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Covariant anomalies and functional determinants

Thesis presented by

Luca Griguolo

for the degree of Magister Philosophiæ

Supervisor: Prof. Antonio Bassetto

S.I.S.S.A. – I.S.A.S.

Elementary Particle Sector

Academic Year 1991 - 92

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Contents

Introduction 3

1 Algebraic characterization of covariant anomalies 5

1.1 Consistent anomaly 5

1.2 Covariant anomalies 7

1.3 The complex extension 9

2 Algebras and functional determinants in gauge theories 13

2.1 The problem of the Weyl determinant 13

2.2 The definition of Weyl determinants 15

3 Gravitational anomalies 21

3.1 General properties 21

3.2 The covariant Lorentz anomaly 25

3.3 The Weyl determinant in curved space 28

3.4 Covariant solution 30

3.5 Consistent solution 35

4 Two dimensional Weyl determinant 36

4.1 Covariant solution 40

4.2 Consistent solution 41

4.3 Conclusions 46

A Heat-kernel expansion 48

B The variation of the determinant 50

C Decoupling technique 51

Introduction

In field theory anomalies appear as a breakdown at quantum level of classical conservation laws: we can group them into two families according to whether the symmetry they break is a rigid or a local one. From another point of view we have the family of *consistent anomalies*: they are characterized by the fact that they satisfy an algebraic equation – the Wess-Zumino consistency condition – derived by the group theoretical properties of the classical symmetry. They split in two different types: chiral anomalies (gauge, local Lorentz, diffeomorphism anomalies and their supersymmetric version), which appear only in chirally asymmetric theories, and conformal anomalies (conformal and superconformal anomalies) which live in chiral and non chiral theories as well. A general feature of the consistent anomaly is the loss of the classical tensorial properties of the currents related to the broken symmetries: consistent anomalies do not transform covariantly under the symmetry transformation of the classical theory. As regards chiral theories we can define a second family, the *covariant anomalies*: they are linked with a redefinition of the currents in such a way that the classical transformation laws are recovered. From the physical point of view the two families are strongly distinguished by the fact that the consistent anomalies derive from dinamical currents, coupled to the potentials of the theory, while the covariant ones do not. In this sense the presence of consistent anomalies destroies the perturbative consistency of the quantum theory: they reveal a conflict between renormalizability and unitarity. When anomalies was discovered [1] they apperead rather as a calculation puzzle. In the first eighties it was realized that a rich algebraic and geometrical structure subtends the existence of consistent anomaly: its characterization as solution of a cohomological problem [2] and its relation with deep algebraic geometric theorems, as the index theorem [3], were crucial tools in understanding and solving many problems. Much less interest was given to an equivalent study of the covariant anomaly, that has not been considered a fundamental object. Only recently, after few years the seminal suggestions of [4], people realized that even for covariant anomalies does exist a deep mathematical framework describing their structure [5].
A complementary approach to the problem of the anomaly relies on the construction of some representations of the anomaly algebra: as concearns theories describing gauge or gravitational interaction of spin one-half fermions this is obtained by means of the so called *fermionic determinant*. In gauge theory different definitions of this object were given by the use of ζ-function technique [6], finite mode regularization [7], regularization of fermion propagator [8], reproducing the correct perturbative results for the consistent anomalies. In the gravitational case, at least at our knowledge, the

only analytical definition and explicit computation of the chiral determinant appeared in [8, 9], claiming the absence of diffeomorphism anomaly: no definition in term of ζ- function was given until now. In this framework covariant anomalies sporadically appeared as a mistake in the regularization procedure [10]: a first step toward a sistematic construction of functional representing the algebra of covariant anomaly is present in [11]. In this notes we extend those results for gauge theory to arbitrary even dimensions: consistent and covariant gauge anomalies are different solutions of an extended cohomological problem and they can be obtained from different fermionic determinants. Then we try to generalize these results to the gravitational case: the situation is more subtle, due to the difficult use of ζ-function technique. We are able to solve completely the covariant problem and, using the properties of the extended algebra, we find the correct operator whose ζ-function determinant gives the consistent anomaly in $d = 2$. We perform the calculation of this determinant, that agrees with Leutwyler's result [9], obtained in a more indirect way: really we have found a one-parameter family of operators realizing the same functional up to local terms. It represents also a manifestly diffeomorphism-invariant calculation of the determinant (in [9] polynomial counterterms are needed to achive the general covariance).
We do not consider here the case of non trivial fiber bundles (non trivial principal bundles for gauge theories and non parallelizable manifold for spinors in curved space), but we hope in future to extend our results to this interesting situations: anyway some hints are given about it. We close remarking that there is some more physical interest for the study of the covariant structure of the anomalies: in a recent paper [12] strong relations between the quantization of anomalous theories and the covariant currents has been found. Also the link between Chern-Simons theory in $2n - 1$ dimensions and covariant anomalies in $2n - 2$ presented in [13] stimulated our studies. The complexification of the gauge group, that is the main tool in our description of covariant anomalies, has also been used, in different context, by Witten [14].

Chapter 1

Algebraic characterization of covariant anomalies

In this section we briefly review the definitions of consistent and covariant anomaly for a gauge theory in flat d-dimensional euclidean space-time. We recall the algebraic description of the consistent anomaly and we show how to extend this characterization to the covariant one.

1.1 Consistent anomaly

Let Γ[A] the vacuum functional in presence of the external gauge field $A = A^a_\mu T_a dx^\mu$ and $J^a_\mu(x) = \frac{\delta\Gamma[A]}{\delta A^\mu_a(x)}$ the gauge field current. In presence of chiral fermions the functional $\Gamma[A]$ is not gauge invariant. If

$$\delta_\lambda A_\mu = D_\mu \lambda = \partial_\mu \lambda + [A_\mu, \lambda]$$

describes the infinitesimal gauge variation of the Lie algebra valued parameter $\lambda = \lambda_a T_a$ and

$$\delta(\lambda) = \int d^{2n-2}x(\delta_\lambda A_\mu(x))_a \frac{\delta}{\delta A^a_\mu(x)}$$

is the operator realizing the transformation on some functional of $A_\mu(x)$, the symmetry breaking manifests itself by the occurence of an (integrated) anomaly $\Delta(\lambda, A)$

$$\begin{aligned} \delta(\lambda)\Gamma[A] = \Delta(\lambda, A) = \textstyle\int d^{2n-2}x Tr[\lambda G](x) \\ D_\mu J^\mu_a = -G_a[A] \end{aligned} \tag{1.1}$$

Obviously $\delta(\lambda)$ represents the Lie algebra

$$[\delta(\lambda_1), \delta(\lambda_2)] = \delta([\lambda_1, \lambda_2]) \tag{1.2}$$

forcing the Wess-Zumino (W-Z) consistency condition for $\Delta(\lambda, A)$

$$\delta(\lambda_1)\Delta(\lambda_2, A) - \delta(\lambda_2)\Delta(\lambda_1, A) = \Delta([\lambda_1, \lambda_2], A) \tag{1.3}$$

Any local functional of A_μ, linear in λ, which satisfies (2.3) is called consistent anomaly. One can put (1.3) into a more compact form by turning λ into a ghost field c: the W-Z condition is equivalent to

$$s\Delta(c, A) = 0 \tag{1.4}$$

with the definition for s

$$\begin{aligned} sA &= -Dc \\ sc &= -\tfrac{1}{2}[c, c] \\ s^2 &= 0 \end{aligned} \tag{1.5}$$

s is the B.R.S.T. operator, characterizing the anomaly as solution of a cohomological problem. Defining the strength field

$$F = dA + AA = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$$

one can prove the "Russian formula"

$$\tilde{F} = F \tag{1.6}$$

where

$$\begin{aligned} \tilde{F} &= \tilde{d}\tilde{A} + \tilde{A}\tilde{A} \\ \tilde{d} &= d + s \\ \tilde{A} &= A + c \end{aligned} \tag{1.7}$$

In the case of trivial bundle (I recall that the relevant structure for gauge theory is a principal bundle $P(M, G)$ where M is the compactified euclidean space-time and G is the Lie group underlying the theory) a solution of (1.4) is easily given applying simultaneosly the "Russian formula" and the "transgression formula" [15]

$$P(F^n) = d\omega^0_{2n-1}(A, F) \tag{1.8}$$

P is a symmetric ad-invariant polynomial on the Lie algebra and $\omega^0_{2n-1}(A, F)$ is the Chern-Simons form:

$$P(\tilde{F}^n) = \tilde{d}\omega_{2n-1}(\tilde{A}, \tilde{F}) \tag{1.9}$$

Expanding the (2n-1) form $\omega_{2n-1}(\tilde{A})$ in powers of c

$$\omega_{2n-1}(\tilde{A}) = \sum_{k=0}^{2n-1} \omega^k_{2n-1-k}(A, c) \tag{1.10}$$

the lower index giving the form degrees and the upper one the ghost number. The so-called "descent equations" are obtained matching the powers of c

$$d\omega^0_{2n-1}(A) = P(F_n)$$
$$s\omega^k_{2n-1-k}(A,c) + d\omega^{k+1}_{2n-2-k}(A,c) = 0 \tag{1.11}$$

For $k = 1$ the integration gives

$$s \int \omega^1_{2n-2}(A,c) = 0 \tag{1.12}$$

identifying $\Delta(A,c) = \int \omega^1_{2n-2}(A,c)$ as a good candidate for the consistent anomaly. An explicit expression is

$$\int \omega^1_{2n-2}(A,c) = n(n-1)\int_0^1 dt\, P(dc + [A,c], tA, F_t^{n-2}) \tag{1.13}$$

with $F_t = t\,dA + t^2 AA$
We remark that the W-Z condition is linear in Δ so in order to find the correct coefficient we need more informations about the matter content of the theory, or, equivalently, about the definition of $\Gamma[A]$. We remember that one can always add to Δ a term of the type $s\alpha(A) + d\beta(A)$ of the correct canonical dimensions: this freedom corresponds to the choice of the representative down the cohomology class.
For non-trivial principal bundle the situation is sligthly complicated: it can be obtained a good solution [4] fixing a background connection A_0, belonging to the same bundle of A. The formula (1.7) is modified as

$$P(F^n) - P(F_0^n) = d\omega^0_{2n-1}(A, A_0) \tag{1.14}$$

showing that $P(F^n)$ is not exact as in the case of trivial bundle: actually we consider $P(F^n)$ as a form on the base manifold, the projection of a (exact) form defined on the whole principal bundle.The anomaly acquires a dependence on A_0: it corresponds, as we will see, to an equivalent obstruction in defining a Weyl determinant in this non topological trivial situation.

1.2 Covariant anomalies

One can easily show that if the consistent anomaly is different from zero the current $J^\mu_a(x)$ does not transform covariantly under gauge transformation. In general it is possible to find a polynomial $X^\mu_a(A)$ which makes the current covariant [16]. One defines a new current

$$\hat{J}^\mu_a = J^\mu_a + X^\mu_a \tag{1.15}$$

with the property

$$\delta(\lambda)\hat{J}_\mu = -[\lambda, \hat{J}_\mu] \tag{1.16}$$

recovering the classical tensorial transformation. The polynomial $X^\mu_a(A)$ is called the Bardeen-Zumino counterterm and the covariant divergence of J^μ_a is known as the covariant anomaly:

$$D_\mu \hat{J}^\mu_a = -\hat{G}_a(A). \tag{1.17}$$

In order to understand this operation two remarks are useful: firstly the redefinition of J^μ_a does not correspond to an allowed one of the vacuum functional. In other words $\hat{G}_a(A)$ does not belong to the same cohomological class of $G_a(A)$ (and it does not satisfy the W-Z condition). Secondly, even in the abelian case does exist a difference between covariant and consistent anomaly: the functional form is equal but the numerical coefficient is different.
In $d = 2n-2$ space-time dimensions the explicit expression for $\hat{G}(A)$ is known [16] (for a simple group)

$$\Delta_{cov}(A,c) = \int dx^{2n-2} Tr[c\,\hat{G}] = n\int P[c\,, F^{n-1}]. \tag{1.18}$$

Anyway also Δ_{cov} can be algebraically characterized, embedding it into the solution of a cohomological problem [4].
We suppose that there is a subgroup K of a Lie group $\tilde{G}$ with the properties that the invariant symmetric polynomial P vanishes when its argoments are restricted to $LieK$ (we assume again that $P(M,\hat{G})$ is trivial). It is possible to decompose A and c along $LieK$ and an invariantly defined orthogonal complement $(LieK)_\perp$

$$\begin{aligned} A &= A_K + A_\perp \\ c &= c_K + c_\perp \end{aligned} \tag{1.19}$$

with $A_K, c_K \in LieK$ and $A_\perp, c_\perp \in (LieK)_\perp$ We remember that $(LieK)_\perp$ is not in general a Lie algebra.
The consistent anomaly $\Delta(c,A)$ as it stands does not vanish along $LieK$ but it reduces to $\Delta(c_K, A_\perp)$: in reference [4] it was proved the existence of a general counterterm $\Gamma_B[A] = \int X(A_\perp, A_K)$, local polynomial in $A_\perp$ and A_K, called the Bardeen counterterm, that added to the vacuum functional gives

$$\hat{\Delta}(c_\perp, A_\perp, A_K) = \Delta(c,A) + s\,\Gamma_B[A]$$

$$s\hat{\Delta}(c_\perp, A_\perp, A_K) = 0$$

$$\hat{\Delta}(c_\perp, A_\perp, A_K) = n\int\int_0^1 dt P(c_\perp, F^{n-1}(A_K + tA_\perp) + n(n-1)$$

$$\int\int_0^1 dt P(A_\perp,\, t^2[A_\perp, c_\perp] - t[A_\perp - c_\perp]_\perp - [A_\perp, c_\perp]_K,\, F^{n-2}(A_K + tA_\perp)). \tag{1.20}$$

Projecting $\hat{\Delta}(c_\perp, A_\perp, A_K)$ on $A_K = 0$

$$\hat{\Delta}(c_\perp, 0, A_K) = n \int P(c_\perp, F^{n-1}(A_K). \tag{1.21}$$

The comparison with the consistent solution

$$\int \omega^1_{2n-2} = \int P(c, F^{n-1}) + ... \tag{1.22}$$

shows the appearence of the factor n [16]. Now if K is the structure group G, $\hat{\Delta}(c_\perp, 0, F^{n-1})$ is the covariant anomaly. So taking a suitable embedding of G in some larger group $\tilde{G}$, it is possible to obtain a solution of the cohomological problem (1.4) that reduces to the covariant anomaly after a projection. The same is true also in presence of a non trivial $P(M, \tilde{G})$, requiring that the bundle is reducible to $P(M, G)$. An interesting feature is that, in this case, the covariant anomaly on the contrary of the consistent one does not depend on some fix background connection: we will discuss these difference in the framework of the functional approach. All these properties have been further explored, and new descent equations, describing the covariant anomalies, have been obtained using concepts like vertical cohomology and local B.R.S.T. symmetry [5]. For our purposes it is sufficient the just described characterization: our choice of the group in which to embed G is different from the original proposal [4] and it was firstly discuss in [11]. As we will see after it is directly related to a functional approach and it is also possible to obtain the consistent anomaly. We will take $G = SU(N)$.

1.3 The complex extension

Let us take complex values to the gauge potentials

$$A = A_a T_a \Longrightarrow \hat{A} = (A^1_a + iA^2_a)T_a = \hat{A}_a \hat{T}_a \tag{1.23}$$

where now $\hat{T}_a$ is a basis for $SL(2N, C)$: applying the previously derived formalism we identify

$$\begin{aligned} \tilde{G} = SL(2N, C) \quad & A_K = A^1_a T_a = A_1 \\ K = SU(N) \quad & A_\perp = A^2_a T_a = A_2 \end{aligned} \tag{1.24}$$

with $c_k = c_1$ and $c_\perp = c_2$.
Then we have to exhibit an invariant symmetric polynomial on $SL(2N, C)$ vanishing on $SU(N)$: the correct choice is

$$P(A_1, A_2) = \frac{1}{2i}[P(F^n(\hat{A}) - P^*(F^n(\hat{A})] \tag{1.25}$$

We can specialize equation (1.20) with the result

$$s\tilde{\Delta}(A_1, A_2; c_2) = 0$$

$$\tilde{\Delta}_{cov}(A_1, A_2; c) = \frac{n}{2i}\int_0^1 dt \int P(c_2,\, F^{n-1}(A_1 + tA_2)) +$$
$$+\frac{n(n-1)}{2i}\int_0^1 dt \int P(A_2, (t^2-1)\,[A_2, c_2],\, F^{n-2}(A_1 + tA_2)) -$$
$$-\,(c_2 \rightarrow -c_2\,;\; A_2 \rightarrow -A_2) \qquad (1.26)$$

For $A_2 = 0$ (real projection), putting $c_2 = i\hat{c}_2$ we obtain

$$\Delta_{cov}(A_1, 0, c_2) = n \int P(\hat{c}_2, F^{n-1}(A_1)) \qquad (1.27)$$

But we can derive from the present formalism also the usual consistent anomaly: if we do not require the vanishing of the symmetric invariant polynomial on $SU(N)$ and we choose

$$P(A_1, A_2) = P(F^n(A))$$

or

$$P(A_1, A_2) = P^*(F^n(A))$$

we obtain a solution of the cohomological problem under the form

$$\tilde{\Delta}_{con}(A_1, A_2, c_1, c_2) = \tilde{\Delta}^1_{con} = \int \omega^1_{2n-2}(A, c)$$

or

$$\tilde{\Delta}_{con}(A_1, -A_2, c_1, -c_2) = \tilde{\Delta}^2_{con} = \int \omega^1_{2n-2}(A^\dagger, c^\dagger)$$

It is rather clear that the projection on $SU(N)$ ($c_2 = 0$) reproduces, after the limit $A_2 = 0$, the usual expression for the consistent anomaly $\Delta_{con}(A_1, c_1)$. This is not really the more general way to embed the consistent solution into the extended problem, although is the basic one: we are going to see that any solution that reduces to the consistent one on $SU(N)$ is built in term of a covariant solution and the just found $\tilde{\Delta}_{con}(A_1, A_2, c_1, c_2)$.
Any invariant polynomial

$$\alpha P(F^n(A)) + \beta P^*(F^n(A)) \qquad (1.28)$$

with $\alpha + \beta = 1$ reduces to the usual $SU(N)$ polynomial $P(F^n(A_1))$ on $A_2 = 0$; conversely applying the descent equation to (1.28) one recovers a solution $\tilde{\Delta}^{\alpha,\beta}(A_1, A_2, c_1, c_2)$ that gives

$$\tilde{\Delta}^{\alpha,\beta}(A_1, 0, c_1, 0) = \Delta_{con}(A_1, c_1). \qquad (1.29)$$

Let us see how $\tilde{\Delta}^{\alpha,\beta}$ is made: we rewrite (2.28) using $\beta = 1 - \alpha$

$$2\alpha\,[\frac{1}{2}(P(F^n) - P^*(F^n))] + P(F^n) \tag{1.30}$$

so immediately it results

$$\tilde{\Delta}^{\alpha,\beta} = 2\alpha\tilde{\Delta}_{cov}(A_1, A_2, c_2) - 2\alpha s\Gamma_B(A_1, A_2) + \tilde{\Delta}_{con}(A_1, -A_2, c_1, -c_2) \tag{1.31}$$

$\tilde{\Delta}^{\alpha,\beta}$ appears (up to coboundary terms) to be the sum of a covariant like solution $2\alpha\tilde{\Delta}_{cov}$ (vanishing on $SU(N)$) and the consistent solution $\tilde{\Delta}^1_{con}$: in other words two solutions of the extended cohomological problem that reduce to the usual consistent on $SU(N)$ (and $A_2 = 0$) differ for a piece proportional to the covariant solution. So all the informations about the consistent anomaly are encoded on Δ^1_{con} or Δ^2_{con}. The natural question that now arises is: in the extended situation

$$s\tilde{\Delta}_{cov} = 0$$

$$s\tilde{\Delta}^i_{con} = 0$$

$\tilde{\Delta}_{cov}$ and $\tilde{\Delta}^i_{con}$ could differ by some term of the type $sH(A)$?
One can prove with some explicit example (for istance in the simplest case $d = 2$) that the answer is not. The two solutions belong to different cohomology classes of the s operator defined on $SL(2N, C)$.
Before closing the section we rewrite the equation for covariant and consistent anomalies, embedded on $SL(2N, C)$, in a more handful way for the future functional applications. Coming back to the original Ward operator $\delta(\lambda)$ for $SL(2n, C)$, $\lambda = \lambda_a\hat{T}_a$ we write

$$\delta(\lambda) = \delta_1(\lambda_1) + \delta_2(\lambda_2) \tag{1.32}$$

with $\delta_1(\lambda_1)$ generating gauge transformations of $SU(N)$ type and $\delta_2(\lambda_2)$ living on the orthogonal invariant complemement. The algebra of δ_1 and δ_2 is

$$\begin{aligned}
[\delta_1(\lambda_1), \delta_1(\lambda_2)] &= \delta_1([\lambda_1, \lambda_2]) \\
[\delta_2(\lambda_1, \delta_2(\lambda_2)] &= -\delta_1([\lambda_1, \lambda_2]) \\
[\delta_1(\lambda_1), \delta_2(\lambda_2)] &= \delta_2([\lambda_1, \lambda_2])
\end{aligned} \tag{1.33}$$

giving the W-Z conditions

$$\begin{aligned}
\delta_1(\lambda_1)a_1(\lambda_2) - \delta_1(\lambda_2)a_1(\lambda_1) &= a_1([\lambda_1, \lambda_2]) \\
\delta_2(\lambda_1)a_2(\lambda_2) - \delta_2(\lambda_2)a_2(\lambda_1) &= -a_1([\lambda_1, \lambda_2]) \\
\delta_1(\lambda_1)a_2(\lambda_2) - \delta_2(\lambda_2)a_1(\lambda_1) &= a_2([\lambda_1, \lambda_2])
\end{aligned} \tag{1.34}$$

where $a_1 = \delta_1\Gamma$ and $a_2 = \delta_2\Gamma$.
The covariant solution corresponds

$$\begin{aligned} a_1 &= 0 \\ a_2 &\neq 0 \end{aligned} \tag{1.35}$$

The consistent one, as it was shown in reference [11], is

$$a_2 = \pm i a_1 \tag{1.36}$$

The sign relies on the choice of P or P^* as invariant polynomial. Obviously this is not the only way to characterize the consistent solution: a_2 can always differ, as we have seen, for a piece proportional to the covariant anomaly

$$\bar{a}_2 = \pm i a_1 + \alpha\, a_2^{cov} \tag{1.37}$$

So it is sufficient looking for solutions respecting (1.36): all the others correspond to sum to the functional Γ a functional $\alpha\Gamma_{cov}$ with

$$\delta_1\Gamma_{cov} = 0$$

$$\delta_2\Gamma_{cov} = a_2^{cov}$$

At the end we remark that (1.35) and (1.36) are particular choices into the cohomology classes giving in the limit $A_2 = 0$ the canonical form for the anomalies: so the equal is to understand there modulo coboundary terms (of $SL(2N,C)$): the projection on $A_2 = 0$ produces the correct anomaly (on $SU(N)$) only for representatives satisfying exactly (1.35), (1.36).

Chapter 2

Algebras and functional determinants in gauge theories

In the previous sections we did not care about the definition of the vacuum functional $\Gamma[A]$: it must be obtained by some regularization procedure from the formal expression

$$\Gamma[A] = -\ln \int [D\phi] \exp -S_{cl}(A)$$

$S_{cl}(A)$ being the classical action for the field ϕ coupled to an external gauge field A. Perturbation theory induces the fact that two different regularization procedures on $\Gamma[A]$ are distinguished only by some local term in the field A and its derivatives, of the correct canonical dimensions. To choose a different regularization corresponds, for the anomaly, to take a different representative down the cohomology class. If we extend the gauge group we have to find a definition for $\Gamma[\hat{A}]$: but now there are two non trivial indipendent classes of cohomology for the extended problem. This means that we can try to construct two non trivial representations of the algebra (1.33), giving respectively the different solution, characterized by the conditions (1.35) and (1.36). They are not distinguished by coboundary terms so only the second reduces to the correct one, giving the consistent anomaly: (1.36) individualizes the correct vacuum functional for an anomalous $SU(N)$ gauge theory. We will use this property also in the gravitational case in order to find the effective action in $d = 2$.

2.1 The problem of the Weyl determinant

The classical action for spinors coupled to a gauge or a gravitational background is

$$S_{cl} = \int dV \bar{\psi} \not{D} \psi \tag{2.1}$$

with $\not{D}$ some first order operator of Dirac type
Formally it results

$$\Gamma[A] = -\ln Det\not{D}(A) \tag{2.2}$$

Immediately we face up with a basic difficulty in defining the determinant for chiral fermions: in this case $\not{D}$ is the *Weyl operator* that maps a chiral spinor on a spinor of opposite chirality

$$\not{D} : \Gamma(S_+)_M \to \Gamma(S_-)_M \tag{2.3}$$

where $\Gamma(S_+)_M$ ($\Gamma(S_-)_M$) is the space of right (left) sections of the vector bundle associated by the Dirac representation to the Spin-principal bundle on M. $\Gamma(S_+)_M$ and $\Gamma(S_-)_M$ are different Hilbert spaces and does not exist any canonical isomorphism between them: $\not{D}$ does not map an Hilbert space into itself and we have no canonical way to define a meaningful eigenvalues problem for this operator. A way out is to modify the Weyl operator in order to have a good eigenvalues problem but, in general, some of the classical properties are lost after the modification. For the Dirac operator there is no problem

$$\begin{gathered} \not{D}_D : \Gamma(S)_M \to \Gamma(S)_M \\ \Gamma(S)_M = \Gamma(S_+)_M \oplus \Gamma(S_-)_M \end{gathered} \tag{2.4}$$

The eigenvalue problem is meaningful

$$\not{D}_D \Psi_n = \lambda_n \Psi_n$$

n is an integer (we suppose to work in a compact manifold). A gauge transformation on $\not{D}_D$ acts

$$g : \not{D}_D \to U^{-1} \not{D}_D U \tag{2.5}$$

The covariant transformation (2.5) implies that the eigenvalues are costant under gauge action. In general λ_n grows with n so the naive definition

$$Det\not{D}_D = \prod_n \lambda_n \tag{2.6}$$

is meaningless. We can use an analytic regularization, the ζ-function regularization [6], that does not change the eigenvalues but simply cuts the contribution of the higher one. The so defined determinant is invariant under gauge transformations. In the Weyl case one immediately recognizes that even if the covariance is conserved by the original operator, the same may not be true for the modified one. Let us go to see what happens if we try to apply directly the ζ-function machinery to the Weyl operator: we fail because there is no way to define its complex power [17]. As an example we consider the $d = 2$ flat case, where the operator is

$$\not{D} = i\sigma_\mu(\partial_\mu + A_\mu) \tag{2.7}$$

A_μ being an antihermitian potential and σ_μ one of the two inequivalent representations of the Weyl algebra [18]. The principal symbol [17] of $\not{D}$ is

$$a_1(x,\xi) = -\sigma_\mu \xi_\mu = -\xi_1 - i\xi_2 \tag{2.8}$$

as we have choosen $\sigma_1 = 1$ and $\sigma_2 = i$. To define a complex power of $\not{D}$ we need the existence of a ray of minimal growth [17], namely we have to find an angle ϕ in the complex plane of the variable $\lambda = t \exp i\phi$ such that

$$t \exp i\phi - a_1 \neq 0 \qquad \forall t > 0,\ |\xi| = 1 \tag{2.9}$$

This is clearly incompatible with the expression (2.8).We see that already for gauge theory on a flat manifold the Weyl determinant cannot be defined, at least by ζ-function method. The same is true in the curved space. The Dirac case is different: the principal symbol is

$$a_1(x,\xi) = -\gamma_\mu \xi_\mu$$

that do possess the ray (γ_μ are the Dirac matrices).

2.2 The definition of Weyl determinants

In order to obtain a good eigenvalues problem it is quite natural to fix an isomorphism T

$$T : \Gamma(S_-) \rightarrow \Gamma(S_+) \tag{2.10}$$

and to define

$$Det\not{D} \equiv Det(T\not{D}) \tag{2.11}$$

where now the operator $T\not{D}$ admits a well defined eigenvalues problem even if the covariance is, a priori, lost: our task is to find a such T to recover, under gauge variation, the consistent anomaly from the just defined functional. In the gauge case the question is easily solved, also for non-trivial principal bundle, and the embedding into a $SL(2n, C)$ theory determines not only the consistent but even the covariant solution of the extended problem. The physical intuition gives immediately a result: in the case of trivial bundle one can always describe a right fermion interacting with a gauge field as a Dirac fermion in which only the right part is coupled with the gauge field. In $d = 2n$ the Weyl operator is

$$\not{D} = i\sigma_\mu(\partial_\mu + A_\mu)$$

σ_μ are the Weyl matrices obeying

$$\sigma_\mu \tilde{\sigma}_\nu + \sigma_\nu \tilde{\sigma}_\mu = 2\delta_{\mu\nu} \tag{2.12}$$

$\tilde{\sigma}_\mu$ is the other inequivalent representation of the algebra, existing for $d = 2n$ [18]. We can use

$$\hat{\not{D}} = i\gamma_\mu(\partial_\mu + (\frac{1+\gamma_{2n+1}}{2})A_\mu) \qquad (2.13)$$

γ_μ being the Dirac matrices. $\hat{\not{D}}$ has a good eigenvalues problem and admits ζ-function regularization: naturally in this case the covariance is lost. It happens

$$\hat{\not{D}} = \begin{pmatrix} 0 & \not{D} \\ i\tilde{\sigma}_\mu\partial_\mu & 0 \end{pmatrix}. \qquad (2.14)$$

so the determinant is

$$Det\hat{\not{D}} = Det[(i\tilde{\sigma}_\mu\partial_\mu)\not{D}] \qquad (2.15)$$

The isomorphism T appears to be the free kinetic Weyl operator of opposite chirality. In some sense it corresponds to a particular normalization of the determinant: $i\tilde{\sigma}_\mu\partial_\mu$ has no dinamical contribution and we can formally subtract it.
From a more formal point of view, we require (always for trivial bundle)

1. $Det(T\not{D})$ gives rise to the consistent anomaly (solution (1.36) of the W-Z extended)

2. The anomaly must be local in the gauge field

3. $Det(T\not{D})$ is smoothly connected to the free case

4. $Det(T\not{D})$ is a functional of the gauge field A_μ only (for non trivial $P(M,G)$ this is not possible as we will see).

Essentially we are trying to define the determinant of a first order operator in terms of the determinant of a second order one. Moreover we extend the gauge group to explore the possible different representation of the extended anomaly algebra, varying the isomorphism T.
A good general candidate for T is

$$T(r) = \tilde{\not{D}}(r) = i\tilde{\sigma}_\mu(\partial_\mu + r\hat{A}^\dagger_\mu) \qquad r \in \mathbf{R} \qquad (2.16)$$

Now some remarks are in order: We can always take the matrices $\tilde{\sigma}_\mu$ as $\sigma^\dagger_\mu$, from the general properties of the Weyl algebra, getting for $r = 1$:

$$\tilde{\not{D}} = \not{D}^\dagger$$

and:

$$Det(T(1)\not{D}) = Det(\not{D}^\dagger\not{D}) = |Det(\not{D})|^2 \qquad (2.17)$$

For $r = 1$ we lose the phase of the determinant in which the *unextended* anomaly lives [19]: it is well known that the modulus of the Weyl determinant is equal (apart from

some regularization terms) to the Dirac determinant.
Let us study the determinant (2.16): conditions 3. and 4. are obviously satisfied by ζ-function definition. To test the different r we make the variation δ_1 and δ_2 on $\hat{A}_\mu$ and $\hat{A}^\dagger_\mu$: we compute the answers of the just defined determinant and we try to find the values of r for which

$$a_1 = \pm i a_2 \tag{2.18}$$

with

$$a_1 = \delta_1 \ln DetT(r)\not{D}$$

$$a_2 = \delta_2 \ln DetT(r)\not{D}$$

where a_1 and a_2 are local on $\hat{A}$ and $\hat{A}^\dagger$. We will find also $T(r)$ for which

$$a_1 = 0 \qquad a_2 \neq 0 \tag{2.19}$$

Using standard ζ-function formalism [6] we obtain

$$W[\hat{A}, \hat{A}^\dagger; r] = \frac{1}{F(r)}\frac{d}{ds} Tr[(\tilde{\not{D}}(r)\not{D})^{-s}]_{s=0} \tag{2.20}$$

Tr is an operatorial trace and $F(r)$ is some normalization factor that gives the correct weight for the effective action.
After some efforts we have found:

$$\delta_1(\alpha)W = \frac{1}{F(r)}\frac{i}{(4\pi)^n}\int d^{2n}x\, Tr[H_n(r)(1-r)\alpha - \tilde{H}_n(r)(1-r)\alpha] -$$

$$-\frac{1}{F(r)}(r-1)r\frac{d}{ds}[\, s\, Tr\{\tilde{\sigma}_\mu[\hat{A}^\dagger_\mu, i\alpha]\not{D}(\tilde{\not{D}}(r)\not{D})^{-s-1}\}]_{s=0} \tag{2.21}$$

and

$$\delta_2(\alpha)W = \frac{1}{F(r)}\frac{1}{(4\pi)^n}\int d^{2n}x\, Tr[H_n(r)(1+r)\alpha + \tilde{H}_n(r)(1+r)\alpha] -$$

$$-\frac{1}{F(r)}(r-1)r\frac{d}{ds}[\, s\, Tr\{\tilde{\sigma}_\mu[\hat{A}^\dagger_\mu, \alpha]\not{D}(\tilde{\not{D}}(r)\not{D})^{-s-1}\}]_{s=0} \tag{2.22}$$

where

$$H_n = H_n(\tilde{\not{D}}(r)\not{D})$$

$$\tilde{H}_n = H_n(\not{D}\tilde{\not{D}}(r))$$

H_n is the n-th coefficient of the heat kernel expansion of the operator in question.
The second term in the variation is in general non local requiring the use of the inverse of differential operators: only in the abelian case, where the commutators disappear we have a local variation for any value of r. The non local term disappears for $r = 0$

and $r = 1$.

<u>$r = 0$ (the consistent case):</u>

$$a_1(\alpha) = \frac{1}{F(0)} \frac{i}{(4\pi)^n} \int d^{2n}x\, Tr[(-H_n(0) + \tilde{H}_n(0))\alpha]$$

$$a_1(\alpha) = \frac{1}{F(0)} \frac{1}{(4\pi)^n} \int d^{2n}x\, Tr[(-H_n(0) + \tilde{H}_n(0))\alpha] \tag{2.23}$$

So $a_2 = -ia_1$. Therefore $r = 0$ gives the consistent anomaly; the correct choice for $F(0)$ is 1. We note that in this case the relevant operator is

$$(i\tilde{\sigma}_\mu \partial_\mu) \not{D} \Rightarrow T(0) = i\tilde{\sigma}_\mu \partial_\mu$$

as we have guessed from physical arguments. The usual determinant is obtained projecting $A_2 = 0$.

<u>$r = 1$ (the covariant case):</u>

$$a_1(\alpha) = 0$$

$$a_2(\alpha) = \frac{1}{F(1)} \frac{1}{(4\pi)^n} \int d^{2n}x\, 2\, Tr[(-H_n(1) + \tilde{H}_n(1))\alpha] \tag{2.24}$$

$a_1(\alpha) = 0$ and $a_2(\alpha) \neq 0$ obtaining the covariant solution. Taking $F(1) = 2$ one can compute in the limit $A_2 = 0$ the trace [18]: the result is

$$\frac{1}{(4\pi)^n} \frac{sgn(\sigma)}{n!} \int d^{2n}x\, \epsilon_{\mu_1 \ldots \mu_{2n}} Tr[F_{\mu_1\mu_2} \ldots F_{\mu_{2n-1}\mu_{2n}} \alpha] \tag{2.25}$$

which corresponds to the cohomological result (1.21), taking into account the correct normalization factors for the polynomial P. We remark that the covariant anomaly can be, in this way, obtained starting from $A_2 = 0$ varying the modulus of the Weyl determinant with a general transformation of $SL(2N, C)$. For general r we have again solutions of the cohomological problem but they are not local.

We can make another use of the covariant solution [11]. a_2 is in general (before the limit A_2=0) a functional of $\hat{A}^\dagger$ and $\hat{A}$. If we put, formally, $\hat{A}^\dagger$=0 taking $\hat{A} \neq 0$, $\hat{A} \in SU(N)$, it is clear that we recover the consistent anomaly. This is not surprising because $\hat{A}^\dagger$=0 corresponds to r=0. In some sense the covariant solution of the extended problem is more general because, using a formal limit, one can obtain both the anomalies. We will explore this property also in the gravitational case.

At the end of this section we want to discuss briefly what happens in presence of a non trivial principal bundle. In this case we cannot define a determinant smoothly connected to the free one, because there is no connection on $P(M, G)$ that descends to the zero one on the manifold, so a problem of normalization holds. Moreover the operator $T(r)\not{D}$ for $r = 0$ is not the correct one: non trivial $P(M, G)$ means that on the manifold does not exist a global definition for the connections. One has different expressions on different patches covering the manifold: these different expressions are linked by some gauge transformation (transition functions). Conversely an operator depending on the connections does not possess a unique form but in different patches has different representations connected by a gauge transformation. For example let us suppose that on a subset $\alpha \subset M$ the Dirac operator is $\not{D}$ and in $\beta \subset M$ is $\not{D}'$ when $\alpha \cap \beta \neq \phi$ being

$$A' = U^{-1}AU + U^{-1}dU$$

it results

$$\not{D}' = U^{-1}\not{D}U$$

. Now in order to have a determinant we need the eigenvalues, that are global properties of the operator: the transformation law means that if in α

$$\alpha : \not{D}\Psi_n = \lambda_n \Psi_n$$

in β

$$\beta : \not{D}'\Psi'_n = \lambda_n \Psi'_n$$

with $\Psi'_n = U\Psi_n$. In this way the eigenvalues do not depend on the patches.
On a non-trivial $P(M,G)$ the Weyl operator do possess the correct transformation law passing between different patches: but for the eigenvalues problem the relevant operator is $T\not{D}$ and we need

$$(T\not{D})' = U^{-1}(T\not{D})U$$

This relation forces for T

$$T' = U^{-1}TU$$

One immediately realizes that T must depend on some fixed background connection A_0, belonging to $P(M,G)$. The natural choice is

$$T = T(0) = (\not{\partial} + \not{A}_0)$$

Nevertheless we can again define a family of operators $T(r)$ fitting between the consistent and the covariant case

$$T(r) = i\tilde{\sigma}_\mu(\partial_\mu + (1-r)\hat{A}^\dagger_{0\mu} + r\hat{A}^\dagger_\mu)$$

It is very easy to verify that

$$T'(r) = U^{-1}T(r)U$$

For $r=1$ the dependence on A_0 disappears, so the covariant anomaly does not depend on the background fixed connection according to the cohomological argument of reference [4]

Example: $d=2$ non abelian gauge theory
The relevant Seeley-de Witt coefficients are:

$$H_1(r) = -\frac{1}{2}r[\hat{A}_\mu, \hat{A}^\dagger_\mu] + \frac{i}{2}r\epsilon_{\mu\nu}[\hat{A}_\mu, \hat{A}^\dagger_\nu]$$

$$-\frac{1}{2}\partial_\mu[r\hat{A}^\dagger_\mu - \hat{A}^\mu] + \frac{i}{2}\epsilon_{\mu\nu}\partial_\mu(\hat{A}_\nu + r\hat{A}^\dagger_\nu] \quad (2.26)$$

$$\tilde{H}_1(r) = \frac{1}{2}r[\hat{A}_\mu, \hat{A}^\dagger_\mu] + \frac{i}{2}r\epsilon_{\mu\nu}[\hat{A}_\mu, \hat{A}^\dagger_\nu]$$

$$+\frac{1}{2}\partial_\mu[r\hat{A}^\dagger_\mu - \hat{A}^\mu] - \frac{i}{2}\epsilon_{\mu\nu}\partial_\mu(\hat{A}_\nu + r\hat{A}^\dagger_\nu] \quad (2.27)$$

The choice $r=0$ gives:

$$a_1(\alpha) = ia_2(\alpha) = \int d^2x \frac{i}{4\pi} Tr[(\epsilon_{\mu\nu}\partial_\mu\hat{A}_\nu - i\partial_\mu\hat{A}_\mu)\alpha] \quad (2.28)$$

that for $\hat{A}_\mu \to A_\mu$ is the usual consistent anomaly (modulo a coboundary). For $r=1$:

$$a_1(\alpha) = 0$$

$$a_2(\alpha) = \frac{1}{4\pi}\int d^2x Tr([\epsilon_{\mu\nu}\partial_\mu(\hat{A}_\nu + \hat{A}^\dagger_\nu + [\hat{A}^\dagger_\mu, \hat{A}_\mu] -$$

$$- i\partial_\mu(\hat{A}_\mu - \hat{A}^\dagger_\mu) - i[\hat{A}^\dagger_\mu, \hat{A}_\mu]\alpha) \quad (2.29)$$

down the real projection $\hat{A}^\dagger_\mu = \hat{A}_\mu$

$$a_2(\alpha) = \frac{1}{4\pi}\int d^2x Tr[(\epsilon_{\mu\nu}F_{\mu\nu})\alpha] = \hat{a}$$

If we put formally in $a_2(\alpha)$

$$\hat{A}^\dagger_\mu = 0$$

we obtain again the consistent anomaly (2.28) (modulo a factor i)

Chapter 3

Gravitational anomalies

In this chapter we turn our attention to the anomalous behaviour of chiral spinors coupled to a Riemannian background. The occurance of gravitational anomalies has been pointed out by the pioneering work of Alvarez-Gaumè and Witten [19], and it has known many attention in the meddle of the eighties [20]: for more recent studies see [21]. We are interested in generalizing the formalism of covariant anomalies in this context and in finding an analytical definition of the curved Weyl determinant, using the properties of the extended algebra. We are not yet been able to give a general solution out of $d = 2$: nevertheless we have performed an explicit coordinate-invariant calculation of the determinant in the two dimensional space recognizing the correct operator to use on a ζ-function approach.

3.1 General properties

The classical Weyl action is

$$S_{cl} = \int d^{2n}x\sqrt{g}\,\bar{\psi}\,e^{\mu}_{a}\,\sigma_a i\,(\partial_\mu + \frac{1}{4}\Omega_{\mu cd}\bar{\sigma}_c\sigma_d)\,\psi \tag{3.1}$$

where σ^a and $\bar{\sigma}^b$ are the Weyl matrices, $E_{\mu a}$ are the n-beins fields (with inverse e^{μ}_{a}) and $\Omega_{\mu cd}$ is the spin-connection linked to the metric tensor

$$g_{\mu\nu} = E_{\mu a}\,E_{\nu a}$$

$$\Omega_{\mu ab} = e^{\nu}_{a}\,(\partial_\mu E_{\nu b} - \Gamma^{\lambda}_{\mu\nu}E_{\lambda b})$$

We are interested in describing the symmetries of the action (3.1): for the moment we do not worry about the global description of this transformations on which we will make some comments in the next pages. We remark that, on the contrary of the gauge

case, the symmetries act on the fibers as well as on the base manifold.
Coordinate transformations (diffeomorphisms):

$$\delta E_{\mu a}(x) = \partial_\mu \xi^\nu(x)\, E_{\nu a}(x) + \partial_\nu E_{\mu a}(x)\, \xi^\mu(x)$$

$$\delta\psi(x) = \partial_\mu \psi(x)\, \xi^\mu(x) \tag{3.2}$$

ξ^μ generates the infinitesimal diffeomorphism on the base manifold.
Frame rotations ($SO(2n,R)$ transformations):

$$\delta E_{\mu a}(x) = \lambda_{ab}(x) E_{\mu b}(x)$$

$$\delta\psi(x) = \lambda_{ab}(x)\omega_{ab}\psi(x) \tag{3.3}$$

with $\lambda_{ab} = -\lambda_{ba}$ and ω_{ab} the generators of $SO(2n,R)$ in the choosen spinorial representation (really the relevant group is $Spin(2n)$ that is a double covering of $SO(2n,R)$).For the Weyl case

$$\omega_{ab} = \frac{1}{4}\left[\sigma_a\tilde{\sigma}_b - \sigma_b\tilde{\sigma}_a\right]$$

Conformal transformations:

$$\delta E_{\mu a}(x) = \lambda(x) E_{\mu a}(x)$$

$$\delta\psi(x) = \nu\lambda(x)\psi(x) \tag{3.4}$$

where $\nu = n - \frac{1}{2}$.
Let $\Gamma[E]$ the vacuum functional of the theory in the external background: let us study the Ward identity derived from $SO(2n,R)$ symmetry and diffeomorphism invariance. Classically there are two currents:
the spin-current

$$J^\mu_{ab}(x) = \frac{\delta S_{cl}}{\delta\Omega_{\mu ab}(x)} \tag{3.5}$$

and the consistent energy-momentum tensor

$$T^\mu_a(x) = \frac{\delta S_{cl}}{\delta E_{\mu a}(x)} \tag{3.6}$$

From $T^{\mu a}$ one can construct the symmetric energy-momentum tensor

$$T^{\mu\nu} = \frac{1}{2}\left(e^\mu_a T^\nu_a + e^\nu_a T^\mu_a\right) \tag{3.7}$$

and

$$T_{ab} = \frac{1}{2}\left(E_{\mu a} T^\mu_b - E_{\mu b} T^\mu_a\right) \tag{3.8}$$

In absence of torsion

$$T_{ab} = 0 \tag{3.9}$$

and the classical symmetries give the equations:

$$D_\mu J^\mu_{ab} = 0 \tag{3.10}$$
$$D_\mu T^{\mu\nu} = 0 \tag{3.11}$$

The presence of anomalies changes these conservation laws: let us suppose that $\Gamma[E]$ is not invariant under frame rotation. The generator of the $SO(2n, R)$ rotations is

$$\delta(\lambda) = \int d^{2n}x\sqrt{g}\,(\delta_\lambda E_{\mu a})\,\frac{\delta}{\delta E_{\mu a}}$$

$$\delta_\lambda E_{\mu a} = -\lambda_{ab} E_{\mu b}$$

giving

$$\delta(\lambda)\Gamma[E] = \int d^{2n}x\sqrt{g}\,\lambda_{ab}\,G_{ab} = t(\lambda) \tag{3.12}$$

But using the quantum definition of the consistent energy-momentum tensor

$$\frac{\delta\Gamma[E]}{\delta E_{\mu a}} = T_{\mu a}$$

$$\delta(\lambda)\Gamma[E] = \int d^{2n}x\sqrt{g}\,\lambda_{ab}T_{ab} \tag{3.13}$$

The anomaly G_{ab} is identified with the antisymmetric part of the energy-momentum tensor (classically zero): the equation (3.10) is modified:

$$D_\mu J^\mu_{ab} + T_{ab} = 0 \tag{3.14}$$

The covariant divergence of the energy-momentum tensor involves the diffeomorphism invariance: be $\delta(\xi)$ the generator of this symmetry

$$\delta(\xi) = \int d^{2n}x\sqrt{g}\,(\delta_\xi E_{\mu a})\frac{\delta}{\delta E_{\mu a}} =$$

$$= \int d^{2n}x\sqrt{g}\,\frac{1}{2}(\delta_\xi g_{\mu\nu})e^\nu_a\frac{\delta}{\delta E_{\mu a}} + (\delta_{\lambda(\xi)}E_{\mu a})\frac{\delta}{\delta E_{\mu a}} \tag{3.15}$$

with

$$\begin{aligned} \delta_\xi g_{\mu\nu} &= D_\mu\,\xi_\nu + D_\nu\,\xi_\mu \\ \lambda_{ab}(\xi) &= \frac{1}{2}e^\nu_a\,e^\mu_b(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) - \xi^\lambda\Omega_{\lambda ab} \end{aligned} \tag{3.16}$$

Diffeomorphism anomaly means

$$\delta(\xi)\Gamma[E] = \int d^{2n}x\sqrt{g}\,\xi^\mu a_\mu = a(\xi) \tag{3.17}$$

Using again the definition of T_a^μ and supposing the presence of the frame anomaly it is easy to prove

$$D_\mu T^{\mu\nu} + l^\nu = a^\nu \tag{3.18}$$

where

$$l^\mu = -D_\nu \left(e_a^\nu e_b^\mu T_{ab}\right) + \Omega_{ab}^\nu T_{ab} \tag{3.19}$$

We remark the interplay between the two anomalies: in fact a^μ is the genuine diffeomorphism anomaly, manifesting the non-invariance of the Weyl determinant under coordinate transformations. Nevertheless in the Ward identity for the energy-momentum tensor also appears the Lorentz anomaly T_{ab}, that changes the balance equation. Even in absence of diffeomorphism anomaly, say $a^\mu = 0$, the energy-momentum tensor is not covariantly conserved, but this is not a signal of general covariance breaking. It is an effect of the Lorentz anomaly: the n-beins fields, not dynamical at the classical level, acquire a dynamical meaning in presence of the frame anomaly. This mixing is better understood if we write the whole algebra of the two symmetries.

If one try to give a global description of this algebra on the base manifold one must be careful if the topology is not trivial (namely if the manifold is not parallelizable). The key point is that the action of ξ^μ on the tangent plane (on $E_{\mu a}$ and $\Omega_{\mu ab}$) is defined only up to a local rotation of the orthogonal frame. Following [20] we introduce from the beginning a fixed background field $E_{\mu a}^0$ so the transformation laws globally defined are:

$$\begin{aligned}\delta(\lambda)E_{\mu a} &= \lambda_{ab}E_{\mu a}\\ \delta(\lambda)\Omega_{\mu ab} &= D_\mu \lambda_{ab}\end{aligned} \tag{3.20}$$

$$\begin{aligned}\delta(\xi)E_{\mu a} &= L_\xi E_{\mu a} + (\xi^\lambda \Omega_{\lambda ab}^0)E_{\mu a}\\ \delta(\xi)\Omega_{\mu ab} &= L_\xi \Omega_{\mu ab} + D_\mu[\xi^\nu(\Omega_{\nu ab} - \Omega_{\nu ab}^0)]\end{aligned} \tag{3.21}$$

The following commutation rules hold:

$$\begin{aligned}[]
[\delta(\lambda_1), \delta(\lambda_2)] &= \delta([\lambda_1, \lambda_2])\\
[\delta(\xi_1), \delta(\lambda_2)] &= \delta(\xi^\mu D_\mu^0 \lambda)\\
[\delta(\xi_1), \delta(\xi_2)] &= \delta([\xi_1, \xi_2]_L) - \delta(\xi_1^\mu \xi_2^\nu R_{\mu\nu}^0)
\end{aligned} \tag{3.22}$$

where L_ξ is the usual Lie derivative, $D_\mu^0 = D_\mu + [\Omega_\mu^0, \ ]$, $[\xi_1, \xi_2]_L^\mu = \xi^\nu \partial_\nu \xi_2^\mu - \xi_2^\nu \partial_\nu \xi_1^\mu$ and $R_{\mu\nu ab} = \partial_\mu \Omega_{\nu ab}^0 - \partial_\nu \Omega_{\mu ab}^0 + [\Omega_\mu^0, \Omega_\nu^0]_{ab}$.

We note that the global gravitational algebra is much more complicated than the gauge one: the W-Z consistency conditions are:

$$\delta(\lambda_1)t(\lambda_2) - \delta(\lambda_2)t(\lambda_1) = t([\lambda_1, \lambda_2])$$

$$\delta(\xi_1)t(\lambda_2) - \delta(\lambda_2)a(\xi_1) = t(\xi^\mu D_\mu^0 \lambda)$$

$$\delta(\xi_1)t(\lambda_2) - \delta(\xi_2)a(\xi_1) = a([\xi_1, \xi_2]_L) - t(\xi_1^\mu \xi_2^\mu R^0_{\mu\nu}) \tag{3.23}$$

In presence of non-trivial topology is not consistent to assume that $a = 0$ while it is possible put $t = 0$. In reference [20] a solution of the full cohomological problem (3.23) has been given, using the same method outlined for the gauge theory: this solution, in the case of parallelizable manifold, gives zero for the diffeomorphism anomaly. It is also possible to construct a Wess-Zumino-Witten action [16], using the n-bein $E_{\mu a}$, in order to compensate the Lorentz anomaly: the prize to pay is anyway the occurrence of a coordinate anomaly. In the following we suppose, unless special remarks, to work with parallelizable manifold, disregarding problems of globality. Within this limitation is consistent to assume both $a^\mu = 0$ or $T_{ab} = 0$ (one can pass from a situation to another with a W-Z-W term): because our operatorial approach will be manifestly coordinate invariant we take $a^\mu = 0$. The problem reduces to find solution of:

$$\delta(\lambda_1)t(\lambda_2) - \delta(\lambda_2)t(\lambda_1) = t([\lambda_1, \lambda_2])$$
$$\delta(\xi)t(\lambda) = -t(\xi^\mu \partial_\mu \lambda) \tag{3.24}$$

The first line is equivalent to a problem similar to the gauge theory one (with gauge group $SO(2n, R)$) while the second equation shows that $t(\lambda)$ transform as a scalar under diffeomorphism: consistent Lorentz anomaly can be obtained from the general solution (1.13) where:

$$A \to \Omega = \Omega_{\mu ab} S_{ab} dx^\mu$$
$$F \to R = d\Omega + [\Omega, \Omega]$$

S_{ab} are the generator of $SO(2n, R)$.

A strong property can be derived from the theory of the ad-invariant polynomial over a Lie algebra: the relevant polynomial to extract the $SO(2n, R)$ anomaly is one of rank $n + 1$ and this does exist only if $n + 1 = 2k$ (it is not difficult to understand this "selection rule" in term of symmetrized trace of the $SO(2n, R)$ generators). The result is the absence of Lorentz anomaly in $d = 4k$.

3.2 The covariant Lorentz anomaly

The presence of the frame anomaly, as in the gauge case, changes the tensorial properties of the spin-current $J_{\mu ab}$ and of the energy-momentum tensor $T^{\mu\nu}$. Classically $J_{\mu ab}$ transforms:

$$\delta(\lambda)J_{\mu ab} = -(\lambda_{ac}J_{\mu cb} + \lambda_{cb}J_{\mu ca}) = \delta_\lambda J_{\mu ab}. \tag{3.25}$$

It is not difficult to show that in presence of anomaly:

$$\delta(\lambda)J_{\mu ab} = \delta(\lambda)\frac{\delta}{\delta\Omega_{\mu ab}}\Gamma[E]$$

$$= \delta_\lambda J_{\mu ab} + \frac{\delta}{\delta \Omega_{\mu ab}} \int d^{2n}x\sqrt{g}\, \lambda_{cd} T_{cd} \quad (3.26)$$

We see the appearance of an inomogeneus term; one of course can define a covariant spin-current

$$\hat{J}_{\mu ab} = J_{\mu ab} + X_{\mu ab}$$

$$\delta(\lambda) X_{\mu ab} = -\frac{\delta}{\delta \Omega_{\mu ab}} \int d^{2n}x\sqrt{g}\, \lambda_{cd} T_{cd} \quad (3.27)$$

modifying the anomaly

$$D_\mu \hat{J}^\mu_{ab} = -T_{ab} + D_\mu X^\mu_{ab} \equiv -\hat{T}_{ab} \quad (3.28)$$

The energy-momentum tensor has the same behaviour:

$$T^{\mu\nu} = \frac{1}{2}(e^\mu_a \frac{\delta}{\delta E_{\nu a}} + e^\nu_a \frac{\delta}{\delta E_{\mu a}})\Gamma[E]$$

$$\delta(\lambda) T^{\mu\nu} = [\delta(\lambda), \frac{1}{2}(e^\mu_a \frac{\delta}{\delta E_{\nu a}} + e^\nu_a \frac{\delta}{\delta E_{\mu a}})]\Gamma[E] = \frac{1}{2}(e^\mu_a \frac{\delta}{\delta E_{\nu a}} + e^\nu_a \frac{\delta}{\delta E_{\mu a}})\delta(\lambda)\Gamma[E] \quad (3.29)$$

The commutator is zero so:

$$\delta(\lambda) T^{\mu\nu} = \frac{1}{2}(e^\mu_a \frac{\delta}{\delta E_{\nu a}} + e^\nu_a \frac{\delta}{\delta E_{\mu a}}) \int d^{2n}x\sqrt{g}\, \lambda_{cd} T_{cd} \quad (3.30)$$

that is in general not zero: $T^{\mu\nu}$ is not a tensor. One can again redefine it, using a local polynomial:

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + l^{\mu\nu} \quad (3.31)$$

recovering the covariance with

$$\delta(\lambda) l^{\mu\nu} = -\delta(\lambda) T^{\mu\nu}$$

The balance equation (3.18) is changed; we will call the covariant divergence of $\hat{T}^{\mu\nu}$ covariant anomaly of the energy-momentum tensor:

$$D_\mu \hat{T}^{\mu\nu} = \hat{a}^\nu \quad (3.32)$$

We remark that the presence of the covariant anomaly does not mean a breaking of the diffeomorphism invariance, as we have seen before.
To obtain algebraically the Lorentz covariant anomaly $\hat{T}_{ab}$ we can work in perfect analogy with the gauge case (the covariant divergence of the energy-momentum tensor actually is not produced by an algebraic method, but we will compute it directly by the use of a functional representation): we extend the orthogonal group $SO(2n, R)$ to the complex orthogonal group $SO(2n, C)$: in order to preserve the relation between

n-bein and metric we choose a particular complexification of the external fields. We start with a complex n-bein:

$$\hat{E}_{\mu a} = E^1_{\mu a} + i\,E^2_{\mu a}$$

with

$$\hat{E}_{\mu a}\hat{E}_{\nu a} = g_{\mu\nu}$$

The simplest way to achieve this is taking

$$\hat{E}_{\mu a} = \Lambda_{ab}E_{\mu b}$$

$$\Lambda_{ab} \in SO(2n, C)$$

The relation between $\hat{E}_{\mu a}$ and $g_{\mu\nu}$ gives the constraints :

$$\begin{aligned} E^1_{\mu a}E^1_{\nu a} - E^2_{\mu a}E^2_{\nu a} &= g_{\mu\nu} \\ e^{1\mu}_a E^1_{\nu a} - e^{2\mu}_a E^2_{\nu a} &= \delta^\mu_\nu \\ e^1_{\mu a}E^1_{\mu b} - E^2_{\mu a}E^2_{\mu b} &= \delta_{ab} \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} E^1_{\mu a}E^2_{\nu a} + E^1_{\nu a}E^2_{\mu a} &= 0 \\ e^1_{\mu a}E^2_{\nu a} + e^2_{\mu a}E^1_{\nu a} &= 0 \\ e^1_{\mu a}E^2_{\mu b} + e^1_{\mu b}E^2_{\mu a} &= 0 \end{aligned} \tag{3.34}$$

We derive the spin-connection

$$\hat{\Omega}_{\mu ab} = \hat{e}^\nu_a(\partial_\mu \hat{E}_{\nu b}) - \Gamma^\lambda_{\mu\nu}\hat{E}_{\lambda b})$$

$$\hat{\Omega}_{\mu ab} = \Omega^1_{\mu ab} + i\,\Omega^2_{\mu ab}$$

$$\begin{aligned} \Omega^1_{\mu ab} &= (e^{1\nu}_a\partial_\mu E^1_{\nu b} - e^{2\nu}_a\partial_\mu E^2_{\nu b} - \Gamma^\lambda_{\mu\nu}(e^{1\nu}_a E^1_{\lambda b} - e^{2\nu}_a E^1_{\lambda b}) \\ \Omega^2_{\mu ab} &= (e^{1\nu}_a\partial_\mu E^2_{\nu b} + e^{2\nu}_a\partial_\mu E^1_{\nu b}) - \Gamma^\lambda_{\mu\nu}(e^{1\nu}_a E^2_{\lambda b} - e^{2\nu}_a E^1_{\lambda b}) \end{aligned} \tag{3.35}$$

If we will call, like in the gauge case, $\delta_1(\lambda)$ the transformation of the maximal compact subgroup ($SO(2n, R)$) and $\delta_2(\lambda)$ the one of its orthogonal complement in $SO(2n, C)$, we obtain:

$$\begin{aligned} \delta_1(\lambda)e^{1\mu}_a &= \lambda_{ab}e^{1\mu}_b \\ \delta_1(\lambda)e^{2\mu}_a &= \lambda_{ab}e^{2\mu}_b \\ \delta_1(\lambda)\Omega^1_{\mu ab} &= -\partial_\mu\lambda_{ab} + \lambda_{ac}\Omega^1_{\mu cb} + \lambda_{bc}\Omega^1_{\mu ac} \\ \delta_1(\lambda)\Omega^2_{\mu ab} &= \lambda_{ac}\Omega^2_{\mu cb} + \lambda_{bc}\Omega^2_{\mu ac} \end{aligned} \tag{3.36}$$

$$\delta_2(\lambda)e^{1\mu}_a = \lambda_{ab}e^{2\mu}_b$$

$$\delta_2(\lambda)e_a^{2\mu} = -\lambda_{ab}e_b^{1\mu}$$
$$\delta_2(\lambda)\Omega^1_{\mu ab} = \lambda_{ac}\Omega^2_{\mu cb} + \lambda_{bc}\Omega^2_{\mu ac}$$
$$\delta_2(\lambda)\Omega^2_{\mu ab} = \partial_\mu\lambda_{ab} - \lambda_{ac}\Omega^1_{\mu cb} - \lambda_{bc}\Omega^1_{\mu ac} \tag{3.37}$$

δ_1 and δ_2 gives the $SO(2n,C)$ algebra

$$[\delta_1(\lambda_1),\delta_1(\lambda_2)] = \delta_1([\lambda_1,\lambda_2])$$
$$[\delta_2(\lambda_1),\delta_2(\lambda_2)] = -\delta_1([\lambda_1,\lambda_2])$$
$$[\delta_1(\lambda_1),\delta_2(\lambda_2)] = \delta_2([\lambda_1,\lambda_2])$$

The W-Z consistency conditions for $SO(2n,C)$ are similar to the $SL(2n,C)$ ones, studied before:

$$\delta_1(\lambda)\Gamma = t_1(\lambda)$$
$$\delta_2(\lambda)\Gamma = t_2(\lambda)$$

with consistent solution:

$$t_1(\lambda) = \pm i t_2(\lambda)$$

and covariant solution:

$$t_1(\lambda) = 0$$
$$t_2(\lambda) \neq 0$$

In the limit $\Omega^2_{\mu ab} = 0$ we recover the known expression for the consistent and covariant Lorentz anomaly.

3.3 The Weyl determinant in curved space

The relevant object in the study of $\Gamma[E]$ is the operator appearing in the action (3.1):

$$\not{D} = i\sigma_a e_a^\mu\,(\,\partial_\mu + \frac{1}{4}\Omega_{\mu cd}\tilde{\sigma}_c\sigma_d\,) \tag{3.38}$$

acting on the space $\Gamma(S_+)$; the invariant measure is $d^{2n}x\sqrt{g}$. We begin studying the transformation properties of $\not{D}$ and $\tilde{\not{D}}$

Frame rotation:

$$\delta(\lambda)\not{D} = \not{D}\tilde{\tau} - \tau\not{D}$$
$$\delta(\lambda)\tilde{\not{D}} = \tilde{\not{D}}\tau - \tilde{\tau}\tilde{\not{D}} \tag{3.39}$$

with

$$\tau = \frac{1}{4}\lambda_{ab}\tilde{\sigma}_a\sigma_b$$

$$\tilde{\tau} = \frac{1}{4}\lambda_{ab}\sigma_a\tilde{\sigma}_b$$

Diffeomorphism:

$$\delta(\xi)\not{D} = \xi^\mu\partial_\mu\not{D} - \not{D}\xi^\mu\partial_\mu$$
$$\delta(\xi)\tilde{\not{D}} = \xi^\mu\partial_\mu\tilde{\not{D}} - \tilde{\not{D}}\xi^\mu\partial_\mu \tag{3.40}$$

As we have seen before it is not possible to define directly, by ζ- function, the determinant of $\not{D}$: in order to have a good eigenvalue problem we would like to find the correct isomorphism. One immedeatly realizes that it is not possible to use the same trick of the gauge theories: from the physical point of view we are tempting to introduce a free partner for the chiral spinor interacting with gravity

$$T = (i\tilde{\sigma}_a\partial_a) \tag{3.41}$$

In so doing we inevitably break both symmetries (we are looking for a diffeomorphism invariant theory): moreover (3.41) is not a fixed isomorphism because a general coordinates transformation changes it and it does not admit a good geometrical interpretation having no invariant meaning in curved space. One may think about a generalization of the Weyl kinetic operator:

$$i\tilde{\sigma}_a e^\mu_a\partial_\mu \tag{3.42}$$

This a good operator, but it is not free, depending crucially on the n-bein field. Anyway we can try to fix the isomorphism T satisfying the properties required to represent a solution of the extended W-Z conditions. Actually we will study a more general operator then $T\not{D}$: we define

$$\Gamma[\hat{E}; r, s] = \frac{1}{k}\ln Det[\,\tilde{\not{D}}_c\,(r)\,\not{D}_c\,(s)\,] \tag{3.43}$$

where

$$\tilde{\not{D}}_c(r) = i\tilde{\sigma}_a\hat{e}^\mu_a\,(\,\partial_\mu + \frac{r}{4}\hat{\Omega}^\dagger_{\mu cd}\tilde{\sigma}_c\sigma_d\,)$$
$$\not{D}_c(s) = i\sigma_a\hat{e}^\mu_a\,(\,\partial_\mu + \frac{s}{4}\hat{\Omega}_{\mu cd}\tilde{\sigma}_c\sigma_d\,) \tag{3.44}$$

where we have used the complexification, deforming both operators with $r, s \in \mathbf{R}$; k is a suitable normalization factor. The functional is defined by ζ-function technique:

$$\Gamma[\hat{E}; r, s] = \frac{1}{k}\frac{d}{dt}Tr[\,(\tilde{\not{D}}_c(r)\not{D}_c(s)\,)^{-t}\,]_{t=0} \tag{3.45}$$

Let us note that $r = s = 1$, $k = 2$ does correspond to the modulus case:

$$\frac{1}{2}\frac{d}{dt}Tr[\,(\not{D}^\dagger_c\not{D}_c)^{-t}\,]_{t=0} = |\ln det\not{D}_c| \tag{3.46}$$

The definition (3.43) gives a functional invariant under diffeomorphism: it is easy to show that

$$\delta(\xi)(\tilde{\not{D}}_c(r)\not{D}_c(s)) = -[(\tilde{\not{D}}_c(r)\not{D}_c(s)), \xi^\mu \partial_\mu]$$

Using the properties of the complex powers of elliptic operators [17], namely the fact they are of trace class

$$Tr[AB] = Tr[BA],$$

this change is seen does not affect the determinant. For frame rotation the situation is not trivial: it is a simple exercise to prove from (3.39)

$$\delta_1(\lambda)\not{D}_c(s) = s[\not{D}_c(s)\tilde{\tau} - \tau\not{D}_c(s)] + (s-1)(\tau - \tilde{\tau})\not{D}_c(s) + (1-s)A(\tilde{\tau})$$

$$\delta_1(\lambda)\tilde{\not{D}}_c(r) = r[\tilde{\not{D}}_c(r)\tau - \tau\tilde{\not{D}}_c(s)] + (1-r)(\tau - \tilde{\tau})\tilde{\not{D}}_c(r) + (1-r)\tilde{A}(\tau) \qquad (3.47)$$

and

$$\delta_2(\lambda)\not{D}_c(s) = is[\not{D}_c(s)\tilde{\tau} - \tau\not{D}_c(s)] + i(s-1)(\tau - \tilde{\tau})\not{D}_c(s) + i(1-s)A(\tilde{\tau})$$

$$\delta_2(\lambda)\tilde{\not{D}}_c(r) = ir[\tilde{\tau}\tilde{\not{D}}_c(s) - \tilde{\not{D}}_c(r)\tau] - i(1-r)(\tau - \tilde{\tau})\tilde{\not{D}}_c(r) - i(1-r)\tilde{A}(\tau) \qquad (3.48)$$

where

$$A(\tilde{\tau}) = ([\sigma_a, \tilde{\tau}]i\hat{e}^\mu_a\partial_\mu + i\frac{s}{4}\hat{\Omega}_{\mu cd}\hat{e}^\mu_a[\sigma_a\tilde{\sigma}_c\sigma_d, \tilde{\tau}])$$

$$\tilde{A}(\tau) = ([\tilde{\sigma}_a, \tau]i\hat{e}^{\dagger\mu}_a\partial_\mu + i\frac{r}{4}\hat{\Omega}^\dagger_{\mu cd}\hat{e}^{\dagger\mu}_a[\tilde{\sigma}_a\sigma_c\tilde{\sigma}_d, \tau])$$

From these we can obtain the whole variation on $\tilde{\not{D}}_c(r)\not{D}_c(s)$: applying ζ-function technique

$$t_1(\tau) = \frac{1}{k}\frac{d}{dt}[t\,Tr\{(\tilde{\not{D}}_c(r)\not{D}_c(s))^{-t}[(s-1)\tilde{\tau} - (r-1)\tau] + (\not{D}_c(s)\tilde{\not{D}}_c(r))^{-t}[(r-1)\tau - (s-1)\tilde{\tau}]\} +$$

$$+ t\,Tr\{[(1-s)\tilde{\not{D}}_c(r)A(\tilde{\tau}) + (1-r)\tilde{A}(\tau)\not{D}_c(s)](\tilde{\not{D}}_c(r)\not{D}_c(s))^{-1-t}\}_{t=0} \qquad (3.49)$$

$$t_2(\tau) = \frac{i}{k}\frac{d}{dt}[t\,Tr\{(\tilde{\not{D}}_c(r)\not{D}_c(s))^{-t}[(s+1)\tilde{\tau} + (r-1)\tau] + (\not{D}_c(s)\tilde{\not{D}}_c(r))^{-t}[-(r+1)\tau - (s+1)\tilde{\tau}]\} +$$

$$+ t\,Tr\{[(1-s)\tilde{\not{D}}_c(r)A(\tilde{\tau}) - (1-r)\tilde{A}(\tau)\not{D}_c(s)](\tilde{\not{D}}_c(r)\not{D}_c(s))^{-1-t}\}_{t=0} \qquad (3.50)$$

3.4 Covariant solution

Putting $r = 1$, $s = 1$ and $k = 2$ we obtain:

$$t_1(\lambda) = 0$$

$$t_2(\lambda) \neq 0$$

$$t_2(\lambda) = \frac{i}{(4\pi)^n} \int d^{2n}x \sqrt{g}\, Tr[\tilde{\tau} H_n(\not{D}_c^\dagger \not{D}_c) - \tau H_n(\not{D}_c \not{D}_c^\dagger)] \tag{3.51}$$

We recall that the heat-kernel coefficients H_n are local in the external field and their derivatives. In the limit $\Omega^2_{\mu ab} = 0$ the covariant Lorentz anomaly is recovered: really until now we have not computed the trace (3.51) for general $d = 2n$, as in the gauge case, because of the greater complexity of the heat-kernel coefficients. The basic reason is that while in the gauge case only the leading contribution in σ matrices to the n-th Seeley-de Witt coefficient survives into the trace, in the gravitational one the subleading term also gives, in general, a non zero effect. So we cannot make a direct computation with the cohomological result given in term of invariant polynomials: nevertheless the explicit calculation in $d = 2, 4, 6$ comforts ourselves.
Now we show how to obtain from this functional

$$\hat{\Gamma}[E^1, E^2] = \frac{1}{2}\frac{d}{dt} Tr[(\not{D}_c^\dagger \not{D}_c)^{-t}]_{t=0} \tag{3.52}$$

the covariant energy-momentum tensor $\hat{T}^{\mu\nu}$ and the covariant spin-current $\hat{J}_{\mu ab}$. Let us define

$$\tilde{T}^{\mu\nu} = -Im(\hat{e}^\mu_a \frac{\delta}{\delta \hat{E}_{\nu a}} + \hat{e}^\nu_a \frac{\delta}{\delta \hat{E}_{\mu a}})\hat{\Gamma}[E^1, E^2] \tag{3.53}$$

where the operator acting on $\hat{\Gamma}[E^1, E^2]$, written in term of $E^1_{\mu a}$ and $E^2_{\mu a}$, is

$$\Theta^{\mu\nu} = \frac{1}{2}(e^{1\mu}_a \frac{\delta}{\delta E^2_{\nu a}} + e^{1\nu}_a \frac{\delta}{\delta E^2_{\mu a}}) - \frac{1}{2}(e^{2\mu}_a \frac{\delta}{\delta E^1_{\nu a}} + e^{2\nu}_a \frac{\delta}{\delta E^1_{\mu a}}) \tag{3.54}$$

We want to prove that:

$$\hat{T}^{\mu\nu} = \tilde{T}^{\mu\nu}|_{e^{2\mu}_a = 0} \tag{3.55}$$

Exactly we will show:

1. $\delta(\lambda)\hat{T}^{\mu\nu} = 0$, with $\delta(\lambda)$ generator of $SO(2n, R)$ rotation: $\hat{T}^{\mu\nu}$ is really a tensor.
2. $D_\mu \hat{T}^{\mu\nu} = \hat{a}^\nu$, with $\hat{a}^\mu$ the covariant anomaly of the energy-momentum tensor.

The point (1) is very simple: we note

$$\delta(\lambda) = \lim_{e^{2\mu}_a \to 0} \delta_1(\lambda) \tag{3.56}$$

where the compact generator has the expression

$$\delta_1(\lambda) = \int d^{2n}x \sqrt{g}\, \lambda_{ab}(e^{1\mu}_b \frac{\delta}{\delta E^2_{\mu a}} + e^{1\mu}_b \frac{\delta}{\delta E^2_{\mu a}})$$

From (3.53)

$$\delta_1(\lambda)\tilde{T}^{\mu\nu} = [\delta_1(\lambda), \Theta^{\mu\nu}]\hat{\Gamma} + \Theta^{\mu\nu}\delta_1(\lambda)\hat{\Gamma} \tag{3.57}$$

Being $\delta_1(\lambda)\hat{\Gamma} = 0$ (we recall that the modulus is invariant under the usual $SO(2n, R)$) and using the explicit result $[\delta_1(\lambda), \Theta^{\mu\nu}] = 0$ we obtain:

$$\delta_1(\lambda)\tilde{T}^{\mu\nu} = 0 \tag{3.58}$$

This is true for any $e_a^{2\mu}$ so in particular for $e_a^{2\mu} = 0$ The second claim is much more subtle. We compute

$$\int d^{2n}x\sqrt{g}\,\xi_\nu D_\mu \tilde{T}^{\mu\nu} \tag{3.59}$$

ξ_ν is an infinitesimal vector field. This expression can be rewrite as

$$\int d^{2n}x\sqrt{g}\,\frac{1}{2}(D_\mu\xi_\nu + D_\nu\xi_\mu)(e_a^{1\mu}\frac{\delta}{\delta E^2_{\nu a}} - e_a^{1\mu}\frac{\delta}{\delta E^2_{\nu a}})\hat{\Gamma} \tag{3.60}$$

Utilizing the expression for the diffeomorphism generated on $\hat{E}_{\mu a}$ and (3.16) one recovers the action of $\delta(\xi)$ on the real and the imaginary part of the n-bein:

$$\delta(\xi)E^1_{\mu a} = \frac{1}{2}(D_\mu\xi^\alpha + D^\alpha\xi_\mu)E^1_{\alpha a} + \lambda^1_{ab}E^1_{\mu b} - \lambda^2 E^2_{\mu b} - \xi^\lambda(\Omega^1_{\lambda ab}E^1_{\mu b} - \Omega^2_{\lambda ab}E^2_{\mu b}) \tag{3.61}$$

$$\delta(\xi)E^2_{\mu a} = \frac{1}{2}(D_\mu\xi^\alpha + D^\alpha\xi_\mu)E^2_{\alpha a} + \lambda^1_{ab}E^2_{\mu b} + \lambda^2 E^1_{\mu b} - \xi^\lambda(\Omega^1_{\lambda ab}E^2_{\mu b} + \Omega^2_{\lambda ab}E^1_{\mu b}) \tag{3.62}$$

with

$$\lambda^1_{ab} = \frac{1}{2}(\partial_\mu\xi_\alpha - \partial_\alpha\xi_\mu)[e_a^{1\alpha}e_b^{1\mu} - e_a^{2\alpha}e_b^{2\mu}]$$

$$\lambda^1_{ab} = \frac{1}{2}(\partial_\mu\xi_\alpha - \partial_\alpha\xi_\mu)[e_a^{1\alpha}e_b^{2\mu} + e_a^{2\alpha}e_b^{1\mu}]$$

The last equations allow us to write (3.60) as:

$$\begin{aligned}
&\int d^{2n}x\sqrt{g}\,[\delta(\xi)E^1_{\mu a}\frac{\delta}{\delta E^2_{\mu a}} - \delta(\xi)E^2_{\mu a}\frac{\delta}{\delta E^1_{\mu a}}]\hat{\Gamma}+ \\
&+\int d^{2n}x\sqrt{g}\,[\lambda^1_{ab} - \xi^\lambda\Omega^1_{\lambda ab}][E^2_{\mu b}\frac{\delta}{\delta E^2_{\mu a}} - E^1_{\mu b}\frac{\delta}{\delta E^1_{\mu a}}]\hat{\Gamma}+ \\
&-\int d^{2n}x\sqrt{g}\,[\lambda^2_{ab} - \xi^\lambda\Omega^2_{\lambda ab}][E^2_{\mu b}\frac{\delta}{\delta E^2_{\mu a}} + E^1_{\mu b}\frac{\delta}{\delta E^1_{\mu a}}]\hat{\Gamma}
\end{aligned} \tag{3.63}$$

The third term is zero (it is a compact $SO(2n, C)$ rotation on Γ), the second is related to the covariant anomaly $\hat{T}_{ab}$, giving

$$\int d^{2n}x\sqrt{g}\,[\delta(\xi)E^1_{\mu a}\frac{\delta}{\delta E^2_{\mu a}} - \delta(\xi)E^2_{\mu a}\frac{\delta}{\delta E^1_{\mu a}}]\hat{\Gamma} + [\lambda^1_{ab} - \xi^\lambda\Omega^1_{\lambda ab}]\hat{T}^{ab} \tag{3.64}$$

The operator acting on $\hat{\Gamma}$ is:

$$[\delta(\xi)E^1_{\mu a}\frac{\delta}{\delta E^2_{\mu a}} - \delta(\xi)E^2_{\mu a}\frac{\delta}{\delta E^1_{\mu a}}]\hat{E}_{\mu a} = i\delta(\xi)\hat{E}_{\mu a} = \delta(i\xi)\hat{E}_{\mu a} \tag{3.65}$$

It generates the diffeomorphism of imaginary parameter: the functional $\hat{\Gamma}$ is not invariant under this transformations on the contrary of real diffeomorphism. Let us study the variation of $\not{D}_c$ and $\not{D}_c^\dagger$ under $\delta(i\xi)$: after some efforts

$$\delta(i\xi)\not{D}_c = \hat{d}_1 + \hat{d}_2 \tag{3.66}$$

with

$$\hat{d}_1 = \frac{1}{4}[D_\mu, \delta_\xi g_{\mu\nu}\sigma_\nu]_+$$
$$\hat{d}_2 = \not{D}_c\tilde{\tau} - \tau\not{D}_c + \{\frac{1}{4}g^\mu_\mu, \not{D}_c\}$$

and the parameter in τ and $\tilde{\tau}$ is

$$\tilde{\lambda}_{ab} = \frac{i}{2}\hat{E}_{\mu a}\hat{E}_{\nu b}\,(\partial^\mu\xi^\nu - \partial^\nu\xi^\mu) - i\xi^\mu\hat{\Omega}_{\mu ab}$$

Conversely

$$\delta(i\xi)\not{D}_c^\dagger = \tilde{d}_1 + \tilde{d}_2 \tag{3.67}$$

with

$$\tilde{d}_1 = \frac{1}{4}\{D_\mu, \delta_\xi g_{\mu\nu}\tilde{\sigma}_\nu\}$$
$$\tilde{d}_2 = \not{D}_c^\dagger\tau^\dagger - \tilde{\tau}^\dagger\not{D}_c^\dagger + \{\frac{1}{4}g^\mu_\mu, \not{D}_c^\dagger\}$$

The total variation is:

$$\delta(i\xi)(\not{D}_c^\dagger\not{D}_c) = \not{D}_c^\dagger(\tau^\dagger - \tau)\not{D}_c + \not{D}_c^\dagger\not{D}_c\tilde{\tau} - \tilde{\tau}^\dagger\not{D}_c^\dagger\not{D}_c$$
$$\frac{i}{4}[\delta_\xi g^\mu_\mu, \not{D}_c^\dagger\not{D}_c] + \tilde{d}_1 + \hat{d}_1 \tag{3.68}$$

The dilatation part, proportional to $\delta_\xi g^\mu_\mu$, drops out in the variation of the determinant being a commutator. The rotational part, proportional to τ or $\tilde{\tau}^\dagger$, gives contribution only with $Im(\tilde{\lambda}_{ab})$: it is a non compact $SO(2n, C)$ on $\hat{\Gamma}$ of parameter

$$-(\lambda^1_{ab} - \xi^\mu\Omega^1_{\mu ab})$$

Therefore $\delta(i\xi)\hat{\Gamma}$ has a first contribution

$$-\int d^{2n}x\sqrt{g}\,(\lambda^1_{ab} - \xi^\mu\Omega^1_{\mu ab})\hat{T}_{ab} \tag{3.69}$$

that kills with the second part of (3.64). It only remains to calculate the $\hat{d}_1$, $\tilde{d}_1$ contribution: after defining:

$$\Lambda = \frac{i}{4}\{D_\mu, \xi^\mu\} + \frac{1}{4}(\tilde{\xi}\not{D}_c + \not{D}_c^\dagger\xi) \tag{3.70}$$

$$\tilde{\Lambda} = \frac{i}{4}\{D_\mu, \xi^\mu\} + \frac{1}{4}(\tilde{\xi}\not{D}_c + \not{D}_c^\dagger\xi) \tag{3.71}$$

with

$$\xi = \xi^\alpha \sigma_\alpha$$
$$\tilde{\xi} = \xi^\alpha \tilde{\sigma}_\alpha$$

we have the useful formulas:

$$\tilde{d}_1 + \hat{d}_1 = \Lambda^\dagger \not{D}_c^\dagger \not{D}_c - \not{D}_c^\dagger \not{D}_c \Lambda + \not{D}_c^\dagger (\tilde{\Lambda} - \tilde{\Lambda}^\dagger) \not{D}_c \qquad (3.72)$$

deriving from the identity

$$\hat{d}_1 = \tilde{\Lambda} \not{D}_c - \not{D}_c \Lambda$$
$$\tilde{d}_1 = \Lambda^\dagger \not{D}_c^\dagger - \not{D}_c^\dagger \tilde{\Lambda}^\dagger$$

Again the ζ-function feels only the variations coming from imaginary terms

$$-\frac{i}{4}\{\{D_\mu, \xi^\mu\}, \not{D}_c^\dagger \not{D}_c\} + \frac{i}{2} \not{D}_c^\dagger \{D_\mu, \xi^\mu\} \not{D}_c \qquad (3.73)$$

At the end we remain with:

$$\int d^{2n}x \sqrt{g}\, \xi_\nu D_\mu \tilde{T}^{\mu\nu} = -\frac{i}{4}\frac{d}{ds}(sTr\{[(\not{D}_c^\dagger \not{D}_c)^{-s} - (\not{D}_c \not{D}_c^\dagger)^{-s}]\{D_\lambda, \xi^\lambda\}\})_{s=0}$$

Some heat-kernel calculations gives (after the elimination of the parameter ξ and the projection on $E^2_{\mu a}$).

$$D_\mu T^{\mu\nu} = \frac{1}{4(4\pi)^n} Tr[D^x_\lambda H_n(x,y) - D^y_\lambda H_n(x,y)]_{x=y} g^{\lambda\nu} -$$

$$-\frac{1}{4(4\pi)^n} Tr[D^x_\lambda \tilde{H}_n(x,y) - D^y_\lambda \tilde{H}_n(x,y)]_{x=y} g^{\lambda\nu} \qquad (3.74)$$

This expression coincides with the result of [8] and can be solved in term of invariant polynomials recovering the perturbative calculation of [19].
For the spin-current we define

$$\hat{J}^\mu_{ab} = \frac{1}{i}\frac{\delta}{\delta \Omega^2_{\mu ab}} \hat{\Gamma}_{\Omega^2_{\mu ab}=0} \qquad (3.75)$$

Performing an analogous calculation to the energy-momentum tensor one we find:

$$\delta(\lambda) \hat{J}^\mu_{ab} = \delta_\lambda \hat{J}^\mu_{ab} \qquad (3.76)$$

and

$$\int d^{2n}x \sqrt{g}\, \lambda_{ab}(D_\mu \hat{J}^\mu_{ab}) = -\delta_2(\lambda)\hat{\Gamma} = -\int d^{2n}x \sqrt{g}\, \lambda_{ab} \hat{T}_{ab} \qquad (3.77)$$

This complete our analysis of the covariant solution of the extended algebra: from the modulus of the extended Weyl operator we recover the covariant e-m tensor and spin-current with the relative anomalies.

3.5 Consistent solution

We see that, on the contrary of the gauge case, the non local terms do not disappear without an explicit calculation, for some values of r and s, unless $r = 1 = s$ (the covariant solution). At the present day we are not able to find some r and s giving rise in $d = 2n$:

$$t_1(\lambda) = \pm i t_2(\lambda) \tag{3.78}$$

nor if this possible solution is local. What we are able to do is to solve the problem for $n = 1$: the abelian character of the theory makes the commutators A , $\tilde{A}$ zero. The calculation is of interest by itself because we can write explicity the determinant: the first computation [22] was based on a particular choice of coordinates; in [9] Leuterwyler showed that it is possible to renormalize the effective action preserving the general covariance. In our approach diffeomorphism invariance appears naturally and the calculation is performed as the determinant of a well defined elliptic operator, using the ζ-function technique.

Chapter 4

Two dimensional Weyl determinant

We assume from the begining that the base manifold is parallelizable: in this case no problem arises with fixed background connections and the used operators do posses a global expression. Even in this situation, the simplest as possible, the calculation is not completly straightforward, as we will see: some subtle points must be clarified before in order to understand the philosophy of the computation.
Firstly we remark that also in the abelian case of $d = 2$ there are two distinct cohomology class: one can prove that if $t_2(\lambda)$ realizes the covariant solution

$$\begin{aligned} \delta_1(\lambda)t_2(\lambda_2) &= 0 \\ t_1(\lambda) &= 0 \end{aligned} \tag{4.1}$$

it does not exist a local functional P in the complexified 2-beins $\hat{E}_{\mu a}$ and $\hat{E}^*_{\mu a}$ and their derivatives, scalar under general coordinates transformations, giving the consistent solution

$$\hat{t}_2(\lambda) = \pm i\hat{t}_1(\lambda) \tag{4.2}$$

with

$$\begin{aligned} \hat{t}_2(\lambda) &= t_2(\lambda) + \delta_2(\lambda)P \\ \hat{t}_1(\lambda) &= \delta_1(\lambda)P \end{aligned} \tag{4.3}$$

Then we note that the request $t_2(\lambda) = \pm i\, t_1(\lambda)$ concearns the structure of the coboundaries: adding to $\hat{\Gamma}$ some local term P we obtain a different representative of the cohomolgy class:

$$\begin{aligned} t_1(\lambda) \rightarrow \hat{t}_1(\lambda) = t_1(\lambda) + \delta_1(\lambda)P \\ t_2(\lambda) \rightarrow \hat{t}_2(\lambda) = t_2(\lambda) + \delta_2(\lambda)P \end{aligned} \tag{4.4}$$

giving in general $\hat{t}_1(\lambda) \neq \hat{t}_2(\lambda)$. So it is not necessary that $\tilde{\not{D}}_c(r)\not{D}_c(s)$ generates anomalies stricly obeying (4.2): we require the consistent costraint up to local terms in the effective action.

The last remark is about a simplification that our approach takes in two dimensions: in order to exploit the solutions of $SO(2n, C)$ anomaly algebra we had to work with the full complexified Weyl operator. Anyway the case $d = 2$ is very particular and we do not need to make the complexification. Basically we recognize that

$$Lie\, SO(2, C) \simeq Lie\, (SO(2, R) \otimes R_+)$$

and we understand R_+ as the group of conformal transformations, described in (3.4): here conformal means a local dilatation of the orthogonal frame (we do not touch the base manifold). Actually the effect of a $SO(2n, C)$ rotation on the Weyl operator is the same of a real rotation plus a conformal transformation.

In the following we shall use the Weyl matrices $\sigma_1 = \tilde{\sigma}_1 = 1$, $\sigma_2 = -\tilde{\sigma}_2 = i$ and the Weyl operator

$$\not{D} = i\sigma_a e_a^\mu (\partial_\mu + i\Omega_\mu) \tag{4.5}$$

with

$$i\Omega_\mu = \frac{1}{4}\Omega_{\mu ab}\tilde{\sigma}_a\sigma_b$$

A $SO(2, C)$ matrix Λ admits the factorization

$$\Lambda = R\,\hat{\Lambda}$$

where $R \in SO(2n, R)$ and

$$\hat{\Lambda} = \begin{pmatrix} \cosh\phi & -i\sinh\phi \\ i\sinh\phi & \cosh\phi \end{pmatrix} \tag{4.6}$$

from which one can derive the identities

$$(\hat{\Lambda}_{ab} e_b^\mu)\sigma_a = \exp(-\phi)\, e_a^\mu \sigma_a \tag{4.7}$$

$$(\hat{\Lambda}_{ac}\hat{\Lambda}_{bd}\frac{1}{4}\Omega_{\mu cd})\tilde{\sigma}_a\sigma_b = i\Omega_\mu - \frac{1}{2}\partial_\mu\phi \tag{4.8}$$

$$e_a^\mu \sigma_a i\partial_\mu\phi = e_a^\mu\sigma_a(\frac{\epsilon^{\mu\nu}}{\sqrt{g}}\partial_\nu\phi) \tag{4.9}$$

The effect of $\hat{\Lambda}$ on $\not{D}$ is

$$\hat{\Lambda} : \not{D} \rightarrow \not{D}' = i\sigma_a e_a'^\mu(\partial_\mu + i\Omega'_\mu)$$

with

$$e_a'^\mu = \exp(-\phi) e_a^\mu \tag{4.10}$$

$$\Omega'_\mu = \Omega_\mu + \frac{1}{2}\frac{\epsilon_\mu^\nu}{\sqrt{g}}\partial_\nu\phi \tag{4.11}$$

These are the changes of e_a^μ and Ω_μ under a conformal transformation: the non-compact sector of $SO(2n,C)$, represented by $\hat{\Lambda}$, acts on the Weyl operator as a conformal transformation. So we do not need to extend the 2-beins to complex values but we will simply understand $\delta(\lambda)$ the conformal generator: in order to match the parameter $i\tau = \frac{i}{4}\lambda_{ab}\tilde{\sigma}_a\sigma_b$ of the non-compact transformation with the parameter ϕ we define

$$i\tau = \frac{1}{2}\phi \tag{4.12}$$

Now the project is: after the definition

$$\Gamma[E;r,s] = \frac{1}{k-2}\frac{d}{dt}[Tr(\tilde{\not{D}}(r)\not{D}(s))^{-t}]_{t=0} \tag{4.13}$$

with

$$\not{D}(s) = i\sigma_a e_a^\mu[\partial_\mu + \frac{1}{4}(1-s)\Omega_\mu]$$

$$\tilde{\not{D}}(r) = i\tilde{\sigma}_a e_a^\mu[\partial_\mu - \frac{1}{4}(1-r)\Omega_\mu]$$

and

$$\begin{aligned} t_1(\lambda;r,s) &= \delta_1(\lambda)\Gamma[E;r,s] \\ t_2(\lambda;r,s) &= \delta_2(\lambda)\Gamma[E;r,s] \end{aligned} \tag{4.14}$$

we want:

1. To find the values of r, s, k that satisfy the consistent constraint (up to coboundary terms)

$$t_1(\lambda;r,s) = \pm t_2(\lambda;r,s)$$

2. To fix the normalization using the relation between consistent and covariant anomaly: we obtain the last one acting with $\delta_2(\lambda)$ on $\ln\sqrt{det(\not{D}^\dagger\not{D})}$

Really we will do something more: in $d=2$ we can compute $\Gamma[E;r,s]$ for any r and s: so we will normalize the determinant itself using the knowledge of $\ln\sqrt{det(\not{D}^\dagger\not{D})}$ (r=s=0=k). Let us note that (4.13) is invariant under diffeomorphisms, as we have seen, while is clear the origin of the $SO(2,R)$ breaking: $\tilde{\not{D}}(r)\not{D}(s)$ does not change covariantly under rotation of the local frame (in the ζ-function language the infinitesimal variation of $\tilde{\not{D}}(r)\not{D}(s)$ is not a commutator). Writing down the transformation law ($\tilde{\tau} = -\tau$):

$$\begin{aligned} \delta_1(\lambda)(\tilde{\not{D}}(r)\not{D}(s)) &= (r\tau - \tilde{\tau})(\tilde{\not{D}}(r)\not{D}(s)) + \\ &+ (1-s)\tau(\tilde{\not{D}}(r)\not{D}(s)) + \tilde{\not{D}}(r)[-r\tau + s\tilde{\tau}]\not{D}(s) \end{aligned} \tag{4.15}$$

$$\delta_2(\lambda)(\tilde{\not{D}}(r)\not{D}(s)) = -i[r\tau - \tilde{\tau}](\tilde{\not{D}}(r)\not{D}(s)) +$$
$$+ (\tilde{\not{D}}(r)\not{D}(s))(1-s)i\tau + \tilde{\not{D}}(r)i[(2-r)\tau + s\tilde{\tau}]\not{D}(s) \tag{4.16}$$

The infinitesimal variation of $\Gamma[E:r,s]$ is:

$$\delta_1(\lambda)\Gamma[E:r,s] = \frac{1}{2-k}Tr[(\tilde{\not{D}}(r)\not{D}(s))^{-t} - (\not{D}(s)\tilde{\not{D}}(r))^{-t}]_{t=0}(r-s)\tau$$

$$\delta_2(\lambda)\Gamma[E:r,s] = \frac{i}{2-k}Tr[(\tilde{\not{D}}(r)\not{D}(s))^{-t}(s-r) - (\not{D}(s)\tilde{\not{D}}(r))^{-t}(r-s)]_{t=0}\tau$$

Using the heat-kernel representation of the complex powers of elliptic operators [17] we obtain

$$\delta_1(\lambda)\Gamma[E:r,s] = \frac{1}{2-k}\frac{1}{4\pi}\int d^2x\sqrt{g}\,Tr[H_1 - \tilde{H}_1](r+s)\tau \tag{4.17}$$

$$\delta_2(\lambda)\Gamma[E:r,s] = \frac{i}{2-k}\frac{1}{4\pi}\int d^2x\sqrt{g}\,Tr[H_1 - \tilde{H}_1](s-r)\tau \tag{4.18}$$

with $H_1 = H_1(\tilde{\not{D}}(r)\not{D}(s))$ and $\tilde{H}_1 = H_1(\not{D}(s)\tilde{\not{D}}(r))$ H_n being the n-th coefficient of the expansion of the heat-kernel in the limit $t \to 0$ [17].
Generalizing the technique developped in [8] to our case (a deformation of $\not{D}^\dagger\not{D}$) we compute the coefficients H_1 and $\tilde{H}_1$ (we do not report the long but straightforward procedure)

$$H_1 = \frac{1}{6}R - \frac{1}{8}R(2-s+r) + \frac{i}{2}D_\mu\Omega^\mu(s+r) \tag{4.19}$$

$$\tilde{H}_1 = \frac{1}{6}R - \frac{1}{8}R(2+s-r) - \frac{i}{2}D_\mu\Omega^\mu(s+r) \tag{4.20}$$

R is the curvature scalar that in $d=2$ is easily expressed as

$$\frac{1}{4}\epsilon_{\mu\nu}\sqrt{g}R = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu \tag{4.21}$$

Then we put:

$$(s-r) = x$$

$$(s+r) = y$$

With this definition and using the explicit form of H_1 and $\tilde{H}_1$, the "anomalies" appear:

$$t_1(\lambda;r,s) = \frac{1}{4\pi}\frac{1}{2-k}\int d^2x\sqrt{g}\,\tau(\frac{1}{4}xyR + i\,y^2D_\mu\Omega^\mu) \tag{4.22}$$

$$t_2(\lambda;r,s) = \frac{i}{4\pi}\frac{1}{2-k}\int d^2x\sqrt{g}\,\tau(-\frac{1}{3}R + \frac{1}{4}x^2R + i\,xyD_\mu\Omega^\mu) \tag{4.23}$$

Let us look for the covariant solution

4.1 Covariant solution

$$t_1(\lambda; r, s) = 0 \tag{4.24}$$

that implies $y = 0$; on r and s we have the condition:

$$r + s = 0 \tag{4.25}$$

The δ_2 variation is:

$$t_2(\lambda; r, s) = \frac{i}{4\pi}\frac{1}{2-k}\int d^2x\sqrt{g}\,R(-\frac{1}{3}+\frac{1}{4}x^2)\,\tau \tag{4.26}$$

We note that there is not an unique solution: varying r and s along $r{=}{-s}$ we obtain a continuos family of operators whose determinant represents covariant solutions of the extended cohomological problem: they differ for a coefficient so we can choose k as function of x in order to fix the correct normalization. The right value is obtained by the modulus of Weyl determinant, characterized by $r{=}s{=}k{=}0$

$$t_2(\lambda) = -\frac{1}{24\pi}\int d^2x\sqrt{g}\,R\,\tau \tag{4.27}$$

The normalization condition derives from (4.26) and (4.27)

$$\frac{1}{4(2-k)}(-\frac{1}{3}+\frac{1}{4}x^2) = -\frac{1}{24} \tag{4.28}$$

Solving along $r = -s$ we find:

$$s(k) = \pm\sqrt{\frac{k}{6}}$$

$$r(k) = \mp\sqrt{\frac{k}{6}} \tag{4.29}$$

where $k \geq 0$: the δ_2 variation of the functional (4.13) is costant along $[s(k), r(k), k]$. The symmetry $r, s \to -r, -s$ reflects a change of σ representation. As we will see after not only the covariant anomaly but also the functional itself is costant along $s(k), r(k)$. Put in another way

$$Det(\tilde{\not{D}}(r(k))\not{D}(s(k))) = [Det(\not{D}^\dagger\not{D})]^{2-k} \tag{4.30}$$

We have found a continuous deformation of the modulus of Weyl determinant, associated to a continuous family of operators, representing continuous powers of the modulus itself (the determinants are normalized to the free laplacian one). The same feature will appear also for the consistent case. By the way we note that the limit $k = 2$ is not singular: with our normalization we find:

$$Det(\tilde{\not{D}}(\pm\frac{\sqrt{3}}{3})\not{D}(\mp\frac{\sqrt{3}}{3}) = 1$$

4.2 Consistent solution

The equation (4.2) gives the system

$$\frac{1}{4}xy = \pm(\frac{1}{3} - \frac{1}{4}x^2)$$
$$y^2 = \mp xy \qquad (4.31)$$

Unfortunately there is no meaningful solution: the only one is

$$y = 0\,;\; x^2 = \frac{3}{4}$$

reducing $t_1(\lambda)$, $t_2(\lambda)$ to coboundary term. We have to exploit the freedom of inserting a local term in the definition of $\Gamma[E; r, s]$ to recover the consistent solution: in so doing we introduce a new parameter, that will appear only in the intermediate calculation. We add to Γ

$$P(\alpha) = \frac{1}{4\pi(2-k)}\alpha \int d^2x\Omega_\mu\Omega^\mu \qquad (4.32)$$

with $\alpha \in R$ and we define

$$\Gamma' = \Gamma + P(\alpha)$$
$$\delta_1(\lambda)\Gamma' = \hat{t}_1(\lambda)$$
$$\delta_2(\lambda)\Gamma' = \hat{t}_2(\lambda)$$

It is easy to verify that $P(\alpha)$ changes (remembering that δ_2 acts like a conformal transformation):

$$\delta_1(\lambda)P(\alpha) = \frac{1}{4\pi(2-k)}\alpha \int \sqrt{g}\,2\,iD_\mu\Omega^\mu\tau \qquad (4.33)$$

$$\delta_2(\lambda)P(\alpha) = \frac{i}{4\pi(2-k)}\alpha \int (-\frac{1}{2}R)\,\tau \qquad (4.34)$$

The new system is

$$\frac{1}{4}xy = \pm(\frac{1}{3} - \frac{1}{4}x^2 + \frac{1}{2}\alpha)$$
$$y^2 + 2\alpha = \mp xy \qquad (4.35)$$

It holds the relation

$$\frac{1}{2}xy = \pm(\frac{1}{3} - \frac{1}{4}x^2 - \frac{1}{4}y^2) \qquad (4.36)$$

that will be fundamental and does not depend on α. Solving for x and y as function of α

$$y^2 = 3\alpha^2$$
$$x^2 = \frac{4}{3} + 3\alpha^2 + 4\alpha \qquad (4.37)$$

To fix α we use the general form of $\Gamma[E : r, s]$: the calculation is performed in isothermal coordinates: we are allowed to choose a particular coordinate system being our definition of determinant invariant for diffeomorphism. Locally any two dimensional riemannian manifold admits a coordinate system in which the metric tensor has the form [23]

$$g_{\mu\nu} = \exp(4G)\delta_{\mu\nu} \tag{4.38}$$

giving the zwein-bein

$$e_a^\mu = \exp(-2G)\delta_b^\mu(\delta_{ab}\cos 2F - \epsilon_{ab}\sin 2F) \tag{4.39}$$

where $F(x)$ describes the freedom of a local orhogonal rotation. We make the additional assumption to work on a manifold admitting a global system of this coordinate. In this system the spin-connection and the scalar curvature have the simple expression:

$$\Omega_\mu = \partial_\mu F + \epsilon_{\mu\nu}\partial_\nu G$$

$$R = -4\frac{1}{\sqrt{g}}\partial_\mu\partial_\mu G$$

Now the possibility to calculate exactly the determinant relies on the fact that one can write:

$$\begin{aligned} \tilde{\not{D}}(r)\not{D}(s) = \exp[-G(3-r) + iF(r-1)]i\partial_+ \exp[-G(2+r-s) \\ \exp[-iF(r+s)]i\partial_- \exp[G(1-s) - iF(1-s)] \end{aligned} \tag{4.40}$$

where $\partial_\pm = \partial_1 \pm \partial_2$

It is not difficult to find the infinitesimal variation of the determinant for the transformation

$$G \to G - \epsilon G$$

$$F \to F - \epsilon F$$

$\epsilon \to 0$ and to iterate this change leading G and F to zero:

$$Det(\tilde{\not{D}}(r)\not{D}(s)) = \exp(-\Gamma[r,s])det(-\partial^2) \tag{4.41}$$

This is the standard decoupling technique [6] that allows the calculation of two dimensional determinants: we note that the normalization to the free laplacian is a natural bonus of the procedure. Some care is needed on the iteration for the presence of a conformal factor in the measure: anyway we give the result deferring the details in the appendix. In isothermal coordinates:

$$\Gamma'[r,s;\alpha] = \frac{1}{8\pi(2-k)}\int d^2x\,(\frac{1}{3} - \frac{1}{4}x^2)4G\partial_\mu\partial_\mu G + i\,\frac{1}{2}xy\,4F\partial_\mu\partial_\mu G +$$

$$+\frac{1}{8\pi(2-k)}\int d^2x\,4F\partial_\mu\partial_\mu F - \frac{1}{8\pi(2-k)}\alpha\int d^2x[4F\partial_\mu\partial_\mu F + 4G\partial_\mu\partial_\mu G] \tag{4.42}$$

We recognize the contribution of $P(\alpha)$ and $\Gamma[r,s]$. The real part of this action is not gauge invariant, but we can reduce it to a gauge invariant form adding a suitable local term, obtaining:

$$\tilde{\Gamma}[r,s;\alpha] = \frac{1}{8\pi(2-k)}\int d^2x\,(\frac{1}{3}-\frac{1}{4}x^2-\frac{1}{4}y^2)\,4G\partial_\mu\partial_\mu G + i\,(\frac{1}{2}xy)\,4F\partial_\mu\partial_\mu G \tag{4.43}$$

$$\tilde{\Gamma} = \Gamma + \frac{1}{8\pi(2-k)}\int d^2x\,(\frac{1}{4}y^2)\,(4F\partial_\mu\partial_\mu F + 4G\partial_\mu\partial_\mu G) \tag{4.44}$$

In fact it is a general property that the modulus of the Weyl determinant can be always written as a gauge invariant quantity [19]. Now we can compare the real part of $\tilde{\Gamma}$ with:

$$\frac{1}{2}\ln Det(\not{D}^\dagger\not{D}) = \Gamma[r,s]_{r=s=k=0} \tag{4.45}$$

In other words we require that the definition

$$det(\not{D}) = \exp(-\tilde{\Gamma}[r,s])$$

is consistent with the well known relation

$$|Det(\not{D})| = \sqrt{Det(\not{D}^\dagger\not{D})}$$

Along the solution of the system (4.35)

$$\tilde{\Gamma} = \frac{1}{8\pi(2-k)}(\frac{1}{2}xy)\int d^2x[4G\partial_\mu\partial_\mu G \pm i\,4G\partial_\mu\partial_\mu F] \tag{4.46}$$

fixing the relative value between the real and the imaginary part of $\tilde{\Gamma}$ for any α: α appears only as a device to find the correct normalization. Expressing $\frac{1}{2}xy$ as a function of α and requiring

$$Re\,\tilde{\Gamma} = \Gamma[r,s]|_{r=s=k=0} \tag{4.47}$$

we obtain the equation:

$$\frac{1}{2(2-k)}xy = \frac{1}{2-k}(-\alpha-\frac{3}{2}\alpha^2) = \frac{1}{6} \tag{4.48}$$

solved by

$$\alpha = -\frac{1}{3}\pm\frac{1}{3}\sqrt{k-1}\ ;\ k\geq 1$$

Then using the definition of G and F and the coordinate invariance of the determinant we can write $\tilde{\Gamma}$ in a manifestly invariant form

$$\tilde{\Gamma} = \frac{1}{192\pi}\int d^2x\sqrt{g(x)}\int d^2y\sqrt{g(y)}R(x)\Delta_g^{-1}(x,y)R(y)\pm iR(x)\Delta_g^{-1}(x,y)\frac{1}{\sqrt{g}}\partial_\nu(\sqrt{g}\Omega^\nu(y)) \tag{4.49}$$

with $\Delta_g^{-1}(x,y)$ the kernel of the inverse of Beltrami-Laplace operator:

$$\Delta_g = \frac{1}{\sqrt{g}}\partial_\mu(g^{\mu\nu}\sqrt{g})\partial_\nu \tag{4.50}$$

$\tilde{\Gamma}$ is connected to Γ, obtained by a purely ζ-function definition, with a local term

$$\beta(k)\int d^2x\sqrt{g}\,\Omega_\mu\Omega^\mu = y^2\frac{1}{8\pi(2-k)}\int d^2x\sqrt{g}\,\Omega_\mu\Omega^\mu \tag{4.51}$$

so $\tilde{\Gamma}$ is a sort of minimal form of the Weyl effective action, in which the real part is reduced to the gauge invariant form. By the way $\tilde{\Gamma}$ coincides with the result of [8], the sign of the imaginary part being related to the chirality. Coming back to $\Gamma[r,s]$, expressing through (4.37) and (4.48) r and s as function of k, we find a family of $\Gamma[E,k]$, generated by the operators $\tilde{\not{D}}(r(k))\not{D}(s(k))$, that differ for the allowed local term and define the Weyl determinant.

$$\Gamma[E;r(k),s(k)] = \tilde{\Gamma} + \beta(k)\int d^2x\sqrt{g}\Omega^\mu\Omega_\mu \tag{4.52}$$

$\beta(k) = \frac{1}{24\pi}\frac{k-2\sqrt{k-1}}{k-2}$; $k \geq 1$. We have expressed y^2 through k: in order to have a well defined Γ for any $k \geq 0$ we have choosen

$$\alpha = -\frac{1}{3} + \frac{1}{3}\sqrt{k-1}$$

disregarding the other solution. The coefficient of the local term has a contiuos behaviour showing that, in the limit $k \to 2$, Γ coincides with $\tilde{\Gamma}$. Solving for $r(k)$ and $s(k)$ we get:

$$\begin{aligned} r(k) &\doteq -\frac{\sqrt{3}}{3}\ ;\ s(k) = \frac{\sqrt{3}}{3}\sqrt{k-1} \\ r(k) &= -\frac{\sqrt{3}}{3}\sqrt{k-1}\ ;\ s(k) = \frac{\sqrt{3}}{3} \end{aligned} \tag{4.53}$$

showing a sort of symmetry between the inequivalent representation of the Weyl algebra. Again we find a family of operators whose determinants give powers of the Weyl determinant, up to local term.

$$Det(\tilde{\not{D}}(r(k))\not{D}(s(k))) = (Det\not{D})^{k-2}\exp -\gamma(k)\int d^2x\sqrt{g}\,\Omega^\mu\Omega_\mu \tag{4.54}$$

with $\gamma(k) = \frac{1}{24\pi}(k - 2\sqrt{k-1})$
In particular one can choose $k = 1$ so $s = 0$ obtaing the original definition

$$\begin{aligned} Det\not{D} &= Det(T\not{D}) \\ T &= \tilde{\not{D}}(-\frac{\sqrt{3}}{3}) \end{aligned} \tag{4.55}$$

We note that (everything is normalized to the free laplacian) for $k = 2$

$$Det(\tilde{\not{D}}(r(2))\not{D}(s(2))) = 1 \tag{4.56}$$

It seems that for the critical values $r = \mp\frac{\sqrt{3}}{3}$; $s = \pm\frac{\sqrt{3}}{3}$ the dependence of the determinant from the external field desappear. At the moment we do not know if ther is some deep reason for this behaviour.
In this formalism diffeomorphism invariance is manifest; we do not need any counterterm to recover the general covariance. Anyway we can add a Wess-Zumino term to Γ, utilizing the zwein-bein field $E_{\mu a}$ in order to cancel the Lorentz anomaly, but generating a coordinate anomaly: we do not know if is possible to implement it in our operatorial approach. Another direction of work is to extend the computation to non parallelizable manifolds.
At the end we want to show how is possible to recover the determinant in a less rigorous but more direct way, utilizing the formal limit on the complexified connection described in chapter two. In the gauge case, if we start with the complexified action, the Weyl determinant is obtained by the calculation of its modulus putting formally $\hat{A}^{\dagger}_{\mu}$=0

$$Det(\tilde{\not{D}}^{\dagger}_{c}\not{D}_{c}) \to Det(\tilde{\not{\partial}}\not{D})$$

In the gravitational case the same trick does not look to produce the same effect: the presence of a n-bein field does not allow to perform a limit on the spin-connection without touch the n-bein itself, unless to break their canonical relation. To put $\hat{\Omega}^{\dagger}_{\mu ab}$=0 seems to force $\hat{E}^{*}_{\mu a}$=$cost$ while $\hat{E}_{\mu a} \neq 0$: the general covariance is lost. But in d=2 the operation can be made in a particular way to preserve the geometry of the theory. If we complexify the zwei-bein with a non-compact $SO(2C)$ rotation

$$\hat{\Lambda} = \begin{pmatrix} \cosh\phi & -i\sinh\phi \\ i\sinh\phi & \cosh\phi \end{pmatrix} \tag{4.57}$$

the relevant spin-connections are

$$\hat{\Omega}_{\mu} = \Omega_{\mu} + 2i\partial_{\mu}\phi$$

$$\hat{\Omega}^{\dagger}_{\mu} = \Omega_{\mu} - 2i\partial_{\mu}\phi$$

giving

$$\partial_{\mu}\phi = \frac{1}{4i}(\hat{\Omega}_{\mu} - \hat{\Omega}^{\dagger}_{\mu}) = I_{\mu} \tag{4.58}$$

As we have seen in the Weyl operator

$$\hat{E}_{\mu a} = \exp(\phi)E_{\mu a}$$

$$\hat{\Omega}_{\mu} = \Omega_{\mu} + \frac{1}{\sqrt{g}}2\varepsilon^{\nu}_{\mu}\partial_{\nu}\phi$$

so $\hat{E}_{\mu a}$ and $\hat{\Omega}_\mu$ are the conformal transformed objects: here ϕ appars as a "dilaton" field. In this way $\not{D}_c^\dagger \not{D}$ becomes the usual $\not{D}^\dagger \not{D}$ where the geometric fields are the conformal transformed of the original ones. Using the explicit result:

$$\frac{1}{2}\ln Det \not{D}'^{\dagger}\not{D}' = \frac{1}{192\pi}\int d^2x\sqrt{g'(x)}\int d^2y\sqrt{g'(y)}R'(x)\Delta_{g'}^{-1}(x,y)R'(y) \qquad (4.59)$$

with

$$g'_{\mu\nu} = \exp(2\phi)g_{\mu\nu}$$
$$R' = \exp(-2\phi)[R + \Delta_g 2\phi]$$

and $\Delta_{g'}$=Δ_g we get

$$\frac{1}{2}\ln Det \not{D}'^{\dagger}\not{D}' = \int d^2x\sqrt{g(x)}\int d^2y\sqrt{g(y)}[R(x)+2\Delta_g\phi]\Delta_g^{-1}(x,y)[R(y)+2\Delta_g\phi] =$$

$$= \int d^2x\sqrt{g(x)}\int d^2y\sqrt{g(y)}R(x)\Delta_g^{-1}(x,y)R(y) + \frac{1}{48\pi}\int d^2x\sqrt{g}\,[R\phi + \phi\Delta_g\phi] \quad (4.60)$$

At this point we come back to $\hat{\Omega}_\mu$ and $\hat{\Omega}_\mu^\dagger$: from (4.58)

$$\Delta_g\phi = -\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}I^\mu \qquad (4.61)$$

giving

$$\frac{1}{2}\ln Det \not{D}_c^\dagger \not{D}_c = \frac{1}{48\pi}\int d^2x\sqrt{g}\;, I_\mu I^\mu +$$

$$+\frac{1}{192\pi}\int d^2x\sqrt{g(x)}\int d^2y\sqrt{g(y)}R(x)\Delta_g^{-1}(x,y)R(y) - 4iR(x)\Delta_g^{-1}(x,y)\frac{1}{\sqrt{g}}\partial_\nu(\sqrt{g}I^\nu(y)) \qquad (4.62)$$

Now all the dependence on the parameter ϕ of the complex transformation is carried by the connections. It easy to check that putting $\hat{\Omega}_\mu$=0 or $\hat{\Omega}_\mu^\dagger$=0 one recovers the correct determinant (with a different sign on the imaginary part). Really it remains a local term

$$\frac{1}{768\pi}\int d^2x\sqrt{g}\,\Omega_\mu\Omega^\mu \qquad (4.63)$$

that, as usual, can be removed. At the moment we do not know if it gives the correct result in more than two dimensions: out of d=2 it is not possible to represent complex rotation as contormal transformations, complicating a lot the the calculation, but in principle, one can work with the Seeley-de Witt coefficients

4.3 Conclusions

In the case of gauge theory we have shown as to describe consistent and covariant anomalies in an unified scheme: the problem of representing this solutions in a functional approach has been completly understood, and it is not difficult to extend the results to non trivial fiber bundle. The essential role of the complexification of the gauge group appear in both the approach. The gravitational case is more involved, for the presence of two symmetries and a more sophisticated geometrical background. Anyway simplifying the problem to the pure Lorentz anomaly we have studied the covariant sector and we have shown as it is related to the complex $SO(2n, C)$ extension of $SO(2n, R)$. Unfortunately we have not been able to find a functional representation for the consistent vacuum functional in $d = 2n$. Nevertheless the semplicity of $d = 2$ has allowed us to calculate exactly the determinant, showing that a continuous family of operators, admitting as determinant the Weil one, do exist. In some sense they correspond to different regularizations of the theory, changing only for local terms in their effective action: our result agrees with the Leutwyler calculation [9].

Appendix A

Heat-kernel expansion

Let us consider a second order differential operator

$$A = -g^{\mu\nu}\partial_\mu\partial_\nu + f^\mu\partial_\mu + h \tag{A.1}$$

The fields f and h are allowed to be matrix-valued: we assume that $g^{\mu\nu}$ is positive and A acts on a compact manifold: A can be rewritten in the form

$$A = -\frac{1}{\sqrt{g}}(\partial_\mu + \tilde{v}_\mu)\sqrt{g}\, g^{\mu\nu}(\partial_\nu + v_\nu) + V \tag{A.2}$$

The heat-kernel of A is a solution of:

$$\frac{\partial}{\partial t}K(x,y;t) + A_x K(x,y;t) = 0 \tag{A.3}$$

with the boundary condition:

$$K(x,y;0) = \delta^d(x-y)\frac{1}{\sqrt{g}} \tag{A.4}$$

We remark that $K(x,y;t)$ also exists for non definite positive operators [17]: the important point is the positivity of the metric itself, that controls the asymptotic behaviour of the eigenvalues of A. We denote the lenght of the shortest path from x to y by $\sqrt{\eta(x,y)}$ (geodesic distance): for $t \to 0$ K admits the expansion [8]:

$$K(x,y;t) \to (4\pi)^{-\frac{d}{2}} \exp -\frac{\eta(x,y)}{4t} \sum_{n=0}^{\infty} t^n H_n(x,y) \tag{A.5}$$

Using the property:

$$g^{\mu\nu}\partial_\mu\eta\partial_\nu\eta = 4\eta \tag{A.6}$$

one can verify that the heat-kernel equation is satisfied order by order in t, provided that the coefficients obey to a recursive differential equation:

$$\frac{1}{2}\partial^\mu\eta D_\mu H_n + (n + \frac{1}{4}D^\mu D_\mu\eta - \frac{d}{2})H_n = -AH_{n-1} \tag{A.7}$$

where D_μ is formed with the field v_μ and the Levi-Civita connection $\Gamma^\lambda_{\mu\nu}$.
This differential equation implies that, at $x = y$, H_n and their derivatives reduce to local polynomial. Using the property:

$$(D_\mu D_\nu - D_\nu D_\mu)C_\alpha = R^\lambda_{\alpha\mu\nu} C_\lambda + f_{\mu\nu} C_\alpha \tag{A.8}$$

it is possible to express all the quantities in terms of the curvatures $R^\alpha_{\beta\mu\nu}$ and $f_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + [v_\mu, v_\nu]$ and the derivatives of η. It is a matter of patience to obtain the heat-kernel coefficients H_n: as an example we report the computation of H_1 for the operator

$$A = \tilde{\not{D}}(r)\not{D}(s) = \tilde{\sigma}_a e^\mu_a [i\partial_\mu + (1-r)\Omega_\mu]\, \sigma_a e^\nu_a [i\partial_\nu - (1-s)\Omega_\nu]$$

that is the relevant one in the case of $d = 2$ gravitational determinant ($\Omega_\mu = \frac{1}{4i}\Omega_{\mu ab}\tilde{\sigma}_a\sigma_b$).
Using the recursive differential equation at $x = y$ it is easy to prove:

$$H_1 = \frac{1}{6}R - V \tag{A.9}$$

Putting $\tilde{\not{D}}(r)\not{D}(s)$ in the "canonical" form:

$$v_\mu = \frac{1}{2}[(1-r)\Omega_\mu + (1-s)\Omega_\mu + 2\Omega_\mu] + \frac{1}{2}\frac{i}{\sqrt{g}}\epsilon^{\lambda\nu} g_{\mu\nu}[(r-1)\Omega_\lambda + 2\Omega_\lambda + (s-1)\Omega_\lambda]$$

$$\tilde{v}_\mu = -v_\mu \tag{A.10}$$

and

$$V = \frac{1}{8}(2 - s + r)R + \frac{i}{2}(r+s)D_\mu\Omega^\mu \tag{A.11}$$

So

$$H_1(\tilde{\not{D}}(r)\not{D}(s)) = \frac{1}{6}R - \frac{1}{8}(2 + r - s)R + \frac{i}{2}(r+s)D_\mu\Omega^\mu \tag{A.12}$$

To get $\tilde{H}_1$ we have to send $r \to s$ and $\Omega_\mu \to -\Omega_\mu$ All the heat-kernel coefficients used in the various chapters were obtained in an analogous way.

Appendix B

The variation of the determinant

In term of the heat-kernel $K(x,y;t)$ we can express the ζ- function connected to the operator A:

$$\zeta(s;A) = Tr[A^{-s}] = \frac{1}{\Gamma(s)} \int_M dx \int_0^\infty dt\, t^{s-1} tr[K(x,x;t)] \tag{B.1}$$

standing the relation between kernel:

$$< x|A^{-s}|y >= \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} tr[K(x,y;t)] \tag{B.2}$$

If we perturbe the operator A with ϵA_1, $\epsilon \to 0$, where A_1 is a differential operator ($ordA \geq ordA_1$), we obtain [6]:

$$\zeta(s;A+\epsilon A_1) = \zeta(s) + \epsilon f(s) + O(\epsilon^2)$$
$$f(s) = sTr[A^{-s-1}A_1] \tag{B.3}$$

From the definition of determinant [6]:

$$-\ln DetA = \frac{d}{ds}\zeta(s)|_{s=0} \tag{B.4}$$

one recovers:

$$-\ln Det(A+\epsilon A_1) = -\ln DetA + \epsilon \frac{d}{ds}(sTr[A^{-s-1}A_1])_{s=0} \tag{B.5}$$

Using the relation (B.2) and the expansion (A.7), together with the fact we are working with trace-class operators, one could in principle to compute every variations. The fact that in the most of our computations only one heat-kernel coefficient is needed derives from the particular form of the variations:

$$A_1 = \alpha(x)A + A\beta(x)$$

In this case is rather clear that only a coefficient makes contribution in the limit $s = 0$.

Appendix C

Decoupling technique

We want to show how the decoupling technique works in the calculation of (4.41). The relevant finite transformation is:

$$\begin{aligned} G &\to G(1-y) = G_y \\ F &\to F(1-y) = F_y \end{aligned} \tag{C.1}$$

Let us define the quantity:

$$\Gamma'[y;r,s] = \lim_{\delta y \to 0} \frac{1}{\delta y}(\Gamma[y-\delta y;r,s] - \Gamma[y;r,s]) \tag{C.2}$$

where

$$\Gamma[\xi;r,s] = -\frac{1}{2}\frac{d}{dt}Tr[(\tilde{\not{D}}(r;\xi)\not{D}(s;\xi))^{-t}]_{t=0} \tag{C.3}$$

$\tilde{\not{D}}(r;\xi)$ and $\not{D}(s;\xi)$ are obtained by the usual $\tilde{\not{D}}(r)$ and $\not{D}(s)$ with the substitution

$$G \to (1-\xi)G$$

$$F \to (1-\xi)F$$

It is clear that

$$\begin{aligned} \Gamma[1;r,s] &= \Gamma[r,s] \\ \Gamma[0;r,s] &= -\ln Det\partial^2 \end{aligned} \tag{C.4}$$

giving:

$$\int_0^1 dt\, \Gamma'[y;r,s] = \Gamma[r,s] - \ln Det\partial^2 \tag{C.5}$$

Using the expression of $\tilde{\not{D}}(r)$ and $\not{D}(s)$ we obtain in conformal coordinates

$$\Gamma'[y;r,s] = \frac{1}{2}Tr\{(\tilde{\not{D}}(r;y)\not{D}(s;y))^{-t}[(2+s-r)G - i(r+s)F]+$$

$$+ (\not{D}(s;y)\tilde{\not{D}}(r;y))^{-t}[(2+r-s)G + i(r+s)F]\}_{t=0} \qquad (C.6)$$

From the heat-kernel expansion:

$$\Gamma'[y;r,s] = \frac{1}{8\pi}\int \sqrt{g(y)}(H_1(\tilde{\not{D}}(r;y)\not{D}(s;y))[(2+r-s)G - i(r+s)F]+$$

$$+ (\tilde{H}_1(\not{D}(s;y)\tilde{\not{D}}(r;y))[(2-r+s)G + i(r+s)F]) \qquad (C.7)$$

with $\sqrt{g(y)} = \exp(4G(1-y))$.

The explicit calculation of H_1 and $\tilde{H}_1$ in isothermal coordinates gives

$$H_1 = \exp(-4(1-y)G)[(-4(1-y)\partial_\mu\partial_\mu G)[-\frac{1}{12} - \frac{1}{8}(r-s)] - \frac{i}{2}(1-y)\partial_\mu\partial_\mu F(r+s)]$$

$$\tilde{H}_1 = \exp(-4(1-y)G)[(-4(1-y)\partial_\mu\partial_\mu G)[-\frac{1}{12} - \frac{1}{8}(s-r)] + \frac{i}{2}(1-y)\partial_\mu\partial_\mu F(r+s)]$$

A straightforward algebra gives:

$$\Gamma'[y;r,s] = (1-y)\frac{1}{8\pi}\int d^2x[\frac{1}{3}(4G\partial_\mu\partial_\mu G) - \frac{1}{4}(r-s)^2(4G\partial_\mu\partial_\mu G)-$$

$$-\frac{i}{2}(r^2-s^2)(4F\partial_\mu\partial_\mu G) + (2+r-s)^2(F\partial_\mu\partial_\mu F)] \qquad (C.8)$$

and

$$\int_0^1 dy\Gamma'[y;r,s] = \frac{1}{16\pi}\int d^2x(4G\partial_\mu\partial_\mu G)(\frac{1}{3} - \frac{1}{4}(r-s)^2)-$$

$$-\frac{i}{2}(r^2-s^2)(4F\partial_\mu\partial_\mu G) + (2+r-s)^2(4F\partial_\mu\partial_\mu F) \qquad (C.9)$$

.

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