Renormalization aspects of the non linear sigma model

Thesis submitted for the degree

of Magister Philosophiae

Candidate

Silvio Paolo Sorella

Supervisor

Prof. Carlo Becchi.

I am grateful to Prof. Carlo Becchi for suggesting me the subject ,for interesting discussion and the help in reading the manuscript.

Contents

Introduction.

Non linear sigma model were introduced more than 15 years ago to describe the properties of systems with spontaneously broken symmetry according to the Goldstone mechanism .

This led to the study of models based on coset spaces ,i.e. the quotient G/H of a Lie group G by a subgroup H .

An important example of this kind is the Heisemberg model ,described by a non linear two dimensional sigma model with $G = O(n)$ and $H = O(n-1)$.

In this case the model is renormalizable and asymptotically free .

In general the asymptotic freedom is a characteristic of the spaces with negative curvature .

This property allows to distinguish the ultraviolet regime from the infrared one ,making possible a separate analysis of the ultraviolet region.

The recent developments in string theories ,have led to the study of two dimensional non linear sigma models defined on a generic Riemannian manifold .

From the point of view of perturbation theory ,space-time dimension two is particularly relevant ,since the model is power counting renormalizable .

As it is well known ,this means that ,developing the lagrangian density in powers of the field ,one does not find any coefficient with negative mass dimension .

The main aim of this work is the study of the ultraviolet behaviour of a two dimensional sigma model defined on a Riemannian manifold M .

This involves the analysis of the quantum fluctuactions in a neighbourhood of a generic assigned point in M [1] .

Let us consider a chart in M centered in a ,space-time independent ,assigned point m ,and let $\phi^i(x)$ be the fields of the model in local coordinates ,where x is a Minkowski space-time point ,or ,by making a Wick rotation ,a point of the euclidean two dimensional space .

We will assume that the quantum fluctuations at the point x are damped when the point goes to infinity ;i .e . :

$$\lim_{|x| \to \infty} \phi(x) = m \quad ,$$

and that they never exceed the border of the chart centered in m [3] .

If one neglects the infrared effects there are no particular obstructions to this hypothesis .

In the ultraviolet region ,indeed ,only the small distance effects enter in the game ; this involves only the local properties of M .

On the contrary ,the infrared behaviour of the model would require global informations .

This is due to the fact that infrared fluctuations ,being long distance effects ,would exceed the border of the considered chart involving the whole manifold .

The infrared problem would require a detailed treatment and would exceed from the main aim of this work .

Chapter one.

The model and its quantization .

Let M_2 be the two dimensional space-time and M a differentiable connected n dimensional Riemannian manifold with metric g .

The field of the sigma model is a map

$$\phi : M_2 \to M$$

represented in local coordinates by $\phi^i(x)$.

The action of the model is :

$$I = \frac{1}{2} \int d^2x \; g_{ij}(\phi)(\partial_\mu \phi^i)(\partial^\mu \phi^j) \tag{1.1}$$

where the fields ϕ^i are dimensionless.

The ultraviolet behaviour is governed by the small field fluctuations;this involves only the local properties of M .

Let us consider a point on M and let us study the quantum ultraviolet fluctuations on a neighbourhood .

In this frame ,the fields ϕ^i split as :

$$\phi^i(x) = \varphi^i(x) + \pi^i(x) \tag{1.2}$$

where the φ^i are the coordinates of the reference point on M ,and π^i are the corresponding quantum fluctuations.

In this case the coordinates of the background point φ are space-time dependent .

We will assume that the φ space-time dependence is weak

;i.e. ,that for each point x in M_2 the corresponding point φ in M varies smoothly in a small neighbourhood of m .

The condition

$$\lim_{x \to \infty} \phi(x) = m \tag{1.3}$$

implies that the quantum fluctuations are zero at infinity

$$\lim_{x \to \infty} \pi(x) = 0 \tag{1.4}$$

and that :

$$\lim_{x \to \infty} \varphi(x) = m \tag{1.5}$$

Formally the quantization of the model is based on the measure [2,3] :

$$\prod_i \prod_x d_g \pi^i(x)\, e^{\frac{i}{\hbar} I(\phi)} \tag{1.6}$$

where $d_g \pi^i(x)$ is the volume element on M :

$$d_g \pi^i(x) = \sqrt{\det g(\phi)}\; \mathcal{D}\pi^i(x) \tag{1.7}$$

It is convenient ,in order to have a manifestly covariant formalism ,to use the geodesic normal coordinates [2] and to express the π^i field in terms of the vector ξ^i tangent to the geodesic through the points $\{\varphi^i\}$ and $\{\varphi^i + \pi^i\}$ in the point $\{\varphi^i\}$;let us furthemore assume that ξ^i is the real quantum field.

One gets :

$$\pi^i = \xi^i + \chi^i(\varphi, \xi) = \xi^i - \sum_{n=2}^{\infty} \frac{1}{n!}\, \xi^{j_1} \cdots \xi^{j_n}\, \Gamma^i_{j_1 \cdots j_n} \tag{1.8}$$

with

$$\Gamma^{i}_{j_1 \cdots j_n} = \left(\tilde{\nabla}_{(j_1} \cdots \tilde{\nabla}_{j_{n-2}} \Gamma^{i}_{j_{n-1}, j_n)} \right)(\varphi) \qquad (1.9)$$

where $\tilde{\nabla}$ indicates that the covariant derivatives is taken only with respect to the lower indices of the Riemann connection Γ^{i}_{jk} .

For the total field ϕ^i one has :

$$\phi^i = \varphi^i + \pi^i(\xi) = \varphi^i + \xi^i + \chi^i(\varphi, \xi) \qquad (1.10)$$

We have to note that the splitting of the total field ϕ^i into a classical and quantum part is not linear ; this does not guarantee the correctness of the usual background method for the ultraviolet divergences analysis. With the introduction of the field ξ^i the quantum measure becomes :

$$\prod_i \prod_x \mathcal{D}\xi^i \sqrt{\det g(\phi)} \left(\det j(\varphi, \xi)\right) e^{\frac{i}{\hbar} I(\phi)} \qquad (1.11)$$

where $(\det j)$ is the Jacobian of the transformation $\pi^i \rightarrow \xi^i$:

$$\det j(\varphi, \xi) = \det \left(\frac{d\pi^i}{d\xi^j} \right) \qquad (1.12)$$

By taking the Fourier transform of the quantum measure one introduces the functional generator $Z(\varphi, J)$:

$$Z(\varphi, J) = e^{\frac{i}{\hbar} W(\varphi, J)} = \int \tilde{\mathcal{D}}\xi \, e^{\frac{i}{\hbar}\left(I(\phi) + \int d^2x \, J_i \xi^i \right)} \qquad (1.13)$$

where

$$\tilde{\mathcal{D}}\xi = \prod_i \prod_x \mathcal{D}\xi^i \sqrt{\det g}\ (\det j) \qquad (1.14)$$

Before to go beyond ,it is necessary to spend some words on the quantum measure ,and particularly on the quantity :

$$\prod_x \sqrt{\det g}\quad (\det j) \qquad (1.15)$$

If one writes :

$$\prod_x \sqrt{\det g}\ (\det j) = e^{\delta^2(0)\int d^2x\ \lg(\sqrt{\det g}\ \det j)} \qquad (1.16)$$

one sees that the effect of the volume element is equivalent to a further interaction term in the initial lagrangian[1].

In general one can see that ,by formulating the model on a lattice ,such interaction term behaves like a counterterm for quadratic divergences coming from the initial lagrangian[5].

Then the simplest thing to do is to define the model by using dimensional regularization ,in which the quadratic divergences are suppressed .

If one uses dimensional regularization :

$$\delta^2(0) \rightarrow \delta^n(0) = 0$$

and one can ignore the effect due to the volume element .

Making reference to the dimensional regularization for the functional $W(\varphi, \mathcal{J})$ one gets :

$$e^{\frac{i}{\hbar} W(\varphi, J)} = \int \prod_{i,x} \mathcal{D} \xi^i \, e^{\frac{i}{\hbar} \left(I(\phi) + \int d^n x \, J_i \, \xi^i \right)} \qquad (1.17)$$

with

$$d^n x = \mu^{\varepsilon} \, d^{2+\varepsilon} x \qquad (1.18)$$

The functional $W(\varphi, J)$ has only a local meaning : it is defined in a neighbourhood of the background point φ .
We will see later a possible way in order to define a more general functional W ; this will require the introduction of an infinite set of sources coupled to all powers of the quantum field ξ^i .

Chapter two .

B.R.S. symmetry

The splitting $\phi^i = \varphi^i + \pi^i$,with $\pi^i = \xi^i + \chi^i$ induces a symmetry for the action $I(\phi)$.

This symmetry derives from the possibility of varying ,in a neighbourhood of the point φ ,the φ^i and the quantum fields ξ^i in such a way that the total variation of ϕ^i is zero[4] .

Let us consider ,indeed ,the transformation :

$$\begin{cases} \delta_\eta \varphi^i(x) = \eta^i(x) \\ \delta_\eta \xi^i(x) = F^i{}_j(\varphi,\xi)\,\eta^j(x) \end{cases} \tag{2.1}$$

where $F^i{}_j$ is determined by the requirement for ϕ^i to be invariant ,i.e. :

$$\delta_\eta \phi^i = \frac{\partial \phi^i}{\partial \varphi^j}\eta^j + \frac{\partial \phi^i}{\partial \xi^k} F^k{}_m \eta^m = 0 \tag{2.2}$$

From $(\delta_\lambda \delta_\eta - \delta_\eta \delta_\lambda)\phi^i = 0$,one gets the condition :

$$\left(\frac{\partial F^i{}_k}{\partial \varphi^j} - \frac{\partial F^i{}_j}{\partial \varphi^k}\right) + \left(\frac{\partial F^i{}_k}{\partial \xi^\ell} F^\ell{}_j - \frac{\partial F^i{}_j}{\partial \xi^\ell} F^\ell{}_k\right) = 0 \tag{2.3}$$

By $\delta_\eta \phi^i = 0$ it follows that :

$$\delta_\eta I(\phi) = 0 \tag{2.4}$$

One introduces a ghost field $C^i(x)$ and an operator s defined by:

$$\left\{\begin{array}{l} s\varphi^i = c^i \\ s\xi^i = F^i_j c^j \\ s c^i = 0 \end{array}\right. \tag{2.5}$$

The s operator can be written as :

$$s = \int d^2x \left(c^i \frac{\delta}{\delta\varphi^i} + F^i_j c^j \frac{\delta}{\delta\xi^i} \right) \tag{2.6}$$

The c^i field is a dimensionless classical field .

It is easy to verify ,by using (2.3) ,that s is nilpotent :

$$s^2 = 0 \tag{2.7}$$

From $s\phi^i = 0$ it follows that $s\,I(\phi) = 0$.

In order to get the Ward identities associated to the B.R.S. symmetry (2.5) and to discuss them at the quantum level ,we need to add to the initial action the source terms coupled to the various operators introduced by the symmetry .

Bearing in mind that the sources must be coupled only to the operators containing quantum fields ,one has :

$$I(\phi) \rightarrow I(\phi) + \int d^2x\; L_i\,(s\xi^i) \tag{2.8}$$

However ,since $F^i_j c^j$ has power counting weight zero it will mix with all other possible dimension zero operators .

It is covenient to introduce the action :

$$\Sigma(\varphi, \xi, L, c) = I(\phi) + \int d^2x\; L_{\alpha i}\, N^i_\alpha \tag{2.9}$$

where

$$\begin{cases} N^i_\alpha = s\Lambda^i_\alpha & \alpha = 0, 1, \ldots, \infty \\ \Lambda^i_0 = \xi^i & L_{0i} = L_i \end{cases} \tag{2.10}$$

The set $\{\Lambda^i_\alpha\}$ are all the possible dimension zero functions of φ and ξ.

If one assigns ghost number $n_g(c) = 1$ to c, $n_g(\varphi) = n_g(\xi) = 0$ to φ and ξ, then L's have dimension two and ghost number $n_g(L) = -1$.

The sources L are needed only to discuss the renormalizability and will be put equal to zero at the end.

From $s\phi^i = 0$ and $s^2 = 0$, it follows :

$$s\,\Sigma(\varphi, \xi, L, c) = 0 \tag{2.11}$$

Since $s\xi^i = \dfrac{\delta\Sigma}{\delta L_i}$, the equation $s\Sigma = 0$ reads :

$$\int d^2x \quad c^i \frac{\delta\Sigma}{\delta\varphi^i} + \frac{\delta\Sigma}{\delta L_i}\frac{\delta\Sigma}{\delta\xi^i} = 0 \tag{2.12}$$

One defines the operators :

$$\begin{cases} A = \displaystyle\int d^2x \quad c^i \frac{\delta}{\delta\varphi^i} \\ B_X = \displaystyle\int d^2x \quad \frac{\delta X}{\delta\xi^i}\frac{\delta}{\delta L_i} + (-)^x \frac{\delta X}{\delta L_i}\frac{\delta}{\delta\xi^i} \end{cases} \tag{2.13}$$

where x is the Grassman parity of the functional X.

In the Σ case, $n_g(\Sigma) = 0$:

$$B_\Sigma = \int d^2x \quad \frac{\delta\Sigma}{\delta\xi^i}\frac{\delta}{\delta L_i} + \frac{\delta\Sigma}{\delta L_i}\frac{\delta}{\delta\xi^i} \tag{2.14}$$

and

$$s\Sigma = A\Sigma + \frac{1}{2} B_\Sigma \Sigma = 0 \qquad (2.15)$$

The B_X operator has the following properties :

$$\begin{cases} B_X Y = (-)^{x+y+xy} B_Y X \\ B_X B_Y Z + (-)^{x+y+z(x+y)} B_Z B_X Y + (-)^{y+z+x(y+z)} B_Y B_Z X = 0 \end{cases} \qquad (2.16)$$

If ,in addition

$$A X + \frac{1}{2} B_X X = 0 \qquad (2.17)$$

then the operator D_X defined by

$$D_X = A + B_X \qquad (2.18)$$

is nilpotent ,i.e.

$$D_X D_X Y = 0 \qquad \forall \quad Y \qquad (2.19)$$

Clearly Σ satisfies (2.17) ,so D_Σ is nilpotent .

Let us consider ,then the functional $W(\varphi, J, L, c)$:

$$e^{\frac{i}{\hbar} W(\varphi, J, L, c)} = \int \prod_{i,x} D\xi^i \; e^{\frac{i}{\hbar}\left(\Sigma + \int d^n x \; J_i \xi^i \right)} \qquad (2.20)$$

Making a B.R.S. transformation one gets the Ward identity :

$$\int d^n x \left(c^i \frac{\delta W}{\delta \varphi^i} - \frac{\delta W}{\delta L_i} J_i \right) = 0 \tag{2.21}$$

Passing to the functional Γ :

$$\begin{cases} \Gamma(\varphi, \xi, L, c) = W(\varphi, J, L, c) - \int d^n x \, J_i \xi^i \\ \xi^i = \dfrac{\delta W}{\delta J_i} \end{cases} , \tag{2.22}$$

$$\int d^n x \left(c^i \frac{\delta \Gamma}{\delta \varphi^i} + \frac{\delta \Gamma}{\delta L_i} \frac{\delta \Gamma}{\delta \xi^i} \right) = 0$$

The Ward identity (2.22) allows a renormalization analysis of the model based on the B.R.S. symmetry (2.5) .
Since we have used an invariant regularization scheme for the B.R.S. symmetry ,namely the dimensional one which does not modify the structure of the manifold M ,the renormalization analysis becomes a stability problem for the classical action Σ (see appendix).

Chapter three .

Renormalization .

As we said ,the invariance of the regularization scheme reduces the renormalization analysis to a stability problem for the classical action Σ .

This means that if one perturbes the action Σ by a term $\varepsilon \Sigma'$,where Σ' is the integral of a local formal series in the fields ,their derivatives ,the external sources $L_{i\alpha}$ and the ghost field c^i ,with dimension two and ghost number zero ;and if one impose that :

$$s (\Sigma - \varepsilon \Sigma') = 0 \tag{3.1}$$

to the first order in ε ,then the quantity $(\Sigma - \varepsilon \Sigma')$ can be obtained by redefining the fields and the parameters of the action Σ .

To the first order in ε one has :

$$\mathcal{D}_\Sigma \Sigma' = 0 \tag{3.2}$$

$$\left\{ \begin{array}{l} \mathcal{D}_\Sigma = c^i \dfrac{\delta}{\delta \rho^i} + \dfrac{\delta \Sigma}{\delta \xi^i} \dfrac{\delta}{\delta L_i} + \dfrac{\delta \Sigma}{\delta L_i} \dfrac{\delta}{\delta \xi^i} \\[2ex] \Sigma = I(\phi) + L_{\alpha i} (s \Lambda^i_\alpha) \end{array} \right. \tag{3.3}$$

where we have omitted a space-time integration .

By power counting :

$$\Sigma' = A(\varphi, \xi) + L_{\alpha k} B^k_\alpha \tag{3.4}$$

where A has ghost number zero and canonical dimension two, B has ghost number $n_g(B) = 1$ and dimension zero .

Inserting (3.4) in (3.2) :

$$\begin{cases} s A + \dfrac{\delta I}{\delta \xi^i} B^i = 0 \\[2ex] \dfrac{\delta (s \xi^k)}{\delta \xi^i} B^i - s B^k = 0 \qquad B^k = B^k_0 \\[2ex] \dfrac{\delta (s \Lambda^k_\rho)}{\delta \xi^i} B^i - s B^k_\rho = 0 \qquad \rho = 1, \ldots, \wp \end{cases} \tag{3.5}$$

In order to find a solution of (3.5) it is convenient to build an operator $X = X(\varphi, \xi, L, c)$ such that :

$$D_\Sigma X = C(\varphi, \xi) + L_{\alpha k} B^k_\alpha \tag{3.6}$$

The X operator has ghost number $n_g(X) = -1$.

By power counting :

$$X = L_{\alpha k} \Omega^k_\alpha \tag{3.7}$$

where Ω^k_α are arbitrary functions of φ and ξ with ghost number $n_g(\Omega) = 0$.

Bearing in mind that the set $\{\Lambda^k_\alpha(\varphi, \xi)\}$ are all possible dimension zero functions of φ and ξ ,it will be possible to write :

$$\Omega^k_\alpha = z_{\alpha\beta} \Lambda^k_\beta \tag{3.8}$$

where $z_{\alpha\beta}$ are constant .

One gets :

$$X = L_{\alpha k}\, \mathcal{Z}_{\alpha\beta}\, \Lambda^{k}{}_{\beta} \tag{3.9}$$

Imposing that :

$$D_{\Sigma} X = C + L_{\alpha k}\, B^{k}_{\alpha} \tag{3.10}$$

one has :

$$\begin{cases} C = \dfrac{\delta I}{\delta \xi^{i}}\, \mathcal{Z}_{o\beta}\, \Lambda^{i}{}_{\beta} \\[2ex] B^{k}_{\alpha} = \dfrac{\delta (\sqrt{\ } \Lambda^{k}_{\alpha})}{\delta \xi^{i}}\, \mathcal{Z}_{o\beta}\, \Lambda^{i}{}_{\beta} - \mathcal{Z}_{\alpha\beta} (\sqrt{\ } \Lambda^{k}_{\beta}) \end{cases} \tag{3.11}$$

It is easy to verify that the B^{k}_{α} in (3.11) are solutions of :

$$\begin{cases} \dfrac{\delta (\sqrt{\ } \xi^{k})}{\delta \xi^{i}}\, B^{i} - \sqrt{\ } B^{k} = 0 \\[2ex] \dfrac{\delta (\sqrt{\ } \Lambda^{k}_{\rho})}{\delta \xi^{i}}\, B^{i} - \sqrt{\ } B^{k}_{\rho} = 0 \end{cases} \tag{3.12}$$

For A one gets :

$$\sqrt{\ } \left(A - \mathcal{Z}_{o\beta} \frac{\delta I}{\delta \xi^{i}} \Lambda^{i}{}_{\beta} \right) = 0 \tag{3.13}$$

This equation tells us that the quantity

$$\left(A - \mathcal{Z}_{o\beta} \frac{\delta I}{\delta \xi^{i}} \Lambda^{i}{}_{\beta} \right) \tag{3.14}$$

is a function of the total field ϕ ,i.e. :

$$A(\varphi,\xi) = \mathcal{G}(\phi) + z_{o\beta}\,\frac{\delta I}{\delta \xi^{i}}\,\Lambda^{i}{}_{\beta} \tag{3.15}$$

where $\mathcal{G}$ has canonical dimension two and ghost number zero .

Without loss of generality :

$$\mathcal{G}(\phi) = \frac{1}{2}\,T_{ij}(\phi)\,(\partial_{\mu}\phi^{i})(\partial^{\mu}\phi^{j}) \tag{3.16}$$

and :

$$\Sigma'(\varphi,\xi,L,c) = \mathcal{G}(\phi) + \mathcal{D}_{\Sigma}\,X \tag{3.17}$$

One can prove that (3.17) is the most general solution for (3.2) ; explicitly :

$$\begin{cases} \Sigma' = \dfrac{1}{2}\,T_{ij}(\phi)\,(\partial_{\mu}\phi^{i})(\partial^{\mu}\phi^{j}) - z_{\beta\alpha}\,L_{\beta k}(\partial\Lambda^{k}_{\alpha}) \\ \qquad + z_{o\beta}\,\dfrac{\delta\Sigma}{\delta\xi^{i}}\,\Lambda^{i}{}_{\beta} \end{cases} \tag{3.18}$$

The $\varepsilon\,\Sigma'$ is obtained from the classical action Σ by redefining the metric ,the fields and the sources in the following way :

$$\begin{cases} g^{o}_{ij}(\phi) = g^{o}_{ij}(\varphi,\xi) = g_{ij}(\phi) - \varepsilon\,T_{ij}(\phi) \\ L^{o}_{\alpha k} = L_{\alpha k} + \varepsilon\,z_{\beta\alpha}\,L_{\beta k} \\ \xi^{oi} = \xi^{i} - \varepsilon\,z_{o\beta}\,\dfrac{\delta\,\Sigma(\varphi,\xi,g,L,c)}{\delta\xi^{i}}\,\Lambda^{i}{}_{\beta} \end{cases} \tag{3.19}$$

This means that :

$$\begin{cases} \Sigma(\varphi, \xi, g, L, c) - \varepsilon\, \Sigma'(\varphi, \xi, g, L, c) \\ \quad = \Sigma(\varphi, \xi^{\circ}, g^{\circ}(\varphi, \xi^{\circ}), L^{\circ}, c) + O(\varepsilon^2) \end{cases} \tag{3.20}$$

One sees that the renormalization involves a redefinition of the metric tensor g ,a multiplicative renormalization of the infinite set of sources $\{ L_{\alpha i} \}$,and a non linear renormalization of the quantum field ξ (4) .

To be precise ,the model needs a mass term to regularize the infrared divergences .

In principle one could introduce a B.R.S. compatible infrared regularization by adding to the action $I(\phi)$ a potential :

$$\int d^2x \; m^2 \, V(\phi)$$

where $V(\phi)$ is a scalar dimensionless function of the total field ϕ .

Such regularization is B.R.S. invariant ,since :

$$s\, V(\phi) = 0 \tag{3.21}$$

The function $V(\phi)$ will have also to be renormalized ; nevertheless since the renormalization will be proportional to m^2 ,by using minimal subtraction ,it will be clearly distinguishable from the other ultraviolet divergences .

In pratical calculations (7) one follows a criterion of minimal disturbance ,i.e. ,one introduces as many infrared cutoff as necessary for an unambiguous determination of the ultraviolet divergent terms .

Typically one considers the term :

$$\frac{m^2}{2} g_{ij}(\varphi)\, \xi^i \xi^j \tag{3.22}$$

Such a term would produce a soft breaking of the Ward identities not ruining the ultraviolet analysis .

Once the correlation functions have been computed one should take the zero mass limit .

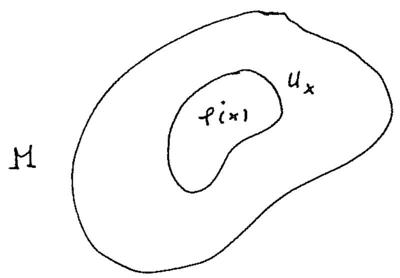
The existence of this limit is a delicate question ,it is sufficient to note that in the case of a coset space this limit exists only for a restricted class of correlation functions [5, 8] .

Chapter four .

The functional $\mathcal{W}(h)$ and its properties .

In this chapter we discusse the possibility of introducing a more general functional generator with global meaning .

The analysis of quantum fluctuations in a neighbourhood of the background point $\varphi(x)$ requires essentially the use of a single chart U_x ,sufficiently big in order to contain all the ultraviolet effects (fig 1.) .



(fig 1.)

In principle the point $\varphi(x)$ could be a generic point of U_x ;thus we have independently assigned the metric tensor $g_{ij}(\varphi,\xi)$ in every point φ (3) .

It is necessary to verify that the local assignement of the metric tensor corresponds to a unique tensorial field defined on the whole domain U_x .

We find in Friedan thesis (2) how this condition can be written in terms of a " non linear connection " Q .

Let us consider a point p in U_x and let v be a vector in the tangent space of p .

One introduces the compatibility operator :

$$\mathcal{D}_i = \frac{\partial}{\partial p^i} - Q^j_i (p, v) \frac{\partial}{\partial v^j} \tag{4.1}$$

The " non linear connection " Q^j_i is such that :

$$\mathcal{D}^2_{ij} = [\mathcal{D}_i , \mathcal{D}_j] = 0 \tag{4.2}$$

One can show that if :

$$\mathcal{D}_i f (p, v) = 0 \tag{4.3}$$

where $f (p, v)$ is a scalar function defined in a neighbourhood of p , then f is the local restriction of a unique global function .

It is easy to recognize in the B.R.S. operator

$$s = \int d^2x \; c^i \left(\frac{\delta}{\delta \varphi^i} + F^k_i (\varphi , \xi) \frac{\delta}{\delta \xi^k} \right) \tag{4.4}$$

the $\mathcal{D}_i$ operator introduced by Friedan .

The equation :

$$s \, \gamma (\varphi , \xi) = 0 \tag{4.5}$$

where γ is a generic function of φ and ξ ,tells us that γ has a global extension .

Let us consider now the functional $\mathcal{W} (h)^{(4)}$:

$$e^{\frac{i}{\hbar} W(h)} = \int \prod_{i,x} \mathcal{D}\xi^i \, e^{\frac{i}{\hbar}\left(I(\phi) + \int d^n x \, h(x;\phi) \right)} \tag{4.6}$$

where $h(x;\phi)$ is a two dimensional scalar quantity which , for every point x in M_2 , is a function on M .
The introduction of this unconventional source term is equivalent to a sum of an infinite number of terms containing all the powers of ϕ .
Using the splitting $\phi^i = \varphi^i + \xi^i + \chi^i(\varphi, \xi)$ one has :

$$\begin{cases} h(\phi) = h(\varphi) + \sum_{n=1}^{\infty}{}' \frac{1}{n!} J_{i_1 \ldots i_n}(\varphi) \, \xi^{i_1} \ldots \xi^{i_n} \\ J_{i_1 \ldots i_n}(\varphi) = \left(\nabla_{i_1} \ldots \nabla_{i_n} \right) h \Big|_{\phi = \varphi} \end{cases} \tag{4.7}$$

In practice one introduces an infinite set of sources coupled to all powers of the quantum field .
However the introduction of h gives a global meaning to the functional $W(h)$, indeed ,being h a function of the total field ϕ , it will be :

$$s \, h(\phi) = 0 \tag{4.8}$$

The functional $W(h)$ can be renormalized repeating the standard arguments of chapter three .
The renormalized $W^r(h)$ can be written :

$$e^{\frac{i}{\hbar} W^{\varepsilon}(h)} = \int \prod_{i,x} \mathcal{D}\xi^{i} \, e^{\frac{i}{\hbar}\left(I(\varphi, \xi^{\circ}, g^{\circ}(\varphi, \xi^{\circ})) + \int d^{2+\varepsilon}x \; h^{\circ}(\varphi, \xi^{\circ}) \right)} \tag{4.9}$$

with :

$$\begin{cases} g^{\circ}_{ij}(\phi) = g^{\circ}_{ij}(\varphi, \xi) = \mu^{\varepsilon}\left(g_{ij}(\phi) - T_{ij}(\phi) \right) \\ h^{\circ}(\phi) = \mu^{\varepsilon} h(\phi) \left(1 - f(\phi) \right) \\ \xi^{\circ i} = \xi^{i} - z_{0\beta} \dfrac{\delta \left(I(\phi) + \int h(\phi) \right)}{\delta \xi^{i}} \Lambda^{i}{}_{\beta} \end{cases} \tag{4.10}$$

where f is a generic dimensionless function of ϕ .

If in (4.9) one makes the change of variables :

$$\xi^{i} \to \xi^{i} + z_{0\beta} \frac{\delta \left(I(\phi) + \int h(\phi) \right)}{\delta \xi^{i}} \Lambda^{i}{}_{\beta} \tag{4.11}$$

one gets :

$$e^{\frac{i}{\hbar} W^{\varepsilon}(h)} = \int \prod_{i,x} \mathcal{D}\xi^{i} \, e^{\frac{i}{\hbar}\left(I(\phi, g^{\circ}(\phi)) + \int d^{2+\varepsilon}x \; h^{\circ}(\phi) \right)} \tag{4.12}$$

One sees that the wave function renormalization cancels out ,being a change of variables .

Appendix .

Stability ,regularization ,renormalization (9) .

In this appendix we want to discuss briefly the main points of the renormalization of a theory with a given symmetry .

The discussion is divided in three parts , in the first one we consider the classical stability of the theory ,in the second one we examine the invariance of the regularization scheme to the quantum level ,and in the last part we give the conditions and we sketch the proof of the renormalizability of the theory .

a) Stability .

Let A be a given classical action ; A depends on a set of parameters y .

y contains masses , coupling constants , a set of fields $\{\varphi^i\}$ and external sources $\{\gamma^i\}$.

According to the power-counting $A(y)$ is the integral of a polynomial ,with a certain canonical mass dimension ,local in the sources γ^i ,in the fields φ^i and their derivatives ;which does not contain constants with negative dimensions .

The canonical dimension of such a polynomial depends on the dimensions of the space-time in which the model is formulated ,and on the requirement that the action $A(y)$ is dimensionless .

Let us suppose now that $A(y)$ is symmetric ; i.e. there exists an operator $\mathcal{S}$, such that :

$$\mathcal{S}(A(y)) = 0$$

Typically $\mathcal{S}$ is not linear ; for example :

$$\mathcal{S}(A(y)) = \frac{\delta A}{\delta \varphi^i} \frac{\delta A}{\delta \gamma^i}$$

where we have omitted a space-time integration .

One introduces the linearized operator :

$$\mathcal{S}'(A(y)) = \frac{\delta A}{\delta \varphi^i} \frac{\delta}{\delta \gamma^i} + \frac{\delta A}{\delta \gamma^i} \frac{\delta}{\delta \varphi^i}$$

In most of the cases $\mathcal{S}'$ is nilpotent .

In general y belongs to a domain D ,called stability domain , of points such that the classical action is symmetrical for every point in D ; i.e. :

$$\mathcal{S}(A(y)) = 0 \quad \forall \; y \in D$$

Let us consider ,for example , a model with an $O(n)$ global symmetry :

$$A(y) = \int d^4x \;\; \varphi^i \Box \varphi^i + b\,(\varphi^i \varphi^i) + c\,(\varphi^i \varphi^i)^2$$

where the fields φ^i are real ,with canonical dimension one , and transform under an $O(n)$ representation ; and b , c are constant .

In this case $y = (b, c, \varphi)$.

The action $A(y)$ is invariant under the transformation :

$$\delta\varphi^i = T_{ij}\,\varphi^j \qquad \text{where} \quad T_{ij} = -\,T_{ji}$$

are the generators of $O(n)$.

The operator $\mathcal{J}$,in this case linear ,reads :

$$\mathcal{J} = \delta\varphi^i \frac{\delta}{\delta\varphi^i} = T_{ij}\,\varphi^j \frac{\delta}{\delta\varphi^i}$$

and

$$\mathcal{J}\,(\,A(y)\,) = 0$$

If we move from y to z ,where z is defined by :

$$b \to b' , \quad c \to c' , \quad \varphi \to d'\varphi$$

with b' , c' , d' constant ,one has :

$$A(y) \to A(z)$$

$$A(z) = \int d^4x \;\; d'^2 \varphi^i \Box \varphi^i + b' d'^2 (\varphi^i\varphi^i) + c' d'^4 (\varphi^i\varphi^i)^2$$

and

$$\mathcal{J}\,(\,A(z)\,) = 0 \qquad ;$$

then y and z are both points of D .

Let us suppose ,now , that the action $A(y)$ is perturbed by a term $\varepsilon\, A^1(y)$,where $A^1(y)$ is a local polynomial with canonical dimension ,and let us impose to the first order in ε that :

$$\mathcal{J}\,\left(\,A(y) + \varepsilon\, A^1(y)\,\right) = 0 + O(\varepsilon^2)$$

i.e. :

$$\mathcal{J}'(\,A(y)\,)\; A^1(y) = 0 \qquad .$$

The theory is said stable if the quantity :

$$\left(A(y) + \varepsilon A^{1}(y) \right)$$

can be obtained by redefining the initial parameters of the classical action $A(y)$,i.e. :

$$A(y) + \varepsilon A^{1}(y) = A(\hat{y}) + O(\varepsilon^{2})$$

with

$$\hat{y} = y + \varepsilon y^{\varepsilon}$$

Viceversa ,if $\hat{y} = y + \varepsilon y^{\varepsilon}$ belongs to the stability domain $\mathcal{D}$,then :

$$\mathcal{J}\left(A(\hat{y}) \right) = 0$$

to all order in ε ,where :

$$A(\hat{y}) = A(y) + \varepsilon A^{1}(y) + \sum_{\ell>1}^{n^{*}} \varepsilon^{\ell} A^{\ell}$$

with A^{ℓ} local and of canonical dimension .

b) Regularization .

At the quantum level ,the action $A(y)$ is replaced by the effective action Γ :

$$\Gamma = \sum_{n=0}^{\infty} \hbar^n \Gamma^n \qquad \Gamma^0 = A(y)$$

Γ contains ultraviolet divergences which ,in some way , have to be regularized .

Let Λ be the ultraviolet cutoff introduced by the regularization scheme and let Γ_Λ be the regularized effective action .

The regularization scheme is said invariant if it preserves at the quantum level the symmetry of the classical action $A(y)$; i.e. :

$$\mathcal{S}(\Gamma_\Lambda) = 0 \qquad \text{i)}$$

In the perturbative expansion of Γ in terms of Feynman graphs it is helpful to analyse the superficial degree of divergence of the single graphs .

Let G be a connected $1PI$ graph and let us consider the family $\mathcal{Y}$ of all its connected subdiagrams .

$\mathcal{Y}$ contains G itself .

Let g be an element of $\mathcal{Y}$ and $\omega(g)$ its superficial degree of divergence .

One has the following convergence theorem :

Theorem .

If $\omega(g) < 0$ for all $g \in \mathcal{Y}$,then the Feynman integral corresponding to G is absolutely convergent .

A useful corollary of the previous theorem is the following :
if G has no superficially divergent subdiagrams , $\omega(g)<0$ for all subdiagrams $g \neq G$, but is itself superficially divergent , $\omega(G) \geqslant 0$, then the divergent part of its amplitude is a polynomial of degree less or equal to $\omega(G)$ in the external momenta .
Making reference to the dimensional regularization with minimal subtraction ,one can say that the divergent part of G ,when $\omega(g)<0$ $\forall$ $g \neq G$,is a Laurent series in ε whose coefficients are polynomials in the external momenta and in the internal masses of degree less or equal to $\omega(g)$.
By using the convergence theorem one can assert that if Γ_Λ is finite up to the order $\hbar^{n-1}$,then its divergent part to the order $\hbar^{n}$ is ,making always reference to the dimensional regularization , a Laurent series in ε whose coefficients are given by an integral of a polynomial ,with canonical dimension ,local in the fields and their derivatives .
In some cases it is not possible to find an invariant regularization scheme which preserves at the quantum level the symmetry of the classical action .
In this cases there could be some anomalous term which could destroy the equation i) .
In this frame the study of such a term is a cohomology problem for the B.R.S. exterior derivative associated to the symmetry .

c) Renormalization .

If the theory is stable and there exists an invariant regularization scheme ,then the theory is renormalizable . This means that ,order by order in $\hbar$,there is a renormalized action A_z such that ,to the considered order $\mathcal{J}(A_z) = 0$; and whose corresponding functional $\widetilde{\Gamma}_\Lambda$ generates a perturbative series finite up to the considered order .

At every order A_z can be obtained from $A(y)$ by redefining the initial parameters y .

The proof is by induction ,showing that at each order the counterterms can be reabsorbed by redefining the initial parameters y of the action $A(y)$.

Let $\hat{y}_n = y - \sum_{\nu=1}^{n} \hbar^\nu y_\Lambda^\nu$ be a stability point for the action A ,i.e. :

$$\mathcal{J}(A(\hat{y}_n)) = 0$$

with

$$A(\hat{y}_n) = A(y) - \sum_{\nu=1}^{n^*} \hbar^\nu A_\Lambda^\nu \qquad\qquad n^* > n$$

where A_Λ^ν are local polynomials of canonical dimension such that the functional $\widetilde{\Gamma}_\Lambda$ which corresponds to $A(\hat{y}_n)$ is finite to the order $\hbar^n$.

Being the regularization scheme invariant ,it will be :

$$\mathcal{J}(\widetilde{\Gamma}_\Lambda) = 0$$

to all orders in $\hbar$.

$$\tilde{\Gamma}_\Lambda = \Gamma^0 + \hbar \tilde{\Gamma}^1_\Lambda + \cdots + \hbar^n \tilde{\Gamma}^n_\Lambda + \sum_{\ell=n+1}^{\infty} \hbar^\ell \tilde{\Gamma}^\ell_\Lambda$$

with

$$\Gamma^0 = A(y)$$

and

$$\lim_{\Lambda \to \infty} \tilde{\Gamma}^j_\Lambda = \text{finite} \qquad 1 \le j \le n$$

Let us consider now $\tilde{\Gamma}^{n+1}_\Lambda$.

$$\tilde{\Gamma}^{n+1}_\Lambda = \tilde{\Gamma}^{n+1}_{\Lambda\, div} + \tilde{\Gamma}^{n+1}_{\Lambda\, fin}$$

where ,by power-counting , $\tilde{\Gamma}^{n+1}_{\Lambda\, div}$ is a local polynomial of canonical dimension .

Indeed, $\tilde{\Gamma}^{n+1}_{\Lambda\, div}$ can contain only superficial divergences ,being the lower orders finite .

Let us consider the restriction of $\mathcal{S}(\tilde{\Gamma}_\Lambda)$ to the order $\hbar^{n+1}$; i.e. :

$$\mathcal{S}(\tilde{\Gamma}_\Lambda)\Big|_{\hbar^{n+1}}$$

It will be :

$$\mathcal{S}(\tilde{\Gamma}_\Lambda)\Big|_{\hbar^{n+1}} = 0$$

$$\mathcal{S}(\tilde{\Gamma}_\Lambda)\Big|_{\hbar^{n+1}} = \mathcal{S}'(A(y))\, \tilde{\Gamma}^{n+1}_\Lambda + \mathcal{S}'(\tilde{\Gamma}^1_\Lambda)\, \tilde{\Gamma}^n_\Lambda + \mathcal{S}'(\tilde{\Gamma}^2_\Lambda)\, \tilde{\Gamma}^{n-1}_\Lambda + \cdots = 0$$

Since

$$\left(\mathcal{S}'(\tilde{\Gamma}^1_\Lambda)\, \tilde{\Gamma}^n_\Lambda + \mathcal{S}'(\tilde{\Gamma}^2_\Lambda)\, \tilde{\Gamma}^{n-1}_\Lambda + \cdots \right)$$

is finite , and being

$$\mathcal{S}(\tilde{\Gamma}_\Lambda)\Big|_{\hbar^{n+1}} = 0$$

, it follows that :

$$\mathcal{S}'(A(y))\, \tilde{\Gamma}^{n+1}_\Lambda$$

is finite .

Isolating the divergent term (in dimensional regularization is a Laurent series in ε) ,one gets :

$$\mathcal{S}'(A(y))\, \tilde{\Gamma}^{n+1}_{\Lambda\, div} = 0$$

From the stability of the action $A(y)$ it follows that :

$$A(y) - \hbar^{n+1}\, \tilde{\Gamma}^{n+1}_{\Lambda\, div} = A(\hat{y}_{n+1}) + O(\hbar^{n+2})$$

with

$$\hat{y}_{n+1} = y - \hbar^{n+1}\, y^{n+1}_\Lambda$$

The action $A(\hat{y}_{n+1})$ generates a perturbative expansion finite to the order $\hbar^{n+1}$.

Then we have proved that it is possible ,supposing that it is true to the order $\hbar^n$,to reabsorb the $\hbar^{n+1}$ counterterm by redefining the initial parameters of $A(y)$.

Since the recurrence hypothesis is true to the order $\hbar^0$,it will be true to all orders .

References .

1) Brezin ,E.,Zinn-Justin , J. ,le Guillou , J.C. :
Phys.Rew .D14 .985 (1976).

A.Blasi and R.Collina , Nucl.Phys. B (1987) in pub.

J.Honerkamp , Nucl.Phys .B 36 (1972) 130 .

2) D.Friedan , Phys .Rev.Letters 45 (1980) 1057 .

Ann. Phys.(N.Y.) 163 (1985) ,318 .

3) C.Becchi " The non linear sigma model " .
preprint (1987) .

4) P.S.Howe ,G.Papadopoulos and K.S.Stelle .
"The background field method and the non linar σ - model " .
Preprint (1987) .

5) S.Elitzur , Nucl. Phys. B212 (1983) ,501 .

6) C.Becchi ,A. Rouet and R.Stora ,
Ann .Phys .98 (1976) 287 .

G.Bandelloni, C.Becchi , A. Blasi ,R.Collina
Ann .IHP XXVIII, (1978) 225 and 255 .

7) L.Alvarez-Gaume ,D.Freedman and S. Mukhi
Ann . Phys .(N.Y.) 134 (1981),85 .

8) F.David , Comm.Math.Phys. 81(1981) ,149 .

9) C.Becchi , Les Houches, Session XL,1983
" Relativity ,groups and topology " ,
B.S.DeWitt and R.Stora , eds.