SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

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CONSERVATIVE DYNAMICAL SYSTEMS WITH NEWTONIAN-TYPE POTENTIALS

Candidate:
Susanna TERRACINI

Supervisor:
Prof. Sergio SOLIMINI

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TRIESTE

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INTRODUCTION

The main purpose of this work is to present some recent results on the existence of periodic solutions of second order conservative systems in the case of singular potentials.

The classical example of a singular potential is the Keplerian potential $F(x) = \frac{-1}{|x|}$. As is well known, Newton solved the associated equation of motion:

$$(K) \qquad -\ddot{x} = \frac{1}{|x|^2}\frac{x}{|x|}$$

in the early 17th century, and found that, for every period, (K) has a continuum of periodic solutions which do not pass through the origin -in fact every solution of (K) with negative energy is periodic.

From the point of view of modern functional analysis, the main reason of our interest in the Kepler problem lies in its strong degeneracy. As we shall see, the counter part to the existence of many periodic solutions, is their instability under small perturbations of the potential.

Most of this thesis is devoted to the study of dynamical systems with Keplerian type potentials. We shall also discuss the case of forces with other kind of singularity: in particular the repulsive electrostatic force between two charges of the same sign and the case of a dynamical system constrained in a potential well.

Our objective is to discuss the existence of solutions of the fixed period problem:

$$(P) \qquad \begin{cases} -\ddot{x} = \nabla_x F(x,t) & x \in \mathbb{R}^N \\ x(t+T) = x(t) \\ x(t) \in \Omega, \ \forall \ t \in \mathbb{R}, \end{cases}$$

where $F \in C^1(\Omega \times \mathbb{R}; \mathbb{R})$ is T-periodic in t, $\Omega \subset \mathbb{R}^N$ is open and $F(x,t) \to \pm\infty$ as $x \to \partial\Omega$.

The variational approach to this problem consists in looking for critical points of the associated functional

$$I_T(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\,dt - T^2\int_0^1 F(x,tT)\,dt$$

$$x \in \Lambda = \{\ y \in H \,/\, y(t) \in \Omega,\ \forall\ t \in [0,1]\ \},$$

where H is a suitable function space and Λ is open in H. We shall choose H in such a way that $I_T \in C^1(\Lambda;\mathbb{R})$ and that each possible critical point of I_T in Λ is a solution, in the classical sense, of the problem

(P')
$$\begin{cases} -\ddot{x} = T^2\nabla_x F(x,Tt) & x \in \mathbb{R}^N \\ x(t+1) = x(t) \\ x(t) \in \Omega, & \forall\ t \in \mathbb{R}. \end{cases}$$

In this setting, up to the rescaling of the period, finding critical points of I_T is in fact equivalent to solving (P).

Finally we will look for critical points of I_T by means of topological variational methods as the Mountain Pass Lemma, Rabinowitz's Saddle Point Theorem and suitable variants.

Chapter 1, Chapter 2 and Chapter 3 are devoted to the study of problem (P) when $F = F(x)$ behaves like $\frac{-1}{|x|^\alpha}$ near the origin, for some $\alpha \geq 1$.

More precisely, in Chapter 1 we apply the methods of classical mechanics to the discussion of the equation

$$-\ddot{x} = \nabla F(x)$$

when $F(x) = \frac{-1}{|x|^\alpha}$.

First we show an important difference between the case $\alpha \geq 2$ and the case $1 \leq \alpha < 2$. In the first case all periodic solutions are circular with constant angular speed, and every other solution either passes through the origin or it is unbounded. In the case $1 \leq \alpha < 2$ if the system has negative energy then every solution is bounded, hence periodic or quasi-periodic; moreover a solution passes through the origin only if the angular momentum is zero.

Thus we will refer to the first case as the <u>strong force case</u>, while the second will be called the <u>weak force case</u>.

Finally, the case of Keplerian potential ($\alpha = 1$) is shown to have some analogies with the resonance case (i.e. $F(x) = \lambda|x|^2$). In fact they are the only two cases for which every bounded solution is periodic.

In Chapter 2 problem (P) is examined from a variational point of view. There we consider potentials F such that $\lim_{|x|\to\infty} F(x) = 0$ and $\lim_{x\to 0} F(x) = -\infty$.

First a suitable min-max method is introduced in order to overcome a lack of compactness due to the vanishing at infinity of the potential. As a first application we obtain the existence of solutions of (P) when, roughly speaking, F(x) behaves like $\frac{-1}{|x|^\alpha}$ ($\alpha \geq 2$) near the origin (Strong force condition). Hence in the strong force case, the solutions of (P) seem to have a very good stability property under perturbations not involving the behaviour near the origin.

The situation is different for the weak force case: one can still obtain an existence result for potentials F such that $\frac{-a}{|x|^\alpha} \leq F(x) \leq \frac{-b}{|x|^\alpha}$ ($1 \leq \alpha < 2$), under a suitable assumpion on $\frac{a}{b}$, but, when $\alpha = 1$, this assumption leads to $a = b$. So this method does not allow us to treat perturbations of the potential $\frac{-1}{|x|}$ and, as is shown in Chapter 3, the reason for this failure is not a weakness of the method, but an actual instability of the periodic solutions of the Kepler problem under perturbations.

Finally, if the potential F is even (i.e. $F(-x,t) = F(x,t)$, $\forall$ x, $\forall$ t), the existence of solutions of (P) is proved by a minimizing argument. Moreover, since the evennes of the potential allows the introduction of some simmetry constraints on the function space, one can treat in this way even perturbations of the Keplerian potential.

In Chapter 3 we show the existence of a sequence of potentials $(F_n)_n$ such that $\frac{-\varepsilon_n}{|x|^\alpha} \leq F_n(x) \leq \frac{-\varepsilon'_n}{|x|^\alpha}$, with both $\varepsilon_n, \varepsilon'_n \to 1$ as $n \to \infty$, and such that every possible sequence $(x_n)_n$ of solutions of the fixed period problem

$$\begin{cases} -\ddot{x} = \nabla F_n(x) \\ x(t+T) = x(t) \\ x(t) \neq 0, \ \forall\, t \in \mathbb{R}, \end{cases}$$

has $I_T^n(x_n) \to +\infty$. This fact shows a strong limitation of the variational approach to Keplerian-like problems.

Chapter 4 is devoted to the study of problem (P) in the presence of other kinds of singularities.

First we discuss problem (P) when the potential F has a singularity at the origin of repulsive type: i.e. $\lim_{x\to 0} F(x) = +\infty$. We show there that in this case there are no solutions of (P) unless F becomes attractive at infinity.

Finally we treat the problem of a dynamical system constrained in a bounded set with an inward directed force, going to infinity on the boundary. We show that this problem is just a limiting case of the so called superquadratic case and we prove the existence of solutions of (P) by a truncation argument.

Preliminaires.

Throughout this thesis $H^1 = H^1([0,1]/\{0,1\};\mathbb{R}^N)$ denotes the Sobolev space of all L^2 functions which derivatives -in the sense of distributions- are regular and belong to L^2, with periodic boundary constraints.

H^1 provided with the inner product

$$(x,y) = \int_0^1 \dot{x}\cdot\dot{y} + \int_0^1 x\cdot y$$

is an Hilbert space, and we shall denote by H^{-1} its dual. We shall often split H^1 in the orthogonal sum:

$$H^1 = E_0 \oplus E_N,$$

where $E_N \cong \mathbb{R}^N$ is the space of constant functions, and $E_0 = \{\, x\in H^1 \,/\, \int_0^1 x = 0\}$.

We recall that H^1 is compactly embedded in C^0 and in every L^p. Moreover C^0 is embedded in H^{-1}.

We shall denote by $-(\ddot{\ })$: $H^1\to H^{-1}$ the (unique) selfadjoint extention on H^1 of the operator $-\frac{d^2}{dt^2}: C^2\to C^0$; by Wirtinger inequality $-(\ddot{\ })$ is an isomorphism between E_0 and H^{-1}.

Let $I: H^1\to\mathbb{R}$: we shall write $I\in C^1(H^1;\mathbb{R})$ if I is Frechet differentiable in each point of H^1, $I'(x)\in H^{-1}$(for every $x\in H^1$) and its derivative $I': H^1\to H^{-1}$ is continuous.

We remark that if $I: H^1\to\mathbb{R}$ has the form

$$I_T(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\, dt - T^2 \int_0^1 F(x,tT)dt,$$

its Frechet derivative is given by

$$I_T'(x) = -\ddot{x} - T^2\nabla F(x,tT).$$

Finally we recall that, if $F\in C^1$ and x is a critical point of I_T (i.e. $I_T'(x) = 0$), then $x\in C^2$ and it satisfies $-\ddot{x} = T^2\nabla F(x,tT)$ pointwise.

1 - CENTRAL FORCES.

1.1 - POTENTIALS OF FORM $\frac{-1}{|x|^\alpha}$: STRONG FORCES AND WEAK FORCES.

In this section we are going to investigate on the existence of periodic solutions which do not cross the origin of the system

(1.1) $$-\ddot{x} = \nabla F(x), \qquad x \in \mathbb{R}^N$$

where $F(x) = \frac{-1}{|x|^\alpha}$, and then

(1.2) $$\nabla F(x) = \alpha \frac{1}{|x|^{\alpha+2}} x.$$

First, let us remark that, since F has a radial symmetry, each solution of every Cauchy problem

$$\begin{aligned} -\ddot{x} &= \nabla F(x) \\ x(0) &= x_0 \\ \dot{x}(0) &= \dot{x}_0 \end{aligned}$$

lies on the plane spanned by $(x_0, \dot{x}_0)$. Hence we can restrict our discussion to planar systems of (1.1) type. Throughout this chapter we will refer to a system of planar coordinates (ρ,θ) of $\mathbb{R}^2$. In thIse coordinate system (1.1) becomes

(1.1') (1.3) $$\ddot{\rho} = \rho\dot{\theta}^2 - \frac{\alpha}{\rho^{\alpha+1}}$$

(1.4) $$2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = 0$$

By a direct integration of (1.3) and (1.4) we obtain the two first integrals:

(1.5) $$\rho^2\dot{\theta} = B$$

(1.6) $$\frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\frac{B^2}{\rho^2} - \frac{1}{\rho^\alpha} = E$$

We observe that the first order system is not actually equivalent to the second order one (1.1'). However, if $(\rho(t),\theta(t))\in C^2(\mathbb{R}, \mathbb{R}^2)$ solves the first order system and $\dot{\rho}$ does not vanish identically then (ρ,θ) solves (1.1').

From (1.3) and (1.6) we get

$$\ddot{(\rho^2)} = 2\dot{\rho}^2 + 2\rho\ddot{\rho} =$$

$$= 4E - 2\frac{B^2}{\rho^2} + \frac{4}{\rho^\alpha} + 2\frac{B^2}{\rho^2} - \frac{2\alpha}{\rho^\alpha}$$

and then

(1.7) $$\ddot{(\rho^2)} = 2(2E + (2 - \alpha)\frac{1}{\rho^\alpha}).$$

A direct consequence of (7) is the following

Proposition 1.1. *The following conditions are necessary for the boundness of the solutions of (1') which do not cross the origin:*

i) if $\alpha > 2$, $E > 0$

ii) if $\alpha = 2$, $E = 0$

iii) if $\alpha < 2$, $E < 0$.

Proof. Let us prove, for example i). If $E \leq 0$, then $\ddot{(\rho(t)^2)} \leq 0$ for every t; morever, if ρ is bounded ($\rho(t) \leq R$) then (7) implies $\ddot{(\rho^2)} \leq 2(2 - \alpha)\frac{1}{R^\alpha} < 0$. Reversing the time (if it is necessary) we can assume $\dot{\rho}(0) \leq 0$, getting in this way that the solution has to reach the origin in a finite interval of time.∎

Of course (1) always admits some periodic solutions: the circular ones. Indeed the periodic functions ($\rho = R$, $\theta = \frac{2\pi}{T}t$) solve (1) if and only if $R^{\alpha+2} = \alpha\,(\frac{2\pi}{T})^2$.
Next problem will be to investigate on the existence of non circular periodic solutions of (1.1). The following Proposition 1.1.2 and Proposition 1.1.3 show a remarkable difference between the case $\alpha \geq 2$ (which will be called strong force case) and $\alpha < 2$.

Proposition 1.1.2. *If $\alpha \geq 2$ the only periodic solutions of (1) are the circular ones. Every other solution either crosses the origin or is unbounded.*

Proof.
Case $\alpha = 2$). Let (ρ,θ) be a bounded solution: then from Proposition 1 , $E = 0$ and hence, from (1.7), $(\ddot{\rho^2}) = 0$; therefore ρ is constant and, from (1.5) so is $\dot{\theta}$, that is (ρ,θ) is a circular solution.

Case $\alpha > 2$). We can assume $E > 0$ because of Proposition 1.1. Hence, a necessary condition for the boundness of the solution is that

$$\inf_{t\in\mathbb{R}} \{E - \frac{1}{2}\frac{B^2}{\rho^2} + \frac{1}{\rho^\alpha}\} \leq 0$$

and this leads to the compatibility condition

$$2E \leq (\frac{\alpha - 2}{2})\, B^2 (\frac{B^2}{\alpha})^{\frac{2}{\alpha-2}}$$

Let (ρ_-, ρ_+) be the unique pair of solutions of

$$E - \frac{1}{2}\frac{B^2}{\rho^2} + \frac{1}{\rho^\alpha} = 0$$

$\rho_- \leq \rho_+$; then, for every $t \in \mathbb{R}$ we have

$$\rho(t) \in \{\rho / E - \frac{1}{2}\frac{B^2}{\rho^2} + \frac{1}{\rho^\alpha} \geq 0\} = \{\rho/\rho \leq \rho_-\} \cup \{\rho/\rho \geq \rho_+\}$$

a) $\rho_- < \rho_+$; then, since $\rho_-^\alpha < \frac{\alpha - 2}{2E} < \rho_+^\alpha$, from (1.7) we obtain that, for every t,

$$(\ddot{\rho(t)^2}) \leq 2(2E + (2 - \alpha)\frac{1}{\rho_-^\alpha}) < 0$$

if $\rho(t) \in \{\rho / \rho \leq \rho_- \}$

and

$$(\ddot{\rho(t)^2}) \geq 2(2E + (2 - \alpha)\frac{1}{\rho_+^\alpha}) > 0$$

if $\rho(t) \in \{\rho / \rho \geq \rho/s(,+) \}$

so in the first case the solution has to cross the origin, while in the second one the solution is unbounded.

b) $\rho_-^\alpha = \frac{\alpha - 2}{2E} = \rho_+^\alpha$; then either

$$\rho(t)^\alpha = \frac{\alpha - 2}{2E}, \text{ for every } t \in \mathbb{R}$$

or

$$\rho(t)^\alpha \neq \frac{\alpha - 2}{2E}, \text{ for every } t \in \mathbb{R}$$

in fact, the unique solution of the Cauchy problem

$$(\ddot{\rho^2}) = 2(2E + (2 - \alpha)\frac{1}{\rho^\alpha}).$$

$$\rho(0)^\alpha = \frac{\alpha - 2}{2E}$$

$$\dot{\rho}^2(0) = 0$$

is the constant solution $\rho(t)^\alpha = \frac{\alpha - 2}{2E}$. Then if the solution is not circular we can conclude as in the case a).∎

Proposition 1.1.3. *If* $\alpha < 2$ *then* $E < 0$ *is a necessary and sufficient condition for the boundness of the solutions. Moreover, a noncircular solution is periodic if and only if it satisfies*

$$\frac{1}{\pi}\int_{\rho_-}^{\rho_+} \frac{B}{\rho\sqrt{2E\rho^2 + 2\rho^{2-\alpha} - B^2}}\,d\rho \in \mathbb{Q}.$$

where (ρ_-, ρ_+) *is the unique pair of distinct solutions of*

$$E - \frac{1}{2}\frac{B^2}{\rho^2} + \frac{1}{\rho^\alpha} = 0.$$

Proof. If $\alpha < 2$ and $E < 0$, the condition for the solvability of (1.6) is that

$$B^2 < \alpha\left(\frac{2-\alpha}{-2E}\right)^{\frac{2-\alpha}{2}}.$$

Of course, every solution has

$$\rho(t) \in \{\rho / E - \frac{1}{2}\frac{B^2}{\rho^2} + \frac{1}{\rho^\alpha} \geq 0\} = \{\rho / \rho_- \leq \rho \leq \rho_+\}$$

hence every solution is bounded.

Moreover, let us consider an interval $[t_-, t_+]$ such that $\dot{\rho}(t_-) = \dot{\rho}(t_+) = 0$ and $\rho(t_-) = \rho_-$, $\rho(t_+) = \rho_+$: then the angle descibed by the solution in the interval of time $[t_-, t_+]$ is given by

$$\int_{\rho_-}^{\rho_+} \frac{B}{\rho\sqrt{2E\rho^2 + 2\rho^{2-\alpha} - B^2}}\, d\rho$$

∎

2.1. THE KEPLER LAWS

As everyone knows, when Kepler wanted to descibe the motion of the planets around the Sun, he formulated his famous three laws:

1) *The planets revolve in elliptic orbits, the Sun occupying one of the two foci,*

2) *The radius joining the Sun to the planet sweeps out equal areas in equal interval of time.*

3) *The square of the periods of revolution are as the cubes of the mean distances of the planets from the Sun.*

1) and 2) were formulated for the first time in 1609 (Astronomia Nova), and 3) in 1619 (Harmonica Mundi).

We assume that the Sun lies at the origin of an euclidean system of coordinates in $\mathbb{R}^3$, and that the position x of the planet satisfies the differential equation

(2.1) $$-\ddot{x} = a\frac{1}{|x|^3}x$$

where a does not depend on the planet. Denoting by $\times$ the usual exterior product in $\mathbb{R}^3$, we have from (2.1) that

$$x\times\ddot{x} = 0$$

and that

$$\frac{d}{dt}[(x\times\dot{x})\times\dot{x} + \frac{a}{|x|}] = (x\times\dot{x})\times\ddot{x} + \frac{a}{|x|}\dot{x} - \frac{a}{|x|^2}(x\cdot\dot{x})x = 0$$

Therefore there are two constant vectors B, P $\in \mathbb{R}^3$ such that

(2.2) $$B = x\times\dot{x}$$

(2.3) $$P = B\times\dot{x} + a\frac{x}{|x|}$$

morever,since the system (2.1) is conservative, we have the energy integral

$$\frac{1}{2}|\dot{x}|^2 - \frac{a}{|x|} = E.$$

We remark that if $B = 0$, then $P = a\frac{x}{|x|}$, so the trajectory lies on a straight line, while, if $P = 0$,

(2.3) gives

$$B\times\dot{x} + a\frac{x}{|x|} = 0$$

and then, by multiplying both sides by x, and setting $B^2 = |B|^2$,we get

$$-B^2 = -a\ |x|$$

therefore the motion is circular and periodic, since x solves the equation

$$-\ddot{x} = \frac{a^3}{B^6}x.$$

Hence let us assume that $P \neq 0$ and $B \neq 0$; if θ is the angle between x and P we get from (2.3), by multiplying both sides by x,

$$x\cdot P = -B^2 + a\ |x|$$

and therefore

(2.4) $$(a - |P|\cos\theta)\, |x| = B^2.$$

We remark that (2.4) is the equation of a conic section in polar coordinates; by an easy computation we find that

$$|P|^2 = a^2 + 2EB^2;$$

hence three cases are possible:

i) $E > 0$: therefore $|P| > a$ and the trajectory lies on an hyperbola.

ii) $E = 0$: therefore $|P| = a$ and the trajectory lies on a parabola.

iii) $E < 0$: then $|P| < a$ and the trajectory lies on an ellipse with semiaxsis $(\frac{a}{-2E}, \frac{B}{\sqrt{-2E}})$.

Therefore, since from Proposition 1.1.2 x is bounded iff $E < 0$, 1) is proved. Moreover,by definition, $\frac{1}{2}B$ is the areal speed,and it is constant. Hence, if T is the period and A the area of the ellipse we have

$$A = \frac{1}{2}BT,$$

and, since

$$A = \pi\frac{aB}{(-2E)\sqrt{-2E}}$$

we obtain

$$(\frac{T}{2\pi})^2 = \frac{a}{(-2E)^3}.$$

1.3 - THE CASE $\alpha = 1$ AS A RESONANCE CASE.

The results contained in Section 2 show that the case $\alpha = 1$ possesses some interesting properties: first of all every periodic solution, as function of the angle θ, has period exactly 2π; secondwise the period of a solution as function of the time, depends only on the energy.

Indeed, from (1.3) and (1.5) we obtain

$$\frac{d^2}{d\theta^2}\left(\frac{1}{\rho}\right) = \frac{1}{\rho} - \frac{\alpha}{B^2}\frac{1}{\rho^{\alpha-1}}$$

which, for $\alpha = 1$ is the linear equation

$$\frac{d^2}{d\theta^2}\left(\frac{1}{\rho}\right) = \frac{1}{\rho} - \frac{1}{B^2}. \tag{3.1}$$

It is well known that the general solution of (3.1) is

$$\frac{1}{\rho} - \frac{1}{B^2} = R\cos(\theta - \theta_0)$$

and it has period exactly 2π.

The reason of the correspondance between the period and the energy is a bit more complicated: in fact a change of variables in the phase space $(\rho,\dot{\rho})$ is needed in order to obtain some linear equation.
We set

$$\phi(\rho,\dot{\rho}) = \cos(\sqrt{-2E}\rho\dot{\rho})(\sqrt{-2E}\rho - \frac{1}{\sqrt{-2E}}) - \sin(\sqrt{-2E}\rho\dot{\rho})\rho\dot{\rho} \tag{3.2}$$

$$\psi(\rho,\dot{\rho}) = \sin(\sqrt{-2E}\rho\dot{\rho})(\sqrt{-2E}\rho - \frac{1}{\sqrt{-2E}}) + \cos(\sqrt{-2E}\rho\dot{\rho})\rho\dot{\rho}; \tag{3.3}$$

an easy computation shows that

$$|\mathrm{Jac}(\phi,\psi)| = \frac{\sqrt{(-2E)^3}}{2}\rho$$

that is, the change of variables is admissible whenever $\rho \neq 0$.
Let (ρ,θ) be a solution of (1') which does not cross the origin: by derivating (3.2) and (3.3), since ρ satisfies (1.7), we obtain that (ϕ,ψ) satisfies the linear system

$$\dot{\phi} = \sqrt{(-2E)^3}\,\psi$$

$$\dot{\psi} = -\sqrt{(-2E)^3}\ \phi.$$

Therefore (ϕ,ψ) has period exactly

$$T = \frac{2\pi}{\sqrt{(-2E)^3}}$$

and so is the period of ρ, because of the injectivity of the change of variables (3.2),(3.3).∎

1.4 - A MINIMIZING PROPERTY OF KEPLERIAN ORBITS.

Assume that x solves

(4.1)
$$-\ddot{x} = \frac{1}{|x|^3}x$$

$$x(t) \neq 0\,, \quad \forall\ t \in [t_0, t_1]$$

As we have seen in Section 1.1 and in Section 1.2, if $E < 0$ and $B \neq 0$, then x is the restriction to $[t_0, t_1]$ of a T-periodic function which does not cross the origin and such that $T = 2\pi\,(-2E)^{\frac{-3}{2}}$. Moreover, from (1.7), since $x \in C^2$,

$$0 = \int_0^T \frac{d^2}{dt^2}|x|^2 = 2\Big(2ET + \int_0^T \frac{1}{|x|}\Big),$$

and therefore

(4.2) $$\int_0^T \frac{1}{|x|} = -2ET = (4\pi^2 T)^{\frac{1}{3}},$$

(4.3) $$\frac{1}{2}\int_0^T |\dot{x}|^2 = -ET = \frac{1}{2}\,(4\pi^2 T)^{\frac{1}{3}}.$$

On the other side, if B = 0, (1.6) and (1.7) show that every extension of x satisfying (1.4) whenever $x(t) \neq 0$, has to cross the origin and it is unbounded if $E \geq 0$. Therefore we will assume that $E < 0$ and that $x(t_0) = x(t_1) = 0$, $x(t) \neq 0$, $\forall\ t \in (t_0, t_1)$.

Taking in to account that x satisfies (1.6) and (1.7) in (t_0, t_1), we have

$$t_0 - t_1 = \int_{t_0}^{t_1} \frac{1}{\dot{\rho}}\, d\rho = 2\pi(-2E)^{\frac{-3}{2}}.$$

Therefore x is periodic with period $T = 2\pi(-2E)^{\frac{-3}{2}}$. We remark that x is continuous, but it is not differentiable in zero; nevertheless, since from (1.6)

$$\left(\frac{1}{2}\frac{d}{dt}|x|^2\right)^2 = |x|^2|\dot{x}|^2 = |x|^2\left(2E + \frac{1}{|x|}\right) =$$

$$= 2E|x|^2 + 2|x|,$$

that is $|x|^2$ is differentiable in zero, by integrating (1.7) we get that (4.2) and (4.3) still hold even if x crosses the origin.

Now let us consider the action integral

$$I_T(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\, dt + T^2 \int_0^1 \frac{1}{|x|}\, dt$$

defined on the set

$$x \in \Sigma = \{\ y \in H^1([0,T]/\{0,T\};\mathbb{R}^2) \ /\ \int_0^T \frac{1}{|y|} < +\infty\ \},$$

and let us consider the two sets

$$\Gamma^* = \{\ x \in \Sigma \ /\ x(t) \neq 0,\ \forall t \in [0,T] \text{ and } x \text{ is not homotopic to a constant map in } \mathbb{R}^2\backslash\{0\}\ \}$$

$$X_0 = \{\ x \in \Sigma \ /\ x(0) = x(T) = 0\ \}.$$

As a consequence of the above discussion we have the following

THEOREM 1.4.1.

$$\inf_{x \in \Gamma^*} I_T(x) = \inf_{x \in X_0} I_T(x) = \frac{3}{2}(4\pi^2 T)^{\frac{1}{3}}$$

Moreover, if $x \in \Gamma^* \cup X_0$ *satisfies* $I_T(x) = \frac{3}{2}(4\pi^2 T)^{\frac{1}{3}}$, *then (4.2) and (4.3) hold.*

Proof. Indeed if $x \in \Gamma^*$ minimizes I_T, since Γ^* is open in $H^1([0,T]/\{0,T\};\mathbb{R}^2)$, and $I_T \in C^1(\Gamma^*,\mathbb{R})$, then by standard arguments one prove that x satisfies (4.1) and therefore, from (4.2) and (4.3) we obtain the thesis.

On the other hand, if $x \in X_0$, for every interval $[t_0, t_1]$ such that $x(t) \neq 0$, $\forall t \in [t_0, t_1]$, since x minimizes the problem

$$\inf \left\{ \frac{1}{2} \int_{t_0}^{t_1} |\dot{y}|^2 + \int_{t_0}^{t_1} \frac{1}{|y|},\ y(t_0) = x(t_0),\ \ y(t_1) = y(t_1) \right\},$$

then x solves (4.1), therefore x satisfies (4.2) and (4.3).∎

2 - EXISTENCE THEOREMS

The contents of this chapter have been taken from some recent papers: in particular from [8] and [33] for Section 2.1 (see also [2] for a different approach). For Section 2.2 we refer to [2] and [32] for the strong force case, and to [20], [25], [26] and [27] for the weak force case (see [13] for the fixed energy problem). Finally, for Section 2.3 we refer to [3] and [5].

2.1 - A MINIMAX METHOD FOR SINGULAR POTENTIALS OF ATTRACTIVE TYPE.

The purpose of this section is to find a minimax argument which allows us to treat in a variational way conservative problems of type

$$\begin{cases} -\ddot{x} = \nabla F(x) \\ x(t+T) = x(t) \end{cases}$$

when the potential F behaves in some sense like $\frac{-1}{|x|^\alpha}$. We are interested on solutions which do not cross the set where F is singular: namely the noncollision solutions.

Assume that F satisfies:

F1) $F \in C^2_1(\mathbb{R}^N \setminus \{0\} \times \mathbb{R}; \mathbb{R}],\ N \geq 2$

$F(x,t+T) = F(x,t) \ , \ \forall x \in \mathbb{R}^N \setminus \{0\}, \forall t \in \mathbb{R}$

F2) $F(x,t) < 0 \ , \ \forall x \in \mathbb{R}^N \setminus \{0\}, \forall t \in \mathbb{R}$

$\lim_{|x| \to \infty} F(x,t) = 0$ (uniformly)

$\lim_{|x| \to \infty} \nabla F(x) = 0$ (uniformly)

F3) $\lim_{x\to 0} F(x,t) = -\infty$

Our goal is to find solutions of the problem

(P) $$\begin{cases} -\ddot{x} = \nabla F(x) \\ x(t+T) = x(t) \\ x(t) \neq 0. \end{cases}$$

Consider the open set

$$\Lambda = \{ x \in H^1([0,1]/\{0,1\}) / \ x(t) \neq 0 \ , \forall t \in [0,1]\}:$$

every solution of (P) correspond, after the rescaling of the period, to a critical point of the functional

$$I_T(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\, dt \ - \ T^2 \int_0^1 F(x,tT)dt$$

$x \in \Lambda$.

Because of the regularity assumption F1) on F, $I_T \in C^2(\Lambda;\mathbb{R})$.

Anyway we remark that, if we consider $I_T : H^1([0,1]/\{0,1\}) \to \mathbb{R} \cup \{+\infty\}$, I_T is sequentially weakly lower semicontinuous.

By F2), I_T has a lower bound in Λ: indeed

$$\inf_{x\in\Lambda} I_T = 0,$$

but of course it does not attain its infimum since F is always strictly less than zero.

However, if one consider

$$X_0 = \{\ x \in H^1([0,1]/\{0,1\}) \setminus \Lambda \text{ such that } I_T(x) < +\infty\ \},$$

(possibly X_0 empty), and

$$c_0 = \inf_{x \in X_0} I_T(x),$$

I_T has some compactness property at any level c, $0 < c < c_0$. More precisely we have

Proposition 2.1.1. *Assume F1), F2) and F3) hold. Then* $c_0 > 0$ *and, for every* c, $0 < c < c_0$, *every Palais-Smale sequence at level* c *in* Λ *possesses a subsequence converging to some limit in* Λ.

Proof. It follows straightforward from F3) that $c_0 > 0$. Let $0 < c < c_0$ and let $(x_n)_n$ be a Palais-Smale sequence at level c in Λ, that is

(1.1) $$x_n \in \Lambda$$

(1.2) $$I_T(x_n) = c_n \to c$$

(1.3) $$-\ddot{x}_n - T^2\nabla F(x_n, tT) = h_n \to 0 \quad \text{in } H^{-1}.$$

We write

$$x_n = w_n + \xi_n\ ,\ \xi_n = \int_0^1 x_n \text{ and } w_n = x_n + \xi_n;$$

since (1.2) together with F2) implies that $(\dot{x}_n)_n$ is bounded in L^2, up to a subsequence $(w_n)_n$ converges weakly in $H^1([0,1]/\{0,1\})$ and strongly in C^0 to some w. Therefore $(-\ddot{x}_n)_n$ converges weakly in H^{-1} to $-\ddot{w}$.

We claim that $(x_n)_n$ is bounded in the C^0 norm. Indeed assume the contrary: then denoting by

$$(M_n)_n = \max_{t \in \mathbb{R}} |x_n(t)|$$

$$(m_n)_n = \min_{t \in \mathbb{R}} |x_n(t)|,$$

the unboundness of $(x_n)_n$ implies

(1.4) $$\lim_{|x| \to \infty} m_n = +\infty;$$

in fact from (1.2)

$$c_n \geq \frac{1}{2} \int_0^1 |\dot{x}_n|^2 \, dt \geq \frac{1}{2} (M_n - m_n)^2.$$

Therefore

(1.5) $$\lim_{n \to \infty} \int_0^1 F(x_n, tT) dt = 0$$

and

(1.6) $$\lim_{n \to \infty} \nabla F(x_n, t) = 0 \quad \text{(uniformly)}.$$

Since C^0 is embedded in H^{-1}, and since (1.6) holds, (1.3) becomes

(1.7) $$\lim_{n \to \infty} (-\ddot{x}_n) = 0 \quad \text{strongly in } H^{-1},$$

that is $-\ddot{w} = 0$ and $w_n \to 0$ strongly in $H^1([0,1]/\{0,1\})$.
Therefore

(1.8) $$\lim_{n \to \infty} \frac{1}{2} \int_0^1 |\dot{x}_n|^2 \, dt = 0$$

which, together with (1.5) leads to $c = 0$.

Hence $(x_n)_n$ is bounded in C^0. Therefore, up to a subsequence it converges weakly in H\o(,\s\up6(1))([0,1]/{0,1}) and strongly in H^{-1} to $v \in \Lambda$. Since C^0 is embedded in H^{-1}, (1.3) says that $(-\ddot{x}_n)_n$ converges strongly in H^{-1} and then $(x_n)_n$ converges strongly in $H^1([0,1]/\{0,1\})$.∎

In order to obtain some strictly positive critical levels we introduce the following classes. Consider the (N - 2)-dimensional sphere

$$S^{N-2} = \{\ x \in \mathbb{R}^{N-1} / |x| = 1\ \}$$

$(S^0 = \{0\})$: we denote by Γ_N the set of all the continuous functions from S^{N-2} to Λ, that is

(1.9) $$\Gamma_N = \{\ \gamma: S^{N-2} \to \Lambda,\ \gamma \text{ continuous }\}.$$

Identifying 0 and 1 and the interval [0,1] with S^1, we can associate with each $\gamma \in \Gamma_N$ a function

(1.10) $$\tilde{\gamma}: S^{N-2} \times S^1 \to \mathbb{R}^N \backslash \{0\}: (x,t) \to \gamma(x)(t).$$

Up to this correspondance we can define

(1.11) $$\Gamma_N^* = \{\ \gamma \in \Gamma_N \text{ such that } \tilde{\gamma} \text{ is not homotopic to a constant map in } \mathbb{R}^N \backslash \{0\}\}$$

Proposition 2.1.2.

i) $\Gamma_N^* \neq \emptyset$ *for every* $N \geq 2$.

ii) Γ_N^* *is invariant under homotopies in* Λ.

Proof.

ii) is obvious. Let un prove i). Indeed $S^{N-2}\times S^1$ can be identified with the boundary of an open set $\Omega\subset\mathbb{R}^N$, and we can assume $0\in\Omega$. Then $\deg(\Omega,id|\Omega,0) = 1$, that is $id:S^{N-2}\times S^1\to S^{N-2}\times S^1$ is not homotopic to a constant map in $\mathbb{R}^N\setminus\{0\}$.∎

Now a minimax level of I_T can be defined as

(1.12) $$c = \inf_{\gamma\in\Gamma_N^*} \sup_{\gamma(S^{N-2})} I_T.$$

Proposition 2.1.3. *Assume F1), F2), F3) hold and that* $c < c_0$. *Then* $c > 0$.

Proof. If not there exists a sequence $(\gamma_n)_n$ in Γ_N^* such that

(1.13) $$\lim_{n\to\infty} \sup_{\gamma_n(S^{N-2})} I_T = 0.$$

Let

$$\psi_n(x) = \int_0^1 \gamma_n(x)dt,$$

since (1.13) together with F2) implies

$$\lim_{n\to\infty}\int_0^1 |\dot{\gamma}_n(x)|^2 dt = 0 \text{ (uniformly for } x\in S^{N-2}),$$

we have

(1.14) $$\lim_{n\to\infty} \|\gamma_n(x) - \psi_n(x)\|_\infty = 0$$

and therefore, for n large γ_n is homotopic to ψ_n as functions from S^{N-2} to $H^1([0,1]/\{0,1\})$. Moreover (1.13) implies that

$$\lim_{n\to\infty}\int_0^1 F(\gamma_n(x),tT)dt = 0 \text{ uniformly in } x$$

and, from (1.14) it follows that

$$\lim_{n\to\infty}|\psi_n(x)| = \infty.$$

Hence, for n large,

$$\psi_n : S^{N-2} \to \mathbb{R}^N\backslash\{0\}.$$

and so, ψ_n is homotopic to a constant map in $\mathbb{R}^N\backslash\{0\}$, that is, as function from S^{N-2} into Λ, $\psi_n \notin \Gamma_N^*$.
Therefore, from (1.11),

$$\lim_{n\to\infty}\sup_{x\in S^{N-2}} d(\gamma_n, \partial\Lambda) = 0$$

and then, from the lower semicontinuity of I_T

$$0 = \lim_{n\to\infty}\sup_{\gamma_n(S^{N-2})} I_T \geq c_0,$$

that is $c_0 = 0$.∎

Before proving the main theorem of this section, let us recall the general minimax principle.

MINIMAX PRINCIPLE. *Consider a C^2 functional defined on an Hilbert space and let η be the flow defined by the evolution problem*

$$\begin{cases} \dfrac{d\eta}{d\sigma} = -\psi(I(\eta)) \dfrac{\nabla I(\eta)}{\|\nabla I(\eta)\|} \min(1, \|\nabla I(\eta)\|) \\ \eta(x,0) = x \end{cases}$$

where ψ *is a cut-off function* $\psi(t) \geq 0$, $\psi(t) > 0$ if $t > a$. *Let* $\mathcal{A}$ *be a class of compact subsets of* H, *invariant under the flow* η, *that is* $\eta(A,\sigma) \in \mathcal{A}$, $\forall\ A \in \mathcal{A}$, $\forall\ \sigma \geq 0$ *.If*

$$c = \inf_{A \in \mathcal{A}} \sup_{x \in A} I_T(x) < a$$

then, for every sequence $(A_n)_n$ *in* $\mathcal{A}$ *such that*

$$\lim_{n\to\infty} \sup_{x \in A_n} I_T(x) = c$$

there is a sequence of points $(x_n)_n$ *such that*

$$d(x_n, A_n) \to 0$$

$$I(x_n) \to c$$

$$\nabla I(x_n) \to 0.$$

THEOREM 2.1.1. *Assume F1), F2), F3) and* $c < c_0$. *Then* (P) *has at least one solution.*

Proof. We apply the minimax principle with $H = H^1([0,1]/\{0,1\})$, $\psi(t) = 1\ \forall t \in \mathbb{R}$, I_T. The main problem is that I_T is defined only on an open set of the space $H^1([0,1]/\{0,1\})$. Anyway, let us take $\varepsilon > 0$ such that $c + \varepsilon < c_0$, and consider

$$\overline{\Gamma}_N^* = \{ \gamma \in \Gamma_N^* \ / \ \sup_{\gamma(S^{N-2})} I_T \leq c + \varepsilon \}.$$

From the lower semicontinuity of I_T we have that

$$\inf_{\gamma\in\overline{\Gamma}_N^*} \min_{x\in S^{N-2}} d(\gamma(x),\partial\Lambda) > 0.,$$

hence $\overline{\Gamma}_N^*$ is invariant under the flow η ,since $I_T \circ \eta$ is decreasing. So we can apply the Minimax Principle obtaining a Palais-Smale sequence in Λ; finally, from Proposition 2.1.3., since $c < c_0$, this sequence admits a subsequence converging to some limit in Λ, and this limit is a critical point of I_T at level c.∎

2.2. SOME APPLICATIONS: STRONG FORCES AND WEAK FORCES

Throughout this section we will assume the hypotesis F1),F2), and F3) of Section 2.1.

We remark that, in order to apply Theorem 2.1.1, some assumptions on the behaviour of the potential F near 0 are required.

For example the Keplerian potential $-1/|x|$ does not satisfy the hypotesis $c < c_0$ (actually $c = c_0$) .As we will see in the next chapter this fact implies the instability of the solutions under some kinds of perturbations of the potential.

The first assumption in order to get $c < c_0$ was introduced by Gordon, who called it strong force condition and implies $c_0 = +\infty$.
If $F(x) = -1/|x|^{\alpha}$ the strong force condition is satisfied if and only if $\alpha \geq 2$.

Some weaker assumptions, introduced latter by Degiovanni, Giannoni and Marino, allow to apply Theorem 2.1.1. to potentials which behave like $-1/|x|^{\alpha}$ with $1 \leq \alpha < 2$, but of course, in the case $\alpha = 1$ this method apply only if the potential is exactly the gravitational one.

On the other hand, in the case of even potentials also the case $\alpha = 1$ presents some stability properties, because of the symmetry constraints that one can introduce.

THEOREM 2.2.1. (Strong force case). *Assume that F satisfies F1), F2), F3) of Theorem 2.1.1. and moreover that there exists a neighborhood* W *of* 0 *in* $\mathbb{R}^N$ *and a function* $U \in C^0(W\setminus\{0\};\mathbb{R})$ *such that*

$$(SF)\quad \begin{cases} (2.1) & \lim_{x\to 0} U(x) = +\infty \\ (2.2) & -F(x,t) \geq |\nabla U(x)|^2 \quad \forall\, x\in W,\ \forall t\in \mathbb{R}\ . \end{cases}$$

Then one has $c_0 = +\infty$, *and* (P) *has at least one noncollision solution.*

Proof. Let $x\in \partial\Lambda$; we claim that

$$-\int_0^1 F(x,tT)dt = +\infty.$$

Since x cannot vanish identically, there exists t_0 such that $x(t_0) \neq 0$, $x(t_0)\in W$. Let

$$t_1 = \sup \{\ t > t_0 \ /\ x(s) \neq 0\ \forall\ s\in [t_0,t]\ \};$$

(we can assume $x([t_0,t_1]) \subset W$), and let

$$\mu^2 = \int_0^1 |\dot{x}|^2\, dt > 0.$$

Then (2.2) implies:

$$U(x(t_1 - \delta)) - U(x(t_0)) = \int_{t_0}^{t_1-\delta} \nabla U(x(s))\cdot \dot{x}(s)\, ds \leq$$

$$\leq \{ \int_{t_0}^{t_1-\delta} |\nabla U(x(s))|^2\, ds\}^{\frac{1}{2}} \{ \int_{t_0}^{t_1-\delta} |\dot{x}|^2(s)\, ds\}^{\frac{1}{2}} \leq$$

$$\leq \mu \left\{ - \int_{t_0}^{t_1-\delta} F(x(s),s)\, ds \right\}^{\frac{1}{2}} .$$

Therefore from (2.2) $I_T(x) = +\infty$ and hence $c_0 = +\infty$. Finally we get the existence of a solution applying Theorem 2.1.1.∎

As we have point out before, in the case of potentials $F(x) = \frac{-1}{|x|^{\alpha}}$, the strong force condition implies $\alpha \geq 2$. Nevertheless we will show that, when $\alpha > 1$, $c_0 < +\infty$, but $c < c_0$ still holds.

Let us introduce the following notations: for $1 \leq \alpha < 2$ we set

(2.3) $$\theta_0(\alpha,T^2) = \inf \left\{ \int_0^1 \left[\frac{1}{2}(\dot{\gamma})^2 - T^2 \frac{1}{\gamma^{\alpha}}\right]dt \, , \gamma \in H_0^1([0,1]/\{0,1\};\mathbb{R}), \gamma \geq 0 \right\}$$

(2.4) $$\theta_1(\alpha,T^2) = \min_{R>0}\left\{2\pi^2 R^2 + T^2 \frac{1}{R^{\alpha}}\right\}.$$

(2.5) $$\theta_0(\alpha) = \theta_0(\alpha,1)$$

(2.6) $$\theta_1(\alpha) = \theta_1(\alpha,1)$$

(2.7) $$\Phi(\alpha) = \left\{\frac{\theta_0(\alpha)}{\theta_1(\alpha)}\right\}^{\frac{\alpha+2}{2}}.$$

Proposition 2.2.1.

i) $\theta_0(\alpha,aT^2) = a^{\frac{\alpha+2}{2}}\,\theta_0(\alpha,T^2)$

ii) $\theta_1(\alpha, aT^2) = a^{\frac{\alpha+2}{2}}\,\theta_1(\alpha, T^2)$

iii) $\Phi(1) = 1$.

Proof.

i) By the change of variables $\gamma = a^{\frac{1}{\alpha+2}}\eta$, we have that

$$\inf\Big\{ \int_0^1 [\tfrac{1}{2}(\dot{\gamma})^2 - aT^2\frac{1}{\gamma^\alpha}]dt\ ,\ \gamma\in H_0^1([0,1]/\{0,1\};\mathbb{R}),\ \gamma \geq 0 \Big\} =$$

$$= a^{\frac{\alpha+2}{2}} \inf\Big\{ \int_0^1 [\tfrac{1}{2}(\dot{\eta})^2 - T^2\frac{1}{\eta^\alpha}]dt\ ,\ \gamma\in H_0^1([0,1]/\{0,1\};\mathbb{R}),\ \gamma \geq 0 \Big\}.$$

ii) By the direct computation of $\theta_1(\alpha, aT^2)$ we find that

$$\theta_1(\alpha, aT^2) = \left(\frac{\alpha+2}{2}\right)(aT^2)^{\frac{\alpha+2}{2}}(4\pi^2)^{\frac{\alpha}{\alpha+2}}.$$

iii) As we have seen in Section 1.4, $\theta_0(1) = \frac{3}{2}(2\pi)^{\frac{2}{3}}$; the direct computation of $\theta_1(1)$ shows that $\theta_1(1) = \theta_0(1)$.∎

Proposition 2.2.2

i) $\theta_0(\alpha, T^2) = c_0$

ii) $\theta_1(\alpha,T^2) = \inf\{ \frac{1}{2}\int_0^1 (\dot{\gamma})^2 - T^2\{ \int_0^1 \frac{1}{\gamma}dt\}^{\alpha} , \gamma\in H_0^1([0,1]/\{0,1\};\mathbb{R}), \gamma \geq 0 \}$

iii) $\Phi(\alpha) > 1 \ \forall \ \alpha > 1.$

Proof.

i) The inequality $\leq$ is obvious. On the other hand, for every $x\in X_0$, we have

$$\int_0^1 (\frac{d}{dt}|x|)^2 \leq \int_0^1 |\dot{x}|^2,$$

proving the other inequality.

ii) Let $\rho\in H_0^1([0,1]/\{0,1\};\mathbb{R})$, $\rho \geq 0$ such that

$$\frac{1}{2}\int_0^1 (\dot{\rho})^2 - T^2\{ \int_0^1 \frac{1}{\rho}dt\}^{\alpha} =$$

$$= \inf\{ \frac{1}{2}\int_0^1 (\dot{\gamma})^2 - T^2\{ \int_0^1 \frac{1}{\gamma}dt\}^{\alpha} , \gamma\in H_0^1([0,1]/\{0,1\};\mathbb{R}), \gamma \geq 0 \}$$

then $\rho(t) > 0$, $\forall \ t\in (0,1)$, $\rho\in C^1((0,1);\mathbb{R})$ and satisfies

$$-\ddot{\rho}(t) = \alpha T^2 (\int_0^1 \frac{1}{\rho})^{\alpha-1} \frac{1}{\rho^2}.$$

Hence ρ is the unique solution of the problem

$$-\ddot{\gamma} = \tau^2 \frac{1}{\gamma^2}$$

$$\dot{\gamma}(\tfrac{1}{2}) = 0$$

$$\gamma(0) = \gamma(1) = 0$$

for $\tau^2 = \alpha T^2 \left(\int_0^1 \frac{1}{\rho} \right)^{\alpha-1}$. Therefore, from Theorem 1.4.1, we have

$$\frac{1}{2} \int_0^1 (\dot{\rho})^2 = \frac{1}{2} (2\pi\tau^2)^{\frac{2}{3}}$$

$$\tau^2 \int_0^1 \frac{1}{\rho} = (2\pi\tau^2)^{\frac{2}{3}},$$

which give

$$\frac{1}{2} \int_0^1 (\dot{\rho})^2 - T^2 \left\{ \int_0^1 \frac{1}{\rho} dt \right\}^{\alpha} = \theta_1 (\alpha, T^2) .$$

iii) It is a straightforward consequence of the strict convexity of the function $f(s) = s^{\alpha}$ when $\alpha > 1$.∎

THEOREM 2.2.2. *Assume F1), F2), F3) hold and moreover that, for some* $1 \le \alpha < 2$, (2.8) $$\frac{a}{|x|^\alpha} \le -F(t,x) \le \frac{b}{|x|^\alpha}$$

(2.9) $$a \le b \le a\Phi(\alpha).$$

Then $c \le c_0$ *and there exists at least one solution of* (P).

Proof. Setting

$$I_T^a(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\,dt - T^2\int_0^1 \frac{a}{|x|^\alpha}dt$$

$$I_T^b(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\,dt - T^2\int_0^1 \frac{b}{|x|^\alpha}dt,$$

we have $I_T^a(x) \le I_T(x) \le I_T^b(x)$ for every x such that $\int_0^1 \frac{1}{|x|^\alpha}dt < +\infty$. Therefore, from Proposition 2.2.2. i), we have

(2.10) $$c_0 \ge \theta_0(\alpha, aT^2) = a^{\frac{2}{\alpha+2}}\,\theta_0(\alpha, T^2).$$

Now, let R_b such that

$$\theta_1(\alpha, bT^2) = 2\pi^2 R_b^2 + bT^2\frac{1}{R_b^\alpha}$$

and let $x_b = R_b\, e^{2\pi i t}$ in some 2-dimensional space of $\mathbb{R}^N$. We obtain from (2.9) and (2.10) that

(2.11) $$I_T(x_b) \le I_T^b(x_b) = \theta_1(\alpha, bT^2) = b^{\frac{2}{\alpha+2}}\,\theta_1(\alpha, T^2) \le$$

$$\leq a^{\frac{2}{\alpha+2}} \frac{\theta_0(\alpha)}{\theta_1(\alpha)} \theta_1(\alpha,T^2) = a^{\frac{2}{\alpha+2}} \theta_0(\alpha,T^2) = c_0 .$$

Let us consider the sphere S^{N-2} in $\mathbb{R}^{N-1}$ as the union of the two semispheres

$$S_+^{N-2} = \{ x \in S^{N-2} \ / \ x_{N-1} \geq 0 \}$$

$$S_-^{N-2} = \{ x \in S^{N-2} \ / \ x_{N-1} \leq 0 \};$$

one has $S_+^{N-2} \cap S_-^{N-2} = S^{N-3}$. We call P the projection (it is actually a homeomorphism) from S_+^{N-2} into the unit ball $B^{N-2} \subset \mathbb{R}^{N-2}$

$$P: S_+^{N-2} \to B^{N-2} : (x_1,\dots,x_{N-1}) \to (x_1,\dots,x_{N-2})$$

and we observe that P: $S^{N-3} \to S^{N-3}$ is the identity on S^{N-3}.

We define the function $\tilde{h}: S^{N-2} \times S^1 \to S^{N-1}$ as

$$(2.12) \qquad \tilde{h}(x) = \begin{cases} (P(x), \sqrt{1 - |P(x)|^2} \cos 2\pi t, \sqrt{1 - |P(x)|^2} \sin 2\pi t) \\ \quad \text{if } x \in S_+^{N-2} \\ (x,0) \\ \quad \text{if } x \in S_-^{N-2} . \end{cases}$$

$\tilde{h}$ is a continuous function having $\deg(S^{N-2} \times S^1, \tilde{h}, 0) = 1$. Therefore, taking the corresponding h: $S^{N-2} \to \Lambda$, we have $h \in \Gamma_N^*$.

Moreover, from (2.10), (2.11), and (2.12)

$$\sup_{x \in S^{N-2}} I_T(R_b h(x)) \leq \sup_{x \in S^{N-2}} I_T^b(R_b h(x)) =$$

$$= I_T^b(R_b x_b) = \theta_1(\alpha, bT^2) \leq c_0 ,$$

that is $c \leq c_0$.

Now, if $c < c_0$ we can apply Theorem 2.1.1. getting a noncollision solution. On the other hand, if $c = c_0$ it follows that there exists $x \in S^{N-2}$ such that $\nabla I_T(R_b h(x)) = 0$. Indeed it follows from the Minimax Principle that if

$$\inf_{x \in S^{N-2}} \|\nabla I_T(R_b h(x))\| > 0,$$

then, for some $\sigma > 0$

$$\sup_{x \in S^{N-2}} I_T(\eta(R_b h(x),\sigma)) < c$$

in contradiction with the definition of c, since for every $\sigma \in \mathbb{R}$, $\eta(R_b h(S^{N-2}),\sigma) \in \Gamma_N^*$. ∎

Remark 2.2.1. In the autonomous case, since T can be arbitrarily choosen, we obtain the existence of a solution of (P) for every $T > 0$; indeed both the strong force assumption (SF) and assumptions (2.8)-(2.9) do not involve the period but only the behaviour of the two functions

$$\phi_1(x) = \inf_{t \in \mathbb{R}} F(x,t)$$
$$\phi_2(x) = \sup_{t \in \mathbb{R}} F(x,t).$$

When F depends explicitly on the time, for the same reason we obtain the existence of noncollision kT-periodic solutions of (P) for every integer $k \geq 1$. ∎

Remark 2.2.2. In the autonomous case, if x_T is the critical point of I_T found by Theorem 2.2.3, then

$$\lim_{T \to 0} \|x_T\|_\infty = 0.$$

Indeed, we have

$$\frac{1}{2}\int_0^1 |\dot{x}_T|^2 \le I_T(x_T) \le \theta_0(\alpha, bT^2) = T^{\frac{4}{\alpha+2}}\,\theta_0(\alpha,b),$$

which gives

$$\lim_{T\to 0}\frac{1}{2}\int_0^1 |\dot{x}_T|^2 = 0.\blacksquare$$

Remark 2.2.3. If $(x_k)_k$ is the sequence of critical points of I_{kT} found by Theorem 2.2.3, then

$$\lim_{k\to\infty} \|x_k\|_\infty = +\infty.$$

In fact let $M_k = \max_{t\in[0,1]} |x_k(t)|$: we have

$$k^{\frac{4}{\alpha+2}}\theta_0(\alpha, bT^2) = \theta_0(\alpha, bk^2T^2) \ge -k^2T^2 \int_0^1 F(x_k, tkT) \ge$$

$$\ge ak^2T^2 \int_0^1 \frac{1}{|x_k|^\alpha} \ge ak^2T^2 \frac{1}{M_k^\alpha},$$

that is

$$M_k^\alpha \ge \frac{aT^2}{\theta_0(\alpha, bT^2)}\, k^{\frac{2\alpha}{\alpha+2}}.$$

Therefore $\lim_{k\to\infty} M_k = +\infty.\blacksquare$

Remark 2.2.4. If $N = 2$, we have

$$c = \inf \{ I_T(x) \,/\, x \in \Lambda \text{ and } x \text{ is not homotopic to a constant function in } \mathbb{R}^2 \setminus \{0\} \}.$$

Hence, in the autonomous case, for every T fixed, the solution found by Theorem 2.2.2. or by Theorem 2.2.3. has T as its minimal period. Indeed, if not, its minimal period has to be T/k for some $k \geq 2$. Let $x \in H^1([0,1]/\{0,1\})$ be the solution after the rescaling of the period: then x has minimal period 1/k . Consider

$$y(t) = x(t/k), \; t \in [0,1]$$

since x is not homotopic to a constant function in $\mathbb{R}^2 \setminus \{0\}$, so is y; moreover, since $k \geq 2$,

$$I_T(y) = I_T(x) + \left(\frac{1}{k^2} - 1\right)\frac{1}{2}\int_0^1 |\dot{x}|^2 \, dt \; < c$$

and this contradicts the definition of c.∎

We end this section with the discussion of the case when F is even. Let us assume F1), F2),F3) and moreover

F4) $$F(-x,t) = F(x,t), \; \forall x \in \mathbb{R}^N \setminus \{0\}, \; \forall t \in \mathbb{R} .$$

The eveness of F implies that the space of the functions symmetric with respect to the origin

(2.13) $$H^S = \{ x \in H^1([0,1]/\{0,1\}) \,/\, x(t + \tfrac{1}{2}) = -x(t), \; \forall \, t \in \mathbb{R} \}$$

is invariant under the flow η; therefore the critical points of $I_T |_{\Lambda \cap H^S}$ in $\Lambda \cap H^S$ are critical points of I_T in Λ.

We consider

(2.14) $$\Lambda^S = H^S \cap \Lambda$$

(2.15) $$X_0^S = \{ x \in H^S \setminus \Lambda^S \text{ such that } -\int_0^1 F(x,t) < +\infty \}$$

and

(2.16) $$c_0^S = \inf_{x \in X_0^S} I_T(x).$$

We remark that I_T has a strictly positive lower bound in Λ^S; indeed let $x \in \Lambda^S$: from its symmetry we have

(2.17) $$\int_0^1 |\dot{x}|^2 dt \geq 4M^2,$$

where

$$M = \max_{t \in [0,1]} |x(t)|,$$

and hence

$$c^S = \inf_{x \in \Lambda^S} I_T(x) \geq$$

$$\geq \inf_{x \in \Lambda^S} \{ 4M^2 - T^2 \int_0^1 F(x,t)\, dt \} > 0.$$

Now we are going to investigate on the existence of a minimum of I_T in Λ^S.

THEOREM 2.2.3. *Assume that F1), F2), F3) and F4) hold and moreover that there exists* $x \in \Lambda^S$ *such that* $I_T(x) \leq c_0^S$. *Then there exists at least one solution of (P) in* Λ^S.

Proof. In fact if $c^s = c_0^s$ and $x \in \Lambda^S$ is such that $I_T(x) = c_0^S$, then x is a minimum of I_T in Λ^S. On the other hand, if $c^S < c_0^S$, taking a minimizing sequence $(x_n)_n$ in Λ^S, from Proposition 2.2.1 we can find a subsequence converging to some limit y and from the lower semicontinuity of I_T, since $c^S < c_0^S$, we can conclude that $y \in \Lambda^S$.∎

A first assumption that one can make in order to get $c^S < c_0^S$ is the strong force condition (SF) obtaining in this way the analogue of THeorem 2.2.2. for solution simmetric with respect to the origin.

More interesting is the fact that one can apply Theorem 2.2.3. to obtain noncollision periodic solutions for even perturbations of the Keplerian potential. More precisely we can state:

THEOREM 2.2.4. *Assume F1), F2), F3), F4) and moreover that*

(2.18) $$\frac{a}{|x|^\alpha} \leq -F(x,t) \leq \frac{b}{|x|^\alpha}$$

(2.19) $$a \leq b \leq a2^\alpha \Phi(\alpha).$$

Then there exists at least one csolution of (P) in Λ^S.

Proof. In fact from (2.18), (2.19) and Proposition 2.2.1, one has:

$$c^s = \inf \{I_T(x),\ x \in X_0^s\} = 4 \inf \{I_{\frac{T}{2}}(x),\ x \in X_0\} = 4\theta_0(\alpha . a\frac{T^2}{4}) =$$

$$= 2^{\frac{2\alpha}{\alpha+2}} a^{\frac{2}{\alpha+2}} \theta_0(\alpha, T^2) \leq c_0^s.$$

Hence, for $x_b = R_b e^{2\pi i t}$, we have $I_T(x_b) \leq c_0^s$ as in (2.11). Finally we apply Theorem 2.2.3. to end the proof.∎

Remark 2.2.5. In the even autonomous case, since the critical level is found by a minimization argument, the corresponding solutions have minimal period exactly T. (see Remark 2.2.1. and Remark 2.2.4.).∎

2.3 - OTHER PERTURBATIONS OF KEPLERIAN TYPE POTENTIALS.

The assumptions for the existence of solutions of (P) that we have seen in the previous section, impliy some strong restrictions on the way that the potential goes to $-\infty$ at 0. However,we can get the existence of noncollision solutions under another kind of hypothesis which do not involve the behaviour of the potential near zero, but far from it.

We assume

F1) $F \in C^2(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$

F2) $\lim_{x \to 0} F(x) = -\infty$

F3) There exists an open set Ω, strictly starshaped with respect to the origin, such that
$F(x) = b = \max_{x \in \mathbb{R}^N} F(x)\ , \ \forall\ x \in \Omega.$

F4) $\limsup_{|x| \to \infty} F(x) = \beta < b.$

According with the notation of Section 2.1. we consider

$$X_0 = \{\ x \in H^1([0,1]/\{0,1\}) \setminus \Lambda \text{ such that } I_T(x) < +\infty\ \},$$

but, since we are concerned only with autonomous potentials, and we are looking for the existence of long periodic solutions of (P), we set

$$c_0(T) = \inf_{x \in X_0} I_T(x).$$

We remark that assumption F4) leads to a good compactness property of I_T, that is

Proposition 2.3.1. *For every*

(3.1) $$c < \min \{ c_0(T) , -\beta T^2 \}$$

and for every Palais-Smale sequence al level c in Λ, there exists a subsequence converging to some limit in Λ.

Proof. Let $(x_n)_n$ be the Palais-Smale sequence at level c in Λ: that is

(3.2) $$x_n \in \Lambda, \forall n$$

(3.3) $$I_T(x_n) = c_n \to c$$

(3.4) $$-\ddot{x}_n = T^2 \nabla F(x_n) + h_n, \; h_n \to 0 \text{ in } H^{-1} .$$

Since F is bounded from above by b, $(\dot{x}_n)_n$ is bounded in the L^2 norm. Hence, if $(x_n)_n$ is bounded in C^0 we end the proof by standard arguments (see for example Proposition 2.1.1.). Assuming by the contrary that $(x_n)_n$ is unbounded in C^0, we have from the boundness of $(\dot{x}_n)_n$ in L^2, that

$$\lim_{n\to\infty} \min_{t \in \mathbb{R}} |x_n(t)| = +\infty$$

and therefore F4) leads to

(3.5) $$\lim_{n\to\infty} T^2 \int_0^1 -F(x_n)\, dt \geq -T^2\beta ;$$

using (3.2), (3.3) and (3.5) we get

$$\lim_{n\to\infty} I_T(x_n) \geq - T^2\beta$$

which contradicts (3.1).∎

Proposition 2.3.2. *Assume that F1), F2) and F3) hold. Then there is a* T_0 *and a* $c > 0$ *such that, for every* $T \geq T_0$, (3.6) $$c_0(T) \geq cT - bT^2.$$

Proof. Without any restriction we can prove Proposition 2.3.1 for a nondecreasing $\varphi(|x|)$ such that $F(x) \leq \varphi(|x|)$ for every x, $\varphi(s) \leq b$, and $\lim_{s\to 0} \varphi(s) = -\infty$.

We write $\psi(s) = \varphi(s) - b$ and we have:

(3.7) $$c_0(T)= \inf \{ \frac{1}{2}\int_0^1 (\dot{\gamma})^2 - T^2 \int_0^1 \varphi(\gamma)dt , \gamma\in H_0^1([0,1]/\{0,1\};\mathbb{R}), \gamma \geq 0 \} =$$

$$= \inf \{ \frac{1}{2}\int_0^1 (\dot{\gamma})^2 - T^2 \int_0^1 \psi(\gamma)dt , \gamma\in H_0^1([0,1]/\{0,1\};\mathbb{R}), \gamma \geq 0 \} - bT^2.$$

Let $\rho\in H_0^1([0,1]/\{0,1\};\mathbb{R})$, $\rho\geq 0$ such that

$$c_0(T) = \frac{1}{2}\int_0^1 (\dot{\rho})^2 - T^2 \int_0^1 \psi(\rho)dt - bT^2.$$

For $\varepsilon > 0$ fixed, if $|\rho(t)| \leq \varepsilon$ for every $t\in [0,1]$ we have

$$c_0(T) \geq -\psi(\varepsilon)T^2 - bT^2,$$

since $\psi \leq 0$ and it is nondecreasing. On the other side, if

$$\max_{t\in\mathbb{R}} |\rho(t)| \geq \varepsilon$$

we consider

$$A_\varepsilon = \{ t\in[0,1] \,/\, \psi(\rho(t)) < -\varepsilon^2\}$$

and we have

$$c_0(T) \geq \frac{1}{2}\int_{A_\varepsilon} (\dot\rho)^2 - T^2 \int_{A_\varepsilon} \psi(\rho)dt - bT^2 \geq$$

$$\geq 2\pi^2\frac{\varepsilon^2}{|A_\varepsilon|} + T^2\varepsilon^2|A_\varepsilon| - bT^2 \geq \inf_{\mu>0}\ \{2\pi^2\frac{\varepsilon^2}{\mu} + T^2\varepsilon^2\mu - bT^2\} =$$

$$= cT - bT^2. ■$$

THEOREM 2.3.1. *Assume F1),F2),F3) and F4) hold. Then there exists a* $T_0 > 0$ *such that, for every* $T \geq T_0$, (P) *has at least one solution.*

Proof. Let g be the radial diffeomorphism between the unit sphere in $\mathbb{R}^N$ and $\partial\Omega$; namely

$$g: S^{N-1} \to \partial\Omega : x \to \alpha(x)x$$

with $0 < \alpha(x) \leq a$, $\alpha \in C^1(S^{N-1})$. We take $h \in \Gamma_N^*$ as is defined in (2.12), that is

$$h(x)(t) = \begin{cases} (P(x), \sqrt{1 - |P(x)|^2}\cos 2\pi t, \sqrt{1 - |P(x)|^2}\sin 2\pi t) \\ \quad \text{if } x \in S^{N-2}_+ \\ (x,0) \\ \quad \text{if } x \in S^{N-2}_- . \end{cases}$$

and we define $f = g \circ h$, $f: S^{N-2} \to \Lambda$; we have $f \in \Gamma_N^*$ since g is a diffeomorphism with $g(0) = 0$. Moreover, for every $x \in S^{N-2}$ we have

$$\left|\frac{d}{dt} f(x)\right| \leq 2\pi \sup_{y \in S^{N-1}} [|\nabla\alpha(y)| + |\alpha(y)|] = M.$$

Therefore

$$I_T(f(x)) \leq M^2 - bT^2, \quad \forall\ x \in S^{N-2}$$

and hence, by Proposition 2.3.2, we get that, for T large enough,

$$c(T) = \inf_{\gamma \in \Gamma_N^*} \sup_{\gamma(S^{N-2})} I_T < c_0(T).$$

Since moreover by Proposition 2.3.1,since $b > \beta$, for T large I_T satisfies the Palais-Smale compactness condition at level c(T), we can end the proof as in Theorem 2.1.1.∎

3 - ON THE INSTABILITY OF THE MINIMA IN THE KEPLERIAN CASE.

In this chapter we will show that the periodic solutions of the system

$$(0.1) \qquad -\ddot{x} = \nabla F(x)$$

when F is the Keplerian potential $-1/|x|$ are not stable under some kinds of perturbations. This result has been obtained in a joint paper with Capozzi and Solimini [18].

As we have seen in Section 1.4, the T-periodic solutions of the Kepler problem are the minima of the action integral on the space of H^1 function with periodic boundary conditions. We are going to show here that one can make an arbitrary small perturbation of the potential $-1/|x|$ which implies that the only minimum of the action integral has to cross the origin, and moreover that the action integral evaluated on each possible T-periodic solution has to be very far from the minimum.

More precisely let us give the following

Definition 1. We suppose that V is given by

$$(0.2) \qquad -V(x) = \frac{M(\theta)}{\rho}$$

where (ρ, θ) are the polar coordinates of $\mathbf{x} \in \mathbb{R}^2$ and M is a continuous function defined in the following way. Let us consider the partition of $\mathbb{R}^2$ induced by two straight lines forming an angle of amplitude 2μ, $0 < 2\mu < \pi$, and let $\nu = \pi - 2\mu$. We denote the sectors of amplitude 2μ (resp. ν) by I_- and I_+ (resp. I_1 and I_2) and we set $M(\theta) = M_-$ in I_- and $M(\theta) = M_+$ in I_+, where M_-, M_+ are positive constants such that $M_- < M_+$. In the sectors I_1 and I_2 M varies in a regular and monotonic way between M_- and M_+. Throughout this paper a potential of this kind will be called <u>N-type potential.</u>

We shall prove the following theorems:

Theorem 3.1. *If V is a N-type potential there are no periodic solutions of (1) with angular speed of constant sign if* ν *is sufficiently small.*

Theorem 3.2. *For every fixed* $T > 0$ *there exists a sequence of N-type potentials* (V_n)

such that the corresponding sequence $(M_n)_n$ *converges uniformly to one and such that every sequence* (x_n) *of T-periodic solutions of*

(0.3) $$-\ddot{x} = \nabla V_n(x)$$

(if there any exists) converge uniformly to zero. Moreover the associated sequence of the energies (E_n) *converges to* $-\infty$.

Observe that in polar coordinates the energy integral has the form

(0.4) $$\dot{\rho}^2 + \rho^2\dot{\theta}^2 - 2\frac{M}{\rho} = 2E$$

We set ($\times$ denotes the usual exterior product in $\mathbb{R}^3$) :

(0.5)
$$\text{i) } \mathbf{B} = x \times \ddot{x}$$
$$\text{ii) } \mathbf{P} = \mathbf{B} \times \dot{x} + \frac{M}{\rho}\, x$$
$$\text{iii) } B^2 = \mathbf{B} \cdot \mathbf{B}$$
$$\text{iv) } \cos\varphi = \frac{x \cdot \mathbf{P}}{\rho\, |\mathbf{P}|}$$

Observe that **B** and **P** are not constant along the trajectories if the potential V is dependent on θ, and

(0.6)
$$\text{i) } \dot{\mathbf{B}} = x \times \ddot{x}$$
$$\text{ii) } \dot{\mathbf{P}} = \dot{\mathbf{B}} \times \dot{x}$$

By (0.5.i) we have that

(0.7) $$\mathbf{B} = \rho^2\dot{\theta}\, j$$

and by (0.6.i), (0.1), (1.3) we get

(0.8) $$\dot{\mathbf{B}} = \frac{1}{\rho}\,\frac{dM}{d\theta}\,\mathbf{j}$$

where

$$\mathbf{j} = \frac{\mathbf{x}\times\dot{\mathbf{x}}}{|\mathbf{x}\times\dot{\mathbf{x}}|}$$

From (0.5) it follows that $|\mathbf{P}|^2 = M^2 + 2EB^2$, and then

(0.9) $$\frac{1}{\rho} = \frac{M}{B^2} - \frac{\sqrt{M^2 + 2EB^2}}{B^2}\cos\varphi$$

is always true.
From (0.5i) and (0.8) we get

(0.10) $$\frac{dB^2}{d\theta} = 2\rho\,\frac{dM}{d\theta}$$

and from (0.6)

(0.11) $$\left|\frac{d}{dt}\frac{\mathbf{P}}{|\mathbf{P}|}\right| = \left|\frac{1}{M^2 + 2EB^2}\,\frac{dM}{d\theta}\,\frac{\dot{\rho}}{\rho}\right|$$

From (0.4) and (0.9) it follows that

(0.12) $$\frac{B^2}{2M} \le \rho \le \frac{M}{-E}$$

and

(0.13) $$\left|\frac{d}{d\theta}\log\rho\right| = \frac{\rho}{|B|}\sqrt{2E + \frac{2M}{\rho} - \frac{B^2}{\rho^2}} \le \sqrt{\rho\,\frac{2M}{B^2}}$$

In the sequel we shall use the subscripts with the same meaning for M (cf. introduction).

3.1 . Proof of Theorem 3.1.

In order to prove the theorem, we initially prove the following lemma:

Lemma 3.1.1.. *For any couple of fixed constants* $a > 1$ and $C > 0$, *we have that*

$$\rho < aC\frac{B_-^2}{M_-} \tag{1.1}$$

holds in the whole sector I_1, *provided that*

$$\text{i)}\quad \nu < \frac{\log a}{\sqrt{2aC}}$$

(1.2)

$$\text{ii)}\quad \rho(\theta_1) < C\frac{B_-^2}{M_-}$$

where θ_1 *is the value of* θ *at the border between* I_- *and* I_1 .

Proof. Indeed by (1.2.ii) if $\theta = \theta_1$ the inequality (1.1) holds for every $a > 1$. Let

$$\theta_2 = \sup\{\, \theta' > \theta_1 \,/\, \forall\, \theta \in [\theta_1, \theta'],\ \rho(\theta) < aC\frac{B_-^2}{M_-}\}$$

and assume by contradiction that $\theta_2 - \theta_1 < \nu$. From (0.13) and (15.i) (observe that by (0.10) it can be obtained that M/B^2 is monotonically decreasing as M is increasing), we get:

$$\frac{\rho(\theta_2)}{\rho(\theta_1)} \leq \exp\left(\int_{\theta_1}^{\theta_2}\sqrt{\rho\frac{2M}{B^2}}\, d\theta\right) \leq \exp\left(\int_{\theta_1}^{\theta_2}\sqrt{aC\frac{B_-^2}{M_-}\frac{2M}{B^2}}\, d\theta\right)$$

$$\leq \exp(\sqrt{2aC}\ \nu) < a$$

and this contradicts the definition of θ_2. ∎

Lemma 3.1.2.. *Assume that*

(1.3) $$\nu < \frac{\log a}{a}\sqrt{\frac{1-\cos\mu}{2}}$$

and,setting

$$\lambda = \frac{M_+}{M_-}$$

(1.4) $$\lambda^{\alpha} \geq \lambda^{\beta} \geq a^2\,\frac{1+\cos\mu}{1-\cos\mu}$$

$$\text{where}\quad \alpha = \frac{2}{a\,(\,1+\cos\mu)} - 1 \geq \frac{2}{a^3\,(\,1+\cos\mu)} - 1$$

Then whenever the trajectory leaves the sector I_- with

(1.5) $$\rho \leq \frac{1}{1-\cos\mu}\,\frac{B_-^2}{M_-}\,.$$

then it reches the next sector I_+ *forming an angle* η *with* **-P** *such that* $\eta \leq \mu$.

Proof. Since (1.3) and (1.5) hold, we can apply Lemma 3.1.1. with $C = a\,(\,1 - \cos\mu\,)^{-1}$ and we get

$$\rho < \frac{a}{1-\cos\mu}\,\frac{B_-^2}{M_-}$$

in the whole I_1.
Of course, two cases are possible: either

a) $\rho\,(\theta) \leq \frac{1}{1+\cos\mu}\,\frac{B_+^2}{M_+}$

for every θ in I_1, and in this case the claim is obviously true; or

b) there exists θ in I_1 such that $\rho\,(\theta) \geq \frac{1}{(\,1+\cos\mu\,)}\,\frac{B_+^2}{M_+}$

In this case it follows from Lemma 3.1.1. that for every θ in I_1

$$\rho(\theta) \geq \frac{1}{a(1+\cos\mu)} \frac{B_+^2}{M_+} \geq \frac{1}{a(1+\cos\mu)} \frac{B^2}{M}$$

From (0.10) we get

$$\frac{d B^2}{d\theta} = 2\rho \frac{d M}{d\theta} \geq \frac{2}{a(1+\cos\mu)} \frac{B^2}{M} \frac{d M}{d\theta}$$

and hence

$$\frac{B_+^2}{B_-^2} \geq \frac{M_+}{M_-} \lambda^\alpha , \qquad \text{where} \qquad \lambda = \frac{M_+}{M_-} , \ \alpha = \frac{2}{a(1+\cos\mu)} - 1 .$$

Since (1.4) holds, we get

$$\frac{B_+^2}{M_+} \geq \frac{B_-^2}{M_-} \frac{1+\cos\mu}{1-\cos\mu} a^2.$$

By definition of η and from the above inequality it follows that

$$\frac{1}{\rho} \leq a(1+\cos\eta) \frac{M_+}{B_+^2} \leq \frac{M_-}{B_-^2} \frac{1-\cos\mu}{1+\cos\mu} \frac{1}{a} (1+\cos\eta)$$

and then, from (1.5)

$$\frac{1+\cos\mu}{1+\cos\eta} \leq 1$$

that is $\eta \leq \mu$. ∎

Proof of Theorem 3.1.

Consider first the sector I_-. Since M is constant we are moving along an arc of the ellipse defined by

$$\frac{1}{\rho} = \frac{M_-}{B_-^2} - \frac{\sqrt{M_-^2 + 2EB_-^2}}{B_-^2}\cos\varphi$$

where φ is the angle between **x** and the Lenz vector **P** (see (0.5.i)), and $B^2_- = B^2(\theta)$ (defined in (0.5.iii)) is constant in I_-. Remark that **P** gives the direction of the axe of the ellipse. We can fix the time direction in order to have $\varphi \geq \mu$ when we are leaving I_- . Then (1.5) holds at the border of I_- and I_1; from Lemma 3.1.2. it follows that when the trajectory reaches I_+ tha angle η between -**P** and **x** is larger than μ.

Therefore when we leave I_+ the angle η' between **x** and -**P** is larger than μ.

Now, crossing I_2, we reach another time I_-. If at the entrance point (1.5) holds, we could change the time versus and, by repeating the same arguments with I_2 at the place of I_1, we should get into I_+ with $\eta' \leq \mu$, which is absurde.

Now we are moving in I_- on the arc of the ellipse defined by

$$\frac{1}{\rho'} = \frac{M_-}{B'^2_-} - \frac{\sqrt{M_-^2 + 2EB'^2_-}}{B'^2_-}\cos\varphi$$

where $\rho' = |\mathbf{x}|$ in I_- and B'_- is the new constant analogous to B_- as defined in (0.6). We claim that $B'_- < B_-$. Indeed assume the contrary. From the above discussion and Lemma 3.1.1. we have

$$\rho' \geq \frac{1}{a}\,\frac{1}{1-\cos\mu}\,\frac{B'^2_-}{M_-}\,. \tag{1.6}$$

in the whole I_1.

Assume first that

$$\frac{B_+^2}{M_+} \geq \frac{B_-^2}{M_-}\,\frac{1+\cos\mu}{1-\cos\mu}\,a^2.$$

holds: if there exists θ in I_2 such that

$$\rho'(\theta) \leq \frac{a}{1-\cos\mu}\,\frac{B_-^2}{M_-}$$

from Lemma3.1.1. (observe that $B'_- \geq B_-$) we have

$$\rho'(\theta) < \frac{a^2}{1-\cos\mu}\,\frac{B_-^2}{M_-} \leq \frac{a^2}{1-\cos\mu}\,\frac{1-\cos\mu}{1+\cos\mu}\,\frac{1}{a^2}\,\frac{B_+^2}{M_+} < \frac{1}{1+\cos\mu}\,\frac{B_+^2}{M_+}$$

in the whole I_2, and then also in the border of I_+, in contradiction with the above discussion. So we can assume that $\rho' > \rho$. Moreover (0.9) implies that

$$B_+^2 - B_-^2 = \int_{\theta_1}^{\theta_1+\nu} 2\rho(\theta)\,\frac{dM}{d\theta}\,d\theta = \int_{M_-}^{M_+} 2\rho(\theta(M))\,dM$$

$$B_+^2 - B_-'^2 = \int_{M_-}^{M_+} 2\rho'(\theta(M))\,dM$$

and hence $B_- > B'_-$.
So we can assume that

$$\frac{B_+^2}{M_+} < \frac{B_-^2}{M_-}\,\frac{1+\cos\mu}{1-\cos\mu}\,a^2$$

from (1.6) and the above inequality we get

$$\rho'(\theta) > \frac{1}{a^3}\,\frac{1}{1+\cos\mu}\,\frac{B_+^2}{M_+}$$

which leads to a contradiction (by using (0.9) and (1.5)).

This discussion ends the proof: it shows that every time the trajectory turns around the origin and get another time I_- the corrensponding B^2_- decreases. Since the trajectory is compact , B^2_- must get zero in a finite number of rounds. ∎

Remark 3.1.1.

Since in the conditions (1.2), (1.3) and (1.4) a can be taken arbitrarily close to 1, Lemma 3.1.1. and Theorem 3.1 are always true if one take a sequence of N-type potentials V_n such that M_n converges to 1 and ν_n converges to 0 .

3.2 - Proof of Theorem 3.2.

Theorem 2.1 says that all possible T-periodic solutions of (1) with N-type potential must have at least one bounce if ν is sufficiently small. Of course the number of bounces is even and it is easy to see that the trajectory can bounce only while it is leaving I_+ (so we can suppose that it happens for example in I_1).

We are going to prove now that the bounce is possible only if ρ is sufficiently large. Indeed in order to prove this fact we need the following

Lemma 3.2.1. *For every $L > 0$, for suitable values of λ and ν (namely $\lambda e^{\nu/2}$ sufficiently close to 1), the condition*

$$\rho \geq \frac{LB^2}{M} \tag{2.1}$$

at the border of I_+ and I_1 is necessary to have a bounce.

Proof. Assume by contradiction that

$$\rho < \frac{LB^2}{M} \tag{2.2}$$

holds at the border of I_+ and I_1. Let θ_1 be the value of θ at the border of I_+ and I_1 and let

$$\bar{\theta} = \sup \{ \theta' / \ \forall \ \theta \in [\theta_1, \theta'] \ , \ \rho(\theta) < \frac{NB^2(\theta)}{2M(\theta)} \}$$

it follows from (0.13) that

$$\int_{\theta_1}^{\bar{\theta}} \frac{d \log B^2}{d\theta} \, d\theta < N \int_{\theta_1}^{\bar{\theta}} \frac{d \log M}{d\theta} \, d\theta$$

and then

$$\frac{B^2(\theta_1)}{B^2(\bar{\theta})} \leq \lambda^N$$

Remark that in I_1 M is decreasing, therefore

$$\frac{M(\bar{\theta})}{B^2(\bar{\theta})} \le \lambda^N \frac{M(\theta_1)}{B^2(\theta_1)}$$

Using the above inequalities we get

$$2\rho(\bar{\theta})\frac{M(\bar{\theta})}{B^2(\theta)} = 2\frac{\rho(\bar{\theta})}{\rho(\theta_1)}\frac{M(\bar{\theta})}{B^2(\bar{\theta})}\rho(\theta_1) \le$$

$$2\frac{\rho(\bar{\theta})}{\rho(\theta_1)}\frac{M(\theta_1)}{B^2(\theta_1)}\lambda^N\rho(\theta_1) \le 2\lambda^N L\frac{\rho(\bar{\theta})}{\rho(\theta_1)} \le$$

$$2e^{N\nu/2}\lambda^N L\,.$$

We will choose ν so small and λ so close to 1 in such a way the inequality

$$N > 2L\left(\lambda e^{\nu/2}\right)^N$$

admits a solution N. In such a case the estimate (2.2) should hold true in the whole I_1 and the bounce could not happen.∎

Lemma 3.2.2. *For every $n \in \mathbb{N}$ we can define a N-type potential V_n such that every solution of (0.3) has to turn at least n times around the origin between two bounces.*

Proof. Let us observe that if there exists $k<1$ such that

$$M^2 + 2EB^2 \le k^2 M^2$$

then it follows from (0.9) that

$$\rho \leq \frac{1}{1-k}\frac{B^2}{M}$$

Then condition (2.1) implies

(2.3) $$M^2 + 2EB^2 > \left(1-\frac{1}{L}\right)^2 M^2$$

Now assume that we are in I_+ just after the bounce. Repeating the above reasoning with the time versus inverted, we conclude that (2.3) is satisfied in I_+ and (2.1) must hold on the border of I_+ and I_1. Therefore, (see (0.9)) on the border of I_+ and I_1, the angle φ between **x** and **P** satisfies

(2.4) $$\cos\varphi \geq \left(1-\frac{1}{L}\right).$$

Since we will fix L very large, φ will be very small: we suppose that $\varphi \leq 2\mu/m$ (m is a suitably large natural number which will be fixed in the following). Therefore, when we leave the sector I_+ getting I_2, the angle between **x** and -**P** is very small (remember that, as ν is very small, 2μ is very close to π). So we can suppose without any restriction that

$$\rho \leq \frac{LB^2}{M}$$

on the border of I_+ and I_2 .

From (0.11) it follows that

(2.5) $$\left|\frac{d}{d\theta}\frac{\mathbf{P}}{|\mathbf{P}|}\right| = \left|\frac{\rho}{M^2+2EB^2}\frac{dM}{d\theta}\frac{d}{d\theta}\frac{1}{\rho}\right|.$$

Two cases are possible:

a) if

(2.6) $$M^2 + 2EB^2 \geq \frac{1}{L}M^2$$

and

(2.7) $$\rho \leq \frac{LB^2}{M}$$

from (2.4) and (0.9) we get

$$\left|\frac{d}{d\theta}\frac{\mathbf{P}}{|\mathbf{P}|}\right| \leq \frac{1}{M}\frac{dM}{d\theta}\sqrt{2L^3}$$

and then

$$\left|\frac{\mathbf{P}_+}{|\mathbf{P}_+|} - \frac{\mathbf{P}_-}{|\mathbf{P}_-|}\right| \leq \sqrt{2L^3}\ \log\lambda$$

where $\mathbf{P}_+$, $\mathbf{P}_-$ are the values of $\mathbf{P}$ in I_+ and I_- rispectively. We will choose λ so close to 1 in order to make the angle between $\mathbf{P}_+$and $\mathbf{P}_-$ less than $2\mu/m$. Also ν is supposed to be less than $2\mu/m$. Then when we reach I_- the angle η between $\mathbf{x}$ and $-\mathbf{P}_-$ must be less than $6\mu/m$, and when we leave it, the angle φ between $\mathbf{x}$ and $\mathbf{P}_-$ has to be less than $8\mu/m$.

Now we are crossing another time I_1: if (2.6) and (2.7) hold in the whole I_1, by the above estimates we conclude that, reaching I_+, the angle φ between $\mathbf{x}$ and $\mathbf{P}$ is less than $12\mu/m$. If (2.5) holds but (2.6) is false in some point of I_1 then we apply Lemma 3.1.1. (we can always assume that condition (1.2) holds with $C = L$) and we get

$$\rho > \frac{1}{a}\frac{LB^2}{M}$$

on the whole I_1. Therefore at the border of I_1 and I_+ the angle φ between $\mathbf{x}$ and $\mathbf{P}$ is less than $2\mu/m$.

Hence we can say that, under condition (2.6), whenever the trajectory describes a full 2π angle around the origin, the vector $\mathbf{P}/|\mathbf{P}|$ describes an angle less than $12\mu/m$. Since condition (2.4) is necessary in order to have a bounce, the vector $\mathbf{P}/|\mathbf{P}|$ has to describe an angle of amplitude at least $\pi - 4\mu/m$ before the next bounce. Setting $m = 12n$ we can conclude that the trajectory must turn at least n times around the origin between the two bounces.

b)There exists some θ_1 such that

$$M^2(\theta_1) + 2EB^2(\theta_1) < \frac{1}{L}M^2(\theta_1). \tag{2.8}$$

From (0.9) we get

$$\frac{d}{d\theta}\log\frac{B^2}{M^2} = 2\left(\rho - \frac{B^2}{M}.\right)\frac{M}{B^2}\frac{d}{d\theta}\log M. \tag{2.9}$$

We claim that the inequality

$$(2.10) \qquad M^2(\theta) + 2EB^2(\theta) < \frac{a^2}{L} M^2(\theta)$$

is satisfied in the whole I_1.

Indeed assume the contrary: from (0.8) it follows that

$$\left| \rho(\theta) - \frac{B^2(\theta)}{M(\theta)} \right| < \frac{a}{\sqrt{L} - a} \frac{B^2(\theta)}{M(\theta)}$$

holds for every $\theta \in [\theta_1, \theta_2]$; moreover (2.9) and the above inequality imply:

$$\frac{B^2(\theta_2)}{M^2(\theta_2)} \geq \lambda^\gamma \frac{B^2(\theta_1)}{M^2(\theta_1)} > \lambda^\gamma \frac{1}{-2E} \left(1 - \frac{1}{L}\right)$$

where

$$\gamma = \frac{-2a}{\sqrt{L} - a}.$$

Since $a > 1$ we can fix λ so close to 1 in order to have

$$(2.11) \qquad 1 - \frac{a^2}{L} < \lambda^\gamma \left(1 - \frac{1}{L}\right).$$

that is

$$M^2(\theta_2) + 2EB^2(\theta_2) > \frac{a^2}{L} M^2(\theta_2)$$

in contradiction with the definition of θ_2.

Therefore we can say that under condition (2.11), if (2.8) holds then (2.10) is true in the next sector of amplitude ν. Since a is arbitrary we will think $a = b^{2n} < L - 1$. By repeating this arguments we get that the trajectory has to describe at least n full 2π angles around the origin before the condition (2.3) (which is necessary in order to have the next bounce) can be again satisfied.∎

End of the proof of Theorem 3.2.

Now we can prove the theorem. Given $T > 0$, for any $n \in \mathbb{N}$ we can fix (by the above arguments) λ_n and ν_n in such a way that every possible T-periodic solution of the problem must turn around the origin at least n times.We are going to give an estimate on the "period" to describe a 2π angle around the origin.

Observe that by the definition of N-type potential and by the choice of λ_n and v_n (v_n is close to 0 and,as λ_n is close to 1, M_+ is close to M_-) we can deduce that the path described around the origin is "like-ellipse".

Let us go back to Lemma 3.2.2; if case b) holds, the trajectory describes a full 2π angle aruond the origin in a time τ_n wich satisfies (from (0.12) and (2.10)):

$$\tau_n = \int_0^{2\pi} \frac{1}{\dot\theta}\, d\theta = \int_0^{2\pi} \frac{\rho^2}{B}\, d\theta \geq \int_0^{2\pi} \frac{B^2}{4M^2}\frac{1}{B}\, d\theta \geq \int_0^{2\pi} \Big[\Big(1-\frac{a^2}{L}\Big)\frac{M^2}{-2E} \Big]^{3/2} \frac{1}{4M^2}\, d\theta$$

$$\geq 2\pi C_n \frac{M_-}{4} (-2E_n)^{-3/2}$$

where C_n converges to 1 when n goes to infinity.

If case a) holds we can say that $\mathbf{P}_+$ lies in I_+ almost n-2 times the path crosses I_+.This means that

$$\tau_n \geq C(v)\pi\,(-2E_n)^{-3/2}$$

and C(v) converges to 1as v goes to 0.

Finally, since the period T is larger than $n\tau_n$, the sequence (τ_n) must converge to 0 and hence the sequences of the energies (E_n) converges to $-\infty$. From (0.12) this fact implies that the possibly sequence of T-periodic solutions converge uniformly to 0.∎

4 - OTHER SINGULAR POTENTIALS.

In this chapter we collect the results contained in some recent papers. For the case of repulsive forces we refer to [24] and [47], and for the results of Section 4.2, we refer to [4] and [12].

4.1 - REPULSIVE FORCES.

We consider the autonomous problem

$$\text{(P1)} \qquad \begin{cases} -\ddot{x} = \nabla F(x) \\ x(t+T) = x(t) \\ x(t) \neq 0 \end{cases}$$

or the forced one

$$\text{(P2)} \qquad \begin{cases} -\ddot{x} = \nabla F(x) + h(t) \qquad h(t+T) = h(t) \\ x(t+T) = x(t). \\ x(t) \neq 0 \end{cases}$$

where F has a singularity in zero of repulsive type. More precisely we will consider the following assumptions:

F1) $\qquad F \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$

F2) $\qquad \lim\limits_{x \to 0} F(x) = +\infty$

F3) $\qquad$ There exists c_1, c_2 such that $\nabla F(x) \cdot x \leq c_1 |x| + c_2$

F4) $\qquad \lim\limits_{|x| \to \infty} \nabla F(x) = +\infty.$

As one can expect, when the force $-\nabla F(x)$ is "everywhere repulsive" the autonomous problem (P1) has no periodic solution. More precisely if we assume

F5) $\limsup_{x \to 0} \nabla F(x) \cdot \frac{x}{|x|} < 0$

F6) $\exists\, \varepsilon > 0$, such that , $\forall\, x,y, \left|\frac{x}{|x|} - \frac{y}{|y|}\right| < \varepsilon \Rightarrow \nabla F(x) \cdot y < 0,$

we have

THEOREM 4.1.1. *Assume that F5), F6) hold. Then there exists* h_0 *such that for every* $h \in C^1([0,T]/\{0,T\})$ *with* $\|h\|_\infty < h_0$ *and* $\int_0^T h = 0$, *(P2) has no solution.*

Anyway we can get solutions of (P2) for every forcing term with nonzero mean:

THEOREM 4.1.2. *Assume that F1), F2) and F3) hold. Then for every* $h \in C^1([0,T]/\{0,T\})$ *such that*

(1.1) $$\int_0^T h \neq 0,$$

(P2) *has at least one solution.*

On the other hand, if the potential F becomes attractive far away from zero, we can obtain the existence of solutions of (P1) or of (P2) for every forcing terms. More precisely we can state

THEOREM 4.1.3. *Assume that F1), F2), F3) hold and assume moreover*

F7) $\exists\, R > 0$ such that $\forall\, y \in \mathbb{R}^N$, $|y| > R$, $F(y) \leq N = \lim_{|x| \to \infty} F(x)$ and $F(y) < N$ if $|y| = R$.

Then for every $h \in C^1([0,T]/\{0,T\})$ *(P2) has at least one solution.*

Proof of Theorem 4.1.1.. Assume by the contrary that there is a sequence $(h_n)_n \in C^1([0,T]/\{0,T\})$ such that

(1.2) $$\int_0^T h_n = 0$$

(1.3) $$\lim_{n\to\infty} \|h_n\|_\infty = 0$$

and (P2) with $h = h_n$ has a solution x_n.

Let t_n be a point ot maximum for $|x_n|$. We have

$$|\dot{x}_n(t_n)|^2 + x_n(t_n)\cdot\ddot{x}_n(t_n) \le 0$$

and hence, since x_n solves (P2)

$$-[h_n(t_n) + \nabla F(x_n(t_n))]\cdot x_n(t_n) \le -|\dot{x}_n(t_n)|^2 \le 0.$$

Therefore

(1.4) $$\nabla F(x_n(t_n)) \cdot \frac{x_n(t_n)}{|x_n(t_n)|} \ge -h_n(t_n).$$

If $|x_n(t_n)|$ is bounded, up to a subsequence we obtain

$$\lim_{n\to\infty} x_n(t_n) = x_0$$

and, since from (1.3) and F5) $x_0 \ne 0$, (1.4) leads to

$$\nabla F(x_0) \cdot \frac{x_0}{|x_0|} \ge 0$$

which contradicts F6).

So we have

$$\lim_{n\to\infty} x_n(t_n) = +\infty.$$

Moreover, since x_n solves (P2), F6) implies that

(1.5) $$\int_0^T |\dot{x}_n|^2 = \int_0^T [\nabla F(x_n) + h_n] \cdot x_n \leq \int_0^T h_n \cdot x_n \leq |x_n(t_n)| \int_0^T |h_n|.$$

From (1.5) and Hölder inequality we find that the diameter d_n of the orbit x_n satisfies

$$d_n \leq T\sqrt{\|h_n\| \, |x_n(t_n)|}$$

and therefore

$$\limsup_{n\to\infty} \sup_{t\in\mathbb{R}} \left| \frac{x_n(t)}{|x_n(t)|} - \frac{x_n(t_n)}{|x_n(t_n)|} \right| = 0.$$

Hence, for n large we have

$$\nabla F(x_n(t)) \cdot x_n(t_n) < 0, \quad \forall\ t \in [0,T],$$

and then, from the above inequalitity and (1.2) we get

$$x_n(t_n) \cdot \int_0^T -\ddot{x}_n = x_n(t_n) \cdot \int_0^T [\nabla F(x_n) + h_n] =$$

$$x_n(t_n) \cdot \int_0^T \nabla F(x_n) < 0,$$

that is $\int_0^T -\ddot{x}_n \neq 0$, which contradicts the fact that x_n is T-periodic.∎

As usual we will find the solutions of (P1) or (P2), up to the the rescaling of the period, as critical points of the functional

$$I_T(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\, dt - T^2 \int_0^1 [F(x)dt + x \cdot h(tT)\, dt$$

$x \in H^1([0,1]/\{0,1\};\mathbb{R}^N)$.

Proposition 4.1.1. *Assume F1) and F3) and consider*

$$U = \{ x \in H^1([0,1]/\{0,1\};\mathbb{R}^N) \;/\; \min_{x\in[0.1]} |x(t)| = 1\}$$

$$c_0 = \inf_{x\in U} I_T(x)$$

then $c_0 > -\infty$.

Proof. See Appendix A.

Proposition 4.1.2. (A priori estimate) *Assume F1), F2), F4) and that*

$$\limsup_{x\to 0} F(x) = M.$$

Then one can find a constant γ *such that, if* $M > \gamma$, *every solution* x *of (P2) has* $|x(t)| \geq \lambda$, $\forall$ $t\in[0,T]$, *where* $\lambda = \sup\{ \rho \;/\; F(x) \geq M, \forall\, |x| \leq \rho \}$.

Proof. See Appendix A.

Proof of Theorem 4.1.2. Let us consider a sequence (φ_k) of smooth real functions such that

$$\varphi_k(s) = s \qquad \forall\, s < k$$

$$0 \leq \varphi_k'(s) \leq 1 \qquad \forall\, s \in \mathbb{R}$$

$$\varphi'_k(s) = 0 \qquad \forall\ s \geq k$$

and we set

$$F_k(y) = \begin{cases} F(y) & \text{if } |y| \geq 1 \\ \varphi_k(F(y)) & \text{if } |y| \leq 1. \end{cases}$$

It is easy to see that, for large k, F_k satisfies the assumptions of Proposition 4.1.2, therefore, for the a priori estimate, the solutions of $(P2)_k$ (that is (P2) with F replaced by F_k) are actually solutions of (P2). We consider the action integral corresponding to the problem $(P2)_k$:

$$I_T^k(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\, dt - T^2 \int_0^1 [F_k(x)dt + x \cdot h(tT)]\, dt,$$

and we observe that, for k sufficiently large, there exists a constant x_1, $|x_1| < 1$ such that

$$I_T^k(x_1) < c_0$$

Moreover, since from F3) F grows sublinearly, and $\int_0^1 h(tT) \neq 0$, we can fix a constant x_2, $|x_2| > 1$ such that

$$I_T^k(x_2) = I_T(x_2) < c_0.$$

Now we consider that Mountain Pass class:

$$\Gamma = \{\ \gamma:[0,1] \to H^1([0,1]/\{0,1\};\mathbb{R}^N) \text{ continuous } /$$

$$\gamma(0) = x_1,\ \gamma(1) = x_2\ \}$$

we have $I_T^k(x_1), I_T^k(x_2) < c_0$. Moreover, since for every $\gamma \in \Gamma$, if

$$s = \inf\{ s \in [0,1] \,/\, \exists\, t \in [0,1],\ |\gamma(s)(t)| \leq 1\}$$

obviously $\gamma(s) \in U$, we obtain that

$$c = \inf_{\gamma \in \Gamma} \sup_{0 \leq s \leq 1} I_T^k(\gamma(s)) \geq c_0 .$$

Now we can find a critical point of I_T^k by the application of Rabinowitz's Mountain Pass Theorem, provided that I_T^k satisfies the Palais-Smale condition at level c.

Indeed let $(x_n)_n$ be a Palais-Smale sequence at level c, that is

$$I_T^k(x_n) \to c \tag{1.7}$$

$$-\ddot{x}_n = T^2 \nabla F(x_n) + T^2 h + h_n, \text{ with } h_n \to 0 \text{ in } H^{-1} \tag{1.8}$$

From F4) thare exists $d > 0$ such that

$$|F_n(y)| \leq d(1 + |y|), \ \forall\, y \in \mathbb{R}^N. \tag{1.9}$$

Let

$$M_n = \max_{t \in [0,1]} |x_n(t)|,$$

from (1.7) and (1.9) we obtain

$$\frac{1}{2} \int_0^1 |\dot{x}_n|^2 \leq c + T^2 d(1 + M_n) + M_n T^2 \int_0^1 |h(tT)|$$

and therefore, if $m_n = \min_{t \in [0,1]} |x_n(t)|$, we have

$$(M_n - m_n)^2 \leq d_1(1 + M_n). \tag{1.10}$$

Hence, assuming that $(M_n)_n$ is unbounded, it follows from (1.10) that $m_n \to +\infty$, and then, in view of F4)

(1.11) $$\lim_{n\to\infty} \nabla F(x_n) = 0 \text{ uniformly.}$$

Taking the duality product of (1.8) with the constant function 1 we get from (1.11) that

$$\lim_{n\to\infty} \int_0^1 [\ddot{x}_n + T^2 h(tT)]) = 0$$

which contradicts (1.1).

So we can assume that $(x_n)_n$ is bounded in the C^0norm, and we end the proof in a standard way (see for example Proposition 2.1.1)∎

Proof of THEOREM 4.1.3. We make a perturbation of (P2) with small constants and, applying Teorem 4.1.2, we find a sequence $(x_n)_n$ of solutions of

(1.12) $$-\ddot{x}_n = T^2 \nabla F(x_n) + T^2 h(tT) + h_n, \quad x_n \in H^1([0,1]/\{0,1\};\mathbb{R}^N).$$

As in the proof of the Palais-Smale condition (see Proposition 2.1.1), in order to obtain a converging subsequence we only need to prove that $(x_n)_n$ is bounded in the C^0 norm.

So we assume that $M_n = \max_{t\in[0,1]} |x_n(t)| \to +\infty$ and we consider the two opposite cases:

a) $\limsup_{|x|\to\infty} F(x) < +\infty$. Then we can assume from F7) that

(1.13) $$\limsup_{|x|\to\infty} F(x) = 0 = \sup_{|x|\geq 1} F(x)$$

and

(1.14) $$F(y) < 1 \text{ if } |y| = 1.$$

First we claim that

(1.15) $$c_0 > -\frac{1}{2}T^2 \int_0^1 |g(tT)|^2$$

where g is the primitive of h of mean value zero. Indeed we can find $\delta > 0$ such that

$$1 \le |y| \le 1 + \delta \Rightarrow F(y) < -\delta$$

Let $x \in U$ and let τ be the time that x has to stay in the anulus $\{ y / 1 \le |y| \le 1 + \delta \}$: by Hölder inequality,

$$\delta \le \mu\sqrt{\tau}$$

where
$$\mu^2 = \frac{1}{2}\int_0^1 |\dot{x}|^2 .$$

Hence, from (1.13) and (1.14) we get

(1.16)
$$\int_0^1 F(x(t)) \le -\frac{\delta^3}{\mu^2}.$$

Moreover, by the definition of g,

(1.17)
$$T\left|\int_0^1 h(tT)\cdot x\right| = \left|\int_0^1 g(tT)\cdot \dot{x}\right| \le \mu\left[\int_0^1 |g(tT)|^2\right]^{\frac{1}{2}};$$

therefore

(1.18)
$$c_0 \ge \inf_{\mu>0}\{ \frac{1}{2}\mu^2 - T\mu\left[\int_0^1 |g(tT)|^2\right]^{\frac{1}{2}} + T^2\frac{\delta^3}{\mu^2} \} >$$

$$> \inf_{\mu>0}\{ \frac{1}{2}\mu^2 - T\mu\left[\int_0^1 |g(tT)|^2\right]^{\frac{1}{2}} \} = -\frac{T^2}{2}\int_0^1 |g(tT)|^2$$

From F3) and the unboundness of the sequence in C^0, by Holder inequality we find that, up to a subsequence,

$$\lim_{n\to\infty} \min_{t\in[0,1]} |x_n(t)| = +\infty,$$

and then from F4)

(1.19)
$$\lim_{n\to\infty} \nabla F(x_n) = 0 \text{ uniformly}$$

and finally, since from (1.19) and (1.13),

$$\lim_{n\to\infty} \int_0^1 [\ddot{x}_n + T^2 h(tT)] = 0$$

we obtain that $(\dot{x}_n)_n$ converges to $-Tg(tT)$ in L^2. Therefore

(1.20) $$c_0 \leq I_T(x_n) =$$

$$= \frac{1}{2}\int_0^1 |\dot{x}_n|^2 dt - T^2 \int_0^1 [F(x_n) + h\cdot x_n] dt \to -\frac{T^2}{2}\int_0^1 |g(tT)|^2 < c_0 .$$

b) $\lim_{|x|\to\infty} F(x) = +\infty$. Then, if $(x_n)_n$ is unbounded in C^0, we find from (1.20) that

$$c_0 \leq \lim_{n\to\infty} I_T(x_n) = -\infty. \blacksquare$$

4.2 - DYNAMICAL SYSTEMS IN A POTENTIAL WELL

Let $\Omega \subset \mathbb{R}^N$ be a bounded set strictly starshaped with respect to the origin. We consider a potential $F:\Omega \to \mathbb{R}$ which satisfies the following assumptions:

F1) $$F \in C^2(\Omega;\mathbb{R})$$

F2) $$0 = \min_{x\in\Omega} F(x)$$

F3) $$\lim_{x\to\partial\Omega} F(x) = +\infty$$

F4) $\exists\, \alpha > 2, \exists\, \varepsilon > 0$ such that $\forall\, x \in \Omega$, $d(x.\partial\Omega) < \varepsilon$,
$\nabla F(x) \cdot x \geq \alpha F(x)$.

Our goal is to find solutions of the problem

$$\text{(P)} \qquad \begin{cases} -\ddot{x} = \nabla F(x) \\ x(t+T) = x(t) \\ x(t) \in \Omega,\ \forall\, t \in [0,T] \end{cases}$$

for $T > 0$ given.

The purpose of this section is to show that this problem is just a limiting case of the superquadratic one. As usual we will find the solutions of (P) as critical points of the functional

$$I_T(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\, dt \ - \ T^2 \int_0^1 F(x)dt$$

$$x \in \{\ y \in H^1([0,1]/\{0,1\};\mathbb{R}^N) \ /\ y(t) \in \Omega,\ \forall\, t \in [0,1]\ \},$$

up to the rescaling of the period.

Since we are going to find critical points of I_T by a truncation argument, let us recall

THEOREM 4.2.1 (Potential superquadratic at infinty). *Assume that G satisfies:*

F5) $G \in C^2(\mathbb{R}^N;\mathbb{R})$

F6) $G(0) = \min_{x \in \mathbb{R}^N} G(x)$

F7) $\exists\, \alpha > 2, \exists\, K \in \mathbb{R}$ such that, $\forall\, x \in \mathbb{R}^N$,
$\nabla G(x) \cdot x \geq \alpha G(x) - K$.

Then for every $T > 0$ problem

F4) $\exists\ \alpha > 2, \exists\ \varepsilon > 0$ such that $\forall\ x \in \Omega,\ d(x.\partial\Omega) < \varepsilon,$
$\nabla F(x) \cdot x \geq \alpha F(x).$

Our goal is to find solutions of the problem

(P)
$$\begin{cases} -\ddot{x} = \nabla F(x) \\ x(t+T) = x(t) \\ x(t) \in \Omega,\ \forall\ t \in [0,T] \end{cases}$$

for $T > 0$ given.

The purpose of this section is to show that this problem is just a limiting case of the superquadratic one. As usual we will find the solutions of (P) as critical points of the functional

$$I_T(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2\, dt - T^2 \int_0^1 F(x)dt$$

$$x \in \{\ y \in H^1([0,1]/\{0,1\}\mathbb{R}^N) \ /\ y(t) \in \Omega,\ \forall\ t \in [0,1]\ \},$$

up to the rescaling of the period.

Since we are going to find critical points of I_T by a truncation argument, let us recall

THEOREM 4.2.1 (Potential superquadratic at infinty). *Assume that G satisfies:*

F5) $G \in C^2(\mathbb{R}^N;\mathbb{R})$

F6) $G(0) = \min_{x \in \mathbb{R}^N} G(x)$

F7) $\exists\ \alpha > 2, \exists\ K \in \mathbb{R}$ such that, $\forall\ x \in \mathbb{R}^N,$
$\nabla G(x) \cdot x \geq \alpha G(x) - K.$

Then for every $T > 0$ problem

F4) $\exists\ \alpha > 2, \exists\ \varepsilon > 0$ such that $\forall$ $d(x.\partial\Omega) < \varepsilon$,
$\nabla F(x) \cdot x \geq \alpha F(x)$.

Our goal is to find solutions of the problem

$$(P)\qquad \begin{cases} -\ddot{x} = \nabla F(x) \\ x(t+T) = x(t) \\ x(t) \in \Omega,\ \forall\ t \in [0,T] \end{cases}$$

r $T > 0$ gi en.

The p rpose of this ion is to show that this lem is just a limiting case of the perquadra c one. As usu e will find the solutions as critical points of the functional

$$I_T(\quad \frac{1}{2}\int_0^1 |\dot{x}|^2\,dt - T^2\int_0^1 F(x)dt$$

$$x \in \quad H^1([0,1]/\{0,1\} \mathbb{R}^N) \,/\, y(t \quad , \forall\ t \in [0,1]\ \},$$

up to the rescaling of the pe .

Since we are going to l critical points of I_T by a t ation argument, let us recall

THEOREM 4.2.1 otential superquadrat at infinty). *Assume that G satisfies:*

F5) $G \in C \quad {}^N; \mathbb{R}\ )$

F6) $G(0) \quad \min_{\in \mathbb{R}^N} G(x)$

F7) $\exists\ \alpha \quad \exists\ K \in \mathbb{R}$ such that, $\forall\ x \in \mathbb{R}^N$,
$\nabla G(\quad \geq \alpha G(x) - K$.

Then for every $T > 0$ proble

(P') $$\begin{cases} -\ddot{x} = \nabla G(x) \\ x(t+T) = x(t) \end{cases}$$

has at least one solution.

Proof. See Appendix B.

The main result of this section is the following

THEOREM 4.2.2. *Assume that F1), F2), F3), and F4) hold. Then, for every T > 0 fixed, problem (P) has at least one solution.*

Before proving Theorem 4.2.2, we need some a priori estimate on the solution found by applying Theorem 4.2.1.

Proposition 4.2.1. *Under the assumptions of Theorem 4.2.1, for every* $c \in \mathbb{R}$ *, there exists a constant* a, *depending on* c, *such that if x is a critical point of*

$$J_T(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2 dt - T^2 \int_0^1 G(x)dt$$

$$x \in H^1([0,1]/\{0,1\};\mathbb{R}^N)$$

with $J_T(x) = c$, *then* $G(x(t)) \leq a, \ \forall\ t \in [0,1]$.

Proof. If x is a critical point of I_T at level c we have

(2.1) $$\frac{1}{2}\int_0^1 |\dot{x}|^2 - T^2 \int_0^1 G(x) = c$$

(2.2) $$-\ddot{x} = T^2\nabla G(x).$$

Multiplying both sides of (2.2) by x and integrating we get

(2.3) $$\frac{1}{2}\int_0^1 |\dot{x}|^2 = \frac{1}{2}T^2\int_0^1 \nabla G(x)\cdot x \geq \frac{\alpha}{2}T^2\int_0^1 G(x) - \frac{K}{2}T^2,$$

and by substituting (2.3) in (2.1) we obtain

(2.4) $$\int_0^1 G(x) \leq c_1$$

and

(2.5) $$\int_0^1 |\dot{x}|^2 \leq c_2.$$

Moreover, from (2.2) x satisfies the energy integral:

(2.6) $$\frac{1}{2}|\dot{x}|^2 + T^2G(x) = E;$$

integrating (2.6), we find from (2.4) and (2.5) that $E \leq \frac{1}{2}c_1 + T^2c_2$. Therefore, (2.6) implies that $G(x) \leq a = T^2(\frac{1}{2}c_1 + T^2c_2)$.

Proof of Theorem 4.2.2. Let f: $\mathbb{R}^N \to \mathbb{R}$ be the function homogeneus of degree one such that $\partial\Omega = \{ x \in \mathbb{R}^N \ / \ f(x) = 1 \}$, and let us fix $\eta > 0$, such that $\{ x\in \mathbb{R}^N / f(x) = 1 - \eta \} \subset \{ x \in \mathbb{R}^N / d(x,\partial\Omega) < \varepsilon \}$ where ε is defined in F4). Let G: $\mathbb{R}^N \to \mathbb{R}$ be defined as

$$G(x) = F(x) \qquad \text{if } f(x) \leq 1 - \eta$$

$$G(x) = \frac{f(x)^\alpha}{(1-\eta)^\alpha} F\left(\frac{(1-\eta)x}{f(x)}\right) \qquad \text{if } f(x) \geq 1 - \eta.$$

From F4), G satisfies the assumptions of Theorem 4.2.1. Therefore we can find a critical level c and a critical point at level c for the functional J_T. From Proposition 4.2.1 we have an a priori estimate on $\|G(x)\|_\infty$. Hence, taking a smaller η (if necessary) we can conclude that $x(t)\in\Omega$, $\forall\ t\in[0,1]$.∎

Remark 4.2.1. If the potential F is convex, one can prove the existence of solutions of (P) having T as their minimal period, for every $T<\frac{2\pi}{\sqrt{\omega}}$, where ω is the largest eigenvalue of $\nabla^2F(0)$ (T arbitrary if $\omega = 0$). This fact can be proved by the use of the Ekeland-Hofer index theory (see[19] and[27]), as it is shown in [4].

APPENDIX A. Proof of Propositions 4.1.1, 4.1.2.

Proposition 4.1.1. By F4) there exists $c \geq 0$ such that

$$|F(y)| \leq c(1 + |y|) \quad , \forall\ |y| \geq 1;$$

moreover, if $x\in U$ we find that, setting $\mu^2 = \int_0^1 |\dot{x}|^2$,

$$\max_{t\in[0,1]} |x(t)| \leq 1 + \mu.$$

Therefore

$$\text{(A.1)} \qquad I_T(x) \geq \frac{1}{2}\mu^2 - cT^2(2 + \mu) - T^2(1 + \mu)\left(\int_0^1 |h(tT)|\right) \geq$$

$$\geq \inf_{\mu>0} \left\{ \frac{1}{2}\mu^2 - cT^2(2 + \mu) - T^2(1 + \mu)\left(\int_0^1 |h(tT)|\right) \right\} > -\infty. ∎$$

Proposition 4.1.2. Let x be a critical point of I_T such that $I_T(x) \geq c_0$. We set

$$\lambda = \sup \{ \sigma > 0 \,/\, \forall\ y, |y| \leq \sigma \Rightarrow F(y) \geq M \}$$

$$m = \max_{t\in[0,1]} |x(t)|.$$

First we assume that $m \leq \lambda$; then we have

$$c_0 \leq I_T(x) = \frac{1}{2}T^2 \int_0^1 \{\nabla F(x) \cdot x + h(tT) \cdot x - 2F(x)\} \leq$$

$$\leq \frac{1}{2}T^2 (c_1 + mc_2 + m (\int_0^1 |h(tT)|) - T^2 M,$$

that is

(A.2) $$m \geq \alpha = \frac{\frac{c_0}{T^2} + M - \frac{c_1}{2}}{c_2 + \int_0^1 |h(tT)|}$$

then

(A.3) $$m \geq \min(\alpha, \lambda).$$

We remark that if F' satisfies F3) with the same (c_1, c_2) and has $F'(x) \geq M$, $\forall\ |x| \leq \lambda$, then every solution of (P') with $F = F'$ at level larger than c_0 satisfies the above estimate on the C^0 norm.

From F4) we can find a constant c_3 such that

(A.4) $$F(y) \geq -c_3(1 + |y|), \ \forall\ y \in \mathbb{R}^N \setminus \{0\},$$

and a constant $c_4(\alpha)$ such that

(A.5) $$F(y) \leq c_4(1 + |y|), \ \forall\ y, |y| \geq \alpha.$$

Let us assume that

$$\min_{t \in [0,1]} |x(t)| \leq 1:$$

then

(A.6) $$|x(t)| \leq 1 + \int_0^1 |\dot{x}|, \ \forall\ t \in [0,1].$$

Moreover is t_0 is such that $|x(t_0)| = M$, we obtain from (A.4), (A.5), (A.6) that

(A.7) $$|\dot{x}(t_0)|^2 \le (\nabla F(x(t_0)) + h(Tt_0)) \cdot x(t_0) \le c_1 + (c_2 + \|h\|_\infty)|x(t_0)| \le$$

$$\le c_1 + (c_2 + \|h\|_\infty)(1 + \int_0^1 |\dot{x}|).$$

From the energy estimate

(A.8) $$\frac{1}{2}|\dot{x}(t)|^2 = \frac{1}{2}|\dot{x}(t_0)|^2 - F(x(t_0)) + F(x(t)) - \int_{t_0}^{t} h(tT) \cdot \dot{x},$$

and from A.4), A.5), A.6) and A.7) we deduce that, if $\xi = \max_{t\in[0,1]} |\dot{x}(t)|$,

(A.9) $$\frac{1}{2}\xi^2 \le \frac{1}{2}[c_1 + (c_2 + \|h\|_\infty)(1 + \xi)] +$$

$$+ c_3(2 + \xi) + c_4(\alpha)(2 + \xi) + \xi \int_0^1 |h(tT)|.$$

Therefore, $\xi \le K(c_1, c_2, c_3, c_4)$ and, by substituting in (A.8) we find

$$F(x(t)) \le \frac{1}{2}K^2 + K\int_0^1 |h(tT)| + c_4(1 + K) .$$

If we assume that $M > \gamma = \frac{1}{2}K^2 + K\int_0^1 |h(tT)| + c_4(1 + K)$, we get that

$$\min_{t\in[0,1]} |x(t)| \geq \min(1,\lambda). \blacksquare$$

APPENDIX B. Proof of Theorem 4.2.1.

First we remark that F6) implies that G has s superquadratic growth at infinity: namely

$$\lim_{|x|\to\infty} \frac{G(x)}{|x|^2} = +\infty.$$

Let ω be the greatest eigenvalue of $\nabla^2 G(0)$ (possibly $\omega = 0$) and let the period T be fixed with $T < \frac{2\pi}{\sqrt{\omega}}$ (T arbitrary if $\omega = 0$). We consider the functional

$$J_T(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2 dt - T^2 \int_0^1 G(t)dt$$

$$x \in E = H^1([0,1]/\{0,1\};\mathbb{R}^N),$$

which critical points correspond to the solutions of (P2) after the rescaling of the period. Let E_0 be the subspace of E of zero mean functions, and E_N be the space of constant functions. We have:

$$E = E_0 \oplus E_N$$

and

(B.1) $$J_T(x) \leq 0, \ \forall \ x \in E_N$$

(B.2) $$J_T(x) > 0, \ \forall \ x \in E_0, \ \|x\|_E = \rho$$

(B.3) $$\lim_{\|x\| \to \infty} J_T(x) = -\infty \text{ in every finite dimentional subspace of E.}$$

In fact, (B1) simply follows from F6), (B.3) follows from the superquadratic growth of G at infinity. Let us prove (B.2): for $x \in E_0$ we have

$$J_T(x) \geq \frac{1}{2}(4\pi^2 - \omega T^2) \int_0^1 |x|^2 + o(|x|^2) > 0$$

if $\|x\|_E$ is small enough.

Now let φ be any fixed function in E_0, $\|\varphi\|_E = 1$, and let $E_{N+1} = \text{span } \{E_N; \varphi\}$. From (B.1) and (B.3), we can find a $R > 0$ such that

(B.4) $$J_T(x) \leq 0, \quad \forall \ x \in E_{N+1}, \ \|x\|_E \geq R.$$

We consider the surface S in E_{N+1}, defined by $S = S_1 \cup S_2 \cup S_3$, where

$$S_1 = \{x \in E_N \ / \ \|x\|_E \leq R \ \}$$

$$S_2 = \{ \ y = x + t\varphi \ / \ x \in E_N, \ \|x\|_E = R, \ 0 \leq t \leq R \ \}$$

$$S_3 = \{ \ y = x + R\varphi \ / \ x \in E_N, \ \|x\|_E \leq R\}$$

From (B.4) we have that

(B.5) $$J_T \leq 0, \ \forall \ x \in S.$$

Moreover, (B.2) imples that

(B.6) $$J_T > 0, \ \forall \ x \in S_0 = \{x \in E_0 \ / \ \|x\|_E = \rho \ \}.$$

Since S links with S_0, we can apply Rabinowitz's Saddle Point Theorem getting a critical point of the functional J_T, provided that the Palais-Smale condition is fulfilled.

Indeed let $(x_n)_n$ be a Palais-Smale sequence at any level c, that is

(B.7) $$J_T(x_n) = c_n \rightarrow c$$

(B.8) $$-\ddot{x}_n = T^2 \nabla G(x_n) + h_n, \quad h_n \rightarrow 0 \text{ in } H^{-1} .$$

From (B.8) and (F7) we get

(B.9) $$\int_0^1 |\dot{x}|_n = T^2 \int_0^1 \nabla G(x_n) \cdot x_n + h_n(x_n) \geq$$
$$\geq \alpha T^2 \int_0^1 G(x_n) - KT^2 + h_n(x_n) .$$

By substituting (B.9) in (B.7) we obtain

$$c_n \geq \frac{\alpha - 2}{2} T^2 \int_0^1 G(x_n) + \frac{1}{2} KT^2 - \frac{1}{2} h_n(x_n) .$$

Therefore

(B.10) $$\int_0^1 G(x_n) \leq c_1 + h_n(x_n)$$

and

(B.11) $$\int_0^1 |\dot{x}_n|^2 \leq c_2 - h_n(x_n) .$$

Now assuming that $(x_n)_n$ is unbounded in the C^0 norm, (B.11) implies that, up to a subsequence, $\lim_{n \to \infty} \min_{t \in [0,1]} |x_n(t)| = +\infty$.

By substituting in (B.10) we find a contradiction, because of the superquadraticity of G at infinity.
So we can assume that $(x_n)_n$ is bounded in C^0, and therefore, up to a subsequence, it converges weakly in $H^1([0,1]/\{0,1\})$ and strongly in C^0 to some limit w. From (B.8) and the compact embedding of C^0 in H^{-1}, we deduce the strong convergence of $(x_n)_n$. ∎

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