CP-Violation and the present of heavy quarks: the case of $K_0 - \bar{K}_0$ mixing

Thesis presented by

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Introduction

The interest in *CP-violation* is both theoretical and phenomenological. From the study of particle-antiparticle mixing and other flavour changing neutral current processes (FCNC) with CP-violation, many important constraints on the physics beyond the standard model have been imposed. Moreover, the Standard Model physics itself has been improved, because many parameters of the model, as the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements, have been bounded or fixed. $K_0-\overline{K}_0$ mixing alone led, in the sixties and early seventies, to the conjecture of the existence of the charm quark and allowed an accurate estimate of its mass prior of its discovery. In the same way, nowaday, some interesting informations on the top quark mass can be derived both from $K_0-\overline{K}_0$ and $B_0-\overline{B}_0$ mixing.

These FCNC processes originate from weak interactions, but they are influenced by strong interactions both at short and at long distances (weak interactions are described in the framework of the Standard Model and strong interactions are accounted for by QCD). The long distance strong interactions are presently still a problem and must be evaluated by non-perturbative techniques. On the other hand, the short distance sector of the theory is actually well known and can be sistematically described by means of renormalization group improved perturbation theory, thank to the asymptotic freedom of QCD.

Every low energy weak process can be described in the effective hamiltonian formalism. The basic expression of the effective hamiltonian for a given physical process contains its tree level weak amplitude. Then, QCD corrections can be added, in a given approximation (leading logarithms, next-to-leading logarithms, etc.) and summed up to all orders, using the standard renormalization group (RG) techniques. It's really important, for the predictivity of the theory, to evaluate how much QCD corrections modify the effective hamiltonian.

In particular, the theoretical expressions of particle-antiparticle mixing and all the CP-violation parameters, computed in the QCD improved effective hamiltonian framework, depend on the top quark mass (m_t) and can be used to improve the present bounds on m_t. However, in order to compare our predictions with the experimental results coming from LEP or SLC, we must describe the m_t-dependence of each of the previous quantities in a rigorous theoretical way.

The computation of QCD improved weak effective hamiltonians for particle-antiparticle mixing, etc. has already been performed in the case of a light top quark ($m_t \leq 60$ Gev), while not in the case of a heavy top quark. As I will explain in more detail later on, a certain number of people has been concerned with this problem, but what is present in literature is still confused. Also, a clear phenomenological analysis of the result is lacking. In spite of the theoretical problems still existing (f.i. in the evaluation of the hadronic matrix elements and generally in the whole long-distance strong interaction physics), it is very interesting to dispose of a general coherent analysis, which uses the more updated experimental and phenomenological inputs of the CP-violation scenario. In this way, it would be possible to clarify what can already be considered as well established and what, instead, need still to be investigated.

Our efforts have been driven just in this direction and we have worked so far on two main subjects:

(i) the explicit calculation of the QCD corrections to the $K_0 - \overline{K}_0$ mixing amplitude, which can be easily generalized to the case of a $F_0 - \overline{F}_0$ mixing process (where F denotes a generic flavour);

(ii) the numerical implementation of the results of (i).

In this context, first of all, we have computed analytically all the one-loop QCD corrections to the box-diagram for the $K_0 - \overline{K}_0$ mixing. Then, we have selected the leading logarithmic contributions both in the case of a light and in the case of a heavy top quark, and we have summed them up, by using renormalization group techniques. We have found that the results obtained for a light top quark can be safely extrapolated to the region where the top quark is heavy, while the viceversa is not true. The QCD corrections obtained in the case of a heavy top quark do not describe with sufficient accuracy the region where the top quark is light. The results obtained in the two regions match quite well in the intermediate region.

It seems relevant to us to have clarified the "quantitative" discrepances between: (i) the tree level weak amplitude (originally computed by Inami and Lim [26]); (ii) the QCD improved weak amplitude for low values of the top quark mass (computed in a definite way by Gilman and Wise [23]) and, finally, (iii) the QCD improved weak amplitude for large values of the top quark mass (initially proposed by Datta et al. [14]).

In a second step we have written a program, which, on the basis of our analytical results, produces a plot of the "ϵ"-parameter (from $K_0 - \overline{K}_0$ mixing) and a plot of the f_B B-meson decay constant (from $B_0 - \overline{B}_0$ mixing) as functions of $\cos\delta$ (where δ is the CP-violating complex phase in the CKM matrix). The use of the experimental value of the ϵ-parameter, which is presently known with great accuracy, allows to restrict, for a given value of the top quark mass, the range of f_B to only two regions in the $(f_B, \cos\delta)$-plane. A better knowledge of f_B could determine a definite value of $\cos\delta$. This seems to us a very interesting and quite new point of view in discussing CP-violation and we would like to further investigate in this direction.

In the future, we are planning to improve our results, by considering also the "ϵ'"-parameter in CP-violating processes, both from a theoretical and a numerical point of view. It is also among our future activities, the more ambitious project to perform a complete two-loop calculation of the effective weak hamiltonian, including heavy top "*penguin*"-diagrams. This calculation is urgently needed to put many theoretical predictions, which involve "*penguin*"-diagrams, on a more solid basis.

My thesis consists of four chapters and two appendices. In the first chapter I give a general description of CP-violation in the K_0–$\overline{K}_0$ mixing and of its justification in the contest of the Standard Model. At the end, the present status of art in the field is generally summarized. The second chapter is devoted to introduce the "*effective hamiltonian*" approach to some particular physical problem, more specifically to the case of weak interactions. In particular, the more technical aspects are illustrated with two instructive examples: (i) the partial explanation of the "*octet enhancement*" or $\Delta I = 1/2$ rule, and (ii) the study of the $\Delta S = 1$ non-leptonic decays. On the latter case, more details, which will be useful for Chapter 3, are also given in Appendix A. The third chapter is concerned with the discussion of QCD corrections to the $K_0 - \overline{K}_0$ mixing amplitude, i.e. the box-diagram amplitude, both for the $m_t \ll M_W$ and the $m_t \gg M_W$ case. The detailed calculations about this second case are reported in Appendix B. Finally, in Chapter 4 it is given the discussion of the numerical implementation of the results presented in Chapter 3.

Chapter 1

CP-violation and the $K_0 - \bar{K}_0$ system in the six quark Standard Model

1.1 ϵ and ϵ' parameters in the $K_0 - \bar{K}_0$ system: general discussion

CP-violation is one of the most interesting phenomena in particle physics. Its existence is well-documented in kaon-decays, but its origin is not completely clear up to now. It is possible to explain CP-violation in the context of the Standard Model, but it could also be sensitive to physics beyond it.

There exists a lot of good reviews about the "*history*" and the "*present status*" of CP-violation [15,27,32]. I will try, here, to summarize the principal features of CP-violation in the $K_0 - \overline{K}_0$ system, because my thesis will concern mainly this particular aspect of the problem.

The phenomenon of CP-violation was first discovered in $K \rightarrow 2\pi$ and $K \rightarrow 3\pi$ decays and these decay modes has been subject to intense scrutiny ever since. Originally, it was thought that weak interactions conserved CP, but not flavour quantum numbers and parity. For this reason $K_0 - \overline{K}_0$ transitions could occur. If CP was conserved, physical states had to coincide with CP-eigenstates. Choosing, as a convention, that $|K_0\rangle = CP|\overline{K}_0\rangle$, CP-eigenstates resulted to be of the form:

$$
\begin{aligned}
|K_1\rangle &= \frac{|K_0\rangle + |\overline{K}_0\rangle}{\sqrt{2}} \qquad CP|K_1\rangle = |K_1\rangle \\
|K_2\rangle &= \frac{|K_0\rangle - |\overline{K}_0\rangle}{\sqrt{2}} \qquad CP|K_2\rangle = -|K_2\rangle
\end{aligned}
\tag{1.1}
$$

Everything seemed to agree with phenomenological results. It was known that the neutral K-meson should have had a long-lived partner and this resulted to be actually the case. If CP had to be a conserved quantum number, K_1 could decay into two pions, but not K_2, because $CP(2\pi)=+1$ while $CP(3\pi)=-1$. Thus K_2 was identified with the long-lived partner of K_1. Only few years later, however, it was discovered that also the long-lived K (K_2) could decay into a pair of pions, with a branching-ratio of about $2 \cdot 10^{-3}$. Thus the long-lived kaon is called K_L and the short-lived one K_S, in order to distinguish them from the CP-eigenstates K_1 and K_2. This was a clear evidence of CP-violation and further stimulated the interest in studying the $K_0 - \overline{K}_0$ system.

Let us now introduce the necessary formalism to discuss the neutral kaon system. The time-evolution of the $K_0 - \overline{K}_0$ is described, using second order perturbation theory, in terms of an effective hamiltonian. That is, if we write the wave function of the system in two components as:

$$\psi(t) = a_K(t)|K_0\rangle + a_{\overline{K}}(t)|\overline{K}_0\rangle = \begin{pmatrix} a_K(t) \\ a_{\overline{K}}(t) \end{pmatrix} \tag{1.2}$$

we have the following equation of motion:

$$i\frac{d\psi(t)}{dt} = \hat{\mathbb{H}}(t) = \left(M - i\frac{\Gamma}{2}\right)\psi(t) \tag{1.3}$$

where

$$\begin{aligned}\hat{\mathbb{H}}_{ij} = \left(M - i\frac{\Gamma}{2}\right)_{ij} &= m_K^{(0)}\delta_{ij} + \langle K_i^{(0)}|\mathcal{H}_W\,|K_j^{(0)}\rangle \\ &+ \sum_n (2\pi)^3\delta^3(\vec{p}_n)\frac{\langle K_i^{(0)}|\mathcal{H}_W|n\rangle\langle n|\mathcal{H}_W|K_j^{(0)}\rangle}{m_K^{(0)} - E_n + i\eta}\,. \end{aligned} \tag{1.4}$$

is usually called the "*mass matrix*". It can be written as:

$$\begin{aligned}\hat{\mathbb{H}} &= \begin{pmatrix} \langle K_0|\mathcal{H}|K_0\rangle & \langle K_0|\mathcal{H}|\overline{K}_0\rangle \\ \langle \overline{K}_0|\mathcal{H}|K_0\rangle & \langle \overline{K}_0|\mathcal{H}|\overline{K}_0\rangle \end{pmatrix} = \\ &= \begin{pmatrix} M_{11} - i\Gamma_{11}/2 & M_{12} - i\Gamma_{12}/2 \\ M_{21} - i\Gamma_{21}/2 & M_{22} - i\Gamma_{22}/2 \end{pmatrix} \end{aligned} \tag{1.5}$$

where M_{ij} and Γ_{ij} are transition matrix elements from virtual and physical intermediate states respectively and can be complex. Moreover:

(i) $M_{11} = M_{22} = M$ and $\Gamma_{11} = \Gamma_{22} = \Gamma$

due to CPT-invariance;

(ii) $M_{21} = M_{12}^*$ and $\Gamma_{21} = \Gamma_{12}^*$

due to the hermiticity of $\hat{\mathbb{H}}$

In this way $\hat{\mathbb{H}}$ assumes the more familiar expression:

$$\begin{pmatrix} M - i\Gamma/2 & M_{12} - i\Gamma_{12}/2 \\ M^\star_{12} - i\Gamma^\star_{12}/2 & M - i\Gamma/2 \end{pmatrix} \tag{1.6}$$

We note that since $|K_0\rangle$ and $|\overline{K}_0\rangle$ do not communicate through strong interactions, their relative phase is not specified. $|K_0\rangle$ is related to $|\overline{K}_0\rangle$ via CP-transformation, up to an arbitrary phase $e^{2i\xi}$:

$$|K_0\rangle = e^{2i\xi} CP |\overline{K}_0\rangle = -e^{2i\xi} C |\overline{K}_0\rangle \tag{1.7}$$

The physical states are the eigenstates of $\hat{\mathbb{H}}$. Clearly, they "*intrinsically*" contain CP-violation and we will see how it is possible to "localize" it in a single parameter, the "ϵ"-parameter.

In order to find the eigenvalues and eigenstates of $\hat{\mathbb{H}}$, we have to solve the equation:

$$det(\hat{\mathbb{H}} - \lambda_\pm 1) = 0 \tag{1.8}$$

which becomes:

$$\begin{pmatrix} \mp(\Delta m - i\Delta\Gamma/2)/2 & M_{12} - i\Gamma_{12}/2 \\ M^\star_{12} - i\Gamma^\star_{12}/2 & \mp(\Delta m - i\Delta\Gamma/2)/2 \end{pmatrix} = 0 \tag{1.9}$$

or:

$$[(\Delta m - i\Delta\Gamma/2)/2]^2 = (M_{12} - i\Gamma_{12}/2)(M^\star_{12} - i\Gamma^\star_{12}/2) \tag{1.10}$$

where we have defined:

$$\mp(\Delta m - i\Delta\Gamma/2)/2 = (M - i\Gamma/2) - \lambda_\pm \tag{1.11}$$

After diagonalizing $\hat{\mathbb{H}}$, the eigenstates corresponding to $\lambda_\pm$ turn out to be respectively:

$$\begin{aligned} |K_L\rangle &= \frac{|K_2\rangle + \bar{\epsilon}|K_1\rangle}{\sqrt{1+|\bar{\epsilon}|^2}} = \frac{(1+\bar{\epsilon})|K_0\rangle - (1-\bar{\epsilon})|\overline{K}_0\rangle}{\sqrt{2(1+|\bar{\epsilon}|^2)}} \\ |K_S\rangle &= \frac{|K_1\rangle + \bar{\epsilon}|K_2\rangle}{\sqrt{1+|\bar{\epsilon}|^2}} = \frac{(1+\bar{\epsilon})|K_0\rangle + (1-\bar{\epsilon})|\overline{K}_0\rangle}{\sqrt{2(1+|\bar{\epsilon}|^2)}} \end{aligned} \tag{1.12}$$

From the eigenvalue equation:

$$(\hat{\mathbb{H}} - \lambda_\pm 1)\begin{pmatrix} 1-\bar{\epsilon} \\ 1+\bar{\epsilon} \end{pmatrix} \tag{1.13}$$

$$= \begin{pmatrix} \mp(\Delta m - i\Delta\Gamma/2)/2 & M_{12} - i\Gamma_{12}/2 \\ M^\star_{12} - i\Gamma^\star_{12}/2 & \mp(\Delta m - i\Delta\Gamma/2)/2 \end{pmatrix}\begin{pmatrix} 1-\bar{\epsilon} \\ 1+\bar{\epsilon} \end{pmatrix} = 0$$

we obtain:

$$\frac{1-\bar{\epsilon}}{1+\bar{\epsilon}} = \frac{(\Delta m - i\Delta\Gamma/2)/2}{M_{12} - i\Gamma_{12}/2} = \frac{M^\star_{12} - i\Gamma^\star_{12}/2}{(\Delta m - i\Delta\Gamma/2)/2} \tag{1.14}$$

This quantity is particularly meaningful, because it does not depend on any phase convention. On the contrary $\bar{\epsilon}$ depends on the phase convention between $|K_0\rangle$ and $|\overline{K}_0\rangle$ and it is not a physical quantity. For this reason, we will introduce a specific quantity η defined as:

$$\eta = \left|\frac{1-\bar{\epsilon}}{1+\bar{\epsilon}}\right| = \left|\frac{M_{12}^{\star} - i\Gamma_{12}^{\star}/2}{M_{12} - i\Gamma_{12}/2}\right|^{1/2} \tag{1.15}$$

This is a physical quantity and the deviation from $\eta = 1$ specifies the amount of CP-violation.

In order to relate $\bar{\epsilon}$ to measurable quantities, we need also to consider the decay of the $K_0 - \overline{K}_0$ system. K_0 and $\overline{K}_0$ decay mainly into two pions. We will use the following notation for the decay amplitudes:

$$\begin{aligned} \langle\pi\pi(I=0)|\mathcal{H}_\mathrm{W}|K_0\rangle &= a_0 e^{i\delta_0} \qquad \langle\pi\pi(I=2)|\mathcal{H}_\mathrm{W}|K_0\rangle = a_2 e^{i\delta_2} \\ \langle\pi\pi(I=0)|\mathcal{H}_\mathrm{W}|\overline{K}_0\rangle &= a_0^{\star} e^{i\delta_0} \qquad \langle\pi\pi(I=2)|\mathcal{H}_\mathrm{W}|\overline{K}_0\rangle = a_2^{\star} e^{i\delta_2} \end{aligned} \tag{1.16}$$

(1.17)

where the phases from strong interactions, $e^{i\delta_0}$ and $e^{i\delta_2}$, are factored out explicitly and the phases of a_o and a_2 are all from weak interactions:

$$\begin{aligned} a_0 &= |a_0|\, e^{i\theta_0} \\ a_2 &= |a_2|\, e^{i\theta_2} \end{aligned} \tag{1.18}$$

Note that the strong interaction phases are the same for K_0 and $\overline{K}_0$, due to CPT-invariance of strong interactions. From the previous definitions, we derive that:

$$\begin{aligned} a_{0,\mathrm{S(L)}} &\equiv \langle\pi\pi(I=0)|\mathcal{H}_\mathrm{W}|K_{\mathrm{S(L)}}\rangle = \frac{1}{\sqrt{2(1+|\bar{\epsilon}|^2)}}\left[(1+\bar{\epsilon})a_0 \pm (1-\bar{\epsilon})a_0^{\star}\right] e^{i\delta_0} \\ a_{2,\mathrm{S(L)}} &\equiv \langle\pi\pi(I=2)|\mathcal{H}_\mathrm{W}|K_{\mathrm{S(L)}}\rangle = \frac{1}{\sqrt{2(1+|\bar{\epsilon}|^2)}}\left[(1+\bar{\epsilon})a_2 \mp (1-\bar{\epsilon})a_2^{\star}\right] e^{i\delta_2} \end{aligned} \tag{1.19}$$

Moreover, relating the isospin states to the physical states as follows:

$$\begin{aligned} \langle\pi^0\pi^0| &= \frac{1}{\sqrt{3}}\langle\pi\pi(I=0)| - \frac{\sqrt{2}}{\sqrt{3}}\langle\pi\pi(I=2)| \\ \langle\pi^+\pi^-| &= \frac{\sqrt{2}}{\sqrt{3}}\langle\pi\pi(I=0)| - \frac{1}{\sqrt{3}}\langle\pi\pi(I=2)| \end{aligned} \tag{1.20}$$

we have that:

$$\begin{aligned} A_{00,\mathrm{S(L)}} &\equiv \langle\pi^0\pi^0|\mathcal{H}_\mathrm{W}|K_{\mathrm{S(L)}}\rangle = \frac{1}{\sqrt{3}}a_{0,\mathrm{S(L)}} - \frac{\sqrt{2}}{\sqrt{3}}a_{2,\mathrm{S(L)}} \\ A_{+-,\mathrm{S(L)}} &\equiv \langle\pi^+\pi^-|\mathcal{H}_\mathrm{W}|K_{\mathrm{S(L)}}\rangle = \frac{\sqrt{2}}{\sqrt{3}}a_{0,\mathrm{S(L)}} + \frac{1}{\sqrt{3}}a_{2,\mathrm{S(L)}} \end{aligned} \tag{1.21}$$

Then, it is usual to define the following measurable quantities:

$$\begin{aligned} \eta_{00} &= \frac{A_{00,L}}{A_{00,S}} = \frac{a_{0,L} - \sqrt{2}a_{2,L}}{a_{0,S} - \sqrt{2}a_{2,S}} = \epsilon - \frac{2\epsilon'}{1 - \sqrt{2}\omega} \\ \eta_{+-} &= \frac{A_{+-,L}}{A_{+-,S}} = \frac{a_{0,L} + 1/\sqrt{2}a_{2,L}}{a_{0,S} + 1/\sqrt{2}a_{2,S}} = \epsilon + \frac{\epsilon'}{1 + (1/\sqrt{2})\omega} \end{aligned} \tag{1.22}$$

where:

$$\begin{aligned} \epsilon &\equiv \frac{a_{0,L}}{a_{0,S}} \quad ; \quad \omega \equiv \frac{a_{2,S}}{a_{0,S}} \quad ; \\ \epsilon' &\equiv \frac{1}{\sqrt{2}}\omega \left(\frac{1}{\omega} \frac{a_{2,L}}{a_{0,S}} - \epsilon \right) \end{aligned} \tag{1.23}$$

Experimentally, it is known that $|a_2/a_0|$ is very small:

$$\left| \frac{a_2}{a_0} \right| \approx \frac{1}{20} \tag{1.24}$$

thus ω can be neglected comparing to one and the approximated expressions for η_{00} and η_{+-}

$$\begin{aligned} \eta_{00} &\approx \epsilon - 2\epsilon' \\ \eta_{+-} &\approx \epsilon + \epsilon' \end{aligned} \tag{1.25}$$

can be used. Note that this is the only point where we make some approximations, using the experimental information on ω. Except for this point, everything up to here is completely general.

Since both η_{00} and η_{+-} are physically measurable quantities, ϵ and ϵ' are too and they are phase convention independent (this is the reason why we denoted with $\bar{\epsilon}$, which is a non-physical quantity, the parameter appearing in th $|K_L\rangle$ and $|K_S\rangle$ definition). ϵ and ϵ' can be cast into the form:

$$\begin{aligned} \epsilon &= \eta_{00} + 2\eta_{+-}/3 \\ \epsilon' &= \eta_{+-} - \eta_{00}/3 \end{aligned} \tag{1.26}$$

and using all the previous formalism, namely eqs. (1.19) and (1.23), we get that:

$$\epsilon = \frac{\bar{\epsilon}\, Rea_0 + i\, Ima_0}{Rea_0 + i\bar{\epsilon}\, Ima_0} = \frac{\bar{\epsilon} + it_0}{1 + i\bar{\epsilon}t_0} \tag{1.27}$$

and

$$\epsilon' = \frac{i}{\sqrt{2}} \left(\frac{Rea_2}{Rea_0} \right) \left(\frac{1 + i\bar{\epsilon}t_2}{1 + i\bar{\epsilon}t_0} \right) (1 - \bar{\epsilon}^2) e^{i(\delta_2 - \delta_0)} \frac{t_2 - t_0}{(1 + i\bar{\epsilon}t_2)(1 + i\bar{\epsilon}t_0)} \tag{1.28}$$

with:

$$t_i \equiv \frac{Ima_i}{Rea_i} \qquad \text{for} \qquad i = 0, 2 \tag{1.29}$$

In a phase convention in which $|\bar{\epsilon}| \ll 1$ (as measured experimentally) eqs. (1.27) and (1.28) become respectively:

$$\begin{aligned} \epsilon &\approx \bar{\epsilon} + it_0 \\ \epsilon' &\approx \frac{i}{\sqrt{2}} \left(\frac{Rea_2}{Rea_0}\right) (t_2 - t_0) e^{i(\delta_2 - \delta_0)} \end{aligned} \tag{1.30}$$

Here we can explicitly check, with some examples, how the dependence of ϵ and ϵ' on a_i, t_i, δ_i $(i = 0, 2)$ and $\bar{\epsilon}$ can vary, because these parameters depend on the phase convention. For instance:

(a) if we chose a phase convention on $|K_0\rangle$ and $|\overline{K}_0\rangle$ such that a_0 is real, then $t_0 = 0$ and:

$$\begin{aligned} \epsilon &= \bar{\epsilon} \\ \epsilon' &\approx \frac{i}{\sqrt{2}} \left(\frac{Rea_2}{Rea_0}\right) t_2 e^{i(\delta_2 - \delta_0)} \end{aligned} \tag{1.31}$$

(b) if, on the contrary, we adopt, as usually done in modern literature, the Kobayashi-Maskawa phase convention:

$$|K_0\rangle = CP|\overline{K}_0\rangle \tag{1.32}$$

we get:

$$\epsilon = \frac{e^{i\pi/4}}{\sqrt{2}} \left[\frac{ImM_{12}}{2ReM_{12}} + t_0\right] = \frac{e^{i\pi/4}}{\sqrt{2}} \left[\frac{t_M}{2} + t_0\right] \tag{1.33}$$

where

$$t_M \equiv \frac{ImM_{12}}{ReM_{12}} \tag{1.34}$$

and

$$\frac{\epsilon'}{\epsilon} = \frac{i}{\sqrt{2}} \frac{1}{\epsilon} e^{i(\delta_2 - \delta_0)} \left[\frac{Ima_2}{Rea_0} - \frac{Rea_2}{Rea_0} t_0\right] \tag{1.35}$$

This phase convention has the advantage that, when you calculate weak interaction amplitudes (within the three families Standard Model), all the complexities come from the phase in the Kobayashi-Maskawa matrix (as we will see in the next section).

In summary, we have to investigate CP-violation on two fronts:

- "*indirect*" CP-violation $\rightarrow$ "*ϵ- parameter*": CP-violation in the $K_0 - \overline{K}_0$ mass matrix;
- "*direct*" CP-violation $\rightarrow$ "*ϵ'-parameter*":CP-violation in the direct $K \rightarrow 2\pi$ transitions.

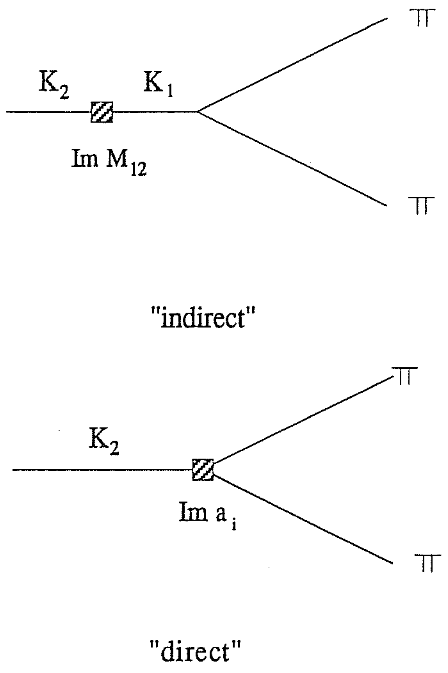

Figure 1.1: "*Indirect*" and "*direct*" CP-violation

as illustrated in fig.(1.1).

In the Standard Model, each of these contributions to CP-violation is related to a particular class of diagrams, to be evaluated at the lowest order in weak and electromagnetic interactions, but to all orders in strong interactions, in a given approximation (leading logarithms, next-to-leading logarithms, etc.). For this reason, the problem of understanding the structure of the effective hamiltonian which describes $|\Delta S| = 1$ and $|\Delta S| = 2$ processes is of fundamental importance. To this aim, we will discuss in detail in the following section, the baeis of the effective hamiltonians and how to account in a rigorous way for QCD radiative corrections.

1.2 CP-violation in the six quark Standard Model

Assuming CPT-invariance of all the interactions, the observed CP-violating effects in the kaon-decay can be described in theories which are:

(a) T-conserving, C-violating, P-violating;

(b) T-violating, C-conserving, P-violating;

(c) T-violating, C-violating, P-conserving;

(d) T-violating, C-violating, P-violating.

From many experiments of similar nature, one can infer that strong and electromagnetic interactions are not of type $(a),(b)$ or (d). Therefore, if the source of CP-violating phenomena is located in strong or electromagnetic phenomena, there must be a part of those interactions belonging to class (c), i.e. C- and T-violating, but P-conserving. Many models have been proposed:

- *Millistrong CP-violation models* [34,40,42], which postulate the existence of C- and T-violating terms of order 10^{-3} in the strong interactions. The $K_L \to \pi^+\pi^-$ would be a two-step process, through an intermediate state X, where $K_L \to X$ via normal CP-conserving weak interaction with $\Delta S = 1$ and $X \to \pi^+\pi^-$ by this T-violating strong interaction. The amplitude of the process would be of order $G_F \cdot a$, where G_F is the Fermi coupling constant and "a" is the coupling of the CP-violating strong interaction. From the experimental value of $|\eta_{+-}|$ one concludes that $a \sim 10^{-3}$.

- *Electromagnetic CP-violation models* [6,7,9,44], which require large part of the electromagnetic interactions of hadrons to be C- and T- violating, but P-conserving. The decay $K_L \to \pi^+\pi^-$ would be again a two-step process: $K_L \to X \to \pi^+\pi^-$, which could occur through interference of weak and electromagnetic CP-violating amplitudes. The product $G_F \cdot \alpha$, where α is the fine structure constant, is not too far from $G_F \cdot 10^{-3}$, as required by the magnitude of $|\eta_{+-}|$.

- *Milliweak models* [1,24,33,35,36,38,41,43,54,52], which assume that a part of the order of 10^{-3} in the weak interaction is CP-violating and responsible for the observed effects. The decay $K_L \to \pi^+\pi^-$ would then be a one-step process.

- *Superweak models* [51], which postulates a new $\Delta S = 2$ CP-violating interaction, with a coupling constant "g" smaller than second-order weak interactions. This interaction could induce a $K_L \to K_S$ transition, with a subsequent decay $K_L \to \pi^+\pi^-$.

Experimentally, there has been no evidence for any CP-violating effect in strong and electromagnetic interactions [25,31]. Thus we are left with "*superweak*" or "*milliweak*" models. These two classes of models can be distinguished by experiments. The "*superweak*" model predicts that $\epsilon' = 0$ and consequently $\eta_{+-} = \eta_{00} = \epsilon$. For the "*milliweak*" models, on the contrary, ϵ' does not vanish and can at most be of the order of the violation of the $\Delta I = 1/2$ rule in non-leptonic weak decays, i.e. $|\epsilon'/\epsilon| < 5 \cdot 10^{-2}$.

One particularly attractive "*milliweak*" model is the one proposed by Kobayashi and Maskawa (KM)[33]. In the framework of the Standard Model, the KM-model describes CP-violation as due to a complex phase in the six quark mixing matrix. It is then possible to construct CP-violating weak amplitudes from "*box-diagrams*" like those in fig.(1.2), which we will consider in details later on.

A necessary feature of this model of CP-violation is the non-equality of the decay rates for $K_L \to \pi^+\pi^-$ and $K_L \to \pi^0\pi^0$. This CP-violation "*of the second kind*"

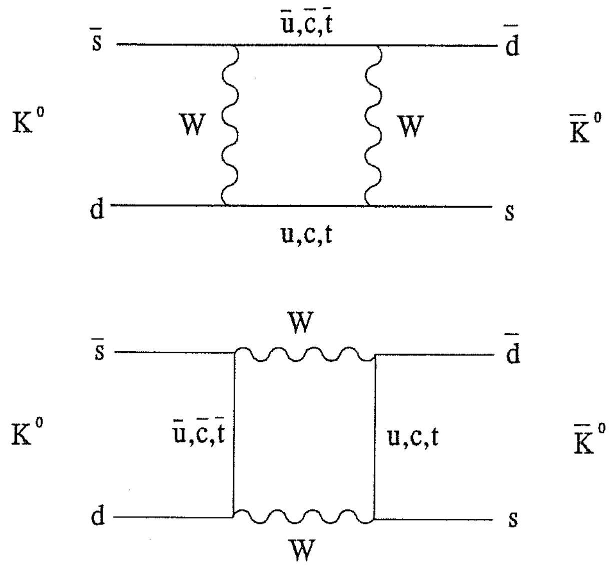

Figure 1.2: $K_0 - \overline{K}_0$ "*box-diagrams*" in the six quark Standard Model

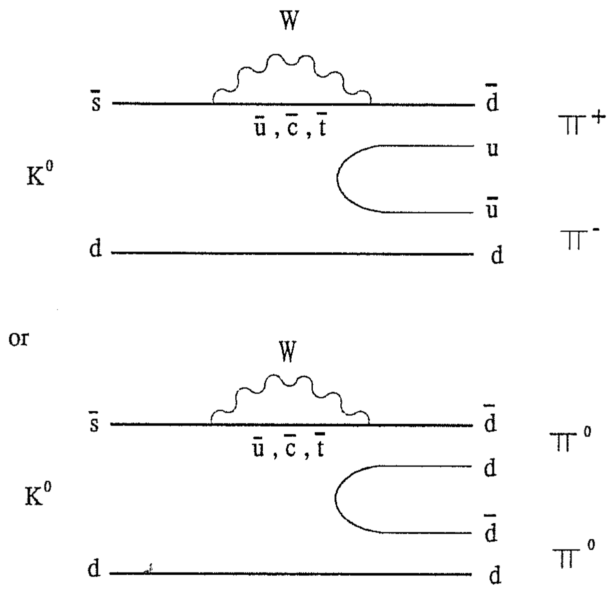

Figure 1.3: $K_L \rightarrow 2\pi$ "*penguin-diagrams*" in the six quark Standard Model

is due to "*penguin-diagrams*" like those in fig. (1.3), which lead to CP-violation in the decay amplitudes (ϵ').

Thus, ϵ and ϵ' can be computed in the Standard Model.

Let us analyze a little bit closer how CP-violation is described in the Standard Model "*a la*" Kobayashi and Maskawa proposal. Neutral currents are flavour-conserving and helicity-conserving, so that passing from the mass eigenstates to the current or "physical" eigenstates does not involve any mixing-matrix between quark states. On the other hand, charged currents are only helicity conserving, but they do mix up-type and down-type quarks (as the $W^{\pm}$ vector bosons carry one unit of charge) and this introduces a quark mixing matrix in a natural way.

Consider, f.i., one of the terms in the lagrangian which generates neutral currents and its transformation properties under a redefinition of the quark fields:

$$\begin{aligned} E_{\rm N}(\vec{x},t) &= \bar{q}_{j\rm R}\gamma^{\mu}\left[\partial_{\mu}-ig_1\left(\frac{2}{3}\right)B_{\mu}\right]q_{j\rm R} = \bar{q}_{\rm R}\gamma^{\mu}\left[\partial\mu-ig_1\left(\frac{2}{3}\right)B_{\mu}\right]q_{\rm R} \\ &= \bar{q}_{\rm R}^{phys}U_{\rm R}\gamma^{\mu}\left[\partial_{\mu}-ig_1\left(\frac{2}{3}\right)B_{\mu}\right]U_{\rm R}^{\dagger}q_{\rm R}^{phys} \\ &= \bar{q}_{\rm R}^{phys}\gamma^{\mu}\left[\partial_{\mu}-ig_1\left(\frac{2}{3}\right)B_{\mu}\right]q_{\rm R}^{phys} \end{aligned} \tag{1.36}$$

We see that the rotation matrix $U_{\rm R}$, which allows to pass from $q_{j\rm R}$ to $q_{j\rm R}^{phys}$, is the same on both sides and no mixing matrix is present at all. On the other hand, the transformation of the term which generates charged currents gives origin to a quark mixing matrix, because in this case left- and right-quark fields are "*rotated*" in a different way, by the matrices $U_{\rm L}$ and $U_{\rm L}'$. In general $U_{\rm L} \neq U_{\rm L}'$ and we have:

$$\begin{aligned} E_{\rm C}(\vec{x},t) &= \left[W_{\mu}^{1}-iW_{\mu}^{2}\right]\bar{q}_{\rm L}\gamma^{\mu}q_{\rm L}' + h.c. \\ &= \left[W_{\mu}^{1}-iW_{\mu}^{2}\right]\bar{q}_{\rm L}^{phys}\gamma^{\mu}U_{\rm L}U_{\rm L}'^{\dagger}q_{\rm L}^{phys} + h.c. \\ &= \left[W_{\mu}^{1}-iW_{\mu}^{2}\right]\bar{q}_{\rm L}^{phys}\gamma^{\mu}V\,q_{\rm L}^{phys} + h.c. \\ &= \left[W_{\mu}^{1}-iW_{\mu}^{2}\right]J_{\rm C}^{\mu} + h.c. \end{aligned} \tag{1.37}$$

where

$$V = U_{\rm L}U_{\rm L}'^{\dagger} \tag{1.38}$$

and $J_{\rm C}^{\mu}$ denotes the charged current:

$$J_{\rm C}^{\mu} = \begin{pmatrix} \bar{u} & \bar{c} & \bar{t} & \cdot & \cdot & \cdot \end{pmatrix}_{\rm L}\gamma^{\mu}\begin{pmatrix} V_{ud} & V_{us} & V_{ub} & \cdot & \cdot & \cdot \\ V_{cd} & V_{cs} & V_{cb} & \cdot & \cdot & \cdot \\ V_{td} & V_{ts} & V_{tb} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} d \\ s \\ b \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}_{\rm L} \tag{1.39}$$

To be more explicit, $E_C(\vec{x},t)$ can be written as:

$$\begin{aligned} E_C(\vec{x},t) &= \left[W_\mu^1 - iW_\mu^2\right] \bar{q}_j \gamma^\mu V_{jk} (1-\gamma_5) q_k' \\ &+ \left[W_\mu^1 + iW_\mu^2\right] \bar{q}_k' \gamma^\mu V_{jk}^\star (1-\gamma_5) q_j \quad . \end{aligned} \tag{1.40}$$

and under CP it transforms as:

$$\begin{aligned} E_C(\vec{x},t) &\overset{CP}{\rightarrow} \left[W_\mu^1 + iW_\mu^2\right] \bar{q}_k' \gamma^\mu V_{jk} (1-\gamma_5) q_j \\ &+ \left[W_\mu^1 - iW_\mu^2\right] \bar{q}_j \gamma^\mu V_{jk}^\star (1-\gamma_5) q_k' \end{aligned} \tag{1.41}$$

where also $(\vec{x},t) \rightarrow (-\vec{x},t)$ is understood.

Thus, in order to have CP-conservation, the matrix V should be real. This reality condition means that the matrix should be real modulus unmeasurable phases. What matters in field theory is not the absolute phase but the relatives phase of fields. Therefore, we must examine which phases in V are measurable and which ones are not. The phases of the fields in J_C^μ are arbitrary. We may redefine them by letting:

$$\begin{aligned} u_L &\rightarrow e^{i\phi(u)} u_L & c_L &\rightarrow e^{i\phi(c)} c_L \\ d_L &\rightarrow e^{i\phi(d)} d_L & s_L &\rightarrow e^{i\phi(s)} s_L \end{aligned} \tag{1.42}$$

where the quantities $\phi(f)$ for $f = u, d, \ldots$ are arbitrary real numbers. There are $2N$ such quantities if there are N families. Under the above phase transformation we have that:

$$V \rightarrow \begin{pmatrix} e^{-i\phi(u)} & 0 & 0 & \cdot & \cdot \\ 0 & e^{-i\phi(c)} & 0 & \cdot & \cdot \\ 0 & 0 & e^{-i\phi(t)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \cdot \begin{pmatrix} V_{ud} & V_{us} & V_{ub} & \cdot & \cdot \\ V_{cd} & V_{cs} & V_{cb} & \cdot & \cdot \\ V_{td} & V_{ts} & V_{tb} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$
$$\cdot \begin{pmatrix} e^{i\phi(d)} & 0 & 0 & \cdot & \cdot \\ 0 & e^{i\phi(s)} & 0 & \cdot & \cdot \\ 0 & 0 & e^{i\phi(b)} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{1.43}$$

therefore, for any number of phases it happens that:

$$V_{\alpha j} \rightarrow \exp[i(\phi(j) - \phi(\alpha))] V_{\alpha j} \tag{1.44}$$

where "α" and "j" denote the up-kind and the down-kind quarks respectively.

A general $(N \times N)$ unitary matrix has N^2 parameters. $N(N-1)/2$ of these parameters may be taken as Euler angles. The remaining parameters are "*phases*". But, from the previous consideration about the measurable phases of the KM-matrix, we see that $(2N-1)$ of them can be eliminated. Therefore V has $(N^2-(2N-1)) = (N-1)^2$ parameters, among which $N(N-1)/2$ are rotation angles. The number of phases will be $(N-1)(N-2)/2$.

It is clear that in a model with only two families, no phase is present in the KM-matrix parametrization; but in a model with three families a complex phase appears. This is exactly the model proposed originally by Kobayashi and Maskawa, in which V is parametrized by three angles (the KM-angles) and one complex phase (the KM-phase). In particular, this phase enters the couplings of quarks to weak vector bosons and it is responsible for the presence of a CP-violating imaginary part in the off-diagonal elements of the mass-matrix in the $K_0 - \overline{K}_0$ system and in the $K \to 2\pi$ decay amplitude.

Many parametrizations of the KM-matrix exist. It is only a matter of choice and I will report here the more usual and useful ones. In the three families Standard Model, the most convenient one is the parametrization proposed by Maiani [37]. In comparison, the original one proposed by Kobayashi and Maskawa leads to cumbersome expressions for the physical relevant transition amplitudes.

In order to understand how the Maiani parametrization arises, let us denote by d', s', b' the three down partners of u, c and t respectively, in the weak charged current left-doublets. Then we chose, following Maiani's prescription:

$$|d'\rangle = c_\beta|d_c\rangle + s_\beta e^{i\delta}|b\rangle \tag{1.45}$$

where $c_\beta = cos\beta$, $s_\beta = sin\beta$ and $|d_c\rangle$ is the Cabibbo-rotated down-quark:

$$|d_c\rangle = c_\theta|d\rangle + s_\theta|s\rangle \tag{1.46}$$

where the same shorthand notation for c_θ and s_θ is used. Two orthonormal vectors, both orthogonal to $|d'\rangle$, are:

$$\begin{aligned} |s_c\rangle &= -s_\theta|d\rangle + c_\theta|s\rangle \\ |v\rangle &= -s_\beta e^{-i\delta}|d_c\rangle + c_\beta|b\rangle \end{aligned} \tag{1.47}$$

The angle γ is now defined by the physical combinations of $|s_c\rangle$ and $|v\rangle$ coupled to c- and t-quark:

$$\begin{aligned} |s'\rangle &= c_\gamma|s_c\rangle + s_\gamma|v\rangle = c_\gamma|s_c\rangle + s_\gamma(-s_\beta e^{-i\delta}|d_c\rangle + c_\beta|b\rangle) \\ |b'\rangle &= -s_\gamma|s_c\rangle + c_\gamma|v\rangle = -s_\gamma|s_c\rangle + c_\gamma(-s_\beta e^{-i\delta}|d_c\rangle + c_\beta|b\rangle) \end{aligned} \tag{1.48}$$

From experiments:

$$s_\theta \equiv \lambda = 0.221 \pm 0.002 \tag{1.49}$$

and also

$$s_\gamma \sim \lambda^2 \qquad \text{and} \qquad s_\beta \sim \lambda^3 \tag{1.50}$$

Thus, empirically, s_γ and s_β are very small in the Maiani convention. Neglecting terms of order λ^4, one obtain a considerable simplification. The weak "*rotated*" doublets are now:

$$\begin{pmatrix} |u\rangle \\ |d_c\rangle + s_\beta e^{i\delta}|b\rangle \end{pmatrix}, \begin{pmatrix} |c\rangle \\ |s_c\rangle + s_\gamma|b\rangle \end{pmatrix}, \begin{pmatrix} |t\rangle \\ |b\rangle - s_\gamma|s_c\rangle - s_\beta e^{-i\delta}|d\rangle \end{pmatrix} \tag{1.51}$$

and the quark mixing matrix V is:

$$\begin{pmatrix} c_\theta & s_\theta & s_\beta e^{i\delta} \\ -s_\theta & c_\theta & s_\gamma \\ s_\gamma s_\theta - s_\beta e^{-i\delta} & -s_\gamma & 1 \end{pmatrix} \tag{1.52}$$

Finally, it is convenient to chose, following Wolfenstein [53]:

$$\begin{aligned} s_\gamma &= A\lambda^2 \\ s_\beta &= A\lambda^3\rho \end{aligned} \tag{1.53}$$

Then V takes the simple form:

$$\begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3\rho e^{i\delta} \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-e^{-i\delta}) & -A\lambda^2 & 1 \end{pmatrix} \tag{1.54}$$

known as the "*Wolfenstein parametrization*" of the KM-matrix. The available experimental informations on A, ρ, δ and other parameters appearing in the previous expressions will be given in Chapter 4, when we discuss the numerical implementation of our results.

Just to conclude, it is interesting to note that CP-violation effects result always proportional, in the small angle approximation, to the quantity:

$$J = s_\theta s_\gamma s_\beta sin\delta \tag{1.55}$$

The quantity J is universal [27], because it does not depend on the parametrization and it is also phase convention independent. It is interesting to note that $J \propto \lambda^6$, so that $j \equiv 10^{-3}$ and this fixes the order of magnitude of CP-violation in the Standard Model.

1.3 Present status of art

The increasing evidence, coming from LEP and SLC experiments, is that there are indeed only three generations of quarks. This implies that CP-violation is governed by one parameter only, i.e. the phase "δ" in the above conventions, and that ϵ and ϵ' are both different from zero. This can be stressed if we restate the definition of the two CP-violation parameters already introduced in the previous section as follows:

- ϵ →"*indirect*" **CP-violation**

$$\epsilon \;=\; s_1 s_2 s_3 \sin\delta \cdot T(m_t, s_{ij}, \delta) \tag{1.56}$$

- ϵ' → "*indirect*" **CP-violation**

$$\epsilon' \;=\; s_1 s_2 s_3 \sin\delta \cdot H(m_t) \tag{1.57}$$

where s_1, s_2 and s_3 correspond to s_θ, s_β and s_γ of the Maiani parametrization; while $T(m_t, s_i, \delta)$ and $H(m_t)$ are functions of the top-quark mass and of the other parameters of the theory. ϵ and ϵ' are zero at the tree level and are suppressed at the one-loop level (box- and penguin-diagrams respectively) by GIM mechanism. However, due to the fact that: $m_u \neq m_c \neq m_t$, both T and H are non-zero and in addition they exhibit a strong dependence on the unknown top-mass. This is particularly true in the $m_t \gg M_W$ region, to be considered in view of the recent experimental lower bound on the top mass ($\sim 60\,GeV$).

The theoretical analysis of ϵ and ϵ' has come to a very advanced stage. It enables us to point out the two main shortcomings of the problem:

- the unknown value of m_t,
- the large theoretical uncertainties in the computation of hadronic matrix elements.

In spite of these problems, the phenomenological analysis of ϵ and ϵ' is still appealing and mandatory.

The actual entity of CP-violation is usually determined as follows:

- for fixed values of m_t, s_1 (with s_2 and s_3 approximately known) and given hadronic matrix elements, the phase δ (which is the last parameter in the CKM matrix) can be found by comparing the ϵ theoretical expression with the recently celebrated experimental value:

$$\epsilon \;=\; (2.258 \pm 0.018) \cdot 10^{-3} \exp\left(i\frac{\pi}{4}\right) \tag{1.58}$$

  One finds two solutions for δ (one in the first and one in the second quadrant), but one of this solutions or even both could be excluded on the basis of ϵ alone and of the observed size of $B_0 - \overline{B}_0$ mixing.

- then the ratio ϵ'/ϵ can be predicted through its theoretical expression ($\sim 10^{-3}$).

A very nice analysis of the consistency of the observed CP-violation, in the six quark Standard Model context, is given in ref. [11]. There, they extensively discuss the allowed ranges for m_t, τ_B and $R = \Gamma(b \to u)/(\Gamma \to c)$ for which the six-quark

Standard Model is consistent with the observed CP-violation in the K-system and in B-meson decay. Unfortunately, this beautiful analysis is not updated and the values of m_t considered are very small (up to 80 Gev). Nevertheless, we think that one could inspire oneself to it, in order to extend the phenomenological analysis to the present available physical regions, mainly the one of heavy masses for the top quark. This is what we would like to state in a more definite way in our research program, after having achieved a clear theoretical understanding of the problem.

In the present work we will be concerned only with the ϵ-parameter, by determining how the box-diagram could be affected by large value of m_t.

The ϵ' case seems to be a little more involved. A lot of work has already been done on the subject [12,8,16], but a clear understanding is not at hand yet. It was thought, initially, that only the QCD "*gluon-penguins*", already mentioned in a previous section, were responsible for the "*direct*" CP-violation. However, recent studies point out that there exist many more relevant contributions, which could change *dramatically* some fundamental physical predictions. The full analysis of the problem is therefore mandatory. Something has been done in ref.[10], but not to a definitive extent.

This is the present status of art and these are our present aims.

Chapter 2

Effective hamiltonians and short distance analysis of weak interactions

2.1 Effective hamiltonians: general meaning

Effective (non-renormalizable) theories are extremely important and have played an important role in the case of weak interactions. The interest in effective theories was initially only phenomenological and had origin with the Fermi theory, which was formulated to describe neutron and nuclei β-decay. The local Fermi lagrangian:

$$\mathcal{L}_{eff} = -\frac{G_{\mathrm{F}}}{\sqrt{2}} J^{\mu} \cdot J_{\mu} \tag{2.1}$$

with

$$J^{\mu} = \bar{\psi}_L \gamma^{\mu} \psi_L = \bar{\psi} \gamma^{\mu} \frac{(1-\gamma_5)}{2} \psi \tag{2.2}$$

was able to give many interesting predictions, experimentally confirmed with great success in later years. It has also been the first pioneering step in our understanding of weak interactions, on the long way which has driven us to the Standard Model theory and beyond it. I shall try to illustrate what I mean, by discussing the problem of effective theories from a more general point of view. The idea of *effective field theories* is basically related to the existence in nature of different mass scales. Obviously, this is the case of weak interactions, in which scales from few Mev's, the leptonic one, up to the weak-vector boson or Higgs mass scales are present. Analogous situations arise in Grand Unified Theories (GUT), Supersymmetric Theories and Supergravity.

There are two different approaches to the problem.

- In the first one, you proceed "*from the top down*", as to say that you completely know the renormalizable theory at higher energy and you need to derive its low energy approximation. In studying the $K_0 - \overline{K}_0$ mixing problem we will adopt precisely this method. Thus it is better to have from now on the feeling of the main peculiarities of this approach.

Given a certain mass scale as UV cut-off, we can formulate an effective field theory involving only those particle which have masses below the UV cut-off. With only these fields, by adding suitable non-renormalizable interactions, we can describe the most general possible interactions consistent with relativistic invariance, unitarity of the S-matrix, CPT invariance and other general properties like these ones. Thus we do not miss anything of the "*descriptive*" power of a theory, in going to the related effective theory. Apparently, due to the introduction of an infinite number of non-renormalizable interactions, corresponding to an infinite number of arbitrary parameters ($\leftrightarrow$ coupling constants of each interaction), we give up the "*predictivity*" of the theory. Actually, this turns out not to be the case, because of "*quantitative*" and "*qualitative*" reasons. "*Quantitatively*" , by knowing the underlying high energy theory, we can calculate all the non-renormalizable interactions and "*quantitative*" calculations can be performed in the spirit of an effective theory (up to a certain order in the inverse UV cut-off); on the other hand "*qualitatively*", all the non-renormalizable interactions in the effective hamiltonian appear with coefficient parameters suppressed by inverse powers of the UV cut-off. Thus, not only we do not lose any quantitative predictivity in going to the effective theory, but also we learn that, when the UV cut-off (e.g. the mass of some very heavy particle) is large enough with respect to the typical mass scales of the effective theory at hand, then the effective theory itself result to be almost renormalizable. There are many examples, which illustrate this kind of approach. You may think to a theory in which a very heavy particle, such that its mass M is completely out of the range of energies experimentally allowed, is present. Then, the only thing you can do is to take the M-mass as an UV cut-off and to develop an effective theory which describes the physics below the UV cut-off. The effective theory will not include the heavy particle of mass M, while some non-renormalizable interactions, proportional to inverse powers of the UV cut-off M, will appear in the effective hamiltonian. This could be the case of GUT's or SUSY theories or technicolor theories and so on so for. All these theories predict the existence of very heavy particles, but we cannot directly verify it. On the other hand, we can derive a low energy approximation of the same theories, which can be predictive and verifiable in the range of energies experimentally allowed. We will apply an "*extreme*" version of this effective field theory language. In fact, we will consider, step by step, the masses of the different heavy particles present in the theory as boundaries between two different effective theories, starting from the mass of the heaviest particle down to the scale at which a given physical process does occur. At each step a given particle mass is treated as an UV cut-off and one can integrate out the related "*heavy*" particle, obtaining a formal expansion in the mass parameter (this is exactly what happens in the case of the Fermi lagrangian with respect to the

W-mass). At the same time, at each threshold, you are supposed to get equivalent physical predictions from the two theories which match at that threshold. Thus you have to connect in a suitable way the parameters of the theory just below the threshold with those of the theory just above. At lowest order, this simply implies that the coupling constants of the light fields are continuous across the boundary. At higher order, due to non-renormalizable interactions, provided by loop-corrections, it imposes a "matching" of all the corresponding couplings. At each threshold "*matching conditions*" have to be imposed in order to eliminate the large logarithms which arise in perturbation theory.

To proceed sistematically, one should start with the renormalization scale μ equal to the mass M of the heaviest particle in the theory and calculate the matching conditions for the parameters of the effective theory with that particle omitted. Then one should rescale μ down to M', the next heaviest mass in the theory, using renormalization group (RG) techniques. At this new threshold M', one has to match the evolved parameters with those of the effective theory below the new threshold M'. One continues, using the same method, in a descending sequence of effective theories, down to the physical scale of interest.

- Another possible approach is to proceed "*from the bottom up*". This is certainly a more physical point of view, the one, f.i., which suggested the presence of a renormalizable gauge field theory of the electro-weak interactions at higher energy, which had as low energy limit exactly the Fermi effective theory.

  In this case, one introduces a non-renormalizable effective theory which accounts for some phenomenological facts. At the same time, one tries to catch all the "*signals*" of a hypothetical underlying renormalizable theory at higher energies, which has the effective one as low energy limit. Typically, every times you have in your effective theory (describing physics up to a given scale) a non-renormalizable interaction with dimensional coupling of order $1/M$ raised to a suitable power (f.i., $1/M^{(D-4)}$ for a D- dimensional operator), you expect that a heavy particle with mass $m \leq M$ exists, responsible for it, so that in the effective theory including also this particle the non-renormalizable interaction disappears. Thus, going up step by step, at each threshold we have that the effects of non-renormalizable interactions grow till they are replaced by renormalizable ones, involving new heavy particles.

  It is just in this spirit that GUT's, SUSY-theories or Supergravity have been considered in recent years.

2.2 Short distance analysis

The strategy "*from the top down*", sketched in the previous section, needs some more details. In particular, I will concentrate on weak processes at low energy, considered at the lowest order in weak interactions, but to all orders in strong interactions (in a given approximation: leading logarithms, next-to-leading logarithms, etc.). The final result of our analysis should be the correct effective lagrangian or hamiltonian describing the low energy process at hand. It is my purpose here to give only a schematic description of this subject and to provide more details on some specific examples [21,22,45,46].

One starts with a basic hamiltonian, which satisfies all the most important properties of the theory. Then one derives an effective hamiltonian from it and studies how this last one is modified by strong interaction radiative corrections. QCD radiative corrections generally modify the lowest order weak effective hamiltonian: if at the tree level you can express it as a linear combination of a given set of operators, generally many more operators will be generated as one-loop QCD effects. In other words, many more new non-renormalizable interactions are introduced, each one with a given coupling.

From a slightly different point of view, you can operatorially expand the product of currents present in the weak effective hamiltonian, at each order of QCD corrections, by writing it as a linear combination of local operators with suitable coefficients. The dependence of the coefficients on the scale can be found by using RG techniques, just as it is done in deep inelastic scattering (DIS).

As we will see in some examples later on, the effective weak hamiltonian receives dominant contributions from dimension-six operators, like the four-fermion operators induced by the W-exchange. At scales below M_W, $\mathcal{H}_{eff}$ is a linear combination of all the operators that mix with the four-quark operator. Call them O_i. Then $\mathcal{H}_{eff}$ will be of the form:

$$\sum_i C_i(\mu) O_i(\mu) \tag{2.3}$$

where $C_i(\mu)$ are the couplings we talked about before (or the Wilson coefficients in an operator product expansion (OPE)). In general, both the coefficients $C_i(\mu)$ and the operators $O_i(\mu)$ depend on the renormalization scale μ, but their product must be renormalization-scale independent. This is the condition which gives the renormalization group equation (RGE) which governs the coefficient evolution. Imposing that:

$$\mu \frac{d}{d\mu} (C_i O_i) = 0 \tag{2.4}$$

you derive that:

$$\left(\mu \frac{\partial}{\partial \mu} \delta_{ij} - \gamma_{ij}(g) \right) C_j(\mu) = 0 \tag{2.5}$$

where g is the strong interaction coupling constant. If you take the lowest order approximation for γ_{ij}:

$$\gamma_{ij} = A_{ij} g^2 + O(g^4) \tag{2.6}$$

it is easy to solve the evolution equation:

$$\mu \frac{d}{d\mu} C_i(\mu) = A_{ij} g^2(\mu) C_i(\mu) \tag{2.7}$$

Denoting with $\hat{A}$ the matrix with components A_{ij} and with $C(\mu)$ the column vector of components $C_i(\mu)$, then we can write the solution of the previous equation as:

$$C(\mu) = \exp\left\{\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \hat{A} g^2(\mu')\right\} C(\mu_0) \tag{2.8}$$

The β-function of the strong coupling constant, defined as:

$$\beta(g) = \mu \frac{d}{d\mu} g(\mu) \tag{2.9}$$

is given at the lowest order by:

$$\beta(g) = -b_0 g^3 \tag{2.10}$$

Thus:

$$g^2 = -\frac{1}{b_0} \frac{1}{g} \mu \frac{d}{d\mu} g \tag{2.11}$$

at the lowest order. By using this relation in eq.(2.8), with a change of variables, you finally get the solution of the RGE for the coefficients $C(\mu)$ in the form:

$$C(\mu) = C(\mu_0) \exp\left\{-\frac{\hat{A}}{b_0} \ln\left(\frac{g(\mu)}{g(\mu_0)}\right)\right\} \tag{2.12}$$

Note that linear combinations of C_i's corresponding to eigenvalues of $\hat{A}$ are multiplicatively renormalized. Thus, in any case, the first thing to do will be to pass to a basis of operators which could give a diagonal $\hat{A}$, in order to have a multiplicative renormalization.

Finally, note that the previous sketch is valid in any effective theory. Depending on the physical problem at hand, you can divide the whole mass range in a suitable arbitrary number of intervals, imposing mass threshold and matching conditions at each one of them and evolving the coefficients in the OPE from one threshold to the other.

2.3 First example: the $\Delta I = 1/2$ rule

The use of short-distance analysis of weak interactions combined with QCD has given a first important result with the partial explanation of the "*octet-enhancement*", or $\Delta I = 1/2$ rule, observed in strangeness-changing decays. In the famous two papers by Altarelli and Maiani and by Gaillard and Lee [19,2], the "*octet-enhancement*" of weak non-leptonic amplitudes is found to occur in asymptotically free gauge theories of strong interactions, combined with unified weak and electromagnetic interactions. In particular, they discuss the structure of strangeness-violating amplitudes, of order αM_W^{-2} ($\sim G_F$, the Fermi constant), that is of the genuine non-leptonic amplitudes.

They assume the effective local form of non-leptonic weak interactions in models in which all interactions are mediated by gauge fields corresponding to a gauge group $G_S \otimes G_W$ (S=strong, W=weak). Moreover they consider models in which weak interactions are described by a Weinberg-Salam type gauge theory and strong interactions by an exactly conserved colour gauge symmetry group.

The discussion is based on the operator product expansion (OPE) of the product of two weak currents. When the short-distance behaviour of the coefficient functions in the OPE is computed, the $\Delta I = 1/2$ rule in $\Delta S \neq 0$ decays can be explained with a more singular short-distance behaviour of the $\Delta I = 1/2$ part of the interaction itself.

Let us summarize the main discussion points. The lowest order contribution of weak and electromagnetic interactions to the transition amplitude $A(i \to f)_{\Delta S \neq 0}$, between two hadronic states, can be written as:

$$A(i \to f)_{\Delta S \neq 0} \sim \alpha \int d^4x D_F(x^2, M_W^2) g^{\mu\nu} \langle f | T \left(J^{\dagger}_{\mu \mathrm{N}}(x) J^{\nu}_{\mathrm{S}}(0) \right) | i \rangle \tag{2.13}$$

where M_W is the charged weak boson mass and $D_F(x^2, M_W^2)$ the charged weak boson propagator. The written currents J^{μ}_{N} and J^{μ}_{S} correspond to strangeness-conserving and strangeness-changing charged currents given by:

$$\begin{aligned} J^{\mu}_{\mathrm{N}} &= (\bar{u} \cos\theta - \bar{c} \sin\theta) \gamma_{\mu} (1 - \gamma_5) d + \cdots \\ J^{\mu}_{\mathrm{S}} &= (\bar{u} \sin\theta + \bar{c} \cos\theta) \gamma_{\mu} (1 - \gamma_5) s + \cdots \end{aligned} \tag{2.14}$$

The full $A(i \to f)_{\Delta S \neq 0}$ transition amplitude would include also some other terms, as tadpole terms or Higgs scalar contributions. However, tadpole terms cannot induce weak interactions; while Higgs exchange terms are negligible, because they are of order $\alpha\, m^2/M_W^2$, where m is a characteristic hadronic mass scale.

The time-ordered product of the two weak currents can be expanded, using OPE, as:

$$T \left[J^{\dagger}_{\mu \mathrm{N}}(x) J^{\mu}_{\mathrm{S}}(0) \right] = \sum_k \mathcal{F}_k(x^2) O_k \tag{2.15}$$

where the $\mathcal{F}_k(x^2)$ are c-number functions of the separation distance and the O_k's are local operators of quarks. This is exactly what we have seen in eq. (2.3). The OPE in (2.15) will contain all the operators needed to account for the tree level process and for the QCD corrections to the process itself, up to a given order. At each order in strong interactions, new operators contribute to the OPE. They naturally arise when you compute QCD corrections to the fundamental four-fermion vertex and their coefficients in the OPE are determined in the same calculation. Thus, generally speaking, using (2.15), you can cast (2.13) in a more explicit form given by:

$$\begin{aligned} A(i \to f)_{\Delta S \neq 0} \;\sim\; & \alpha M_W^{-2} \sum_k D_k \left(\ln \frac{M_W^2}{m^2} \right)^{d_k} \langle f | O_k | i \rangle \\ & + O\left[\alpha M_W^{-2} \frac{m^2}{M_W^2} \ln \left(\frac{M_W^2}{m^2} \right)^{d_k} \right] \end{aligned} \tag{2.16}$$

where O_k are local operators of dimension five or six (see later discussion) and d_k are related to the operator anomalous dimensions. Operators of higher dimension only contribute to the remaining terms, suppressed at least as m^2/M_W^2.

It is clear from (2.13) and (2.16) that the amplitude $A(i \to f)_{\Delta S \neq 0}$ is dominated by the matrix elements of those operators with $d_k < 0$. This gives rise to a possible mechanism to enhance contributions with definite quantum numbers, as f.i. $\Delta I = 1/2$ versus $\Delta I = 3/2$ contributions.

The operators O_k could be composed of two or four quarks. But the operators in the first class either are removed by renormalizing the fields and the parameters in the lagrangian; or are suppressed by very small coefficients [2,19,45] and can be neglected. Thus, only four-quark operators remain. It is important to observe, at this point, that the expansion:

$$\sum_k D_k(\mu) O_k(\mu) \tag{2.17}$$

is scale independent; while the coefficients $D_k(\mu)$ and the operators $O_k(\mu)$ in the expansion are scale dependent. This means that the number and kind of four fermion operators present in the OPE change with the scale μ at which I chose to make the expansion. In a four quark theory (as the one considered in [2,19]), as far as I consider a scale $\mu^2 \gg m_c^2$, only the $d = 6$ operators:

$$\begin{aligned} O_1 &= \bar{q}_1 \gamma^\mu (1-\gamma_5) q_2 \; \bar{q}_3 \gamma_\mu (1-\gamma_5) q_4 \\ O_2 &= \sum_a \bar{q}_1 \gamma^\mu (1-\gamma_5) t^a q_2 \; \bar{q}_3 \gamma_\mu (1-\gamma_5) t^a q_4 \end{aligned} \tag{2.18}$$

contributes (where we have denoted with t^a the usual Gell-mann $SU(3)$ colour matrices). However, when I consider regions in which $\mu^2 \ll m_c^2$, new $d = 6$ operators appear, the famous "*penguin*"-operators, of the form:

$$\sum_a (\bar{q}_1 t^a q_2)_{\rm L} \sum_f (\bar{q}_f t^a q_f)_{\rm (L+R)} \tag{2.19}$$

where $\sum_a$ and $\sum_f$ are sums over colour and flavour respectively. They can mix with $O_1$and O_2 to order α through the diagrams in fig.(2.1), when the quark masses are non degenerate, that is when the GIM cancellation is active.

However, they arise in a region in which perturbative theory start lacking. If we chose not to work in that region, "*penguin*"-diagrams are automatically ruled out and they will not be considered in what follows.

If we consider only O_1 and O_2, we find that they mix under renormalization. On the other hand, operators with definite anomalous dimension can be found diagonalizing the matrix of the renormalization constants, computed from diagrams where a gluon is exchanged in all the possible ways between two fermion lines. Explicit calculations show that the eigenvectors of the anomalous dimension matrix are:

$$O_\pm = \frac{N \pm 1}{N} O_1 \pm \frac{1}{2} O_2$$

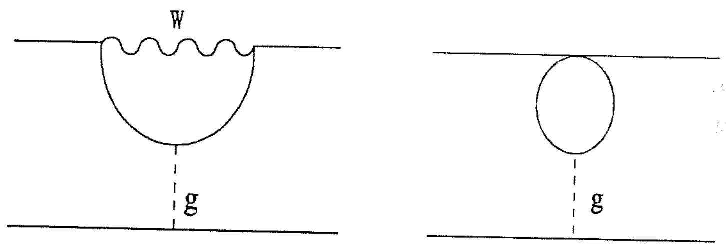

Figure 2.1: "*Penguin*"-diagrams

$$= \frac{1}{2}[(\bar{q}_1 q_2)_L(\bar{q}_3 q_4)_L \pm (\bar{q}_1 q_4)_L(\bar{q}_3 q_2)_L] \quad (2.20)$$

with anomalous dimension:

$$\gamma_\pm = \mp\frac{3}{4\pi}\left(\frac{N\mp 1}{N}\right) \quad (2.21)$$

respectively. Note that we have used the shorthand notation:

$$(\bar{q}_1 q_2)_L \equiv \bar{q}_1\gamma^\mu(1-\gamma_5)q_2 \quad (2.22)$$

and the Fierz identities:

$$\begin{aligned}(\bar{q}_1 q_2)_L(\bar{q}_3 q_4)_L &= \frac{1}{N}(\bar{q}_1 q_4)_L(\bar{q}_3 q_2)_L + 2\sum_a(\bar{q}_1 t^a q_4)_L(\bar{q}_3 t^a q_2)_L \\ \sum_a(\bar{q}_1 t^a q_2)_L(\bar{q}_3 t^a q_4)_L &= \left(\frac{N^2-1}{2N^2}\right)(\bar{q}_1 q_4)_L(\bar{q}_3 q_2)_L - \frac{1}{N}\sum_a(\bar{q}_1 t^a q_4)_L(\bar{q}_3 t^a q_2)_L\end{aligned} \quad (2.23)$$

$O_\pm$ contain the terms $(\bar{s}u)_L(\bar{u}d)_L$ in a combination which is symmetric and antisymmetric under exchange of u and d respectively. Thus, in O_-, u and d are in a $(I=0)$ combination and the operator can mediate only pure $\Delta I = 1/2$; while in O_+, u and d are in a $(I=1)$ combination and the operator can mediate both $\Delta I = 1/2$ and $\Delta I = 3/2$ transitions. Now, the exponents corresponding to the eigenvectors $O_\pm$ are respectively:

$$d_\pm = \mp\frac{1}{2b}\left(\frac{3}{8\pi^2}\right)\left(\frac{N\mp 1}{N}\right) \quad (2.24)$$

where $2b = (24\pi^2)^{-1}(11N - 2n_f)$ ($N \leftrightarrow SU(N)$ n_f =number of flavours) is the first coefficient of the strong coupling constant β-function.

Thus, the result $d_+ < 0$ shows that $\Delta I = 3/2$ transitions can never be enhanced; while $d_- > 0$ implies that "*octet enhancement*" is always obtained.

Moreover, $O_\pm$ form a complete set of operators, as a consequence of the Fierz identities, and they renormalize multiplicatively. It means that we can rewrite (2.13) as:

$$A(i \to f)_{\Delta S \neq 0} \sim \alpha M_W^{-2} \{C_+(t,\alpha)\langle f|O_+(0)|i\rangle + C_-(t,\alpha)\langle f|O_-(0)|i\rangle + \cdots\} \tag{2.25}$$

where

$$t = \ln \frac{M_W^2}{\mu^2} \tag{2.26}$$

with μ a reference mass scale, $\alpha = \alpha(\mu)$ being the renormalized QCD coupling at scale μ:

$$\alpha(\mu) = \frac{g^2(\mu^2)}{4\pi} = \frac{1}{b\ln(\mu^2/\Lambda^2)} \tag{2.27}$$

The dots stand for non-leading terms from operators of higher dimension and also for "*penguin*"-operators of dimension six. Eq. (2.25) strictly resembles eq. (2.16), if we only take care of the scale evolution of the $C_\pm(t,\alpha)$ coefficients. The $C_\pm(t,\alpha)$ coefficients satisfy a RGE of the form:

$$\left(-\frac{\partial}{\partial t} + \beta(\alpha)\frac{\partial}{\partial \alpha} - \gamma_\pm(\alpha)\right) C_\pm(t,\alpha) = 0 \tag{2.28}$$

where $\beta(\alpha)$ and $\gamma_\pm(\alpha)$ are defined at the leading order as follows:

$$\begin{aligned} \beta(\alpha) &= \frac{\partial \alpha}{\partial \ln \mu^2} = -b\alpha^2 \\ \gamma_\pm(\alpha) &= \gamma_\pm^{(1)}\alpha + \cdots \end{aligned} \tag{2.29}$$

and are respectively the coupling constant β-function and the anomalous dimension of the operators $O_\pm$. It is well-known that the solution of the RGE is of the form:

$$C_\pm(t,\alpha) = C_\pm(\alpha(t)) \exp\left\{\int_\alpha^{\alpha(t)} d\alpha' \frac{\gamma_\pm(\alpha')}{\beta(\alpha')}\right\} \tag{2.30}$$

where $\alpha(t)$ is the running coupling constant at scale M_W^2, while $\alpha = \alpha(\mu^2)$, and

$$C_\pm(\alpha(t)) = C_\pm(0,\alpha(t)) \tag{2.31}$$

The coefficient evolution then will be:

$$C_\pm(t,\alpha) \longrightarrow C_\pm(\alpha(t)) \left[\frac{\alpha}{\alpha(t)}\right]^{\gamma_\pm^{(1)}/b} \tag{2.32}$$

and eq. (2.25) becomes now:

$$A(i \to f)_{\Delta S \neq 0} \sim \alpha M_W^{-2} \sum_{i=\pm} C_i(\alpha(t)) \left[\frac{\alpha}{\alpha(t)}\right]^{\gamma_i^{(1)}/b} \langle f|O_i|i\rangle + \cdots \tag{2.33}$$

Following (2.27), we take α to be of the form:

$$\alpha(t) = \frac{1}{b\ln(M_W^2/\Lambda^2)} \tag{2.34}$$

then the term in the square bracket is exactly of the form:

$$\left(\ln\frac{M_W^2}{\mu^2}\right)^{d_i} \qquad \text{for} \qquad d_i = \frac{\gamma_i^{(1)}}{b} \tag{2.35}$$

modulus constant terms and we obtain:

$$A(i \to f)_{\Delta S \neq 0} \sim \alpha M_W^{-2} \sum_{i=\pm} D_i \left(\ln\frac{M_W^2}{\mu^2}\right)^{d_i} \langle f|O_i|i\rangle \cdots \tag{2.36}$$

as in eq. (2.16). From here, as we have already discussed, it is possible to derive a partial explanation of the $\Delta I = 1/2$ rule.

2.4 Second example: weak decays

Some years later, Witten has stated in a more definite way the approach to weak interactions using effective theories, in a famous paper: "*Short distance analysis of weak interactions*", followed by an analogous work on DIS. [49,50]. The basic ideas and the method used in ref. [49] are very instructive, because he takes, as a specific example, precisely the $K_0 - \overline{K}_0$ mixing problem, in a three- and in a four-quark theory. The method to calculate the S-matrix elements to the lowest order in weak and electromagnetic interactions, while to all orders in the strong ones (at a given approximation), is elucidated and two main points are stressed:

- weak interactions can be replaced by an effective hamiltonian,
- QCD corrections can be summed up to all orders using RG techniques.

There is nothing new in that with respect to the ideas already present in other precedent works, as the ones we already mentioned. However the proof of the theoretical correctness of the method used seems here to be better stated. The proof proceed by induction, starting from the weak and electromagnetic interaction lowest order contribution to the $K_0 - \overline{K}_0$ mixing, the box-diagram in fig.(2.2), without QCD corrections and verifying, at the one-loop order of QCD corrections, that the theory can still be described by an effective local hamiltonian of the form:

$$\mathcal{H}_{eff} = k \cdot (\bar{s}\gamma_\mu(1-\gamma_5)d\,\bar{s}\gamma^\mu(1-\gamma_5)d \;+\; h.c.) \tag{2.37}$$

Assuming the form of the $\mathcal{H}_{eff}$ in eq. (2.37), the result is proven at n-loop in perturbation theory and a short distance analysis of the obtained effective hamiltonian

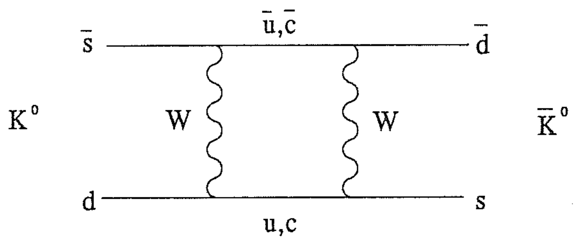

Figure 2.2: $K_0 - \overline{K}_0$ box-diagram

improved. Later on, some very interesting technical applications has also been developed in the study of non-leptonic and semi-leptonic weak decays. An exhaustive description of the problem is present in literature, especially in the complementary papers by Vaĭnshteĭn et al. [45,46] and by Gilman et al. [21,22]. The first one gives a very detailed treatment of the general structure of the effective hamiltonian to be used, by choosing the operators in the OPE on the basis of symmetry principles and flavour quantum numbers. The authors confirm the result that $d = 6$ four-quark operators are the dominant ones and classify the variety of such operators in terms of "*unitary spin*" and "*isotopic spin*", in view of their applications to specific physical problems (they suppose chiral symmetry to be broken explicitly at the $SU(3)_L \otimes SU(3)_R$ level, leaving only an $SU(3)$ of "*unitary spin*", while it survives at the $SU(2)_L \otimes SU(2)_R$ level). Moreover, they treat in a certain detail the real problem of how to deal with the "long-distance" sector of the theory, namely the evaluation of hadronic matrix elements.

On the other hand, I would like to spend some more words on the paper by Gilman and Wise [21]. Later on, in the discussion of the central argument of my thesis, we will meet with another fundamental work by them [23], which will apply again the same techniques, except for a few necessary modifications.

In ref. [21], they discuss $\Delta S = 1$ non-leptonic decays in a six-quark model, starting from a $(V-A)\otimes(V-A)$-type effective hamiltonian, at the lower order in weak interactions and at zero-order in strong interactions, of the form:

$$\mathcal{H}_{eff} = \frac{g^2}{8M_W^2} J_\mu^+(0) \cdot J_\mu^-(0) + h.c. \tag{2.38}$$

where

$$J_\mu^+(0) = (\bar{u}d')_{V-A} + (\bar{c}s')_{V-A} + (\bar{t}b')_{V-A} \tag{2.39}$$

d',s' and b' are the current quark-eigenstates and

$$(qq')_{V-A} = \bar{q}\gamma_\mu(1-\gamma_5)q' \tag{2.40}$$

The strangeness-changing part in the effective hamiltonian $\mathcal{H}_{eff}$ can also be expanded, because of GIM mechanism, on the operator basis costituted by:

$$O_q^{(\pm)} = [(\bar{s}_\alpha u_\alpha)_{V-A}(\bar{u}_\beta d_\beta)_{V-A} \pm (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A}] - [u \rightarrow q] \tag{2.41}$$

with $q = c, t$, as

$$\mathcal{H}_{eff}^{(\Delta S=1)} = -\frac{G_F}{2\sqrt{2}}\left[A_c\left(O_c^{(+)} + O_c^{(-)}\right) + A_t\left(O_t^{(+)} + O_t^{(-)}\right)\right] \tag{2.42}$$

where

$$\begin{aligned} A_c &= s_1 c_2 (c_1 c_2 c_3 - s_2 s_3 e^{-i\delta}) \\ A_t &= s_1 s_2 (c_1 s_2 c_3 + c_2 s_3 e^{-i\delta}) \end{aligned} \tag{2.43}$$

with: $s_i = \sin\theta_i$, $c_i = \cos\theta_i$ and δ are the CKM matrix parameters. This expression for the effective hamiltonian is particularly suitable to account for strong interaction QCD corrections, because $O_q^{(\pm)}$ are multiplicatively renormalizable.

They analyze the problem in the leading logarithmic approximation (LL), setting a certain number of mass thresholds, namely M_W, m_t, m_b and m_c (other quarks are considered as massless), and developing the theory step by step as explained in section (2.2), starting from $\mu = M_W$ at highest mass scale. I will summarize here only the main results, referring to Appendix A for the details.

Step 1 At this step we take M_W as much heavier than any other mass scale and we get at the leading order in the W-boson mass:

$$\begin{aligned} \mathcal{H}_{eff}^{(\Delta S)=1} &= -\frac{G_F}{2\sqrt{2}}\sum_{q=c,t}\left[A_q^{(+)}\left(\frac{M_W}{\mu}, g\right)O_q^{(+)} + A_q^{(-)}\left(\frac{M_W}{\mu}, g\right)O_q^{(-)}\right] \\ &= -\frac{G_F}{2\sqrt{2}}\left\{\left[\frac{\alpha(M_W^2)}{\alpha(\mu^2)}\right]^{a^{(+)}}\left(A_c O_c^{(+)} + A_t O_t^{(+)}\right) + \right. \\ &+ \left.\left[\frac{\alpha(M_W^2)}{\alpha(\mu^2)}\right]^{a^{(-)}}\left(A_c O_c^{(-)} + A_t O_t^{(-)}\right)\right\} \end{aligned} \tag{2.44}$$

where

$$a^{(+)} = \frac{6}{33 - 2N_f} \qquad \text{and} \qquad a^{(-)} = -\frac{12}{33 - 2N_f} \tag{2.45}$$

Step 2 Now m_t is taken as the highest mass scalar and a different treatment for $O_c^{(\pm)}$ and $O_t^{(\pm)}$ is required.

For the first one, $O_c^{(\pm)}$, we are exactly in the same situation as before, with the only difference that now we shall work in an effective field theory with only five quark fields, instead of six. This requires the introduction of a new β-function, new anomalous dimensions and so on so for. $O_c^{(\pm)}$ result to be multiplicatively renormalized as before and only the dependence of a^+, a^- and $\alpha(\mu)$ on the number of flavours is altered.

On the other hand, the case of $O_t^{(\pm)}$ is more complicated. Thank to the Appelquist and Carazzone theorem [5], we can express this part of $\mathcal{H}_{eff}$ on a suitable basis of operators $\{O_i\}$, which mix under renormalization. They depend on five quark fields only and they are given by:

$$\begin{aligned}
O_1 &= (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A} \\
O_2 &= (\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A} \\
O_3 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V-A} + \ldots + (\bar{b}_\beta b_\beta)_{V-A}] \\
O_4 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V-A} + \ldots + (\bar{b}_\beta b_\alpha)_{V-A}] \\
O_5 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V+A} + \ldots + (\bar{b}_\beta b_\beta)_{V+A}] \\
O_6 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V+A} + \ldots + (\bar{b}_\beta b_\alpha)_{V+A}]
\end{aligned} \tag{2.46}$$

$\mathcal{H}_{eff}$, at a scale μ such that $m_b \ll \mu \ll m_t$, results to be:

$$\begin{aligned}
\mathcal{H}_{eff}^{(\Delta S=1)} &= -\frac{G_F}{2\sqrt{2}}\left\{\left[\frac{\alpha(m_t^2)}{\alpha'(\mu^2)}\right]^{a'^{(+)}}\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a^{(+)}} A_c O_c^{(+)} + \right. \\
&+ \left[\frac{\alpha(m_t^2)}{\alpha'(\mu^2)}\right]^{a'^{(-)}}\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a^{(-)}} A_c O_c^{(-)} + \\
&+ \left.\sum_k \left(\mathrm{V}_k^{(+)}\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a^{(+)}} A_t O_k + \mathrm{V}_k^{(-)}\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a^{(-)}} A_t O_k\right)\right\}
\end{aligned} \tag{2.47}$$

where

$$\begin{aligned}
\mathrm{V}_k^{(+)} &\equiv \sum_{i,j} \mathrm{V}_{kj}\left[\frac{\alpha(m_t^2)}{\alpha'(\mu^2)}\right]^{a'_j} \mathrm{V}_{ji}^{-1} B_i^{(+)} \\
\mathrm{V}_k^{(-)} &\equiv \sum_{i,j} \mathrm{V}_{kj}\left[\frac{\alpha(m_t^2)}{\alpha'(\mu^2)}\right]^{a'_j} \mathrm{V}_{ji}^{-1} B_i^{(-)}
\end{aligned} \tag{2.48}$$

All the operators on the r.h.s. of eq. (2.47) should have their matrix elements evaluated in a five-quark effective theory (all the notations used are explained in App. A).

Step 3 When the heaviest mass scale of the theory becomes m_b, a new effective theory with only four flavours exist, while b-quark is removed from the theory. The same argument seen at *Step 2* holds both for $O_c^{(\pm)}$ and $\{O_i\}_{i=1,\ldots,6}$: the first ones are multiplicatively renormalized, while the second ones require the introduction of a new set of linearly independent operators $\{P_j\}$, of the same kind of the $\{O_i\}$'s but with the b-quark removed:

$$\begin{aligned} P_1 &= (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A} \\ P_2 &= (\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A} \\ P_3 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V-A} + \ldots + (\bar{c}_\beta c_\beta)_{V-A}] \\ P_4 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V-A} + \ldots + (\bar{c}_\beta c_\alpha)_{V-A}] \\ P_5 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V+A} + \ldots + (\bar{c}_\beta c_\beta)_{V+A}] \\ P_6 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V+A} + \ldots + (\bar{c}_\beta c_\alpha)_{V+A}] \end{aligned} \tag{2.49}$$

After RGE's for the coefficients have been solved, you get $\mathcal{H}_{eff}^{(\Delta S=1)}$, at a scale μ such that $m_c \ll \mu \ll m_b$, in the form:

$$\begin{aligned} \mathcal{H}_{eff}^{(\Delta S=1)} = & -\frac{G_F}{2\sqrt{2}} \left\{ \left[\frac{\alpha'(m_b'^2)}{\alpha'(\mu^2)}\right]^{a''(+)} \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{a'(+)} \left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a(+)} A_c O_c^{(+)} + \right. \\ & + \left[\frac{\alpha'(m_b'^2)}{\alpha'(\mu^2)}\right]^{a''(-)} \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{a'(-)} \left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a(-)} A_c O_c^{(-)} + \\ & \left. + \sum_{k,n} \mathcal{W}_{nk} \mathbb{V}_{ki} \left(\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a(+)} B_i^{(+)} A_t + \left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a(-)} B_i^{(-)} A_t \right) P_n \right\} \end{aligned} \tag{2.50}$$

where

$$\begin{aligned} \mathcal{W}_{nk} &\equiv \sum_{l,m} W_{nm} \left[\frac{\alpha'(m_b'^2)}{\alpha''(\mu^2)}\right]^{a''_m} W_{ml}^{-1} C_k^l \\ \mathbb{V}_{ki} &\equiv \sum_{i,j} V_{kj} \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{a'_j} V_{ji}^{-1} \end{aligned} \tag{2.51}$$

Step 4 Finally, when the heaviest mass scale coincides with m_c and a three-quark effective theory enters the game, also $O_c^{(\pm)}$ stop being multiplicatively renormalizable and we need to expand its matrix elements too on a complete set of linearly independent operators $\{Q_i\}$, which again mix under renormalization, given by:

$$\begin{aligned} Q_1 &= (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A} \\ Q_2 &= (\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A} \\ Q_3 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V-A} + (\bar{d}_\beta d_\beta)_{V-A} + (\bar{c}_\beta c_\beta)_{V-A}] \\ Q_5 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V+A} + (\bar{d}_\beta d_\beta)_{V+A} + (\bar{c}_\beta c_\beta)_{V+A}] \\ Q_6 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V+A} + (\bar{d}_\beta d_\alpha)_{V+A} + (\bar{c}_\beta c_\alpha)_{V+A}] \end{aligned} \tag{2.52}$$

where Q_4 has been dropped to recover linear independence. After having solved the RGE's for their coefficients too, we get:

$$
\begin{aligned}
\mathcal{H}_{eff}^{(\Delta S=1)} = & -\frac{G_F}{2\sqrt{2}}\Bigg\{\sum_r \mathbb{X}_r^{(+)}\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{a''(+)}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{a'(+)}\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a(+)} A_c Q_r + \\
& + \sum_r \mathbb{X}_r^{(-)}\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{a''(-)}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{a'(-)}\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a(-)} A_c Q_r + \\
& + \sum_{k,n,r,i} \mathbb{X}_{rn}\mathbb{W}_{nk}\mathbb{V}_{ki}\left(\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a(+)} B_i^{(+)} A_t + \right. \\
& + \left.\left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)}\right]^{a(-)} B_i^{(-)} A_t\right) Q_r \Bigg\}
\end{aligned}
\tag{2.53}
$$

where

$$
\begin{aligned}
\mathbb{X}_r^{(+)} &\equiv \sum_{p,q} \mathbb{X}_{rq}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{a_q'''} \mathbb{X}_{qp}^{-1} D_p^{(+)} \\
\mathbb{X}_r^{(-)} &\equiv \sum_{p,q} \mathbb{X}_{rq}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{a_q'''} \mathbb{X}_{qp}^{-1} D_p^{(-)} \\
\mathbb{X}_{rn} &\equiv \sum_{p,q} \mathbb{X}_{rq}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{a_q'''} \mathbb{X}_{qp}^{-1} D_n^p \\
\mathbb{W}_{nk} &\equiv \sum_{l,m} \mathbb{W}_{nm}\left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{a_m''} \mathbb{W}_{ml}^{-1} C_k^l \\
\mathbb{V}_{ki} &\equiv \sum_{j} \mathbb{V}_{kj}\left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{a_j'} \mathbb{V}_{ji}^{-1}
\end{aligned}
\tag{2.54}
$$

and at each threshold suitable matching conditions are imposed.

Some interesting considerations can be done on the structure of the effective hamiltonian at low energy, mainly about CP-violation. Among the $\{Q_r\}$ operators:

- Q_1 and Q_2 are usual four-fermions operators, while
- Q_3,Q_5 and Q_6 are "*penguin*" operators

and only Q_2 is present at the zero-order in strong interactions. The "*penguin*"-diagrams give rise to new effective "interactions" described by Q_3, Q_5 and Q_6 and generate for all the coefficients an imaginary part, which is at the origin of CP-violation in the Standard Model. In particular "*penguin*"-operators Q_3, Q_5 and Q_6 have small coefficients, but their matrix elements are expected to be larger than for Q_1 and Q_2 , so to to give a global non-negligible CP-violating effect.

Chapter 3

$K_0 - \bar{K}_0$ mixing: a detailed analysis

3.1 The box-diagram and CP-violation

In the general context of the Standard Model, I have already described the basic features of $K_0 - \overline{K}_0$ mixing and its relation to the *ϵ-parameter*. ϵ is given by the following expression:

$$\begin{aligned}\epsilon &= \frac{\exp i\frac{\pi}{4}}{\sqrt{2}\Delta M}\left(ImM_{12} + 2\xi ReM_{12}\right) \\ &\simeq \frac{\exp i\frac{\pi}{4}}{\sqrt{2}}\left(\xi + \frac{ImM_{12}}{\Delta M}\right)\end{aligned} \tag{3.1}$$

where

$$\xi = \frac{ImA_0}{ReA_0} \ , \qquad \text{with} \qquad \langle\pi\pi(I=0)|\mathcal{H}_W|K\rangle = A_0 e^{i\delta_0} \ , \tag{3.2}$$

is only a very small correction. ΔM is the mass difference between K_L and K_S states (experimentally: $\Delta M = 3.5 \cdot 10^{-15}$ GeV) and M_{12} is the dispersive part of the off-diagonal elements of the K_0–$\overline{K}_0$ mixing mass-matrix:

$$M_{12} = \langle\overline{K}_0|\mathcal{H}_{eff}^{|\Delta S|=2}|K_0\rangle \tag{3.3}$$

At lowest order in weak interactions, there are two box-diagrams (see fig. (3.1)) which contribute to the $|\Delta S| = 2$ amplitude and similarly for any other $|\Delta F| = 2$ process (F =flavour) (e.g., the B_0–$\overline{B}_0$ mixing).

Our main purpose will be to get a full understanding of the principal features of the $|\Delta S| = 2$ amplitude, renormalized by QCD perturbative corrections. The aim is to obtain an effective lagrangian where the radiative effects are summed up to all orders, in the leading logarithmic (LL) approximation.

The theoretical expression of $\mathcal{H}_{eff}^{|\Delta S|=2}$ is of the form:

$$\mathcal{H}_{eff}^{|\Delta S|=2} = = \frac{G_F^2}{16\pi^2}M_W^2(\bar{d}\gamma^\mu(1-\gamma_5)s)^2\left\{\lambda_c^2 F(x_c) + \lambda_t^2 F(x_t) + 2\lambda_c\lambda_t F(x_c, x_t)\right\} \tag{3.4}$$

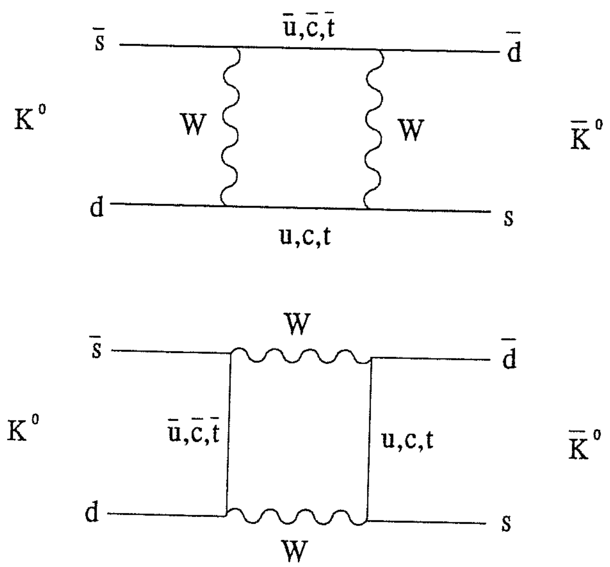

Figure 3.1: Two possible box-diagrams for the $K_0 - \overline{K}_0$ mixing

which, in the case of a light top quark, can also be written as:

$$\mathcal{H}_{eff}^{|\Delta S|=2} = \frac{G_F^2}{16\pi^2} M_W^2 (\bar{d}\gamma^\mu(1-\gamma_5)s)^2 \left\{\lambda_c^2\eta_1 E(x_c) + \lambda_t^2\eta_2 E(x_t) + 2\lambda_c\lambda_t\eta_3 E(x_c, x_t)\right\} \tag{3.5}$$

because QCD corrections factorize. In equations (3.4) and (3.5), the notations used have the following meaning:

- G_F is the Fermi coupling constant,
- λ_q's are related to the CKM matrix element by

$$\lambda_q = V_{qi}^\star V_{qf} \tag{3.6}$$

  where 'i' and 'f' are the labels of the initial and final states respectively (for example, $i = s$ and $f = d$ in the K_0–$\overline{K}_0$ mixing case),
- $x_q = m_q^2/M_W^2$,
- the functions $E(x_i)$ and $E(x_i, x_j)$ are the so-called *Inami-Lim* functions [26], obtained from the calculation of the basic box-diagram (as we will see in App. B) and are explicitly given by:

$$E(x_i) = E(x_i, x_i) = x_i \left[\frac{1}{4} + \frac{9}{4}\frac{1}{(1-x_i)} - \frac{3}{2}\frac{1}{(1-x_i)^2}\right] + \frac{3}{2}\left(\frac{x_i}{x_i - 1}\right)\ln x_i \tag{3.7}$$

$$E(x_i,x_j) = x_i x_j \left\{ \left[\frac{1}{4} + \frac{3}{2}\frac{1}{(1-x_j)} - \frac{3}{4}\frac{1}{(1-x_j)^2}\right] \frac{\ln x_j}{x_j - x_i} + (x_j \leftrightarrow x_i) - \frac{3}{4}\frac{1}{(1-x_i)(1-x_j)} \right\} \quad (3.8)$$

$E(x_i,x_j)$ is evaluated from the box–diagram in which a i-quark is exchanged along one internal fermion line of the box and a j-quark along the other, and $E(x_i)$ refers to the case in which the two exchanged quarks have the same flavour. The Inami-Lim results for the E-functions are valid for arbitrary values of the internal quark masses. It will be very useful to consider their expressions for $x_i \ll 1$:

$$E(x_i) \simeq x_i \quad (3.9)$$

and for $x_j \ll x_i \ll 1$:

$$E(x_i,x_j) \simeq x_j \ln \frac{x_i}{x_j} \quad (3.10)$$

- the functions $F(x_i,x_j)$ and $F(x_i)$ are only a shorthand notation to indicate the *QCD-corrected Inami-Lim* function. In particular, when the top quark is light, they are related to the E-functions by the following simple relation:

$$\begin{aligned} F(x_c,x_t) &= \eta_3 E(x_c,x_t) \\ F(x_c) &= \eta_1 E(x_c) \\ F(x_t) &= \eta_2 E(x_t) \end{aligned} \quad (3.11)$$

The physical value of M_{12} is obtained from the following amplitude:

$$\begin{aligned} M_{12} &= \langle \overline{K}_0 | \mathcal{H}_{eff}^{|\Delta S|=2} | K_0 \rangle \\ &= \frac{G_F^2}{16\pi^2} M_W^2 \left\{ \lambda_c^2 F(x_c) + \lambda_t^2 F(x_t) + 2\lambda_c\lambda_t F(x_c,x_t) \right\} \\ &\quad \cdot \langle \overline{K}_0 | (\bar{d}\gamma^\mu(1-\gamma_5)s)^2 | K_0 \rangle \end{aligned} \quad (3.12)$$

The crucial point is the evaluation of the matrix element of the $(V-A)\otimes(V-A)$ product of currents. Usually the "*vacuum saturation*" approximation is used. This approximation allows us to use a well known result about the matrix element of a $(V-A)$ current between the vacuum and a pseudoscalar meson state. In that case, only the axial current contributes:

$$\langle 0 | \bar{s}\gamma_\mu(1-\gamma_5)d | K_0 \rangle = \frac{i p_\mu f_K}{\sqrt{2E_K}} \quad (3.13)$$

where p_μ and E_K are the momentum and the energy of the meson and f_K its decay constant. The "*vacuum saturation*" approximation consists in restricting the sum over a complete set of intermediate states to the vacuum state only, obtaining, in the K_0 rest frame:

$$\begin{aligned} \langle \overline{K}_0 | (\bar{d}\gamma^\mu(1-\gamma_5)s)^2 | K_0 \rangle &= 2\,\frac{4}{3} B_K \frac{1}{m_K} {m_K}^2 f_K^2 \\ &= \frac{4}{3} B_K f_K^2 m_K \end{aligned} \quad (3.14)$$

where:

(i) a factor 2 has been introduced to account for the possibility to contract in two different ways the currents on the K_0-$\overline{K}_0$ mesons;

(ii) the factor 4/3 arises because the vacuum state can be inserted into two ways. This corresponds to the two possible box-diagrams in fig. (3.1), which differ by the exchange of two s-fields. As well known, the $(V-A)\otimes(V-A)$ product of the two currents can be Fierz-rearranged in the following way:

$$(\bar{d}_1 s_1)_L(\bar{d}_2 s_2)_L = \frac{1}{3}(\bar{d}_1 s_2)_L(\bar{d}_2 s_1)_L + 2\sum_a(\bar{d}_1 t^a s_2)_L(\bar{d}_2 t^a s_1)_L \tag{3.15}$$

where:

$$(\bar{d}_i s_j)_L \equiv \bar{d}_i \gamma^\mu(1-\gamma_5)s_j \tag{3.16}$$

and

$$(\bar{d}_i t^a s_j)_L \equiv \bar{d}_i t^a \gamma^\mu(1-\gamma_5)s_j \tag{3.17}$$

with t^a being the colour $SU(3)$ Gell-Mann matrices. The octet-octet term cannot contribute when each octet is sandwiched between the vacuum and a colour singlet K-state. Thus, the two vacuum state insertions, which differ by the $s_1 \leftrightarrow s_2$ exchange, contribute with the factor:

$$\left(1+\frac{1}{3}\right) = \frac{4}{3} \tag{3.18}$$

(iii) m_K is the mass of the K-meson;

(iv) f_K is the decay constant for the K-meson;

(v) B_K is a factor which is inserted to take into account all the possible deviations from the vacuum saturation approximation. In other words, it parametrizes our ignorance of the matrix element of $(\bar{d}\gamma^\mu(1-\gamma_5)s)^2$ between K_0 and $\overline{K}_0$, with $B_K = 1$ corresponding to an exact vacuum insertion approximation.

Finally, collecting all the results, we will have that:

$$M_{12} = \frac{G_F^2}{12\pi^2}M_W^2 B_K f_K^2 m_K \left\{\lambda_c^2 F(x_c) + \lambda_t^2 F(x_t) + 2\lambda_c\lambda_t F(x_c, x_t)\right\} \tag{3.19}$$

and

$$\begin{aligned} |\epsilon|_{\xi=0} &= \frac{1}{\sqrt{2}}\frac{1}{\Delta M}\left[\frac{G_F^2}{12\pi^2}M_W^2 B_K f_K^2 m_K\right] \\ &\quad \cdot Im\left\{\lambda_c^2 F(x_c) + \lambda_t^2 F(x_t) + 2\lambda_c\lambda_t F(x_c, x_t)\right\} \end{aligned} \tag{3.20}$$

where $\xi = 0$ has been chosen, in view of the smallness of the ξ parameter. We now give the expression of λ_c and λ_t in the Wolfenstein parametrization, as given in [53]:

$$\begin{aligned} \lambda_c &= U_{cs}U_{cd}^* = -\lambda\left(1-\frac{\lambda^2}{2}\right) \simeq -\lambda + \cdots \\ \lambda_t &= U_{ts}U_{tc}^* = -A^2\lambda^5(-1+\rho e^{i\delta}) \end{aligned} \tag{3.21}$$

Due to the unitarity of the KM matrix, by taking λ_u real, we must have:

$$Im\,\lambda_c = -Im\,\lambda_t \tag{3.22}$$

because:

$$Im(\lambda_u + \lambda_c + \lambda_t) = 0 \tag{3.23}$$

This implies:

$$\begin{aligned} Im(\lambda_c^2) &= 2\,Re\,\lambda_c Im\,\lambda_c = -2\lambda Im\,\lambda_c \\ &= 2\lambda Im\,\lambda_t = 2\rho\sin\delta A^2\lambda^6 \end{aligned} \tag{3.24}$$

$$Im(\lambda_t^2) = 2\rho\sin\delta A^4\lambda^{10}(1-\rho\cos\delta) \tag{3.25}$$

$$Im(\lambda_c\lambda_t) = -2\lambda Im\,\lambda_t = -2\rho\sin\delta A^2\lambda^6 \tag{3.26}$$

and we get:

$$\begin{aligned} & Im\left\{\lambda_c^2 F(x_c) \;+\; \lambda_t^2 F(x_t) + 2\lambda_c\lambda_t F(x_c, x_t)\right\} \\ &= -2A^2\lambda^6\rho\sin\delta\left\{F(x_c,x_t) \;+\; F(x_t)[A^2\lambda^4(1-\rho\cos\delta)] - F(x_c)\right\} \end{aligned} \tag{3.27}$$

Eq. (3.20) can now be written as:

$$\begin{aligned} |\epsilon|_{\xi=0} &= \frac{1}{\sqrt{2}\Delta M}\left[\frac{G_F^2}{12\pi^2}M_W^2 B_K f_K^2 m_K\right] 2A^2\lambda^6\rho\sin\delta \\ &\quad\cdot\left\{F(x_c,x_t) + F(x_t)[A^2\lambda^4(1-\rho\cos\delta)] - F(x_c)\right\} \end{aligned} \tag{3.28}$$

To account for QCD corrections, is necessary to compute the perturbative factors η_1, η_2 and η_3. Two different cases must be considered:

(i) the $M_W^2 \gg m_t^2 \gg m_c^2$ case or $x_t \ll 1$ case;

(ii) the $m_t \gg M_W^2 \gg m_c^2$ case or $x_t \gg 1$ case.

where m_t and m_c are the top- and the charm-quark masses respectively. In the first case, which is now excluded experimentally, very detailed calculations are already present in literature. In the second case, on the other hand, there are a few recent contributions. However, we believe that the final result has to be checked and the structure of the leading corrections in the case of a heavy top quark clarified.

I will consider case (i) and (ii) separately and discuss the main results in sections (3.2) and (3.3) respectively. Finally in Chapter 4 I will present a numerical discussion of the results obtained using different approximations on $\mathcal{H}_{eff}^{|\Delta S|=2}$, with or without QCD corrections, in the two different ranges for m_t.

3.2 Case of a light top quark

The analysis of the corrections from QCD to the effective weak hamiltonian, for small values of the t-quark mass ($m_t \ll M_W$), is nowadays well established. Quite a few papers [23,39,47,48] in literature describe how to discuss the problem in a very systematic way. Among these contributions, I think that it is important to quote those by M.I. Vysotskij [48] and by F.J. Gilman and M.B. Wise [23].

I will essentially follow the second of these papers. Nevertheless, I want to stress that interesting hints for the case of a large top quark mass are implicitly contained in the first one, although the general discussion appears a little bit rough.

The starting point is the $\Delta S = 2$ effective hamiltonian without QCD corrections:

$$\mathcal{H}_{eff}^{|\Delta S|=2} = \frac{G_F^2}{16\pi^2} M_W^2 (\bar{d}\gamma^\mu(1-\gamma_5)s)^2 \left\{\lambda_c^2 E(x_c) + \lambda_t^2 E(x_t) + 2\lambda_c\lambda_t E(x_c, x_t)\right\} \tag{3.29}$$

obtained from eq. (3.4) or from eq. (3.5), when $\eta_1 = \eta_2 = \eta_3 = 1$. As in section (3.1), $x_i = m_i^2/M_W^2$ for $i = c, t$. Eq. (3.29) can also be written as:

$$\mathcal{H}_{eff}^{|\Delta S|=2} = \lambda_c^2 \mathcal{H}_1 + \lambda_t^2 \mathcal{H}_2 + 2\lambda_c\lambda_t \mathcal{H}_3 \tag{3.30}$$

where $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ correspond respectively to the three cases in which: 1) two c-quarks, 2) two t-quarks or 3) one t- and one c-quark are exchanged along the internal fermion lines of the box-diagram. For small values of the t-quark mass, we can take the $x_c \ll x_t \ll 1$ approximation for $E(x_c)$, $E(x_t)$ and $E(x_c, x_t)$. As already seen in section (3.1), in this approximation, the *Inami-Lim* functions become:

$$\begin{aligned} E(x_c) &\simeq x_c \\ E(x_t) &\simeq x_t \\ E(x_c, x_t) &\simeq x_c \ln \frac{x_t}{x_c} \end{aligned} \tag{3.31}$$

and $\mathcal{H}_1$,$\mathcal{H}_2$ and $\mathcal{H}_3$ correspond to the following expressions:

$$\mathcal{H}_1 = \frac{G_F^2}{16\pi^2} m_c^2 [\bar{d}_\alpha \gamma_\mu (1-\gamma_5) s_\alpha][\bar{d}_\beta \gamma^\mu (1-\gamma_5) s_\beta] \tag{3.32}$$

$$\mathcal{H}_2 = \frac{G_F^2}{16\pi^2} m_t^2 [\bar{d}_\alpha \gamma_\mu (1-\gamma_5) s_\alpha][\bar{d}_\beta \gamma^\mu (1-\gamma_5) s_\beta] \tag{3.33}$$

$$\mathcal{H}_3 = \frac{G_F^2}{16\pi^2} m_c^2 \ln \frac{m_t^2}{m_c^2} [\bar{d}_\alpha \gamma_\mu (1-\gamma_5) s_\alpha][\bar{d}_\beta \gamma^\mu (1-\gamma_5) s_\beta] \tag{3.34}$$

The above result is modified by the presence of strong interactions and we can take them into account with the "*from top down*" method, explained in section (2.1). In view of the present discussion, in section (2.1) I have explicitly illustrated the construction of the effective hamiltonian with QCD corrections for $\Delta S=1$ non-leptonic weak decays (with details given in Appendix A). The method to be used here is exactly the same, so I will skip some points already emphasized in the $\Delta S=1$ case.

The UV cut-off of the theory is assumed equal, step by step, to one of the masses of the physical particles present in the theory, namely M_W, m_t, m_b and m_c respectively. Different effective theories arise at each step and suitable matching conditions must be imposed at the thresholds. Moreover, the effective hamiltonian should be expressed at each step as a linear combination of different sets of linearly independent operators, with expansion coefficients which are evolved from the highest limit to the lowest one (the physical hadronic mass scale under consideration) by the action of the renormalization group.

The three terms in the effective hamiltonian involve different fields, which become more or less important at different mass scales. Thus, except for Step 1, $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ behave in a different way at each step. We will examine what happens at each term separately, step by step.

At **Step 1** , the UV cut-off is set by M_W and one develops an effective theory from which the W-field has been removed. $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ involve the W-field in an equivalent way and they can be treated on an equal foot. When the W-boson becomes very heavy, each of the two $\Delta S=1$ W-vertices is shrunk to a point (see fig. (3.2)) and you can image to apply to each of them the operatorial analysis already detailed in the $\Delta S=1$ case. The product of the two weak currents at each vertex can be expressed as the linear combination of two operators, which do not mix under renormalization and are respectively half the sum of colour symmetrized (+) and anti-symmetrized (−) pieces:

$$\begin{aligned} O_q^{(\pm)} = & [(\bar{s}u)_{V-A}(\bar{u}d)_{V-A} \pm (\bar{s}d)_{V-A}(\bar{u}u)_{V-A}] \\ & - [(\bar{s}q)_{V-A}(\bar{q}d)_{V-A} \pm (\bar{s}d)_{V-A}(\bar{c}c)_{V-A}] \end{aligned} \tag{3.35}$$

where "q"can be a c- or a t-quark. We have used the following shorthand notation for the product of $(V-A)$-currents:

$$(\bar{s}d)_{V-A}(\bar{s}d)_{V-A} = \bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha \bar{s}_\beta \gamma^\mu (1-\gamma_5) d_\beta \tag{3.36}$$

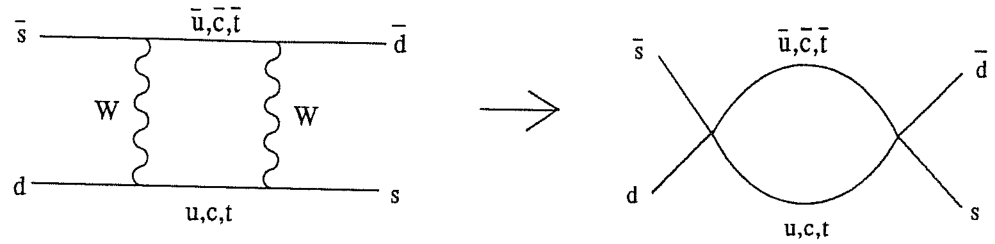

Figure 3.2: $\Delta S = 2$ amplitude in effective theory with a very heavy W-boson

As long as the internal fermion lines are not shrunk to a point, the $\Delta S=2$ vertex in the effective hamiltonian is equivalent to the product of the two $\Delta S=1$ vertices, as emphasized in fig. (3.2). Thus, each of the effective hamiltonians $\mathcal{H}_j$ for $j = 1,2,3$ can be obtained as the product of two $\Delta S=1$ effective hamiltonians. Each $\Delta S=1$ vertex is the linear combination of two operators: $O_q^{(\pm)}$, where q is the quark of one of the internal fermion lines. Thus, the product of two $\Delta S=1$ vertices gives origin to four terms and we can write each $\mathcal{H}_j$ for $j = 1,2,3$ as:

$$\mathcal{H}_j = \mathcal{H}_j^{(++)} + \mathcal{H}_j^{(+-)} + \mathcal{H}_j^{(-+)} + \mathcal{H}_j^{(--)} \qquad (3.37)$$

where $\mathcal{H}_j^{(++)}$ is obtained from the product of two O^+ operators, $\mathcal{H}_j^{(+-)}$ from the product of an O^+ and an O^- operator and so on so for.

We can account for QCD corrections to the global $\Delta S=2$ effective hamiltonian, by simply multiplying two *QCD-corrected* $\Delta S=1$ effective hamiltonians. We have seen in sections (2.3) and (2.4), how for a $\Delta S=1$ vertex QCD corrections, computed in the (LL) approximation, can be summed up to al orders in strong interactions (using RG techniques). The product of two QCD-corrected $\Delta S=1$ vertices, then, gives the QCD-corrected expression for each term $\mathcal{H}_j$ in $\mathcal{H}_{eff}^{(|\Delta S|=2)}$ as follows:

$$\begin{aligned}\mathcal{H}_j &= \left[\frac{\alpha_s(M_W^2)}{\alpha_s(\mu^2)}\right]^{2a^{(+)}} \mathcal{H}_j^{(++)} + \left[\frac{\alpha_s(M_W^2)}{\alpha_s(\mu^2)}\right]^{a^{(+)}+a^{(-)}} \mathcal{H}_j^{(+-)} + \\ &+ \left[\frac{\alpha_s(M_W^2)}{\alpha_s(\mu^2)}\right]^{a^{(-)}+a^{(+)}} \mathcal{H}_j^{(-+)} + \left[\frac{\alpha_s(M_W^2)}{\alpha_s(\mu^2)}\right]^{2a^{(-)}} \mathcal{H}_j^{(--)} \end{aligned} \qquad (3.38)$$

where $a^{(+)} = 6/21$ and $a^{(-)} = -12/21$; μ^2 is renormalization scale and $\alpha(m_i^2) = g^2(m_i^2)/4\pi$ the strong interaction running coupling constant in a theory with six quarks, evaluated at a scale $\mu^2 = m_i^2$.

From **Step 2**, when the UV cut-off of the new effective theory coincides with the top quark mass and the top quark is removed from the theory, the three factors of the effective hamiltonian start behaving in a different way, due to the different role played by the t-quark in each of them. We must treat them separately.

- Consider first $\mathcal{H}_1$.

  Both at **Step 2** and at **Step 3**, when the UV cut-off of the theory coincides respectively with m_t and m_b, there are no problems, because $\mathcal{H}_1$ does not involve directly either the t-quark or the b-quark. The $\Delta S=2$ vertex in the effective hamiltonian can again be interpreted as the product of two $\Delta S=1$ vertices, which renormalize separately. Thus, the basic structure of the hamiltonian does not change with respect to the one in eq. (3.38), except for the presence of RG-evolved coefficients. The RG-evolved coefficients will be the product of three terms of the form $[\alpha(m_i^2)/\alpha(m_j^2)]^\gamma$, obtained as at Step 1, but in different effective theories. The explicit calculation for each $\Delta S=1$ vertex is exactly the same one explained in more detail in Appendix A. The only difference is that we have here the product of two $\Delta S=1$ vertices. However, this product can be treated exactly as at Step 1. Namely, taking into account that we have passed two mass thresholds and we have changed our effective theory at each step, down to a four-quark effective theory, we get for $\mathcal{H}_1$:

$$\begin{aligned}
\mathcal{H}_1 = & \left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{12/21}\left[\frac{\alpha_s(m_t^2)}{\alpha_s'(m_b'^2)}\right]^{12/23}\left[\frac{\alpha_s'(m_b'^2)}{\alpha_s''(\mu^2)}\right]^{12/25}\mathcal{H}_1^{(++)} \\
+ & \left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{-6/21}\left[\frac{\alpha_s(m_t^2)}{\alpha_s'(m_b'^2)}\right]^{-6/23}\left[\frac{\alpha_s'(m_b'^2)}{\alpha_s''(\mu^2)}\right]^{-6/25}\mathcal{H}_1^{(+-)} \\
+ & \left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{12/21}\left[\frac{\alpha_s(m_t^2)}{\alpha_s'(m_b'^2)}\right]^{12/23}\left[\frac{\alpha_s'(m_b'^2)}{\alpha_s''(\mu^2)}\right]^{12/25}\mathcal{H}_1^{(+-)} \\
+ & \left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{-24/21}\left[\frac{\alpha_s(m_t^2)}{\alpha_s'(m_b'^2)}\right]^{-24/23}\left[\frac{\alpha_s'(m_b'^2)}{\alpha_s''(\mu^2)}\right]^{-24/25}\mathcal{H}_1^{(--)}
\end{aligned} \tag{3.39}$$

Eq. (3.39) has exactly the same structure as eq. (3.38), as we expected. Masses, coupling constants and anomalous dimensions have been evaluated, step by step, in a six-quark, five-quark and four-quark effective theory. I have denoted them with non primed, primed and doubled primed variables, referring to the case of a six-quark, five-quark and four-quark effective theory respectively.

Finally, at **Step 4**, when the UV cut-off of the theory becomes m_c, also the c-quark is removed from explicitly appearing in $\mathcal{H}_1$. The two intermediate quark-lines of the box-diagram are shrunk to a point , as shown in fig. (3.3) and the resulting effective interaction will be expressed in terms of the $\Delta S=2$ four-fermion vertex operator only:

$$(\bar{s}d)_{V-A}(\bar{s}d)_{V-A} = [\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma^\mu(1-\gamma_5)d_\beta] \tag{3.40}$$

with anomalous dimension:

$$\gamma'''^{(+)}(g''') = \frac{g'''^2}{4\pi^2} + O(g'''^4) = 4\frac{\alpha'''}{4\pi} + O(\alpha'''^2) \tag{3.41}$$

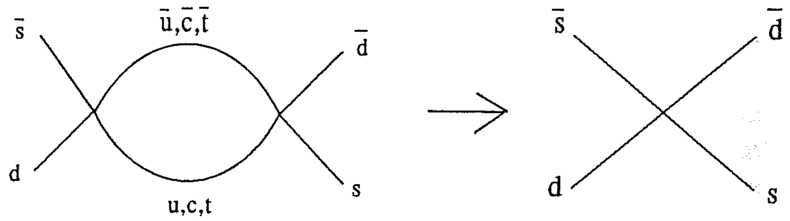

Figure 3.3: $\Delta S = 2$ amplitude in a three-quark effective theory

Each operator in eq. (3.39) can be "projected" on the four-fermion operator in eq. (3.40). As we learn from Appendix A, the matrix elements of each term in $\mathcal{H}_1$ (which appear in evaluating explicitly weak amplitudes) can be expanded as:

$$\langle\,|\,\mathcal{H}_1^{(\pm\pm)}\,|\,\rangle'' = L^{(\pm\pm)}\left(\frac{m_c''}{\mu}, g''\right) m_c''^2 \langle\,|\,(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}\,|\,\rangle''' \tag{3.42}$$

with the usual meaning of all the non primed, primed and double-primed variables. The triple-primed variables refer now to the three-quark effective theory and the formalism used in eq.(3.42) has the following meaning:

$$\langle\,|\,\mathcal{H}_1^{(\pm\pm)}\,|\,\rangle'' = \langle\,|\,\mathcal{H}_1^{(\pm\pm)}\,|\,\rangle(g'', \mu', m_u'', \cdots, m_c'') \tag{3.43}$$

and equally for $\langle\,|\,(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}\,|\,\rangle'''$, except for obvious modifications.

$\mathcal{H}_1^{(\pm\pm)}$-type operators have well-defined anomalous dimensions, thus it is not difficult to state the RGE's for the $L^{(\pm\pm)}$ coefficients, imposing that both sides of eq.(3.42) are μ-independent. The RG formalism to be used here is strictly analogous to the one reported in Appendix A. Thus, I will give only the final expressions for the $L^{(\pm\pm)}$ coefficients, which read:

$$\begin{aligned}
L^{(++)}\left(\frac{m_c''}{\mu}, g\right) &= -\left[\frac{3}{2}\right]\frac{1}{\pi^2}\left[\frac{\alpha''(\mu^2)}{\alpha''(m_c''^2)}\right]^{12/25}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27}\left[\frac{\alpha''(m_c''^2)}{\alpha''(\mu^2)}\right]^{24/25} \\
L^{(\pm\mp)}\left(\frac{m_c''}{\mu}, g\right) &= -\left[-\frac{1}{2}\right]\frac{1}{\pi^2}\left[\frac{\alpha''(\mu^2)}{\alpha''(m_c''^2)}\right]^{-6/25}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27}\left[\frac{\alpha''(m_c''^2)}{\alpha''(\mu^2)}\right]^{24/25} \\
L^{(--)}\left(\frac{m_c''}{\mu}, g\right) &= -\left[\frac{1}{2}\right]\frac{1}{\pi^2}\left[\frac{\alpha''(\mu^2)}{\alpha''(m_c''^2)}\right]^{-24/25}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27}\left[\frac{\alpha''(m_c''^2)}{\alpha''(\mu^2)}\right]^{24/25}
\end{aligned} \tag{3.44}$$

where the initial conditions

$$L^{(++)}(1,0) = -\frac{1}{\pi^2}\left[\frac{3}{2}\right]$$

$$
\begin{aligned}
L^{(\pm\mp)}(1,0) &= -\frac{1}{\pi^2}\left[-\frac{1}{2}\right] \\
L^{(--)}(1,0) &= -\frac{1}{\pi^2}\left[\frac{1}{2}\right]
\end{aligned}
\tag{3.45}
$$

are assumed. The final expression of $\mathcal{H}_1$ is now at hand and is:

$$
\begin{aligned}
\mathcal{H}_1 = & - \frac{G_F^2}{16\pi^2} m_c^{\star 2} [\bar{d}_\alpha \gamma_\mu (1-\gamma_5) s_\alpha][\bar{d}_\beta \gamma^\mu (1-\gamma_5) s_\beta] \cdot \left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27} \\
& \cdot \left[\frac{3}{2}\left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{12/21} \left[\frac{\alpha_s(m_t^2)}{\alpha_s'(m_b'^2)}\right]^{12/23} \left[\frac{\alpha_s'(m_b'^2)}{\alpha_s''(m_c''^2)}\right]^{12/25}\right. \\
& - \left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{-6/21} \left[\frac{\alpha_s(m_t^2)}{\alpha_s'(m_b'^2)}\right]^{-6/23} \left[\frac{\alpha_s'(m_b'^2)}{\alpha_s''(m_c''^2)}\right]^{-6/25} \\
& \left. + \frac{1}{2}\left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{-24/21} \left[\frac{\alpha_s(m_t^2)}{\alpha_s'(m_b'^2)}\right]^{-24/23} \left[\frac{\alpha_s'(m_b'^2)}{\alpha_s''(m_c''^2)}\right]^{-24/25}\right]
\end{aligned}
\tag{3.46}
$$

where

$$
m_c^\star = m_c'' \left[\frac{\alpha''(m_c''^2)}{\alpha''(\mu^2)}\right]^{12/25}
\tag{3.47}
$$

- Consider now $\mathcal{H}_2$.

  Its case is strictly analogous, with the only difference that now, instead of two c-quarks, two t-quarks are exchanged. Thus, the $\Delta S=2$ four-fermion operator in eq. (3.40) already appears at **Step 2**, when the UV cut-off is set by m_t and the top quark is explicitly removed from the theory. After this threshold, the four-fermion operator renormalizes multiplicatively, as a $\Delta S = 1$ vertex operator, so that the final result for $\mathcal{H}_2$ reads:

$$
\begin{aligned}
\mathcal{H}_2 = & - \frac{G_F^2}{16\pi^2} m_t^{\star 2} [\bar{d}_\alpha \gamma_\mu (1-\gamma_5) s_\alpha][\bar{d}_\beta \gamma^\mu (1-\gamma_5) s_\beta] \\
& \times \left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27} \left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)}\right]^{6/25} \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)}\right]^{6/23} \\
& \times \left[\frac{3}{2}\left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{12/21} - \left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{-6/21} + \frac{1}{2}\left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{-24/21}\right]
\end{aligned}
\tag{3.48}
$$

- Finally, consider $\mathcal{H}_3$.

  This case is a little bit more tricky, because $\mathcal{H}_3$ involves both t- and c-quark. At **Step 2**, when m_t sets the UV cut-off, only one of the two internal quark lines is shrunk to a point and we cannot describe anymore the global $\Delta S=2$ effective hamiltonian as the product of two $\Delta S = 1$ effective hamiltonians. On the contrary, by explicitly calculating the renormalization of the $\Delta S=2$ effective hamiltonian, one finds that a new complete set of linearly independent operators is required in order to expand the matrix elements of each term in $\mathcal{H}_3$. They are given by:

$$
\begin{aligned}
O_1^{(\pm\pm)} &= i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A}\} \\
O_2^{(\pm\pm)} &= i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A}\} \\
O_3^{(\pm\pm)} &= i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V-A} + \ldots + (\bar{b}_\beta b_\beta)_{V-A}]\} \\
O_4^{(\pm\pm)} &= i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V-A} + \ldots + (\bar{b}_\beta b_\alpha)_{V-A}]\} \\
O_5^{(\pm\pm)} &= i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V+A} + \ldots + (\bar{b}_\beta b_\beta)_{V+A}]\} \\
O_6^{(\pm\pm)} &= i\int d^4x\, T\{O_c^{(\pm)}(x)(\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V+A} + \ldots + (\bar{b}_\beta b_\alpha)_{V+A}]\}
\end{aligned}
\tag{3.49}
$$

which are well-known from the $\Delta S = 1$ case, plus two additional ones

$$
\begin{aligned}
O_7^{(\pm\pm)} &= i\int d^4x\, T\,\{[(\bar{s}_\alpha u_\alpha)_{V-A}(\bar{c}_\beta d_\beta)_{V-A} \pm (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{c}_\beta u_\beta)_{V-A}](x) \\
&\times \quad [(\bar{s}_\lambda c_\lambda)_{V-A}(\bar{u}_\delta d_\delta)_{V-A} \pm (\bar{s}_\lambda d_\lambda)_{V-A}(\bar{u}_\delta c_\delta)_{V-A}](x)\}
\end{aligned}
\tag{3.50}
$$

$$
O_8 = \frac{m_c'^2}{g'^2}[\bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha][\bar{s}_\beta \gamma^\mu (1-\gamma_5) d_\beta] \tag{3.51}
$$

They mix under renormalization and we can compute the matrices of their anomalous dimensions, at the leading order in the strong coupling constant of the five-quark effective theory, g' (or α'). The same happens at **Step 3**, when the UV cut-off coincides with m_b. We need a new set of eight operators to expand the operators in $\mathcal{H}_3$. They are of the same form of the operators in eqs. (3.49) and (3.50), but with the b-quark dropped wherever it appears. Finally, at **Step 4**, when m_c sets the UV cut-off, both the quark internal lines are shrunk. We are left, as in the case of $\mathcal{H}_1$ and $\mathcal{H}_2$, with the usual four fermion operator only, namely O_8.

At each of the previous steps, we should only generalize, in a straightforward way, the technique of the $\Delta S = 1$ case. However, this operation results to be really complex and an analytic expression for $\mathcal{H}_3$ can be derived only by making some suitable approximations. The authors propose, at this point, some sensible approximations, looking at the structure of the bare coefficients of each operator and at the expressions of their anomalous dimension matrices. They find that operators O_3,...,O_6 can be neglected with respect to operators O_1, O_2, O_7 and O_8. With this assumption, the problem greatly simplifies. At Step 2 and Step 3, only O_7 and O_8 operators are considered, while, at Step 4, as usual, only a term proportional to O_8 occurs. Assuming all that, the derived analytical expression for $\mathcal{H}_3$ looks like [23]:

$$
\mathcal{H}_3 = \frac{G_F^2}{64\pi\alpha''(m_c''^2)} m_c^{*2}[\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha][\bar{s}_\beta\gamma^\mu(1-\gamma_5)d_\beta] \cdot \left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27}
$$

$$
\begin{aligned}
\times & \left\{ \frac{72}{35} \left[5 \left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)} \right]^{12/25} \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)} \right]^{12/23} + 2 \left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)} \right]^{5/25} \right. \right. \\
& \cdot \left. \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)} \right]^{12/23} - 7 \left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)} \right]^{5/25} \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)} \right]^{7/23} \right] \left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)} \right]^{12/2} \\
& + \frac{48}{143} \left[13 \left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)} \right]^{-6/25} \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)} \right]^{-6/23} - 2 \left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)} \right]^{5/25} \right. \\
& \cdot \left. \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)} \right]^{-6/23} - 11 \left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)} \right]^{5/25} \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)} \right]^{7/23} \right] \left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)} \right]^{-6/21} \\
& + \frac{24}{899} \left[-31 \left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)} \right]^{-24/25} \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)} \right]^{-24/23} + 2 \left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)} \right]^{5/25} \right. \\
& \cdot \left. \left. \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)} \right]^{-24/23} + 29 \left[\frac{\alpha'(m_b'^2)}{\alpha''(m_c''^2)} \right]^{5/25} \left[\frac{\alpha(m_t^2)}{\alpha'(m_b'^2)} \right]^{7/23} \right] \left[\frac{\alpha(M_W^2)}{\alpha(m_t^2)} \right]^{-24/2} \right\}
\end{aligned}
\tag{3.52}
$$

This completes the theoretical analysis of the problem and the effects of QCD can be verified by comparing $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ as given in eqs.(3.46), (3.48), (3.52), with their free quark values.

3.3 Case of a heavy top quark

In the present section, we will consider the effective $\Delta S = 2$ hamiltonian in the case of a very heavy top quark.

The most complete calculation available in literature is the one contained in the paper by Datta et al. [14] I have checked their calculation and I agree with the results.

I have also to mention other people which have worked on the same problem [29,30,18,17]. The results contained in the latter calculations are incomplete or only partially correct, as is very well discussed in a recent preprint by Buras et al. [13].

Let us now return to formulate the problem as Datta et al. [14] have made. As I have already said, I have reproduced the whole calculation and tried, in the meanwhile, to confirm the correctness of the underlying logical scheme. Let us start considering the $\Delta S = 2$ *bare* effective hamiltonian:

$$
\mathcal{H}_{eff}^{\Delta S=2} = \frac{G_F^2}{16\pi^2} (\bar{d}_\alpha \gamma_\mu (1-\gamma_5) s_\alpha)^2 \left(\lambda_c^2 E(x_c) + \lambda_t^2 E(x_t) + 2\lambda_c \lambda_t E(x_c, x_t) \right) \tag{3.53}
$$

and the $\Delta S = 2$ *QCD-corrected* effective hamiltonian:

$$
\mathcal{H}_{eff}^{\Delta S=2} = \frac{G_F^2}{16\pi^2} (\bar{d}_\alpha \gamma_\mu (1-\gamma_5) s_\alpha)^2 \left(\lambda_c^2 F(x_c) + \lambda_t^2 F(x_t) + 2\lambda_c \lambda_t F(x_c, x_t) \right) \tag{3.54}
$$

with the aim to compute QCD corrections in the presence of a very heavy top quark (all the notations used should already be clear). In eq. (3.53), one has formally the same effective hamiltonian considered by Gilman and Wise [23] and discussed in the previous section. However, here we do not make any approximation on the *Inami-Lim* functions $E(x_i, x_j)$ and $E(x_i)$.

The calculation is made in the leading logarithmic (LL) approximation and, as mentioned in the previous section, many hints can be found in ref. [48]. In particular:

(i) the idea to write each contribution in the *bare* effective hamiltonian (or the *Inami-Lim* functions themselves) in a suitable way for the subsequent inclusion of QCD corrections (namely, for the computation of the loop integrals appearing at that point);

(ii) the idea to analyze the two-loop structure of the box-diagram integration, when QCD-corrections are present, in three steps: (1) fix the external loop momentum (the p-momentum) and integrate on the gluon-loop momentum (the q-momentum) first, then (2) keep only the (LL) terms and (3) finally reduce all the loop-integrals to the only two possible integrals which may appear. These last two integrations result, thank to the particular structure of the effective hamiltonian;

(iii) the idea to collect the graphs obtained by exchanging a gluon in all the possible ways and compute separately the contribution of each of them, selecting only those which give a (LL) contribution. Then you have only to sum up all the corrections in order to define the anomalous dimension at order $O(\alpha_s)$;

(iv) finally the method to deal with the running of quark masses.

However, in [48] these ideas applied only to the case of a light top-quark, while here we will analyze the whole range of m_t, from the region in which $m_t \ll M_W$ up to the region in which $m_t \gg M_W$. Obviously, some new features will appear. Mainly, a whole class of graphs (those in which a gluon is exchanged between an external leg and an internal heavy-quark leg) does not contribute to the anomalous dimension in the case of a heavy top quark.

Here following, I will give separately a summary of the expressions for $F(x_c)$, $F(x_t)$ and $F(x_c, x_t)$ to be used, in the case when $m_t \ll M_W$ and in the case when $m_t \gg M_W$. From the comparison, the difference between the two cases will be clear. The $m_t \ll M_W$ results are taken from the paper of Gilman and Wise [23], while the $m_t \gg M_W$ results come from a very careful analysis of the work made by Datta et al. [14]. In Chapter 4, a numerical comparison of the two cases will be given.

(1) $F(x_c)$ case

$F(x_c)$ remains the same both for $m_t \ll M_W$ and for $m_t \gg M_W$, because it does not involve the top-quark operator. Namely, it is given by:

$$F(x_c) = \eta_1 E(x_c) \simeq \eta_1 x_c \tag{3.55}$$

where the $x_c \ll 1$ approximation of the *Inami-Lim* functions has been used. However, different authors give different expressions for η_1, because they attribute different relevance to the threshold structure of the underlying effective theory. Thus, following **Gilman and Wise** :

$$\begin{aligned}
\eta_1 \;=\; & \left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{6/27} \\
& \cdot \left[\frac{3}{2}\left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{12/21}\left[\frac{\alpha_s(m_t^2)}{\alpha_s'(m_b'^2)}\right]^{12/23}\left[\frac{\alpha_s'(m_b'^2)}{\alpha_s''(m_c''^2)}\right]^{12/25}\right. \\
& - \left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{-6/21}\left[\frac{\alpha_s(m_t^2)}{\alpha_s'(m_b'^2)}\right]^{-6/23}\left[\frac{\alpha_s'(m_b'^2)}{\alpha_s''(m_c''^2)}\right]^{-6/25} \\
& \left. + \frac{1}{2}\left[\frac{\alpha_s(M_W^2)}{\alpha_s(m_t^2)}\right]^{-24/21}\left[\frac{\alpha_s(m_t^2)}{\alpha_s'(m_b'^2)}\right]^{-24/23}\left[\frac{\alpha_s'(m_b'^2)}{\alpha_s''(m_c''^2)}\right]^{-24/25}\right]
\end{aligned} \tag{3.56}$$

where no primed, primed, double-primed, etc. quantities refer to different effective theories, with six, five, four, etc. flavours only.

On the other hand, following **Datta et al.** we have:

$$\begin{aligned}
\eta_1 &= \left[\frac{\alpha(m_c^2)}{\alpha(\mu^2)}\right]^{2/b_3} F(m_c^2, M_W^2; 8/b_n, 2/b_n, -4/b_n) \\
&= \left[\frac{\alpha(m_c^2)}{\alpha(\mu^2)}\right]^{6/27} F(m_c^2, M_W^2; 24/25, 6/25, -12/25)
\end{aligned} \tag{3.57}$$

where:

$$\begin{aligned}
& F(p^2, M_W^2; 4/b_n, 1/b_n, -2/b_n) \;\equiv \\
& \frac{1}{2}\left[\frac{\alpha_s(p^2)}{\alpha_s(M_W^2)}\right]^{4/b_n} - \left[\frac{\alpha_s(p^2)}{\alpha_s(M_W^2)}\right]^{1/b_n} + \frac{3}{2}\left[\frac{\alpha_s(p^2)}{\alpha_s(M_W^2)}\right]^{-2/b_n}
\end{aligned} \tag{3.58}$$

(2) $\boldsymbol{F(x_t)}$ case

In the $m_t \gg M_W$ region, Datta et al. compute $F(x_t)$ numerically, starting from its integral expression:

$$\begin{aligned}
F(x_t) \;=\; & M_W^2 \int dp^2 \left\{ \frac{p^4}{p^2 + M_W^2}\left[\frac{1}{p^2 + M_W^2}\left(g_t^2 + \frac{p^2}{M_W^2}(2g_t g_1 - g_1^2)\right)\right.\right. \\
& \left.\left. + \frac{1}{M_W^2} g_1(-2g_t z' + g_1 z)\right] + m_t^4 \frac{p^2}{(p^2 + M_W^2)^2} g_t^2 \left[\frac{p^2}{4M_W^2} + 2\right]\right\} x
\end{aligned} \tag{3.59}$$

where

$$x = \left[\frac{\alpha_s(p^2)}{\alpha_s(\mu^2)}\right]^{2/b_n} \tag{3.60}$$

$$z = F(p^2, p^2 + M_W^2; 8/b_n, 2/b_n, -4/b_n) \tag{3.61}$$

$$z' = F(p^2, p^2 + M_W^2; 4/b_n, 1/b_n, -2/b_n) \tag{3.62}$$

$$g_i = \frac{1}{p^2 + m_i^2} \tag{3.63}$$

and the threshold structure in the integrations is accounted for in the $\alpha_s(p^2)$ computation.

In the $m_t \ll M_W$ region, the Gilman and Wise result is recovered by setting $z = z'$ only.

(3) $F(x_c, x_t)$ case

Also in this case, the $m_t \gg M_W$ region is described by the Datta et al. result. They integrate numerically the following expression for $F(x_c, x_t)$:

$$\begin{aligned} F(x_c, x_t) &= \int dp^2 \left\{ \frac{p^4(g_c - g_1)}{p^2 + M_W^2} \left[g_1 \left(\frac{p^2}{p^2 + M_W^2} - z \right) - g_t \left(\frac{p^2}{p^2 + M_W^2} - z' \right) \right] \right. \\ &+ m_c^2 m_t^2 g_c g_t \frac{p^2}{p^2 + M_W^2} \left[\frac{1}{p^2 + M_W^2} + \frac{1}{M_W^2} z' \right. \\ &- \left. \left. \frac{3}{4} \frac{p^2}{M_W^2 (p^2 + M_W^2)} \right] \right\} xy \end{aligned} \tag{3.64}$$

where x, z and z' are as before, while y is defined by:

$$y = \left[\frac{\alpha_s(p^2)}{\alpha_s(m_c^2)}\right]^{8/b_n} \tag{3.65}$$

On the other hand, in the $m_t \ll M_W$ region the Gilman and Wise result holds, obtained from the previous one by setting again $z = z'$.

Actually, in the numerical analysis, we have improved the results given by Datta et al., because we have fully accounted for the threshold structure of the effective theory, while they did not in their paper. On the other hand, Gilman and Wise have considered in their work all the thresholds of the effective theory. Thus, our results are better comparable with those given by Gilman and Wise.

We will give now a description of the analysis step by step, reminding for a certain number of detailed calculations to Appendix B.

3.3.1 The standard box-diagram

When we consider the *Inami-Lim* functions, automatically we have included in the amplitude all the four different box-diagrams, in which: *a*) two W-bosons, *b*) one Higgs boson and one W-boson (two possibilities for two legs) and *c*) two Higgs bosons are exchanged in the box (see fig.(B.1) in Appendix B. Each case can be separately computed and the relative contribution can be cast in a suitable form to account for QCD corrections at a subsequent stage.

We work in the Feynman gauge for the weak interactions, while in the Landau gauge for the strong ones. GIM mechanism is introduced from the very beginning, as it is well known from the box-diagram amplitude evaluation, whose main features are collected in Appendix B.

As an example, consider the contribution coming from the exchange of two W-bosons, $\mathcal{H}_{WW}$ say. As can be read in Appendix B, it gives:

$$\mathcal{H}_{WW} = \frac{G_F^2}{16\pi^2} M_W^2 (\bar{d}_\alpha \gamma_\mu (1-\gamma_5) q_\alpha)^2 \sum_{i,j} \lambda_i \lambda_j M_W^2 m_i^2 m_j^2 I(m_i^2, m_j^2) \tag{3.66}$$

where indices i and j refers to the quarks of the internal fermionic lines, namely u, c or t, while the function $I(m_i^2, m_j^2)$ is given by the extrnal loop integration

$$I(m_i^2, m_j^2) = \int dp^2 \frac{1}{(p^2 + M_W^2)^2 (p^2 + m_i^2)(p^2 + m_j^2)} \tag{3.67}$$

As well known from the light-top theory, in the approximation of a massless u-quark, only three term in the previous sum contribute, those in which: *a*) two c-quark, *b*) two t-quark or *c*) one c- and one t-quark are exchanged. In the present approach, it is essential to define the dominant regions in the integration over the external loop $p-$momentum. Thus, we switch to the Euclidean region and then we integrate over the four dimensional angles. All the integrals can be expressed, in this way, as linear combinations of the following two elementary integrals:

$$B(m_i^2) = \int dp^2 \frac{1}{(p^2 + m_i^2)^2} = 1 \tag{3.68}$$

and

$$\begin{aligned} C(m_i^2, m_j^2) &= \int dp^2 \frac{1}{(p^2 + m_i^2)(p^2 + m_j^2)} \\ &= \frac{1}{(m_j^2 - m_i^2)} \int_{m_i^2}^{m_j^2} \frac{dp^2}{p^2} = \frac{1}{m_j^2 - m_i^2} \ln \frac{m_i^2}{m_j^2} \end{aligned} \tag{3.69}$$

with coefficients which depend on functions of the masses of the particles in the game. The original expressions given in [26] as functions of the x_i variables: $x_i = m_i^2/M_W^2$

can also be rearranged in such a way to give

$$\begin{aligned} A_{WW,ii} &= \frac{m_i^2 M_W^2}{(M_W^2 - m_i^2)^2}\left(B(m_i^2) + \frac{m_i^2}{M_W^2}B(M_W^2) - 2m_i^2 C(m_i^2, M_W^2)\right) \\ &= \frac{x_i^2}{(1-x_i)^2}\left(1 + \frac{1}{x_i} + \frac{2}{1-x_i}\ln x_i\right) \end{aligned} \quad (3.70)$$

for $i = c, t$ and

$$\begin{aligned} A_{WW,ct} &= m_c^2 m_t^2 \left(\frac{1}{M_W^2 - m_c^2}\left(\frac{B(M_W^2)}{M_W^2 - m_t^2} + \frac{M_W^2}{m_t^2 - m_c^2}C(m_c^2, M_W^2)\right)\right. \\ &\quad \left. - \frac{M_W^2}{(M_W^2 - m_t^2)(m_t^2 - m_c^2)}C(m_t^2, M_W^2)\right) \\ &= x_c x_t \left(\frac{1}{(1-x_c)(1-x_t)} - \frac{1}{(1-x_c)^2(x_t - x_c)}\ln x_c \right. \\ &\quad \left. + \frac{1}{(1-x_t)^2(x_t - x_c)}\ln x_t\right) \end{aligned} \quad (3.71)$$

In a quite analogous way we get for the other term in the global box-diagram amplitude:

$$\begin{aligned} A_{HH,tt} &= \frac{m_t^4}{4(M_W^2 - m_t^2)^2}\left(B(M_W^2) + \frac{m_t^2}{M_W^2}B(m_t^2) - 2m_t^2 C(m_t^2, M_W^2)\right) \\ &= \frac{x_t^2}{4(1-x_t^2)}\left(1 + x_t + 2\frac{x_t}{1-x_t}\ln x_t\right) \end{aligned} \quad (3.72)$$

$$\begin{aligned} A_{WH,tt} &= \frac{2m_t^4}{(M_W^2 - m_t^2)^2}\left((m_t^2 + M_W^2)C(m_t^2, M_W^2) - B(m_t^2) - B(M_W^2)\right) \\ &= -2\frac{x_t^2}{(1-x_t)^2}\left(\frac{1+x_t}{1-x_t}\ln x_t + 2\right) \end{aligned} \quad (3.73)$$

$$\begin{aligned} A_{WH,ct} &= \frac{2m_c^2 m_t^2}{m_t^2 - m_c^2}\left(-\frac{m_c^2}{M_W^2 - m_c^2}C(m_c^2, M_W^2) + \frac{m_t^2}{M_W^2 - m_t^2}C(m_t^2, M_W^2)\right. \\ &\quad \left. + \frac{m_t^2 - m_c^2}{(m_t^2 - M_W^2)(M_W^2 - m_c^2)}B(M_W^2)\right) \\ &= 2\frac{x_c x_t}{x_t - x_c}\left(\frac{x_c}{(1-x_c)^2}\ln x_c - \frac{x_t}{(1-x_t)^2}\ln x_t \right. \\ &\quad \left. - \frac{x_t - x_c}{(1-x_t)(1-x_c)}\right) \end{aligned} \quad (3.74)$$

$$A_{HH,ct} = \frac{m_c^2 m_t^2}{4M_W^2(m_t^2 - m_c^2)}\left(\frac{m_c^4}{M_W^2 - m_c^2}C(m_c^2, M_W^2)\right.$$

$$
\begin{aligned}
& -\frac{m_t^4}{M_W^2-m_t^2}C(m_t^2,M_W^2)+\frac{M_W^2(m_t^2-m_c^2)}{(m_t^2-M_W^2)(m_c^2-M_W^2)}B(M_W^2)\Bigg) \\
= \; & \frac{x_c x_t}{4(x_t-x_c)}\left(-\frac{x_c^2}{(1-x_c^2)^2}\ln x_c+\frac{x_t^2}{(1-x_t)^2}\ln x_t\right. \\
& \left.+\frac{x_t-x_c}{(1-x_t)(1-x_c)}\right) \qquad (3.75)
\end{aligned}
$$

and for the effective hamiltonian in (1):

$$
\begin{aligned}
\mathcal{H} \; = \; & k\left(\lambda_c^2 A_{WW,cc}+\lambda_t^2(A_{WW,tt}+A_{WH,tt}+A_{HH,tt})+\right. \\
& +2\lambda_c\lambda_t(A_{WW,ct}+A_{WH,ct}+A_{HH,ct})) \qquad (3.76)
\end{aligned}
$$

where k is given by

$$
k=\frac{G_F^2}{16\pi^2}M_W^2(\bar{d}_\alpha\gamma_\mu(1-\gamma_5)q_\alpha)^2 \qquad (3.77)
$$

For comparison with the previous light-top effective hamiltonian, it suffices to say that Gilman and Wise [23] use the following approximation

$$
\begin{aligned}
A_{WW,cc}+A_{WH,cc}+A_{HH,cc} &\approx x_c \\
A_{WW,tt}+A_{WH,tt}+A_{HH,tt} &\approx x_t \\
A_{WW,ct}+A_{WH,ct}+A_{HH,ct} &\approx x_c\ln(x_t/x_c) \qquad (3.78)
\end{aligned}
$$

3.3.2 QCD corrections

Now QCD corrections can be included. The whole problem reduces to define only how to treat two possible cases. After the gluon-loop integration, we are generally left with an integral expression of the form:

$$
\int dp^2 f(p^2)\left[\frac{\alpha_s^2(p^2)}{\alpha_s^2(p_1^2)}\right]^\gamma \qquad (3.79)
$$

where p_1 is a fixed momentum scale. Here, two possibilities arise:

(i) the integral $\int dp^2 f(p^2)$ after the integration has a power dependence, as is the case for $B(m_i^2)$ terms. In this case, an accurate result is obtained just by multiplying the integral $\int dp^2 f(p^2)$ by $[\alpha_s^2(m_i^2)/\alpha_s^2(p_1^2)]^\gamma$;

(ii) the integral $\int dp^2 f(p^2)$ after the integration has a logarithmic dependence, as is the case for $C(m_i^2, m_j^2)$ terms. In this case, the previous approximation is not suitable anymore and a careful treatment requires the direct evaluation of the following integral:

$$
\frac{1}{(m_j^2-m_i^2)}\int_{m_i^2}^{m_j^2}\frac{dp^2}{p^2}\left[\frac{\alpha_s^2(p^2)}{\alpha_s^2(p_1^2)}\right]^\gamma \qquad (3.80)
$$

as originally well-stated in a paper by Novikov et al [39].

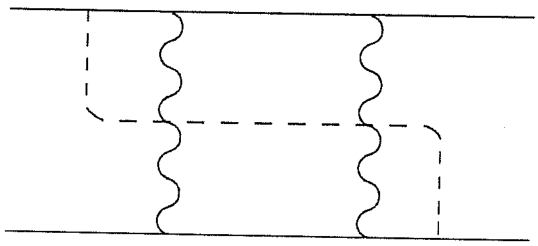

Figure 3.4: QCD corrections: gluon exchanged between external quark legs

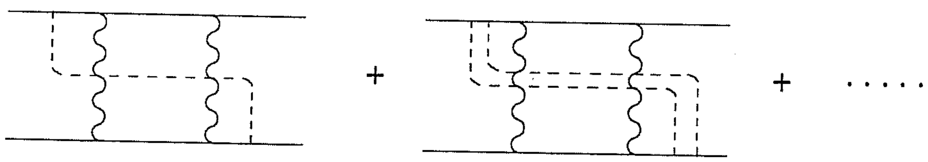

Figure 3.5: QCD correction are summed up to all orders, in the case of a diagram in fig. (3.4)

This is the basic statement of the problem. Let us now give some more details for the different cases which may occur.

Gluons connecting the external quarks We start considering the case of corrections to the whole $\Delta S = 2$ vertex operator, in which a gluon connects two equal external legs, as in fig. (3.4). The main features of this computation are reported in Appendix B, while I will comment here only about the result, which is:

$$\begin{aligned}\mathcal{H}_{WW} &= kM_W^2 \frac{\alpha_s}{4\pi} \sum_{i,j=3}^{3} \lambda_i \lambda_j \int dp^2 \frac{p^4}{(p^2+M_W^2)^2(p^2+m_i^2)(p^2+m_j^2)} \\ &\cdot \left(\ln\left(\frac{\mu^2}{p^2}\right) + O\left(\ln \frac{p^2+m^2}{m^2}, \ln \frac{p^2}{m^2}\right)\right) \end{aligned} \quad (3.81)$$

We recognize a (LL) contribution which can be summed up at all the orders in strong interactions, as sketched in fig. (3.5) using the renormalization group equation:

$$\begin{aligned} \mathcal{H}_{WW} &= kM_W^2\frac{\alpha_s}{4\pi}\sum_{i,j=1}^{3}\lambda_i\lambda_j\int dp^2\frac{p^4}{(p^2+M_W^2)^2(p^2+m_i^2)(p^2+m_j^2)} \\ &\cdot \left[\frac{\alpha_s(p^2)}{\alpha_s(\mu^2)}\right]^{2/b_n} \end{aligned} \tag{3.82}$$

In other words, an external gluon exchange does not see the presence of an internal heavy or light top quark, but is a correction to the whole vertex-operator (obtained in the last case when m_t starts to be considered the heaviest mass scale in an effective five-quark theory).

Similar results holds for the diagrams with the Higgs bosons. Using the GIM mechanism, we can summarize our results as follows:

$$\mathcal{H}_{WH} = kM_W^2\sum_{i,j=2}^{3}\lambda_i\lambda_j\int dp^2\frac{p^4}{(p^2+M_W^2)^2}(g_i-g_1)(g_j-g_1)x \tag{3.83}$$

$$\mathcal{H}_{WH} = k\sum_{i,j=2}^{3}\lambda_i\lambda_j 2\int dp^2\frac{p^2m_i^2m_j^2}{(p^2+M_W^2)^2}g_ig_jx \tag{3.84}$$

$$\mathcal{H}_{HH} = k\sum_{i,j=2}^{3}\lambda_i\lambda_j\frac{1}{4M_W^2}\int dp^2\frac{p^4m_i^2m_j^2}{(p^2+M_W^2)^2}g_ig_jx \tag{3.85}$$

Gluons connecting an external to an internal quark A lot of diagrams do not give any (LL) contribution when the gluon is treated in the Landau gauge, namely those in fig. (3.6). This was originally suggested in the Vysotskij paper [48] and we have checked it explicitly, confirming it (see Appendix B).

On the other hand, all the "crossed" graphs of the type reported in fig. (3.7) give a (LL) contribution, but only in the case in which the gluon is attached to an internal light-quark line, as is clearly explained in Appendix B.

For this kind of diagrams, we can see that each $\Delta S=2$ contribution is equivalent to the product of two $\Delta S=1$ vertices, as illustrated in fig. (3.8).

Therefore, everything is based on the computation of the (LL) QCD corrections to the $\Delta S=1$ vertex, in which one or two external legs may also be heavy-quark legs. Finally the two halves are composed again to give the $\Delta S=2$ amplitude. The basic calculations of the $\Delta S=2$ amplitude and of its $\Delta S=1$ sub-amplitude are given in Appendix B. In that context, it is pointed out that the dominant logarithmic terms are of the form $\ln((p^2+m^2)/p^2)$, in contrast to $\ln(*p^2+M_W)M_W)$ or $\ln((p^2+m_t^2)m_t^2)$ terms, which appear only at a sub-leading order. Unfortunately, the (LL) term does not reproduce the exact propagator structure in the integral expression for the amplitude.

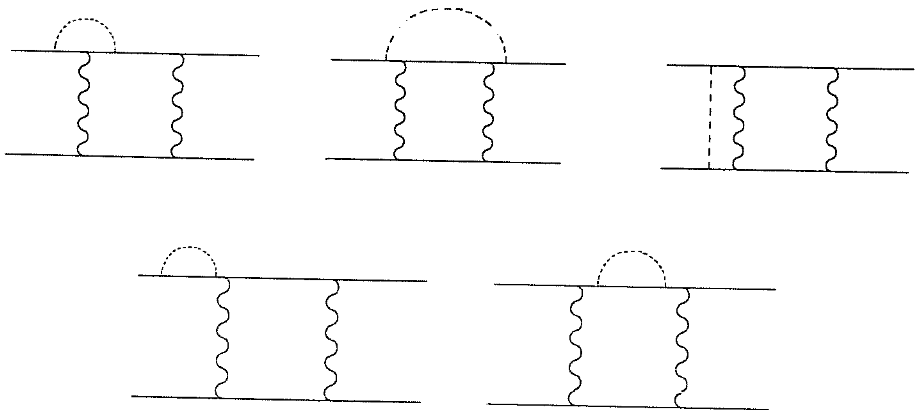

Figure 3.6: QCD corrections which do not give any (LL) contribution

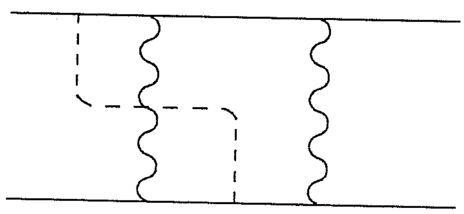

Figure 3.7: QCD corrections: gluon starting from an external quark leg and landing on an internal one

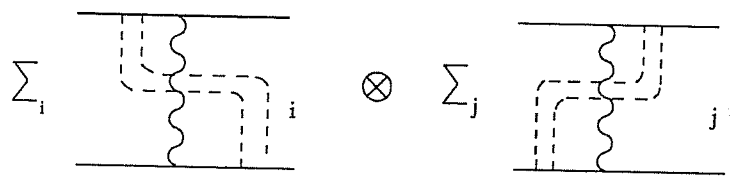

Figure 3.8: ($\Delta S = 2$) diagram as the product of two ($\Delta S = 1$) vertices

For the WW-box diagram, f.i., the result reported in Appendix B can also be cast in the form:

$$\mathcal{H}_{WW} = k\frac{\alpha_s}{4\pi}M_W^2 \sum_{i,j=1}^{3} \lambda_i\lambda_j \int dp^2 \frac{p^4}{(p^2+M_W^2)} g_i g_j (R_i + R_j) \tag{3.86}$$

where:

$$R_i = \frac{1}{M_W^2 - m_i^2} \ln\frac{p^2+m_i^2}{p^2+M_W^2} \tag{3.87}$$

There are two possible cases:

(a)$m_i^2 \ll M_W^2$

In the $m_i^2 \ll M_W^2$ case, due to the fact that the integral is already UV convergent, we can replace, to a good approximation:

$$\frac{1}{M_W^2 - m_i^2} \longrightarrow \frac{1}{p^2+M_W^2} \tag{3.88}$$

and

$$\ln\frac{p^2+m_i^2}{p^2+M_W^2} \longrightarrow \ln\frac{p^2}{M_W^2} \tag{3.89}$$

so that:

$$\mathcal{H}_{WW} \longrightarrow k\frac{\alpha_s}{4\pi}M_W^2 2 \sum_{i,j=1}^{3} \lambda_i\lambda_j \int dp^2 \frac{p^4}{(p^2+M_W^2)^2} g_i g_j \ln\frac{p^2}{M_W^2} \tag{3.90}$$

where the correct propagator structure in the amplitude is recovered. Thus, the QCD corrections, after they have been summed up using the RGE formalism, reduce to multiplicative factors. These factors can be computed in a standard way, by using the $(\Delta S=1)\otimes(\Delta S=1)$ structure of the whole $\Delta S=2$ amplitude. The well-known multiplicative factor, in this case, is given by:

$$\begin{aligned} F(p^2, M_W^2; 8/b_n, 2/b_n, -4/b_n) \equiv \\ \frac{1}{2}\left[\frac{\alpha_s(p^2)}{\alpha_s(M_W^2)}\right]^{8/b_n} - \left[\frac{\alpha_s(p^2)}{\alpha_s(M_W^2)}\right]^{2/b_n} + \frac{3}{2}\left[\frac{\alpha_s(p^2)}{\alpha_s(M_W^2)}\right]^{-4/b_n} \end{aligned} \tag{3.91}$$

It is structure can be derived from the renormalization of the $(\Delta S=1)$ vertex, in terms of the $O_\pm$ operators and their anomalous dimension factors $d_\pm$ ($d_+ = -\frac{2}{b_n}$, $d_- = \frac{4}{b_n}$), as we have seen in section (2.3). When you compose two $(\Delta S=1)$ vertices, you obtain that the $(\Delta S=2)$ amplitude and hence the $(\Delta S=2)$ effective hamiltonian is expressible as the sum of four terms:

$$\mathcal{H}^{\Delta S=2} \sim C_+^2 O_+ O_+ + 2C_+C_- O_+ O_- + C_-^2 O_- O_- \tag{3.92}$$

To each coefficient will be associated an anomalous dimension factor in terms of d_+ and d_-:

$$C_+^2 \rightarrow 2d_+ = -\frac{4}{b_n}$$

$$
\begin{aligned}
C_+C_- \rightarrow d_+ + d_- &= \frac{2}{b_n} \\
C_-^2 \rightarrow 2d_- &= \frac{8}{b_n}
\end{aligned} \tag{3.93}
$$

Thus, up to here, the general QCD correction factor has the form:

$$
\eta_{QCD} \sim a \left[\frac{\alpha_s(p^2)}{\alpha_s(\mu^2)}\right]^{8/b_n} + b \left[\frac{\alpha_s(p^2)}{\alpha_s(\mu^2)}\right]^{2/b_n} + c \left[\frac{\alpha_s(p^2)}{\alpha_s(\mu^2)}\right]^{-4/b_n} \tag{3.94}
$$

with a,b and c coefficients to be determined. Clearly: $a+b+c=1$, because $\eta_{QCD} \sim 1$ when $p^2 \sim \mu^2$. Their relative value can be determined by colour traces. When you combine the two $(\Delta S=1)$ vertices, you get:

$$
\mathcal{H} \sim \bar{q}_a C_{ab} q_b \bar{q}_c C_{cd} q_d \; \bar{q}_s D_{st} q_t \bar{q}_u D_{uv} q_v \tag{3.95}
$$

where $C_{ab}C_{cd}$ (or $D_{st}D_{uv}$) are the colour matrices in the two currents previously joined by a W-line:

$$
\begin{aligned}
C_{ab}C_{cd} &= C \otimes C = A 1 \otimes 1 + B \sum_a t^a \otimes t^a \\
D_{st}D_{uv} &= D \otimes D = A' 1 \otimes 1 + B' \sum_b t^b \otimes t^b
\end{aligned} \tag{3.96}
$$

C or D can correspond either to O_+ or to O_-. For instance, for $C \sim O_+$, $A = 4/3$ and $B = 2$; while for $C \sim O_-$, $A = 2/3$ and $B = -2$. When the intermediate quark line is contracted to give a four-quark $(\Delta S=2)$ vertex, we get:

$$
\mathcal{H} \sim \bar{q}_a (CD)_{av} q_v \; \bar{q}_s (DC)_{sd} q_d \tag{3.97}
$$

At this point, if you take the two possible projections over colour singlets, by contracting either with $\delta_{av}\delta_{sd}$ or with $\delta_{ad}\delta_{sv}$, you obtain that:

$$
\mathcal{H} \sim 12AA' + 4(BA' + AB') + \frac{22}{3} BB' \tag{3.98}
$$

and

$$
O_+O_+ : O_+O_- : O_-O_- = \frac{3}{2} : (-1) : \frac{1}{2} \tag{3.99}
$$

or

$$
a = \frac{3}{2} \quad , \qquad b = -1 \quad , \qquad c = \frac{1}{2} \tag{3.100}
$$

so that $F(p^2, M_W^2; 8/b_n, 2/b_n, -4/b_n)$ assume the aspect as in (3.91).

(b) $m_t^2 \gg M_W^2$

Clearly, in the $m_t^2 \gg M_W^2$case, the previous replacements are no longer justified, because now:

$$
\lim_{m_t \rightarrow \infty} R_t = 0 \tag{3.101}
$$

Thus, terms containing R_t are omitted. Moreover, R_c is taken to be:

$$R_c \simeq R_1 \tag{3.102}$$

In order to obtain the corrected perturbative series, the lowest order propagator is replaced by:

$$\frac{1}{p^2 + M_W^2} = \frac{1}{M_W^2} - \frac{p^2}{M_W^2(p^2 + M_W^2)} \tag{3.103}$$

Where the first term can be summed with the higher order QCD corrections associated with it, while the second one has no QCD correction factor. This technical trick is equivalent to the assertion that such kind of diagrams are suppressed by a power m_i^2/M_W^2 in the main integration. In the case in which the two internal quark lines are both of heavy-quark, there will be no contribution; while in the case in which only one of them is a heavy-quark one, the multiplicative F-factor will be:

$$F(p^2, M_W^2; 4/b_n, 1/b_n, -2/b_n) \tag{3.104}$$

where the anomalous dimension factors are half of the previous ones, because half of the graphs does not contribute.

Finally, we observe that for box-diagrams containing Higgs bosons, the only contributions come from diagrams with two top-quarks in the internal loop, because the relevant coupling is proportional to m_i^2/M_W^2. They give:

$$\begin{aligned}
\mathcal{H}_{HW} &= k\frac{\alpha_s}{4\pi}\sum_{i,j=2}^{3}\lambda_i\lambda_j 2\int dp^2 \frac{p^2 m_i^2 m_j^2}{(p^2+M_W)} g_i g_j (R_i + R_j) \\
\mathcal{H}_{HH} &= k\frac{\alpha_s}{4\pi}\sum_{i,j=2}^{3}\lambda_i\lambda_j \frac{1}{4M_W^2}\int dp^2 \frac{p^4 m_i^2 m_j^2}{(p^2+M_W)} g_i g_j (R_i + R_j)
\end{aligned} \tag{3.105}$$

Mass corrections Finally, we have to consider also that the introduced QCD corrections depend to a large extent from the masses of the internal quarks. Masses are "running-objects" in the context of the RGE analysis and they have to be evaluated, in our case, at the dominant momentum of the integrals. The running masses [39] are of the form:

$$m_q(p^2) = m_q\left(1 - 4\frac{\alpha_s(m_q^2)}{4\pi}\ln\frac{p^2}{m_q^2}\right) = m_q\left[\frac{\alpha_s(p^2)}{\alpha_s(m_q^2)}\right]^{\frac{4}{b_n}} \tag{3.106}$$

The mass corrections enter the calculation depending on the integral over the internal momenta. There are the following three possible integration cases:

(i) **Type 1**: $m_q^N B(m_q^2)$ where N is an arbitrary exponent. These terms give no large logarithms (because the dominant momentum region in the loop sits around m_q^2);

(ii) **Type 2:** $m_q^N B(M_W^2)$ In this case, when $m_q \ll M_W$, we have a correction, because m_q must be evaluated at the dominant momentum of the integral, i.e. M_W. The evolution gives as correction factor $[\alpha_s(m_q^2)/\alpha_s(M_W^2)]^{4N/b}$;

(iii) **Type 3:** $m_q^N C(m_q^2, M_W^2)$ Being the whole range between m_q and M_W (for $m_q \ll M_W$) involved in the integration, a correction is possible, in the form: $[\alpha_s(p^2)/\alpha_s(M_W^2)]^{4N/b}$. However it has to be included in the integral, due to the logarithmic behaviour of the initial integration ($C(m_q^2, M_W^2)$).

In all the cases in which the mass correction is present, it results to be very small for heavy quarks and we will account for running mass corrections only in the c-quark case.

3.3.3 Analysis of the results

Here, I would like to organize in a detailed summary all the possible QCD corrections in the two cases : (1) the $m_t \ll M_W$ and (2) the $m_t \gg M_W$ case. The case in which $m_t \sim M_W$, can be obtained by matching the other two cases. These results will be introduced, later on, in the numerical discussion.

1. **$m_t \ll M_W$ region : light top region.**

   In this region we are concerned with QCD corrections to a box-diagram with two internal light quarks "i" and "j", assuming $m_i \ll m_j$ if they are different. Clearly, here the analysis could have also be performed as Gilman and Wise [23] have done in their paper and a direct comparision is possible and required. As we saw, the Inami-Lim functions are approximated by their light top form and up to here the two approaches agree, while they depart one from each other in going further. Let us see how in each case:

   (i) when the two quarks are identical:

$$\mathcal{H}_i = k\lambda_q^2 F(x_q) = k\lambda_q^2 E(x_q)\eta_i \simeq k\lambda_q^2 x_q \eta_i \tag{3.107}$$

   for $i = 1, 2$ and $q = c, t$ respectively. Each η_i receives contributions from both the kinds of graphs we have described in the previous subsection. No running mass correction is present, because the integration structure is of *Type 1*. Thus, at the end:

$$\eta_i = \left[\frac{\alpha_s(m_q^2)}{\alpha_s(\mu^2)}\right]^{2/b_3} F(m_q^2, M_W^2; 8/b_n, 2/b_n, -4/b_n) \tag{3.108}$$

   for $i = 1, 2$ and two principal differences stand out with respect to ref. [23]: (a) there is no account for running mass corrections, and (b) the threshold structure of the effective theory is roughly treated. In fact, the

b-threshold is neglected, but $n = 4$ is chosen, as if the b-threshold itself had already been accounted for. Datta et al. apologize for that, justifying the choice of b_n with a one-unit uncertainty by saying that it could produce only a negligible numerical difference.

(ii) when the two quarks are different:

$$\begin{aligned} \mathcal{H}_3 &= 2k\lambda_c\lambda_t F(x_c, x_t) = 2k\lambda_c\lambda_t E(x_c, x_t)\eta_3 \\ &\simeq 2k\lambda_c\lambda_t x_c \ln\left(\frac{x_c}{x_t}\right)\eta_3 \end{aligned} \tag{3.109}$$

where

$$\begin{aligned} \eta_3 &= \frac{1}{\ln x_c/x_t}\int_{m_c^2}^{m_t^2}\frac{dp^2}{p^2}\left[\frac{\alpha_s(p^2)}{\alpha_s(\mu^2)}\right]^{2/b_3}\left[\frac{\alpha_s(p^2)}{\alpha_s(m_c^2)}\right]^{8/b_3} \\ &\quad \cdot F(m_q^2, M_W^2; 8/b_n, 2/b_n, -4/b_n) \end{aligned} \tag{3.110}$$

We see that mass corrections are accounted for, being the integral expression of *Type*2, so that the only difference with ref. [23] is in the roughly discussion of the threshold structure of the effective theory.

In both of the previous cases there are no Higgs contributions, because we are considering light quarks only.

2. $m_t \gg M_W$ **region : heavy top region.**

Here the new features discovered for QCD corrections play an important role and account for the differences in the η_i's behaviour with respect to the light top region. Basically we have that:

(i) η_1 remain unchanged from (1), because no top-quark is involved in $\mathcal{H}_1$;

(ii) In order to compute η_2, let us summarize all the contributions proportional to the term λ_t^2 in the hamiltonian. $\mathcal{H}_{WW}$, $\mathcal{H}_{HW}$ and $\mathcal{H}_{HH}$ give different results.

(ii.a) $\mathcal{H}_{WW}$

In the case of the box with two W-bosons we have:

- the tree level hamiltonian contribution;
- QCD corrections from gluons exchanged between external legs;
- QCD corrections from gluons exchanged between external and internal legs.

They give respectively:

$$\mathcal{H}_{WW} = kM_W^2\sum_{i,j=2}^{3}\lambda_t^2\int dp^2\frac{p^4}{(p^2+M_W^2)^2}(g_t - g_1)^2 \tag{3.111}$$

$$\delta\mathcal{H}_{WW}^{(1)} = k\frac{\alpha_s}{4\pi}M_W^2\lambda_t^2\int dp^2\frac{p^4}{(p^2+M_W^2)^2}(g_t - g_1)^2 2\ln\frac{p^2}{\mu^2} \tag{3.112}$$

$$\delta\mathcal{H}^{(2)}_{WW} = k\frac{\alpha_s}{4\pi}M_W^2 2\sum_{i,j=1}^{3}\lambda_t^2\int dp^2\frac{p^4}{(p^2+M_W^2)}(-2g_tg_1R_1+2g_1^2R_1) \quad (3.113)$$

where the notation used should be already familiar.

(ii.b) $\mathcal{H}_{HW}$

In the case of $\mathcal{H}_{HW}$ there are:

- the tree level contribution;
- QCD corrections from gluons exchanged between external quarks;
- no QCD corrections from gluon exchanged between external and internal quarks, because they give zero contribution for $R_t = 0$.

They are respectively:

$$\mathcal{H}_{HW} = k\sum_{i,j=2}^{3}\lambda_t^2 2\int dp^2\frac{p^2m_t^4}{(p^2+M_W^2)^2}g_t^2 \quad (3.114)$$

$$\delta\mathcal{H}^{(1)}_{HW} = k\frac{\alpha_s}{4\pi}\sum_{i,j=2}^{3}\lambda_t^2 2\int dp^2\frac{p^2m_t^4}{(p^2+M_W^2)^2}g_t^2 2\ln\frac{p^2}{\mu^2} \quad (3.115)$$

(ii.c) $\mathcal{H}_{HH}$

In the case of $\mathcal{H}_{HH}$ we have:

- the tree level contribution;
- QCD corrections from gluons exchanged between external quarks;
- no QCD corrections from gluon exchanged between external and internal quarks, because they give sub-leading contributions.

They are respectively:

$$\mathcal{H}_{HH} = k\sum_{i,j=2}^{3}\lambda_t^2\frac{1}{4M_W^2}\int dp^2\frac{p^4m_t^2}{(p^2+M_W^2)^2}g_t^2 \quad (3.116)$$

$$\delta\mathcal{H}^{(1)}_{HH} = k\sum_{i,j=2}^{3}\lambda_t^2\frac{1}{4M_W^2}\int dp^2\frac{p^4m_t^4}{(p^2+M_W^2)^2}g_t^2 2\ln\frac{p^2}{\mu^2} \quad (3.117)$$

Adding all the previous contributions and summing up the (LL) QCD corrections we get for $F(x_t)$ the following integral expression:

$$\begin{aligned} F(x_t) &= M_W^2\int dp^2\left\{\frac{p^4}{p^2+M_W^2}\left[\frac{1}{p^2+M_W^2}\left(g_t^2+\frac{p^2}{M_W^2}(2g_tg_1-g_1^2)\right)\right.\right. \\ &+ \left.\left.\frac{1}{M_W^2}g_1(-2g_tz'+g_1z)\right] + m_t^4\frac{p^2}{(p^2+M_W^2)^2}g_t^2\left[\frac{p^2}{4M_W^2}+2\right]\right\}x \end{aligned} \quad (3.118)$$

where

$$x = \left[\frac{\alpha_s(p^2)}{\alpha_s(\mu^2)}\right]^{2/b_n} \quad (3.119)$$

$$(3.120)$$

$$z = F(p^2, p^2 + M_W^2; 8/b_n, 2/b_n, -4/b_n)$$

$$(3.121)$$

$$z' = F(p^2, p^2 + M_W^2; 4/b_n, 1/b_n, -2/b_n)$$

$$(3.122)$$

$$g_i = \frac{1}{p^2 + m_i^2}$$

η_2 can be computed numerically from (3.118), as the ratio between (3.118) and (3.118) itself but with no QCD corrections, that is with $x = z = z' = 1$.
[(iii)] η_3 can be computed numerically as η_2. We have to collect all the terms proportional to $\lambda_c \lambda_t$ in the hamiltonian. Again $\mathcal{H}_{WW}$, $\mathcal{H}_{HW}$ and $\mathcal{H}_{HH}$ give different results. The contributions to each one of the previous terms in the hamiltonian are exactly as in case *(ii)*, except for $\mathcal{H}_{HW}$, which receives also contributions from gluons exchanged between an external and an internal leg. Moreover, there is an overall running mass corrections, due to the logarithmic nature of the tree level integral expression of the amplitude.
The final result for $F(x_c, x_t)$ is easily obtainable and reads:

$$\begin{aligned} F(x_c, x_t) = & \int dp^2 \left\{ \frac{p^4(g_c - g_1)}{p^2 + M_W^2} \left[g_1 \left(\frac{p^2}{p^2 + M_W^2} - z \right) \right.\right. \\ & \left. - g_t \left(\frac{p^2}{p^2 + M_W^2} - z' \right) \right] \\ & + m_c^2 m_t^2 g_c g_t \frac{p^2}{p^2 + M_W^2} \left[\frac{1}{p^2 + M_W^2} + \frac{1}{M_W^2} z' \right. \\ & \left.\left. - \frac{3}{4} \frac{p^2}{M_W^2 (p^2 + M_W^2)} \right] \right\} xy \end{aligned} \quad (3.123)$$

where x, z and z' are as before, while y, defined by:

$$y = \left[\frac{\alpha_s(p^2)}{\alpha_s(m_c^2)}\right]^{8/b_n} \quad (3.124)$$

is the running mass correction.

Chapter 4

Numerical analysis of the results

I will describe in the present chapter the numerical implementation of the theoretical results discussed in Chapter 3. The general purpose of our analysis is to obtain a global description of CP-violation, using all the theoretical and experimental informations at hand. In this context, large uncertainties still exist, mainly the unknown value of the top quark mass and the large theoretical uncertainties in the calculation of hadronic matrix elements. Nevertheless, the whole space of parameters involved in the description of CP-violation can be greatly constrained, if we use simultaneously the informations coming from three principal fields:

- $K_0 - \overline{K}_0$ mixing,
- $B_0 - \overline{B}_0$ mixing,
- ϵ'/ϵ ratio.

In the previous chapter, I have described how the theoretical analysis of F_0–$\overline{F}_0$ mixing, for "F" a generic flavour, has been developed up to now. The problem of QCD corrections for large values of the top quark mass seems now to be understood. Thus, the indications coming from $K_0 - \overline{K}_0$ mixing and $B_0 - \overline{B}_0$ mixing can be considered quite accurate. On the other hand, there are still large uncertainties in the theoretical and experimental prescriptions on the ϵ'/ϵ ratio. It would be our aim for the future to proceed to a careful analysis of the problem, but for the moment it is only a work in progress. Thus, we have implemented numerically only the results about $K_0 - \overline{K}_0$ mixing (ϵ-parameter) and $B_0 - \overline{B}_0$ mixing (f_B decay constant).

First of all, we have verified how much QCD corrections for a large value of the top quark mass (m_t) differ from those for a small value of the top quark mass. We have compared the *bare* (without QCD corrections) *Inami-Lim* functions with the *QCD-corrected Inami-Lim* functions, predicted by Gilman and Wise [23] for small values of m_t and by Datta et al. [14] for large values of m_t.

Remember that the effective hamiltonian for $F_0 - \overline{F}_0$ mixing is of the form:

$$\mathcal{H}_{eff}^{|\Delta S|=2} = k\left\{\lambda_c^2 F(x_c) + \lambda_t^2 F(x_t) + 2\lambda_c\lambda_t F(x_c, x_t)\right\} \tag{4.1}$$

where

$$k = \frac{G_F^2}{16\pi^2} M_W^2 (\bar{d}\gamma^\mu(1-\gamma_5)s)^2 \tag{4.2}$$

$F(x_c)$ does not depend on m_t, because it is related to the amplitude of a box-diagram without any top quark in the internal fermion lines. On the other hand, $F(x_t)$ and $F(x_c, x_t)$ depend on m_t explicitly. They are given respectively by:

(i) the *bare Inami-Lim* functions $E(x_t)$ and $E(x_c, x_t)$, in the absence of QCD corrections;

(ii) the product of the *bare Inami-Lim* functions $E(x_t)$ and $E(x_c, x_t)$ times a factor due to QCD corrections, in the case of a theory with a light top quark, as in the paper by Gilman and Wise:

$$\begin{aligned} F(x_t) &= \eta_2 E(x_t) \\ F(x_c, x_t) &= \eta_3 E(x_c, x_t) \end{aligned} \tag{4.3}$$

where η_2 and η_3 are the QCD correction factors discussed in section 3.2;

(iii) the integral expressions given in eqs. (3.118) and (3.123), in the case of a theory with a very heavy top quark, as in the work by Datta et al.

In the numerical program we have used the complete expressions for the *bare Inami-Lim* functions, that is:

$$E(x_t) = E(x_t, x_t) = x_t\left[\frac{1}{4} + \frac{9}{4}\frac{1}{(1-x_t)} - \frac{3}{2}\frac{1}{(1-x_t)^2}\right] + \frac{3}{2}\left(\frac{x_t}{x_t - 1}\right)\ln x_t \tag{4.4}$$

$$\begin{aligned} E(x_c, x_t) &= x_c x_t \left\{\left[\frac{1}{4} + \frac{3}{2}\frac{1}{(1-x_t)}\frac{3}{4}\frac{1}{(1-x_t)^2}\right]\frac{\ln x_t}{x_t - x_c}\right. \\ &+ \left.(x_t \leftrightarrow x_c) - \frac{3}{4}\frac{1}{(1-x_c)(1-x_t)}\right\} \end{aligned} \tag{4.5}$$

On the other hand, in order to account for QCD corrections in a more straightforward way,we have improved the results given by Datta et al. In their paper [14], they have proposed for $F(x_t)$ and $F(x_c, x_t)$ the integral expressions in eqs. (3.118) and (3.123), which we report also here for sake of clearness:

$$\begin{aligned} F(x_t) &= M_W^2 \int dp^2 \left\{\frac{p^4}{p^2 + M_W^2}\left[\frac{1}{p^2 + M_W^2}\left(g_t^2 + \frac{p^2}{M_W^2}(2g_t g_1 - g_1^2)\right)\right.\right. \\ &+ \left.\left.\frac{1}{M_W^2} g_1(-2g_t z' + g_1 z)\right] + m_t^4 \frac{p^2}{(p^2 + M_W^2)^2} g_t^2 \left[\frac{p^2}{4M_W^2} + 2\right]\right\} x \end{aligned} \tag{4.6}$$

and

$$
\begin{aligned}
F(x_c, x_t) &= \int dp^2 \left\{ \frac{p^4(g_c - g_1)}{p^2 + M_W^2} \left[g_1 \left(\frac{p^2}{p^2 + M_W^2} - z \right) \right. \right. \\
&\quad \left. - g_t \left(\frac{p^2}{p^2 + M_W^2} - z' \right) \right] \\
&\quad + m_c^2 m_t^2 g_c g_t \frac{p^2}{p^2 + M_W^2} \left[\frac{1}{p^2 + M_W^2} + \frac{1}{M_W^2} z' \right. \\
&\quad \left. \left. - \frac{3}{4} \frac{p^2}{M_W^2 (p^2 + m_w^2)} \right] \right\} xy \qquad (4.7)
\end{aligned}
$$

where

$$x = \left[\frac{\alpha_s(p^2)}{\alpha_s(\mu^2)} \right]^{2/b_n} \qquad (4.8)$$

$$z = F(p^2, p^2 + M_W^2; 8/b_n, 2/b_n, -4/b_n) \qquad (4.9)$$

$$z' = F(p^2, p^2 + M_W^2; 4/b_n, 1/b_n, -2/b_n) \qquad (4.10)$$

$$g_i = \frac{1}{p^2 + m_i^2} \qquad (4.11)$$

$$y = \left[\frac{\alpha_s(p^2)}{\alpha_s(m_c^2)} \right]^{8/b_n} \qquad (4.12)$$

However they have not accounted in a systematic way for the thresholds in the effective theory.

On the contrary, we have implemented in the program the whole threshold structure of the effective theory, using a specific subroutine. This subroutine calculates $\alpha(\mu^2)$ at a given scale, after having satisfied the whole set of matching conditions present at each threshold above the given scale μ^2. A detailed description of the method used in the subroutine is given in Appendix C.

With this improvement, the result obtained in a theory with a very heavy top quark become directly comparable with those obtained in a theory with a light top quark, as given by Gilman and Wise [23], who have considered the whole threshold structure of the theory. Their results can also be obtained from the integral expressions in (4.6) and (4.7), by setting $z = z'$, because in the case of light top quark all the possible QCD radiative corrections contribute.

We have reported in Table 4.1 and Table 4.2 the obtained numerical results. For each value of m_t, Table 4.1 gives the value of the *bare Inami-Lim* function, say $E_I(x_t)$; the value of the *QCD-corrected Inami-Lim* function obtained from a theory

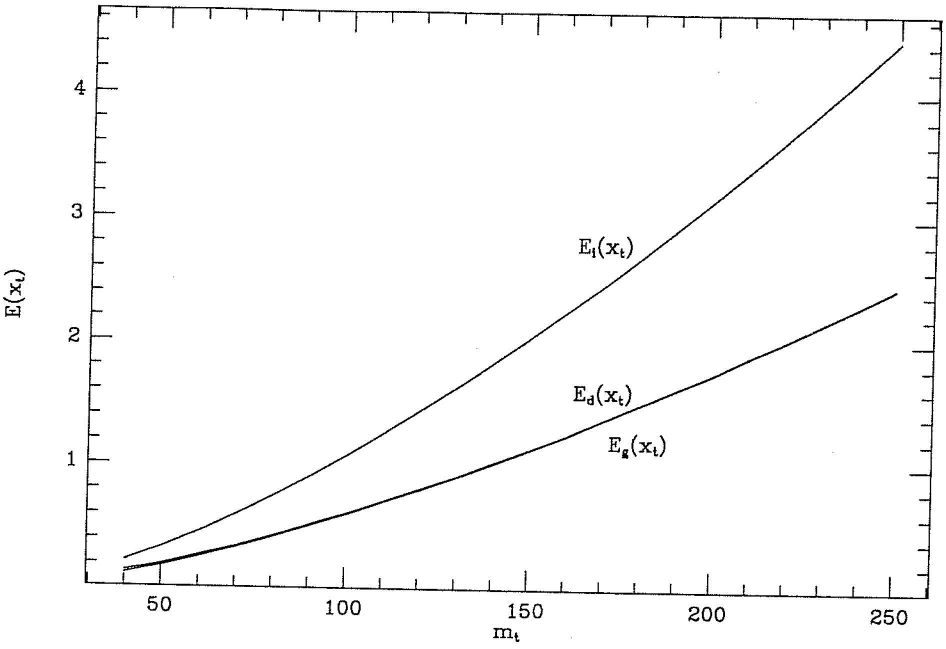

Figure 4.1: $E_I(x_t)$, $E_G(x_t)$ and $E_D(x_t)$ as functions of m_t

with a light top quark, say $E_G(x_t)$, (extrapolating the results in the heavy top region) and the value of the *QCD-corrected Inami-Lim* function obtained from a theory with a heavy top quark, say $E_D(x_t)$, (extrapolating the results in the light top region). Table 4.2 is analogous, but it gives, for each value of m_t, $E_I(x_c, x_t)$, $E_G(x_c, x_t)$ and $E_D(x_c, x_t)$ respectively.

We can observe that, while the Gilman and Wise results, extrapolated at large values of m_t are quite accurate, the improved Datta et al. results, extrapolated at small values of m_t are not so accurate. Moreover, as you can see also from figs. (4.1) and (4.2), the two theories differ only in the small m_t region, while it is impossible to distinguish them in the large m_t region.

See also figs. (4.3) and (4.4), where we have plotted the ratios $E_D(x_t)/E_G(x_t)$ and $E_D(x_c, x_t)/E_G(x_c, x_t)$: except for the small m_t region, each ratio results to be very closed to one.

Finally, we have plotted in figs. (4.5) and (4.6) $F(x_t)$ and $F(x_c, x_t)$, using the values of $E_G(x_t)$ or $E_G(x_c, x_t)$ for small m_t and the values of $E_D(x_t)$ or $E_D(x_c, x_t)$ for large m_t. As one can see, the two curves match very well at the threshold ($\approx$ 80 GeV).

In each case, the exact values of the threshold and the value of m_b and m_c are not influent.

m_t	$E_I(x_t)$	$E_G(x_t)$	$E_D(x_t)$
40	0.221	0.138	0.118
50	0.331	0.197	0.180
60	0.456	0.267	0.252
70	0.595	0.342	0.328
80	0.745	0.424	0.411
90	0.906	0.515	0.503
100	1.075	0.607	0.597
110	1.253	0.709	0.700
120	1.439	0.811	0.802
130	1.632	0.916	0.908
140	1.832	1.025	1.017
150	2.038	1.137	1.130
160	2.251	1.252	1.245
170	2.471	1.381	1.375
180	2.696	1.504	1.498
190	2.928	1.629	1.623
200	3.166	1.757	1.752
210	3.411	1.896	1.891
220	3.661	2.030	2.026
230	3.918	2.168	2.164
240	4.181	2.314	2.310
250	4.450	2.464	2.460

Table 4.1: $E_I(x_t)$, $E_G(x_t)$ and $E_D(x_t)$ for different values of m_t

m_t	$E_I(x_c, x_t)$	$E_G(x_c, x_t)$	$E_D(x_c, x_t)$
40	2.00 e-03	6.86 e-04	6.84 e-04
50	2.14 e-03	7.27 e-04	7.25 e-04
60	2.25 e-03	7.50 e-04	7.49 e-04
70	2.33 e-03	7.67 e-04	7.67 e-04
80	2.40 e-03	7.81 e-04	7.81 e-04
90	2.46 e-03	7.93 e-04	7.93 e-04
100	2.51 e-03	8.03 e-04	8.03 e-04
110	2.56 e-03	8.11 e-04	8.12 e-04
120	2.60 e-03	8.18 e-04	8.19 e-04
130	2.63 e-03	8.32 e-04	8.33 e-04
140	2.66 e-03	8.38 e-04	8.39 e-04
150	2.69 e-03	8.44 e-04	8.45 e-04
160	2.72 e-03	8.49 e-04	8.50 e-04
170	2.74 e-03	8.53 e-04	8.54 e-04
180	2.77 e-03	8.57 e-04	8.58 e-04
190	2.79 e-03	8.60 e-04	8.62 e-04
200	2.80 e-03	8.64 e-04	8.65 e-04
210	2.82 e-03	8.67 e-04	8.68 e-04
220	2.84 e-03	8.70 e-04	8.71 e-04
230	2.85 e-03	8.72 e-04	8.74 e-04
240	2.87 e-03	8.75 e-04	8.76 e-04
250	2.88 e-03	8.77 e-04	8.79 e-04

Table 4.2: $E_I(x_c, x_t)$, $E_G(x_c, x_t)$ and $E_D(x_c, x_t)$ for different values of m_t

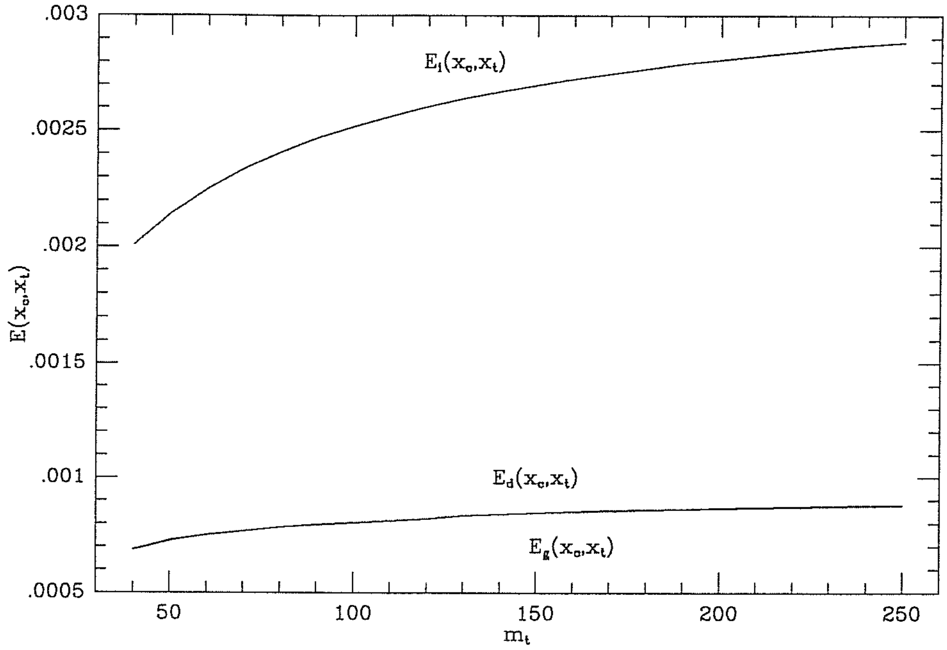

Figure 4.2: $E_I(x_c, x_t)$, $E_G(x_c, x_t)$ and $E_D(x_c, x_t)$ as functions of m_t

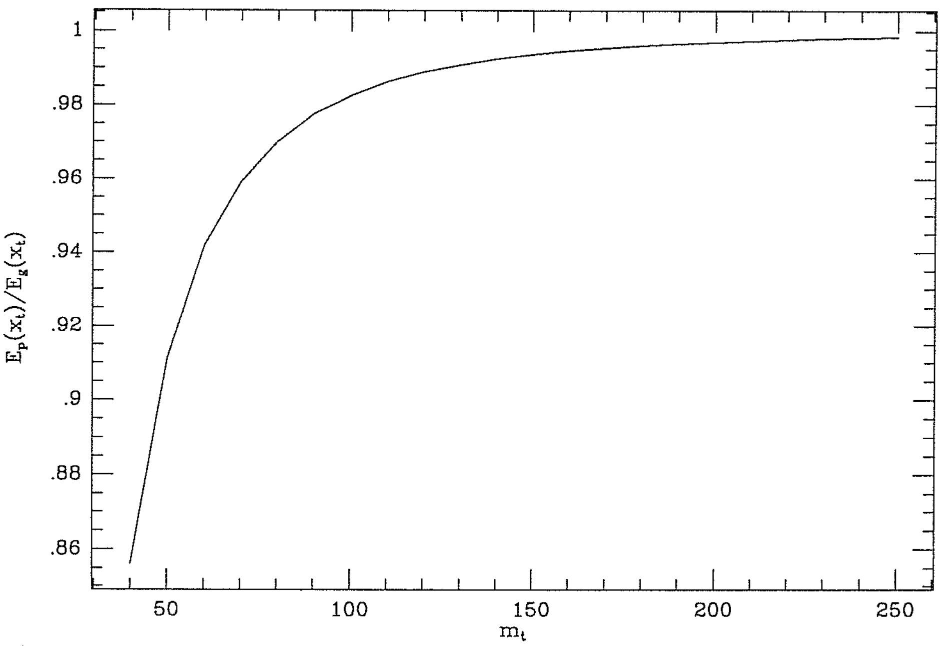

Figure 4.3: The ratio $E_D(x_t)/E_G(x_t)$ as a function of m_t

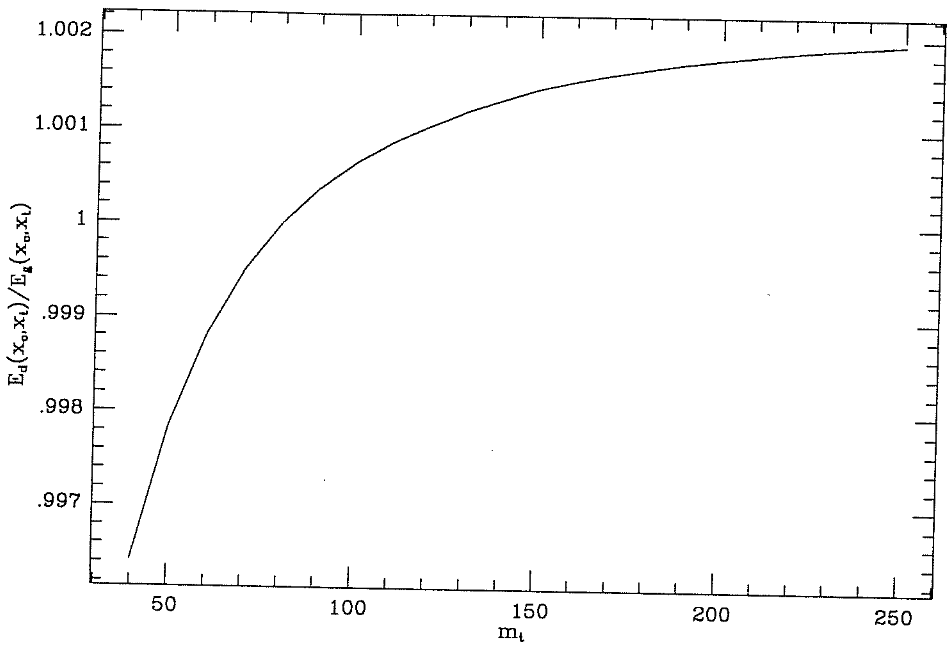

Figure 4.4: The ratio $E_D(x_c, x_t)/E_G(x_c, x_t)$ as a function of m_t

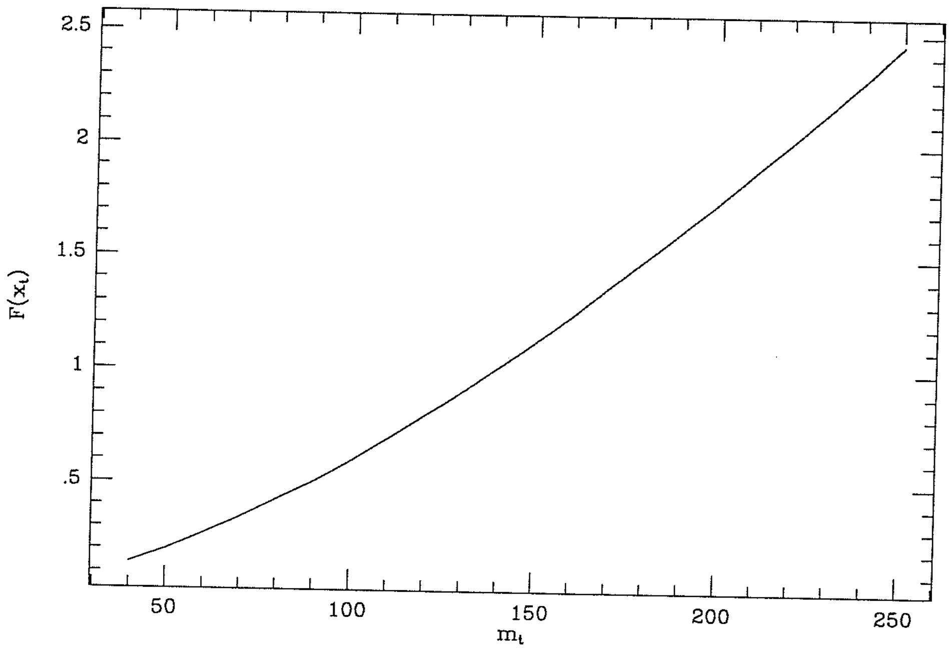

Figure 4.5: Plot of $F(x_t)$ without extrapolation

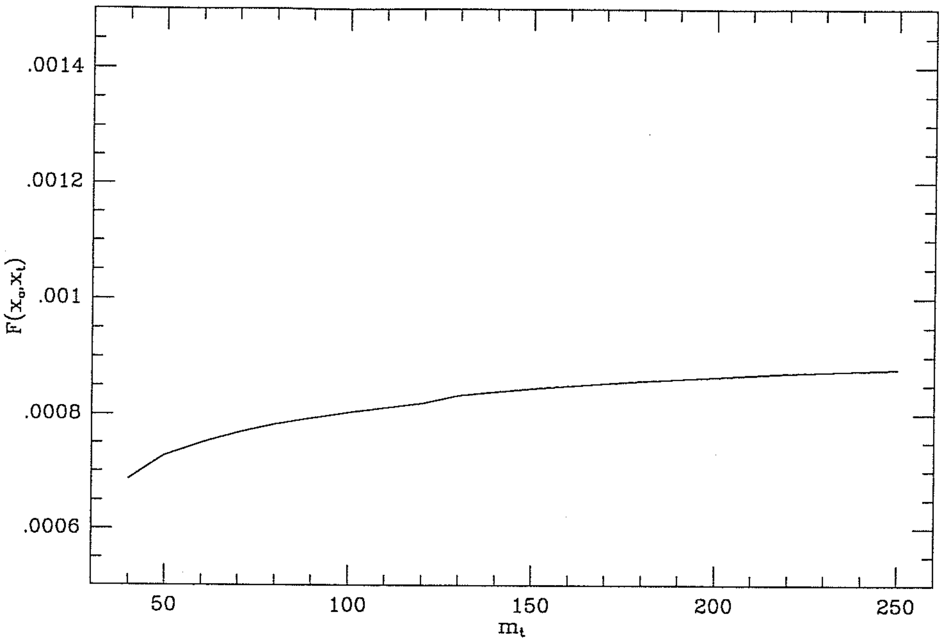

Figure 4.6: Plot of $F(x_c, x_t)$ without extrapolation

In a second time, we have used our QCD-improved results to see how the space of parameters which describes CP-violation can be restricted by only using $K_0-\overline{K}_0$ mixing (ϵ-parameter) and $B_0-\overline{B}_0$ mixing (f_B decay constant). We have considered the following expressions for the ϵ-parameter and for the f_B^2 decay constant:

$$|\epsilon|_{\xi=0} = \left[\frac{G_F^2 M_W^2 M_K f_K^2}{12\sqrt{2}\pi^2 \Delta M}\right] B_K 2A^2\lambda^6\rho\sin\delta\left[F(x_c,x_t) + A^2\lambda^4(1-\rho\cos\delta)F(x_t) - F(x_c)\right] \tag{4.13}$$

and

$$f_B^2 = x_B\left[\frac{G_F^2 M_W^2 M_B\lambda^6}{6\pi^2}\right]^{-1}\left[\tau_B A^2(1+\rho^2-2\rho\cos\delta)f_B^2 F(x_t)\right]^{-1} \tag{4.14}$$

They depend on a set of parameters which can be fixed, because a variation of them couldn't produce a significative effect. We have fixed these quantities as reported in Table 4.3.

Let us comment some of these choices. We have fixed $\Lambda_{QCD} = 0.1, 0.2, 0.3$ in three different runnings of the program, in order to verify that the dependence on Λ_{QCD} is negligible. The values of the other parameters are taken from the data of the Madrid Conference of the last year, except for B_K, taken from the data given by Sharpe during his communication at the meeting in Capri, last year too.

Both B_K and B_B values are from lattice calculations, also if there are also theoretical arguments for $B_B = 1$. A very important comment about the B-parameter must occur at this point. The B-parameters reported in Table 4.3 does not depend on the renormalization scale μ. However, the B-parameter defined as the ratio between

parameter	*value*
G_F	$1.16634 \cdot 10^{-5}$ GeV^{-2}
Λ_{QCD}	0.1 / 0.2 / 0.3 GeV
m_c	1.5 GeV
m_b	4.5 GeV
M_W	80.6 GeV
B_K	0.94 ± 0.02
B_B	1
M_K	0.49 GeV
M_B	5.287 GeV
ΔM	$3.521 \cdot 10^{-15}$
f_K	0.165 GeV
$\lambda = \sin\theta_c$	0.221 ± 0.002
x_B	0.73
ϵ_{exp}	$2.28 \cdot 10^{-3}$

Table 4.3: Used values of the fixed parameters

the matrix element of the *bare* $\Delta S = 2$ effective hamiltonian between $|K_0\rangle$ and $|\overline{K}_0\rangle$ states and the same matrix element evaluated in the vacuum insertion approximation is μ-dependent. Usually one writes that:

$$\langle\overline{K}_0|(\bar{s}d)_{(V-A)}(\bar{s}d)_{(V-A)}|K_0\rangle = \frac{8}{3}B_K(\mu)f_K M_K^2 \tag{4.15}$$

or:

$$\begin{aligned} B_K(\mu) &= \frac{\langle\overline{K}_0|(\bar{s}d)_{(V-A)}(\bar{s}d)_{(V-A)}|K_0\rangle}{\frac{8}{3}f_K M_K^2} \\ &= \frac{\langle\overline{K}_0|(\bar{s}d)_{(V-A)}(\bar{s}d)_{(V-A)}|K_0\rangle}{\langle\overline{K}_0|(\bar{s}d)_{(V-A)}|0\rangle\langle 0|(\bar{s}d)_{(V-A)}|K_0\rangle} \end{aligned} \tag{4.16}$$

as we have seen in Section 3.1. However, also the QCD correction factors are μ-dependent. Precisely, when the UV cut-off is made to coincide with the mass of the charm quark, an overall factor:

$$\left[\frac{\alpha_s(m_c^2)}{\alpha_s(\mu^2)}\right]^{2/b_3} \tag{4.17}$$

arises in all the kinds of QCD corrections to the $\Delta S = 2$ box-diagram, both in a theory with a heavy top quark and in a theory with a light top quark. Thus, if one defines a B-factor (for the $K_0 - \overline{K}_0$ mixing as well as for the $B_0 - \overline{B}_0$ mixing) as:

$$B = B(\mu)\cdot[\alpha_s(\mu)]^{-2/b_3} = B(\mu)\cdot[\alpha_s(\mu)]^{-2/9} \tag{4.18}$$

this is μ-independent and the μ-dependence is cancelled also in the QCD correction factors, compensated by the μ-dependence of $B(\mu)$. We have assumed the B-factor defined as in eq. (4.18) and, coherently, we have put $\alpha_s(\mu^2) = 1$ in the computation of the QCD correction factors.

To estimate the uncertainty from the experimental measurements, we have let m_t, A, ρ and τ_B to vary in the following intervals:

$$\begin{aligned} 90 &\le m_t \le 160 \\ 1.0 &\le A \le 1.10 \\ .40 &\le \rho \le .50 \\ 1.13 &\le \tau_B \le 1.61 \end{aligned} \tag{4.19}$$

They are justified by experimental and phenomenological analysis. I will try here to sketch it briefly.

- τ_B is quite well measured experimentally. The reported value is:

$$\tau_B = 1.18 \pm 0.14\, ps. \tag{4.20}$$

- A is determined from the B-lifetime and the semi-leptonic branching ratio $B_{SL} = B(b \to e\nu X)$:

$$B_{SL} = \Gamma_{SL} \cdot \tau_B \tag{4.21}$$

The semi-leptonic width Γ_{SL} can be computed in the parton model improved by QCD and phase space corrections, as in the ACCMM model [3]. One could also add non-perturbative corrections to the spectator picture, typical of the parton model. These terms are model dependent, but are small for the "fully inclusive" semi-leptonic width. One obtains:

$$\tau_B|V_{bc}|^2 = \tau_B|A\lambda^2|^2 = \frac{B_{SL} \cdot 10^{-13} sec}{Z_c\left(1 + \frac{\Gamma(b\to u)}{\Gamma(b\to c)}\right)} \tag{4.22}$$

The parameter Z_c (proportional to $\Gamma(b \to ce\nu)$) has been carefully evaluated in [3], where the value $Z_c = 4.0 \pm 0.6$ was determined. The value of B_{SL} is taken from experimental data to be: $B_{SL} = 0.0109 \pm 0.006$. Inserting Z_c and B_{SL} in eq. (4.22) (for $\Gamma(b \to u)/\Gamma(b \to c) < 0.08$), one obtains:

$$A \simeq 1.05 \pm 0.17 \tag{4.23}$$

and our interval for A is well centered on this value.

- The present constraints on ρ are obtained from the experimental limit on $R = \Gamma(b \to u)/\Gamma(b \to c)$ (where $\Gamma(b \to u(c)$ means $\Gamma(b \to e\nu X_{u(c)})$, the semi-leptonic width into charmless (charmed) final states), trough the expression:

$$(0.47 \pm 0.02)R = \frac{|V_{ub}|^2}{|V_{uc}|^2} = (\lambda\rho)^2 \tag{4.24}$$

where

$$\lambda = \sin\theta_c = 0.221 \pm 0.002 \tag{4.25}$$

The numerical factor:

$$(0.47 \pm 0.02) \tag{4.26}$$

is obtained from the parton model plus phase space and QCD corrections (as in the ACCMM model [3]). R is determined by the electron spectrum near the endpoint. We have:

$$\frac{d\Gamma}{dE_e} = \Gamma(b \to u)\left[\frac{1}{\Gamma(b \to u)}\frac{d\Gamma(b \to u)}{dE_e}\right] + \Gamma(b \to c)\left[\frac{1}{\Gamma(b \to c)}\frac{d\Gamma(c \to u)}{dE_e}\right] \tag{4.27}$$

and the derivation of a limit on R necessarily involves some model dependence. It seems quite difficult to improve the systematic error involved in the theoretical predictions for the inclusive rates. Tentatively, iti s given:

$$0.04 \le \frac{|V_{ub}|}{|V_{uc}|} \le 0.13 \tag{4.28}$$

or

$$0.2 \leq \rho \leq 0.6 \tag{4.29}$$

which seems consistent with other results of CP-violation (ϵ, ϵ'/ϵ). Our interval is well centered on these values.

- Experimental indications coming from LEP and SLC suggest for m_t the range between 60 and 220 GeV. Presently, the lower bound has been fixed at 89 GeV, by CDF results (if $B(t \to e\nu X) \sim 11\%$). Also from LEP, we should have some bound indications, expecially from the analysis of the invisible and hadronic width of the Z_0.

However, the parameters A, ρ and τ_B are not independent and for this reason we have thought to modify our program as follows.

1. We have rewritten the expression for ϵ as follows:

$$|\epsilon| = cost_1 \cdot C_1(A,\rho) \cdot \sin\delta \, [F(x_c,x_t) - F(x_c) + C_2(A)F(x_t)(1-\rho\cos\delta)] \tag{4.30}$$

where:

$$\begin{aligned} cost_1 &= \frac{G_F^2 M_W^2 M_K f_K^2 \lambda^6}{6\pi^2} \\ C_1(A,\rho) &= B_K \rho [A^2\lambda^4] \\ C_2(A) &= A^2\lambda^4 \end{aligned} \tag{4.31}$$

Then we have let only C_1 and ρ varying (with an infinitesimal increment around a given value). For each set of values (m_t, $\cos\delta$, C_1, ρ) we have considered only the maximum and the minimum values of $|\epsilon|$, among all the values obtained by varying C_1 and ρ. Repeating this procedure for a fixed value of m_t and for $\cos\delta$ which varies from -1 to 1 with regular steps, we have obtained a band and the central value in the ($\cos\delta$, ϵ) plane.

2. In an analogous way, we have rewritten the expression for f_B^2 as follows:

$$f_B^2 = \frac{x_B}{cost_2} \frac{1}{F(x_t)C_3(A,\tau_B)[1-\rho^2-2\rho\cos\delta]} \tag{4.32}$$

where

$$\begin{aligned} cost_2 &= \frac{G_F^2 M_W^2 M_B \lambda^2}{6\pi^2} \\ C_3(A,\tau_B) &= \tau_B A^2 \lambda^4 \end{aligned} \tag{4.33}$$

With the same method used for $|\epsilon|$, we have obtained two sets of values for f_B, a minimal and a maximal one, plus a set of central values, as plotted in figs. (4.7), (4.8) and (4.9). Also in this case, a band and a central value is defined in the $(\cos\delta,\ f_B)$ plane.

Finally, by combining the results in (1) and (2) and by using the experimental value of ϵ, presently known with a great accuracy, we have determined, for different values of m_t, the allowed regions for f_B. In figs. (4.7), (4.8) and (4.9) we have plotted f_B and ϵ as functions of $\cos\delta$, one under the other, for three different values of m_t. In the ϵ-plot, the dotted line stands for the experimental value of ϵ ($\epsilon = 2.28 \cdot 10^{-3}$). As one can see, if the mass of the top quark is small or if f_B is small, no interesting information can be derived from the plots. On the other hand, for large values of the top quark mass ($m_t \simeq 160$ GeV or more), two narrow regions on the $cos\delta$ axis are determined, by fitting the theoretical ϵ-curve with the experimental value. Moreover, was f_B quite large or small, only one of these two regions of $\cos\delta$ could be selected. This would be a great improvement in our knowledge of CP-violation.

Recent indications coming from lattice [4] calculations push up the value of f_B. Up to here, not so much attention have been paid to the value of f_B. On the other hand, it seems to us a very important point. A great value of f_B would imply a different "morphology" of the description of CP-violation in the Standard Model. As reported in [28], the amount of CP-violation in the Standard Model is proportional to the area of the triangle, in the complex plane, of sides:

$$\begin{aligned} a_1^{jk} &= V_{1j}V_{1k}^\star \\ a_2^{jk} &= V_{2j}V_{2k}^\star \\ a_3^{jk} &= V_{3j}V_{3k}^\star \end{aligned} \tag{4.34}$$

This is due to the unitarity of the CKM matrix, for which:

$$\sum_{i=1}^{3} V_{ij}V_{ik}^\star = 0 \tag{4.35}$$

This is equivalent to say that:

$$a_1^{jk} + a_2^{jk} + a_3^{jk} = 0 \tag{4.36}$$

and this relation define exactly a triangle in the complex plane, the so called *unitarity triangle*. If no CP-violation exists, all the a_i^{jk} for $i = 1, 2, 3$ are real and the triangle degenerates to a line (area=0).

The present estimation of CP-violation suggests that the triangle is a very flat one, with a very small area. However, a large value of f_B would imply a large value of $|V_{tb}V_{td}^\star|$ and this would change the shape of the unitarity triangle. The triangle would have a larger area and a larger amount of CP-violation could be argued. This is a quite new point of view on the problem and it would also have a great experimental importance.

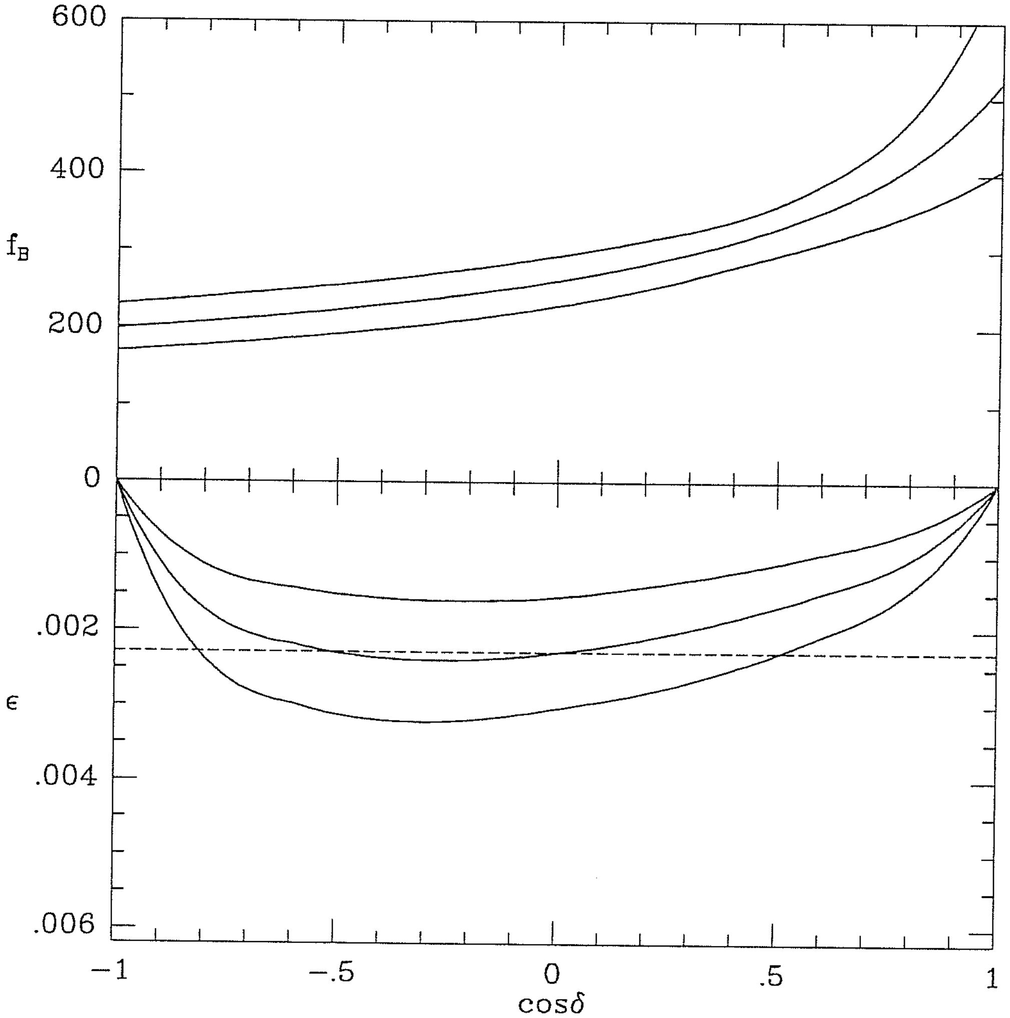

Figure 4.7: ϵ and f_B as functions of $\cos\delta$ for m_t =100 GeV

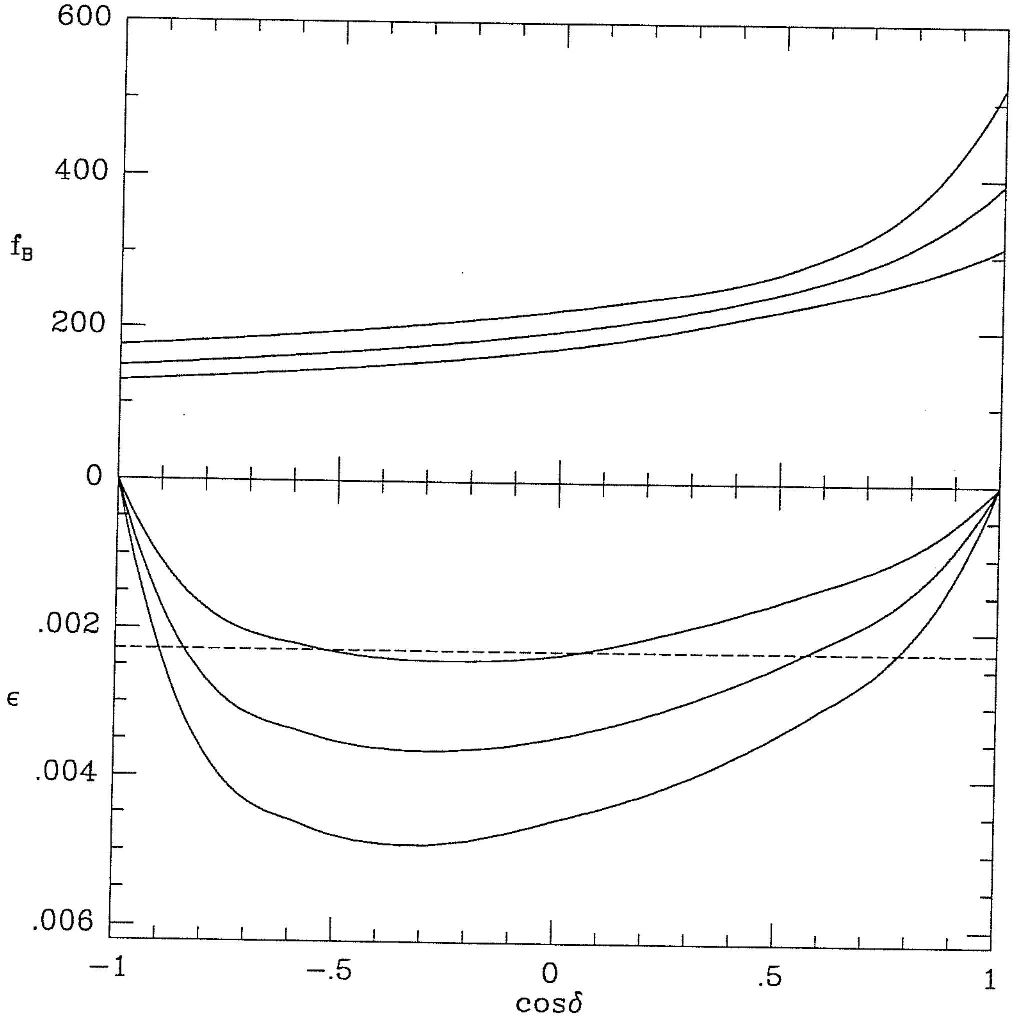

Figure 4.8: ϵ and f_B as functions of $\cos\delta$ for m_t =140 GeV

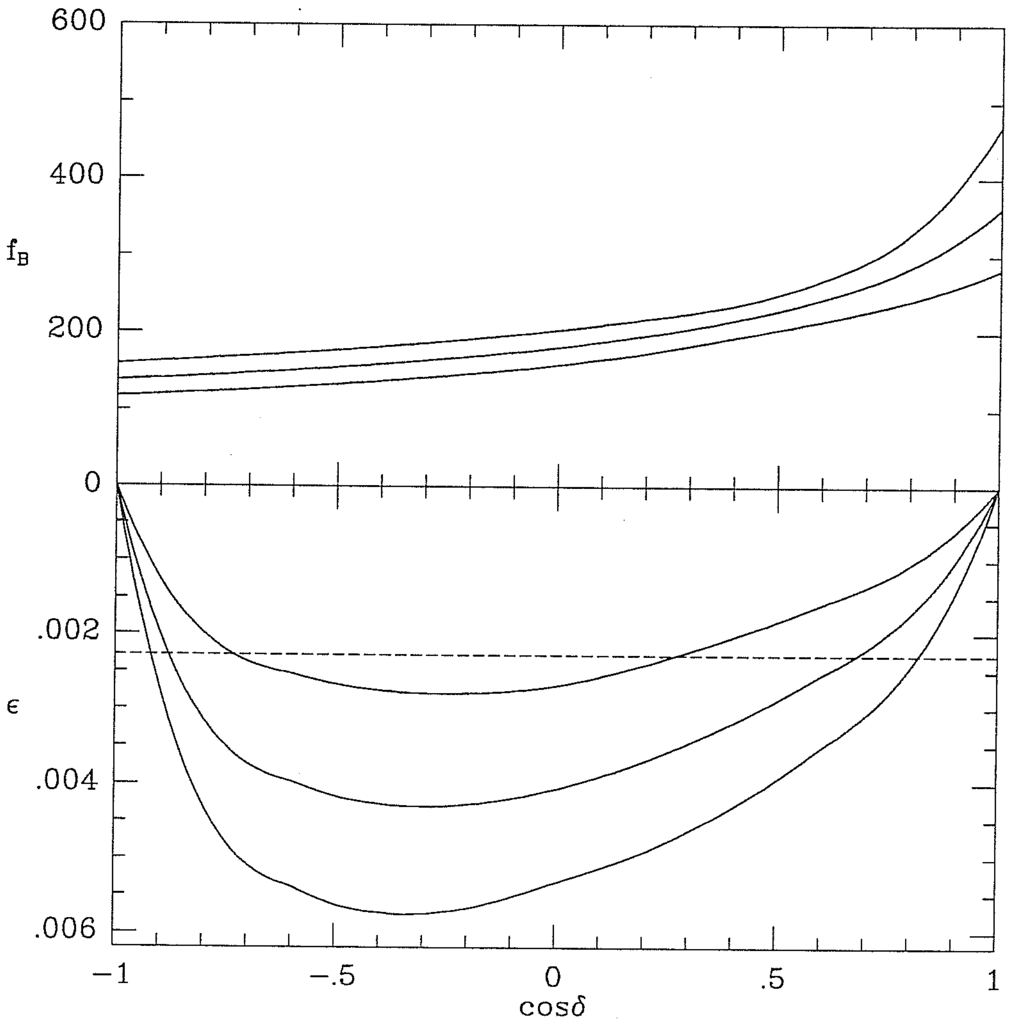

Figure 4.9: ϵ and f_B as functions of $\cos\delta$ for m_t =160 GeV

Conclusions

The study of non-leptonic weak decays and mixing processes has been and is also now very important for a deep understanding of the Standard Model. It does represent one of the most powerful tools in the analysis of the subtle points of the Standard Model and could also give advices of new physics.

In this context, we believe that the explanation of CP-violation is a fundamental step. As I have discussed in my thesis, the field is still wide open. Our future research program will be oriented in this direction. We would like to complete the theoretical analysis of CP-violation, by studying the case of the ϵ'/ϵ ratio. Our project is to clarify the theoretical features of the problem, reproducing a next-to-leading calculation of ϵ'/ϵ. In a second time, we would like to insert the found results in a numerical program which optimizes all the theoretical and experimental informations at hand. Using $K_0 - \overline{K}_0$ mixing, $B_0 - \overline{B}_0$ mixing and ϵ'/ϵ ratio, we should be able to better describe the physics of CP-violation and to better state our knowledge of the Standard Model.

Appendix A

Effective hamiltonian for $\Delta S = 1$ processes

I would like to give here some computative details about each of the steps trough which the final expression for $\mathcal{H}_{eff}^{(\Delta S=1)}$ is derived.

At **Step 1** evolved $A_q^{(\pm)}\left(\frac{M_W}{\mu}, g\right)$ coefficients are introduced. Namely, in a mass-independent minimal subtraction scheme, they must satisfy a RGE of the form:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma^{(\pm)}(g)\right) A_q^{(\pm)}\left(\frac{M_W}{\mu}, g\right) = 0 \tag{A.1}$$

where $\gamma^{(\pm)}(g)$ are the anomalous dimensions of the operators $O_q^{(\pm)}$ for $q = c, t$, which, calculated at one-loop give:

$$\begin{aligned}\gamma^{(+)}(g) &= \frac{g^2}{4\pi^2} + O(g^4) \\ \gamma^{(-)}(g) &= -\frac{g^2}{2\pi^2} + O(g^4)\end{aligned} \tag{A.2}$$

the standard solution of eq. . (A.1) gives:

$$A_q^{(\pm)}\left(\frac{M_W}{\mu}, g\right) = \exp\left(\int_g^{\bar{g}(M_W/\mu, g)} -\frac{\gamma^{(\pm)}(x)}{\beta(x)}dx\right) A_q^{(\pm)}\left(1, \bar{g}\left(\frac{M_W}{\mu}, g\right)\right) \tag{A.3}$$

where $A_q^{(\pm)}\left(1, \bar{g}\left(\frac{M_W}{\mu}, g\right)\right)$ can be replaced by their free-field values in a LL calculation. Setting:

$$-\frac{\gamma^{(\pm)}(x)}{\beta(x)} = \frac{2a^{(\pm)}}{x} + \text{ (finite terms at x=0)} \tag{A.4}$$

with

$$a^{(+)} = \frac{6}{33 - 2N_f} \qquad \text{and} \qquad a^{(-)} = \frac{-12}{33 - 2N_f} \tag{A.5}$$

we get:

$$A_q^{(\pm)}\left(\frac{M_W}{\mu},g\right)=\left[\frac{\bar{g}^2(M_W/\mu,g)}{\bar{g}^2(1,g)}\right]^{a(\pm)}A_q=\left[\frac{\alpha(M_W^2)}{\alpha(\mu^2)}\right]^{a(\pm)}A_q \tag{A.6}$$

At **Step 2** we start dealing separately with $O_c^{(\pm)}$ and $O_t^{(\pm)}$:

(•) $O_c^{(\pm)}$ are multiplicatively renormalized, as to say that:

$$\langle\,|O_c^{(\pm)}|\,\rangle=B^{(\pm)}\left(\frac{m_t}{\mu},g\right)\langle\,|O_c^{(\pm)}|\,\rangle' \tag{A.7}$$

where

$$\langle\,|O_c^{(\pm)}|\,\rangle'=\langle\,|O_c^{(\pm)}|\,\rangle(g',\mu,m_u',\cdots,m_b') \tag{A.8}$$

is the matrix element evaluated in the new effective theory, while $B^{(\pm)}\left(\frac{m_t}{\mu},g\right)$ satisfy now to the following RGE:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\gamma_t(g)m_t\frac{\partial}{\partial m_t}+\gamma^{(\pm)}(g)-\gamma'^{(\pm)}(g)\right)\cdot$$
$$\cdot B^{(\pm)}\left(\frac{m_t}{\mu},g\right)=0 \tag{A.9}$$

as is easy derivable. The solution of eq. . (A.9) can be found defining the running coupling constant through:

$$\ln y=\int_g^{\bar{g}(y,g)}\frac{1-\gamma_t(x)}{\beta(x)}dx \tag{A.10}$$

instead of

$$\ln y=\int_g^{\bar{g}(y,g)}\frac{dx}{\beta(x)} \tag{A.11}$$

where the two definitions agree because they have the same small-x behaviour. We get this way that:

$$B^{(\pm)}\left(\frac{m_t}{\mu},g\right)=\exp\left(\int_g^{\bar{g}(m_t/M_W,g)}\frac{\gamma^{(\pm)}(x)}{\beta(x)}dx\right)\cdot$$
$$\cdot\exp\left(\int_{-g}^{\bar{g}(1,\bar{g})}-\frac{\gamma'^{(\pm)}(x)}{\beta'(x)}dx\right)B^{(\pm)}(1,\bar{g})$$
$$=\left[\frac{\alpha(m_t^2)}{\alpha(\mu^2)}\right]^{-a(\pm)}\left[\frac{\alpha(m_t^2)}{\alpha'(\mu^2)}\right]^{a'(-)}B^{(\pm)}\left(\frac{m_t}{\mu},g\right) \tag{A.12}$$

where again the $B^{(\pm)}\left(\frac{m_t}{\mu},g\right)$ can be replaced by their free-field values $(=+1)$.

($\bullet$) $O_t^{(\pm)}$, on the other hand, require the introduction of a set of linearly independent operators $\{O_i\}_{i=1,\dots,6}$:

$$\begin{aligned} O_1 &= (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A} \\ O_2 &= (\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A} \\ O_3 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V-A} + \dots + (\bar{b}_\beta b_\beta)_{V-A}] \\ O_4 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V-A} + \dots + (\bar{b}_\beta b_\alpha)_{V-A}] \\ O_5 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V+A} + \dots + (\bar{b}_\beta b_\beta)_{V+A}] \\ O_6 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V+A} + \dots + (\bar{b}_\beta b_\alpha)_{V+A}] \end{aligned} \tag{A.13}$$

as a basis to expand the $O_t^{(\pm)}$ matrix elements on, as follows:

$$\langle\, |O_t^{(\pm)}|\, \rangle = \sum_i B_i^{(\pm)}\left(\frac{M_W}{\mu}, g\right) \langle\, |O^i|\, \rangle' + O\left(\frac{1}{m_t^2}\right) \tag{A.14}$$

They are not, generally, all multiplicative renormalizable and in order to solve the RGE's for the related coefficients:

$$\sum_j \left[\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_t(g)m_t\frac{\partial}{\partial m_t} + \gamma^{(\pm)}(g)\right)\delta_{ij} - \gamma_{ij}^{'T}(g')\right] B_j^{(\pm)}\left(\frac{m_t}{\mu}, g\right) = 0 \tag{A.15}$$

it's better to diagonalize the matrix $\gamma_{ij}^{'}$ of their anomalous dimensions. The eigenvectors of such diagonalized matrix will be operators: $\tilde{O}_i^{(\pm)}$, multiplicatively renormalizable, whose coefficients $\tilde{B}_i^{(\pm)}\left(\frac{m_t}{\mu}, g\right)$ are related to the old ones through the diagonalizing matrix $\mathbb{V}$ as follows:

$$\tilde{B}_i^{(\pm)}\left(\frac{m_t}{\mu}, g\right) = \sum_j \mathbb{V}_{ij}^{-1} B_j^{(\pm)}\left(\frac{m_t}{\mu}, g\right) \tag{A.16}$$

After having solved the RGE in such a basis, you "*rotate back*" to the old one and finally get for each coefficient that:

$$B_k^{(\pm)}\left(\frac{m_t}{\mu}, g\right) = \left[\frac{\alpha(m_t^2)}{\alpha(\mu^2)}\right]^{-a^{(\pm)}} \sum_i \mathbb{V}_i B_i^{(\pm)}(1, \bar{g}) \tag{A.17}$$

where

$$\mathbb{V}_i \equiv \sum_j \mathbb{V}_{kj}\left[\frac{\alpha(m_t^2)}{\alpha'(\mu^2)}\right]^{a_j'} \mathbb{V}_{ji}^{-1} \tag{A.18}$$

where $a_i^{'}$ is defined in completely analogy to $a^{(\pm)}$, but in the effective theory. As we can see, they are exactly the evolved coefficients inserted in the final expression for $\mathcal{H}_{eff}^{(\Delta S=1)}$ at this stage (2.47). Then, at **Step 3**, we are faced with the same situation, as already said, that is: ($\bullet$) $O_c^{(\pm)}$ are multiplicatively renormalized, so that:

$$B^{(\pm)}\left(\frac{m_b^{'}}{\mu^{'}}, g\right) = \left[\frac{\alpha^{'}(m_b^{'2})}{\alpha^{''}(\mu^2)}\right]^{a^{''\pm}} \left[\frac{\alpha^{'}(m_b^{'2})}{\alpha^{'}(\mu^2)}\right]^{a^{'\pm}} B^{(\pm)}(1, \bar{g}) \tag{A.19}$$

(•) $O_t^{(\pm)}$ require the introduction of a new set of linearly independent operators $\{P_n\}_{n=1,\ldots,6}$, which are obtained from the $\{O_i\}$s by dropping the b-quark dependence only, or explicitly:

$$\begin{aligned} P_1 &= (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A} \\ P_2 &= (\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A} \\ P_3 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V-A} + \ldots + (\bar{c}_\beta c_\beta)_{V-A}] \\ P_4 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V-A} + \ldots + (\bar{c}_\beta c_\alpha)_{V-A}] \\ P_5 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V+A} + \ldots + (\bar{c}_\beta c_\beta)_{V+A}] \\ P_6 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V+A} + \ldots + (\bar{c}_\beta c_\alpha)_{V+A}] \end{aligned} \tag{A.20}$$

such that

$$\langle\, |O^k|\,\rangle' = \sum_n \bar{C}_n\left(\frac{m_b'}{\mu}, g\right) \langle\, |P^n|\,\rangle'' \tag{A.21}$$

whose coefficients should solve the RGE:

$$\sum_{k,n}\left[\left(\mu\frac{\partial}{\partial\mu} + \beta'(g')\frac{\partial}{\partial g'} + \gamma_b'(g')m_b'\frac{\partial}{\partial m_b'}\right)\delta_{jk}\delta_{nm} + \right.$$
$$\left. + \gamma_{jk}'(g')\delta_{nm} - \delta_{jk}\gamma_{nm}''^{T}(g'')\right] C_k^n\left(\frac{m_b'}{\mu}, g\right) = 0 \tag{A.22}$$

Diagonalizing the two anomalous dimension matrices independently, the first one exactly as at *Step 2*, while the second one through a new matrix W, solving the "*rotated*" RGE for certain $\tilde{C}_k^n\left(\frac{m_b'}{\mu}, g\right)$ "*diagonal*" coefficients and finally "*rotating back*" to the original operator basis, we get the evolved coefficients in the form:

$$C_k^n\left(\frac{m_b'}{\mu}, g\right) = \sum_{i,l} \mathrm{V}_{ik} W_{nl} C_i^l\left(1, \bar{g}'\left(\frac{m_b'}{\mu}, g\right)\right) \tag{A.23}$$

where

$$\mathrm{V}_i \equiv \sum_j \mathrm{V}_{ij}\left[\frac{\alpha'(m_b'^2)}{\alpha'(\mu^2)}\right]^{-a_j'} \mathrm{V}_{jk}^{-1} \tag{A.24}$$

$$W_i \equiv \sum_m W_{nm}\left[\frac{\alpha'(m_b'^2)}{\alpha''(\mu^2)}\right]^{a_m''} W_{ml}^{-1} \tag{A.25}$$

which combined with the result at *Step 2* gives exactly the expression of the final $\mathcal{H}_{eff}^{(\Delta S=1)}$ at *Step 3* (2.50).

Finally we come at **Step 4**, where both $O_c^{(\pm)}$ and $O_t^{(\pm)}$ are not multiplicatively renormalizable anymore. A common operator basis $\{Q_r\}_{r=1,\ldots,6}$ is needed to expand both $\langle\, |P^n|\,\rangle''$ and $\langle\, |O_c^{(\pm)}|\,\rangle''$ matrix elements. They are obtained with the usual strategy , dropping the c-quark contribution from the $\{P_n\}$'s, but now they are not linearly independent anymore. Linear independence can be recovered only omitting Q_4, where explicitly:

$$\begin{aligned}
Q_1 &= (\bar{s}_\alpha d_\alpha)_{V-A}(\bar{u}_\beta u_\beta)_{V-A} \\
Q_2 &= (\bar{s}_\alpha d_\beta)_{V-A}(\bar{u}_\beta u_\alpha)_{V-A} \\
Q_3 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V-A} + (\bar{d}_\beta d_\beta)_{V-A} + (\bar{c}_\beta c_\beta)_{V-A}] \\
Q_4 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V-A} + (\bar{d}_\beta d_\alpha)_{V-A} + (\bar{c}_\beta c_\alpha)_{V-A}] \\
Q_5 &= (\bar{s}_\alpha d_\alpha)_{V-A}[(\bar{u}_\beta u_\beta)_{V+A} + (\bar{d}_\beta d_\beta)_{V+A} + (\bar{c}_\beta c_\beta)_{V+A}] \\
Q_6 &= (\bar{s}_\alpha d_\beta)_{V-A}[(\bar{u}_\beta u_\alpha)_{V+A} + (\bar{d}_\beta d_\alpha)_{V+A} + (\bar{c}_\beta c_\alpha)_{V+A}]
\end{aligned} \tag{A.26}$$

In such a basis:

(•) for $O_c^{(\pm)}$ matrix elements we get:

$$\langle\, |O_c^{(\pm)}|\,\rangle'' = \sum_r D_r^{(\pm)}\left(\frac{m_c''}{\mu}, g''\right)\langle\, |Q_r|\,\rangle''' \qquad \text{for } r = 1,2,3,5,6 \tag{A.27}$$

where the $D_r^{(\pm)}$ satisfy the RGE:

$$\begin{aligned}
\sum_r \Bigg[\left(\mu\frac{\partial}{\partial\mu}\right. &+ \left.\beta''(g'')\frac{\partial}{\partial g''} + \gamma_c''(g)m_c''\frac{\partial}{\partial m_c''} + \gamma''^{\pm}(g'')\right)\delta_{pr} - \\
&- \gamma_{pr}'''^{T}(g''')\Bigg] D_r^{(\pm)}\left(\frac{m_c''}{\mu}, g''\right) = 0
\end{aligned} \tag{A.28}$$

and result to be:

$$D_r^{(\pm)}\left(\frac{m_c''}{\mu}, g''\right) = \left[\frac{\alpha''(m_c''^2)}{\alpha''(\mu^2)}\right]^{-a''^{\pm}} \sum_p \mathbb{X}_{rp} D_p^{(\pm)}(1,\bar{g}) \tag{A.29}$$

where

$$\mathbb{X}_{rp} \equiv \sum_q \mathbb{X}_{rq}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{a_q'''} \mathbb{X}_{qp}^{-1} \tag{A.30}$$

(•) for $O_t^{(\pm)}$ matrix elements we have to consider that now:

$$\langle\, |P_n|\,\rangle'' = \sum_r D_n^r\left(\frac{m_c''}{\mu}, g''\right)\langle\, |Q_r|\,\rangle''' \qquad \text{for } r = 1,2,3,5,6 \tag{A.31}$$

where the $D_n^r\left(\frac{m_c''}{\mu}, g''\right)$ satisfy the RGE:

$$\begin{aligned}
\sum_{n,r} \Bigg[\left(\mu\frac{\partial}{\partial\mu}\right. &+ \left.\beta''(g'')\frac{\partial}{\partial g''} + \gamma_c''(g)m_c''\frac{\partial}{\partial m_c''}\right)\delta_{jr}\delta_{nm} + \\
&+ \gamma_{jr}''(g'')\delta_{nm} - \delta_{jk}\gamma_{nm}'''^{T}(g''')\Bigg] D_r^n\left(\frac{m_c''}{\mu}, g''\right) = 0
\end{aligned} \tag{A.32}$$

whose solution, after usual manipulations, results to be:

$$D_r^n\left(\frac{m_c''}{\mu},g''\right)=\sum_{n,p}W_{nl}\mathbb{X}_{rp}D_n^p\left(1,\bar{g}\right) \tag{A.33}$$

where

$$\begin{aligned} W_{nl} &\equiv \sum_m W_{nm}\left[\frac{\alpha''(m_c''^2)}{\alpha''(\mu^2)}\right]^{-a_m''} W_{ml}^{-1} \\ \mathbb{X}_{rp} &\equiv \sum_q \mathbb{X}_{rq}\left[\frac{\alpha''(m_c''^2)}{\alpha'''(\mu^2)}\right]^{a_q'''} \mathbb{X}_{qp}^{-1} \end{aligned} \tag{A.34}$$

and inserting the final form of the evolved coefficients in $\mathcal{H}_{eff}^{(\Delta S=1)}$ at *Step 3* (2.50) we obtained $\mathcal{H}_{eff}^{(\Delta S=1)}$ at *Step 4* as given in (2.53).

Appendix B

One-loop radiative corrections to the box-diagram

B.1 The box-diagram

In this section I would like to summarize the main features of the *box-diagram* amplitude calculation, which gives the correct expression of the *Inami-Lim* [26] functions, discussed in Chapter 3. If we consider a generic ($\Delta F = 2$)(with F a given flavour quantum number) process in the context of the electro-weak Standard Model, at the lowest order in the weak interactions and without QCD strong interaction corrections, there are four box-diagrams which could contribute, leading to the well-known structure of the effective hamiltonian $\mathcal{H}_{eff}^{(\Delta S=2)}$. They are reported in fig. (B.1) and correspond to the case in which:

(a) two W-vector bosons,

(b),(c) one W-vector boson and one Φ-Higgs boson,

(d) two Φ-Higgs bosons

are exchanged along the internal bosonic lines of the box, while the u_i and u_j quarks exchanged along the internal fermionic lines correspond to first elements of the weak fermionic doublets, that is the charge-2/3 elements of each doublet. These four amplitudes present some common computative features:

(i) In each case the amplitude for a given pair of internal quarks u_i and u_k is proportional to a function of the quark masses

$$F(m_{u_i}, m_{u_k}) \tag{B.1}$$

and the total amplitude is the sum over all the possible pairs of exchanged internal quarks, "weighted" by coefficients depending from the CKM matrix

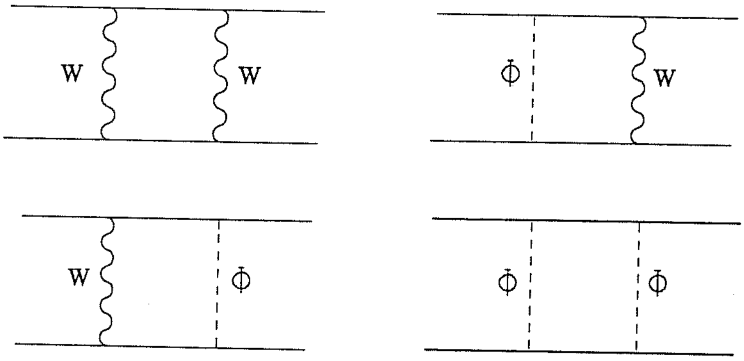

Figure B.1: The four possible box-diagrams

elements associated to each pair of quarks:

$$F = \sum_{j=1}^{N} \sum_{k=1}^{N} \lambda_j \lambda_k F(m_{u_i}, m_{u_k}) \tag{B.2}$$

where N is the number of generations and λ_j is defined as a product of CKM mixing matrix elements of the form:

$$\lambda_j = U_{js}^{*} U_{jd} \tag{B.3}$$

for a u_j quark exchanged along one internal fermion line.

(ii) This amplitude can be cast in a more suitable form by using the unitarity property of the CKM matrix itself, or the derived unitarity relation:

$$\sum_{j=1}^{N} \lambda_j = 0 \quad \longrightarrow \quad \lambda_u = -\sum_{j=2}^{N} \lambda_j \tag{B.4}$$

In fact, using (B.2) and (B.4) the amplitude can be written as the sum of four terms:

$$\begin{aligned} F \;&=\; \sum_{j=1}^{N} \sum_{k=1}^{N} \lambda_j \lambda_k \left[F(m_{u_i}, m_{u_k}) - F(m_{u_1}, m_{u_k}) \right. \\ &\quad \left. - F(m_{u_i}, m_{u_1}) + F(m_{u_1}, m_{u_1}) \right] \end{aligned} \tag{B.5}$$

fact this which turns out to be very uselful in practical calculations too, in order to retrieve the convergence of the integrals in it. By direct computation F results to be explicitly of the form:

$$F \propto \left(\bar{d}_\alpha \gamma_\mu (1 - \gamma_5) q_\alpha \right)^2 \sum_{i,j} \lambda_i \lambda_j M_W^2 m_i^2 m_j^2 \mathcal{I}(m_i^2, m_j^2) \tag{B.6}$$

where

$$\mathcal{I}(m_i^2, m_j^2) = \int dp^2 \frac{1}{(p^2+M_W^2)^2(p^2+m_i^2)(p^2+m_j^2)} \tag{B.7}$$

(iii) Correspondingly, the contributions to the effective hamiltonian (3.5)(with $\eta_1 = \eta_2 = \eta_3 = 1$) from the four diagrams in fig. (B.1), with exchange of quarks u_j and u_k can be written as:

$$\begin{aligned} \bar{E} &= 2\sum_{i=a}^{d}\Big[E_{\Box}^{(i)}(x_j,x_k) - E_{\Box}^{(i)}(x_1,x_k) \\ &\quad - E_{\Box}^{(i)}(x_j,x_1) + E_{\Box}^{(i)}(x_1,x_1)\Big] \end{aligned} \tag{B.8}$$

where $i = a, b, c$ and d and $x_j = m_{u_j}^2/M_W^2$. The point is, thus, to compute these four amplitudes and this is done by using standard algebra, obtaining at the end the following result:

$$\begin{aligned} E_{\Box}^{(a)}(x_j,x_k) &= -\frac{1}{2}g_1(x_j,x_k) \\ E_{\Box}^{(b)}(x_j,x_k) &= E_{\Box}^{(c)}(x_j,x_k) = \frac{1}{2}x_jx_kg_0(x_j,x_k) \\ E_{\Box}^{(d)}(x_j,x_k) &= -\frac{1}{8}x_jx_kg_1(x_j,x_k) \end{aligned} \tag{B.9}$$

where

$$\begin{aligned} g_1(x,y) &= \frac{1}{y-x}\left[\left(\frac{y}{y-1}\right)^2\ln y - \left(\frac{x}{x-1}\right)^2\ln x - \frac{1}{y-1} + \frac{1}{x-1}\right] \\ g_0(x,y) &= \frac{1}{y-x}\left[\frac{y}{(y-1)^2}\ln y - \frac{x}{(x-1)^2}\ln x - \frac{1}{y-1} + \frac{1}{x-1}\right] \end{aligned} \tag{B.10}$$

Summing the four contributions we get:

$$\begin{aligned} E_{\Box}(x_j,x_k) &\equiv \sum_{i=a}^{d} E_{\Box}(x_j,x_k) \\ = \frac{3}{4}g_1(x_j,x_k) &- \left[\frac{x_jx_k}{x_j-x_k}\left(\frac{1}{4} - \frac{3}{2}\frac{1}{x_j-1}\right) + \frac{7}{4}\frac{x_j}{(x_j-1)^2}\right] \cdot \\ &\cdot \ln x_j + \frac{7}{8} + \frac{7}{4}\frac{1}{x_j-1} + (x_j \leftrightarrow x_k) \end{aligned} \tag{B.11}$$

from which the explicit form of the Inami-Lim functions both for different and equal quark exchange are easly derived as:

$$\begin{aligned} E(x_j,x_k) &= -x_jx_k\left\{\frac{1}{x_j-x_k}\left[\frac{1}{4} - \frac{3}{2}\frac{1}{x_j-1} - \frac{3}{4}\frac{1}{(x_j-1)^2}\right]\ln x_j \right. \\ &\quad \left. + (x_j \leftrightarrow x_k) - \frac{3}{4}\frac{1}{(x_j-1)(x_k-1)}\right\} \end{aligned} \tag{B.12}$$

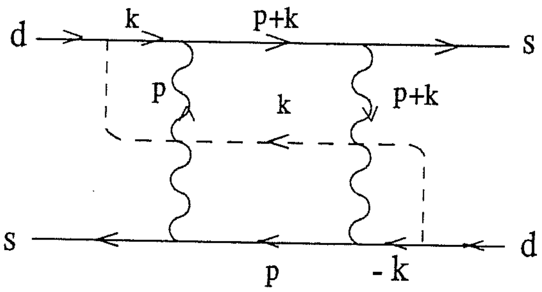

Figure B.2: Gluon exchanged between equal external quark legs

$$E(x_j, x_j) \equiv E(x_j) = -\frac{3}{2}\left(\frac{x_j}{x_j - 1}\right)^2 \ln x_j - x_j\left[\frac{1}{4} - \frac{9}{4}\frac{1}{x_j - 1} - \frac{3}{2}\frac{1}{(x_j - 1)^2}\right] \tag{B.13}$$

The original calculation by Inami and Lim was done in a generalized ξ- gauge, but, as one could expect, the final result given in (B.12) and (B.13) is gauge-independent.

B.2 Radiative QCD corrections

I will summarize here all the possible one-loop QCD radiative corrections to the box-diagram, noting when a possible (LL) term appears, in order to identify the anomalous dimension for the $O(\alpha_s)$ term in the expansion. In each case all the possible four boxes in fig. (B.1) should be examinated. However, in many cases the box-diagrams with one or two Higgs bosons, do not contribute, due to the coupling proportional to the involved fermion-mass.

(a) Gluon exchanged between two external equal quark legs

Let's consider before the box with double-W-boson exchange. We will work in the Feynman gauge for the weak interactions, while in the Landau gauge for the strong ones. With reference to fig. (B.2) using the properties already detailed in section (B), after carrying on some straightforward but lengthy algebra and introducing Feynman parameters we get the amplitude in the form:

$$\mathcal{A}_{(a)}(WW) = i\frac{g_W^2}{2}g_s^2 \sum_{i,j=2}^{3} \lambda_i\lambda_j m_i^2 m_j^2 \int \frac{dp^2}{(2\pi)^4}\frac{1}{(p^2 - m_j^2)(p^2 - M_W^2)}$$

$$\cdot \int dx\, x^2 \int \frac{d^4k}{(2\pi)^4} \left[\left(\frac{9}{2}x - \frac{3}{2} \right) k^2 + 3x(1-x)^2 p^2 \right]$$
$$\cdot \frac{1}{m_i^2 - M_W^2} \frac{1}{m_i^2} \left\{ \left(\frac{1}{[k^2 - M(m_i^2)]^4} - \frac{1}{[k^2 - M(m_i^2 = 0)]^4} \right) \right.$$
$$\left. - \; (m_i \leftrightarrow M_W) \right\} \; (\bar{s}_L \gamma^\mu d_L)(\bar{s}_L \gamma_\mu d_L) \tag{B.14}$$

where

$$M(m_i^2) = x(1-x)p^2 + (1-x)m_i^2 = p^2x^2 - (p^2 + m_i^2)x + m_i^2 \tag{B.15}$$

and

$$(\bar{s}_L \gamma_\mu d_L) = \frac{1}{2}(\bar{d}_\alpha \gamma_\mu (1 - \gamma_5) s_\alpha) \tag{B.16}$$

Transforming the integrals to the euclidean region and performing the inner ones, with the introduction of a suitable IR cut-off (which would naturally have been given by the external quark masses, if we had retained them) we get:

$$\begin{aligned} A_{(a)}(ww) &= \frac{g_W^4}{2} g_s^2 \sum_{i,j} \lambda_i \lambda_j m_i^2 m_j^2 \int \frac{dp^2}{(2\pi)^4} \frac{1}{(p^2 - m_j^2)(p^2 - M_W^2)} \frac{1}{m_i^2 - M^2} \frac{1}{m_i^2} \\ &\cdot \frac{1}{6} \left[\left(\mathcal{I}(p^2, m_i^2) - \mathcal{I}(p^2, 0) \right) - (m_i \leftrightarrow M_W) \right] \; (\bar{s}_L \gamma^\mu d_L)(\bar{s}_L \gamma_\mu d_L) \end{aligned} \tag{B.17}$$

where

$$\begin{aligned} \mathcal{I}(p^2, m^2) &= \int_0^1 dx \left(\frac{9x^3 - 3x^2}{-p^2x - (m^2 - p^2)x + m^2 + \mu^2} \right. \\ &+ \left. \frac{3x^3(1-x)^2p^2}{(-p^2x - (m^2 - p^2)x + m^2 + \mu^2)^2} \right) \\ &= \frac{3}{p^2} \left\{ -3 + \frac{m^2}{p^2} - \frac{m^4}{p^4} + \frac{m^6}{p^4(p^2 + m^2)} \right. \\ &+ \left. 2\frac{m^4}{p^2(p^2 + m^2)} \ln \frac{p^2 + m^2}{m^2} - 2\frac{p^2}{p^2 + m^2} \ln \frac{\mu^2}{p^2 + m^2} \right\} \\ &\simeq \frac{3}{p^2} \left\{ -4\frac{p^2}{m^2} - 2\frac{p^2}{m^2} \ln \frac{\mu^2}{m^2} \right\} \end{aligned} \tag{B.18}$$

when $p^2 \ll m^2$, which is the only region which gives (LL), and:

$$\mathcal{I}(p^2, 0) = \frac{3}{p^2} \left\{ -3 - 2\frac{\ln}{\mu^2} p^2 \right\} \tag{B.19}$$

Thus, in the previous "(LL)-region" the amplitude is:

$$A_{(a)}(ww) = -\frac{\alpha_s}{4\pi} C \sum_{i,j=2}^{3} \lambda_i \lambda_j m_i^2 m_j^2 M_W^2 \int dp^2 \frac{1}{(p^2 + m_j^2)(p^2 + M^2)} \frac{1}{m_i^2 - M_W^2}$$

$$\cdot \quad \frac{1}{m_i^2}\left[\left(-2\frac{p^2}{m_i^2}-\frac{p^2}{m_i^2}\ln\frac{\mu^2}{p^2}+\frac{3}{2}+\ln\frac{\mu^2}{p^2}\right)-(m_i\leftrightarrow M_W)\right] \tag{B.20}$$

with

$$C=-\frac{i}{16\pi^2}\frac{g_W^4}{2M_W^2}(\bar{s}_L\gamma^\mu d_L)(\bar{s}_L\gamma_\mu d_L)=-i\frac{G_F^2}{16\pi^2}M_W^2(\bar{s}_L\gamma^\mu d_L)(\bar{s}_L\gamma_\mu d_L) \tag{B.21}$$

The (LL) contribution is (in the $p^2 \ll m_i^2, M_W^2$ region):

$$\begin{aligned} \mathcal{A}^{(LL)}_{(a)}(WW) &= \frac{\alpha_s}{4\pi}C\sum_{i,j=2}^{3}\lambda_i\lambda_j m_i^2 m_j^2 M_W^4 \\ &\cdot \int dp^2 \frac{1}{(p^2+M_W^2)^2(p^2+m_i^2)(p^2+m_j^2)}2\ln\frac{\mu^2}{p^2} \end{aligned} \tag{B.22}$$

where we have accounted for the crossed graph too. Summing up all these (LL) contributions, (B.22) can be cast as in (3.83).

Then we have to consider the Higgs-boxes. The formalism is exactly the same and I will summarize here only the main results.

We consider first the case of the two boxes with a W-boson and a Higgs boson. The amplitude for the exchange of a gluon between equal external legs is now of the form:

$$\begin{aligned} \mathcal{A}_{(a)}(HW) &= -i\frac{g_W^2}{16M_W^2}\frac{g_s^2}{(4\pi)^2}\sum_{i,j=2}^{3}\lambda_i\lambda_j m_i^2 m_j^2\int\frac{dp^4}{(2\pi)^4}\frac{1}{(p^2-m_j^2)(p^2-M_W^2)} \\ &\cdot \frac{1}{m_i^2-M^2}\left[\left(\mathcal{I}(p^2,m_i^2)-\mathcal{I}(p^2,0)\right)-(m_i\leftrightarrow M_W)\right]\;(\bar{s}_L\gamma^\mu d_L)(\bar{s}_L\gamma_\mu d_L) \end{aligned} \tag{B.23}$$

where:

$$\begin{aligned} \mathcal{I}(p^2,m^2) &= -\frac{1}{p^2}\left\{3+\frac{p^2-m^2}{p^2}\ln\frac{\mu^2}{m^2}+\frac{m^4+p^4}{p^2(p^2+m^2)}\ln\frac{\mu^2}{m^2+p^2}\right. \\ &\left.-\frac{m^4+p^4}{p^2(p^2+m^2)}\ln\left(1+\frac{p^2}{m^2}\right)-2\frac{m^2}{p^2}\ln\left(1+\frac{p^2}{m^2}\right)+\frac{m^2}{m^2+p^2}\right\} \end{aligned} \tag{B.24}$$

Following the same procedure used for $\mathcal{A}_{(a)}(WW)$, we find that the (LL) contribution in this case is:

$$\begin{aligned} \mathcal{A}^{(LL)}_{(a)}(HW) &= \frac{\alpha_s}{4\pi}C\sum_{i,j=2}^{3}2\lambda_i\lambda_j\cdot \\ &\cdot \int dp^2\frac{m_i^2 m_j^2 p^2}{(p^2+M_W^2)^2(p^2+m_i^2)(p^2+m_j^2)}2\ln\frac{\mu^2}{p^2} \end{aligned} \tag{B.25}$$

which, by summing up (LL)'s, can be cast as in (3.84).

Finally, we have to consider the case of the box with two Higgs bosons. When a gluon is exchanged between equal external legs, the amplitudes reads:

$$\begin{aligned} A_{(a)}(HH) &= -i\frac{g_W^2}{16M_W^4}\frac{g_s^2}{(4\pi)^2}\sum_{i,j=2}^{3}\lambda_i\lambda_j m_i^2 m_j^2 \frac{1}{4}\int\frac{dp^2}{(2\pi)^4}\frac{1}{(p^2-m_j^2)(p^2-M_W^2)} \\ &\cdot \frac{1}{m_i^2-M^2}\left[\left(\mathcal{I}(p^2,m_i^2)-\mathcal{I}(p^2,0)\right)-(m_i\leftrightarrow M_W)\right]\ (\bar{s}_L\gamma^\mu d_L)(\bar{s}_L\gamma_\mu d_L) \end{aligned} \tag{B.26}$$

where:

$$\begin{aligned} \mathcal{I}(p^2,m^2) &= \frac{1}{p^2}\left\{2+\frac{m^2}{p^2}-\frac{m^4}{p^2(m^2+p^2)}\frac{1}{2}\left(1+\frac{m^4}{p^4}\right)\ln\frac{\mu^2}{m^2}\right. \\ &+\frac{m^4+p^4}{p^2(m^2+p^2)}\left(\ln\frac{\mu^2}{m^2+p^2}-\ln\frac{m^2+p^2}{m^2}\right)-\frac{1}{2}\frac{m^4}{p^4}\ln\frac{\mu^2}{m^2+p^2} \\ &\left.-\frac{5}{2}\frac{m^4}{p^4}\ln\frac{m^2+p^2}{m^2}\right\} \end{aligned} \tag{B.27}$$

Proceeding as in the previous two cases, we get for the (LL) contribution:

$$\begin{aligned} A_{(a)}^{(LL)}(HH) &= \frac{\alpha_s}{4\pi}C\sum_{i,j=2}^{3}2\lambda_i\lambda_j\frac{1}{4M_W^2} \\ &\cdot \int dp^2\frac{p^4 m_i^2 m_j^2}{(p^2+M_W^2)^2(p^2+m_i^2)(p^2+m_j^2)}2\ln\frac{\mu^2}{p^2} \end{aligned} \tag{B.28}$$

and (3.85) can be recovered, after (LL)'s have been summed up.

(b) Gluon exchanged between an external and an internal quark leg

To fix the point, consider the diagram in fig. (B.3). Its amplitude can be written as:

$$\begin{aligned} A_{(b)} &= -\frac{1}{3}\frac{g_W^4}{64}g_s^2\sum_{i,j}\lambda_i\lambda_j \\ &\cdot \int dp^2\frac{1}{(p^2+M_W^2)(p^2+m_i^2)(p^2+m_j^2)}\cdot\mathcal{I}(p^2,m_i^2,M_W^2) \end{aligned} \tag{B.29}$$

where $\mathcal{I}(p^2,m_i^2,M_W^2)$ is the contribution from the sub-diagram in fig. (B.4), which the only thing we actually have to compute, if we follow the analysis reported in section (3.3). Using the same formal machinery, we arrive at a sub-diagram amplitude of the form:

$$A_{(b)}^{sub} = -\frac{1}{(4\pi)^4}U_{jd}U_{is}^*\left\{3(\bar{q}_j\gamma_\mu(1-\gamma_5)d)(\bar{s}\gamma^\mu(1-\gamma_5)q_i)p^2\right. \tag{B.30}$$

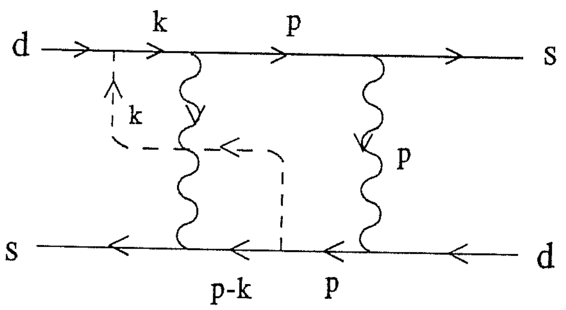

Figure B.3: Gluon exchanged between an internal and an external leg

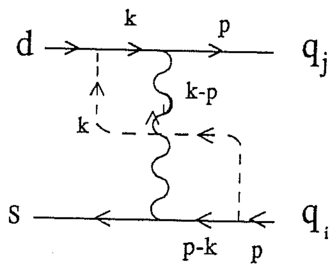

Figure B.4: Sub-diagram relative to the diagram in fig.(B.3)

$$
\begin{aligned}
&\cdot \quad \left[\frac{1}{(M_W^2-m_i^2)}\ln\left(\frac{p^2+m_i^2}{p^2+M_W^2}\right)+\frac{1}{p^2}\right.\\
&- \quad \left.\frac{m_i^4}{M_W^2-m_i^2}\frac{1}{p^4}\ln\left(1+\frac{p^2}{m_i^2}\right)+\frac{M_W^4}{M_W^2-m_i^2}\frac{1}{p^4}\ln\left(1+\frac{p^2}{M_W^2}\right)\right]\\
&+ \quad (\bar{q}_j\gamma_\mu(1-\gamma_5)\not{p}\gamma^\lambda d)(\bar{s}\gamma^\mu(1-\gamma_5)q_i)\not{p}\gamma_\lambda q_i)\\
&\cdot \quad \left[\frac{1}{p^4}-\frac{m_i^4}{M_W^2-m_i^2}\frac{1}{p^6}\ln\left(1+\frac{p^2}{m_i^2}\right)+\frac{M_W^4}{M_W^2-m_i^2}\frac{1}{p^6}\ln\left(1+\frac{p^2}{m_i^2}\right)\right]\\
&+ \quad m_i(\bar{q}_j\gamma_\mu(1-\gamma_5)\not{p}\gamma^\lambda d)(\bar{s}\gamma^\mu(1-\gamma_5)q_i)\gamma_\lambda q_i)\\
&\cdot \quad \left.\left[\frac{M_W^2}{M_W^2-m_i^2}\frac{1}{p^4}\ln\left(1+\frac{p^2}{m_i^2}\right)-\frac{m_i^2}{M_W^2-m_i^2}\frac{1}{p^4}\ln\left(1+\frac{p^2}{m_i^2}\right)\right]\right\}
\end{aligned}
\tag{B.31}
$$

Inserting it in the whole amplitude expression we get:

$$
\begin{aligned}
\mathcal{A}_{(b)} \quad = \quad & -i\frac{\alpha_s}{4\pi}\frac{1}{3}\sum_{i,j}\lambda_i\lambda_j m_i^2 m_j^2 M_W^4\int dp^2\frac{p^2}{(p^2+M_W^2)(p^2+m_i^2)(p^2+m_j^2)}\frac{1}{m_i^2}\\
\cdot \quad & \left\{3\frac{1}{M_W^2-m_i^2}\ln\left(\frac{p^2+m_i^2}{p^2+M_W^2}\right)+7\frac{1}{p^2}-7\frac{m_i^4}{M_W^2-m_i^2}\frac{1}{p^4}\ln\left(1+\frac{p^2}{m_i^2}\right)\right.\\
+ \quad & 7\frac{M_W^4}{M_W^2-m_i^2}\frac{1}{p^4}\ln\left(1+\frac{p^2}{m_i^2}\right)-2\frac{m_i^2M_W^2}{M_W^2-m_i^2}\frac{1}{p^4}\ln\left(1+\frac{p^2}{m_i^2}\right)\\
+ \quad & \left.2\frac{m_i^4}{M_W^2-m_i^2}\frac{1}{p^4}\ln\left(1+\frac{p^2}{m_i^2}\right)\right\}(\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha)(\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha)
\end{aligned}
\tag{B.32}
$$

and is the first term which gives the (LL) contribution in the various regions of integration. Both in the case in which $m_i^2 \ll M_W^2$ and $m_i^2 \gg M_W^2$, the (LL) term appear in the intermediate momentum range of integration, that is, respectively, when $m_i^2 \ll p^2 \ll M_W^2$ or when $M_W^2 \ll p^2 \ll m_i^2$. However, while in the first case the (LL) term:

$$R_i = \frac{1}{M_W^2-m_i^2}\ln\left(\frac{p^2+m_i^2}{p^2+M_W^2}\right) \tag{B.33}$$

is replaced to a good approximation by:

$$
\begin{aligned}
\frac{1}{M_W^2-m_i^2} &\rightarrow \frac{1}{p^2+M_W^2}\\
\ln\left(\frac{p^2+m_i^2}{p^2+M_W^2}\right) &\rightarrow \ln\frac{p^2}{M_W^2}
\end{aligned}
\tag{B.34}
$$

allowing us to retrieve in the amplitude the original right propagator structure and to clearly identify the anomalous dimension at this order; in the second case the approximations made in (B.34) are no longer justified, while new ones occur :

$$\frac{1}{M_W^2-m_i^2} \rightarrow -\frac{1}{m_i^2}$$

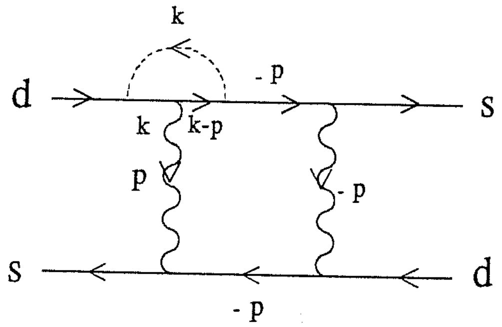

Figure B.5: Gluon exchanged across a W-vertex

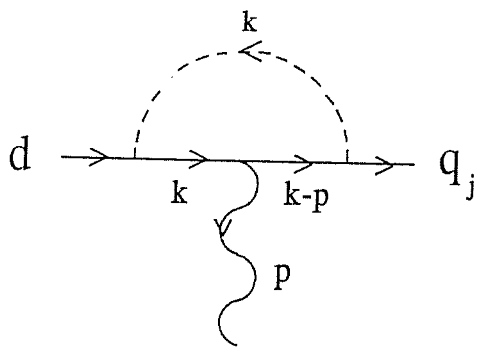

Figure B.6: Sub-diagram included in the diagram of fig.(B.5)

$$\ln\left(\frac{p^2+m_i^2}{p^2+M_W^2}\right) \quad\rightarrow\quad -\ln\frac{p^2}{m_i^2} \tag{B.35}$$

which do not reproduce anymore in the amplitude the correct propagator structure. We can retrieve it "by hand" dividing and multiplying for a W-propagator term, but this causes the (LL) term now obtained to be suppressed by a factor M_W^2/m_i^2 and hence to be negligible. This explain the result mentioned in section (3.3) as "decoupling" of the contributes due to gluons starting from an external leg and landing on an internal heavy quark leg.

The Higgs boxes can be computed in avery similar way and the result is exactly the one reported in (3.105).

(c) Vertex correction 1

Let's consider corrections of the type sketched in fig.(B.5). Also in this case, the real thing is the computation of the sub-diagram in fig.(B.6). Its amplitude results

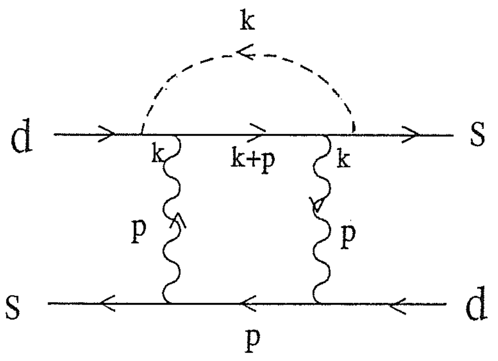

Figure B.7: Gluon exchanged crossing two W-vertices

to be:

$$\begin{aligned}
A^{sub}_{(c)} &= C_F \frac{g_W}{2\sqrt{2}} \frac{g_s}{(4\pi)^2}{}^2 U_{jd} \\
&\cdot \left\{ (4p^\mu \not{p} - p^2\gamma_\mu) \left[\frac{1}{2}\frac{1}{p^2} - \frac{m_j^2}{p^4} + \frac{m_j^2}{p^6} \ln\left(1 + \frac{p^2}{m_j^2}\right) \right] \right. \\
&+ \left. m_j (4p^\mu - \not{p}\gamma_\mu) \left[\frac{1}{p^2} - \frac{m_j^2}{p^4} \ln\left(1 + \frac{p^2}{m_j^2}\right) \right] \right\} (1-\gamma_5) d
\end{aligned} \tag{B.36}$$

When inserted in the computation of the box-diagram amplitude, this gives:

$$\begin{aligned}
A_{(c)} &= i \frac{g_W}{64} \frac{g_s^2}{(4\pi)^2} C_F \sum_{i,j} \lambda_i \lambda_j \int dp^2 \frac{1}{(p^2 + M_W^2)(p^2 + m_i^2)(p^2 + m_j^2)} \\
&\cdot \left\{ \bar{s}\gamma_\mu (1-\gamma_5) \left[-\not{p}(4p^\nu \not{p} - p^2\gamma^\nu) \left(\frac{1}{2}\frac{1}{p^2} - \frac{m_j^2}{p^4} + \frac{m_j^4}{p^6} \ln\left(1 + \frac{p^2}{m_j^2}\right) \right) \right. \right. \\
&+ \left. \left. m_j^2 (4p^\nu - \not{p}\gamma^\nu) \left[\frac{1}{p^2} - \frac{m_j^2}{p^4} \ln\left(1 + \frac{p^2}{m_j^2}\right) \right) \right] (1-\gamma_5) d \right\} \\
&\cdot \{ \bar{s}\gamma_\nu (1-\gamma_5)(-\not{p} + m_i)\gamma_\mu (1-\gamma_5) d \} = \ldots = \\
&= C \sum_{i,j} \lambda_i \lambda_j m_i^2 m_j^2 \int dp^2 \frac{1}{(p^2 + M_W^2)^2 (p^2 + m_i^2)(p^2 + m_j^2)} \\
&\cdot (\bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha)(\bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_\alpha) + \ldots
\end{aligned} \tag{B.37}$$

and no (LL) contribution is present. This will also be true for WH- or HH-boxes.

(d) vertex correction 2

Another possibility to exchange a gluon along a fermion line is the one shown in fig.(B.7) where a gluon is exchanged crossing two W-vertices. As before, we are interested in the sub-diagram shown in fig. (B.8) whose amplitude is:

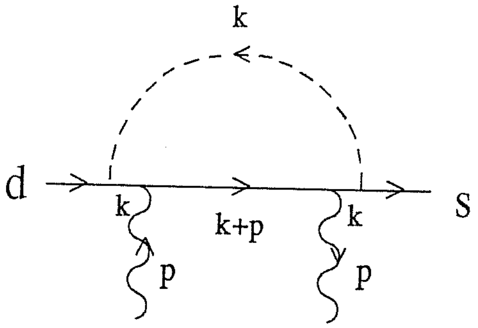

Figure B.8: Sub-diagram included in the diagram of fig(B.7)

$$\begin{aligned}
\mathbb{A}^{sub}_{(d)} &= C_F \frac{g_W^2}{4}\frac{g_s^2}{(4\pi)^2}\lambda_j \left\{[\bar{s}(\not{p}\gamma^\nu\gamma^\mu + \gamma^\mu\not{p}\gamma^\nu + \gamma^\mu\gamma^\nu\not{p})(1-\gamma_5)d]\right. \\
&\cdot \left[\frac{1}{p^2} - \frac{m_j^2}{p^4} + \frac{m_j^4}{p^6}\ln\left(1+\frac{p^2}{m_j^2}\right)\right] + \left[\bar{s}\gamma^\lambda\not{p}\gamma^\mu\not{p}\gamma^\nu\not{p}\gamma_\lambda(1-\gamma_5)d\right] \\
&\cdot \left.\frac{1}{2}\left[\frac{1}{6}\frac{1}{p^4} - \frac{m_j^2}{p^6} - 4\frac{m_j^4}{p^8} + 3\frac{m_j^4}{p^8}\ln\left(1+\frac{p^2}{m_j^2}\right) - 4\frac{m_j^6}{p^{10}}\ln\left(1+\frac{p^2}{m_j^2}\right)\right]\right\}
\end{aligned} \quad (B.38)$$

After inserting it in the main amplitude, we get:

$$\begin{aligned}
\mathbb{A}_{(d)} &= C_F \frac{g_W^4}{4}\frac{g_s^2}{(4\pi)^2}\sum_{i,j}\lambda_i\lambda_j\int dp^2 \frac{1}{(p^2+M_W^2)^2(p^2+m_i^2)} \\
&\cdot \left\{\frac{4}{3} - 2\frac{m_i^2}{p^4} + 4\frac{m_j^4}{p^4} + \frac{m_j^6}{p^6}\ln\left(1+\frac{p^2}{m_j^2}\right)\right\} \\
&\cdot (\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha)(\bar{s}_\alpha\gamma_\mu(1-\gamma_5)d_\alpha) + \ldots
\end{aligned} \quad (B.39)$$

and again no (LL) contribution is present. The same will hold also for Higgs-containing boxes.

(e) Gluon exchanged between different external quark leg

As we can see from fig. (B.9) the only part of the diagram to be computed is the sub-diagram in fig. (B.10) whose amplitude is:

$$\begin{aligned}
\mathbb{A}^{sub}_{(e)} &= -\frac{g_W^2}{24}\frac{g_s^2}{(4\pi)^2}U_{jd}U^\star_{is} \\
&\cdot \left\{-\left[-\frac{1}{p^2} + \frac{M_W^2}{p^4}\ln\left(1+\frac{p^2}{M_W^2}\right)\right](\bar{q}_j\gamma^\mu(1-\gamma_5)d)(\bar{s}\gamma_\mu(1-\gamma_5)q_i)\right. \\
&+ \frac{1}{2}\left[\frac{1}{2}\frac{1}{p^4} + 3\frac{M_W^2}{p^6}\ln\left(1+\frac{p^2}{M_W^2}\right) - 3\frac{M_W^4}{p^8}\ln\left(1+\frac{p^2}{M_W^2}\right)\right]
\end{aligned}$$

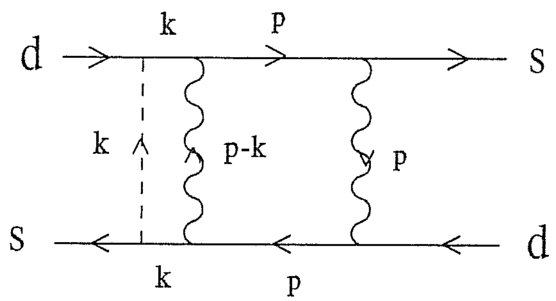

Figure B.9: Gluon exchanged between different external quark legs

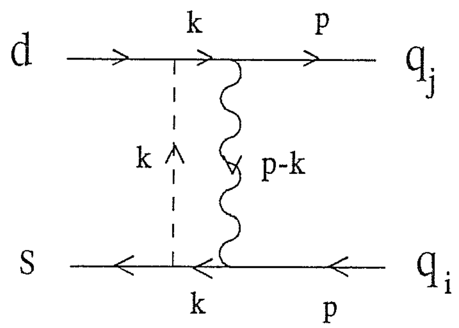

Figure B.10: Sub-diagram included in the diagram of fig.(B.9)

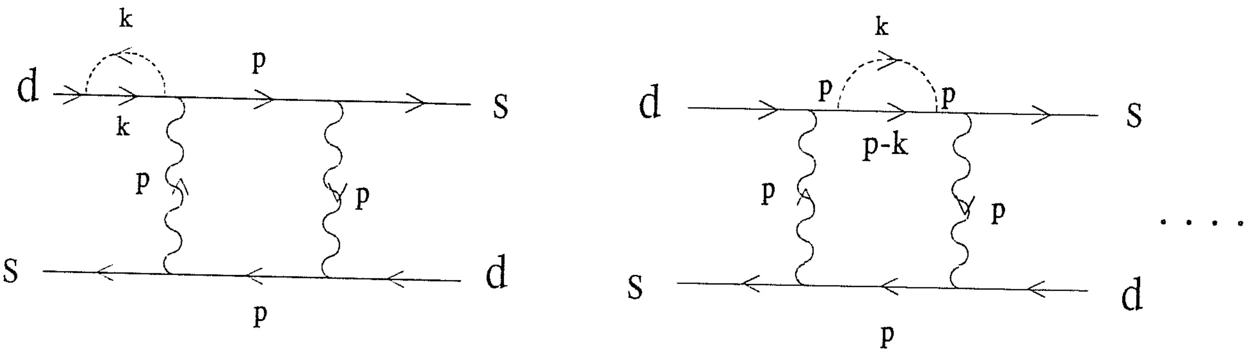

Figure B.11: Possible self-energy corrections on the internal and external quark legs

$$\cdot \quad (\bar{q}_j\gamma^\mu \not{p} \gamma^\lambda(1-\gamma_5)d)(\bar{s}\gamma_\lambda \not{p} \gamma_\mu(1-\gamma_5)q_i)\} \tag{B.40}$$

which, inserted in the main graph gives:

$$\begin{aligned} \mathbb{A}_{(e)} &= i\frac{g_W^4}{16}\frac{g_s^2}{(4\pi)^2}\sum_{i,j}\lambda_i\lambda_j\int dp^2\frac{1}{(p^2+M_W^2)^2(p^2+m_i^2)(p^2+m_j^2)} \\ &\cdot \left\{2\frac{M_W^2}{p^2} - M_W^2 p^2 \ln\left(1+\frac{p^2}{M_W^2}\right) - 2\frac{M_W^4}{p^4}\ln\left(1+\frac{p^2}{M_W^2}\right)\right\} \\ &\cdot (\bar{s}\gamma^\mu(1-\gamma_5)d)(\bar{s}\gamma_\mu(1-\gamma_5)d) \end{aligned} \tag{B.41}$$

with no (LL) contribution.The same for the Higgs-including boxes.

(f) Quark self-energies on the quark legs

All the internal and external quark legs can give a self-energy term, as shown in fig. (B.11). Clearly, we need to compute only the quark self-energy at this order, see fig. (B.12), which gives:

$$\begin{aligned} \mathbb{A}^{sub}_{(f)} &= g_s^2\left\{\frac{i\not{p}}{(4\pi)^2}\left(4+\frac{4}{3}\frac{m_j^2}{p^2}\ln\left(1+\frac{p^2}{m_j^2}\right)\right)\right. \\ &+ \left. 4m_j\frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma_E-\int dx \ln\left[x(1-x)p^2+(1-x)m_j^2\right]+O(\epsilon^2)\right)\right\} \end{aligned} \tag{B.42}$$

The second term is responsible for mass corrections, while the first one has to be considered in our main amplitude. Clearly, when the self-energy is inserted on an external leg, there is no correction at all. Otherwise you get:

$$\begin{aligned} \mathbb{A}_{(f)} &= -i\frac{g_W^4}{16}\frac{g_s^2}{(4\pi)^2}\sum_{i,j}\lambda_i\lambda_j m_i^2 m_j^2\int dp^2\frac{1}{(p^2+M_W^2)^2(p^2+m_i^2)(p^2+m_j^2)} \\ &\cdot \left\{1-\frac{1}{3}\frac{m_j^2}{p^2}\ln\left(1+\frac{p^2}{m_j^2}\right)\right\}(\bar{s}\gamma^\mu(1-\gamma_5)d)(\bar{s}\gamma_\mu(1-\gamma_5)d) \end{aligned} \tag{B.43}$$

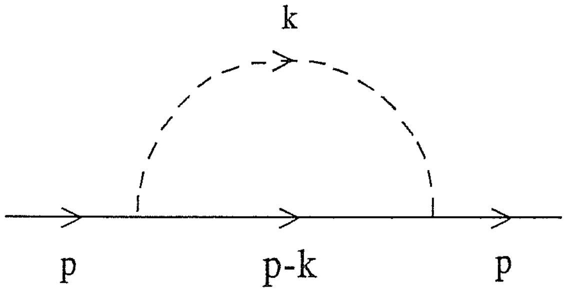

Figure B.12: Quark self-energy

with no (LL) contribution, as for the Higgs-boxes too.

This summarizes all the possible QCD radiative corrections to the box-diagram, where only in case *(a)* and *(b)* we find (LL) contributions.

Appendix C

The value of $\alpha(\mu^2)$ at scale μ^2

I have described in Chapter 4 how we have implemented in a numerical program the theoretical results obtained in the study of $F_0 - \overline{F}_0$ mixing processes for a theory with a very heavy top quark. As I have stressed in that discussion, we have improved the results given by Datta et al. [14], because our computation of the QCD-corrected *Inami-Lim* functions $F(x_t)$ and $F(x_c, x_t)$ takes into account all the thresholds of the effective theory. We have realized this improvement, by introducing a specific subroutine in the numerical program, the subroutine "*alpha*". This subroutine calculates the value of the strong coupling constant $\alpha(\mu^2)$ at a given scale μ^2, after having satisfied the whole set of matching conditions for the thresholds above the given scale μ^2.

Here following, I will describe how the subroutine "*alpha*" works. We have taken the usual definition of the strong coupling constant:

$$\alpha(\mu^2) = b\,\frac{1}{\ln\frac{\mu^2}{\Lambda^2}} \tag{C.1}$$

where $\Lambda = \Lambda_{QCD}$, and "b" is the inverse of the first coefficient of the strong coupling constant β-function, that is :

$$\frac{1}{b} = 11 - \frac{2}{3}N_f \tag{C.2}$$

with N_f the number of flavours.

In the effective hamiltonian approach to the description of $F_0 - \overline{F}_0$ mixing processes, the UV cut-off of the theory is made coinciding step by step with the physical masses of the theory, namely M_W^2, m_t^2, m_b^2 and m_c^2. When the UV cut-off coincides with a certain physical mass m_i^2, all the physical particle with mass larger than m_i^2 are integrated out of the theory. Thus, moving from M_W^2 down to m_c^2, we pass from a six flavour theory to a three flavour theory. Each physical mass m_i^2 is a threshold, at which effective theories with a different number of flavours should match, that is should give equal physical predictions. In particular, we have to impose that the strong coupling constant $\alpha(\mu^2)$ predicted by one theory or by the other is the same. The expression of $\alpha(\mu^2)$ at a given scale μ^2 depends on the number of matching conditions imposed.

The subroutine "*alpha*" starts selecting if a given scale μ^2 is greater than m_c^2 or not. If $\mu^2 \ll m_c^2$, than the strong coupling constant to be considered is simply :

$$\alpha_3(\mu^2) = b_3 \frac{1}{\ln \frac{\mu^2}{\Lambda^2}} \tag{C.3}$$

where $\alpha_3(\mu^2)$ and b_3 must be calculated in an effective theory with only three flavours.

On the other hand, if $\mu^2 \gg m_c^2$, the subroutine "*alpha*" consider three possible cases: $m_c^2 \ll \mu^2 \ll m_b^2$ or $m_b^2 \ll \mu^2 \ll m_t^2$ or $\mu^2 \gg m_t^2$.

If $m_c^2 \ll \mu^2 \ll m_b^2$, the right strong coupling constant is $\alpha_4(\mu^2)$, evaluated in a four quark effective theory. The subroutine calculates $\alpha_4(\mu^2)$ as follows:

$$\alpha_4(\mu^2) = \frac{1}{C_1 + \frac{1}{b_4} \ln \frac{\mu^2}{m_c^2}} \tag{C.4}$$

where the constant C_1 is defined as:

$$C_1 = \frac{1}{b_3} \ln \frac{m_c^2}{\Lambda^2} \tag{C.5}$$

To obtain (C.4), you have only to impose that $\alpha_3(\mu^2)$ and $\alpha_4(\mu^2)$ match, at the m_c^2 threshold. This means that:

$$\frac{1}{\alpha_3(m_c^2)} = \frac{1}{\alpha_4(m_c^2)} \tag{C.6}$$

In order to satisfy this matching condition, suppose that:

$$\frac{1}{\alpha_4(\mu^2)} = \frac{1}{b_4} \ln \frac{\mu^2}{\Lambda^2} + K_1 \tag{C.7}$$

for a given constant K_1. Imposing (C.6), one finds that:

$$K_1 = \left(\frac{1}{b_3} - \frac{1}{b_4}\right) \ln \frac{m_c^2}{\Lambda^2} \tag{C.8}$$

and inserting (C.8) into (C.7), one finds exactly the expression for $\alpha_4(\mu^2)$ given in (C.4).

If $m_b^2 \ll \mu^2 \ll m_t^2$, the strong coupling constant is $\alpha_5(\mu^2)$, evaluated in a five flavour effective theory. Again, one takes $\alpha_5(\mu^2)$ to be of the form:

$$\frac{1}{\alpha_5(\mu^2)} = \frac{1}{b_5} \ln \frac{\mu^2}{\Lambda^2} + K_2 \tag{C.9}$$

for a given constant k_2. By imposing the matching condition:

$$\frac{1}{\alpha_4(m_b^2)} = \frac{1}{\alpha_5(m_b^2)} \tag{C.10}$$

one obtains:

$$K_2 = C_1 + \frac{1}{b_4}\ln\frac{m_b^2}{m_c^2} - \frac{1}{b_5}\ln\frac{\mu^2}{m_b^2} \tag{C.11}$$

and finally:

$$\alpha_5(\mu^2) = \frac{1}{C_2 + \frac{1}{b_5}\ln\frac{\mu^2}{m_b^2}} \tag{C.12}$$

where:

$$C_2 = C_1 + \frac{1}{b_4}\ln\frac{m_b^2}{m_c^2} \tag{C.13}$$

Finally, if $\mu^2 \gg m_t^2$, the subroutine gives $\alpha_6(\mu^2)$, the strong coupling constant in a six flavour effective theory. In this case, again, you start from:

$$\frac{1}{\alpha_6(\mu^2)} = \frac{1}{b_6}\ln\frac{\mu^2}{\Lambda^2} + K_3 \tag{C.14}$$

and imposing that:

$$\frac{1}{\alpha_5(m_t^2)} = \frac{1}{\alpha_6(m_t^2)} \tag{C.15}$$

you get respectively:

$$K_3 = C_2 + \frac{1}{b_5}\ln\frac{m_t^2}{m_b^2} - \frac{1}{b_6}\ln\frac{m_t^2}{\Lambda^2} \tag{C.16}$$

and

$$\alpha_6(\mu^2) = \frac{1}{C_3 + +\frac{1}{b_6}\ln\frac{\mu^2}{m_t^2}} \tag{C.17}$$

with:

$$C_3 = C_2 + \frac{1}{b_5}\ln\frac{m_t^2}{m_b^2} \tag{C.18}$$

In this way, $\alpha(\mu^2)$ can be recursively calculated at a given scale μ^2.

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