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A Baire Category Approach
to the Bang-Bang Property

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Benedetto Piccoli

SUPERVISOR

Prof. Alberto Bressan

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1 - Introduction.

The aim of this paper is to develop a new technique, based on the Baire category theorem, in order to establish the closedness of reachable sets and the existence of optimal trajectories for control systems, without the usual convexity assumptions.

Most of our results will be formulated within the framework of differential inclusions. Let $F : \mathbb{R} \times \mathbb{R}^n \mapsto 2^{\mathbb{R}^n}$ be a continuous multifunction with compact convex values and denote by $extF(t, x)$ the set of extreme points of $F(t, x)$. We say that F has the *bang-bang property* if, for every interval $[a, b]$ and every Caratheodory solution $x(\cdot)$ of

$$\dot{x}(t) \in F(t, x(t)) \qquad t \in [a, b], \qquad (1.1)$$

there exists also a solution of

$$\dot{y}(t) \in extF(t, y(t)) \qquad t \in [a, b] \qquad (1.2)$$

such that

$$y(a) = x(a), \qquad y(b) = x(b). \qquad (1.3)$$

If A, B are respectively $n \times n$ and $n \times m$ matrices, and $U \subset \mathbb{R}^m$ is compact convex, the well known Bang-Bang Theorem [8, 10, 16] implies that the above property holds for the "linear" multifunction

$$F(t, x) = \Big\{ A(t)x + B(t)u; \quad u \in U \Big\} \subset \mathbb{R}^n. \qquad (1.4)$$

From the papers [18,19], however, it is clear that such a theorem cannot hold for a generic nonlinear control system. In the present paper, the bang-bang property is proved for a new class of "concave" multifunctions, characterized by the existence of suitable linear selections. The proofs rely on Lyapunov's theorem in connection with a Baire category argument. As applications, we obtain some closedness theorems for the reachable set of a differential inclusion with non-convex right hand side, and new existence results for optimal control problems in Mayer as well as in Bolza form.

Roughly speaking, the Baire category method consists in showing that the set $\mathcal{S}^{extF}$ of solutions of (1.2) is the intersection of countably many relatively open and dense subsets of the family $\mathcal{S}^F$ of all solutions of (1.1). Since $\mathcal{S}^F$ is closed, Baire's theorem thus implies $\mathcal{S}^{extF} \neq \emptyset$. The effectiveness of such an argument, in connection with the Cauchy problem for a differential inclusion, was suggested by Cellina [5] and demonstrated in [4, 9, 20] and in other papers. Here, this basic technique will be combined with Lyapunov's theorem and applied to the two-point boundary value problem (1.2), (1.3).

The use of a Lyapunov-type theorem, in order to prove existence of optimal solutions for non-convex control problems, was introduced by Neustadt [13] and later applied in [1, 14, 17] to a variety of optimization problems, always in connection with evolution equations and cost functionals which are linear w.r.t. the state variable. In [12] Mordukhovich and in [6] Cellina and Colombo showed that the linear cost functional can be replaced by one which is concave w.r.t. the state variable. Extensions and applications to partial differential

equations have recently appeared in [7, 15]. We remark that, if a variational problem of the type considered in [6] is reformulated as a Mayer problem of optimal control, then the corresponding multifunction satisfies our concavity assumptions. The present results can thus be regarded as a natural extension of the theorem in [6], for optimization problems which are "fully concave": in their dynamics as well as in the cost functional.

2 - Preliminaries.

In this paper, $|\cdot|$ is the euclidean norm in $I\!R^n$, $B(x, r)$ denotes the open ball centered at x with radius r, while $B(A, \varepsilon)$ denotes the open ε-neighborhood around the set A. We write $\overline{A}$ and $\overline{co}A$ respectively for the closure and the closed convex hull of A, while $A \setminus B$ indicates a set-theoretic difference. The Lebesgue measure of a set $J \subset I\!R$ is $meas(J)$. We recall that a subset $A \subseteq S$ is a G_δ if A is the intersection of countably many relatively open subsets of S.

In the following, $\mathcal{K}_n$ denotes the family of all nonempty compact convex subsets of $I\!R^n$, endowed with the Hausdorff metric. A key technical tool used in our proofs will be the function $h : I\!R^n \times \mathcal{K}_n \mapsto I\!R \cup \{-\infty\}$, defined by

$$h(y, K) = sup \left\{ \left(\int_0^1 |f(x) - y|^2 \, dx \right)^{\frac{1}{2}}; \quad f : [0, 1] \to K, \int_0^1 f(x) \, dx = y \right\}, \qquad (2.1)$$

with the understanding that $h(y, K) = -\infty$ if $y \notin K$. Observe that $h^2(y, K)$ can be interpreted as the maximum variance among all random variables supported inside K, whose mean value is y. From the above definition, it is clear that

$$h(\xi + y, \ \xi + K) = h(y, K), \qquad h(\lambda y, \lambda K) = \lambda h(y, K), \qquad \forall \xi \in I\!R^n, \ \lambda > 0. \qquad (2.2)$$

For the basic theory of multifunctions and differential inclusions we refer to [2]. Given a solution $x(\cdot)$ of (1.1), following [3] we define its *likelihood* as

$$L(x) = \left(\int_a^b h^2(\dot{x}(t), F(t, x(t))) dt \right)^{\frac{1}{2}} = \|h(\dot{x}, \ F(\cdot, x))\|_{\mathcal{L}^2}. \qquad (2.3)$$

The following results were proved in [3]:

Lemma 1. *For every y, K, one has $h(y, K) \leq r(K)$, where $r(K)$ is the radius of the smallest ball containing K (i.e., the Cebyshev radius). Moreover, $h(y, K) = 0$ iff $y \in ext K$. Therefore, a solution $x(\cdot)$ of (1.1) satisfies also (1.2) iff $L(x) = 0$.*

Lemma 2. *The map $(y,K) \mapsto h(y,K)$ is upper semicontinuous in both variables and concave w.r.t. y. The map $x(\cdot) \mapsto L(x)$ is upper semicontinuous on the set of solutions of (1.1), endowed with the C^0 norm.*

3 - The main results.

In the following, we denote by $S_{a,p}^{b,q}$ the set of all Caratheodory solutions of the two-point boundary value problem

$$\dot{x}(t) \in F(t,x(t)), \qquad x(a) = p, \qquad x(b) = q. \tag{3.1}$$

Theorem 1. *Let $F : \mathbb{R} \times \mathbb{R}^n \mapsto 2^{\mathbb{R}^n}$ be a continuous multifunction with compact, convex values. The following conditions are equivalent:*
(1) *For every interval $[a,b]$ and every $p,q \in \mathbb{R}^n$, if $S_{a,p}^{b,q} \neq \emptyset$, then the set of solutions of*

$$\dot{y}(t) \in ext F(t,y(t)), \qquad x(a) = p, \quad x(b) = q \tag{3.2}$$

is a dense G_δ in $S_{a,p}^{b,q}$.
(2) *F has the bang-bang property*
(3) *For every interval $[a,b]$ and every $p,q \in \mathbb{R}^n$, if $S_{a,p}^{b,q} \neq \emptyset$, then for every $\varepsilon > 0$ there exists a solution $x(\cdot)$ of (3.1) such that*

$$L^2(x) \doteq \int_a^b h^2\big(\dot{x}(t), F(t,x(t))\big) \, dt \; < \; \varepsilon. \tag{3.3}$$

Proof. (1) $\Rightarrow$ (2)   If $S_{a,p}^{b,q} \neq \emptyset$, then by (1) the set of solutions of (3.2), being dense, is nonempty. Hence (2) holds.

(2) $\Rightarrow$ (3)   If $S_{a,p}^{b,q} \neq \emptyset$ then by (2) there exists a solution $y(\cdot)$ of (3.2). This implies (3), because by Lemma 1

$$\int_a^b h^2(\dot{y}(t), F(t,y(t))) \, dt = 0 < \varepsilon.$$

(3) $\Rightarrow$ (1)   Consider the sets $A_m \doteq \big\{ x \in S_{a,p}^{b,q} ; \; L(x) < \frac{1}{m} \big\}$. By Lemma 2, L is upper semicontinuous, hence each A_m is open. Now fix any $x(\cdot) \in S_{a,p}^{b,q}$, $\varepsilon > 0$. Define

$$\Omega \doteq \big\{ (t,z); \; t \in [a,b], \; |z - x(t)| \leq \varepsilon \big\}$$

and choose a constant M so large that

$$F(t,x) \subseteq \overline{B}(0,M) \qquad \forall (t,x) \in \Omega. \tag{3.4}$$

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Split the interval $[a, b]$ into k equal subintervals $J_i = [t_{i-1}, t_i]$, inserting the points $t_i \doteq a + (i/k)(b - a)$, choosing k so large that $2M(b - a)/k \leq \varepsilon$.

By the assumption (3), for each i there exists a solution $y_i : [t_{i-1}, t_i] \mapsto I\!R^n$ of the two-point boundary value problem

$$\dot{y}(t) \in F(t, y(t)), \qquad y(t_{i-1}) = x(t_{i-1}), \quad y(t_i) = x(t_i), \tag{3.5}$$

with

$$L^2(y_i) = \int_{t_{i-1}}^{t_i} h^2\big(\dot{y}_i(t), \ F(t, y_i(t))\big) \ dt < \frac{1}{m^2 k}. \tag{3.6}$$

Define $y(\cdot)$ as the solution of (3.1) whose restriction to each J_i coincides with y_i. Given any $t \in [a, b]$, if, say, $t \in J_i$, then (3.4), (3.5) imply

$$|y(t) - x(t)| \leq \int_{t_i}^t |\dot{y}_i(s) - \dot{x}(s)| \ ds \leq \frac{2M(b - a)}{k} \leq \varepsilon.$$

Hence $\|y - x\|_{C^0} \leq \varepsilon$. Moreover, $y \in A_m$ because

$$L^2(y) = \sum_{i=1}^k \int_{t_{i-1}}^{t_i} h^2\big(\dot{y}_i(t), \ F(t, y_i(t))\big) \ dt < \frac{k}{m^2 k} = \frac{1}{m^2}.$$

Since $x(\cdot)$ and $\varepsilon > 0$ were arbitrary, this proves that each A_m is dense in $S_{a,p}^{b,q}$. By Baire's theorem, it follows that $A = \bigcap_m A_m$ is a G_δ dense subset of $S_{a,p}^{b,q}$. If $y \in A$, then $L(y) = 0$ and hence $\dot{y}(t) \in ext F(t, y(t))$ almost everywhere.

In the previous theorem, the implication (3) $\Rightarrow$ (1) determines the strength of the category method. In order to prove that "most" solutions of (3.1) actually solve (3.2) as well, it suffices to show (for every a, b, p, q) the existence of some solution of (3.1) with arbitrarily small likelihood. Roughly speaking, this requires the construction of some solution y of (3.1) whose derivative remains close to the extreme points of $F(t, y)$ during most of the time.

In practice, the condition (3) may often be easier to verify. We now show that this is indeed the case, if the multifunction F satisfies suitable concavity conditions.

Theorem 2. *Let $F : I\!R \times I\!R^n \mapsto 2^{I\!R^n}$ be a Hausdorff continuous multifunction with compact, convex values. Assume that:*

(C1) *For each (t, x) and every $y \in F(t, x)$, there exists a linear function $z \mapsto Az + c$ satisfying*

$$y = Ax + c, \qquad Az + c \in F(t, z) \qquad \forall z \in \overline{B}(x, \rho(t, x)), \tag{3.7}$$

where the radius $\rho = \rho(t, x)$ remains uniformly positive on compact sets.

(C2) *For each (t, x), every $y \in F(t, x)$ and $\varepsilon > 0$, there exist $\delta > 0$ and $n + 1$ linear functions $z \mapsto A'z + c_i$, $i = 0, \ldots, n$, such that*

$$y \in \overline{co}\big\{A'x + c_0 , \ \ldots \ , A'x + c_n\big\}, \tag{3.8}$$

$$h(A'x + c_i, \ F(t,x)) \le \varepsilon \qquad \forall i, \tag{3.9}$$

$$A'z + c_i \in F(t,z) \qquad \forall z \in \overline{B}(x,\delta), \ \forall i. \tag{3.10}$$

Then F has the bang-bang property.

We refer to (C1), (C2) as *concavity conditions* because they require, for each point (t,x,y) of the graph of F, the existence of suitable linear (non-homogeneous) selections. A similar property is shared by the epigraph of a concave scalar function, which admits global linear selections through each of its points.

Proof of Theorem 2. We will prove that F has property (3) stated in Theorem 1. Let $x^*(\cdot)$ be a solution of (3.1), for some interval $[a,b]$ and some points $p,q \in I\!\!R^n$. Let any $\varepsilon > 0$ be given, and define

$$\eta = \inf_{t \in [a,b]} \rho(t, x^*(t)), \qquad V = \Big\{(t,z); \ t \in [a,b], \ |z - x^*(t)| \le \eta\Big\}. \tag{3.11}$$

By assumption, $\eta > 0$. Choose M so large that

$$F(t,z) \subseteq \overline{B}(0,M) \qquad \forall (t,z) \in V. \tag{3.12}$$

By Lemma 1, this implies

$$h(y, \ F(t,z)) \le M \qquad \forall y, \ \forall (t,z) \in V. \tag{3.13}$$

1. As a first step, we construct measurable, bounded functions A, c, such that

$$\dot{x}^*(t) = A(t)x(t) + c(t) \qquad \text{for a.e. } t \in [a,b], \tag{3.14}$$

$$A(t)z + c(t) \in F(t,z) \qquad \forall z \in \overline{B}(x^*(t),\eta). \tag{3.15}$$

Since $\dot{x}^*(\cdot)$ is measurable, by Lusin's theorem there exists a sequence of disjoint compact sets $(J_\nu)_{\nu \ge 1}$ such that

$$meas\Big([a,b] \setminus \bigcup_{\nu \ge 1} J_\nu\Big) = 0 \tag{3.16}$$

and such that the restriction of $\dot{x}^*$ to each J_ν is continuous. Define the multifunction $G : [a,b] \mapsto 2^{I\!\!R^{n \times n} \times I\!\!R^n}$ by setting

$$G(t) \doteq \Big\{(A,c): \ \dot{x}^*(t) = Ax^*(t) + c, \quad Az + c \in F(t,z) \quad \forall z \in \overline{B}(x^*(t),\eta)\Big\}.$$

Because of (C1) and of the choice of η, $G(t) \ne \emptyset$ for a.e. t. One easily checks that the restriction of G to each J_ν has closed graph, because of the continuity of $\dot{x}^*$, x^* and F. Hence, G is a measurable multifunction on $[a,b]$ with closed, nonempty values. By [11], it admits a measurable selection $t \mapsto \big(A(t), c(t)\big)$, which clearly satisfies (3.14), (3.15). Observe

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that the matrices $A(t)$ and the vectors $c(t)$ must be uniformly bounded, because of (3.15), (3.12).

2. As a second step, we construct measurable functions $A', c_0, \ldots c_n, \theta_0, \ldots, \theta_n, \delta$, such that, for almost every $t \in [a, b]$, the following holds:

$$\delta(t) > 0, \qquad \theta_i(t) \in [0, 1], \qquad \sum_{i=0}^{n} \theta_i(t) = 1, \tag{3.17}$$

$$\dot{x}^*(t) = A'(t)x^*(t) + \sum_{i=0}^{n} \theta_i(t)c_i(t), \tag{3.18}$$

$$A'(t)z + c_i(t) \in F(t, z), \qquad h^2\Big(A'(t)z + c_i(t), \, F(t, z)\Big) \le \varepsilon \qquad \forall i, \ \forall z \in \overline{B}\Big(x^*(t), \delta(t)\Big). \tag{3.19}$$

By Lemma 2, h is upper semicontinuous, hence there exists a nonincreasing sequence $(h_m)_{m \ge 1}$ of continuous functions such that

$$h(y, K) = \inf_{m \ge 1} h_m(y, K) \qquad \forall (y, K) \in I\!R^n \times \mathcal{K}_n. \tag{3.20}$$

For each $m \ge 1$, define the multifunction

$$H_m(t) \doteq \Big\{ \ \Big(A', c_0, \ldots, c_n, \theta_0, \ldots, \theta_n, \delta\Big); \qquad \delta = \frac{1}{m}, \qquad \theta_i \in [0, 1],$$

$$\dot{x}^*(t) = \sum_{i=0}^{n} \theta_i(A'x^*(t) + c_i), \qquad \sum_{i=0}^{n} \theta_i = 1,$$

$$A'z + c_i \in F(t, z), \qquad h_m^2\Big(A'z + c_i, \, F(t, z)\Big) \le \varepsilon \qquad \forall z \in \overline{B}\Big(x^*(t), \frac{1}{m}\Big) \Big\}.$$

If $(J_\nu)_{\nu \ge 1}$ is the same sequence of compact sets considered at (3.16), the continuity of $\dot{x}^*$, x^*, h_m and F implies that the restriction of H_m to each J_ν has closed graph, with uniformly bounded, possibly empty values.

Defining $I_m = \{t; \ H_m(t) \ne \varnothing\}$, it is clear that on each I_m the multifunction H_m is measurable with closed, nonempty values. By [11], it admits a measurable selection, say $t \mapsto \Phi_m(t)$. By (C2), (3.20) and the continuity of each h_m, for every ν we have $\bigcup_{m \ge 1} I_m \supseteq J_\nu$. Therefore, the selection

$$\Big(A'(t), c_0(t), \ldots, c_n(t), \theta_0(t), \ldots, \theta_n(t), \delta(t)\Big) \doteq \Phi_m(t) \qquad \text{iff} \quad t \in I_m \setminus \bigcup_{\ell < m} I_\ell$$

is measurable and defined for a.e. $t \in [a, b]$. By construction, the conditions (3.17)-(3.19) hold.

3. We can now complete the proof of the theorem. Since $\delta(\cdot)$ is measurable and positive, there exists an integer m^* such that

$$\frac{1}{m^*} < \eta, \qquad meas\Big(J'_{m^*}\Big) \ge (b - a) - \varepsilon, \tag{3.21}$$

where

$$J'_m \doteq \left\{ t \in [a,b]; \quad \delta(t) \geq \frac{1}{m}, \quad |A'(t)|, |c_i(t)| \leq m \right\}. \tag{3.22}$$

Split $[a,b]$ into k equal subintervals $I_j = [t_{j-1}, t_j]$, inserting the points $t_j \doteq a + (j/k)(b-a)$ and choosing k so large that:

$$2M \frac{b-a}{k} < \frac{1}{m^*}. \tag{3.23}$$

Using the selections A, c and A', c_i, θ_i constructed in the previous steps, define:

$$A^*(t) = \begin{cases} A(t) & \text{if} \quad t \notin J'_{m^*}, \\ \\ A'(t) & \text{if} \quad t \in J'_{m^*}, \end{cases}$$

$$f(t) = \begin{cases} c(t) & \text{if} \quad t \notin J'_{m^*}, \\ \\ \sum_{i=0}^n \theta_i(t) c_i(t) & \text{if} \quad t \in J'_{m^*}. \end{cases}$$

By (3.14), (3.18), $\dot{x}^*(t) = A^*(t) x(t) + f(t)$. Calling $W(\cdot, \cdot)$ the matrix fundamental solution of the bounded linear system $\dot{v} = A^*(t)v$, we thus have the representation

$$x^*(t) = W(t, t_{j-1}) x^*(t_{j-1}) + \int_{t_{j-1}}^t W(t,s) f(s) \, ds, \qquad t \in [t_{j-1}, t_j].$$

Applying Lyapunov's theorem on each interval I_j, for every j we obtain a measurable partition $\{I_{j,0}, \ldots, I_{j,n}\}$ of I_j and an absolutely continuous function w_j satisfying the two-point boundary value problem

$$\dot{w}_j(t) = \begin{cases} A^*(t) w_j(t) + c(t) & \text{if} \quad t \in I_j \setminus J'_{m^*}, \\ \\ A^*(t) w_j(t) + c_\ell(t) & \text{if} \quad t \in I_{j,\ell} \cap J'_{m^*}, \quad \ell = 0, \ldots, n, \end{cases}$$

$$w_j(t_j) = x^*(t_j), \qquad w_j(t_{j+1}) = x^*(t_{j+1}).$$

We claim that

$$|w_j(t) - x^*(t)| \leq \frac{1}{m^*} \qquad \forall t \in I_j. \tag{3.24}$$

If not, there would exist a first time $\tau \in I_j$ such that

$$|w_j(\tau) - x^*(\tau)| = \frac{1}{m^*}. \tag{3.25}$$

Recalling (3.15), (3.19) and using (3.12) and (3.23), we then have:

$$|w_j(\tau) - x^*(\tau)| \leq \int_{t_{j-1}}^\tau |\dot{w}_j(t) - \dot{x}^*(t)| \, dt \leq 2M \frac{b-a}{k} < \frac{1}{m^*},$$

a contradiction with (3.25). This proves (3.24). In particular, by (3.15), (3.19) we conclude that $\dot{w}_j(t) \in F(t, w_j(t))$ for a.e. $t \in I_j$.

Now consider the solution $w(\cdot)$ of (3.1) whose restriction to each I_j coincides with w_j. Recalling (3.13), (3.19), (3.21), its likelihood is computed by

$$L^2(w) = \int_{J'_{m^*}} h^2(\dot{w}(t), F(t, w(t)))\, dt + \int_{[a,b]\setminus J'_{m^*}} h^2(\dot{w}(t), F(t, w(t)))\, dt$$

$$\leq \varepsilon \cdot meas\left(J'_{m^*}\right) + M^2 \cdot meas\left([a,b]\setminus J'_{m^*}\right) \leq \varepsilon((b-a) + M^2).$$

Since ε was arbitrary, this establishes the property (3) in Theorem 1, which is equivalent to the bang-bang property.

Remark 1. The previous theorems, with the same proofs, remain valid if F is defined on some open set $\Omega \subset I\!R \times I\!R^n$.

4 - Examples of concave multifunctions.

Aim of this section is to exhibit some classes of multifunctions which satisfy the concavity properties (C1), (C2) stated in Theorem 2.

Proposition 1. *Let $\varphi : I\!R \times I\!R^n \mapsto]0, \infty[$ be a continuous function, with $x \mapsto \varphi(t, x)$ convex for every t. Let $U \subset I\!R^n$ be compact, convex, containing the origin. Then the multifunction*

$$F(t, x) = \varphi(t, x)U \tag{4.1}$$

has the bang-bang property.

Proof. In order to apply Theorem 2, we first verify the concavity condition (C1). Fix any (t, x) and any $y = \varphi(t, x)u \in F(t, x)$. Since φ is continuous and strictly positive and its subdifferential $\partial_x \varphi$ w.r.t. x is uniformly bounded on compact sets, we have

$$\varphi(t, x) + \xi \cdot (z - x) \geq 0 \qquad \forall \xi \in \partial_x \varphi(t, x), \quad \forall z \in \overline{B}(x, \rho(t, x)), \tag{4.2}$$

for some function $\rho = \rho(t, x)$ uniformly positive on compact sets. Choose any vector $\xi \in \partial_x \varphi(t, x)$ and define the linear map

$$z \mapsto Az + c \doteq (\xi \cdot z)u + \left[\varphi(t, x)u - (\xi \cdot x)u\right].$$

If $|z - x| \leq \rho(t, x)$, we need to show the existence of some $\omega \in U$ such that

$$(\xi \cdot z)u + \left[\varphi(t, x)u - (\xi \cdot x)u\right] = \varphi(t, z)\omega. \tag{4.3}$$

From (4.2) and the convexity of φ it follows

$$\omega = \frac{\varphi(t, x) + \xi \cdot (z - x)}{\varphi(t, z)}u = \alpha u \tag{4.4}$$

for some $\alpha \in [0,1]$. The assumptions on U thus imply $\omega \in U$.

We now turn to the condition (C2). Let (t,x), $y = \varphi(t,x)u \in F(t,x)$ and $\varepsilon > 0$ be given. We can assume

$$u = \sum_{i=1}^{\nu} \theta_i u_i, \qquad \theta_i \in (0,1], \qquad \sum_{i=1}^{\nu} \theta_i = 1 \qquad (4.5)$$

for some $\nu \in \{1, \ldots, n+1\}$, $u_i \in extU$. Select $\xi \in \partial_x \varphi(t,x)$ as before and define

$$u_i' = u_i + \varepsilon'(u - u_i), \qquad (4.6)$$

choosing $\varepsilon' \in (0,1)$ so small that

$$h(u_i', \ U) < \frac{\varepsilon}{\varphi(t,x)} \qquad \forall i. \qquad (4.7)$$

This is possible because $h(u_i, U) = 0$ and h is upper semicontinuous. Then define

$$A'z = (\xi \cdot z)u, \qquad c_i = \varphi(t,x)u_i' - (\xi \cdot x)u.$$

Recalling $(2.2)_2$, (4.7) yields

$$h\Big(A'x + c_i, \ F(t,x)\Big) = h\Big(\varphi(t,x)u_i', \ \varphi(t,x)U\Big) < \varepsilon.$$

Moreover, by (4.5), (4.6),

$$y = \varphi(t,x)u = \sum_{i=1}^{\nu} \theta_i \varphi(t,x)u_i' \in \overline{co}\Big\{A'x + c_i, \quad i = 1, \ldots, \nu\Big\}.$$

It remains to prove that $A'z + c_i \in \varphi(t,z)U$ for $|z - x|$ small enough. For each fixed i, define the subspace $E_i \doteq span\{u, \ u_i\}$.

If E_i has dimension 2, consider the triangle $\Delta = \overline{co}\{0, \ u, \ u_i\}$ and call $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ the unit vectors in E_i which are outer normals to the sides $u_i - u$, u_i, u, respectively. Observe that

$$\varphi(t,z)\overline{co}\{0, \ u, u_i\} = \Big\{y \in E_i; \ \ \mathbf{n}_1 \cdot y \le \varphi(t,z)(\mathbf{n}_1 \cdot u), \ \ \mathbf{n}_2 \cdot y \le 0, \ \ \mathbf{n}_3 \cdot y \le 0\Big\} \subseteq F(t,z). \quad (4.8)$$

Since u_i' is a strict convex combination of u and u_i, one has $\mathbf{n}_2 \cdot u_i' < 0$, $\mathbf{n}_3 \cdot u_i' < 0$. By continuity, for z sufficiently close to x we still have

$$\mathbf{n}_j \cdot \Big[\varphi(t,x)u_i' + \xi \cdot (z-x)u\Big] = \mathbf{n}_j \cdot \Big[A'z + c_i\Big] < 0, \qquad j = 2,3. \qquad (4.9)$$

Moreover, since $\mathbf{n}_1 \cdot u = \mathbf{n}_1 \cdot u_i' > 0$, the convexity of φ implies

$$\begin{aligned}
\varphi(t,z)(\mathbf{n}_1 \cdot u) &\ge \Big[\varphi(t,x) + \xi \cdot (z-x)\Big](\mathbf{n}_1 \cdot u) \\
&= \mathbf{n}_1 \cdot \Big[\varphi(t,x)u_i' + \big(\xi \cdot (z-x)\big)u\Big] \qquad (4.10) \\
&= \mathbf{n}_1 \cdot [A'z + c_i].
\end{aligned}$$

By (4.8), the inequalities (4.9), (4.10) together imply $A'z + c_i \in F(t, z)$.

Finally, consider the case where E_i has dimension ≤ 1. Then, either $u = u_i$, or else u'_i lies in the relative interior of the segment $S \doteq \overline{co}\{0, u, u_i\}$. In the former case $\nu = 1$ and $u \in ext F(t, x)$. The same argument as in (4.3), (4.4) can be used. In the latter case, the map $z \mapsto A'z + c_i$ takes values inside E_i, with $\varphi^{-1}(t, x)[A'x + c_i] \in rel\ int S$. By continuity, $\varphi^{-1}(t, z)[A'z + c_i] \in S$ for $|z - x|$ small enough. This proves again that $A'z + c_i \in F(t, z)$ for every z in a neighborhood of x.

An application of Theorem 2 now yields the desired conclusion.

The next application is concerned with a multifunction F whose values are polytopes, with variable shape but constant number of vertices. More precisely, we assume that $F(t, x)$ admits the representations

$$F(t, x) = \overline{co}\{y_1(t, x), \dots, y_N(t, x)\}, \tag{4.11}$$

where $y_1, \dots, y_N$ are the (distinct) vertices of $F(t, x)$, as well as

$$F(t, x) = \left\{ y \in I\!\!R^n \ ; \quad w_j(t) \cdot y \leq \psi_j(t, x) = \max_{\omega \in F(t,x)} w_j(t) \cdot \omega, \quad j = 1, .., k \right\}. \tag{4.12}$$

On the product set of indices $\{1, \dots, N\} \times \{1, \dots, k\}$, we consider the "incidence" relation

$$i \sim j \quad \text{iff} \quad w_j(t) \cdot y_i(t, x) = \psi_j(t, x). \tag{4.13}$$

Proposition 2. *Let F be a multifunction admitting the representations (4.11), (4.12). Assume that*

(i) $w_j : I\!\!R \mapsto I\!\!R^n$, $\psi_j : I\!\!R \times I\!\!R^n \mapsto I\!\!R$ are continuous functions, $|w_j(t)| \equiv 1$, each map $x \mapsto \psi_j(t, x)$ is convex.

(ii) The relation $\sim$ defined in (4.13) is independent of (t, x).

Then F has the bang-bang property.

Proof. For each $i \in \{1, \dots, N\}$, consider the set of indices

$$J_i = \left\{ j; \quad w_j(t) \cdot y_i(t, x) = \psi_j(t, x) \right\}. \tag{4.14}$$

By (ii), this set does not depend on t, x.

We begin by checking the condition (C1) in Theorem 2. First, assume $y \in ext F(t, x)$, say $y = y_i(t, x)$. In this case we can choose n independent vectors, say $w_{j_1}(t), \dots, w_{j_n}(t)$, with $j_1, \dots, j_n \in J_i$. Define the dual vectors $w^*_{j_\ell}$, by requiring that

$$w^*_{j_\ell} \cdot w_{j_m} = \begin{cases} 1 & \text{if} \quad \ell = m, \\ 0 & \text{if} \quad \ell \neq m. \end{cases} \tag{4.15}$$

By convexity, each function $z \mapsto \psi_j(t,z)$ is differentiable almost everywhere. Therefore, there exists a sequence of points $x_\nu \to x$ such that the gradients $\nabla_x \psi(t, x_\nu)$ exist for each ν, together with the limits

$$\lim_{\nu \to \infty} \nabla_x \psi_j(t, x_\nu) = \xi_j \in \partial_x \psi_j(t, x) \qquad \forall j. \tag{4.16}$$

Now define

$$A_i z = \sum_{\ell=1}^{n} (\xi_{j_\ell} \cdot z) w_{j_\ell}^*, \qquad c_i = y_i(t, x) - \sum_{\ell=1}^{n} (\xi_{j_\ell} \cdot x) w_{j_\ell}^*. \tag{4.17}$$

Clearly, $A_i x + c_i = y_i$. Using the representation (4.12) we now check that

$$A_i z + c_i \in F(t, z) \qquad \forall z \in \overline{B}(x, \rho(t, x)) \tag{4.18}$$

for some $\rho = \rho(t, x)$ uniformly positive on compact sets.

If $j \in J_i$, then there exist unique coefficients α_ℓ such that $w_j(t) = \sum \alpha_\ell w_{j_\ell}(t)$. The assumption (ii) together with (4.16) now implies

$$\psi_j(t, z) = w_j(t) \cdot y_i(t, z) = \sum_{\ell=1}^{n} \alpha_\ell \psi_{j_\ell}(t, z) \qquad \forall z,$$

$$\sum_{\ell=1}^{n} \alpha_\ell \xi_{j_\ell} \in \partial_x \psi_j(t, x). \tag{4.19}$$

From (4.19) and the convexity of ψ_j it follows

$$w_j(t) \cdot \left[A_i z + c_i \right] = \sum_{\ell=1}^{n} \alpha_\ell w_{j_\ell}(t) \cdot \left[y_i(t, x) + \sum_{h=1}^{n} \left(\xi_{j_h} \cdot (z - x) \right) w_{j_h}^* \right]$$

$$= \psi_j(t, x) + \left(\sum_{\ell=1}^{n} \alpha_\ell \xi_{j_\ell} \right) \cdot (z - x) \leq \psi_j(t, z).$$

On the other hand, if $j \notin J_i$ then $w_j(t) \cdot \left[A_i x + c_i \right] < \psi_j(t, x)$. Hence, by continuity we still have

$$w_j(t) \cdot \left[A_i z + c_i \right] < \psi_j(t, z) \qquad \forall z \in \overline{B}\left(x, \rho(t, x) \right).$$

By the assumption (ii), the continuity of the functions w_j, ψ_j and the local boundedness of the subgradients $\partial_x \psi_j$, it follows that ρ can be taken uniformly positive on bounded sets. This proves (C1) in the case $y \in ext F(t, x)$.

When y is an arbitrary element in $F(t, x)$, there exist extreme points y_i and coefficients $\theta_i \in [0, 1]$ such that

$$y = \sum_{i=1}^{N} \theta_i y_i(t, x), \qquad \sum_{i=1}^{N} \theta_i = 1.$$

If A_i, c_i are the matrices and vectors defined at (4.17), then the convex combinations $A = \sum \theta_i A_i$, $c = \sum \theta_i c_i$ satisfy

$$Ax + c = y, \qquad Az + c \in F(t, z) \qquad \forall z \in \overline{B}(x, \rho(t, x)).$$

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Next, consider the condition (C2). Let (t,x), $y \in F(t,x)$, $\varepsilon > 0$ be given. Write y as a convex combination of points $y_1, \ldots, y_\nu \in extF(t,x)$, say

$$y = \sum_{i=1}^{\nu} \theta_i y_i \qquad \theta_i \in (0,1] \qquad \sum_{i=1}^{\nu} \theta_i = 1,$$

and define

$$y_i' = y_i + \varepsilon'(y - y_i),$$

choosing $\varepsilon' \in (0,1]$ so small that

$$h\big(y_i', \ F(t,x)\big) < \varepsilon \qquad \forall i. \tag{4.20}$$

Consider the vector space

$$E \doteq \text{span}\big\{w_j(t); \quad w_j(t) \cdot y = \psi_j(t,x)\big\}.$$

Choose a basis $\{w_{j_1}, \ldots, w_{j_\mu}\}$ of E and define the dual basis $\{w_{j_1}^*, \ldots, w_{j_\mu}^*\}$ as in (4.15). Select vectors $\xi_j \in \partial_x \psi_j(t,x)$ as in (4.16) and define

$$A'z = \sum_{\ell=1}^{\mu} (\xi_{j_\ell} \cdot z) w_{j_\ell}^*, \qquad c_i = y_i' - \sum_{\ell=1}^{\mu} (\xi_{j_\ell} \cdot x) w_{j_\ell}^*. \tag{4.21}$$

The above definitions imply

$$y = \sum_{i=1}^{\nu} y_i' \in \overline{co}\big\{A'x + c_i; \quad i = 1, \ldots, \mu\big\}.$$

Using the representation (4.12), we now prove that $A'z + c_i \in F(t,z)$. If $w_j(t) \in E$, then

$$w_j(t) \cdot y = w_j(t) \cdot y_i' = \psi_j(t,x) \qquad \forall i = 1, \ldots, \nu.$$

Moreover, there exist coefficients α_ℓ such that $w_j(t) = \sum \alpha_\ell w_{j_\ell}(t)$. The assumption (ii) together with (4.16) now implies

$$\psi_j(t,z) = w_j(t) \cdot y_i(t,z) = \sum_{\ell=1}^{\mu} \alpha_\ell \psi_{j_\ell}(t,z) \qquad \forall z, \tag{4.22}$$

$$\sum_{\ell=1}^{\mu} \alpha_\ell \xi_{j_\ell} \in \partial_x \psi_j(t,x). \tag{4.23}$$

From (4.22), (4.23) and the convexity of ψ_j it follows

$$w_j(t) \cdot \big[A'z + c_i\big] = \sum_{\ell=1}^{\mu} \alpha_\ell w_{j_\ell}(t) \cdot \left[y_i' + \sum_{h=1}^{\mu} \big(\xi_{j_h} \cdot (z-x)\big) w_{j_h}^*\right]$$

$$= \psi_j(t,x) + \Big(\sum_{\ell=1}^{\mu} \alpha_\ell \xi_{j_\ell}\Big) \cdot (z - x)$$

$$\leq \psi_j(t,z).$$

On the other hand, if $w_j(t) \notin E$, then

$$w_j(t) \cdot [A'x + c_i] = w_j(t) \cdot y_i' < \psi_j(t, x).$$

By continuity, for $|z - x|$ sufficiently small we still have

$$w_j(t) \cdot [A'z + c_i] < \psi_j(t, z).$$

This completes the proof of condition (C2). An application of Theorem 2 now yields the desired result.

Remark 2. Assume that A, b are a $n \times n$ matrix and a n-vector, depending continuously on t, and that F is a continuous, compact convex valued multifunction satisfying the concavity conditions (C1), (C2). Then the multifunction

$$G(t, x) = A(t)x + b(t) + F(t, x)$$

satisfies all assumptions in Theorem 2 as well. In particular, from Proposition 1 it follows that the bang-bang property holds for a control system of the form

$$\dot{x} = A(t)x + b(t) + \varphi(t, x)u, \qquad u(t) \in U,$$

with U compact, convex, containing the origin and $\varphi > 0$ convex w.r.t. x.

5 - A nonconvex optimal control problem.

This section is concerned with an application of Theorem 2 to an optimal control problem of Bolza. The analysis will clarify the connections between the concavity conditions (C1), (C2) and the assumptions made in [6, 12,13]. Given the linear control system on $I\!R^n$

$$\dot{x}(t) = A(t)x + f(t, u(t)) \qquad u(t) \in U \tag{5.1}$$

with initial and terminal constraints

$$x(0) = \bar{x} \qquad (T, x(T)) \in S, \tag{5.2}$$

consider the minimization problem:

$$\min \left\{ \int_0^T [\alpha(t, x(t)) + \beta(t, u(t))] \, dt \right\}. \tag{5.3}$$

Theorem 3. *Let the functions A, f, α, β be continuous, with α concave w.r.t. x. Assume that the control set $U \subseteq I\!R^m$ is compact and that the terminal set S is closed and contained*

in $[0, T_0] \times I\!R^n$, *for some* T_0. *If some solution of (5.1), (5.2) exists, then the minimization problem (5.3) admits an optimal solution.*

Proof. We begin by adding an extra variable x_0, writing the problem in Mayer form:

$$\min x_0(T) \tag{5.4}$$

$$\begin{cases} \dot{x}(t) = A(t)x(t) + f(t, u(t)) \\ \dot{x}_0(t) = \alpha(t, x(t)) + \beta(t, u(t)) \end{cases} \qquad u(t) \in U, \tag{5.5}$$

$$(x, x_0)(0) = (\bar{x}, 0) \qquad (T, x(T)) \in S. \tag{5.6}$$

The continuity of A, f and the compactness of U imply that all trajectories of the differential inclusion

$$\dot{x}(t) \in A(t)x(t) + \overline{co}\{f(t, u); \ u \in U\}, \qquad x(0) = \bar{x}, \qquad t \in [0, T_0], \tag{5.7}$$

are contained in some bounded open set $\Omega \subset I\!R \times I\!R^n$. Define the constant

$$M \doteq 1 + \sup\{\alpha(t, x) + \beta(t, u); \qquad (t, x) \in \Omega, \ u \in U\} \tag{5.8}$$

and the multifunction (independent of x_0)

$$F(t, x, x_0) \doteq \overline{co}\{(y, y_0); \quad y = A(t)x + f(t, u), \quad \alpha(t, x) + \beta(t, u) \le y_0 \le M \ \text{for some} \ u \in U\}. \tag{5.9}$$

Observe that F admits the representation

$$F(t, x) = \Big\{(y, y_0); \quad y = A(t)x + \sum_{i=0}^{n+1} \theta_i f(t, u_i), \quad y_0 = \Big[\alpha(t, x) + \sum_{i=0}^{n+1} \theta_i \beta(t, u_i)\Big](1 - v) + Mv,$$

$$(\theta_0, \ldots, \theta_{n+1}) \in \Delta_{n+1}, \ u_0, \ldots, u_{n+1} \in U, \ v \in [0, 1]\Big\}, \tag{5.10}$$

where

$$\Delta_{n+1} = \Big\{(\theta_0, \ldots, \theta_{n+1}); \quad \theta_i \in [0, 1], \quad \sum_{i=0}^{n+1} \theta_i = 1\Big\}.$$

Since F is continuous with compact convex values, the optimization problem (5.4) subject to the boundary conditions (5.6) with dynamics

$$(\dot{x}, \dot{x}_0) \in F(t, x) \tag{5.11}$$

admits an optimal solution. The existence of a solution to the original problem (5.1)-(5.3) will be proved by showing that the multifunction F has the bang-bang property, for $(t, x, x_0) \in \Omega \times I\!R$.

To verify the concavity condition (C1), define the constant $\rho > 0$ by

$$\frac{1}{\rho} = \sup\Big\{|\xi|; \quad \xi \in \partial_x \alpha(t, x), \quad (t, x) \in \Omega\Big\}. \tag{5.12}$$

Given $(t,x) \in \Omega$, $(y, y_0) \in F(t,x)$, assume first $y_0 < M - 1$. Choose any $\xi \in \partial_x \alpha(t,x)$ and consider the linear map

$$\Phi(z) \doteq \big(y + A(t)(z - x),\ y_0 + \xi \cdot (z - x)\big). \tag{5.13}$$

From (5.12) and the assumption on y_0 it follows that

$$y_0 + \xi \cdot (z - x) \leq M \qquad \forall z \in \overline{B}(x, \rho). \tag{5.14}$$

Let y, y_0 be as in (5.10), for some θ_i, u_i, v. Using the concavity of α we then obtain

$$y + A(t)(z - x) = A(t)z + \sum_{i=0}^{n+1} \theta_i f(t, u_i),$$

$$y_0 + \xi \cdot (z - x) \geq \alpha(t,x) + \sum_{i=0}^{n+1} \theta_i \beta(t, u_i) + \xi \cdot (z - x)$$

$$\geq \alpha(t,z) + \sum_{i=0}^{n+1} \theta_i \beta(t, u_i).$$

This, together with (5.14), implies $\Phi(z) \in F(t,z)$.

Next, assume $y_0 \in [M - 1,\ M]$. Then the definition (5.8) implies that the map

$$\Psi(z) \doteq \big(y + A(t)(z - x),\ y_0\big)$$

is again a selection of $F(t, \cdot)$.

We now turn to the condition (C2). Fix $(t,x) \in \Omega$, $\tilde{y} = (y, y_0) \in F(t,x)$, $\varepsilon > 0$, and choose $\xi \in \partial_x \alpha(t,x)$. Write $\tilde{y}$ as a convex combination

$$\tilde{y} = \sum_{j=1}^{\nu} \vartheta_j \tilde{y}_j, \qquad \vartheta_j \in (0,1], \qquad \sum_{j=1}^{\nu} \vartheta_j = 1,$$

with $\tilde{y}_j \in ext F(t,x)$, and define

$$\tilde{y}_j' = \tilde{y}_j + \varepsilon'(\tilde{y} - \tilde{y}_j),$$

choosing $\varepsilon' \in (0,1]$ such that

$$h\big(\tilde{y}_j',\ F(t,x)\big) < \varepsilon \qquad \forall j. \tag{5.15}$$

We now distinguish two cases.

If $y_0 < M$, then $\tilde{y}_j' = (y_j', y_{0,j}')$ satisfies $y_{0,j}' < M$ for all j. Hence, for $|z - x|$ sufficiently small, the maps

$$\Phi_j(z) \doteq \big(y_j' + A(t)(z - x),\ y_{0,j}' + \xi \cdot (z - x)\big)$$

are affine selections of F.

On the other hand, if $y_0 = M$, then $y_{0,j} = y'_{0,j} = M$ for all j. Hence the maps

$$\Phi_j(z) \doteq \Big(y'_j + A(t)(z - x), \ M \Big)$$

are affine selections of F.

Applying Theorem 2, we now obtain the existence of an optimal solution $(x^*, x_0^*) : [0, T] \mapsto$ $I\!\!R^n \times I\!\!R$ to (5.4), (5.6), (5.11), with the additional property

$$(\dot{x}^*, \ \dot{x}_0^*)(t) \in extF(t, x^*(t)) \qquad \text{for a.e.} \quad t \in [0, T].$$

The representation (5.10) and the selection theorem [11] now yield the existence of some measurable $u^* : [0, T] \mapsto U$ such that

$$\begin{cases} \dot{x}^*(t) = A(t)x^*(t) + f(t, u^*(t)), \\ x_0^*(t) \in \Big\{ \alpha(t, x^*(t)) + \beta(t, u^*(t)), \ M \Big\} \end{cases}$$

for almost every t. Since the terminal value $x_0^*(T)$ is minimized, by (5.8) we must have $\dot{x}_0^*(t) < M$ almost everywhere. Therefore, x^* is an optimal trajectory for the original system (5.1), corresponding to the control u^*.

6 - A counterexample.

The following example shows how the bang-bang property may fail, if some of the assumptions in Theorem 2 or in Proposition 2 are not satisfied. More general results concerning systems of this form can be found in [18, 19].

On $I\!\!R^2$, consider the control system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \qquad u(t) \in [-1, 1], \tag{6.1}$$

with

$$f(x_1, x_2) = (x_1, x_2), \qquad g(x_1, x_2) = (1, x_1).$$

For $t \in [0, 1]$, the trajectory $t \mapsto (0, e^t)$, corresponding to the control $u(t) \equiv 0$, steers the system from $p = (0, 1)$ to $q = (0, e)$.

Defining the auxiliary function

$$V(x_1, x_2) = x_2 - \frac{x_1^2}{2},$$

a straightforward computation yields

$$\frac{d}{dt}V(x(t)) = V(x(t)) - \frac{x_1^2(t)}{2}$$

for every solution of (6.1). This implies

$$V(x(t)) \leq e^t V(x(0)),$$

with equality holding if and only if $x_1(s) = 0$ for all $s \in [0,t]$. In particular, the control $u \equiv 0$ is the only one which steers the system from p to q in minimum time. Hence the multifunction

$$F(x_1, x_2) = \left\{ (x_1 + u, \ x_2 + x_1 u); \ |u| \leq 1 \right\}$$

does not have the bang-bang property. Observe that

(i) For each $y = f(x) + g(x)\omega \in F(x)$, defining

$$A = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix} \qquad c = \begin{pmatrix} \omega \\ 0 \end{pmatrix},$$

one checks that the condition (C1) in Theorem 2 holds. However, the condition (C2) here fails.

(ii) The multifunction $extF(x) = \left\{ f(x) + g(x), \ f(x) - g(x) \right\}$ satisfies both (C1) and (C2) in Theorem 2, but its values are not convex.

(iii) Each set $F(x)$ is a segment. Moreover, F admits the representation

$$F(x) = \left\{ y; \ w \cdot y \leq \psi_w(x) \doteq \max_{|u| \leq 1} \ w \cdot \left(f(x) + g(x)u \right) \right\}.$$

Since f, g are linear, each ψ_w is convex. However, this representation does not satisfy all assumptions in Proposition 2.

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