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Anyon, Chern-Simons Theory, and FQHE

Thesis Submitted for the Degree of
Magister Philosophiae

Candidate:
Li Dingping

Supervisor:
Prof. R. Iengo

Academic Year 1990/91

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Abstract

Anyon has fractional spin and statistics. In recent years, anyon physics becomes a rather hot topic. For example, high temperature superconductivity maybe relates with some anyon mechanism. However in this thesis, we will mainly concentrate on anyon in fractional quantum hall effect. we try to present some topics as clear as possible. The organization is as follows. In chapter 1, we will give an introduction to fractional spin and statistics. In chapter 2, some anyon models are considered. The spin-statistics relation is emphasized. In chapter 3, we consider Chern-Simons Maxwell theory on torus, and take limit to get pure Chern-Simons theory. In chapter 4, we discuss Ginzburg-Landau theory of fractional quantum hall effect and some related topics. Finally in chapter 5, we discuss anyon and fractional quantum hall effect on sphere.

INDEX

1. Fractional Spin and Statistics

As we know that the fundamental particles in nature are bosons or fermions, like quarks are fermions, photons and gluons are bosons. Even composite particles are bosons or fermions, for example, protons are fermions, π mensons are bosons. That the relativistic quantum field theory in $3+1$ can only have bosons and fermions is a well known fact. It seems that there exist only bosons and fermions, and so the statistics can only be bosonic (symmetric) and fermionic (anti symetric) Statistics. Also spin should be only integer or half integer according to the spin- statistics relation theorem. Then people suppose that in other dimension of space time, maybe there also only exist fermions and bosons. But unexpected things happened. The possibility of fractional statistics was found theoretically $2+1$ in space time. The quasi particles of fractional quantum hall effect (FQHE) have fractional statistics. It is the first physical example in which we find fractional statistics. The particles with fractional statistics and spin are called anyons [3]. Nowadays, it seems the fact there are fractional spin and statistics in $2+1$ dimension of space time is a little bit trivial. In the history of physics, such kinds of events are not rare. Like the Berry phase, although it is very fundamental, and should be found much earlier, but only was caught eight years ago by Berry.

The unexpected fact that fractional statistics and spin can appears in $2+1$ dimension space time also is very fundamental, and it is really a very important new phenomena, has a wide applications to physics. Below, we will give a careful scrutiny of those fundamental facts.

Let us brief recall some history. It is Leinaas and Myrheim who first discussed the possibility of fractional statistics in $2+1$ [52]. But unfortunately this paper did not absorb much notice in that time. Then Goldin, Menikoff and Sharp by using differmorphism group and current algebra also found the possibility of fractional statistics [53]. One of key figures in anyon business is certainly Frank Wilczek, who found it independently, and moreover discussed the possible generalization of spin and statistics relation in $2+1$ space time [70,3]. Wilczek and Zee [71] first discovered that in non linear sigma model in $2+1$ space time with Hopf term, the soliton excitations have fractional spin and statistics, and spin-statistics relation is standard one. Also in this paper, they found that Chern-Simons (Hopf term) was related to fractional spin and statistics. Really, the particles coupled with $U(1)$ Chern- Simons theory are anyons. Wu formulated an anyon theory directly from the lagrangian and by using path integral and braid group language (first reference in [4]). One of important contribution is from D. P. Arovas, R. Schrieffer and F. Wilczek who demonstrated the quasi particles in Fractional Quantum Hall Effect obey fractional statistics and also have fractional charge by using Berry phase method, confirmed the assumption of Halperin and Laughling [77]. This gave people a strong believe anyons are really physical relevant, not academic game. After [77], the research about anyon is intensified. Anyon theory now has a wide application to physics, for example FQHE, possible anyon superconductivity, *etc.* The material about this subject is so wealth, we can not mention all cover all important articles about this subject. For recent review, see for example, [2] [25] *etc.* In following section and chapter, we will only focus on some subjects which is interesting to us.

1.1. Fractional Statistics in Quantum Mechanics

First we exam the possible value of spin in $2+1$ dimension. Before do that, let one recall some basic facts in quantum mechanics. In quantum mechanics, symmetries are given by the unitary (or anti-unitary, like time reversal symmetry) transformations on Hilbert space. But the representation of this transformation need not to be a faithful group representation in usual sense, but can be up to a unitary phase to the faithful representation. This is due to the fact that the state in Hilbert space is defined uniquely only up to a unitary phase. We call this kind representation as a ray representation.

In $3+1$ dimension, one have $SO(3)$ rotational symmetry. The ray representation of $SO(3)$ is the $SU(2)$ ($SU(2)$ is the simple connected covering group of

$SO(3)$). So the spin not only can be integer (if one only has $SO(3)$ symmetry), but also can be half integer because we should consider ray representation of $SO(2)$, $SU(2)$ which contains both integer and half integer [27]. And also we have spin–statistics relation theorem in this case. Corresponding to half integer spin, we have antisymmetric statistics. We call this kind particle as fermion. Corresponding to integer spin, we have symmetric statistics (about statistics in $3+1$ space time, see also following discussion). We call it boson. If the dimension of the space is bigger than 3, also there are only fermions and boson.

In the dimension 1 space, the spin is described by helicity, and it can be arbitrary number. This is because there is no axis to rotate, and so there is no symmetry operator as in $3+1$ dimension space time we have $SO(3)$ rotational symmetry, in $2+1$ we have $SO(2)$ rotational symmetry. Also there is a problem about the definition of statistics in 1 dimension, since the particles will pass each other when they are exchanged. If we assume the particles are boson with the scattering amplitude of two particles as -1, then also we can equally say we have fermions with the scattering amplitude 1. So the intrinsic spin in 1 dimension is mixed with local interaction. In 1 space dimension, Coleman-Mandelstam construction of bosonization can be used to transformed fermion to boson. So in 1 dimension, there is no spin-statistics relation.

Fortunately, interesting things happen in 2 dimension space. In 2 dimension space, the group of rotation is $SO(2)$. The ray representation is a real line. So we can have arbitrary spin. Also in following discussion we will show the statistics in 2 dimension can be a arbitrary real number. This is exactly what we want if the spin-statistics relation is still valid.

What is most relevant for discussing statistics? The answer is the topology of coordinate space. That the topology maybe relate with spin and statistics is not new fact. For example, some topological solitons can be fermions even the fundamental field is boson. Topological solitons appear due to nontrivial topology. And particles in the presence of monopole can change their spin and statistics. Indeed the most convenient way to demonstrate the possibility of fractional statistics in 2 space dimension is to argue in terms of topology of coordinate space.

For following discussion, first a brief recall about path integral in multiply connected space is needed. For details, the book by Schulman is excellent [26]. Assuming we have a multiply connected space X. So the path integral is given by

$$K(x_2, t_2; x_1, t_1) = \sum_{x(t_1)=x_1, x(t_2)=x_2, \alpha} c_\alpha e^{iS_\alpha}, \tag{1.1}$$

α corresponds to a homotopy class which the path from x_1 to x_2 lies. The c_α is the corresponding weight of every homotopy class. This generalization of path integral is justified as one can see that, if we just only sum path within one homotopy class, the amplitude satisfies Schrödinger equation. Schrödinger equation does not depend on which homotopy path we choose. Moreover we can give a restriction to α. In quantum mechanics, we should have a rather general properties

$$< A|B >= \sum_n < A|n >< n|B >, \tag{1.2}$$

n is a complete set of wave function at some middle time. In the language of path integral, it means

$$K(x_3, t_3; x_1, t_1) = \int dx_2 K(x_3, t_3; x_2, t_2) K(x_2, t_2; x_1, t_1). \tag{1.3}$$

From this equation, one gets

$$c_\alpha c_\beta = c_{\alpha o \beta}. \tag{1.4}$$

The function of c only depends on the homotopy class of path. One can define the multiplication of path in the sense of homotopy concept. Then function c will be a representation of homotopy group if the path is closed. Still the function is not unique defined. If we have a $c_\alpha(x_1, x_2)$, then we can define another function which have the same group property. Assuming we have a close path p_1 in point x_1 , close path p_2 in point x_2, we can define a new function c'

$$c'_\alpha(x_1, x_2) = c(p_1)c_\alpha(x_1, x_2)[c(p_2)]^{-1}, \tag{1.5}$$

which still satisfies the properties we want (representation of homotopy group). Using this new defined function, we find path integral changed up to a number $c(p_1)[c(p_2)]^{-1}$. We should restrict this number as a unitary phase because wave function in quantum mechanics should be defined uniquely up to a phase. The condition will be hold if $c_\alpha(x_1, x_2)$ is a unitary phase. To summary, we have

$$|c_\alpha| = 1, \;\; c_\alpha c_\beta = c_{\alpha o \beta}. \tag{1.6}$$

One point we want to point out is that relation of those phases are pure quantum mechanical. In classical level, we only can pick the stational solution of the action within one homotopy path. There is no way to relate the weight of different homotopy path in classical mechanics.

Now let us consider the statistics of indistinguishable particles which is the our main concern. In most all standard text book, the way to deal the indistinguishable particles is using the coordinates R^{3N} for N particles (now we consider 3 space dimension following those standard books). And because the Hamiltonian commutes with the exchange operator, we can find eigenvectors which also form the representation of symmetric group. If eigenvectors form a identity representation of permutation group, then the particle is boson. If eigenvectors form a total antisymmetric representation of permutation group, then the particle is fermion. But from reasoning above, we can also have other statistics, not only fermionic and bosonic statistics which are special representation of permutation group. So this kind of approach to identical particle is not good.

Another way to deal the indistinguishable particles was given first by Laidlaw and De Witt [100]. Consider a set of point in R^{3N}, $y = (x_1, x_2, \cdots, x_N)$. From

R^{3N}, we exclude the points in which any pair of x_i, x_j coincide. The correct way to describe the N indistinguishable particle is described as

$$X = [R^{3N} - coincidence\, points]/S_N. \tag{1.7}$$

The reason why the coincidence points can be excluded is, if we allow two particles occupy same point, the statistics must be bosonic. Because other statistics exclude the possibility of a pair particles occupying the same position. This is one of possibility that particle is boson. In order to discuss other possibility, we can consider space X. One finds that the homotopy group of X is S_N from the standard book of topology. The exchange of 2 particles represents a closed path on space X. The corresponding phase c of that closed path determines the statistics of the particle. The one dimension representation of permutation group can only be 2 case. One is identity representation, $c_\alpha = 1$, α is in some element of homotopy class of the corresponding permutation element in space X. This actually corresponds bosonic statistics we consider above. Another is $c_\alpha = -1$ if the α corresponds to odd permutation. So in $3+1$ dimension space time, we only have bosonic and fermionic statistics. In the space dimension larger than 3, one will get same conclusion.

However in 2 space dimension, we will have coordinate space,

$$X^2 = [R^{2N} - coincidence\, points]/S_N. \tag{1.8}$$

A very naive and elegant argument about the homotopy group of X is given by R. Mackenzie and F. Wilczek in [27]. One finds the homotopy group of this space is braid group. So c_α only needs to be a representation of braid group which is larger than permutation group. The one dimensional representation of braid group can be an arbitrary unitary phase. This means we can have fractional statistics. This is why in 2 space dimension, exotic statistics appear. From the fact the spin can also be fractional, so one conclude in $2+1$ dimension, that it is possible to have fractional statistics and spin in $2+1$ space time dimension. Moreover the representation of braid group can be also matrix which will give nonablien statistics. After those happy results, now we would like to discuss some special examples.

1.2. Do really fractional spin and statistics exist?

cyons

In two seminal papers by F. Wilczek [70,3], he gave the first concrete example of anyon with fractional spin and statistics (which is called by Goldhaber as cyon). However, there are some delicate point about the definition of spin of the particle in dimension 2 space [33]. It is until very recently it was resolved by A. S. Goldhaber and R. Mackenzie, and T. H. Hansson, *etc* [32]. It is worth to clarify those points because even now there are some misunderstanding and confusing in some articles.

Before going to these subtlety point, we present original argument by Wilczek[70,3]. Following Wilczek, we consider a particle interacting with magnetic field. Then the lagrangian is given

$$L = \frac{1}{2}m\sum_i(\frac{dx^i}{dt})^2 + ex^i \cdot A^i, \tag{1.9}$$

with $i = 1, 2$, A is a magnetic potential of flux perpendicular to the surface. We assume the flux Φ concentrated on the origin. A is

$$A^i = -\frac{\epsilon^{ij}}{2\pi}\frac{x^j}{r}\Phi. \tag{1.10}$$

Suppose the flux is turned on slowly from zero, we use the Faraday law, an induced electric field should be

$$E^i = \frac{1}{2\pi}\epsilon^{ij}x^j\frac{d\Phi(t)}{dt}. \tag{1.11}$$

The mechanical angular momentum

$$l = m(x^1 \frac{dx^2}{dt} - x^2 \frac{dx^1}{dt}). \tag{1.12}$$

It will be changed due to the electric force

$$\frac{dl}{dt} = x^1 F^2 - x^2 F^1 = x^1 eE^2 - x^2 eE^1 = -\frac{e}{2\pi}\frac{d\Phi(t)}{dt}. \tag{1.13}$$

If the initial l is integer, the final l will be

$$l = n - \frac{e\Phi}{2\pi},\ n \subset integer, \tag{1.14}$$

Φ is final value of flux. So we find the mechanical angular momentum can be fractional in 2 dimension space. Wilczek take this mechanical momentum as the spin of composite object which includes the particle and magnetic flux. Wilczek called this kind of particle tightly bounded with flux as anyon in [3]. Another way to see the above result is using singular gauge transformation.

$$A^{i'} = A^i - \frac{\partial \Lambda}{\partial x^i} = 0, \tag{1.15}$$

With

$$\Lambda = \frac{\varphi}{2\pi}\Phi. \tag{1.16}$$

Now the wave function becomes singular

$$\Psi' = e^{-i\Lambda}\Psi. \tag{1.17}$$

In this case, we have no magnetic flux, but the wave function does not satisfy the periodic condition as the particle goes around the origin. Now we consider 2 anyons. Let one anyon goes around another anyon. In ordinary QED, the phase is

$$\Psi(R,\theta;r,\varphi+2\pi)=e^{4\pi ie\Delta}\Psi(R,\theta;r,\varphi), \tag{1.18}$$

Where $\Delta=-\frac{e\Psi}{2\pi}$. Now if two anyons are identical particle, then when we change two anyons, wave function will get a phase (keep in mind that we still work in *singular gauge*)

$$\Psi(R,\theta;r,\varphi+\pi)=e^{2\pi i\Delta}\Psi(R,\theta;r,\varphi), \tag{1.19}$$

R,θ is the center coordinate of two particles in polar coordinate, and r,φ is relative coordinate. To summary the results the spin of anyon is given by

$$s=l=n-\frac{e\Phi}{2\pi},\ n\subset integer. \tag{1.20}$$

The statistics is given by

$$\Delta=-\frac{e\Phi}{2\pi}. \tag{1.21}$$

The relation of spin and statistics is a generalization of ordinary relation in which we have fermion with half spin and boson with integer spin. So really we get a consistent picture of spin and statistics in 2 dimension space.

The story does not end here. There is a delicate point as we said above. Jackiw and Redlich [33] show that the mechanical and canonical momentum are different. What was calculated by Wilczek is mechanical momentum. Wilczek said this mechanical momentum is also the spin of anyon, but Jackiw and Redlich however had a different opinion. The full understanding of this dispute was obtained in [32].

We know mechanical momentum is

$$L_m=\mathbf{r}\times m\mathbf{v}. \tag{1.22}$$

The canonical momentum is

$$L_c = \mathbf{r} \times [m\mathbf{v} + e\mathbf{A}] = \mathbf{r} \times \mathbf{p}. \tag{1.23}$$

In quantum mechanics, $\mathbf{p}$ becomes $\mathbf{p} = -i\nabla$. It is easy to see the canonical momentum is not gauge invariant, but the mechanical angular momentum is gauge invariant. The magnetic field also carries angular momentum, which is given by (gauge invariant part)

$$L_e = -\int d^2x x^i E^i B. \tag{1.24}$$

The total gauge invariant angular momentum of particle and magnetic field is

$$L_t = L_m + L_e, \tag{1.25}$$

which is not conserved Noether current. The conserved current is

$$L = L_t + L_s = L_s + \int d^2x \partial_i (E^i (x^1 A^2 - x^2 A^1)). \tag{1.26}$$

The difference between of conserved Noether current and non conserved gauge invariant current is a surface term L_s which can not be omitted due to the long range property of electric magnetic field. In the case of present problem, it is showed by Jackiw and Redlich, $L = L_c$, with L_c as canonical momentum. So the "total" angular momentum L remains integer, because L_c is an integer (in non anyon gauge). Also it is easy to show $L_e = 0$ if the flux only concentrate in origin. Now assuming initially there is no flux. Then, we slowly turn the flux on in origin. The total angular momentum L should be conserved, which is integer because $L = L_c$, and L_c is an integer. Then slowly, some of L is carried by radiated field to the infinity (surface term). But after some time, this term will not be relevant in local physics (physics around the flux). On the other hand, we can say the relevant angular momentum will be the gauge invariant angular momentum, $L_m + L_e$ which can be fractional (although L_c is integer, but $L_c = L_m + L_e + L_s$ in the present problem, so $L_m + L_e$ and L_s can be fractional). So it is appropriate to consider $L_m + L_e$ as the spin of anyon (however in the cyon model $L_e = 0$) [32].

Anyon from braid group representation

We can write the wave function of a group particles which satisfy fractional statistics as two parts (in anyon gauge)

$$\Phi = c(x_1, \cdots, x_N)\Phi_{bosonic}(x_1, \cdots, x_N). \tag{1.27}$$

The bosonic part satisfies the bosonic statistics. Only c will responsible for the fractional statistics. One of simple representation of c on the plane is

$$c(x_1, \cdots, x_N) = e^{2i\alpha \sum_{i<j} \Theta(x_i - x_j)}, \tag{1.28}$$

$\Theta(x_i - x_j)$ is the angle of two points x_i and x_j. When two particles exchanged, and we assuming there is no other particle inside the exchange trajectory, then the wave function changes by a phase $e^{2\alpha i\pi}$ (We assume x_j is turned anti clock around x_i, and $j > i$). Of course the exchange phase depends on the exchange trajectory. The Hamiltonian of the anyon is

$$H = \sum_i \frac{1}{2m} \mathrm{p}_i^2. \tag{1.29}$$

We make a singular gauge transformation to transform the wave function to bosonic wave function. The Hamiltonian is changed to

$$H = \sum_i \frac{1}{2m} (\mathrm{p}_i + \mathrm{A}(x_i))^2, \tag{1.30}$$

with

$$\mathrm{A}(x_i) = 2\alpha \sum_{j \neq i} \frac{\partial}{\partial x_i} \Theta(x_i - x_j), \tag{1.31}$$

and the corresponding magnetic field of A is $B(x_i) = 4\pi\alpha \sum_{j\neq i} \delta^2(x_i - x_j)$. And the corresponding lagrangian is given by

$$
\begin{aligned}
L &= \sum_i \frac{1}{2m}(\frac{dx_i}{dt})^2 - 2\alpha \sum_{i \neq j} \frac{dx_i}{dt}\frac{\partial}{dx_i}\Theta(x_i - x_j) \\
&= \sum_i \frac{1}{2m}(\frac{dx_i}{dt})^2 - 2\alpha \sum_{i<j} \frac{d}{dt}\Theta(x_i - x_j)
\end{aligned}
\tag{1.32}
$$

This lagrangian was first obtained by Y. S. Wu [4].

In above model, we have particles with fractional statistics. One of interesting question is that what is the spin of one particle. Now we will discuss this problem

Following Thouless and Y. S. Wu [34], we consider indistinguishable particles in 2 space dimension with spin s. And the particles also obey fractional statistics as described above.

$$
\Phi = c(x_1, \cdots, x_N)\Phi_{bosonic}(x_1, \cdots, x_N)\chi_s, \tag{1.33}
$$

here an extra χ_s is the spin part of the wave function. And

$$
c(x_1, \cdots, x_N) = e^{i\frac{\theta}{\pi}\sum_{i<j}\Theta(x_i - x_j)}. \tag{1.34}
$$

Consider p particles as a cluster. We exchange two cluster, the exchange phase is $e^{ip^2\theta}$ which will give us statistics of the cluster. Now we consider the spin of cluster. In order to get the spin of cluster, one rotates the cluster 2π around some point. We get a phase $e^{2i\theta}$ from every of the $\frac{p(p-1)}{2}$ pair of the particle (the term Θ). We also get a phase $e^{2\pi is}$ from the spin part of the wave function of every particle. So final phase we get is

$$
e^{ip(p-1)\theta + 2\pi ips}. \tag{1.35}
$$

So the spin of the cluster is

$$
S = \frac{p(p-1)\theta}{2\pi} + ps + integer. \tag{1.36}
$$

If the particle has ordinary spin-statistics relation, that is $s = \frac{\theta}{2\pi}$, then also the cluster obeys ordinary spin-statistics relation as can be easily verified, or verse versa. Really in many interesting system, for example some quantum field theory, in which the particles have the generalized spin-statistics [93]. However about anyon spin in fractional quantum hall effect, the spin statistics relation is a delicate issue. We will discuss it later.

Now we come back to the model of equation (1.32). Now we calculate the angular momentum of the whole system. The gauge invariant angular momentum is (we write in bosonic gauge, but the result is independent on gauge)

$$J = \sum_i \mathbf{x_i} \times (\mathbf{p_i} + \mathbf{A}(\mathbf{x_i})). \tag{1.37}$$

It can be shown

$$J = \alpha N(N-1) + integer. \tag{1.38}$$

So it is the spin of whole system (now the cluster is whole system). So compare (1.38)with (1.36), in the model we consider, the spin is zero, and the spin-statistics will not be ordinary one. This is expected because the model is not derived from any quantum field theory. One way to get the right spin-statistics relation is we can add extra term, which is self linking term of the path of the particle (if a particle goes around and comes back, this term will gives the self linking number of the closed path). It is Polyakov in a profound paper [35] who first realized such kind of term can change the statistics and spin of particle in the quantum mechanics frame work. Nowadays, it is easy to understand it as we know that Chern-Simons term can transmute both spin and statistics of the particle. When one integrates Chern-Simons term, one gets the linking term between different particles which will transmute the statistics of the particles, and self linking term of the same particle which will transmute the spin of the particles. However the Polyakov's work has some tiny dispute (the second reference in [32]). We will come back to this point later.

In above model, a statistic flux is bounded to every particle. One notices the difference between the statistic flux and real magnetic flux in QED, if both fluxes have same value, then one finds from the discussion of last subsection and present subsection, that the statistic parameter of the particle bounded with statistic flux is only half the statistics parameter of the particle bounded with same value of magnetic flux. We will come this point later.

1.3. Chern-Simons Term, Fractional Spin and Statistics

from path integral point of view

In a beautiful paper, Wilczek and Zee [71] found the fractional statistics and spin is related to Chern-Simons term. It was further elaborated in [77] that the particles interacting with Chern-Simons gauge field will have fractional statistics. Now we consider the action of point particles

$$S = S_m + S_i + S_c, \tag{1.39}$$

with

$$S_c = \frac{k}{4\pi} \int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{1.40}$$

and

$$S_i = \int d^3x j^\mu A_\mu. \tag{1.41}$$

The current of particles is

$$j^\mu = \sum_i \int ds \delta^{(3)}(x - x_i) \frac{dx^\mu}{ds}, \tag{1.42}$$

where s parametrizs the trajectories of the particles, for example time.

Because Chern-Simons term is quadratic in A. So field A can be exactly integrated out to get an effective action. We will work on the theory which always lives on the plane except some special mention. Then we get effective non local action

$$S_e = -i\ln\int DA^\mu e^{i(S_i+S_c)} \\ = \frac{\pi}{2k}\int d^3x\int d^3y j^\mu K_{\mu\nu}(x,y)j^\nu, \tag{1.43}$$

$K_{\mu\nu}$ is given by

$$K_{\mu\nu}(x,y) = -\frac{1}{2\pi}\epsilon_{\mu\lambda\nu}\frac{(x-y)^\lambda}{|x-y|^3}. \tag{1.44}$$

We can write it explicitly

$$S_e = -\frac{1}{4k}\sum_{i,j}\int dsds'\frac{dx_i^\mu}{ds}\epsilon_{\mu\lambda\nu}\frac{(x_i^\lambda(s)-x_j^\lambda(s'))}{|x_i(s)-x_j(s')|^3}\frac{dx_j^\nu}{ds'}. \tag{1.45}$$

We need to do regularization when $i = j$ [35]. This term is responsible for the fractional spin of particle as it was first pointed out by Polyakov. This self linking term can change the spin of the particles. This kind of spin is not like Pauli spin. And this self linking term is a topological term which will not change the equation of motion, like θ vacuum term, but can change the spin of the particle (for a comprehensive reference, see [36]).

Now we will brief summarize the result. In this formalism, we can easy understand how fractional spin not only fractional statistics appears from Chern-Simons Theory (see the paper [25] for a extensive review about this interesting topic, However some coefficients in effective action we do not agree with him). One gets the effective action

$$S_e = \sum_i S_i + \sum_{i\neq j} S_{i,j}. \tag{1.46}$$

The linking term $S_{i,j}$ is given by

$$S_{i,j} = -\frac{1}{2k}\int dt\frac{d}{dt}\Theta(x_i - x_j) + I_g. \tag{1.47}$$

I_g is a term which depends also on the boundary condition. For close path (or periodic boundary condition) it disappears. It can be shown [25] this term will not change the statistics and spin of particle which is our main interests. This can be understood that this term disappears if we choose close path (periodic boundary condition), but the spin and statistics of particle should be independent on the boundary condition. The other term in $S_{i,j}$ is the same term we get from the braid group approach to anyon, which is responsible for fractional statistics. The self linking term S_i responsible for the fractional spin of the particle is

$$S_i = \frac{\pi}{k} w. \tag{1.48}$$

To calculate S_i, one needs regularization. One regularization is to thick the curve to a ribbon. There are different ways to thick it, or to frame it. However there exists a framing independent regularization, so S_i is framing independent (for example see [25]). One finds w is the writhing number of the curve. The writhe number can be written as

$$w = l - \tau. \tag{1.49}$$

The l is self linking number which is always an integer for closed path, and τ is the torsion of the path. Although l and τ is framing dependent, but the difference of them is framing independent. Moreover the torsion term is

$$\tau = \int \frac{ds}{2\pi} \epsilon^{\mu\nu\lambda} e_\nu n_\lambda \dot{n}_\mu. \tag{1.50}$$

The s is the parametrization of the path, for example, $ds^2 = dx^\mu dx^\nu g_{\mu\nu}$. The dot means the derivative with s. And n, e are

$$n^\mu = \frac{\dot{e}^\mu}{|\dot{e}|};\; e^\mu = \frac{\dot{x}^\mu}{|\dot{x}|}. \tag{1.51}$$

It can be proved that τ will give the fractional spin for particles, which is $\frac{1}{2k}$ (for example, [35] for the special case $k = 1$, and [25] which includes arbitrary k). Also one sees from $S_{i,j}$ the statistics parameter is $\frac{1}{k}$ (which means one changes two particle, the phase is $e^{\frac{i\pi}{k}}$. Moreover spin-statistics relation is standard one, it fits to the general frame work in [34].

from canonical point of view

We consider action, $S = S_m + S_i + S_c$, with $S_c = \frac{k}{4\pi}\int d^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$, $S_i = \int d^3x j^\mu A_\mu$ as last subsection. The equations with respect to A is

$$\epsilon^{\mu\nu\lambda} F_{\nu\lambda} = -\frac{2\pi}{k} j^\mu. \tag{1.52}$$

In plane, the gauge field can be exactly solved from equation of motion. In high genus, there exists extra topological component which also contribute to dynamics. At present we only consider the theory on the plane. Now one solves gauge field in Coulomb gauge [37] $\partial_i A_i = 0$. To write the equation of motion more explicitly

$$\epsilon^{0ij} \partial_i A_j = -\frac{2\pi}{k} j^0, \tag{1.53}$$

and

$$\epsilon^{i0j}(\partial_0 A_j - \partial_j A_o) = -\frac{2\pi}{k}. \tag{1.54}$$

Then A_i is solved as

$$A_i = \frac{2\pi}{k} \epsilon_{0ij} \partial_j \frac{1}{\partial^2} j^0. \tag{1.55}$$

Insert A_i to equation (1.54), and also use current conservation condition $\partial_\mu j^\mu$, one obtains the equation for A_0

$$\epsilon^{ij} \partial_j A_0 = -\frac{2\pi}{k}(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2}) j^j. \tag{1.56}$$

The solution is

$$A_0 = \frac{2\pi}{k} \epsilon_{0ij} \frac{1}{\partial^2} \partial_i j^j. \tag{1.57}$$

Now assuming we have nonrelativistic point particles. Then the currents are

$$j^0(x,t) = \sum_i \delta^2(x - x_i),\ j^i = \sum_l \frac{dx_l^i}{dt} \delta^2(x - x_l). \tag{1.58}$$

We substitute the solutions back into the action. After some calculation (assuming we have periodic boundary condition for simplicity in order to avoid boundary term), one finds the same effective action term $S_{i,j}$, an interaction term between different particles, which we also get from the path integral approach. However, there is also a problem of regularization if we consider self interaction term. This is related to the spin of particles as we know it from path integral approach.

How to understand factor "half" physically?

We mention before that different from ordinary QED, in Chern-Simons theory the statistics parameter is half the statistics parameter of ordinary QED if particles are bounded with the same gauge flux. Moreover, even in $2+1$ QED, if there is heavy fermion, when the heavy fermion field is integrated which means we only consider energy scale much smaller than the mass of the heavy fermion, then we can get Chern-Simons term. Then in this case, every particle bounded with some magnetic flux which will change the spin and statistics of the particle. One will ask what is the statistics and spin in the presence of both Chern-Simons and Maxwell term. It was found no matter how small Chern-Simons term, the statistics parameter is half of the one without Chern-Simons term. It seems very strange. The complete understanding of this problem was obtained just three years ago in [32,38]. In this subsection, we give a naive argument to understand this subtlety for the Chern-Simons theory without Maxwell term following Wen and Zee in [38]. We will come back to this problem in next chapter when we consider $2+1$ QED in which both Chern-Simons and Maxwell term present.

Now we define effective current in the theory considered in above subsection

$$j_e^\mu = j^\mu + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda, \tag{1.59}$$

which is the coefficient of $\partial_\mu f$ in the lagrangian when we do gauge transformation A_μ to $A_\mu + \partial_\mu f$. The term of gauge part in effective current may be thought as the effect of vacuum polarization in the presence of charge particles [38]. The effective charge from effective current is

$$q = \int d^2x (j_0 + \frac{k}{8\pi}\epsilon_{0\nu\lambda}F^{\nu\lambda}) = \frac{1}{2}q_0. \tag{1.60}$$

The q_0 is the original charge of the particle. Now let us turn one particle anti clock around another one. we get a phase

$$e^{iq\int dx^\mu A_\mu} = e^{iq(-\frac{2\pi q_0}{k})} = e^{\frac{-i\pi q_0^2}{k}}. \tag{1.61}$$

The essential point is that we should consider the effective current due to vacuum polarization. However there is another phase due to the effect of field generated by moving particle on the static one according to Faraday theorem. It equals above phase we just calculated. Then when we exchange two particle, we get half of above total phase. The final phase for exchanging two particles is $e^{\frac{-i\pi q_0^2}{k}}$. However, in ordinary QED (there are no heavy fermion, and so there is not vacuum polarization), if the particle bounds with same gauge flux, $-\frac{2\pi q_0}{k}$, when we exchange two particles, then the phase is $e^{\frac{-2\pi i q_0^2}{k}}$ (see the first subsection in the last section). The statistics parameter of Chern-Simons theory is the half value of statistics parameter of cyon if bounded with same gauge flux. One needs very careful about the value of the spin and statistics. And one must be careful about how those gauge flux comes from.

2. Some Anyon Models

In the first chapter, basically we only consider anyons in quantum mechanical frame work. In present chapter, some models in the second quantized form will be considered. They will include nonrelativistic quantum field theory of anyons, $2+1$ QED, and nonlinear sigma model.

2.1. Quantum Field Theory of Anyon with only Chern-Simons term

We consider bosonic particles interacting with Chern-Simons term. For fermionic case, however see a recent paper by Semenoff [39]. The action we consider [84] is

$$\begin{aligned} S = \int d^3r[i\Psi^\dagger D_0\Psi - \frac{1}{2m}|\mathbf{D}\Psi|^2 \\ + \frac{g}{2}(\Psi^\dagger(\mathbf{r})\Psi(\mathbf{r}))^2 + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda] \end{aligned} \quad . \tag{2.1}$$

The covariant derivative is

$$D_0 = \partial_0 - iA_0, \; D_i = \partial_i - iA_i. \tag{2.2}$$

This action is exactly corresponding to the action of point particles in last chapter.

When quantize the theory, the commutation of bosonic field at equal time is

$$[\Psi(\mathbf{r}), \Psi(\mathbf{r}')] = 0 \; [\Psi^\dagger(\mathbf{r}), \Psi^\dagger(\mathbf{r}')] = 0 \;, [\Psi(\mathbf{r}), \Psi^\dagger(\mathbf{r}')] = \delta^2(\mathbf{r}-\mathbf{r}')$$

Now the density j_0 in second quantize form is

$$j_0 = \Psi^\dagger\Psi. \tag{2.3}$$

In analogue with point particle case in the last chapter, one has relation from the equation of motion

$$\epsilon^{0ij}\partial_i A_j = -\frac{2\pi}{k} j_0. \tag{2.4}$$

We use the identity

$$\partial^2 \frac{1}{2\pi} \ln r = \delta^2(\mathbf{r}), \tag{2.5}$$

with $r = (x^2 + y^2)^{\frac{1}{2}}$. The solution can be written as (in Coulomb gauge and keep in mind we work in a plane)

$$A_i = \frac{2\pi}{k}\epsilon_{ij}\partial_j \frac{1}{\partial^2} j^0 = \frac{1}{k}\epsilon_{ij}\partial_j \int d^2\mathbf{r}' \ln|\mathbf{r} - \mathbf{r}'| j_0(\mathbf{r}'). \tag{2.6}$$

The commutation of A and Ψ is

$$[A_i(\mathbf{r}), \Psi(\mathbf{r}')] = -\frac{1}{k}\epsilon_{ij}\partial_j \ln|\mathbf{r} - \mathbf{r}'|\Psi(\mathbf{r}'). \tag{2.7}$$

We would like also to calculate the commutation of $A_i(\mathbf{r})$ and $\Psi(\mathbf{r})$ at a same point. But $\partial_i \ln|r|$ is not well defined when $r = 0$. One know $\partial_i \ln|r|$ is odd function of its variables. So one of consistent regularization of $\partial_i \ln|r|$ is that it is zero at origin. Using this kind regularization, $\Psi(\mathbf{r})$ and $\Psi^\dagger(\mathbf{r})$ commute with $A_i(\mathbf{r})$. So there is no ordering ambiguity in operator $D_i\Psi$. And one finds the Hamiltonian density H_d from the lagrangian density is

$$H_d(\mathbf{r}) = \frac{1}{2m}(D\Psi)^\dagger D\Psi - \frac{g}{2}(j_o)^2. \tag{2.8}$$

In equation (2.8), We take a special ordering for H_d because $(D\Psi)^\dagger$ and $D\Psi$ do not commute. It is easy to see H_d is an Hermitian operator. Because Hamiltonian is nonlinear, so due to ordering of operator, the equation of operator at quantum level is different from the classical equation. The equation of motion in quantized form is

$$i\partial_0\Psi(\mathbf{r}) = [\Psi(\mathbf{r}, H], \tag{2.9}$$

with $H = \int d^2\mathbf{r}H_d$. It is

$$i\partial_0\Psi(\mathbf{r}) = -\frac{1}{2m}D^2\Psi(\mathbf{r}) - A_0\Psi(\mathbf{r}) - gj_0\Psi(\mathbf{r}) + \frac{1}{2k}\int d^2\mathbf{r}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}j_0(\mathbf{r}')\Psi(\mathbf{r}), \tag{2.10}$$

and A_0 is (it is consistent with last chapter, section two in the first quantized form)

$$A_0(\mathbf{r}) = \frac{2\pi}{k}\int \frac{1}{2\pi}\ln|\mathbf{r}-\mathbf{r}'|\epsilon_{ij}\partial_i j^j(\mathbf{r}'). \tag{2.11}$$

The j^k, $k = 1, 2$ current in second quantized form is

$$j^k = \frac{1}{2mi}(\Psi^\dagger D_k\Psi - (D_k\Psi)^\dagger\Psi). \tag{2.12}$$

The appearing of the last term in equation (2.9)is due to operator ordering. In classical equation, it is absent.

Now we discuss anyon gauge in second quantized form. In order to do that, we need the identity

$$\epsilon^{ij}\frac{r^j}{r^2} = -\partial_i\tan^{-}\frac{x^2}{x^1} = -\partial_i\theta, \tag{2.13}$$

and θ is the polar angle of the coordinate. Then one can get from the equation (2.6)

$$A_i = \frac{-1}{k}\partial_i\int d^2\mathbf{r}'\theta(\mathbf{r}-\mathbf{r}')j_0(\mathbf{r}'). \tag{2.14}$$

Then one makes a singular gauge transformation to eliminate A

$$\Psi(\mathbf{r}) = e^{-\frac{i}{k}\int d^2\mathbf{r}'\theta(\mathbf{r}-\mathbf{r}')j_0(\mathbf{r}')}\Psi'(\mathbf{r}) = e^{i\omega}\Psi'(\mathbf{r}). \tag{2.15}$$

One can calculate the commutation

$$\Psi'(\mathbf{r})\Psi'(\mathbf{r}') = e^{\frac{i\pi}{k}}\Psi'(\mathbf{r}')\Psi'(\mathbf{r}). \tag{2.16}$$

This is the manifestation fractional statistics in second quantized form. Now we insert this singular gauge transformation into the equation (2.10)to get the equation of motion in anyon gauge. the result (we expected) is

$$i\partial_0\Psi'(\mathbf{r}) = -\frac{1}{2m}D^2\Psi'(\mathbf{r}) - gj_0\Psi'(\mathbf{r}). \tag{2.17}$$

Now we consider the relation between first quantize form and second quantize form. In second quantized form, we have

$$H|E,N>=E|E,N>. \tag{2.18}$$

and

$$N|E,N>=N|E,N>. \tag{2.19}$$

N is the operator of the total number of particles

$$N = \int d^2\mathbf{r}j_0(\mathbf{r}). \tag{2.20}$$

H and N commute with each other. On the other hand, in first quantized form, the wave function $\psi_{E,N}$ is related to the second quantized state $|E,N>$

$$<\Omega|\Psi(\mathbf{r}_1)\cdots\Psi(\mathbf{r}_N)|E,N>=\psi_{E,N}(\mathbf{r}_l). \tag{2.21}$$

Ω is vacuum in second quantized form. By taking inner product of the equation of motion in second quantized form between vacuum and eigenstates, we can get Schrödinger equation. It is

$$[\frac{-1}{2m}\sum_l \mathbf{D}_l^2 - g\sum_{l<k}\delta(\mathbf{r}_l-\mathbf{r}_k)]\psi_E(\mathbf{r}_l) = E\psi_E(\mathbf{r}_l), \tag{2.22}$$

with

$$D_{il} = \partial_{il} - iA_{il}, \tag{2.23}$$

and

$$A_{il} = \frac{1}{k}\sum_{k\neq l}\epsilon_{ij}\frac{(\mathbf{r}_l-\mathbf{r}_k)^j}{|\mathbf{r}_l-\mathbf{r}_k|^2}. \tag{2.24}$$

This will be the results of last chapter if we forget the hard core interaction (related with g term).

Finally we make some remarks. In the present regularization, no self interaction exists, however which is important for the fractional spin. In plane, the self interaction maybe can be omitted. However, in a finite compact surface, however, self interaction must consider in order to have a consistent theory. This is similar with the fact that Dirac quantization condition for magnetic flux out of closed surface must be satisfied. But in the plane, there is no need for Dirac quantization.

2.2. $2+1$ QED, a Good Candidate for Anyon

$2+1$ QED is a theory in which we can do many theoretical experiments. Many interesting things appears in this theory. The topics related $2+1$ QED are extremely wealth and very helpful for understanding some interesting physical problem. For example, magnetic flux can induce fractional fermion number and fractional charge to the state of vacuum. For a comprehensive reference, see for example [40] and reference there in. And also indeed fractional spin and statistics naturally appear in the theory.

One knows that in $2+1$ dimension, the mass term of fermion will violate T and P invariance. Now assuming we have heavy fermion, we can integrate fermion field out to get effective action. It is a plausible procedure because heavy fermion should decouple from the scale of the physics we are interested in. However, one finds the effective action will contain a finite Chern-Simons term which also does not respect T and P symmetry (TP is a good symmetry however). There are various way to get this term [40]. It is found one loop calculation of the effective action is nonperturbative and high loop correction is zero as proved by Coleman and Hill [41]. However in the following discussion, we will not specify how the Chern-Simons term comes from.

Now we consider a lagrangian density which includes matter field term L_m, gauge field term L_g, and the interacting term between them

$$L_d = L_m + L_i + L_g. \tag{2.25}$$

For convenience, we change some convention of last chapter to more standard convention in literature [88]. L_g is

$$L_g = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}. \tag{2.26}$$

Compare to the notation of last chapter, $k = 2\mu\pi$. And

$$L_i = -A_\mu J^\mu. \tag{2.27}$$

(We change the sign of interaction term compare to last chapter. We apologize for those inconvenience. But this change is not difficult to trace and easy to transfer between different notation). The lagrangian density of fermion field is

$$L_f = i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi, \tag{2.28}$$

with $\not{\partial} = \gamma^\mu\partial_\mu$. The γ matrix in $2+1$ dimension can take representation by using Pauli matrices

$$\gamma^0 = \sigma^3,\ \gamma^1 = i\sigma^1,\ \gamma^2 = i\sigma^2, \tag{2.29}$$

and the current is

$$J^\mu = -e\bar{\psi}\gamma^\mu\psi. \tag{2.30}$$

Now let us show Chern-Simons term and fermion mass term (in 2+1 space time) is P,T violation term [88]. Discrete transformations leave the form of the equation of motion invariant. It is not difficult to find those discrete transformations following standard text book as we do in $3+1$ dimension (for example, Itzykson and Zuber [42]). However the form of discrete transformations In $2+1$ dimension are rather different from $3+1$ dimension. They are given by [88]

$$\begin{aligned} PA^0(t,\mathbf{r})P^{-1} &= A^0(t,\mathbf{r}') \\ PA^1(t,\mathbf{r})P^{-1} &= -A^1(t,\mathbf{r}') \\ PA^2(t,\mathbf{r})P^{-1} &= A^2(t,\mathbf{r}') \quad , \\ P\psi(t,\mathbf{r})P^{-1} &= \sigma^1\psi(t,\mathbf{r}') \\ \mathbf{r} = (x,y),\ \mathbf{r}' &= (-x,y) \end{aligned} \tag{2.31}$$

as parity transformation

$$\begin{aligned} TA^0(t,r)T^{-1} &= A^0(-t,r) \\ TA^i(t,r)T^{-1} &= -A^i(-t,r)\,, \\ T\psi(t,r)T^{-1} &= \sigma^2\psi(-t,r) \end{aligned} \tag{2.32}$$

as time reverse transformation. It is easy to check that the fermion mass term and Chern-Simons term change sign under the transformations of P,T. This is one of reason there are some exotic phenomena in $2+1$ QED.

<u>Quantization of Chern-Simons and Maxwell Theory</u>

There are many ways to quantize this theory [88,97]. In the following discussion, we will use the approach in [88] to quantize the theory. First, let us only consider the pure gauge theory (in the absence matter field) which has the lagrangian density L_g. Let us first work on Weyl gauge $A^0 = 0$, Then the canonical momentums are

$$\Pi^i = \frac{\delta L_g}{\delta \dot{A}^i} = -E^i + \frac{\mu}{2}\epsilon^{ij}A^j,\ E^i = F^{i0}. \tag{2.33}$$

The Hamiltonian is

$$H = \frac{1}{2}\int d^2\mathbf{r}[\mathbf{E}^2 + B^2], \tag{2.34}$$

with $B = \epsilon^{ij}\partial_i A^j$. The Hamiltonian can be written as

$$H = \frac{1}{2}\int d^2\mathbf{r}[(\Pi^i - \frac{\mu}{2}A^j)^2 + B^2]. \tag{2.35}$$

The theory can be quantized if we impose commutation relations at equal time

$$i[\Pi^i(\mathbf{r}), A^j(\mathbf{r}')] = \delta^{ij}\delta^2(\mathbf{r}-\mathbf{r}'). \tag{2.36}$$

When we impose gauge $A^0 = 0$, it does not still fix gauge because we can do time independent gauge transformation which will not violate Weyl gauge condition. One also needs to fix the residue gauge freedom. It can be solved as following way.

Now in Weyl gauge, time component field equation will become a constraint. The time component field equation is

$$G = \partial_i\Pi^i + \frac{\mu}{2}B = 0. \tag{2.37}$$

One finds G commute with Hamiltonian. Moreover, G is the generator for doing time independent gauge transformation. Because we have the commutation of G with A as

$$[G(\mathbf{r}), A^j(\mathbf{r})] = -i\partial_j\delta(\mathbf{r}-\mathbf{r}'). \tag{2.38}$$

So one should impose the condition

$$G|\Psi >= 0. \tag{2.39}$$

Because the physical state must be gauge invariant. The condition is consistent because G commutes with H. We can solve the constraint on physical state by doing following transformation

$$\Psi(\mathbf{A}) = e^{i\chi}\Phi(\mathbf{A}), \tag{2.40}$$

with

$$\chi = \frac{\mu}{2}\int d\mathbf{r} B(\nabla^{-}\cdot\mathbf{A}), \tag{2.41}$$

with $\nabla^i = \partial_i$ and ∇^- is the inverse of ∇. Inserting this transformation to the constraint equation, we get

$$\partial_i \Pi^i \Phi(\mathbf{A}) = 0. \tag{2.42}$$

It is actually Coulomb gauge condition. (2.42)can be written as

$$\Phi(\mathbf{A}) = \Phi(\mathbf{A} + \partial\Omega), \tag{2.43}$$

for arbitrary function Ω. So Φ only depends on gauge invariant quantity from A, B. Or we can say it depends on the transverse part of A

$$A_T^i = \epsilon^{ij}\partial_i \frac{1}{\nabla^2} B. \tag{2.44}$$

When Π operate on the function of A_T^i, it will becomes Π_T in the projected Hilbert space Φ, with commutation relation

$$i[\Pi_T^i(\mathbf{r}), A_T^j(\mathbf{r}')] = (\delta^{ij} - \frac{\partial_i \partial_j}{\nabla^2})\delta(\mathbf{r} - \mathbf{r}'). \tag{2.45}$$

Correspondingly the Hamiltonian changes in the Hilbert space Φ to

$$H_c = e^{-i\chi} H E^{i\chi}, \tag{2.46}$$

which will be

$$H_c = \frac{1}{2}\int d^2\mathbf{r}[\mathbf{\Pi}_T^2 + A_T^i(-\nabla^2 + \mu^2)A_T^i]. \tag{2.47}$$

Actually now, we work on the Coulomb gauge. Moreover there exists a simple way to get the result in Coulomb gauge. Now we will show how it works. Directly, we choose Coulomb gauge. Using path integral quantization, in order to fix gauge, one can put gauge fixing condition as δ function inserted in path integral. The ghost terms need not to appear because our gauge theory is $U(1)$ gauge theory. The lagrangian density is

$$L = \frac{1}{2}(\mathbf{E}^2 - B^2) + \frac{\mu}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}. \tag{2.48}$$

The action is

$$S = \int dt d^2\mathbf{r} L, \tag{2.49}$$

and $\mathbf{E}^2 = \dot{\mathbf{A}}^2 + (\nabla A^0)^2 - 2\partial^i A^0 \partial^0 A^i$. In Coulomb gauge, the last term will disappear in action S by doing partial integration. We get action

$$S = \int dt d^2\mathbf{r}[\frac{1}{2}[\dot{\mathbf{A}}^2 + (\nabla A^0)^2] - \frac{1}{2}B^2 - \mu A_0 B - \frac{\mu}{2}\epsilon^{ij}A_i\dot{A}_j]. \tag{2.50}$$

One can integrate the component A_0 because the action is only up to the square of A_0 and does not contain time derivative. The equation for A_0 is

$$\nabla^2 A_0 = -B\mu, \tag{2.51}$$

or

$$A_0 = -\frac{\mu}{\nabla^2}B. \tag{2.52}$$

Substitute back to action, the result is

$$S = \int d\mathbf{r}[\frac{1}{2}\dot{\mathbf{A}}^2 - \frac{1}{2}B^2 - \frac{\mu}{2}\epsilon^{ij}A_i\dot{A}_j + \frac{\mu^2}{2}B\nabla^{-2}B]. \tag{2.53}$$

Keep in mind that A_i satisfy the Coulomb constraint, $A^i = \epsilon^{ij}\partial_i\omega$. The term $\int \epsilon^{ij}A_i\dot{A}_j$ disappears when one does partial integration. Finally, the action is

$$S = \int d\mathbf{r}[\frac{1}{2}\dot{\mathbf{A}}^2 - \frac{1}{2}B^2 + \frac{\mu^2}{2}B\nabla^{-2}B]. \tag{2.54}$$

The Hamiltonian from this lagrangian is

$$H = \int d\mathbf{r}[\frac{1}{2}\dot{\mathbf{A}}^2 + \frac{1}{2}B^2 - \frac{\mu^2}{2}B\nabla^{-2}B]. \tag{2.55}$$

It is easy to verify the Hamiltonian in the equation (2.55)is the same Hamiltonian in the equation (2.47)by using identity $A_T^i = -\epsilon^{ij}\frac{\partial_j}{\nabla^2}B$.

Now we switch to the theory with matter field. The full Hamiltonian is

$$H = H_g + H_m - \int d\mathbf{r} \mathbf{J} \cdot \mathbf{A}. \tag{2.56}$$

H_g is the Hamiltonian of pure gauge theory. H_m is the Hamiltonian of matter field. The Gauss constraint becomes

$$G = \nabla \cdot \Pi + \frac{\mu}{2} B + J_0 = 0. \tag{2.57}$$

One can exactly follow the same way of quantizing pure gauge theory, in Weyl gauge, first solving the Gauss constraint of the wave function which is now a little bit more complicated than pure gauge theory, then one can quantize the theory in Coulomb gauge. Another easier way is using the method just described above. One finds

$$H_g - \int d\mathbf{r} \mathbf{J} \cdot \mathbf{A} = \frac{1}{2} \int d\mathbf{r} [\Pi^2 + A_T^i(-\nabla^2 + \mu^2) A_T^i + J_0 \frac{-1}{\nabla^2} J_0] - \int d\mathbf{r} [J^i + \frac{\mu}{\nabla^2} \epsilon^{ij} \partial_j J_0] A_T^i. \tag{2.58}$$

Then make the transformation

$$A_T^i = \epsilon^{ij} \hat{\partial}_j \varphi, \ B = \sqrt{-\nabla^2} \varphi, \tag{2.59}$$

and also similar form for canonical momentum

$$\Pi_T^i = -\epsilon^{ij} \hat{\partial}_j \Pi. \tag{2.60}$$

The commutation relation between Π and φ is

$$i[\Pi(\mathbf{r}), \varphi(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}'). \tag{2.61}$$

The Hamiltonian can be written in Π and φ

$$\begin{aligned} H_g - \int d\mathbf{r} \mathbf{J} \cdot \mathbf{A} = \frac{1}{2} \int d\mathbf{r} [\Pi^2 + (\sqrt{-\nabla^2 + \mu^2} \varphi + \frac{\mu}{\sqrt{-\nabla^2}} \frac{1}{\sqrt{-\nabla^2 + \mu^2}} J_0)^2 \\ + J_0 \frac{1}{-\nabla^2 + \mu^2} J_0] - \int d\mathbf{r} \epsilon^{ij} \hat{\partial}_i J^i \varphi \end{aligned} \tag{2.62}$$

One can use canonical transformation to shift φ by $\frac{\mu}{(-\nabla^2 + \mu^2)\sqrt{-\nabla^2}} J_0$ (we will show

how to do it in a easy way). We get a Hamiltonian which is convenient for the discussion of next subsection

$$
\begin{aligned}
H_g - \int d\mathbf{r}\mathbf{J}\cdot\mathbf{A} = \frac{1}{2}\int d\mathbf{r}[\Pi^2 + \varphi(-\nabla^2+\mu^2)\varphi + J_0\frac{1}{-\nabla^2+\mu^2}J_0] \\
+ \int d\mathbf{r}[\mu\hat{\partial}_i J^i \frac{-1}{-\nabla^2+\mu^2}\Pi - \epsilon^{ij}\hat{\partial}_i J^j \varphi \\
- \mu\epsilon^{ij}J^i\frac{\partial_j}{\nabla^2(-\nabla^2+\mu^2)}J_0]
\end{aligned} \quad . \qquad (2.63)
$$

One would wonder how to get this canonical transformation. Of course in Hamiltonian form, when you do a canonical transformation to shift a field, it is not easy. One way to go around it for the present problem is we change the Hamiltonian in equation (2.62)back to the lagrangian form, and can do the shift of the field. After the shift we get the new lagrangian. Then from the new lagrangian, we get a Hamiltonian which is exactly equation (2.63). (Frankly, the Hamiltonian here is different from [88] in a sign. It maybe is a misprint or my mistake).

About the theory on the torus, we will discuss it in next chapter.

Fractional Statistics and Spin in Chern-Simons-Maxwell theory

In 2+1 QED, heavy fermion can be integrated out to give finite Chern-Simons term. In last chapter, it is shown that pure Chern-Simons term can give particles fractional statistics and spin. One would ask what will happen if a Maxwell term is added. This is the task of present subsection.

We consider a group of particles with mass $m \gg \mu$, but much smaller that the heavy fermion, so we can use nonrelativistic limit. Further we assume those particles are wildly separated with distance $\gg \frac{1}{\mu}$ in order to avoid some unnecessary process, for example, creation of particles, photon or other process which are no important to the statistics and spin of particles we are interested in. From the Hamiltonian (2.63)obtained in the last subsection, it is easy to see that only last term survives in the long range limit. The last term is

$$H_L = \int d^2[\mathbf{r} - \mu\epsilon^{ij}J^i \frac{\partial_j}{\nabla^2(-\nabla^2+\mu^2)}J_0]. \tag{2.64}$$

In the nonrelativistic limit, the currents are (for electron $q=-e$)

$$J^0 = q\sum_l \delta(\mathbf{r}-\mathbf{r}_l), \;\; J^i = q\sum_l \frac{dr_l^i}{dt}\delta(\mathbf{r}-\mathbf{r}_l). \tag{2.65}$$

We get a action in the nonrelativistic and long distance limit

$$S_e = \int d^3xL_e = \int d^3x[L_m - H_L] = \int dt[\sum_l \frac{1}{2}m(\frac{d\mathbf{r}_l}{dt})^2 + \sum_{l\neq n}\frac{q^2}{2\mu\pi}\epsilon^{ij}\frac{dr_l^i}{dt}\frac{r_l^j - r_n^j}{|\mathbf{r}_l-\mathbf{r}_n|^2}], \tag{2.66}$$

or

$$S_e = \int dt[\sum_l \frac{1}{2}m(\frac{d\mathbf{r}_l}{dt})^2 - \sum_{l<n}\frac{q^2}{2\mu\pi}\frac{d\Theta(\mathbf{r}_l-\mathbf{r}_n)}{dt}]. \tag{2.67}$$

So in the long range limit, we get a theory as pure Chern-Simons theory. The statistics parameter is $\Delta = \frac{q^2}{4\mu\pi}$ (we define Δ as, when one changes two particle, one obtains a phase $e^{2\pi i\Delta}$). Now we can make conclusion that, for the Chern-Simons Maxwell theory, the fractional statistics exists as if without Maxwell term. In the long range limit, Maxwell term disappears. One notes that the naive argument for factor half of the pure Chern-Simons theory (compare to cyons of Wilczek) in the last chapter can still be used for Chern-Simons Maxwell theory. More physically, we can understand it in following way [32]. Charge can be defined by using Gauss law or in terms of the coupling constant of gauge potential with the current. However, in Chern-Simons Maxwell theory, because the gauge field becomes massive, the electric field and magnetic field are short range(one can derive the equation motion for them) . So there is no charge given by Gauss law, $\int E_i dS_i = 0$. However the gauge potential A still can be long range (in the infinity, $A \sim \frac{1}{r}$). So when two particles are turned around each other, we get a phase from the particle moving around a flux because the gauge potential is long range (the phase is $e^{iq\oint dx_\mu A^\mu}$). But due to the electric field and magnetic is not long range, that moving a flux around a charge will not produce any effect by Faraday theorem, because the gauge field is short range. On the other hand, in the ordinary QED, the field is long range. When you move a flux around a charge particle, according Faraday theorem, one can get an induced electric field which is long range and can influence the charge particle. This effect will give a phase which equals the other one. In Chern-Simon Maxwell theory, one only can get one phase, due to that a charge moves around a flux. This can explain a strange factor half appears in Chern-Simons Maxwell theory.

What is the spin of the particle? This is a more difficult problem. Until now, there is not a every convincing result. However, some plausible conjectures exist.

One still considers nonrelativistic limit. For discussing spin, one can consider just one particle. The current is

$$J^\mu = q\delta^{\mu 0}\delta(\mathbf{r}), \tag{2.68}$$

which is the current of a static particle. The equations of motion for gauge field are

$$\partial_\nu F^{\nu\mu} + \frac{1}{2}\mu\epsilon^{\mu\nu\lambda}F_{\nu\lambda} - q\delta^{\mu 0}\delta(\mathbf{r}) = 0. \tag{2.69}$$

The solution is (J. F. Schonfeld in [43])

$$B = \frac{q\mu}{2\pi}K_0(\mu r),\ \mathbf{E} = \frac{q}{2\pi}\nabla K_0(\mu r), \tag{2.70}$$

and the function K is Bessel function. The gauge invariant angular momentum of gauge field is

$$J_e = -\int d^2\mathbf{r}B(\mathbf{r}\cdot\mathbf{E}). \tag{2.71}$$

Then for the static point particle [43]

$$J_e = \frac{q^2}{4\pi\mu}. \tag{2.72}$$

What kind of angular momentum is mattered in considering the spin of the particle? It was shown in the last chapter that total gauge invariant angular momentum is responsible for the spin of particle. Now one needs to calculate the gauge invariant angular momentum which is carried by gauge field and the gauge invariant angular momentum carried by particle itself, the mechanical momentum. Then the two angular momentums sum up, what we get is the spin of the particle. Now assume spin statistics relation holds, so the spin should equal to $\frac{q^2}{4\pi\mu}$ (keep in mind that from the result of last subsection, the statistical parameter is $\frac{q^2}{4\pi\mu}$). Then J_m must be an integer. However, no one can calculate J_m directly. However some people argued $J_m = 0$ [32].

Here, we give another argument based under very general assumption. The assumption is the angular momentum given by Noether current always is an integer. Until now, no examples are found to violate this assumption. The Noether current J_n is a sum of three parts

$$J_n = J_m + J_e + J_s, \tag{2.73}$$

with J_s is the surface term

$$J_s = \int d^2\mathbf{r}\partial_\mu(r^1 T^{\mu 02} - r^2 T^{\mu 01}), \tag{2.74}$$

and

$$T^{\mu\nu\lambda} = (\frac{1}{2}\mu\epsilon^{\nu\mu\sigma}A_\sigma - F^{\nu\mu})A^\lambda. \tag{2.75}$$

It was calculated in [43], $J_s = -J_e$. So from the assumption, $J_n = integer$, one concludes $J_m = integer$. But $J_m + J_e$ can be considered as the spin of the particle, so up to a integer, $J_m + J_e = \frac{q^2}{4\pi\mu}$. So statistics spin relation is standard one in the present theory.

Another approach to fractional statistics and spin is by using canonical quantum field theory as shown in [97]. In analogue to first quantized form, one can do the singular gauge transformation. The operators in the singular gauge have a generalized commutation relation with fractional statistics.

Finally, we come to the real $2+1$ QED model. Assuming we have n massive two component fermion with charge e. After integrating out the heavy fermions, one has $\mu = \frac{ne^2}{4\pi}$ [40]. From the equation of motion with respect to A_0, one finds if a vortex has basic flux $\frac{2\pi}{e}$, then it gets an induced charge $\frac{ne}{2}$. Then the statistics parameter of the vortices is $\frac{\Delta=n^2}{4}$ up to an integer. So naturally fractional statistics and spin appear if n is an odd number.

When one reads the paper of Polyakov, one will find the result of Polyakov is not consistent with the present results (which was noted in [32]). Assuming now the charge $e = 1$, if we want a boson transmutes to a fermion, one needs $\mu = \frac{1}{2\pi}$. However in the paper by Polyakov [35], $\mu = \frac{1}{8\pi}$ is responsible for the transmutation. Which one is correct? We just state our idea about it. Our results for pure Chern-Simons theory can be confirmed by many different methods, path integral, canonical approach, *etc.* We agree with the all things in [35] except the equation (3) (the coefficient in (3) in [35] is different from our results).

2.3. Nonlinear σ Model ($O(3)$ or CP^1) and Anyon

Wilczek and Zee [71] found the soliton in the nonlinear σ $O(3)$ model of $2+1$ space time plusing some topological term is anyon. In this section, we would like see this interesting subject.

First, let us consider two dimension σ model. The lagrangian density of the model is

$$L = \partial_\mu \phi_i \partial_\mu \phi_i, \ \phi_i \phi_i = 1, \ i = 1, 2, 3, \tag{2.76}$$

with $\mu = 1, 2$ and the metric is Euclidean metric. We can use the map z to ϕ, $\phi_i = z^\dagger \sigma_i z$, with $z = \binom{z_1}{z_2}$, and $|z_1|^2 + |z_2|^2 = 1$. Then the lagrangian can be written as (for example, see [44,46]. For an extensive review, see [45])

$$L = Tr(D_\mu z)^\dagger D_\mu z, \tag{2.77}$$

with $D_\mu = \partial_\mu - A_\mu$, and $A_\mu = z^\dagger \partial_\mu z$. The equation of motion is

$$D_\mu D_\mu z + z(D_\mu z)^\dagger D_\mu z = 0. \tag{2.78}$$

Using the identity

$$\int d^2x (D_\mu z \pm i\epsilon_{\mu\nu} D_\nu z)^\dagger (D_\mu z \pm i\epsilon_{\mu\nu} D_\nu z) \geq 0. \tag{2.79}$$

It is

$$2\int d^2x [(D_\mu z)^\dagger (D_\mu z) \pm i\epsilon_{\mu\nu}(D_\mu z)^\dagger (D_\nu z)] \geq 0. \tag{2.80}$$

However $Q = -\frac{i}{2\pi}\int d^2x \epsilon_{\mu\nu}(D_\mu z)^\dagger \cdot (D_\nu z)$ is a topological invariant which describes the mapping of internal space S_2 to compactified coordinate space S_2 of R_2. So one has $S \geq 2\pi|Q|$. The condition for equality in equation (2.80)is the self dual equation (anti self dual)

$$D_\mu z = \pm i\epsilon_{\mu\nu} D_\nu z. \tag{2.81}$$

This is a first order equation. If equation (2.81)is satisfied, the equation of motion (2.78)also is satisfied. This is exactly the analogue of the instanton in $3+1$ dimension QCD. For the following discussion, one defines holomorphic and antiholomorphic coordinate, $x_\pm = x_1 \pm ix_2$. Then the self-dual (anti seldual) equation is

$$D_\mp z = 0. \tag{2.82}$$

Due to the gauge invariance of the model (for example, [44]) we can fix the gauge by parametrizing $z = \frac{1}{\sqrt{1+|f|^2}}\binom{1}{f}$. The general instanton solution (equation

(2.82)) for $Q = k$ is [45] (now we just consider instanton, for anti instanton is exact parallel)

$$f = c\frac{\prod_{i=1}^{k}(x_+ - a_i)}{\prod_{i=1}^{k}(x_+ - b_i)}, \tag{2.83}$$

with c, a_i, b_i are constants of some complex numbers. Above the model is in two dimension. Now, we consider an action in $2+1$ space time

$$S = \int dtd^2x(D^\mu z)^\dagger D_\mu z. \tag{2.84}$$

The definition in equation (2.84)is exactly as before except we add an extra coordinate t. The metric $g^{\mu\nu}$ is

$$g^{00} = 1,\ g^{11} = -1,\ g^{22} = -1. \tag{2.85}$$

Instanton solution in 2 dimension nonlinear σ model is the static soliton solution in 3 dimension nonlinear σ model. Now we consider the k soliton solution $z = \frac{1}{\sqrt{|f_1|^2+|f_1|^2}}\binom{f_1}{f_2}$ with

$$\begin{aligned} f_1 &= c_1(x_+ - a_1)\cdots(x_+ - a_k) \\ f_2 &= c_2(x_+ - b_1)\cdots(x_+ - b_k) \end{aligned}, \tag{2.86}$$

and for simplicity, let $c_i = 1$. One soliton is made of one particle and one antiparticle. a_i, b_i is the coordinates of the i th soliton or Skyrmion because of the following reasoning. When $a_i = b_i$, the i th soliton will disappear as it can be easy to see from the equation (2.86). Now assuming the coordinates of the solitons it slowly depend on time, then equation (2.86)still is a good approximate solution of the equation of motion. One can define $R_i = \frac{a_i+b_i}{2}$ is the coordinate of the i th soliton, $r_i = \frac{a_i-b_i}{2}$ is the size of the i th soliton. r_i should not be zero, otherwise the particle and anti particle will annihilate. Also we suppose solitons are wildly separated with each other. Insert the weak time dependent solution into action, one gets

$$S = \int_0^T dt \int d^2x \left| \frac{dz}{dt} - (z^\dagger \frac{dz}{dt}) z \right|^2 - 2\pi k T, \tag{2.87}$$

with T is the interval of time coordinate. In order to have a finite action, c_i must be independent on time (in the following discussion, we will take $c_i = 1$), and $\sum_i (a_i - b_i) = constant$ (for example see [45]). Insert the weak time dependent solution to (2.87), the action becomes an action of k particle and anti particle pairs (k solitons) moving on two dimension

$$S = \int dt g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}, \tag{2.88}$$

here x^α are the coordinates a_i, b_i. We will not calculate $g_{\alpha\beta}$ because it will not be used for the following discussion.

In order to have fractional statistics and spin, an extra term needs added. Wilczek and Zee [71] realized that a topological term, Hopf term can alter the statistics and spin of solitons. Let us assume periodic condition of field on time. Then the coordinate space of our model is $S^2 \times S^1$, which is locally isomorphic to S^3. The field space of ϕ_i is S^3. But, $\Pi_3(S^2) = Z$, *integer*, as one can check mathematical book . So we can classify the space of the model (mapping from the base manifold to field manifold) by some topological invariant numbers. One finds the space of model is infinite connected. The Hopf term just describe this topological number.

We introduce a topological current

$$J^\mu = -\frac{i}{2\pi} \epsilon^{\mu\nu\lambda} (D_\nu z^\dagger \cdot D_\lambda z). \tag{2.89}$$

The topological currents are conserved without considering the equation of motion. The current can be written as

$$J^\mu = \epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda, \tag{2.90}$$

with the A is given as before, $A_\mu = iz^\dagger \cdot \partial_\mu z$. And the Hopf term is

$$H = \int dt d^2 x A_\mu J^\mu. \tag{2.91}$$

Now, we consider the action

$$S = \int dt d^2 x ((D^\mu z)^\dagger \cdot D_\mu z + \frac{\theta}{2\pi} A_\mu J^\mu)$$

It is easy to verify that

$$h = 2\pi A_\mu J^\mu = \epsilon^{\mu\nu\lambda}(z^\dagger \cdot \partial_\mu z)(\partial_\nu z^\dagger \cdot \partial_\lambda z). \tag{2.92}$$

Now let $z_1 = y_1 + iy_2$, $z_2 = y_3 + iy_4$, y_i is real number, with $\sum_i y_i^2 = 1$. It is not difficult to show

$$h = \epsilon^{abcd}\epsilon^{\mu\nu\lambda} y_a \partial_\mu y_b \partial_\nu y_c \partial_\lambda y_d. \tag{2.93}$$

One will find that h is a total divergence. So this term should not affect the equation of motion. Classically it is not relevant. But it will be relevant at quantum level. We will find the Hopf term will change the statistics and spin of the solitons.

Insert the solitons solution to our new action with Hopf term, after a long calculation (for the details of the calculation, see [45]), The action corresponding to the topological term (Hopf term) will be

$$S_h = \int dt - \frac{i\theta}{4\pi}\frac{d}{dt}\sum_{i,j} \ln \frac{a_i - b_j}{\bar{a}_i - \bar{b}_j}, \tag{2.94}$$

or in coordinate R_i, r_i ($R_i = \frac{a_i + b_i}{2}$ and $r_i = \frac{a_i - b_i}{2}$)

$$S_h = \int dt - \frac{i\theta}{4\pi}\frac{d}{dt}\sum_{i,j} \ln \frac{R_i + r_i - R_j + r_j}{\bar{R}_i + \bar{r}_i - \bar{R}_j + \bar{r}_j}. \tag{2.95}$$

One assumes the solitons are wildly separated, and the size of every soliton is much smaller than the distance of two solitons. So when one exchanges two solitons, from equation (2.95)one gets a phase $e^{i\theta}$. Then the statistics parameter is $\Delta = \frac{\theta}{2\pi}$ (in the process of exchange, we still keep two solitons wildly separated, and for simplicity, no other solitons are inside the exchange path).

Now we consider the spin of the soliton. The term related with the spin of the soliton in the action (2.95)is the term of $i = j$ in the summation. It is

$$S_s = \int dt - \frac{i\theta}{4\pi} \frac{d}{dt} \sum_i \ln \frac{r_i}{\bar{r}_i}. \tag{2.96}$$

When one rotates the internal coordinates r_i a 2π, one will get a phase, $e^{i\theta}$. So the spin of every soliton is $s = \frac{\theta}{2\pi}$. So one sees that statistics-spin relation is standard one.

If the coordinate of soliton is described by the coordinate of sphere, compactified R^2, we can give a restriction on the value of θ. As a famous fact, the statistics parameter of anyons on compact Riemann surface is subjected a restriction. How the restriction on θ appears in the present model? We give an naive argument here. Let one soliton go to infinity, a point on the north pole of compact Riemann sphere. Then let the soliton rotates, one can say it rotates around the other $k-1$ soliton, also can say it rotates around nothing just itself if looking at it from the rear side. Physically the two phase should be equal. The first phase is given by the phase of the rotation of the soliton around every other soliton and the rotation around themselves. It is $e^{2(k-1)i\theta} \cdot e^{i\theta}$. Another phase is just given by the rotation around themselves, but in opposite direction (from the point of view of the rear side), the phase is $e^{-i\theta}$. Let two phases equal. One finds

$$e^{2k\theta} = 1. \tag{2.97}$$

3. Chern-Simons-Maxwell Theory on Torus*

In the three dimensional space–time a Chern–Simons gauge field coupled with a matter field turns the ordinary statistics of the latter into "strange" statistics [77,13]; the properties of the resulting particles, called anyons, have been analysed in many papers, see for example [2] and references therein. The dynamics of the gauge vector potential in many of the treatments is governed by the Chern–Simons action $\frac{k}{4\pi}\int FA$ and not by the usual Maxwell term $-\frac{g}{4}\int F^2$. The reasons for this are essentially that it is the Chern–Simons action which induces the statistics flip and that the Maxwell term, being quadratic in the derivatives, can be neglected at low energies.

In this chapter we consider a theory which contains both terms in the Action defined on a torus i.e. on a boundaryless surface which is geometrically translation invariant. Our main interest will be to investigate how translation invariance can be implemented in the dynamics, that is to construct the Hamiltonian and Momentum operators and to compute their commutation relations. It is known that the implementation of the translation invariance is a central point in the discussion about the possible superconducting properties of a system of anyons [13,2,19]. Let us stress that one can obviously think of the torus as a plane with suitable boundary conditions, which make the mathematical definitions of the Hilbert space and the operators acting in it precise.

An interesting aspect of the resulting theory arises also from the following considerations. It is well known that it is possible to couple a Maxwell field to a system of charged particles on a compact surface only if the total charge of the particles is zero; this excludes in particular the coupling of a system of just N electrons, without positive charges. Now, adding the Chern–Simons term to the Action allows, in particular, to couple a system of arbitrary integer total charge; the total charge will be equal, up to a constant, to the total flux of the gauge field

* This chapter is adopted from [82]

over the torus. Quantum mechanically this system can be consistent provided a Dirac quantization condition is satisfied. In our non–relativistic treatment we will be able indeed to couple a system of N identical charged anyons.

The theory of free anyons on a torus has been investigated in detail in refs. [19,20], where, in particular, the fundamental role of the topological components of the gauge field, arising on the torus, has been pointed out. Here we present an extension of this theory to include in the Action also a term of the Maxwell form. One of the main results of the present chapter is the non–trivial realization of translational invariance in the resulting theory. It turns out that the Hamiltonian and the two components of the momentum operator commute among them due to a subtle interplay between the topological components of the gauge field and its total flux. It is interesting to note that a similar relation between the two was found also in the theory of free anyons [19] implying a kind of universality of this mechanism.

Finally we show how in the limit in which the Maxwell term gets decoupled one gets back the theory of free anyons on the torus. It turns out that the Hilbert space for free anyons is just the ground–state sector of the "topological" part of the Hamiltonian of the Chern–Simons–Maxwell theory. This result can be viewed as an extension of an analogous result for the "pure" Chern–Simons–Maxwell theory, i.e. the one without coupling to a matter field, see ref [66].

We consider the following $2+1$ dimensional non–relativistic action

$$S = \int d^3x \left(\frac{k}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - \frac{g}{4} F_{\mu\nu} F^{\mu\nu} + i \Psi^\dagger D_0 \Psi + \frac{1}{2m} \Psi^\dagger \vec{D}^2 \Psi \right). \qquad (3.1)$$

Here A_μ is the vector potential and Ψ is the matter field, k and g are the Chern-Simons and electromagnetic coupling constants respectively and the covariant derivative is given by $D_\mu = \partial_\mu - iA_\mu$. The field strength is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The indices μ, ν etc. take the values $0, 1, 2$.

First we want to derive, in a second quantized formalism, the canonical Hamiltonian and Momentum corresponding to this action on the torus. We take for simplicity a torus with unit area on which the coordinates x_i are defined modulo the periodicity relation $x_i \simeq x_i + m_i$, where the m_i are integers. The principal difference with respect to the treatment on the plane is the appearance in the spatial vector potential A_i of independent topological components, the zero–modes of the gauge field on the torus (flat connections), and of a constant "background" which is due to the fact that the total flux $\int F_{12} d^2x$ of the vector potential has to be necessarily different from zero if we want to couple the gauge field non–trivially to the matter field Ψ, see below.

The equations of motion corresponding to (3.1)are given by

$$\begin{aligned} \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} + g\partial_\rho F^{\rho\mu} + j^\mu = 0 \\ iD_0\Psi + \frac{1}{2m}\vec{D}^2\Psi = 0 \end{aligned}, \tag{3.2}$$

where we defined the current

$$j_i = \frac{i}{2m}\left(\Psi^\dagger D_i\Psi - D_i^\dagger\Psi^\dagger\Psi\right), \qquad j_0 = \rho = \Psi^\dagger\Psi. \tag{3.3}$$

We will work in the Coulomb gauge $\partial_i A_i = 0$ which we found to be convenient for our purposes. The zero-th component of the equation in the first line of eq. (3.2)is actually a constraint, given more explicitly by $(F \equiv F_{12})$

$$\frac{k}{2\pi}F + \rho = g\nabla^2 A_0. \tag{3.4}$$

Integration of this equation over the torus yields the important relation

$$\int F\, d^2x = -\frac{2\pi}{k}\int \rho\, d^2x = -\frac{2\pi}{k}\,N, \tag{3.5}$$

where N is the (integer) total number of charged particles, corresponding to the matter field Ψ, which in our non–relativistic theory is a conserved fixed quantity.

The Hamiltonian and Momentum can be derived from the energy–momentum tensor associated to (3.1). Noting that the Chern-Simons term, being intrinsically metric independent, does not contribute to the energy–momentum tensor we have

$$\begin{aligned} H &= \int d^2x \left(-\frac{1}{2m} \Psi^\dagger \vec{D}^2 \Psi + \frac{g}{2}(F^2 + F_{0i}^2) \right) \\ P_j &= \int d^2x \left(i\Psi^\dagger D_j \Psi + g F_{0i} F_{ji} \right) . \end{aligned} \tag{3.6}$$

To go on we want now to solve the constraint (3.4)for the vector potential. Due to the non trivial topology of the torus and eq.(3.5)we can write:

$$A_i(x) = a_i + \frac{\pi}{k} \epsilon_{ij} x_j N + \tilde{A}_i . \tag{3.7}$$

In (3.7)the a_i are the topological components of the gauge field (flat connections) corresponding to the two non trivial homology cycles of the torus, which constitute new degrees of freedom and have to be quantized independently. The term proportional to N carries the mean total flux associated to the vector potential and the term $\tilde{A}_i$ represents the periodic (around the torus) fluctuation of the vector potential with the properties

$$\partial_j \tilde{A}_j = 0, \qquad \tilde{F} \equiv \partial_1 \tilde{A}_2 - \partial_2 \tilde{A}_1 = F + \frac{2\pi}{k} N, \tag{3.8}$$

so that $\int d^2x\, \tilde{F} = 0$. We can then solve the constraint (3.4)in the Coulomb gauge by

$$A_0(x) = \frac{1}{4\pi g} \int d^2y\, P(x-y) \left(\frac{k}{2\pi} F(y) + \rho(y) \right) . \tag{3.9}$$

The function $P(x)$ is the (periodic) Green–function for the Laplacian on the torus which is given in terms of the standard θ–functions, with period $\tau = i$, by

$$P(x) = \ln \left| \frac{\theta \left[{}^{1/2}_{1/2} \right] (x_1 + i x_2 | i)}{\theta' \left[{}^{1/2}_{1/2} \right] (0 | i)} \right|^2 - 2\pi\, x_2^2, \tag{3.10}$$

and satisfies $\nabla^2 P(x) = 4\pi(\delta^2(x) - 1)$. Note that $\tilde{A}_i$ is of the particular form $\tilde{A}_i = \epsilon_{ij} \partial_j V$, where the (periodic) function V can be read off from eqs. (3.4)and

(3.8), i.e.

$$V(x) = \frac{1}{2k}\int d^2y P(x-y)(\rho(y) - g\nabla^2 A_0(y)). \tag{3.11}$$

In the following we will indicate with $\tilde{F}_{0i}$ the curvature associated to $\tilde{A}_i$, $\tilde{F}_{0i} = \partial_0 \tilde{A}_i - \partial_i A_0$. Notice that the term proportional to N in eq. (3.7)does not contribute to $\tilde{F}_{0i}$ nor to F_{0i}. Substituting the expression (3.7)for A_j into eq. (3.6)we get for the Hamiltonian and Momentum on the torus

$$\begin{aligned} H &= \int d^2x \left(-\frac{1}{2m}\Psi^\dagger \vec{D}^2 \Psi + \frac{g}{2}(F^2 + \tilde{F}_{0i}^2) + \frac{g}{2}\dot{a}_i^2\right) \\ P_j &= \int d^2x \left(i\Psi^\dagger D_j \Psi + g\tilde{F}_{0i}F_{ji} - \frac{2\pi}{k} g\epsilon_{ji}\dot{a}_i N\right). \end{aligned} \tag{3.12}$$

To write the "canonical" Hamiltonian we have to determine the momenta conjugate to the independent variables $\Psi, \tilde{A}_i$ and a_i respectively. Observing that the contribution of the topological components of the gauge field to the kinetic part ΔS of the action, i.e. the first two terms of eq. (3.1), is given by

$$\Delta S = \frac{g}{2}\dot{a}_i^2 - \frac{k}{4\pi}\epsilon_{ij}\, a_i \dot{a}_j. \tag{3.13}$$

We can write the conjugate momenta as (by using the notation: degree of freedom $\rightarrow$ corresponding conjugate momentum)

$$\begin{aligned} &\Psi \rightarrow \Pi = i\Psi^\dagger \\ &\tilde{A}_j \rightarrow \Pi_j = g\tilde{F}_{0j} + \frac{k}{4\pi}\epsilon_{ji}\tilde{A}_i \\ &a_i \rightarrow b_i = g\dot{a}_i + \frac{k}{4\pi}\epsilon_{ij}a_j. \end{aligned} \tag{3.14}$$

In deriving the contribution to Π_j coming from the Chern–Simons term we made use of the fact that $\int d^2x \tilde{A}_i = 0$. Remember that A_0 is not an independent dynamical degree of freedom in that $\dot{A}_0$ does not appear in the Action (3.1).

Substituting everything in eqs. (3.12)we get the canonical Hamiltonian and Momentum:

$$
\begin{aligned}
H &= \int d^2x \left(\frac{i}{2m}\Pi\vec{D}^2\Psi + \frac{g}{2}F^2 + \frac{1}{2g}\left(\Pi_i - \frac{k}{4\pi}\epsilon_{ij}\tilde{A}_j\right)^2 + \frac{1}{2g}\left(\frac{k}{4\pi}a_j + \epsilon_{ji}b_i\right)^2 \right) \\
P_j &= \int d^2x\, \Pi \left(\partial_j - i\left(a_j + \frac{\pi N}{k}\epsilon_{jh}x_h\right)\right)\Psi - \frac{2\pi N}{k}\left(\frac{k}{4\pi}a_j + \epsilon_{ji}b_i\right) + \int d^2x\, \Pi_i \partial_j \tilde{A}_i \cdot \\
&= \int d^2x\, \Pi \left(\partial_j - i\frac{\pi N}{k}\epsilon_{jh}x_h\right)\Psi + \frac{N}{2}a_j - \frac{2\pi N}{k}\epsilon_{ji}b_i + \int d^2x\, \Pi_i \partial_j \tilde{A}_i .
\end{aligned}
\tag{3.15}
$$

It is possible to give to the Hamiltonian a more standard form by the following considerations. Note that while $\tilde{A}_i$ is transverse the momentum Π_j is not. Eqs. (3.4)and (3.14)give in fact

$$
\partial_j \Pi_j = N - \rho - \frac{k}{4\pi}\tilde{F}, \tag{3.16}
$$

remembering that $\tilde{F} = F + \frac{2\pi}{k}N$. It is then convenient to define the following transverse momentum

$$
\tilde{\Pi}_j(x) = \Pi_j(x) - \frac{1}{4\pi}\partial_j \int d^2y\, P(x-y)\left(N - \rho(y) - \frac{k}{4\pi}\tilde{F}(y)\right), \tag{3.17}
$$

with the properties

$$
\partial_j \tilde{\Pi}_j = 0, \qquad \int d^2x\, \tilde{\Pi}_j = 0.
$$

After some algebra, using in particular eq. (3.11), one can obtain the following expressions for the Hamiltonian and Momentum operators, in which the usual Coulomb interaction appears explicitly (see refs. [88,97] for the analogous expressions on the plane)

$$
\begin{aligned}
H &= \int d^2x \left(\frac{i}{2m}\Pi\vec{D}^2\Psi + \frac{g}{2}F^2 + \frac{1}{2g}\tilde{\Pi}_i^2 \right) + \frac{1}{2g}\left(\frac{k}{4\pi}a_j + \epsilon_{ji}b_i\right)^2 \\
&\quad - \frac{1}{8\pi g}\int d^2x\, d^2y \left(\rho(x) + \frac{k}{2\pi}F(x)\right) P(x-y) \left(\rho(y) + \frac{k}{2\pi}F(y)\right) \quad . \\
P_j &= \int d^2x \Pi \left(\partial_j - i\frac{\pi N}{k}\epsilon_{jh}x_h\right)\Psi + \frac{N}{2}a_j - \frac{2\pi N}{k}\epsilon_{ji}b_i + \int d^2x\, \tilde{\Pi}_i \partial_j \tilde{A}_i .
\end{aligned}
\tag{3.18}
$$

The quantization is now performed by imposing the following commutation relations

$$
\begin{aligned}
[\Psi(x), \Pi(y)] &= i\,\delta^2(x-y) \\
[\tilde{A}_i(x), \tilde{\Pi}_j(y)] &= i\,\delta^{tr}_{ij}(x-y)\,, \\
[a_i, b_j] &= i\,\delta_{ij}
\end{aligned}
\tag{3.19}
$$

while all other commutators vanish. In the second relation a transverse δ–function appears consistently with the transversality of $\tilde{A}_i$ and $\tilde{\Pi}_j$. Actually the matter field Ψ could be quantized also with Fermi–statistics, here for definiteness we have chosen Bose–statistics.

Due to the intrinsic translational invariance of the underlying surface, i.e. the torus, our theory, living on this surface, should be translational invariant too. It is, in fact, not difficult, but technically a little bit involved, to show that the expressions for H, P_1 and P_2 we derived above commute among them. The mechanism which gives rise to this result is based on a peculiar interplay between the topological components of the gauge field a, b and the mean total flux (see eq.(3.7)). We will check these commutation relations and illustrate this mechanism - for simplicity - in a first quantized formalism which we will now present briefly.

In first quantization we introduce the coordinates x_i^α of the particles described by the matter field Ψ, where α runs from 1 to N. The vector potential felt by the α-th particle becomes then

$$
A_i^\alpha = a_i + \frac{\pi}{k}\epsilon_{ij}\, x_j^\alpha N + \tilde{A}_i(x^\alpha). \tag{3.20}
$$

Defining the covariant derivatives as $D_j^\alpha = \partial_j^\alpha - iA_j^\alpha$ we can now write the first quantized Hamiltonian and Momentum as

$$H = -\frac{1}{2m}\sum_{\alpha=1}^{N}\vec{D}_\alpha^2 + \frac{1}{2g}\left(\frac{k}{4\pi}a_j + \epsilon_{ji}b_i\right)^2 + \int d^2x\left(\frac{g}{2}F^2 + \frac{1}{2g}\tilde{\Pi}_i^2\right)$$
$$- \int d^2x\, d^2y\left(\sum_\alpha \delta^2(x - x^\alpha) + \frac{k}{2\pi}F(x)\right)\frac{P(x-y)}{8\pi g}\left(\sum_\beta \delta^2(y - x^\beta) + \frac{k}{2\pi}F(y)\right)$$
$$P_j = \sum_\alpha\left(i\partial_j^\alpha + \frac{\pi N}{k}\epsilon_{jh}\, x_h^\alpha\right) + \frac{N}{2}a_j - \frac{2\pi N}{k}\epsilon_{ji}\, b_i + \int d^2x\, \tilde{\Pi}_i\partial_j\tilde{A}_i. \tag{3.21}$$

These operators act on a Hilbert space which is spanned, in coordinate representation, by the function(al)s of x_i^α, a_i and $\tilde{A}_i$: $\psi(\vec{x}^\alpha, \vec{a}, \tilde{A}_i)$. In this representation the conjugate momenta are represented by

$$\begin{aligned}\tilde{\Pi}_i(x) &= -i\int d^2y\, \delta_{ij}^{tr}(x-y)\frac{\delta}{\delta\tilde{A}_j(y)}\\ b_i &= -i\frac{\partial}{\partial a_i}.\end{aligned} \tag{3.22}$$

Not all of the functions $\psi(\vec{x}^\alpha, \vec{a}, \tilde{A}_i)$ belong to the Hilbert space. Restrictions come from the requirements of covariance under "large" gauge transformations, $\vec{a} \to \vec{a} + 2\pi\vec{p}$, and of quasi–periodicity of the wave–function under shifts $\vec{x} \to \vec{x} + \vec{n}$, where p_i and n_i are integers. The covariance of the Hamiltonian and Momentum requires the transformation properties

$$\psi(\vec{a} + 2\pi\vec{p}) = \exp\left(-\frac{ik}{2}\epsilon_{ij}p_i a_j\right)\exp\left(2\pi i\sum_\alpha \vec{x}^\alpha\cdot\vec{p}\right)\exp\left(i\pi k(p_1p_2 + \alpha(p_1+p_2))\right)\psi(\vec{a})$$
$$\psi(\vec{x}^\alpha + \vec{n}^\alpha) = \exp\left(-\frac{i\pi N}{k}\epsilon_{ij}\sum_\alpha n_i^\alpha\cdot x_j^\alpha\right)\exp\left(i\pi\frac{N}{k}\sum_\alpha(n_1^\alpha n_2^\alpha + \beta(n_1^\alpha + n_2^\alpha))\right)\psi(\vec{x}^\alpha), \tag{3.23}$$

while ψ can be an arbitrary function of $\tilde{A}_i$. The phases proportional to p_1p_2 and $n_1^\alpha n_2^\alpha$ ensure the group property of the transformations, while α and β are arbitrary. In order to save the modular invariance (which is broken in our present theory due to the presence of the Maxwell term in the Action (3.1)) in the limit $g \to 0$ we will take $\alpha = \beta = 1$ [20]. If we impose that the "large" gauge transformations along the two handles of the torus commute, then k has to be an *integer*, and if we require, moreover, the compatibility of the transition functions in the second relation of eq. (3.23)then we find the Dirac quantization condition

$$\frac{N}{k} = \text{integer.}$$

It is interesting to note that these quantization conditions for k and N/k are precisely the ones which are also found in the theory on the torus without the Maxwell term in the action (3.1), i.e. at $g = 0$, see for example refs. [20,61,68]. Actually, the Dirac quantization condition is compatible in general with *rational* k. This general case is more complicated and can be discussed following ref [20].

Let us now see how it comes that the commutator $[P_i, P_j]$ vanishes. The terms of the form $\tilde{\Pi}\partial\tilde{A}$ commute among them and with the rest, see eq. (3.21). The topological components give a contribution to the commutator which is $\frac{2\pi i N^2}{k}\epsilon_{ij}$ and this contribution is precisely cancelled by the action of the derivative operator $\sum_\alpha \partial_i^\alpha$ on the mean total flux $\epsilon_{jh}\sum_\alpha x_h^\alpha$. An analogous cancellation between the topological components of the gauge field and the mean flux occurs in the commutator $[H, P_j]$. In summary we get $[H, P_j] = 0 = [P_i, P_j]$, as expected.

Actually we can give an explicit representation of the topological sector of the Hilbert space defined in (3.23). The "free" topological Hamiltonian H_0 can, in fact, be read off from eq. (3.21)as

$$\begin{aligned} H_0 &= \frac{1}{2g}\left(\frac{k}{4\pi}a_j + \epsilon_{ji}b_i\right)^2 = \frac{1}{2g}\left(\frac{k}{2}\bar{a} - \frac{1}{\pi}\frac{\partial}{\partial a}\right)\left(\frac{k}{2}a + \frac{1}{\pi}\frac{\partial}{\partial \bar{a}}\right) + c_0 \\ &\equiv \frac{1}{2g}A^\dagger A + c_0 \end{aligned}, \tag{3.24}$$

where we introduced complex topological components $a, \bar{a}$ via the definition

$$a = \frac{1}{2\pi}(ia_1 + a_2),$$

and c_0 is some constant. The algebra $[A, A^\dagger] = \frac{k}{\pi}$ holds, meaning that H_0 corresponds to a harmonic oscillator. The ground–states of H_0 are given by the functions which are annihilated by the operator A

$$A\,\psi^{(0)} = 0$$

in compatibility with the defining eq. (3.23). One gets

$$\psi^{(0)} = \exp\left(-\frac{\pi k}{2} a\bar{a}\right)\phi(a)$$

where $\phi(a)$ has to be a holomorphic function of a. Imposing now the first condition of eq. (3.7)furnishes the k independent ground–state solutions $\psi_m^{(0)} = \exp\left(-\frac{\pi k}{2} a\bar{a}\right)\phi_m$ $(m = 1, ..., k)$ where

$$\phi_m = \exp\left(\pi\left(a - \frac{\bar{z}}{k}\right)z + \frac{\pi k}{2}\left(a - \frac{\bar{z}}{k}\right)^2 + \frac{\pi|z|^2}{2k}\right)\overline{\theta\begin{bmatrix}\frac{m}{k} + \frac{1}{2}\\ \frac{k}{2}\end{bmatrix}(k\bar{a} - z|ki)}. \quad (3.25)$$

Here we introduced the center–of–mass coordinate $z = \sum_\alpha z^\alpha$ using a complex notation also for the particles' coordinates: $z^\alpha = x_1^\alpha + ix_2^\alpha$; the (covariant) derivatives become $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$, $D = \partial - iA, \bar{D} = \partial - i\bar{A}$, where $A = \frac{1}{2}(A_1 - iA_2)$, $\bar{A} = \frac{1}{2}(A_1 + iA_2)$ etc. This notation will be convenient in what follows.

Due to the covariance of the operators $A, A^\dagger$ the general state in the topological sector of the Hilbert space can be written as a superposition of the states

$$\psi_{m,l}^{(0)} = (A^\dagger)^l\,\psi_m^{(0)}, \quad (3.26)$$

where l is an integer. The generic state in the total Hilbert space can be written as $\psi = \sum_{m=1}^{k}\sum_{l=0}^{\infty} C_{m,l}\,\psi_{m,l}^{(0)}$ where the $C_{m,l}$ are functions which depend on z^α and $\tilde{A}$, but not on $a, \bar{a}$.

An interesting question is in which sense one can get from the theory under investigation here the one describing free anyons on the torus. The relevant limit is of course $g \to 0$, which we will now investigate in some detail in the final part of this chapter.

To get under the limit $g \to 0$ a finite Hamiltonian in eq. (3.21), we see that first of all $\tilde{\Pi}_i$ should go to zero. This implies that the wave function does no longer depend on the dynamical part $\tilde{A}$ of the vector potential, but only on its topological components and the particles' coordinates. Second, in this limit (in the first quantized language we are using here $\rho(x) = \sum_\alpha \delta^2(x - x_\alpha)$) we must have

$$\rho(x) + \frac{k}{2\pi} F(x) = 0$$

Notice that this relation insures that the second row in the expression for the Hamiltonian eq. (3.21)goes to zero as it is necessary to keep the Hamiltonian finite. It then follows that we can write (see eq.(3.11)):

$$-iA_\alpha = -\pi a + \frac{\pi}{2k} N \bar{z}_\alpha + \frac{1}{2k} \sum_{\beta \neq \alpha} \partial_\alpha P(z^\alpha - z^\beta)$$

with an analogous expression for $\overline{A}$ (some subtleties concerning a normal ordering ambiguity are explained in [81]). For what concerns the topological sector we observe that in order to keep H_0 in eq. (3.24)finite we have to restrict the Hilbert space to the states satisfying $A\psi = 0$. But this means that the topological sector is projected on his ground–states $\psi_m^{(0)}$, which we determined above, and that $H_0 \to 0$. Putting everything together we obtain for the Hamiltonian and (complex) Momentum for the theory at $g = 0$

$$\begin{aligned} H &= -\frac{1}{m} \sum_\alpha \left(\bar{D}_\alpha D_\alpha + D_\alpha \bar{D}_\alpha \right) \\ P &= \sum_\alpha \left(\partial_\alpha + \frac{\pi N}{2k} \bar{z}_\alpha - \frac{\pi}{2} a + \frac{1}{k} \frac{\partial}{\partial \bar{a}} \right) \\ \bar{P} &= \sum_\alpha \left(\bar{\partial}_\alpha - \frac{\pi N}{2k} z_\alpha + \frac{\pi}{2} \bar{a} + \frac{1}{k} \frac{\partial}{\partial a} \right), \end{aligned} \tag{3.27}$$

where the covariant derivatives are given by

$$
\begin{aligned}
D_\alpha &= \partial_\alpha - \pi a + \frac{\pi}{2k} N \bar{z}_\alpha + \frac{1}{2k} \sum_{\beta \neq \alpha} \partial_\alpha P(z^\alpha - z^\beta) \\
\bar{D}_\alpha &= \bar{\partial}_\alpha + \pi \bar{a} - \frac{\pi}{2k} N z_\alpha - \frac{1}{2k} \sum_{\beta \neq \alpha} \bar{\partial}_\alpha P(z^\alpha - z^\beta)
\end{aligned} \tag{3.28}
$$

These operators act on the (projected) Hilbert space spanned by the states $\psi_m^{(0)} = \exp\left(-\frac{\pi k}{2} a\bar{a}\right) \phi_m$. If we factorize the exponential $\exp\left(-\frac{\pi k}{2} a\bar{a}\right)$ to let these operators act directly on the ϕ the Hamiltonian remaines the same, while the Momenta turn into

$$
\begin{aligned}
P &= \sum_\alpha \left(\partial_\alpha + \frac{\pi N}{2k} \bar{z}_\alpha - \pi a \right) \\
\bar{P} &= \sum_\alpha \left(\bar{\partial}_\alpha - \frac{\pi N}{2k} z_\alpha + \frac{1}{k} \frac{\partial}{\partial a} \right)
\end{aligned} \tag{3.29}
$$

As last step we observe that in the ϕ–space the scalar product is now defined in terms of the coherent state measure

$$
\langle \phi' | \phi \rangle = \int d^2 a \exp\left(-\pi k a \bar{a}\right) \overline{\phi'(a)} \phi(a),
$$

and in the Hilbert space endowed with this measure the operator $\bar{a} = a^\dagger$ can be represented as the derivative with respect to a:

$$
\bar{a} = \frac{1}{\pi k} \frac{\partial}{\partial a}. \tag{3.30}
$$

As a consequence we get the non–trivial commutator

$$
[\bar{a}, a] = \frac{1}{\pi k}. \tag{3.31}
$$

Notice that in the original theory this commutator is zero in that a_1 and a_2 were independent commuting variables. The fact that under the limit $g \to 0$ a non zero commutator is gotten is explained observing that the Hilbert space of the final theory has been obtained by a suitable projection of the original one and that the operators of the new theory have to be restricted in this projected space; but a projection is an operation which in general does not preserve algebraic relations, see also 68. We remark, moreover, that also the final theory, as the initial one, should be translational invariant in that also the final theory lives on a torus. The relation (3.31)is precisely the one which insures translational invariance. In fact, after taking it into account one verifies easily that $P = \sum_\alpha D_\alpha$, $\bar{P} = \sum_\alpha \bar{D}_\alpha$ and that the relations $[P, H] = [\bar{P}, H] = [P, \bar{P}] = 0$ are indeed satisfied.

The Hilbert space of the theory at $g \to 0$ is given by the functions $\psi = \sum_{m=1}^{k} G_m \phi_m$, where the G_m are functions only of the coordinates $z^\alpha, \bar{z}^\alpha$ which have to be determined such that the second relation of (3.23)is satisfied. All these results, and in particular eq. (3.25), are indeed consistent with what has been found in refs. [191,192] for the Chern–Simons theory coupled to a non relativistic matter field; we refer to those papers for further developments of the theory at $g = 0$.

4. Anyon and Chern-Simons Theory in Fractional Quantum Hall Effect (FQHE)

4.1. Ginzburg-Landau Theory of FQHE and Anyon

That anyons exist on FQHE is a well established fact. By using Berry phase, from the Laughling wave function, one can show that the quasi particles are anyons, which obey fractional statistics. Recently, people by studying Ginzburg-Landau theory of FQHE, obtained many results which were gotten from the wave function approach. These two methods compensate with each other. In this chapter, we will concentrate on GL theory of FQHE, and see how anyons appear in FQHE. For references, for example, [14,21,17] *etc*, and one of the recent reference [47],

We start from Hamiltonian [14]

$$H=\int d^2x-\frac{1}{2m}|\mathbf{D}\Psi(x)|^2+\frac{1}{2}\int d^2xd^2x\prime(\Psi(x)^\dagger\Psi(x)-\rho)V(\mathrm{x}-\mathrm{x}')(\Psi(x')^\dagger\Psi(x')-\rho). \tag{4.1}$$

We consider the case that the spins of the electrons are polarized. So ϕ is one component fermion. However for nonpolarized case, see the paper of D. H. Lee and C. L. Kane [48]. The covariant derivative defined here is, $D_i=\partial_i+ieA_i$, A is the magnetic field. And V is the interaction potential between electrons, for example, Coulomb potential. In $2+1$ dimension, it is possible to bosonize fermions (for example, [31] and many others)

$$\Psi(x)=e^{-ik\int d^2x'\theta(\mathrm{x}-\mathrm{x}')j_0(x')}\phi(x), \tag{4.2}$$

with $j_0(x)=\phi^\dagger(x)\phi(x)$, ϕ is a boson field, $\theta(\mathrm{x})$ is the polar angle of x, and k is an odd integer we will specify below. Substitute those equation to the Hamiltonian above, one gets a Hamiltonian in boson field

$$H=\int d^2x-\frac{1}{2m}|(D_i-ia_i)\phi(x)|^2+\frac{1}{2}d^2xd^2x\prime(\phi(x)^\dagger\phi(x)-\rho)V(\mathrm{x}-\mathrm{x}')(\phi(x')^\dagger\phi(x')-\rho), \tag{4.3}$$

with

$$a_i(x) = -\int d^2x k \epsilon^{ij} \frac{x^j - x'^j}{|\mathbf{x} - \mathbf{x}'|^2} j_0(x'). \tag{4.4}$$

And one can show

$$\epsilon^{ij} \partial_i a_j = 2k\pi j_0(x). \tag{4.5}$$

This constrain can be realized by using Chern-Simons term. The lagrangian with Chern-Simons term is

$$\begin{aligned} L = \int d^2x \phi \dagger i(D_0 - ia_0)\phi - \frac{1}{2m}|(D_i - ia_i)\phi|^2 - \frac{1}{4k\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \\ - \frac{1}{2} d^2x d^2x \prime (\phi(x)^\dagger \phi(x) - \rho) V(\mathbf{x} - \mathbf{x}')(\phi(x')^\dagger \phi(x') - \rho) \end{aligned} . \tag{4.6}$$

FQHE happens when the boson field is condensed to the ground state (see [47], and reference there in). The classical solution of the equation of motion corresponding the condensation state is

$$\phi = \sqrt{\rho},\ a_0 = 0,\ \mathbf{a} = e\mathbf{A},\ \epsilon^{ij}\partial_i a_j = 2k\pi\rho. \tag{4.7}$$

We take $A_0 = 0$. If there exists a solution, we need $2k\pi\rho = B$, this corresponding to filling factor $\nu = \frac{1}{k}$. The corresponding state is Laughling state at filling $\nu = \frac{1}{k}$. On the other hand, for the GL theory of FQHE, k equals to $\frac{1}{\nu}$. Then the physical picture of FQHE is, every electron is bounded with k fundamental quanta flux of statistical field and magnetic field. This composite object is a boson, because when two objects are exchanged, one will get an extra minus sign from the *statistical* flux (not magnetic flux). On this point, there are some chaos in literature. The ordinary magnetic flux bounded with electrons can not be used for transmutation of the statistics of the particles. The statistical flux will do the job for transmutation of statistics as one can remember the factor half for the calculation of statistical parameter when the gauge field has Chern-Simons term (for details, see the chapter *Some Anyon Models*). Now, let electrons move, then the magnetic flux bounded with the electrons also move, from Faraday theorem, the moving flux will induce electric field. Then one can get quantized Hall resistance $\sigma = \frac{kh}{e^2}$ from Faraday theorem [47].

Now one considers the vortex excitation in the condensed state (Z. F. Ezawa and A. Iwazaki, [21]). The one vortex excitation solution above condensation state in the asymptotic behavior (r goes to infinity, or the distance much larger than magnetic length) is

$$\phi \simeq \sqrt{\rho} e^{i\theta}, \ a_i \simeq eA_i + \partial_i \theta, \tag{4.8}$$

with $a_0 = 0$. From (4.5), the number of electron will increase a number

$$\begin{aligned} \Delta N &= \frac{1}{2k\pi} \int d^2x \epsilon^{ij} \partial_i \Delta a_j \\ &= \frac{1}{2k\pi} \int d^2x \epsilon^{ij} \partial_i \partial_j \theta = \frac{1}{k} \end{aligned} \tag{4.9}$$

One should be careful with $\epsilon^{ij}\partial_i\partial_j\theta(\mathbf{x})$, because $\theta(\mathbf{x})$ is a multivalued function, it will not be zero, but $2\pi\delta(\mathbf{x})$. So if one wants to create a vortex excitation, then will create $\frac{1}{k}$ electron, which agrees with the wave function approach. Actually the vortex has a finite size in FQHE, but it can be approximated as a point particle because strong magnetic field (then one has smaller magnetic length). One can study the statistics and spin of those vortices, or quasi particles. Due to $\Delta N = \frac{1}{k}$, when a quasi rotates around other particle with fundamental flux of statistics field, we get a phase $e^{\frac{2\pi i}{k}}$ (keep in mind the factor half because a is statistical field). So the statistical parameter is $\Delta = \frac{1}{2k}$. Finally, one can conclude the quasi particles obey fractional statistics. There is some delicate points about GL theory of FQHE. We will discuss it in the following chapters.

4.2. Hierarchy Structure in GL theory of FQHE

Now let us consider more general vortices excitation. With vortices excitation, the field is

$$\phi = \phi^b e^{i\chi}, a_\mu = a_\mu^b + \partial_\mu \chi. \tag{4.10}$$

The meaning of bar is that the phase angle of bar field is single valued. χ describes the multivalued phase of the field. Substitute above equation into lagrangian density, one gets the lagrangian density

$$L_d = L_d^b - \frac{1}{2k\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu \partial_\lambda \chi - \frac{1}{4k\pi}\epsilon^{\mu\nu\lambda}\partial_\mu \partial_\nu \partial_\lambda \chi, \tag{4.11}$$

here L_d^b is the lagrangian density with bare fields. Following [21] we define

$$k^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu \partial_\lambda \chi \ , G = \frac{1}{4\pi^2}\epsilon^{\mu\nu\lambda}\partial_\mu \chi \partial_\nu \partial_\lambda \chi. \tag{4.12}$$

Now we consider N vortices and anti vortices, with coordinates x_k, $k = 1, \cdots, N$. Then $\chi = \sum_k \epsilon_k \theta_k$, $\epsilon = 1$ corresponds a vortex, $\epsilon = -1$ corresponds an anti vortex, and $\theta_k = \theta(\mathrm{x} - \mathrm{x}_k)$. It is not difficult to verify the following equation

$$\begin{aligned} k^\mu &= \frac{1}{2\pi}\sum_{k=1}^{N} \epsilon_k \epsilon^{\mu\nu\lambda}\partial_\nu \partial_\lambda \theta_k \\ &= \sum_{k=1}^{N} \int dx_k^\mu \delta^3(x - x_k) \end{aligned} . \tag{4.13}$$

Remember that particle propagate forwards and anti particle propagate backwards along time. Another quantity G is related to the linking number of vortices by using equation (4.13)

$$\begin{aligned} \int d^3x G &= \int d^3x \frac{1}{2\pi} k^\mu \partial_\mu \chi \\ &= \frac{1}{2\pi}\sum_{k_1,k_2}^{N} \int dx_{k_1}^\mu \partial_\mu \theta(\mathrm{x}_{k_1} - \mathrm{x}_{k_2})\epsilon_{k_2} \end{aligned} . \tag{4.14}$$

Two vortices or two anti vortices will give above equation a positive integer. One vortex and one anti vortex will give a negative integer. But there are some ambiguities in the formula because self linking term, when $k_1 = k_2$. It can be overcome by introducing framing of the path. One will find $\int d^3xG$ is an integer. Now the lagrangian density (4.11)can be written as

$$L_d = L_d^b - \frac{1}{k} a_\mu^b k^\mu - \frac{\pi}{k} G. \tag{4.15}$$

The topological term G can be discarded if only if $\frac{\pi}{k} = 2\pi \cdot integer$ as one can see it by using path integral quantization. For the problem of FQHE, this term can not be thrown away. L_d describes a series of equivalent theory with lagrangian density

$$L_d(p) = L_d + 2\pi p G, \tag{4.16}$$

with p an integer. Because, different p has same physics, like θ vacuum in QCD, θ and $\theta + 2\pi$ have same vacuum structure. Physically, this lagrangian describes a theory that quasi particle are bounded with even number $2p$ flux quanta (statistical flux) as one can see it from the meaning G.

Now use identity (it is easy to check it)

$$e^{i\int d^3x\alpha G} = \int Db_\mu e^{-i\int d^3x(b_\mu k^\mu + \frac{1}{4\alpha}\epsilon^{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda)}. \tag{4.17}$$

One can eliminate the term G. From the lagrangian density $L_d(p)$, one gets the equivalent lagrangian density

$$L_d(p) = L_d^b - (\frac{1}{k} a_\mu^b + b_\mu) k^\mu - \frac{1}{4\pi(-\frac{1}{k} + 2p)} \epsilon^{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda. \tag{4.18}$$

In order to get a quantum field theory of vortexes, one uses identity

$$\sum_{N=1}^{\infty} \frac{1}{N!} \int \prod_k^N Dx_k e^{(-i\int d^3x c_\mu k^\mu)} = \int D\phi e^{i\int d^3x L_d}, \tag{4.19}$$

here L_d is (see [49])

$$L_d = |i\partial_\mu + c_\mu|\phi^2 - M^2 \phi^\dagger \phi$$

The above equation can be understood as following way. We consider a scalar theory interacting with a nondynamical gauge field c_μ. Non dynamical means there is not Yang-Mills term or other terms which contain time derivative of the field in the theory. Now we consider partition function, the probability of transition from vacuum to vacuum. In the process, the number of particles is conserved, the theory is essential free theory. Consider N fixed particle, then sum all amplitude with different N. For fixed N process, the only amplitude is contributed from the coupling of particle with gauge field. There are no loop diagrams in this free theory. The above equation exactly is due to this reason.

So by using equation (4.19), we can get a lagrangian density L_d^v with vortices

$$\begin{aligned} L_d(p) = L_d^b &- \frac{1}{4\pi(-\frac{1}{k}+2p)}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda \\ &+ |(i\partial_\mu + \frac{1}{k}a_\mu^b + b_\mu)|\phi_q{}^2 - m_q^2\phi_q^\dagger\phi_q \end{aligned} , \tag{4.20}$$

here ϕ_q is the field of vortices, m_q is the mass of vortex (it was claimed in [21], that Ezawa, Iwazaki and Hotta in a unpublished report, they got (in the nonrelativistic limit) $m_q = \frac{m}{k}$ as expected because the quasi particle has $\frac{1}{k}$ electron number).

Now we do the following transformation for the lagrangian density when $p = 0$

$$c_\mu = b_\mu + \frac{1}{k}a_\mu^b. \tag{4.21}$$

a field remains unchanged. So there is not nontrivial Jacobian. One gets a lagrangian density when $p = 0$

$$\begin{aligned} L_d = &\ \phi_e^\dagger i(\partial_0 - ia_0 + ieA_0)\phi_e + \frac{1}{2m}\phi_e^\dagger(\partial_i - ia_i + ieA_i)^2\phi_e \\ &+ \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu a_\lambda \\ &+ i\phi_q^\dagger(\partial_0 - ic_0)\phi_q + \frac{1}{2m_q}\phi_q^\dagger(\partial_i - ic_i)^2\phi_q - V_i \end{aligned} , \tag{4.22}$$

here ϕ_e is the electron field, ϕ_q is the vortex field, and V_i is the lagrangian density due to Coulomb interaction. For simplicity, now we will omit superscript b for the field a^b_μ (but should remember it is bar field), And have written the lagrangian density in nonrelativistic limit.

For $p \neq 0$, one has (we can put it on path integral, and do some manipulations in the path integral)

$$\begin{aligned} L^v_d(p) &= \phi^\dagger_e i(\partial_0 - ia_0 + ieA_0)\phi_e + \frac{1}{2m}\phi^\dagger_e(\partial_i - ia_i + ieA_i)^2\phi_e \\ &+ \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu a_\lambda - \frac{1}{8\pi p}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda \\ &+ i\phi^\dagger_q(\partial_0 - ic_0 - ib_0)\phi_q + \frac{1}{2m_q}\phi^\dagger_q(\partial_i - ic_i - ib_i)^2\phi_q - V_i \end{aligned} \quad . \tag{4.23}$$

(we should write b field as b', because it will confuse with equation (4.21). To get equation (4.23), one do transformation (4.21), and introduce another field b'. But for the simplicity of notation, we write b instead b'. We hope it will not be confused. We deeply apologize for the similar problems in other place due to our lake of experience about how to let the notation clear and consistent). One can also get the statistics of vortex directly from the lagrangian (4.23). For the detailed discussion about how to get the statistics of vortex or quasi particle from GL theory of FQHE, see [17].

Now we consider the condensation state of the lagrangian (4.23). From the equation of motion, *one* of condensation state is (another is Laughling state without vortex excitation we describe before)

$$\begin{aligned} & b_\mu + c_\mu = a_\mu - eA_\mu = 0 \\ & n_e = \frac{1}{2\pi}\epsilon^{ij}\partial_i c_j \\ & n_q = -\frac{k}{2\pi}\epsilon^{ij}\partial_i c_j + \frac{1}{2\pi}\epsilon^{ij}\partial_i a_j = \frac{1}{4\pi p}\epsilon^{ij}\partial_i b_j \end{aligned} \quad . \tag{4.24}$$

We get the parameter for this state

$$\nu = \frac{2\pi n_e}{B} = \frac{1}{k - \frac{1}{2p}}\,,\ n_q = \frac{n_e}{2p}. \tag{4.25}$$

This is the second level hierarchical state of FQHE (we define that Laughling state is the first level). Still we can consider the vortex excitation around this state by following exactly the approach to Laughling state [21,17]. General level hierarchical GL theory of FQHE can be obtained in this way. A nice formula for n level hierarchical GL theory of FQHE obtained in [17]

$$
\begin{aligned}
L_{hn} = & i\phi_e^\dagger(\partial_0 - ia_0^0 + ieA_0)\phi_e + \frac{1}{2m_e}\phi_e^\dagger(\partial_i - ia_i^0 + ieA_i)^2\phi_e + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_\mu^0\partial_\nu a_\lambda^1 \\
& + \sum_{k=1}^{n-1}[i\phi_q^{\dagger k}(\partial_0 + ia_0^{2k-1} + ia_0^{2k})\phi_q^k + \frac{1}{2m_q^k}\phi_q^{\dagger k}(\partial_i + ia_i^{2k-1} + ia_i^{2k})^2\phi_q^k \\
& + \frac{p_k}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu^{2k-1}\partial_\nu a_\lambda^{2k-1} + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_\mu^{2k}\partial_\nu a_\lambda^{2k+1}] \\
& + i\phi_q^{\dagger n-1}(\partial_0 + ia_0^{2n-3} + ia_0^{2n-2})\phi_q^{n-1} \\
& + \frac{1}{2m_q^{n-1}}\phi_q^{\dagger n-1}(\partial_i + ia_i^{2n-3} + ia_i^{2n-2})^2\phi_q^{n-1} \\
& + \frac{p_{n-1}}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu^{2n-3}\partial_\nu a_\lambda^{2n-3} - \frac{1}{4\pi p_n}\epsilon^{\mu\nu\lambda}a_\mu^{2n-2}\partial_\nu a_\lambda^{2n-2}
\end{aligned}
\qquad , \qquad (4.26)
$$

here p_1 is an odd positive number, $p_i, i = 2, \cdots, n$ are even integers. Also one should keep in mind that except last two gauge fields, $a_\mu^{2n-3}, a_\mu^{2n-2}$, all other fields are bar fields. This means that in the vortex excitation around $n-1$ level hierarchy condensation state, there are no vortex excitation of gauge fields except last two gauge fields. However, we can consider nontrivial topology sector for all gauge fields when we consider the vortex excitation induced by adiabatic turning of magnetic flux. Because when you are turning on some magnetic flux, due to a_μ^0 always companied with A_μ, you can equally say a_μ^0 is added some flux, and A_μ is unchanged. So a_μ^0 is in nontrivial topological sector. Same reasoning can be proceeded to other a_μ^k gauge fields.

One can also study the condensation state of above lagrangian, for example, statistics of vortex excitation around the condensation state [17]. The corresponding studies from microscopic point of view (directly from wave function) can be found in second paper in [17]. The condensate state of above theory has filling factor (it is not difficult to work it out by following the example of level 2 hierarchical state)

$$\nu = \cfrac{1}{p_1 - \cfrac{1}{p_2 - \cfrac{1}{\cdots - \cfrac{1}{p_n}}}}. \tag{4.27}$$

The state is determined by p_i and n_e, all other parameters are dependent on them. We will study hierarchical state on the sphere from microscopic point of view in the late chapters.

4.3. VORTEX-CHARGE DUALITY

We consider other version of effective theory of FQHE, dual form of above Ginzburg-Landau theory of FQHE. We start from the lagrangian (4.22)of Laughling state with vortex excitation (we change c to $-c$ compare the lagrangian (4.22). The notation here is the same as (4.26))

$$\begin{aligned} L_d = & \phi_e^\dagger i(\partial_0 - ia_0 + ieA_0)\phi_e + \frac{1}{2m}\phi_e^\dagger(\partial_i - ia_i + ieA_i)^2\phi_e \\ & + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu a_\lambda \\ & + i\phi_q^\dagger(\partial_0 + ic_0)\phi_q + \frac{1}{2m_q}\phi_q^\dagger(\partial_i + ic_i)^2\phi_q - V_i \end{aligned} \quad . \tag{4.28}$$

Doing transformation, shift a_μ field to $a_\mu + eA_\mu$, new lagrangian is

$$\begin{aligned} L_d = & \phi_e^\dagger i(\partial_0 - ia_0)\phi_e + \frac{1}{2m}\phi_e^\dagger(\partial_i - ia_i)^2\phi_e \\ & + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda + \frac{e}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu A_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu a_\lambda \, . \\ & + i\phi_q^\dagger(\partial_0 + ic_0)\phi_q + \frac{1}{2m_q}\phi_q^\dagger(\partial_i + ic_i)^2\phi_q - V_i \end{aligned} \tag{4.29}$$

Now,one integrates ϕ_e and a_μ fields to get an effective action. First to integrate the lagrangian density

$$
\begin{aligned}
L_e = \phi_e^\dagger i(\partial_0 - ia_0)\phi_e + \frac{1}{2m}\phi_e^\dagger(\partial_i - ia_i)^2\phi_e \\
+ \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu a_\lambda - V_i
\end{aligned}
. \tag{4.30}
$$

First note that for the condensation state, one takes $a_\mu = 0$ in (4.30). It is a vacuum state which respects rotational symmetry *etc.* When integrates a, One gets a lagrangian density which only depends on $\epsilon^{\mu\nu\lambda}\partial_\nu c_\lambda$, field of c_μ. Also due to rotational invariance (the above lagrangian has rotational symmetry. it is a nonrelativistic limit of other relativistic theory. We suppose the basic theory is Lorentz invariant), effective lagrangian should depend on $(\partial_1 c_2 - \partial_2 c_1)^2$ and $\sum_{i=1}^{2}(\partial_0 c_i - \partial_i c_0)^2$. Further suppose there is Lorentz invariance, then the effective lagrangian is $\frac{1}{g^2}(c_{\mu\nu})^2$, $c_{\mu\nu} = \partial_\mu c_\nu - \partial_\nu c_\mu$. Here g is some coefficient which we do not need to know for the following discussion. We also omit higher order terms because it is irrelevant in long distance limit (compare to magnetic length) which we are interested in.

Before doing integration, one has the equation of motion from the lagrangian (4.29)

$$
j_e^\mu = -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu c_\lambda. \tag{4.31}
$$

It is a useful equation for us tracing back to original theory from the dual form theory. Also the fancy name, charge-vortex duality is due to this reason. Now to summarize, one obtains a lagrangian density

$$
\begin{aligned}
L_d = \frac{1}{g^2}(c_{\mu\nu})^2 + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda + \frac{e}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu A_\lambda \\
+ i\phi_q^\dagger(\partial_0 + ic_0)\phi_q + \frac{1}{2m_q}\phi_q^\dagger(\partial_i + ic_i)^2\phi_q
\end{aligned}
, \tag{4.32}
$$

with the charge-vortex duality relation (4.31). For the lagrangian $L_d(p)$ (vortex binding with $2p$ quanta, then condensate vortex excitation, it is a level two hierarchical state), the same step leads to

$$
\begin{aligned}
L_d = & \frac{1}{g^2}(c_{\mu\nu})^2 + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda + \frac{e}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu A_\lambda \\
& + i\phi_q^\dagger(\partial_0 + ic_0 + ib_0)\phi_q + \frac{1}{2m_q}\phi_q^\dagger(\partial_i + ic_i + ib_i)^2\phi_q - \frac{1}{8p}\epsilon^{\mu\nu\lambda}b_\mu\partial_{nu}b_\lambda
\end{aligned} . \quad (4.33)
$$

Then one would like to integrate ϕ_q field. One introduces another field d_μ, changes the term

$$
\begin{aligned}
& -\frac{1}{8p}\epsilon^{\mu\nu\lambda}b_\mu\partial_\nu b_\lambda \\
& \simeq \frac{2p}{4\pi}\epsilon^{\mu\nu\lambda}d_\mu\partial_\nu d_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}d_\mu\partial_\nu b_\lambda
\end{aligned} . \quad (4.34)
$$

Then make a transformation b_μ to $b_\mu - c_\mu$, and finally integrate b_μ, we get a effective lagrangian density

$$
\begin{aligned}
L_d = & \frac{1}{g^2}(c_{\mu\nu})^2 + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda + \frac{e}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu A_\lambda \\
& + \frac{2p}{4\pi}\epsilon^{\mu\nu\lambda}d_\mu\partial_\nu d_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}d_\mu\partial_\nu c_\lambda + \frac{1}{g_q^2}(d\mu\nu)^2
\end{aligned} , \quad (4.35)
$$

with vortex-charge duality for the quasi particles from the equation of motion of (4.33)

$$
j_q^\mu = -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu d_\lambda. \quad (4.36)
$$

If we add vortex excitation around condensation state, one would have a lagrangian density

$$
\begin{aligned}
L_d = & \frac{1}{g^2}(c_{\mu\nu})^2 + \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu c_\lambda + \frac{e}{2\pi}\epsilon^{\mu\nu\lambda}c_\mu\partial_\nu A_\lambda \\
& + \frac{2p}{4\pi}\epsilon^{\mu\nu\lambda}d_\mu\partial_\nu d_\lambda + \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}d_\mu\partial_\nu c_\lambda + \frac{1}{g_q^2}(d\mu\nu)^2 \\
& + i\phi_{q1}^\dagger(\partial_0 + ic_0)\phi_{q1} + \frac{1}{2m_{q1}}\phi_{q1}^\dagger(\partial_i + ic_i)^2\phi_{q1} \\
& + i\phi_{q2}^\dagger(\partial_0 + id_0)\phi_{q2} + \frac{1}{2m_{q2}}\phi_{q2}^\dagger(\partial_i + id_i)^2\phi_{q2}
\end{aligned} , \quad (4.37)
$$

with the ϕ_{q1} is the vortex excitation due to adiabatic adding the magnetic flux, ϕ_{q2} is the ordinary vortex excitation around condensation state. Because the

Yang-Mills term is not relevant for discussing the statistics and filling, which is due to Chern-Simons term, the gauge fields are screened, for convenience, we will not write them in the following formula.

One can generalize above argument to n level hierarchical state. The Chern-Simons term in the lagrangian density is

$$L_{dc} = \frac{e}{2\pi}\epsilon^{\mu\nu\lambda}a_{1\mu}\partial_\nu A_\lambda + \sum_{k,l}^{n}\frac{1}{4\pi}\Lambda_{kl}\epsilon^{\mu\nu\lambda}a_{k\mu}\partial_\nu a_{l\lambda}, \tag{4.38}$$

with matrix

$$\Lambda = \begin{pmatrix} p_1 & 1 & 0 & \cdots & 0 \\ 1 & p_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & p_n \end{pmatrix}. \tag{4.39}$$

And the charge-duality relation

$$j_k^\mu = -\frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_{k\mu}\partial_\nu a_{k\lambda}. \tag{4.40}$$

The lagrangian density of the vortex excitation of the theory

$$L_{dv} = \sum_k i\phi_k^\dagger(\partial_0 + ia_{k0})\phi_k + \frac{1}{2m_k}\phi_k^\dagger(\partial_i + ia_{ki})^2\phi_k. \tag{4.41}$$

From the equation of motion, we find

$$-\Lambda_{kl}j_l^\mu + \frac{e}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda\delta_{01} - j_{qk}^\mu = 0. \tag{4.42}$$

j_l^μ is the currents of the condensed quasi particle, and j_{qk}^μ is the currents of the vortex excitation. Now if let $j_{qk}^\mu = 0$, we can find the filling of the condensation state of FQHE, the formula is the same as before (4.27). The number of other quasi particle also can be solved from this equation. The statistics of the vortex excitation can be also easily discussed in this form. So the dual form of GL theory has a rather nice physical picture.

4.4. Universality and Symmetry of FQHE

One can study the incompressible FQHE from very general point of view, which is based on rather elementary physical requirement, for example, current conservation, gauge invariance, positive energy gap of excitation *etc* [89,50]. It will give a compensate to the approach of GL theory of FQHE.

Now assume that electrons are polarized (for nonpolarized case, this still has not been worked out by using this approach. It will be a very interesting problem) with the magnetic potential is A_μ. one has current conservation of electrons

$$\partial_\mu J^\mu = 0. \tag{4.43}$$

However in $2+1$ dimension, we can solve above equation

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda. \tag{4.44}$$

Above equation is invariant with the transformation, $a_\mu \to a_\mu + \partial_\mu f$, so it is gauge invariant. Due to the dimension of the current in $2+1$ dimension is 2, so the dimension of a_μ is 1. Now we would like to write the lagrangian of electrons in gauge field a_μ. The gauge invariant term in $2+1$ dimension will include Chern-Simons term, Yang-Mills term and other higher dimension term. In the scaling limit (long range limit), only Chern-simons term survives. Also the Chern-Simons term is responsible for the energy gap because it will give a mass to gauge field. So the lagrangian density in the scaling limit will be the form

$$\begin{aligned}\mathcal{L} &= \frac{p}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - J^\mu A_\mu \\ &= \frac{p}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu A_\lambda\end{aligned}, \tag{4.45}$$

with $-J_\mu A_\mu$ is the interaction term between electron and electromagnetic field. We have done partial integration in the above lagrangian density. One will find this is the lagrangian density of level 1 hierarchical state, Laughling state. Now integrate the field a_μ, then the effective lagrangian density is

$$\mathcal{L}_{eff} = -\frac{1}{4\pi p}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda. \tag{4.46}$$

Then one find the current is

$$J^\mu = -\frac{\delta\mathcal{L}_{eff}}{\delta A_\mu} = \frac{1}{2\pi p}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda. \tag{4.47}$$

So one finds the filling is $\nu = \frac{1}{p}$. However p can not be an arbitrary number, because the electron has fermion statistics, p must be an odd number, as one can see it from the lagrangian, the equation (4.45).

Now one considers the simple case, $p = 1$, which corresponds to fill just only 1 Landau level. And one can generalize the above argument to the case there are m branch electron currents, for example, the electrons can be in m Landau level. One assumes the gap of Landau level is so large, so it is possible to think each level is dynamical independent. The currents are

$$J_i = \frac{1}{2\pi} \star da_i. \tag{4.48}$$

We write above equation in differential form. The total electron current is

$$J = \sum_{i=1}^{m} J_i. \tag{4.49}$$

If there is no interaction between the electrons in different Landau Level, one will lagrangian density (without magnetic field)

$$\mathcal{L} = \frac{1}{4\pi}\sum_i a_i da_i. \tag{4.50}$$

It corresponds to fill m Landau level ($\nu = m$). Now suppose we add an interaction between electrons in different Landau level. (one will find that it is similar to

Jain's construction of FQHE). One makes a physical assumption, the interaction term only made from total electron current. Now the lagrangian in the scaling limit will be

$$\mathcal{L} = \frac{1}{4\pi}\sum_i a_i da_i + p(\sum_i a_i)d(\sum_i a_i). \tag{4.51}$$

Write it in matrix form

$$\mathcal{L} = \frac{1}{4\pi}\alpha K d\alpha. \tag{4.52}$$

With $K = I + pC$, C is the matrix with all element 1, and α is a matrix with $\alpha_{i1} = a_i$. One notes that the above lagrangian just gives only one kind of Quantum Hall fluid, various generalizations are possible [50]. We will also show that the above theory is equivalent to an hierarchical theory. Now write the lagrangian density in the magnetic field and with vortex excitation current

$$\mathcal{L} = \frac{1}{4\pi}\alpha K d\alpha - J \cdot A + \sum_i a_{i\mu} j^{i\mu}, \tag{4.53}$$

with $j^{i\mu}$ is vortex current. Now integrate all gauge field a_i to get an effective lagrangian density. Due to the lagrangian density only contains the power of a up to two, so a can be exactly integrated. From the equation of motion with respect to a

$$\frac{1}{2\pi}\sum_{l=1}^{m} \epsilon^{\mu\nu\lambda} K_{k,l}\partial_\nu a_{l\lambda} = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda - j_k^\mu. \tag{4.54}$$

In order to solve (4.54), One should get the inverse operator of $B^{\mu\lambda} = \epsilon^{\mu\nu\lambda}\partial_\nu$ (Green function). The operator

$$G^{\mu\lambda} = -\epsilon^{\mu\nu\lambda}\partial_\nu \Delta^-, \tag{4.55}$$

is the inverse operator. It satisfies

$$B^{\mu\nu}G_{\nu\lambda}(x) = \delta_{\mu\lambda}\delta(x) + \partial_\lambda f^\mu, \tag{4.56}$$

With f^μ which we will not specify here, and Δ is the Laplacian operator. The term $\partial_\lambda f^\mu$ can be in (4.56), and $G_{\mu\nu}$ still is a good inverse operator, because the currents in the right side of the equation (4.54)are conserved. Then the solution of the equation of motion

$$a_k^\mu = -K_{kl}^{-}G^{\mu\lambda}(j_{l\lambda} - \frac{1}{2\pi}\epsilon_\lambda^{\nu\beta}\partial_\nu A_\beta). \tag{4.57}$$

Substitute back to the lagrangian density (4.53), one gets an effective action

$$\mathcal{L} = \mathcal{L}_{AA} + \mathcal{L}_{Aq} + \mathcal{L}_{qq}, \tag{4.58}$$

With

$$\mathcal{L}_{AA} = -\frac{1}{4\pi}\sum_{k,l} K_{kl}^{-}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda, \tag{4.59}$$

and

$$\mathcal{L}_{Aq} = A_\mu \sum_{k,l} K_{kl}^{-} j_l^\mu, \tag{4.60}$$

and

$$\mathcal{L}_{qq} = \pi\sum_{k,l}\epsilon^{\mu\nu\lambda}j_{k\mu}K_{kl}^{-}\partial_\nu\Delta^{-}j_{l\lambda}. \tag{4.61}$$

Varying $\mathcal{L}$ with respect to A, one obtains the total electron current related with magnetic field

$$J^\mu = \frac{1}{2\pi}\sum_{k,l} K_{kl}^{-}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda. \tag{4.62}$$

So the filling factor is

$$\nu = \sum_{k,l} K_{kl}^{-}. \tag{4.63}$$

Suppose one has a vortex excitation with a vortex charges Φ_m from j_m^0. From

$\mathcal{L}_{Aq}$, one sees the vortex has an electromagnetic charge

$$q = \sum_k q_k = \sum_{kl} K^-_{kl}\Phi_l. \tag{4.64}$$

From $\mathcal{L}_{qq}$, one gets the statistics of the vortex

$$\theta = \pi \sum_{k,l} \Phi_k K^-_{kl} \Phi_l = \pi \sum_k q_k \Phi_k = \pi \sum_{k,l} q_k K_{kl} q_l, \tag{4.65}$$

Which means that, when one exchanges two vortices, one will get a phase $e^{i\theta}$. One notes that the statistical parameter defined before is $\Delta = \frac{\theta}{2}$. From $K = I + pC$, one finds from the formula (4.63)

$$\nu = \frac{m}{mp+1}. \tag{4.66}$$

Now one can still put some more physical requirements. The first one is that there are excitations with same quantum number of electrons in every Landau level (for example, with only the excitation $j_m = 0, m \neq l$ in one Landau level l). Let $q_i = \delta_{il}$ (an excitation in l^{th} Landau level), it will have electron charge 1 from the equation (4.64). It should be an electron. Then it has fermion statistics. From the equation (4.65), the statistics of this excitation is

$$\theta = \pi K_{ll}. \tag{4.67}$$

So K_{ll} must be odd number.

The second condition one would like to put is the wave function of any physical excitation with coordinate x_q is single valued with the coordinate of any electron x_e as it happens in FQHE. The monodromy of the excitation 1 with the excitation 2, $2\theta_{12}$ is

$$2\theta_{12}/\pi = 2 < q_1|K|q_2 >= 2 < q_1|\Phi_2 > . \tag{4.68}$$

and we write the equation (4.68)in Dirac braket. And the monodromy means when the excitation 1 turns around the excitation 2, one gets a phase $e^{i\theta_{12}}$. Now let the excitation 1 has the quantum number of electron, for example $q_{1m} = \delta_{ml}$, which is an excitation of the electrons in the l^{th} Landau level. Then $\theta_{12} = \Phi_{2l}$ should be an integer. So Φ_l must be an integer for the arbitrary excitation. Let the excitation 2 is the excitation with the quantum number of the electron in the k^{th} Landau level, one can find K_{lk} must be an integer.

Now we summary the main results. K_{ii} must be an odd integer, $K_{ij}, i \neq j$ is an integer, and Φ_i is an integer. In the simple model, $K = I + pC$, one finds p must be an even integer.

Now one would like to see how the simple model is related to a hierarchical model. First from the filling $\nu = \frac{m}{mp+1}$, one can write it as

$$\nu = \cfrac{1}{p_1 - \cfrac{1}{p_2 - \cfrac{1}{\cdots - \cfrac{1}{p_m}}}}, \tag{4.69}$$

With $p_1 = p + 1$ and $p_i = -\frac{1}{2}$, $i = 2, 3, \cdots, m$. This observation was first obtained by N. Read as quoted as private communication in the first reference in [17]. Indeed, one finds transformations

$$K = S^T R S, \tag{4.70}$$

with

$$R = \begin{pmatrix} p+1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & \vdots \\ \dots & \dots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 \end{pmatrix}, \tag{4.71}$$

and

$$S = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & \vdots \\ \dots & \dots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \tag{4.72}$$

and

$$S^{-1} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & \vdots \\ \dots & \dots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{4.73}$$

R has the form of matrix of hierarchical model. But with off diagonal as -1 compare 1 in the last section. One can redefine the sign of the current in the present section, then it can change those -1 to 1. The intrinsic parameters are the diagonal terms in K which will not change in the redefintion of the sign of current. One also note the matrix K is a symmetric matrix. So one finds the model we discussed above is equivalent to an hierarchical model. it also was emphasized by Jain that the filling $\nu = \frac{m}{mp+1}$ contains almost all observed filling. Also because K matrix has no zero eigenvalue (because it is invertible), this implies the fluid is incompressible (there are no massless modes of mass). Various generalization within the basic physical requirement can be made [50]. There are still many interesting problems there.

4.5. Edge State in FQHE and Its Relation to Chern-Simons Theory

The edge state in QHE was first discussed by B. H. Halperin [75]. Recently it became a hot topic because Edge state is tightly related to bulk state and actually it can be observed in experiments. On the other aspect, we know that Chern-Simons theory with boundary also has *edge effect.* The edge states of C-S theory are Chiral Conformal Field Theory. About the relation of C-S theory with CFT, one can see for example [57,58,59,60,63,64,65,66,67] *etc.* FQHE can be described by GL theory which has C-S term. So one naturally guesses that edge state of FQHE can be interpreted by Chiral Conformal Field Theory (CCFT).

First let us consider Chern-Simons theory with lagrangian density

$$\mathcal{L} = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda. \tag{4.74}$$

This is the lagrangian density for the FQHE at filling $\nu = \frac{1}{k}$ as we see from the above sections (k is an odd number). Here A is the magnetic field, the electron current is $J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$. Under gauge transformation

$$\delta a_\mu = \partial_\mu f. \tag{4.75}$$

The action changes to

$$\delta S = \int \frac{k}{4\pi}\partial_\mu(\epsilon^{\mu\nu\lambda} f \partial_\nu a_\lambda). \tag{4.76}$$

If one has a bounded region, then the theory is not gauge invariant because (4.76)will not be zero. There are many ways to go around this problem without disturbing the bulk state (inside the boundary). The first one we will discuss here is that we can introduce another lagrangian with extra field which lives only on the boundary, and let the total lagrangian be a gauge invariant one. Another way is that we take suitable boundary condition which will not change the equation of motion of the bulk state. These two approaches should be equal (we have not seen any prove of it, but by physical reasoning they should be equal).

Now let us first follow F Wilczek in [2] for a very rough discussion, but it has rather clear physical picture. However see also recent interesting paper by D. H. Lee and X. G. Wen [51]. We believe this rough method can be refined, and will give all essential property of edge state (until now we do not see anyone follows this approach to give a complete discussion of the edge state. It will be interesting to do it).

Now consider the edge is a straight line $y = 0$ for simplicity. Then suppose the lagrangian of the scalar field in the edge (this is the simplest possibility, of course there exists many other possible lagrangian) is

$$\mathcal{L}_{ed} = c_i(\partial_i\phi - qa_i)^2 + c_3\epsilon^{ij}\partial_i a_j\phi, \tag{4.77}$$

with i, j is 0 or 1. One can normalize $c_0 = \frac{1}{2}$. However c_1, c_3, q should be determined by the experiment and possible physical restrictions that $\mathcal{L} + \mathcal{L}_{eg}$ is gauge invariant. Under the gauge transformation

$$\begin{aligned} \phi &\to \phi + qf \\ a_i &\to a_i + \partial_i f \end{aligned}. \tag{4.78}$$

The action due to $\mathcal{L}$ changes

$$\begin{aligned} \delta S &= \int \partial_\mu(\epsilon^{\mu\nu\lambda} f \partial_\nu a_\lambda) \\ &= \int dx dt \frac{k}{4\pi} f(\partial_0 a_1 - \partial_1 a_0) \end{aligned}, \tag{4.79}$$

and the action due to $\mathcal{L}_{eg}$ changes to

$$\delta S_{eg} = \int dx dt c_3 q \epsilon^{ij} \partial_i a_j f. \tag{4.80}$$

So in order to have a gauge invariant lagrangian, one needs

$$c_3 q + \frac{k}{4\pi} = 0. \tag{4.81}$$

Moreover, if we assume there is still Lorentz invariance in the edge (it seems to me it is reasonable condition), hence $c_2 = c_1$. So only one parameter q remains to be determined by the experiment (it must be related to the velocity of the edge state in the chiral boson representation in the following discussion). Further exploring along this direction will be interesting. However we will stop at this point and swith to a more common approach to the edge state.

Let us consider the lagrangian density (4.74). Now we consider the disc boundary. The polar angle coordinate is θ. Define $x_\pm = \theta \pm vt$, with v arbitrary constant now. Following [89], one writes the lagrangian by using differential form to simplify the formula. One can write (4.74)as

$$\mathcal{L} = \frac{k}{4\pi}(a - \frac{1}{k}A)d(a - \frac{1}{k}A), \tag{4.82}$$

Up to a constant which dependents on A. $a = a_\mu dx^\mu$ is a standard definition of differential form. Now we need to put the boundary condition. The guiding principle for taking the boundary condition is that there are no boundary correction to the equation of motion of bulk state. About C-S theory with boundary, for example, [57,101,65]. Below we will follow the standard way to quantize the Chern-Simons theory with boundary and see how the edge states come out. First one notes that k is an odd number. But as pointed out by many articles (for example, [65]), in order to have a good theory, for example let the theory have modular invariance, k should be an even integer. It was first obtained in [58] that k can be odd if one puts the theory on spin manifold. The corresponding conformal theory is $U(1)$ level $\frac{k}{2}$. It is a superchiral $c = 1$ theory. For rational k, it was discussed in [67], then some practical calculation and further development were carried out in [20]. The understanding of the relation among the GL-CS theory of FQHE, conformal field theory and the wave function of FQHE (include hierarchical wave function) still has not been completed (however see [102,20] for some related discussion).

Define the coordinates

$$x_\pm = \theta \pm vt, \tag{4.83}$$

θ is the angular coordinate of the disc and we take the radius of the disc as 1 for simplicity. In usual quantization of C-S theory, for example [65], one quantizes the theory on the disc with a lagrangian density $\mathcal{L} = \frac{k}{4\pi}ada$ by using boundary condition $a_0 = 0$ on the boundary. However other good boundary conditions exist. We can use coordinate $x_\pm$, r (θ, r is the polar coordinates). One can decompose exterior derivative d and gauge field a

$$d = dx_- \partial_{x_-} + \bar{d}, \ a = a_- + \bar{a}. \tag{4.84}$$

Then it is simple to check that $a_- = 0$ boundary condition also is a good boundary condition (one can follow [65] in demonstrating this point) which will not modify the equation of motion. For the theory of FQHE, one should consider the lagrangian (4.82). First do transformation, $b = a - \frac{1}{k}A$, then quantize the theory with boundary condition $b_- = 0$. With this boundary condition, the action is

$$S = \frac{k}{4\pi}\int \bar{b}(\partial x_- \bar{b}) dx_- - \frac{k}{2\pi}\int b_- \overline{db}. \tag{4.85}$$

Putting it on path integral and integrating b_-, one gets a constraint equation

$$\overline{db} = 0. \tag{4.86}$$

The solution of the above equation is

$$\bar{b} = \bar{d}\phi. \tag{4.87}$$

Substitute the equation (4.87)to original lagrangian (about the Jacobian due to this change of variable, see [65]), one gets

$$S = \frac{k}{4\pi}\int (\partial x_- \phi)(\partial x_+ \phi) dx_- dx_+. \tag{4.88}$$

(4.88)is the lagrangian of a chiral conformal field theory (for example [65,98]). In the theory with the lagrangian (4.88), one has [65]

$$\partial x_- \phi = 0. \tag{4.89}$$

It is

$$\phi = \phi(x_+). \tag{4.90}$$

Hence one finds the velocity of the edge wave is v. Following [98], one quantize the theory to get

$$[\partial x_+ \phi(x_+), \partial x_+ \phi(x'_+)] = \frac{2\pi}{k} i\delta'(x_+ - x'_+)$$

We define the chiral current

$$J_+ = \frac{k}{2\sqrt{2}\pi} \partial x_+ \phi. \tag{4.91}$$

Hence

$$[J_+(x_+), J_+(x'_+)] = \frac{k}{4\pi} i\delta'(x_+ - x'_+). \tag{4.92}$$

The current algebra is generated by J_+ and $e^{\frac{k\phi}{\sqrt{2}}}$. And the primary fields are $e^{\frac{j\phi}{\sqrt{2}}}$, $j = 0, 1, \cdots, k-1$. The conformal dimensions of those primary fields are $h_j = \frac{j^2}{2k}$. The spectrum of Hilbert space is the spectrum of the excitation of the edge state. Moreover $e^{\frac{k\phi}{\sqrt{2}}}$ is fermionic current. So the current algebra is a super algebra. It is a level $\frac{k}{2}$ $U(1)$ super chiral algebra [58].

5. Anyon and Quantum Hall Effect on Sphere*

Recently, there are a wide interests on anyon physics. Physical anyons have fractional statistics and spin. One of its realization in nature is in the phenomena of fractional quantum hall effect [5,8,1]. Quasiparticles in FQHE are anyons. Moreover, superconductivity due to some mechanism of anyons maybe can explain the high temperature superconductivity [9,10,11,12,79].

In this chapter, we will consider anyons on the sphere [22]. It is not only for academic interests. For example, because sphere is a compact Riemann surface, it is often used by computer simulation [6,1].

The organization of this chapter is as follows. First in 2, we write the lagrangian by using stereographic coordinate of the sphere. It is easy to integrate Chern-Simons term to get the effective action of the anyons in this formalism. Then standard Noether current method is employed to get angular momentum of the system. Some problems of anyon gauge on sphere are discussed, for example, spin statistics relation, and comparison with braid group analysis. Finally we consider anyons on magnetic field, and get some ground states.

In 3, FQHE on the sphere is revisited by using the projective coordinate. Following Blok and Wen, we propose Hierarchical wave function of FQHE on the sphere. By requiring non-degenerate of the state of FQHE, we get the same equation which Blok and Wen (see also [18,21])got on disc geometry of FQHE by using plasma analogue. One of advantage of using of projective coordinate is that it is easy to see the rotational property of the state.

In 4, the relation of FQHE on the sphere with the vortex of 2-dim scalar field on the sphere is discussed, specially we concentrate on the spin singlet FQHE. Some of application of the vortex theory of two dimension scalar field in FQHE are considered.

⋆ This chapter is largely adopted from 104

5.1. Anyons on Sphere

First, we write the lagrangian of anyons on the sphere. It is

$$s = \int dt[\frac{m}{2}\Sigma_{i=1}^{N} g_{\alpha\beta}(x_i)\frac{dx_i^\alpha}{dt}\frac{dx_i^\beta}{dt} + qA_\mu\frac{dx_\mu}{dt}] + \frac{\mu}{4\pi}\int d^3x\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda. \tag{5.1}$$

And the metric g is given by

$$g_{\alpha\beta}(x) = \frac{1}{(1+r'^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.2}$$

The notation is, $x^1 = x$, $x^2 = y$, $r'^2 = \frac{x^2+y^2}{4R^2}$, R is the radii of the sphere. It is easy to see Chern-Simons term has not explicit dependence on the metric.

Now, because C-S term does contribute to dynamics, we can integrate it out. By solving $\frac{\delta S}{\delta A^0} = 0$, we get

$$q\Sigma\delta^2(x - x_i) + \frac{\mu}{2\pi}\epsilon^{ij}\partial_i A_j = 0. \tag{5.3}$$

Because now, we have a compact Riemann surface, equation (5.3)will make sense if it satisfies the condition of Dirac quantization. In our case, it is, $\frac{q^2N}{\mu} = n$. Here, n is a integer. Physically, this means the Dirac string is invisible, or Wu-Yang connection is well defined.

Now, we define, $z = x + \imath y, \bar{z} = x - \imath y, A_1 = A_z + A_{\bar{z}}$, $A_2 = \imath(A_z - A_{\bar{z}})$.

rewrite the formula in the complex coordinate

$$\partial z A_{\bar{z}} - \partial\bar{z}A_z = -\imath\frac{q}{\mu}\pi\sum_{i=1}^{N}\delta^2(z - z_i). \tag{5.4}$$

By using $\partial\bar{z}\frac{1}{z} = \pi\delta^2(z)$, we get

$$A_z = \frac{\imath q}{2\mu}\sum_{i=1}^{N}[\frac{1}{z - z_i} - \frac{1}{z}], \tag{5.5}$$

here, Dirac string is at $z = 0$. If we put Dirac string $z = \infty$, the solution is

$$A_z = \frac{\imath q}{2\mu} \sum_{i=1}^{N} \frac{1}{z - z_i}, \tag{5.6}$$

where Dirac solution of the monopole field is used instead of Wu-Yang connection.

The problem arises here. $A(z_i)$ is not well defined. $\frac{1}{z_i - z_i}$ is divergent. This is related with the fact that $\delta^2(z_i - z_i) = \delta^2(0)$ is not well defined. So the theory need to be regularized. One consistent regularization is this divergent term becomes a constant magnetic field on the sphere in order to respect the rotational invariance. In this regularization, the results is

$$A_{z_i} = \frac{\imath q}{2\mu} \sum_{j \neq i} [\frac{1}{z_i - z_j} - \frac{1}{z_i}] + \frac{\imath q}{2\mu} [\frac{\frac{\bar{z}_i}{4R^2}}{1 + \frac{z_i \bar{z}_i}{4R^2}} - \frac{1}{z_i}]. \tag{5.7}$$

If Dirac string is at $z = \infty$, the solution is

$$A_{z_i} = \frac{\imath q}{2\mu} \sum_{j \neq i} \frac{1}{z_i - z_j} + \frac{iq}{2\mu} \frac{\frac{\bar{z}_i}{4R^2}}{1 + \frac{z_i \bar{z}_i}{4R^2}}. \tag{5.8}$$

Below, for convenience, we rescale $z_i \rightarrow 2Rz_i$, or simply to say, take $R = \frac{1}{2}$ in order to simplify the formula. Finally we get effective lagrangian [13]

$$L = \frac{m}{2} \sum_{i=1}^{N} [\frac{1}{(1 + z_i \bar{z}_i)^2} \frac{dz_i}{dt} \frac{d\bar{z}_i}{dt} + qAz_i \frac{dz_i}{dt} + qA\bar{z}_i \frac{d\bar{z}_i}{dt}], \tag{5.9}$$

and

$$Az_i = \frac{\imath q}{2\mu} \sum_{j \neq i} [\frac{1}{z_i - z_j}] + \frac{\imath q}{2\mu} \frac{\bar{z}_i}{1 + z_i \bar{z}_i}, \tag{5.10}$$

where Dirac string is at $z = \infty$. Now, we quantize the theory. The conjugate momentum of coordinate z_i is

$$P_{z_i} = \frac{m}{2} \frac{1}{(1+z_i\bar{z}_i)^2} \frac{d\bar{z}_i}{dt} + qA_{z_i}. \tag{5.11}$$

In quantized form, then $P_{z_i} = -\imath\partial z_i$ ($P_{\bar{z}_i} = -\imath\partial_{\bar{z}_i}$). If we quantize the theory, the commutation relation is

$$[z_i\ P_{z_j}] = \imath\delta_{ij}. \tag{5.12}$$

The Hamiltonian can be written as

$$H = \frac{2}{m}\sum_{i=1}^{N}(1+z_i\bar{z}_i)^2\,(P_{z_i} - qA_{z_i})(P_{\bar{z}_i} - qA_{\bar{z}_i}) \tag{5.13}$$

In the quantized form, there is a normal ordering problem. In order to discuss this problem, we first define inner product of the state vector. It is

$$<\psi_1(z_i)|\psi_2(z_i)> = \int\prod_{i=1}^{N}\frac{dz_i d\bar{z}_i}{(1+z_i\bar{z}_i)^2}\overline{\psi_1(z_i)}\psi_2(z_i). \tag{5.14}$$

H must be Hermitian with the inner product. The Hamiltonian defined in (5.13)is Hermitian with this inner product. Above we write the Hamiltonian in a special ordering. One can also use the Laplace-Beltrami operator ordering

$$H = \frac{1}{m}\sum_{i=1}^{N}(1+z_i\bar{z}_i)^2\,[(P_{z_i} - qA_{z_i})(P_{\bar{z}_i} - qA_{\bar{z}_i}) + (P_{\bar{z}_i} - qA_{\bar{z}_i})(P_{z_i} - qA_{z_i})], \tag{5.15}$$

it is also Hermitian. Below we will specify different ordering in different case. This means that we only consider suitable hard core interaction [15]. In this case, some exact ground wave functions can be obtained [15].

Now we consider the symmetry of the lagrangian. The lagrangian is rotational invariant. Under the rotation, the coordinates are transformed as

$$z' = \frac{az+b}{cz+d}, \tag{5.16}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(3)$. They are generated by the three rotation about different axes.

$$\begin{aligned} R_x &= \frac{1}{\sqrt{2}} \begin{pmatrix} (1+\cos\alpha)^{\frac{1}{2}} & \imath(1-\cos\alpha)^{\frac{1}{2}} \\ \imath(1-\cos\alpha)^{\frac{1}{2}} & (1+\cos\alpha)^{\frac{1}{2}} \end{pmatrix} \\ R_y &= \frac{1}{\sqrt{2}} \begin{pmatrix} (1-\cos\beta)^{\frac{1}{2}} & (1-\cos\beta)^{\frac{1}{2}} \\ -(1-\cos\beta)^{\frac{1}{2}} & (1-\cos\beta)^{\frac{1}{2}} \end{pmatrix}. \\ R_z &= \begin{pmatrix} \exp\frac{\imath\gamma}{2} & 0 \\ o & \exp\frac{-\imath\gamma}{2} \end{pmatrix} \end{aligned} \tag{5.17}$$

When one does those transformation in the lagrangian, the lagrangian is invariant up to a total time derivative. So the theory is invariant under this transformation [24]. The total derivative is related to the concept of cocycle in Quantum Mechanics [24]. In this case, Noether current can still be obtained. We list some of useful transformation below

$$\begin{aligned} z_i' - z_j' &= \frac{z_i - z_j}{(cz_i+d)(cz_j+d)} \\ 1 + z'\bar{z}' &= \frac{1+z\bar{z}}{(cz+d)(\bar{c}\bar{z}+\bar{d})} \end{aligned} \tag{5.18}$$

If the Dirac singularity is at ∞ (Dirac string at ∞ and 0 are related by a well defined gauge transformation), the Noether currents are

$$\begin{aligned} J_x &= \sum_{i=1}^{N} [\frac{\imath}{2}(1-z_i^2)P_{z_i} - \frac{\imath}{2}(1-\bar{z}_i^2)P_{\bar{z}_i} - \frac{s}{2}(z_i+\bar{z}_i)] \\ J_y &= \sum_{i=1}^{N} [-\frac{1+z_i^2}{2}P_{z_i} - \frac{1+\bar{z}_i^2}{2}P_{\bar{z}_i} - \frac{\imath s}{2}(\bar{z}_i - z_i)] \quad , \\ J_z &= \sum_{i=1}^{N} [\imath z_i P_{z_i} - \imath\bar{z}_i P_{\bar{z}_i} + \frac{s}{2}] \end{aligned} \tag{5.19}$$

where $s = \frac{q^2 N}{2\mu} = \frac{n}{2}$, n is a integer as discussed before. It can be checked that J_i are Hermitian. The commutation relation of J_i is standard as

$$[J_i\, J_j] = \imath \epsilon^{ijk} J_k. \tag{5.20}$$

The particle has spin s from the formula of the angular momentum. Now we want to see how the wave function is transformed in finite rotation. It will give us an idea what is the cocycle in present problem. If the operator of the transformation $R = \exp \imath(\alpha_x J_x + \alpha_y J_y + \alpha_z J_z)$ corresponds to the rotation of the coordinate $z' = \frac{az+b}{cz+d}$. We get

$$R\Psi = \prod_{i=1}^{N} \left(\frac{\bar{c}\bar{z}_i + \bar{d}}{cz_i + d} \right)^s \Psi \left(\frac{az_i + b}{cz_i + d} \right). \tag{5.21}$$

This is very similar to the global conformal transformation of primary field in Conformal Field Theory with spin $s = h - \bar{h}$. h and $\bar{h}$ is the conformal dimensions of the field in holomorphic and antiholomorphic sector.

Now we discuss the problem of what is anyon gauge on the sphere. It is known that anyon gauge is not so simple on the torus [81]. One also finds the anyon gauge on the sphere also is not so straightforward as the anyon gauge on the plane.

When the Dirac singularity is at $z = \infty$, some formula is more simple. But in this case it is a little difficult to trace the Dirac singularity. So now we use the formula with Dirac string at $z = 0$.

The singular gauge transformation of the wave function is

$$\Psi = \prod_{i<j} \left(\frac{\bar{z}_i - \bar{z}_j}{z_i - z_j} \right)^{\frac{\theta}{2}} \prod_{i=1}^{N} \left(\frac{z_i}{\bar{z}_i} \right)^{\frac{(N-1)\theta}{2}} \Psi', \tag{5.22}$$

with $\theta = \frac{q^2}{\mu}$. The gauge transformation is

$$U = \prod_{i<j} \left(\frac{\bar{z}_i - \bar{z}_j}{z_i - z_j} \right)^{\frac{\theta}{2}} \prod_{i=1}^{N} \left(\frac{z_i}{\bar{z}_i} \right)^{\frac{(N-1)\theta}{2}}. \tag{5.23}$$

The reason why we should add an extra term compared to the plane is that gauge flux from this singular gauge must be 0 because now we work on the sphere.

In the anyon gauge, A_{z_i} will change to A'_{z_i}

$$A'_{z_i} = \frac{\imath q}{2\mu}[\frac{\bar{z}_i}{1+z_i\bar{z}_i} - \frac{1}{z_i}]. \quad (5.24)$$

So in anyon gauge, the particles see a gauge field with Dirac string, because the Dirac condition is not satisfied. U also sees Dirac string because it is not single valued around $z_i = 0$. But in non singular gauge, we call it Chern-Simons gauge, wave function will not see Dirac string, because Dirac quantization is hold in this case. So it seems there is no good definition of anyon gauge in the sphere, and it is better to work in the Chern-Simons gauge. What happens if the Dirac singularity is at $z = \infty$. In this case, We have

$$\Psi = \prod_{i<j} \left(\frac{\bar{z}_i - \bar{z}_j}{z_i - z_j}\right)^{\frac{\theta}{2}} \Psi', \quad (5.25)$$

$$U = \prod_{i<j} \left(\frac{\bar{z}_i - \bar{z}_j}{z_i - z_j}\right)^{\frac{\theta}{2}}, \quad (5.26)$$

$$A'_{z_i} = \frac{\imath q}{2\mu}\frac{\bar{z}_i}{1+z_i\bar{z}_i}. \quad (5.27)$$

It seems that the singular transformation is the same as on the plane. But it is very different because we still have Dirac singularity in $z = \infty$. In anyon gauge, the particle also see Dirac string because the Dirac Quantization is not satisfied. U also sees Dirac string because it is not single valued around $z_i = \infty$.

In a paper by Einarsson [103] (the braid group approach to anyon in the torus, see [90] and [19,81]), he gave braid group analysis of anyons on compact Riemann surface (see also reference in [103]). Now we compare his result about anyon on the sphere to our formula. In our case, the braid group generator σ will take value $\sigma = \exp \imath\pi\theta$ ($\theta = \frac{q^2}{\mu}$ as above. and Dirac quantization condition is $\theta N = n$, n is an integer). We find the relation, $\sigma_1\sigma_2 \cdots \sigma_{N-1}^2 \cdots \sigma_2\sigma_1 = \exp -2\imath\pi\theta$. So in our case, Anyon must have spin, because otherwise those product should be 1 if the anyon has not spin. Compare above identity with the similar equation in [103], the spin of the anyon should be equal to $\frac{\theta}{2}$. This is indeed the case in our model. The reason is that in the anyon gauge, the particle still interact with a monopole background due to self interaction of Chern-Simons term. It is a well known fact the monopole background will contribute to the spin of the particle. The value of the spin is related with the flux of the monopole background (we can use Noether current to find the angular momentum in this case as we do in Chern-Simons gauge). The spin is the half of the flux in the fundamental flux unit, which can be verified directly by deriving the Noether Current. From this point of view, we get spin as $\frac{\theta}{2}$. So in this case, the spin-statistics relation is satisfied as the case analysed by Einarsson.

In another naive argument, we can reason in following way

$$\Psi' = \Psi \prod_{i<j} \left(\frac{z_i - z_j}{\bar{z}_i - \bar{z}_j} \right)^{\frac{\theta}{2}} \prod_i \left(\frac{\bar{z}_i}{z_i}\right)^{\frac{(N-1)\theta}{2}}. \tag{5.28}$$

Ψ is single valued under permutation (in our bosonic case). Also in Chern-Simons gauge, particle has integer or half integer spin. So it will not give any phase due to deformation of curve in the sphere with zero area to rear side of the sphere [103]. So when we do the braiding$\sigma_1\sigma_2 \cdots \sigma_{N-1}^2 \cdots \sigma_2\sigma_1$, it is enough to consider the factor of singular gauge transformation. This braiding can be deformed to the rear side. But the Dirac string will be met (U is not single valued around $z_i = 0$). After careful calculation of the phase due to deform around the Dirac string, we find the same equation as Einarsson.

Below a lagrangian is given corresponding to the case of non spinning particle with fractional statistics. It is not derived from any Quantum Field Theory. We should not expect there is the spin-statistics relation. We assume

$$(N-1)\theta = n = \frac{(N-1)q^2}{\mu}, \tag{5.29}$$

where n is a integer. And one has

$$A_{z_i} = \frac{\imath q}{2\mu}\sum_{j\neq i}[\frac{1}{z_i - z_j} - \frac{1}{z_i}]. \tag{5.30}$$

A is well defined on the sphere because the Dirac quantization is hold. The Dirac singularity is at $z = 0$. It is invisible. The anyon gauge will be

$$\begin{aligned}\Psi' &= \Psi \prod_{i<j}\left(\frac{z_i - z_j}{\bar{z}_i - \bar{z}_j}\right)^{\frac{\theta}{2}} \prod_i (\frac{\bar{z}_i}{z_i})^{\frac{(N-1)\theta}{2}} \\ A'_{z_i} &= 0\end{aligned} . \tag{5.31}$$

So particle has not spin in anyon gauge. This is exactly agreed with the results for the nonspinning anyon [103,90].

It is difficult to solve the ground state of the theory except in some special cases. Now we give some examples. The first one is the analogue of the result of [15,19]. In the case of spinning anyon model on the sphere, if we impose hard core attraction (chose some Hamiltonian with special ordering), we consider the Hamiltonian

$$H = \frac{2}{m}\sum_{i=1}^{N}(1 + z_i\bar{z}_i)^2 (P_{\bar{z}_i} - qA_{\bar{z}_i})(P_{z_i} - qA_{z_i}). \tag{5.32}$$

We get the ground state wave function (Dirac string at ∞ in following equation)

$$\Psi = \overline{f(z_i)}\prod_{i<j}\left(\frac{(\bar{z}_i - \bar{z}_j)(z_i - z_j)}{(1 + z_i\bar{z}_i)(1 + z_j\bar{z}_j)}\right)^{\frac{-\theta}{2}} \prod_i (1 + z_i\bar{z}_i)^{\frac{-n}{2}}, \tag{5.33}$$

where n is the integer defined before as the Dirac quantization condition. The $f(z_i)$ is the symmetric analytic function. There is a constraint on the function

$f(z)$. The power of any z_i in $f(z_i)$ must be small than n, in order to have a normalizable wave function. $f(z_i)$ is generated by the symmetric function $f_m = \sum_{z=1}^{n} z_i^m$. For example, $f = \prod_m f_m^{k_m}$, with the restriction $\sum m\, k_m < n$. Here k_m is a positive integer. The number of solution is well known if one is familar with conformal theory about the number of descendent primary state in some level. The conclusion is that the ground state is finite degenerated in hard core attraction model. The other results of Girvin's paper also can be easily generalized to the case on the sphere. We will come to some points later.

About hard core repulsion model, we fail to find ground states. However, if we put anyons on the magnetic field, we do find some normalizable ground states with hard core repulsion interaction. Now suppose we add a electromagnetic potential, eA_{z_i}, e is the electric charge of the anyon. Then $eA'_{z_i} = -\imath\frac{\phi}{2}\frac{\bar{z}_i}{1+z_i\bar{z}_i}$, and ϕ is the magnetic flux. It should be quantized according to Dirac quantization. We consider the hard core repulsion Hamiltonian of anyon in the magnetic field

$$H = \frac{2}{m}\sum_{i=1}^{N}(1+z_i\bar{z}_i)^2\,(P_{z_i} - qA_{z_i} - eA'_{z_i})(P_{\bar{z}_i} - qA_{\bar{z}_i} - eA'_{\bar{z}_i}). \tag{5.34}$$

Notice different ordering of hard core attraction and hard core repulsion Hamiltonian.

We will get the ground state wave function for bosonic statistics in C-S gauge

$$\Psi = f(z_i)\prod_{i<j}[\frac{(z_i - z_j)(\bar{z}_i - \bar{z}_j)}{(1+z_i\bar{z}_i)(1+z_j\bar{z}_j)}]^{\frac{\theta}{2}}\prod_i(1+z_i\bar{z}_i)^{-\frac{\phi-n}{2}}. \tag{5.35}$$

We assume $l = \phi - n \geq 0$, in order to have a normalizable ground state. $f(z_i)$ is a symmetric analytic function of z_i. The power of every z_i in $f(z_i)$ must be small than l in order to have a normalizable wave function. Due to this restriction, the solution of $f(z_i)$ is finite as discussed above in the case of hard core attraction model of free anyon.

If ϕ is large enough, we can get wave function for fermion

$$\Psi = f(z_i)\prod_{i<j}[\frac{(z_i - z_j)(\bar{z}_i - \bar{z}_j)}{(1 + z_i\bar{z}_i)(1 + z_j\bar{z}_j)}]^{\frac{\theta}{2}} \frac{(z_i - z_j)}{(1 + z_i\bar{z}_i)^{\frac{1}{2}}(1 + z_j\bar{z}_j)^{\frac{1}{2}}} \prod_i (1 + z_i\bar{z}_i)^{-\frac{\phi - n - N}{2}}. \tag{5.36}$$

We assume that $l = \phi - n - N$. The $f(z_i)$ is symmetric wave function. The power of every z_i must be small than l. So in this case the ground states still are finite degenerated.

5.2. Fractional Quantum Hall Effect on Sphere

In this section, we will formulate the Laughling wave function on the sphere in projection coordinate. Then the Hamiltonian is (up to a finite constant compared with standard Laplace-Beltrami ordering of Hamiltonian)

$$H = \frac{2}{m}(1 + z\bar{z})^2(P_z - eA_z)(P_{\bar{z}} - eA_{\bar{z}}), \tag{5.37}$$

and

$$eA_z = -\frac{\imath\phi}{2}\frac{\bar{z}}{1 + z\bar{z}}. \tag{5.38}$$

ϕ is the magnetic flux in the fundamental unit of the flux. It is an integer. We put Dirac string at $z = \infty$ (In the paper of Haldane, he put the Dirac string at $z = 0$ and $z = \infty$, in our notation, he took the magnetic field as $eA_z = -\frac{\imath\phi}{2}(\frac{\bar{z}}{1+z\bar{z}} - \frac{1}{2z})$).

By solving the equation $(P_{\bar{z}} - eA_{\bar{z}})\Psi = 0$, the ground state of the electron is obtained as

$$\Psi = z^k(1 + z\bar{z})^{-\frac{\phi}{2}}, \tag{5.39}$$

where $k \le \phi$ is hold in order to have a normalizable ground state. The Laughling wave function for N particle is

$$\Psi = \prod_{i<j}(z_i - z_j)^m \prod_i (1 + z_i\bar{z}_i)^{-\frac{\phi}{2}}, \tag{5.40}$$

with m is an odd number. It is a well known fact that the ground state wave function of FQHE on the sphere must be nondegenerate. A unique ground state will be survived after the breaking of degeneracy due to Coulomb interaction. Also we suppose that the system has rotational symmetry. So from those facts, the ground must be rotational invariant. Under the rotation $z' = \frac{az+b}{cz+d}$, the rotational operator acts this way

$$R\Psi(z_i) = \prod_i (\frac{\bar{c}\bar{z}_i + \bar{z}_i}{cz_i + d})^{\frac{-\phi}{2}} \Psi(\frac{az_i + b}{cz_i + d}). \tag{5.41}$$

The generator of the rotational operator can be obtained by the standard Noether technic. This already is discussed in the section 2. In order that the Laughling wave function is rotational invariant, by simple calculation and using identities of some transformations in section 2, we get a condition

$$\phi = m(N - 1). \tag{5.42}$$

For simplicity, we define $d_{ij} = \frac{z_i - z_j}{(1+z_i\bar{z}_i)^{\frac{1}{2}}(1+z_j\bar{z}_j)^{\frac{1}{2}}}$. So $\Psi = \prod_{i<j} d_{ij}^m$.

Now we consider the Hierarchical wave function. There are several Hierarchical wave functions proposed by Halperin, Blok and Wen, Read *etc* [7,17,18,20]. We follow the approach in [17]. The natural generalization of the Hierarchical wave function on the sphere is (for details, see [17])

$$\Psi(z_i) = \prod_{i<j}^{N_e} d(z_i, z_j)^{p_1} \int \frac{d^2 z_i^2}{1 + z_i^2 \bar{z}_i^2} \cdots \int \frac{d^2 z_i^n}{1 + z_i^n \bar{z}_i^n} \prod_{s=2}^{s=n} \prod_{i,j=1}^{N_s} \mid d(\tilde{z}_i^s, \tilde{z}_j^s) \mid^{2sgn(q_s - 2)\frac{\theta_{s-1}}{\pi}},$$
$$d(\tilde{z}_i^s, \tilde{z}_j^s)^{p_s} \prod_{i,j}^{N_s, N_{s+1}} d(\tilde{z}_i^s, \tilde{z}_j^{s+1}) \tag{5.43}$$

with notations, $\tilde{z}^s = z^s$ if s is odd and $\tilde{z}^s = \bar{z}^s$. And N_e is the number of the electron, N_s, $s = 2, 3, \cdots, n$ is the number of quasi particle of the s-th hierarchy level. z_i^s is the respective coordinate. p_1 is an odd positive number. p_s is even integer with $s = 2, 3, \cdots, n$. n is the level of the hierarchy. Moreover, we have relations

$$q_{n+1} = -\frac{q_n}{p_{n+1} - sgn(q_n)\frac{\theta_n}{\pi}}, \quad (5.44)$$

and

$$\frac{\theta_{n+1}}{\pi} = \frac{sgn(q_n)}{p_{n+1} - sgn(q_n)\frac{\theta_n}{\pi}}, \quad (5.45)$$

where $q_0 = -1$, $\theta_0 = 0$. And p_1 is an odd positive integer, p_i, $i = 2, 3, \cdots n$ is even positive integer (in N. Read paper,some other case also considered for negative p_i). The properties of the system should depend only on N_e, p_i. All other parameters can be calculated from them. As argued above, the wave function of FQHE must be nondegenerate, and so rotational invariant on the sphere. we apply the rotational operator R on the wave function, make a requirement that the wave function is invariant under this operation. Because the coordinates of the quasi particles is a dummy variable, we can also do transformation $z_i^{s\prime} = \frac{az_i^s+b}{cz_i^s+d}$, $s = 2, 3, \cdots, n$. Notice $\frac{dz_i^s d\bar{z}_i^s}{1+z_i^s \bar{z}_i^s}$ is invariant under this transformation, And $d(z_1', z_2') = (\frac{\overline{cz_1+d}}{cz_1+d})^{\frac{1}{2}}(\frac{\overline{cz_2+d}}{cz_2+d})^{\frac{1}{2}} d(z_1, z_2)$. Then after some simple calculation, we have

$$\begin{array}{l} p_1(N_e-1) + (N_2-1) = \phi - 1 \\ (N_e-1) - p_2(N_2-1) - (N_3-1) = 0 \\ \vdots \\ (-1)^s(N_{s-1}-1) - (-1)^s p_s(N_s-1) + (-1)^{s+1}(N_{s+1}-1) = 0 \\ \vdots \\ (-1)^n(N_{n-1}-1) - (-1)^n p_n(N_n-1) = (-1)^{n+1} \end{array} . \quad (5.46)$$

This is the same equation obtained in [21,17,18] It can be written in more compact form

$$\Lambda_{ij}(N-I)_j = \Phi_i \;\; i = 1, \cdots, n. \quad (5.47)$$

The matrix Λ is

$$\Lambda_{ij} = \begin{pmatrix} p'_1 & 1 & \dots & 0 & 0 & 0 \\ 1 & p'_2 & -1 & 0 & 0 & 0 \\ 0 & -1 & p'_3 & 1 & 0 & \dots \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \dots & (-1)^n \\ 0 & 0 & 0 & \dots & (-1)^n & p'_n \end{pmatrix}, \tag{5.48}$$

and $N_1 = N_e$, $I_i = 1$, and $\Phi_1 = \phi - 1$, $\Phi_n = (-1)^{n+1}$, $\Phi_i = 0, i \neq 0$. Moreover, we can solve above equation, to get filling factor $\nu = \frac{N_e}{\phi}$(in the limit of N_e, ϕ much larger than 1)

$$\nu = \cfrac{1}{p_1 + \cfrac{1}{p_2 + \cfrac{1}{\dots + \cfrac{1}{p_n}}}}. \tag{5.49}$$

5.3. Fractional Spin of Anyon in FQHE

In this section, we would like to discuss the spin of the anyon in FQHE. Consider the simplest case, the level 2 hierarchical state, with filling $\nu = \frac{1}{m+\frac{1}{p}}$, and m is a positive odd number, p is an even positive integer. The wave function of the condensation state is given by

$$\Phi(z_i) = \int \prod_\alpha dv_\alpha \Phi_e(z_i, z_\alpha)\Phi_q(z_\alpha), \tag{5.50}$$

where z_i is the coordinate of the electron, $\Phi_e(z_i, z_\alpha)$ is the normalized wave function of the electrons in the presence of quasiparticle with coordinate z_α, and $\Phi_q(z_\alpha)$ is the wave function of the quasiparticles. Following the last section , they are

$$\Phi_e = \prod_{i<j} d_{ij}^m \prod_{i\alpha} d_{i\alpha} \prod_{\alpha<\beta} |d_{\alpha\beta}|^{\frac{1}{m}}, \tag{5.51}$$

and

$$\Phi_q = \prod_{\alpha<\beta} |d_{\alpha\beta}|^{\frac{1}{m}} (\bar{d}_{\alpha\beta})^p, \tag{5.52}$$

with the relation

$$\begin{aligned} m(N_e - 1) + N_q = \phi \\ N_e - p(N_q - 1) = 0 \end{aligned} . \tag{5.53}$$

N_e is the number of electrons, N_q is the number of quasiparticles. Now one should find the Hamiltonian of the quasi particles which will have a ground state Φ_q. Compare Φ_q with the wave function of anyon on the magnetic field, we find that fractional statistics of the quasiparticles is $|\theta| = \frac{1}{m}$ (actually the parameter θ with the notation of the last section will be negative $\frac{1}{m}$ and the charge of the quasiparticles is also negative compare to electron because the quasi particle is the hole of the electron now). And the interaction parameter ϕ' is (one needs a little bit algebraic calculation by using (5.53)).

$$\phi' = \frac{\phi}{m} + 1. \tag{5.54}$$

What is the meaning of the above equation? The first term in (5.54)is easy to understand. This is due to the interaction of the charge $\frac{1}{m}$ quasiparticle with the magnetic field which has ϕ elementary quanta. However the second term seems not much clear to us. If we suppose this term is due to the intrinsic self interaction of the quasi particle which will relate with the spin of the quasi particle, one will find the spin of the quasiparticle is

$$s_q = \frac{1}{2} - \frac{1}{2m}. \tag{5.55}$$

. So the spin-statistics relation for quasiparticle is not standard one. There is an extra term in (5.55), $\frac{1}{2}$. How to explain this strange fact? As one know, in the polarized FQHE, the one component fermion field satisfies fermionic statistics, but is rotational scalar field. Hence this field has fermion statistics without spin. For a consistent check, we consider $m = 1$, which corresponds to an integer quantum hall effect. The excitation or quasiparticle is the electron itself. From (5.55), one gets $s_q = 0$ as expected.

To summary, one finds anyons in FQHE not only have fractional statistics, also have fractional spin. However the spin-statistics relation is not standard one. One notes that the standard GL theory of FQHE is not complete correct, because the boson with Chern-Simons term in GL theory will transmute to a fermion with also spin $\frac{1}{2}$. which is not correct in the problem of polarized FQHE. Actually this is the reason why we have relation $m(N-1) = \phi$ for the Laughling state instead of $mN = \phi$ for the problem of FQHE in the sphere and disc [6,18]. According to the standard GL theory, however one will get relation $mN = \phi$. If we discard the self interaction term due to integrating the Chern-Simons term, then we will have right relation $k(N-1) = \phi$. The self interaction term is responsible for the spin with the corresponding statistics. Without the self interaction term, the fermion field in GL theory will have spin 0, which is what we want. Due to these strange facts, so FQHE can not be correctly described by GL Chern-Simons theory. So spin-statistics relation is not standard one. However FQHE on the torus, the relation of the number of electrons and the magnetic flux for the Laughling state is $mN = \phi$. This is due to the some of flux also needed for the moving of center coordinate [6]. For a useful reference which maybe relate with this problem , see [20] (in this paper, the large component of gauge field also become a dynamical field. This maybe is the reason that [20] gives a correct picture of FQHE). The spin of anyons in FQHE should not depend on the topology of the space. What we discuss about anyon spin and statistics on the sphere should also be correct for the case of FQHE on the torus or other space. However in other space, it is not easy to see what is the spin of the anyon. Now one would like to interpret the above results by using Berry phase [77,106]. Suppose there is only a quasiparticle excitation in Laughling state with coordinate z_α. The wave function is

$$\Phi = \prod_{i<j} d_{ij}^m \prod_i d_{i\alpha}. \tag{5.56}$$

One notes that in one quasiparticle case, the normalization constant of above wave function is constant. Now let z_α depends periodically on time t. In one period, the wave function will get a phase γ_c (Berry phase). The phase $\gamma(t)$ at time t satisfies the equation

$$\frac{d\gamma(t)}{dt} = i < \Phi(t)|\frac{d}{dt}|\Phi(t) > . \tag{5.57}$$

It can be shown

$$\frac{d\gamma(t)}{dt} = i\sum_i < \Phi(t)|\frac{d \ln d_{i\alpha}}{dt}|\Phi(t) > \tag{5.58}$$

One put $|z_\alpha|$ to infinity, and rotate z_α anti clockwise. From (5.58), one has

$$\gamma_c = -2\pi N, \tag{5.59}$$

with N is the number of electron. If the closed path of z_α extends a sphere angle Ω, then

$$\gamma_c = -\frac{2\pi N\Omega}{4\pi} = -\frac{N\Omega}{2}. \tag{5.60}$$

For the Laughling state, one has $m(N-1) = \phi_0$ (ϕ_0 is the original flux), so $N = \frac{\phi_0}{m} + 1$. Now to create a quasiparticle, one needs to add a unit magnetic flux. The wave function of (5.56)corresponds that one has the magnetic flux $\phi = \phi_0 + 1$. So

$$N = \frac{\phi}{m} - \frac{1}{m} + 1. \tag{5.61}$$

Hence

$$\gamma_c = -\frac{\Omega}{2}(\frac{\phi}{m} - \frac{1}{m} + 1). \tag{5.62}$$

The first term is due to that the quasiparticle has charge $\frac{1}{m}$, so it will interact with the magnetic field. The last two term is due to the spin of quasiparticle or these term contribute to the spin of the quasi particle. They will give the quasiparticle spin $\frac{1}{2} - \frac{1}{2m}$. So directly by using Berry phase, one confirms the results about the spin of anyon in FQHE which can also get from the wave function. About how to get the fractional statistics of anyon on the sphere by using Berry phase, everything is exactly parallel to the case in the FQHE on disc. we refer it to any standard reference. Finally, one notes in level n hierarchical state, if anyon has fractional statistics $|\theta|$ (the statistical parameter is $\Delta = \frac{\theta}{2}$), then it will have spin $\frac{1}{2} - \frac{\theta}{2}$. And if one also adds the wave function the spin (polarized) component (here is Pauli spin), the spin-statistics relation will be standard one.

5.4. Spin Singlet FQHE

In this section, we consider the spin singlet FQHE wave function on the sphere (for example, the appendix in [1])

$$\Phi_m([z_i],[\eta_\alpha]) = \prod_{i<j,\alpha<\beta} d(z_i,z_j)^{m+1} d(\eta_\alpha,\eta_\beta)^{m+1} d(z_i,\eta_\alpha)^m, \tag{5.63}$$

where $z_i, i = 1,\cdots,N$ are spin up electron, $\eta_\alpha, \alpha = 1,\cdots,N$ are spin down electron. m is a positive even integer. The complete wave function should be anti symmetrized one of above Φ_m (see for example, the appendix in [1]). The wave function is spin singlet. We call it SQHE. By accounting the power of $(1+z_i\bar{z}_i)$, we get the total flux, $\psi = (m+1)(N-1)+mN$. So the filling factor ν equals $\frac{2}{2m+1}$.

The wave function can be factorized, $\Phi = \Phi_1\Phi_2$, With

$$\Phi_1 = \prod_{i<j,\alpha<\beta} d(z_i,z_j)^m d(\eta_\alpha,\eta_\beta)^m d(z_i,\eta_\alpha)^m \prod_{i<j,\alpha<\beta} |d(z_i,z_j)|^{\frac{1}{2}} |d(\eta_\alpha,\eta_\beta)|^{\frac{1}{2}} |d(z_i,\eta_\alpha)|^{\frac{1}{2}}, \tag{5.64}$$

and

$$\Phi_2 = \prod_{i<j,\alpha<\beta} d(z_i,z_j) d(\eta_\alpha,\eta_\beta) \prod_{i<j,\alpha<\beta} |d(z_i,z_j)|^{-\frac{1}{2}} |d(\eta_\alpha,\eta_\beta)|^{-\frac{1}{2}} |d(z_i,\eta_\alpha)|^{-\frac{1}{2}}. \tag{5.65}$$

First let us consider Φ_2. It is not hard to see that Φ_2 is the wave function of free semions in Chern-Simons gauge with ordinary Pauli spin and fermion statistics (keep in mind the complete wave function should be anti symmetrized) Counting the power of $(1+z_i\bar{z}_i)$ in Φ_2, we get the spin of the semions $s=\frac{1}{4}$ (compare with (5.33)). The power of $(1+z_i\bar{z}_i)$ in Φ_2 is not due to magnetic field, it is due to the spin of semions as required by spin-statistics relation. However we should keep in mind the difference between this spin and ordinary Pauli spin. The Hamiltonian with hard core interaction and with which Φ_2 is a ground state ($H\Phi_2=0$) is

$$H=\frac{2}{m}\sum_{i=1}^{2N}(1+z_i\bar{z}_i)^2(P_{z_i}-qA_{z_i})(P_{\bar{z}_i}-qA_{\bar{z}_i}), \tag{5.66}$$

with $z_{N+i}=\eta_i$, and $\theta=-\frac{1}{2}$ following the notation in section 2.

Now let us consider Φ_1. Compare the wave function with the wave function in section 2 (hard core repulsion wave function in magnetic field), equation (5.35), we get, $\theta=\frac{1}{2}$ and $\psi=N+m(2N-1)$. So the filling factor is $\frac{2}{2m+1}$. Φ_1 is a wave function of fractional quantum hall state of Semions in magnetic field with filling $\frac{2}{2m+1}$ (the Hamiltonian is the same type of Hamiltonian in equation (5.34)).

So Φ_2 represents the spin part of Φ_m, it is a wave function of free spin-$\frac{1}{2}$ semion. On the other hand Φ_1 represents the interaction part of Φ_m, or a charge sector, it is a Laughling type wave function of semions [16]. Moreover, Φ_1 and Φ_2 are rotational invariant.

Now we explore some facts the wave function Φ_2. Consider scalar field on the sphere

$$S=\frac{1}{2}\int d^2x\sqrt{g}g^{\alpha\beta}\partial_\alpha X\partial_\beta X. \tag{5.67}$$

The sphere metric is

$$g_{\alpha\beta}=\frac{1}{2(1+z\bar{z})^2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{5.68}$$

The action becomes

$$S=2\int d^2z\partial_{\bar{z}}X\partial_z X. \tag{5.69}$$

The equation of motion is

$$\partial_{\bar{z}}\partial_z X = 0. \tag{5.70}$$

Now we consider $X = q[Im \ln(z - z_i) - Im \ln(z - z_j)]$. It represents the vortex and anti vortex pair. q is the charge of vortex. which should be an integer [23]. The action of the vortex and anti vortex pair will be (up to a constant which depend on cutoff)

$$S(X) = 2\pi q^2 \ln[\frac{|z_1 - z_2|}{(1 + z_i\bar{z}_i)^{\frac{1}{2}}(1 + z_j\bar{z}_j)^{\frac{1}{2}}}]. \tag{5.71}$$

Now we consider the mapping to the plasma gas

$$\Phi_2\bar{\Phi}_2 = e^{\sum_{i<j} \ln|d(z_i,z_j)| + \sum_{\alpha<\beta} \ln|d(\eta_\alpha,\eta_\beta)| - \sum_{i,\alpha} \ln|d(z_i,\eta_\alpha)|}. \tag{5.72}$$

This is a partition function for a neutral gas of vortices and anti-vortices. It is not only globally neutral but also locally neutral [23]. From these results, we see that Φ_2 is not only spin singlet, but also it is local spin singlet, so does Φ_m. So we generalize the similar results on the plane [15] to the sphere.

Some related discussion about non spinning anyon also can be found in [105].

REFERENCES

1. *The Quantum Hall Effect*, edited by R. Prange and S. Girvin. 2nd ed. Springer-Verlag, New York,Heidelberg, 1990
2. F. Wilczek, *Fractional Statistics and Anyon Superconductivity*, World Scientific, 1990.
3. F. Wilczek, Phys. Rev. Lett. **49**(1982)957.
4. Y. S. Wu, Phys.Rev. Lett. **52**(1984)2103; ibid, bf 53(1984)111.
5. R. B. Laughling, Phys. Rev. Lett.**50**(1983)1395.
6. F. D. M. Haldane, Phys. Rev. Lett.**51**(1983)605.
7. B. I. Halperin, Phys. Rev. Lett. **52**(1984)1583.
8. D. P. Arovas, J. R. Schriffer, and F. Wilczek, Phys. Rev. Lett. **53**(1984)722.
9. R. B. Laughlin, Phys. Rev. Lett. **60**(1988)2677.
10. V. Kalmeyer, R. B. Laughlin, Phys. Rev. Lett. **59** (1988)2095.
11. C. B. Hanna, R. B. Laughlin, and A. L. Fetter, Phys. Rev. **B40**(1989)8745.
12. C. B. Hanna, R. B. Laughlin, and A. L. Fetter, Phys. Rev. **B39**(1989)9679.
13. Y. H. Chen, F. Wilczek, E. Witten, and B. Halperin, Int. J. Mod. Phys. **3**(1989)1001.
14. S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett.**62**(1989)82.
15. S. M. Girvin, A. H. Mac-Donald, M. P. A. Fisher, S. J. Rey and J. Sethna, Phys. Rev. Lett. **65**(1990)1671.
16. S. Balatasky and E. Fradkin, *The Singlet Quantum Hall Effect and Chern-Simons Theories*, Illinois Preprint.
17. Blok, X. G. Wen, Phys. Rev. **B42**(1990)8145; ibid. **43**(1991)8337.
18. N. Read, Phys. Rev. Lett. **65**(1990)1502.

19. R. Iengo and K. Lechner, Nucl. Phys. **B346**(1990)551.

20. R. Iengo and K. Lechner, SISSA Preprint, SISSA-182-90-EP (March 1991). Nucl. Phys. FS in press.

21. Z. F. Ezawa and A. Iwazaki, Phys. Rev. **B43**(1991)2637.

22. K. Lee, *Anyons on Spheres and Tori*, Preprint BU/HEP-89-28.

23. B. A. Ovrut and S. Thomas, Phys. Rev. **D43**(1991)1314.

24. *Current Algebra and Anomalies*, edited by B. Treiman, *etc.* World Scientific, 1985.

25. S. Forte, *Quantum Mechanics and Field Theory with Fractional Statistics*, SPhT/90-180, Saclay Preprint.

26. L. S. Schulman, *Techniques and Applications of Path Integration*, Wiley, New York, 1981.

27. R. Mackenzie and F. Wilczek, Int. J. Mod. Phys.**A3**(1988)2827.

28. S. Coleman, Phys. Rev.**D11**(1975)2088.

29. S. Mandelstam, Phys. Rev.**D11**(1975)3026.

30. T. Matsuyama, Phys. Lett.**B228**(1989)99.

31. M. Lüscher, Nucl. Phys.**B228**(1898)557.

32. A. S. Goldhaber and R. Mackenzie, Phys. Lett.**B214**(1988)471; T. H. Hansson, *etc*, ibid **214**(1988)475.

33. R. Jackiw and A. N. Redlich, Phys. Rev. Lett.**50**(1983)555.

34. D. Thouless and Y. S. Wu, Phys. Rev.**B31**(1985)1191.

35. A. M. Polyakov, Mod. Phys. Lett. **A3**(1988)325.

36. A. M. Polyakov, in *Fields, Strings and Critical phenomena*, Les Houches summer school 1989. edited by E. Brézin and J. Zinn-Justin(North Holland, Amsterdam).

37. C. R. Hagen, Ann. Phys. **157**(1984)342. Phys. Rev.D**31**(1985)1029.

38. X. G. Wen and A. Zee, J. Phys. France. **50**(1989)1623

39. G. M. Semenoff, *Anyons and Chern-Simons Theory: a Review*, Preprint.

40. A. Niemi and G. M. Semenoff, Phys. Rep. **135**(1986)99.

41. S. Coleman and B. Hill, Phys. Lett. **B159**(1985)184.

42. Itzykson and Zuber, *Quantum Field Theory.*

43. J. F. Schonfeld, Nucl. Phys. **B185**(1981)157.

44. R. Rajaraman, *Soiltons and Instantons*, North Holland, Amsterdam, 1982.

45. W. J. Zakrzewski, *Low Dimensional Sigma Models*, Adam Hilger.

46. Y.S Wu and A. Zee, Phys. Lett. **B147**(1984)325.

47. D. H. Lee and M. P. A. Fisher, *Anyon Superconductivity and Charge-Vortex Duality*, Preprint.

48. D. H. Lee and C. L. Kane, Phys. Rev. Lett. **64**(1990)1313.

49. K. Bardakci and S. Samuel, Phys. Rev. **D18**(1878)2849.

50. J. Fröhlich and A. Zee, *Large Scale Physics of the Quantum Hall Fluid*, ETH-Hoenggerberg and Santa Babara Preprint. NSF-ITP-91-31 (1991).

51. Dung-Hai Lee and Xiao-Gang Wen, Phys. Rev. Lett. **66**(1991)1765.

52. J.M. Leinaas and J. Myrheim, Nuovo Cimento **37B** (77) 1

53. G.A. Goldin, R. Menikoff and D.H. Sharp, J. Math. Phys. **22** (81) 1664.

54. D. Finkelstein and Julio Rubinstein, J. Math. Phys. **9** (68) 1762.

55. Y. Aharonov and D. Bohm, Phys. Rev. **115** (59) 485.

56. M.V. Berry, Proc. R. Soc. **A392** (84) 45.

57. E. Witten, Comm. Math. Phys. **121** (89) 351.

58. R. Dijkgraaf and E. Witten, Comm. Math. Phys. **129** (90) 393.

59. M. Bos and V.P. Nair, Phys. Lett. **B223** (89) 61.

60. M. Bos and V.P. Nair, Int. J. Mod. Phys. **A5** (90) 959.

61. Y. Hosotani, Phys. Rev. Lett. **62** (89) 2785.

62. Y. Hosotani and S. Chakravarty, Phys. Rev. **B42** (90) 342.

63. J.M.F. Labastida and A.V. Ramallo, Phys. Lett. **B227** (89) 92.

64. J.M.F. Labastida and A.V. Ramallo, Phys. Lett. **B228** (89) 214.

65. S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. **B326** (89) 108.

66. A.P. Polychronakos, Ann. Phys. **203** (90) 231.

67. A.P. Polychronakos, *Abelian Chern–Simons Theories and Conformal Blocks*, pre- print, UFIFT -89-9.

68. A.P. Polychronakos, Phys. Lett. **B241** (90) 37.

69. A.P. Polychronakos, Nucl. Phys. **B281** (87) 241.

70. F. Wilczek, Phys. Rev. Lett. **48** (82) 1144.

71. F. Wilczek and A. Zee, Phys. Rev. Lett. **51** (83) 2250.

72. R.B. Laughlin, Phys. Rev. **B23** (81) 5632.

73. R.B. Laughlin, Phys. Rev. **B27** (83) 3383.

74. R.B. Laughlin, Sciences **242** (88) 525.

75. B.I. Halperin, Phys. Rev. **B25** (82) 2185.

76. J.K. Jain, Phys. Rev. **B41** (90) 7653.

77. D.P. Arovas, R. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. **B251** [FS13] (85) 117.

78. N. Read, Phys. Rev. Lett. **62** (89) 86.

79. X.G. Wen, F. Wilczek and A. Zee, Phys. Rev. **B39** (89) 11413.

80. X.G. Wen, E. Dagotto and E. Fradkin, *How to put anyons on a torus*, P/90/4/49, Santa Barbara.

81. K. Lechner, *Anyon Physics on the torus*, Ph.D. Thesis, SISSA–ISAS, Trieste (1991).

82. R. Iengo, K. Lechner and Dingping Li, *Chern–Simons–Maxwell theory on the torus*, SISSA .../91/EP, Trieste, to appear in Phys. Lett. B.

83. R. Jackiw and S.Y. Pi, Phys. Rev. Lett. **64** (90) 2969.

84. R. Jackiw and S.Y. Pi, Phys. Rev. **D42** (90) 3500.

85. F.D.M. Haldane and E.H. Rezayi, Phys. Rev. **B31** (85) 2529.

86. D. Mumford, *Tata Lectures on Theta*, Birkhauser Vols. 2 (1983).

87. R. Jackiw, "Topics in planar Physics", lectures given at the XXIX Schladming School (Austria), March 1990.

88. S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48** (82) 975; Ann. Phys. (NY) **140** (82) 372.

89. J. Fröhlich and T. Kerler, Nucl. Phys. **B354**(1991)369.

90. T. Einarsson, Phys. Rev. Lett. **64** (90) 1995.

91. T.–S. Wu, Y. Hatsugai and M. Kohmoto, Phys. Rev. Lett. **66** (91) 659.

92. M.D. Johnson and G.S. Canright, Phys. Rev. **B41** (90) 6870.

93. J. Fröhlich and P.A. Marchetti, Lett. Math. Phys. **16** (88) 347.

94. J. Fröhlich and P.A. Marchetti, *Spin–Statistics Theorem and Scattering in planar Quantum Field Theories with braid statistics*, DFPD/TH/10/90, Padua, May (1990).

95. G. Dunne, A. Lerda and C. Trugenberger, *Landau levels and Vertex Operators for Anyons*, CTP/1938/January 1991.

96. R.E. Prange, Phys. Rev. **B23** (81) 4802.

97. G.W. Semenoff and P. Sodano, Nucl. Phys. **B328** (89) 753.

98. E.Witten, Comm. Math. Phys. **92**(1984)455

99. G. Moore and N. Seiberg, Lecture on RCFT, Spring School, Trieste(1989)

100. M. G. G. Laidlaw and C. M. De Witt, Phys. Rev.**D5**(1971)1357.

101. G. Moore and N. Seinberg, Phys. Lett. **B220**(1989)422.

102. G. Moore and N. Read, Nucl. Phys. B**360**(1991)362.

103. T. Einarsson, Preprint, Göteborg ITP 91-1.

104. D. P. Li, *Anyons and Quantum Hall Effect on the Sphere*, SISSA Preprint. SISSA/129/91/EP.

105. A. Comtet, J. McCabe and S. Ouvry, *Some Remarks on Anyons on the 2-Sphere*, Orsay Preprint IPNO/TH 91-55.

106. D. Arovas in *Geometric Phase in Physics*, edit by F. Wilczek, World Scientific, 1989