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Approximation of Relaxed Dirichlet Problems by Boundary Value Problems in Perforated Domains

Thesis submitted for the degree of
"Magister Philosophiæ"

CANDIDATE
Annalisa Malusa

SUPERVISOR
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Introduction

The notion of "relaxed Dirichlet problem" was introduced in [6] to describe the asymptotic behaviour of the solutions of classical Dirichlet problems in strongly perturbed domains. Given a bounded open subset Ω of $\mathbf{R}^n$, $n \geq 2$, and an elliptic operator L on Ω, a relaxed Dirichlet problem can be written in the form

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where $f \in H^{-1}(\Omega)$ and μ belongs to the space $\mathcal{M}_0(\Omega)$ of all positive Borel measures on Ω which do not charge any set of capacity zero.

The main result concerning relaxed Dirichlet problems is the following compactness theorem (see [6], Theorem 4.14): for every sequence $\{\Omega_h\}$ of open subsets of Ω there exist a subsequence, still denoted by $\{\Omega_h\}$, and a measure $\mu \in \mathcal{M}_0(\Omega)$, such that for every $f \in H^{-1}(\Omega)$ the solutions u_h of the Dirichlet problems

$$\begin{cases} Lu_h = f & \text{in } \Omega_h, \\ u_h = 0 & \text{on } \partial\Omega_h, \end{cases} \tag{0.2}$$

extended to 0 on $\Omega \backslash \Omega_h$, converge in $L^2(\Omega)$ to the unique solution u of (0.1). Moreover, the following density theorem holds (see [6], Theorem 4.16): for every $\mu \in \mathcal{M}_0(\Omega)$ there exists a sequence $\{\Omega_h\}$ of open subsets of Ω such that for every $f \in H^{-1}(\Omega)$ the solution u of (0.1) is the limit in $L^2(\Omega)$ of the sequence $\{u_h\}$ of the solutions of (0.2). The proof of this density theorem provides an explicit approximation only when μ is the Lebesgue measure, while it is rather indirect in the other cases, and does not suggest any efficient method for the construction of the sets Ω_h.

The aim of this paper is to present an explicit approximation scheme for the relaxed Dirichlet problems (0.1) by means of sequences of classical Dirichlet problems of the form (0.2). We assume that $\mu \in \mathcal{M}_0(\Omega)$ is a Radon measure. The sets Ω_h will be obtained by removing an array of small balls from the set Ω. The geometric construction is quite simple. For every $h \in \mathbf{N}$ we fix a partition $\{Q_h^i\}_i$ of $\mathbf{R}^n$ composed of cubes with side $1/h$, and we consider the set I_h of all indices i such that $Q_h^i \subset\subset \Omega$. For every $i \in I_h$ let B_h^i be the ball with the same center as Q_h^i and radius $1/2h$, and let E_h^i be another ball with the same center such that

$$\operatorname{cap}^L(E_h^i, B_h^i) = \mu(Q_h^i).$$

Finally, let $E_h = \bigcup_{i \in I_h} E_h^i$ and $\Omega_h = \Omega \backslash E_h$. Note that the size of the hole E_h^i contained in the cube Q_h^i depends only on the operator L and on the value of the measure μ on Q_h^i.

By using a very general version of the Poincaré inequality proved by P. Zamboni [15], we shall show that, if μ belongs to the Kato space $K_n^+(\Omega)$, i.e., the potential generated by μ is continuous, then the method introduced by D. Cioranescu and F. Murat [4] can be applied, so that for every $f \in H^{-1}(\Omega)$ the solutions u_h of the Dirichlet problems (0.2) converge in $L^2(\Omega)$ to the solution u of the relaxed Dirichlet problem (0.1). To prove that the same result holds also when μ is an arbitrary Radon measure of the class $\mathcal{M}_0(\Omega)$ we use the method of μ-capacities introduced in [6] and [3].

Finally, if μ is a Radon measure and $\mu \notin \mathcal{M}_0(\Omega)$, then we prove that our construction leads to the approximation of the solutions of the relaxed Dirichlet problem

$$\begin{cases} Lu + \mu_0 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ_0 is the greatest measure of the class $\mathcal{M}_0(\Omega)$ which is less than or equal to μ.

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1. Notation and preliminaries

Let Ω be a bounded open subset of $\mathbf{R}^n$, $n \geq 2$. We shall denote by $H^1(\Omega)$ and $H_0^1(\Omega)$ the usual Sobolev spaces, by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$, by $L^p_\mu(\Omega)$, $1 \leq p < \infty$ the usual Lebesgue space with respect to the measure μ; if μ is the Lebesgue measure, we shall use the notation $L^p(\Omega)$.

For every subset E of Ω the (harmonic) capacity of E with respect to Ω is defined by

$$\inf \int_\Omega |\nabla u|^2 \, dx \,,$$

where the infimum is taken over all functions $u \in H_0^1(\Omega)$ such that $u \geq 1$ a.e. in a neighbourhood of E. We say that a property $\mathcal{P}(x)$, depending on a point $x \in \Omega$, holds quasi everywhere (q.e.) in Ω if there exists a set $E \subseteq \Omega$, with $\operatorname{cap}(E, \Omega) = 0$, such that $\mathcal{P}$ holds in $\Omega \backslash E$. It is well known that every $u \in H^1(\Omega)$ admits a quasi-continuous representative, which is uniquely defined up to a set of capacity zero (see, e.g., [16], Theorem 3.1.4). In the sequel we shall always identify u with its quasi-continuous representative.

By a Borel measure on Ω we mean a positive, countably additive set function with values in $\overline{\mathbf{R}}$ defined on the σ-field of all Borel subsets of Ω; by a Radon measure on Ω we mean a Borel measure which is finite on every compact subset of Ω. Finally, by $\mathcal{M}_0(\Omega)$ we denote the set of all positive Borel measures μ on Ω such that $\mu(E) = 0$ for every Borel set $E \subseteq \Omega$ with $\mathrm{cap}(E, \Omega) = 0$. If μ is a Borel measure and E is a Borel subset of Ω, the Borel measure $\mu \llcorner E$ is defined by $(\mu \llcorner E)(B) = \mu(E \cap B)$ for every Borel set $B \subseteq \Omega$. If μ, ν are Radon measures and ν has a density f with respect to μ, we shall write $\nu = f\mu$. For every $E \subseteq \Omega$ we denote by ∞_E the measure of the class $\mathcal{M}_0(\Omega)$ defined by

$$\infty_E(B) = \begin{cases} 0, & \text{if } \mathrm{cap}(B \cap E, \Omega) = 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.1}$$

We shall see later that these measures are used to express the classical Dirichlet problems (0.2) in the form (0.1). This will allow us to treat problems (0.1) and (0.2) in a unified way.

Another class of measures we are interested in is the Kato space.

Definition 1.1 The *Kato space* $K_n^+(\Omega)$ is the cone of all positive Radon measures μ on Ω such that

$$\lim_{r \to 0+} \sup_{x \in \Omega} \int_{\Omega \cap B_r(x)} G_n(y - x)\, d\mu(y) = 0,$$

where G_n is the fundamental solution of the Laplace operator $-\Delta$ in $\mathbf{R}^n$, and $B_r(x)$ denotes the open ball with center x and radius r.

For every $\mu \in K_n^+(\Omega)$ and for every Borel set $A \subseteq \Omega$ we define

$$\|\mu\|_{K_n^+(A)} = \sup_{x \in A} \int_A |y - x|^{2-n}\, d\mu(y), \qquad \text{if } n \geq 3,$$

$$\|\mu\|_{K_n^+(A)} = \sup_{x \in A} \int_A \log\left(\frac{\mathrm{diam}\,(A)}{|y - x|}\right) d\mu(y) + \mu(A), \qquad \text{if } n = 2.$$

For every $\mu \in K_n^+(\Omega)$ it is easy to see that $\|\mu\|_{K_n^+(\Omega)} < +\infty$ and $\|\mu\|_{K_n^+(A)}$ tends to zero as $\mathrm{diam}(A)$ tends to zero. We recall that every measure in $K_n^+(\Omega)$ is bounded and belongs to $H^{-1}(\Omega)$. For more details about this subject we refer to [1], [6], [9], [14]. We shall use in the following a Poincaré inequality involving Kato measures.

Lemma 1.2 *Let A be a Borel subset of a ball $B_R = B_R(x_0)$ such that* $\mathrm{diam}(A) \geq q\,R$ *for some $q \in (0,1)$, and let $\mu \in K_n^+(A)$. Then there exists a positive constant c, depending only on q and on the dimension n of the space, such that*

$$\int_A u^2\, d\mu \leq c\, \|\mu\|_{K_n^+(A)} \int_{B_R} |\nabla u|^2\, dx$$

for every $u \in H^1_0(B_R)$.

Proof. An inequality of this kind was proved by P. Zamboni in the case $n \geq 3$, $A = B_R$, and μ absolutely continuous with respect to the Lebesgue measure. The same arguments can be adapted, up to minor modifications, also to the general case. The main change in the case $n = 2$ is the use of the inequality

$$\int_{B_R} \frac{1}{|x-y|\,|z-y|}\, dy \leq c_q \left(\log\left(\frac{\operatorname{diam}(A)}{|x-z|}\right) + 1 \right) \qquad \forall x, z \in A,$$

which can be proved by direct computation. □

Finally we need a sort of dominated convergence theorem for measures in $H^{-1}(\Omega)$.

Lemma 1.3 *Let $\{\mu_h\}$ be a sequence of positive measures belonging to $H^{-1}(\Omega)$ that converges to 0 in the weak* topology of measures and suppose that there exists $\mu \in H^{-1}(\Omega)$ such that $\mu_h \leq \mu$. Then the sequence $\{\mu_h\}$ converges to 0 strongly in $H^{-1}(\Omega)$.*

Proof. This result could be obtained easily by using the strong compactness of the order intervals in $H^{-1}(\Omega)$. However, we give here a self-contained elementary proof. Let us define $\nu_h = \mu - \mu_h$. Clearly $\|\nu_h\|_{H^{-1}(\Omega)} \leq \|\mu\|_{H^{-1}(\Omega)}$ and so, up to a subsequence, $\{\nu_h\}$ converges to μ weakly in $H^{-1}(\Omega)$. The previous inequality, together with the lower semicontinuity of the norm, implies that $\|\nu_h\|_{H^{-1}(\Omega)}$ converges to $\|\mu\|_{H^{-1}(\Omega)}$. This shows that $\{\nu_h\}$ converges to μ strongly in $H^{-1}(\Omega)$ and concludes the proof of the lemma.

□

Let $L\colon H^1_0(\Omega) \to H^{-1}(\Omega)$ be a linear elliptic operator in divergence form

$$Lu = -\operatorname{div}(A\nabla u),$$

where $A = A(x) = (a_{ij}(x))$ is a symmetric $n \times n$ matrix of bounded measurable functions satisfying, for a suitable constant $\alpha > 0$, the ellipticity condition

$$\alpha |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i\, \xi_j \leq \alpha^{-1} |\xi|^2$$

for a.e. x in Ω, and for every $\xi \in \mathbf{R}^n$.

A set function cap^L_μ can be associated with every measure μ in the class $\mathcal{M}_0(\Omega)$.

Definition 1.4 Let $\mu \in \mathcal{M}_0(\Omega)$. For every open set $A \subseteq \Omega$ and for every Borel set $E \subseteq A$ we define the μ-capacity of E in A corresponding to the operator L as

$$\mathrm{cap}^L_\mu(E, A) = \min\left\{ \langle Lu, u\rangle + \int_E u^2\, d\mu : u - 1 \in H^1_0(A) \right\},$$

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

The μ-capacity corresponding to $L=-\Delta$ will be denoted by cap_μ, while the μ-capacity with respect to $\mu=\infty_\Omega$ will be denoted by cap^L. The latter coincides with the classical capacity relative to the operator L according to the definition of [13] and [12]. If $L=-\Delta$ and $\mu=\infty_\Omega$, then cap_μ^L coincides with the harmonic capacity introduced at the beginning of this section. If $\mu=\infty_F$ for some $F\subseteq\Omega$, and L is any elliptic operator, then $\mathrm{cap}_\mu^L(E,A)=\mathrm{cap}^L(E\cap F,A)$ for every $E\subseteq A$.

Some of the properties of cap_μ^L are stated in the following proposition.

Proposition 1.5 *Let $\mu\in\mathcal{M}_0(\Omega)$, A, B open subsets of Ω and E, F subsets of A. Then*

(i) $\mathrm{cap}_\mu^L(\emptyset,A)=0$;

(ii) $E\subseteq F\Longrightarrow \mathrm{cap}_\mu^L(E,A)\le\mathrm{cap}_\mu^L(F,A)$;

(iii) $\mathrm{cap}_\mu^L(E\cup F,A)\le\mathrm{cap}_\mu^L(E,A)+\mathrm{cap}_\mu^L(F,A)$;

(iv) $A\subseteq B\Longrightarrow \mathrm{cap}_\mu^L(E,A)\ge\mathrm{cap}_\mu^L(E,B)$;

(v) $\alpha\,\mathrm{cap}_\mu(E,A)\le\mathrm{cap}_\mu^L(E,A)\le\alpha^{-1}\,\mathrm{cap}_\mu(E,A)\le\alpha^{-1}\mathrm{cap}(E,A)$;

(vi) if $\{E_h\}$ is an increasing sequence of subsets of A and $E=\cup_h E_h$, then $\mathrm{cap}_\mu^L(E,A)=\sup_h\mathrm{cap}_\mu^L(E_h,A)$.

Proof. See [6], Proposition 3.11, Theorem 3.10 and [5], Theorem 2.9. □

Now we introduce the notion of relaxed Dirichlet problems.

Definition 1.6 Given $\mu\in\mathcal{M}_0(\Omega)$ and $f\in H^{-1}(\Omega)$, we say that a function u is a solution of the *relaxed Dirichlet problem*

$$\begin{cases} Lu+\mu u=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}\tag{1.2}$$

if $u\in H_0^1(\Omega)\cap L_\mu^2(\Omega)$ and

$$\langle Lu,v\rangle+\int_\Omega u\,v\,d\mu=\langle f,v\rangle$$

for every $v\in H_0^1(\Omega)\cap L_\mu^2(\Omega)$.

We recall that for every $f\in H^{-1}(\Omega)$ there exists a unique solution u of problem (1.2) (see [6], Theorem 2.4). It is easy to see that, if E is a closed set, then u is a solution of

$$\begin{cases} Lu+\infty_E\,u=f & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

if and only if $u = 0$ q.e. in $E \cap \Omega$ and $u_{|\Omega \setminus E}$ is a weak solution of the the classical boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega \setminus E, \\ u \in H_0^1(\Omega \setminus E). \end{cases}$$

Definition 1.7 A sequence $\{\mu_h\}$ in $\mathcal{M}_0(\Omega)$ γ^L-converges to $\mu \in \mathcal{M}_0(\Omega)$ if, for every $f \in H^{-1}(\Omega)$, the sequence $\{u_h\}$ of the solutions of the problems

$$\begin{cases} Lu_h + \mu_h u_h = f & \text{in } \Omega, \\ u_h = 0 & \text{on } \partial\Omega, \end{cases}$$

converges strongly in $L^2(\Omega)$ to the solution u of the problem

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

With every $\mu \in \mathcal{M}_0(\Omega)$ we associate the lower semicontinuous quadratic functional on $H_0^1(\Omega)$ defined by

$$F_\mu(u) = \langle Lu, u \rangle + \int_\Omega u^2 \, d\mu .$$

The following theorem shows the connection between γ^L-convergence of the measures μ_h and Γ-convergence of the corresponding functionals F_{μ_h}.

Theorem 1.8 *A sequence $\{\mu_h\}$ in $\mathcal{M}_0(\Omega)$ γ^L-converges to the measure $\mu \in \mathcal{M}_0(\Omega)$, if and only if the following conditions are satisfied for every $u \in H_0^1(\Omega)$:*
(a) for every sequence $\{u_h\}$ in $H_0^1(\Omega)$ converging to u in $L^2(\Omega)$

$$F_\mu(u) \le \liminf_{h\to\infty} F_{\mu_h}(u_h);$$

(b) there exists a sequence $\{u_h\}$ in $H_0^1(\Omega)$ converging to u in $L^2(\Omega)$ such that

$$F_\mu(u) = \lim_{h\to\infty} F_{\mu_h}(u_h).$$

Proof. See [2], Proposition 2.9. □

Our definition of γ^L-convergence coincides with the definition considered in [5]. As shown in [2], Proposition 2.8, if properties (a) and (b) hold on Ω, then they also hold for every open set $\Omega' \subseteq \Omega$. Conversely, if (a) and (b) hold for every open set $\Omega' \subset\subset \Omega$, then they hold on Ω. So our definition of γ^L-convergence differs from the definition given in [3] only in the fact that now the ambient space is Ω instead of $\mathbf{R}^n$. When $L = -\Delta$, our definition coincides with the definition given in [6].

Remark 1.9 Let $\{\lambda_h\}$ and $\{\mu_h\}$ be two sequences in $\mathcal{M}_0(\Omega)$ which γ^L-converge to λ and μ, respectively. If $\lambda_h \le \mu_h$ for every h, by Theorem 1.8 we have $\int_\Omega u^2\, d\lambda \le \int_\Omega u^2\, d\mu$ for every $u \in H_0^1(\Omega)$. In particular, if μ is a Radon measure, then $\lambda \le \mu$.

We briefly recall some properties of the γ^L-convergence of measures in $\mathcal{M}_0(\Omega)$.

Theorem 1.10 *For every sequence $\{\mu_h\}$ in $\mathcal{M}_0(\Omega)$ there exists a subsequence $\{\mu_{h_k}\}$ which γ^L-converges to a measure μ in $\mathcal{M}_0(\Omega)$.*

Proof. The proof for the case $L = -\Delta$, can be found in [7], Theorem 4.14. The proof in the general case is similar. □

Theorem 1.11 *Let $\{\mu_h\}$ be a sequence in $\mathcal{M}_0(\Omega)$ which γ^L-converges to a measure μ in $\mathcal{M}_0(\Omega)$. Then*

$$\mathrm{cap}^L_\mu(A,B) \le \liminf_{h\to\infty} \mathrm{cap}^L_{\mu_h}(A,B),$$

for every pair of open sets A, B, with $A \subseteq B \subseteq \Omega$.

Proof. See [5], Proposition 5.7. □

We consider now a sufficient condition for the γ^L-convergence of a sequence of measures of the form $\{\infty_{E_h}\}$, where $\{E_h\}$ is a sequence of compact subsets of Ω. In this case, if $\Omega_h = \Omega \backslash E_h$, the solution u_h coincides with the solution of the classical problem

$$\begin{cases} L u_h = f & \text{in } \Omega_h, \\ u_h = 0 & \text{on } \partial\Omega, \end{cases}$$

prolonged to zero outside Ω_h.

Assume that $\{E_h\}$ satisfies the following hypotheses, studied by D. Cioranescu and F. Murat: there exist a measure $\mu \in W^{-1,\infty}(\Omega)$, a sequence $\{w_h\}$ in $H^1(\Omega)$, and two sequences of positive measures of $H^{-1}(\Omega)$, $\{\nu_h\}$ and $\{\lambda_h\}$, such that

$$w_h \rightharpoonup 1 \qquad \text{weakly in } H^1(\Omega),$$
$$w_h = 0 \qquad \text{q.e. in } E_h,$$
$$L w_h = \nu_h - \lambda_h,$$
$$\nu_h \to \mu \qquad \text{strongly in } H^{-1}(\Omega),$$
$$\lambda_h \rightharpoonup \mu \qquad \text{weakly in } H^{-1}(\Omega),$$

and $\langle \lambda_h, v\rangle = 0$ for every $h \in \mathrm{N}$ and for every $v \in H_0^1(\Omega)$, with $v = 0$ q.e. in E_h.

Under these hypotheses the sequence $\{u_h\}$ converges weakly in $H_0^1(\Omega)$ to the weak solution u of the problem

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(see [4], Théorème 1.2). Later, H. Kacimi and F. Murat pointed out that the hypothesis $\mu \in W^{-1,\infty}(\Omega)$ can be replaced by $\mu \in H^{-1}(\Omega)$ (see [10], Rémarque 2.4). In conclusion, using the language introduced in Definition 1.7, the following theorem holds.

Theorem 1.12 *If $\{E_h\}$ satisfies the hypotheses considered above, with $\mu \in H^{-1}(\Omega)$, then the sequence of measures $\{\infty_{E_h}\}$ γ^L-converges to the measure μ.*

2. The main results

In this section we prove that for every Radon measure $\mu \in \mathcal{M}_0(\Omega)$ the general approximation rule outlined in the introduction provides a sequence of measures of the form $\{\infty_{E_h}\}$ which γ^L-converges to μ according to Definition 1.7.

To deal with the case $\mu \in K_n^+(\Omega)$, we need the following lemmas.

Lemma 2.1 *Let U and V be open subsets of Ω, with $V \subset\subset U \subset\subset \Omega$, and let w be the L-capacitary potential of V with respect to U, i.e., the unique solution of*

$$\begin{cases} w \in H_0^1(U), & w \geq 1 \text{ q.e. on } V, \\ \langle Lw, v - w\rangle \geq 0, & \forall v \in H_0^1(U),\ v \geq 1 \text{ q.e. on } V. \end{cases}$$

Let us extend w to Ω by setting $w = 0$ on $\Omega \backslash U$. Then $w \in H_0^1(\Omega)$ and $w = 1$ q.e. on V. Moreover there exist two positive Radon measures γ and ν belonging to $H^{-1}(\Omega)$ such that $\operatorname{supp}\gamma \subseteq \partial V$, $\operatorname{supp}\nu \subseteq \partial U$, $Lw = \gamma - \nu$ in Ω, and $\nu(\Omega) = \gamma(\Omega) = \operatorname{cap}^L(V,U)$.

We call γ (resp. ν) the inner (resp. outer) L-capacitary distribution of V with respect to U.

Proof of Lemma 2.1. It is well known (see [13], Section 3) that there exists a positive Radon measure $\gamma \in H^{-1}(U)$, with $\operatorname{supp}\gamma \subseteq \partial V$, such that $Lw = \gamma$ in Ω and $\gamma(\Omega) = \operatorname{cap}^L(V,U)$. Let us consider now the following obstacle problem

$$\begin{cases} z \in H_0^1(\Omega), & z \geq 0 \text{ q.e. in } \Omega\backslash U, \\ \langle Lz + \gamma, v - z\rangle \geq 0 & \forall v \in H_0^1(\Omega),\ v \geq 0 \text{ q.e. in } \Omega\backslash U. \end{cases}$$

It is well known that there exists a unique solution z of this problem, that z is a supersolution of $L+\gamma$, i.e., $Lz+\gamma = \nu \geq 0$ for some positive measure $\nu \in H^{-1}(\Omega)$, and that $z \leq \zeta$ for every supersolution $\zeta \in H^1(\Omega)$ of $L+\gamma$ with $\zeta \geq 0$ q.e. in $\Omega\backslash U$ (see [11], Section II.6).

Since γ is a positive measure, 0 is a supersolution of $L+\gamma$. Consequently $z \leq 0$ q.e. in Ω. As $z \geq 0$ q.e. in $\Omega \backslash U$, we conclude that $z = 0$ q.e. in $\Omega \backslash U$, hence $z \in H_0^1(U)$. On the other hand $Lz + \gamma = 0$ in U. As $Lw = \gamma$ on U, by uniqueness we can conclude that $z = -w$ in U, hence in Ω. This implies $Lw = \gamma - \nu$ in Ω. As $Lw - \gamma = 0$ in U and in $\Omega \backslash \overline{U}$ we conclude that $\operatorname{supp} \nu \subseteq \partial U$. Since $Lw = \gamma - \nu$ in Ω, we have

$$\int_\Omega A \nabla w \cdot \nabla \varphi \, dx = \int_\Omega \varphi \, d\gamma - \int_\Omega \varphi \, d\nu \qquad \forall \varphi \in H_0^1(\Omega).$$

Let ψ be a cut-off function of class $C_0^\infty(\Omega)$ such that $\psi(x) = 1$ in $\overline{U}$. Choosing $\varphi = \psi(w-1)$ as test function we obtain

$$\int_\Omega A \nabla w \cdot \nabla w \, \psi \, dx + \int_\Omega A \nabla w \cdot \nabla \psi \, (w-1) \, dx = \int_\Omega \psi \, (w-1) \, d\gamma + \int_\Omega \psi \, (1-w) \, d\nu$$

and, using the fact that $w = 1$ γ-a.e. in Ω and $\psi(1-w) = 1$ q.e. on $\operatorname{supp} \nu$, we obtain $\int_\Omega A \nabla w \cdot \nabla w \, dx = \nu(\Omega)$. As $\gamma(\Omega) = \operatorname{cap}^L(V, U) = \int_\Omega A \nabla w \cdot \nabla w \, dx$, we conclude that $\nu(\Omega) = \gamma(\Omega) = \operatorname{cap}^L(V, U)$. □

Let us fix $x^0 \in \Omega$. For every $\rho > 0$ let $B_\rho = B_\rho(x^0)$ and let Q_ρ be the open cube $\{x \in \mathbf{R}^n \colon -\rho < x_k - x_k^0 < \rho \text{ for } k = 1, \ldots, n\}$. If $0 < \rho < r$ and $B_r \subset\subset \Omega$, let w_r^ρ be the L-capacitary potential of B_ρ with respect to B_r, and let ν_r^ρ be the corresponding outer L-capacitary distribution.

Lemma 2.2 *For every $q \in (0,1)$ there exists a constant $c = c(q, \alpha, n)$, independent of the operator L, such that, if $B_r \subset\subset \Omega$ and $0 < \rho \leq qr$, then*

$$\frac{1}{\nu_r^\rho(\partial B_r)} \int_{\partial B_r} \varphi \, d\nu_r^\rho \leq c \, \frac{1}{\nu_r^{qr}(\partial B_r)} \int_{\partial B_r} \varphi \, d\nu_r^{qr}$$

for every $\varphi \in H^1(Q_r)$ with $\varphi \geq 0$ q.e. in Q_r.

Proof. Let us fix q, ρ, r, φ as required, and let $u \in H_0^1(\Omega)$ be a function whose restriction to B_r is a solution of the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } B_r, \\ u - \varphi \in H_0^1(B_r). \end{cases}$$

We may assume that $u = \varphi$ q.e. on the annulus $B_R \backslash \overline{B}_r$ for some $R > r$, so that $u = \varphi$ q.e. on $B_R \backslash B_r$. By De Giorgi's theorem, we have $u \in C^0(B_r)$. For every $s \in (0, r)$ we want to prove that

$$\frac{1}{\gamma_r^s(\partial B_s)} \int_{\partial B_s} u \, d\gamma_r^s = \frac{1}{\nu_r^s(\partial B_r)} \int_{\partial B_r} \varphi \, d\nu_r^s, \tag{2.1}$$

where γ_r^s is the inner L-capacitary distribution associated with w_r^s. Using the symmetry of the operator L, we get

$$0 = \int_{B_r} A\nabla u \cdot \nabla w_r^s \, dx = \int_\Omega A\nabla w_r^s \cdot \nabla u \, dx =$$
$$= \int_\Omega u \, d\gamma_r^s - \int_\Omega u \, d\nu_r^s = \int_{\partial B_s} u \, d\gamma_r^s - \int_{\partial B_r} \varphi \, d\nu_r^s.$$

Since $\nu_r^s(\partial B_r) = \mathrm{cap}^L(B_s, B_r) = \gamma_r^s(\partial B_s)$, we obtain (2.1).

Now we remark that, by the maximum principle, $u \geq 0$ on B_r. On the other hand, by Harnack's inequality,

$$\sup_{B_{qr}} u \leq c \inf_{B_{qr}} u,$$

where the constant c depends only on n, q, α, (see [13], Theorem 8.1). If we apply (2.1) with $s = \rho$ and $s = qr$, we obtain

$$\frac{1}{\nu_r^\rho(\partial B_r)} \int_{\partial B_r} \varphi \, d\nu_r^\rho = \frac{1}{\gamma_r^\rho(\partial B_\rho)} \int_{\partial B_\rho} u \, d\gamma_r^\rho \leq \sup_{B_{qr}} u \leq$$
$$\leq c \inf_{B_{qr}} u \leq c \frac{1}{\gamma_r^{qr}(\partial B_{qr})} \int_{\partial B_{qr}} u \, d\gamma_r^{qr} = c \frac{1}{\nu_r^{qr}(\partial B_r)} \int_{\partial B_r} \varphi \, d\nu_r^{qr},$$

and the lemma is proved. □

For every $0 < \rho < r$, with $B_r \subset\subset \Omega$, let $M_r^\rho : H^1(Q_r) \to \mathbf{R}$ be the linear function defined by

$$M_r^\rho u = \frac{1}{\nu_r^\rho(\partial B_r)} \int_{\partial B_r} u \, d\nu_r^\rho, \tag{2.2}$$

where ν_r^ρ is the outer L-capacitary distribution of B_ρ with respect to B_r.

Lemma 2.3 *For every $q \in (0,1)$ there exists a constant $c = c(q, \alpha, n)$ such that, if $Q_r \subset\subset \Omega$ and $0 < \rho \leq qr$, then*

$$\|u - M_r^\rho u\|_{L^2(Q_r)} \leq c\, r \, \|\nabla u\|_{L^2(Q_r)},$$

for every $u \in H^1(Q_r)$.

Proof. Let us fix q, ρ, r as required. It is not restrictive to assume $x^0 = 0$. Let $Q = Q_1$ and $B = B_1$. Let us consider the operator L_r defined by $L_r u = -\mathrm{div}\,(A_r \nabla u)$, where $A_r(y) = A(ry)$. It is easy to check that, if $w_r^\rho(x)$ is the L-capacitary potential of B_ρ with respect to B_r, then $v_r^\rho(y) = w_r^\rho(ry)$ is the L_r-capacitary potential of $B_{\rho/r}$ with respect to B. By Lemma 2.1 we can write $L_r v_r^\rho = \lambda_r^\rho - \mu_r^\rho$, with $\operatorname{supp}\lambda_r^\rho \subseteq \partial B_{\rho/r}$ and $\operatorname{supp}\mu_r^\rho \subseteq \partial B$. We want to prove that for every $u \in H^1(Q_r)$ we have

$$\int_{\partial B_r} u\, d\nu_r^\rho = r^{n-2} \int_{\partial B} u_r\, d\mu_r^\rho, \tag{2.3}$$

where $u_r(y) = u(ry)$. Let us fix $u \in H^1(Q_r)$ and let $\psi \in C_0^\infty(\Omega)$ be a cut-off function such that $\psi = 1$ on ∂B_r and $\psi = 0$ on $\overline{B}_\rho$. If $\psi_r(y) = \psi(ry)$, then

$$\int_{\partial B_r} u\, d\nu_r^\rho = \int_{\partial B_r} u\,\psi\, d\nu_r^\rho = -\int_{B_r} A\,\nabla w_r^\rho \cdot \nabla(u\,\psi)\, dx =$$
$$= -r^{n-2} \int_B A_r\,\nabla v_r^\rho \cdot \nabla(u_r\,\psi_r)\, dy = r^{n-2} \int_{\partial B} u_r\,\psi_r\, d\mu_r^\rho = r^{n-2} \int_{\partial B} u_r\, d\mu_r^\rho,$$

which proves (2.3). Taking $u = 1$ we get $\nu_r^\rho(\partial B_r) = r^{n-2}\,\mu_r^\rho(\partial B)$, so that the previous equality gives

$$\frac{1}{\nu_r^\rho(\partial B_r)} \int_{\partial B_r} u\, d\nu_r^\rho = \frac{1}{\mu_r^\rho(\partial B)} \int_{\partial B} u_r\, d\mu_r^\rho \tag{2.4}$$

for every $u \in H^1(Q_r)$. Finally, we recall that, if P is a projection from $H^1(Q)$ into $\mathbf{R}$, then the following Poincarû inequality holds for every u in $H^1(Q)$:

$$\|u - P(u)\|_{L^2(Q)} \leq \beta\, \|P\|_{(H^1(Q))'}\, \|\nabla u\|_{L^2(Q)},$$

where $(H^1(Q))'$ is the dual space of $H^1(Q)$ and the constant β depends only on the dimension n of the space (see [16], Theorem 4.2.1). Applying this result to

$$P_r^\rho(u) = \frac{1}{\mu_r^\rho(\partial B)} \int_{\partial B} u\, d\mu_r^\rho,$$

and using (2.4), we obtain

$$\begin{aligned} \|u - M_r^\rho u\|^2_{L^2(Q_r)} &= r^n \int_Q \left(u_r - P_r^\rho(u_r)\right)^2\, dy \leq \\ &\leq \beta^2\, r^n \left(\frac{1}{\mu_r^\rho(\partial B)}\, \|\mu_r^\rho\|_{(H^1(Q))'}\right)^2 \int_Q |\nabla u_r|^2\, dy = \\ &= \beta^2\, r^2 \left(\frac{1}{\mu_r^\rho(\partial B)}\, \|\mu_r^\rho\|_{(H^1(Q))'}\right)^2 \int_{Q_r} |\nabla u|^2\, dx\,. \end{aligned} \tag{2.5}$$

It remains to estimate $\frac{1}{\mu_r^\rho(\partial B)}\|\mu_r^\rho\|_{(H^1(Q))'}$. By Lemma 2.2, applied to L_r, we obtain

$$\left|\frac{1}{\mu_r^\rho(\partial B)} \int_{\partial B} \varphi\, d\mu_r^\rho\right| \leq c\, \frac{1}{\mu_r^{qr}(\partial B)} \int_{\partial B} |\varphi|\, d\mu_r^{qr}$$

for every $\varphi \in H^1(Q)$, so that

$$\frac{1}{\mu_r^\rho(\partial B)} \|\mu_r^\rho\|_{(H^1(Q))'} \leq c \frac{1}{\mu_r^{qr}(\partial B)} \|\mu_r^{qr}\|_{(H^1(Q))'}. \tag{2.6}$$

By Proposition 1.5(v) and by Lemma 2.1 we have

$$\mu_r^{qr}(\partial B) = \operatorname{cap}^{L_r}(B_q, B) \geq \alpha \operatorname{cap}(B_q, B). \tag{2.7}$$

Let $\zeta \in C_0^\infty(\mathbf{R}^n)$ be a cut-off function such that $\zeta = 1$ on ∂B, $\zeta = 0$ on $\overline{B}_q$, $0 \leq \zeta \leq 1$ on B, and $|\nabla \zeta| \leq c_q = 2/(1-q)$ on B. Then, using again Proposition 1.5(v), for every $\varphi \in H^1(Q)$ we obtain

$$\begin{aligned} &\int_Q \varphi \, d\mu_r^{qr} = \int_{\partial B} \varphi \zeta \, d\mu_r^{qr} = -\int_B A_r \nabla v_r^{qr} \cdot \nabla(\varphi\zeta) \, dy \leq \\ &\leq c_q \, \alpha^{-1/2} \left(\operatorname{cap}^{L_r}(B_q, B)\right)^{1/2} \|\varphi\|_{H^1(Q)} \leq c_q \, \alpha^{-1} \left(\operatorname{cap}(B_q, B)\right)^{1/2} \|\varphi\|_{H^1(Q)}. \end{aligned} \tag{2.8}$$

From (2.6), (2.7), (2.8) we obtain

$$\frac{1}{\mu_r^\rho(\partial B)} \|\mu_r^\rho\|_{(H^1(Q))'} \leq k(q, \alpha, n),$$

which, together with (2.5), concludes the proof of the lemma. □

For every $r > 0$ let $\hat{Q}_r$ be the cube $\{x \in \mathbf{R}^n \colon -r \leq x_k - x_k^0 < r \text{ for } k = 1, \ldots, n\}$, so that Q_r is the interior of $\hat{Q}_r$.

Lemma 2.4 *Let μ be a measure of $K_n^+(\Omega)$. For every $r > 0$, with $Q_r \subset\subset \Omega$, let $\rho = \rho(r) \in (0, r)$ be the radius such that $\operatorname{cap}^L(B_\rho, B_r) = \mu(\hat{Q}_r)$, and let $M_r = M_r^{\rho(r)}$, where $M_r^{\rho(r)}$ is the average defined in (2.2). Then there exists a function $\omega_\mu \colon \mathbf{R}_+ \to \mathbf{R}_+$, with $\lim_{r \to 0+} \omega_\mu(r) = 0$, such that*

$$\|u - M_r u\|_{L_\mu^2(Q_r)} \leq \omega_\mu(r) \, \|\nabla u\|_{L^2(Q_r)} \tag{2.9}$$

for every $u \in H^1(\Omega)$.

Proof. First of all we prove that for every $q \in (0,1)$ there exists $r_q > 0$ such that $\rho(r) \leq qr$ for $r \leq r_q$. We consider only the case $n \geq 3$; the case $n = 2$ is analogous. Since μ is a Kato measure, for every $r > 0$ we have

$$\mu(\Omega \cap B_r)\, r^{2-n} \leq \int_{\Omega \cap B_r} |y - x^0|^{2-n} \, d\mu(y) \leq \psi(r),$$

where ψ is an increasing function with $\lim_{r\to 0+} \psi(r) = 0$. If $\rho = \rho(r) > qr$, then, recalling that $\mathrm{cap}(B_{qr}, B_r) = c_q\, r^{n-2}$, and using Proposition 1.5(v), we obtain

$$\alpha c_q\, r^{n-2} \le \alpha\, \mathrm{cap}(B_\rho, B_r) \le \mathrm{cap}^L(B_\rho, B_r) = \mu(\hat{Q}_r)\,.$$

So we can write $\alpha c_q\, r^{n-2} \le \mu(\hat{Q}_r) \le \mu(\Omega \cap B_{n\,r}) \le \beta_n\, \psi(n\,r)\, r^{n-2}$. Choosing r_q such that $\psi(n\, r_q) < \alpha c_q/\beta_n$, we obtain a contradiction for $r \le r_q$. Therefore, there exists $r_q > 0$, with $Q_{r_q} \subset\subset \Omega$, such that $\rho(r) \le qr$ for every $r \le r_q$. Since $c_q \to +\infty$ as $q \to 1$, we can choose r_q so that for every $r > 0$, with $Q_r \subset\subset \Omega$, there exists $q \in (0,1)$, with $r \le r_q$.

Let us fix $q \in (0,1)$. It is clearly enough to prove (2.9) for every $r \le r_q$. As $\mu \in K_n^+(\Omega)$, by Lemma 1.2, there exists a constant $c_n > 0$ such that, if $Q_r \subset\subset \Omega$, then

$$\int_{Q_r} u^2\, d\mu \le c_n\, \|\mu\|_{K_n^+(Q_r)} \int_{B_{nr}} |\nabla u|^2\, dx \tag{2.10}$$

for every $u \in H_0^1(B_{nr})$.

Let us fix a bounded extension operator $\Pi\colon H^1(Q_1) \to H_0^1(B_n)$, and for every $r > 0$ let us define the extension operator $\Pi_r\colon H^1(Q_r) \to H_0^1(B_{nr})$ by $(\Pi_r u)(x) = (\Pi u_r)(x/r)$, where $u_r(y) = u(ry)$. It is easily seen that the boundedness of Π implies the existence of a constant $k_n > 0$ such that

$$\int_{B_{nr}} |\nabla(\Pi_r\, v)|^2\, dx \le k_n \left(\int_{Q_r} |\nabla v|^2\, dx + \frac{1}{r^2} \int_{Q_r} v^2\, dx \right) \tag{2.11}$$

for every $v \in H^1(Q_r)$. Note that, if $v \in H^1(\Omega)$ and $Q_r \subset\subset \Omega$, then $v = \Pi_r\, v$ q.e. on $\hat{Q}_r$, since both functions are quasi continuous and coincide on Q_r. Using (2.10) and (2.11), for every $u \in H^1(\Omega)$ we obtain

$$\begin{aligned} \int_{\hat{Q}_r} (u - M_r\, u)^2\, d\mu &\le c_n\, \|\mu\|_{K_n^+(Q_r)} \int_{B_{nr}} \left(\nabla\left(\Pi_r(u - M_r\, u)\right)\right)^2 dx \\ &\le c_n\, k_n\, \|\mu\|_{K_n^+(Q_r)} \left(\int_{Q_r} |\nabla u|^2\, dx + \frac{1}{r^2} \int_{Q_r} (u - M_r\, u)^2\, dx \right). \end{aligned}$$

As $r \le r_q$, we have $\rho = \rho(r) \le qr$, so that Lemma 2.3 implies that

$$\frac{1}{r^2} \int_{Q_r} (u - M_r\, u)^2\, dx \le c^2 \int_{Q_r} |\nabla u|^2\, dx,$$

hence

$$\int_{\hat{Q}_r} (u - M_r\, u)^2\, d\mu \le c_n\, k_n\, (1 + c^2)\, \|\mu\|_{K_n^+(Q_r)} \int_{Q_r} |\nabla u|^2\, dx,$$

for every $r \le r_q$ and for every $u \in H^1(\Omega)$. Since $\|\mu\|_{K_n^+(Q_r)}$ tends to zero as r tends to zero, the statement is proved. □

We are now in a position to prove our result for Kato measures. Let $\{Q_h^i\}_{i\in \mathbf{Z}^n}$ be the partition of $\mathbf{R}^n$ composed of the cubes

$$Q_h^i = \{x \in \mathbf{R}^n \colon\, i_k/h \le x_k < (i_k+1)/h \text{ for } k = 1,\dots,n\}.$$

Theorem 2.5 *Let $\mu \in K_n^+(\Omega)$. Let I_h be the set of all indices i such that $Q_h^i \subset\subset \Omega$. For every $i \in I_h$ let B_h^i be the ball with the same center as Q_h^i and radius $1/2h$, and let E_h^i be another ball with the same center such that*

$$\operatorname{cap}^L(E_h^i, B_h^i) = \mu(Q_h^i)\,.$$

Define $E_h = \bigcup_{i\in I_h} E_h^i$. Then the measures ∞_{E_h} γ^L-converge to μ as $h \to \infty$.

Proof. Let v_h^i be the L-capacitary potential of E_h^i with respect to B_h^i, extended to 0 on Ω, and let $w_h^i = 1 - v_h^i$. By Lemma 2.1, we obtain $Lw_h^i = \nu_h^i - \lambda_h^i$ in Ω, with ν_h^i, $\lambda_h^i \in H^{-1}(\Omega)$, $\nu_h^i \ge 0$, $\lambda_h^i \ge 0$, $\operatorname{supp}\nu_h^i \subseteq \partial B_h^i$, $\operatorname{supp}\lambda_h^i \subseteq \partial E_h^i$, and

$$\nu_h^i(Q_h^i) = \lambda_h^i(Q_h^i) = \operatorname{cap}^L(E_h^i, B_h^i) = \mu(Q_h^i). \tag{2.12}$$

Let us define $w_h \in H^1(\Omega)$ as

$$w_h = \begin{cases} w_h^i & \text{in } B_h^i \setminus E_h^i, \\ 0 & \text{in } E_h^i, \\ 1 & \text{elsewhere} \end{cases} \tag{2.13}$$

and the measures ν_h and λ_h as

$$\nu_h = \sum_{i\in I_h} \nu_h^i, \qquad \lambda_h = \sum_{i\in I_h} \lambda_h^i. \tag{2.14}$$

We want to prove that all hypotheses of Theorem 1.12 hold for w_h and ν_h.

First of all, we prove that w_h converges weakly to 1 in $H^1(\Omega)$. Since, by the maximum principle, $0 \le w_h \le 1$ in Ω, we have that $\{w_h\}$ is bounded in $L^2(\Omega)$. On the other hand,

$$\alpha \int_\Omega |\nabla w_h|^2\, dx \le \sum_{i\in I_h} \operatorname{cap}^L(E_h^i, B_h^i) = \sum_{i\in I_h} \mu(Q_h^i) \le \mu(\Omega)\,.$$

Thus $\{w_h\}$ is bounded in $H^1(\Omega)$ so that there exist a subsequence (still denoted $\{w_h\}$) and a function $w \in H^1(\Omega)$, such that $\{w_h\}$ converges to w weakly in $H^1(\Omega)$, and hence strongly in $L^2(\Omega)$. We are going to show that $w=1$ a.e. in Ω, using the arguments of D. Cioranescu and F. Murat (see [4], Théorème 2.2). Let us consider the family $\{C_h^i\}_{i\in\mathbf{Z}^n}$ of all open balls with radius $(\sqrt{n}-1)/2h$ and centers in the vertices i/h of the cubes Q_h^i. In these balls we have $w_h=1$. Therefore, if we define C_h as the union of the balls C_h^i contained in Ω, we have $w_h\chi_{C_h}=\chi_{C_h}$, where χ_{C_h} is the characteristic function of C_h. Since $\{\chi_{C_h}\}$ converges to a positive constant in the weak* topology of $L^\infty(\Omega)$, passing to the limit in the equality $w_h\chi_{C_h}=\chi_{C_h}$ we obtain $w=1$ a.e. in Ω.

It remains to prove that the measures ν_h defined in (2.14) converge to μ in the strong topology of $H^{-1}(\Omega)$. Indeed, since w_h converges to 1 weakly in $H^1(\Omega)$, this implies also that λ_h converges weakly to μ in $H^{-1}(\Omega)$.

For every $h\in\mathbf{N}$ we introduce the polyrectangle $P_h=\bigcup_{i\in I_h}Q_h^i$ and we define $S_h=\Omega\setminus P_h$. Moreover, for every $\varphi\in H_0^1(\Omega)$, we consider the function

$$\varphi_h=\sum_{i\in I_h}(M_h^i\varphi)\,\chi_{Q_h^i},$$

where, according to (2.2),

$$M_h^i\varphi=\frac{1}{\nu_h^i(\partial B_h^i)}\int_{\partial B_h^i}\varphi\,d\nu_h^i,$$

and we define $\varepsilon_h=\|\mu\llcorner S_h\|_{H^{-1}(\Omega)}$. Note that $\{\varepsilon_h\}$ tends to zero by Lemma 1.3. Recalling that $\mu(Q_h^i)=\nu_h^i(\partial B_h^i)$ and using the Poincarû inequality (2.9), we have that,

$$|\langle\nu_h,\varphi\rangle-\langle\mu,\varphi\rangle|=\left|\sum_{i\in I_h}\frac{\mu(Q_h^i)}{\nu_h^i(\partial B_h^i)}\int_{\partial B_h^i}\varphi\,d\nu_h^i-\sum_{i\in I_h}\int_{Q_h^i}\varphi\,d\mu-\int_{S_h}\varphi\,d\mu\right|\leq$$

$$\leq\int_{P_h}|\varphi-\varphi_h|\,d\mu+\int_{S_h}|\varphi|\,d\mu\leq\left(\mu(\Omega)\int_{P_h}(\varphi-\varphi_h)^2\,d\mu\right)^{1/2}+\|\mu\llcorner S_h\|_{H^{-1}(\Omega)}\|\varphi\|_{H_0^1(\Omega)}=$$

$$=\left(\mu(\Omega)\sum_{i\in I_h}\|\varphi-M_h^i\varphi\|^2_{L^2_\mu(Q_h^i)}\right)^{1/2}+\varepsilon_h\|\varphi\|_{H_0^1(\Omega)}\leq$$

$$\leq\left(\mu(\Omega)\sum_{i\in I_h}\omega(1/h)^2\|\nabla\varphi\|^2_{L^2(Q_h^i)}\right)^{1/2}+\varepsilon_h\|\varphi\|_{H_0^1(\Omega)}\leq\left(\omega(1/h)\,\mu(\Omega)^{1/2}+\varepsilon_h\right)\|\varphi\|_{H_0^1(\Omega)}.$$

Thus we obtain

$$\|\nu_h-\mu\|_{H^{-1}(\Omega)}\leq\mu(\Omega)^{1/2}\omega(1/h)+\varepsilon_h,$$

hence $\{\nu_h\}$ converges to μ strongly in $H^{-1}(\Omega)$. Therefore $\{\infty_{E_h}\}$ γ^L-converges to μ by Theorem 1.12. □

In order to generalize this result to every Radon measure we need the following results.

Proposition 2.6 *For every Radon measure $\mu \in \mathcal{M}_0(\Omega)$ there exist a measure $\nu \in K_n^+(\Omega)$ and a positive Borel function $g: \Omega \to [0, +\infty]$ such that $\mu = g\,\nu$.*

Proof. See [2], Proposition 2.5. □

Proposition 2.7 *Let $\lambda \in \mathcal{M}_0(\Omega)$, let μ be a Radon measure in $\mathcal{M}_0(\Omega)$; for every $x \in \Omega$ let*

$$f(x) = \liminf_{r \to 0} \frac{\mathrm{cap}_\lambda^L(B_r(x), B_{2r}(x))}{\mu(B_r(x))}.$$

Assume that f is bounded. Then λ is a Radon measure and we have $\lambda = f\,\mu$.

Proof. See [3], Theorem 2.3. □

Proposition 2.8 *Let μ be a positive Radon measure on Ω. Then there exists a unique pair (μ_0, μ_1) of Radon measures on Ω such that:*
(i) $\mu = \mu_0 + \mu_1$;
(ii) $\mu_0 \in \mathcal{M}_0(\Omega)$;
(iii) $\mu_1 = \mu \llcorner N$, *for some Borel set N with* $\mathrm{cap}(N, \Omega) = 0$.

Proof. See [8], Lemma 2.1. □

We are now in a position to prove our main result in its most general form.

Theorem 2.9 *Let μ be a positive Radon measure on Ω. Let $\{Q_h^i\}$ and $\{E_h\}$ be defined as in Theorem 2.5. Then the following results hold:*
(i) if μ belongs to $\mathcal{M}_0(\Omega)$, then $\{\infty_{E_h}\}$ γ^L-converges to μ;
(ii) if $\mu = \mu_0 + \mu_1$, with μ_0 and μ_1 as in Proposition 2.8, then $\{\infty_{E_h}\}$ γ^L-converges to μ_0.

Proof. If μ is a Radon measure in $\mathcal{M}_0(\Omega)$, then, by Proposition 2.6, $\mu = g\,\nu$, where $\nu \in K_n^+(\Omega)$ and g is a positive Borel function. By Theorem 1.8, there exists a subsequence, still denoted by $\{E_h\}$, and a measure $\lambda \in \mathcal{M}_0(\Omega)$, such that $\{\infty_{E_h}\}$ γ^L-converges to λ. Let $x \in \Omega$ and let $r > 0$ such that $B_{2r}(x) \subseteq \Omega$. We want to prove that for every Borel set $E \subseteq B_{2r}$

$$\mathrm{cap}_\lambda^L(E, B_{2r}(x)) \leq \mu(E). \tag{2.15}$$

If A and A' are two open sets such that $A' \subset\subset A \subseteq B_{2r}(x)$ and h is small enough we have

$$\bigcup_{E_h^i \cap A' \neq \emptyset} Q_h^i \subseteq A,$$

hence, by Theorem 1.5,

$$\mathrm{cap}^L(E_h \cap A', B_{2r}(x)) \le \sum_{E_h^i \cap A' \ne \emptyset} \mathrm{cap}^L(E_h^i, B_{2r}(x)) \le$$
$$\le \sum_{E_h^i \cap A' \ne \emptyset} \mathrm{cap}^L(E_h^i, B_h^i) = \sum_{E_h^i \cap A' \ne \emptyset} \mu(Q_h^i) \le \mu(A).$$

Using Theorem 1.11 we obtain,

$$\mathrm{cap}_\lambda^L(A', B_{2r}(x)) \le \liminf_{h\to\infty} \mathrm{cap}^L(E_h \cap A', B_{2r}(x)) \le \mu(A)$$

and, as $A' \nearrow A$, we obtain $\mathrm{cap}_\lambda^L(A, B_{2r}(x)) \le \mu(A)$ for every open set $A \subseteq B_{2r}(x)$ (see Theorem 1.5(vi)). Since μ is a Radon measure, this inequality can be easily extended to all Borel subsets of $B_{2r}(x)$. So (2.15) is proved. Choosing $E = B_r(x)$ in (2.15) and applying Proposition 2.7, we obtain that λ is a Radon measure and that $\lambda \le \mu$.

Define, for $k \in \mathbf{N}$, the measures $\mu^k = g^k \nu$, where $g^k(x) = \min(g(x), k)$. As $\mu^k \in K_n^+(\Omega)$, by Theorem 2.5 for every k there exists a sequence $\{E_{k,h}\}_h$ such that $\{\infty_{E_{k,h}}\}_h$ γ^L-converges to μ^k. Since $\mu^k \le \mu$, the construction of Theorem 2.5 implies that $E_{k,h} \subseteq E_h$ for every h and k. By Remark 1.9 this implies $\lambda \ge \mu^k$ for every k, hence $\lambda \ge \mu$. As the opposite inequality has already been proved, we obtain $\lambda = \mu$. Since the γ^L-limit does not depend on the subsequence, the whole sequence $\{\infty_{E_h}\}$ γ^L-converges to μ.

Let now μ be any Radon measure on Ω. By Proposition 2.8, we can write $\mu = \mu_0 + \mu_1$, with $\mu_0 \in \mathcal{M}_0(\Omega)$ and $\mu_1 = \mu \llcorner N$, where N is a Borel set with $\mathrm{cap}(N, \Omega) = 0$. Arguing as before, let λ be the γ^L-limit of a subsequence of $\{\infty_{E_h}\}$. If $x \in \Omega$ and $r > 0$ is such that $B_{2r}(x) \subseteq \Omega$, we have

$$\mathrm{cap}_\lambda^L(B_r(x), B_{2r}(x)) = \mathrm{cap}_\lambda^L(B_r(x) \setminus N, B_{2r}(x)),$$

since $\mathrm{cap}(N, B_{2r}(x)) = 0$ (see Proposition 1.5). Therefore (2.15), applied with $E = B_r(x) \setminus N$, gives

$$\mathrm{cap}_\lambda^L(B_r(x), B_{2r}(x)) \le \mu(B_r(x) \setminus N) = \mu_0(B_r(x)).$$

By applying again Proposition 2.7 we obtain $\lambda \le \mu_0$.

Since μ_0 is a Radon measure of $\mathcal{M}_0(\Omega)$, by the first part of this theorem we can construct the holes $E_{0,h}$ such that $\{\infty_{E_{0,h}}\}$ γ^L-converges to μ_0. Since $\mu(Q_h^i) \ge \mu_0(Q_h^i)$, we have $E_{0,h} \subseteq E_h$, hence, by Remark 1.9, $\lambda \ge \mu_0$. As the opposite inequality has already been proved, we obtain $\lambda = \mu_0$. As before, this implies that the whole sequence $\{\infty_{E_h}\}$ γ^L-converges to μ_0. □

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