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A Brief Introduction to Kac-Moody and Virasoro Algebras .

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TRIESTE

Con mucho cariño a Thalía
que hizo todo.

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INDEX .

NOTATION

$\mathbb{Z}$	set of integer numbers
$\mathbb{Z}+\frac{1}{2}$	set of half-integer numbers
$\mathbb{Z}^+$	set of positive integer numbers
$\forall$	for every
G	Lie group
g	Lie algebra
$\hat{g}$	affine Kac-Moody algebra of g
Φ	set of roots of g
$\hat{\Phi}$	set of roots of $\hat{g}$
$W(g)$	Weyl group of g
$\hat{\Phi}_{(real)}$	set of real roots of $\hat{g}$
$\Omega(g)$	weight lattice of g

1.- INTRODUCTION .

The aim of this work is to provide an introduction to the theory of Kac-Moody and Virasoro algebras in the context of Theoretical physics.

In modern physics, symmetry plays a central role and group theory is in fact the mathematical formulation of the study of symmetry. Up to now the use of finite (discrete) groups is well established, and they are principally applied to crystallography. Also the continuous infinite groups with a finite number of parameters (such as $SU(N)$, $SO(N)$, $U(1)$, etc.) are well known and have been applied with success in atomic, nuclear and particle physics. In fact, the gauge -- theories of elementary particles introduces the interactions among them by making that the theory (described by a Lagrangian density) respect certain kind of symmetries.

It is very usual and some times more easy to study the groups through its associated algebras (Lie groups and Lie algebras) which are in some sense the infinitesimal version of them. To a Lie group described by N parameters one can associate an N-dimensional Lie algebra, whose basis vectors $\{T^a\}$ satisfies the relations

$$[T^a, T^b] = i f^{ab}{}_c T^c \qquad a,b,c = 1,\dots,N \tag{1.1}$$

where $f^{ab}{}_c$ are the structure constants characterizing the Lie algebra.

In the last 20 years, mathematicians have developed a new kind of algebras, the Kac-Moody and the associated Virasoro algebras (1, 2, 3, 4, 5, 6) which are associated to continous groups with an infinite number of parameters. These algebras are infinite dimensional and have the novelty of having a central element, that is, an element that commutes with all the other elements of the algebra. To any finite dimensional Lie algebra g as (1.1), the associated (untwisted) (*) Kac-Moody algebra $\hat{g}$ is defined by

$$\left.\begin{aligned} [T^a_m, T^b_m] &= i f^{ab}_c T^c_{m+n} + K m \delta^{ab} \delta_{m,-n} \\ [T^a_m, K] &= 0 \end{aligned}\right\} \qquad (1.2)$$

where $a,b,c,=1,\dots,N$ ($N=\dim g$) and $m,n \in \mathbb{Z}$. Note that the generators $\{T^a_0\}$ form a subalgebra isomorphic to (1.1).

In the other side, the Virasoro algebra $\hat{V}$ has generators $\{L_n, c\}$ which satisfie

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m,-n}$$

$$[L_n, c] = 0 \qquad (1.3)$$

with $m,n \in \mathbb{Z}$.

The algebras (1.2), (1.3) are not unrelated structures, and as we shall see they form the factors of a semidirect product such that

$$[L_m, T^a_n] = -n T^a_{m+n}$$

$$[L_n, K] = 0 \qquad [T^a_n, C] = 0 \qquad (1.4)$$

(*) In all the work we will only study the untwisted Kac-Moody algebras which are the ones that have importance in physics, and so we will refere to them simply as Kac-Moody algebras.

In chapter 2 we shall give a review of the theory of the finite-dimensional simply compact Lie algebras including the theory of roots and Dynkin diagrams which are used to classify them.

Chapter 3 is the marrow of this work, there we will study the theory of the Kac-Moody and Virasoro algebras. We will also generalize the theory of - roots and Dynkin diagrams to classify them. A short overview of the unitary representations (which are the best known) is also given.

Chapter 4 concludes the work with two simple examples of physical theories (the ordinary relativistic string and the massless fermions in 1 + 1 dimensions) which show a simple manner, the way in which Kac-Moody and Virasoro algebras appear in physical problems.

2.- FINITE DIMENSIONAL LIE ALGEBRAS .

In this chapter we will give a short review of the theory of roots, weights and Dynkin diagrams for the simple and compact finite Lie algebras (*) (7,8,9).

Let us take the Lie algebra g, associated to the simple and compact Lie group G. The generators of g, $\{T^a\}$ satisfy :

$$[T^a, T^b] = i f^{abc} T^c \qquad (2.1)$$

$$a, b, c = 1, \ldots, \dim g$$

where f^{abc} are the structure constants of g which are assumed to be completely antisymmetric because the algebra is simple and compact and then, the T^a can be chosen to obey the ortonormality condition

$$Tr\,(T^a T^b) = A\delta^{ab} \qquad (2.2)$$

where A is a constant that depends on the representation. We use the condition (2.2) to define a scalar product on g. If $x, y \in g$ then:

$$\langle x, y \rangle \equiv \frac{1}{A}\, Tr\,(x y) = x^a y^a \qquad (2.3)$$

where $x = x^a T^a$, $y = y^a T^a$. Note that $\langle x,y \rangle$ do not depend on the representation, and it is invariant under G;

$$\langle \gamma x \gamma^{-1}, \gamma y \gamma^{-1} \rangle = \langle x, y \rangle \qquad (2.4)$$

$$\forall\, \gamma \in G$$

(*) While for a general compact Lie group; its Lie algebra can be splited into the direct sum of simple Lie algebras and abelian Lie algebras, so the results can be extended to it.

which implies that

$$\langle x , [y , z] \rangle + \langle y , [x , z] \rangle = 0 \qquad (2.5)$$

$$\forall x, y, z \in g .$$

A tipical element of G is given by

$$\gamma = e^{-i T^a \theta a} \quad , \theta_a^* = \theta_a \qquad (2.6)$$

and in a unitary representation of G , the $\{T^a\}$ are hermitian

$$T^{a\dagger} = T^a \quad . \qquad (2.7)$$

The standard way of constructing the algebra g , is by choosing a basis in which there is a maximal set of commuting hermitian generators H^i ,

$$[H^i , H^j] = 0 \qquad i, j = 1, \ldots, r \qquad (2.8)$$

The number r of generators is called the rank of g , and the abelian subalgebra generated by the $\{H^i\}$ is called a Cartan subalgebra (CSA). With the CSA, we ex tend it into a basis for the whole g by taking other generators E^α , such that

$$[H^i , E^\alpha] = \alpha^i E^\alpha \qquad (2.9)$$

$$i, = 1, \ldots, r$$

The real non-zero r-dimensional vector α , is called a <u>root</u> and E^α the <u>step operator corresponding to α</u> .

It can be shown that the roots α are non degenerate and that the only multiples of a root α which are roots are $\pm\alpha$ (7). In fact $-\alpha$ is a root with step operator $E^{-\alpha} = E^{\alpha +}$ which follows from (2.9). We denote the set of roots by Φ. Then the number of roots is $|\Phi| = \dim g - r$.

To complete the construction of g it remains to consider $[E^{\alpha}, E^{\beta}]$ for each pair of step operators E^{α} and E^{β}. From the Jacobi identity, we have

$$[H^i, [E^{\alpha}, E^{\beta}]] = (\alpha^i + \beta^i)[E^{\alpha}, E^{\beta}] \tag{2.10}$$

So if $\alpha+\beta \notin \Phi$ and $\alpha+\beta \neq 0$ then $[E^{\alpha}, E^{\beta}] = 0$. If $\alpha+\beta \in \Phi$ then $[E^{\alpha}, E^{\beta}]$ must be proportional to $E^{\alpha+\beta}$. And finaly if $\alpha+\beta = 0$ it is possible to see, using (2.5) that

$$\langle \xi \cdot H, [E^{\alpha}, E^{-\alpha}] \rangle = 0 \quad \text{if} \quad \xi \cdot \alpha = 0.$$

So $[E^{\alpha}, E^{-\alpha}]$ is proportional to $\alpha \cdot H$ and we fix the normalization of E^{α} such that $[E^{\alpha}, E^{-\alpha}] = \frac{2\alpha \cdot H}{\alpha^2}$.

In resumè

$$[E^{\alpha}, E^{\beta}] = \begin{cases} \rho(\alpha,\beta) E^{\alpha+\beta} & \text{if} \quad \alpha+\beta \in \Phi \\ \dfrac{2\alpha \cdot H}{\alpha^2} & \text{if} \quad \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.11}$$

where the constants $\rho(\alpha,\beta)$ are antisymmetric in α and β and can be easily calculated, nevertheless we will

not give their expression (see for instance ref.7).
The basis

$$H^i \quad i=1,\ldots,r \qquad E^\alpha \quad \alpha \in \Phi \qquad (2.12)$$

with the commutation relations (2.8), (2.9) and (2.11) generates g and is called the "Cartan Weyl" basis (8, 10).

For each root α it is possible to associate generators

$$E^\alpha \; , \quad E^{-\alpha} \; , \quad \frac{2\alpha\cdot H}{\alpha^2} \qquad (2.13)$$

which form a SU(2) subalgebra isomorfic to

$$I_+ \; , \quad I_- \; , \quad 2 I_3 \qquad (2.14a)$$

where

$$[I_+, I_-] = 2I_3 \; , \qquad [I_3 \; , I_\pm] = \pm I_\pm \; . \qquad (2.14b)$$

In fact

$$I_\pm = I_1 \pm i I_2 \qquad (2.14c)$$

with

$$[I_m, I_n] = i\,\varepsilon_{mnl}\, I_l \qquad (2.14d)$$

$$m, n, l = 1, 2, 3$$

and

$$I_+^\dagger = I_- \quad , \quad I_m^\dagger = I_m \qquad m = 1,2,3. \tag{2.14e}$$

As a consequence of this, the eigenvalues of $\frac{2\alpha\cdot H}{\alpha^2}$ in any unitary representation are integral. In the adjoint representation the eigenvalues are $\frac{2\alpha\cdot\beta}{\alpha^2}$ $\beta \in \Phi$ together with zero r times. Thus

$$\frac{2\alpha\cdot\beta}{\alpha^2} \in \mathbb{Z} \qquad \forall \alpha, \beta \in \Phi . \tag{2.15}$$

From this it follows that the angles between the roots and the ratios of the lenghts are given by:

$$\frac{|\alpha|}{|\beta|} \begin{cases} \sqrt{3} & \text{for} \quad \theta = 30^\circ, 150^\circ \\ \sqrt{2} & \text{for} \quad \theta = 45^\circ, 135^\circ \\ 1 & \text{for} \quad \theta = 60^\circ, 120^\circ \\ \text{undetermined} & \text{for} \quad \theta = 90^\circ \end{cases} \tag{2.16}$$

Furthermore, the step operators of the form $E^{\beta+m\alpha}$ $m \in \mathbb{Z}$ form a set of irreducible tensor operators belonging to the adjoint representation of the SU(2) subalgebra generated by (2.13), (5,7) thus, because

$$\left[\frac{2\alpha\cdot H}{\alpha^2}, E^\beta\right] = \frac{2\alpha\cdot\beta}{\alpha^2} E^\beta ,$$

E^β, has helicity $m_j = -\frac{\alpha\cdot\beta}{\alpha^2}$, the step operator $E^{\beta+m\alpha}$ with $m = -\frac{2\alpha\cdot\beta}{\alpha^2}$ have opposite helicity $-m_j$, then

$$\nabla_\alpha(\beta) \equiv \beta - \frac{2\alpha\cdot\beta}{\alpha^2}\alpha \in \Phi \qquad \forall \alpha, \beta \in \Phi . \tag{2.17}$$

The linear operator σ_α makes reflections in the hyperplane normal to α , and we have seen that σ_α transforms roots into themselves. The σ's generate a finite group $W(g)$ called the Weyl group of g. Some properties of σ are:

$$\left.\begin{aligned} \sigma(\alpha+\beta) &= \sigma(\alpha)+\sigma(\beta) \\ \sigma(m\alpha) &= m\,\sigma(\alpha) \end{aligned}\right\} \text{ linear} \qquad (2.18a)$$

$$\left(\sigma(\alpha)\right)^2 = \alpha^2 \qquad (2.18b)$$

preserves the lenght of the vectors,

$$\sigma\left(\sigma(\alpha)\right) = \alpha \qquad (2.18c)$$

or in operator form

$$\sigma^2 = \mathbb{1} . \qquad (2.18d)$$

The number of roots in general exceeds the rank r of g, and it is possible to select a basis of roots $\{\alpha_i\}$ $i=1,\dots,r$ called of <u>simple roots</u> (5,7) that generates the r-dimensional space and have the property that any root α can be written as:

$$\alpha = \sum_{i=1}^{r} n_i \alpha_i , \qquad n_i \in \mathbb{Z} \qquad (2.19)$$

where either $n_i \geq 0$ $\forall i$ <u>or</u> $n_i \leq 0$ $\forall i$. In the first case α is said to be positive ($\alpha>0$) and in the second negative ($\alpha<0$) (7). Obviously the Weyl group W transforms a basis of simple roots into another

$$\alpha_i' = \sigma_\alpha(\alpha_i) \quad , \tag{2.20}$$

and it can be used to generate all the roots, just beginning from the simple ones.

The structure of g can be characterized by the following matrix(*)

$$K_{ij} \equiv \frac{2\alpha_i \cdot \alpha_j}{\alpha_i^2} \qquad ij = 1, \dots, r \tag{2.21}$$

called "<u>Cartan</u> <u>matrix</u>" of g . Furthermore, from (2.15) it happens that $K_{ij} \in \mathbb{Z}$.

The following properties of K come directly from its definition (2.21), the property (2.16) of the roots, and the fact for simple roots α_i ,α_j , $\alpha_i \cdot \alpha_j \leq 0$ (see ref.7)

$$K_{ii} = 2 \tag{2.22a}$$

$$K_{ij} K_{ji} = 0, 1, 2, 3 \qquad i \neq j \tag{2.22b}$$

If $\alpha_i^2 = \alpha_j^2 \quad (i \neq j)$ then

$$K_{ij} = K_{ji} = -1, 0 \tag{2.22c}$$

If $\alpha_j^2 > \alpha_i^2 \quad (i \neq j)$ then

$$K_{ji} = -1 \qquad \text{and} \qquad K_{ij} = -2, -3 \tag{2.22d}$$

(*) In the rest of this chapter we will <u>not</u> use the convention of sum under repeated indexes.

$$\det K_{ij} \neq 0, \tag{2.22e}$$

K is invariant under the transformation (2.20), that is:

$$K'_{ij} = 2\frac{\sigma_\alpha(\alpha_i)\cdot\sigma_\alpha(\alpha_j)}{\sigma_\alpha(\alpha_j)^2} = K_{ij}. \tag{2.22f}$$

Also it is possible to see that, there are two root lengths at most (7) so, they are divided into long and short roots. If all the roots have the same length then g is called simple laced. An equivalent way of classifing g, due to Dynkin, is by the socalled Dynkin diagrams (11,12). They are constructed as follows:

For each simple root there is a dot and they are joined by K_{ij} K_{ji} lines with an arrow pointing from the long to the short root, (or without it if they have the same length). The figure 1 shows the Dynkin diagrams for all the simple Lie algebras.

Let us now see how finite-dimensional irreducible representations of g are expressed in the Cartan-Weyl basis. We take a basis $\{|\mu\rangle\}$ of eigenstates of the CSA H^i, that is:

$$H^i|\mu\rangle = \mu^i|\mu\rangle \qquad i=1,\ldots,r \tag{2.23}$$

and call the r-dimensional vector μ of eigenvalues a weight. While the SU(2) subalgebra (2.13); $\frac{2\alpha\cdot H}{\alpha^2}$ must have integral eigenvalues, so:

$$\frac{2\alpha\cdot\mu}{\alpha^2} \in \mathbb{Z} \qquad \forall\alpha\in\Phi \text{ and } \mu \text{ a weight} \tag{2.24}$$

and the states $|\mu\rangle$ must form SU(2) multiplets. While

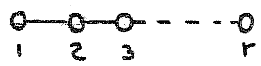
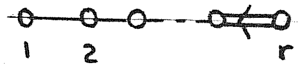
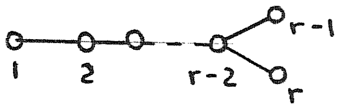
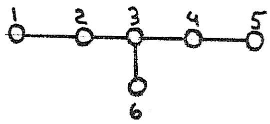
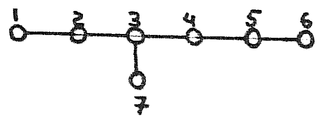
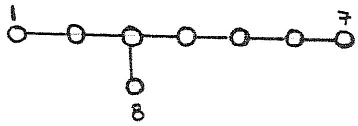
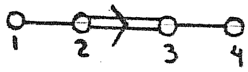
Cartan label	Group label	Order	Dynkin diagram
A_r	$SU(r+1)$	$r(r+2)$	
B_r	$SO(2r+1)$	$r(2r+1)$ $r \geq 2$	
C_r	$Sp(2r)$	$r(2r+1)$ $r \geq 3$	
D_r	$SO(2r)$	$r(2r-1)$ $r \geq 4$	
E_6	E_6	78	
E_7	E_7	133	
E_8	E_8	248	
F_4	F_4	52	
G_2	G_2	14	

Fig. 1 Dynkin Diagrams for simple finite-dimensional Lie algebras.

$|\mu\rangle$ has helicity $m_j = \frac{-\alpha\cdot\mu}{\alpha^2}$ that is

$$\frac{2\alpha\cdot H}{\alpha^2}\,|\mu\rangle = \frac{2\alpha\cdot H}{\alpha^2}\,|\mu\rangle$$

the state $|\bar{r}_\alpha(\mu)\rangle$ has opposite helicity, that is:

$$2\frac{\alpha\cdot H}{\alpha^2}\,|\sigma_\alpha(\mu)\rangle = -2\frac{\alpha\cdot\mu}{\alpha^2}\,|\sigma_\alpha(\mu)\rangle$$

as follows from the definition of σ_α. So the weights of a given representation are mapped into other weight of the same representation by σ_α, and so by the whole Weyl group $W(g)$. The condition (2.24) defines a lattice $\Omega(g)$ called the <u>weight lattice</u> of g.

It is possible to construct a basis for $\Omega(g)$ consisting of <u>fundamental</u> weight $\{\mu_i\}$ which has the property

$$2\frac{\mu_i\cdot\alpha_i}{\alpha_j^2} = \delta_{ij} \qquad (2.25)$$

where α_j is a simple root. They are clearly related to the simple roots, by:

$$\mu_i = \sum_{j=1}^{r} K^{-1}_{ij}\,\alpha_j \qquad (2.26)$$

where K^{-1} is the inverse of K which exists because (2.22e). The weights can then be expressed as

$$\mu = \sum_{i=1}^{r} n_i\,\mu_i \qquad n_i \in \mathbb{Z} \qquad (2.27)$$

If $n_i \geqslant 0 \;\forall i$, the weight is called <u>dominant</u> and it is possible to map every weight into a unique dominant weight by the Weyl group (5). Clearly the weight is dominant, if and only if

$$\mu\cdot\alpha_i \geqslant 0 \qquad i = 1,\dots,r. \qquad (2.28)$$

Let us define a particular dominant weight

$$\rho \equiv \sum_{i=1}^{r} \mu_i \qquad (2.29)$$

which has the property that $\rho\cdot\alpha>0$ if $\alpha>0$ and $\rho\cdot\alpha<0$ if $\alpha<0$. Now we can define an order among the weight saying that

$$\mu_1 > \mu_2 \qquad \text{if} \qquad \mu_1\cdot\rho > \mu_2\cdot\rho \ . \qquad (2.30)$$

In any finite-dimensional irreducible representation of g it is possible to find a unique state (7) $|\mu_0\rangle$ called <u>highest weight</u> state, such that its weight is the biggest one, in the sense of (2.30). It has the property that

$$E^{\alpha}\,|\mu_0\rangle = 0 \qquad \forall\,\alpha>0\,,\ \alpha\in\Phi \qquad (2.31)$$

because, if it were not zero, it would be equal to a state with weight $\mu_0+\alpha$ which is bigger than μ_0.

The space of the irreducible representation will be generated by the action of lowering operators on the highest weight state $|\mu_0\rangle$, that is

$$(E^{-\beta_1})^{n_1}\,(E^{-\beta_2})^{n_2}\,\ldots\,(E^{\beta_m})^{n_m}\,|\mu_0\rangle \qquad (2.32)$$

where the β's are positive roots and $n_i\in\mathbb{Z}^+\cup\{0\}$ $i=1,\ldots,m$. Clearly the weight of the state (2.32) is

$$\mu = \mu_0 - \sum_{i=1}^{m} n_i\,\beta_i \qquad \begin{array}{l} n_i\in\mathbb{Z}^+\cup\{0\} \\ \beta_i\in\Phi\ ,\ \beta_i>0 \end{array} \qquad (2.33)$$

and so all the weights of the representation are expressed in this way. In the adjoint representation of g , the weights are the roots and the highest weight is called the <u>highest</u> <u>root</u> Ψ . Obviously by - (2.33), if $\alpha \in \Phi$, then $\Psi - \alpha$ is a sum of positive roots. Also, because Ψ is dominant,

$$\Psi = \sum_{i=1}^{r} n_i \mu_i \quad , n_i \geqslant 0 \ . \tag{2.34}$$

The highest root Ψ plays an important role in the theory of Kac-Moody algebras, as we will see in the next chapter, where we will generalize the concepts introduced in this one.

3.- KAC-MOODY AND VIRASORO ALGEBRAS .

i) Groups of Infinite Order.

In recent times the mathematicians have been interested in the study of continuous groups of infinite order (that is, each element of the group is characterized by an infinite set of parameters)(*) and the infinite-dimensional algebras associated with them. The Kac-Moody algebras (1, 2, 3, 4) and the associated Virasoro algebras are two examples of them; in the last years they have been applied to physical problems (10, 13) and in particular to string theories (see for instance refs. 14, 15).

Let us take a Lie group G (**) with Lie algebra g, and consider the group $\hat{G}$ consisting in the diffeomorphism from the circle S' to G. So

$$\hat{G} = \{ \gamma : S' \longrightarrow G \} \tag{3.1}$$

with

$$S' = \{ z \in \mathbb{C} \mid |z| = 1 \} \tag{3.2}$$

and the group operation defined by that on G, so if $\gamma_1(z)\ \gamma_2(z) \in \hat{G}$

(*) From now on, we shall refer to them simply as infinite dimensional groups.

(**) For the moment we are only supposing that the group is compact, but it can be non simple.

$$\gamma_1 \cdot \gamma_2 \, (z) = \gamma_1(z) \cdot \gamma_2(z) \qquad (3.3)$$

Note that by $\gamma(z)$ we are denoting an element of $\hat{G}$ to whom there corresponds a set of elements of G obtained by varing z . $\hat{G}$ is called the loop group of G . To calculate the Lie algebra $\hat{g}_0$ of $\hat{G}$, we start from the Lie algebra g of G , with generators $\{T^a\}$ satisfing

$$[T^a, T^b] = i f^{ab}{}_c T^c \qquad (3.4)$$

where the structure constants $f^{ab}{}_c$ are not supposed to be completely antisymmetric at this stage. The elements of G are expressed as:

$$\gamma = e^{-iT^a \theta_a} \qquad a = 1, \ldots, \dim g \qquad (3.5)$$

where $\theta_a^* = \theta_a$ are the parameters of G ; while the elements of $\hat{G}$ (which can be continously deformed to the identity) can be expressed by the function $\theta_a(z)$ in such a way that:

$$\gamma(z) = e^{-iT^a \theta_a(z)} \qquad (3.6)$$

Making a Laurent expansion of $\theta_a(z)$ we have

$$\theta_a(z) = \sum_{n=-\infty}^{+\infty} \theta_a^{-n} \, z^n \qquad (3.7)$$

and we can define the generators

$$T_n^a \equiv T^a z^n \tag{3.8}$$

in terms of which

$$\gamma(z) = e^{-i \sum_{n,a} T_{-n}^a \theta_a^n} \tag{3.9}$$

The generators (3.8) of the loop group $\widehat{G}$ satisfy the Lie algebra

$$[T_m^a, T_n^b] = i f^{ab}{}_c T_{m+n}^c \tag{3.10}$$

as follows from (3.4) and (3.8). The algebra (3.10) is called the <u>untwisted</u> <u>affine</u> <u>Kac-Moody</u> <u>algebra</u> $\hat{g}_0$ (*) associated to g (10), and further we will see how it admits what is called a central extension. Note that $\widehat{G}$ is the group of general gauge transformations. Note also that the generators T_0^a form a subalgebra of $\hat{g}_0$ which is isomorphic to g, and corresponds to the subgroup of $\widehat{G}$ consisting in constant maps $S^1 \longrightarrow G$ so, clearly it is isomorphic to G itself. For G compact and the generator $\{T^a\}$ of g hermitian, that is

$$T^{a\dagger} = T^a, \tag{3.11}$$

it happens that $$T_n^{a\dagger} = T_{-n}^a \tag{3.12}$$

(*) From now on we will refear to it simply as affine Kac-Moody algebra.

as follows from (3.8) and that, for $|z|=1$, $z^* = z^{-1}$.

A representation of $\hat{g}_0$ satisfing (3.12) will be called unitary because then $\hat{G}$ will be unitary.

Let us now consider another infinite-dimensional group; that is, the group V of diffeomorphism $S' \longrightarrow S'$ with the law of multiplication defined by the composition of functions, thus

$$\varphi_1 \circ \varphi_2 (z) = \varphi_1 [\varphi_2 (z)]. \tag{3.13}$$

In order to calculate the algebra associated to V, let us consider a representation of V that acts on the space of functions $f: S' \longrightarrow \mathcal{V}$ where $\mathcal{V}$ is some vector space. Then

$$\hat{\varphi} f(z) \equiv f(\varphi^{-1}(z)) \tag{3.14}$$

For $\varphi \in V$ near to the identity we have

$$\varphi(z) = z\, e^{-i\xi(z)} \tag{3.15}$$

with $\xi^*(z) = \xi(z)$ and $|\xi(z)| \ll 1$. Then

$$\varphi(z) \simeq z - i z \xi(z)$$

and

$$\varphi^{-1}(z) \simeq z + i z \xi(z)^{(*)} \tag{3.16}$$

(*)Note that here φ^{-1} means inverse in the sense of composition of function, thus $\varphi^{-1} \circ \varphi = \mathbb{1}$.

Then substituting on (3.14) we obtain

$$\hat{\varphi} f(z) \simeq f(z) + i z \xi(z) \frac{d}{dz} f(z). \qquad (3.17)$$

Doing a Laurent expansion of

$$\xi(z) = \sum_{n=-\infty}^{+\infty} \xi_{-n} \; z^n \qquad (3.18)$$

we can introduce the generators

$$L_n \equiv -z^{n+1} \frac{d}{dz} \qquad (3.19)$$

and then (3.17) becomes

$$\hat{\varphi} f(z) \simeq \left(1 - i \sum_{n=-\infty}^{+\infty} \xi_{-n} L_n \right) f(z). \qquad (3.20)$$

The $\{L_n\}$ satisfies the Lie algebra $\hat{V}_0$

$$[L_m, L_n] = (m-n) \; L_{m+n} \qquad (3.21)$$

which is the Virasoro algebra (without a central element; we will see later how does it appear). For a unitary representation

$$L_n^+ = L_{-n} \qquad (3.22)$$

because $\xi^*_{-n} = \xi_n$.

The groups $\mathcal{V}$ and $\widehat{G}$ can be related easily to form the factors of a semidirect product. Consider a representation of G that acts on the vector space $\mathcal{V}$, the $\gamma(z) \in \widehat{G}$ acts on the functions $f: S^1 \rightarrow \mathcal{V}$ in the usual way $\gamma(z) f(z)$ and $\varphi \in \mathcal{V}$ acts by (3.14), then we define the convined action of $\mathcal{V}$ and $\widehat{G}$ by

$$(\varphi, \gamma) f(z) = \gamma(\varphi^{-1}(z)) f(\varphi^{-1}(z)) \qquad (3.23)$$

Since in this representation the generators of $\mathcal{V}$ and $\widehat{G}$ are represented by (3.8) and (3.9), it follows that the commutation relation between L_m and T_n^a are

$$[L_m, T_n^a] = -n T^a_{m+n} \;. \qquad (3.24)$$

ii) Central Extensions.

One interesting feature of the algebras (3.10) and (3.21) is that they admit the addition of elements to the algebra that commute with the initial ones. This is what is called a central extension of the algebra, they play an important role in the application of the Kac-Moody algebras to the quantum theories, as we will see in the next chapter.

Let us consider in general an algebra

$$[t^a, t^b] = i f^{ab}{}_c t^c \qquad (3.25)$$

(for the moment we are not supposing that the structure constants f^{ab}_c are totaly antisymmetric). Because of the Jacobi identity

$$[t^a,[t^b,t^c]]+[t^c,[t^a,t^b]]+[t^b,[t^c,t^a]]=0 \qquad (3.26)$$

the stucture constants obey the relation

$$f^{ab}{}_e f^{ec}{}_d + f^{bc}{}_e f^{ea}{}_d + f^{ca}{}_e f^{eb}{}_d = 0 \qquad (3.27)$$

Now we can ask ourselves about the possibility of adding elements K^i to the algebra (3.25) such that

$$[t^a,t^b] = if^{ab}{}_c t^c + ig^{ab}{}_i K^i \qquad (3.28a)$$

$$[t^a, K^i] = 0 \qquad (3.28b)$$

$$[K^i, K^j] = 0 \qquad (3.28c)$$

Because (3.28a) needs to satisfie (3.26) we obtain that the $g^{ab}{}_i$ must satisfie the ecuations

$$f^{ab}{}_e g^{ec}{}_i + f^{bc}{}_e g^{ea}{}_i + f^{ca}{}_e g^{eb}{}_i = 0 \qquad (3.29a)$$

and also $\qquad g^{ab}{}_i = -g^{ba}{}_i \qquad (3.29b)$

The space of solutions of the linear ecuations (3.29) determines the possible central extensions of the algebra. Nevertheless there are some solutions of (3.29) that can be removed by a redefinition of the t^a. If

$$t^a \longrightarrow t^a - \eta^a{}_i K^i \qquad (3.30a)$$

the $g^{ab}{}_i$ transforms as

$$g^{ab}{}_i \longrightarrow g^{ab}{}_i + f^{ab}{}_c \eta^c{}_i \qquad (3.30b)$$

and it is easy to verify that

$$f^{ab}{}_{c}\, \eta^{c}{}_{i} \qquad (3.30c)$$

satisfies (3.29) trivially.

Let us now investigate the possible central extensions that the simple compact finite-dimensional Lie algebra, given by (2.1) admits. While the generators obey (2.2), the structure constants will obey an analogous relation

$$f^{ars} f^{brs} = C\,\delta^{ab} \qquad (3.31)$$

with C a constant. Multipling (3.29a) by f^{abs},using (3.31) and recalling the indexes, we obtain

$$C\, g^{ab}_{i} = 2 f^{ars} f^{bst}\, g^{tr}_{i} \,,$$

using (3.27) and after a little of algebra

$$g^{ab}_{i} = \frac{1}{C} f^{abr} f^{rst}\, g^{st}_{i} \qquad (3.32)$$

which is of the form (3.30c), so all the central extensions of (2.1) are trivial. The same argument can be applied to semi-simple compact finite-dimensional Lie algebras, and for the non-compact case, things can be arranged to obtain the same result (10). So there is no central extension for such algebras.

For the Kac-Moody and Virasoro algebras, things are different and there exist non trivial solutions to(3.29). The algebras with the central element are:

$$\left[T^{a}_{m}, T^{b}_{n}\right] = i f^{abc} T^{c}_{m+n} + K m\, \delta^{ab} \delta_{m,-n} \qquad (3.33a)$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12} m(m^2-1)\delta_{m,-n} \qquad (3.33b)$$

$$[L_m, T_n^a] = -n T_{m+n}^a \qquad (3.33c)$$

Where K and C are the central elements that can be taken as a constant in each irreducible representation of the algebra; and f^{abc} are the antisymmetric structure constants of a compact and simple Lie algebra g. From now on we will denote by $\hat{g}$ and $\hat{V}$ the Kac-Moody and Virasoro algebras (3.33a) and (3.33b) with the central elements added.

In general there is not a method to resolve the equations (3.29) and often one needs to test and error.

iii) Theory of roots and Dynkin Diagrams of $\hat{g}$.

In chapter 2 we have studied the theory of roots and Dynkin diagrams for the simple, compact finite-dimensional Lie algebras g, now we will generalize it to classify the affine Kac-Moody algebras $\hat{g}$.

In the Cartan Weyl basis, the algebra g is described by the commutation relations (2.8),(2.9) and (2.11), in this basis the algebra $\hat{g}$ is

$$[H_m^i, H_n^j] = K m \delta^{ij} \delta_{m,-n} \qquad (3.34)$$

$$[H_m^i, E_n^\alpha] = \alpha^i E_{m+n}^\alpha \qquad (3.35)$$

$$[E^{\alpha}_m, E^{\beta}_n] = \begin{cases} \rho(\alpha,\beta)\, E^{\alpha+\beta}_{m+n} & \text{if} \quad \alpha+\beta \in \Phi \qquad (3.36a) \\ \frac{2}{\alpha^2}(\alpha \cdot H_{m+n} + K m\, \delta_{m,-n}) & \text{if} \quad \alpha = -\beta \qquad (3.36b) \\ 0 & \text{otherwise} \qquad (3.36c) \end{cases}$$

$$[K, E^{\alpha}_n] = [K, H^i_n] = 0. \qquad (3.37)$$

In a unitary representation the generators have the hermiticity conditions

$$H^{i\dagger}_n = H^i_{-n}, \qquad E^{\alpha\dagger}_n = E^{-\alpha}_{-n}, \qquad K^{\dagger} = K \qquad (3.38)$$

Now we will try to construct a CSA for $\hat{g}$. We can start with H^i_0 and K in which the step operators will be E^{α}_n and H^i_n ,

$$[H^i_0, E^{\alpha}_n] = \alpha^i E^{\alpha}_n \qquad (3.39a)$$

$$[K, E^{\alpha}_n] = 0, \qquad (3.39b)$$

$$[H^i_0, H^j_n] = 0 \qquad (3.39c)$$

$$[K, H^i_n] = 0 \qquad (3.39d)$$

Nevertheless the roots $(\alpha, 0)$ are infinitely degenerate and (3.39c) and (3.39d) show us that we have not a finite maximal set of commuting elements in the algebra. In order to take care of these difficulties, we add another element d to the algebra $\hat{g}$ which has the commutation relations

$$[d, T_n^a] = n T_n^a \qquad (3.40a)$$

$$[d, K] = 0 \qquad (3.40b)$$

and if (3.38) holds,

$$d^+ = d \qquad (3.41)$$

In the Cartan Weyl basis (3.40) has the form

$$\left.\begin{aligned} [d, H_n^i] &= n H_n^i \\ [d, E_n^\alpha] &= n E_n^\alpha \\ [d, K] &= 0 \end{aligned}\right\} \qquad (3.42)$$

Note that (3.40) and (3.42) are consistent with the Jacobi identities and in fact d can be identified with $-L_0$ of the Virasoro algebra (see (3.33)).

We can now use as a CSA the operators H_0^i, K, d with the operator E_n^α, H_n^i $(n \neq 0)$ as step operator. So that

$$[H_0^i, E_n^\alpha] = \alpha^i E_n^\alpha, \quad [K, E_n^\alpha] = 0, \quad [d, E_n^\alpha] = n E_n^\alpha$$

and

$$[H_0^i, H_n^j] = 0, \quad [K, H_n^j] = 0, \quad [d, H_n^j] = n H_n^j;$$

and then

E_n^α corresponds to the root $a \equiv (\alpha, 0, n)$ (3.43a)

and

H_n^i $(n \neq 0)$ corresponds to the root $n \equiv (0,0,n)$ (3.43b)

the roots $a=(\alpha,0,n)$, $\alpha \in \Phi$ and $n \in \mathbb{Z}$ are non-degenerated and are called real roots, while the roots $n=(0,0,n)$, $n \in \mathbb{Z}-\{0\}$ are r-fold degenerated and are called imaginary roots. Once again the roots can be divided into positive and negative

$$(\alpha,0,n)>0 \quad \text{if} \quad n>0 \text{ or } n=0 \text{ and } \alpha>0 \tag{3.44}$$

We can expand the new $r+1$-dimensional root space by taking as a basis of simple roots the following

$$a_i \equiv (\alpha_i,0,0) \qquad i=1,\ldots,r \tag{3.45a}$$

(with $\{\alpha_i\}$ simple roots of g) and

$$a_0 \equiv (-\psi,0,1) \tag{3.45b}$$

where ψ is the highest root of g. Then every root a of $\hat{g}$ can be written as:

$$a = \sum_{i=0}^{r} n_i a_i \qquad n_i \in \mathbb{Z} \tag{3.46}$$

with $n_i \geqslant 0$ for $a>0$ and $n_i \leqslant 0$ for $a<0$.

We now want to define an analogous of the scalar product (2.3) for the generator of $\hat{g}$. Nevertheless the non trivial representations of $\hat{g}$ are infinite-dimensional and so, it is difficult to define a trace operation. However it is possible to construct such scalar product, up to a constant, requiring that it satisfies the invariance property (2.5). First of all because $\{T_0^a\}$ forms a subalgebra isomorphic to g, following (2.2) and (2.3), we take

$$\langle T_0^a , T_0^b \rangle = \delta^{ab}$$

taking $x = T_m^a$, $y = K$ and $z = T_n^b$ on (2.5) we obtain

$$\langle K , K \rangle = 0$$

$$\langle K , T_n^a \rangle = 0 .$$

Now taking $x = T_m^a$, $y = T_n^b$ and $z = d$ follows

$$\langle T_m^a , T_n^b \rangle = 0 \qquad \text{if} \qquad m + n \neq 0 .$$

Taking $x = T_m^a$, $y = T_0^b$, $z = T_n^c$ and using the last results

$$\langle T_m^a , T_n^b \rangle = A(m)\, \delta^{ab}\, \delta_{m,-m} .$$

Taking $x = T_m^a$, $y = T_{-m-1}^b$ and $z = T_1^c$ follows that $A(m) = 1$, so

$$\langle T_m^a , T_n^b \rangle = \delta^{ab} \delta_{m,-n}$$

Finally, taking $x = T_m^a$, $y = d$ and $z = T_n^b$;

$$\langle d , K \rangle = 1 \qquad \text{and} \qquad \langle d , T_n^a \rangle = 0$$

The only unconstrained product is

$$\langle d , d \rangle$$

and this is due to the fact that the algebra is unchanged by the transformation $d \longrightarrow d + AK$, A a constant, so we can take $\langle d , d \rangle$ as zero. In resumè we have:

$$\langle T^a_m , T^b_n \rangle = \delta^{ab} \delta_{m,-n} \quad (3.47a)$$

$$\langle K , T^a_n \rangle = 0 \quad (3.47b)$$

$$\langle K , K \rangle = 0 \quad (3.47c)$$

$$\langle d , T^a_n \rangle = 0 \quad (3.47d)$$

$$\langle d , K \rangle = 1 \quad (3.47e)$$

$$\langle d , d \rangle = 0 \quad (3.47f)$$

The relation (3.47) induces a metric in $\hat{g}$. While the metric on g is euclidian in caracter, that is $\langle T^a , T^b \rangle = \delta^{ab}$, on $\hat{g}$ it is Lorentzian as can be seen by considering the scalar product (3.47) on the hermitian basis

$$T^a_0 , \frac{1}{\sqrt{2}} (T^a_m + T^a_{-m}), \frac{1}{\sqrt{2}i} (T^a_m - T^a_{-m}), \frac{1}{\sqrt{2}} (K+d), \frac{1}{\sqrt{2}} (K-d)$$

All these vectors are ortonormal, and the norm of $\frac{1}{\sqrt{2}}(K-d)$ is -1. This will modify the scalar product among the roots α and the new weights $\hat{\mu}$ on $\hat{g}$ as we are going to see.

Let us take a basis $\{|\mu, \mu_K, \mu_d\rangle\}$ of eigenstates for the CSA H^i_0, K, d such that

$$H^i_0 |\mu, \mu_K, \mu_d\rangle = \mu^i |\mu, \mu_K, \mu_d\rangle$$

$$K |\mu, \mu_K, \mu_d\rangle = \mu_K |\mu, \mu_K, \mu_d\rangle \quad (3.48)$$

$$d |\mu, \mu_K, \mu_d\rangle = \mu_d |\mu, \mu_K, \mu_d\rangle ,$$

and call the $r+2$ dimensional vector

$$\hat{\mu} \equiv (\mu, \mu_K, \mu_d) \tag{3.49}$$

a weight of $\hat{g}$. For the CSA, the products (3.47) are:

$$\langle H_0^i, H_0^j \rangle = \delta^{ij}, \quad \langle H_0^i, K \rangle = \langle H_0^i, d \rangle = 0$$

$$\langle K, K \rangle = \langle d, d \rangle = 0 \qquad \langle K, d \rangle = 1$$

and so the metric tensor in the weight and root space is

$$g = \left(\begin{array}{c|cc} \mathbb{1} & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & 1 \\ & 1 & 0 \end{array} \right)$$

where $\mathbb{1}$ is the $r \times r$ unitary matrix. The product of two vectors $\hat{\mu}$ and $\hat{\mu}'$ is then

$$\hat{\mu} \cdot \hat{\mu}' = \mu \cdot \mu' + \mu_K \mu'_d + \mu_d \mu'_K , \tag{3.50}$$

meanwhile for two real roots $a = (\alpha, 0, n)$ and $a' = (\alpha', 0, n)$

$$a \cdot a' = \alpha \cdot \alpha' \tag{3.51}$$

whilst the imaginary roots $n = (0, 0, n)$ and $n' = (0, 0, n)$ have

$$n \cdot n' = 0 \tag{3.52a}$$

and also $n \cdot a = 0$. (3.52b)

Now we can define in a straightforward way the Cartan matrix of $\hat{g}$ to be: (*)

$$\hat{K}_{ij} \equiv 2\frac{\alpha_i \cdot \alpha_j}{\alpha^2} \qquad i, j = 0, 1, \ldots, r. \tag{3.53}$$

From (3.45) and (3.51), it happens that

$$\hat{K}_{ij} = 2\frac{\alpha_i \cdot \alpha_j}{\alpha_j^2} \qquad i, j = 1, \ldots, r \tag{3.54a}$$

$$\hat{K}_{oi} = -2\frac{\psi \cdot \alpha_i}{\alpha_i^2} \qquad i = 1, \ldots, r \tag{3.54b}$$

$$\hat{K}_{io} = -2\frac{\alpha_i \cdot \psi}{\psi^2} \qquad i = 1, \ldots, r \tag{3.54c}$$

And by (2.34)

$$\psi = -\sum_{i=1}^{r} \hat{K}_{oi}\, \mu_i$$

so that $$\hat{K}_{oi} \leqslant 0 \tag{3.55}$$

Now, because $\hat{K}_{oi}\, \hat{K}_{io} = 0, 1, 2, 3, 4$
and $\psi^2 \geqslant \alpha_i^2$ $i = 1, \ldots, r$ we see that

$$\hat{K}_{io} = -1 \qquad \text{for} \qquad K_{oi} \neq 0 \tag{3.56a}$$

$$K_{io} = 0 \qquad \text{for} \qquad K_{oi} = 0 \tag{3.56b}$$

(*) In the rest of this chapter there is no sum over repeated indexes.

(provided that ψ is not a simple root which only happens for SU(2)).

Now we can construct the Dynkin diagrams for $\hat{g}$. They are the same as those for g, with an extra point added corresponding to the root a_0 and joined by $K_{0i}\ K_{i0}$ lines to the other points (which corresponds to the other roots). An arrow pointing from a_0 towards a_i is added if $K_{0i}\,K_{i0} > 1$. Figure 2 shows the Dynkin diagram for the affine simple Kac-Moody algebras. The diagram for $\hat{A}_1$=SU(2) is an exceptional one, because there, the simple roots are $a_0=(-\alpha,0,1)$ and $a_1=(\alpha,0,0)$ so that $K_{01}\,K_{10}=4$ (see the remark after (3.56)), and the fact that $a_0^2=\alpha_1^2$ in this case, it is indicated by the doble arrows in the diagram.

To round out, we note that it is possible to construct a Weyl group $\hat{W}$ for $\hat{g}$, it is defined to be generated by the reflexions in the hyperplanes, normal to the real roots, thus:

$$\sigma_a(v) = v - 2\,\frac{a\cdot v}{a^2}\;a \tag{3.57}$$

The imaginary roots cannot be included because they have zero square (see (3.52)). Note also that

$$\sigma_a(n) = n \tag{3.58}$$

with n an imaginary root and that for two real roots $a=(\alpha,0,n_\alpha)$ and $b=(\beta,0,n_\beta)$;

$$\sigma_a(b) = \left(\sigma_\alpha(\beta),\ 0\ ,\ n_\beta - 2\frac{\alpha\cdot\beta}{\alpha^2}\;n_\alpha\right) \tag{3.59}$$

is a root. So as before, $\hat{W}$ transforms roots into

Cartan label | Dynkin diagram

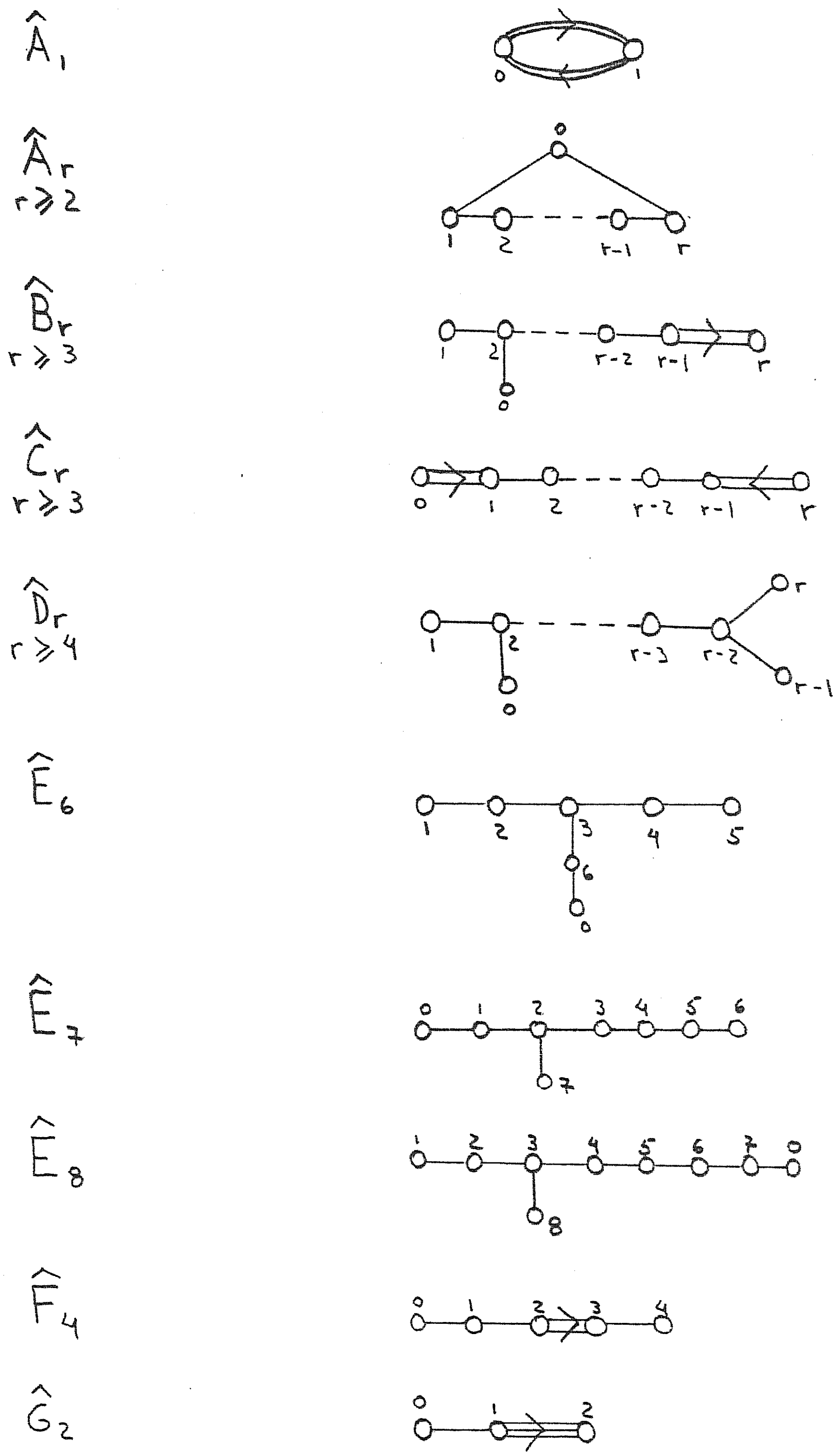


Fig. 2 Dynkin Diagrams for untwisted affine simple Kac-Moody algebras.

themselves.

To any root $a=(\alpha,0,n)$ of $\hat{g}$ we can associate the SU(2) subalgebra generated by

$$E^{\alpha}_{n}\,,\ E^{-\alpha}_{-n}\,,\ \frac{2a\cdot\hat{H}}{a^2} = \frac{2(\alpha\cdot H+nK)}{\alpha^2} \qquad (3.60)$$

where $\hat{H}\equiv(H^i_0,K,d)$, which is isomorphic to (2.14a). Then, the states of any unitary representation of $\hat{g}$ must form SU(2) multiplete of (3.60) and this implies that $\sigma_a(\hat{\mu})$ mapps weights of $\hat{g}$ into themselves.

iv) Highest Weight Representations.

In physical applications of the Kac-Moody ($\hat{g}$) and Virasoro ($\hat{V}$) algebras, the operator $-d=L_0$ is usually associated with a physical quantity such as the energy or the mass, and then its eigenvalues are bounded below for physical reasons. Such a kind of representations are called <u>highest weight</u>. Now we are going to study them in some detail.

Let us consider the Kac-Moody algebra $\hat{g}$. For a unitary highest weight representation, the vacuum states $|\hat{\mu}_0\rangle$ satisfies

$$d\,|\hat{\mu}_0\rangle = \mu_d\,|\hat{\mu}_0\rangle \qquad (3.61a)$$

(see ec.(3.49)) and

$$T^a_n\,|\hat{\mu}_0\rangle = 0 \qquad n>0 \qquad (3.61b)$$

as follws from (3.40a). For an irreducible representation, the states $|\hat{\mu}_0\rangle$ form an irreducible representation of g, so each irreducible highest weight representation of $\hat{g}$ is characterized by μ_0 (the highest weight of g), the value of K and the maximum value of d; that is

$$\hat{\mu}_0 = (\mu_0, K, \mu_{0d}) \tag{3.62}$$

characterizes the representation. Another way of writting the conditions (3.60) is in the Cartan Weyl basis, there:

$$H^i_n\,|\hat{\mu}_0\rangle = 0 \qquad \text{for} \quad n>0 \tag{3.63a}$$

and

$$E^\alpha_n\,|\hat{\mu}_0\rangle = 0 \tag{3.63b}$$

for $n>0$
or $n=0$ and $\alpha>0$

Let us take some weight $\hat{\mu}=(\mu, K, \mu_d)$ belonging to the irreducible unitary representation of $\hat{g}$ characterized by (3.62) then

$$\hat{\mu}_0 - \hat{\mu} = (\mu_0 - \mu, 0, \mu_{0d} - \mu_d)$$

$$= \left(\sum_{i=1}^{r} n_i \alpha_i, 0, \mu_{0d} - \mu_d\right)$$

where $n \geqslant 0$ and $\mu_{0d} - \mu_d \geqslant 0$, and in the last step we have used (2.33). Then we see that $\hat{\mu}_0 - \hat{\mu}$ is a sum of positive roots and as a consequence, also

$$\hat{\mu}_0 - \sigma_\alpha(\hat{\mu}) \qquad \alpha \in \hat{\Phi} \text{ (real)} \tag{3.64}$$

is a sum of positive roots. If we take $\alpha > 0$ and $\hat{\mu} = \mu_0$ in (3.64) it happens that

$$2 \frac{\alpha \cdot \hat{\mu}_0}{\alpha^2} \in \mathbb{Z} \cup \{0\} \tag{3.65}$$

now taking $\alpha = \alpha_0 = (-\psi, 0, 1)$ we obtain that

$$2 \frac{-\psi \cdot \mu_0 + K}{\psi^2} \in \mathbb{Z}^+ \cup \{0\}$$

Then from (2.24), and (2.28) and (2.34)

$$2 \frac{\psi \cdot \mu_0}{\psi^2} \in \mathbb{Z}^+ \cup \{0\}$$

then
$$\frac{2K}{\psi^2} \in \mathbb{Z}^+ \cup \{0\} \tag{3.66a}$$

and
$$K \geqslant \psi \cdot \mu_0 \geqslant 0 \tag{3.66b}$$

This two relations are the conditions that K must satisfy in order to have a unitary highest weight representation of $\hat{g}$. The equations (3.66) show us essentially that the values of K are quantised in the highest weight representations. This two relations are the necessary (they are also sufficient (5,10)) conditions that K must satisfy in order to have a unitary highest weight representation of $\hat{g}$. From (3.66) we also see that the representations (3.9) and (3.10) with $K = 0$ are not highest weight, except for the trivial one with $\mu_0=0$.

Similar things happens for the unitary representations of the Virasoro algebra (3.33b), there the vacuum state $|\ell\rangle$ is unique in any irreducible highest weight representation. $|\ell\rangle$ satisfies the relations

$$L_0 |\ell\rangle = \ell |\ell\rangle \tag{3.67a}$$

and $$L_n |\ell\rangle = 0 \qquad n>0 \tag{3.67b}$$

While the representation is unitary, L_n satisfies $L_n^+ = L_{-n}$, so we have, for $n>0$ that:

$$\begin{aligned}
\| L_{-n} |\ell\rangle \|^2 &= \langle \ell | L_n L_{-n} |\ell\rangle \\
&= \langle \ell | [L_n , L_{-n}] |\ell\rangle \\
&= \langle \ell | 2n L_0 + \tfrac{c}{12} n(n^2-1) |\ell\rangle \\
&= (2n\ell + \tfrac{c}{12} n(n^2-1) \langle \ell|\ell\rangle \\
&= (2n\ell + \tfrac{c}{12} n(n^2-1) \| |\ell\rangle \|^2
\end{aligned}$$

and then

$$2n\ell + \frac{c}{12} n(n^2-1) \geq 0 \qquad \forall n > 0 .$$

if we take first $n=1$ and then n large, we obtain:

$$\ell \geqslant 0 \qquad \text{and} \qquad c \geqslant 0$$

as the necessary condition that ℓ and c must satisfie in order that the representation be highest -- weight. The necessary and sufficient conditions for the unitary representation to be highest weight are:
Either

$$\ell \geqslant 0 \qquad \text{and} \qquad c \geqslant 1 \tag{3.68a}$$

or

$$\left.\begin{aligned} c &= 1 - \frac{5}{(m+2)(m+3)} \\ \text{and} \quad \ell &= \frac{[(m+3)p-(m+2)q]^2-1}{4(m+2)(m+3)} \end{aligned}\right\} \tag{3.68b}$$

where $m=0,1,2,\ldots$; $p=1,2,\ldots,m+1$; $q=1,2,\ldots,p$.
(See ref. (5,16) for the demostration).

4.- CONCLUSION .

In this work we have reviewed the basical mathematical framework of the theory of Kac-Moody and Virasoro algebras, and some aspects of the representations of them. In particular we have seen in detail how the unitary representations can be constructed by means of the theory of weights and roots.

In recent times several applications of the Kac-Moody and Virasoro algebras to different problems have appeared to be usefull. We will limit to show in the following how they appear in two physical problems: The ordinary free relativistic string and the theory of N free massless fermions, in two space time dimensions.

i) The Free Relativistic String. (14,17,18)

It is described by the Lagrangian density

$$\mathcal{L}(\tau,\sigma)=\frac{1}{2}\sqrt{-g}\;g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X_\mu \qquad (4.1)$$

where $g_{\alpha\beta}$ is the metric tensor of the internal τ,σ space, $\partial_0\equiv\frac{\partial}{\partial\tau}$, $\partial_1\equiv\frac{\partial}{\partial\sigma}$ and $g\equiv\det g_{\alpha\beta}$. The variation on $X^\mu(\tau,\sigma)$ gives

$$\partial_\alpha\left(\sqrt{-g}\;g^{\alpha\beta}\,\partial_\beta\,X^\mu\right)=0 \qquad (4.2)$$

while the variation on $g_{\alpha\beta}$ makes the energy-momentum tensor to be zero

$$T_{\alpha\beta}=\frac{1}{2}\left(\partial_\alpha X^\mu\partial_\beta X_\mu-\frac{1}{2}g_{\alpha\beta}\,\partial^\gamma X^\mu\partial_\gamma X_\mu\right)=0 \qquad (4.3)$$

from this it follows that

$$g_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x_\mu \tag{4.4}$$

In the conformal gauge

$$g_{\alpha\beta} = e^\phi \eta_{\alpha\beta} \qquad \eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.5}$$

The equations of motion (4.2) and (4.3) simplify to

$$\ddot{x} - x'' = 0 \tag{4.6a}$$

and

$$T_{\alpha\beta} = \frac{1}{2} \begin{pmatrix} \frac{1}{2}(\dot{x}^2 + x'^2) & \dot{x} \cdot x' \\ \dot{x} \cdot x' & \frac{1}{2}(\dot{x}^2 + x'^2) \end{pmatrix} = 0 \tag{4.6b}$$

where $\dot{x} \equiv \partial_0 x$ and $x' \equiv \partial_1 x$. Introducing light-cone coordinates in the τ, σ space, such that, for $V = (V^0, V^1)$ we have

$$\left.\begin{aligned} V^\pm &\equiv \frac{1}{\sqrt{2}}(V^0 \pm V^1), \\ V_\pm &\equiv \frac{1}{\sqrt{2}}(V_0 \pm V_1), \end{aligned}\right\} \tag{4.7a}$$

$$V^\pm = V_\mp \tag{4.7b}$$

and the scalar product is

$$V^{\mu} W_{\mu} = V^{+} W^{-} + V^{-} W^{+}, \qquad (4.7c)$$

We obtain for (4.6b) that

$$T_{\pm\pm} = \frac{1}{4}\begin{pmatrix} (\dot{x} + x')^2 & 0 \\ 0 & (\dot{x} - x')^2 \end{pmatrix} = 0$$

Let us take $L_{+} \equiv T_{++}$ and $L_{-} \equiv T_{--}$, then

$$L_{\pm}(\tau, \sigma) = \frac{1}{4}(\dot{x} \pm x')^2 = 0 \qquad (4.8)$$

Going to the hamiltonian formulation with $p_{\mu} = \dot{x}_{\mu}$ and the canonical Poisson braket given by

$$\left\{ x^{\mu}(\sigma), P_{\nu}(\sigma') \right\}_{P.B} = \delta^{\mu}_{\nu}\, \delta(\sigma - \sigma')$$

One obtains that (4.8) satisfies the algebra

$$\left\{ L_{\pm}(\sigma), L_{\pm}(\sigma') \right\}_{P.B.} = \pm\left(L_{\pm}(\sigma) + L_{\pm}(\sigma') \right) \delta'(\sigma - \sigma') \qquad (4.9a)$$

and $$\left\{ L_{+}(\sigma), L_{-}(\sigma') \right\}_{P.B} = 0 \qquad (4.9b)$$

By means of the equations of motion and the boundary conditions, one is lead to the Fourier decompositions

$$L_{\pm}(\tau, \sigma) = \frac{1}{2\pi} \sum_{n} L^{\pm}_{n}\, e^{-in\xi^{\pm}} \qquad (4.10)$$

where $\xi^{\pm} \equiv \tau \pm \sigma$

The algebra (4.9) is then turned into

$$\left\{ L^{\pm}_{m} , L^{\pm}_{n} \right\}_{P.B.} = i(n-m) \; L^{\pm}_{m+n} \tag{4.11a}$$

$$\left\{ L^{\pm}_{m} , L^{\mp}_{n} \right\}_{P.B.} = 0 \tag{4.11b}$$

The algebra (4.11a) is the Virasoro algebra (3.33b) with $C=0$. On quantisation $i\{\,\}_{P.B} \to [\;]$ commutator $(\hbar=1)$ and due to the normal ordering, one obtains (14,15)

$$\left[L^{\pm}_{m} , L^{\pm}_{n} \right] = (m-n) L^{\pm}_{m+n} + \frac{D}{12} m(m^2-1) \delta_{m,-n} \tag{4.12}$$

where D is the space-time dimension. This is (3.33b) with $C = D$.

ii) N Free Massless Fermions in two Space-Time Dimensions.

The Lagrangian density of this theory is: (13)

$$\mathcal{L} = \frac{1}{2} \bar{\Psi}_i \, i \, \not{\partial} \, \Psi_i \tag{4.13}$$

(i is sumed from 1 to N)
We take the γ-matrices to be given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.14}$$

and $$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \tag{4.15}$$

The equation of motion following from (4.13) are

$$\not{\partial}\, \psi_i = 0 \qquad i = 1, \ldots, N \tag{4.16}$$

and from (4.14) and (4.15) we have

$$\begin{aligned} (\partial_0 + \partial_1)\, \psi_{i-} &= 0 \\ (\partial - \partial_1)\, \psi_{+i} &= 0 \end{aligned} \tag{4.17}$$

which implies that

$$\psi_+ = \psi_+(t + x) \qquad \psi_- = \psi_-(t - x) \tag{4.18}$$

The canonical anticommutation relation of the quantised theory are

$$\{\psi_{\pm i}(x), \psi_{\pm j}(y)\} = \delta_{ij}\, \delta(x - y) \tag{4.19a}$$

and $$\{\psi_{\pm i}(x), \psi_{\pm j}(y)\} = 0 \tag{4.19b}$$

$(\hbar = 1)$.

Let us consider the group G generated by $T^a = i M^a$ where M^a are real $N \times N$ antisymmetric matrices, which satisfy

$$[M^a, M^b] = f^{abc} M^c \tag{4.20}$$

Under such a representation G is a subgroup of the $O(N)$ symmetry group of the theory. Associated with the symmetry group G there are currents

$$J_\mu^a = \frac{1}{2\sqrt{2}} \Psi M^a \gamma_\mu \Psi \tag{4.21}$$

which satisfy

$$\partial^\mu J_\mu^a = 0 \tag{4.22}$$

Expressing J_μ^a in the hight-cone coordinates (4.7), we have

$$J_\pm^a = \frac{i}{2} \Psi_\pm^t M^a \Psi_\pm$$

and due to (4.18):

$$J_\pm^a = J_\pm^a (x^\pm). \tag{4.23}$$

Because of (4.19) we have

$$[J_+^a(x^+), J_-^b(x^-)] = 0$$

and

$$\left[J_{\pm}^{a}(x^{\pm}), J_{\pm}^{b}(y^{\pm}) \right] = i f^{abc} J_{\pm}^{c}(x^{\pm}) \delta(x^{\pm}-y^{\pm}) + \frac{iK_{\lambda}}{4\pi} \delta^{ab} \delta'(x^{\pm}-y^{\pm}) \qquad (4.24)$$

(See ref. (19)). The second term on the right hand side is called Schwinger Term and the constant K_{λ} is called the Dynkin index (see ref; (13)) and is given by

$$t_r (M^a M^b) = -K_{\lambda} \delta^{ab} . \qquad (4.25)$$

From now on we will denote $J_{\pm}^{a}(x^{\pm})$ simply as $J^{a}(x)$. Let un impose periodic boundary condition with period L such that

$$J^{a}(x+L) = J^{a}(x), \qquad (4.26)$$

then we can express $J^{a}(x)$ by its Fourier expansion

$$\left.\begin{aligned} &J^{a}(x) = \frac{1}{L} \sum_n J_{-n}^{a} \; Z^{n} \\ \text{where} \qquad &Z = e^{2\pi i x/L} \end{aligned}\right\} \qquad (4.27)$$

Sustituting (4.26) on (4.24) we obtain

$$\left[J_{m}^{a}, J_{n}^{b} \right] = i f^{abc} J_{m+n}^{c} + \frac{K_{\lambda}}{2} m \, \delta^{ab} \delta_{m,-n} \qquad (4.28)$$

which is the Kac-Moody algebra (3.33a) with $K = K_{\lambda}/2$.

The symetric energy-momentum tensor of the theory (4.13) is given by

$$T^{\mu\nu} = \frac{1}{4} \bar{\Psi}_i \left(\gamma^\mu \overleftrightarrow{\partial^\nu} + \gamma^\nu \overleftrightarrow{\partial^\mu} \right) \Psi_i \qquad (4.29)$$

and it is traceless due to the equations of motion (4.16). Using light-cone coordinates we obtain

$$T_{\pm\pm} = \frac{1}{\sqrt{2}} \Psi_{i\pm} \overleftrightarrow{\partial_\pm} \Psi_{i\pm} \qquad (4.30)$$

and $\qquad T_{\pm} = 0$

Making a Fourier expansion of $\Psi_{i\pm}(x^\pm)$ and suppressing the $\pm$ subindexes to abbreviate we have

$$\Psi_i(x) = \frac{1}{\sqrt{L}} \sum_r b^i_{-r} \, z^r \qquad (4.31)$$

where $r \in \mathbb{Z}$ for $\Psi(x+L) = \Psi(x)$ and $r \in \mathbb{Z} + \frac{1}{2}$ for $\Psi(x+L) = -\Psi(x)$ which corresponds to have periodic and antiperiodic fields which are consistent with (4.26). Due to (4.19) we have

$$\{ b^i_r , b^j_s \} = \delta^{ij} \delta_{r,-s} \qquad (4.32)$$

Sustituting (4.31) on (4.30) with the normal ordering prescription

$$:b_r b_s: = \begin{cases} b_r b_s & \text{if} \quad r < 0 \\ \frac{1}{2}[b_r, b_s] & \text{if} \quad r = 0 \\ -b_s b_r & \text{if} \quad r > 0 \end{cases} \qquad (4.33)$$

and defining

$$L(z) \equiv T_{\pm\pm} = \sum_n L_{-n} \; z^n \qquad (4.34)$$

one can find that (see for instance ref. (5,13))

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{N}{24}\, m\,(m^2-1)\; \delta_{m,-n} \qquad (4.35)$$

which is (3.33b) with $C = \frac{1}{2} N$. Further, taking J_n^a given by (4.27) one obtains that

$$[L_m, J_n^a] = -n\, J_{m+n}^a \qquad (4.36)$$

which is just (3.33c).

For a more elavorated list of physical applications of the Kac-Moody algebras, the interested reader can see references (5,10,13).

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