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**Gravitational Instantons
in String Theory**

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Introduction

The main argument of this thesis is the study of gravitational instantons [1,2] in the context of string theory.

They have an intrinsic interest and, in the context of supergravity, may provide a mechanism for the non perturbative breaking of local supersymmetry [3]. In string theory, they have to appear as the non trivial four dimensional background for the propagation of the string; so, the first and in general unresolvable problem to face is the identification of the Conformal Field Theories corresponding to the geometries under consideration. As we will argue, the natural context is a $N=4$ superconformal field theory with $c=6$.

Significant advances have been recently made in identifying conformal field theories related with black-hole space-times in unphysical dimensions ($D=2$) [4,5]. A limit case of instanton conformal field theory has been discussed in a seminal paper by Callan, Harvey and Strominger [6]. The great novelty in string theory is the existence of the dilaton and axion field. In Einstein gravity instantons are not asymptotically flat; the best requirement we can impose is that they are asymptotically locally euclidean, i.e. , at infinity, they approach the flat metric of R^4 only after some finite group quotient. This is why the Konishi et al. [3] mechanism for supergravity spontaneous breaking is considered somewhat unphysical. The problem is solved in string theory by the introduction of the new fields in the effective lagrangian.

We will investigate, in complete generality, the question of modular invariance of the heterotic superstring, compactified on a Calabi-Yau manifold and with a non trivial four dimensional background with $c=6$ and $N=4$, and, in the second part of this thesis, we will focus on the specific example of the Callan et al. instanton and its realization as a conformal field theory.

The knowledge of all the emission vertices in such a theory suggest a more careful investigation of the possibility to compute scattering amplitude and to explore non perturbative effects in string theory.

The thesis is organized in six chapter plus two appendices.

Chapter one deals with the possibility of constructing consistent heterotic vacua where the usual four dimensional flat space is replaced by some new $c=6$ theory, describing string propagation on a non trivial four dimensional geometry. In the case of a gravitational instanton it's well known that the geometry is that of an HyperKahler manifold and so for a general argument the associated sigma model supports a $N=4$ SUSY. The result is the proposal of a *generalized h-map*, according to which the propagation on an instanton background of a heterotic superstring compactified on a Calabi-Yau space is given by the tensor product of three conformal theories: a $c = 6$ (4,4)-theory, a $c = 9$ (2,2)-theory and the $c = 11$ right-moving current algebra of $SO(6) \times E_8'$. A general prediction of our framework is that all particle modes in any instantonic background of this type are classified by an $SU(6)$ symmetry group. This prediction is absolutely analogous to the prediction that, in Calabi-Yau compactifications, massless particles fall into F_4 representations.

In chapter two we explicitly construct all the vertices of massless string excitations on this nontrivial background, comparing the result with the corresponding Kaluza-Klein analysis. In appendix A, the reader will find the complete list of particles, organized in $SU(6)$ representations.

Having solved the general problem of consistence of our construction and indicated the modification of the gauge group coming from Calabi-Yau compactification to some non trivial $N=4$ background, we will examine a particular example of stringy instanton due to Callan. These configurations were originally introduced by D'Auria and Regge [7] long time ago and were more recently rediscovered in string theory by Callan, Harvey and Strominger [6] and by Rey [8]. Their distinguished feature is asymptotic flatness which is an essential feature in order to utilize the instanton in any supersymmetry breaking mechanism à la Konishi et al [3]. Indeed, for the instanton to contribute to a scattering amplitude, the asymptotic states must be the same in flat space and in the instanton background. Such asymptotic flatness requires an unsoldering of the Lorentz-bundle from the tangent bundle which is indeed what the axionic and dilatonic fields combined are able to realize.

In chapters 3,4 we will set the general framework and will study the technology for $N=4$ supersymmetric sigma-models. We study in depth the relation between $(4,4)$ world-sheet supersymmetry and the self-duality (respectively antiselfduality) of the curvatures $R(\omega_R \pm T)$, where T is the torsion and ω_R is the Riemannian connection. Spaces with this property correspond, as we show, to a *generalization of HyperKähler manifolds* and have the proper geometry to describe axionic-dilatonic instantons.

Unfortunately, the Callan et al. instanton is easily realized as a conformal field theory only when reduce to $SU(2) \times R$, exactly the case in which asymptotic flatness is lost. Nevertheless, in chapter 5, we will study it as a toy model and discuss the $(4,4)$ -moduli of this specific model showing that they are four as for flat space. We will explicitly exhibit the infinitesimal deformations of the metric and of the torsion, finding apparently that the $SU(2) \times R$ instanton is a point in a 16-dimensional space. In the same way, if one counts the deformations of the flat space R^4 one finds that they depend on 16 parameters. However, although one gets the correct counting of zero-modes, from the geometrical point of view all the deformations can be reabsorbed by diffeomorphisms (any constant metric can be diagonalized and rescaled to unity in this way). The same does not hold true if the space has the topology of a torus, since global diffeomorphisms must respect the fundamental identifications. In the case of the instanton, as we will see, in general the deformations are significative; however further study is required to decide if someone (and which one) of the deformations can be reabsorbed by diffeomorphisms for certain values of the moduli. The global characterization of this moduli space and the geometric interpretation of the deformed space is therefore still an outstanding problem.

In chapter 6 we will further elucidate the structure of the Callan et al. solution by relating it to a different background geometry via a duality transformation. Duality has been for a long time an efficient method for obtaining new Kahler or HyperKahler manifold from old one, and, nowadays, it has been realized that it's also a symmetry of conformal field theory. This is to say that dual model represents completely different geometry corresponding to the same conformal field theory. Applying the duality transformation to the $SU(2) \times R$ instanton we will have the surprise to obtain a Kahler geometry describing

continuation of a black hole.

Chapter 1

Gravitational instantons and (4,4)-superconformal theories

1.1 The basic idea

We want to investigate the possibility of constructing consistent heterotic string vacua where the usual $c = (6, 4)$ conformal field-theory (CFT) that represents four dimensional flat space is replaced by some new $c = (6, 6)$ theory describing string propagation on a non-trivial four-dimensional geometry. For many reasons, that will become clear in the sequel, particularly appealing are the possibilities offered by $c = 6$ theories possessing $N = 4$ world-sheet supersymmetry. We focus on these theories.

The first part of our discussion is somehow heuristic: we use, as a guideline, the analogy of the scheme we propose with the procedure utilized to compactify string-theory on 6-dimensional manifolds of $SU(3)$ holonomy [9]. As it is well-known [10], from the abstract point of view, this operation is represented as the replacement of the $c = (9, 6)$ theory, corresponding to six flat dimensions, by a $(9, 9)_{2,2}$ conformal theory *

Let us briefly review the process of this compactification, in order to proceed in analogy with it also for the space-time part.

The “initial” situation is that required by critical heterotic string theory [11] in $d=10$, namely the vacuum is a CFT of central charges $(15, 26)$ that can be realized as

$$(15, 26) = (15, 10) \oplus (0, 16) \tag{1.1}$$

The $(15, 10)$ theory is generated by 10 left-moving \oplus 10 right-moving world-sheet bosons, together with 10 left-moving fermions: it represents the heterotic σ -model on flat 10-dimensional space. The $(0, 16)$ theory is that generated by 32 right-moving fermions de-

* From now on we use the notation $(c_L, c_R)_{n_L, n_R}$ to mean a CFT of central charges c_L, c_R in the left (right) sector, possessing n_L, n_R left (right) supersymmetries.

cribing the gauge group G_{gauge} degrees of freedom, namely those of the Kač-Moody algebra \hat{G}_{gauge} . The choice of G_{gauge} is determined by the enforcement of modular invariance and we consider the version of the theory where $G_{gauge} = E'_8 \times E_8$. We consider the 10-dimensional space to be split in a 6-dimensional internal submanifold and a 4-dimensional space-time manifold. At the level of conformal field-theories this means:

$$(15, 26) = (6, 4) \oplus (9, 6) \oplus (0, 16)$$

The main point of the h -map construction [10] is the possibility of considering heterotic string vacua where the above situation is modified as follows:

$$(15, 26) = (6, 4) \oplus (9, 9) \oplus (0, 13) \quad (1.2)$$

This is commonly expressed by saying that six of the heterotic fermions have been “eaten up” by the internal theory (which becomes left-right symmetric); the remaining thirteen generate the current algebra of $E'_8 \times SO(10)$.

From the Kaluza-Klein viewpoint, one is considering a 10-dimensional manifold with the following structure:

$$\mathcal{M}_{10} = \mathcal{M}_6 \times \mathcal{M}_4^{\text{flat}} \quad (1.3)$$

The “eating” of six heterotic fermions is due to the 10-dimensional axion Bianchi identity $dH = 0$ which (at 1st order) requires

$$0 = \text{Tr } F \wedge F - \text{Tr } R_{(6)} \wedge R_{(6)} \quad (1.4)$$

and is solved by embedding the spin connection into the gauge connection. In this way the gauge group is broken to the normalizer of the internal manifold holonomy group $\mathcal{Hol}(\mathcal{M}_6)$.

In the particular case of manifolds with $SU(3)$ holonomy (Calabi-Yau manifolds), the residual gauge group is $E_6 \otimes E'_8$, as it follows from the maximal subgroup embedding:

$$E_6 \times SU(3) \longrightarrow E_8 \quad (1.5)$$

Thus Kaluza-Klein analysis shows that the massless fields on $\mathcal{M}_4^{\text{flat}}$ are organized in E_6 representations. From the abstract point of view, the case of $SU(3)$ holonomy corresponds to the particular case of the decomposition (1.2) in which the internal theory has (2, 2)-supersymmetries:

$$(15, 26) = (6, 4) \oplus (9, 9)_{2,2} \oplus (0, 13)$$

One can show [10] that the $U(1)$ current appearing in the $N = 2$ algebra and the $SO(10)$ currents of the heterotic fermions combine, together with suitable spin fields, to yield the current algebra of E_6 , in due agreement with the maximal subgroup embedding:

$$SO(10) \times U(1) \longrightarrow E_6 \quad (1.6)$$

Hence the emission vertices of the 4-dim fields are organized in E_6 -representations as it is required by Kaluza-Klein analysis.

The question of consistency of these compactified theories and, in particular, the question of their (1-loop) modular invariance is better addressed by looking at their construction from a different viewpoint. Consider a modular-invariant type II superstring vacuum: for what concerns central charges we have:

$$(15, 15) = (6, 6) \oplus (9, 9) \quad (1.7)$$

the (6,6)-theory corresponding to flat 4-dim space and the (9,9)-theory describing some non-trivial “internal” manifold. One shows that the “ h -mapped” heterotic vacuum, obtained by replacing, in the partition function of (1.7), the subpartition function of the two right-moving transverse fermions with that of 2 + 24 fermions (generating a $E'_8 \times SO(2+8) = E'_8 \times SO(10)$ current algebra) is also modular invariant.

When the internal theory has $N = 2$ supersymmetry, the fundamental implication of modular invariance is the projection onto odd-integer charge states with respect to the diagonal $U(1)$ group obtained by summing the $U(1)$ of the $N = 2$ algebra with the $SO(2)$ generated by the transverse space-time fermions. This is just the rephrasing in the present context of the GSO projection [10,13].

Let's now consider an extension of the above described mechanism. We start from the conformal field-theory describing the heterotic string compactified on a Calabi-Yau manifold,

$$(15, 26) = (6, 4) \oplus (9, 9)_{2,2} \oplus (0, 13)$$

and we let the four-dimensional theory eat four of the heterotic fermions, so that

$$(15, 26) = (6, 6) \oplus (9, 9)_{2,2} \oplus (0, 11) \quad (1.8)$$

The remaining heterotic fermions generate a current algebra $E'_8 \times SO(6)$.

From the geometrical σ -model point of view, what we have done is to consider a target space of the form

$$\mathcal{M}_{10} = \mathcal{M}_6 \times \mathcal{M}_4$$

where \mathcal{M}_6 is still a manifold of $SU(3)$ holonomy but \mathcal{M}_4 is no longer flat space. Condition (1.4) extends to

$$0 = \text{Tr} F \wedge F - \text{Tr} R_{(6)} \wedge R_{(6)} - \text{Tr} R_{(4)} \wedge R_{(4)}$$

which can be solved by embedding also the holonomy group $\mathcal{H}ol(\mathcal{M}_4)$ into the gauge group. In particular consider the case where $\mathcal{H}ol(\mathcal{M}_4) \subset SU(2)$: this happens for gravitational instantons, whose curvature is either self-dual or anti-self-dual. In this situation the gauge group is broken to $SU(6)$, as it follows from the maximal subgroup embedding:

$$SU(6) \times SU(3) \times SU(2) \longrightarrow E_8 \quad (1.9)$$

From the abstract viewpoint, this is reproduced if the $c = (6, 6)$ theory possesses a (4,4) supersymmetry:

$$(15, 26) = (6, 6)_{+,+} \oplus (9, 9)_{2,2} \oplus (0, 11) \quad (1.10)$$

Indeed the $U(1)$ current of the $N = 2$ algebra associated with \mathcal{M}_6 , the $SU(2)$ currents of the $N = 4$ algebra associated with \mathcal{M}_4 and the $SO(6)$ currents of the heterotic fermions

combine together with suitable spin fields to yield the $SU(6)$ -current algebra, according to the maximal embedding

$$U(1) \times SU(2) \times SO(6) \longrightarrow SU(6) \quad (1.11)$$

Thus, on this background, the emission vertices for particle-modes (both massive and massless) are organized in $SU(6)$ -representations, as it is requested by Kaluza-Klein analysis.

1.2 The issue of modular invariance

As we already recalled, in the case of compactifications on Calabi-Yau manifolds, that is by means of a $(9,9)_{2,2}$ theory, one starts from a type II modular invariant partition function, corresponding to a CFT:

$$(15, 15) = (6, 6) \oplus (9, 9)$$

The $(6, 6)$ part, corresponding to 4-dimensional flat space, contains the world-sheet bosons X^μ, \tilde{X}^μ and the fermions $\psi^\mu, \tilde{\psi}^\mu$ and the complete partition function has the structure:

$$Z_{tot} = \sum_{i, \bar{i}} Z_{i, \bar{i}}^{(9,9)} Z(X^\mu, \tilde{X}^\mu) B_i^{(4)} \left(B_{\bar{i}}^{(4)} \right)^* B_i^{(-2)} \left(B_{\bar{i}}^{(-2)} \right)^* \quad (1.12)$$

where

- i) $Z(X^\mu, \tilde{X}^\mu)$ is the usual partition function for the four free bosons,
- ii) $B_i^{(4)}$ are the $SO(4)$ -characters in which we can organize the partition function for the four free fermions $\psi^\mu, \tilde{\psi}^\mu$ (the index i taking the values $0, v, s, \bar{s}$, for the scalar, vector and spinor conjugacy class, respectively),
- iii) $B_i^{(-2)}$ are the partition functions for the superghosts,
- iv) $Z_{i, \bar{i}}^{(9,9)}$ is the partition function for the internal theory which couples to reps (i, \bar{i}) of the space-time $SO(4)$ and of the superghosts.

The reason why we have denoted as $B_i^{(-2)}$ the superghost partition function becomes clear from the following considerations. If the superghosts have boundary conditions $\begin{bmatrix} a \\ b \end{bmatrix}$, ($\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$) their partition function can be computed to be[12]:

$$Z^{sg} \begin{bmatrix} a \\ b \end{bmatrix} (\tau|z) = \frac{\eta(\tau)}{\theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau|z)} = \frac{1}{Z \begin{bmatrix} a \\ b \end{bmatrix} (\tau|z)} \quad (1.13)$$

which is exactly the reciprocal of the partition function for two free fermions with spin-structure $\begin{bmatrix} a \\ b \end{bmatrix}^*$

* This reciprocity holds only at genus $g=1$. For higher genera it is amended by a phase factor that amounts to a correct assignment of spin statistics[12]. In all known constructions if one fixes 1-loop modular invariance plus spin statistics, higher loop modular invariance is also ensured. We assume that this will go through also in our construction.

Since the superghosts are forced by world-sheet supersymmetry to have the same spin-structure as the space-time fermions, dealing with the theory described by eq.(1.12) one can use the cancellation of the superghost partition function with the partition function for two fermions. Instead of (1.12) one can simply write

$$Z_{tot} = \sum_{i,\bar{i}} Z_{i,\bar{i}}^{(9,9)} Z(X^\mu, \tilde{X}^\mu) B_i^{(2)} \left(B_{\bar{i}}^{(2)} \right)^* \quad (1.14)$$

that is, one considers only the transverse fermions.

The h -map construction of the associated heterotic theory is based on an isomorphism between the $SO(2n)$ -characters and those of $SO(2n+24)$ or $E'_8 \times SO(2n+8)$. It works as follows. The action of the modular transformations S and T on the characters of $SO(2n)$ in the basis labeled by $0, v, s, \bar{s}$ is given by

$$\begin{aligned} B_i^{(2n)} &\xrightarrow{T} T_{ij}^{(2n)} B_j^{(2n)} \\ B_i^{(2n)} &\xrightarrow{S} S_{ij}^{(2n)} B_j^{(2n)} \end{aligned} \quad (1.15)$$

where

$$\begin{aligned} T^{(2n)} &= \text{diag}(1, 1, e^{in\frac{\pi}{4}}, e^{in\frac{\pi}{4}}) e^{-in\frac{\pi}{12}} \\ S^{(2n)} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & e^{i\frac{n\pi}{12}} & -e^{i\frac{n\pi}{12}} \\ 1 & -1 & -e^{i\frac{n\pi}{12}} & e^{i\frac{n\pi}{12}} \end{pmatrix} \end{aligned} \quad (1.16)$$

The isomorphism is realized by

$$\begin{aligned} T^{(2n)} &= M T^{(2n+24)} M \\ S^{(2n)} &= M S^{(2n+24)} M \end{aligned} \quad (1.17)$$

where the idempotent matrix M is given by:

$$M = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & -1 & 0 \\ & & 0 & -1 \end{pmatrix}$$

It interchanges the scalar and the vector characters, besides flipping the sign of the spinor and antispinor characters. The characters of $E'_8 \times SO(2m)$ transform as those of $SO(2m+16)$, so that the isomorphism permits also to reach these groups.

Due to (1.17), if one replaces the two right-moving transverse ψ^μ fermions with 26 heterotic fermions that generate the gauge group $E'_8 \times SO(10)$, by taking proper account of the matrix M , the resulting theory has a modular invariant partition function.

In eq.(1.12) we made no explicit use of the cancellation between superghosts and longitudinal fermions for the following reason. We wanted to emphasize the possibility of constructing, out of the $Z^{sg} \begin{bmatrix} a \\ b \end{bmatrix}$ and by means of combinations analogous to those used

for the free fermions, new characters labeled by an index $i = 0, v, s, \bar{s}$, whose modular transformations are very similar to those in eq.(1.16).

Indeed, in analogy with the characters of $2n$ fermions let the superghost characters be:

$$\begin{aligned} B_0^{(-2)} &= \frac{1}{Z \begin{bmatrix} 0 \\ 0 \end{bmatrix}} + \frac{1}{Z \begin{bmatrix} 0 \\ 1 \end{bmatrix}} ; B_v^{(-2)} = \frac{1}{Z \begin{bmatrix} 0 \\ 0 \end{bmatrix}} - \frac{1}{Z \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \\ B_s^{(-2)} &= \frac{1}{Z \begin{bmatrix} 1 \\ 0 \end{bmatrix}} + \frac{1}{Z \begin{bmatrix} 1 \\ 1 \end{bmatrix}} ; B_{\bar{s}}^{(-2)} = \frac{1}{Z \begin{bmatrix} 1 \\ 0 \end{bmatrix}} - \frac{1}{Z \begin{bmatrix} 1 \\ 1 \end{bmatrix}} \end{aligned} \quad (1.18)$$

Eq.(1.18) is obtained from the definition of the $B_i^{(2n)}$ characters [10,13] by the replacement $(Z \begin{bmatrix} a \\ b \end{bmatrix})^n \rightarrow 1/Z \begin{bmatrix} a \\ b \end{bmatrix}$. Using the modular transformations of the $Z \begin{bmatrix} a \\ b \end{bmatrix}(\tau)$, already utilized to obtain eq.(1.16), we find :

$$B_i^{(-2)} \xrightarrow{T} T_{ij}^{(-2)} B_j^{(-2)} B_i^{(-2)} \xrightarrow{S} S_{ij}^{(-2)} B_j^{(-2)} \quad (1.19)$$

where

$$\begin{aligned} T^{(-2)} &= \text{diag}(1, 1, e^{-i\frac{\pi}{4}}, e^{-i\frac{\pi}{4}}) e^{i\frac{\pi}{12}} \\ S^{(-2)} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & e^{-i\frac{\pi}{12}} & -e^{-i\frac{\pi}{12}} \\ 1 & -1 & -e^{-i\frac{\pi}{12}} & e^{-i\frac{\pi}{12}} \end{pmatrix} \end{aligned} \quad (1.20)$$

Formally, these matrices are obtained from those in eq.(1.16) by setting $2n = -2$, which explains the chosen notation. Moreover it is manifest that we can use the h -map isomorphism to substitute the characters of the superghosts with those of 22 heterotic fermions with gauge group $E'_8 \times SO(-2 + 8) = E'_8 \times SO(6)$.

Consider now a modular invariant type II vacuum in which the $c = (6, 6)$ part represents a four-dimensional space with non trivial geometry. The partition function of such a theory is

$$Z_{tot} = \sum_{i, \bar{i}} Z_{i, \bar{i}}^{(9,9)} Z_{i, \bar{i}}^{(6,6)} B_i^{(-2)} \left(B_{\bar{i}}^{(-2)} \right)^* \quad (1.21a)$$

$Z_{i, \bar{i}}^{(6,6)}$ being the partition function for the $(6, 6)$ theory which couples to the characters (i, \bar{i}) of the superghosts.

Although the $SO(4)$ characters have disappeared from the game, we can still perform the h -map construction of an associated modular invariant heterotic theory. The result is just of the form (1.8). If, in addition we choose a space-time with $SU(2)$ holonomy, the result is of the form (1.10). After h -map the partition function (1.21a) becomes

$$Z_{tot} = \sum_{i, \bar{i}} Z_{i, \bar{i}}^{(9,9)} Z_{i, \bar{i}}^{(6,6)} B_i^{(-2)} \left(B_i^{(E'_8 \times SO(6))} \right)^* \quad (1.21b)$$

Chapter 2

Zero modes and vertex operators

The next step in the analysis of the heterotic theory (1.21b) is the construction of the corresponding vertex operators. This construction summarizes many aspects of the theory under discussion. Furthermore physical amplitudes are expressed in terms of the vertex correlators, so that knowledge of the vertices is an essential ingredient to extract any physical information.

In the construction of the relevant vertices we proceed in analogy with what one does for the internal dimensions. We relate the counting and the group-theoretical indexing of the possible conformal operators that possess the correct dimensions and charges to the counting of zero-modes for the fields appearing in the low-energy effective supergravity, when this latter is expanded around the particular background, abstractly described by the CFT under investigation. The procedure is like a Kaluza-Klein compactification to zero dimensions. On the other hand, in order to gain a more intuitive comprehension of the role of the operators appearing in the $(6,6)_{+,+}$ theory, it is instructive to compare the vertices with those of flat space. To this purpose it is useful to recall that flat four-dimensional space possesses an $N=4$ world-sheet supersymmetry. Hence we can recast the operators appearing in the vertices in a form suitable of generalization to any $(6,6)_{+,+}$. As pointed out, in what follows we try to establish a general procedure; yet we choose to illustrate it in terms of an example, namely using the $K3$ manifold. The convenience of this choice is manifest. Indeed we want to proceed in analogy with Calabi-Yau compactifications and $K3$ is the unique non-trivial compact Calabi-Yau space in four dimensions. This makes the analogy closer. Furthermore compactification on $K3$ -surfaces has been extensively studied in the past [14] and it is known to be represented by a $(6,6)_{+,+}$ theory, which in some points of moduli space is even solvable, being given by a tensor product of $N = 2$ minimal

models. The knowledge of $K3$ cohomology, described by the Hodge diamond

$$\begin{array}{ccccc}
 & & & & 1 \\
 & & & 0 & 0 \\
 & 1 & & 20 & 1 \\
 & & 0 & & 0 \\
 & & & & 1
 \end{array} \tag{2.2}$$

makes the counting of the zero-modes easy yielding non-trivial results that can be compared with the CFT counting of vertices.

On the contrary, for physical reasons, $K3$ is not the most appealing possibility. It is a gravitational instanton, but it is compact. Our goal is to extend the same techniques to four-dimensional instantons of the effective lagrangian that are asymptotically flat (this last feature seems to be realizable only with torsion[7]). An instantonic solution with the desired properties has appeared frequently in the recent literature [6,8], and in later sections we focus on it. Unfortunately an exact and solvable $N=4$ SCFT corresponding to this solution is known only in a particular limit in which the asymptotic flatness is lost. However the theory remains interesting in its own right as a case study. We may also stress that one of the results of the present thesis is the explicit deformation of this solution by means of its moduli that we discuss later on. In this way we are able to extend the particular background of [6] to a class of solutions depending on certain parameters whose geometrical interpretation is still an open problem.

2.1 The vertex operators

In order to analyse the vertex-operators for the zero-modes we need the field content of the effective four-dimensional theory, which is a matter-coupled $D=4, N=1$ supergravity arising from compactification on the internal Calabi-Yau manifold. This field content is described in the next section.

We begin with the E_6 charged fields given by the gauge multiplet (gauge bosons and gauginos, transforming in the 78 representation), by $h^{2,1}$ WZ multiplets transforming in the 27 and $h^{1,1}$ transforming in the $\overline{27}$ -representation (these Hodge numbers being those of the compactified CY space). We consider the zero-modes of these fields in the classical background provided by a $K3$ manifold. As already emphasized, we embed the space-time spin connection into the gauge connection, breaking the gauge group as follows:

$$E_6 \longrightarrow SU(6) \times SU(2) \tag{2.3}$$

To investigate the zero-modes we must take into account the branching of the representations of E_6 under (2.3).

The adjoint representation is decomposed as

$$78 = (35, 1) + (1, 3) + (20, 2) \tag{2.4}$$

Consider the gaugino field. Its index in the adjoint of E_6 is split accordingly to eq.(2.4); it also has a spinorial index on $K3$. Thus the possible cases are :

- λ_α^A A being an index in the adjoint (35) of $SU(6)$, α being the spinorial index. The zero-modes are in correspondence with the Dolbeaut cohomology $H^{0,q}$. Since the chirality is given by $(-1)^q$, looking at the Hodge diamond (2.3) we see that there are two zero-modes both of the same chirality.
- λ_α^X X in the adjoint of the $SU(2)$ holonomy group of $K3$. The zero-modes should be related to the cohomology groups $H^{0,q}(EndT)$ of Endomorphism-(of the tangent bundle)-valued antiholomorphic forms. By the explicit realization of $K3$ as an algebraic surface one can evaluate the dimension of this cohomology group, case by case.
- $\lambda_\alpha^{a,x}$ a belongs to the 20 of E_6 ; x in the 2 of $SU(2)$ is the same as a contravariant holomorphic index which can be lowered by means of the holomorphic $(2,0)$ form. Because of the spinorial index, the zero-modes correspond to $(1,q)$ harmonic forms. We can therefore have just $h^{1,1} = 20$ zero-modes with the opposite chirality with respect to those in the adjoint of $SU(6)$

Consider then the gauge bosons. According to the decomposition (2.3) we have:

- A_μ^A μ can be a holomorphic or antiholomorphic index. Since, due to the vanishing of $h^{1,0}$ and $h^{0,1}$ the holomorphic $SU(6)$ bundle is trivial, there is no zero-mode of this kind.
- A_μ^X Zero-modes are related to the Dolbeaut cohomology $H^1(EndT)$.
- $A_\mu^{a,x}$ Again, x behaves as a holomorphic index that can be lowered by the holomorphic $(2,0)$ form or by the metric according to the necessity to obtain again an antisymmetric form. Then the zero-modes can be set in correspondence with $(1,1)$ forms, for both the type of μ . We have thus $2 h^{1,1} = 40$ zero-modes of this kind.

The 27 of E_6 is decomposed as

$$27 = (15, 1) + (6, 2) \quad (2.5)$$

Consider the fermion field belonging to any of the WZ multiplets that transform in the 27 representation (these are the charged fields paired to the complex structure deformations of the Calabi-Yau manifold). The decomposition (2.5) gives rise to the following cases:

- χ_α^A A belonging to the 15 of $SU(6)$, α the spinorial index. Zero-modes correspond to the Dolbeaut cohomology $H^{0,q}$ so there are two zero-modes of the same chirality
- $\chi_\alpha^{a,x}$ a is in the 6 of $SU(6)$; x in the 2 of $SU(2)$ is like a holomorphic index; once lowered by the $(2,0)$ form the zero-modes are put into correspondence with $H^{1,q}$ so that there are $h^{1,1} = 20$ modes, of opposite chirality with respect to the previous ones.

The possibilities for the scalars of these 27 families are:

- φ^A for which there is just $h^{0,0} = 1$ zero mode.
- $\varphi^{a,x}$ Lowering the index, the correspondence is with $H^{0,1}$ and so no zero-modes exist

The $\overline{27}$ of E_6 decomposes as

$$\overline{27} = (\overline{15}, 1) + (\overline{6}, \overline{2}) \quad (2.6)$$

For the $\overline{27}$ -spinors we have, analogously to the 27-ones, two zero-modes in the $\overline{15}$ and twenty in the $\overline{6}$, of opposite chiralities; for the $\overline{27}$ -scalars, one mode in the $\overline{15}$.

Consider now the fields of the gravitational multiplet

To look for the zero-modes of the graviton field, i.e. of the metric, means to look for the solutions of the Lichnerowicz equation on $K3$, which are known to be 58. This number is of course determined by the cohomology of $K3$, and to this purpose a discussion is necessary, about the separated counting of the metric and torsion zero-modes. It goes as follows. From the K_3 Hodge diamond (2.2) we know that $h^{2,0} = 1$ and $h^{1,1} = 20$. Let Ω_{ij} be the (2,0)-holomorphic form and let g_{ij^*} be the fiducial Ricci flat Kähler metric ($i, j = 1, 2$) that, for each Kähler class, is guaranteed to exist by the Calabi-Yau condition $c_1(K_3) = 0$. Furthermore let $U_{ij^*}^{(\alpha)}$ be a basis for the (1,1)-forms ($\alpha = 0, 1, \dots, 19$). A variation of the reference metric which keeps it Ricci-flat is given by:

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \delta g_{\mu\nu} \quad ; \quad \delta g_{\mu\nu} = \begin{cases} \delta g_{ij} \\ \delta g_{ij^*} \\ \delta g_{i^*j^*} \end{cases} \quad (2.7)$$

where δg_{ij} , δg_{ij^*} and $\delta g_{i^*j^*}$ are harmonic tensors of the type specified by their indices. Hence we can immediately write:

$$\delta g_{ij^*} = c_\alpha U_{ij^*}^{(\alpha)} \quad (2.8)$$

where c_α are 20 real coefficients. They parametrize the deformations of the Kähler class. On the other hand, using the holomorphic 2-form, any harmonic tensor with two antiholomorphic indices $t_{i^*j^*}$ can be written as the following linear combination:

$$t_{i^*j^*} = -d_\alpha^* \frac{1}{\|\Omega\|^2} \overline{\Omega}_{i^*}^k U_{kj^*}^\alpha \quad (2.9)$$

where raising and lowering of the indices is performed by means of the fiducial metric and where d_α^* are constant complex coefficients. Since $h^{2,0} = 1$ it follows that, of the 20 independent linear combinations appearing in (2.9), only one leads to an antisymmetric $t_{i^*j^*}$; all the other combinations produce a symmetric tensor $t_{i^*j^*}$. Hence we can choose a basis of the (1,1)-harmonic forms such that:

$$\overline{\Omega}_{i^*j^*} = -\frac{1}{\|\Omega\|^2} \overline{\Omega}_{i^*}^k U_{kj^*}^0 \quad (2.10)$$

$$-\frac{1}{\|\Omega\|^2} \overline{\Omega}_{i^*}^k U_{kj^*}^a = S_{i^*j^*}^a = S_{j^*i^*}^a \quad (a = 1, \dots, 19) \quad (2.11)$$

The 19 symmetric tensors $S_{i^*j^*}^a$ provide a basis for the expansion of the antiholomorphic part of the metric deformation

$$\delta g_{i^*j^*} = d_a S_{i^*j^*}^a \quad (2.12)$$

The holomorphic part just is the complex conjugate and it is expanded along the complex conjugate basis $S_{ij}^a : \delta g_{ij} = d_a S_{ij}^a$. The 19 complex coefficients d_a parametrize the complex structure deformations of the K_3 manifold. Summarizing the 58 zero-modes of the metric emerge from the following counting:

$$\# \text{ metric zero - modes} = h^{1,1} + 2 (h^{1,1} - 1) \quad (2.13)$$

This formula is just a consequence of $h^{2,0} = 1$ and it has a meaning also for non-compact manifolds, like the instanton we consider later in this paper, as a counting of local deformations. For global deformations one has still to check if they can be reabsorbed by diffeomorphisms.

In string-theory, the metric is not the only background field. We have also the antisymmetric axion $B_{\mu\nu}$, whose curl $H_{\lambda\mu\nu}$ is identified with the torsion $T_{\lambda\mu\nu}$, as we are going to see while discussing the σ -model formulation (see section 4). The zero-modes of the field $B_{\mu\nu}$ are counted in a similar way to the case of the metric. From the linearized field equation around the reference background, one concludes that δB_{ij} , $\delta B_{i^*j^*}$ and δB_{ij^*} must be harmonic tensors. Because of the different symmetry of the indices, this time we have:

$$\delta B_{ij} = A \Omega_{ij} \quad (2.14)$$

$$\delta B_{ij^*} = b_\alpha U_{ij^*}^\alpha \quad (2.15)$$

where A is a complex parameter and b_α are real parameters. Hence we have 22 axion zero-modes that emerge from the following counting:

$$\# \text{ axion zero - modes} = h^{1,1} + 2 \quad (2.16)$$

Altogether there are $58 \oplus 22 = 80 = 4h^{1,1}$ zero modes of the field $g_{\mu\nu} + iB_{\mu\nu}$. In the next subsection we see that this counting agrees with the counting of $N=4$ preserving marginal operators in a $(6,6)_{4,4}$ -theory.

The gravitino zero-modes are the zero-modes of the Rarita-Schwinger operator. Utilizing the standard trick of writing spinors as differential forms we can relate the number of these modes to the dimensions of the cohomology groups. Let

$$\begin{aligned} \{ \Gamma_i, \Gamma_j \} &= 0 \quad ; \quad \{ \Gamma_{i^*}, \Gamma_{j^*} \} = 0 \\ \{ \Gamma_i, \Gamma_{j^*} \} &= 2 g_{ij^*} \end{aligned} \quad (2.17)$$

be the Clifford algebra written in a well-adapted basis. A spin $\frac{3}{2}$ field ψ_μ can be written as follows:

$$\begin{aligned} \psi_i &= (\omega_i \mathbf{1} + \omega_{ij^*} \Gamma^{j^*} + \omega_{ij^*k^*} \Gamma^{j^*k^*}) | \zeta \rangle \\ \psi_{i^*} &= (\omega_{i^*} \mathbf{1} + \omega_{i^*j^*} \Gamma^{j^*} + \omega_{i^*j^*k^*} \Gamma^{j^*k^*}) | \zeta \rangle \end{aligned} \quad (2.18)$$

where the spinor $|\zeta\rangle$ satisfies the condition:

$$\Gamma_{i^*} | \zeta \rangle = \Gamma^i | \zeta \rangle = 0 \quad (2.19)$$

The field ψ_μ is a zero mode if the coefficients ω_{\dots} in (2.18) are harmonic tensors. Hence from ω_i and ω_{i^*} we get $h^{1,0}$ and $h^{0,1}$ zero-modes respectively. From ω_{ij^*} and $\omega_{i^*j^*}$ we obtain $h^{1,1} + h^{1,1}$ zero-modes. Finally 2 $h^{1,2}$ zero-modes arise from $\omega_{ij^*k^*}$ and $\omega_{i^*j^*k^*}$. In view of the symmetries of the Hodge diamond the total number of zero-modes for the gravitino field is given by the formula

$$\# \text{ gravitino zero - modes} = 2 h^{1,1} + 4 h^{1,0} \quad (2.20)$$

In the case of K_3 the above number is 40.

Finally, for the E_6 neutral WZ multiplets, the fermion has two zero-modes of the same chirality (in correspondence with $H^{0,q}$), while the scalar has just the (trivial) zero-mode corresponding to $H^{0,0}$.

2.2 The $(6,6)_{\pm,\pm}$ theory and the moduli operators

A fundamental role is played by those fields of the $N=4$ theory that can be identified with the abstract (1,1)-forms of the associated manifold [13,sec.VI.10]. The $N=4$ algebra contains the stress-energy tensor, four supercurrents and three currents A^i that close an $SU(2)_1$ algebra. In a (4,4)-theory there is a realization of these operators both in the left and in the right sector. The fields of the theory are organized in representations of $SU(2)_L \otimes SU(2)_R$. We denote by $\Phi \left[\begin{smallmatrix} h, \bar{h} \\ J, \bar{J} \end{smallmatrix} \right]^{m, \bar{m}}$ a primary conformal field with left and right dimensions h, \bar{h} and isospins J, \bar{J} , and with third components m, \bar{m} . Consider for example the left sector. The $SU(2)_1$ can be bosonized in terms of a single free boson $\tau(z)$:

$$A^3 = \frac{i}{\sqrt{2}} \partial\tau \quad ; \quad A^\pm = e^{\pm i\sqrt{2}\tau} \quad (2.21)$$

The spectral flow of the $N=2$ theories is extended to a “multiplets of spectral flows”:

$$\Phi \left[\begin{smallmatrix} h \\ J \end{smallmatrix} \right]^m = e^{im\sqrt{2}\tau} \hat{\Phi}^{(h-m^2)} \quad (2.22)$$

where $\hat{\Phi}^{(h-m^2)}$ is a singlet of $SU(2)$ of conformal weight $h - m^2$.

For example a doublet of $SU(2)$, $\Psi \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right]$, made of an $N=2$ chiral and an antichiral field of weight $1/2$, (note that the charge respect to the $U(1)$ of the $N=2$ contained in the $N=4$ is twice the third component of the isospin) in the NS sector is related by the spectral flow (2.22) to an $SU(2)$ singlet in the R sector:

$$\Psi \left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix} \right]^{\pm \frac{1}{2}} = e^{\pm i\frac{\tau}{\sqrt{2}}} \Psi \left[\begin{smallmatrix} 1/4 \\ 0 \end{smallmatrix} \right] \quad (2.23)$$

We use the convention of giving the same name to fields related by spectral flow, distinguishing them when necessary by their weight and isospin.

As explained in [13], the $N=4$ analogues of the (c, c) and (c, a) fields of weight $(\frac{1}{2}, \frac{1}{2})$, which play the role of “abstract” $(1,1)$ - and $(2,1)$ -forms in the $(9,9)_{2,2}$ theory, is given by those primary fields of the $(6,6)_{\pm,\pm}$ CFT that are of the form

$$\Psi_A \left[\begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \end{array} \right] \quad (2.24)$$

and correspond to the lowest components in a short representation of the $N=4$ algebra. In (2.24) the index A runs on $h^{1,1}$ values. Focusing on the left sector a short representation is made of the following set of fields

$$\Psi \left[\begin{array}{c} 1/2 \\ 1/2 \end{array} \right]^a (z), \quad \Phi \left[\begin{array}{c} 1 \\ 0 \end{array} \right] (z), \quad \Pi \left[\begin{array}{c} 1 \\ 0 \end{array} \right] (z)$$

satisfying the OPEs

$$\begin{aligned} \mathcal{G}^a(z)\Psi^b(w) &= \frac{\epsilon^{ab}\Phi(w)}{z-w} + reg. \\ \bar{\mathcal{G}}^a(z)\Psi^b(w) &= \frac{\delta^{ab}\Pi(w)}{z-w} + reg. \\ \mathcal{G}^a(z)\Phi(w) &= \bar{\mathcal{G}}^a(z)\Pi(w) = 0 \\ \bar{\mathcal{G}}^a(z)\Phi(w) &= 2\epsilon^{ab}\partial \left(\frac{\Psi^b(w)}{z-w} \right) + reg. \\ \mathcal{G}^a(z)\Pi(w) &= -2\delta^{ab}\partial \left(\frac{\Psi^b(w)}{z-w} \right) + reg. \end{aligned} \quad (2.25)$$

where $\mathcal{G}^a(z), \bar{\mathcal{G}}^a(z)$, $a=1,2$ denote the supercurrents organized in two $SU(2)$ doublets. The fields Φ and Π have dimension 1 and, being the last components of an $N=4$ representation (see the last two of the OPEs (2.25)), when added (in suitable combinations of the left and right sectors) to the Lagrangian they don't break its $N=4$ invariance. We call them the “ $N=4$ moduli”.

As already hinted, the fields $\Psi_A \left[\begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \end{array} \right]$ represent the abstract $(1,1)$ -forms on the manifold described by the $(6,6)_{\pm,\pm}$ -theory.

As a first example of $(6,6)_{\pm,\pm}$ theory let's briefly consider that associated with flat space. The $N=4$ algebra (as an illustration we consider the left moving sector) is realized by the stress-energy tensor

$$T(z) = -\frac{1}{2}\partial X^\mu \partial X^\mu + \psi^\mu \partial \psi^\mu \quad (2.26)$$

by the supercurrents

$$\begin{aligned} G^0(z) &= \sqrt{2}\psi^\mu \partial X^\mu \\ G^x(z) &= \sqrt{2} \left(\hat{\mathcal{J}}^x \psi \right)^\mu \partial X^\mu \end{aligned} \quad (2.27)$$

and by the $SU(2)$ currents

$$A^i(z) = -\frac{i}{2}\psi^\mu \hat{\mathcal{J}}_{\mu\nu}^i \psi^\nu = i(\psi^0 \psi^i + \frac{1}{2}\epsilon^{ijk} \psi^j \psi^k) \quad (2.28)$$

The three tensors $\hat{\mathcal{J}}_{\mu\nu}^x$ are constant complex structures and satisfy the quaternionic algebra; the role of their generalizations to any $N=4$ theory will be discussed at length in section 5. For the moment it suffices to know that they can be explicitly constructed, so that from their expression (implicitly exhibited in eq.(2.28)) we get

$$\begin{aligned} G^1 &= \sqrt{2} \{ \psi^0 \partial X^1 - \psi^3 \partial X^2 + \psi^2 \partial X^3 - \psi^1 \partial X^0 \} \\ G^2 &= \sqrt{2} \{ \psi^3 \partial X^1 + \psi^0 \partial X^2 - \psi^1 \partial X^3 - \psi^2 \partial X^0 \} \\ G^3 &= \sqrt{2} \{ -\psi^2 \partial X^1 + \psi^1 \partial X^2 + \psi^0 \partial X^3 - \psi^3 \partial X^0 \} \end{aligned}$$

In the left sector we can find two short representations, given by

$$\begin{aligned} \Psi_1 \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} &= \begin{pmatrix} \psi^0 + i\psi^3 \\ \psi^2 + i\psi^1 \end{pmatrix} \quad ; \quad \Phi_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\partial X^2 - i\partial X^1 \quad ; \quad \Pi_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\partial X^0 - i\partial X^3 \\ \Psi_2 \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} &= \begin{pmatrix} \psi^2 - i\psi^1 \\ -(\psi^0 - i\psi^3) \end{pmatrix} \quad ; \quad \Phi_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \partial X^0 - i\partial X^3 \quad ; \quad \Pi_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\partial X^2 + i\partial X^1 \end{aligned} \quad (2.29)$$

Two analogous ones exist in the right sector. Multiplying them in all possible ways we obtain four abstract (1,1)-forms $\Psi_A \begin{bmatrix} 1/2, 1/2 \\ 1/2, 1/2 \end{bmatrix}$. For instance we can set:

$$\begin{aligned} \Psi_1 \begin{bmatrix} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \end{bmatrix} (z, \bar{z}) &= \Psi_1 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (z) \Psi_1 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (\bar{z}) \\ \Psi_2 \begin{bmatrix} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \end{bmatrix} (z, \bar{z}) &= \Psi_1 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (z) \Psi_2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (\bar{z}) \\ \Psi_3 \begin{bmatrix} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \end{bmatrix} (z, \bar{z}) &= \Psi_2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (z) \Psi_1 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (\bar{z}) \\ \Psi_4 \begin{bmatrix} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \end{bmatrix} (z, \bar{z}) &= \Psi_2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (z) \Psi_2 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} (\bar{z}) \end{aligned} \quad (2.30)$$

This number of $N = 4$ -moduli agrees with the Hodge diamond of flat space *

$$\begin{array}{ccccc} & & & & 1 \\ & & & & 2 & 2 \\ & & & & 1 & 4 & 1 \\ & & & & 2 & 2 \\ & & & & 1 \end{array} \quad (2.31)$$

* We refer by this to the Hodge diamond of the flat space compactified to a torus.

According to (2.31) we have also two holomorphic 1-forms and two antiholomorphic ones. At the level of CFT they are represented by operators of the form $\Psi_{\mathcal{A}} \left[\begin{smallmatrix} \frac{1}{2}, 0 \\ \frac{1}{2}, 0 \end{smallmatrix} \right]$ and $\Psi_{\mathcal{A}^*} \left[\begin{smallmatrix} 0, \frac{1}{2} \\ 0, \frac{1}{2} \end{smallmatrix} \right]$, respectively. The index $\mathcal{A}(\mathcal{A}^*)$ runs on $2=h^{1,0}$ ($h^{0,1}$) values. The explicit expression of the two (0,1)-forms can be taken to be

$$\begin{aligned} \Psi_{1^*} \left[\begin{smallmatrix} 0, \frac{1}{2} \\ 0, \frac{1}{2} \end{smallmatrix} \right] &= \mathbf{1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z) \Psi_1 \left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix} \right] (\bar{z}) \\ \Psi_{2^*} \left[\begin{smallmatrix} 0, \frac{1}{2} \\ 0, \frac{1}{2} \end{smallmatrix} \right] &= \mathbf{1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} (z) \Psi_2 \left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix} \right] (\bar{z}) \end{aligned} \quad (2.32)$$

The two (1,0)-forms have an analogous expression with the role of the left and right-moving sectors interchanged.

Another interesting point is the identification of the spin fields with the spectral flows of the identity operator and of the lowest component of the short representations (2.30). This is a very important point because the spin fields appear in the fermion emission vertices. If we are able to recast these vertices in an abstract $(6,6)_{\pm,4}$ language the extension from flat space to an instanton background is guaranteed. The gravitino emission vertex, for instance, that includes the proper gravitino and dilatino vertices, in flat space has the following expression [13, page 2063]:

$$V_{\mu}^{\alpha} (k, z, \bar{z}) = e^{\frac{1}{2} \phi^{sg}(z)} S_{\alpha}(z) \bar{\partial} \bar{X}(\bar{z}) e^{ik \cdot X(z, \bar{z})} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix} \quad (2.33)$$

$$V_{\dot{\alpha}\mu} (k, z, \bar{z}) = e^{\frac{1}{2} \phi^{sg}(z)} S^{\dot{\alpha}}(z) \bar{\partial} \bar{X}(\bar{z}) e^{ik \cdot X(z, \bar{z})} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix} \quad (2.34)$$

the two formulae referring to the two possible chiralities. The last operator in the above formulæ is a spectral flow of the identity in the internal theory. In order to convert these expressions to an abstract $N=4$ notation we need the interpretation of the operators $S_{\alpha}(z) \bar{\partial} \bar{X}(\bar{z}) e^{ik \cdot X(z, \bar{z})}$ and $S^{\dot{\alpha}}(z) \bar{\partial} \bar{X}(\bar{z}) e^{ik \cdot X(z, \bar{z})}$.

To this effect we note that $\tilde{P}^{\mu}(\bar{z}) = \bar{\partial} X^{\mu}(z, \bar{z})$ is expressed by linear combinations of the operators $\tilde{\Pi}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\tilde{\Pi}_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\tilde{\Phi}_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\tilde{\Phi}_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the right-sector counterparts of those appearing in (2.30). It remains to consider the spin fields. The four free fermions can be bosonized in terms of two free bosons as

$$\begin{aligned} \psi^0 \pm i\psi^3 &= \pm c^{\pm} e^{\pm i\varphi_2} \\ \psi^2 \pm i\psi^1 &= \pm c^{\pm} e^{\pm i\varphi_1} \end{aligned} \quad (2.35)$$

where the signs and the cocycle factor ($c^{\pm} = e^{\mp \pi p^1}$) are arranged to reproduce the anticommutation properties of the fermions. The $SU(2)$ currents of eq.(2.28) can be reexpressed via eq.(2.35). In particular

$$A^{\pm} = \pm c^{\pm} e^{\pm i\varphi_2} e^{\mp i\varphi_1}$$

However, we can rephrase all the algebra in terms of the vertex operators $e^{\pm i\varphi_1}$, $e^{\pm i\varphi_2}$, eliminating the need of preserving anticommutation relations (these operators anticommute

with themselves and commute with each other). Then the $SU(2)$ currents are simply given by

$$\begin{aligned} A^3 &= \frac{i}{2}(\partial\varphi_2 - \partial\varphi_1) \\ A^\pm &= e^{\pm i\varphi_2} e^{\mp i\varphi_1} \end{aligned} \quad (2.36)$$

Comparison with the standard bosonized form (2.21) is immediate. We get:

$$\tau = \frac{1}{\sqrt{2}}(\varphi_2 - \varphi_1)$$

so that the spectral flow of eq.(2.22) is rewritten as

$$\Phi \begin{bmatrix} h \\ J \end{bmatrix}^m = e^{im(\varphi_2 - \varphi_1)} \hat{\Phi}(h - m^2) \quad (2.37)$$

The fields Ψ_1, Ψ_2 of eq.(2.30), as doublets with respect to the currents (2.36) are given by

$$\Psi_1 = \begin{pmatrix} e^{i\varphi_2} \\ e^{i\varphi_1} \end{pmatrix} ; \Psi_2 = \begin{pmatrix} e^{-i\varphi_1} \\ e^{-i\varphi_2} \end{pmatrix} \quad (2.38)$$

We can single out the spectral flow and find their Ramond partners:

$$\begin{aligned} \Psi_1 \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} &= \begin{pmatrix} e^{i\varphi_2} \\ e^{i\varphi_1} \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}(\varphi_2 - \varphi_1)} \\ e^{-\frac{i}{2}(\varphi_2 - \varphi_1)} \end{pmatrix} e^{\frac{i}{2}(\varphi_2 + \varphi_1)} = \text{spectral flow} \cdot \Psi_1 \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \\ \Psi_2 \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} &= \begin{pmatrix} e^{-i\varphi_1} \\ e^{-i\varphi_2} \end{pmatrix} = \begin{pmatrix} e^{\frac{i}{2}(\varphi_2 - \varphi_1)} \\ e^{-\frac{i}{2}(\varphi_2 - \varphi_1)} \end{pmatrix} e^{-\frac{i}{2}(\varphi_2 + \varphi_1)} = \text{spectral flow} \cdot \Psi_2 \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \end{aligned}$$

Therefore we see that these R fields are just the two spin fields of positive chirality. Indeed, once the fermions have been bosonized as in eq.(2.35), the spin fields, corresponding to the weights of the $SO(4)$ spinor (s) and antispinor (\bar{s}) representations, are expressed as follows

$$\begin{aligned} + \text{chirality } (s \text{ rep}) : \quad & S^1 = e^{\frac{i}{2}\varphi_2 + \frac{i}{2}\varphi_1} \\ & S^2 = e^{-\frac{i}{2}\varphi_2 - \frac{i}{2}\varphi_1} \\ - \text{chirality } (\bar{s} \text{ rep}) : \quad & S^1 = e^{\frac{i}{2}\varphi_2 - \frac{i}{2}\varphi_1} \\ & S^2 = e^{-\frac{i}{2}\varphi_2 + \frac{i}{2}\varphi_1} \end{aligned} \quad (2.39)$$

Finally note that the spin fields of negative chirality form a doublet under the $SU(2)_L$ and are related through spectral flow to the identity operator:

$$\begin{pmatrix} S^1 \\ S^2 \end{pmatrix} \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix} = \begin{pmatrix} e^{\frac{i}{2}(\varphi_2 - \varphi_1)} \\ e^{-\frac{i}{2}(\varphi_2 - \varphi_1)} \end{pmatrix} = \text{spectral flow} \cdot \mathbf{1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.40)$$

Comparing these results with equations (2.33,34) we see that, in flat space, the 8 gravitino zero-modes of positive chirality are given by the left spectral flow of the abstract (1,1)-forms (2.30), while the 8 zero-modes of negative chirality are given by the left spectral flow

of the $(0,1)$ -forms (2.32). In both cases, the right-moving part of the operator is SUSY-transformed to the last multiplet-components. This is in perfect agreement with formula (2.20) and with the Hodge diamond of flat space (2.31). In the case of the K_3 -manifold only the positive chirality zero-modes are present, since the analogues of the $(0,1)$ -forms (2.32) do not exist ($h^{1,0} = 0$).

This general discussion suffices to illustrate the idea of the generalized h -map and of the use of $(6,6)_{\pm,\pm}$ theories as a description of gravitational instantons backgrounds. In Appendix A we give an exhaustive list of the emission vertices for all particle zero-modes in a generic $(6,6)_{\pm,\pm}$ case.

Chapter 3

Rheonomic description of (1,1) sigma-models

We turn now to the main part of this thesis: a specific example of stringy instanton on which to test the above general framework. A natural candidate is the following solution [6,7] of the beta-functions equation for the effective string theory:

$$ds^2 = e^{-2\Phi}(dx)^2$$
$$e^{-2\Phi} = A + \frac{2k}{r^2}$$

$$\Phi = H_{abc} = \frac{1}{3} \epsilon_{abcd} \partial_d \Phi$$

In any case, although a solution to order $O(\alpha')$ of the equations of motion is not usually an all order solution, the N=4 SUSY guarantees the all order exactness of the solution.

In this chapter and the next we will set the general language for supersymmetric sigma-models.

3.1 The Bosonic σ -model

In correspondence with a solution of the equations of motion derived from the effective bosonic lagrangian:

$$\mathcal{L}_{eff} = e^{-2\Phi}(\mathcal{R} - 4D_\mu \Phi D^\mu \Phi + \dots)$$

that contains the metric $G_{\mu\nu}$ (i.e. equivalently the vielbeins V^a), the three form H and the dilaton Φ , we write the action for the correspondent bosonic σ -model utilizing a geometric

first order formalism:

$$\begin{aligned} & \frac{1}{4\pi} \int_{\partial\mathcal{M}} V^a (\Pi_+^a e^+ - \Pi_-^a e^-) + \Pi_+^a \Pi_-^a e^+ e^- - 2\Phi R^{(2)} + p_+ T^+ + p_- T^- + \\ & + \frac{1}{4\pi} \int_{\mathcal{M}} H_{abc} V^a V^b V^c \end{aligned} \quad (3.1)$$

Once rewritten in the 2^{nd} order formalism, this action takes more familiar form

$$S = \frac{-1}{4\pi} \int_{\partial\mathcal{M}} dz d\bar{z} [G_{\mu\nu}(X) \partial X^\mu \bar{\partial} X^\nu + B_{\mu\nu}(X) \partial X^\mu \bar{\partial} X^\nu] \quad (3.2)$$

where $ds^2 = G_{\mu\nu} dX^\mu \otimes dX^\nu = V^a \otimes V^a$ is the target space line element and the antisymmetric tensor $B_{\mu\nu}$ is such that $H = 2dB$.

Note that when the action is written as in eq.(3.2) it keeps no tracks of the dilaton which contributes only to the classical stress-energy tensor of the model. This contribution is obtained in a simple way in the 1^{st} order formulation. In a similar way, when we consider the supersymmetric extensions of the above model, the contributions of the dilaton to the supercurrents are also easily retrieved from the 1^{st} order formulation.

Let's briefly explain the somewhat unusual notations and the meaning of the quantities appearing in eq.(3.1)[13,23]. In particular e^+ and e^- are the vielbein on the world-sheet $\partial\mathcal{M}$, whose geometry is described by the structure equations

$$\begin{aligned} de^+ - \omega^{(2)} e^+ &= T^+ \\ de^- + \omega^{(2)} e^- &= T^- \\ d\omega^{(2)} &= R^{(2)} \end{aligned} \quad (3.3)$$

$\omega^{(2)}, T^\pm, R^{(2)}$ are the two-dimensional spin connection, torsion and curvature respectively. Classical conformal invariance of the model allows the choice of the ‘‘special conformal gauge’’:

$$e^+ = dz \quad ; \quad e^- = d\bar{z} \quad ; \quad \omega^{(2)} = R^{(2)} = 0 \quad (3.4)$$

where $z = x^0 + x^1$ and $\bar{z} = x^0 - x^1$. This is the choice we have used to obtain the 2^{nd} order form of the action (2). More specifically ‘‘after variation’’ we can use eq.(3.4). Π_\pm^a, p_\pm are ‘‘ 1^{st} order fields’’: they can be reexpressed in terms of the usual dynamical fields upon use of the equations obtained by varying in $\Pi_\pm^a, p_\pm, \omega^{(2)}$.

$$\begin{aligned} \text{Varying in } \Pi_\pm^a &: \quad \Pi_\pm^a = V_\pm^a \\ \text{Varying in } \omega^{(2)} &: \quad p_\pm = \mp 2\partial_\pm \Phi = \mp 2\partial_a \Phi V_\pm^a \\ \text{Varying in } p_\pm &: \quad T_+ = T_- = 0 \end{aligned} \quad (3.5)$$

In the present formalism, the general recipe to obtain the components of the stress-energy tensor is to vary the action with respect to the w.s. vielbein, defining

$$\delta S = \frac{-1}{2\pi} \int \mathcal{T}_+ \delta e^+ + \mathcal{T}_- \delta e^- \quad (3.6)$$

and to consider the expansion $\mathcal{T}_+ = \mathcal{T}_{++}e^+ + \mathcal{T}_{+-}e^-$ (and the analogous one for \mathcal{T}_-). The conformal invariance of the model implies that $\mathcal{T}_{+-} = \mathcal{T}_{-+} = 0$ and one defines the usual holomorphic and antiholomorphic part of the stress-energy tensor to be

$$T(z) = \mathcal{T}_{++} \quad ; \quad \tilde{T}(\bar{z}) = \mathcal{T}_{--} \quad (3.7)$$

For the model described by the action (4.1) varying, for example, in e^+ , we obtain:

$$\delta S = \frac{-1}{2\pi} \left(-\frac{1}{2} \right) \int (V^a \Pi_+^a - \Pi_+^a \Pi_-^a e^-) \delta e^+ + p_+ d\delta e^+$$

Substituting eqs.(3.5), we obtain:

$$T(z) = \mathcal{T}_{++} = -\frac{1}{2} V_z^a V_z^a - \partial\partial\Phi \quad (3.8)$$

In view of our specific interest in the instanton configuration of eq.(3.28), let us consider the above σ -model in the case where $d=4$ and the metric, dilaton and axion fields are chosen as follows:

$$\begin{aligned} ds^2 &= e^{-2\Phi} (dx)^2 & V^a &= e^{-\Phi} e^a \\ e^{-2\Phi} &= \frac{2k}{r^2} & \Leftrightarrow & \Phi = \log \frac{r}{\sqrt{2k}} \end{aligned} \quad (3.9)$$

$$\Phi = H_{abc} = \frac{1}{3} \epsilon_{abcd} \partial_d \Phi$$

The above eq.s correspond to the limit $A = 0$ of (3.28). In this limit the manifold has the curious and somewhat unwanted topology of $R \otimes SU(2)$, which is not asymptotically flat. Asymptotic flatness is instead ensured when A is non vanishing. Yet as we are going to see at $A = 0$ the corresponding σ -model defines a solvable conformal and superconformal field-theory. Hence this limit is quite worth to be considered. In (3.9) $\{e^a\}$ is a set of vielbein for the flat 4-dimensional space, r being a radial coordinate and the remaining 3 coordinates being the coordinates of a 3-sphere. Indeed we choose to write the flat metric as follows

$$dx^2 = dr^2 + \frac{r^2}{2} \Omega^i \otimes \Omega^i$$

where Ω^i are the Maurer-Cartan forms of $SU(2)$ which satisfy the equations

$$d\Omega^i = -\frac{\epsilon^{ijk}}{\sqrt{2}} \Omega^j \Omega^k$$

so that $\frac{1}{2} \Omega^i \otimes \Omega^i$ is the metric on the three-sphere of unit radius. The metric of the configuration (3.9) becomes

$$ds^2 = 2k \frac{dr^2}{r^2} + k \Omega^i \otimes \Omega^i \quad (3.10)$$

Redefining the radial coordinate as follows: $t = \sqrt{2k} \log(r/\sqrt{2k})$ we obtain:

$$ds^2 = dt^2 + k \Omega^i \otimes \Omega^i \quad (3.11)$$

(showing that the singularity in (3.10) is a coordinate artifact), while the dilaton is linear in the coordinate t :

$$\Phi = \frac{t}{\sqrt{2k}} \quad (3.12)$$

In correspondence with eq.(3.11) we choose the vielbeins as follows:

$$\begin{aligned} V^0 &= dt \\ dV^i &= -\sqrt{k} \Omega^i \end{aligned} \quad (3.13)$$

The only non-zero components of H in the Maurer-Cartan basis $\{\Omega^i\}$ turn out to be

$$H_{ijk} = \frac{1}{3} \epsilon_{ijk} (-\sqrt{k})^3 \frac{\partial \Phi}{\partial t} = -\frac{k}{3} \frac{\epsilon_{ijk}}{\sqrt{2}} \quad (3.13)$$

(note that, with our choice of Maurer-Cartan forms, $\sqrt{2} \epsilon_{ijk}$ are just the structure constants of $SU(2)$).

The σ -model action corresponding to the configuration we have described is $S = S_t + S_{\text{WZW}}$ where

$$\begin{aligned} S_t &= \frac{1}{4\pi} \int_{\partial\mathcal{M}} dt (\Pi_+ e^+ - \Pi_- e^-) + \Pi_+ \Pi_- e^+ e^- - \sqrt{\frac{2}{k}} t R^{(2)} + p_+ T^+ + p_- T^- \\ S_{\text{WZW}} &= \frac{1}{4\pi} \int_{\partial\mathcal{M}} -\sqrt{k} \Omega^i (\Pi_+^i e^+ - \Pi_-^i e^-) + \Pi_+^i \Pi_-^i e^+ e^- - \frac{1}{3} \frac{k}{4\pi} \int_{\mathcal{M}} \frac{\epsilon_{ijk}}{\sqrt{2}} \Omega^i \Omega^j \Omega^k \end{aligned} \quad (3.14)$$

Once rewritten in 2^{nd} order formalism, these two actions take the simpler form

$$\begin{aligned} S_t &= \frac{-1}{4\pi} \int_{\partial\mathcal{M}} dz d\bar{z} \partial t \bar{\partial} t \\ S_{\text{WZW}} &= \frac{-k}{4\pi} \int_{\partial\mathcal{M}} dz d\bar{z} \Omega^i_+ \Omega^i_- - \frac{1}{3} \frac{k}{4\pi} \int_{\mathcal{M}} \frac{\epsilon_{ijk}}{\sqrt{2}} \Omega^i \Omega^j \Omega^k \end{aligned} \quad (3.15)$$

S_{WZW} is the correct expression for the action of the WZW model realized at level k , and corresponds to a CFT of central charge

$$c_{\text{WZW}} = \frac{3k}{k+2} \quad (3.16)$$

The field $\varphi = -it$ is a free scalar boson with background charge $Q_{bk} = -i\sqrt{\frac{2}{k}}$. Indeed from the action (3.14), using the general recipe provided by eq. (3.8), we obtain the following stress-energy tensor:

$$T_t(z) = -\frac{1}{2} (\partial t)^2 - \frac{1}{\sqrt{2k}} \partial \bar{\partial} t \quad (3.17)$$

which corresponds to a central charge

$$c_t = 1 + \frac{6}{k} \quad (3.18)$$

This follows from the general formula

$$c = 1 - 3Q_{bk}^2 \quad (3.19)$$

substituting the value of the background charge.

As one sees the σ -model on the configuration (3.9) is exactly conformal invariant at the quantum level and leads to a solvable conformal field theory: namely a tensor product of a Feigin-Fuchs model with a WZW-model (see ref.s [24,25,26]). This is the reason why we are particularly interested in this specific instanton that provides a good toy-model.

Actually, in order to discuss superstring theory, we rather need the supersymmetric version of the just described σ -model . Strictly speaking, heterotic string theory would require a (1,0) supersymmetrization; however, in view of the h -map, we can go one step beyond and consider the case of (1,1) local supersymmetry. Therefore we recall now some essential features of the geometrical formulation of (1,1) supersymmetric σ -model [15] and we include dilaton contributions.

3.2 The (1,1) locally supersymmetric σ -model

One realizes a classical superconformal invariant theory in terms of fields living on a super-world-sheet with two fermionic coordinates θ and $\bar{\theta}$ besides the two bosonic ones z and \bar{z} . The cotangent basis on the super world-sheet(the “supervielbein”) is given by the already introduced 1-forms e^+, e^- and two bidimensional gravitinos ζ, χ .

The structure equations (3.3) are enlarged by the appearance of two fermionic torsion 2-forms:

$$\begin{aligned} T^\bullet &= d\zeta - \frac{1}{2}\omega^{(2)}\zeta \\ T^\circ &= d\chi + \frac{1}{2}\omega^{(2)}\chi \end{aligned} \quad (3.20)$$

The “curvatures” $T^+, T^-, T^\bullet, T^\circ, R^{(2)}$ must satisfy the Bianchi identities obtained by exterior differentiation of eqs.(3.3) and (3.20). This imposes a certain form for their parametrization, whose most relevant part is:

$$\begin{aligned} T^+ &= \frac{i}{2}\zeta\zeta \\ T^- &= -\frac{i}{2}\chi\chi \end{aligned} \quad (3.21)$$

The superconformal invariance of this construction allows for the choice of a “special superconformal gauge” where

$$\begin{aligned} e^+ &= dz + \frac{i}{2}\theta d\theta & ; & & e^- &= d\bar{z} + \frac{i}{2}\bar{\theta} d\bar{\theta} \\ \zeta &= d\theta & ; & & \chi &= d\bar{\theta} \end{aligned} \quad (3.22)$$

This is the choice we always use in 2^{nd} order formalism (see discussion after eq.(3.4)). We describe superstring propagation on an arbitrary target manifold $\mathcal{M}_{\text{target}}$ by means of an embedding function $X^\mu(z, \bar{z}, \theta, \bar{\theta})$ mapping the super world-sheet into $\mathcal{M}_{\text{target}}$. We consider the quantities defining the geometry of $\mathcal{M}_{\text{target}}$, such as its vielbeins and spin-connection, as superfield on the super world-sheet, and thus they can be expanded on the cotangent basis of this latter. In particular we set

$$V^a = V_+^a e^+ + V_-^a e^- + \lambda^a \zeta + \mu^a \chi \quad (3.23)$$

Also the torsion and curvature 2-forms of $\mathcal{M}_{\text{target}}$ can be expanded in the various ‘‘sectors’’ on the super world-sheet. For example, the torsion, defined by:

$$dV^a + \omega^{ab} V^b = T^a = T^{abc} V^b V^c \quad (3.24a)$$

yields

$$e^+ e^- : \quad -\nabla_- V_+^a + \nabla_+ V_-^a - 2T^{abc} V_+^b V_-^c = 0 \quad (3.24b)$$

.....

(relations that we are always free to use because they are just the ‘‘pull-back’’ of the original definitions).

The key point are the Bianchi identities of $\mathcal{M}_{\text{target}}$ which become differential equations for V^a as a super-worldsheet function; that is, they determine the eqs. of motion for $V_+^a, V_-^a, \lambda^a, \mu^a$ [15]. The B.I. for the torsion of $\mathcal{M}_{\text{target}}$ is $\nabla T^a = \nabla^2 V^a = R^{ab} V^b$ or, explicitly

$$\nabla^2 V_+^a = R^{ab} V_+^b \quad (3.25.a)$$

$$\nabla^2 V_-^a = R^{ab} V_-^b \quad (3.25.b)$$

$$\nabla^2 \lambda^a = R^{ab} \lambda^b \quad (3.25.c)$$

$$\nabla^2 \mu^a = R^{ab} \mu^b \quad (3.25.d)$$

Each of these equations can be analyzed in its various sectors. In particular the λ^a field equation, setting $\nabla_o \lambda^a = 0$, constraint compatible with the Bianchi identity:

$$-\frac{i}{2} \nabla_- \lambda^a = -R^{ab}{}_{cd} \lambda^b \mu^c \mu^d \quad (3.26a)$$

is retrieved in the $\chi\chi$ sector of eq.(3.25.c) and the μ^a field equation

$$\frac{i}{2} (\nabla_+ \mu^a + 2T^{abc} \mu^b V_+^c) = -R^{ab}{}_{cd} \mu^b \lambda^c \lambda^d \quad (3.26b)$$

is retrieved in the $\zeta\zeta$ sector of eq.(3.25d). Bianchi identities for the curvature R^{ab} do not give any new information.

Next one tries to write down an action defined on super world-sheet from which both the definitions (3.24) and the field equations follow as variational equations. To this

purpose one starts writing down the most general geometrical action defined on the super world-sheet which respects invariance under Weyl rescalings and two-dimensional Lorentz transformations, with undetermined coefficients; these latter are fixed by comparing the variational equations with parametrizations (3.24) and field equations.

It turns out that the projections of the variational equations in $\delta\lambda^a$ and $\delta\mu^a$ is sufficient to fix all the coefficients. The super world-sheet action takes then the form

$$\begin{aligned}
S = \frac{1}{4\pi} \int_{\partial\mathcal{M}} & (V^a - \lambda^a\zeta - \mu^a\chi)(\Pi_+^a e^+ - \Pi_-^a e^-) + \Pi_+^a \Pi_-^a e^+ e^- + 2i\lambda^a \bar{\nabla} \lambda^a e^+ + \\
& + 2i\mu^a \bar{\nabla} \mu^a e^- + \lambda^a V^a \zeta - \mu^a V^a \chi - \lambda^a \mu^a \zeta \chi + \frac{4}{3} iT_{abc} \lambda^a \lambda^b \lambda^c \zeta e^+ - \\
& - \frac{4}{3} iT_{abc} \mu^a \mu^b \mu^c \chi e^- + 4R_{abcd} \lambda^a \lambda^b \mu^c \mu^d e^+ e^- + \\
& - 2\Phi R^{(2)} + p_+ T^+ + p_- T^- + p_\bullet T^\bullet + p_\circ T^\circ + \frac{1}{4\pi} \int_{\mathcal{M}} H
\end{aligned} \tag{3.27}$$

The variation in δX^μ , restricted to the sectors $\zeta\zeta, \chi\chi$, where it really corresponds to a supersymmetry variation, fixes

$$T_{abc} = -3H_{abc} \tag{3.28}$$

justifying our assumption that T_{abc} is completely antisymmetric in its indices.

The action (3.26) is a geometrical one on the super world-sheet, and is therefore invariant against super-world-sheet diffeomorphisms. Its expression is however uniquely determined by its ‘‘bosonic’’ section $\zeta = \chi = 0$, due to the fact that the components of the curvatures along the ‘‘fermionic’’ directions are expressed by eqs.(3.24) in terms of those along the ‘‘bosonic’’ (or ‘‘inner’’) ones. This property is called ‘‘rheonomy’’. One can forget, if he wants to, about the super world-sheet and then the would-be diffeomorphisms in fermionic directions appear as supersymmetry transformations. For $\zeta = \chi = 0$ the action reduces to

$$\begin{aligned}
S = \frac{1}{4\pi} \int_{\partial\mathcal{M}} & V^a(\Pi_+^a e^+ - \Pi_-^a e^-) + \Pi_+^a \Pi_-^a e^+ e^- + 2i\lambda^a \bar{\nabla} \lambda^a e^+ + \\
& + 2i\mu^a \bar{\nabla} \mu^a e^- + 4R^{ab}_{cd} \lambda^a \lambda^b \mu^c \mu^d e^+ e^- + \\
& - 2\Phi R^{(2)} + p_+ T^+ + p_- T^- + \frac{1}{4\pi} \int_{\mathcal{M}} H
\end{aligned} \tag{3.29}$$

The above action possesses a global (1,1) supersymmetry that is the remainder of the local one present when the gravitino fields are switched on. In the next section we recall how, for suitable target manifolds, this global (1,1) SUSY extends to a global (4,4) supersymmetry.

From the complete form (3.26) of the action, one can derive the super-stress-energy tensor (i.e. the stress-energy tensor and the supercurrent) extending eq.(3.6) to

$$\delta S = \frac{-1}{2\pi} \int \mathcal{T}_+ \delta e^+ + \mathcal{T}_- \delta e^- + \mathcal{T}_\bullet \delta \zeta + \mathcal{T}_\circ \delta \chi \tag{3.30}$$

Superconformal invariance requires

$$\begin{aligned}
\mathcal{T}_{+-} = \mathcal{T}_{-+} = \mathcal{T}_{\bullet-} = \mathcal{T}_{-\bullet} = \mathcal{T}_{\circ+} = \mathcal{T}_{+\circ} = 0 \\
\mathcal{T}_{+\bullet} = \frac{1}{2} \mathcal{T}_{\bullet+} \quad ; \quad \mathcal{T}_{-\circ} = -\frac{1}{2} \mathcal{T}_{\circ-}
\end{aligned} \tag{3.31}$$

The surviving four independent components define the classical holomorphic and antiholomorphic parts of stress-energy tensor and supercurrent:

$$\begin{aligned} T(z) &= \mathcal{T}_{++} & \tilde{T}(\bar{z}) &= \mathcal{T}_{--} \\ G(z) &= 2\sqrt{2}e^{-i\frac{\pi}{4}}\mathcal{T}_{+\bullet} & \tilde{G}(\bar{z}) &= 2\sqrt{2}e^{-\frac{3i\pi}{4}}\mathcal{T}_{-\circ} \end{aligned} \quad (3.32)$$

In the action (3.26) or (3.28) two different covariant derivatives appear, ∇^{\pm} and $\bar{\nabla}$, constructed with the two spin-connections ω^{\pm} , defined as

$$\begin{aligned} \omega_{ab}^{-} &= \omega_{ab}^{\text{R}} - T_{abc}V^c \\ \omega_{ab}^{+} &= \omega_{ab}^{\text{R}} + T_{abc}V^c = \omega_{ab}^{-} + 2T_{abc}V^c \end{aligned} \quad (3.33)$$

where ω_{ab}^{R} is the Riemannian connection, i.e. is such that $dV^a + \omega_{ab}^{\text{R}}V^b = 0$. The connection appearing in eq.(3.23) (the one for which the torsion is T^a) is $\omega_{ab} = \omega_{ab}^{-}$. These connections play an important role in the sequel.

Chapter 4

N=4 sigma-models and HyperKähler manifolds

In the first part of this Chapter we review the conditions for the existence of additional global supersymmetries in the (1,1)-locally supersymmetric σ -model [19].

In the second part we discuss the specific conditions for (4,4) supersymmetry: they imply a very particular structure of the target manifold that corresponds to a generalization with torsion of HyperKähler geometry. This structure implies self-duality and antiself-duality, respectively, for the two curvatures $R(\omega^-)$ and $R(\omega^+)$ and, as such, it is the proper geometry for an instanton with torsion.

Finally in the third part we show how to construct the classical supercurrents generating these additional supersymmetries.

4.1 Complex structures and extended SUSY

To discuss the additional supersymmetries, we formally introduce new fermionic directions of the super world-sheet, adding to the cotangent basis new “gravitinos” ζ^x and χ^x so that the parametrization (3.21) is extended to

$$\begin{aligned} T^+ &= \frac{i}{2}(\zeta\zeta + \zeta^x\zeta^x) \\ T^- &= -\frac{i}{2}(\chi\chi + \chi^x\chi^x) \end{aligned} \tag{4.1}$$

while the embedding of the extended super world-sheet in $\mathcal{M}_{\text{target}}$ is described by expanding the target-space vielbeins as follows:

$$V^a = V_+^a e^+ + V_-^a e^- + \lambda^a \zeta + \bar{\mathcal{J}}_{ab}^x \lambda^b \zeta^x + \mu^a \chi + \mathcal{J}_{ab}^x \mu^b \chi^x \tag{4.2}$$

(Note that the new terms do not introduce any new dynamical quantities).

Consistency with the torsion definition and implementation of the Bianchi Identities leads to constraints on the tensors \mathcal{J}_{ab}^\pm , and therefore to a characterization of $\mathcal{M}_{\text{target}}$.

The torsion definition (3.24a): $\bar{\nabla} V^a = T_{abc} V^b V^c$ can now be expanded in many sectors*. Using the sectors

$$\begin{aligned}\zeta\zeta : & \quad \frac{i}{2}V_+^a + \nabla \cdot \lambda^a + T_{abc}\lambda^b\lambda^c = 0 \\ \zeta\zeta^x : & \quad \nabla \cdot (\mathcal{J}_{ab}^x \lambda^b) + \nabla \cdot \lambda^a + 2T_{abc}\lambda^b \mathcal{J}_{cr}^x \lambda^r = 0 \\ \zeta^x \zeta^y : & \quad iV_+^a \delta^{xy} + \nabla \cdot (\mathcal{J}_{ab}^y \lambda^b) + \nabla \cdot (\mathcal{J}_{ab}^x \lambda^b) + 2T_{abc}\mathcal{J}_{br}^x \mathcal{J}_{cs}^y \lambda^r \lambda^s = 0\end{aligned}\tag{4.3}$$

by looking at terms containing V_+^a , one finds:

for $x = y$

$$\mathcal{J}_{ab}^x \mathcal{J}_{br}^x = -\delta_{ar} \quad \text{i.e.} \quad (\mathcal{J}^x)^2 = -1\tag{4.4a}$$

for $x \neq y$

$$\{\mathcal{J}^x, \mathcal{J}^y\}_{ar} = 0\tag{4.4b}$$

It follows that the \mathcal{J}^x form a representation of the Clifford algebra. From the remaining terms in these equations, after some manipulations, one gets :

for $x = y$, the condition that the usual Nijenhuis tensor relative to each \mathcal{J}^x should vanish:

$$N_{abn}(\mathcal{J}^x, \mathcal{J}^x) = \overset{\text{R}}{\nabla}_m \mathcal{J}_{a[b}^x \mathcal{J}_{mn]}^x + \mathcal{J}_{am}^x \overset{\text{R}}{\nabla}_{[b} \mathcal{J}_{mn]}^x = 0\tag{4.5}$$

and for $x \neq y$ analogous non-diagonal Nijenhuis conditions [19].

From sectors $\chi\chi, \chi\chi^x, \chi^x\chi^y$ the same relations for \mathcal{J}^\pm are retrieved:

$$\begin{aligned}(\mathcal{J}^\pm)^2 &= -1 \quad ; \quad \{\mathcal{J}^x, \mathcal{J}^y\} = 0 \\ N_{abc}(\mathcal{J}^\pm, \mathcal{J}^\pm) &= 0\end{aligned}\tag{4.6}$$

Starting from the sector

$$\zeta^x \chi^y : \quad \nabla \cdot (\mathcal{J}_{ab}^y \mu^b) + \nabla \cdot (\mathcal{J}_{ab}^x \lambda^b) + 2T_{abc}\mathcal{J}_{br}^x \lambda^r \mathcal{J}_{cs}^y \mu^s = 0$$

and substituting the relations that follows from the other sectors $\zeta\chi^x, \chi\zeta, \zeta\chi$ by considering the terms that contain $\nabla \cdot \mu^a$ we come to the conclusion that the two set of tensors should commute:

$$[\mathcal{J}^x, \mathcal{J}^y] = 0\tag{4.7}$$

Now we can also consider the various sectors of the torsion Bianchi identities. In particular from eq.(3.25.c) in the sector $\zeta^x \zeta^x$:

$$\frac{i}{2}\nabla_+ \mu^a + \nabla \cdot \nabla \cdot \mu^a = -R^a{}_{cd} \mathcal{J}_{cr}^x \mathcal{J}_{ds}^x \lambda^r \lambda^s \mu^b$$

* From now on we drop in all calculations the superscript - for $\bar{\mathcal{J}}^x, \bar{\nabla}$ etc.

looking at the terms involving V_+^T and using the field equation (3.26a) one ends with

$$\nabla_m \mathcal{J}_{ab}^x = 0 \quad (4.8)$$

while the other terms impose the condition:

$$R = \mathcal{J}^{xT} R \mathcal{J}^x$$

$R^{ab} = R_{cd}^{ab} V^c V^d$ being the curvature two-form. This coincides with the integrability condition for eq.(4.8), namely

$$[R, \mathcal{J}^x] = 0 \quad (4.9)$$

if

$$\mathcal{J}^{xT} = -\mathcal{J}^x \quad (4.10)$$

which is just the hermiticity condition expressed in tangent indices.

Considering the sector $\chi^x \chi^x$ of eq.(3.26.d) and analyzing terms proportional to V_-^a one sees that the torsion terms are such that the analogue of eq.(4.8) is given by:

$$\overset{+}{\nabla}_m \overset{+}{\mathcal{J}}_{ab}^x = 0 \quad (4.11)$$

At the same time in order for the other terms to reproduce the integrability condition

$$[\overset{+}{R}, \overset{+}{\mathcal{J}}^x] = 0 \quad (4.12)$$

(where $\overset{+}{R}_{ab}$ is the curvature 2-form of the connection ω_{ab}^+) the hermiticity condition

$$\overset{+}{\mathcal{J}}^{xT} = -\overset{+}{\mathcal{J}}^x \quad (4.13)$$

must be verified.

Summarizing: *The condition to have (N, N) supersymmetries is the existence of two sets of $N - 1$ complex structures on the target space (whose Nijenhuis tensors vanish), each set realizing a representation of the Clifford algebra, and the two sets commuting with each other. One of the two sets, namely $\overset{-}{\mathcal{J}}^x$, must be covariantly constant with respect to the connection ω^- , while the other one, $\overset{+}{\mathcal{J}}^x$ is covariantly constant with respect to ω^+ . The target space metric should be hermitean with respect to all complex structures.*

4.2 (4, 4) SUSY and generalized HyperKähler Manifolds

Consider the case of exactly 3+3 additional supersymmetries. It is easy to see that if \mathcal{J}^1 and \mathcal{J}^2 are two complex structures satisfying the above requirements then $\mathcal{J}^3 = \mathcal{J}^1 \mathcal{J}^2$ is another one. Due to the Clifford algebra requirement the set \mathcal{J}^x closes a quaternionic algebra:

$$\mathcal{J}^x \mathcal{J}^y = -\delta^{xy} + \epsilon^{xyz} \mathcal{J}^z \quad (4.14)$$

The same holds true for $\hat{\mathcal{J}}^x$.

In the case of zero torsion, a manifold $\mathcal{M}_{\text{target}}$ with three covariantly constant complex structures that realize a quaternionic algebra, and respect to which the metric is hermitean is said to be a HyperKähler manifold. Indeed on $\mathcal{M}_{\text{target}}$ there exist three globally defined 2-forms $\Omega^x = \mathcal{J}_{ab}^x V^a V^b$ which are closed: $d\Omega^x = 0$. The role of these forms is the generalization of that played for a Kähler space by the Kähler form $\Omega = \mathcal{J}_{ab} V^a V^b$.

Choosing a well-adapted basis of vielbeins one can show that the holonomy group $\text{Hol}(HK_m)$ of a HyperKähler space HK_m with dimension $4m$ is contained in $Sp(2m)$ [20]. In particular a four-dimensional HyperKähler space has a holonomy group contained in $SU(2)$: the curvature 2-form is selfdual or anti-selfdual. Note that this is the requirement a manifold must satisfy in order to be a gravitational instanton.

Let us what happens when we introduce torsion in the game.

Recall that the \mathcal{J}^x and $\hat{\mathcal{J}}^x$ complex structures with both indices lowered are antisymmetric matrices.

In 4-dimensions we can construct a basis for 4×4 antisymmetric matrices made by the following two sets of three constant matrices, respectively named $\hat{\mathcal{J}}^x$ and $\tilde{\mathcal{J}}^x$ ($x = 1, 2, 3$)*:

$$\begin{aligned}\hat{\mathcal{J}}_{ab}^x &= -(\delta_{a0}\delta_{bx} - \delta_{b0}\delta_{ax} + \epsilon_{xab}) \\ \tilde{\mathcal{J}}_{ab}^x &= (\delta_{a0}\delta_{bx} - \delta_{b0}\delta_{ax} - \epsilon_{xab})\end{aligned}\tag{4.15}$$

that is

$$\begin{aligned}\hat{\mathcal{J}}^1 &= \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} & ; & \hat{\mathcal{J}}^2 = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} & ; & \hat{\mathcal{J}}^3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \\ \tilde{\mathcal{J}}^1 &= \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} & ; & \tilde{\mathcal{J}}^2 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} & ; & \tilde{\mathcal{J}}^3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}\end{aligned}\tag{4.16}$$

These matrices have the following properties:

- each set $\hat{\mathcal{J}}^x, \tilde{\mathcal{J}}^x$ gives a representation of quaternionic algebra (4.14)
- they are selfdual (resp anti-selfdual):

$$\begin{aligned}\hat{\mathcal{J}}_{ab}^x &= \frac{1}{2} \epsilon_{abcd} \hat{\mathcal{J}}_{cd}^x & \leftrightarrow & \epsilon_{ijk} \hat{\mathcal{J}}_{0=}^k - \hat{\mathcal{J}}_{ij}^x \\ \tilde{\mathcal{J}}_{ab}^x &= -\frac{1}{2} \epsilon_{abcd} \tilde{\mathcal{J}}_{cd}^x & \leftrightarrow & \epsilon_{ijk} \tilde{\mathcal{J}}_{0=}^k \tilde{\mathcal{J}}_{ij}^x\end{aligned}\tag{4.17}$$

- all the $\hat{\mathcal{J}}^x$ commute with all the $\tilde{\mathcal{J}}^x$.

On a 4-manifold with (4,4) extended SUSY, we have two sets of complex structures, \mathcal{J}^x and $\hat{\mathcal{J}}^x$, that are covariantly constant under the connection $\omega^- = \omega^R - T$ and $\omega^+ = \omega^R + T$, respectively, different for non-zero torsion, so that \mathcal{J}^x and $\hat{\mathcal{J}}^x$ cannot coincide. A priori the matrices of both these sets can be expanded along the basis given by $\hat{\mathcal{J}}^x, \tilde{\mathcal{J}}^x$:

$$\begin{aligned}\mathcal{J}^x &= s_y^x \hat{\mathcal{J}}^y + a_y^x \tilde{\mathcal{J}}^y \\ \hat{\mathcal{J}}^x &= s_{+y}^x \hat{\mathcal{J}}^y + a_{+y}^x \tilde{\mathcal{J}}^y\end{aligned}\tag{4.18}$$

* ϵ symbol vanishes on the index 0

For all the coefficients in the expansion (4.18) there are two possibilities: they can be zero or, in order for the \mathcal{J}^x and $\hat{\mathcal{J}}^x$ to satisfy the quaternionic algebra, must be such that

$$s_p^x s_p^y = \delta^{xy} \quad (4.19)$$

$$\epsilon^{xyz} s_t^z = \epsilon^{pqt} s_p^x s_q^y \quad (4.20)$$

The same conditions hold for $a_y^x, s_{+y}^x, a_{+y}^x$. Relations (4.19-20) mean that each of these 3×3 matrices is orthogonal, namely they belong to the adjoint representation of $SO(3)$. We can use a vector notation \vec{s}^x (\vec{s}_+^x) for the rows of the matrix s_y^x (s_{+y}^x); if they are non zero, these vectors constitute an orthonormal basis in three-dimensional space.

Let us then consider the consequences of the fact that all the \mathcal{J}^x must commute with all the $\hat{\mathcal{J}}^x$. Using the expansion (4.18) this means that

$$\vec{s}^x \wedge \vec{s}_+^y = 0 \quad ; \quad \vec{a}^x \wedge \vec{a}_+^y = 0 \quad \forall x, y \quad (4.21)$$

(here the symbol \wedge denotes the usual exterior product of three-dimensional vectors). We can expand the \vec{s}_+^y in the basis $\{\vec{s}^x\}$:

$$\vec{s}_+^y = c_p^y \vec{s}^p$$

Suppose now that the \vec{s}^x are different from zero. Then the condition (4.21), upon use of eq.(4.20), states that

$$c_p^y \vec{s}^x \wedge \vec{s}^p = \epsilon^{xpq} c_p^y \vec{s}^q = 0$$

implying $c_p^y = 0$, that is $\vec{s}_+^y = 0$.

If the \vec{a}^x were non-zero, then an analogous argument would constrain the \vec{a}_+^y to vanish as well; then the $\hat{\mathcal{J}}^x$ would just be zero, which cannot be. The only allowed situation is the following

$$\begin{aligned} \mathcal{J}^x &= s_y^x \hat{\mathcal{J}}^y \\ \hat{\mathcal{J}}^x &= a_{+y}^x \tilde{\mathcal{J}}^y \end{aligned} \quad (4.22)$$

that is, the \mathcal{J}^x are selfdual while the $\hat{\mathcal{J}}^x$ antiselfdual (or viceversa).

Consider the curvature 2-form R^{ab} relative to the connection ω_{ab}^- . Let R be the matrix of components R^{ab} . It is an antisymmetric matrix and as such it can be expanded as follows:

$$R = A_x \hat{\mathcal{J}}^x + B_x \tilde{\mathcal{J}}^x$$

It must satisfy the integrability condition (4.9) for the covariant constancy of \mathcal{J}^x ,

$$[R, \mathcal{J}^x] = 0$$

Inserting the expansions of R and \mathcal{J}^x this means

$$A_p s_y^x [\hat{\mathcal{J}}^p, \hat{\mathcal{J}}^y] = 2 \epsilon^{pyt} A_p s_y^x \hat{\mathcal{J}}^t = 0$$

The unique solution of this constraint is $A_p = 0$; this implies that R is antiselfdual.

Repeating an analogous argument for the curvature $\overset{+}{R}_{ab}$ of the connection ω_{ab}^+ we find that $\overset{+}{R}$ is selfdual.

Summarizing: a 4-dimensional target manifold $\mathcal{M}_{\text{target}}$ of a (4,4)-supersymmetric σ -model is what we name a generalized an HyperKähler manifold with torsion

DEFINITION: A Generalized HyperKähler manifold with torsion admits two sets of mutually commuting complex structures that separately close the quaternionic algebra (4.14) and that are covariantly constant, one set with respect to the ω^- connection, the other set with respect to the ω^+ connection.

In 4-dimensions the above definition implies that the curvatures constructed from ω^- and ω^+ are one antiselfdual and the other selfdual (or viceversa). Because of this fact we can *identify* the concept of 4-dimensional generalized HyperKähler manifolds with torsion with the concept of *axionic gravitational instantons*

4.3 The classical supercurrents

Suppose that a (1,1) σ -model on a manifold $\mathcal{M}_{\text{target}}$ admits an extended (4,4) supersymmetry. The 3+3 additional supersymmetries are just global ones. The action on the bosonic world-sheet, namely eq.(3.29) is not modified at all: we just find that it is invariant against additional transformations. The novelty is that we can now search for the complete form of the action on the extended super world-sheet, i.e. the analogue of eq.(3.27) including terms proportional to ζ^x and χ^x . One should repeat the same steps needed to fix the form (3.27) taking into account all the possible new terms. Since from our point of view the only relevance of such an expression would be its use in the derivation of the classical supercurrents, we will confine ourselves to the terms involved in this derivation. Let us note that the “dilaton” terms will be enlarged to

$$\Phi R^{(2)} + p_+ T^+ + p_- T^- + p_\bullet T^\bullet + p_x^\bullet T_x^\bullet + p_\circ T_\circ^+ p_\circ^x T_x^\circ \quad (4.23)$$

where (in perfect analogy with eq.(3.20)) T_x^\bullet, T_x° are the fermionic torsion two-forms relative to the new super world-sheet gravitinos:

$$T_x^\bullet = d\zeta^x - \frac{1}{2}\omega^{(2)}\zeta^x \quad ; \quad T_x^\circ = d\chi^x + \frac{1}{2}\omega^{(2)}\chi^x \quad (4.24)$$

Variations in the 1st order fields p 's sets all the torsions to zero. This allows the choice of an “enlarged” special superconformal gauge (the extension of eq.(3.22)).

Variation in the two-dimensional spin-connection $\omega^{(2)}$ yields

$$\begin{aligned} p_+ &= -2\partial_a \Phi V_+^a & ; & & p_- &= 2\partial_a \Phi V_-^a \\ p_\bullet &= -4\partial_a \Phi \lambda^a & ; & & p_\circ &= 4\partial_a \Phi \mu^a \\ p_x^\bullet &= -4\partial_a \Phi (\mathcal{J}^x \lambda)^a & ; & & p_\circ^x &= 4\partial_a \Phi (\overset{+}{\mathcal{J}}^x \mu)^a \end{aligned} \quad (4.25)$$

The fermionic torsion terms in (4.23) will contribute to the variation of the action in the new gravitinos, as it is seen from expression (4.24). After variation we make use of eqs.(4.25).

The supercurrents are obtained by obvious extensions of eqs.(3.30) and following ones. Let

$$\delta S = \frac{-1}{2\pi} \int T_+ \delta e^+ + T_- \delta e^- + T_\bullet \delta \zeta + T_\circ \delta \chi + T_{\bullet}^x \delta \zeta^x + T_{\circ}^x \delta \chi^x \quad (4.26)$$

Then superconformal invariance imposes on the 1-forms T_{\bullet}^x and T_{\circ}^x the analogue of conditions (3.31), namely:

$$T_{+\bullet}^x = \frac{1}{2} T_{\bullet+}^x \quad ; \quad T_{-\circ}^x = -\frac{1}{2} T_{\circ-}^x$$

All the other components are zero.

Definition (3.32) is enlarged to include also the supercurrents G^x :

$$G^x(z) = 2\sqrt{2}e^{-\frac{i\pi}{4}} T_{+\bullet}^x \quad ; \quad \tilde{G}^x(\bar{z}) = -2\sqrt{2}e^{-\frac{3i\pi}{4}} T_{-\circ}^x \quad (4.27)$$

From the action (3.27) we can extract $G^0(z) = G(z)$ and $\tilde{G}^0(\bar{z}) = \tilde{G}(\bar{z})$. For example, to get $G^0 \propto T_{\bullet+}$ we vary in $\delta \zeta$ and we look for the terms proportional to e^+ ; the relevant terms are :

$$\begin{aligned} \delta S &\rightarrow \frac{1}{4\pi} \int_{\partial \mathcal{M}} \delta \zeta \left\{ -\lambda^a \Pi_+^a e^+ - \lambda^a V_+^a e^+ + \frac{4}{3} i T_{abc} \lambda^a \lambda^b \lambda^c \right\} + \delta(p_\bullet T^\bullet) + \dots = \\ &= \frac{-1}{2\pi} \int_{\partial \mathcal{M}} \delta \zeta \left\{ \lambda^a V_+^a e^+ - \frac{2}{3} i T_{abc} \lambda^a \lambda^b \lambda^c e^+ - \frac{1}{2} \partial_+ p_\bullet e^+ + \dots \right\} \end{aligned}$$

where we have integrated by parts the last term after use of the definition (3.20). Using eq.(4.25) we get

$$T_{+\bullet} = \frac{1}{2} T_{\bullet+} = \frac{1}{2} \lambda^a V_+^a - \frac{i}{3} T_{abc} \lambda^a \lambda^b \lambda^c + \partial_+ (\partial_r \Phi \lambda^r) \quad (4.28)$$

so we finally obtain the expression for $G^0 = -2\sqrt{2}e^{-i\frac{\pi}{4}} T_{+\bullet}$. In a similar way one obtains $\tilde{G}^0(\bar{z})$.

To derive the other supercurrents we must analyze the possible new terms that contribute to the relevant variations, and fix their coefficients by comparing the variational equations with the projections of the equation defining the target torsion (3.24a).

For example to get $G^x(z)$ through the computation of $T_{\bullet+}^x$ the relevant terms in the extended super world-sheet action are (compare with eq.(3.27)):

$$\begin{aligned} S &= \int_{\partial \mathcal{M}} (V^a - \lambda^a \zeta - (\mathcal{J}^x \lambda)^a \zeta^x - \dots) (\Pi_+^a e^+ - \dots) + 2i \lambda^a \nabla \lambda^a e^+ + \dots + \lambda^a V^a \zeta + \\ &+ (\mathcal{J}^x \lambda)^a V^a \zeta^x + \dots + \frac{4}{3} i T_{abc} \lambda^a \lambda^b \lambda^c \zeta e^+ + n_1 T_{abc} (\mathcal{J}^x \lambda)^a \lambda^b \lambda^c \zeta^x e^+ + \dots \\ &+ p_\bullet T^\bullet + p_\bullet^x T_x^\bullet + \dots \end{aligned} \quad (4.29)$$

A priori, besides the term of the form $T(\mathcal{J}\lambda)\lambda\lambda$, we could add to eq. (4.29) also two other kind of terms, namely $T(\mathcal{J}\lambda)(\mathcal{J}\lambda)\lambda$ and $T(\mathcal{J}\lambda)(\mathcal{J}\lambda)(\mathcal{J}\lambda)$. The reason why it suffices to add only the first term is the vanishing of the Nijenhuis tensor. Indeed the diagonal Nijenhuis tensor constructed from \mathcal{J}^x or $\mathcal{J}^{\dagger x}$, (see eq.(4.5)), upon use of the covariant constancy condition $\bar{\nabla}_m \mathcal{J}_{ab}^x = 0$, or $\nabla_m^{\dagger} \mathcal{J}_{ab}^{\dagger x} = 0$ can be rewritten as follows:

$$N_{abc}(\mathcal{J}, \mathcal{J}) = 3T_{rm[a}\mathcal{J}_{rb}\mathcal{J}_{mc]} - T_{abc} \quad (4.30)$$

(the antisymmetrization in abc is understood). By use of the Nijenhuis condition $N_{abc} = 0$ it is easy to show that

$$\begin{aligned} T(\mathcal{J}\lambda)(\mathcal{J}\lambda)\lambda &\propto T\lambda\lambda\lambda \\ T(\mathcal{J}\lambda)(\mathcal{J}\lambda)(\mathcal{J}\lambda) &\propto T(\mathcal{J}\lambda)\lambda\lambda \end{aligned}$$

Hence there is only one coefficient to fix in (4.29), namely n_1 . To obtain its value, we consider the equation that follows from varying the action (4.29) in $\delta\lambda^a$. Focusing on its $\zeta^x e^+$ sector and comparing with the $\zeta^x \zeta^x$ sector of the torsion definition (see eq.(4.3)), we obtain

$$n_1 = 4i$$

Varying now (4.29) in $\delta\zeta^x$ and searching for $\mathcal{T}_{\bullet+}^x$, in analogy with the procedure utilized for G^0 , we get

$$\begin{aligned} \mathcal{T}_{\bullet+}^x &= \left\{ \frac{1}{2}(\mathcal{J}^x\lambda)^a \Pi_+^a e^+ + \frac{1}{2}(\mathcal{J}^x\lambda)^a V_+^a e^+ - 2iT_{abc}(\mathcal{J}^x\lambda)^a \lambda^b \lambda^c e^+ + \partial p_{\bullet+}^x e^+ + \dots \right\} \\ \mathcal{T}_{\bullet+}^x &= \{(\mathcal{J}^x\lambda)^a V_+^a - 2iT_{abc}(\mathcal{J}^x\lambda)^a \lambda^b \lambda^c + 2\partial(\partial_a \Phi(\mathcal{J}^x\lambda)^a)\} \end{aligned}$$

Thus $G^x(z) = 2\sqrt{2}e^{-i\frac{\pi}{4}}\mathcal{T}_{\bullet+}^x$ is determined.

In a similar way one can calculate $\tilde{G}^x(\bar{z})$.

Summarizing: when a (1,1) supersymmetric σ -model described by the action (3.27) admits a global (4,4) supersymmetry, its classical supercurrents have the following expression in terms of the 3+3 complex structures of $\mathcal{M}_{\text{target}}$:

$$G^0(z) = \sqrt{2}e^{-i\frac{\pi}{4}} \left\{ \lambda^a V_z^a - \frac{2}{3}iT_{abc}\lambda^a \lambda^b \lambda^c + 2\partial[\partial_a \Phi \lambda^a] \right\} \quad (4.31a)$$

$$G^x(z) = \sqrt{2}e^{-i\frac{\pi}{4}} \left\{ (\mathcal{J}^x\lambda)^a V_z^a - 2iT_{abc}(\mathcal{J}^x\lambda)^a \lambda^b \lambda^c + 2\partial[\partial_a \Phi(\mathcal{J}^x\lambda)^a] \right\}$$

$$\tilde{G}^0(\bar{z}) = \sqrt{2}e^{-\frac{3i\pi}{4}} \left\{ \mu^a V_{\bar{z}}^a - \frac{2}{3}iT_{abc}\mu^a \mu^b \mu^c + 2\bar{\partial}[\partial_a \Phi \mu^a] \right\} \quad (4.31b)$$

$$\tilde{G}^x(\bar{z}) = \sqrt{2}e^{-\frac{3i\pi}{4}} \left\{ (\mathcal{J}^{\dagger x}\mu)^a V_{\bar{z}}^a - 2iT_{abc}(\mathcal{J}^{\dagger x}\mu)^a \mu^b \mu^c + 2\bar{\partial}[\partial_a \Phi(\mathcal{J}^{\dagger x}\mu)^a] \right\}$$

Chapter 5

The Callan et al. instanton

Equipped with the general results of the previous sections, we now focus on the *DRCHS* instanton and we consider the limit $A \rightarrow 0$, where asymptotic flatness is lost but superconformal solvability is gained. This limit corresponds to the (1,1) locally supersymmetric σ -model on the background (3.9), that can be advantageously rewritten as in eq.s (3.10-3.12), leading to the vielbein (3.13). As already pointed out, the σ -model that emerges from this choice describes the direct product of a supersymmetric Feigin-Fuchs model with a supersymmetric WZW model of the group $SU(2)$. This theory has a (4,4) global supersymmetry since it satisfies all the conditions described in the previous sections. Let us see how this happens.

5.1 The classical σ -model

Using the vierbein (3.13), we can write the Maurer-Cartan equations of the group-manifold $SU(2) \times \mathbb{R}$ as follows:

$$dV^a = \frac{1}{2} \sqrt{\frac{2}{k}} f_{abc} V^b V^c \quad a = 1, 2, 3, 0 \quad (5.1)$$

where the totally antisymmetric structure constants f_{abc} are given by

$$\begin{aligned} f_{0ab} &= 0 \\ f_{ijk} &= \epsilon_{ijk} \end{aligned} \quad (5.2)$$

With these notations, an $SU(2) \times \mathbb{R}$ element in the adjoint representation is given by the 4×4 matrix

$$\Gamma_{ab} = \begin{pmatrix} \Gamma_{ij} & 0 \\ 0 & 1 \end{pmatrix} \quad (5.3)$$

the 3×3 submatrix Γ_{ij} being an $SU(2)$ element in its own adjoint representation. As such the matrix Γ has the properties that

$$\begin{aligned}\Gamma^T \Gamma &= \mathbf{1} \\ (\Gamma^T d\Gamma)_{ab} &= \sqrt{\frac{2}{k}} f_{abc} V^c\end{aligned}\tag{5.4}$$

In 2^{nd} order formalism the action (3.29) for the (1,1)-supersymmetric σ -model on a generic manifold is written as follows:

$$\begin{aligned}S &= \frac{-1}{4\pi} \int_{\partial\mathcal{M}} dz d\bar{z} \left\{ V_z^a V_{\bar{z}}^a + 2i\lambda^a \bar{\nabla}_z \lambda^a - 2i\mu^a \nabla_{\bar{z}} \mu^a - 4R_{cd}^{ab} \lambda^a \lambda^b \mu^c \mu^d \right\} + \\ &\quad + \frac{1}{4\pi} \int_{\mathcal{M}} H\end{aligned}\tag{5.5}$$

For our particular background, as for any other group-manifold, this expression simplifies, due to the existence of two non-Riemannian spin-connections (the “zero” and the “one” connection in Cartan terminology [13]) that are proportional to the structure constants and that parallelize the manifold. These two connections coincide exactly with the ω^- and ω^+ discussed in the previous section. Indeed, utilizing the expression of the torsion that follows from eq.(5.1), we find:

$$\begin{aligned}\omega_{ab}^- &= 0 \\ \omega_{ab}^+ &= \sqrt{\frac{2}{k}} f_{abc} V^c\end{aligned}\tag{5.6}$$

so that

$$R_{ab}^- = R_{ab}^+ = 0\tag{5.7}$$

The “minus” covariant derivative is just an ordinary derivative, so the fermions λ^a are just free left-moving fermions. The μ^a , instead, are neither free nor right-moving. However we can rewrite the action in terms of right-moving quantities, using the 1-forms $\tilde{V}^a = \Gamma^{ab} V^b$, that provide an alternative set of vielbein for our manifold. They are given (compare with eq.(3.13)) by:

$$\begin{aligned}\tilde{V}^0 &= dt \\ \tilde{V}^i &= -\sqrt{k} \tilde{\Omega}^i\end{aligned}\tag{5.8}$$

where the forms $\tilde{\Omega}^i$ are the components, along a Lie-algebra basis, of the right-invariant form on the group manifold: $\tilde{\Omega} = dg g^{-1}$. We expand these right-moving vielbeins on the superworld-sheet as follows:

$$\tilde{V}^a = \tilde{V}_+^a e^+ + \tilde{V}_-^a e^- + \tilde{\lambda}^a e^+ + \tilde{\mu}^a e^-$$

Relying on the relation

$$\tilde{\mu}^a = \Gamma^{ab} \mu^b,\tag{5.9}$$

on the definition of $\overset{\dagger}{\nabla}$ and on the properties (5.4) of the adjoint matrix we find that

$$-2i\mu^a \overset{\dagger}{\nabla}_z \mu^a = -2i\bar{\mu}^a \bar{\partial} \bar{\mu}^a$$

Hence for group-manifolds the action (5.5) can be rewritten in such a way that involves only free fermions:

$$S = \frac{-1}{4\pi} \int_{\partial\mathcal{M}} dz d\bar{z} \{ V_z^a V_{\bar{z}}^a + 2i\lambda^a \partial \lambda^a - 2i\bar{\mu}^a \bar{\partial} \bar{\mu}^a \} + \frac{1}{24\pi} \sqrt{\frac{2}{k}} \int_{\mathcal{M}} f_{abc} V^a V^b V^c \quad (5.10)$$

On the four-dimensional group-manifold $SU(2) \times \mathbb{R}$ it is now easy to show that the conditions for (4,4) supersymmetry are matched. Due to the vanishing of the ω^- connection, the set of complex structures \mathcal{J}^x must be constant, and we can choose them to coincide with the $\hat{\mathcal{J}}^x$ of eqs.(4.15-16):

$$\mathcal{J}^x = \hat{\mathcal{J}}^x \quad (5.11)$$

The complex structures $\overset{\dagger}{\mathcal{J}}^x$, that commute with the previous set and are covariantly constant with respect to ω^+ connection, are given by

$$\overset{\dagger}{\mathcal{J}}^x = \Gamma^T \tilde{\mathcal{J}}^x \Gamma \quad (5.12)$$

This easily follows from the properties of the adjoint matrix. Substituting eqs.(5.11-6.12) into the general expression (4.31), we can write down the explicit classical expression of the supercurrents in the case of the $SU(2) \times \mathbb{R}$ background.

Before doing this, we find it convenient to reformulate the theory in terms of free fermions $\psi^a(z), \tilde{\psi}^a(\bar{z})$ that satisfy the standard OPEs:

$$\psi^a(z)\psi^b(w) = -\frac{1}{2} \frac{\delta^{ab}}{z-w} \quad (5.13)$$

(and the same for the $\tilde{\psi}^a(\bar{z})$). This involves a simple renormalization of the original free fermions. Indeed from the classical Dirac brackets of the fields λ^a and the $\bar{\mu}^a$, that translate into their quantum OPEs, we have:

$$\begin{aligned} \lambda^a(z)\lambda^b(z) &= -\frac{i}{2} \frac{\delta^{ab}}{z-w} \\ \bar{\mu}^a(\bar{z})\bar{\mu}^b(\bar{z}) &= \frac{i}{2} \frac{\delta^{ab}}{z-w} \end{aligned}$$

Hence it suffices to set:

$$\lambda^a = e^{i\pi/4} \psi^a \quad ; \quad \bar{\mu}^a = e^{i3\pi/4} \tilde{\psi}^a \quad (5.14)$$

For the left supercurrents, recalling the form of the dilaton, (eq.(3.12)), which implies that

$$\partial_a \Phi = \frac{\delta_{a0}}{\sqrt{2k}}$$

we immediately obtain the following expressions

$$\begin{aligned}
G^0(z) &= \sqrt{2} \left\{ \psi^a V_z^a + \frac{1}{3} \sqrt{\frac{2}{k}} \epsilon_{ijk} \psi^i \psi^j \psi^k + \sqrt{\frac{2}{k}} \partial \psi^0 \right\} \\
G^x(z) &= \sqrt{2} \left\{ (\hat{\mathcal{J}}^x \psi)^a V_z^a + \sqrt{\frac{2}{k}} \epsilon_{ijk} (\hat{\mathcal{J}}^x \psi)^i \psi^j \psi^k + \sqrt{\frac{2}{k}} \partial (\hat{\mathcal{J}}^x \psi)^0 \right\}
\end{aligned} \tag{5.15}$$

For the right supercurrents, we must, first of all, give their expression in terms of right-moving quantities. To this purpose it suffices to make use of the properties (5.4) of the adjoint matrix and of the additional one

$$f_{abc} \Gamma_{ar} \Gamma_{bs} \Gamma_{ct} = f_{rst} \tag{5.16}$$

corresponding to the invariance of the group structure constants. These properties imply

$$\begin{aligned}
\mu^a V_{\bar{z}}^a &= \tilde{\mu}^a \tilde{V}_{\bar{z}}^a & ; & & (\hat{\mathcal{J}}^x \mu)^a V_{\bar{z}}^a &= (\tilde{\mathcal{J}}^x \tilde{\mu})^a \tilde{V}_{\bar{z}}^a \\
\epsilon_{ijk} \mu^i \mu^j \mu^k &= \epsilon_{ijk} \tilde{\mu}^i \tilde{\mu}^j \tilde{\mu}^k & ; & & \epsilon_{ijk} (\hat{\mathcal{J}}^x \mu)^i \mu^j \mu^k &= \epsilon_{ijk} (\tilde{\mathcal{J}}^x \tilde{\mu})^i \tilde{\mu}^j \tilde{\mu}^k \\
(\hat{\mathcal{J}}^x \mu)^0 &= (\tilde{\mathcal{J}}^x \tilde{\mu})^0
\end{aligned}$$

so that, in our case, from eq.(4.31) we obtain, in terms of the fermions $\tilde{\psi}^a$:

$$\begin{aligned}
\tilde{G}^0(\bar{z}) &= \sqrt{2} \left\{ \tilde{\psi}^a \tilde{V}_{\bar{z}}^a - \frac{1}{3} \sqrt{\frac{2}{k}} \epsilon_{ijk} \tilde{\psi}^i \tilde{\psi}^j \tilde{\psi}^k + \sqrt{\frac{2}{k}} \bar{\partial} \tilde{\psi}^0 \right\} \\
\tilde{G}^x(\bar{z}) &= \sqrt{2} \left\{ (\tilde{\mathcal{J}}^x \tilde{\psi})^a \tilde{V}_{\bar{z}}^a - \sqrt{\frac{2}{k}} \epsilon_{ijk} (\tilde{\mathcal{J}}^x \tilde{\psi})^i \tilde{\psi}^j \tilde{\psi}^k + \sqrt{\frac{2}{k}} \bar{\partial} (\tilde{\mathcal{J}}^x \tilde{\psi})^0 \right\}
\end{aligned} \tag{5.17}$$

5.2 Quantization and abstract conformal field-theory

In the case of supersymmetric WZW models [21], the analysis of extended global SUSY can be also performed in purely algebraic terms; a complex structures is in one-to-one correspondence with a Cartan decomposition of the Lie algebra. The group $SU(2) \times U(1)$ (this is our case) has actually three complex structures and so $N=4$ SUSY follows. We arrive at this algebraic description by quantizing our theory.

The quantization of the supersymmetric WZW on any group manifold and in particular on $SU(2) \times U(1)$ is straightforward [13,23]. Focusing on the left sector (we write the formulas for the right sector only when some difference is present) and using the currents J^a such that

$$\partial t = -i\sqrt{2}J^0 \tag{5.18a}$$

$$\Omega^i = \frac{i\sqrt{2}}{k} J^i \tag{5.18b}$$

we find, as result of a standard procedure,

$$J^i(z)J^j(w) = \frac{k}{2} \frac{\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}J^k}{z-w} \quad (5.19a)$$

$$J^0(z)J^0(w) = \frac{1}{2(z-w)^2} \quad (5.19b)$$

We will use also the notation $j^a = (J^0, \frac{J^i}{\sqrt{k+2}})$.

The correct quantum expression for the stress-energy tensor includes the Sugawara form for the level k $SU(2)$ WZW model, and is given by

$$T(z) = J^0J^0 + \frac{1}{k+2}J^iJ^i + \frac{i}{\sqrt{k+2}}\partial J^0 + \psi^a\partial\psi^a \quad (5.20)$$

Comparing this expression to eq.(3.17) we see that at quantum level a shift $k \rightarrow k+2$ is necessary in the background charge term. This shift of two unities in the value of k can be understood in the following way.

The term responsible for the background charge couples to the supersymmetrized version of the WZW-model at level k . From a purely algebraic point of view it is well known that a super Kac-Moody algebra of level k corresponds to an ordinary bosonic Kac-Moody algebra of level $k - C_V$ (where C_V is the value of the quadratic Casimir) plus a set of free fermions having regular OPEs with the Kac-Moody currents. The shift in k is due to this fact: the relevant value of k for the computation of the background charge is the central charge of the super Kac-Moody currents:

$$j_{super}^a = j^a + const. f^{abc} \psi^b \psi^c$$

and not the central charge of the Kac-Moody currents j^a .

The central charge attributed to the Feigin-Fuchs boson t is shifted to $c_{FF} = 1 + 6/(k+2)$, the only value for which the total central charge sums up to 6, the correct one for a four dimensional supersymmetric solution:

$$c = c_{FF} + c_{WZW} + c_{ff} = 1 + \frac{6}{k+2} + \frac{3k}{k+2} + 2 = 6 \quad (5.21)$$

where c_{WZW} is the ordinary central charge of the bosonic $SU(2)$ WZW at level k and $c_{ff}=2$ is the contribution of the four free fermions.

In other words we have a $(6,6)_{4,4}$ in agreement with the general set up of section 2. Note that the dilaton, not necessary to obtain $N=4$ supersymmetry at the classical level, is essential at the quantum level to fix the central charge to its correct value.

The quantum expressions of the supercurrents (the classical ones were given in eq.(5.15-17)) are:

$$\begin{aligned} G^0(z) &= 2 \left\{ -ij^a\psi^a + \frac{1}{3} \frac{\epsilon_{ijk}}{\sqrt{k+2}} \psi^i\psi^j\psi^k + \frac{\partial\psi^0}{\sqrt{k+2}} \right\} \\ G^x(z) &= 2 \left\{ -ij^a\hat{\mathcal{J}}_{ab}^x\psi^b + \frac{\epsilon_{ijk}}{\sqrt{k+2}} (\hat{\mathcal{J}}^x\psi)^i\psi^j\psi^k + \frac{\partial(\hat{\mathcal{J}}^x\psi)^0}{\sqrt{k+2}} \right\} \end{aligned} \quad (5.22a)$$

$$\begin{aligned}
\tilde{G}^0(z) &= 2 \left\{ -i\tilde{j}^a\tilde{\psi}^a - \frac{1}{3} \frac{\epsilon_{ijk}}{\sqrt{k+2}} \tilde{\psi}^i\tilde{\psi}^j\tilde{\psi}^k + \frac{\partial\tilde{\psi}^0}{\sqrt{k+2}} \right\} \\
\tilde{G}^x(z) &= 2 \left\{ -i\tilde{j}^a\tilde{J}_{ab}^x\tilde{\psi}^b - \frac{\epsilon_{ijk}}{\sqrt{k+2}} (\tilde{J}^x\tilde{\psi})^i\tilde{\psi}^j\tilde{\psi}^k + \frac{\partial(\tilde{J}^x\tilde{\psi})^0}{\sqrt{k+2}} \right\}
\end{aligned} \tag{5.22b}$$

Without the dilatonic contributions (the last terms in the above eqs.), as already stressed, $N=4$ symmetry would be still present, but the supercurrents would not close the standard algebra; they would rather close the so called $N=4$ extended algebra [22], based on the Kac-Moody algebra of $SU(2) \times SU(2) \times U(1)$. The canonical way to reduce this extended algebra to the standard one is to add a background charge with a particular value. The solution we are considering automatically performs this reduction, assigning the needed background charge to the field t .

The supercurrents (5.22) close thus the standard algebra, which requires c to be a multiple of six:

$$T(z)T(w) = \frac{c}{2} \frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + reg. \tag{5.23a}$$

$$T(z)\mathcal{G}^a(w) = \frac{3}{2} \frac{\mathcal{G}^a(w)}{(z-w)^2} + \frac{\partial\mathcal{G}^a(w)}{(z-w)} + reg. \tag{5.23b}$$

$$T(z)\bar{\mathcal{G}}^a(w) = \frac{3}{2} \frac{\bar{\mathcal{G}}^a(w)}{(z-w)^2} + \frac{\partial\bar{\mathcal{G}}^a(w)}{(z-w)} + reg. \tag{5.23c}$$

$$T(z)A^i(w) = \frac{A^i(w)}{(z-w)^2} + \frac{\partial A^i(w)}{(z-w)} + reg \tag{5.23d}$$

$$A^i(z)\mathcal{G}^a(w) = \frac{1}{2} \frac{\mathcal{G}^b(w)(\sigma^i)^{ba}}{(z-w)} + reg. \tag{5.23e}$$

$$A^i(z)\bar{\mathcal{G}}^a(w) = \frac{-1}{2} \frac{\bar{\mathcal{G}}^b(w)(\sigma^i)^{ab}}{(z-w)} + reg \tag{5.23f}$$

$$\mathcal{G}^a(z)\bar{\mathcal{G}}^b(w) = \frac{2c}{3} \frac{\delta^{ab}}{(z-w)^3} + \frac{4(\sigma_i^*)^{ab}A^i(w)}{(z-w)^2} + \frac{2\delta^{ab}T(w) + 2\partial A^i(w)(\sigma_i^*)^{ab}}{(z-w)} + reg. \tag{5.23g}$$

$$A^i(z)A^j(w) = \frac{c}{12} \frac{\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}A^k(w)}{(z-w)} + reg. \tag{5.23h}$$

The same holds for the right sector.

The $SU(2)_1$ currents of the two sectors are realized entirely in terms of free fermions:

$$\begin{aligned}
A^i(z) &= -\frac{i}{2}\psi^a\hat{J}_{ab}^i\psi^b = i(\psi^0\psi^i + \frac{1}{2}\epsilon^{ijk}\psi^j\psi^k) \\
\bar{A}^i(z) &= -\frac{i}{2}\tilde{\psi}^a\tilde{J}_{ab}^i\tilde{\psi}^b = -i(\tilde{\psi}^0\tilde{\psi}^i - \frac{1}{2}\epsilon^{ijk}\tilde{\psi}^j\tilde{\psi}^k)
\end{aligned} \tag{5.24}$$

i.e. they have the same expression as for the flat space (see eq.(2.28)), except that, due to the non-vanishing torsion we are forced to use the two different sets of complex structures

in the two sectors. The supercurrents \mathcal{G}^a , $\bar{\mathcal{G}}^a = (\mathcal{G}^a)^*$, organized in $SU(2)$ doublets as dictated by the above OPEs, are given by

$$\mathcal{G} = \frac{1}{\sqrt{2}} \begin{pmatrix} G^0 - iG^3 \\ -(G^2 + iG^1) \end{pmatrix} \quad ; \quad \bar{\mathcal{G}} = \frac{1}{\sqrt{2}} \begin{pmatrix} G^0 + iG^3 \\ -(G^2 - iG^1) \end{pmatrix} \quad (5.25)$$

for the left sector, and by the same tilded expressions in the right one.

Substituting the explicit form (4.15-16) of the complex structures into eqs.(5.22) we get

$$\begin{aligned} G^0 &= 2 \left[-iJ^0\psi^0 - \frac{i}{\sqrt{k+2}} J^i\psi^i + \frac{2}{\sqrt{k+2}} \psi^1\psi^2\psi^3 + \frac{\partial\psi^0}{\sqrt{k+2}} \right] \\ G^1 &= 2 \left[iJ^0\psi^1 - \frac{i}{\sqrt{k+2}} (J^1\psi^0 - J^2\psi^3 + J^3\psi^2) + \frac{2}{\sqrt{k+2}} \psi^0\psi^2\psi^3 - \frac{\partial\psi^1}{\sqrt{k+2}} \right] \\ G^2 &= 2 \left[iJ^0\psi^2 - \frac{i}{\sqrt{k+2}} (J^1\psi^3 + J^2\psi^0 - J^3\psi^1) + \frac{2}{\sqrt{k+2}} \psi^1\psi^0\psi^3 - \frac{\partial\psi^2}{\sqrt{k+2}} \right] \\ G^3 &= 2 \left[iJ^0\psi^3 - \frac{i}{\sqrt{k+2}} (-J^1\psi^2 + J^2\psi^1 + J^3\psi^0) + \frac{2}{\sqrt{k+2}} \psi^1\psi^2\psi^0 - \frac{\partial\psi^3}{\sqrt{k+2}} \right] \end{aligned} \quad (5.26)$$

while the \tilde{G} have analogous but slightly different expressions. This algebra was first obtained by Kounnas, Porrati and Rostand [23] and used in this specific framework by Callan, Harvey and Strominger [6].

The doublets of supercurrents can be written as

$$\begin{aligned} \mathcal{G} &= \left[-i\sqrt{2}(j^0 - ij^3) + 2\sqrt{\frac{2}{k+2}} i(\psi^0\psi^3 - \psi^1\psi^2) + \sqrt{\frac{2}{k+2}} \partial \right] \begin{pmatrix} \psi^0 + i\psi^3 \\ \psi^2 + i\psi^1 \end{pmatrix} + \\ &\quad - i\sqrt{2}(j^2 + ij^1) \begin{pmatrix} \psi^2 - i\psi^1 \\ -(\psi^0 - i\psi^3) \end{pmatrix} \end{aligned} \quad (5.27a)$$

$$\bar{\mathcal{G}} = (\mathcal{G})^*$$

and by

$$\begin{aligned} \tilde{\mathcal{G}} &= \left[i\sqrt{2}(\tilde{j}^0 + i\tilde{j}^3) + 2\sqrt{\frac{2}{k+2}} i(\tilde{\psi}^0\tilde{\psi}^3 + \tilde{\psi}^1\tilde{\psi}^2) - \sqrt{\frac{2}{k+2}} \tilde{\partial} \right] \begin{pmatrix} -(\tilde{\psi}^0 - i\tilde{\psi}^3) \\ \tilde{\psi}^2 + i\tilde{\psi}^1 \end{pmatrix} + \\ &\quad - i\sqrt{2}(\tilde{j}^2 + i\tilde{j}^1) \begin{pmatrix} \tilde{\psi}^2 - i\tilde{\psi}^1 \\ \tilde{\psi}^0 + i\tilde{\psi}^3 \end{pmatrix} \end{aligned} \quad (5.27b)$$

$$\tilde{\bar{\mathcal{G}}} = (\tilde{\mathcal{G}})^*$$

The relevant point is that we can easily obtain now the explicit form of the moduli operators for the conformal field theory we have just described. We need primary fields of dimension one which are the same time last components of an $N=4$ representation, namely we have to find solutions to the OPEs (2.25). Remarkably, in our case the solution of these OPEs

is very similar in form to the solution (2.29) one obtains in the flat space case. Indeed consider the $SU(2)$ doublets:

$$\Psi_1(z) = e^{-\sqrt{\frac{2}{k+2}}t} \begin{pmatrix} \psi^0 + i\psi^3 \\ \psi^2 + i\psi^1 \end{pmatrix} ; \Psi_2(z) = e^{-\sqrt{\frac{2}{k+2}}t} \begin{pmatrix} \psi^2 - i\psi^1 \\ -(\psi^0 - i\psi^3) \end{pmatrix} \quad (5.28a)$$

$$\tilde{\Psi}_1(\bar{z}) = e^{-\sqrt{\frac{2}{k+2}}\bar{t}} \begin{pmatrix} -(\tilde{\psi}^0 - i\tilde{\psi}^3) \\ \tilde{\psi}^2 + i\tilde{\psi}^1 \end{pmatrix} ; \tilde{\Psi}_2(\bar{z}) = e^{-\sqrt{\frac{2}{k+2}}\bar{t}} \begin{pmatrix} \tilde{\psi}^2 - i\tilde{\psi}^1 \\ \psi^0 + i\tilde{\psi}^3 \end{pmatrix} \quad (5.28b)$$

These operators satisfy eq.s (2.25) with as last components the operators

$$\Phi_1(z) = e^{-\sqrt{\frac{2}{k+2}}t} \left\{ i\sqrt{2}(j^2 + ij^1) + 2\sqrt{\frac{2}{k+2}}(\psi^0 + i\psi^3)(\psi^2 + i\psi^1) \right\}$$

$$\Pi_1(z) = e^{-\sqrt{\frac{2}{k+2}}t} \left\{ i\sqrt{2}(j^0 + ij^3) + 2i\sqrt{\frac{2}{k+2}}(\psi^0\psi^3 - \psi^1\psi^2) \right\}$$

$$\Phi_2(z) = e^{-\sqrt{\frac{2}{k+2}}t} \left\{ -i\sqrt{2}(j^0 - ij^3) + 2i\sqrt{\frac{2}{k+2}}(\psi^0\psi^3 - \psi^1\psi^2) \right\}$$

$$\Pi_2(z) = e^{-\sqrt{\frac{2}{k+2}}t} \left\{ i\sqrt{2}(j^2 - ij^1) + 2\sqrt{\frac{2}{k+2}}(\psi^0 - i\psi^3)(\psi^2 - i\psi^1) \right\} \quad (5.29a)$$

and

$$\tilde{\Phi}_1(z) = e^{-\sqrt{\frac{2}{k+2}}\bar{t}} \left\{ i\sqrt{2}(\tilde{j}^2 + i\tilde{j}^1) + 2\sqrt{\frac{2}{k+2}}(\tilde{\psi}^0 - i\tilde{\psi}^3)(\tilde{\psi}^2 + i\tilde{\psi}^1) \right\}$$

$$\tilde{\Pi}_1(z) = e^{-\sqrt{\frac{2}{k+2}}\bar{t}} \left\{ i\sqrt{2}(\tilde{j}^0 - i\tilde{j}^3) + 2i\sqrt{\frac{2}{k+2}}(\tilde{\psi}^0\tilde{\psi}^3 + \tilde{\psi}^1\tilde{\psi}^2) \right\}$$

$$\tilde{\Phi}_2(z) = e^{-\sqrt{\frac{2}{k+2}}\bar{t}} \left\{ i\sqrt{2}(\tilde{j}^0 + i\tilde{j}^3) + 2i\sqrt{\frac{2}{k+2}}(\tilde{\psi}^0\tilde{\psi}^3 + \tilde{\psi}^1\tilde{\psi}^2) \right\}$$

$$\tilde{\Pi}_2(z) = e^{-\sqrt{\frac{2}{k+2}}\bar{t}} \left\{ i\sqrt{2}(\tilde{j}^2 - i\tilde{j}^1) + 2\sqrt{\frac{2}{k+2}}(\tilde{\psi}^0 + i\tilde{\psi}^3)(\tilde{\psi}^2 - i\tilde{\psi}^1) \right\} \quad (5.29b)$$

Note that, as expected from the purely fermionic form of the currents of $SU(2)$, the doublets are quite completely expressed in terms of the free fermions, the exponential term being only needed to cancel some unwanted poles. We stress that, due to the existence of the background charge, the operator of the F.F. theory

$$: e^{-\sqrt{\frac{2}{k+2}}t} : \quad (5.30)$$

has conformal dimension zero. Indeed in a F.F. theory with stress-energy tensor $T(z) = -\frac{1}{2}\partial t\partial t - \frac{i}{2}Q_{bk}\partial^2 t$ the vertex operators $:\exp(i\alpha t):$ have a conformal weight $\Delta_\alpha = \frac{1}{2}\alpha(\alpha + Q_{bk})$ and in our case $Q_{bk} = -i\sqrt{\frac{2}{k+2}}$.

This factor is the counterpart of the plane-wave factor $e^{[ik \cdot X(z, \bar{z})]}$ appearing in the flat space case. Also there the exponential factor has conformal weight zero since $k^2 = 0$. Indeed we can say that $k_0 = \sqrt{\frac{2}{k+2}}$ is the energy component of the four-momentum. It is fixed to a constant value in terms of the space-like components \mathbf{k} . The difference resides in that \mathbf{k} is a continuous variable for flat space, while its analogue is quantized to fixed values for the $SU(2) \times \mathbb{R}$ background, namely there is a finite number of zero-mode operators rather than a continuous infinity as in flat-space. This difference follows from the different topology of the constant-time slices in the two cases: noncompact \mathbb{R}^3 for flat-space, compact S^3 for the case under consideration.

The four fields Φ_a, Π_a , $a = 1, 2$ are the moduli of our conformal theory. Combining left and right fields, we find 16 infinitesimal deformations of our theory that preserve the $N=4$ superconformal algebra. These combinations are formally the same as the combinations (2.29). Moreover it is possible to construct two abstract $(0,1)$ -forms $\Psi_{\mathcal{A}^*}^* \left[\begin{smallmatrix} 0, \frac{1}{2} \\ 0, \frac{1}{2} \end{smallmatrix} \right]$, analogously to eq.(2.32).

As an abstract $(6,6)_{+,+}$ -theory the $SU(2) \times \mathbb{R}$ background has the same Hodge-diamond as flat space (compare with eq.(2.31)). However since the torsion is different from zero, these abstract Hodge numbers are not the usual ones of compactified version of the underlying manifold $S^1 \times S^3$, whose Betti numbers

$$b^0 = 1, \quad b^1 = 1, \quad b^2 = 0, \quad b^3 = 1, \quad b^4 = 1$$

are obviously incompatible with such an Hodge decomposition.

5.3 Deformations of $\mathcal{M}_{\text{target}}$ geometry in the $SU(2) \times \mathbb{R}$ case

The existence of non-trivial $N=4$ moduli implies that the geometrical data of the σ -model, namely its metric $g_{\mu\nu}$ and torsion (related to the axion $B_{\mu\nu}$) can be deformed in such a way as to maintain $N=4$ supersymmetry. In other words the existence of $h^{1,1}$ moduli implies that the generalized HyperKähler manifold we have considered is just an element in a continuous family of generalized HyperKähler manifolds, parametrized by 4 $h^{1,1}$ parameters. For instance in the case of the K_3 manifold the existence of 20 $N=4$ moduli follows from the fact that, as an algebraic surface, K_3 is described by a homogeneous equation with 19 nontrivial complex coefficients fixing the complex structure and that, for fixed complex structure, we still have a one parameter family of deformations for the Kähler class. These deformations of the metric and of the torsion fill an 80-dimensional moduli space whose global structure turns out to be $\mathcal{M}_{K_3} = SO(4, 20)/SO(4) \times SO(20)/SO(4, 20; \mathbb{Z})$. In a similar way flat space has four $N=4$ moduli because the constant metrics and constant torsions fills a space of dimension 16, namely the space of all 4×4 matrices (the symmetric part is the metric, the antisymmetric part is the axion).

The geometrical interpretation of the moduli and the knowledge of the moduli space is very important because in the functional integral we are supposed to integrate over all

geometries. In practice the use of the instanton conformal field theory to calculate physical amplitudes is the following. Given a scattering process with N external legs ($i = 1, \dots, N$) there is an expression for the emission vertex of the i -th particle in each conformally flat background Bk corresponding to a specific $(6,6)_{\pm, \pm}$: let us name this vertex $V_{Bk}(i)$. At every number of loops in string perturbation theory the true scattering amplitude is obtained by calculating the correlators for a fixed background, by integrating on the Riemann surface and on its moduli space and then by summing over the backgrounds. Schematically, if we disregard the Riemann surface integration we can write:

$$A(1, 2, \dots, N) = \sum_{Bk} \langle V_{Bk}(1) V_{Bk}(2) \dots V_{Bk}(N) \rangle$$

The sum on Bk has a discrete and a continuous part. On one side we have to sum over the various topologies, namely flat space and all the possible instantons. On the other side at fixed topology we have to integrate on the instanton moduli space.

For the limit case of the $SU(2) \times \mathbb{R}$ instanton we have discovered from the algebraic approach that there are four $N=4$ moduli just as for flat space. Their geometrical interpretation, however, is less clear. In this section we explore the consequences of the $N=4$ moduli on the geometry of the target space. Namely we calculate the explicit form of the infinitesimal deformations of the metric and of the torsion due to these moduli. We show that the deformed space is still generalized HyperKähler as expected: the curvatures of ω^+ and ω^- are no longer zero but still self-dual (respectively antiselfdual) after the deformation and there exist deformed complex structures fulfilling all the requirements. A global characterization of this space of metrics and torsions is still an open and interesting problem. Let us discuss the infinitesimal deformations obtained by inserting the moduli operators in the σ -model Lagrangian.

We focus on the bosonic sector which suffices to give us informations about the new metric, new torsion and new complex structures. The bosonic parts of the moduli, reshift-ing the background charge to its classical value, are expressed, for the left sector, in terms of the components of the left-moving vielbeins:

$$\begin{aligned} \Phi_1(z) &= (V_z^2 + iV_z^1)e^{-\sqrt{\frac{2}{k}}t} & \Phi_2(z) &= (V_z^0 + iV_z^3)e^{-\sqrt{\frac{2}{k}}t} \\ \Pi_1(z) &= -(V_z^0 - iV_z^3)e^{-\sqrt{\frac{2}{k}}t} & \Pi_2(z) &= (V_z^2 - iV_z^1)e^{-\sqrt{\frac{2}{k}}t} \end{aligned} \quad (5.31)$$

and for the right sector, in terms of the right-moving ones:

$$\begin{aligned} \bar{\Phi}_1(\bar{z}) &= (\bar{V}_{\bar{z}}^2 + i\bar{V}_{\bar{z}}^1)e^{-\sqrt{\frac{2}{k}}\bar{t}} & \bar{\Phi}_2(\bar{z}) &= -(\bar{V}_{\bar{z}}^0 - i\bar{V}_{\bar{z}}^3)e^{-\sqrt{\frac{2}{k}}\bar{t}} \\ \bar{\Pi}_1(\bar{z}) &= (\bar{V}_{\bar{z}}^0 + i\bar{V}_{\bar{z}}^3)e^{-\sqrt{\frac{2}{k}}\bar{t}} & \bar{\Pi}_2(\bar{z}) &= (\bar{V}_{\bar{z}}^2 - i\bar{V}_{\bar{z}}^1)e^{-\sqrt{\frac{2}{k}}\bar{t}} \end{aligned} \quad (5.32)$$

Now we can construct conformal operators of weights $(1, 1)$ to insert into the Lagrangian combining these $(1, 0)$ and $(0, 1)$ ones in all possible ways. Hence the most general expression we can add to the Lagrangian is simply

$$e^{-\sqrt{\frac{2}{k}}t(z, \bar{z})} V_z^a M_{ab} \bar{V}_{\bar{z}}^b \quad (5.33)$$

M_{ab} being a constant matrix. The reality condition for this expression imposes $M \in GL(4, \mathbb{R})$. Thus our deformations depend on 16 real parameters as anticipated from the abstract counting.

In terms of the components of the undeformed vielbein, which we have chosen to be the left-moving ones, the term in (5.33) has the form:

$$e^{-\sqrt{\frac{2}{k}}t} V_z^a (M\Gamma)_{ab} V_{\bar{z}}^b \quad (5.34)$$

Γ being the variable $SU(2) \times \mathbb{R}$ element (point in the manifold) in the adjoint representation.

It is useful to separate the symmetric and antisymmetric part of the matrix $M\Gamma$ and to this purpose we introduce the notation

$$\begin{aligned} h_{ab} &= \frac{1}{2} e^{-\sqrt{\frac{2}{k}}t} (M\Gamma + \Gamma^T M^T)_{ab} \\ b_{ab} &= -\frac{1}{2} e^{-\sqrt{\frac{2}{k}}t} (M\Gamma - \Gamma^T M^T)_{ab} \end{aligned} \quad (5.34)$$

The overall normalization of the new term is of course irrelevant, since M is arbitrary, and we choose it in such a way that the bosonic part of the deformed σ -model action is:

$$S = \frac{-1}{4\pi} \int_{\partial\mathcal{M}} dz d\bar{z} \{ V_z^a V_{\bar{z}}^a + 2V_z^a h_{ab} V_{\bar{z}}^b - 2V_z^a b_{ab} V_{\bar{z}}^b \} + \frac{1}{4\pi} \int_{\mathcal{M}} H \quad (5.36)$$

The torsion deformation (parametrized by the antisymmetric matrix b) can be recast in a shift of the 3-form H :

$$S = \frac{-1}{4\pi} \int_{\partial\mathcal{M}} dz d\bar{z} \{ V_z^a V_{\bar{z}}^a + 2V_z^a h_{ab} V_{\bar{z}}^b \} + \frac{1}{4\pi} \int_{\mathcal{M}} (H + \delta H) \quad (5.37)$$

where $\delta H = dB$ with $B = b_{ab} V^a V^b$.

Following [24] it is also convenient to use the combinations:

$$\begin{aligned} G_{ab}^+ &= h_{ab} + b_{ab} = e^{-\sqrt{\frac{2}{k}}t} (\Gamma^T M^T)_{ab} = (G^-)_{ab}^T \\ G_{ab}^- &= h_{ab} - b_{ab} = e^{-\sqrt{\frac{2}{k}}t} (M\Gamma)_{ab} \end{aligned} \quad (5.38)$$

Relying on the properties of the adjoint matrix (see eq.s(5.4)) simple expressions are obtained for the derivatives of the above matrices:

$$\begin{aligned} \partial_a G_{bc}^- &= \sqrt{\frac{2}{k}} (-\delta_{a0} G_{bc}^- + G_{br}^- f_{rca}) \\ \partial_a G_{bc}^+ &= -\sqrt{\frac{2}{k}} (\delta_{a0} G_{bc}^+ + f_{abr} G_{rc}^+) \end{aligned} \quad (5.39)$$

So far we have identified the deformation of the vielbein (i.e. of the metric):

$$V'^a = V^a + \delta V^a = V^a + h_{ab} V^b \quad (5.40)$$

and the components of the new torsion in the old basis which are given (this follows from the same supersimmetry variation argument as in the undeformed case) as

$$(T + \delta T)_{abc} = -3(H + \delta H)_{abc} \quad (5.41)$$

Now we must solve the relevant torsion equations for the two non-Riemannian connections we are interested in, these latter, in the undeformed situation, being given by eq.(5.6) The two torsion equations are, working at 1st order in the moduli:

$$dV'^a + (\omega^\pm + \delta\omega^\pm)^{ab} V'^b = \mp(T')^a \quad (5.42)$$

The solutions of eqs.(5.42) are given by

$$\delta\omega_{ab|c}^\pm = -2\overset{\pm}{\nabla}_{[a}G_{b]c}^\pm \mp 4G_{[ar}^\pm T_{rb]c} - \overset{\pm}{\nabla}_c G_{[ab]}^\pm \pm 2G_{rc}^\pm T_{rab} \quad (5.43)$$

Making the covariant derivatives explicit, using eqs.(5.39) and the undeformed connections, we finally get

$$\delta\omega_{ab|c}^\pm = \sqrt{\frac{2}{k}} \left\{ 2\delta_{[a0}G_{b]c}^\pm + \delta_{c0}G_{[ab]}^\pm \pm G_{[ar}^\pm f_{rb]c} \pm G_{rc}^\pm f_{rab} \right\} \quad (5.44)$$

Next we look for the deformations of the associated curvatures. From the general formula $\delta R = \nabla \delta\omega$ we have

$$\begin{aligned} \delta R_{ab}^- &= d\delta\omega_{ab}^- = (\partial_p \delta\omega_{ab|q}^- + \frac{1}{2}\sqrt{\frac{2}{k}} \delta\omega_{ab|r}^- f_{rpq}) V^p V^q \\ \delta R_{ab}^+ &= (\partial_p \delta\omega_{ab|q}^+ + \frac{1}{2}\sqrt{\frac{2}{k}} \delta\omega_{ab|r}^+ f_{rpq} + 2\sqrt{\frac{2}{k}} f_{[arp} \delta\omega_{rb]|q}^+) V^p V^q \end{aligned} \quad (5.45)$$

Using eqs.(5.44) and (5.39), after some algebra one ends up with the following results:

$$\begin{aligned} \delta R_{0i}^\pm &= -\frac{2}{k} \{ G_{ij}^\pm V^0 V^j \mp G_{il}^\pm \epsilon_{ljk} V^j V^k \} \\ \delta R_{jk}^\pm &= \pm \epsilon_{ijk} \delta R_{0i}^\pm \end{aligned} \quad (5.46)$$

We have that *the curvature of $\omega^+ + \delta\omega^+$ (which is δR^+) is selfdual*, while *that of $\omega^- + \delta\omega^-$ (which is δR^-) is antiselfdual*:

$$\delta R_{ab}^\pm = \pm \frac{1}{2} \epsilon_{abcd} \delta R_{cd}^\pm \quad (5.47)$$

Recall that this is a necessary condition for the deformed $\mathcal{M}_{\text{target}}$ to have $N=4$ supersymmetry, as we discussed in sec.5.

5.4 Deformations of the Complex Structures

Having singled out the deformations of the vielbein, of the torsion and, consequently, of the two non-Riemannian connections, our aim is now to find the deformations of the complex structures corresponding to the insertion of the $N=4$ moduli in the lagrangian. Indeed they must exist since $N=4$ symmetry is maintained. Any infinitesimal deformation of one of the sets of complex structures must be such that the quaternionic algebra is preserved:

$$(\mathcal{J}^x + \delta\mathcal{J}^x)(\mathcal{J}^y + \delta\mathcal{J}^y) = -\delta^{xy} + \epsilon^{xyz}(\mathcal{J}^z + \delta\mathcal{J}^z)$$

that is

$$\delta\mathcal{J}^x \mathcal{J}^y + \mathcal{J}^x \delta\mathcal{J}^y = \epsilon^{zxy} \delta\mathcal{J}^z \quad (5.48)$$

The general ansatz solving this requirement is

$$\delta\mathcal{J}^x = [\mathcal{J}^x, F] + \sum_z M_z \epsilon^{zxy} \mathcal{J}^y \quad (5.49)$$

F being a generic (infinitesimal) matrix and M_z generic infinitesimal parameters. We have to impose the “deformed” covariant-constancy conditions, different for the two sets of complex structures relevant in the left and in the right sector:

$$\overset{\pm}{\nabla} \delta \overset{\pm}{\mathcal{J}}_{ab}^x + 2\delta\omega_{[ar}^{\pm} \overset{\pm}{\mathcal{J}}_{rb]}^x = 0 \quad (5.50)$$

Inserting the ansatz (5.49) with $M_z=0$ into eq.(5.50) we get

$$\overset{\pm}{\mathcal{J}}_{[ar}^x (\overset{\pm}{\nabla} F^{\pm} - \delta\omega^{\pm})_{rb]} = 0 \quad (5.51)$$

Note that the deformations of the connections, (see eq.s(5.44)) can also be written as

$$\begin{aligned} \delta\omega_{ab}^+ &= -\overset{+}{\nabla} G_{[ab]}^+ - \sqrt{\frac{2}{k}} \hat{\mathcal{J}}_{ab}^x G_{xr}^+ V^r \\ \delta\omega_{ab}^- &= -\overset{-}{\nabla} G_{[ab]}^- + \sqrt{\frac{2}{k}} \tilde{\mathcal{J}}_{ab}^x G_{xr}^- V^r \end{aligned} \quad (5.52)$$

where $\hat{\mathcal{J}}^x$ and $\tilde{\mathcal{J}}^x$ are the two sets of constant complex structures introduced in cap. 4. (see eq.(4.16)). Therefore if we start from the ansatz (17) with

$$F_{ab}^{\pm} = G_{[ab]}^{\pm} \quad (5.53)$$

and $M_z=0$, the requirement (5.51) reduces to

$$\begin{aligned} \overset{+}{\mathcal{J}}_{[ar}^x \hat{\mathcal{J}}_{rb]}^y G_{yr}^+ V^r &= 0 \\ \overset{-}{\mathcal{J}}_{[ar}^x \tilde{\mathcal{J}}_{rb]}^y G_{yr}^- V^r &= 0 \end{aligned} \quad (5.54)$$

Recall that for our undeformed manifold, $\bar{\mathcal{J}}^x \equiv \hat{\mathcal{J}}^x$. The above equations hold then true due to the commutations relations (see sec. 5)

$$\begin{aligned} \left[\hat{\mathcal{J}}^x, \bar{\mathcal{J}}^y \right] &= 0 \\ \left[\hat{\mathcal{J}}^x, \tilde{\mathcal{J}}^y \right] &= 0 \end{aligned} \quad \forall x, y \quad (5.55)$$

Summarizing, we have obtained that the deformations of the left- and right-moving complex structures due to the insertion of the moduli in the original $N=4$ theory are given by

$$\delta \hat{\mathcal{J}}^x = \left[\hat{\mathcal{J}}^x, \pm bJ \right] \quad (5.56)$$

where

$$b_{ab} = \mp G_{[ab]}^\pm = e^{-\sqrt{\frac{2}{k}}t} (M\Gamma - \Gamma^T M^T)_{ab}$$

Chapter 6

A dual model

To further elucidate the structure of $SU(2) \times U(1)$ instanton solution, we will "identify" it to an apparently unrelated Kahler geometry, which it's believed to represent the same conformal field theory. A powerful method to obtain new Kahler and HyperKahler manifold from old ones is that to perform a duality transformation [25,26] on the N=2 superfield formulation of the action. Furthermore, it's interesting to realize that the duality is actually a symmetry of conformal field theory, which generalizes the well known $R \rightarrow 1/R$ duality [27].

6.1 An N=2 superfield formulation of the $SU(2) \times U(1)$ instanton.

We first describe a different formulation of the $SU(2) \times U(1)$ WZW model in N=2 superfields. We will work in minkowskian N=2 superspace with complex superderivatives satisfying the algebra

$$\{D_{\pm}, \bar{D}_{\pm}\} = \pm i\partial_{\pm}, \quad \partial_{\pm} = \partial_1 \pm \partial_0 \quad (6.1)$$

all other anticommutators vanish.

The general N=2 sigma model without torsion requires a Kahler target manifold and can be simply realized in N=2 superspace by the action

$$\int d^2x d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi}) \quad (6.2)$$

where K is a Kahler potential for the manifold and Φ are complex chiral superfields

$$\bar{D}_{\alpha}\Phi = D_{\alpha}\bar{\Phi} = 0 \quad (6.3)$$

K is defined modulo the gauge transformation $\delta K = \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi})$.

The reduction to N=1 superfield formalism is realized by defining the Majorana N=1 superderivative \tilde{D} and the generator of the hidden supersymmetry \tilde{Q}

$$\begin{aligned} \tilde{d} &= \sqrt{\frac{1}{2}}(D + \bar{D}), \\ \tilde{Q} &= \sqrt{\frac{1}{2}}(D - \bar{D}), \quad \{\tilde{D}, \tilde{Q}\} = 0 \end{aligned} \tag{6.4}$$

With this formulation the N=2 formalism cannot describe a manifold with torsion (a group manifold, for example); the way to introduce in the action WZW terms relies on the combined use of two different kind of supermultiplets: the chiral (defined above) and the twisted chiral ones [26]. The latter are defined by the conditions:

$$\tilde{D}_+ \Lambda = D_- \Lambda = 0 \quad \text{and} \quad c.c. \tag{6.5}$$

Defining the action of the hidden supersymmetry as $\delta X = i[\eta^\alpha \tilde{Q}_\alpha, X]$ we find

$$\begin{aligned} \delta \Phi &= i\eta^\alpha \tilde{D}_\alpha \Phi, \\ \delta \bar{\Phi} &= -i\eta^\alpha \tilde{D}_\alpha \bar{\Phi} \\ \delta \Lambda &= -i(\eta\gamma_5)^\alpha \tilde{D}_\alpha \Lambda \\ \delta \bar{\Lambda} &= i(\eta\gamma_5)^\alpha \tilde{D}_\alpha \bar{\Lambda} \end{aligned} \tag{6.6}$$

where $\gamma_5 = \sigma_3$.

From the point of view of the N=1 formalism the extended supersymmetry is realized via the complex structure as

$$\delta \Phi^i = (\eta^\alpha D_\alpha \Phi^j) f_j^i \tag{6.7}$$

In a theory with only chiral superfields the complex structure can be read from eq. (6.6)

$$f_i^j = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

If, otherwise, our theory involves only twisted chiral fields, γ_5 can be into η and the theory is equivalent to one with only chiral fields. The novelty comes when we use both chiral and twisted chiral fields: the purely bosonic part of the action after usual reduction reads

$$\begin{aligned} &\int d^2 x [K_{u\bar{v}} \partial^a \Phi^u \partial_a \bar{\Phi}^{\bar{v}} - K_{p\bar{q}} \partial^a \Lambda^p \partial_a \bar{\Lambda}^{\bar{q}} \\ &+ \epsilon_{ab} (K_{u\bar{p}} \partial_a \Phi^u \partial_b \bar{\Lambda}^{\bar{p}} + K_{p\bar{u}} \partial_a \bar{\Phi}^{\bar{u}} \partial_b \Lambda^p)] \end{aligned} \tag{6.8}$$

The ϵ term has a structure similar to a two-dimensional WZW term.

Using this ideas Rocek et al. [5] had been able to write down an explicitly N=2 superfields action for the $SU(2) \times U(1)$ WZW model.

We parametrize the group by a chiral superfield Φ and a twisted chiral one Λ

$$g = \frac{e^{i\theta}}{\sqrt{\Phi\bar{\Phi} + \Lambda\bar{\Lambda}}} \begin{pmatrix} \Lambda & \bar{\Phi} \\ -\Phi & \bar{\Lambda} \end{pmatrix}$$

where $\theta = -\frac{1}{2}\ln(\Phi\bar{\Phi} + \Lambda\bar{\Lambda})$.

In these coordinates the metric on the group manifold is

$$ds^2 = \frac{d\Phi d\bar{\Phi} + d\Lambda d\bar{\Lambda}}{\Phi\bar{\Phi} + \Lambda\bar{\Lambda}}$$

and we find for the Kahler potential

$$K(\Phi, \bar{\Phi}, \Lambda, \bar{\Lambda}) = - \int^{\frac{\Lambda\bar{\Lambda}}{\Phi\bar{\Phi}}} \frac{dx}{x} \ln(1+x) + \ln\Phi \ln\bar{\Phi}$$

the torsion part of the action can also be read from K as in eq. (6.8).

We already know that $SU(2) \times U(1)$ admits N=4 supersymmetry. In this formalism, the necessary condition for N=4 SUSY is [26]:

$$K_{\Phi\bar{\Phi}} + K_{\Lambda\bar{\Lambda}} = 0$$

which is easily verified in our case.

At the level of current algebra our model closes the N=4 extended superconformal algebra [21]. The Callan et al. model is simply obtained, as before, by adding a background charge to the U(1) field.

6.2 Duality transformation

We will show that every sigma model with an isometry can be related to a dual geometry which describes the same conformal field theory. This procedure has a natural extension to the N=2 supersymmetric case. For our model has a non trivial killing vector well adapted to be use in this formalism, we will be able to derive the dual geometry [5].

We consider the sigma model action

$$\frac{1}{2\pi} \int d^2z [(g_{\mu\nu} + b_{\mu\nu})\partial x^\mu \bar{\partial} x^\nu + \Phi(x)R^{(2)}] \quad (6.12)$$

where the metric, the torsion and the dilaton field are invariant under the isometry $\delta x^\mu = \epsilon k^\mu$. We choose also coordinates (x^0, x^i) such as the isometry acts by translation of x^0 , i.e. g, b and Φ are independent of x^0 .

In the action

$$\begin{aligned} \frac{1}{2\pi} \int d^2z [g_{00}A\bar{A} + (g_{0a} + b_{0a})A\bar{\partial}x^a + (g_{a0} + b_{a0})\partial x^a\bar{A} \\ + (g_{ab} + b_{ab})\partial x^a\bar{\partial}x^b + \tilde{\theta}(\partial\bar{A} - \bar{\partial}A)] \end{aligned} \quad (6.13)$$

the lagrange multiplier $\bar{\theta}$ forces the gauge fields to be pure gauge $A = \partial x^0, \bar{A} = \bar{\partial} x^0$ and we recover the original action. The dual model is obtained by integrating out the gauge fields A, \bar{A} ; because of the isometry, x^0 does not explicitly appear in the coefficients of the action, so that the integration is purely gaussian. We obtain a new sigma model with the new metric and torsion

$$\begin{aligned}\tilde{g}_{00} &= \frac{1}{g_{00}}, \tilde{g}_{0a} = \frac{b_{0a}}{g_{00}}, \tilde{g}_{ab} = g_{ab} - \frac{g_{a0}g_{0b} + b_{a0}b_{0b}}{g_{00}} \\ \tilde{b}_{0a} &= \frac{g_{0a}}{g_{00}}, \tilde{b}_{ab} = b_{ab} + \frac{g_{a0}b_{0b} + b_{a0}g_{0b}}{g_{00}}\end{aligned}\tag{6.14}$$

The quantum mechanical corrections coming from the measure in the gaussian integration amount, at one loop, to a simple shift of the dilaton [28]

$$\Phi \rightarrow \Phi + \ln(g_{00})\tag{6.15}$$

The general claim is that the couple of dual model constructed with these prescriptions are conformally equivalent. The proof of this statement relies on the construction of auxiliary models with chiral isometries, following a well known example in the recent literature [4]: two different quotients of $SL(2, \mathbb{R})$ by $U(1)$ subgroups (resulting from the vector and the axial gauging) give rise to equivalent conformal field theory with manifold related by the above duality transformation (eq (6.14)) (the cigar and the trumpet black hole). Following this example, we will construct an arbitrary pair of dual models in d dimension as different chiral quotients of $d+1$ dimensional sigma model with a left and a right chiral current:

$$\begin{aligned}S &= \frac{1}{2\pi} \int d^2 z [\partial\theta_L \bar{\partial}\theta_L + \partial\theta_R \bar{\partial}\theta_R + 2B(x)\partial\theta_R \bar{\partial}\theta_L + G_a^R \partial\theta_R \bar{\partial}x^a \\ &\quad + G_a^L \partial x^a \bar{\partial}\theta_L + (G_{ab} + B_{ab})\partial x^a \bar{\partial}x^b]\end{aligned}\tag{6.16}$$

The chiral currents associated with the symmetry

$$\delta\theta_R = \alpha_R(\bar{z}), \delta\theta_L = \alpha_L(z)$$

are

$$\begin{aligned}J^L &= \partial\theta_L + B\partial\theta_R + \frac{1}{2}G_a^L \partial x^a \\ -J^R &= \bar{\partial}\theta_R + B\bar{\partial}\theta_L + \frac{1}{2}G_a^R \bar{\partial}x^a\end{aligned}\tag{6.17}$$

We gauge an axial or a vector subgroup

$$\delta\theta_R = \alpha, \delta\theta_L = \pm\alpha$$

obtaining the gauged action

$$S + \frac{1}{2} \int d^2 z [A\bar{J}^R \pm \bar{A}J^L + \frac{1}{2}A\bar{A}(1 \pm B)]$$

Integrating out the gauge fields one obtains for the vector gauging

$$S_V + \frac{1}{2\pi} \int d^2z \left[\frac{1+B}{1-B} \partial\bar{\theta}\bar{\theta} + \frac{G_a^R}{1-B} \partial\bar{\theta}\bar{\theta}x^a + \frac{G_a^L}{1-B} \partial x^a \bar{\theta}\bar{\theta} \right. \\ \left. + \frac{G_a^L G_b^R}{2(1-B)} \partial x^a \bar{\theta}x^b + (G_{ab} + B_{ab}) \partial x^a \bar{\theta}x^b \right] \quad (6.18)$$

and for the axial gauging

$$S_A + \frac{1}{2\pi} \int d^2z \left[\frac{1-B}{1+B} \partial\theta\bar{\theta} + \frac{G_a^R}{1+B} \partial\theta\bar{\theta}x^a - \frac{G_a^L}{1+B} \partial x^a \bar{\theta}\theta \right. \\ \left. - \frac{G_a^L G_b^R}{2(1+B)} \partial x^a \bar{\theta}x^b + (G_{ab} + B_{ab}) \partial x^a \bar{\theta}x^b \right] \quad (6.19)$$

where

$$\theta = \theta_R - \theta_L, \bar{\theta} = \theta_R + \theta_L$$

The translations in $\theta, \bar{\theta}$ are isometries of the two models. It's easy to verify that S_A and S_V are related by the duality transformation (eq.(6.14)). The shift in the dilaton due to the jacobian in the integration of gauge fields

$$\Phi \rightarrow \Phi + \ln(1 \pm B) \quad (6.20)$$

is also in accordance with eq.(6.15) because

$$g_{00} = \frac{1+B}{1-B}$$

The approach via chiral quotient produces conformally invariant theory from conformally invariant ones, so the d+1 dimension model is conformally invariant if and only if the dual pair are. The dual models are related by reversing the sign of B and G_a^L , i.e. by the coordinate transformation $\theta_L \rightarrow -\theta_L$. At the level of conformal field theory it simply amounts to reverse the sign of the left charge of fields under the $U(1)_L \times U(1)_R$: this is an isomorphism of conformal field theory.

Thus we conclude that duality is a symmetry of CFT.

We have so concluded that for every bosonic sigma model with an isometry we are able to construct a new sigma model with a dual geometry which describes the same conformal field theory.

There exist a simple generalization to the N=2 superfield case. The N=2 quotient is the trivial copy of the bosonic one. We gauge the isometry, not including the kinetic term for the gauge field, and integrate out the gauge field by its equation of motion.

We define also the real linear multiplet G and the real twisted linear multiplet H by the conditions [26]

$$\begin{aligned} \bar{D}_+ \bar{D}_- G &= D_+ D_- G = 0, \\ D_+ \bar{D}_- H &= \bar{D}_+ D_- H = 0 \end{aligned} \quad (6.21)$$

The relation between the two set of multiplets is

$$\begin{aligned} G &= \Lambda + \bar{\Lambda} \\ H &= \Phi + \bar{\Phi} \end{aligned} \tag{6.22}$$

We start with an action of the form

$$\int d^2x d^4\theta K(\Lambda + \bar{\Lambda}) = \int d^2x d^4\theta K(G) \tag{6.23}$$

where the isometry corresponds to the imaginary part of Λ ; we can rewrite it in first order formalism as

$$\int d^2x d^4\theta [K(G) - (\eta + \bar{\eta})G] \tag{6.24}$$

with $\eta, \bar{\eta}$ chiral fields.

The variation in the lagrange multiplier η reproduce the original action. The only subtle point is that η is a constrained field; fortunately, the constrain can be solved by writing η in terms of an unconstrained complex superfield

$$\eta = i\bar{D}_+ \bar{D}_- \psi$$

Therefore, integrating by part,

$$\begin{aligned} \frac{\delta S}{\delta \eta} &= 0 \rightarrow \bar{D}_+ \bar{D}_- G = 0 \\ \frac{\delta S}{\delta \bar{\eta}} &= 0 \rightarrow D_+ D_- G = 0 \end{aligned}$$

so $G = \Lambda + \bar{\Lambda}$, and we recover the original action.

On the other hand , we can eliminate G from the first order action

$$\frac{\delta S}{\delta G} = 0 \rightarrow \frac{\partial K}{\partial G} = \eta + \bar{\eta}$$

We can solve for G and obtain the dual action in terms of chiral fields only. The lagrangian is the Legendre transform of K . Starting with our action with a chiral and a twisted chiral field and performing the duality transformation on the twisted chiral one, we end with an action with only chiral fields, i.e. with a Kahler geometry.

6.3 The dual model

We will apply the previous discussion to the case of the $SU(2) \times U(1)$ instanton. From eq. (6.11) we see that the Kahler potential is a function of $\ln\Lambda + \ln\bar{\Lambda}$ in the twisted chiral sector

$$K = K(\Lambda + \bar{\Lambda}, \Phi, \bar{\Phi}) \tag{6.25}$$

so the metric has an isometry corresponding to the phase of Λ .

Defining $G = \ln\Lambda + \ln\bar{\Lambda}$ the kähler potential is

$$K(G, \ln\Phi, \ln\bar{\Phi}) = - \int^{\frac{e^G}{\Phi\bar{\Phi}}} \frac{dx}{x} \ln(1+x) + \ln\Phi \ln\bar{\Phi} \quad (6.26)$$

In first order form the action reads

$$K(G, \Phi, \bar{\Phi}) - (\eta + \bar{\eta})G$$

with $\eta, \bar{\eta}$ chiral superfields. By eliminating the lagrange multiplier, $G = \chi + \bar{\chi}$, with χ twisted chiral ($\chi = \ln\Lambda$) we recover the initial action.

By the G equation of motion

$$\frac{\partial K}{\partial G} = \eta + \bar{\eta}$$

we obtain

$$e^G = (e^{-\eta-\bar{\eta}} - 1)\Phi\bar{\Phi} \quad (6.27)$$

and defining $e^{-\eta} = \chi$

$$K = - \int^{\chi\bar{\chi}-1} \frac{dx}{x} \ln(1+x) + \ln\Phi \ln\bar{\Phi} + \ln\chi\bar{\chi}[\ln\Phi\bar{\Phi} + \ln(\chi\bar{\chi} - 1)]$$

we finally define $\tilde{\Phi} = \ln\Phi + \ln\chi$, so we can rewrite in the previous equation

$$\ln\Phi \ln\bar{\Phi} + (\ln\chi + \ln\bar{\chi})(\ln\Phi + \ln\bar{\Phi}) = \tilde{\Phi}\bar{\tilde{\Phi}} - \ln\chi\ln\bar{\chi} + (\ln\chi\ln\Phi + \ln\bar{\chi}\ln\bar{\Phi})$$

the third term is a gauge transformation and can be eliminated, and the second term can be rewritten, modulo gauge transformations, as

$$-\frac{1}{2}(\ln\chi\bar{\chi})^2$$

The final form of the Kahler potential is

$$K = - \int^{\chi\bar{\chi}-1} \frac{dx}{x} \ln(1+x) + \ln\chi\bar{\chi}(\ln(\chi\bar{\chi} - 1)) - \frac{1}{2}(\ln\chi\bar{\chi})^2 + \tilde{\Phi}\bar{\tilde{\Phi}} \quad (6.28)$$

This Kahler potential describes the dual theory of the $SU(2) \times U(1)$ instanton. The two theories are equivalent at the level of classical equation of motion, but their geometric interpretation is very much different: the original model describes a group manifold with torsion, whereas the dual model describes a Kahler manifold (torsion free).

From the explicit expression of the Kahler potential we can derive the metric of our manifold: it is clearly the direct product of a torus ($\tilde{\Phi}$ part of the potential: a free complex boson) and the two-dimensional manifold described by the metric

$$d^2s = \frac{d\chi d\bar{\chi}}{\chi\bar{\chi}(\chi\bar{\chi} - 1)} \quad (6.29)$$

The physical meaning of this metric is manifest after analytic continuation

$$\begin{aligned}\chi &= \chi_1 + i\chi_2 \rightarrow \chi_1 + \chi_2 \\ \bar{\chi} &= \chi_1 - i\chi_2 \rightarrow \chi_1 - \chi_2\end{aligned}$$

in the new variables $u = 1/\chi, v = 1/\bar{\chi}$

$$d^2s = \frac{dudv}{1-uv} \quad (6.30)$$

We have again to take into account the shift in the dilaton field due to the integration measure in the change of variable. We already know that

$$\phi \rightarrow \phi - \frac{1}{2} \ln g_{00}$$

Changing variables to $\lambda = \ln \Lambda$ we obtain

$$d^2s = \frac{d\Phi d\bar{\Phi} + e^{2\lambda_1} d\lambda d\bar{\lambda}}{\Phi\bar{\Phi} + e^{2\lambda_1}}$$

The isometry reads $\lambda_2 \rightarrow \lambda_2 + \alpha$, so

$$g_{00} = \frac{e^{2\lambda_1}}{\Phi\bar{\Phi} + e^{2\lambda_1}}$$

and for the dilaton shift

$$\phi \rightarrow \phi - \lambda_1 + \frac{1}{2} \ln(\Phi\bar{\Phi} + e^{2\lambda_1})$$

with the dual change of variables

$$\begin{aligned}\chi\bar{\chi} &= 1 + \frac{\Lambda\bar{\Lambda}}{\Phi\bar{\Phi}} \\ \bar{\Phi} &= \ln\Phi + \ln\chi\end{aligned}$$

we obtain

$$\begin{aligned}\lambda_1 &= \frac{1}{2}\bar{\Phi} + \frac{1}{2}\Phi - \frac{1}{2}\ln\chi\bar{\chi} + \frac{1}{2}\ln(\chi\bar{\chi} - 1) \\ \ln(\Phi\bar{\Phi} + e^{2\lambda_1}) &= \bar{\Phi} + \Phi\end{aligned}$$

so finally the shift reads

$$-\frac{1}{2}\ln\left(1 - \frac{1}{\chi\bar{\chi}}\right)$$

and in the new variables u, v

$$\phi \rightarrow \phi - \frac{1}{2}\ln(1-uv) \quad (6.31)$$

The combined set of eq. (6.30,31) is well known in the recent literature: modulo a minus sign in front to the metric this background describes the two-dimensional black hole discovered by Witten [4], based on the gauging of a $U(1)$ subgroup in the $SL(2,R)$ WZW model. The wrong sign is related (as more detailed in the appendix B) to the change of sign in the level k of the WZW model: to be more precise, we are gauging the group $SU(2)$. The final effect is to obtain again a region of the two-dimensional Schwarzschild black hole.

The nature of our solution is manifest also in the euclidean signature after the change of variables:

$$\chi_1 = \frac{cht}{senr}, \quad \chi_2 = \frac{sht}{senr}$$

after which the metric reads

$$d^2 = (dr)^2 + \tan^2 r (dt)^2 \quad (6.32)$$

i.e. the euclidean $SU(2)$ version of the black hole (see appendix B).

This example provides a check of the above claim that duality is a symmetry of conformal field theory: starting with a conformal model we have obtained a new manifest conformal background, as it's evident from the WZW interpretation of the final combined system of metric and dilaton. In other words, we have related the $SU(2) \times U(1)$ and $SU(2)/U(1) \times U(1)^2$ WZW models. Moreover the form (6.32) of the metric is known to be exact only in the large $1/k$ limit in the bosonic case, while it's exact to *all* order in k in the supersymmetric case [29], where it is an exact conformal background at all order.

The last step is the inclusion of the original dilaton field. So far we have worked with the $SU(2) \times U(1)$ model which is not the instanton background proposed by Callan et al., still lacking the right background charge to the free $U(1)$ boson.

We find the correspondence

$$d^2 s = \frac{d\Phi d\bar{\Phi} + d\Lambda d\bar{\Lambda}}{\Phi\bar{\Phi} + \Lambda\bar{\Lambda}} = \frac{1}{r^2} (d^2 s)_{flat}$$

so for the dilaton field we find (modulo coefficients)

$$t = -\ln r = -\frac{1}{2} \ln(\Phi\bar{\Phi} + \Lambda\bar{\Lambda}) = \theta$$

As we have explained the duality transformation does not involve explicitly the dilaton fields: once we have checked that the dilaton is preserved by the isometry (i.e. is killed by the killing vector, or, equivalently, is independent from what we called x^0) we perform the above transformation on metric and torsion and shift the dilaton by the factor due to the measure, regardless of the initial value of the dilaton field.

In our case eq. shows that the dilaton is preserved by the isometry (the phase change of Λ) and all we have to do is the change to the new variables:

$$t = -\frac{1}{2} (\bar{\Phi} + \bar{\Phi})(6.33)$$

As we expected, the complete dual model is obtained by adding a background charge to one of the free fields of the final model.

This completes the alternative dual description of the Callan et al. instanton background: after an analytic continuation to Minkowski signature it describes some region of a black hole geometry. Unfortunately, the final geometry factorizes in two two-dimensional independent parts, so it's not very interesting as a four-dimensional background. However, we know also some deformation of the Callan et al. instanton. Modulo the solution of some subtle points like the fate of $N=4$ superconformal symmetry in the dual model (we have a manifest $N=2$ lagrangian which should be automatically promoted to $N=4$ superconformal symmetry after the inclusion of the background charge realizes $c=6$) we should be able to rewrite the deformation in the new coordinates: we expect to obtain a highly non-trivial mixing of the four dimensions and so a new interesting stringy black hole (?) solution.

Appendix A

We list now the vertices that correspond to emission of particle zero-modes for the various fields appearing in the effective 4-dimensional lagrangian in terms of the conformal fields of the generic "instantonic" CFTs described in sec. 2. The notations used in this list are essentially all explained in sec.2, for what concerns the space-time operators, that in the following expressions are distinguished by the square brackets. The $\Psi_{\mathcal{A}}, \Psi_{\mathcal{A}}^*$ are the operators correspondent to the (1,0) and (0,1) forms, $\Phi_{\mathcal{A}}, \Pi_{\mathcal{A}}$ (and the starred analogues) the correspondent "upper components". $\hat{1}$ is the identity times maybe the dimension zero operator that plays the same role as $e^{ik \cdot X(z, \bar{z})}$ in the flat space case, and whose presence depends on the uncompactified geometry represented by the abstract $(6,6)_{\pm, \pm}$.

The operators in the internal theory are labeled by their left and right conformal weights and $U(1)$ charges. In particular, the (chiral, chiral) and (chiral, antichiral) fields Ψ_k^{\pm} are lowest components of short $N=2$ reps. and play the role of abstract (1,1) and (2,1) forms of the compactifying Calabi-Yau manifold. The internal fields Ω_i are all the possible primary fields with the specified weights and charges, including more than the Ψ_k^{\pm} . We refer for more extensive exposition of the notation to [13, sec.VI.10].

The branchings of the $SU(6)$ reps into $SO(6) \times SU(2) \times U(1)$ which explains how the $SU(6)$ index of the charged vertices is reconstructed are indicated in the form

$$rep_{SU(6)} = (rep_{SO(6)}, rep_{SU(2)}, \bar{q})$$

(we omit the $U(1)$ charge \bar{q} when it is equal to zero). We list the vertices referring to their interpretation as zero-modes of the various E_6 -neutral and E_6 -charged fields of the effective $d=4$ theory (see sec.2), just to facilitate the comparison with the zero modes counting of that section; we count them for each field of the effective lagrangian of a certain kind; the number of these fields depends of course in the usual way from the topological numbers of the internal CY manifold.

In the specific case of $K3$ the abstract $(0,1)$ -forms are not in the spectrum of the theory ($h^{0,1}=0$).

Gravitational multiplet:

Graviton

$$e^{i\phi_{sg}(z)} \Phi_{\mathcal{A}} \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$e^{i\phi_{sg}(z)} \Pi_{\mathcal{A}} \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$4h^{1,1}$ zero modes

Gravitino

$$e^{\frac{i}{2}\phi_{sg}(z)} \Phi_{\mathcal{A}} \begin{bmatrix} 1/4 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)} \Pi_{\mathcal{A}} \begin{bmatrix} 1/4 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)} \Phi_{\mathcal{A}^*} \begin{bmatrix} 1/4 & 1 \\ 1/2 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)} \mathbf{1} \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}^a \Pi_{\mathcal{A}^*} \begin{bmatrix} 1/4 & 1 \\ 1/2 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix}$$

$2h^{1,1}$ zero modes of (+) chirality and $4h^{0,1}$ zero modes of (-) chirality

Neutral WZ multiplets

SU(6)-singlet scalars

$$e^{i\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Omega_i \begin{pmatrix} 1/2 & 1 \\ 1 & 0 \end{pmatrix}$$

One zero mode.

SU(6)-singlet fermions

$$e^{\frac{i}{2}\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 1/4 & 0 \\ 1/2 & 0 \end{bmatrix}^a \Omega_i \begin{pmatrix} 3/8 & 1 \\ -1/2 & 0 \end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)} \Psi_{\mathcal{A}} \begin{bmatrix} 1/4 & 0 \\ 0 & 0 \end{bmatrix} \Omega_i \begin{pmatrix} 3/8 & 1 \\ 1/2 & 0 \end{pmatrix}$$

2 zero modes of (-) chirality and $h^{1,0}$ of (+) chirality.

E_6 Gauge bosons:

Vertices in the 35 of SU(6), "SU(6)-gauge bosons"

$$35 = (15, 1) + (1, 1) + (1, 3) + (4, 2, \bar{q} = 3/2) + (\bar{4}, 2, \bar{q} = -3/2)$$

$$\begin{aligned}
& e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}} \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} J^A(\bar{z}) \\
& e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}} \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \partial\phi(\bar{z}) \\
& e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}} \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tilde{A}^i(\bar{z}) \\
& e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}} \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}^{a\tilde{a}} \mathbf{1} \begin{pmatrix} 0 & 3/8 \\ 0 & 3/2 \end{pmatrix} \Sigma_{\alpha}(\bar{z}) \\
& e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}} \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}^{a\tilde{a}} \mathbf{1} \begin{pmatrix} 0 & 3/8 \\ 0 & -3/2 \end{pmatrix} \Sigma_{\dot{\alpha}}(\bar{z})
\end{aligned}$$

$2h^{1,0}$ zero modes.

J are the currents in the adjoint (15) of $SO(6)$, \tilde{A} the currents of the $SU(2)$ of the right $N=4$ algebra, $\partial\phi$ is the internal $U(1)$ current expressed in term of a free boson.

SU(6)-singlet scalars

$$e^{i\phi_{sg}(z)}\Omega_i \begin{bmatrix} 1/2 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The Ω_i are all the ‘‘space-time’’ fields with the indicated weights and isospins, so they are more than the $\Psi_{\mathcal{A}}$ (see sec.2) and their number should correspond to $\#End(T_{K3})$

‘‘Scalar’’ vertices in the 20 of SU(6)

$$20 = (6, 2) + (4, 1, \tilde{q} = 3/2) + (\bar{4}, 1, \tilde{q} = -3/2)$$

$$\begin{aligned}
& e^{i\phi_{sg}(z)}\Psi_{\mathcal{A}} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}^{a\tilde{a}} \mathbf{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \theta_P(\bar{z}) \\
& e^{i\phi_{sg}(z)}\Psi_{\mathcal{A}} \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 0 & 3/8 \\ 0 & 3/2 \end{pmatrix} \Sigma_{\alpha}(\bar{z}) \\
& e^{i\phi_{sg}(z)}\Psi_{\mathcal{A}} \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 0 & 3/8 \\ 0 & -3/2 \end{pmatrix} \Sigma_{\dot{\alpha}}(\bar{z})
\end{aligned}$$

$2h^{(1,1)}$ zero modes.

The heterotic fermions $\theta_P(\bar{z})$ transform in the fundamental of $SO(6)$; Σ_{α} and $\Sigma_{\dot{\alpha}}$ are the $SO(6)$ spin fields of the two chiralities.

E_6 Gauginos:

Vertices in the 35, "SU(6) gauginos"

$$\begin{aligned}
 & e^{\frac{i}{2}\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 1/4 & 0 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix} J^A(\bar{z}) \\
 & e^{\frac{i}{2}\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 1/4 & 0 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix} \partial\phi(\bar{z}) \\
 & e^{\frac{i}{2}\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 1/4 & 0 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix} \tilde{A}^i(\bar{z}) \\
 & e^{\frac{i}{2}\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 1/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}^{a\bar{a}} \mathbf{1} \begin{pmatrix} 3/8 & 3/8 \\ -3/2 & 3/2 \end{pmatrix} \Sigma_\alpha(\bar{z}) \\
 & e^{\frac{i}{2}\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 1/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}^{a\bar{a}} \mathbf{1} \begin{pmatrix} 3/8 & 3/8 \\ -3/2 & -3/2 \end{pmatrix} \Sigma_{\dot{\alpha}}(\bar{z})
 \end{aligned}$$

Two zero modes of (-) chirality.

$$e^{\frac{i}{2}\phi_{sg}(z)} \Psi_{\mathcal{A}} \begin{bmatrix} 1/4 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix} J^A(\bar{z})$$

.....

$h^{1,0}$ of (+) chirality.

SU(6) singlets fermions)

$$e^{i\phi_{sg}(z)} \Omega_i \begin{bmatrix} 1/4 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix}$$

The numbers of these vertices is again related to $\#End(T_{K3})$.

Fermions in the 20 of SU(6)

$$\begin{aligned}
 & e^{\frac{i}{2}\phi_{sg}(z)} \Psi_{\mathcal{A}} \begin{bmatrix} 1/4 & 1/2 \\ 0 & 1/2 \end{bmatrix}^{\bar{a}} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix} \theta_P(\bar{z}) \\
 & e^{\frac{i}{2}\phi_{sg}(z)} \Psi_{\mathcal{A}} \begin{bmatrix} 1/4 & 1/4 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 3/8 \\ 3/2 & 3/2 \end{pmatrix} \Sigma_\alpha(\bar{z}) \\
 & e^{\frac{i}{2}\phi_{sg}(z)} \Psi_{\mathcal{A}} \begin{bmatrix} 1/4 & 1/4 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 3/8 \\ 3/2 & -3/2 \end{pmatrix} \Sigma_{\dot{\alpha}}(\bar{z})
 \end{aligned}$$

$h^{1,1}$ zero modes, all of (+) chirality.

27-charged Scalars:

Scalars in the 15 of SU(6)

$$15 = (6, 1, \tilde{q} = 1) + (4, 2, \tilde{q} = -\frac{1}{2}) + (1, 1, \tilde{q} = -2)$$

$$e^{i\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Psi_k^+ \begin{pmatrix} 1/2 & 1/2 \\ 1 & 1 \end{pmatrix} \theta_P(\bar{z})$$

$$e^{i\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 0 & 1/4 \\ 0 & 1/2 \end{bmatrix}^{\tilde{a}} \Psi_k^+ \begin{pmatrix} 1/2 & 3/8 \\ 1 & -1/2 \end{pmatrix} \Sigma_\alpha(\bar{z})$$

$$e^{i\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Psi_k^+ \begin{pmatrix} 1/2 & 1 \\ 1 & -2 \end{pmatrix}$$

one zero mode.

scalars in the 6 of SU(6)

$$6 = (1, 2, \tilde{q} = 1) + (4, 1, \tilde{q} = -\frac{1}{2})$$

$$e^{i\phi_{sg}(z)} \Psi_{\mathcal{A}^*}^* \begin{bmatrix} 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}^{\tilde{a}} \Psi_k^+ \begin{pmatrix} 1/2 & 1/2 \\ 1 & 1 \end{pmatrix}$$

$$e^{i\phi_{sg}(z)} \Psi_{\mathcal{A}^*}^* \begin{bmatrix} 0 & 1/4 \\ 0 & 0 \end{bmatrix} \Psi_k^+ \begin{pmatrix} 1/2 & 3/8 \\ 1 & -1/2 \end{pmatrix} \Sigma_\alpha(\bar{z})$$

$h^{0,1}$ zero modes.

27-charged fermions

fermions in the 15

$$e^{\frac{i}{2}\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 1/4 & 0 \\ 1/2 & 0 \end{bmatrix}^a \Psi_k^+ \begin{pmatrix} 3/8 & 1/2 \\ -1/2 & 1 \end{pmatrix} \theta_P(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 1/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}^{a\tilde{a}} \Psi_k^+ \begin{pmatrix} 3/8 & 3/8 \\ -1/2 & -1/2 \end{pmatrix} \Sigma_\alpha(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)} \hat{\mathbf{1}} \begin{bmatrix} 1/4 & 0 \\ 1/2 & 0 \end{bmatrix}^a \Psi_k^+ \begin{pmatrix} 3/8 & 1 \\ -1/2 & -2 \end{pmatrix}$$

Two zero modes of (-) chirality.

$$e^{\frac{i}{2}\phi_{sg}(z)} \Psi_{\mathcal{A}} \begin{bmatrix} 1/4 & 0 \\ 0 & 0 \end{bmatrix} \Psi_k^+ \begin{pmatrix} 3/8 & 1/2 \\ 1/2 & 1 \end{pmatrix} \tilde{\theta}_P(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)} \Psi_{\mathcal{A}} \begin{bmatrix} 1/4 & 1/4 \\ 0 & 1/2 \end{bmatrix}^{\tilde{a}} \Psi_k^+ \begin{pmatrix} 3/8 & 3/8 \\ 1/2 & -1/2 \end{pmatrix} \tilde{\Sigma}_\alpha(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix} 1/4 & 0 \\ 0 & 0 \end{bmatrix}\Psi_k^+\begin{pmatrix} 3/8 & 1 \\ 1/2 & -2 \end{pmatrix}$$

$h^{1,0}$ zero modes of (+) chirality.

fermions in the 6

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix} 1/4 & 1/2 \\ 0 & 1/2 \end{bmatrix}^{\bar{a}}\Psi_k^+\begin{pmatrix} 3/8 & 1/2 \\ 1/2 & 1 \end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix} 1/4 & 1/4 \\ 0 & 0 \end{bmatrix}\Psi_k^+\begin{pmatrix} 3/8 & 3/8 \\ 1/2 & -1/2 \end{pmatrix}\Sigma_{\alpha}(\bar{z})$$

$h^{1,1}$ zero modes of (+) chirality.

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}^*}^*\begin{bmatrix} 1/4 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}^{a\bar{a}}\Psi_k^+\begin{pmatrix} 3/8 & 1/2 \\ -1/2 & 1 \end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}^*}^*\begin{bmatrix} 1/4 & 1/4 \\ 1/2 & 0 \end{bmatrix}^a\Psi_k^+\begin{pmatrix} 3/8 & 3/8 \\ -1/2 & -1/2 \end{pmatrix}\tilde{\Sigma}_{\alpha}(\bar{z})$$

$2h^{0,1}$ zero modes of (-) chirality.

$\overline{27}$ -charged Scalars:

scalars in the $\overline{15}$ of SU(6)

$$\overline{15} = (6, 1, \tilde{q} = -1) + (\bar{4}, 2, \tilde{q} = \frac{1}{2}) + (1, 1, \tilde{q} = 2)$$

$$e^{i\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\Psi_k^-\begin{pmatrix} 1/2 & 1/2 \\ 1 & -1 \end{pmatrix}\theta_P(\bar{z})$$

$$e^{i\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix} 0 & 1/4 \\ 0 & 1/2 \end{bmatrix}^{\bar{a}}\Psi_k^-\begin{pmatrix} 1/2 & 3/8 \\ 1 & 1/2 \end{pmatrix}\Sigma_{\dot{\alpha}}(\bar{z})$$

$$e^{i\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\Psi_k^-\begin{pmatrix} 1/2 & 1 \\ 1 & 2 \end{pmatrix}$$

one zero mode.

scalars in the $\bar{6}$ of SU(6)

$$6 = (1, 2, \tilde{q} = -1) + (\bar{4}, 1, \tilde{q} = \frac{1}{2})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}^*}^*\begin{bmatrix} 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}^{\bar{a}}\Psi_k^-\begin{pmatrix} 3/8 & 1/2 \\ 1/2 & -1 \end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}^*}^*\begin{bmatrix}0 & 1/4 \\ 0 & 0\end{bmatrix}\Psi_k^-\begin{pmatrix}3/8 & 3/8 \\ 1/2 & 1/2\end{pmatrix}\Sigma_\alpha(\bar{z})$$

$h^{0,1}$ zero modes

$\overline{27}$ -charged fermions:

fermions in the $\overline{15}$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix}1/4 & 0 \\ 1/2 & 0\end{bmatrix}^a\Psi_k^-\begin{pmatrix}3/8 & 1/2 \\ -1/2 & -1\end{pmatrix}\theta_P(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix}1/4 & 1/4 \\ 1/2 & 1/2\end{bmatrix}^{a\bar{a}}\Psi_k^-\begin{pmatrix}3/8 & 3/8 \\ -1/2 & 1/2\end{pmatrix}\Sigma_\alpha(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix}1/4 & 0 \\ 1/2 & 0\end{bmatrix}^a\Psi_k^-\begin{pmatrix}3/8 & 1 \\ -1/2 & 2\end{pmatrix}$$

Two zero modes of (-) chirality.

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 0 \\ 0 & 0\end{bmatrix}\Psi_k^-\begin{pmatrix}3/8 & 1/2 \\ 1/2 & -1\end{pmatrix}\tilde{\theta}_P(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 1/4 \\ 0 & 1/2\end{bmatrix}^{\bar{a}}\Psi_k^-\begin{pmatrix}3/8 & 3/8 \\ 1/2 & 1/2\end{pmatrix}\tilde{\Sigma}_\alpha(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 0 \\ 0 & 0\end{bmatrix}\Psi_k^-\begin{pmatrix}3/8 & 1 \\ 1/2 & 2\end{pmatrix}$$

$h^{1,0}$ zero modes of (+) chirality.

fermions in the $\overline{6}$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 1/2 \\ 0 & 1/2\end{bmatrix}^{\bar{a}}\Psi_k^-\begin{pmatrix}3/8 & 1/2 \\ 1/2 & -1\end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 1/4 \\ 0 & 0\end{bmatrix}\Psi_k^-\begin{pmatrix}3/8 & 3/8 \\ 1/2 & 1/2\end{pmatrix}\Sigma_\alpha(\bar{z})$$

$h^{1,1}$ zero modes of (+) chirality.

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}^*}^*\begin{bmatrix}1/4 & 1/2 \\ 1/2 & 1/2\end{bmatrix}^{a\bar{a}}\Psi_k^-\begin{pmatrix}3/8 & 1/2 \\ -1/2 & -1\end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}^*}^*\begin{bmatrix}1/4 & 1/4 \\ 1/2 & 0\end{bmatrix}^a\Psi_k^-\begin{pmatrix}3/8 & 3/8 \\ -1/2 & 1/2\end{pmatrix}\tilde{\Sigma}_\alpha(\bar{z})$$

$2h^{0,1}$ zero modes of (-) chirality.

Appendix B

We briefly review the construction of the stringy black hole based on the coset $SL(2, R)/U(1)$ [4].

The conformal field theory describing the black hole is obtained from the WZW $SL(2, R)$ model by gauging the symmetry

$$g \rightarrow hgh$$

with h in some abelian subgroup H of $SL(2, R)$. We will assume that H is a compact $U(1)$ subgroup and we will obtain the euclidean version of the black hole. The minkoswkian version of the black hole can be obtained by taking a $U(1)$ non compact subgroup, while the dual black hole model [] by gauging the symmetry

$$g \rightarrow hgh^{-1}.$$

We parametrize the group manifold as

$$g = e^{\frac{i}{2}\theta_L\sigma_2} e^{\frac{1}{2}r\sigma_1} e^{\frac{i}{2}\theta_R\sigma_2}$$

We will take the generator of the abelian subgroup H to be σ_2 , so that the symmetry correspond to shifting $\theta_{L,R} \rightarrow \theta_{L,R} + \alpha$. The gauged WZW action reads

$$S[g, A] = S[g] + \frac{k}{2\pi} \int d^2z [A(\bar{\partial}\theta_R + \cosh r \bar{\partial}\theta_L) + \bar{A}(\partial\theta_L + \cosh r \partial\theta_R) - \bar{A}A(\cosh r + 1)]$$

with

$$S[g] = \frac{k}{4\pi} \int d^2z (\bar{\partial}r\partial r - \bar{\partial}\theta_L\partial\theta_L - \bar{\partial}\theta_R\partial\theta_R - 2\cosh r \bar{\partial}\theta_L\partial\theta_R)$$

with the gauge fixing

$$\theta_L = -\theta_R$$

Solving for A via the equations of motion (the WZW topological term is trivial after this gauge fixing) we finally obtain

$$S = \frac{k}{4\pi} \int d^2z [\partial r \bar{\partial} r + th^2 r \partial \theta \bar{\partial} \theta]$$

there is also a contribution from the integration measure for the dilaton field: the best way to compute it is via the one loop β - function equations [4]

$$\phi = 2lncoshr + cost.$$

This is the stringy black hole solution proposed by Witten. Asymptotically, it approaches the liouville version of c=1 string. The black hole interpretation is more manifest after analytic continuation in Kruskal-like coordinates.

Our solution is simply related to the Witten one by a change of sign in front of the action, i.e. the change of sign of k; this correspond to gauging SU(2) instead of SL(2,R). The only change we have to do in our formulas is

$$r \rightarrow ir$$

So our euclidean version of the metric reads

$$d^2s = dr^2 + \tan^2 r dt^2$$

and after analytic continuation in t

$$d^2s = dr^2 - \tan^2 r dt^2$$

In kruskal-like coordinates

$$\bar{r} = lnsenr$$

$$v = e^{\bar{r}+t}$$

$$u = e^{\bar{r}-t}$$

the metric reads

$$d^2s = \frac{dudv}{1-uv}.$$

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