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Macroscopic Gravity and Averaging

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Contents

Abstract

The present thesis is concerned broadly with the averaging problem in General Relativity. The important observation is, that there are some smoothing procedures implicit in the standard Friedmann-Lemaître-Robertson-Walker models of cosmology, taken to be isotropic and homogeneous.

The aim of this thesis is then to discuss this problem of fitting of the idealised models to the inhomogeneous real universe, and to present a critical review of some approaches to tackle it.

In the introduction we discuss the theoretical basis of cosmological observations in context of modelling the real universe.

In chapter 2 we describe the main problem from different viewpoints and point out the difficulties arising if one wants to solve it.

In chapter 3, some explicit ways for constructing the smoothed out universe models are considered, in particular, the perturbation approaches are contrasted with the approaches based on averaging.

In chapter 4, the issue of smoothing of cosmological spacetimes is put in a differential geometry setting, enabling a new way of smoothing of Ricci-positive metrics via the Ricci-Hamilton flow. Finally, in the last section we shortly discuss Thurston's conjecture.

Chapter 1

Introduction

The aim of cosmology is an investigation of the structure of space-time in the largest possible scale. One can think of two ways of doing this [1]:

"top-down" way, whereby one assumes *a priori* something about the large scale of the universe and tries to deduct local physics (Milne, Bondi and Gold); and

"bottom-up" way, whereby one extrapolates given local physics as far as possible, but the generalization involved here, also (implicitly) makes some *a priori* assumptions about the universe as a whole or cosmology itself.

In order for cosmology to be concerned with global structure of space-time one has to make some non-empirical assumptions. From the point of view of field theory, local extrapolations into the future can be valid only under the so called no-interference assumption (see e.g. [2]). If one wants to interpret modern cosmological observations the theory of space-time is needed. The conclusions one arrives at about the properties of the universe depend on integration of differential equations on non-local regions of space-time. However, usually it is the simplest cosmological models that are implicitly used in practice (highly symmetrical) which hide the above fact.

In General Relativity (GR henceforth), the space-time manifold with its various geometrical structures has very special, almost axiomatic status. The fields (represented also by various geometrical structures) are on the other hand, introduced by hand. Consequently, through the Einstein equations all the fields (the gravitational field already is in the geometry) are non-local as well. In the early universe, close to the initial singularity each physical property implied global consequences, and the singularity itself (initial) is related to the space-time being geodesically incomplete. The singularity theorems [3] show that geodesic incompleteness of the considered space-time cannot coexist with some of its properties, like, compactness, non-existence of closed time-like curves, existence of Cauchy hypersurface, etc., which are non-local ones.

The global understanding of Einstein's equations can be gained by studying the space of their solutions, the so called *ensemble* of universes[1]. The field equations act on and at the same time create, in a sense, the space-time manifold. This is what a construction of cosmological models refers to, and this is the point of view of observational cosmology. One simply assumes that each solution to the Einstein field equations describes a universe, and cosmology should but specify boundary or initial conditions relevant to the universe resulting from the astronomical observations.

Of interest, is rather more the issue of how a given property (e.g. the existence of singularities) appears in the ensemble, than the structure of particular models. This is supported by the observation that the cosmological model can describe the real universe only within some limits of accuracy, and such a model should have the property of structural stability. In order to study this kind of problems it is necessary to understand the equivalence of models and the "distance" between them. This calls for the introduction of the appropriate topology and metrical structure on the ensemble, which is a difficult and till now an open problem.

Usually, one starts with $Lor(\mathcal{M})$, the space of all Lorentz metrics on the manifold $\mathcal{M}$. But this space admits infinitely many topologies, none of which is "natural". However, what one is really interested in, is a subset of $Lor(\mathcal{M})$, namely, those metrics that are the solutions of Einstein's equations. Fortunately, this set usually is a smooth manifold with a local representation in the space of four functions of three variables, and Einstein's equations acting as a hamiltonian system. Further, the space of linearized solutions of

[1]E.g., the Cauchy problem in a cosmological context naturally leads to the ensemble of possible universes: the set of all admissible kinds of initial data (on a space-like hypersurface) is equivalent to the set of universes evolving from it.

the Einstein equations is tangent to it (this is so only, if our set of metrics is a smooth manifold in the neighborhood of a given solution) and in this region Einstein's equations are stable with respect to the linearization[2].

In practice however, we have at our disposal a procedure for testing the cosmological models. Let us emphasize again, the interpretation of cosmological observations is impossible without assuming a working model of space-time. The choice for this model (or a class of models) is eventually made on the basis of postulates, principles and even philosophical tastes. The comparison is then made between the relations amongst observables in the model and these same relations implied by observations. This line of approach was paved by the beautiful paper of Kristian and Sachs [4], and further the problem of observational basis of cosmology was treated in a series of papers by G. Ellis and collaborators (see [5] and references therein).

The general conclusion is that the ideal observations on our past light cone are not sufficient to uniquely construct its space-time geometry. In other words, without dynamical equations (field equations) we can only reach a conclusion of consistency of many different cosmological models with the observations. If we take into account the field equations the situation gets much more involved. However, it is possible to ascertain, that the maximal observable data set $D(w_o, z^*)$, is at the same time, the minimal observable data set needed to uniquely determine the geometry of our light cone, up to z^* (the redshift of the last scattering surface) [5].

Given this, what can be said is that the standard cosmological model, the Friedmann-Lemaître-Robertson-Walker (FLRW henceforth) solutions, that in fact make up a set of measure zero in the space of solutions of the Einstein field equations, quite well, but of course only approximately describe the real universe. How well and how bad, is a matter of investigation in the present thesis (work is in progress). In particular, is the FLRW line element a good representation of the geometry of the universe [21], and are the equations of General Relativity satisfied - on what scale?

[2] In general, this is not true for space-times with Killing vectors.

Chapter 2

Modelling the real universe

The standard FLRW universe models are homogeneous and isotropic, therefore one can question their applicability for modelling the universe accurately, as it is manifestly not a FLRW universe (on at least some scales). However, these standard models are usually taken to represent the structure of the universe on some averaged scale. This implied averaging procedure is of fundamental importance in cosmology, both in terms of interpreting the meaning of FLRW models, and in relation to galaxy formation studies, as well. However, it hardly ever receives a due attention, even though it underlies the geometric and physical applications of FLRW metrics to describing the real universe.

On local scales, the space-time geometry determined by Einstein's equations is very complex, and thus not very useful from the point of view of cosmology. Therefore, when considering the kinematics and dynamics of the universe as a whole, one usually ignores the fine-graining due to the local inhomogeneities and deals with the simpler structure of space-time geometry, which is usually more illuminating for cosmology.

In practice it means, that the matter inhomogeneities have been averaged (or smoothed) out and redistributed homogeneously (e.g., in the form of a perfect fluid). Then the basic and tacit assumption which underlies this procedure, is that such a smoothed out universe and the actual, inhomogeneous one, behave identically under their own gravitation. This is usually taken for granted, but does by no means, need to be true.

Problems arise when we start to relate the realistic inhomogeneous universe models to these smoothed out, idealised models. The very relation between them is not clear, in particular, it is not quite obvious how the galaxies or clusters of galaxies are related to the comoving coordinates of the averaged idealised models, nor how particular light rays correspond to the idealised geodesics of these models. The standard approach in this respect, is the theory of perturbed FLRW models (see e.g. [7]) and their relation to observations of galaxies and background radiation [17, 34], which is still a matter of investigation [37, 38].

In observational cosmology it is standard to follow this scheme: firstly - to observe the distribution, masses and velocities relative to us of neighbouring galaxies; secondly - to calculate the averaged quantities assuming the Hubble law, i.e., the isotropy (on average) of the relative velocity field, and homogeneity of the distribution of galaxies; thirdly - to compare these mean properties with those of FLRW models, having the same density as that of the total mass of the galaxies uniformly distributed in the observed region. As a result, one could in principle get in this way, the best-fit parameters of the FLRW models. Unfortunately, the discrepancy between the observational data and the properties of FLRW models is such, that it is usually necessary to introduce an additional matter contents besides that, estimated from the visible matter, the so called dark matter, or even a cosmological constant. This might suggest that FLRW models are too crude an approximation to use in modelling the real universe (see e.g. [39]).

Alternative analyses of homogeneity based on "almost Killing vectors" [43] or "observational homogeneity" [44], do not yet seem able to resolve these issues.

Above all, a reliable description of the inhomogeneities in the expanding universe is wanting, since the inhomogeneities influence the propagation of light.

2.1 The "fitting" problem in cosmology

The basic idea here is that we do not *a priori* assume that the universe is necessarily well described at all times by the FLRW model, but nevertheless decide to use such a model for, say, pragmatic reasons [42].

The problem is then how to determine a "best-fit" between a clumpy cosmological model $\mathcal{U}$, which is supposed to give a realistic representation of the universe, including all inhomogeneities down to some specified length scale, and a smoothed out, idealised FLRW model $\mathcal{U}'$. The focus in this approach is on the relation between the idealised model and more realistic descriptions of reality. Therefore, one should also be able to specify details of that fitting, including the issue of how good the fit is. Basically, one could aim at the repeated use of the smoothing procedure, i.e., to consider a best fit between any lumpy universe model and a model $\mathcal{U}''$, which gives an even better description of the real universe than $\mathcal{U}'$, by describing the inhomogeneities at an even more detailed level.

In principle, this process would allow one to determine the best description at any prescribed level of detail.

The above approach can be implemented in many ways, based on:

(i) the space of spacetimes,

(ii) initial data for spacetimes,

(iii) the "gauges" adopted in perturbation studies,

(iv) near equivalence,

(v) average behaviour,

(vi) null data,

(vii) normal coordinates.

See [42] for more details and a specific null data approach.

Although, the (i) approach gives a useful concept of the fitting idea, it does not take into account the dynamics of General Relativity. In order to do so, we will advocate the averaging procedure within (ii) approach. We consider the space S^* of initial data for a spacetime, with initial data given on a spacelike hypersurface Σ : $(g_{ab}, K^{ab}, \mu, q^a)$, where, g_{ab} is the 3-metric on Σ, K^{ab} its second fundamental form, μ the matter energy density, q^a a 3-momentum relative to Σ. Given this, each point Q in S^* will correspond to a specification of all of these quantities at each point on the initial hypersurface Σ, chosen so as to satisfy the Einstein constraints on Σ.

The problem then can be stated in the following way: to define an appropriate distance function on S^*, such that when given $\mathcal{U}$, we choose $\mathcal{U}'$, so as to minimize the distance in S^* between the given point Q and Q'. Note, that one has to take into account here, the fact that different data can represent the same cosmology $\mathcal{U}$ (e.g., choose two different spacelike slicing Σ of $\mathcal{U}$ to get different 3-space metrics g_{ab}). However, involved here is the Hamiltonian structure of GR, and the symplectic form of that structure gives a way of comparing metrics on two different spacelike hypersurfaces, to see if they represent slicings of the same space-time. Therefore, in principle, once the fit of 3-metrics is determined, the 4-dimensional fit would be as well, even if not explicitly known.

2.2 The "averaging" problem in cosmology

One important way of thinking of the use of a smoothed-out model is that it represents the average properties of the inhomogeneous model. If $\mathcal{U}^*$ is the smoothed-out model universe, obtained form a clumpy one $\mathcal{U}$ by an averaging procedure, then it represents the nature of $\mathcal{U}$, when described over some averaging length scale L. The best-fit FLRW model $\mathcal{U}'$ should be the same as the averaged model $\mathcal{U}^*$, if one can indeed describe the large scale nature of $\mathcal{U}$ by a FLRW space-time [42].

We want to average the geometry and matter present, i.e., to determine both sides of Einstein's field equations. Considering the above kind of averaging, one has to determine a relation between the manifold structures and corresponding points in the two models, that we deal with at each step of averaging. Finally, we would like to explicitly determine the averaged field equations responsible for the large-scale, average dynamics, in order to study the observational properties of the average universes and the relation between more detailed and the averaged behaviour in them (see [35] and particularly [36]).

In the weak-field or almost FLRW cases, one can use direct methods to define the averaging, and in this thesis we also give a review of some of them (see section (3.2)).

The issue of averaging raise significant problems about the effective field equations at a cosmological scale, which are expected not to be the simple Einstein equations, since the process of averaging does not commute with taking the field equations [36]. Einstein's field equations are tested and believed to hold on a particular scale and it is not *a priori* clear, that in the inhomogeneous situation the geometrically averaged space-time metric representing the situation on some other scales will obey the same field equations.

It is conceivable that the Einstein theory of gravity might be a sort of "mean field" theory, that when applied on some other large scale should be corrected to take into account small scale inhomogeneities. Indeed, this is a problem of the **macroscopic** description of the gravitational field, that is to say, a problem of averaging out Einstein's equations which are microscopic in a physical sense.

The widely used approach in cosmology is not strictly speaking correct, since there one uses the so called perfect fluid form of the energy - momentum tensor. This means, that an effective averaging of real inhomogeneities has been carried out, while the unchanged left hand side of Einstein's equations is taken to describe the averaged gravitational field. However, we should bear in mind that the Einstein equations are highly non-linear, which is why any averaging process is far from trivial in general[1], in other words the averaging process may change their structure and consequently the geometric and physical meaning of the macroscopic gravitational field would be changed. In particular, Ellis conjectured [36] that, upon smoothing out the spacetime geometry, there would appear geometric correction terms in

[1]Unlike in electrodynamics, where the macroscopic Maxwell equations can be derived by averaging out the microscopic Maxwell-Lorentz equations over 4-regions in Minkowski space-time.

the sources to Einstein equations, that may have influence on the dynamics of the universe. In general, there would always be a non-zero backreaction of inhomogeneities on the dynamic behaviour at the smoothed out scales affecting the expansion rate and the estimate of age for the universe.

At the end, we would like to comment on a statistical kind of averaging. Suppose, that we have a statistical ensemble containing all possible density and velocity distribution of fluid elements (representing galaxies) with some constraints, which characterize the universe we wish to approximate. Choosing the particular ensemble in which the density and velocity distribution satisfy the condition $< \delta\mu > = < v^i > = 0$, and the averaging of any quantity with spatial derivative vanishes, then the averaging procedure obtained could be also appropriate to treat the situation where there are singularities, unlike within the spatial averaging concept [32].

Finally, the whole problem is the problem of choice of gauge in disguise, and it cannot be avoided if one is to investigate non-linear gravitational processes, either in the context of determining the effective field equations at cosmological scales, or of investigating the structure formation problem in the non-linear regime.

Chapter 3

Approximation schemes for constructing inhomogeneous models of the universe in General Relativity

3.1 Perturbation approaches

There is the "usual" perturbation approach, by which we mean here that one first introduces a fixed, i.e., unaffected by the perturbations, background space-time, e.g. the FLRW metric, and assumes that the perturbation variables in the given background are small. With this assumption, one can expand these variables to higher orders, keeping only the zero and first order terms. This allows to tackle only weakly non-linear situations. The important fact is ignored in this approach, namely, that the material distribution itself determines the geometry and thus one cannot specify the background metric independently from the inhomogeneities.

The study of perturbations of the Einstein equations in the cosmological context started with the pioneering work of Lifshitz [6]. Of particular interest, are the scalar perturbations since they are directly related to density fluctuations in the early universe and are thus relevant to the structure formation. Lifshitz's theory is however not easy to interpret due to its gauge ambiguity. This ambiguity is eliminated in the theory of gauge invariant perturbations due to Bardeen [7]. Both approaches are in reality valid only in the linear regime.

In principle, there exists a general method of determining the equations for any order of perturbations, but in practice the generalization of these schemes to the non-linear situations is not straightforward.

Recently, a new gauge invariant version of perturbation theory has been given [8, 9] (henceforth we call it EBH scheme) and we review it below (see also [13]).

The standard $\delta\mu/\mu$ approach compares two evolutions (the actual one and a fictitious comparison one) along a world line. The covariant and gauge-invariant EBH scheme compares evolutions along neighbouring worldlines in the actual universe, reflecting thus the spatial density variation in the fluid.

The advantages of this scheme are, firstly that it does not necessarily assume the background geometry *a priori*, since exact equations are given governing the evolution of density inhomogeneities in arbitrary space-time without any reference to a background space-time. Secondly, it deals with exact quantities, like e.g. the comoving fractional gradient of the energy density orthogonal to the fluid flow (spatial projection of the energy density gradient). These are both directly observable and gauge invariant in the case of linear perturbations about FLRW universe. It has been shown that the linearized EBH equations are equivalent to the Bardeen gauge invariant equations [10]. The EBH equations are, however, not restricted to the linear case[1].

We consider the exact covariant fluid equations for a general fluid flow in a curved space-time [11]. The 4-velocity vector tangent to the flow lines (the world-lines of fundamental observers in the universe, which are at rest w.r.t. our volume element of fluid) is $u^a = dx^a/d\tau$ $(u^a u_a = -1)$, where τ

[1] An extension of EBH scheme, combined with the spatial averaging to tackle non-linear case will be reported elsewhere.

is the proper time along the fluid flow lines. The projection tensor into the tangent 3-spaces orthogonal to u^a (into the local rest frame of a comoving observer) is

$$h_{ab} \equiv g_{ab} + u_a u_b \tag{3.1}$$

and $h_b{}^a h_c{}^b = h_c{}^a, h_a{}^b u_b = 0$.

The time derivative of any tensor $T^{ab}{}_{cd}$ along the fluid flow lines is $\dot{T}^{ab}{}_{cd} \equiv T^{ab}{}_{cd;e}u^e$, the covariant derivative along u^a (the rate of change of $T^{ab}{}_{cd}$ as measured by a fundamental observer).

The first covariant derivative of the 4-velocity vector is

$$u_{a;b} \;=\; \omega_{ab} + \sigma_{ab} + \frac{1}{3}\Theta h_{ab} - \dot{u}_a u_b \tag{3.2}$$

where $\Theta \equiv u^a{}_{;a}$ is the expansion, $\omega_{ab} = \omega_{[ab]}$ is the vorticity tensor ($\omega_{ab}u^b = 0$), and $\sigma_{ab} = \sigma_{(ab)}$ is the shear tensor ($\sigma_{ab}u^b = 0,\ \sigma^a{}_a = 0$). A representative length scale S along the flow lines is defined by

$$\dot{S}/S = \frac{1}{3}\Theta. \tag{3.3}$$

The vorticity and shear magnitudes are defined by $\omega^2 = \frac{1}{2}\omega_{ab}\omega^{ab}$, $\sigma^2 = \frac{1}{2}\sigma_{ab}\sigma^{ab}$.

As we restrict our attention to the case of a perfect fluid, the matter stress tensor takes the form

$$T_{ab} = \mu u_a u_b + p h_{ab} \tag{3.4}$$

where, μ is the energy density, $\mu = T_{ab}u^a u^b$ and the pressure $p = \frac{1}{3}h^{ab}T_{ab}$ (in the local rest frame of a comoving observer). In general, μ and p will be related through an equation of state.

In a FLRW universe model the shear, vorticity, acceleration, and Weyl tensor vanish, and the energy density μ, the pressure p and expansion Θ are functions of the cosmic time t only. Three simple gauge invariant quantities give us the information we need to discuss the time evolution of density fluctuations.

The first is the *comoving fractional density gradient*

$$\mathcal{D}_a \equiv S h_a{}^b \frac{\mu_{,b}}{\mu} \tag{3.5}$$

which is gauge-invariant and dimensionless, and represents the spatial density variation over a fixed comoving scale [8]. Note that S, and so $\mathcal{D}_a$, is defined only up to a constant along each world-line by equation (3.5); this allows it to represent the density variation between any neighbouring world-lines. The time variation of this quantity precisely reflects the relative growth of density in neighbouring fluid comoving volumes.

The second is the *pressure gradient*

$$\mathcal{Y}_a \equiv h_a{}^b p_{,b} \,. \tag{3.6}$$

The third is the *comoving expansion gradient*

$$\mathcal{Z}_a \equiv S h_a{}^b \Theta_{,b} \,. \tag{3.7}$$

We can determine exact propagation equations along the fluid flow lines for these quantities, and then linearize these to the almost-FLRW case. The linearized equations are those given in [12] (see equations (13) to (19) there) plus the linearized propagation equations for the gauge-invariant spatial gradients defined above (see [8, 9] and [13]).

The basic equations are: the energy and momentum-conservation equations (the time and space components of the 4-dimensional equation $T^{ab}{}_{;b} = 0$)

$$\dot{\mu} + (\mu + p)\Theta = 0, \tag{3.8}$$

$$(\mu + p)\dot{u}_a + \mathcal{Y}_a = 0; \tag{3.9}$$

the Raychaudhuri equation

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) + \frac{1}{2}\kappa(\mu + 3p) - \dot{u}^a{}_{;a} = 0 \tag{3.10}$$

where, $\dot{u}^a{}_{;a}$ is the acceleration divergence; and the propagation equations for the gauge-invariant variables $\mathcal{D}_a$, $\mathcal{Z}_a$:

$$h_c{}^a(\mathcal{D}_a)^{\cdot} = -\mathcal{D}_a(\omega^a{}_c + \sigma^a{}_c) + \frac{p}{\mu}\Theta\mathcal{D}_c - (1 + \frac{p}{\mu})\mathcal{Z}_c, \tag{3.11}$$

$$\begin{aligned} h_c{}^a(\mathcal{Z}_a)^{\cdot} \;=\; & -\Theta\mathcal{Z}_a + h_c{}^a(\frac{1}{3}\Theta\mathcal{Z}_a - \frac{1}{2}\mu\kappa\mathcal{D}_a - 2S(\sigma^2)_{,a} + 2S(\omega^2)_{,a} + \\ & S\dot{u}^b{}_{;ba}) - \mathcal{Z}_b(\sigma^b{}_c + \omega^b{}_c) + \dot{u}_c S\mathcal{R}, \end{aligned} \tag{3.12}$$

where, $\mathcal{R} \equiv \frac{1}{3}\Theta^2 - 2\sigma^2 + 2\omega^2 + \kappa\mu + \dot{u}^a{}_{;a}$.

In the above, $\dot{u}^b{}_{;ba}$ stands for the gradient of the acceleration divergence, and $\kappa = 8\pi G$.

Once the equation of state of the fluid is known, the evolution of $\mathcal{Y}_a$ will follow from that for $\mathcal{D}_a$.

For completeness, we give also the propagation equation for the acceleration $a_c \equiv \dot{u}_c$

$$h_a{}^c(a_c)^{\cdot} = a_a\Theta(\frac{dp}{d\mu} - \frac{1}{3}) + h_a{}^b\left(\frac{dp}{d\mu}\Theta\right)_{,b} - a_c(\omega^c{}_a + \sigma^c{}_a) \tag{3.13}$$

($\frac{dp}{d\mu}$ is taken along the fluid flow lines).

Further equations can be used as the basis of various systematic approximation schemes. The major point to notice is that in using equations (3.11) and (3.12) in an approximation scheme to determine propagation of density inhomogeneities to the nth. order, we only need solve the other equations of the model to the $(n-1)$th. order [33]. This gives the behaviour of the coefficients in these equations (and the Christoffel terms implied in the covariant derivatives on the left) to that order; then they directly determine the behaviour of inhomogeneities at the nth. order.

As equations (3.11) and (3.12) are gauge-invariant as well as covariant, we can use any coordinates and any convenient choice of background FLRW model in their further investigation. However equations (3.8) and (3.10) are not gauge-invariant; we can deal with this by using an averaging procedure to determine a background model (see footnote[1] on page 11).

3.2 Approaches based on averaging

There is no covariant definition of spatial averaging available in a general space-time. The issue of fundamental importance for cosmology, e.g., in the context of structure formation and the interpretation of distance measurements or gravitational lensing, is the transition from an inhomogeneous model to an averaged (or smoothed) FLRW model. However, this averaging procedure is still not a fully understood problem. Since Einstein's equations are non linear, any averaging process is far from trivial in general.

The usual perturbation approaches are not suitable to tackle the non-linear case. In the presence of inhomogeneities, one cannot specify the background metric independently, since the material distribution itself determines the geometry (the backreaction problem). Recently, this fact has been taken into account, thus enabling the construction of realistic universe models by means of spatial averaging [30, 31, 32, 16, 29][2].

Below we give an overview of several known approaches.

3.2.1 Futamase's approach

In a series of papers [30, 31, 32] Futamase developed an approximation scheme for describing an inhomogeneous universe, valid in non-linear case, basically with an arbitrary density contrast. This is a perturbative approach and the averaging introduced, gives a clean separation between the global and local quantities.

The aim is to construct the approximate, reliable metric representing the real, clumpy universe in General Relativity. Obviously, the averaged, smooth metric coincides nowhere with the real inhomogeneous metric, but we know that, the FLRW description is valid, only in some averaged sense (if so). It seems then natural, to suppose that the space-time is close to a FLRW space-time, i.e., the inhomogeneous spacetime is in a sense a small deviation away from the averaged smooth spacetime, which is not *a priori* given.

The crucial observation is the fact, that the size of the metric perturbation and that of the density contrast are independent of each other in the exact theory, as well as in post-Newtonian approximations. For example, in the Solar System the metric coefficients in nearly orthonormal coordinates deviate from their special relativistic values, by no more than

[2] Whether the spatial averaging only is a fully successful scheme to tackle this problem will be discussed elsewhere; see also [46].

$\sim 2GM_\odot/c^2R_\odot \sim 10^{-6}$, whereas the density contrast between the interior of the Sun, planets and interplanetary space is $> 10^{20}$.

The *ansatz* for the metric is taken as:

$$g_{\mu\nu} = a^2(\eta)(\tilde{g}^{(b)}_{\mu\nu} + h_{\mu\nu}), \tag{3.14}$$

where, $h_{\mu\nu}$ are generated by local matter inhomogeneities and the gravitational waves (we neglect the latter), assumed to be small; this does not imply the smallness of density contrast. The scale factor $a(\eta)$ describes as usual, the global FLRW expansion (averaged). In other words, $g_{\mu\nu}$ is the standard FLRW metric when $h_{\mu\nu} = 0$, with $\tilde{g}^{(b)}_{\mu\nu} = -d\eta^2 + d\Omega^2_3(k)$, where $d\Omega^2_3(k)$ is the standard metric on S^3 if $k = 1$ and on $\Re^3$ if $k = 0, -1$.

The *ansatz* for the metric is such, that the deviations from the FLRW models are small, but this does not imply that the zero-th order space-time is the FLRW one. It depends on the approximation chosen, e.g., within the linear approximation the zero-th order space-time is indeed taken to be the FLRW space-time. The approximation chosen depends on the kind of physical situation that one deals with, for the case at hand, what we have in mind, is the matter clumps of various scales interacting gravitationally with each other and the density contrast between them and the mean density is $\gg 1$.

Above all, we have to restrict our space-times to those in which there is a well-defined meaning of the spatial average. The spatial averaging is therefore defined in a family of geometrically preferred slices, i.e., such that the metric deviation away from the FLRW metric is small everywhere on them.

The scheme is worked out in harmonic gauge:

$$\overline{h}^{\mu\nu}_{|\nu} = 0, \tag{3.15}$$

where, "$|$" stands for covariant derivation with respect to $\tilde{g}^{(b)}$, and $\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\tilde{g}^{(b)}_{\mu\nu}h$, with $h = \tilde{g}^{(b)\mu\nu}h_{\mu\nu}$, is the so called trace reversed perturbation.

The equations are derived under the assumption, that h is small and that the scale on which h varies is small compared to that of a and $\tilde{g}^{(b)}$.

The above *ansatz* is used to calculate the Einstein equations as follows (a prime stands for the derivative w.r.t. the conformal time η)

$$\begin{aligned}(\tfrac{a'}{a})^2(4\tilde{g}^{(b)\mu\eta}\tilde{g}^{(b)\nu\eta} - \tilde{g}^{(b)\eta\eta}\tilde{g}^{(b)\mu\eta}) - 2\tfrac{a''}{a}(\tilde{g}^{(b)\mu\eta}\tilde{g}^{(b)\nu\eta} - \tilde{g}^{(b)\eta\eta}\tilde{g}^{(b)\mu\nu}) + A^{\mu\nu} + \\ \tfrac{a'}{a}(2\overline{h}^{\eta(\mu|\nu)} - \overline{h}^{\mu\nu|\eta} - \tilde{g}^{(b)\eta(\mu}\overline{h}^{|\nu)} + \tfrac{1}{2}\tilde{g}^{(b)\mu\nu}\overline{h}^{|\eta}) - \tfrac{1}{2}\overline{h}^{\mu\nu|\rho}{}_{|\rho} = 8\pi G\tau^{\mu\nu}, \quad (3.16)\end{aligned}$$

where, $A^{\mu\nu}$ is the background spatial curvature term, given by $A^{\eta\eta} = -3k\tilde{g}^{(b)\eta\eta} = 3k$, $A^{ij} = -k\tilde{g}^{(b)ij}$, and $A^{\eta i} = 0$. $\tau^{\mu\nu} = a^4 T^{\mu\nu} + t^{\mu\nu}$ may be regarded as the effective (total) stress-energy tensor, i.e., material stress-energy tensor $T^{\mu\nu}$ plus gravitational stress-energy pseudotensor $t^{\mu\nu}$, consisting of terms quadratic in $\overline{h}$.

Effectively, this means that the Einstein equations are expanded in terms of two small parameters, ϵ and κ, whose meaning is the following:

ϵ is the size (amplitude) of the metric perturbation h and it is assumed that h, $h_{\mu\nu} \simeq \mathcal{O}(\epsilon^2)$, $h_{\mu\nu,\rho} \simeq \mathcal{O}(\epsilon^2/l)$; The ϵ defined this way is also an amplitude of the peculiar gravitational potential Φ, the gravitational potential generated by the inhomogeneous distribution of matter, where $\Phi \simeq \mathcal{O}(\epsilon^2)$, as well.

$\kappa = \frac{l}{L}$ is the ratio between the scale of the variation of h, and that of a and $\tilde{g}^{(b)}$, where for L we can take the present horizon size $\sim 10^4\, Mpc$, and for l the typical scale of inhomogeneities.

Note, that $\kappa \in [0, 1]$, with small κ indicating condensed density contrast, large κ - diffused density contrast.

The relative size of κ and ϵ depends on the physical system considered. It is straightforward to see, that the density contrast is of the order of ϵ^2/κ^2, and consequently the linear stage is characterised by $\kappa \gg \epsilon$, and in the non-linear stage we have $\epsilon \gg \kappa$. For galaxies we typically have $\epsilon > \kappa$, and the ratio ϵ/κ increases, when we consider smaller regions. Basically, if the typical size of inhomogeneities is larger than a galactic scale, the approximation is valid in the parameter range $\epsilon^2 \ll \kappa^3$.

[3] In [32] an approximate metric is constructed in the situation with strong gravity and/or smaller regions of inhomogeneity, where $\epsilon^2 \gg \kappa$, but we will not discuss this case here.

In deriving (3.16), terms like $\frac{a''}{a}\overline{h} \simeq \mathcal{O}(\epsilon^2/L^2)$, $\frac{a'}{a}\overline{h}_{|\rho}\overline{h} \simeq \mathcal{O}(\epsilon^4/lL)$, $\overline{h}_{|\rho}\overline{h}_{|\sigma}\overline{h} \simeq \mathcal{O}(\epsilon^6/l^2)$ and of higher orders, were neglected, whereby it was taken, that $\frac{a'}{a} \simeq \mathcal{O}(1/L)$ and $\frac{a''}{a} \simeq \mathcal{O}(1/L^2)$. Physically, it means that the effect of the self-gravity of clumps on their dynamics is more important than the expansion of the universe. The neglected terms are negligible as far as $\epsilon, \kappa \ll 1$ and $\kappa \gg \epsilon^2$.

A perfect fluid is taken for the material source, which is characterized by the density field ρ, its peculiar velocity, $\vec{v}$, and peculiar gravitational potential Φ;

$$T^{\mu\nu} = [\rho + p(\rho)]u^\mu u^\nu + p(\rho)g^{\mu\nu}. \tag{3.17}$$

One works with conformally rescaled variables: $\tilde{u}^\mu = au^\mu$, $\tilde{g}^{\mu\nu} = a^2 g^{\mu\nu}$. Then $\tau^{\mu\nu} = a^2\tilde{T}^{\mu\nu} + t^{\mu\nu}$, where, $\tilde{T}^{\mu\nu} = [\rho + p]\tilde{u}^\mu\tilde{u}^\nu + p\tilde{g}^{\mu\nu} = a^2 T^{\mu\nu}$.

On very large scales the universe is assumed to be homogeneous in space. In the next step the spatial averaging is applied to the truncated Einstein field equations (3.16), assuming spatial periodicity of the material initial data[4] (as well as of the free data for the gravitational field) and no coherent motion over the volume to be averaged[5].

The spatial average over a volume V is defined as:

$$< Q >= \frac{1}{V}\int_V Q\sqrt{\tilde{g}^{(b)}}d^3x, \tag{3.18}$$

where, $\tilde{g}^{(b)}$ is the determinant of the spatial part of the background metric $\tilde{g}^{(b)}_{\mu\nu}$ and $\sqrt{\tilde{g}^{(b)}}d^3x$ is the invariant volume element in the background space, and for the density we have $< \rho >= \rho_b$ (background density). The only property used in the calculation is $< Q_{|i} >= 0$, which is implied by the spatial periodicity. Also, $< \tau^{\eta i} >= 0$, which just means no coherent motion over the averaging volume.

The spatial average of (3.16) under the above requirements, implies $< \overline{h}^{\eta i} >= 0$, and of (3.15) $< \overline{h}_{\eta\eta} >= const.$, which under a redefinition of the

[4]Space-time averaging may be another possibility; in [40] temporal periodicity for the inhomogeneities had to be assumed, in order to get a separation between global and local evolution.

[5]This is almost always safe, i.e., with large enough averaging volumes and randomly distributed perturbations.

time variable and scale factor can be put to zero. Also, $<\overline{h}^k{}_k>$ can be put to zero, since it expresses an additional isotropic expansion, which can be absorbed into the scale factor upon its redefinition.

The averaged sources are used to calculate the global expansion and the following averaged Einstein equations are obtained from (3.16), to the first non-trivial order (by non-trivial order, we mean the order at which the first backreaction effect due to inhomogeneities on the expansion of the universe appears)

$$\begin{aligned} (\tfrac{a'}{a})^2 &= \tfrac{8\pi G}{3} < \tau^{\eta\eta} > -k \\ \tfrac{a''}{a} &= \tfrac{4\pi G}{3} < \tau^{\eta\eta} - \tau^k{}_k > -k \\ \tfrac{1}{a^2}(a^2 < \overline{h}^{ij} >_{|\eta})_{|\eta} &= 16\pi G < \hat{\tau}^{ij} >, \end{aligned} \tag{3.19}$$

where, $\hat{\tau}^{ij} = \tau^{ij} - \frac{1}{3}\tilde{g}^{(b)ij}\tau^k{}_k$ is the trace free part of τ^{ij}.

The averaged line element

$$< ds^2 >= a^2(\eta)(-d\eta^2 + (\delta_{ij} + < \overline{h}_{ij} >)dx^i dx^j), \tag{3.20}$$

tells us, that the averaged spacetime expands anisotropically, except when $< \overline{h}_{ij} >$ vanishes identically, since $< \overline{h}_{ij} >$ expresses the deviation from the isotropic expansion, due to the inhomogeneities $< \hat{\tau}^{ij} >$. Note, that the first two equations of (3.19) are the Friedmann equations with source terms replaced by the effective stress-energy tensor, therefore the effect of the local inhomogeneities on the global expansion can be partly expressed by the effective density and pressure $\rho_{eff} = a^2 < \tau^{\eta\eta} >$, $p_{eff} = \frac{1}{3}a^2 < \tau^k{}_k >$.

One can integrate the last equation of the system (3.19) to obtain the expression for $< \overline{h}^{ij} > (\eta)$, and see that a sufficient condition for global isotropic expansion is given by $< \overline{h}^{ij} >_{,\eta} (\eta_o) \equiv 0$ and $< \hat{\tau}^{ij} >\equiv 0$. The equations determining the evolution of the local inhomogeneities are derived by substituting the above equations into (3.16), additionally we have also the equations of motion (derived from the conservation of the stress-energy tensor).

Now, we can employ a particular approximation scheme. The evolution of the density perturbations in our picture is sufficiently well described by means of a post-Newtonian approximation. The post-Newtonian approximation is characterized by small parameter ϵ, of the order of a typical peculiar velocity divided by the speed of light. ϵ is introduced by a coordinate transformation $\eta_N = \epsilon\eta$, η_N is the Newtonian time, which means physically that a typical time scale gets longer as ϵ^{-1} as the velocity goes to zero. This parameter is identified with the already introduced ϵ.

The orders for material variables are assumed to be $\rho_N = \epsilon^{(-2)}\rho$, $v^i_N = \epsilon^{(-1)}v^i$, $p_N = \epsilon^{(-4)}p$.

The other small parameter, κ, is associated with global cosmic expansion (when $\kappa \to 0$, the expansion of the universe slows down). ϵ and κ are our order parameters, in the sense, that they parameterize a sequence of spacetimes and we study the Newtonian limit on that sequence.

The evolution equations for the local inhomogeneities are solved perturbatively up to the first non-trivial order.

An interesting outcome of the application of this approximation scheme is, that the backreactions lead to an underestimation of the age of the universe as inferred from a measurement of today's Hubble constant [22] (also [18]). For a simple model within the framework of pancake theory for structure formation on a flat expanding background, it is shown in [22] that the age problem (severe in view of the recent determinations of globular cluster ages) may be solved by taking into account the backreactions of inhomogeneities in an averaged sense.

This scheme can also be used for the correct interpretation of observations of gravitational lenses.

3.2.2 Kasai's approach

Kasai's scheme [16] to construct inhomogeneous relativistic universes which are homogeneous and isotropic on average, goes a bit farther than Futamase's approach. It is not assumed here that the deviations from a FLRW model are small to acquire FLRW-like behaviour on average.

Here also, spatial averaging is introduced, but the description is based on the deformation tensor and can give yet another possibility to obtain solutions to describe more realistic situations, and on the other hand to formulate a relativistic version of the Zel'dovich approximation, used to handle the evolution of the large scale structure in Newtonian cosmology.

In [16] the inhomogeneous irrotational dust universe models are constructed in the framework of General Relativity, with the property of being homogeneous an isotropic on average.

The averaging is introduced for matter only, on the hypersurfaces Σ_t orthogonal to dust motion, and the mean ("background") density for the inhomogeneous universe model is written as

$$\rho_b =< \rho >\equiv \lim_{V\to\Sigma_t} \frac{1}{\int_V [det(g_{ij})]^{1/2} d^3x} \int_V \rho [det(g_{ij})]^{1/2} d^3x, \tag{3.21}$$

for $V \subset \Sigma_t$ (assuming that this limit exists).

The "scale factor" (averaged) is then defined by

$$\dot{\rho}_b + 3(\frac{\dot{a}}{a})\rho_b = 0. \tag{3.22}$$

The peculiar deformation tensor is introduced as

$$V^i_j \equiv u^i_j - \frac{\dot{a}}{a}\delta^i_j. \tag{3.23}$$

This quantity describes the deviation from a uniform Hubble expansion. As usual, u^μ is a 4-velocity and comoving coordinates are used.

The deformation tensor $u^i_j \equiv u^i_{;j}$, describing the change of the relative position X^i between the world lines of neighbouring "particles" (galaxies), $\dot{X}^i = u^i_j X^j$, is at the same time, the extrinsic curvature of the $t = const.$ hypersurfaces Σ_t.

Another quantity introduced is a density contrast[6]

$$\Delta \equiv \frac{\rho - \rho_b}{\rho}. \tag{3.24}$$

Given this, the Einstein equations are the following:

$$\dot{\Delta} + (1 - \Delta)V^i_i = 0 \tag{3.25}$$

[6]Note, that it differs form the conventionally adopted $\delta \equiv \frac{\rho-\rho_b}{\rho_b}$.

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\frac{\rho_b}{1-\Delta} - \frac{1}{6}{}^{(3)}R - \frac{2}{3}\frac{\dot{a}}{a}V^i_i - \frac{1}{6}[(V^i_i)^2 - V^i_jV^j_i] \qquad (3.26)$$

$$\ddot{\Delta} + 2(\frac{\dot{a}}{a})\dot{\Delta} - 4\pi G\rho_b\Delta = -(1-\Delta)[(V^i_i)^2 - V^i_jV^j_i] \qquad (3.27)$$

where, $\dot{} \equiv \frac{\partial}{\partial t}$, ${}^{(3)}R = {}^{(3)}R^i_i$ and $i, j = 1, 2, 3$.

FLRW model (with ρ_b and a) is the background model. Note, that (3.26) reduces to the Friedmann equation $(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3}\rho_b - \frac{k}{a^2}$, when there are no inhomogeneities , if and only if the condition

$$\frac{8\pi G}{3}\rho_b\frac{\Delta}{1-\Delta} - \frac{1}{6}{}^{(3)}R + \frac{k}{a^2} - \frac{2}{3}\frac{\dot{a}}{a}V^i_i - \frac{1}{6}[(V^i_i)^2 - V^i_jV^j_i] = 0 \qquad (3.28)$$

holds.

On the other hand, when the LHS of (3.27) is zero, one gets the evolution equation for $\delta \equiv \frac{\rho-\rho_b}{\rho_b}$ in the linear perturbation theory.

The nice property of this approach is the fact, that equations similar to (3.25) and (3.27) appear in Newtonian cosmology in the context of extending the Zel'dovich-type approximations, with V^i_j spatial gradient of peculiar velocity $v^i_{,j}$.

In particular, when $(V^i_i)^2 - V^i_jV^j_i = 2(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) = 0$, where λ_i are eigenvalues of V^i_j, Δ obeys the same equation as δ and the solutions can be extrapolated from the result of the linear perturbation theory.

Therefore the present construction can represent "relativistic pancake solutions" analogous to those in Newtonian cosmology.

3.2.3 Zotov and Stoeger's approach

The scheme in this approach is the following: the parameter values characterizing the averaged model are the averages (over some volume on a chosen set of space-like hypersurfaces) of the values of the same parameters in the clumpy model [29].

In this approach one can see what the effect of smoothing is on the form of the field equations, since we do not appeal to some standard form of Einstein's equations at each stage of smoothing.

On a scale in which galaxies are the points of inhomogeneity, we have

$$\overline{G}_{\mu\nu} = \kappa\overline{T}_{\mu\nu} + P_{\mu\nu}, \tag{3.29}$$

where, $\overline{G}_{\mu\nu}$ is the Einstein tensor calculated from the averaged metric components, independently from the averaged stress-energy tensor, and $P_{\mu\nu}$ represents the effect of small scale inhomogeneities on the dynamics of the large-scale universe.

Since, $\overline{G}_{\mu\nu} = G_{\mu\nu} + \delta G_{\mu\nu}$, we can write

$$P_{\mu\nu} = \kappa(T_{\mu\nu} - \overline{T}_{\mu\nu}) + \delta G_{\mu\nu}. \tag{3.30}$$

The interpretation of $P_{\mu\nu}$ is then such that, it represents the difference between the energy-momentum at a point and the averaged energy momentum, minus the difference between the Einstein tensor there and the averaged Einstein tensor. It contains a part due to internal energy of gravitational sources and a part due to changes in geometry.

It is not assumed in the present scheme that the difference between the averaged metric and the real inhomogeneous one, $g_{\mu\nu}$, is everywhere small. The aim is spelled out clearly, namely, to construct the background on some definite length scale, using the inhomogeneities and an averaging procedure. However, it has to be stressed that it is necessary to assume some initial background in order to be able to define a volume element for averaging.

Consider a space-like distribution of inhomogeneities (here, galaxies of equal mass). We have then the Schwarzschild metrics embedded in FLRW background (another reason for assuming an initial background is that some background has to be assigned to the space between non-overlapping Schwarzschild regions before averaging). Clearly, the background is expanding, and because the background density is $\ll$ than the critical density of the universe, one can take the curvature parameter $k = -1$ for the background geometry. The averaging volume is a sphere. The metric is taken to be isotropic, as then only the first metric coefficient g_{rr} has to be averaged.

Galaxies are assumed to be non-interacting, as a consequence the background will be a linear superposition of Schwarzschild metrics due to the individual galaxies.

In the first step, the $r - r$ component of Schwarzschild metric is averaged out (in the usual way), giving $< g_{rr} >$ in a small coordinate volume surrounding a galaxy, within a sphere of a certain radius up to which the gravitational influence of the galaxy is significant.

The next averaging step is done over a volume containing many galaxies, and we get

$$\overline{g}_{rr} = [< g_{rr} > V_1 N + g^o_{rr}(V_2 - V_1 N)]/V_2, \tag{3.31}$$

where, g^o_{rr} refers to the background metric and $g^o_{rr} = \frac{R^2(t)}{(1-\frac{r^2}{4})^2}$ (with R the scale factor), V_1, V_2 are the first and second averaging volumes respectively, and N is the coordinate density of galaxies = *constant*.

In the limit of $R(t) \rightarrow \infty$ for large $t \rightarrow \infty$, one can write the line element as:

$$ds^2 = \frac{S^2(t)}{(1 - \frac{r^2}{4})^2}(dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2) - c^2 dt^2, \tag{3.32}$$

where, $S^2(t) = R^2(t)(1 - K)$, with $K \equiv \frac{NV_1}{V_2}$.

Because the formula (3.32) still has FLRW form, the forms of the Einstein tensor $(\overline{G}_{rr}, \overline{G}_{tt})$ obtained from $S(t)$ will remain the same. It can be explicitly shown that $\delta G_{rr}, \delta G_{tt} \neq 0$.

The conclusion therefore, is that for large t, $S(t) < R(t)$, and so the rate of expansion of the inhomogeneous universe is smaller than the background rate. Futamase arrived also to similar conclusions about the effect of inhomogeneities on the rate of expansion of the universe, as was discussed previously.

The quantitative results are uncertain due to the fact that the present density of galaxies is not well known. Also, N is not really a *const.* As the density of inhomogeneities grows, either their number in the comoving volume must decrease, or the background density must decrease at a faster rate than that due to cosmic expansion, but this has not been taken into account here.

3.3 The Green Function approach

With the aim of studying the effects of a given matter distribution on the metric, and hence on the radiation, Jacobs and colleagues put forward a new scheme of determining the realistic metric of our universe [27, 28].

The idea is of solving the field equations through the use of scalar harmonics as spatial basis functions, while avoiding the use of any averaging procedure for the metric perturbations. Small metric perturbations are assumed (again, this does not restrict the size of perturbations to the matter variables), and the global expansion rate is that of FLRW model. Assumed is also, the matter distribution and its evolution (known from observations and/or theory). The results do not assume a particular model for the formation of structure in the matter distribution, and are valid everywhere in our universe outside of strong field regions.

In the presence of inhomogeneities we have

$$ds^2 = a^2(\eta)[\gamma_{\mu\nu}(\vec{x}) + h_{\mu\nu}(\eta, \vec{x})]dx^\mu dx^\nu, \tag{3.33}$$

where, $h_{\mu\nu}$ describes the metric perturbations. It is assumed, that $h_{\mu\nu} \equiv \mathcal{O}(\epsilon^2) \ll \gamma_{\mu\nu} \equiv \mathcal{O}(1)$ (background terms are of order 1), then $\Delta_\delta h_{\mu\nu} \simeq \mathcal{O}(\epsilon^2/\kappa)$. [7] Also, $\epsilon^2 \ll 1$ and $\epsilon^2 \ll \kappa$. The latter means that the matter inhomogeneities move non-relativistically and the effective stress-energy of metric perturbations is small.

Since the background is homogeneous and isotropic one can perform a separation of space and time dependencies in the field equations, enabling perturbations to be written not as functions of $h_{\mu\nu}$, but as harmonic decomposition. The spatial dependence of perturbations is then expanded as eigenfunctions (normal modes) of the covariant Laplacian ${}^{(3)}\nabla^2$ on the 3-dimensional static background γ_{ij}. The field equations are reduced this way to the equations for the time dependent amplitudes of the modes.

Only scalar harmonics (scalar modes Q) are considered, in terms of solutions of

$${}^{(3)}\nabla^2 Q(\vec{x}, \vec{q}) = -q^2 Q(\vec{x}, \vec{q}) \tag{3.34}$$

[7] The parameters ϵ and κ are the same as described above in the Futamase approach.

and obviously, $Q \simeq \mathcal{O}(1)$ and $\nabla_i Q = \mathcal{O}(q) \simeq \mathcal{O}(\kappa^{-1})$.

The metric perturbations (h_{oo}, h_{oi}, h_{ij}) are then expanded in terms of scalar harmonics Q [7]. The longitudinal gauge is assumed, and the Einstein tensor is written, including terms linear in $h_{\mu\nu}$ and its derivatives. Non-linear terms of $\mathcal{O}(\epsilon^4)$, $\mathcal{O}(\epsilon^4/\kappa)$, $\mathcal{O}(\epsilon^4/\kappa^2)$, or smaller are neglected. With this, we retain non-linear interactions of energy density of perturbations and their backreaction on the FLRW component. The stress-energy tensor is constructed in the usual way, taking perfect fluid as the background model, and perturbations (scalar) to the energy density μ, pressure p, and velocity v_i. The components of $T_{\mu\nu}$ are then written to the first order in the velocity.

By exploiting the harmonic decomposition of the field equations one can solve them by taking their spatial projections against different scalar modes. The important result is then

$$ds^2 = a^2[-(1+2\phi)d\eta^2 + (1-2\phi)\gamma_{ij}dx^i dx^j], \tag{3.35}$$

where, $\phi(\eta, \vec{x}) = -\frac{1}{2}h_{oo} = -\int d\mu(\vec{q})Q(\vec{x}, \vec{q})H(\eta, \vec{q}) + \mathcal{O}(\epsilon^4)$, and $H \simeq \mathcal{O}(\epsilon^2)$ is the amplitude ($d\mu$ is the measure associated with the eigenvalue spectrum).

$\phi(x^\mu)$ is the effective quasi-Newtonian potential of the inhomogeneities, characterizing metric perturbations.

In terms of the matter variables the equation for ϕ can be obtained from the following equation for H:

$$3(\frac{a'}{a})H' + (q^2 + 8\pi a^2\mu - 6k)H = 4\pi a^2 \mu \Delta, \tag{3.36}$$

where, $\Delta(\eta, \vec{q})$ is a suitable density fluctuation variable describing perturbations to the energy density μ, and *a priori* $|\Delta| > 1$; as usual $' = \frac{d}{d\eta}$.

To this level of approximation the Friedmann equation holds as well, and there are two additional equations relating the metric perturbations to the pressure and velocity perturbations.

The estimation of the orders of magnitude of matter variables perturbations allows us to conclude, that in any allowed regime (linear $\epsilon/\kappa \ll 1$, non-linear $\epsilon/\kappa \gg 1$) the pressure and velocity perturbations are much weaker than the density fluctuations. In other words, the metric perturbation $H(\eta, \vec{q})$ are determined primarily by $\Delta(\eta, \vec{q})$, i.e., hydrodynamically the density fluctuations can be treated as the source.

To any order of magnitude arguments we have to consider effects on the scale factor a, since it makes an implicit contribution. On physical grounds:

$$a(\eta) = a_{FLRW}[1 + \mathcal{O}(< \epsilon^4/\kappa^2 >)], \qquad (3.37)$$

and clearly, $\mathcal{O}(< \epsilon^4/\kappa^2 >) \ll \epsilon^4/\kappa^2 \ll 1$, so using the background scale factor does not alter the arguments about the matter variables perturbations.

Solving equation (3.36) for $H(\eta, \vec{q})$ we can obtain the pseudo-Newtonian potential:

$$\begin{aligned} \phi(\eta, \vec{q}) \;\; = \;\; & \int dV(\vec{y}) G(\eta_o, \eta, \vec{x}, \vec{y}) \phi(\eta_o, \vec{y}) - \frac{4\pi}{3} \int_{\eta_o}^{\eta} du \frac{a^3 \mu}{a'} \int dV(\vec{y}) \\ & G(u, \eta, \vec{x}, \vec{y}) \Delta(u, \vec{y}) + \mathcal{O}(\epsilon^4), \end{aligned} \qquad (3.38)$$

where, dV is a coordinate volume element, and $G(u, \eta, \vec{x}, \vec{y})$ is a Green function for metric perturbations due to the scalar density fluctuations in a FLRW background.

In case of flat spatial sections ($k = 0$) it can be proved that

$$G_{k=0}(u, \eta, \vec{x}, \vec{y}) = \frac{a(u)}{a(\eta)} \frac{1}{[4\pi C(u, \eta)]^{\frac{3}{2}}} \exp[-\frac{|\vec{y} - \vec{x}|^2}{4C(u, \eta)}]. \qquad (3.39)$$

The latter gives an interesting analogue with the diffusion [28].

The Green function expression for the potential can be reduced to a Newtonian form $\phi_{Newt} \simeq - \int dV (\frac{3}{8\pi} (\frac{a'}{a})^2 \frac{\Delta}{|\vec{y} - \vec{x}|}) \simeq - \int a^3 dV \frac{\mu \Delta}{a|\vec{y} - \vec{x}|}$ under appropriate conditions. But more interesting are situations where the time evolution of density fluctuations makes a significant contribution to the metric, e.g. post-Newtonian ones.

Formula (3.38) offers a relativistically correct way of calculating the metric perturbations, taking into account the cosmological expansion, non-linear density evolution and also, e.g. deviations from the thin-lens approximation.

Chapter 4

Smoothing of cosmological spacetimes. The averaging problem revisited.

We are interested in setting up a program for approximating the evolution of cosmological space-times solutions of Einstein's equations via the development of a procedure for "smoothing" sets of initial data for such space-times.

Looked at this way, smoothing is equivalent to a physical approximation scheme for particular space-times. The idea is the following. Given an initial data set: the spatial metric g, the extrinsic curvature K, and matter fields ψ, the aim is to build a new smooth initial data set $(\overline{g}, \overline{K}, \overline{\psi})$ (smooth means spatially homogeneous), so that the new initial data is more easily evolved than the old one, and at the same time the evolution of new initial data models certain aspects of the evolution of the original initial data.

On the other hand, one can think of smoothing as a mathematical method for making general statements about a collection of space-times. The smoothing procedure could then be used as a map from general space-times to the spatially homogeneous ones, in order to study the space of space-times, in particular, the collection of space-times whose large scale dynamics are closely represented by the dynamics of spatially homogeneous space-times [15].

The flows of the metrics are an important part of the smoothing we have in mind[1].

In the following, after giving some necessary mathematical preliminaries, we expand on the above ideas in more detail.

4.1 Mathematical preliminaries

4.1.1 Some Riemannian Geometry

In this chapter we recall some basic concepts in Riemannian Geometry [19, 20] (see also [41]).

Let $\mathcal{M}$ be a smooth C^∞, Hausdorff, connected, oriented, compact n-dimensional manifold without boundary and let g be a Riemannian metric on $\mathcal{M}$, i.e., a smoothly varying family of inner products G_x on the tangent spaces $T_x\mathcal{M}$, $x \in \mathcal{M}$.

A metric g on $\mathcal{M}$ is called an *Einstein* metric if the Ricci curvature $Ric(g) = \lambda g$ for some constant λ. By normalization, one can always assume to be in one of the three cases: $Ric(g) = g$ (when $\lambda > 0$), $Ric(g) = 0$ ($\lambda = 0$) or $Ric(g) = -g$ (when $\lambda < 0$). We use the term "Einstein manifolds" for Riemannian manifolds of constant Ricci curvature.

Let $c:\ [0, a] \to \mathcal{M}$ be a curve, and $0 = a_o < a_1 < \ldots < a_n = a$ be a partition of $[0, a]$ such that $c_{|[a_i, a_{i+1}]}$ is of class C^1. The *length of* c is defined by

$$L(c) = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} |c'(t)| dt, \tag{4.1}$$

where, $|c'(t)| = \sqrt{g(c'(t), c'(t))}$.

The length of a curve does not depend on the choice of a regular parameterization.

[1]In the present thesis we discuss only the Ricci-Hamilton flow; the results on the renormalization group flow of metrics and its relation to the Ricci-Hamilton flow will be reported elsewhere.

The *Riemannian distance* between two points x and y in $\mathcal{M}$ is defined to be the infimum of the length (w.r.t. g) of the curves from x to y. The *diameter* D of $(\mathcal{M}, g)$ is the diameter of $\mathcal{M}$ for the Riemannian distance.

The *geodesics* are the curves which satisfy the Euler-Lagrange equation of the problem of minimization of the energy of a curve. In particular, given any point x in $\mathcal{M}$ and any unit vector $u \in T_x\mathcal{M}$, there is (locally) one and only one geodesic $c_{x,u}$ parameterized by arc length t, such that $c_{x,u}(0) = x$ and $\dot{c}_{x,u}(0) = u$ (such a geodesic is defined for all values of t when $\mathcal{M}$ is closed).

We define the *exponential map* $\exp_x : T_x\mathcal{M} \to \mathcal{M}$, by $\exp_x(tu) = c_{x,u}(t)$, for any $t \geq 0$ and any unit tangent vector u. The exponential map is a local diffeomorphism from a neighbourhood of 0 in $T_x\mathcal{M}$ to a neighbourhood of x in M, its derivative at 0 is the identity map.

An *isometry* f between two Riemannian manifolds $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ is a smooth map $f : \mathcal{M} \to \mathcal{N}$ whose derivative induces isometries between the tangent spaces, with respect to the inner products g and h, respectively. In particular, the two Riemannian manifolds are isometric if there exists some diffeomorphism $f : \mathcal{M} \to \mathcal{N}$, which transfers h into g, i.e., $f^*h = g$.

A Riemannian *structure* is a class of isometric Riemannian manifolds. In other words, if $Riem(\mathcal{M})$ denotes the set of Riemannian metrics on $\mathcal{M}$, the set of Riemannian structures on $\mathcal{M}$ is the quotient $Riem(\mathcal{M})/Diff(\mathcal{M})$ of $\mathcal{M}$ by the group of diffeomorphisms $Diff(\mathcal{M})$ of $\mathcal{M}$.

The various notions of *curvature* measure how the exponential maps differ from being isometries (at least locally). Let P be a 2-plane in $T_x\mathcal{M}$. Given a small enough r, consider the image under the exponential map $\exp_x$ of a circle of radius r and centre 0 in the plane P. This is a closed curve in $\mathcal{M}$ with length $L(r)$. When $r \to 0$ we have Puiseux' formula:

$$L(r) = 2\pi r(1 - \frac{1}{6}\sigma(x, P)r^2 + \mathcal{O}(r^3)). \tag{4.2}$$

The number $\sigma(x, P)$ is called the *sectional curvature* of the 2-plane P at x (see [50] for a lucid exposition).

An oriented Riemannian manifold is also equipped with a natural *Riemannian measure* v_g, whose expression in a local coordinate system $\{x_i\}$ is $det(g_{ij})^{\frac{1}{2}} dx$, where dx is the Lebesgue measure and where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. The volume of $(\mathcal{M}, g)$ is denoted by $V(g) = \int_{\mathcal{M}} dv_g$.

We can write the pull-back $\exp_x^* v_g$ of the Riemannian measure v_g by the exponential map in polar coordinates in $T_x\mathcal{M}$ by $\exp_x^* v_g = \Theta_x(t,u)dtdu$, where $t \geq 0$, dt is the Lebesgue measure on $\Re_+$, u is a unit vector and du is the canonical measure on the unit sphere.

When $t \to 0$, we have

$$\Theta_x(t,u) = t^{n-1}(1 - \frac{1}{6}\rho_x(u)t^2 + \mathcal{O}(t^3)). \tag{4.3}$$

The number $\rho_x(u)$ is a quadratic form on $T_x\mathcal{M}$ which defines a symmetric bilinear form called the *Ricci curvature*, $Ric(g)$ of $\mathcal{M}$ at the point x.

If $\{u, e_2, \ldots, e_n\}$ is an orthonormal basis in $T_x\mathcal{M}$ and if P_i is the 2-plane spanned by u and e_i, we have the formula

$$Ric(g)(u,u) = \sum_{i=2}^{n} \sigma(x, P_i), \tag{4.4}$$

so the Ricci quadratic form is essentially a sum of sectional curvatures.

4.1.2 The Ricci-Hamilton Flow

From now on, $n = 3$, unless explicitly stated otherwise.

Let $Riem(\mathcal{M})$ denotes the space (infinite dimensional) of smooth Riemannian metrics on $\mathcal{M}$, this set has a natural structure of Frechet manifold; and $S^2\mathcal{M}$ the set of smooth bilinear forms on $\mathcal{M}$.

The diffeomorphisms act by pull-back, i.e., $Diff(\mathcal{M}) : S^2\mathcal{M} \to S^2\mathcal{M}$.

The Riemannian structure underlying $(\mathcal{M}, g)$ is described by the orbit O_g of g, in $Riem(\mathcal{M})$ under $Diff(\mathcal{M})$ defined as

$$O_g = \{g' \in Riem(\mathcal{M}) \,/\, g' = \varphi^* g \text{ for some } \phi \in Diff(\mathcal{M})\}. \tag{4.5}$$

The tangent space to $Riem(\mathcal{M})$ in the point g, i.e., $T_g Riem(\mathcal{M})$ is the set of infinitesimal deformations of the given g, and is isomorphic to $S^2\mathcal{M}$. In particular, it contains as the subspace a tangent space to the orbit O_g, which is an image in $S^2\mathcal{M}$ of a certain linear differential operator.

One can prove (using the decomposition theorems; see [51] for details) that any infinitesimal deformation $h \in S^2\mathcal{M}$, can be decomposed into a longitudinal deformation mapping g into, say, g' within the same orbit, i.e., $g \to g' \in O_g$, and a transversal one, $g \to g'' \notin O_g$, which takes g to the other orbit and provides thus an infinitesimally deformed new Riemannian structure on $\mathcal{M}$.

We observe, that for any $k \in \Re$, $[Ric(g) - kgR(g)]$, is never tangent, at g, to the orbit O_g, unless it vanishes [2]. In other words, the deformation $g \to g'$, such that $g' = g - \xi[Ric(g) - kgR(g)] + \mathcal{O}(\xi^2)$, defines a new Riemannian structure on $\mathcal{M}$, since such $g' \notin O_g$.

We conclude, that in $Riem(\mathcal{M})$ there exists a naturally defined $Ric(g)$-generated field of non-trivial infinitesimal deformations provided by associating with g, the tensor field $[Ric(g) - kgR(g)]$. One can investigate therefore, the question of existence and behaviour of the integral curves (if any) of this vector field[3].

The answer was given by R. Hamilton [47], that for a vector field given by $-2Ric(g)$ there exists a continuous local flow of metrics (on $Riem(\mathcal{M})$), which is global on condition the Ricci tensor associated with the metric is a positive bilinear form.

By deforming or smoothing flow of metrics, we mean a curve $g_{ab}(\beta)$, such that $g_{ab}(0)$ is the original given metric, and $g_{ab}(\beta)$ becomes smooth for $\beta \to \infty$. To see how it comes about, consider the general infinitesimal deformation of a metric

$$g_{ab} \to g_{ab} + \Delta\beta h_{ab}, \tag{4.6}$$

where, h_{ab} is any symmetric rank two tensor. Since, in appropriate coordinates, the leading term in the Ricci curvature $Ric(g)$, is $\nabla^c\nabla_c g_{ab}$, a natural choice for h_{ab} is

$$h_{ab} = -2Ric(g). \tag{4.7}$$

[2]Ricci tensor $Ric(g) : Riem(\mathcal{M}) \to S^2\mathcal{M}$; $R(g)$ stands for the scalar curvature.

[3]It is not evident *a priori*, that it can give rise to smooth trajectories due to the Frechet structure of $Riem(\mathcal{M})$.

Writing this in the form of a differential equation, and adding a term responsible for preserving the volume of $(\mathcal{M}, g_{ab})$ along the flow, results in the Ricci-Hamilton flow equation (4.8).

Theorem 1 *Let $(\mathcal{M}, g)$ be a closed (compact and without boundary) Riemannian 3-manifold, such that its Ricci tensor, $Ric(g)$, is a positive definite bilinear form (i.e., $[Ric(g)]_{ab}v^a v^b > 0 \ \forall v \neq 0$ vector field), then the given metric g can be uniformly deformed into a constant curvature metric $\overline{g}$.*

In this case, the universal simply connected cover of $\mathcal{M}$ is the 3-sphere S^3 and the pull back of $\overline{g}$ to S^3 via the covering map $S^3 \rightarrow \mathcal{M}$ is the standard metric[4].

The flow of metrics $(g, \beta) \rightarrow g(\beta)$ (with $\beta \geq 0$ the deformation parameter) realizing the above deformation is the unique solution to the weakly parabolic initial value problem

$$\frac{\partial}{\partial \beta} g_{ab}(\beta) = \frac{2}{3} < R(\beta) >_\beta g_{ab}(\beta) - 2R_{ab}(\beta), \tag{4.8}$$

with the initial data $g_{ab}(\beta = 0) = g_{ab}$ $(a, b = 1, 2, 3)$, where $R_{ab}(\beta)$ are the components of the Ricci tensor $Ric(g(\beta))$, and $< R(\beta) >_\beta$ denotes the average scalar curvature

$$< R(\beta) >_\beta = \frac{1}{V(g(\beta))} \int_{\mathcal{M}} R(\beta) dv_\beta \tag{4.9}$$

The Ricci-Hamilton flow equation is a heat-like equation (weakly parabolic) and thus results in a smoothing deformation of the initial data $g(x, \beta = 0) = g(x)$.

This equation has been studied by mathematicians and a number of relevant results are available [47] (for a comprehensive review see [23]).

The flow $(g, \beta) \rightarrow g(\beta)$ defined by (4.8) preserves the total volume of $(\mathcal{M}, g)$, i.e., $V(\mathcal{M}, g(\beta)) = V(\mathcal{M}, g)$, $\beta \geq 0$. Any isometries of the original metric are preserved under the flow.

[4]Notice, that the theorem in fact, forces $\mathcal{M}$ to be topologically S^3/Γ, i.e., S^3 possibly quotiented by a discrete group.

The linearized Ricci-Hamilton flow evolves a given infinitesimal deformation yielding a β-parameterised family of vectors $h(\beta) \in TRiem(\mathcal{M})$, connecting two neighbouring flows of metrics.

Upon the formal linearization of the initial value problem (4.8) around a given solution $g(\beta)$, we obtain (the β's in the brackets are suppressed)

$$\frac{\partial}{\partial\beta}h_{ab} = \frac{2}{3} < R > h_{ab} + \frac{2}{3}g_{ab}[\frac{1}{2} < Rg^{ab}h_{ab} > -\frac{1}{2} < R >< g^{ab}h_{ab} > - < R^{ab}h_{ab} >] - \Delta_L h_{ab} + 2[div^*(div(h - \frac{1}{2}(Trh)\, g))]_{ab}, \quad (4.10)$$

with the initial data $h_{ab}(\beta = 0) = h_{ab}$, where, $h \in S^2\mathcal{M}$ is a given symmetric bilinear form, Δ_L is the Lichnerowicz-DeRham Laplacian on bilinear forms, and the operators Δ_L, div^*, div and Tr are considered with respect to the flow of metric $(g, \beta) \to g(\beta)$, solution of (4.8). The div (here, minus the usual divergence) is the divergence operator on $S^2\mathcal{M}$, div^* is the L^2 adjoint of div, acting from the space of vector fields on $\mathcal{M}$ to $S^2\mathcal{M}$ (it can be identified with $\frac{1}{2}[Liederivative]$ of the metric tensor along a vector field).

Note, that a $h(\beta)$ solution of the initial value problem (4.10) always exists and is unique, and evolves a given infinitesimal deformation yielding a β-parameterised family of vectors $h(\beta)$ in $TRiem(\mathcal{M})$ connecting two neighbouring flows of metrics $g(\beta)$ and $g'(\beta)$ (obtained as solutions of problem (4.8) with initial data $g(\beta = 0) = g$ and $g'(\beta = 0) = g(\beta = 0) + \epsilon h(\beta = 0) + \mathcal{O}(\epsilon^2)$, respectively).

According to the $Diff(\mathcal{M})$ equivariance of the Ricci-Hamilton flow, a trivial deformation $h_{ab} = L_X g_{ab}$ is always mapped by (4.10) into a trivial deformation (where, X is a smooth vector field in $\mathcal{M}$), in other words, the solution to the linearized Ricci-Hamilton initial value problem is determined up to the infinitesimal diffeomorphism.

This actually implies, that all symmetries which the original metric may be endowed with are preserved by the Ricci-Hamilton flow.

In general, a β-dependent longitudinal term will be generated by the very process of deforming the given h_{ab} according to (4.10), i.e., the deformation does not stay divergence free along the flow. What it means is, that there arises a non-trivial action of the diffeomorphism group, $Diff(\mathcal{M})$.

If we want to smooth out lumpy 3-geometries, the linearized Ricci-Hamilton flow appears as a natural candidate for averaging out the second fundamental form K_{ab}, associated with the initial data set whose Cauchy development characterizes the space-time to be smoothed out. As is known, the momentum constraint connects the divergence of K_{ab} to the momentum density T^{oa} $(a = 1, 2, 3)$ of the external sources of the gravitational field (see section (4.2) for an explicit smoothing procedure).

Since we want to end up with an isotropic K_{ab}, we need to control the divergence of K_{ab} as we deform it, which is not an easy task, as a "coordinate shear" builds up during the deformation.

However, according to the results of DeTurk the (reduced) Ricci-Hamilton flow

$$\frac{\partial}{\partial\beta} g_{ab}(\beta) = -2R_{ab}(\beta) \tag{4.11}$$

with the initial data $g_{ab}(\beta = 0) = g_{ab}$, and the flow solution of the manifestly parabolic initial value problem, can be connected by a family of β-dependent diffeomorphisms. Choosing suitably the diffeomorphism generating vector field $w(\beta)$ (responsible for the parabolic character of the appropriate initial-value problem), one can always control the longitudinal part of the flow $(h_{ab}, \beta) \rightarrow \tilde{h}_{ab}(\beta)$.

In other words, if we evolve two metrics $g \in O_g$, $g' \in O_{g'}$ in two Riemannian manifolds with positive definite Ricci form and the same volumes, according to the Ricci-Hamilton flow we generically end up with two metrics $\overline{g}, \overline{g}'$, lying in the same orbit $O_{\overline{g}}$. In order to have the same 3-metric, $\overline{g}$ say, we have to act upon one of the 3-manifolds with a β-dependent family of diffeomorphisms.

This is implemented by evolving g according to the parabolic initial-value problem $({g_{ab}}', \beta) \rightarrow \tilde{g}'_{ab}(\beta)$, exhibiting the diffusive nature of the Ricci-Hamilton flow. In that case, the infinitesimal deformation $\tilde{h}$ (properly normalized), connecting $g(\beta)$ with $\tilde{g}'(\beta)$ evolves again according to the linearized the parabolic initial-value problem.

Given this, the Ricci-Hamilton flow appears as a good candidate for associating with a given metric g an constant curvature "model geometry", fixed by the normalization $V(\mathcal{M}, \overline{g}) = V(\mathcal{M}, g)$ and obtained by smoothing out the original metric.

In practice, according to Hamilton's theorem we can smooth only a subset of the Riemannian structures a closed manifold can carry (these with $Ric(g) > 0$), and this allows us to smooth out locally inhomogeneous and anisotropic space-times, sufficiently near (but not necessarily infinitesimally near in the sense of perturbation theory) to a closed FLRW, into a dynamically equivalent closed FLRW space-time.

The natural problem to address, is therefore that of generalizing the Hamilton theorem as much as possible.

For the understanding, why there is a positivity requirement on the Ricci tensor, and to what extend it can be weakened - from the point of view of solvability the initial-value problem (4.8), as well as of the topological obstructions to positive Ricci curvature - see [23] (see also [24]).

What we want to stress, is that the positivity condition is not a necessary one [15, 26]. As explicitly proved in [26], the 3-torus T^3 provides a non-trivial example of Ricci-Hamilton flow (with the Ricci tensor being non-positive), such that the Hamilton initial-value problem admits a global solution. The T^3 cosmology, with a 3-space in form of a 3-torus is the simplest inhomogeneous empty universe.

4.1.3 Geodesic balls covering of manifolds of bounded geometries. Compactness properties of $Riem(\mathcal{M})$.

Consider two Riemannian manifolds, and let $i_1(\mathcal{M}_1), i_2(\mathcal{M}_2)$ stand for two isometric embeddings of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, in some metric space (A, d).

For more details of what follows, one can consult [25, 23] and [52].

A *Hausdorff distance* in (A, d) between $i_1(\mathcal{M}_1)$, $i_2(\mathcal{M}_2)$ can be introduced as follows:

$$\begin{aligned} d_H^A[i_1(\mathcal{M}_1), i_2(\mathcal{M}_2)] &= inf\{\epsilon > 0/ \ U_\epsilon(i_1(\mathcal{M}_1)) \supset i_2(\mathcal{M}_2), \\ &\quad U_\epsilon(i_2(\mathcal{M}_2)) \supset i_1(\mathcal{M}_1)\}, \end{aligned} \tag{4.12}$$

where, the ϵ-neighbourhood $U_\epsilon(i_i(\mathcal{M}_i))$ of $i_i(\mathcal{M}_i)$, $i = 1, 2$ is defined as

$$U_\epsilon(i_i(\mathcal{M}_i)) = \{z \in A/ \ d(z, i_i(\mathcal{M}_i)) \leq \epsilon\}. \tag{4.13}$$

The Hausdorff distance thus defined is the lower bound of the ϵ, such that $i_1(\mathcal{M}_1)$ is contained in the ϵ-neighbourhood of $i_2(\mathcal{M}_2)$, and vice versa.

The *Gromov distance* $d_G(\mathcal{M}_1, \mathcal{M}_2)$ provides a natural generalization of the Hausdorff distance, and it is defined as the lower bound of the Hausdorff distances, as A varies in the set of metric spaces, and i_1, i_2 vary in the set of all isometric embeddings of $\mathcal{M}_1$ and $\mathcal{M}_2$ in (A, d).

We define the set of open *geodesic balls* as

$$B_{\mathcal{M}}(p_i, \epsilon) = \{x \in \mathcal{M} |\ d(x, p_i) \leq \epsilon\},\ i = 1, \ldots, N \tag{4.14}$$

where, $d(.,.)$ denotes the distance function of $\mathcal{M}$.

The Gromov distance provides us with a sense of geometric nearness for Riemannian structures, which is related to a classification of Riemannian manifolds, according to how they can be covered by small geodesic balls. Coverings with the balls packed in similar configurations are possible for the Riemannian manifolds, that can be considered close to each other in the sense of Gromov distance.

In particular, the Gromov distance between two compact manifolds is always finite, and $d_G(\mathcal{M}_1, \mathcal{M}_2) = 0$, implies that the two manifolds are isometric.

Let us introduce the following class of Riemannian structures:

for $k \in \Re$ and $D \in \Re_+$, let $Ric[n, k, D]$ denote the space of isometry classes of closed, connected n-dimensional Riemannian manifolds $(\mathcal{M}, g)$ (without any preassumption on their topology) with Ricci curvature $Ric(g) \geq (n-1)kg$ and diameter $\leq D$.

Remark

Recall, that if we define

$$k(x) \equiv inf\{inf\ Ric(u, u) |\ u \in T_x\mathcal{M},\ |u_x| = 1\} \tag{4.15}$$

then, the lower bound of the Ricci tensor of $\mathcal{M}$ is defined as the lower bound of $k(x)$ as x varies in $\mathcal{M}$.

The best such $k = k(g)$ is just the lowest eigenvalue of the Ricci curvature $Ric(g)$. It is a fundamental numerical invariant of a compact Riemannian manifold.

For any manifold $\mathcal{M} \in Ric[n, k, D]$, it is possible to introduce the covering by geodesic balls, providing a coarse classification of Riemannian structures in $Ric[n, k, D]$.

Taking any $\epsilon > 0$, it is always possible to find an ordered set of points $\{p_1, \ldots, p_N\}$ in $\mathcal{M}$ from the above class, so that:

i) the balls $B_{\mathcal{M}}(p_i, \epsilon), i = 1, \ldots, N$ cover $\mathcal{M}$, i.e., the collection $\{p_1, \ldots, p_N\}$ is an ϵ-*net* in $\mathcal{M}$.

ii) the balls $B_{\mathcal{M}}(p_i, \epsilon/2)$, $i = 1, \ldots, N$ are disjoint, i.e., $\{p_1, \ldots, p_N\}$ is a *minimal* ϵ-net in $\mathcal{M}$.

A *filling function* $N_\epsilon^{(0)}(\mathcal{M})$ of the covering is defined as the function, which associates with $\mathcal{M}$ the maximum number of geodesic balls realizing a minimal ϵ-net in $\mathcal{M}$.

Any minimal net is characterized by its *intersection pattern*, defined as the set of indices pairs

$$I_\epsilon(\mathcal{M}) \equiv \{(i,j) |\, i,j = 1, \ldots, N |\, B(p_i, \epsilon) \cap B(p_j, \epsilon) \neq \emptyset\} \qquad (4.16)$$

Two manifolds $\mathcal{M}_1, \mathcal{M}_2 \in Ric[n, k, D]$ with minimal ϵ-nets $\{p_1, \ldots, p_N\}$, and $\{q_1, \ldots, q_N\}$, respectively, are said to be equivalent, if and only if $N = L$ and if they have the same intersection pattern, i.e., if the equivalence relations

$$N_\epsilon^{(o)}(\mathcal{M}_1) = N_\epsilon^{(o)}(\mathcal{M}_2) \qquad (4.17)$$

$$I_\epsilon(\mathcal{M}_1) = I_\epsilon(\mathcal{M}_2), \qquad (4.18)$$

are true (up to combinatorial isomorphism).

In fact, the above relations partition $Ric[n, k, D]$ into disjoint equivalence classes, whose finite number can be estimated in terms of the parameters n, k, D.

Two Riemannian manifolds in $Ric[n,k,D]$ get closer and closer to each other in d_G, if we can cover them with finer and finer minimal ϵ-nets of geodesic balls with the same intersection patterns.

In order to have $d_G(\mathcal{M}_1, \mathcal{M}_2) < \epsilon$, for any two compact Riemannian manifolds, it is sufficient to show that there exist an $\epsilon/2$ lattice in $\mathcal{M}_1$ and an $\epsilon/2$ lattice in $\mathcal{M}_2$, and two isometric embeddings $i_j : \mathcal{M}_j \to Z$ in some metric space (Z, d), such that the distance between the corresponding points of the embedded lattices is $< \epsilon/2$.

When discussing the convergence of a sequence $\{\mathcal{M}_i\}$ of Riemannian manifolds with respect to the Gromov distance, there is no need to refer to isometric embeddings in metric spaces. The sequence $\{\mathcal{M}_i\}$ admits a convergent subsequence, if and only if $\forall \epsilon > 0$, $\exists$ a number $N_\epsilon^{(o)}$ providing for each i, an upper bound to the maximum number of disjoint geodesic balls of radius ϵ, filling up each $\mathcal{M}_i$, i.e., $N_\epsilon(\mathcal{M}_i) \leq N_\epsilon^{(o)}$, for each i.

Stated differently, the convergence with respect to d_G is related to a uniform control of the "geometric size" of the manifolds $\mathcal{M}_i$, as indicated by the number of balls of a given radius, that is needed to fill up each $\mathcal{M}_i$ in the considered sequence. For example, in the case of a sequence of compact surfaces of bounded curvature converging in d_G to S^2, each of them can be filled up by $N_\epsilon(\mathcal{M}_i)$ maximum number of disjoint geodesic balls of radius ϵ, such that $N_\epsilon(\mathcal{M}_i) \leq N_\epsilon^{(o)}(S^2)\ \forall i$.

Definition A metric space E is said to be *precompact*, if $\forall \epsilon > 0$, $\exists$ a finite (open) covering B_j of E, such that the sets B_j have diameter $< \epsilon$. Equivalently, $\forall \epsilon > 0$, there exist a finite set $F \subset E$, such that $d(x, F) < \epsilon, \forall x \in E$. A stronger notion is that of compactness, yielded by the closure of the considered space.

Theorem 2 *The set $Ric[n,k,D]$ of isometry classes of compact manifolds, with the Ricci tensor satisfying $Ric(g) \geq (n-1)kg$ and diameter $\leq D$, $(k \in \Re, D \in \Re_+)$, is precompact when endowed with the Gromov distance d_G.*

The theorem states that in the set of closed Riemannian manifolds (with Ricci curvature bounded below, diameter bounded above) there is a subset, let us call it $\tilde{Ric}$, containing for each ϵ, a finite number of Riemannian manifolds $\tilde{\mathcal{M}}_j$, such that for any $\mathcal{M} \in Ric[n, k, D]$, we have $d_G(\mathcal{M}, \tilde{\mathcal{M}}_i) < \epsilon$ for some $\tilde{\mathcal{M}}_i \in \tilde{R}ic$.

What this means is, that for each "length scale ϵ", there exists a finite number of "model" geometries, which describes with an ϵ-approximation any given Riemannian geometry. So, given a ball of a certain radius $> \epsilon$, in any Riemannian manifold $\mathcal{M}$ in $Ric[n, k, D]$ there exists a ball metrically similar (up to an ϵ scale) in one of the model geometries, which does not retain the details of the original manifold on scales smaller than ϵ. Roughly speaking, ϵ is a measure of the typical curvature inhomogeneity with respect to the model background.

Let us stress, that this is a highly non-trivial result, in the sense that the metrical properties of the manifolds in the infinite dimensional set $Ric[n, k, D]$ (This set is of infinite dimension because a point $\mathcal{M}$ in its interior, remains on the set under small perturbations of the metric, so locally it is in principle as complicated as the set of all Riemannian metrics.) are up to an ϵ scale described by the metrical properties of just a finite number of model Riemannian manifolds.

However, since $Ric[n, k, D]$ is only precompact, and not compact we can have for instance a situation where a sequence of manifolds in $Ric[n, k, D]$ converges under d_G, to a manifold of lower dimension[5], or to a space with singularities[6].

Therefore below we will limit ourselves to the subset of $Ric[n, k, D]$, generated by those Riemannian manifolds with sectional curvatures bounded in absolute value[7].

Theorem 3 *The set $(\tilde{R}ic[n, k, D], d_G)$ of Riemannian structures having diameter $\leq D$, volume $\geq V$ and sectional curvatures bounded below in absolute value, is compact.*

One can think of the model manifolds $\tilde{\mathcal{M}}_i \in \tilde{R}ic$ as the "smoothed out" counterparts of the manifolds in $\tilde{R}ic[n, k, D]$.

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[5]The dimension of a manifold is not continuous for the topology defined by d_G.

[6]Phenomena of this kind are, for example the pinching of a geodesic in a torus.

[7]We will address the issue of the above bounds in Gromov spaces elsewhere. Clearly, they are unnatural in relativity.

For the dimension $n = 3$, the sectional curvature tensor is completely determined by the Ricci tensor, and therefore the 3-manifolds with constant sectional curvatures are necessarily the Einstein manifolds. Manifolds, that can appear are that of $S^3, \Re^3$ or H^3 and the Einstein metrics are locally one of these three types.

These results suggest a connection with Ricci-Hamilton deforming flow (see [23] for details).

4.2 Smoothing out spatially closed cosmologies

In [14] (see also [24]) a specific smoothing out procedure was put forward, deforming a family of locally inhomogeneous and anisotropic spatially closed space-times into closed FLRW universes. These space-times are associated with gravitational configurations, that can be considered near to the standard ones generating closed FLRW cosmological models. This class is large, it contains solutions to the Einstein field equations that are not just perturbations of closed FLRW space-times.

The smoothing out procedure is employed in the full theory, and a precise content to the averaging hypothesis, by providing explicitly the correction terms to the physical sources induced upon smoothing out the space-time geometry, can thus be given.

The idea is the following. We pick up an appropriate initial data set, which upon the Cauchy evolution is going to be the space-time to be averaged out. Such data set is then smoothly deformed into a FLRW initial data set, by the action of parabolic-type operators. This deformation is constructed in such a way as to make the deformed data satisfy the four constraints associated with Einstein's equations. It follows then, that the flow of deformed initial data generates a one parameter family of solutions to the field equations, which interpolates between the original space-time and a closed FLRW space-time, considered to be the smoothed out counterpart of the given universe model.

To make the above precise, let $({}^{(4)}V \overset{\phi}{\simeq} \mathcal{M} \times I, {}^{(4)}g)$ be a space-time manifold, the Cauchy evolution of a regular initial data set $(\mathcal{M}, g, K)$, where

ϕ is a diffeomorphism mapping ${}^{(4)}V$ to $\mathcal{M} \times I$ ($I \subset \Re$), with $\mathcal{M}$ the (closed) 3-manifold carrier of the initial data (i.e., a space-like 3-hypersurface in the space-time manifold) and $g, K \in S^2\mathcal{M}$, representing in the final space-time the induced Riemannian 3-metric on $\mathcal{M}$, and the second fundamental form of the embedding $\mathcal{M} \rightarrow ({}^{(4)}V, {}^{(4)}g)$, respectively.

We assume, that $\mathcal{M}$ is topologically a 3-sphere S^3 and that the class of initial data supported by $\mathcal{M}$ is such, that $Ric(g)$ is a positive definite bilinear form for them (recall, that $Ric(g)$ is the Ricci tensor associated with g).

Due to the results of Hamilton (see section (4.1.2)), $({}^{(4)}V, {}^{(4)}g)$ resulting form the time evolution of data from the above class, can be taken as modelling a locally anisotropic and inhomogeneous universe, not too far from a closed FLRW space-time. A smoothing out mapping associates with the given initial data set a one parameter family $(\mathcal{M}, g(\beta), K(\beta))$ with $0 \leq \beta < \infty$, $g(0) = g, K(0) = K$, approximating closer and closer, the standard initial data set for a closed FLRW model and reaching it uniformly as $\beta \rightarrow \infty$.

According to the Hamilton theorem, we can deform the metric g into the constant-curvature metric $\overline{g}$ on S^3, by the flow of metrics $g(\beta)$, $0 \leq \beta < \infty$, solution to the non-linear, weakly parabolic, initial value problem

$$\frac{\partial}{\partial \beta} g_{ab}(\beta) = \frac{2}{3} < R(\beta) >_\beta g_{ab}(\beta) - 2R_{ab}(\beta) \tag{4.19}$$

with $g_{ab}(0) = g_{ab}$, $(a, b = 1, 2, 3)$, where, $< R(\beta) >_\beta$ is the average scalar curvature over $(\mathcal{M}, g(\beta))$, and $R_{ab}(\beta)$, $R(\beta)$ are the components of the Ricci tensor and the scalar curvature associated with $g(\beta)$, respectively. For the properties of the Ricci-Hamilton flow see section (4.1.2).

In order to smooth out the whole data set, we need to average out the second fundamental form K, as well.

Obviously, we have for the smoothed metric $\overline{g}_{ab} = \lim_{\beta \rightarrow \infty} g_{ab}(\beta)$ (presuming that the flow converges), where $g_{ab}(\beta)$ satisfies the Ricci flow equation (4.19). Given (g, K), let us define then a nearby flow $\tilde{g}_{ab}(\beta; \epsilon)$, with initial condition

$$\tilde{g}_{ab}(\beta = 0; \epsilon) \equiv g_{ab}(\beta = 0) + \epsilon K_{ab}(\beta = 0). \tag{4.20}$$

These flows evolve with β yielding as "connecting vector" the bilinear form

$$K_{ab}(\beta) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(\tilde{g}_{ab}(\beta;\epsilon) - g_{ab}(\beta)), \tag{4.21}$$

so we can define

$$\overline{K}_{ab} = \lim_{\beta \to \infty} K_{ab}(\beta). \tag{4.22}$$

The β evolution of $K(\beta)$ is found by linearizing (4.19). Formally we define it as

$$\frac{\partial}{\partial\beta}K_{ab}(\beta) = \frac{2}{3}g_{ab}(\beta)(\frac{1}{2} < R(\beta)K^c_c(\beta) >_\beta - \frac{1}{2} < R(\beta) >_\beta < K^c_c(\beta) >_\beta -$$
$$< R_{ab}(\beta)K^{ab}(\beta) >_\beta) + \frac{2}{3} < R(\beta) >_\beta K_{ab}(\beta) - \Delta_\beta K_{ab}(\beta) - L_Y g_{ab}(\beta) \tag{4.23}$$

with $K_{ab}(0) = K_{ab}$, and where Δ_β denotes the DeRham-Lichnerowicz Laplacian associated with $g(\beta)$, $\Delta_\beta K_{ab}(\beta) = -\nabla^c\nabla_c K_{ab} + R_{ac}K^c_b + R_{bc}K^c_a - R^d_{abc}K^c_d$, and L_Y is the Lie derivative along the vector field Y.

It can be shown, that the flow $K(\beta)$, solution of (4.23), is such that $\frac{\partial}{\partial\beta}(< K^a_a(\beta) >_\beta) = 0$, i.e., the average over $(\mathcal{M}, g(\beta))$ of the trace of K is constant during the deformation.

Also, as (4.23) is the formal linearization of (4.19), we have $\lim_{\beta\to\infty} K_{ab}(\beta) = \frac{1}{3} < K^a_a >_o \overline{g}_{ab}$; $< \cdots >_o$ stands for the space average of the original physical quantity. Thus, the flow $K(\beta)$ deforms the given K by eliminating its shear: $K_{ab} - \frac{1}{3}K^c_c g_{ab}$, and replacing the original (position-dependent) rate of volume expansion K^a_a with its average value.

The smoothing flow of regular initial data sets has to be such, that for each value of β, the four constraints of the Einstein equations:

$$R(\beta) - K_{ab}(\beta)K^{ab}(\beta) + (K^a_a(\beta))^2 = 2\rho(\beta) \tag{4.24}$$

$$\nabla_a K^{ab}(\beta) - \nabla^b K^a_a(\beta) = J^b(\beta), \tag{4.25}$$

have to be satisfied; where $\rho(\beta), J(\beta)$ are the mass and momentum density, respectively (of the external sources as described by a system of observers instantaneously at rest on $\mathcal{M}$) referred to the β dependent measure associated with $g(\beta)$.

For $\beta = 0$, (4.24) and (4.25) hold true, since $\rho(\beta = 0) = \rho$ and $J(\beta = 0) = J$ are the physical densities of sources of a given gravitational configuration $(\mathcal{M}, g, K)$. The averaging flows $\rho(\beta)$ and $J(\beta)$ cannot be defined independently, once g and K are given and deformed according to (4.19) and (4.23), in order for the constraints (4.24) and (4.25) to remain valid. In other words, to properly average the sources one has to take into account the backreaction of the geometry, determined by the constraints.

Therefore, we interpret the constraints as actually defining $\rho(\beta)$ and $J(\beta)$. Indeed, then $\overline{\rho} \equiv \lim_{\beta\to\infty} < \rho(\beta) >_\beta$ and $\lim_{\beta\to\infty} < J(\beta) >_\beta \equiv \overline{J} = 0$ (from (4.24) and (4.25) and the properties of the Ricci-Hamilton flow). One can show explicitly, that

$$\overline{\rho} = [< \rho >_o + \frac{1}{2} < (K_{ab} - \frac{1}{3}K^c_c g_{ab})(K^{ab} - \frac{1}{3}K^c_c g^{ab}) >_o + \frac{1}{2}\overline{R}(\eta + \sigma^2)]/(1+\sigma^2) \tag{4.26}$$

where, $\sigma \equiv (< (K^a_a)^2 >_o - < (K^a_a) >^2_o)/ < K^a_a >_o$, $\eta \equiv (\overline{R} - < R >_o)/\overline{R}$, i.e., σ is the standard deviation describing the fluctuations of the original (position dependent) value of K^a_a with respect to its average (conserved) value $< K^a_a >_o$; η, $0 \leq \eta < 1$ denotes the relative function of the physical scalar curvature with respect to the averaged one $\overline{R}$.

Now, we can build an effective stress tensor modelling the dynamical effects of deviations from a spatially homogeneous geometry (also those which are too big to be handled by perturbation techniques) that have been smoothed out.

If $(\mathcal{M}, g_t)$ defines a normal geodesic slicing of $({}^{(4)}V, {}^{(4)}g)$ (for sufficiently small t) the stress tensor enters only into the evolution part of Einstein's equations,

$$\frac{\partial}{\partial t}K_{ab} = R_{ab} + K^c_c K_{ab} - 2K_{ac}K^c_b - (T_{ab} - \frac{1}{2}T^c_c g_{ab}) - \frac{1}{2}\rho g_{ab}.\text{œ[5} \tag{4.27}$$

The smoothing flow $T(\beta)$ of the spatial stress tensor is defined by requiring that, for each t for which the evolution of the data $(g(\beta), K(\beta))$ is defined, the flows $(g_t(\beta), K_t(\beta))$ (resulting from the evolution equations) are Ricci-Hamilton flows, with initial conditions $g_t(0) = g_t$, $K_t(0) = K_t$, respectively.

The physical meaning of the presented results enables us to state precisely what is meant by the requirement that the original physical model universe and its FLRW smoothed-out ideal should behave as close as possible under their own gravitation. Namely, for $\beta \to \infty$, the volume $V(S^3, \overline{g}_t) = V(\mathcal{M}, g_t)$; this shows how the dynamics of the closed FLRW model is related to the dynamics of the original space-time. The fact that $V(\mathcal{M}, g_t) = V(\mathcal{M}, g_t(\beta))$ implies, that $\frac{\partial}{\partial t}(< K^a_a >_o) = \frac{\partial}{\partial t}(< K^a_a(\beta) >_\beta)$ along the flow $(g_t(\beta), K_t(\beta))$.

The smoothed out pressure $\overline{p} \equiv \lim_{\beta\to\infty} < p(\beta) >_\beta$ (in the final FLRW model on the surface of homogeneity $t = 0$; $\overline{p}_t$ (as well as $\overline{\rho}_t$, if the equation of state is known) can be determined by the evolution equation) is shown to be (taking into account (4.27));

$$\begin{aligned} \overline{p} &= \frac{1}{3} < T^a_a >_o + \frac{2}{3} < (K_{ab} - \frac{1}{3}K^c_c g_{ab})(K^{ab} - \frac{1}{3}K^c_c g^{ab}) >_o - \frac{4}{9}\sigma^2 < K^a_a >^2_o \\ &\quad + \frac{1}{3}(< \rho >_o - \overline{\rho}) \end{aligned} \tag{4.28}$$

with $\overline{\rho}$ given by (4.26). We see from (4.26) and (4.28) that $\overline{\rho} > 0$ and $|\overline{p}| \leq \overline{\rho}$, i.e., the dominant energy condition is satisfied.

In case when $\sigma^2 \approx 0$ (homogeneous expansion) and $\eta \ll 1$ (fluctuations of the physical curvature w.r.t. FLRW background curvature small, on average) the closed FLRW universe is the right model, only if we add to the physical sources $< \rho >_o$ and $< T^a_a >_o$ the term: $< (K_{ab} - \frac{1}{3}K^c_c g_{ab})(K^{ab} - \frac{1}{3}K^c_c g^{ab}) >_o$, taking this way into account the contribution of cosmological gravitational radiation. This term can influence the dynamics of the universe, and there is no evidence by now, that the relative magnitude of this term with respect to $< \rho >_o$ is $\ll 1$.

Let us note, that the smoothed stress tensor is defined by requiring that the smoothing commutes with the Einstein evolution, i.e., at each stage of this smoothing we in fact, appeal to some standard form of Einstein's equations. Therefore in this scheme one cannot say anything about what the effect of smoothing is on the form of the equations[8].

[8] We will show elsewhere how one can try to fix this problem, using smoothing procedure over the geodesic balls and arguments a lá renormalization group.

Let us also stress, that this smoothing program has a few unresolved issues, like, firstly the problem of making an identification between hypersurfaces of the original space-time and those of the smoothed space-time[9]. Secondly, the issue is how to define the smoothed stress tensor T_{ab} if one uses a more complicated form for $T_{\mu\nu}$ than a perfect fluid.

One possible application of the averaging procedure would be also to examine the conjecture, that all cosmological models with S^3 spatial topology have a time of maximum expansion.

4.3 Ricci-Hamilton flow vs. Thurston conjecture

Definition A Riemannian metric g on a compact 3-manifold $\mathcal{M}$ is defined to be *locally homogeneous*, if and only if for every pair (x, y) of points of $\mathcal{M}$, there exist neighbourhoods U_x of x and V_y of y, such that there is an isometry $\psi : U_x \to V_y$ with $\psi(x) = y$.

Remarks

Generally, these local isometries do not extend to isometries of the whole space $(\mathcal{M}, g)$. If the local isometries do extend, then the geometry is defined to be homogeneous, i.e., $(\mathcal{M}, g)$ is *homogeneous* if for every pair of points x, y in $\mathcal{M}$, there exists an isometry $\phi : \mathcal{M} \to \mathcal{M}$ with $\phi(x) = y$. In this case the group of isometries of $\mathcal{M}$ acts transitively. For every locally homogeneous geometry the universal cover is homogeneous. We say, that the locally homogeneous geometry is modelled by the homogeneous geometry.

Thurston's conjecture claims, that any closed (orientable) 3-manifold can be decomposed into pieces, such that each of them admits a locally homogeneous geometry [48] (for a semi-popular account, see also [49]). This decomposition is done by cutting $\mathcal{M}$ along 2-spheres and tori, and gluing 3-balls to the resulting boundary spheres on each piece. The components (the simpler manifolds) are conjectured to admit one and only one of eight possible geometric structures. But the situation is quite complicated in reality, firstly, by the fact that there are locally homogeneous structures on 3-manifolds that are not isometric to one of the three constant curvature

[9] We can hope this problem to be relaxed by adapting the averaging to a past null cone.

spaces: S^3, E^3, H^3, and secondly, that direct sums of two or more of these (eight) geometries may not admit one of the eight geometries.

Clearly, Thurston's proposition of a geometric classification of the topologies of 3-manifolds is of relevance to theoretical physics. Locally homogeneous models in relativistic cosmology (Bianchi, Kantowski-Sachs) basically make use of the eight Thurston's geometries. Also, in the functional integral approach to calculating amplitudes in quantum gravity one should integrate over spatial topologies, taking into account that in quantum gravity the spatial topology is not fixed.

Hamilton's program for using the Ricci-Hamilton flow to study Thurston's 3-dimensional geometrization conjecture requires one to understand the Ricci-Hamilton flow of all locally homogeneous geometries on closed 3-manifolds.

The idea to prove this conjecture using Ricci-Hamilton flow is the following: choose an arbitrary metric on $\mathcal{M}$ and deform it via the Ricci-Hamilton flow equation (normalized). One hopes to relate the local singularities of the flow to the manifold decomposition in Thurston's conjecture, and then to show that the Ricci-Hamilton flow of the geometry away from each of the singularities approaches that of a locally homogeneous geometry in each disconnected piece.

However, 3-dimensional Ricci-Hamilton flows do not necessarily converge to the Einstein metrics (the zeroes of the RHS of the Ricci-Hamilton flow equation), and most 3-manifolds do not admit Einstein's metrics.

Ricci-Hamilton flow commutes with the cover map $\mu : \hat{\mathcal{M}} \to \mathcal{M}$, $\hat{g} \equiv \mu^* g$, so one can study the Ricci-Hamilton flow of any locally homogeneous geometry by examining that of its homogeneous model.

In [45], the characteristic behaviour of the Ricci-Hamilton flows, depending upon the geometry type was described.

- Whenever the flow can converge (i.e., in those classes which admit the Einstein metrics) it does converge. The convergence is exponential.

- Curvature singularities in the class $S^2 \times \Re$ may be averted by changing the Ricci-Hamilton flow normalization.

- In some classes, the curvature dies along the flow as $\frac{1}{t}$ ($||Ric|| \simeq \frac{1}{t}$) and the flows approach either pancake or cigar degeneracies.

- In all cases, except Nil, the generic flow approach the flows of the maximally symmetric members of the class.

No way is yet known to extend the solutions past the singularities which do occur. To get around this one can look at families of inhomogeneous metrics which have certain symmetries and topological conditions preventing the occurrence of pinches.

Finally, let us remark in advance that, metrics deformation a lá the renormalization group combined with the manifold covering with geodesic balls, may shed a new light upon the means hopefully useful for proving Thurston's conjecture.

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