) is not ensured if

$$(5.10) \qquad \lim_{\substack{|x| \to +\infty \\ x \in C}} \Phi(x) = +\infty$$

is assumed instead of (Φ3). For instance, it is easy to prove that when $C \setminus B(0,R)$ (R large enough) is contractible, then (Φ1) and (5.10) imply $J_C(\Phi,\infty) = 0$.

We finally notice that in the proof of lemma 5.2 we have shown that the flow π^{Φ} is dissipative; indeed, one can prove (suitably modifying the arguments in [49,29]) that $I(\Pi, B(0,R) \cap C) = \chi(C)$, for $R > 0$ large, whenever Π is a dissipative semi-flow on C.

If, furthermore, $g: \mathbf{R}_+ \times \mathbf{R}^m \to \mathbf{R}^m$ is ω-periodic in the t-variable, then, with a few changes in the proof of Lemma 5.2, one can also show that the process induced by (5.1) is dissipative on C, provided that (Φ1) and (Φ3) are satisfied. Hence, in such a particular case the existence of an ω-periodic solution may be obtained using some extensions to ENRs of the known theorems for periodic dissipative processes (see [27, Ch.4]).

CHAPTER 6

APPLICATIONS

6.1. In this chapter we apply the results of Chapter 4. More precisely, in Section 6.2 we prove the existence of non-negative solutions to first order differential systems by means of the results of section 4.2; in Section 6.3 we examine a situation in which it is natural to choose the set C as a domain with holes, and we use the theorems of section 4.3.

6.2. We deal with the problem of the existence of periodic solutions to the differential system

(6.1) $$\dot{x} = F(t,x) = f(t,x;1)$$

lying in the convex set

$$C = R^m_+ = \{x \in R^m : x_i \geq 0 \,,\, i = 1, \ldots, m\}.$$

Throughout this section, we will assume that $F : R_+ \times R^m \to R^m$ is continuous and ω-periodic in the first variable, i.e. $F(t+\omega,x) = F(t,x)$. Then, the solutions of (6.1) such that $x(0) = x(\omega)$ (obtained by means of Theorem 4.1 and its corollaries) may be extended to R_+ as classical (C^1) ω-periodic solutions.
Observe that in our case each $u \in \mathrm{fr}C$ is $\hat{u}_i$ for some $i \in \{1,\ldots,m\}$ and $\eta \in \mathcal{N}(u)$ if and only if $\eta = \sum_{i\in J} -\mu_i e_i \,,\, \mu_i \geq 0,\ \sum_{i\in J} \mu_i > 0,\ J = \{i : u_i = 0\}$. Hence, (e1) of theorem 4.1 holds for all $u \in \mathrm{fr}C$ if and only if $f_i(t,\hat{u}_i;\lambda) \geq 0$ for all $i \in \{1,\ldots,m\}$, $t \in [0,\omega]$ and $\lambda \in [0,1]$.
Our first application may be considered as a variant of a theorem by R. Reissig (see [43, th.2]) in the context of positive solutions, and it applies for $F(t,x) = g(x) + h(t)$, where $g : R^m \to R^m$, $h : R_+ \to R^m$ are continuous and h is ω-periodic. Then we have :

EXAMPLE 6.1. *Suppose that*

(6.2) $$g_i(\hat{x}_i) + h_i(t) \geq 0 \quad,\quad i = 1,\ldots,m \quad \textit{for all } t \textit{ and } x \geq 0 \;;$$

(6.3) $$g_i(x) < -\bar{h}_i \;, \qquad i = 1,\ldots,m \quad \textit{for all } x \geq 0 \textit{ such that } x_i \geq R.$$

Then, equation $\dot{x} = g(x) + h(t)$ *has a non-negative* ω*-periodic solution.*

Proof. We define $f(t,x;\lambda) := (g(x) + h(t))$ and we want to apply again theorem 4.1. Since assumption (e1) is trivially satisfied, we want to find a suitable set G in order to obtain (e2). This is achieved by means of the a-priori bounds performed by Reissig. Indeed, for the i-th component x_i of a ω-periodic solution of the equation

$$(6.1_\lambda) \qquad \dot{x} = f(t,x;\lambda),$$

$\lambda \in (0,1)$, let $M_i = x_i(t^*) = \max\{x_i(t)\}$. Assume $M_i > R$, and observe (taking mean values on $\dot{x}_i = \lambda(g_i(x) + h_i(t))$ that $x_i(\tilde{t}) < R$ for some $\tilde{t} \in [0,\omega]$. Let t_1 be such that $x(t_1) = R$ and $x(t) > R$ for all $t \in (t_1, t^*]$. Using again (6.3), we obtain

$$x_i(t^*) \leq x_i(t_1) + \lambda \int_{t_1}^{t^*} (h_i(s) - \bar{h}_i)ds \;\leq\; R + |h - \bar{h}|_{L^1} := c .$$

Therefore, $M_i \leq c$. Then, we have proved that, for any $c' > c$, the set $G = [0,c')^m$ is suitable for the validity of (e2).

The proof of (e3) is straightforward, since, by (6.3), we have

$$u_i + g_i(u) + \bar{h}_i = u_i + (\bar{f}_0(u))_i < u_i$$

for all $u \in fr_C G$ and therefore there is a compact m-dimensional rectangle $\mathcal{R} \subset G$ such that $r(I + \bar{f}_0) : fr_C G \to \mathcal{R}$. Then, $i_C\,(r(I + \bar{f}_0)\,, G) = 1$ and the proof is complete. ∎

Using a standard perturbation argument, the inequality (6.3) can be easily relaxed.

Secondly, we consider the non-autonomous differential system of Lotka-Volterra type

$$(6.4) \qquad \dot{x}_i = x_i(e_i(t) - g_i(x)) \;, \quad i = 1, ..., m \;,$$

which represents a possible model for the interaction of m species in a seasonal varying environment.

Systems of the form (6.4) have been widely examined in the recent years ([47,48]), expecially in the case of g linear (see [1,16]). Here we confine ourselves to the case in which the functions e_i are periodic of a common period $\omega > 0$ and we look for non-negative and non-trivial ω-periodic solutions to (6.4). More precisely, we suppose that $e := col(e_i) : \mathbf{R}_+ \to \mathbf{R}^m$ is continuous and ω-periodic and $g : \mathbf{R}^m \to \mathbf{R}^m$ is a continuous function. We further assume that

(6.5) $$g(0) = 0 \; ,$$

(6.6) $$\bar{e}_i > 0 \; , \; i = 1, \ldots, m \; .$$

These conditions imply that the intraspecific interactions and the limiting factors have negligible effect at small densities of the populations and that the (possibly non constant) growth rates have positive average during, for example, the year. Such conditions are usually considered in the most natural models.

Observe that in such a situation $(F(t,u) \mid \eta) = 0$ for $\eta \in frC$ and $\eta \in \mathcal{N}(u)$ and (e1) holds for any possible choice of the set G. Adapting again the proof of Reissig theorem, the following result can be achieved, under (6.5)-(6.6).

EXAMPLE 6.2. *Suppose there is* $R > 0$ *such that for each* $i = 1,\ldots,m$

(6.7) $$g_i(x) \geq \bar{e}_i \; , \; \textit{for all} \; x \in \mathbf{R}^m_+ \; \textit{with} \; x_i \geq R \; .$$

Then, equation (6.4) *has at least one* ω*-periodic solution* $x(\cdot)$ *with* $x(t) \geq 0$ *and* $|x(t)| > 0$ *for every* t.

Proof. It is sufficient to define $f_i(t,x;\lambda) := x_i(e_i(t) - g_i(x) - (1-\lambda)\delta x_i)$, where $\delta > 0$ is a fixed but as small as necessary constant, and construct an appropriate set

$$G = [0,M)^m \setminus B[0, \varepsilon] \; ,$$

for suitable $M \geq R$ and $\varepsilon > 0$. The detailed calculations are left out for brevity, since they do not differ from the argument exposed above. ∎

We remark that, under the same assumptions, it can be proved that the one-dimensional subsystems

$$\dot{x}_i = x_i(e_i(t) - g_i(0, \ldots, 0, x_i, 0, \ldots, 0)) \ , \ i = 1, \ldots, m$$

have non-trivial periodic solutions. Therefore, a drawback of our result is that we cannot prevent that the solutions we find in $cl_C G$ are different from these elementary ones. However, there are some examples in which (under suitable quite reasonable hypotheses) the existence of non-elementary solutions is ensured. In this direction, we have :

EXAMPLE 6.3. *Consider the two-dimensional system :*

$$(6.8) \qquad \begin{aligned} \dot{x}_1 &= x_1(e_1(t) - g_1(x_1, x_2)) \\ \dot{x}_2 &= x_2(e_2(t) - g_2(x_1, x_2)) \ , \end{aligned}$$

where $g = (g_1, g_2) : \mathbf{R}^2 \to \mathbf{R}^2$ *and* $e = (e_1, e_2) : \mathbf{R}_+ \to \mathbf{R}^2$ *are continuous functions, with* $g(\cdot,\cdot)$ *locally lipschitzian and* $e(\cdot)$ ω*-periodic. Besides* (6.5) *and* (6.6), *we suppose that there are positive constants* R, K, ε *such that*

(6.9) $\quad |g_i(x)| \leq K , \quad \forall x \in \mathbf{R}^2_+ , i = 1,2 ;$

(6.10) $\quad g_i(x) < \bar{e}_i , \quad$ *for* $x_i \geq R , \ x_j = 0 , \ j \neq i ;$

(6.11) $\quad g_i(x) > \bar{e}_i , \quad$ *for* $x_i \geq R , \ x_j \geq \varepsilon , \ j \neq i ;$

(6.12) $\quad g_i(x) > |e_i|_\infty , \quad$ *for* $x_i \leq \varepsilon , \ x_j \geq R , \ j \neq i .$

Finally, suppose that there are two continuous functions $\gamma_1 , \gamma_2 : [R, +\infty) \to \mathbf{R}^+$ *such that*

$\bar{e}_i = g_i(x)$ *for* $x_i \geq R$ *if and only if* $x_j = \gamma_i(x_i), j \neq i$.

Then, system (6.8) *has at least one non-trivial* ω*-periodic solution which is non-negative. If, furthermore,* $g_i(x) < \bar{e}_i$ *for all* $x_i \geq 0$, $x_j = 0$ $(i = 1,2)$, $j \neq i$, *then such solution a is positive in both components.*

Proof. First of all we remark that from all the assumptions we have

$$(6.13) \qquad g_i(x) > \bar{e}_i\,, \qquad \text{for } x_i \geq R\ ,\ x_j > \gamma_i(x_i)\ ,\ j \neq i\ ,$$

$$(6.14) \qquad g_i(x) < \bar{e}_i, \qquad \text{for } x_i \geq R\ ,\ 0 \leq x_j < \gamma_i(x_i)\ ,\ j \neq i\ .$$

We define $f_i(t,x;\lambda) = x_i(e_i(t) - g_i(x))$, $i = 1,2$, so that system (6.1_λ) becomes

$$(6.15) \qquad \dot{x}_i = \lambda x_i(e_i(t) - g_i(x))\ ,\ i = 1, 2\ , \lambda \in (0,1)\ .$$

Along the proof, $x(\cdot)$ is a ω-periodic solution of (6.15) with $x(t) \geq 0$ for all t and $x(\cdot) \not\equiv 0$. By the uniqueness for the Cauchy problems, we know that $x(t) \neq 0$ for all t. Moreover, if $x_i(t) > 0$ for some t, then the same inequality holds for all t. Let $\delta > 0$ be such that $g_i(y) < (\bar{e}_i / 2)$ for $y \geq 0$, $|y| \leq \delta$. Suppose $0 < x_i(t) < \delta$ for all t. Then by $\dot{x}_i / \lambda x_i = e_i - g_i(x) \geq e_i - \bar{e}_i / 2$, we get immediately a contradiction taking the mean value in a period. Therefore, $x_i(t_i) \geq \delta$ for some t_i, provided that $x_i(\cdot) \not\equiv 0$. Then, arguing like in [43], we have that

$$(6.16) \qquad \dot{x}_i / x_i = \lambda(e_i - g_i(x)) \geq -c\ ,\ \text{with } c := |e|_\infty + \max\{|g(x)|\,,\ |x| \leq \delta\}\ ,$$

holds, for all t such that $0 < x_i(t) \leq \delta$. Hence, we immediately get $x_i(t) > m_i > 0$ for all t, with $m_i < \delta\exp(-c\omega)$.
Now, assume that $x_i(t) > R$ for all t. We distinguish the following cases : either $x_j(t) > \varepsilon$ for all t or $x_j(\tilde{t}) \leq \varepsilon$ for some $\tilde{t}$ (with $j \neq i$). In the latter situation, (6.12) implies that $x_j(t) \leq \varepsilon$ for all t, and so, if $x_j(t) > 0$, we get a contradiction computing the j-th equation at the minimum point of x_j. Therefore, the preceding alternative turns to : either $x_i(t) > R$ and $x_j(t) > \varepsilon$ for all t, or $x_i(t) > R$ for all t and $x_j \equiv 0$. In both cases, we reach a contradiction taking the mean value of (6.16) and using (6.11) and (6.10), respectively.

Thus, we have proved that there is t_i such that $x_i(t_i) \le R$, and hence $x_i(t) < M_i$ for all t, with $M_i > R\cdot\exp\omega(|e_i|_\infty + K)$, by (6.9). Therefore, the set

$$G = ([0,M_1)\times[0,M_2)) \setminus ([0,m_1]\times[0,m_2])$$

fulfils condition (e2).

Finally, (e3) follows from (6.13)-(6.14) using theorem 2.1. Indeed,

$$i_C\,(r(I + \bar{f}_0)\,,\,G) \;=\; \chi(B) - \chi(b^-) = -\,1\,,$$

as $B = ([0,M_1] \times [0,M_2]) \setminus ([0,m_1) \times [0,m_2))$ and b^- is the disjoint union of two closed intervals on the lines $x_1 = M_1$, $x_2 = M_2$ respectively (see remark 4.2).

The final claim in our statement follows by direct computation. ∎

REMARK 6.1. We point out that through the assumptions considered in example 6.3 the existence of periodic solutions on the axis cannot be predicted anymore; $\bar{f}_0$ may even vanish on $\mathrm{fr}C\cap G$. Hence, the usual approach based on topological degree cannot be directly applied. We finally note that neither theorem 3.3 by Srzednicki can be used in this example (see also example 4.1).

6.3. In this section we present some applications of the results of section 4.3 to the periodic BVP

(6.17) $\qquad \dot{x} = F(t,x)$

(6.18) $\qquad x(0) = x(\omega)$,

with $F:\mathbf{R}_+\times(\mathbf{R}^m\setminus S)\to\mathbf{R}^m$ continuous, ω-periodic in the first variable and S a closed subset of $\mathbf{R}^m$. We recall that the solutions of (6.17)-(6.18) may be extended to $\mathbf{R}_+$ as classical ω-periodic solutions.

We examine the case in which it is natural to choose C as a domain with holes. Such situation occurs, for example, in hydrodynamic applications (see [32,42]); for instance, F may denote the velocity field of the flow and $x = x(\mathbf{a},t)$ the position vector at various times t of the "element" of fluid identified by the label **a**.

For simplicity, we confine ourselves to the case

(6.19) $\quad S = \{x_1, x_2, \ldots, x_n\}$.

We point out that, with simple changes in the proofs, the case in which S is the finite union of disjoint compact sets can be considered as well.
In order to apply our theorems, we also suppose that F is locally lipschitzian in x. However, we stress the fact that all the results contained in this section are still true even if F is just continuous, as can be seen by standard perturbation arguments.
For any S like in (6.19), we define

$$\eta := \max\{|x_i|,\ i = 1,\ldots,n\},$$
$$\delta := \min\{|x_i - x_j|,\ i,j = 1,\ldots,n,\ i \neq j\}.$$

The first result of this section is the following:

EXAMPLE 6.4. *Let* $A \cup B = \{1,\ldots,n\}$, $A \cap B = \emptyset$ *and* $\varepsilon \in (0, \delta/2)$ *be such that for all* $t \in [0,\omega]$ *and* $|x - x_i| = \varepsilon$:

(6.20) $\quad (F(t,x) \mid (x - x_i)) \geq 0,$ *for* $i \in A$

(6.21) $\quad (F(t,x) \mid (x - x_i)) \leq 0,$ *for* $i \in B$.

Let $R > \eta + \varepsilon$ *be such that for all* $t \in [0,\omega]$ *and* $|x| = R$,

(6.22) $\quad (F(t,x) \mid x) \geq 0$

holds.

Then, (6.17)-(6.18) *has at least one solution* $x(\cdot)$ *with* $|x|_\infty \leq R$, *provided that one of the following conditions holds:*

(6.23) $\quad m$ *even,* $n \neq 1$

(6.24) $\quad m$ *odd,* $\mathrm{card}(A) \neq \mathrm{card}(B) + 1$.

Roughly speaking, our hypotheses mean that the flow enters in the holes surrounding x_i, $i \in B$, and escapes from B(0,R) and from the holes around x_i, $i \in A$. In such a situation, we only need conditions on the number of such holes.
Our example is a generalization of a similar one considered in [19, p.169], where m = 3 and n = cardB = 1.

Proof. We apply Corollary 4.3. First of all, we note that there exist two lipschitzian functions

$$k : \mathbf{R}^m \to \mathbf{R}^m, \qquad \rho : \mathbf{R}^m \to [0,1]$$

such that

$$k(x) = \begin{cases} x - x_i & \text{for } i \in A \text{ , } |x - x_i| = \varepsilon \\ x - x_i & \text{for } i \in B \text{ , } |x - x_i| = \varepsilon \\ x & \text{for } |x| = R \text{ ,} \end{cases}$$

$$\rho(x) = \begin{cases} 0 & \text{if } \exists i \in B : |x - x_i| \le \varepsilon/2 \text{ ,} \\ 1 & \text{if } \forall i \in B \ |x - x_i| \ge \varepsilon \text{ .} \end{cases}$$

Then, we define:

$$C := \mathbf{R}^m \setminus \Big(\bigcup_{i \in A} B(x_i,\varepsilon) \cup \bigcup_{i \in B} B(x_i,\varepsilon/2) \Big) \text{ ,}$$

$$G := \Big[C \setminus \big(\bigcup_{i \in B} B[x_i,\varepsilon] \big) \Big] \cap B(0,R)$$

and

$$f(t,x;\lambda) := \rho(x)(\lambda F(t,x) + (1 - \lambda)k(x)).$$

We observe that, using (6.20) and the definitions of k and ρ:

$$(f(t,x;\lambda) \mid x - x_i) \geq 0 \quad \text{for } i \in A, |x - x_i| = \varepsilon,$$

$$(f(t.x;\lambda) \mid x - x_i) = 0 \quad \text{for } i \in B, \ |x - x_i| = \varepsilon/2.$$

Then, the set C is positively invariant and (g1) is satisfied.
We note that

$$fr_C G = S(0,R) \cup (\bigcup_{i \in B} S(x_i,\varepsilon))$$

and, by (6.21), (6.22) and the choice of k and ρ,

$$(f(t,x;\lambda) \mid x) > 0 \qquad \text{for } |x| = R,$$

$$(f(t,x;\lambda) \mid x - x_i) < 0 \qquad \text{for } i \in B, \ |x - x_i| = \varepsilon$$

hold for all $t \in [0,\omega]$ and $\lambda \in (0,1)$. Hence, the homotopized field λf is transversal at $fr_C G$ and so, by standard arguments, (h2) is satisfied.

Moreover, $\overline{f}_0(z) = \rho(z)k(z) = k(z)$ for all $z \in fr_C G$ so that (h3) holds.

Finally, (h4) may be computed by means of theorem 2.1 as clG is a block, with $fr_C G$ its set of "egress points". Therefore

$$I(\overline{\pi}, G) = \chi(clG) - \chi(fr_C G).$$

As clG is a closed ball with n holes, we have:

$$\chi(clG) = \begin{cases} 1 - n \ , & m \ \text{even} \\ 1 + n \ , & m \ \text{odd} \ . \end{cases}$$

On the other hand,

$$\chi(fr_C G) = \begin{cases} 0 \ , & m \ \text{even} \\ 2 + 2\text{card}(B) \ , & m \ \text{odd} \ ; \end{cases}$$

hence,

$$I(\bar{\pi}, G) = \begin{cases} 1 - n , & m \text{ even} \\ \text{cardA} - \text{cardB} - 1 , & m \text{ odd} \end{cases}$$

and, using (6.23)-(6.24), (h4) is proved.
Thus, we can apply Corollary 4.3 and we obtain the existence of a solution $x(\cdot)$ of $\dot{x} = f(t,x;1)$ satisfying (6.18) and such that $x(t) \in \text{cl}G$ for all $t \in [0,\omega]$. At last, we observe that $\rho \equiv 1$ on clG, so that $f(t,x;1) = F(t,x)$ and the proof is complete. ∎

It is easy to see that if we reverse the inequality in (6.22), then the result is true provided that (6.24) changes to

$$m \text{ odd}, \quad \text{cardA} \neq \text{cardB} - 1.$$

As a second example in this section, we consider the case in which the nonlinear field F splits as

(6.25) $F(t,x) = g(x) + e(t).$

Without loss of generality (possibly adding to g and subtracting to $e(\cdot)$ the mean value $\bar{e}$), we can assume

(6.23) $\bar{e} = 0 .$

We study a situation in which (6.22) is no longer satisfied but, nevertheless, the existence of periodic solutions is ensured for *any* forcing term e.
We confine ourselves to the case m = 2 in order to make the geometry of the problem more transparent.

EXAMPLE 6.5. *Let* m = 2. *Suppose that, for each* i = 1,...,n,

(6.27) $$\lim_{x \to x_i} (g(x) \mid x - x_i) \,/\, |x - x_i| = +\infty \;;$$

assume there is $R > \eta$ *such that*

(6.28) $\qquad g_j(x) \neq 0, \qquad$ *for* $\ |x_j| \geq R,\ j = 1,2.$

Then, (6.17)-(6.18) *has at least one solution, provided that*

(6.29) $\qquad n \geq 2.$

Proof. We apply Proposition 4.1 with $f(t,x;\lambda):=F(t,x)$. By (6.27), there is $\varepsilon > 0$ with $2\varepsilon < \min\{\delta, R - \eta\}$ such that

$$(F(t,x) \mid x - x_i) = (g(x) + e(t) \mid x - x_i) > 0$$

for $|x - x_i| = \varepsilon$ and all t.
Then, the set

$$C := \mathbb{R}^2 \setminus \Big(\bigcup_{i=1}^{n} B(x_i,\varepsilon)\Big)$$

is positively invariant for (6.17). Now, we produce some a-priori bounds in order to find a compact set K as in Proposition 4.1. Let $x(\cdot)$ be an ω-periodic solution of $\dot{x} = \lambda f(t,x;\lambda) = \lambda(g(x) + e(t))$, for some $\lambda \in (0,1)$. Fix $j \in \{1,2\}$; taking the mean value on $\dot{x}_j = \lambda(g_j(x) + e_j(t))$ and using (6.26), we get $g_j(x(\tilde{t})) = 0$ for some $\tilde{t} \in [0,\omega]$. Then, (6.28) implies $|x_j(\tilde{t})| < R$. Let $M_j = \max\{x_j(t)\} = x_j(t^*)$, with $t^* > \tilde{t}$. If $M_j \leq R$, an upper bound for the j-th component of the solution is found. Assume $M_j > R$; then, there are t_1, t_2 with $t_1 < t^* < t_2 \leq t_1 + \omega$, such that $x_j(t_1) = x_j(t_2) = R$ and $x(t) > R$ for $t \in (t_1,t_2)$. Now, we use an argument from [43]. Indeed, for $\sigma \in \{t_1,t_2\}$ we get:

$$x_j(t^*) \leq x_j(\sigma) + \int_{\sigma}^{t^*} g_j(x(s))ds + \omega|e|_\infty ;$$

by (6.28) and choosing, alternatively, $\sigma = t_1$ if $g_j(x) < 0$ and $\sigma = t_2$ if $g_j(x) > 0$, it follows

$$x_j(t^*) \leq R + \omega|e|_\infty : = c.$$

Then, in any case, we have proved that $M_j \leq c$.
If we repeat the same calculations for $m_j = \min\{x_j(t)\}$, we get $m_j \geq -c$.
Thus, if we set $K := [-c,c]^2 \cap C$ and, for any $d > c$, $G := (-d,d)^2 \cap C$, then we are in the setting of Proposition 4.1. Observe that $fr_C G$ is the perimeter of $[-d,d]^2$. Now, it is enough to prove that $I(\bar{\pi},G) \neq 0$, when $\bar{\pi}$ is the flow induced by

$$\dot{x} = g(x). \tag{6.30}$$

Since clG is a block, the index $I(\bar{\pi},G)$ may be computed by means of theorem 2.1. By (6.28), we know that g_1 (respectively, g_2) has constant sign on each vertical (resp., horizontal) side of $fr_C G$. Then, if we denote by b^- the set of "egress points" of clG with respect to (6.30), we realize (by an exhaustive investigation of all the possible cases) that

$$\chi(b^-) \in \{0,1,2\}.$$

Therefore

$$I(\bar{\pi},G) = \chi(clG) - \chi(b^-) = 1 - n - \chi(b^-).$$

Then, by (6.29), we can apply Proposition 4.1 and the proof is complete. ∎

REMARK 6.2. The result is still true in the more general situation in which $\{1,\dots,n\} = A \cup B$ $(A \cap B = \emptyset)$ and, instead of (6.27), we require

$$\lim_{x \to x_i} (g(x) \mid x - x_i) \,/\, |x - x_i| = +\infty , \qquad i \in A$$

and

$$\lim_{x \to x_i} (g(x) \mid x - x_i) \,/\, |x - x_i| = -\infty , \qquad i \in B .$$

It seems also worthy to notice that the set clG defined along the proof of Example 6.5, which is a block for the "averaged" flow $\bar{\pi}$ induced by (6.30), need not be a block for the process induced by (6.17). Consequently, the results in [50] cannot be applied to this situation.

REFERENCES

[1] C. ALVAREZ and A. C. LAZER, An application of topological degree to the periodic competing species problem, *J. Austral. Math. Soc.* (Ser.B), **28** (1986), 202-218.

[2] J. P. AUBIN and A. CELLINA, "Differential Inclusions", Springer Verlag, Berlin, 1984.

[3] H. AMANN, A note on degree theory for gradient maps, *Proc. Amer. Math. Soc.*, **85** (1982), 591-595.

[4] J. G. BORISOVIC and J. E. GLIKLIH, Fixed points of mappings of Banach manifolds and some applications, *Nonlinear Analysis, TMA* **4** (1980), 165-192.

[5] R. BROWN, "The Lefschetz fixed point theorem", Scott, Foresman & Co., London, 1970.

[6] A. CAPIETTO and F. ZANOLIN, An existence theorem for periodic solutions in convex sets with applications, *Results in Mathematics,* **14** (1988), 10-29.

[7] A. CAPIETTO and F. ZANOLIN, A continuation theorem for the periodic BVP in flow-invariant ENRs with applications. Preprint SISSA-ISAS **55 M** (1988).

[8] L. CESARI, Functional Analysis, nonlinear differential equations and the alternative method, in "Nonlinear Functional Analysis and Differential Equations" (L. Cesari, R. Kannan and J. D. Schuur, eds), pp.1-197, Dekker, New York, 1977.

[9] C. C. CONLEY, "Isolated invariant sets and the Morse index", CBMS **38**, Amer. Math. Soc., Providence, R.I., 1978.

[10] M. G. CRANDALL, A generalization of Peano's existence theorem and flow-invariance. *Proc. Amer. Math. Soc.* **36** (1972), 151-155.

[11] E. N. DANCER, Multiple fixed points of positive mappings, *J. für Reine und Angewandte Math.*, **371** (1986), 46-66.

[12] K. DEIMLING, Cone-valued periodic solutions of ordinary differential equations, in "Applied Nonlinear Analysis" (V. Lakshmikanthan, Ed.), Proc. Conf. Arlington, pp.127-142, Academic Press, New York, 1978.

[13] A. DOLD, "Lectures on Algebraic Topology", Springer Verlag, Berlin, 1972.

[14] J. DUGUNDJI and A. GRANAS, "Fixed point theory", Polish Scientific Publishers, Warszawa, 1982.

[15] V. A. EFREMOVIC and Yu. B. RUDJAK, On the concept of the Euler characteristic, *Uspehi Mat. Nauk* **31**, 1986 5 (191), 239-240. (Russian)

[16] P. DE MOTTONI and A. SCHIAFFINO, Competition systems with periodic coefficients: a geometric approach, *J. Math. Biol.,* **11** (1981), 319-335.

[17] M. L. C. FERNANDES and F. ZANOLIN, Repelling conditions for boundary sets using Liapunov-like functions.1: flow-invariance, terminal value problem and weak persistence, *Rend. Sem. Mat. Univ. Padova,* in print.

[18] M. L. C. FERNANDES and F. ZANOLIN, On periodic solutions, in a given set, for differential systems. Trieste, 1987, preprint.

[19] M. FURI and M. P. PERA, Global branches of periodic solutions for forced differential equations on nonzero Euler characteristic manifolds, *Boll. Un. Mat. Ital.* 3-C (6) (1984), 157-170.

[20] M. FURI and M. P. PERA, A continuation principle for forced oscillations on differentiable manifolds, *Pacific J. Math.,* **121** (1986), 321-338.

[21] R. E. GAINES and J. MAWHIN, "Coincidence degree and nonlinear differential equations", Lecture Notes in Math., **586**, Springer-Verlag, Berlin, 1977.

[22] R. E. GAINES and J. SANTANILLA, A coincidence theorem in convex sets with applications to periodic solutions of ordinary differential equations, *Rocky Mountain J. Math.*, **12** (1982), 669-678.

[23] A. GRANAS, The Leray-Schauder index and the fixed point theory for arbitrary ANRs, *Bull. Soc. Math. France*, **100** (1972), 209-228.

[24] A. GRANAS, "Point fixes pour les applications compactes: espaces de Lefschetz et la théorie de l'indice", Les Presses de l'Université de Montréal, 1980.

[25] H. GROEMER, On the Euler characteristic in spaces with a separability property, *Math. Ann.*, **211** (1974), 315-321.

[26] G. B. GUSTAFSON and K. SCHMITT, A note on periodic solutions for delay differential systems, *Proc. Amer. Math. Soc.*, **42** (1974), 161-166.

[27] J. K. HALE, "Theory of functional differential equations", Springer Verlag, Berlin, 1977.

[28] O. HANNER, Retraction and extension of mappings of metric and nonmetric spaces, *Arkiv Mat., Svenska Vetens. Akad.*, **2** (1952), 315-360.

[29] J. HOFBAUER and K. SIGMUND, Permanence for replicator equations, in "Dynamical systems" (A.B. Kurzhanski and K. Sigmund eds) Proc. Conf. Sopron, Lecture Notes in Economics and Mathematical Systems **287**, pp.70-92, Springer Verlag, Berlin, 1988.

[30] H. HOFER, Variational and topological methods in partially ordered Hilbert spaces, *Math. Ann.*, **261** (1981), 493-514.

[31] S. T. HU, "Theory of retracts", Wayne State University Press, Detroit, 1965.

[32] T. KATO, On classical solutions of the two-dimensional non-stationary Euler equation, *Arch. Rational Mech. Anal.* **25** (1967), 188-200.

[33] M. A. KRASNOSEL'SKII, "The operator of translation along the trajectories of differential equations", Amer. Math. Soc., Providence, R.I., 1968.

[34] M. A. KRASNOSEL'SKII and P. P. ZABREIKO, "Geometrical methods of nonlinear Analysis", Springer-Verlag, Berlin, 1984.

[35] J. LERAY, Théorie des points fixes: indice total et nombre de Lefschetz, *Bull. Soc. Math. France,* **87** (1959), 221-233.

[36] J. MAWHIN, Recent results on periodic solutions of differential equations, in "International Conference on Differential Equations" (H. A. Antosiewicz, ed.), Proc. Conf. Southern California University, 1974, pp. 537-556, Academic Press, NewYork, 1975.

[37] J. MAWHIN, Equations intégrales et solutions périodiques des systèmes différentiels non linéaires. *Acad. Roy. Belg. Bull. Cl. Sci.* **55** (1969), 934-947.

[38] J. MAWHIN, "Topological degree methods in nonlinear boundary value problems", CBMS **40**, Amer. Math. Soc., Providence, R.I., 1979.

[39] M. NAGUMO, Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen, *Proc. Phys.-Math. Soc. Japan,* (3) **24** (1942), 551-559.

[40] R. D. NUSSBAUM, "The fixed point index and some applications", Les Presses de l'Université de Montréal, 1985.

[41] J. P. PENOT, A characterization of tangential regularity, *Nonlinear Analysis, TMA* **5** **(6)** (1981), 625-643.

[42] R. RAUTMANN, A criterion for global existence in case of ordinary differential equation, *Applicable Analysis* **2** (1972), 187-194.

[43] R. REISSIG, Periodic solutions of a nonlinear n-th order vector differential equation, *Ann. Mat. Pura Appl.*, (4) **87** (1970), 111-123.

[44] K. P. RYBAKOWSKI, "The homotopy index and partial differential equations", Springer-Verlag, Berlin, 1987.

[45] J. SANTANILLA, Some coincidence theorems in wedges, cones and convex sets, *J. Math. Anal. Appl.*, **105** (1985), 357-371.

[46] J. SANTANILLA, Nonnegative solutions to boundary value problems for nonlinear first and second order ordinary differential equations, *J. Math. Anal. Appl.*, **126** (1987), 397-408.

[47] H. L. SMITH, Periodic competitive differential equations and the discrete dynamics of competitive maps, *J. Differential Equations*, **64** (1986), 165-194.

[48] H. L. SMITH, Periodic solutions of periodic competitive and cooperative systems, *SIAM J. Math. Anal.*, **17** (1986), 1289-1318.

[49] R. SRZEDNICKI, On rest points of dynamical systems, *Fund. Math.*, **126** (1985), 69-81.

[50] R. SRZEDNICKI, Periodic and constant solutions via topological principle of Wažewski, *Acta Math. Univ. Iag.*, **26** (1987), 183-190.

[51] J. W. VICK, "Homology theory", Academis Press, New York, 1973.

[52] T. WAŽEWSKI, Sur un principe topologique de l'examen de l'allure asymptotique des intégrales des équations différentielles, *Ann. Soc. Polon. Math.*, **20** (1947), 279-313.

[53] J. E. WEST, Compact ANRs have finite type, *Bull. Amer. Math. Soc.*, **81** (1975), 163-165.

[54] J. E. WEST, Mapping Hilbert cube manifolds to ANRs. A solution to a conjecture of Borsuk, *Annals of Mathematics*, **106** (1977), 1-18.

[55] J. A. YORKE, Invariance for ordinary differential equations, *Math. Systems Theory*, 1 (1967), 353-372.