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A Critique of Semiclassical Gravity

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A Rossella

Fools had ne'er less grace in a year;
For wise men are grown foppish,
And know not how their wits to wear,
Their manners are so apish.

Shakespeare, *King Lear* **1**.*iv*

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Chapter 1

Introduction

Gravity, as described by general relativity, is by far the most peculiar among the known fundamental interactions. This status of things has its deepest origin in the geometrization lying at the basis of Einstein's theory, which makes it the most elegant and formally perfect of all the viable theoretical physical constructions. In this visual, gravity is not described by some object (e.g. a field) defined *on* spacetime, but rather it *is* spacetime itself: the concept of gravitational field is then reduced to that of curvature, and the role of the field equations is to relate the geometry of spacetime to its matter content.

A particularly striking consequence of this theory, which heavily relies on its geometrical character, is the prediction of the existence of "regions" to which the theory itself cannot be applied. More precisely, theorems have been proved [1,2] asserting that a sufficiently general and causally well behaving spacetime, whose energy content satisfies a positivity condition, and in which a convergence criterion is satisfied, must admit a singularity, defined in terms of incompleteness of causal geodesics; since these curves represent the world lines of classical free-falling particles, it follows that, in the framework of general relativity, there is a large variety of realistic situations in which particles (and observers) reach the boundary of spacetime in a finite proper time. The implications of such a result are quite intriguing: in fact, if one believes that singularities do really exist, then he/she will have serious problems in the specification of boundary conditions; alternatively, it is possible to look at singularity theorems as at a *reductio ad absurdum* of some hypothesis underlying their proof: then they can be used as evidence for the breakdown of the energy condition or for the replacement of general relativity by another theory of spacetime structure. Unfortunately, very little significant work has been done in either of these research areas, and the physical nature and meaning of singularities are still mysterious, despite a conspicuous amount of technical advances in their study.

It is a widespread conviction that the problems raised by the singularity theorems could be solved by taking into account an element which is absent from general relativity: the quantum principle. The common opinion about the practical achievement of this purpose is that it will emerge as a byproduct of an even more ambitious realization, namely of a quantum theory of gravity. After forty years of efforts made by considerable physicists in all the world, such a theory does not exist yet in a satisfactory form; it seems therefore a reasonable proposal to forget, for the moment, the extreme requirements underlying it, and to search for a weaker way to insert the quantum principle in a gravitational theory. It is good to stress, at this point of the discussion, that the idea that gravity and quantum mechanics should, somehow, cohabit, is more a consequence of the philosophical belief known as "unity of physics" than of some experimental result: both classical gravity and quantum mechanics have been successfully tested separately, but there is complete lack of experiments devoted to investigate their reciprocal relations. This shows how much a wildly speculative territory is the one we are now going to explore.

There are three levels at which the quantum principle can be introduced into general relativity, which lead to theories that can be classified as quantum theory in curved spacetime, semiclassical gravity, and quantum gravity; we shall now briefly analyse the status of these three classes of theories.

We have already said something about quantum gravity [3,4,5]: there have been many technical advances, but very little conceptual understanding; the huge number of competing theories is contrasted by the lack of strong physical principles as heuristic guide in the choice between them. For these reasons, we prefer not to work formally on some quantum theory of gravity, but rather to try to understand, gradually, the implications of quantum mechanics on the structure of spacetime [6], and, in particular, to which extent is gravity forced to be quantized.

Quantum theory in curved spacetime [7] is the study of quantum matter in a fixed background gravitational field; trivial (from the point of view of gravity!) examples of these theories are QED and QCD in Minkowski spacetime. The research in this sector has produced most interesting results (particle creation in dynamical universes; Hawking effect; connection between black holes physics and thermodynamics; criticism to the concept of particle); however, this kind of approach is clearly suitable for the analysis of specific effects, but not for the formulation of a fundamental theory: there is in fact no prescription about the back reaction of quantum matter on the spacetime, which is supposed fixed *a priori*.

As a nice example of investigation in this frame of ideas, we shall briefly mention to the possibility of enlarging our understanding of spacetime singularities by using quantum matter as a probe. The key idea of all the discussion can be quickly expressed as follows: while singularity theorems are proved in terms of incompleteness of causal geodesics, these latter represent physical particles only in a classical approximation; even in Minkowski spacetime a particle can be approximately represented by a curve only for a finite part of its existence, because wavepackets spread. How much does then a geodesic represent a physical particle near, say, a collapse or cosmological singularity? and which is the meaning of the "mathematically defined" singularities in terms of the behaviour of particles in its neighbourhood? The relevance of these questions suggests that the analysis of a quantum field, rather than that of causal geodesics, could be suitable for a classification of singularities which displays their physical properties.

It is important to remark that, near curvature singularities, vacuum polarization effects will become not negligible, and their backreaction on the metric will probably have to be taken into account: an analysis based on quantum theory in curved spacetime can therefore shed light on the problems and clarify some concept, but does not provide the framework for the formulation of a consistent theory. We are then led to tackle the controversial problem of how does quantum matter act as source of gravity, namely how to construct a semiclassical theory.

In semiclassical gravity, the gravitational field is treated classically again, by means of general relativity, but the backreaction of quantum matter on the geometry is taken into account by semiclassical field equations. These latter represent, to our mind, the crux of the theory: the problems arising when a classical system is coupled to a quantum one are in fact deeply intertwined with the chosen interpretation of the quantum theoretical formalism, which is itself a still open foundational subject. Let us try to explain this point with a simple example, constructed out of the theory on which almost all the results obtained so far in semiclassical gravity (mainly in the study of black holes evaporation and of inflationary cosmology) are based: the field equations are chosen to be [8,9,10]

$$G_{ab} = \frac{8\pi G}{c^4} \langle\psi|\hat{T}_{ab}|\psi\rangle \ , \tag{1.1}$$

where $\hat{T}_{ab}$ and $|\psi\rangle$ are, respectively, the stress-energy-momentum tensor operator and the (normalized to unity) state vector of quantum matter.

In this context, we can imagine a situation in which a particle has the same probability 1/2 to be in two disjoint regions of space, far from each other; moreover, let us suppose that the weak field limit holds, so that (1.1) become

$$\nabla^2 \phi = 4\pi G m |\psi|^2 , \tag{1.2}$$

where $\psi(x)$ is the Schrödinger wave function and m is the mass of the particle: then, according to (1.2), the gravitational field should be the one produced by two particles with mass $m/2$, placed in the two regions. Performing a measurement into one of these regions, the particle will be found or not, changing abruptly the right hand side of (1.2): near the region where the particle is now known to be present the field will be increased to the one generated by a mass m, while at a distance d from it, it will be the same as before for a time d/c; moreover, there are no masses out of the region. Such a situation is not allowed by the weak field equation (1.2) (nor by (1.1)), as can be immediately seen from its integral formulation.

We think this paradox to be particularly instructive, because it shows clearly the need to achieve physical understanding and insight in the subject, rather than to rely on a purely formal treatment, which can hardly deal with conceptual problems; moreover, it provides a simple example of the kind of difficulties arising when the nonlocal features of quantum theory and the typically relativistic requirements of causality are considered at the same time.

Which of the hypothesis underlying the reasoning in the previous thought experiment have to be changed in order to remove inconsistencies? It seems that the paradoxical result has its deep origin in the requirement for the field to have a classical behaviour even when coupled to a quantum particle: an obvious solution could then be to abandon the idea of a classical field, admitting a weakly semiclassical approximation, where the field is allowed to have quantum features, but only those which are induced by the source. Unfortunately, it will be shown that this approximation holds only for linear field equations, which seems too much a strong requirement from the physical point of view.

We are therefore led to explore the opposite scenario, where the field is supposed to behave classically, while matter is allowed to exhibit a nonstandard quantum behaviour; the coupling between matter and field provides in fact an interaction between a quantum and a classical system, which could be considered as a measurement on some observables of the former, with consequent state vector reduction. The previous paradox would then be solved because the position measurements performed by the experimenters occur only after the wavepacket has already been reduced by the continuous interaction with classical gravity: this "spontaneous localization" effect can also be introduced, phenomenologically, as a correction in the Schrödinger equation, and it is worth to be noted that models including it have already been suggested and studied [11,12] in some detail.

There is another problem in a theory based on equations like (1.1), which is also related to the interpretation of quantum theory. Usually the state vector $|\psi\rangle$ is not supposed to describe a single system, but rather an ensemble of identically prepared systems: then the right hand side of (1.1) must be intended as an average over such an ensemble. On the contrary, the left hand side is referred to a well precise spacetime, so that it seems that equations (1.1) make no sense at all! There are possible solutions to this puzzle: in a weakly semiclassical context, for example, the left hand side is related to an ensemble of spacetimes, too, so there is no more interpretative inconsistence; however, if gravity is considered as truly classical, then we need to confront us with the uneasy task of formulating a quantum theory of individual systems. Theories of this kind are, for example, those based on hidden variables [13,14], in which a particle is supposed to have, at each time, well defined values of position and momentum: in such theories it is reasonably easy to construct physically meaningful source terms for the field equations. Unfortunately, it has been proven experimentally [15] that any viable hidden variables theory must have a nonlocal character [16,17], and it is very difficult to incorporate this feature into a relativistic context.

As a natural consequence of this discussion, we shall divide the present thesis into two parts, one concerning with conceptual and interpretative aspects of quantum physics, the other devoted to study how does quantum matter act as source of gravity. We want to warn here that, even if Chapter 2 is organized in a "mathematical" style, this has been done only in order to gain in clearness, and there is no claim of formal rigour.

We shall use spacetime metrics with signature +2; in tensors, the indices a, b, c, ... run from 0 to 3, while i, j, k, ... run from 1 to 3. Other notations will be defined in due place.

Part I

Conceptual Aspects of Quantum Theory

In this part we shall critically review the foundations of standard quantum theory. It is particularly important, according to the general philosophy underlying this work, to distinguish, when speaking about a physical theory, between the mathematical formalism and its interpretation; in the particular case of quantum mechanics, these two aspects are so difficult to relate each other, that the character of the resulting theory has some features resembling schizophrenia. For this reason, we shall split the material into two part: Chapter 2 is devoted to a brief summary of the basic axioms, also for the purpose of establishing our "technical language", while in Chapter 3 it will be discussed the problem of connecting the mathematical concepts to the world of experience. About the formalism, we make a distinction between kinematics and dynamics: by kinematics we shall mean all the paraphernalia needed for the description of states and observables, while dynamics will be related to time evolution.

In Chapter 4 we shall explore the compatibility between quantum theory and relativity.The general formalism of the former gives to the concept of time a very peculiar role, thus entering in conflict with the relativistic requirements of covariance: the only way to retain the quantum dynamical law is therefore to write it with respect to a reference system, i.e. a congruence of future directed timelike worldlines, for which time is a well defined concept. Unfortunately, in so doing, observers belonging to different systems give essentially different descriptions of physical phenomena, and concepts like that of particle become ill-defined. The main root of these difficulties has to be identified, to our mind, in the nonlocal features of quantum theory, and we believe that an acceptable solution to the problem could be found only after a thorough understanding of these unusual properties.

Chapter 2

The Mathematical Formalism

a) Kinematics

The mathematical framework of quantum theory is devoted to the description of the concepts [18,19] of *state* and *observable*. The physical context in which these ideas are meaningful will be discussed in the next chapter: here we want only to associate them to some mathematical objects which are precisely defined. The main concept on which all the following definitions rely is that of a Hilbert space [20,21] H.

Definition 2.1 *A state is represented by a linear operator* [1] $\hat{\rho} : \mathrm{H} \longrightarrow \mathrm{H}$ *such that:*

i) $\hat{\rho}$ *is self-adjoint;*

ii) $\hat{\rho}$ *is nonnegative definite (i.e.* $\langle\psi|\hat{\rho}|\psi\rangle \geq 0, \qquad \forall|\psi\rangle \in \mathrm{H}$ *);*

iii) $tr\hat{\rho} = 1$.

Proposition 2.2 *There exists an orthonormal basis* $\{|n\rangle\}$ *of* H, *in which* $\hat{\rho}$ *is diagonal:*

$$\hat{\rho} = \sum_n p_n |n\rangle\langle n|. \tag{2.1}$$

For the eigenvalues p_n, $p_n \in [0,1]$ *and*

$$\sum_n p_n = 1. \tag{2.2}$$

Proof. Trivial, remembering the properties of self-adjoint operators. □

Definition (2.1) provides us with the most general notion of state in quantum theory; it will prove useful in the following, however, to distinguish a class of particular states, known as *pure cases*: they can be characterized

[1] Density, or statistical, operator.

by adding to the properties in Definition (2.1) the requirement that $\hat{\rho}$ be a projection operator, namely that

$$\hat{\rho}^2 = \hat{\rho}. \tag{2.3}$$

Proposition 2.3 *In the basis of Proposition(2.2), $\hat{\rho}$ is a projection operator iff*

$$\hat{\rho} = |n\rangle\langle n|, \tag{2.4}$$

for some n.

__**Proof.**__ If (2.3) is applied to (2.1), we get

$$p_n^2 = p_n, \qquad \forall n,$$

which gives either $p_n = 0$ or $p_n = 1$. By (2.2) it is clear that there must exist one, and only one, n such that $p_n = 1$, from which (2.4) follows. The converse is trivial. □

It is now clear that a pure case can be described not only by the statistical operator $\hat{\rho}$, but also by the normalized vector $|n\rangle \in$H; this correspondence, however, is not one to one, because $\hat{\rho}$ is not altered under the replacement of $|n\rangle$ by $e^{i\alpha}|n\rangle$, with $\alpha \in$R. Bearing in mind this arbitrariness, we can give the following

Definition 2.4 *A __pure state__ is associated to a unit vector of* H. *Vectors differing by a phase factor define the same pure state.*

Remark 2.5 *Of course, a pure state can be defined also specifying that $\hat{\rho}$ is a projection operator. The cases for which (2.3) does not hold are called __mixed states__ or __mixtures__.*

Remark 2.6 *It is important to notice that to a pure state it is associated a state vector of* H, *but the converse is not always true: there are unit vectors of* H *that cannot be associated to any physical state: they are excluded by the so called superselection rules.*

Definition 2.7 *An __observable__ is represented by a linear self-adjoint operator in* H. *If A denotes an observable, then its associated operator will be written as $\hat{A}$.*

Another concept, here introduced as primitive, and whose physical meaning will be clarified in the next chapter, is that of measurement of an observable on a given state.

If $|a_n, r\rangle$ is an eigenvector of $\hat{A}$, in the sense that

$$\hat{A}|a_n, r\rangle = a_n|a_n, r\rangle, \tag{2.5}$$

where r represents a degeneracy index, we can construct [1] an orthonormal basis $\{|a_n, r\rangle\}$:

$$\langle a_n, r|a_m, s\rangle = \delta_{nm}\delta_{rs}. \tag{2.6}$$

Then the projection operator

$$\hat{P}(a_n) \equiv \sum_r |a_n, r\rangle\langle a_n, r|, \tag{2.7}$$

which projects on the subspace of H corresponding to the eigenvalue a_n, can be constructed for each n, and $\hat{A}$ can be decomposed as

$$\hat{A} = \sum_n a_n \hat{P}(a_n). \tag{2.8}$$

Now it is possible to state two most important assumptions of quantum theory:

Axiom 2.8 *The only possible results of the measurement of an observable A are the eigenvalues $\{a_n\}$ of the operator $\hat{A}$.*

Axiom 2.9 *If the observable A is measured on a state described by the density operator $\hat{\rho}$, then the probability that a_n be the outcome of the measurement is*

$$p(a_n) \equiv tr(\hat{\rho}\hat{P}(a_n)). \tag{2.9}$$

Remark 2.10 *The previous axioms are well posed. In fact, it is well known that the eigenvalues of a self-adjoint operator are real, so they are suitable to represent outcomes of measurements. Moreover, $p(a_n)$ defined by (2.9) satisfies all the requirements for a probability:*

i) $p(a_n) \in \mathsf{R}^+$;

in fact, working in the basis of Proposition(2.2),

$$\begin{aligned} p(a_n) &= \sum_i \langle i|\hat{\rho}\hat{P}(a_n)|i\rangle = \sum_{ij}\langle i|\hat{\rho}|j\rangle\langle j|\hat{P}(a_n)|i\rangle = \\ &= \sum_{ijk} p_k \langle i|k\rangle\langle k|j\rangle\langle j|\hat{P}(a_n)|i\rangle = \\ &= \sum_k p_k\langle k|\hat{P}(a_n)|k\rangle \geq 0; \end{aligned}$$

[1] Since $\hat{A}$ is self-adjoint, eigenvectors corresponding to different eigenvalues are mutually orthogonal.

ii) $\sum_n p(a_n) = 1$;
in fact

$$\sum_n p(a_n) = \sum_n tr(\hat{\rho}\hat{P}(a_n)) = tr(\hat{\rho}\sum_n \hat{P}(a_n)) = tr\hat{\rho} = 1;$$

iii) $p(a_n) \leq 1$;
follows from i) and ii).

Proposition 2.11 *The average value of the measurements of A on a state described by $\hat{\rho}$ is*

$$\langle A\rangle = tr(\hat{\rho}\hat{A}). \tag{2.10}$$

Proof. By (2.9) and the definition of average value,

$$\langle A\rangle = \sum_n a_n p(a_n) = \sum_n a_n tr(\hat{\rho}\hat{P}(a_n)).$$

Using the basis $\{|a_n, r\rangle\}$ in evaluating the trace, and remembering (2.7) and (2.6), we have

$$\begin{aligned}\langle A\rangle &= \sum_{nmr} a_n \langle a_m, r|\hat{\rho}\hat{P}(a_n)|a_m, r\rangle = \sum_{nr} a_n \langle a_n, r|\hat{\rho}|a_n, r\rangle = \\ &= \sum_{nr} \langle a_n, r|\hat{\rho}a_n|a_n, r\rangle.\end{aligned}$$

Using now (2.5), we get (2.10). □

Proposition 2.12 *If A is measured on a pure state described by $|\psi\rangle$, then (2.9) and (2.10) become, respectively:*

$$p(a_n) = \sum_r |\langle a_m, r|\psi\rangle|^2 : \tag{2.11}$$

$$\langle A\rangle = \langle\psi|\hat{A}|\psi\rangle. \tag{2.12}$$

Proof. The proof of (2.11) is trivial. To prove (2.12) it is sufficient to remember the spectral representation of the identity operator

$$\hat{1} = \sum_n \hat{P}(a_n) = \sum_{nr} |a_n, r\rangle\langle a_n, r|. \tag{2.13}$$

□

Remark 2.13 *Equation (2.9) can be written also in the interesting form*

$$p(a_n) = \langle P(a_n)\rangle. \tag{2.14}$$

The probability $p(a_n)$, in the sense of statistical frequency, can then be considered as the average of an observable $P(a_n)$ to which the self-adjoint operator $\hat{P}(a_n)$ is associated; it must be noticed that the possible results of a measurement of $P(a_n)$ are 1 or 0, corresponding to the result a_n of the measurement of A or to any other result.

Until now, our discussion has been conducted only for operators with a discrete spectrum; however, in the future we shall deal with such observables like position, whose spectrum is generally a continuous one, so an extension of the formalism should be necessary. Since this turns out to be technically nontrivial, but has no relevance in the treatment of the conceptual problems we are interested in, we shall not include it here, remembering that it is thoroughly examined in ref. 24 and 25.

It is useful to consider the case in which H is the direct product [21] of other two Hilbert spaces H_1 and H_2, i.e. $H=H_1\otimes H_2$. If $\hat{A}$ is an operator on H representing an observable, and $\hat{\rho}$ represents a state, equation (2.9) will certainly hold again; an interesting problem is, however, how to extract out of $\hat{\rho}$ the informations characterizing the states associated to, say, H_1: this can be done by introducing the concept of partial trace as in the following

Definition 2.14 *Let $\hat{A}$ be a linear operator on H. The "tracing over H_2" operation* [1]

$$tr_2\hat{A} \equiv \sum_{n_2}\langle n_2|\hat{A}|n_2\rangle \tag{2.15}$$

produces a new operator on H_1.

Proposition 2.15 *Let $\hat{A}_1$ be an operator on H_1 representing an observable A_1, and let $\hat{\rho}$ be a density operator on H. Then, if a_n^1 is an eigenvalue of $\hat{A}_1$, the probability that a measurement of A_1 give the result a_n^1 is*

$$p(a_n^1) = tr_1(\hat{\rho}_1\hat{P}_1(a_n^1)), \tag{2.16}$$

where

$$\hat{P}_1(a_n^1) \equiv \sum_r |a_n^1, r\rangle\langle a_n^1, r|, \tag{2.17}$$

and

$$\hat{\rho}_1 \equiv tr_2\,\hat{\rho}. \tag{2.18}$$

[1] Equation (2.15) can be properly understood thinking to $\hat{A}$ as expanded in its spectral decomposition.

Proof. If $\hat{A}_2$ is an arbitrary operator in H_2, representing an observable A_2, we have

$$p(a_n^1) = \sum_m p(a_n^1, a_m^2) = \sum_m tr(\hat{\rho}\hat{P}(a_n^1, a_m^2)),$$

where $p(a_n^1, a_m^2)$ is the probability that a simultaneous [1] measurement of A_1 and A_2 gives the results a_n^1 and a_m^2, and $\hat{P}(a_n^1, a_m^2)$ is the operator which projects in the subspace of the eigenvectors corresponding to a_n^1 and a_m^2:

$$\begin{aligned} \hat{P}(a_n^1, a_m^2) &= \sum_{rs} |a_n^1, r\rangle\langle a_n^1, r| \otimes |a_m^2, s\rangle\langle a_m^2, s| = \\ &= \hat{P}_1(a_n^1) \otimes \hat{P}_2(a_m^2). \end{aligned} \tag{2.19}$$

Now, using the obvious relation

$$tr = tr_1 \circ tr_2, \tag{2.20}$$

and the cyclic property of the trace, we obtain:

$$\begin{aligned} p(a_n^1) &= \sum_m tr(\hat{\rho}\hat{P}(a_n^1, a_m^2)) = \sum_m tr(\hat{P}(a_n^1, a_m^2)\hat{\rho}) = \\ &= \sum_m tr_1(tr_2(\hat{P}_1(a_n^1) \otimes \hat{P}_2(a_m^2)\hat{\rho})) = \\ &= \sum_m tr_1(\hat{P}_1(a_n^1) tr_2(\hat{1}_1 \otimes \hat{P}_2(a_m^2)\hat{\rho})) = tr_1(\hat{P}_1(a_n^1) tr_2\hat{\rho}) = \\ &= tr_1(\hat{P}_1(a_n^1)\hat{\rho}_1) = tr_1(\hat{\rho}_1\hat{P}_1(a_n^1)). \end{aligned}$$

□

The meaning of Proposition (2.15) is that, if $\hat{\rho}$ describes a state in $H_1 \otimes H_2$, the corresponding state in, say, H_1, can be consistently described by $\hat{\rho}_1$, obtained by tracing $\hat{\rho}$ over H_2. Since $\hat{\rho}_1$ verifies *i)*- *ii)*-*iii)* of Definition (2.1), it can be identified as a statistical operator.

b) Dynamics

The results of the measurements of an observable change as time passes; this can be expressed by saying that the probability $p(a_n)$ is actually a function of time, $p(a_n, t)$. By equation (2.9) it is evident that either $\hat{\rho}$, or $\hat{P}(a_n)$, or both of them must be function of time, too; the dynamical problem in quantum theory is therefore to determine this dependence. The most important general property of time evolution is formulated in the following

[1] A_1 and A_2 can be certainly measured simultaneously, being associated to the operators $\hat{A}_1 \otimes \hat{1}_2$ and $\hat{1}_1 \otimes \hat{A}_2$ of H, which commute.

Axiom 2.16 *Time evolution is represented as*

$$\hat{\rho}(t) = \hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}(t,t_0)^\dagger, \tag{2.21}$$

where $\hat{U}(t,t_0)$ *is a unitary linear operator on* H *such that* $\hat{U}(t,t) = \hat{1}$.

Remark 2.17 *Unitarity of the transformation (2.21) is needed in order to guarantee that properties i)-ii)-iii) of Definition (2.1) are preserved by time evolution, that is, that* $\hat{\rho}(t)$ *is a density operator if* $\hat{\rho}(t_0)$ *is. Linearity is a central requirement in the ordinary quantum theory, which has been tested experimentally to a high degree of precision [22]: however, theoretical investigations about deviations from the linear evolution are currently pursued [23].*

Axiom (2.16) represents time evolution in the Schrödinger picture, where the state is supposed to evolve, but observables are not. Since all what can be measured experimentally is only the statistical frequency of a result, $p(a_n)$, equation (2.9) suggests, as an alternative, to consider the state fixed, allowing observables to evolve: this is formally expressed by

$$\begin{aligned} p(a_n,t) &= tr(\hat{\rho}(t)\hat{P}(a_n,t_0)) = tr(\hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}(t,t_0)^\dagger \hat{P}(a_n,t_0)) = \\ &= tr(\hat{\rho}(t_0)\hat{U}(t,t_0)^\dagger \hat{P}(a_n,t_0)\hat{U}(t,t_0)) = tr(\hat{\rho}(t_0)\hat{P}(a_n,t)), \end{aligned}$$

where

$$\hat{P}(a_n,t) \equiv \hat{U}(t,t_0)^\dagger \hat{P}(a_n,t_0)\hat{U}(t,t_0). \tag{2.22}$$

Being $\hat{P}(a_n,t)$ constructed from the eigenvectors $\{|a_n,r,t\rangle\}$ of the operator associated to the observable A like in (2.7), we see that (2.22) involves a time evolution of these eigenvectors as

$$|a_n,r,t\rangle = \hat{U}(t,t_0)^\dagger |a_n,r,t_0\rangle, \tag{2.23}$$

which, by (2.5), implies a time evolution of $\hat{A}$:

$$\hat{A}(t) = \hat{U}(t,t_0)^\dagger \hat{A}(t_0)\hat{U}(t,t_0). \tag{2.24}$$

The scheme of time evolution expressed in (2.23) and (2.24), where observables and their eigenstates evolve in time, but the state do not, takes the name of Heisenberg picture.

Proposition 2.18 *For a pure case, the Schrödinger evolution (2.21) becomes simply*

$$|t\rangle = \hat{U}(t,t_0)|t_0\rangle. \tag{2.25}$$

There is now clear evidence for a first problem of compatibility between quantum theory and relativity: when dynamics is taken into account, a particular time enters in the description, and this sounds as a highly non-covariant feature of the theory.

There are, apparently, two possible ways to solve this problem. One relies on the idea that the spacetime coordinates must appear all together, on the same footings[1], in the fundamental equations; this has been, historically, the first attempt to formulate a relativistic quantum theory, but it has shown to be essentially unsuccessful [24]. Alternatively, it is possible to think that the time t in the dynamical equations is referred to a particular reference frame, that is, to a congruence [25] of future directed timelike curves with a normalized tangent vector, each of those represents an observer: quantum theory would involve therefore the concept of observer in its very basic formulation. This last approach is the one currently used in studies of quantum mechanics on a fixed background spacetime [7,26], and it has led to remarkable results [27,28,29]; nevertheless, there are still serious problems, both in the formalism [7] and in its interpretation [30,31,32]. We shall return to this topic in Chapter 4 of the present thesis.

Coming back to the general formulation of quantum theory, we expect $\hat{U}(t,t_0)$ to depend on the details of the specific system considered. The unitarity of $\hat{U}(t,t_0)$ allows, however, to write

$$\hat{U}(t,t_0) = \exp\left(i\hat{\Omega}\,(t,t_0)\right), \tag{2.26}$$

with $\hat{\Omega}\,(t,t_0)$ a self-adjoint linear operator on H such that $\hat{\Omega}\,(t,t) = \hat{0}$; moreover, let us give the following, obviously well posed,

Definition 2.19 *The* <u>*hamiltonian*</u> *is the self-adjoint linear operator on* H

$$\hat{H}(t) \equiv -\hbar\frac{d}{dt'}\,\hat{\Omega}\,(t',t)|_{t'=t}\,, \tag{2.27}$$

where $\hbar$ *is Planck's constant.*

Proposition 2.20 *The evolution equations (2.21), (2.24) and (2.25) are solutions of the following differential equations:*

$$i\hbar\frac{d\hat{\rho}\,(t)}{dt} = [\hat{H}(t),\hat{\rho}\,(t)]; \tag{2.28}$$

$$i\hbar\frac{d\hat{A}(t)}{dt} = -[\hat{H}(t),\hat{A}(t)]; \tag{2.29}$$

$$i\hbar\frac{d|t\rangle}{dt} = \hat{H}(t)|t\rangle. \tag{2.30}$$

[1] Apart from obvious differences due to the metric of spacetime.

Proof. Equations (2.26) and (2.27), together with the conditions $\hat{U}(t,t) = \hat{1}$ and $\hat{\Omega}(t,t) = \hat{0}$, allow to write

$$\hat{U}(t+\varepsilon, t) = \exp\left(-\frac{i\varepsilon}{\hbar}\hat{H}(t)\right) = \hat{1} - \frac{i\varepsilon}{\hbar}\hat{H}(t) + \cdots . \tag{2.31}$$

By definition of derivative, and using (2.31), our new equations follow trivially from (2.21), (2.24) and (2.25). □

A general survey of the mathematical formalism of quantum theory cannot go much further, because the next step, that is, the specification of $\hat{H}(t)$, requires the detailed knowledge of the system under study. Our present treatment, however, is sufficient to undertake an extensive discussion about the interpretation problems, which will be examined in the next chapter.

Chapter 3

The Interpretation

a) The Statistical Interpretation

The time has come to establish a correspondence between the formalism developed so far and what is generally understood as physical reality, that is, the properties of the external world which can be explored by objective experiments.

The main interpretative problem to solve concerns the concept of state; since this is formalized, in Definition (2.1), by a density operator, we can ask, more precisely: "what does actually a density operator describe?". Intuitively, the answer would be that it describes the state of a physical system, about which information can be obtained by means of measurements of some quantities (which have been called observables in Chapter 2).

As reasonable as this statement could seem, it nevertheless contains an element which is responsible for several difficulties when a more detailed study is performed: the idea that standard quantum theoretical formalism be suitable to describe a *single* physical system. It is commonly believed this to be possible, provide a deterministic behaviour be replaced by an intrinsically probabilistic (or, more technically, "stochastic") one: this because the fundamental laws of chapter 2 do not establish a unique correspondence between a state and the results of possible measurements on it: (2.9) gives only the probability of an outcome, and differs therefore drastically from the classical laws which are characterized by unique predictions, once the initial conditions (i.e. the state) are completely specified.

The flaw of this argument can be easily discovered by noticing that the predictions of quantum formalism cannot be tested on a single system: equation (2.9) can be checked only if many systems to which it corresponds the same $\hat{\rho}$ are available. The strategy for an experimental control of quantum theoretical results is then to prepare a big number of identical systems all in the same state, which is supposed to be described by a statistical operator $\hat{\rho}$, and to test (2.9) according to the frequentistic interpretation of probability. It becomes then pretty clear that $\hat{\rho}$ fully describes the behaviour of an *ensemble* of similarly prepared systems, but not of an *individual* one; this leads to the obvious conclusion that quantum mechanics is a theory of ensembles: in fact its predictions cannot be tested on a single system, which the formalism is thus inadequate to describe.

The argument against an interpretation in terms of individuals relies therefore on the fact that there are features of a state (i.e. of a density operator $\hat{\rho}$) which cannot at all be detected by any conceivable experiment performed on a single system. A typical example is provided by the phenomenon of interference, which we shall now treat in some detail because of its relevance to the understanding of the concept of classical behaviour.

The most straightforward approach to the analysis of interference effects consists in the study of the difference between a pure case and a mixture; let us suppose, to fix ideas, that a state is described by the vector

$$|\psi\rangle = \sum_n p\,(a_n)^{1/2}|a_n\rangle, \tag{3.1}$$

where $|a_n\rangle$ represent eigenstates of an observable A; from $|\psi\rangle$ it can be constructed the statistical operator

$$\hat{\rho} = |\psi\rangle\langle\psi| = \sum_{nm} p\,(a_n)^{1/2} p\,(a_m)^{1/2}|a_n\rangle\langle a_m|. \tag{3.2}$$

The problem is now how to distinguish between the state described by $\hat{\rho}$ and the one described by

$$\hat{\rho}\,' = \sum_n p\,(a_n)|a_n\rangle\langle a_n|, \tag{3.3}$$

which differs by $\hat{\rho}$ in that it does not contain off-diagonal elements: a simple series of measurements of A is certainly not suitable for this purpose, because

$$tr(\hat{\rho}\hat{P}(a_n)) = tr(\hat{\rho}'\hat{P}(a_n)) = p\,(a_n), \tag{3.4}$$

so that $\hat{\rho}\,'$ reproduces all the features of the pure state (3.1) that can be tested measuring A; however, $\hat{\rho}\,'$ represents a mixture, because

$$\hat{\rho}\,'^2 = \sum_n p\,(a_n)^2|a_n\rangle\langle a_n| \neq \hat{\rho}\,'.$$

Let then B be another observable, with spectrum $\{b_{n'}\}$; now,

$$\begin{aligned} tr(\hat{\rho}\hat{P}(b_{n'})) &= tr(\hat{\rho}\,'\hat{P}(b_{n'})) + \\ &+ \sum_{n\neq m} p\,(a_n)^{1/2} p\,(a_m)^{1/2}\langle a_n|\hat{P}(b_{n'})|a_m\rangle, \end{aligned} \tag{3.5}$$

and, if $\langle a_n|\hat{P}(b_{n'})|a_m\rangle \neq 0$ for some m, n, n', it is possible to discriminate between (3.2) and (3.3) with measurements of B, the difference arising just from the off-diagonal terms in (3.2): since these involve different eigenstates of A, they are said to exhibit interference effects. While in the state represented by (3.3) we have

$$\begin{aligned} tr(\hat{\rho}\,'\hat{P}(b_{n'})) &= \sum_n p(a_n)\langle a_n|\hat{P}(b_{n'})|a_n\rangle = \sum_n |\langle b_{n'}|a_n\rangle|^2 p(a_n) = \\ &= \sum_n p(b_{n'}, a_n)p(a_n), \end{aligned} \tag{3.6}$$

which can be realized supposing that, during each measurement of B, the system is in a well defined state $|a_n\rangle$ with probability $p(a_n)$, this is not conceivable for the state (3.2), where different eigenstates $|a_n\rangle$ are considered at the same time.

In the statistical interpretation [18,19] of quantum theory, therefore, (3.2) and (3.3) describe ensembles of differently prepared systems; more precisely, (3.3) describes an ensemble E$'$ of systems, a fraction $p(a_n)$ of which has been prepared according to the state $|a_n\rangle$[1], but this partition cannot be performed at all for (3.2), which describes an ensemble E of systems _all_ prepared according to $|\psi\rangle$. In other words, a mixture is an ensemble admitting subensembles which are pure cases; this can be expressed, schematically, as in Figure (3.1) and Figure (3.2).

It is useful to notice that the statistical character of the outcomes of measurements has two essentially different origins; in the situation described by $\hat{\rho}\,'$, there is nothing intrinsically quantum in the probabilistic distribution, which derives simply by the fact that the measurements are performed on systems which have been prepared in different ways. In the case of (3.2), however, this is not true, and the uncertainty in the results is quantum in character: interference represents therefore the way to discriminate between quantum and classical behaviours, an issue to which we shall soon return.

We want here to stress on the fact that in the statistical interpretation it is meaningless to speak about the state of a system, this concept being referred only to ensembles: such expressions like "the system is in a superposition state" lose therefore any significance, and the idea of superposition means only that the ensemble cannot be split in subensembles of the type occurring in E$'$ of Figure (3.2). This appears then a very reasonable and economic solution to one aspect of the state vector reduction problem[2], which we shall now treat in some detail.

[1] For each n.

[2] Particular cases of which are the famous "measurement problem" and "Schrödinger's cat paradox" [33].

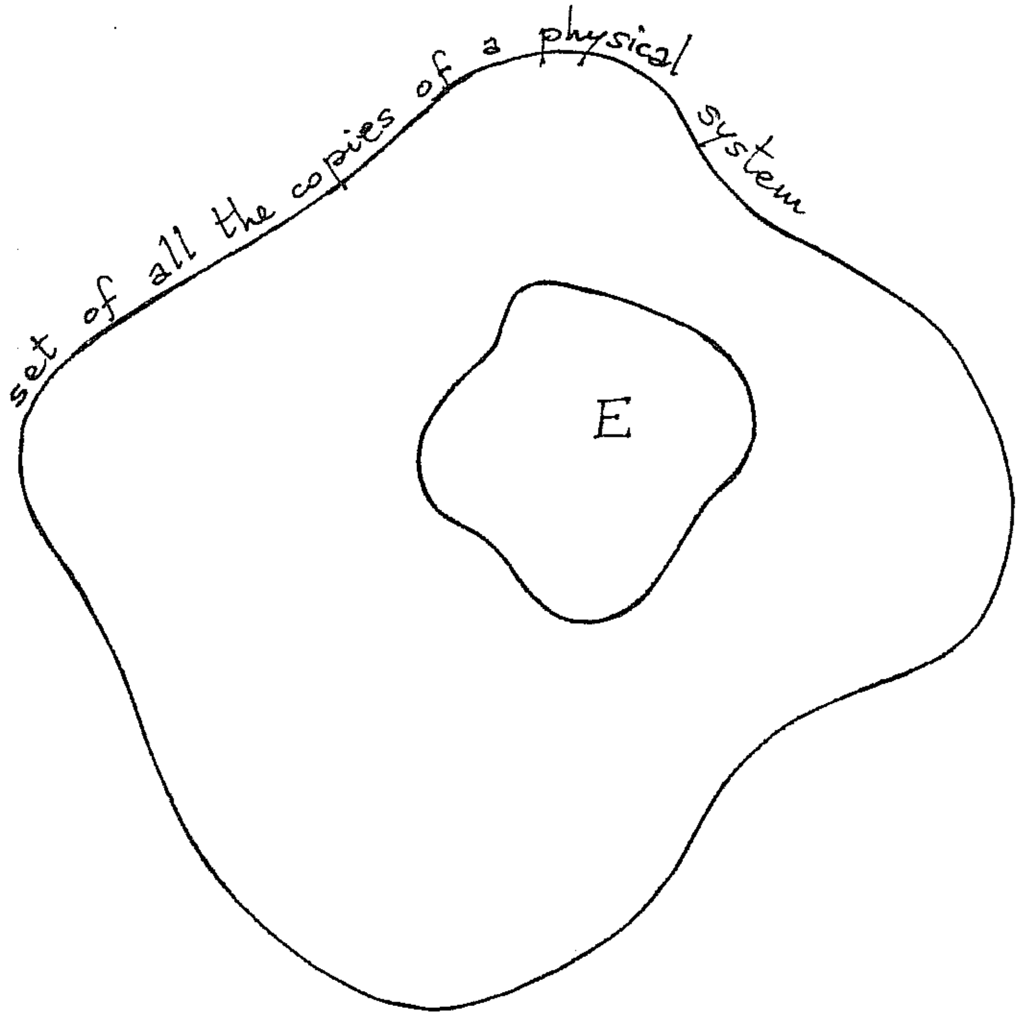

Figure 3.1: **Pure case :** $|\psi\rangle$ describes the ensemble E of *identically prepared* systems.

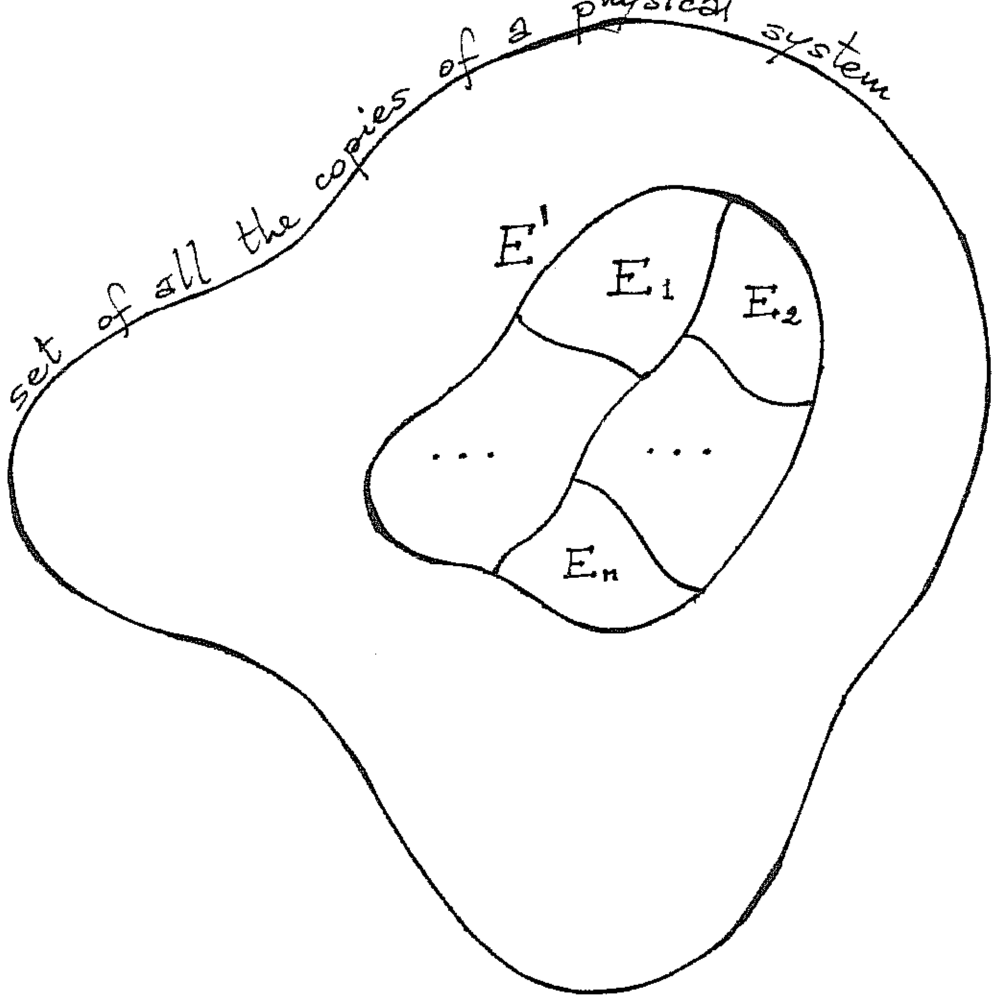

Figure 3.2: Mixture : $\hat{\rho}'$ describes the ensemble E$'$ of *nonidentically prepared* systems, a fraction $p(a_n)$ of which are in the subensemble E_n described by $|a_n\rangle$.

Let us consider a system composed of a microobject m and a macroscopic one (e.g. a measuring device) M; moreover, let M be coupled to m in such a way that an observable A_M of M is triggered by an observable A_m of m as follows: if $|\psi\rangle$ is an eigenvector of the operator $\hat{1}_M \otimes \hat{A}_m$ corresponding to the eigenvalue $a^{(m)}$, it is also an eigenvector of $\hat{A}_M \otimes \hat{1}_m$ with eigenvalue $a^{(M)}$ (if M has to act as a measuring device of A_m on m, for example, part of the spectra of A_m and A_M must be putted in a one-to-one correspondence, in order that the knowledge of $a^{(M)}$ (e.g. the position of a pointer) could be uniquely related to the value of $a^{(m)}$, thus obtaining a realization of the intuitive notion of measurement of A_m by M).

This kind of situation creates the following problem: if we start with an initial state

$$|t_0\rangle = |a_0^{(M)}, a_0^{(m)}\rangle, \tag{3.7}$$

where $|a^{(M)}, a^{(m)}\rangle \in \mathrm{H}_M \otimes \mathrm{H}_m$ are common eigenvectors of $\hat{1}_M \otimes \hat{A}_m$ and $\hat{A}_M \otimes \hat{1}_m$ corresponding, respectively, to the eigenvalues $a^{(m)}$ and $a^{(M)}$, and we leave it to evolve in time assuming that quantum theory, supposed to be universal and complete, holds both for m and M, we get at time t the Schrödinger evolved state

$$|t\rangle = \sum_a c_a(t) |a^{(M)}, a^{(m)}\rangle. \tag{3.8}$$

The tricky point is that (3.8) represents again a pure state, so it displays both superposition and interference also in H_M, as can be easily seen from the associated density operator

$$\begin{aligned} \hat{\rho}(t) = |t\rangle\langle t| &= \sum_a |c_a(t)|^2 \, |a^{(M)}, a^{(m)}\rangle\langle a^{(M)}, a^{(m)}| + \\ &+ \sum_{a \neq a'} c_a(t)\, c_{a'}(t)^* \, |a^{(M)}, a^{(m)}\rangle\langle a'^{(M)}, a'^{(m)}|; \end{aligned} \tag{3.9}$$

but it is well known to be characteristic of macroscopic objects (and in particular of measuring devices and cats!) that they have well defined values of their observables, and exhibit no interference phenomena at all, so that the statistical operator at time t would rather be expected to be

$$\hat{\rho}'(t) = \sum_a |c_a(t)|^2 \, |a^{(M)}, a^{(m)}\rangle\langle a^{(M)}, a^{(m)}|, \tag{3.10}$$

which corresponds to a mixture and cannot therefore result from (3.7) under Schrödinger evolution, that preserves the condition of equation (2.3). In order to get rid of this paradox, it has been suggested [20] to replace equation (2.21) with the "projection postulate" (or "state vector reduction")

$$\hat{\rho}(t) \longrightarrow \hat{\rho}'(t)$$

whenever a "measurement" is performed; this proposal, however, raises more problems than it solves: in fact, which criterion has to be applied in order to decide that a physical process is a measurement? and, what happens to a system which is continuously observed [33,34]?

It is important to remark that there are two aspects of this paradox, both of interpretative nature, though the second is more technical:

(1) how can a macroscopic object, whose observables have always well defined values, be in a superposition state?

(2) why is interference never observed in macroscopic systems?

For what concerns (1), as for the problems related to the state vector reduction, it must be said that the incongruence is present only if a state vector is supposed to describe a single system: in the statistical interpretation, there is no meaning to the statement that "an individual is in a superposition state", the concept of superposition being applicable only to ensembles.

In trying to solve (2), we need again to rely on the statistical interpretation, but the arguments involved are a little more subtle. First of all, let us simplify the problem by considering only a macroscopic system M, to which a Hilbert space H is associated; in order to observe the typically quantum phenomenon of interference, we need to prepare <u>*in the same way*</u> an ensemble E of copies of M, which is described by the vector of H

$$|\psi\rangle = \sum_n c_n |n\rangle = \sum_n e^{i\alpha_n} |c_n||n\rangle, \tag{3.11}$$

where $\{|n\rangle\}$ is an orthonormal basis of H. By its very definition, however, a macrosystem has a huge number of degrees of freedom, so that the preparation of many of its copies in the same way, leads to actually different ensembles, and a practically realizable experiment requires therefore to treat with a mixture rather than with a pure state. To formulate better this concept, let us suppose that M is constituted of N elementary (i.e. with very few degrees of freedom) subsystems S_i, so that $\mathrm{H}=\otimes_{i=1}^{N} \mathrm{H}_i$. A member of an orthonormal basis in H will then be

$$|n\rangle \equiv |n_1\rangle_1 \otimes \cdots \otimes |n_N\rangle_N \equiv \bigotimes_{i=1}^{N} |n_i\rangle_i, \tag{3.12}$$

and a state vector of H will have the form (3.11), where each coefficient $c_n \in \mathrm{C}$ is associated to a well precise set of vectors $\{|n_1\rangle_1, ..., |n_N\rangle_N\}$, the index n standing for the set $\{n_1, ..., n_N\}$. Our idea is now that the "coarse-graining inability" in preparing systems with a large number of degrees of freedom makes the phases α_n uncontrollable [35]; more precisely, we shall show that if the states described by the vectors

$$\sum_n e^{i\alpha_n} |c_n| |n\rangle \tag{3.13}$$

and

$$\sum_n e^{i\alpha'_n} |c_n| |n\rangle \tag{3.14}$$

describe ensembles of M obtained as results of the same macroscopic preparation process, then the experiments must deal with the mixture described by

$$\sum_n |c_n|^2 |n\rangle\langle n|. \tag{3.15}$$

In other words, the origin of the classical behaviour (i.e. of the lack of interference) in complex systems has to be identified in the "thermodynamical" difficulty of preparing them in the same microscopic way.

To show this, let E be an ensemble of M such that $E=\bigcup_\mu E_\mu$, where E_μ are pure cases composed of a fraction p_μ of copies of M in E, with $p_\mu = p_{\mu'} \ll 1$[1], described by

$$|\mu\rangle = \sum_n e^{i\alpha_n^{(\mu)}} |c_n||n\rangle : \tag{3.16}$$

physically, each E_μ represents an ensemble for a microstate (in the thermodynamical, not quantum, sense!), and the coarse-graining of the preparation process does not allow to decide to produce an element of E_μ rather than of $E_{\mu'}$ (this is why $p_\mu \mp p_{\mu'}$). E will then be a mixture described by the density operator

$$\hat{\rho} = \sum_\mu p_\mu |\mu\rangle\langle\mu|; \tag{3.17}$$

but, by (3.16),

$$|\mu\rangle\langle\mu| = \sum_{nm} e^{i(\alpha_n^{(\mu)} - \alpha_m^{(\mu)})} |c_n||c_m| |n\rangle\langle m|,$$

so

$$\hat{\rho} = \sum_{nm} |c_n||c_m| |n\rangle\langle m| \sum_\mu p_\mu e^{i(\alpha_n^{(\mu)} - \alpha_m^{(\mu)})}. \tag{3.18}$$

Now, for randomly distributed phases $\alpha_n^{(\mu)}$, and remembering the condition $p_\mu \ll 1$,

$$\sum_\mu p_\mu e^{i(\alpha_n^{(\mu)} - \alpha_m^{(\mu)})} \approx \delta_{nm} , \tag{3.19}$$

[1] All the microstates are equally probable, and there are many macroscopically indistinguishable microstates.

which, substituted into (3.18), gives

$$\hat{\rho} \approx \sum_n |c_n|^2 \, |n\rangle\langle n| \,, \tag{3.20}$$

which is the result we claimed before.

It must be pointed out that the statistical interpretation is central to this solution of problem (2): in fact, our result is that, because of coarse-graining reasons, every practically realizable ensemble for a macrosystem will be described by (3.20) and not by (3.13); the flaw in the argument about (3.9) and (3.10) was therefore the hypothesis underlying (3.7), that it is possible to assume a pure state to describe the behaviour of an ensemble for mUM: even if conceivable in principle, such a situation is practically unrealisable.

b) Alternative Theories

Standard quantum theory, as we have presented it, does not deal with predictions about single systems, but only about ensembles; this restriction, even if it is not relevant for what concerns laboratory experiments of microphysics, becomes highly unsatisfactory when the theory, supposed to be universal, is applied to bigger and bigger systems, whose ensembles are more and more difficult to prepare: a complete failure of its predictive power is reached if we pretend to apply it to the biggest existing system - the universe - only one copy of which is available, making thus impossible any test of frequentistic character. If we do not want to give up the task of treating the universe in terms of fundamental physics, we need therefore to change quantum theory, in its interpretation or, even more deeply, in its mathematical formalism.

An alternative theory which retains all the formalism developed in Chapter 2 is the one based on the *relative state*[1] interpretation [18,36,37,38], which we shall now sketch very briefly, since it is extensively discussed in the literature. The main hypothesis is that there is a one-to-one correspondence between quantum formalism and physical reality, in the sense that a state vector fully describes an individual. Let us now consider the system mUM treated at page 23, whose state vector evolves from $|t_0\rangle$ of (3.7) to $|t\rangle$ in (3.8): if we combine the experimental fact that macrosystems always have well defined values of their observables (so that, in the case of (3.8), the compound system mUM can only be found in one of the states $|a^{(M)}, a^{(m)}\rangle$) with the previously stated hypothesis (which implies

[1] Or "many-worlds".

that all the states $|a^{(M)}, a^{(m)}\rangle$ present in (3.8) must have a counterpart in the real world), we are led to the conclusion that the Schrödinger evolution from (3.7) to (3.8) involves a splitting of the initial system $m \cup M$ in several mutually noninteracting copies of it, each one corresponding to a well defined state $|a^{(M)}, a^{(m)}\rangle$. The most attractive features of this theory are certainly its capability to treat with the entire universe and an economy in the axioms: it can be proved [18,36], in fact, that the probabilistic interpretation of the coefficients c_a in (3.8) is a consequence of the theory, without requiring an axiom like (2.9). It seems therefore that the relative state interpretation, providing by itself, through the process of splitting, a representation of the ensembles of systems which are used in the statistical interpretation, constitutes an extension of this latter, reducing to it in the limit of laboratory situations. Since, in this thesis, we shall not deal with quantum cosmology, we shall continue to rely on the statistical interpretation, whose pragmatic nature allows to avoid some tricky problems involved with the many-worlds idea [18].

An interpretation of quantum theory which makes reference to individuals, necessarily has to deal with the riddle of state vector reduction; in order to apply the theory to single systems, there have been suggestions [11,39] about some changes to bring to the formalism. The starting point is to introduce nonlinearity in the evolution equation, in such a way as to produce the collapse of the state vector; in ref. [11,12] this has been done by adjusting the nonlinear term so that, for simple systems, the reduction effects are negligible (as they should be, on the basis of experience), while increasing the number of degrees of freedom they become more and more important, until they lead to a complete "measurement collapse" when a macroscopic system is involved. Other models of dynamical reduction have been investigated [39,40], and it has also been suggested [41,42] that these violations of quantum mechanics have their origin in the coupling with gravity; all these theories share the feature of considering the state vector as a real entity, and not only as a mathematical tool.

Once the statistical character of standard quantum theory has been recognized, the most straightforward generalization toward a theory of individuals is probably the hidden variables approach. In fact, if the outcomes of experiments on equally prepared systems are predictable only in their statistical distribution, it is natural to think that the behaviour of individuals is determined by some "hidden" variables, whose values are not controllable during the preparation process. An interesting example of such theories is Bohm's [13,14], in which a particle is supposed to have, at each time, well defined values of position and momentum, and to obey classical laws of motion, but under the action of an additional "quantum potential"

$$Q(\mathbf{x},t) \equiv -\frac{\hbar^2}{2m}\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}} , \tag{3.21}$$

where m is the mass of the particle and $\rho(\mathbf{x},t)$ satisfies the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 , \tag{3.22}$$

with $\mathbf{v}$ the particle's velocity. In order that the theory be able to reproduce the results of quantum mechanics, it is easy to see that $\rho(\mathbf{x},t)$ must be interpreted as density of probability that the particle be found at time t in the point $\mathbf{x}$: $\rho(\mathbf{x},t)$ is therefore a quantity which refers to an ensemble of particles, whose position is not exactly known because of its dependence from hidden parameters[1]; but then it is hard to understand how such a function, expressing the behaviour of the ensemble, can enter in the determination of the motion of a single particle through (3.21). This objection [21] is certainly one of the most serious to the viability of Bohm's theory, and it is still under debate [43]. Other problems which any theory based on hidden variables must face with, are those related to the nonlocal features of quantum mechanics; however, since these aspects are treated extensively in the literature [14,17,18,44], and they do not play a crucial role in our next discussion, we shall not concern with them in this thesis.

[1]The position of the particle at an initial time t_0.

Chapter 4

Quantum Theory and Relativity

As we have already pointed out after equation (2.25), the formalism of quantum theory is not manifestly covariant, since it singles out a particular time: the Hilbert space of states and the algebra of observables are specified at one instant of time, and the dynamical equations make these objects to evolve along all the history of the system under study. It has been suggested, in order to reconcile this state of affairs with a relativistic treatment, that quantum theory requires the concept of observer to be introduced from the beginning, and that t in (2.21) is nothing but the proper time of this observer; we now want to investigate this idea in more detail.

First of all, we must ask for the class of spacetimes which allow the introduction of a global notion of time; the answer is straightforwardly given in the literature [1,2], and indicates that we need to restrict ourselves to a globally hyperbolic spacetime (M, g), which admits Cauchy hypersurfaces. Let therefore Σ be such an hypersurface, with future directed normal vector $\mathbf{u}$ such that

$$u_a u^a = -1 \; ; \tag{4.1}$$

let us choose a congruence of observers γ whose tangent vector on Σ coincides with $\mathbf{u}$ (Figure (4.1)), and such that the reference frame so constructed is proper time synchronizable [45] (the existence of such frames follows from the global hyperbolicity of (M, g), which guarantees the existence of a global function τ such that $u^a = -\nabla^a \tau$ is a future directed timelike vector field). Let us choose, moreover, the origin of the proper time τ of these observers so that $\gamma(\tau_0) \in \Sigma$, $\forall \gamma$; at a time $\tau \neq \tau_0$, Σ will be evolved, along the γ curves, to another Cauchy hypersurface $\Sigma' = \{\gamma(\tau)\}$, orthogonal to the observers and representing thus their transverse space.

Now it is possible to construct a quantum theory referred to the

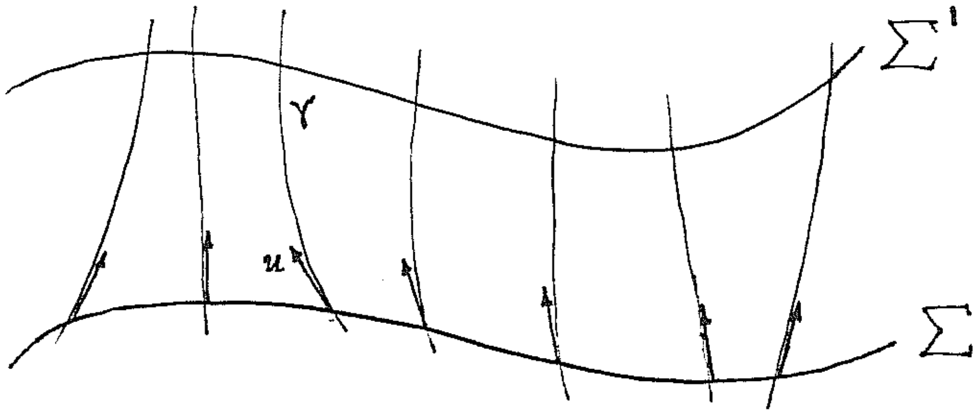

Figure 4.1:

observers γ, where the states are related to systems on one hypersurface of the foliation $\{\Sigma\}$, and the dynamics is described by a unitary operator $\hat{U}_u(\tau, \tau_0)$ such that $\hat{U}_u(\tau, \tau) = \hat{1}$ and

$$\frac{d\hat{U}_u(\tau, \tau_0)}{d\tau} = -\frac{i}{\hbar}\hat{H}_u(\tau)\hat{U}_u(\tau, \tau_0), \tag{4.2}$$

according to (2.26) and (2.27), where $\hat{H}_u(\tau)$ is the hamiltonian of the system on the hypersurface Σ'. It is easy to verify that the analogous of equations (2.28), (2.30) and (2.29) are, respectively:

$$i\hbar\frac{d\hat{\rho}_u(\tau)}{d\tau} = [\hat{H}_u(\tau), \hat{\rho}_u(\tau)]\ ; \tag{4.3}$$

$$i\hbar\frac{d|\tau;u\rangle}{d\tau} = \hat{H}_u(\tau)|\tau;u\rangle\ ; \tag{4.4}$$

$$i\hbar\frac{d\hat{A}_u(\tau)}{d\tau} = -[\hat{H}_u(\tau), \hat{A}_u(\tau)]\ ; \tag{4.5}$$

where the first two equations hold in the Schrödinger picture (hence with $\hat{A}_u(\tau) = \hat{A}_u(\tau_0)$) while the third holds in the Heisenberg picture (with $\hat{\rho}_u(\tau) = \hat{\rho}_u(\tau_0)$ or $|\tau;u\rangle = |\tau_0;u\rangle$).

A particular case occurs when the system under study is a quantum field, and the observables A are therefore the intensity of the field components, satisfying a system of field equations (for example, Klein-Gordon equation for a scalar field, or Maxwell equations for the electromagnetic field); the operators associated to A will then be functions of the spacetime point $x \in M$, and the derivative $d/d\tau$ will be replaced by the derivative along $\mathbf{u}$, L_u. If, in the Schrödinger picture, $L_u \hat{A}^S_u(x) = \hat{0}$ (that is, if A does not depend explicitly on the $\mathbf{u}$-time), then (4.5) becomes

$$i\hbar\, L_u \hat{A}^H_u(x) = -[\hat{H}_u(\tau(x)), \hat{A}^H_u(x)] \ , \tag{4.6}$$

where $\hat{A}^H_u(x)$ is the field in Heisenberg picture:

$$\hat{A}^H_u(x) = \hat{U}(\tau(x), \tau_0)^\dagger \hat{A}^S_u(x) \hat{U}(\tau(x), \tau_0). \tag{4.7}$$

This method of extending the formalism of Chapter 2 to the case of a more general reference frame in a curved spacetime is quite straightforward and naive, but has led to important theoretical results [7,27,28,29]; nevertheless, it appears somewhat artificial and not sufficiently general, since it relies on some hypothesis, such as the global hyperbolicity, which are not necessarily satisfied in a generic spacetime (and, moreover, whose validity is very difficult to test in our own universe!). For this and other reasons, it has been recently suggested by Hartle [46], to consider the Hilbert space formulation of quantum theory only as a particular case of the more general sum-over-histories formulation, which is an extension of Feynman's celebrated path integral formalism [47]. In this scheme, the fundamental element is no more the intuitive concept of state, but rather that of *history* H of a system; the theory asks therefore directly for probabilities of measurement's results, without involving all the machinery of Hilbert spaces and so on.

To be more explicit, we need to distinguish between *conditions* C, which are the observables fixed by the experimental design, and *observations* O, which are the possible results of the experiments; the all theory relies then on the following

Axiom 4.1 *The probability that it will be observed O under conditions C is*

$$p(O|C) = \frac{p(O,C)}{\sum_{O'} p(O',C)} \ , \tag{4.8}$$

where

$$p(O,C) = |\Phi(O,C)|^2 \ , \tag{4.9}$$

and

$$\Phi(O,C) = \sum_{\mathsf{H}} \Phi[\mathsf{H}] \ , \tag{4.10}$$

with

$$\Phi[\mathsf{H}] = \exp\left(\frac{i}{\hbar} S[\mathsf{H}]\right) \ ; \tag{4.11}$$

in (4.11) S is the action functional for the system, and the sum in (4.10) is over all the histories H compatible with the prescribed O and C.

What is remarkable in this formulation, is that it reproduces all the results of the theory developed in Chapter 2, without giving any special role to the concept of time; this makes it particularly suitable for the analysis of quantum mechanical effects in a relativistic context. Moreover, it has been proved [46] that the sum-over-histories approach allows a Schrödinger-Heisenberg formulation on a hypersurface if this latter satisfies some conditions: since this is, in general, not guaranteed, we are led to conclude that the sum-over-histories method could well be applied in situations in which the one based on equations (4.4) and (4.5) cannot, thus preserving the predictability of quantum theory. It should be remembered, however, that these conclusions, having been tested only on a rather limited variety of cases, must still be considered quite speculative: we can therefore conclude that, at the present time, a satisfactory, covariant formulation of quantum theory does not exist.

Part II

Quantum Matter as Source of Gravity

The very fundamental structure of every geometrical theory of gravity can be epitomized in the relation

$$\text{geometry} \longleftrightarrow \text{matter}.$$

In Einstein's theory [1,2], the left hand side corresponds to the well known tensor[1]

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R \, ,$$

where R_{ab} is the Ricci curvature tensor and $R \equiv g^{ab} R_{ab}$ is the scalar curvature; the properties of classical matter are represented, in the right hand side, by the stress-energy-momentum tensor T_{ab}. The problem of writing the field equations (or their equivalent) when quantum behaviour is allowed is, as we have already mentioned in the Introduction, an extraordinarily difficult one, and it is still open.

In this part, we shall critically analize the problems which arise when the quantum behaviour is introduced in an Einstein-like theory of gravity: this is different from what it has been done in Chapter 4, because now it is the dynamical part of gravity -the field equations- that is examined; of course, the difficulties already met, related to a covariant formulation of quantum theory, will combine (certainly in a nonlinear way!) to the new one.

Chapter 5 deals mainly with a critical review of the most pregnant physical arguments which have been advanced in favour of the quantization of the gravitational field: our conclusion is that they are mainly inconclusive, being based on theoretical prejudices rather than on experimental evidence. Therefore, we support the less ambitious semiclassical program, where the spacetime geometry is treated classically, while matter is quantum: even in a fully quantum theory of gravity, in fact, it is possible to envisage situations in which this is a fairly good approximation. In this context, the main problem one must deal with is, essentially, the one concerning the choice of the right hand side in the field equations. The physical structure of the generally accepted source term (see equations (1.1)) is investigated in Chapter 6, while Chapter 7 is devoted to a detailed discussion of the conceptual consistence of the resulting theory. The rather disappointing result we obtain is that a semiclassical theory requires, for its formulation, such foundational changes in quantum mechanics, that it is more convenient to face the problem of quantization of the field; on these grounds, a weakly semiclassical approach is suggested, showing how it allows, for linear situations, to recover the expected field equations without running into inconsistences. Further improvements of this idea are discussed, finally, in Chapter 8.

[1] We do not care, here, about the possible presence of a cosmological constant, which is of no relevance to our later discussion.

Chapter 5

Is It Necessary to Quantize Gravity?

Classical general relativity [1,2], as discussed in the Introduction, replaces the concept of gravitational field by that of riemannian curvature in the spacetime manifold, described by a tensor R^d_{abc} whose contractions $R_{ab} \equiv R^c_{acb}$ and $R \equiv g^{ab} R_{ab}$ satisfy the Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab} \ , \tag{5.1}$$

where $\kappa \equiv 8\pi G/c^4$ is a constant of nature and the stress-energy-momentum tensor T_{ab} can be obtained from a classical action S_m for matter by functional derivation with respect to the metric g^{ab}:

$$T_{ab} = -\frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{ab}} \ . \tag{5.2}$$

The great amount of efforts devoted to the construction of a quantum version of this theory, has led to only few remarkable results [3], and the lack of a general, reliable theoretical framework, induces to think that probably it is wrong to try to extrapolate our current methods of investigation to such an exotic field, and that a critical analysis of the foundations of modern physics must precede any further theoretical study [6]. For these reasons, we shall not take as granted the need to create a quantum theory of gravity, but rather we shall try to review the motivations for its construction, trying to maintain an objective attitude, in which only the experimental results are considered unquestionable.

Let us first ask for the motivations of essentially theoretical nature: the most serious of all is, in our opinion, the use of quantum gravity as a remedy to cure the disease of other field theories [48,49]. It is well known, in fact, that quantum theories of matter fields lead to ultraviolet divergences when treated in a classical background spacetime; now, it happens that, in a quantum theory of gravity, the one-loop contributions from gravitons would be comparable to the vacuum polarization effects of matter, leaving thus the possibility for a mutual cancellation.

There are other, more vague arguments in favour of quantum gravity; one of the most remarkable results of the study of quantum theory in curved spacetime is Hawking's discovery [27] that to a black hole can be associated a physically meaningful temperature and entropy, making therefore possible a well posed study of black holes' thermodynamics. These quantities are intrinsically quantum in their definition, as it is evident from their expressions for, e.g., a Schwarzschild black hole of mass M:

$$T = \frac{\hbar c^3}{8\pi k G}\frac{1}{M} , \tag{5.3}$$

$$S = \frac{4\pi k G}{c\hbar} M^2 , \tag{5.4}$$

where k is Boltzmann constant. The appearance of Planck constant $\hbar$ in (5.3) and (5.4) emphasizes their quantum origin, which has been clarified by Bekenstein [50] on the ground of the equality between entropy increase and information loss, occurring during the collapse process of a matter configuration to a black hole. The key of his argument is that, accordingly to relativistic quantum theories, a particle cannot be localized more precisely than within its Compton length $\lambda \equiv \hbar/mc$ [24]: in order to fit into the hole during collapse, therefore, λ must be smaller than the Schwarzschild radius, and this imposes a lower limit on the mass m of the particle; consequently, the number of possible configurations that can give origin to a black hole of given mass M, is limited by quantum nature of matter, which allows thus to speak meaningfully of the quantities (5.3) and (5.4).

The thermodynamics of black holes has proved to be so rich, simple and physically interesting [51,52], that it is natural to ask about the possibility that it has a more fundamental meaning, in which the entropy (5.4) corresponds to "internal microstates" of the hole. It has also been suggested [53] that the transformation of pure states into mixtures associated to the process of black hole radiation, provides evidence for the crucial role of gravity in the puzzle of state vector reduction. Personally, we believe these arguments, though attractive, too vague and cloudy to compel us to undertake a task like that of the construction of a quantum theory of gravity.

Another context where quantum gravity is often invoked as a cure, is that of spacetime singularities, which we have quickly described in the Introduction: however, even here the attitude is to use handwaving arguments, based on a theory which does not yet exist, to guess what such a theory could do in order to remove pathologies of the existing models. Until now, these studies have given a negligible contribution both to the understanding of singularities and to the formulation of quantum gravity: therefore, we cannot consider them differently from what they are, that is, only intuitive guesses.

We remain, thus, only with the argument related to the role of gravity in the cancellation of divergences occurring in field theories; this seems a more strong and rigorous reason to accept the quantization of the gravitational field, but it is more fair to admit that, on experimental grounds, it looks very speculative, too. The formalism of quantum field theories, in fact, is quite messy and inelegant; the complexity of the mathematical machinery employed in the derivation of experimentally testable results, should make us open to the possibility that another formalism could exist, which leads to nearly the same numerical conclusions, without running into the technical problems of cancellation of infinities. That such alternative approaches are possible, in the sense that the currently available experimental data do not uniquely fix the theory, can be understood thinking to the case of electrodynamics, where the amount of data is extremely high, with respect to the situation for other interactions, because of the possibility to perform significant experiments at low energies. It is astonishing that most of these results can be understood and theoretically predicted by a very simple semiclassical model [54], and that a semiclassical nonlinear theory can reproduce all the observed details, like those related to the anomalous magnetic moment of the electron [55], which are commonly believed to provide unquestionable evidence for the quantization of the electromagnetic field. These examples make us very suspicious about the validity of the previously advanced argument for quantization of gravity, and, in general, of all the arguments based on the internal consistency of theories which are, experimentally, very ill-founded.

On the other side, there are no experiments which could be used as evidence, even indirect, in favour or against quantum gravity; however, if the problem has to be tackled (and it *should* be, being a fundamental one), it is necessary to rely on some leading principle. The lack of experimental data suggests therefore to try to involve the very fundamental principles of physics in the discussion, through the use of some *gedanken* experiments. There is a statement by Unruh [56] which we believe worth to be quoted here: "[Gedanken experiments] serve not to test nature but rather to present the *a priori* prejudices of the theorist in their simplest physical guise. They highlight the beliefs and prejudices the theorist has about the physical world - beliefs which could well be proven wrong by true experiments, but which seem necessary to limit the infinite range of possible theories in the absence of experiment."

Unruh itself suggests a gedanken experiment to support the view that the gravitational field should be quantized: the experimental setup consists of a neutron star of mass oscillating in its fundamental quadrupole mode, with consequent emission of gravitational radiation; this would damp both amplitude Q and momentum P of the vibration in a time of the order of a second, with consequent decay to zero of the commutator $[\hat{Q}, \hat{P}]$. Such a conclusion seems to be in contrast with the principles of quantum theory, and the natural conclusion of the argument would be therefore that gravity should be quantized, in order to provide an additional force which, taking into account vacuum fluctuations of gravity itself, restores $[\hat{Q}, \hat{P}]$ to the value $i\hbar \neq 0$. This argument, which could be also used to prove that the electromagnetic field must be quantized, seems, at first sight, very convincing; however, a deeper analysis shows that it could well be inexact, so that it is not so compelling as it looks, after all.

In fact, it is reasonable to accept that, if we allow the existence of a physical system which is not quantum and which can interact with other, quantum, systems, then quantum theory has certainly to suffer of some changes [57]; therefore, it should not be a surprise that $[\hat{Q}, \hat{P}]$ is found to be damped to zero by such an interaction: the only thing we must be sure about, is that such a damping cannot be observed in laboratory systems, which are the only one over which such experiments have been performed. The results of the gedanken experiment cannot be used as a proof of the need for the gravitational field to be quantized, since such an experiment has never been carried on for a neutron star: they only prove that a classical field is not compatible with a theory in which $[\hat{Q}, \hat{P}] = i\hbar$ forever. It is easy to check, with a rough calculation, that for the electron in an hydrogen atom, the gravitational damping time is of the order of 10^{39} seconds, which is about 10^{22} times the life of the universe accordingly to standard cosmology! There is no doubt that such small deviations from quantum theory would be practically impossible to detect.

Another interesting attempt to establish the quantum nature of gravity by means of a gedanken experiment, is due to Eppley and Hannah [58]. They consider the scattering of classical gravitational wave packets by a quantum particle prepared in a state with spatial localization Δx; such scattering can take place in two, mutually exclusive, possible ways. Either it produces a wave function collapse, and it can be considered as a position measurement (scattering by a pointlike object), or it does not produce any collapse at all (scattering by an extended object). In the first case, it is easy to realize that, using gravitational wave packets of sufficiently small width and little amplitude (there is no lower limit to these quantities, because gravity is supposed to behave classically), and starting with a particle with well defined momentum (i.e. with a great value of Δx), we are allowed, detecting the position of classical waves after scattering, to infer both the values of position and momentum for the particle, thus violating the uncertainty principle. If, in the other case, scattering does not collapse the state vector, then it provides a way for observing the wave function without reduction: it is shown in [58] that this leads to a violation of causality, in the sense that signals can be transmitted at a speed larger than c. Eppley and Hannah conclude therefore that the assumption that gravity is classical violates very fundamental principles of physics, and that semiclassical theories must be rejected.

In our opinion, this thought experiment proves nothing else than the inconsistency of the interpretations of quantum theory which suppose a state vector to describe a single system. In the statistical interpretation, as we know from Chapter 3, the wave function has no meaning for a single particle, and there is thus no question about the occurrence of collapse; moreover, it must be pointed out that the uncertainty relation $\Delta x \Delta p_x \geq \hbar/2$ is also meaningful only for measurements performed on an ensemble. Therefore it is possible to conclude, remembering the quotation from Unruh, that the thought experiment due to Eppley and Hannah only displays their prejudices in favour of one interpretation of quantum mechanics. Let us remind, as a side remark, that the statistical interpretation is probably the most pragmatic and moderate of all, since it does not introduce in the theory any arbitrary or metaphysical element; moreover, it solves easily problems which, in other interpretations, lead to insurmountable paradoxes.

We believe this situation not to be limited to the few examples discussed here, but to be much more general: it is very difficult to devise a gedanken experiment which, involving only features of the present theories which have already been tested experimentally, could prove in a convincing way something about a field so far from standard physics as this is. We prefer therefore another line of attack, trying to analyze explicitly theories which could be considered as alternative to quantum gravity; the rest of our thesis is devoted to the study of one of them.

Chapter 6

The Semiclassical Einstein Equations

As we have discussed in Chapter 5, there is no evidence, neither of experimental nor of theoretical nature, for the quantization of gravity: all the thought experiments so far devised in order to show that the presence of a classical gravitational field would lead to some changes in quantum theory, cannot be used as arguments in favour of quantum gravity, since the deviations they predict are so difficult to measure that they could well be present, but have never been detected until now.

This "conservative" attitude in favour of the construction of a theory where gravity is not quantized receives further encouragement when, more pragmatically, the order of magnitude of the scale at which quantum gravity effects should become important is computed: the result turns out to be the ridiculously small Planck length

$$l_P \equiv \left(\frac{G\hbar}{c^3}\right)^{1/2} \approx 1.6 \cdot 10^{-33}\, cm \,, \tag{6.1}$$

which, compared to the ordinary (atomic, nuclear, or even of particles) scales of quantum systems, induces to think that a regime is conceivable which plays an intermediate role between the "rigid" scheme of quantum theory in a fixed background spacetime and the still unknown full quantum gravity. It is possible, in fact, to envisage experimental situations in which the gravitational field is generated by matter behaving quantum mechanically, but it is measured averaging over regions whose typical size is much greater than l_P; in such conditions it is therefore meaningful, even if a quantum theory of gravitation is available, to ask for the formulation of a consistent semiclassical treatment, where classical gravity is coupled to quantum matter.

Despite of the theoretical and practical interest of such a program, there has been an extremely little amount of work about it, and the only concrete theory investigated so far is the one constructed from the field equations [8,9]

$$G_{ab}(x) = \kappa \langle \psi | \hat{T}_{ab}(x) | \psi \rangle \ , \tag{6.2}$$

where $x \in M$ is a point of spacetime, $|\psi\rangle$ represents the state of quantum matter, and $\hat{T}_{ab}$ is its stress-energy-momentum tensor operator. It is easy to understand that the (6.2) reduce, in the limit of macroscopic systems, to the ordinary Einstein equations (5.1): in fact the statistical dispersion of T_{ab} around the average value $\langle \psi | \hat{T}_{ab} | \psi \rangle$ will become negligible, thus allowing the stress-energy-momentum features of matter to be treated classically.

Equations (6.2) have some intriguing consequences, both of conceptual and technical nature. First of all, it should be clear that their solution is not an easy task at all: not only $\hat{T}_{ab}(x)$ will contain explicitly (as usual in general relativity) the metric tensor $g_{ab}(x)$, but also $|\psi\rangle$ can be completely characterized only once the entire spacetime structure is known; the Hilbert space formulation of quantum theory, in fact, strongly relies on the global properties of M, as it has been discussed in Chapter 4. This fact implies also that, in the context of (6.2), the superposition principle of quantum theory is violated [10,59], since different $|\psi\rangle$ are compatible with different spacetimes.

It must be pointed out that a straightforward calculation of $\langle \psi | \hat{T}_{ab} | \psi \rangle$ for quantum matter fields will give, in general, a divergent expression: the right hand side of (6.2) is therefore the result of complex regularization techniques [7]. This aspect of the problem is complicated further if the theory based on (6.2) is not regarded as exact, but only as an approximation to quantum gravity, along the line of thought we have explained commenting (6.1): in this case the one-loop corrections due to gravitons will produce effects comparable with those of the matter fields, so they also have to be considered in the computation of $\langle \psi | \hat{T}_{ab} | \psi \rangle$.

An interesting analysis, which we shall pursue in some detail, concerns the physical structure of the source term in (6.2): this can be studied by exploring the newtonian limit for the case of a single particle; the semiclassical gravity problem can be therefore formulated, in this approximation, as follows:

Problem: *We know that the particle, through Poisson equation, behaves as the source of a gravitational field. But how can an expression for the source term be extracted only from the knowledge of the probability density for the particle's position?*

The approach based on equation (6.2) tries to solve this problem with the following

Hypothesis: *The field is not generated by a point mass, but by a mass density* $\langle\psi|\hat{\mu}(\mathbf{x},t)|\psi\rangle$ *,where* $\hat{\mu}(\mathbf{x},t)$ *is a mass density operator*[1].

In order to study the consequences of this hypothesis, we need to specify $\hat{\mu}(\mathbf{x},t)$; let therefore $|\mathbf{y},t\rangle$ be an eigenstate of position, such that

$$\hat{\mathbf{z}}(t)|\mathbf{y},t\rangle = \mathbf{y}\,|\mathbf{y},t\rangle \,, \tag{6.3}$$

where $\hat{\mathbf{z}}(t)$ is the position operator of the particle: we require

$$\hat{\mu}(\mathbf{x},t)|\mathbf{y},t\rangle \equiv m\,\delta^3(\mathbf{x}-\mathbf{y})|\mathbf{y},t\rangle \,, \tag{6.4}$$

with m the particle's mass. Equation (6.4) can be justified by noticing that very narrow wavepackets peaked on $\mathbf{y}$ approximate the classical concept of point particle located in $\mathbf{y}$, to which a classical mass density

$$\mu(\mathbf{x},t) = m\,\delta^3(\mathbf{x}-\mathbf{y})$$

is associated; in fact, (6.3) and (6.4) together imply the symbolical relation

$$\hat{\mu}(\mathbf{x},t) = m\,\delta^3(\mathbf{x}\,\hat{1} - \hat{\mathbf{z}}(t)) \,. \tag{6.5}$$

The state vector $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \int d^3y\,\langle\mathbf{y},t|\psi\rangle\,|\mathbf{y},t\rangle = \int d^3y\,\psi(\mathbf{y},t)\,|\mathbf{y},t\rangle \,, \tag{6.6}$$

finding

$$\begin{aligned}\hat{\mu}(\mathbf{x},t)|\psi\rangle &= \int d^3y\,\psi(\mathbf{y},t)\,\hat{\mu}(\mathbf{x},t)|\mathbf{y},t\rangle = \\ &= m\psi(\mathbf{x},t)|\mathbf{x},t\rangle \,,\end{aligned} \tag{6.7}$$

and

$$\langle\mu(\mathbf{x},t)\rangle \equiv \langle\psi|\hat{\mu}(\mathbf{x},t)|\psi\rangle = m|\psi(\mathbf{x},t)|^2 \,. \tag{6.8}$$

We can therefore conclude that, in the semiclassical approach, a nonrelativistic particle is treated as a spread mass with density $m|\psi(\mathbf{x},t)|^2$.

There is an immediate objection which could be raised against (6.8): in the theory of the Schrödinger field, energy density has the expression

$$\frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi \,, \tag{6.9}$$

[1] We are working here in the Heisenberg picture.

which differs totally from (6.8). However, it is easy to realize that (6.9) corresponds only to kinetic energy which, in our approximation, is negligible with respect to (6.8): the reason why only (6.9) is generally written in the Schrödinger hamiltonian, is that the volume integral of (6.8) gives simply m, which does not contribute to the dynamics of the particle.

The term (6.9), representing kinetic energy, is of order v^2, where v is the particle's speed: it can therefore be neglected also in the weak field approximation, which retains terms of order v and generalizes thus the newtonian limit, allowing "gravomagnetic" effects to be treated. In this approximation, Einstein's theory reduces to a form which closely resembles Maxwell's one [2], and energy current has to be taken into account as source of gravity, while material stresses are still negligible. The current term can be found by noticing that, classically,

$$\mathbf{j} = \mu \mathbf{v} = \frac{1}{m} \mu \mathbf{p} \ ,$$

so we are led to construct some operator like $\hat{\mu}(\mathbf{x}, t)\hat{\mathbf{p}}(t)$; let us then analyze the combined actions of $\hat{\mu}(\mathbf{x}, t)$ and $\hat{\mathbf{p}}(t)$. Since

$$[\hat{\mu}(\mathbf{x}, t), \hat{\mathbf{p}}(t)] \neq \hat{0} \ ,$$

neither $\hat{\mu}(\mathbf{x}, t)\hat{\mathbf{p}}(t)$ nor $\hat{\mathbf{p}}(t)\hat{\mu}(\mathbf{x}, t)$ are self-adjoint, but their sum is; let us therefore define

$$\hat{\mathbf{j}}(\mathbf{x}, t) \equiv \frac{1}{2m} (\hat{\mu}(\mathbf{x}, t)\hat{\mathbf{p}}(t) + \hat{\mathbf{p}}(t)\hat{\mu}(\mathbf{x}, t)) \ . \tag{6.10}$$

What interest us about the operator (6.10) is, according to (6.2), its expectation value, which is calculated in the following

Proposition 6.1 *If $\psi(\mathbf{x}, t)$ is the Schrödinger wave function of the state $|\psi\rangle$, then*

$$\begin{aligned} \langle \mathbf{j}(\mathbf{x}, t) \rangle &\equiv \langle \psi | \hat{\mathbf{j}}(\mathbf{x}, t) | \psi \rangle = \\ &= \frac{\hbar}{2i} \psi(\mathbf{x}, t)^* \, \underline{\nabla} \psi(\mathbf{x}, t) \ , \end{aligned} \tag{6.11}$$

where

$$\alpha \, \underline{\nabla} \beta \equiv \alpha \, \nabla \beta - \nabla \alpha \, \beta \ .$$

Proof : Remembering (6.4) and the similar relation

$$\hat{\mathbf{p}}(t) | \mathbf{k}, t \rangle = \mathbf{k} \, | \mathbf{k}, t \rangle \ , \tag{6.12}$$

and writing the completeness relations for $\{|\mathbf{y},t\rangle\}$ and $\{|\mathbf{k},t\rangle\}$,

$$\int d^3y\,|\mathbf{y},t\rangle\langle\mathbf{y},t| = \int d^3k\,|\mathbf{k},t\rangle\langle\mathbf{k},t| = \hat{1}\ , \tag{6.13}$$

we get easily, remembering that

$$\langle\mathbf{x},t|\mathbf{k},t\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\,e^{i\frac{\mathbf{k}\cdot\mathbf{x}}{\hbar}}\ \ : \tag{6.14}$$

$$\hat{\mu}(\mathbf{x},t)\hat{\mathbf{p}}(t)|\psi\rangle = \frac{m}{(2\pi\hbar)^{3/2}}\int d^3k\,\mathbf{k}\,e^{i\frac{\mathbf{k}\cdot\mathbf{x}}{\hbar}}\,\langle\mathbf{k},t|\psi\rangle\,|\mathbf{x},t\rangle\ ; \tag{6.15}$$

$$\hat{\mathbf{p}}(t)\hat{\mu}(\mathbf{x},t)|\psi\rangle = \frac{m}{(2\pi\hbar)^{3/2}}\int d^3k\,\mathbf{k}\,e^{-i\frac{\mathbf{k}\cdot\mathbf{x}}{\hbar}}\,\langle\mathbf{x},t|\psi\rangle\,|\mathbf{k},t\rangle\ . \tag{6.16}$$

From (6.15) and (6.16), equation (6.10) becomes

$$\begin{aligned}\hat{\mathbf{j}}(\mathbf{x},t) &= \frac{1}{2(2\pi\hbar)^{3/2}}\int d^3k\,\mathbf{k}\left(e^{i\frac{\mathbf{k}\cdot\mathbf{x}}{\hbar}}\,|\mathbf{x},t\rangle\langle\mathbf{k},t| + e^{-i\frac{\mathbf{k}\cdot\mathbf{x}}{\hbar}}\,|\mathbf{k},t\rangle\langle\mathbf{x},t|\right) = \\ &= \frac{\hbar}{2i(2\pi\hbar)^{3/2}}\int d^3k\left(\nabla e^{i\frac{\mathbf{k}\cdot\mathbf{x}}{\hbar}}\,|\mathbf{x},t\rangle\langle\mathbf{k},t| - \nabla e^{-i\frac{\mathbf{k}\cdot\mathbf{x}}{\hbar}}\,|\mathbf{k},t\rangle\langle\mathbf{x},t|\right).\end{aligned} \tag{6.17}$$

Taking now the expectation value of (6.17) in the state $|\psi\rangle$, and noticing that

$$\frac{1}{(2\pi\hbar)^{3/2}}\int d^3k\,e^{i\frac{\mathbf{k}\cdot\mathbf{x}}{\hbar}}\,\langle\mathbf{k},t|\psi\rangle = \psi(\mathbf{x},t)\ , \tag{6.18}$$

we easily obtain equation (6.11). □

Summarizing these results, we see that, for a nonrelativistic quantum particle, the right hand side of (6.2) must reduce, in the weak field limit, to the source terms

$$\langle\mu\,(\mathbf{x},t)\rangle = m\psi(\mathbf{x},t)^*\,\psi(\mathbf{x},t) \tag{6.19}$$

and

$$\langle\mathbf{j}(\mathbf{x},t)\rangle = \frac{\hbar}{2i}\,\psi(\mathbf{x},t)^*\,\underline{\nabla}\psi(\mathbf{x},t)\ . \tag{6.20}$$

It is very interesting to notice that, writing

$$\psi(\mathbf{x},t) = |\psi(\mathbf{x},t)|\,e^{\frac{i}{\hbar}S(\mathbf{x},t)}\ , \tag{6.21}$$

(6.20) becomes

$$\langle\mathbf{j}(\mathbf{x},t)\rangle = \langle\mu\,(\mathbf{x},t)\rangle\,\frac{1}{m}\,\nabla S(\mathbf{x},t)\ , \tag{6.22}$$

which, defining

$$\mathbf{v}(\mathbf{x},t) \equiv \frac{1}{m}\,\nabla S(\mathbf{x},t)\ , \tag{6.23}$$

becomes at the end

$$\langle \mathbf{j}(\mathbf{x},t)\rangle = \langle \mu\,(\mathbf{x},t)\rangle \mathbf{v}(\mathbf{x},t)\ . \tag{6.24}$$

Let us notice that, assuming Schrödinger equation[1] to hold, $\langle\,\mu\rangle$ and $\mathbf{v}$ satisfy the continuity equation

$$\frac{\partial\,\langle\,\mu\rangle}{\partial t} + \nabla\cdot(\langle\,\mu\rangle\mathbf{v}) = 0\ . \tag{6.25}$$

For what concerns its action as source of gravity, therefore, our quantum particle acts as a fluid whose density and velocity are given, respectively, by equations (6.19) and (6.23); this is a nice result, but we can do even better, remembering that the stress-energy-momentum tensor not only represents the energy density and current of matter, but also its stress content. Surprisingly enough, a stress tensor p_{ij} can be extracted out of the wave function ψ by the use of Schrödinger equation: this tensor is of no use in the weak field limit of semiclassical field equations (6.2), but it gives an insight on the kind of source terms one must expect in the fully relativistic theory.

Euristically, a stress tensor should act on the fluid characterized by $\langle\,\mu\rangle$ and $\mathbf{v}$ in such a way as to make possible to write down an Euler equation

$$\langle\,\mu\rangle\,\frac{dv_i}{dt} = -\partial_j p_{ij} - \frac{\langle\,\mu\rangle}{m}\partial_i V\ , \tag{6.26}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\ , \tag{6.27}$$

and V is a real potential energy, which can also depend on ψ (see previous footnote). Introducing (6.21) into Schrödinger equation, and making use of (6.25), we find

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2 = \frac{\hbar^2}{2m}\frac{\nabla^2|\psi|}{|\psi|} - V\ ; \tag{6.28}$$

applying now (6.27) to (6.23), and using (6.28) and (6.19), we get

$$\frac{d\mathbf{v}}{dt} \;=\; \frac{1}{m}\frac{d}{dt}(\nabla S) = \frac{1}{m}\nabla\left(\frac{\partial S}{\partial t} + \frac{1}{2m}(\nabla S)^2\right) =$$

[1]Even with nonlinear corrections of the form $V[\psi]\psi$, with $V[\psi]$ real.

$$= \frac{\hbar^2}{2m^2}\nabla\left(\frac{\nabla^2|\psi|}{|\psi|}\right) - \frac{1}{m}\nabla V =$$
$$= -\frac{1}{\langle\mu\rangle}\frac{\hbar^2}{2m}\left(\nabla|\psi|\nabla^2|\psi| - |\psi|\nabla\nabla^2|\psi|\right) - \frac{1}{m}\nabla V \,. \qquad (6.29)$$

Requiring (6.26) to be satisfied, we are led to write

$$\begin{aligned} \partial_j p_{ij} &= \frac{\hbar^2}{2m}\left(\partial_i|\psi|\partial_j\partial_j|\psi| - |\psi|\partial_i\partial_j\partial_j|\psi|\right) = \\ &= \frac{\hbar^2}{2m}\partial_j\left(\partial_i|\psi|\partial_j|\psi| - |\psi|\partial_i\partial_j|\psi|\right) = \\ &= -\frac{\hbar^2}{4m^2}\partial_j\left(\langle\mu\rangle\partial_i\partial_j\ln\langle\mu\rangle\right) \,, \end{aligned} \qquad (6.30)$$

and finally

$$p_{ij} = -\frac{\hbar^2}{4m^2}\langle\mu\rangle\partial_i\partial_j\ln\langle\mu\rangle + C_{ij} \,, \qquad (6.31)$$

where $\partial_j C_{ij} = 0$: from now on, we shall take $C_{ij} = 0$.

We have thus obtained the noticeable result that the system of equations (6.25), (6.26) and (6.31) is equivalent to the Schrödinger equation, once a correspondence between $\langle\mu\rangle$ and v on one side and ψ on the other is established through equations (6.19), (6,21) and (6.23). It is therefore possible to think to a quantum particle with spin zero as an irrotational[1] perturbation [60] in a fluid whose pressure is defined by (6.31)!

This conclusion is very exciting, but we should be careful about it: it cannot be taken seriously unless we show that p_{ij} acts as a true physical pressure, because we are interested in p_{ij} as source of gravity. In order to decide about this, we shall now work out a specific simple example, calculating p_{ij} on the wall of a rectangular box of sides a, b, c, containing a particle. Inserting the wave function [61]

$$\psi(x,y,z) = \sqrt{\frac{8}{abc}}\sin\frac{\pi n_1}{a}x\,\sin\frac{\pi n_2}{b}y\,\sin\frac{\pi n_3}{c}z \,, \qquad (6.32)$$

which corresponds to an eigenstate of energy with eigenvalue

$$E = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{a^2} + \frac{n_2^2}{b^2} + \frac{n_3^2}{c^2}\right) \,, \qquad n_i \in \mathsf{N}^+ \,,$$

[1] It is easy to see that, treating in the same way a particle with spin, the perturbation is no more irrotational.

into (6.31), we find p_{ij} to be diagonal, with

$$p_{11} = \frac{4\pi^2\hbar^2}{ma^3bc} n_1^2 \sin^2 \frac{\pi n_2}{b} y \, \sin^2 \frac{\pi n_3}{c} z \ , \tag{6.33}$$

and similar results for p_{22} and p_{33}. This means that pressure along the i-th direction does not depend on x_i: on a face normal to x_i (e.g. the box wall) it is maximum in the centre, and becomes zero at the edges; the mean value of p_{11} on such a face is

$$\langle p_{11} \rangle = \frac{\pi^2\hbar^2}{ma^3bc} n_1^2 \ . \tag{6.34}$$

The striking point is that (6.34) agrees completely with the result of the standard, physically meaningful, way to calculate pressure [61, page 67]!

Another specific example which is worth to be analyzed is that of a free particle with gaussian wavepacket centered in $\mathbf{x} = \mathbf{0}$ at time t_0:

$$|\psi(\mathrm{x}, t_0)|^2 = \frac{1}{(2\pi\hbar)^3} \exp\left(-\frac{\mathbf{x}^2}{2R^2}\right) \ , \tag{6.35}$$

where R gives a measure of the gaussian's width. We know that Schrödinger theory with $V = 0$ predicts the packet to spread in such a way that it doubles its width in a time

$$t_{Schr} \sim \frac{mR^2}{\hbar} \ .$$

In our "fluiddynamical model", it is straightforward to compute

$$p_{ij} = \frac{\hbar^2}{4m^2R^2} \langle \mu \rangle \delta_{ij} \equiv p\delta_{ij} \ : \tag{6.36}$$

then, there will be a "dynamical" spread of the perturbation, according to (6.26) which becomes now

$$\langle \mu \rangle \frac{d\mathbf{v}}{dt} = -\frac{\hbar^2}{4m^2R^2} \nabla \langle \mu \rangle \ . \tag{6.37}$$

A rough order of magnitude estimate gives to the acceleration the value

$$\frac{\hbar^2}{m^2R^3} \ ,$$

so the perturbation will increase its size, in time t, by

$$\frac{\hbar^2}{m^2R^3} t^2 \ ;$$

requiring this length to be equal to R we obtain the dynamical time t_{dyn} of spreading

$$t_{dyn} \sim t_{Schr} \ :$$

the spreading of wavepackets can be therefore considered as a pressure effect!

It is interesting to write, from (6.36),

$$p = \left(\frac{\lambda}{2R}\right)^2 \langle \mu \rangle c^2 \ , \tag{6.38}$$

where $\lambda \equiv \hbar/mc$ is the Compton length of the particle and c is the speed of light: then, by noticing that, when $R \leq \lambda/2$, it is $p \geq \langle \mu \rangle c^2$, we are led to two conclusions:

i) for very localized particles, quantum stress is important as source of gravity;

ii) in the relativistic case, the "causality" requirement $p \leq \langle \mu \rangle c^2$ should lead to $R \geq \lambda/2$, which means that particles cannot be localized with a precision greater than $\sim \lambda$.

It turns out that *ii*) is a general property of relativistic quantum theory [24], and this supports the idea that the quantum stress p_{ij} plays a physically significant role in the field equations (6.2). As a result of our analysis, we can therefore state that in the weak field limit, for $v \ll c$, the source term $\langle \psi | \hat{T}_{ab} | \psi \rangle$ has the structure

$$\begin{pmatrix} \langle \mu \rangle c^2 & \langle \mu \rangle v_i c \\ \langle \mu \rangle v_i c & p_{ij} + \langle \mu \rangle v_i v_j \end{pmatrix} ,$$

where the term $\langle \mu \rangle v_i v_j$ has been added to p_{ij} in the spatial part, as usual, in order to guarantee that the complete 4-tensor is divergenceless. It must be remarked that the three kinds of terms present in the previous matrix are of different order in v, so their contributions as sources of gravity are comparable only for a relativistic particle, whose treatment is beyond the approximations performed here, and requires therefore a more detailed analysis [62].

Chapter 7

Criticism of the Semiclassical Einstein Equations

The analysis of the equations (6.2), which we have started in the previous chapter, could be pursued further, generalizing it to the fully relativistic case [62] and exploring the mathematical structure of the semiclassical problem [10]. Before starting to carry on such a sophisticated program, however, we believe it is necessary to be sure that it is *physically* well posed, that is, that it does not lead to conceptual inconsistences: for this reason, we shall devote this chapter to the physical understanding of (6.2).

As mentioned in Chapter 6, a technical difficulty of the semiclassical approach is nonlinearity: usually, the origin of this feature is identified in the structure of Einstein equations, but we want to stress here that there is another, more subtle, source of nonlinearity, which has to be ascribed to the fact that, in (6.2), the particle is treated as an extended object. In order to investigate this point, we need to complete (6.2) by adding to them the quantum mechanical evolution equation which, in the Schrödinger picture, is given by (4.4); however, we need some care in specifying the hamiltonian $\hat{H}_u(\tau)$, because it will depend on the spacetime metric g_{ab}, which, by (6.2), will be a functional of $|\psi\rangle$: therefore, $\hat{H}_u(\tau)$ will be a functional of $|\psi\rangle$, too, and this will lead to a right hand side of (4.4) which is nonlinear in $|\psi\rangle$. It is easy to see how this nonlinearity is increased by, but not due to, the nonlinearity of Einstein equations; this can be realized, again, by studying the newtonian limit for a single particle, when equations (6.2) and (4.4) assume the form:

$$\nabla^2 V = 4\pi G m^2 |\psi|^2 , \tag{7.1}$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V[\psi]\psi . \tag{7.2}$$

Equation (7.2) looks quite strange, since it contains a self-interaction term for the particle; the physical origin of this unusual "correction" can be understood thinking to the ψ-wave as propagating in a field created by the particle through (7.1): the potential energy will be therefore a functional $V[\psi]$ of the wave function, which, in turn, will act on the behaviour of the wave function itself. It should be clear that (7.2) is really the correct newtonian limit for the dynamical evolution of $|\psi\rangle$; in fact, in the relativistic formulation (4.4), the hamiltonian $\hat{H}_u$ is not only function of τ, but also of g_{ab}, which embodies the concept of gravitational potential: the self-interaction of quantum matter, mediated by the classical field, is therefore present also at this level, and has to be considered seriously in a theory based on (6.2). It is interesting to notice that the same problem arises in a semiclassical theory of electromagnetism, but since, in general, $e^2 \gg Gm^2$, the nonlinear effects should be much more relevant than what they are for gravity, and it would be interesting to have some specific result which could be compared with experimental data, in order to check the validity of an *ansatz* like (7.1) [63].

There is another peculiar feature of a semiclassical system like that described by equations (6.2) and (4.4), which has been recently analyzed by Boucher and Traschen [64], and which can be shortly summarized by saying that (6.2) and (4.4) do not allow to the gravitational field to "feel" the quantum fluctuations of matter: in fact, being gravity coupled to the expectation value of T_{ab}, its behaviour will be driven by $\langle T_{ab}\rangle$, regardless of any fluctuation around it, which could be, in principle, quite big. This point is intriguing enough by itself, but it becomes particularly disconcerting when (6.2) and (4.4) are applied to the study of cosmological models whose matter content is a quantum field; in most of the current literature [65] on this topic, the background spacetime is assumed to be represented, initially, by a spatially homogeneous manifold, satisfying Einstein equations (6.2) with the right hand side constructed out of a scalar field $\phi(x)$, which is, of course, initially homogeneous and isotropic, too. The reasoning which is generally performed is that, as time (referred to the fundamental observers of our model) passes, $\phi(x)$ will develop spatial perturbations, due to quantum fluctuations; these, by the gravitational field equations, will induce perturbations in the metric of spacetime, which will later act as concentration centers for matter, after other physical processes will have taken place. This picturesque scenario, which should provide a physical mechanism for the origin of the inhomogeneities (clusters of galaxies) that are observed in the present universe, cannot unfortunately rely on (6.2), because these equations do not couple quantum fluctuations of matter to the geometry of spacetime: if the initial conditions are spatially homogeneous both in g_{ab} and in ϕ, they will remain so for all the time.

It is obvious, at this point, that the equations (6.2) seem to involve an objective interpretation for $|\psi\rangle$; in equation (7.1), for example, $|\psi|^2$ acts as a source of the gravitational field (the particle is treated as an extended system), and this is hard to reconcile with an interpretation where ψ is not a physical field, but represents only the probability of a configuration. On the other side, in Chapter 3 we have given extensive support to the idea that $|\psi\rangle$ fully describes only an ensemble of equally prepared systems, being it inadequate to predict the behaviour of an individual (remember that quantum theoretical predictions cannot be tested on an individual). Moreover, as it has been explained in the Introduction, equations (6.2) are inconsistent with an interpretation in which the state vector collapses when a measurement is performed; we can analyse this point writing the state vector collapse as

$$\hat{\rho} \longrightarrow \hat{\rho}\,' , \tag{7.3}$$

where

$$\hat{\rho} = |\psi\rangle\langle\psi| , \tag{7.4}$$

$$|\psi\rangle = \sum_n c_n \, |n\rangle , \tag{7.5}$$

$$\hat{\rho}\,' = \sum_n |c_n|^2 \, |n\rangle\langle n| , \tag{7.6}$$

and $|n\rangle$ are orthonormal vectors representing eigenstates of the observable which is measured (let us suppose, for sake of simplicity, that there is no degeneracy). According to (6.2), before the measurement has been performed, the source term for gravity is

$$\langle\psi|\hat{T}_{ab}|\psi\rangle = tr\,(\hat{\rho}\hat{T}_{ab}) , \tag{7.7}$$

while, after the measurement, it becomes

$$\langle n|\hat{T}_{ab}|n\rangle , \tag{7.8}$$

with probability $|c_n|^2$. As in the thought experiment described in the Introduction, the various $|n\rangle$ can correspond to different spatial (i.e. on Σ of Chapter 4) distributions of the sources; if the measurement has a duration $\Delta\tau$, the effect of the state vector reduction will be to change the source from (7.7) to (7.8) in a time $\sim \Delta\tau$: the inconsistency between this process and the equations (6.2) is emphasized in Figure (7.1), which shows a space-time diagram of the situation envisaged in the Introduction. At time τ_1 a particle is in a state such that it has the same probability 1/2 to be in two disconnected spatial regions A and B; at time τ_2 a position measurement

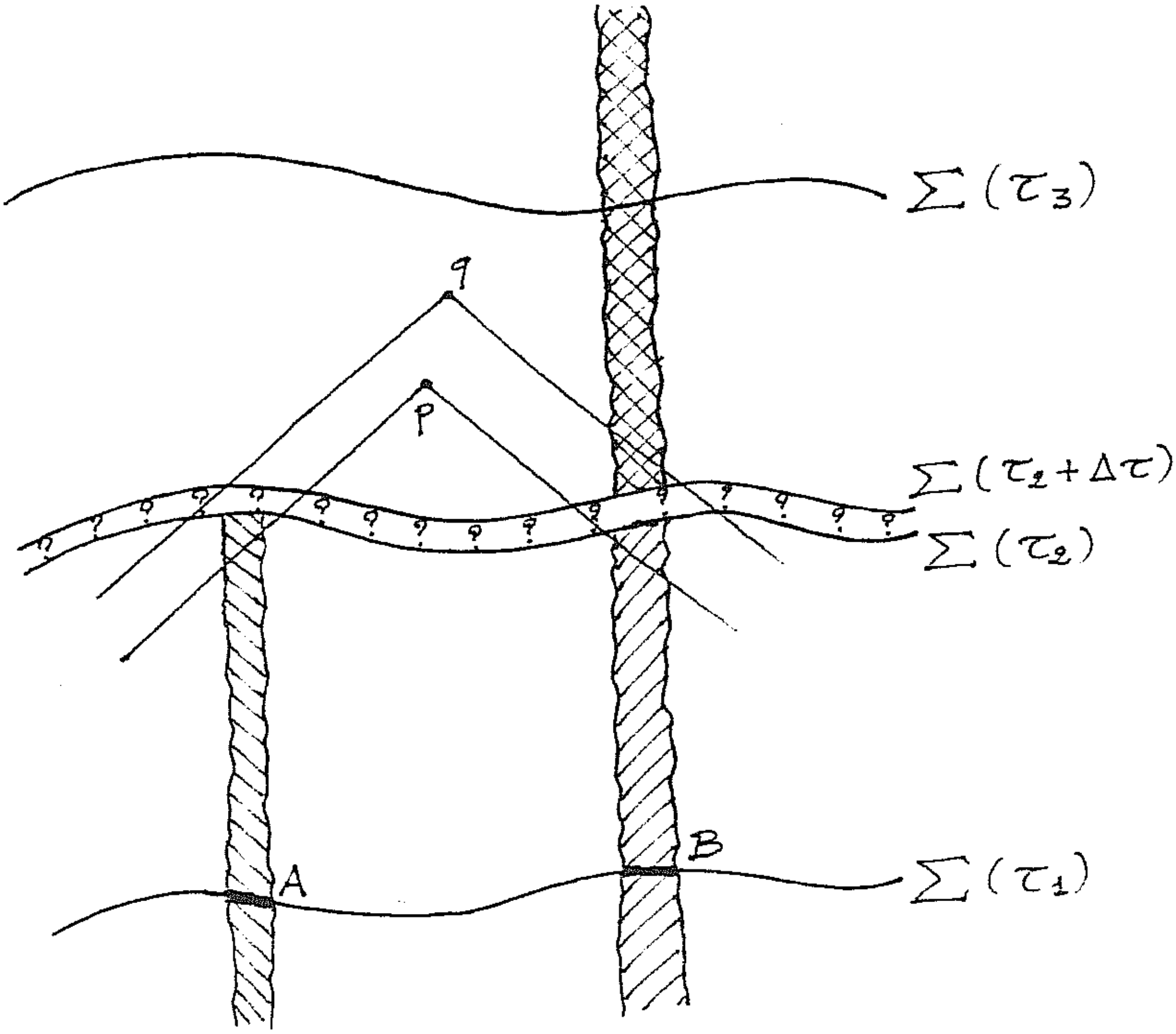

Figure 7.1:

on the particle is performed, which takes a time $\Delta\tau$; after time $\tau_2 + \Delta\tau$ (e.g. at time τ_3), as a result of the measurement, the particle is known to be in the region B, with probability 1. If equations (6.2) are assumed to hold, the gravitational field in the spacetime point p will have matter contributions from both regions A and B and, possibly, from the region labeled by question marks in Figure (7.1), while in the point q it will have contributions only from region B and, possibly, from the "?-region"; since the time $\Delta\tau$ can be made reasonably small, and p and q can therefore be chosen very close each other, it is clear that, in order to guarantee the validity of (6.2), we are forced to admit that matter is present in the ?-region, so that the act of measurement induces an acausal (i.e. spacelike) matter flow from A to B. This physically unacceptable conclusion leads to drop either equations (6.2) or the hypothesis that the state vector collapse.

As a last remark on this thought experiment, we want just to stress on the fact that the occurrence of a spacelike matter flow is a straightforward consequence of the mathematical structure of (6.2); in fact, the identity

$$\nabla^b G_{ab} = 0 \tag{7.9}$$

implies

$$\nabla^b \langle\psi|\hat{T}_{ab}|\psi\rangle = 0 \; ; \tag{7.10}$$

equation (7.10), as known, has the meaning of a conservation law, in the sense that any change of the matter content[1] inside a closed spatial 2-surface implies a flow through the 2-surface itself: the negative result of the measurement inside A must be therefore accompanied by a matter flow from A to B, as explained before.

These arguments lead quite naturally to the conclusion that (6.2) require, in order to be consistent, an interpretation of quantum theory where the state vector does not collapse; as we know from Chapter 3, the most reliable of such interpretations are the statistical one and the many-worlds: we shall now show that both of them are incompatible with equations (6.2).

In the relative state formulation, the vector (7.5) is supposed to describe faithfully the matter content of space, under the hypothesis of simultaneous existence of different copies of the material system, each of them described by one of the vectors $|n\rangle$; moreover, these copies do not interact each other. When semiclassical gravity is taken into account by mean of (6.2), however, this last condition holds no more, because the gravitational field in a point of spacetime is generated by the source (7.7), which

[1]Here we do not care about the intricate problems concerning a rigorous definition of energy in a curved spacetime [66], since they are irrelevant to our discussion.

takes into account all the copies of the system through the vectors $|n\rangle$ contained in $|\psi\rangle$; this situation provides therefore a coupling between different copies, mediated by classical gravity. An observer's state vector, in fact, is contained in one of the components of $|\psi\rangle$, but by a measurement of the gravitational field he/she can be aware of the presence of other components; it is easy to realize, however, that such a possibility is in conflict with observations. An experiment has been performed [67] which is a gravitational analogue of the Schrödinger's cat paradox, where the positions of macroscopic bodies are triggered by a quantum mechanical process; as expected, the gravitational field has been found to correspond only to one component of the matter's state vector, against the predictions of (6.2). Of course, this result could be considered as an indirect evidence for the quantization of gravity [67], but we prefer to adopt a more moderate attitude, and to take it only as a further proof of the non viability of the equations (6.2) to describe semiclassical gravity.

We are left therefore, in our criticisms to (6.2), with the statistical interpretation of quantum theory: unfortunately, in this visual, the situation is even worse. In fact, the left hand side

$$G_{ab}[g] \tag{7.11}$$

is referred to a single spacetime, while the right hand side (7.7) represents an average over an ensemble: equations (6.2) look then conceptually inconsistent. It is possible to try to balance the situation by considering the term (7.11) as an average, too, in the following sense: an ensemble E_m of identically prepared material systems is described by a state vector $|\psi\rangle$; each of these systems is supposed to correspond to a gravitational field, so that an ensemble E_g will be constructed for gravity. But this means that gravity also must be treated quantum mechanically, and it seems therefore that a purely semiclassical theory which does not change quantum mechanics very deeply is not viable.

Having achieved such a negative result, it is natural to ask for the conditions under which (6.2) hold in a quantum theory of gravity, in the sense of a conveniently defined average. Since such a theory, as we have often stressed, is not presently available, a rigourous answer cannot be given; however, we can try to investigate the problem in a restricted context, which will be called *weakly semiclassical* approximation. The idea underlying this approach is to allow the field to have quantum features, but only those which are induced by the matter: as can be easily realized, this hypothesis removes all the previous inconsistences and paradoxes, and does not require to deal with the most tricky problems of field quantization,since "self-quantization" processes are neglected.

In order to be more explicit about this point, we shall now work out in detail a "toy model" of weakly semiclassical gravity, where a scalar field ϕ in Minkowski spacetime is coupled to nonrelativistic matter according to the classical equation

$$\nabla_a \nabla^a \phi = \mu \ , \tag{7.12}$$

where μ is a source term. This model simplifies the problem very much, since it reduces to one the number of field and source components, while preserving some crucial features of Einstein's theory, because (7.12) is a second order, hyperbolic, equation for ϕ; we must remind, however, that (7.12) differs from (5.1) in that it is linear in the field. Let us now suppose matter to be quantum: the arguments performed throughout this chapter imply the field to be quantum, too. Its value at a spacetime point x will therefore be, in general, not defined: however, it is possible to argue as follows in order to succeed in defining a probability distribution for $\phi(x)$.

Classically, (7.12) can be solved by the use of the Green function $D(x, x')$, which satisfies the equation [68]

$$\nabla_a \nabla^a D(x, x') = -\delta^4(x, x') \ ; \tag{7.13}$$

in fact, writing

$$\mu(x) = \int_{\mathrm{R}^4} d^4x' \delta^4(x, x') \mu(x') \tag{7.14}$$

in (7.12), we obtain, by linearity,

$$\phi(x) = -\int_{\mathrm{R}^4} d^4x' D(x, x') \mu(x') \ . \tag{7.15}$$

As it is well known, $D(x, x')$ is determined by (7.13) only up to a solution $\tilde{D}(x, x')$ of the homogeneous equation

$$\nabla_a \nabla^a \tilde{D}(x, x') = 0 \ : \tag{7.16}$$

to remove this arbitrariness corresponds to choose a Green function in agreement with some physical requirements. The choice that is generally made is the one which leads to the retarded Green function

$$\begin{aligned} D^{ret}(x, x') &= \frac{1}{2\pi} \Theta(t - t') \delta\left((x - x')^2\right) = \\ &= \frac{1}{4\pi |\mathrm{x} - \mathrm{x}'|} \delta\left(|\mathrm{x} - \mathrm{x}'| - c(t - t')\right) \ , \end{aligned} \tag{7.17}$$

where δ and Θ are, respectively, the Dirac and Heaviside functions. It is evident, from (7.17), that the support of $D^{ret}(x, x')$ is the past light cone of x, and that (7.15) becomes thus

$$\phi(x) = -\int_{\mathrm{R}^3} d^3x' \frac{1}{4\pi |\mathrm{x} - \mathrm{x}'|} \mu\left(\mathrm{x}', t - \frac{1}{c} |\mathrm{x} - \mathrm{x}'|\right) \ . \tag{7.18}$$

The physical meaning of (7.18) is that a classical pointlike distribution located in $\mathbf{y}$ at time $t - \frac{1}{c}|\mathbf{x} - \mathbf{y}|$, produces in $\mathbf{x}$ at time t a field of intensity

$$\phi(\mathbf{x}, t; \mathbf{y}) \equiv -\frac{\alpha}{4\pi|\mathbf{x} - \mathbf{y}|} , \tag{7.19}$$

where α is a constant depending on the nature of field and matter: this can be easily seen by substituting

$$\mu(\mathbf{x}', t - \frac{1}{c}|\mathbf{x} - \mathbf{x}'|) = \alpha\delta^3(\mathbf{x}' - \mathbf{y}) \tag{7.20}$$

into (7.18).

Now we can state the weakly semiclassical hypothesis, by requiring that the probability that the field in $\mathbf{x}$ at time t is found to have the value (7.19) be equal to the probability that the particle is found in $\mathbf{y}$ at time $t - \frac{1}{c}|\mathbf{x} - \mathbf{y}|$. More precisely, if ψ is the wave function for matter, we have

$$p(\phi(\mathbf{x}, t; \mathbf{y})) = |\psi(\mathbf{y}, t - \frac{1}{c}|\mathbf{x} - \mathbf{y}|)|^2 . \tag{7.21}$$

Let us notice that, provide matter motion is restricted to be timelike (which is certainly true, since we are considering a nonrelativistic model),

$$\int_{\mathsf{R}^3} d^3y\, p(\phi(\mathbf{x}, t; \mathbf{y})) = \int_{\mathsf{R}^3} d^3y\, |\psi(\mathbf{y}, t - \frac{1}{c}|\mathbf{x} - \mathbf{y}|)|^2 = 1 , \tag{7.22}$$

because $|\psi|^2$ obeys a continuity equation, and no flowlines can avoid the past $\mathbf{x}$-lightcone, or intersect it twice. This allows to define an average value for $\phi(x)$ as

$$\begin{aligned} \langle\phi(x)\rangle &\equiv \int_{\mathsf{R}^3} d^3y\, p(\phi(x; \mathbf{y}))\phi(x; \mathbf{y}) = \\ &= \int_{\mathsf{R}^3} d^3y\, |\psi(\mathbf{y}, t - \frac{1}{c}|\mathbf{x} - \mathbf{y}|)|^2\phi(x; \mathbf{y}) = \\ &= -\alpha \int_{\mathsf{R}^4} d^4x'\, |\psi(x')|^2 D(x, x') . \end{aligned} \tag{7.23}$$

Now comes the crucial point of all the discussion: we want in fact to check if it is possible to write the weakly semiclassical equation for ϕ as

$$\nabla_a\nabla^a\langle\phi\rangle = \langle\psi|\hat{\mu}|\psi\rangle , \tag{7.24}$$

where $\hat{\mu}$ is the source operator, such that (in the Heisenberg picture)

$$\hat{\mu}(\mathbf{x}, t)|\mathbf{y}, t\rangle = \alpha\delta^3(\mathbf{x} - \mathbf{y})|\mathbf{y}, t\rangle . \tag{7.25}$$

Applying the linear differential operator $\nabla_a \nabla^a$ to $\langle \phi(x) \rangle$ we get easily

$$\begin{aligned}\nabla_a \nabla^a \langle \phi(x) \rangle &= \alpha \int_{\mathrm{R}^4} d^4x' |\psi(x')|^2 \delta^4(x, x') = \\ &= \alpha |\psi(x)|^2 = \langle \psi | \hat{\mu}(x) | \psi \rangle \ , \end{aligned} \tag{7.26}$$

which is exactly (7.24).

The result obtained is very interesting and enlightening; our toy model admits a well posed and consistent semiclassical problem, but only in its weak formulation: the term $\langle \psi | \hat{\mu} | \psi \rangle$ behaves as source not for the ϕ-field, but rather for its expectation value.

Before to comment a little more about the weakly semiclassical approximation, we shall put the previous discussion on more precise grounds. Since the field is considered quantum, we associate to it, in each point of space, x, a Hilbert space H(x), and in H(x) we consider a self-adjoint field operator $\hat{\phi}(\mathrm{x}, t)$ associated to the observable "field intensity at x". Let us suppose, moreover, that there exist a set of vectors[1]

$$|\phi, \mathrm{x}, t; \mathrm{y}\rangle \in \mathbf{H}(\mathrm{x}) \ , \qquad \forall \mathrm{x} \in \mathrm{R}^3 \ , \tag{7.27}$$

which satisfy the eigenvalue equation

$$\hat{\phi}(\mathrm{x}, t) |\phi, \mathrm{x}, t; \mathrm{y}\rangle = \phi(\mathrm{x}, t; \mathrm{y}) |\phi, \mathrm{x}, t; \mathrm{y}\rangle \ , \tag{7.28}$$

with $\phi(\mathrm{x}, t; \mathrm{y})$ given by (7.19); the vectors (7.27) will be required also to satisfy the generalized orthonormality property

$$\langle \phi, \mathrm{x}, t; \mathrm{y} | \phi, \mathrm{x}, t; \mathrm{z}\rangle = \delta^3(\mathrm{y} - \mathrm{z}) \ . \tag{7.29}$$

A general state vector of H(x) will be written as

$$|\phi, \mathrm{x}\rangle = \int_{\mathrm{R}^3} d^3y \, \Phi(\mathrm{x}, t; \mathrm{y}) |\phi, \mathrm{x}, t; \mathrm{y}\rangle \ , \tag{7.30}$$

where

$$\Phi(\mathrm{x}, t; \mathrm{y}) \equiv \langle \phi, \mathrm{x}, t; \mathrm{y} | \phi, \mathrm{x}\rangle \tag{7.31}$$

is the "wave function" of the ϕ-field at x. The key idea of the weakly semiclassical approximation is that the $\Phi(\mathrm{x}, t; \mathrm{y})$, which determine the quantum behaviour of the field, must be entirely defined by the source; the most natural hypothesis is that

$$\Phi(\mathrm{x}, t; \mathrm{y}) = e^{i\alpha(\mathrm{x}, \mathrm{y}, t)} \, \psi(\mathrm{y}, t - \frac{1}{c}|\mathrm{x} - \mathrm{y}|) \ , \tag{7.32}$$

[1] Of the rigged Hilbert space.

with α an arbitrary real function: this leads trivially to (7.21).

We would like to stress on the fact that, in this approach, the field is a quantum object; therefore, equation (7.30) *does not* imply that, in the point x at time t, the field is a weighted sum of the fields $\phi(\mathrm{x}, t; \mathrm{y})$ (which is what strongly semiclassical theory states)! The meaning of (7.30) is that, if an observer performs a measurement of ϕ in x at time t, he/she has a probability density $|\Phi(\mathrm{x}, t; \mathrm{y})|^2$ to find the value $\phi(\mathrm{x}, t; \mathrm{y})$.

For what regards the normalization of $|\phi, \mathrm{x}\rangle$, it is

$$\langle \phi, \mathrm{x} | \phi, \mathrm{x} \rangle = \int_{\mathrm{R}^3} d^3y \, |\Phi(\mathrm{x}, t; \mathrm{y})|^2 = \int_{\mathrm{R}^3} d^3y \, p\,(\phi(\mathrm{x}, t; \mathrm{y})) = 1 \; , \qquad (7.33)$$

accordingly to (7.22). The expectation value of $\phi(x)$ in the state (7.30) is therefore

$$\langle \phi(x) \rangle \equiv \langle \phi, \mathrm{x} | \phi(x) | \phi, \mathrm{x} \rangle \; , \qquad (7.34)$$

which leads again to (7.23) and (7.24).

It is very satisfactory and exciting that we have finally succeeded in the attempts to give a meaningful formulation of a semiclassical problem, but care is needed again for two reasons:

i) (7.24) is an equation which relates the *ensemble averages* of ϕ and μ: it is useless, therefore, for purposes like that of describing the field created by a single quantum system;

ii) the validity of the semiclassical field equation has been established only for a toy model: we must still understand how the features of gravity which were not embodied in (7.12) can change our results.

Let us comment separately about these two points. For what concerns *i*), it must be remembered that the semiclassical approach makes sense, as we have extensively discussed throughout this chapter, only in its weak formulation, where the field is also a quantum system; for this reason, all that can be computed about it, are, accordingly to Chapter 2, the probability for the outcoming of measurements, and the consequent average values of observables. These predictions, of course, will be testable (and meaningful!) only for an ensemble of equally prepared fields; since we have supposed the behaviour of the field to be completely determined by the behaviour of the matter which creates it (this is the essence of the weakly semiclassical approximation) the way to construct such an ensemble is simply to construct an ensemble E_m for matter, described by a state vector $|\psi\rangle$, which will uniquely define an ensemble E_f for the field. The averages $\langle \phi(x) \rangle$ and $\langle \mu(x) \rangle$ are thus performed, respectively, on E_f and E_m. If we prefer, we can also consider field and matter as a single system, and say that, in the weakly semiclassical approach, an ensemble E for the total system can be obtained simply preparing matter: the state vector describing E will be therefore uniquely determined by $|\psi\rangle$.

It is important, to our mind, to make at this point a comment about a topic that is sometimes misunderstood. In the standard treatment of elementary quantum mechanics, it is straightforward to derive the classical limit of the behaviour of a particle subject to an external potential energy $U(\mathrm{x}, t)$ as

$$m\frac{d^2\langle t|\hat{\mathrm{x}}|t\rangle}{dt^2} \approx -\nabla\langle t|U(\mathrm{x},t)|t\rangle \qquad (7.35)$$

(Ehrenfest theorem), where the Schrödinger picture is used. What we want to make clear is that the physical situations underlying (7.24) and (7.35) are precisely the opposite! In fact, (7.35) holds in an external field approach, where the particle's behaviour is driven by a classical field: the appearance of the expectation values in (7.35) is a consequence of the little dispersion of x around $\langle t|\hat{\mathrm{x}}|t\rangle$, a fact which depends much on the preparation process for the particle at an initial time t_0, and very little on the features of the external potential. Contrarywise, in (7.24) is the matter's behaviour to determine the field's one, and both of them are quantum in character; moreover, it is not possible to assign independently the initial states of matter and field: if, in the framework of applicability of (7.35), the state $|t\rangle$ is determined by $U(\mathrm{x}, t)$ and $|t_0\rangle$, but there is arbitrariness in the choice of this latter, for (7.24) it is not conceivable to make a separation between $|\psi\rangle$ and $|\phi, \mathrm{x}\rangle$: if the matter is quantum, then the field must be quantum, too.

Due to their nature of ensemble average, the weakly semiclassical field equations for gravity, even if they will prove to be correct[1], lose much of their charm. This interpretation, in fact, makes them useful only when a collection of equally prepared, identical matter systems is available as a field source; such a situation can be easily realized in the domain of atomic physics (and, in fact, semiclassical calculations for the electromagnetic field are often carried on successfully [54]), but in the context of general relativity (particularly in cosmology), the relevant physical systems are generally present in a single copy, und this makes therefore equations like (6.2) uninteresting. We can compare the gravitational field in the weakly semiclassical approximation to the macroscopic device M of page 23, which is triggered by the microsystem m; since the behaviour of m is quantum in character, it is impossible to predict exactly the values of the observables A_m and, consequently, A_M: only the probabilities $p(a^{(M)})$ and the expectation value $\langle A_M\rangle$ can be computed, but they have no predictive power if only one measurement is performed.

If only one matter system is considered at any time, it is easy to

[1]Until now, we have proved only (7.24), which holds for a linear toy model!

understand that $V[\psi]$ in (7.1) has no operative meaning, being only an average potential energy over an ensemble; as such, it will have no relevance on the physical behaviour of the system, and equation (7.2) will be replaced by the ordinary Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi \ , \tag{7.36}$$

thus removing all the problems about nonlinearity, at least in the newtonian limit. In the full relativistic theory, the situation is not so clear, because the hamiltonian $\hat{H}_u$ in (4.4) is still function of the metric: this metric, however, cannot be simply the solution to equations (6.2), but must be the one determined by the single system we are dealing with: since this cannot be known *a priori*, all the problem seems to become ill-posed.

Let us now face point *ii*), concerning the generalization of our conclusions from the toy model for the ϕ-field to Einstein gravity. The main difference we have to concern about, regards the linearity of the field equations; this feature played a crucial role in the proof of (7.24), since the existence of a Green function and the calculation (7.26) depend heavily on it. It would be very difficult to repeat these arguments in the case of general relativity, but fortunately we can use the ensemble interpretation developed while commenting point *i*), in order to discuss possible differences. Let us suppose the field equations (6.2) to hold in the sense of average; this means that, if $|\psi\rangle$ describes an ensemble of matter systems, each of them producing a metric tensor g_{ab} and a related Einstein tensor $G_{ab}[g]$, it will be

$$\langle G_{ab}[g]\rangle_{ens} = \kappa\langle\psi|\hat{T}_{ab}|\psi\rangle \ . \tag{7.37}$$

But, because of the nonlinear dependence of G_{ab} from g_{ab}, it is clear that

$$\langle G_{ab}[g]\rangle_{ens} \neq G_{ab}[\langle g\rangle_{ens}] \ , \tag{7.38}$$

so the metric tensor which solves (7.37) (or, equivalently, (6.2)) is not $\langle g_{ab}\rangle_{ens}$, and has therefore no physical meaning, neither in an ensemble context.

Chapter 8

Outlooks and Conclusions

The analysis performed throughout this thesis has led to several stimulating results, the most important of which is, in our opinion, the conclusion that a semiclassical problem can be consistent only in a weak formulation (unless quantum theory is deeply changed in its foundations); as a logical consequence, systems like the gravitational field, which interact with quantum matter, must behave in a quantum mechanical way, even on scales much greater than Planck's[1]. Since such a behaviour, as far as we know, is intrinsically probabilistic, it is meaningless to speak about the value of the field in some point of spacetime; semiclassical field equations makes thus sense only as ensemble averages, and can be consistently applied only to situations where many equally prepared, identical matter systems, act together to produce the field. Such a condition is generally not satisfied for physically interesting problems involving gravity, where usually only a single system is considered.

As an example, let us consider again the case of a quantum scalar field in an initially spatially homogeneous universe; from our "ensembles" point of view, it is clear that inhomogeneities will be developed, both in the ϕ-field and in the metric of spacetime, but there is no way, even in principle, to predict exactly their occurrence. Equations (6.2), as we have already said, cancel all the effects of quantum fluctuations, but we can hope that a better theory would be able to retain some of their features, for example predicting a spectrum of perturbations in $g_{ab}(x)$.

Once it has been established that a field coupled with quantum matter must be quantum, too, even if in a weak sense, it is natural to ask for the possibility of "self-quantization" effects. There is a rather natural way to tackle this problem in our approach: the key idea is to consider vacuum as a

[1]Planck length, time, mass, temperature and so on, only give the scale at which self-quantization effects (if they exist) become non negligible.

form of matter, which is, operationally, a very reasonable point of view. In fact, accordingly to the interpretation of quantum formalism explained in Chapter 3, a state vector describes an ensemble of equally prepared systems: therefore, when speaking of a vacuum state $|0\rangle$, we necessarily mean an ensemble of "systems", all prepared in such a way as to correspond to an idea of "vacuum", which we are trying to concretely realize. Maintaining the attitude which ascribes the origin of the field's quantum properties to the matter, we are led to the conclusion that to an ensemble E_m described by $|0\rangle$, it corresponds an ensemble E_f of fields in vacuum, whose elements can be, in principle, different from each other. Physically, this means that when a vacuum is prepared, it determines the field consistent with it and subject to some prescribed boundary conditions; to different vacua of E_m correspond different fields of E_f, but it is experimentally impossible to decide to prepare one or another of them.

Now, the field corresponding to one matter system will be affected by the arbitrariness in the preparation of the vacuum copy: to each copy of the system in an ensemble E'_m, will correspond many different copies of the field in E'_f. Even if definite estimates are lacking at the moment, one can hope that this could account for the so called "vacuum polarization" effects, which could be therefore considered in the context of a weak, matter driven, field quantization. Needless to say, however, that these ideas are very speculative, and have a high probability to be proved incorrect; as examples of the many problems which must be solved in order to allow a more concrete discussion to take place, we can ask the following, until now unanswered, two questions:

1) which criterions has to be applied in order to decide what is matter and what is field?

2) which is the prescription to adopt (analogously to (7.32)) in order to associate a field's state to $|0\rangle$?

There is another improvement suggested by our analysis, based on the remark that a field, being a system with infinite degrees of freedom, is very likely to exhibit a classical behaviour, accordingly to our considerations about complex systems, developed in Chapter 3. We can understand better this point thinking that the Hilbert space for the all field should be an object like

$$\bigotimes_{x \in R^3} \mathbf{H}(x) \,, \tag{8.1}$$

where, as explained in the previous chapter, H(x) is the Hilbert space associated to the field at the spatial point x. (8.1) is, clearly, a monster from the mathematical point of view, providing evidence for the need to replace our naive formulation with a more rigorous one; anyway, it makes intuitively clear that the field is a complex system of the kind treated at page 24. This idea can be made more precise with the help of equation (7.32), suggested during the treatment of the weakly semiclassical approach: in fact, the arbitrariness of the function $\alpha(\mathbf{x},\mathbf{y},t)$ could introduce random phase relations between the state vectors of the field at different points of space, thus verifying the hypothesis under which equation (3.20) is proved. An ensemble for the field would then be a mixture, described by the statistical operator

$$\hat{\rho}(t) = \int_{\mathsf{R}^3} d^3y\, |\Phi(t;\mathbf{y})|^2\, |\phi,t;\mathbf{y}\rangle\langle\phi,t;\mathbf{y}| \ , \tag{8.2}$$

where

$$|\Phi(t;\mathbf{y})|^2 \sim \prod_{\mathbf{x}\in\mathsf{R}^3} |\Phi(\mathbf{x},t;\mathbf{y})|^2 \tag{8.3}$$

and

$$|\phi,t;\mathbf{y}\rangle \sim \bigotimes_{\mathbf{x}\in\mathsf{R}^3} |\phi,\mathbf{x},t;\mathbf{y}\rangle \tag{8.4}$$

should be better defined in a more rigorous formulation. It is important to remark that, when there is phase coherence between the different field wave functions Φ in a region of space, equation (8.2) does not hold any more, and the field's behaviour becomes typically quantum, exhibiting interference phenomena; the temptation to identify these regions of coherence with the quanta of the field is difficult to resist.

As it is possible to understand from the suggestions in this last chapter, our study has led well beyond the original subject of semiclassical field theory, providing physical insight for a new, alternative, way to look at field quantization. Even if the probability that it will eventually prove to be wrong is very high, nevertheless we believe this approach to be worth to be pursued further, in order to obtain more definite results, whose validity could be tested comparing them with the currently available experimental data.

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