Integrability of continuous bundles and applications to dynamical systems

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Contents

Chapter 1

Introduction

In this dissertation we study the problem of integrability of bundles with low regularities. It is organized into the next three chapters:

- Chapter 2 is a co-authored paper with Stefano Luzzatto and Sina Türeli which is published in Ergodic Theory and Dynamical Systems. In this Chapter we investigate the integrability of 2-dimensional invariant distributions (tangent sub-bundles) which arise naturally in the context of dynamical systems on 3-manifolds. In particular we prove unique integrability of dynamically dominated and volume dominated Lipschitz continuous invariant decompositions as well as distributions with some other regularity conditions.

- Chapter 3 is also a co-authored paper with Stefano Luzzatto and Sina Türeli and is accepted for publication in J. Reine. [Crelle's Journal]. We give new sufficient conditions for the integrability and unique integrability of *contin-*

uous tangent sub-bundles on manifolds of arbitrary dimension, generalizing Frobenius' classical Theorem for C^1 sub-bundles. Using these conditions we derive new criteria for uniqueness of solutions to ODE's and PDE's and for the integrability of invariant bundles in dynamical systems. In particular we give a novel proof of the Stable Manifold Theorem and prove some integrability results for dynamically defined dominated splittings.

Chapter 2

Integrability of dominated splittings

2.1 Introduction and Statement of Results

Let M be a smooth manifold and $E \subset TM$ a distribution of tangent hyperplanes. A basic question concerns the (unique) integrability of the distribution E, i.e. the existence at every point of a (unique) local embedded submanifold everywhere tangent to E. For one-dimensional distributions it follows from classical results on the existence and uniqueness of solutions of ODE's that regularity conditions suffice: existence is always guaranteed for continuous distributions and uniqueness for Lipschitz continuous distributions. For higher dimensional distributions the situation is more complicated and regularity conditions alone cannot guarantee integrability, indeed there exists arbitrarily smooth distributions which are not integrable [36]. It turns out however, that if the distributions are realized as $D\varphi$-invariant distributions for some diffeomorphisms φ then some conditions can be formulated which imply integrability. More precisely, let M be a Riemannian 3-manifold, $\varphi : M \to M$ a C^2 diffeomorphism and $E \oplus F$ a continuous $D\varphi$-invariant tangent bundle decomposition. For definiteness we shall always assume, without loss of generality, that $dim(E) = 2$ and $dim(F) = 1$. We state our results in the following subsections.

2.1.1 Dynamical domination and robust transitivity

A $D\varphi$-invariant decomposition $E \oplus F$ is *dynamically dominated* if there exists a Riemannian metric such that

$$\frac{\|D\varphi_x|_{E_x}\|}{\|D\varphi_x|_{F_x}\|} < 1 \tag{2.1.1}$$

for all $x \in M$. Then we have the following result.

Theorem 1. *Let M be a Riemannian 3-manifold, $\varphi : M \to M$ a volume-preserving or transitive C^2 diffeomorphism and $E \oplus F$ a $D\varphi$-invariant, Lipschitz, dynamically dominated, decomposition. Then E is uniquely integrable.*

Remark 2.1.1. Our dynamical domination condition is usually referred to in the literature simply as *domination*, we use this non-standard terminology to avoid confusion in view of the fact that we will introduce below another form of domination. We remark also that the dynamical domination condition is usually formulated with the co-norm $m(D\varphi_x|_{F_x}) := \min_{v\in F, v\neq 0} \|D\varphi_x(v)\|/\|v\|$ instead of $\|D\varphi_x|_{F_x}\|$ but of course the two definitions are equivalent when F is one-dimensional, as here. We will also occasionally call E, the dominated bundle.

Remark 2.1.2. Under the stronger assumption that φ is *robustly transitive* (i.e. φ is transitive and any C^1 sufficiently close diffeomorphism is also transitive) instead of just transitive, the dynamical dominated condition is automatically satisfied [20] and so integrability follows under the additional assumption that the decomposition is Lipschitz.

Remark 2.1.3. We could replace the transitivity assumption in Theorem 1 by chain recurrence or even just the absence of sources, see Section 2.2.2.

2.1.2 Volume domination

We will obtain Theorem 1 as a special case of the following more general result which replaces the volume preservation/transitivity and dynamical domination conditions with a single "volume domination" condition. A $D\varphi$-invariant decomposition $E \oplus F$ is *volume dominated* if there exists a Riemannian metric such that

$$\frac{|det(D\varphi_x|_{E_x})|}{|det(D\varphi_x|_{F_x})|} < 1 \tag{2.1.2}$$

for all $x \in M$. Then we have the following result.

Theorem 2. *Let M be a Riemannian 3-manifold, $\varphi : M \to M$ a C^2 diffeomorphism and $E \oplus F$ a $D\varphi$-invariant, Lipschitz continuous, volume dominated decomposition. Then E is uniquely integrable.*

Theorems 1 and 2 extend analogous statements in [37] obtained using different arguments, in arbitrary dimension but under the assumption that the decomposition is C^1. They also extend previous results of Burns and Wilkinson [16], Hammerlindl and Hertz-Hertz-Ures [27, 55] and Parwani [44] who prove analogous results[1] for respectively C^2, C^1 and Lipschitz distributions under the assumption of *center-bunching* or *2-partial hyperbolicity*:

$$\frac{\|D\varphi_x|_{E_x}\|^2}{\|D\varphi_x|_{F_x}\|} < 1 \tag{2.1.3}$$

for every $x \in M$. In the 3-dimensional setting condition (2.1.3) clearly implies volume domination and is therefore more restrictive. In Section 2.1.3 we sketch an example of a diffeomorphism and an invariant distribution E which does not satisfy condition (2.1.3) but does satisfy the dynamical domination and volume domination assumptions we require in Theorem 2. This particular example is uniquely integrable by construction and so is not a "new" example, but helps to justify the observation that our conditions are indeed less restrictive than center-bunching

[1] In some of the references mentioned, the relevant results are not always stated in the same form as given here but may be derived from related statements and the technical arguments. In some cases the setting considered is that of partially hyperbolic diffeomorphisms with a tangent bundle decomposition of the form $E^s \oplus E^c \oplus E^u$ where E^s is uniformly contracting and E^u uniformly expanding. In this setting, one considers the integrability of the sub-bundles $E^{sc} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$ and it is not always completely clear to what extent the existence of a uniformly expanding sub-bundle is relevant to the arguments. We emphasize that the setting we consider here does not require the invariant distribution E to contain any further invariant sub-bundle.

(2.1.3).

The techniques we employ here are similar to those of Parwani but to relax the center-bunching condition one needs a more careful analysis of the behaviour of certain Lie brackets, this is carried out in Section 2.3.

A more sophisticated version of our arguments also yields an alternative sufficient condition for integrability which is related to bundles which are Lipschitz along a transversal direction. Since the conditions are somewhat technical and it is not completely clear if they are satisfied by any natural examples we have "relegated" the precise formulation and proof to the Appendix. We do think nevertheless that both the statement and the techniques used are of some independent interest and discuss this further below.

Remark 2.1.4. The assumption that the diffeomorphisms in the Theorems above are C^2 is necessary for the arguments we use in the proofs. In the proof of Theorem 2, we need to be able to compute the Lie brackets of iterates of certain sections from E by $D\varphi$. For this reason $D\varphi$ needs to be C^1 to keep the regularity of a section along the orbit of a initial point p.

Remark 2.1.5. A stronger version of volume domination exists in some previous related literature. A decomposition $E \oplus F$ with $dim(E) = 2$, $dim(F) = 1$ is called *volume hyperbolic* if there exists some constants $C > 0$ and $r < 1$ such that for all $k \in \mathbb{Z}^+$ and $x \in M$

$$|det(D\varphi_x^k|_{E_z})| \leq Cr^k \qquad ||D\varphi_x^{-k}|_{F_x}|| < Cr^k.$$

Volume hyperbolicity is clearly stronger than volume domination. This property may also be obtained from other topological properties of the map φ in some generic setting, for example C^1 generically non-wandering maps are volume hyperbolic, see for example [21, 7, 56] for this and other related results. It is not clear to what extent the results in the cited papers may improve this chapter's results since we require our maps to be C^2, however we mention them as it seems interesting that weaker forms of partial hyperbolicity seem to be relevant in different settings.

2.1.3 Volume Domination versus 2-Partially Hyperbolic

In this section we are going to sketch the construction of some non-trivial examples which satisfy the volume domination condition (2.1.2) but not the center-bunching condition (2.1.3). This is a variation of the "derived from Anosov" construction due to Mañé [41] (see [9] for the volume preserving case, which is what we use here). We are very grateful to Raúl Ures for suggesting and explaining this construction to us. Consider the matrix

$$\begin{pmatrix} -3 & 0 & 2 \\ 1 & 2 & -3 \\ 0 & -1 & 1 \end{pmatrix}$$

This matrix has determinant 1 and has integer coefficients therefore induces a volume preserving toral automorphism on $\mathbb{T}^3$. It is Anosov since its eigenvalues are $r_1 \sim -0.11, r_2 \sim 3.11, r_3 \sim -3.21$. Note that $r_1 r_2 / r_3 < 1$ but $r_2^2 / r_3 > 1$. Hence (2.1.2) is satisfied but (2.1.3) is not. Now take a fixed point p and a periodic point q

and a neighbourhood U of p so that forward iterates of q never intersect U. One can apply Mané's construction to perturb the map on U as to obtain a new partially hyperbolic automorphism of $\mathbb{T}^3$ which is still volume preserving. Such a perturbation is not a small one and therefore one can not claim integrability of the new system trivially by using standard theorems as in [33]. Since the perturbation is performed on U and orbit of q never intersects U, the perturbation does not change the splitting and the contraction and expansion rates around q and in particular (2.1.3) is still not satisfied on the orbit of q. Yet the new example is volume preserving therefore it is necessarily the case that (2.1.2) is satisfied.

2.2 Strategy and overview of the proof

We will first show that Theorem 1 is a special case of Theorem 2. We will consider the volume preserving setting and the transitive setting separately. We then discuss the proof of Theorem 2.

2.2.1 Volume preserving implies volume domination

We show that when φ is volume preserving, dynamical domination implies volume domination. Indeed, notice that $|det(D\varphi_x|_{F_x})| = \|D\varphi_x|_{F_x}\|$ since F is one-dimensional, so the difference between dynamical domination and volume domination consists of the difference between $\|D\varphi_x|_{E_x}\|$ and $|det(D\varphi_x|_{E_x})|$. These two quantities are in general essentially independent of each other; indeed considering the singular value decomposition of $D\varphi_x|_{E_x}$ and letting $s_1 \le s_2$ denote the two singular values (since we assume E is 2-dimensional), we have that $\|D\varphi_x|_{E_x}\| = s_2$ and $|det(D\varphi_x|_{E_x})| = s_1 s_2$. If $\|D\varphi_x|_{E_x}\| = s_2 < 1$ then we have a a straightforward inequality $|det(D\varphi_x|_{E_x})| = s_1 s_2 < s_2 = \|D\varphi_x|_{E_x}\|$ but this is of course not necessarily the case in general. However there is a relation in the volume preserving setting as this implies $|detD\varphi_x|_E| \cdot |detD\varphi_x|_F| = 1$ and so (2.1.1) implies $|detD\varphi_x|_F| > 1$ (arguing by contradiction, $|detD\varphi_x|_F| = \|D\varphi_x|_F\| \le 1$ would imply $|detD\varphi_x|_E| \ge 1$ by the volume preservation, and this would imply $\|D\varphi_x|_E\|/\|D\varphi_x|_F\| \ge 1$ which would contradict (2.1.1)). Dividing the equation $|detD\varphi_x|_E| \cdot |detD\varphi_x|_F| = 1$ through by $(|detD\varphi_x|_F)^2|$ we get (2.1.2).

2.2.2 Transitivity implies volume domination

We show that when φ is transitive (or, as mentioned in Remark 2.1.3 when φ is chain-recurrent or just φ has no sources), dynamical domination implies volume domination. We are grateful to the referee for pointing out this fact and explaining the proof. The argument is based on the following Lemma whose proof we sketch below and which follows closely arguments in [6, 18].

Lemma 2.2.1. *Let Λ be a compact invariant set with a continuous splitting $E \oplus F$ with $\dim E = 2$ and $\dim F = 1$. Then the splitting is volume dominated if and only if the Lyapunov exponents $\lambda_1 \le \lambda_2 \le \lambda_3$ of any invariant ergodic measure μ satisfy $\lambda_1 + \lambda_2 \le \lambda_3 - a$ for some uniform value of a (depending on the constants* *of domination).*

Sketch of proof. One direction is trivial: if the splitting is volume dominated clearly the condition on the Lyapunov exponents satisfies the stated bounds. The other direction is non-trivial and we argue by contradiction. Suppose that Λ is not volume dominated, this means that there exists a sequence of points x_n and a sequence $k_n \to \infty$ such that for every $0 \leq j \leq k_n$ one has

$$|det(D\varphi^j|_E(x_n))| \geq \frac{1}{2}\|D\varphi^j|_F(x_n)\|. \tag{2.2.1}$$

Now consider the sequence of probability measures

$$\mu_n = \frac{1}{k_n}\sum_{i=0}^{k_n-1}\delta_{\varphi^i(x_n)}.$$

Up to passing to a subsequence if necessary, we can assume that μ_n is convergent to an invariant probability measure μ. Since the splitting is continuous we have

$$\int \log(det(D\varphi|_E)d\mu_n \to \int \log(det(D\varphi|_E))d\mu$$

and,

$$\int log(\|D\varphi|_F\|)d\mu_n \to \int log(\|D\varphi|_F\|)d\mu.$$

Since the determinant is multiplicative and F is one dimensional, the integrals with respect to μ_n are Birkhoff sums and therefore converge exactly to the sum of Lyapunov exponents (in the E case) and the Lyapunov exponent (in the F case). Using the ergodic decomposition and (2.2.1) it follows that there is an ergodic invariant measure with $\lambda_1 + \lambda_2 \geq \lambda_3$. This contradicts the assumption and does proves that Λ is volume dominated. $\square$

To complete the proof of volume domination, notice that dynamical domination implies $\lambda_2 < \lambda_3 - a$ for all invariant ergodic probability measures for some a independent of μ. Assume by contradiction that φ is transitive and dynamically dominated but not volume dominated. Then, by Lemma 2.2.1, it admits a measure μ such that $\lambda_1 + \lambda_2 \geq \lambda_3 - a$. But dynamical domination implies $\lambda_2 < \lambda_3 - a$ and so we get $\lambda_1 > 0$. Thus all Lyapunov exponents of μ are strictly positive and therefore μ is supported on a source, contradicting transitivity (or chain-recurrence, or that f does not have sources).

2.2.3 Volume domination implies integrability

From now on we concentrate on Theorem 2 and reduce it to a key technical Proposition. Therefore the assumptions are that $E \oplus F$ is Lipschitz and volume dominated. We don't require or use dynamical domination for any of the propositions that we prove here. We fix an arbitrary point $x_0 \in M$ and a local chart $(\mathcal{U}, x^1, x^2, x^3)$ centered at x_0. We can assume (up to change of coordinates) that $\partial/\partial x^i, i = 1, 2, 3$ are transverse to E and thus we can define linearly independent vector fields X and Y, which span E and are of the form

$$X = \frac{\partial}{\partial x^1} + a\frac{\partial}{\partial x^3} \qquad Y = \frac{\partial}{\partial x^2} + b\frac{\partial}{\partial x^3}.$$

where a and b are Lipschitz functions. Notice that it follows from the form of the vector fields X, Y that at every point of differentiability the Lie bracket is well defined and lies in the x^3 direction, i.e.

$$[X, Y] = c\frac{\partial}{\partial x^3}$$

for some L^∞ function c. In Section 2.3 we will prove the following

Proposition 2.2.2. *There exists $C > 0$ such that for every $k > 1$ and $x \in \mathcal{U}$, if the distribution E is differentiable at x then we have*

$$\|[X, Y]_x\| \leq C\frac{|det(D\varphi^k_x|_{E_x})|}{|det(D\varphi^k_x|_{F_x})|}.$$

Substituting the volume domination condition (2.1.2) into the estimate in Proposition 2.2.2, we get that the right hand side converges to 0 as $k \to \infty$, and therefore $\|[X, Y]_x\| = 0$ and so the distribution E is involutive at every point x at which it is differentiable. Theorem 2 is then an immediate consequence of the following general result of Simić [58] which holds in arbitrary dimension and a generalization of a well-known classical result of Frobenius proving unique integrability for involutive C^1 distributions.

Theorem ([58])**.** *Let E be an m dimensional Lipschitz distribution on a smooth manifold M. If for every point $x_0 \in M$, there exists a local neighbourhood $\mathcal{U}$ and a local Lipschitz frame $\{X_i\}_{i=1}^m$ of E in $\mathcal{U}$ such that for almost every point $x \in \mathcal{U}$, $[X_i, X_j]_x \in E_x$, then E is uniquely integrable.*

Remark 2.2.3. We mention that there are some versions of Proposition 2.2.2 in the literature for C^1 distributions and giving an estimate of the the $\|[X, Y]_x\| \leq$

$|D\varphi_x^k|_{E_x}|^2/m(D\varphi_x|_{F_x})$, , see e.g. [55, 44],. For this quantity to go to zero, one needs the center bunching assumption (2.1.3). In our proposition, through more careful analysis, we relax the condition of center bunching (2.1.3) to volume domination (2.1.2).

2.3 Lie bracket bounds

This section is devoted to the proof of Proposition 2.2.2, which is now the only missing component in the proof of Theorems 1 and 2. As a first step in the proof, we reduce the problem to that of estimating the norm of a certain projection of the bracket of an orthonormal frame. More specifically, let π denote the orthogonal projection (with respect to the Lyapunov metric which orthogonalizes the bundles E and F) onto F.

Lemma 2.3.1. *There exists a constant $C_1 > 0$ such that if $\{Z, W\}$ is an orthonormal Lipschitz frame for E and differentiable at $x \in \mathcal{U}$ then we have*

$$\|[X, Y]_x\| \leq C_1 \|\pi[Z, W]_x\|.$$

Proof. Notice that since F and $\frac{\partial}{\partial x_3}$ are transverse to E, then one has that $K_1 \leq ||\pi \frac{\partial}{\partial x^3}|| \leq K_2$ for some constants $K_1, K_2 > 0$. Moreover since $\|\pi[X, Y]\| = |c|.\|\pi\partial/\partial x^3\|$ and $\|[X, Y]\| = |c|$ then it is sufficient to get an upper bound for $\|\pi[X, Y]\|$. Writing X, Y in the local orthonormal frame $\{Z, W\}$ we have

$$X = \alpha_1 Z + \alpha_2 W \quad \text{and} \quad Y = \beta_1 Z + \beta_2 W.$$

By bilinearity of the Lie bracket and the fact that $\pi(Z) = \pi(W) = 0$ since π is a projection along E, straightforward calculation gives

$$\|\pi[X,Y]\| = |\alpha_1\beta_2 - \alpha_2\beta_1|.\|\pi[Z,W]\|$$

By orthonormality of $\{Z,W\}$, we have $|\alpha_i| \leq \|X\|, |\beta_i| \leq \|Y\|$ and since these are uniformly bounded, the same is true for $|\alpha_1\beta_2 - \alpha_2\beta_1|$ and so we get the result. □

By Lemma 2.3.1 it is sufficient to obtain an upper bound for the quantity $\|\pi[Z,W]\|$ for some Lipschitz orthonormal frame. In particular we can (and do) choose Lipschitz orthonormal frames $\{Z,W\}$ of E such that for every $x \in \mathcal{U}$ and every $k \geq 1$ we have

$$\|D\varphi_x^k Z\|\|D\varphi_x^k W\| = |det(D\varphi^k|_E)|.$$

For these frames will we prove the following.

Lemma 2.3.2. *There exists $C_2 > 0$ such that for every $k \geq 1$ and $x \in \mathcal{U}$, if the distribution E is differentiable at x we have*

$$\|\pi[Z,W]_x\| \leq C_2 \frac{|det(D\varphi^k|_{E_x})|}{||D\varphi^k|_{F_x}||}.$$

Combining Lemma 2.3.2 and Lemma 2.3.1 and letting $C = C_1C_2$ we get:

$$\|[X,Y]_x\| \leq C_1\|\pi[Z,W]_x\| \leq C_1C_2\frac{|det(D\varphi_x^k|_{E_x})|}{\|D\varphi_x^k|_{F_x}\|} = C\frac{|det(D\varphi_x^k|_{E_x})|}{|det(D\varphi_x^k|_{F_x})|}$$

which is the desired bound in Proposition 2.2.2 and therefore completes its proof.

To prove Lemma 2.3.2, observe first that for every $y \in M$ there exist 2 orthonormal Lipschitz vector fields A_y, B_y that span E in a neighborhood of y and by compactness we can suppose that we have finitely many pairs, say $(A_1, B_1), ..., (A_\ell, B_\ell)$ of such vector fields which together cover the whole manifold. We denote by $\mathcal{U}_i$ the domain where the vector fields A_i, B_i are defined and let

$$C_2 := \sup\{|\pi[A_i, B_i](x)| : 1 \leq i \leq l \text{ and almost every } x \in \mathcal{U}_i\}.$$

Note this constant C_2 is finite. In fact, by the standard fact that Lipschitz functions have weak differential which is essentially bounded (or L^∞), then for every $i \in \{1, ..., l\}$ the function $|[A_i, B_i]|$ is bounded. To complete the proof we will use the following observation.

Lemma 2.3.3. *For any Lipschitz orthonormal local frame $\{Z, W\}$ for E which is differentiable at $x \in M$, we have*

$$|\pi[Z, W]| \leq C_2$$

Proof. Write $Z = \alpha_1 A_i + \alpha_2 B_i$ and $W = \beta_1 A_i + \beta_2 B_i$ for some $1 \leq i \leq \ell$. Using the bilinearity of the Lie bracket and the fact that $\pi(A_i) = \pi(B_i) = 0$ we get $|\pi[Z, W]| = |\alpha_1\beta_2 - \alpha_2\beta_1||\pi[A_i, B_i]|$. Since $\{A_i, B_i\}$ and $\{Z, W\}$ are both orthonormal frames, we have $|\alpha_1\beta_2 - \alpha_2\beta_1| = 1$, and so we get result. □

Proof of lemma 2.3.2. For $k > k_0$ and $x \in \mathcal{U}$ such that E is differentiable at x, Let

$$\tilde{Z}(\varphi^k x) = \frac{D\varphi^k_x Z}{\|D\varphi^k_x Z\|} \quad \text{and} \quad \tilde{W}(\varphi^k x) = \frac{D\varphi^k_x W}{\|D\varphi^k_x W\|}$$

Recall that $D\varphi_x^k(E_x) = E_{\varphi^k(x)}$. Therefore, since Z, W span E in a neighborhood of x, then $\tilde{Z}, \tilde{W}$ span E in a neighbourhood of $\varphi^k(x)$ and in particular $\pi(\tilde{Z}) = \pi(\tilde{W}) = 0$. Therefore we get

$$\|\pi[D\varphi^k Z, D\varphi^k W]\| = |det(D\varphi^k|_{E^{(k)}})| \|\pi[\tilde{Z}, \tilde{W}]\| \tag{2.3.1}$$

Note that $\|\pi[D\varphi^k Z, D\varphi^k W]\| = \|\pi D\varphi^k [Z, W]\|$. Then by the invariance of the bundles we have

$$\|\pi D\varphi^k [Z, W]\| = \|D\varphi^k \pi[Z, W]\|. \tag{2.3.2}$$

Since F is one dimensional,

$$\|D\varphi^k \pi[Z, W]\| = \|D\varphi^k|_F\| \|\pi[Z, W]\| \tag{2.3.3}$$

Combining (3.2.16) and (2.3.3) we get

$$\|D\varphi^k|_F\| \|\pi[Z, W]\| = \|\pi D\varphi^k [Z, W]\|.$$

Putting this into equation (3.2.11) and using the fact that $\|\pi[\tilde{Z}, \tilde{W}]\|$ is uniformly bounded by lemma 2.3.3 one gets

$$\|\pi[Z, W]\| \leq C_2 \frac{|det(D\varphi^k|_E)|}{\|D\varphi^k|_F\|}$$

This concludes the proof of Lemma 2.3.2. □

2.4 Sequential transversal regularity

A counter example in [54] shows that the Lipschitz regularity condition in our Theorems cannot be fully relaxed, without additional assumptions, in order to guarantee unique integrability. Nevertheless the kind of techniques we use lead naturally to the formulation of a somewhat unorthodox regularity condition, which we call "sequentially transversal Lipschitz regularity". The main reason that we choose to present this result is that the techniques used in the proof, especially those in Section 2.4.4, generalize naturally to yield continuous Frobenius-type theorems, such as those given in forthcoming papers [39, 63]; we also believe that there are some interesting questions to be pursued regarding the relation between Lipschitz regularity and transversal Lipschitz regularity, we discuss these in Section 2.4.2 below. We will give a "detailed sketch" of the arguments concentrating mostly on techniques which are novel, the full arguments can be found in a previous version of this paper [38].

2.4.1 Definition and statement of result

As above, let M be a 3-manifold, $\varphi : M \to M$ be a C^2 diffeomorphism, and $E \oplus F$ a continuous $D\varphi$-invariant tangent bundle decomposition with $dim(E) = 2$. We say that E is *sequentially transversally Lipschitz* if there exists a C^1 line bundle Z, everywhere transverse to E, and a C^1 distribution $E^{(0)}$ such that the sequence of C^1 distributions $\{E^{(k)}\}_{k>1}$ given by

$$E_x^{(k)} = D\varphi^{-k}_{\varphi^k x} E^{(0)}_{\varphi^k x}, \forall x \in M, k > 1 \tag{2.4.1}$$

are equi-Lipschitz along Z, i.e. there exists $K > 0$ such that for every $x, y \in M$ close enough and belonging to the same integral curve of Z, and every $k \geq 0$, we have $\measuredangle(E_x^{(k)}, E_y^{(k)}) \leq Kd(x,y)$.

Theorem 3. *Let M be a Riemannian 3-manifold, $\varphi : M \to M$ a C^2 diffeomorphism and $E \oplus F$ a $D\varphi$-invariant, sequentially transversally Lipschitz, dynamically and volume dominated, decomposition. Then E is uniquely integrable.*

2.4.2 Relation Between Lipschitzness and Transversal Lipschitzness

Before starting the sketch of the proof of Theorem 3 we discuss some general questions concerning the relationships between various forms of Lipschitz regularity. We say that a sub-bundle E is *transversally Lipschitz* if there exists a C^1 line bundle Z, everywhere transverse to E, along which E is Lipschitz. The relations between Lipschitz, transversally Lipschitz, and sequentially transversally Lipschitz are not clear in general. For example it is easy to see that sequentially transversally Lipschitz implies transversally Lipschitz but we have not been able to show that transversally Lipschitz, or even Lipschitz, implies sequentially transversally Lipschitz. Nevertheless certain equivalence may exist under certain forms of dominations for bundles which occur as invariant bundles for diffeomorphisms. We formulate the following question:

Question 1. *Suppose $E \oplus F$ is a $D\varphi$-invariant decomposition satisfying* (2.1.3). *Then is E transversally Lipschitz if and only if it is Lipschitz ?*

One reason why we believe this question is interesting is that that transversal Lipschitz regularity is a-priori strictly weaker than full Lipschitz regularity. Thus a positive answer to this question would imply that transversal Lipschitzness of center-bunched dominated systems becomes in particular, by Theorem 2, a criterion for their unique integrability. More generally, a positive answer to this question would somehow be saying that one only needs some domination condition and transversal regularity to prevent E from demonstrating pathological behaviours such as non-integrability or non-Lipschitzness.

The notion of sequential transversal regularity and the result of Theorem 3 may play a role in a potential solution to the question above. Indeed, if E is sequentially transversally Lipschitz and volume dominated, then by Theorem 3 it is uniquely integrable. Then, under the additional assumption of centre-bunching, by arguments derived from theory of normal hyperbolicity (see [33]) it is possible to deduce that E is Lipschitz along its foliation $\mathcal{F}$. We also know that there is a complementary transversal foliation given by integral curves of Z along which E is sequentially Lipschitz and therefore Lipschitz. This implies that E is Lipschitz.

Thus center-bunching and sequential transverse regularity implies Lipschitz. The missing link would just be to show that if E is transversally Lipschitz along a direction then it is also sequentially transversally Lipschitz along that direction. This would yield a positive answer to the question.

2.4.3 General philosophy and strategy of proof

Since our distribution is no longer Lipschitz we are not able to apply any existing general involutivity/integrability result, such as that of Simić quoted above[2]. Instead we will have to essentially construct the required integral manifolds more or less explicitly "by hand".

The standard approach for this kind of construction is the so-called *graph transform* method, see [33], which takes full advantage of certain hyperbolicity conditions and consists of "pulling back" a sequence of manifolds and showing that the sequence of pull-backs converges to a geometric object which can be shown to be a unique integral manifold of the distribution. This method goes back to Hadamard and has been used in many different settings but, generally, cannot be applied in the partially hyperbolic or dominated decomposition setting where the dynamics is allowed to have a wide range of dynamical behaviour and it is therefore impossible to apply any graph transform arguments to E^{sc} under our assumptions. This is perhaps one of the main reasons why this setting has proved so difficult to deal with.

The strategy we use here can be seen as a combination of the Frobenius/Simić involutivity approach and the Hadamard graph transform method. Rather than approximating the desired integral manifold by a sequence of manifolds we ap-

[2]Some notion of Lie bracket can be formulated in lower regularity, see for example [13, Proposition 3.1], but it is not clear to us how to obtain a full unique integrability result using these ideas.

proximate the continuous distribution E by a sequence $\{E^{(k)}\}$ of C^1 distributions obtained dynamically by "pulling back" a suitably chosen initial distribution. Since these approximate distributions are C^1, the Lie brackets of C^1 vector fields in $E^{(k)}$ can be defined. *If the $E^{(k)}$ were involutive*, then each one would admit an integral manifold $\mathcal{E}^{(k)}$ and it is fairly easy to see that these converge to an integral manifold of the original distribution E. However this is generally not the case and we need a more sophisticated argument to show that the distributions $E^{(k)}$ are "asymptotically involutive" in a particular sense which will be defined formally below. For each k we will construct an "approximate" local center-stable manifold $\mathcal{W}^{(k)}$ which is not an integral manifold of $E^{(k)}$ (because the $E^{(k)}$ are not necessarily involutive) but is "close" to being integral manifolds. Further estimates, using also the asymptotic involutivity of the distributions $E^{(k)}$, then allow us to show that these manifolds converge to an integral manifold of the distribution E. We will then use a separate argument to obtain uniqueness, taking advantage of a result of Hartman.

2.4.4 Almost involutive approximations

In this section we state and prove a generalization of Proposition 2.2.2 which formalizes the meaning of "almost" involutive. We consider the sequence of C^1 distributions $\{E^{(k)}\}_{k>1}$ as in the definition of sequential transverse regularity in (2.4.1). We fix a coordinate system $(x^1, x^2, x^3, \mathcal{U})$ so that $\partial/\partial x^i$ are all transverse to E and therefore to $E^{(k)}$ for k large enough since $E^{(k)} \to E$ uniformly in angle. Then thanks to this transversality assumption we can find vector fields defined on $\mathcal{U}$ of the form

$$X^{(k)} = \frac{\partial}{\partial x^1} + a^{(k)}\frac{\partial}{\partial x^3} \quad \text{and} \quad Y^{(k)} = \frac{\partial}{\partial x^2} + b^{(k)}\frac{\partial}{\partial x^3}. \tag{2.4.2}$$

that span $E^{(k)}$ and converge to vector fields of the form

$$X = \frac{\partial}{\partial x^1} + a\frac{\partial}{\partial x^3} \quad \text{and} \quad Y = \frac{\partial}{\partial x^2} + b\frac{\partial}{\partial x^3}. \tag{2.4.3}$$

that span E for $a^{(k)}$, a, $b^{(k)}$, b everywhere non-vanishing functions. Moreover we can choose $\partial/\partial x^3$ to be the direction where sequential transversal Lipschitzness holds true (since it is already transversal to E) so we have the property that there exists $C > 0$

$$\left|\frac{\partial a^{(k)}}{\partial x^3}\right| < C \quad \text{and} \quad \left|\frac{\partial b^{(k)}}{\partial x^3}\right| < C \tag{2.4.4}$$

for all k. It is easy to check that $[X^{(k)}, Y^{(k)}]$ lies in the $\partial/\partial x^3$ direction. As before we will have some estimates about how fast the Lie brackets of these vector fields decay to 0. For the following let $F^{(k)}$ be the continuous bundle which is orthogonal to $E^{(k)}$ with respect to Lyapunov metric on E so that $F^{(k)}$ goes to F in angle (since F is orthogonal to E with respect to the Lyapunov metric). We have the following analogue of Proposition 2.2.2

Proposition 2.4.1. *There exists $C > 0$ such that for every $k > 1$ and $x \in \mathcal{U}$, we have*

$$\|[X^{(k)}, Y^{(k)}](x)\| \leq C\frac{|det(D\varphi^k_x|_{E^{(k)}_x})|}{\|D\varphi^k_x|_{F^{(k)}_x}\|}.$$

Sketch of proof. The proof of Proposition 2.4.1 is very similiar to that of Proposition 2.2.2 and it is not hard to get the result with the difference that E and F in the right hand side of the estimate in Proposition 2.2.2 are replaced by $E^{(k)}$ and $F^{(k)}$. In this case we choose our collection of C^1 orthonormal collection of frames $\{Z^{(k)}, W^{(k)}\}$ of $E^{(k)}$ so that $||D\varphi^k Z^{(k)}||||D\varphi^k W^{(k)}|| = det(D\varphi^k|_{E^{(k)}})$ Then exactly as in lemma 2.3.3, to get an upper bound on $|[X^{(k)}, Y^{(k)}]|$, it is enough to bound $[Z^{(k)}, W^{(k)}]$. The proof of the inequality $\|[Z^{(k)}, W^{(k)}]\| \leq det(D\varphi^k|_{E^{(k)}})/\|D\varphi^k|_{F^{(k)}}\|$ follows quite closely the proof of lemma 2.3.2 where the projection π is replaced by $\pi^{(k)}$ which the projection to $F^{(k)}$ along $E^{(k)}$ at relevant places. □

The next, fairly intuitive but in fact quite technical, step is to replace the estimates on the approximations with estimates on the limit bundle.

Proposition 2.4.2. *There exists $C > 0$ such that for every $k > 1$ and $x \in \mathcal{U}$, we have*

$$|det(D\varphi_x^k|_{E_x^{(k)}})| \leq C|det(D\varphi_x^k|_{E_x})| \tag{2.4.5}$$

and

$$\|D\varphi_x^k|_{F_x^{(k)}}\| \geq C\|D\varphi_x^k|_{F_x}\| \tag{2.4.6}$$

Sketch of proof. (2.4.6) is fairly easy since for any vector $v \notin E$, $|D\varphi^k v| \geq CD\varphi_x^k|_{F_x}|v|$ (since F has dimension 1). The real technical estimate is (2.4.5). One first needs to make the observation that there exists a cone $C(\alpha)$ of angle α around E such that $D\varphi^k E^{(k)} \subset C(\alpha)$. This is the main observation that allows to relate two determinants independent of k. Indeed given a basis v_1^k, u_1^k of $E^{(k)}$, then $K|det(D\varphi_x^k|_{E_x^{(k)}})| = |D\varphi_x^k v_1^k \wedge D\varphi_x^k u_1^k| = |\Lambda(D\varphi_x^k)v_1^k \wedge u_1^k|$ where $\Lambda(D\varphi_x^k)$ is the induced action of $D\varphi_x^k$

on $TM \wedge TM$. But then $\Lambda(D\varphi_x^k)$ allows a dominated splitting of $TM \wedge TM$ whose invariant spaces are $E_1 = E \wedge E$, $E_2 = E \wedge F$ where E_1 is dominated by E_2. We have that for all k, $E_1^{(k)} = \text{span}(v_1^k \wedge u_1^k)$ is a space which is inside a cone $C(\alpha)$ around E_1. Therefore usual dominated splitting estimates give that $|\Lambda(D\varphi_x^k)|_{E_1^{(k)}(x)}| \leq K|\Lambda(D\varphi_x^k)|_{E_1(x)}|$ which proves the claim about determinants since $|\Lambda(D\varphi_x^k)|_{E_1(x)}| = |det(D\varphi_x^k|_{E_x})|$. $\square$

2.4.5 Almost Integral Manifolds

We will use the local frames $\{X^{(k)}, Y^{(k)}\}$ to define a family of local manifolds which we will then show converge to the required integral manifold of E. We emphasize that these are *not* in general integral manifolds of the approximating distribution $E^{(k)}$. We will use the relatively standard notation $e^{tX^{(k)}}$ to denote the flow at time $t \in \mathbb{R}$ of the vector field $X^{(k)}$. Then we let

$$\mathcal{W}_{x_0}^{(k)}(t,s) := e^{tX^{(k)}} \circ e^{sY^{(k)}}(x_0).$$

This map is well defined for all sufficiently small s, t so that the composition of the corresponding flows remains in the local chart $\mathcal{U}$ in which the vector fields $X^{(k)}, Y^{(k)}$ are defined. Since the vector fields $X^{(k)}, Y^{(k)}$ are uniformly bounded in norm, choosing ϵ sufficiently small the functions $\mathcal{W}_{x_0}^{(k)}$ can be defined in a fixed domain $U_\epsilon = (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ independent of k such that $\mathcal{W}_{x_0}^{(k)}(U_\epsilon) \subset \mathcal{U}$. By a direct application of the chain rule and the definition of $\mathcal{W}_{x_0}^{(k)}$, for every $(t,s) \in U_\epsilon$ we have

$$\tilde{X}^{(k)}(t,s) = \frac{\partial \mathcal{W}_{x_0}^{(k)}}{\partial t}(t,s) = X^{(k)}(\mathcal{W}_{x_0}^{(k)}(t,s))$$

and

$$\tilde{Y}^{(k)}(t,s) = \frac{\partial \mathcal{W}_{x_0}^{(k)}}{\partial s}(t,s) = (e^{tX^{(k)}})_* Y^{(k)}(\mathcal{W}_{x_0}^{(k)}(t,s)).$$

where for two vector fields V, Z and $t \in \mathbb{R}$, $(e^{tV})_* Z$ denotes that pushforward of Z by the flow of V defined by

$$[(e^{tV})_* Z]_p = De^{tV}_{e^{-tV}(p)} Z_{e^{-tV}(p)}.$$

The following lemma gives a condition for this family of maps to have a convergent subsequence whose limits becomes a surface tangent to E:

Lemma 2.4.3. *If $\tilde{X}^{(k)} \to X$ and $\tilde{Y}^{(k)} \to Y$ then the images of $\mathcal{W}_{x_0}^{(k)}$ are embedded submanifolds and this sequence of submanifolds has a convergent subsequence whose limit is an integral manifold of E.*

Proof. Since X and Y are linearly independent by assumption of convergence, the differential of the map $\mathcal{W}_{x_0}^{(k)}$ is invertible at every point $(t,s) \in U_\epsilon$, i.e. the partial derivatives $\partial \mathcal{W}_{x_0}^{(k)}/\partial s$ and $\partial \mathcal{W}_{x_0}^{(k)}/\partial t$ are linearly independent for every $(t,s) \in U_\epsilon$. Thus the maps $\mathcal{W}_{x_0}^{(k)}$ are embeddings and define submanifolds through x_0 (which are not in general integral manifolds of $E^{(k)}$). Moreover, since $X^{(k)}, Y^{(k)}$ have uniformly bounded norms, it follows by Proposition 2.4.4 that $D\mathcal{W}_{x_0}^{(k)}$ has bounded norm uniformly in k and therefore the family $\{\mathcal{W}_{x_0}^{(k)}\}$ is a compact family in the C^1 topology. By the Arzela-Ascoli Theorem this family has a subsequence converging to some limit

$$\mathcal{W}_{x_0} : U_\epsilon \to \mathcal{U}. \tag{2.4.7}$$

We claim that $\mathcal{W}_{x_0}(U_\epsilon)$ is an integral manifold of E. Indeed, as $k \to \infty$, $X^{(k)} \to X$, $Y^{(k)} \to Y$ and $\{X, Y\}$ is a local frame of continuous vector fields for E, in particular X, Y are linearly independent and span the distribution E. Moreover, by Proposition 2.4.4, the partial derivatives $\partial\mathcal{W}_{x_0}^{(k)}/\partial t$ and $\partial\mathcal{W}_{x_0}^{(k)}/\partial s$ are converging uniformly to X and Y and therefore

$$\frac{\partial \mathcal{W}_{x_0}}{\partial t} = X \quad \text{and} \quad \frac{\partial \mathcal{W}_{x_0}}{\partial s} = Y.$$

This shows that $\mathcal{W}_{x_0}(U_\epsilon)$ is a C^1 submanifold and its tangent space coincides with E and thus $\mathcal{W}_{x_0}(U_\epsilon)$ is an integral manifold of E, thus proving integrability of E under these assumptions. $\square$

It therefore just remains to verify the assumptions of Lemma 2.4.3, i.e. to show that the vectors $X^{(k)}$ and $(e^{tX^{(k)}})_* Y^{(k)}$ converge to X and Y. The first convergence is obviously true. Thus it remains to show the latter which we show in the next result, thus completing the proof of the existence of integral manifolds.

Proposition 2.4.4. *For all $t \in (-\epsilon, \epsilon)$ we have*

$$\lim_{k\to\infty} \|(e^{tX^{(k)}})_* Y^{(k)} - Y^{(k)}\| = 0.$$

Proof. To obtain this proposition one uses the following standard property of the pushforward (see proof of Proposition 2.6 in [1] for instance):

$$\frac{d}{dt}[(e^{tX^{(k)}})_* Y^{(k)} - Y^{(k)}] = (e^{tX^{(k)}})_*[X^{(k)}, Y^{(k)}] \tag{2.4.8}$$

together with the following proposition:

Proposition 2.4.5. *There exists $C > 0$ such that for every $k \geq 1, x \in \mathcal{U}$ and $|t| \leq \epsilon$, we have*

$$||(e^{tX^{(k)}})_* \frac{\partial}{\partial x^3}|_x|| = \exp \int_0^t \frac{\partial a^{(k)}}{\partial x^3} \circ e^{-\tau X^{(k)}}(x) d\tau$$

This latter proposition follows by integrating the equality

$$\frac{d}{dt}((e^{tX^{(k)}})_* \frac{\partial}{\partial x^3}|_x) = \left(e^{tX^{(k)}}\right)_* [X^{(k)}, \frac{\partial}{\partial x^3}]|_x$$

Once this is established since we know that $\partial a^{(k)}/\partial x^3$ is uniformly bounded (2.4.4), we obtain that the effect of $(e^{tX^{(k)}})_*$ on $\partial/\partial x^3$ is bounded. Since $[X^{(k)}, Y^{(k)}]$ is a vector in this direction whose norm goes to 0 we directly obtain by equation (2.4.8) that $\frac{d}{dt}[(e^{tX^{(k)}})_* Y^{(k)} - Y^{(k)}]$ goes to 0 uniformly and hence by mean value theorem that $|(e^{tX^{(k)}})_* Y^{(k)} - Y^{(k)}|$ goes to 0 which is the proposition. □

Remark 2.4.6. From the proof, it is seen that the most crucial ingredient is for the approximations to satisfy the pushforward bound in Proposition 2.4.4. One can generalize this observation to get geometric theorems about integrability of continuous sub-bundles, not just those arising in dynamical systems, with some additional assumptions such as the Lie brackets going to 0.

2.4.6 Uniqueness

To get uniqueness of the integral manifolds we will take advantage of a general result of Hartman which we state in a simplified version which is sufficient for our purposes.

Theorem 4 ([30], Chapter 5, Theorem 8.1)**.** *Let* $X = \sum_{i=1}^{n} X^i(t,p)\frac{\partial}{\partial x^i}$ *be a continuous vector field defined on* $I \times U$ *where* $U \subset \mathbb{R}^n$ *and* $I \subset \mathbb{R}$*. Let* $\eta_i = X^i(t,p)dt - dx^i$*. If there exists a sequence of* C^1 *differential forms* η_i^k *such that* $|\eta_i^k - \eta_i|_\infty \to 0$ *and* $d\eta_i^k$ *are uniformly bounded then* X *is uniquely integrable on* $I \times U$*. Moreover on compact subset of* $U \times I$ *the integral curves are uniformly Lipschitz continuous with respect to the initial conditions.*

We recall that a two form being uniformly bounded is equivalent to each of its component is being uniformly bounded.

Corollary 2.4.7. *Vector fields* X *and* Y *defined in* (2.4.3) *are uniquely integrable.*

Proof. We will give the proof for X, that for Y is exactly the same. Since X has the form

$$X = \frac{\partial}{\partial x^1} + a\frac{\partial}{\partial x^3},$$

its solutions always lie in the $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3}$ plane. Therefore given a point (x_0^1, x_0^2, x_0^3), it is sufficient to consider the restriction to such a plane. Then the C^1 differential 1-forms defined in Theorem 4 are

$$\eta_1 = dt - dx^1 \qquad \eta_2 = a(x)dt - dx^3$$

where $x = (x^1, x_0^2, x^3)$, and for the approximations we can write

$$\eta_1^k = dt - dx^1 \qquad \eta_2^k = a^{(k)}(x)dt - dx^3$$

where $a^{(k)}(x)$ are functions given in equation (2.4.2), again for some fixed x_0^2. But then by sequential transversal Lipschitz assumption and choice of coordinates we have that $|\frac{\partial a^{(k)}}{\partial x^3}| < C$ for all k and

$$d\eta_1^k = 0 \qquad d\eta_2^k = \frac{\partial a^{(k)}}{\partial x^3} dx^3 \wedge dt$$

(since we restrict to $x^2 = $ const planes) and the requirements of Theorem 4 are satisfied which proves that X is uniquely integrable. □

Now we have that X and Y are uniquely integrable at every point. Assume there exist a point $p \in \mathcal{U}$ such that through p there exist two integral surfaces $\mathcal{W}_1, \mathcal{W}_2$. This means both surfaces are integral manifolds of E and in particular the restriction of X and Y to their tangent space are uniquely integrable vector fields. Therefore there exists ϵ_1 such that the integral curve $e^{tX}(p)$ for $|t| \leq \epsilon_1$ belongs to both surfaces. Now consider an integral curve of Y starting at the points of $e^{tX}(p)$, that is $e^{sY} \circ e^{tX}(p)$. For ϵ_1 small enough, there exists ϵ_2 small enough such that for every $|t| < \epsilon_1$ and $|s| < \epsilon_2$ this set is inside both surfaces since Y is also uniquely integrable (ϵ_i only depend on norms $|X|$, $|Y|$ and size of $\mathcal{W}_i$ and therefore can be chosen uniformly independent of point). This set is a C^1 disk and therefore $\mathcal{W}_1$ and $\mathcal{W}_2$ coincide on an open domain. Applying this to every point $p \in U$ we obtain that through every point in U there passes a single local integral manifold. This concludes the proof of the uniqueness.

Chapter 3

Integrability of continuous bundles

3.1 Introduction and statement of results

In this chapter we address the question of integrability and unique integrability of *continuous* tangent sub-bundles on C^r manifolds with $r \geq 1$. A continuous m-dimensional tangent sub-bundle (or a distribution) E on a $m+n$ dimensional C^r manifold M is a continuous choice of m-dimensional linear subspaces $E_p \subset T_pM$ at each point $p \in M$. A C^1 sub-manifold $N \subset M$ is a local integral manifold of E if $T_pN = E_p$ at each point $p \in N$. The distribution E is *integrable* if there exists local integral manifolds through every point, and *uniquely integrable* if these integral manifolds are unique in the sense that whenever two integral manifolds intersect, their intersection is relatively open in both integral manifolds.

The question of integrability and unique integrability is a classical problem that goes back to work of Clebsch, Deahna and Frobenius [17, 19, 24] in the mid 1800's. Besides their intrinsic geometric interest, integrability results have many applications to various areas of mathematics including the existence and uniqueness of solutions for systems of ordinary and partial differential equations and to dynamical systems. The early results develop conditions and techniques to treat cases where the sub-bundles, or the corresponding equations, are at least C^1 and, notwithstanding the importance and scope of these results, it has proved extremely difficult to relax the differentiability assumptions completely. Some partial generalizations have been obtained by Hartman [29] and other authors [60, 42, 52] but all still require some form of weak differentiability, e.g. a Lipschitz condition.

The main point of our results is to formulate new integrability conditions for purely *continuous* equations and sub-bundles. We apply these conditions to obtain results for examples which are not more than Hölder continuous and for which the same statements cannot be obtained by any other existing methods. Our Main Theorem which contains the most general version of our results is stated in Section 3.1.5. All other results are essentially corollaries and applications of the Main Theorem and are stated in separate subsections of the introduction. We begin with the results for ODE's and PDE's which are of independent interest and easy to formulate.

3.1.1 Uniqueness of solutions for ODE's

In this section we consider continuous ordinary differential equations of the form

$$\frac{dy^i}{dt} = F^i(t, y(t)), \qquad (t_0, y(t_0)) \in V \tag{3.1.1}$$

where $i = 1, ..., n$ and $F = (F^1(t, y), ..., F^n(t, y)) : V \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a continuous vector field. By Peano's theorem, an ODE with a continuous vector field always admits solutions and by Picard-Lindelöf-Cauchy-Lipschitz theorem, an ODE with Lipschitz vector fields admits unique solutions through every point. The Lipschitz condition is however not necessary and a lot of work exists establishing weaker regularity conditions which imply uniqueness, see [2] for a comprehensive survey. One such condition is Osgood's criterion (see Theorem 1.4.2 in [2]) where the modulus of continuity w (we give the precise definition below) with respect to the space variables satisfies

$$\lim_{\epsilon \to 0} \int_0^\epsilon \frac{1}{w(s)} ds < \infty. \tag{3.1.2}$$

This condition, albeit much weaker than Lipschitz, is also not necessary as there are examples of uniquely integrable ODE's that do not satisfy Osgood's criterion, such as the case $F(t, x) = e^t + x^\alpha$ for $\alpha < 1$ which was studied in [3]. We give here a new condition for uniqueness of solutions for continuous ODE's and a simple example of an ODE which satisfies our conditions, and thus is uniquely integrable, but does not satisfy any previously known condition for uniqueness.

Definition 3.1.1. Let $w : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing, continuous function such that

$\lim_{s\to 0} w(s) = 0$. A function $F : U \subset \mathbb{R}^n = (\xi^1, ..., \xi^n) \to \mathbb{R}^m$ is said to have *modulus of continuity* w *with respect to variable* ξ^i if there exists a constant $K > 0$ such that for all $(\xi^1, ..., \xi^n) \in U$ and for all s small enough so that $(\xi^1, ..., \xi^i + s, ..., \xi^n) \in U$,

$$|F(\xi^1, ..., \xi^i + s, ..., \xi^n) - F(\xi^1, ..., \xi^n)| \leq Kw(|s|).$$

We say that F has modulus of continuity w if it has modulus of continuity w with respect to all variables. We denote by C^{r+w} functions whose r^{th} order partial derivatives have modulus of continuity w (in the case $r = 0$ they will simply be functions with modulus of continuity w).

We denote the extended version of the vector-field $F(t, x)$ by

$$\tilde{F} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} F^i(t, y)\frac{\partial}{\partial x^i} = \sum_{i=1}^{n+1} \tilde{F}^i(\xi)\frac{\partial}{\partial \xi^i}$$

where $(\xi^1, ..., \xi^{n+1})$ is a collective tag for the coordinates t and $(y^1, ..., y^n)$. Note that $\tilde{F}(\xi) \neq 0$ for all ξ since $\tilde{F}^1 = 1$.

Theorem 5. *Consider the ODE in* (3.1.1) *with* $F : V \subset \mathbb{R}^{n+1}$ *a continuous function with modulus of continuity* w_1. *Let* $\xi \in V$ *and* $i \in \{1, ..., n+1\}$ *be such that* $\tilde{F}^i(\xi) \neq 0$ *and suppose that* $\tilde{F}$ *has modulus of continuity* w_2 *with respect to variables* $(\xi^1, ..., \xi^{i-1}, \xi^{i+1}, ...\xi^{n+1})$ *and*

$$\lim_{s\to 0} w_1(s)e^{w_2(s)/s} = 0. \tag{3.1.3}$$

Then the ODE (3.1.1) *with initial condition* (t_0, ξ) *has a unique solution.*

Example 3.1.2. Consider the ODE

$$F(t,x,y) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (-tlog(t^{\beta}) - xlog(x^{\gamma}), 1 + y^{\alpha} - xlog(x^{\delta})) \qquad (3.1.4)$$

for $0 < \beta, \gamma, \delta < \alpha < 1$ (the requirement that they are less than 1 is not necessary but we only impose it to make the examples non-Lipschitz and therefore more interesting). As far as we know, the uniqueness of solutions for (3.1.4) cannot be verified by any existing method except ours. In particular, Osgood criterion does not hold due to the term y^{α} and the main result of [3] would require the right hand side to be Lipschitz with respect to x and t which is not the case.

To see that (3.1.4) satisfies the assumptions of Theorem 5 we will use certain elementary properties of the modulus of continuity, which for convenience we collect in 3.5.1 in the Appendix. Notice first of all that $F(0,0,0) = (0,1)$ so that its y component is non-zero. Moreover by items 4 and 5 of Proposition 3.5.1, F^1 has modulus of continuity $-slog(s^{\beta})$ with respect to variable t for $s \leq \frac{1}{e}$ and has modulus of continuity $-slog(s^{\gamma})$ with respect to variable x for $s \leq \frac{1}{e}$ and is constant with respect to y; F^2 has modulus of continuity $-slog(s^{\delta})$ for $s \leq \frac{1}{e}$ with respect to variable x and has modulus of continuity s^{α} with respect to variable y and is constant with respect to t. Therefore by Item 7 of proposition 3.5.1, letting $\sigma = \max\{\beta, \gamma, \delta\} < \alpha$, F has modulus of continuity s^{α} with respect to y, modulus of continuity $w_2(s) = -slog(s^{\sigma})$ with respect to x and t, and finally by Item 6 of proposition 3.5.1 and the fact that $s^{\alpha} \geq -slog(s^{\sigma})$ for $|s| \leq 1$, and s^{α} is convex, F has modulus of continuity $w_1(s) = s^{\alpha}$. Since $\lim_{s\to 0} w_1(s)e^{w_2(s)/s} = \lim_{s\to 0} Kx^{\alpha-\sigma} =$

0, it has unique solutions by Theorem 5.

Remark 3.1.3. Theorem 5 and Example 3.1.2 also describe a qualitative way in which regularities between variables can be traded. If a vector field F has its i^{th} component non-zero then you can decrease the vector field's regularity with respect to the i^{th} variable as long as you increase the others in a way described by equation (3.1.3). It is a more flexible criterion than both Osgood and that of [3]. Condition (3.1.3) is satisfied for example if the vector field is only Hölder continuous overall, i.e. $w_1(s) = s^\alpha$, but, restricting to all but one variables is just a little bit better than Hölder continuous, such as for example $w_2(s) = -slog(s^\beta)$ for some $\beta \in (0, \alpha)$.

Remark 3.1.4. Part of the interest in condition (3.1.3) lies in the fact that two different regularities w_1, w_2 to come into play. Clearly we always have $w_2 \leq w_1$ and therefore Theorem 5 holds under the stronger condition obtained by using the overall regularity, i.e. letting $w = w_1 = w_2$ to get

$$\lim_{s \to 0} w(s) e^{w(s)/s} = 0. \tag{3.1.5}$$

Thus, as an immediate corollary of Theorem 5, *a vector field F with modulus of continuity w such that w satisfies equation* (3.1.5) *is uniquely integrable.* In view of this, it is natural to search for, and try to describe and characterize, functions w satisfying condition (3.1.5) and in particular to compare this condition with Osgood's. One can check that many functions w verify both (3.1.2) and (3.1.5), such as $w_1(s) = slog^{1+s}(s)$ or $w_1(s) = sloglog....(s)$, and many others satisfy neither condition, such as $w_1(s) = s^\alpha log(s)$ for $\alpha < 1$. So far we have not however been able to show that the two conditions are equivalent nor to find any examples of functions which satisfy one and not the other. In any case, this simplified version also gives an interesting way to replace Osgood's criterion with a relatively easy limit condition, at least for the most relevant examples that we know.

Question 3.1.5. *Are conditions* (3.1.2) *and* (3.1.5) *equivalent?*

3.1.2 Uniqueness of solutions for PDE's

In this section we consider linear partial differential equations of the form

$$\frac{\partial y^i}{\partial x^j} = F^{ij}(x, y(x)), \qquad (x, y(x)) \in V \tag{3.1.6}$$

where $i = 1, ..., n$, $j = 1, ..., m$ and $F^{ij} : V \subset \mathbb{R}^{n+m} \to \mathbb{R}$ are continuous functions. Note that the ODE (3.1.1) is a special case of (3.1.6), the case where $m = 1$. We again denote the collective coordinates $(\xi^1, ..., \xi^{n+m}) = (x^1, ..., x^m, y^1, ..., y^n)$. In this case we define the $n \times (m + n)$ matrix extension $\hat{F}$ of the $n \times m$ matrix F^{ij} by

$$\hat{F}^{ij}(\xi) = \delta_{ij} \text{ for } 1 \leq i \leq n,\ 1 \leq j \leq m$$

and

$$\hat{F}^{ij}(\xi) = F^{ij}(\xi) \text{ for } 1 \leq i \leq n, m + 1 \leq j \leq m + n.$$

Given any set of indices $I = (i_1, ..., i_n)$, we denote the submatrix of $\hat{F}(\xi)$ which corresponds to the $i_1, ..., i_n$'th columns by $\hat{F}^I(\xi)$.

The existence of solutions for PDE's is not automatic as it is in the ODE setting, and in particular is not a direct consequence of their regularity, and so the following result concerns the uniqueness of solutions while assuming their existence. Our most general results below also give conditions for existence of solutions, but they require a more geometric "involutivity" condition which is independent of the regularity and thus not so easy to state in this setting.

Theorem 6. *Consider the PDE in* (3.1.6) *with* F^{ij} *continuous with modulus of continuity* w_1. *Let* $\xi \in V$ *be a point such that for some* $I = (i_1, ..., i_n)$, $det(\hat{F}^I(\xi)) \neq 0$ *and suppose that* F^{ij} *has modulus of continuity* w_2 *with respect to the variables* $\{\xi^{i_1}, ..., \xi^{i_n}\}$ *and that*

$$\lim_{s \to 0} w_1(s) e^{w_2(s)/s} = 0. \tag{3.1.7}$$

Then if the PDE (3.1.6) *admits a solution at* ξ, *that solution is unique.*

Remark 3.1.6. Notice that if $I = (m + 1, ..., m + n)$ then $det(\hat{F}^I(\xi)) = 1 \neq 0$ and therefore if $F^{ij}(x, y)$ has modulus of continuity w_2 with respect to variables $(y^1, ..., y^n)$ so that (3.1.7) is satisfied, then the solutions of (3.1.6), whenever they exists, are unique.

Constructing examples of PDE's satisfying our conditions is more complicated than constructing examples of ODE's because of the problem of existence of solutions mentioned above. We therefore formulate our example within a special class of equations for which existence can be verified directly. Suppose the functions F^{ij}

are of the form

$$F^{ij}(x,y) = G_i(y^i)\frac{\partial H_i}{\partial x^j}(x) \tag{3.1.8}$$

with $(x,y) \in V = V_1 \times V_2$, for continuous functions $G_i : U_i \subset \mathbb{R} \to \mathbb{R}$, $H_i : V_2 \subset \mathbb{R}^m \to \mathbb{R}$ for $j = 1, ..., m$, $i = 1, ..., n$, $V_1 = U_1 \times ... \times U_n$. In this case the next proposition (proved in Section 3.4) tells us that we can get existence of solutions with no additional regularity assumptions.

Proposition 3.1.7. *Consider a partial differential equation* (3.1.6) *where the functions* F^{ij} *are of the form* (3.1.8). *Then* (3.1.6) *admits solutions through every point.*

This allows us to give examples of PDE's to which we can apply our uniqueness criterion.

Example 3.1.8. Consider the PDE

$$\frac{\partial y^i}{\partial x^j}(x,y) = -(x^j)^{\alpha_{ij}} y^i log((y^i)^{\beta_i}) \tag{3.1.9}$$

for parameters $0 < \beta_i, \alpha_{ij} < 1$ and also $\beta = \max_i \beta_i < \alpha = \min_{i,j} \alpha_{ij}$. This equation can be written in the form (3.1.8) with

$$H_i(x) = \sum_{j=1}^{n} \frac{(x^j)^{\alpha_{ij}+1}}{\alpha_{ij}+1} \quad \text{and} \quad G_i(y^i) = -y^i log((y^i)^{\beta_i})$$

Therefore by Proposition 3.1.7 it admits solutions. For uniqueness, we have that the regularity of the H_i's is C^{1+w_3} with $w_3(s) = s^\alpha$ and the regularity of the G_i's is C^{w_2} with $w_2(s) = -slog(s^\beta)$. Therefore, letting $w_1(s) = \max\{w_2(s), w_3(s)\}$, we have $\lim_{s\to 0} w_1(s)e^{w_2(s)/s} = \lim_{s\to 0} Kx^{\alpha-\beta} = 0$ and using Theorem 6 we have that this system has unique solutions.

Example 3.1.9. Consider the PDE

$$\frac{\partial y^1}{\partial x^1}(x,y) = (1 - x^1 ln((x^1)^{\alpha_{11}}))((y^1)^{\beta_1} + 1)$$
$$\frac{\partial y^1}{\partial x^2}(x,y) = (x^2)^{\alpha_{12}}(1 + (y^1)^{\beta_1})$$
$$\frac{\partial y^2}{\partial x^1}(x,y) = y^2 ln((y^2)^{\beta_2}) x^1 ln((x^1)^{\alpha_{21}})$$
$$\frac{\partial y^2}{\partial x^2}(x,y) = -y^2 ln((y^2)^{\beta_2})(x^2)^{\alpha_{22}}$$

with $0 < \alpha_{12}, \alpha_{22}, \beta_1 < \alpha_{11}, \alpha_{21}, \beta_2 < 1$. Set $\alpha = \max\{\alpha_{11}, \alpha_{21}, \beta_2\}$ and $\beta = \min\{\alpha_{12}, \alpha_{22}, \beta_1\}$. The right-hand side again has the form (3.1.8) and so the PDE has solutions by Proposition 3.1.7. For the uniqueness, one considers the matrix F_{ij} at $(x,y) = (0,0)$, which is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

and so the sub-matrix corresponding to columns $i_1 = 2$ and $i_2 = 3$ is invertible. But then with respect to variables $\xi^2 = x^2$ and $\xi^3 = y^1$, F^{ij} has modulus of continuity $w_2(s) = -sln(s^\alpha)$ and in general has modulus of continuity $w_1(s) = s^\beta$. But $w_1(s)$ and $w_2(s)$ satisfy condition (3.1.7) and so, by Theorem 6, the PDE has unique solutions in a neighborhood of the origine $(0,0)$.

These are only two particularly simple examples one can construct using Proposition 3.1.7. Here the forms of $H_i(y)$ are quite simple in the sense that $F_i(y) = \sum_j G_{ij}(y^j)$ and more complicated examples can be achieved with more work.

3.1.3 Unique Integrability of Continuous Bundles

We will derive the results above on existence and uniqueness of solutions for ODE's and PDE's from more general and more geometric results about the integrability and unique integrability of tangent bundles on manifolds. In this section we state Theorem 7 which can be seen as a mid step between the ODE and PDE theorems stated in the previous sections and the more general results in Theorem 8 and the Main Theorem in the following sections. Throughout this section we assume that E is a continuous tangent sub-bundle on a manifold M. First we need to generalize certain classical definitions of modulus of continuity to bundles.

Definition 3.1.10. A bundle E of rank m is said to have modulus of continuity w, where w is a continuous, increasing function $w : I \subset \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{s\to 0} w(s) = 0$, if in every sufficiently small neighbourhood, it can be spanned by linearly independent vector fields $X_1, ..., X_m$ such that in local coordinates $|X_i(p) - X_i(q)| \leq w(|p-q|)$. E is said to have *transversal modulus of continuity* w if for every $x_0 \in M$, there exists a coordinate neighbourhood $(V, (x^1, ..., x^m, y^1, ..., y^n))$ around x_0 so that E is transverse to $\text{span}\{\frac{\partial}{\partial y^i}\}_{i=1}^n$ and with respect to coordinates $\{y^1, ..., y^n\}$ E has modulus of continuity $w(s)$.

Theorem 7. *Let E be a rank n bundle with modulus of continuity w_1 and transversal modulus of continuity w_2. Assume E is integrable and*

$$\lim_{s\to 0} w_1(s) e^{w_2(s)/s} = 0 \tag{3.1.10}$$

Then E is uniquely integrable.

The scope of Theorem 7 is possibly limited because integrability is assumed. However we have stated it here because it gives uniqueness as a result of a natural regularity condition and in general existence is a highly non trivial property which cannot be reduced to any regularity condition. We will show that Theorem 7 is a corollary of the more general results below which address the problem of existence as well as uniqueness, and that it easily implies Theorems 5 and 6.

3.1.4 Asymptotic involutivity and exterior regularity

We now formulate the special case of our most general result, addressing the problem of existence and uniqueness of integral manifolds for continuous tangent bundle distributions. Since integrability of tangent bundles is a local property, we assume from now on that U is a Euclidean ball in $\mathbb{R}^{n+m}$ and E is a continuous tangent bundle distribution of rank n on U. We let $|\cdot|$ denote the (induced) Euclidean norm on sections of the tangent bundle and of k-differential forms for all $0 \leq k \leq n+m$. $|\cdot|_\infty$ denotes the sup-norm over U, which gives the aforementioned sections a Banach space structure. We also employ , point-wise, tangent vectors with the induced Euclidean metric. Letting $\mathcal{A}^1(E)$ denote the space of all 1-forms η defined on U with $E \subset \ker(\eta)$, the distribution E is completely described by any set $\{\eta_i\}_{i=1}^n$ of n linearly independent 1-forms in, i.e. any basis of, $mathcal A^1(E)$. If the distribution E is differentiable, the forms $\{\eta_i\}_{i=1}^n$ can also be chosen differentiable and the classical Frobenius theorem [17, 19, 24] states that E is *uniquely integrable* if, for any basis of differentiable 1-forms $\{\eta_i\}_{i=1}^n$ of $\mathcal{A}^1(E)$, the *involutivity condition*

$$|\eta_1 \wedge \cdots \wedge \eta_n \wedge d\eta_i(p)| = 0 \tag{3.1.11}$$

holds for all $i = 1, ..., n$ and $p \in U$. Several generalizations of this Theorem exist in the literature, including results which weaken the differentiability assumption, we mention for example results by Hartman [29], Simic [58] and other authors [42, 52], but which still essentially use the fact that the exterior derivative $d\eta_i$ exists, for example if E is Lipschitz then the η_i are differentiable almost everyhere and $d\eta_i$ exists almost everywhere, and therefore such results can be formulated in essentially the same way as Frobenius, using condition (3.1.11).

One of the first stumbling blocks in obtaining some integrability criteria for general *continuous* distributions is that the exterior derivatives of the forms $\{\eta_i\}_{i=1}^n$ which define E do not in general exist and it is thus not even possible to state condition (3.1.11). Our strategy for resolving this issue is to consider a sequence $\{\eta_i^k\}_{i=1}^n$ of C^1 differential forms, for which therefore the exterior derivatives $d\eta_i^k$'s do exist, which converge to $\{\eta_i\}_{i=1}^n$ and satisfy certain conditions which we define precisely below and which imply that the sequence is in some sense "asymptotically involutive" and which will allow us to deduce that E is integrable without having to define an involutivity condition directly on E. A quite interesting by-product of this approach is a clear distinction between the involutivity conditions required for integrability and the regularity conditions required for unique integrability. In the

C^1 case these regularity conditions are automatically satisfied and thus the involutivity condition (3.1.11) is sufficient to guarantee both integrability and unique integrability.

Definition 3.1.11. A continuous tangent sub-bundle E of rank n is *strongly asymptotically involutive* if there exist a basis $\{\eta_i\}_{i=1}^n$ of $\mathcal{A}^1(E)$, a constant $\epsilon_0 > 0$, and a sequence of C^1 differential 1-forms $\{\eta_i^k\}_{i=1}^n$ such that $\max_i |\eta_i^k - \eta_i|_\infty \to 0$ as $k \to \infty$ and

$$\max_j |\eta_1^k \wedge ... \eta_n^k \wedge d\eta_j^k|_\infty e^{\epsilon_0 \max_i |d\eta_i^k|_\infty} \to 0 \quad \text{as } k \to \infty. \tag{3.1.12}$$

Definition 3.1.12. A continuous tangent sub-bundle E of rank n is *strongly exterior regular* if there exist a basis $\{\beta_i\}_{i=1}^n$ of $\mathcal{A}^1(E)$, a constant $\epsilon_1 > 0$, and a sequence of C^1 differential 1-forms $\{\beta_i^k\}_{i=1}^n$ such that

$$\max_j |\beta_j^k - \beta_j|_\infty e^{\epsilon_1 \max_i |d\beta_i^k|_\infty} \to 0 \quad \text{as } k \to \infty. \tag{3.1.13}$$

We note that we refer to these conditions as "strong" since we will define some more general versions below.

Theorem 8. *Let E be a continuous tangent subbundle. If E is strongly asymptotically involutive then it is integrable. If E is integrable and strongly exterior regular then it is uniquely integrable.*

Notice that if E is C^1, the strong exterior regularity is trivially satisfied by choosing $\beta_i^k = \eta_i$ and the Frobenius involutivity condition (3.1.11) is equivalent to the strong asymptotic involutivity condition (3.1.12) by choosing $\eta_i^k = \eta_i$. We remark that a version of Theorem 8 in dimension ≤ 3 was obtained in [39] by using different arguments.

Remark 3.1.13. One can also combine the asymptotic involutivity condition of Theorem 8, which gives integrability, with the condition on the modulus of continuity of E in Theorem 7 which then gives uniqueness (indeed, we will show below that the condition of Theorem 7 implies exterior regularity) as this last condition may be easier to check in some situations. As we discuss in more details below, conditions such as those of asymptotic involutivity and exterior regularity, which are based on a sequence of approximations, are actually quite natural. It would also be interesting however to know whether there is any way to formulate the existence conditions without recourse to approximations, directly in terms of properties of the bundle E (or $\mathcal{A}^1(E)$ to be more precise).

Question 3.1.14. *Can the strong asymptotic involutivity condition in Theorem 8 be replaced by a condition that can be stated only in terms of geometric and analytic properties of the bundle E rather than a sequence of approximations?*

An answer to this question would be a natural form of Peano's Theorem in higher dimensions.

3.1.5 The Main Theorem

In this section we state our most general theorem, which contains Theorem 8 as a special case and also implies all the other results stated above. This more general result will be important for applications to the tangent bundles which arise in the context of Dynamical Systems.

Let $U \subset \mathbb{R}^{m+n}$. Given two tangent bundles E^1 and E^2 on U, for $x \in U$, we denote by $\angle(E^1_x, E^2_x)$ the maximum angle between all possible rays $R_1 \subset E^1, R_2 \subset E^2$ orthogonal to $E^1_x \cap E^2_x$ (with respect to the induced point-wise metric at x). A sequence of bundles $\{E^k\}$ is said to converge to E^1 in angle if $\sup_{x\in U} \angle(E^1_x, E^k_x) \to 0$ as $k \to \infty$. This also means that the Haussdorff distance between the unit spheres inside E^k_x and E^1_x goes to zero for all x.

Now assume we are given E a continuous tangent bundle of rank n on U. We choose a coordinate system $(x^1, ..., x^m, y^1, ..., y^n)$ in U so that the y^i coordinates are transverse to E and if E^k is any sequence of bundles of rank n which converge in angle to E, then we can assume without loss of generality that they are also transverse to the y^i coordinates. We denote the subspace

$$\mathcal{Y}_p := \text{span}\left\{\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\right\}|_p.$$

By the transversality condition, we can span E by vectors of the form

$$X_i = \frac{\partial}{\partial x^i} + \sum_{j=1}^{n} a_{ij}(x,y)\frac{\partial}{\partial y^j}$$

for some C^0 functions $a_{ij}(x, y)$, and if E^k is a sequence of C^1 bundles converging to E then each E^k can be spanned by vectors of the form

$$X^k_i = \frac{\partial}{\partial x^i} + \sum_{j=1}^{n} a^k_{ij}(x,y)\frac{\partial}{\partial y^j}$$

for some sequence of C^1 functions $a^k_{ij}(x, y)$ so that X^k_i converges to X_i as $k \to \infty$.

Note that a basis of sections $\{\alpha_1^k, ..., \alpha_n^k\}$ of $\mathcal{A}^1(E^k)$ defined on U, gives a non-vanishing section of the frame bundle $F(\mathcal{A}^1(E^k))$ of $\mathcal{A}^1(E^k)$. We denote this section by A^k, which in local coordinates is the matrix of 1-forms whose j^{th} row is α_j^k. More explicitly if evaluated at a point p it is the map $A_p^k : \mathbb{R}^{m+n} \to \mathbb{R}^n$ defined by

$$A_p^k(v) := (\alpha_1^k(v), ..., \alpha_n^k(v))|_p.$$

Sometimes if we evaluate it along a curve γ then we denote $A_s^k = A_{\gamma(s)}^k$. By our assumptions A_p^k has rank n (since it has n rows made from a linearly independent set of 1-forms) and $ker(A_p^k) = E_p^k$. Therefore restricted to $\mathcal{Y}_p$ which is transverse to E_p^k these maps are invertible and we write

$$A_p^{-k} := (A^k|_{\mathcal{Y}_p})^{-1}.$$

In the statement and proof of the theorem, we will use a sequence of open covers $\{U^{k,i}\}_{i=1}^{s_k}$ of U associated to sequence E^k of approximating bundles and a corresponding sequence of sections $\{A^{k,i}\}_{i=1}^{s_k}$ defined on the elements of these covers. We will use the notation $A_p^{-k,j} := (A^{k,j}|_{\mathcal{Y}_p})^{-1}$. Since the elements of these covers overlap we will need the following compatibility condition.

Definition 3.1.15. A finite open cover $\{U^{k,i}\}_{i=1}^{s_k}$ of U is a *compatible cover* for non-vanishing sections $\{A^{k,i}\}_{i=1}^{s_k}$ of the frame bundle $F(\mathcal{A}^1(E^k))$ defined on $U^{k,i}$ if for all $i, j = 1, ..., s_k$, $p \in U_i^k \cap U_j^k$ we have

$$||A_p^{k,i} \circ A_p^{-k,j}|| = 1.$$

Given a compatible cover, we also define the maps $dA^{k,i} : \mathbb{R}^{2(n+m)} \to \mathbb{R}^n$ by

$$dA_p^{k,i}(u,v) = (d\alpha_1^{k,i}(u,v), ..., d\alpha_n^{k,i}(u,v))|_p$$

for $u, v \in \mathbb{R}^{n+m}$. We denote by $dA^{k,i}|_{E_p^k}$ the restriction of this map to $E_p^k \times E_p^k$. We also define the following constant depending on k and i

$$M_A^{k,i} := \sup_{\substack{v\in E, w\in\mathbb{R}^n \\ |v|=|w|=1 \\ p\in U^{k,i}}} |dA_p^{k,i}(A_p^{-k,i}w, v)|. \tag{3.1.14}$$

Definition 3.1.16. A continuous tangent subbundle E on $U \subset \mathbb{R}^{n+m}$ is *asymptotically involutive* if there is a sequence of C^1 subbundles E^k that converge to E, $\epsilon > 0$ and, for all k, a compatible open cover $\{U^{k,i}\}_{i=1}^{s_k}$ of U with non-vanishing sections $\{A^{k,i}\}_{i=1}^{s_k}$ of $F(\mathcal{A}^1(E^k))$ defined on $U^{k,i}$ such that

$$\max_{i,j,\ell\in\{1,...,s_k\}} ||dA^{k,i}|_{E^k}||_\infty \ ||A^{-k,j}||_\infty \ e^{\epsilon M_A^{k,\ell}} \to 0 \ \text{ as } k \to 0.$$

Definition 3.1.17. A continuous tangent subbundle E on $U \subset \mathbb{R}^{n+m}$ is *exterior regular* if there is a sequence of C^1 bundles E^k that converge to E, $\epsilon > 0$ and, for all k a compatible open cover $\{U^{k,i}\}_{i=1}^{s_k}$ of U and non-vanishing sections $B^{k,i}$ of $F(\mathcal{A}^1(E^k))$ defined on $U^{k,i}$ such that

$$\max_{i,j,\ell\in\{1,...,s_k\}} \|B^{k,i}|_E\|_\infty \ ||B^{-k,j}||_\infty \ e^{\epsilon M_B^{k,\ell}} \to 0 \text{ as } k \ \to 0.$$

Main Theorem. *Let E be a continuous tangent subbundle. If E is asymptotically involutive then E is integrable. If E is integrable and exterior regular then it is uniquely integrable*

Remark 3.1.18. The proof of Theorem 8 consists of verifying that the strong asymptotic involutivity and strong exterior regularity conditions are simply special cases of their more general versions given here. There are two main differences which make the general versions more general, and more applicable, than the strong versions. The first is that in the general versions of asymptotic involutivity and exterior regularity the forms defining the sub-bundles are only defined locally. The second, more important, difference is that in the the strong versions, the differential forms $\{\eta_1^k, ..., \eta_n^k\}$ are assumed to converge to a set of linearly independent forms, whereas this is not required by the general version. Indeed, multiplying a form by a constant or even by a function, does not change its kernel and thus does not change the bundle that it defines, and what one really needs is the convergence of a sequence of approximating bundles not necessarily the forms defining these bundles. Thus assuming the convergence of the forms, while allowing for a tidier formulation of the conditions, is an unnecessary restriction. This more general formulation allows us in particular to obtain an application to dynamical systems, including the well known Stable Manifold Theorem, which would not follow from Theorem 8.

3.1.6 Stable Manifold Theorem

A rich supply of continuous, integrable and non-integrable distributions come from dynamical systems where some dynamically defined tangent bundles occur naturally. The integrability (or not) of these subbundles has implications for the study of statistical and topological properties of such systems [27, 28] and there is a rich literature going back to Hadamard and Perron [25, 45, 46] concerning techniques for studying the problem, see also [14, 33, 35, 48] for classical results going back to the 1970's and [12, 28, 49, 53] for an overview of recent approaches. We give here a fairly general class of dynamical systems, which in particular includes classical uniform hyperbolic systems and certain partially hyperbolic systems, for which the assumptions of the Main Theorem can be readily verified. This gives a unification of many results, which have so far been proved by a variety of techniques, as a direct corollary of a single abstract Frobenius type integrability result.

Throughout this section M denotes an $(n+m)$-dimensional compact manifold and $\phi : M \to M$ denotes a C^2 diffeomorphism. The diffeomorphism ϕ is said to admit a *dominated splitting* if there exists a $D\phi$-invariant continuous decomposition $E \oplus F$ of TM such that

$$\sup_{x\in M} \|D\phi_x|_{E_x}\| < \inf_{x\in M} m(D\phi_x|_{F_x}). \tag{3.1.15}$$

Here $m(\cdot)$ denotes the conorm of an operator, that is $m(D\phi|_F(x)) = \inf_{v\in F(x)} \frac{|D\phi v|}{|v|}$.

Note that (3.1.15) is a purely dynamical condition and there is no a priori reason why such condition, or any other similar dynamical condition, should have a bearing on the question of integrability. However, remarkably, stronger domination conditions such as uniform hyperbolicity, where $\|D\phi|_E\| < 1 < m(D\phi|_F)$, do imply integrability of both subbundles [33], though there are counterexamples which show that weaker dominated splittings as in (3.1.15) do not [62, 65] and also that systems with dominated splitting may be integrable but not uniquely [54]. We give here a sufficient condition for unique integrability for a class of systems with dominated splitting which contains the uniformly hyperbolic diffeomorphisms but significantly relaxes the contraction of the subbundle E to allow for neutral or mildly expanding behavior (including, for example, the time one map of uniformly hyperbolic flows).

Definition 3.1.19. E is called at most *linearly growing* for ϕ if there exists constants C, D such that $|D\phi^k|_{E(x)}| \leq kC + D$ for all $x \in M$ and $k \geq 0$.

Theorem 9. *Let $\phi : M \to M$ be a C^2 diffeomorphism with an invariant dominated splitting $E \oplus F$. If E is at most linearly growing then E is uniquely integrable.*

A particular case of diffeomorphisms with dominated splitting are *partially hyperbolic systems*, which have a $D\phi$-invariant splitting $E^s \oplus E^c \oplus E^u$ where

$$\|D\phi|_{E^s}\| < 1 < m(D\phi|_{E^u}) \text{ and } \|D\phi|_{E^s}\| < m(D\phi|_{E^c}) \leq \|D\phi|_{E^c}\| < m(D\phi|_{E^u}).$$

Corollary 3.1.20. *Let $\phi : M \to M$ be a C^2 partially hyperbolic diffeomorphism then if E^c grows at most linearly for ϕ and ϕ^{-1} then it is uniquely integrable.*

Corollary 3.1.20 generalizes a result in [11] that gives unique integrability for E^c under the stronger assumption that ϕ is *center-isometric*, i.e $\|D\phi v\| = \|v\|$ for every $v \in E^c$. Note that partially hyperbolic systems are special cases of dominated splitting in (3.1.15) where $E = E^s \oplus E^c$ and $F = E^u$ or $E = E^s$ and $F = E^c \oplus E^u$. Therefore Corollary 3.1.20 is a direct application of Theorem 9, by showing that both $E^s \oplus E^c$ and $E^c \oplus E^u$ are uniquely integrable.

3.1.7 Philosophy and overview of the paper

Our main result is the Main Theorem, whose proof will occupy Sections 3.2 and 3.3, and all other results are, directly or indirectly, corollaries of the Main Theorem and will be proved in Section 3.4. In the Appendix we prove some basic lemmas required from analysis. The proof of the Main Theorem can be divided, as usual, into two parts: The existence of integral manifolds, which will be proved in Section 3.2, and the uniqueness, which will be proved in Section 3.3.

The key idea in the proof of existence is the following. Given a set of m linearly independent differentiable vector fields $X_1, ..., X_m$, there is a canonical way of constructing an m-dimensional manifold W by successive integration of the vector fields, see (3.2.3). In the case where the Frobenius involutivity (3.1.11) is satisfied, W can be shown to define an integral manifold of the span of $X_1, ..., X_m$, and this is indeed one possible strategy to prove Frobenius theorem. Our main idea is to give a quantitative estimate of how "non-integrable" the manifold W is in the general case in terms of certain quantities which come into our definition of asymptotic involutivity, see Proposition 3.2.1. We then apply Proposition 3.2.1 to our sequence E^k to get that the corresponding manifolds W^k are getting closer to being integral manifold and we show that the limit defines an integral manifold, see Section 3.2.3.

The proof of Proposition 3.2.1 relies on the crucial observation that the involutivity is essentially related to the pushforward of vector fields along flows. Indeed, one way to write the involutivity of a bundle E is that there is a choice of vector fields $X_1, ..., X_m$ that span E such that $[X_i, X_j] = 0$ or, equivalently, that the pushforward along the flow of X_i leaves X_j invariant, i.e

$$[X_i, X_j] = 0 \iff De^{tX_i} X_j = X_j$$

where e^{tX_i} denotes the flow of X_i. The quantitative measurement of non-integrability of E mentioned above is thus essentially given by the quantity $De^{tX_i} X_j - X_j$. This difference can further be expressed by the pushforward of the Lie bracket $[X_i, X_j]$ along the flow of X_i, see (3.2.7), which reduces the problem of that of estimating the norm of the pushforward.

The method by which we achieve this is perhaps the main technical innovation in the paper. Standard techniques give estimates of the form

$$||De_p^{tX}||_\infty \lesssim e^{t|X|_{C^1}}. \tag{3.1.16}$$

However this is not useful when X approximates a continuous vector field, as in our case, since the C^1 norm of X might blow-up. In certain settings, using the notation of differential forms, there is a better estimate by Hartman (see section 9 of chapter 5 in [29]), who gives

$$||De_p^{tX}|| \lesssim e^{t|d\eta|_\infty} \tag{3.1.17}$$

where $X \in \ker \eta$. It is easy to see that (3.1.16) is much weaker than (3.1.17). For example we consider the simple case where $X = \partial_x + b\partial_y$ and $\eta = dy - bdx$. In this case, $|X|_{C^1}$ involves both $|\frac{\partial b}{\partial x}|_\infty$ and $|\frac{\partial b}{\partial y}|_\infty$ whereas $|d\eta|_\infty$ involves only $|\frac{\partial b}{\partial y}|_\infty$ since $d\eta = \frac{\partial b}{\partial y} dx \wedge dy$. Another example is where $\eta = df$ for some C^1 function f and X

is any vector field in the kernel of η, in this case (3.1.17) is always satisfied while (3.1.16) may not even make sense since X may not be differentiable. In our case, see Proposition 3.2.2, we obtain an even weaker condition,

$$||De_p^{tX}||_\infty \lesssim e^{tM} \tag{3.1.18}$$

where M is $d\eta$ evaluated at two specific directions, one in $\ker(\eta)$ and the other in the transverse subspace of $\ker(\eta)$, see (3.1.14). In particular, the fact that $d\eta$ is evaluated at a vector in the kernel of η plays an important role in bounding the value of M in specific applications.

The bound (3.1.18) also comes into play in the proof of uniqueness under the exterior regularity condition. The key point of the proof is first of all to reduce the problem to that of uniqueness of solutions for ODE's, as we show below. To prove the uniqueness for ODE's we use an innovative argument based on Stoke's Theorem rather than the more standard approach based on Gronwall's inequality. To present a brief conceptual overview of the argument, we consider for simplicity a vector field X on a surface.

For smooth vector fields we can define a change of coordinates that straightens out the integral curves and we can define a differential 1-form α with $X \in ker(\alpha)$ and $d\alpha = 0$. Uniqueness of solutions is then an easy consequence of Stoke's Theorem: the integration of α along any closed curve is zero and so, by contradiction, if X is not uniquely integrable at a point there is a closed curve γ formed by two integral curves of X and a curve λ transversal to X (as in Figure 3.2). The integral of α

along γ is non-zero because $\alpha(X) = 0$ and only one piece, λ, of γ is transverse to X, and thus we get a contradiction. In the case of *continuous* vector fields we consider a sequence of smooth approximations X^k of X and corresponding differential 1-forms α^k (which do not necessarily have to converge to α). Integrating these forms α^k along the very same closed curve γ we cannot apply the exact same argument because we may have $\alpha^k(X) \neq 0$ but, using Equation (3.1.18), we can show that $|\alpha^k(X)| \to 0$ as $k \to \infty$ and we then show that this is sufficient to obtain uniqueness for X.

3.2 Existence of Integral Manifolds

In this section we are going to prove the existence of integral manifolds under the asymptotic involutivity, thus proving the first part of the Main Theorem. The general strategy is quite geometric and intuitive. We construct a sequence of local integral manifolds W^k and show that they converge to a manifold which is an integral manifold of the distribution E. The approximating manifolds W^k will be constructed in terms of the approximating C^1 distributions E^k but are of course in general not integral manifolds of these distributions since the E^k are not in general integrable. We can measure how far these manifolds are from being integral manifolds by comparing their tangent spaces to the distributions E^k and the key step in the proof will consist of relating this "distance" to the quantities involved in the definition of asymptotic involutivity in terms of the forms which define E^k

and their derivatives.

To emphasize the generality of our approach, we work first in the context of an arbitrary C^1 distribution. In section 3.2.1 we define a C^1 manifold $\mathcal{W}$ associated to this distribution, and state the key estimate in Proposition 3.2.1 which bounds the "non-integrability" of $\mathcal{W}$. We reduce the proof of Propositon 3.2.1 to that of a more technical Proposition 3.2.2 which uses the pushforward of vector-fields insides this distribution. In Subsection 3.2.2 we prove Proposition 3.2.2 and then in Subsection 3.2.3 we apply the estimates to our sequence of approximations.

3.2.1 Almost integral manifolds

Let Δ be a C^1 m-dimensional bundle on an open set $U \subset \mathbb{R}^{n+m}$ for $m, n \geq 1$. Fix a point $x_0 \in M$. We can choose a coordinate system $(x^1, ..., x^m, y^1, ..., y^n, U)$ centered at $x_0 \in U$ so that Δ is spanned by vector fields of the form

$$X_i := \frac{\partial}{\partial x^i} + \sum_{j=1}^{n} a_j^i \frac{\partial}{\partial y^j} \tag{3.2.1}$$

for some C^1 functions a_i^j for $i = 1, ..., m$. For later on use we also define

$$\mathcal{Y}_p := \text{span} \left\{ \frac{\partial}{\partial y^\ell}|_p, \ell = 1, ..., n \right\} \tag{3.2.2}$$

One of the most useful properties of such vector-fields is that $[X_i, X_j]_p \in \mathcal{Y}_p$ for all $i, j = 1, ..., m$ and $p \in U$. This property will be used repeatedly all through out the paper.

Since the vector fields X_i are C^1 in U, they are uniquely integrable and we let $e^{tX_i}(p)$ denote the flow associated to X_i starting at the point p. Then, for $\epsilon_1 > 0$ sufficiently small, we define the map $W : (-\epsilon_1, \epsilon_1)^m \to U$ by

$$W(t_1, ..., t_m) = e^{t_m X_m} \circ \cdots \circ e^{t_1 X_1}(x_0). \tag{3.2.3}$$

The set

$$\mathcal{W} := W((-\epsilon_1, \epsilon_1)^m)$$

is our candidate manifold that "integrates" the set of vector fields $\{X_i, i = 1, ..., m\}$. In general it is not an integral manifold of Δ.

Let $\{U_i\}_{i=1}^{\ell}$ be any open cover of U, $\eta_1^i, ..., \eta_n^i$ a basis of sections of $\mathcal{A}^1(\Delta)$ on U_i and let A^i be the section of the $F(\mathcal{A}^1(\Delta))$ on U_i formed by these sections. We adopt all the notations given in section 3.1.5 for these objects (but we drop the index k). We also denote by $A_p^{-1,i}$ the inverse of A_p^i restricted to $\mathcal{Y}_p$

Proposition 3.2.1. *For every* $t = (t_1, ..., t_m) \in (-\epsilon_1, \epsilon_1)^m$ *and* $i = 1, ..., m$ *we have*

$$\left| \frac{\partial W}{\partial t_i}(W(t)) - X_i(W(t)) \right| \leq m\epsilon_1 \sup_{r,s,j \in \{1,...,\ell\}} ||dA^r|_\Delta||_\infty ||A^{-1,s}||_\infty e^{m\epsilon_1 M_A^j}. \tag{3.2.4}$$

Notice that if the distribution Δ satisfies the usual Frobenius involutivity condition then $dA^r|_\Delta = 0$ for all r and then Proposition 3.2.1 implies that $\partial W / \partial t_i = X_i$ which implies that $\mathcal{W}$ is an integral manifold of Δ. In our setting, the distributions E^k are not involutive but the weak asymptotic involutivity condition implies that they are increasingly "almost involutive" and thus, by Proposition 3.2.1, "almost integrable". In Section 3.2.3 we will show that this implies that we can pass to the limit and obtain an integral manifold for our initial distribution E of the Main Theorem.

Proposition 3.2.1 follows from the next proposition which we prove in Section 3.2.2.

Proposition 3.2.2. *Let Δ be a C^1, rank m distribution on U, $X_1, ..., X_m$ a basis of Δ of the form* (3.2.1) *and $\mathcal{Y}$ the complementary distribution of the form* (3.2.2). *Let $\{U_i\}_{i=1}^{q}$ be an open cover of U, $\{\eta_1^i, ..., \eta_n^i\}$ basis of sections of $\mathcal{A}^1(\Delta)$ defined on U_i and A^i be the section of $F(\mathcal{A}^1(\Delta))$ on U_i formed by these differential 1-forms so that they form a compatible cover. Then for all $(t_1, ..., t_m) \in (-\epsilon_1, \epsilon_1)^m$ and $Y \in \mathcal{Y}$ we have*

$$|De^{t_m X_m} \circ ... \circ De^{t_1 X_1}_{x_0} Y| \leq \sup_{s \in \{1,...,\ell\}} |A^i_x(Y)|| |A^{-j}_{x_m}||e^{m\epsilon_1 M^s_A} \tag{3.2.5}$$

where $x_m = e^{t_m X_m} \circ ... \circ e^{t_1 X_1}(x_0)$ and i, j are such that $x_0 \in U_i$, $x_m \in U_j$.

Proof of Proposition 3.2.1 assuming Proposition 3.2.2. Observe first that by the chain rule, for $i = 1, ..., m$, we have

$$\frac{\partial W}{\partial t_i} = (De^{t_m X_m} \circ ... \circ De^{t_{i+1} X_{i+1}})X_i. \tag{3.2.6}$$

where $De^{t_i X_i}$ denotes the differential of the flow with respect to the spatial coordinates (to simplify the notation we omit the base points at which the derivatives are calculated because our estimates will be uniform in U and so the specific base points do not matter). By a relatively standard result on the calculus of vectors (see [1, Chapter 2]), for any two vector fields Z, X on U we have

$$(De^{tZ}X)(x) - X(x) = \int_0^t (De^{sZ}[X, Z])(x)ds. \tag{3.2.7}$$

Thus, for $t = (t_1, ..., t_m)$, using (3.2.6) and (3.2.7), we have

$$\begin{aligned}\frac{\partial W}{\partial t_i}(W(t)) - X_i(W(t)) &= (De^{t_m X_m} \circ ... \circ De^{t_{i+1}X_{i+1}})X_i - X_i \\ &= \sum_{j=i+1}^{m} De^{t_m X_m} \circ ... \circ De^{t_{j+1}X_{j+1}} \left(De^{t_j X_j} X_i - X_i\right) \\ &= \sum_{j=i+1}^{m} De^{t_m X_m} \circ ... \circ De^{t_{j+1}X_{j+1}} \int_0^{t_j} De^{sX_j}[X_i, X_j]ds \\ &= \sum_{j=i+1}^{m} \int_0^{t_j} De^{t_m X_m} \circ ... \circ De^{t_{j+1}X_{j+1}} De^{sX_j}[X_i, X_j]ds\end{aligned}$$

Then taking norms on both sides we get

$$\left|\frac{\partial W}{\partial t_i}(W(t)) - X_i(W(t))\right| \leq m\epsilon_1 \max_{\substack{(t_m,...,t_1)\in[-\epsilon_1,\epsilon_1]^m \\ s,r\in\{1,...,m\}}} |De^{t_m X_m^k} \circ ... \circ De^{t_1 X_1^k}[X_s, X_r]|_\infty \tag{3.2.8}$$

Note that by the choice of X_i, the brackets $[X_s, X_r]$ lie in $\mathcal{Y}$ so we can apply Proposition 3.2.2 with Y replaced by $[X_s, X_r]$ to get

$$|De^{t_m X_m} \circ ... \circ De_x^{t_1 X_1}[X_s, X_r]| \leq \sup_{s,r,j\in\{1,...,\ell\}} |A_x^i([X_s, X_r])| \; ||A^{-r}||_\infty e^{m\epsilon_1 M_A^j} \tag{3.2.9}$$

Then using Cartan's formula

$$|A_x^i([X_s, X_r])| = |dA_x^i(X_s, X_r)|$$

we get

$$|De^{t_m X_m} \circ ... \circ De_x^{t_1 X_1}[X_s, X_r]| \leq \sup_{s,r,j\in\{1,...,\ell\}} ||dA^s|_\Delta||_\infty ||A^{-r}||_\infty e^{m\epsilon_1 M_A^j}. \tag{3.2.10}$$

By inserting Equation (3.2.10) into Equation (3.2.8) we get Proposition 3.2.1. □

3.2.2 Proof of Proposition 3.2.2

Proposition 3.2.2 is the technical heart of the proof of existence of integral manifolds where we use the exterior derivative of the annihilator differential forms to control the push forwards of our vector-fields. We first define a non-autonomous flow which corresponds to flowing along a direction X_i and then switching to X_{i+1} and so on. Let $(t_1, ..., t_m) \in (-\epsilon_1, \epsilon_1)^m$ and $T_i = \sum_{\ell=1}^{i} |t_\ell|$. We define the non-autonomous piecewise smooth vector field

$$X_t := \sigma(t_i)X_i \quad \text{if} \quad T_i \leq t < T_{i+1}$$

where σ is the sign function. Its associated non-autonomous flow is denoted by $\phi(t)$. With this notation we have that for any $T_i < t < T_{i+1}$

$$\phi(t) = e^{\sigma(t_i)(t-T_i)X_i} \circ ... \circ e^{\sigma(t_1)T_1X_1}(x),$$

$$Y_t = D\phi(t)Y = De^{\mathrm{s}(t_i)(t-|t_i|)X_i} \circ ... \circ De_x^{t_1X_1}Y.$$

Let ϕ be the piecewise smooth curve which is the image of the map $\phi : [0, T_m] \to U$. Recall now that we had a cover $\{U_i\}_{i=1}^q$ of U. We can take the intersection of ϕ with these open sets and consider the connected components of these intersections, which are curves in ϕ, and which we denote by $\{I_j\}_{j=1}^u$. By shrinking and reindexing I_i we can assume that $I_{i+1} = \phi([s_i, s_{i+1}])$ with $s_0 = 0$, $s_u = T_m$, $s_i < s_{i+1}$ and that each I_i is inside one of the elements U_{ℓ_i} of the covering $\{U_i\}_{i=1}^q$. We let $\{A^{\ell_i}\}_{i=1}^u$ denote restrictions of the sections of $F(\mathcal{A}^1(\Delta))$ defined on the U_{ℓ_i}'s.

Lemma 3.2.3. *For every $i = 1, ..., u,\ s_i > s \geq s_{i-1}$ and $Y \in \mathcal{Y}$ we have*

$$A_s^{\ell_i}(Y_s) = A_{s_{i-1}}^{\ell_i}(Y_{s_{i-1}}) + \int_{s_{i-1}}^{s} dA^{\ell_i}(X_\tau, Y_\tau)(\phi(\tau))d\tau \tag{3.2.11}$$

Proof of Proposition 3.2.2 assuming Lemma 3.2.3. Equation 3.2.11 can be rewritten as

$$A_s^{\ell_i}(Y_s) = A_{s_{i-1}}^{\ell_i}(Y_{s_{i-1}}) + \int_{s_{i-1}}^{s} dA_\tau^{\ell_i}(X_\tau, A_\tau^{-1,\ell_i} \circ A_\tau^{\ell_i} Y_\tau)(\phi(\tau))d\tau \tag{3.2.12}$$

This tells us that $A_t^{\ell_i}(Y_t)$ is the solution of the ODE

$$\frac{dF}{dt} = dA_t^{\ell_i}(X_t, A_t^{-1,\ell_i} \circ F_t) \quad \text{for} \quad s_{i-1} < t < s_i$$

with initial condition

$$F(s_{i-1}) = A_{s_{i-1}}^{\ell_i}(Y_{s_{i-1}}).$$

This ODE is linear and piecewise C^1 in t and C^1 in other variables so has unique solutions. Let G_t^i be the fundamental matrix of this ODE which satisfies (see [29] for instance)

$$|G_t^i| \leq e^{|t-s_i|\ ||dA^{\ell_i}(X_t, A_t^{-1,\ell_i}\cdot)||_\infty} \leq e^{|s_{i+1}-s_i| M_A^{\ell_i}}. \tag{3.2.13}$$

Moreover

$$A_{s_i}^{\ell_i}(Y_s) = G_s \circ A_{s_{i-1}}^{\ell_i}(Y_{s_{i-1}}),$$

and so we have

$$Y_s = A_{s_i}^{-1,\ell_i} \circ G_s \circ A_{s_{i-1}}^{\ell_i}(Y_{s_{i-1}}). \tag{3.2.14}$$

So repeatedly applying (3.2.14) and using (3.2.13), we get

$$|Y_{s_u}| \leq ||A_{s_u}^{-1,\ell_u}||\ |A_{s_0}^{\ell_1}(Y)| \prod_{i=2}^{u-1} ||A_{s_i}^{\ell_i} \circ A_{s_i}^{-1,\ell_{i-1}}|| \prod_{i=1}^{u} ||G_{s_{i+1}}^i||. \tag{3.2.15}$$

But now, by assumption of compatible cover we get $||A_{s_i}^{\ell_i} \circ A_{s_i}^{-1,\ell_{i-1}}|| = 1$, and by (3.2.13) we get

$$\prod_{i=1}^{u} ||G_{s_{i+1}}^i|| \leq e^{m\epsilon_1 M^{\ell_i}}.$$

We remind the reader that $s_0 = 0$, $s_u = T_m$, $Y_0 = Y$, $Y_{T_m} = De^{t_m X_m} \circ ... \circ De_x^{t_1 X_m} Y$, and so from equation (3.2.15) we get

$$|De^{t_m X_m} \circ ... \circ De_x^{t_1 X_m} Y| \leq \sup_{s \in \{1,...,\ell\}} ||A_{x_m}^{-1,\ell_u}|| |A_x^{\ell_1}(Y)| e^{m\epsilon_1 M_A^s}$$

□

To prove Lemma 3.2.3, first note that

$$A_s^{\ell_i}(Y_s) = (\eta_1^{\ell_i}(Y_s), ..., \eta_n^{\ell_i}(Y_s)) \text{ and } dA_s^{\ell_i}(X_s, Y_s) = (d\eta_1^{\ell_i}(X_s, Y_s), ..., d\eta_n^{\ell_i}(X_s, Y_s)).$$

So it is sufficient to prove (3.2.11) for a fixed differential form $\eta_j^{\ell_i}$ defined on U_{ℓ_i}. For convenience in this part we will drop the index ℓ_i and denote the evaluation points as subscripts. Therefore we need to prove

$$\eta(Y_s)_{\phi(s)} = \eta(Y_{s_{i-1}})_{\phi(s_{i-1})} + \int_{s_{i-1}}^{s} d\eta(X_\tau, Y_\tau)(\phi(\tau))d\tau. \tag{3.2.16}$$

We will first consider the case when the flow $\phi(t)$ is obtained from a single vector field, that is $\phi(t) = e^{tX_i}(x)$. The general case will be deduced from this one.

Lemma 3.2.4. *For every $x \in U$, $Y \in \mathcal{Y}$ and $|t_i|$ small enough so that $x_i = e^{t_i X_i}(x) \in U$ we have*

$$\eta_j(De_x^{t_i X_i} Y)_{x_i} = \eta_j(Y)_x + \int_0^{t_i} d\eta_j(X_i, De_x^{sX_i} Y)_{e^{sX_i}(x)} ds$$

for all $i = 1, ..., m$, $j = 1, ..., n$.

Proof. Let γ be a curve defined on $[0,\tilde{t}]$ such that $\gamma(0) = x$, $\gamma'(0) = Y$ and $e^{t_iX_i}(\gamma) \subset U$. Note that X_i is always transverse to γ. Denote $y = \gamma(\tilde{t})$. Define the parameterized surface Γ by

$$r(s_1, s_2) = e^{s_2X_j} \circ \gamma(s_1)$$

for $0 < s_1 \leq \tilde{t}$ and $0 < s_2 \leq t_i$. Then the boundary of Γ is composed of the curve γ and the following piecewise smooth curves (see Figure 3.1):

$$\xi_1(s) = e^{sX_i}(x) \qquad \xi_2(s) = e^{sX_i}(y) \qquad \beta(s) = e^{t_iX_i} \circ \gamma(s).$$

Since $\eta_j(X_i) = 0$ for all i, j, using Stoke's theorem we have

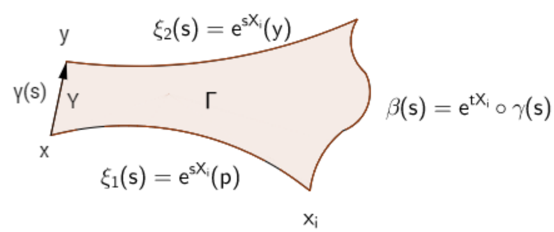

Figure 3.1: Applying Stoke's Theorem

$$\int_\beta \eta_j - \int_\gamma \eta_j = \int_\Gamma d\eta_j$$

which gives

$$\begin{aligned} \int_0^{\tilde{t}} \eta_j(\beta'(s_1))ds_1 = \int_0^{\tilde{t}} \eta_j(\gamma'(s_1))ds_1 + \\ \int_0^{\tilde{t}} \int_0^{t_i} d\eta_j(\frac{\partial r}{\partial s_1}, \frac{\partial r}{\partial s_2})(s_1, s_2)ds_1ds_2. \end{aligned} \tag{3.2.17}$$

Differentiating (3.2.17) with respect to $\tilde{t}$ at $\tilde{t} = 0$ we have

$$\eta_j(\beta'(0)) = \eta_j(\gamma'(0)) + \int_0^{t_i} d\eta_j(\frac{\partial r}{\partial s_1}, \frac{\partial r}{\partial s_2})(0, s_2) ds_2.$$

Using chain rule we have

$$\beta'(0) = De^{t_i X_i} Y|_x \qquad \gamma'(0) = Y$$

and

$$\frac{\partial r}{\partial s_1}(0, s_2) = De^{s_2 X_i} Y|_x \qquad \frac{\partial r}{\partial s_2}(0, s_2) = X_i(e^{s_2 X_i}(x))$$

one can write the equality (3.2.17) as

$$\eta_j(De_x^{t_i X_i} Y_{x_i}) = \eta_j(Y)_x + \int_0^{t_i} d\eta_j(De_x^{s_2 X_i} Y, X_{s_2})_{r(0,s_2)} ds_2$$

which concludes the proof of the lemma. $\square$

Our next step is to generalize Lemma 3.2.4 for the composition of differentials.

Lemma 3.2.5. *For every* $(t_1, ..., t_m) \in (-\epsilon_1, \epsilon_1)^m$*,* $Y \in \mathcal{Y}$ *and* $j = 1, ..., n$ *we have*

$$\eta_j(De^{t_m X_m} \circ \cdots \circ De^{t_1 X_1} Y)_{x_m} = \eta_j(Y)_{x_0} + \sum_{i=1}^{m} \int_0^{t_i} d\eta_j(X_i, De^{s X_i} \circ \cdots \circ De^{t_1 X_1} Y)(x_i(s)) ds$$

where $x_m = e^{t_m X_m} \circ \cdots \circ e^{t_1 X_1}(x_0)$ *and* $x_i(s) = e^{s X_i} \circ \cdots \circ e^{t_1 X_1}(x_0)$*.*

Lemma 3.2.5 will follow by successive applications of Lemma 3.2.4. However we need to check first that the pushforward of the flows leaves the $\mathcal{Y}$ subspace invariant.

Lemma 3.2.6. *For every $i = 1, ..., m$, $p \in U$, $|t_i| < \epsilon_1$ and $Y \in \mathcal{Y}$ we have*

$$De_p^{t_i X_i} Y \in \mathcal{Y}.$$

Proof. Let $p = (x_0^1, ..., x_0^m, y_0^1, ..., y_0^n)$. Recall that $X_i = \frac{\partial}{\partial x^i} + \sum_{j=1}^n a^{ij}(x,y)\frac{\partial}{\partial y^j}$ and so the flow of this vector field is $e^{tX_i}(x_0^1, ..., x_0^m, y_0^1, ..., y_0^n) = (x_0^1, ..., x_0^i + t, ..., x_0^m, y^1(t, x_0, y_0), ..., y^n(t, x_0, y_0))$ where $y^i(t,x,y)$ are functions C^1 in their variables. Since the differential has the form

$$D(e^{tX_i})|_{(x,y)} = \begin{pmatrix} Id_{m\times m} & 0_{m\times n} \\ A_{n\times m} & B_{n\times n} \end{pmatrix}$$

for some matrices A and B, the invariance of vectors in $\mathcal{Y}$ follows directly.

□

Proof of Lemma 3.2.5. Let $(t_1, ..., t_m) \in (-\epsilon_1, \epsilon_1)^m$ and $Y \in \mathcal{Y}$, we first carry out the proof for $t_1, t_2 \geq 0$, $t_j = 0$ for $j > 2$. First note that by Lemma 3.2.6 $De_p^{t_1 X_1} Y \in \mathcal{Y}$. So applying Lemma 3.2.4 twice, one has that

$$\eta_j(De^{t_2 X_2} \circ De^{t_1 X_1} Y)_{x_2} = \eta_j(Y)_{x_2} + \int_0^{t_1} d\eta_j(X_1, De^{sX_1}Y)(x_1(s))ds$$
$$+ \int_0^{t_2} d\eta_j(X_2, De^{sX_2} \circ De_p^{t_1 X_1} Y)(x_2(s))ds.$$

The general case follows in the same way, by applying Lemma 3.2.4 repeatedly. □

3.2.3 Convergence to Integral Manifolds

In this section we show that an asymptotic involutive bundle is integrable, thus proving the first part of the Main Theorem. We suppose that the asymptotic involutivity in Definition 3.1.16 is satisfied, in particular we have the sequences of differential forms $\{\eta_1^{k,i}, ..., \eta_n^{k,i}\}$ defined on open sets U_i^k of the covering of U and the sequence of bundles E^k defined on U which converges to the continuous bundle E. As before we can choose a coordinate system (x, y) independent of k where E^k and E are spanned by vector fields of the form (3.2.1) (though for E the vector fields are only continuous). We recall that $A_p^{-k,i}$ denotes the inverse of $A_p^{k,i}$ restricted to $\mathcal{Y}_p$. For $k > 1$, let W^k be the analogous of the map defined in (3.2.3) for Δ. By Proposition 3.2.1, for every $k > 1$ we have

$$\left|\frac{\partial W^k}{\partial t_i}(W^k(t)) - X_i^k(W(t))\right| \leq m\epsilon_1 sup_{s,r,j\in\{1,...,\ell\}}||dA^{k,s}|_\Delta||_\infty||A^{-k,r}||_\infty e^{m\epsilon_1 M_A^{k,j}}$$

Choosing ϵ_1 small enough so that $\epsilon_1 m \leq \epsilon$ and using asymptotic involutivity we have

$$\lim_{k\to 0}\left|\frac{\partial W^k}{\partial t_i}(W^k(t)) - X_i^k(W(t))\right| = 0.$$

In particular we have uniformly sized manifolds $\mathcal{W}^k$ whose tangent spaces converge to E in angle as $k \to \infty$. This is enough to show that these manifolds converge to some manifold $\mathcal{W}$ and that this is an integral manifold of E. This fact is quite intuitive but we give a proof of such a statement in a more abstract setting for completeness.

Proposition 3.2.7. *Let E be a continuous tangent subbundle of rank m defined on U and E^k be a C^1 approximation of E. Let $\mathcal{V}^k \subset U$ be a sequence of C^1 manifolds of dimension m and of uniform size with a point p in common. Assume that*

$$\lim_{k\to 0} \sup_{q\in\mathcal{V}^k} \angle(T_q\mathcal{V}^k, E_q^k) = 0 \tag{3.2.18}$$

Then there exists a subsequence of submanifolds $\mathcal{W}^k \subset \mathcal{V}^k$, which converges to an m-dimensional manifold $\mathcal{W}$, which is an integral manifold of E passing through p.

Proof. We choose coordinates $(x^1, ..., x^m, y^1, ..., y^n)$ so that $E_p = \text{span}\{\frac{\partial}{\partial x^i}\}_{i=1}^m$ and denote $\mathcal{Y} = \text{span}\{\frac{\partial}{\partial y^i}\}_{i=1}^n$. We shrink U if necessary so that each E_q is transverse to $\mathcal{Y}_q$ for all $q \in U$. Since E^k converges to E,

$$\lim_{k\to 0} \sup_{q\in\mathcal{V}^k} \angle(T_q\mathcal{V}^k, E_q) = 0.$$

Along with this, one has that $\mathcal{V}^k$ have uniform size so for k large enough we have submanifolds $\mathcal{W}^k \subset \mathcal{V}^k$ which can be written as graphs of functions $G^k : V \subset E_p \to U$. Thus we can write $\mathcal{W}^k$ as the images of the functions:

$$W^k(x^1, ..., x^m) = (x^1, ..., x^m, G^k(x^1, ..., x^m))$$

where

$$X_i^k = \frac{\partial W^k}{\partial x^i} = \frac{\partial}{\partial x^i} + \sum_{j=1}^m \frac{\partial G_j^k}{\partial x^i}\frac{\partial}{\partial y^j}$$

span the tangent space of $\mathcal{W}^k$. Note first that the differential DW^k has X_i^k as its columns which are linearly independent so $\mathcal{W}^k$ are C^1 embedded manifolds. Since the tangent space of $\mathcal{W}^k$ converges in angle to E, which is transverse to $\mathcal{Y}$, one has that $\sup_{k,i} |X_i^k|_\infty \leq C_1$ for some constant $C_1 > 0$. Since the differential DW^k is a matrix whose columns are X_i^k, we get that $\sup_{k,i} |DW_i^k|_\infty \leq C_2$ for some constant $C_2 > 0$. Therefore the sequence of functions W_i^k is equi-Lipschitz and equi-bounded and so up to choosing a subsequence, converges to a continuous function $W : V \to U$.

We are left to prove that $\mathcal{W} := W(V)$ is a C^1 m-dimensional manifold tangent to E. For this it is sufficient to prove that X_i^k's converges to some linearly independent X_i's that span E. This implies that DW^k is a matrix which converges to another matrix, say A whose columns are X_i. Thus we have that $p \in \mathcal{W}^k$ for all k, $W^k \to W$ and $DW^k \to A$. Therefore in fact $W : V \to U$ is a C^1 function whose derivative is a matrix whose columns are X_i. Therefore $\mathcal{W}$ is a C^1 manifold that is tangent to E and passes through p.

To prove the convergence of X_i^k's, we first observe that $\text{span}\{X_i^k\}_{i=1}^m$ at $q_k = W^k(x^1, ..., x^m)$ converges to E at $q = W(x^1, ..., x^m)$. Moreover E can be spanned by vector fields of the form

$$X_i = \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} H_{ij}(x, y) \frac{\partial}{\partial y^j}.$$

Since the tangent space of $\mathcal{W}^k$ converges to E, there exist vector fields Y_i^k inside the tangent space that converges to each X_i. But $Y_i^k = \sum_{j=1}^n a_{ij}^k X_j^k$ and by the form of X_i we see that $a_{ii}^k \to 1$ while $a_{ij}^k \to 0$ for $j \neq i$ as $k \to \infty$. And so in particular $|X_i^k - Y_i^k|_\infty \to 0$ as $k \to \infty$ which implies that X_i^k converges to X_i. □

3.3 Uniqueness of Local Integral Manifolds

In the previous section we have proven that asymptotic involutivity implies integrability of E. In this section we assume that E is integrable and show that if E is exterior regular then it is uniquely integrable which will then conclude the proof of the Main Theorem. We remind that uniquely integrable means the following: When ever two integral manifolds of E intersect, they intersect in a relatively open (in both) set. First we reduce the question of unique integrability to the following:

Proposition 3.3.1. *Assume E is integrable and exterior regular then E is spanned by a linearly independent set of vector fields X_i which are uniquely integrable.*

This immediately implies uniqueness for the integral manifolds.

Proof of Uniqueness part of Main Theorem assuming Proposition 3.3.1. Assume that there exist two integral manifolds $\mathcal{W}_1, \mathcal{W}_2$ of E such that $z \in \mathcal{W}_1 \cap \mathcal{W}_2$. By assumption we have that E is spanned by uniquely integrable vector fields X_i. Moreover since $\mathcal{W}_j$ for $j = 1, 2$ are integral manifolds of E, for any $q_j \in \mathcal{W}_j$ $X_i(q_j) \in T_{q_j}\mathcal{W}_j$ for all i and j. Now for ϵ small enough the m-dimensional surface $\mathcal{W} = \{e^{t_m X_m} \circ ... \circ e^{t_1 X_1}(z) : |t_i| \leq \epsilon\}$ is well defined. Moreover by unique integrability of X_i restricted to each $\mathcal{W}_j$, $\mathcal{W}$ is a subset of both surfaces. This means that the intersection of $\mathcal{W}_1$ with $\mathcal{W}_2$ is relatively open in both surfaces. □

To prove Proposition 3.3.1 we define quite explicitly the linearly independent vector fields X_i which span E and show that they are uniquely integrable. We first introduce some notation which we will need for the argument. Note that the assumption of exterior regularity means that there exists a sequence of approximations E^k of E defined on U and for each $k > 0$, there exists a covering $\{U_i^k\}_{i=1}^{n_k}$ of U, a basis of sections $\{\beta_j^{k,i}\}_{j=1}^{n}$ of $\mathcal{A}^1(E^k)$ defined on each U_i^k and the section $B^{k,i}$ of $F(\mathcal{A}^1(E^k))$ formed by these differential 1-forms. As in the previous sections we can find $\{X_i^k\}_{i=1}^m$ a basis of vector fields which span E^k such that they converge to $\{X_i\}_{i=1}^m$ which is a basis of vector fields for E such that they have the form

$$X_i^k = \frac{\partial}{\partial x^i} + \sum_{j=1}^{n} b_j^{i,k} \frac{\partial}{\partial y^j} \quad \text{and} \quad X_i = \frac{\partial}{\partial x^i} + \sum_{j=1}^{n} b_j^{i} \frac{\partial}{\partial y^j}.$$

Note that these approximations E^k and X_i^k may be different from the ones used in the previous sections. Let $i \in \{1, ..., m\}$ and $x_0 \in U$ and consider integral curves of X_i passing through $x_0 = (x_0^1, ..., x_0^m, y_0^1, ..., y_0^n)$. Due to the form of X_i, any integral curve $\gamma(t)$ can be written as

$$\gamma(t) = (x_0^1, ..., x_0^i + t, ..., x_0^m, y^1(t), ..., y^n(t))$$

where $y^j(t)$ are differentiable functions in t. In particular if an integral curve passes through the point x_0 then it necessarily always remains inside the $n+1$ dimensional plane $P_i = \{x^j = x_0^j \text{ for } j \neq i\}, x_0 \in P_i$ passing through x_0. Therefore it is sufficient just to prove uniqueness restricting to each such subspace. So given such an x_0 and P_i we restrict $X_i^k, X_i, \{\beta_j^{k,i}\}_{j=1}^n, U^{k,i}, B^{k,i}$ to $V_i = U \cap P_i$ with coordinates $(x^i, y^1, ..., y^n)$. For simplicity we will omit the index i. Note that $\{\beta_j^k\}_{j=1}^n$ are all non-vanishing and linearly independent and X_i^k is in the kernel of β_j^k's. Moreover B^k and X^k restricted to these subspaces still satisfy the exterior regularity conditions.

3.3.1 A Condition for Unique Integrability of X

We can now start the proof of the proposition. By contradiction, we assume X admits two integral curves $\gamma_1(t), \gamma_2(t) \subset U$ with $0 \leq t \leq t_1$ which intersect but whose intersection is not relatively open. Under this assumption, without loss of generality we can assume that $\gamma_1(0) = \gamma_2(0) = x_0$ for some $x_0 \in V_i$ and that $\gamma_1(t) \neq \gamma_2(t)$ for $0 < t \leq t_1$. Notice that for $0 \leq t \leq t_1$, $\gamma_1(t)$ and $\gamma_2(t)$ have the form

$$\gamma_1(t) = (t, y_1^1(t), ..., y_1^n(t)) \quad \text{and} \quad \gamma_2(t) = (t, y_2^1(t), ..., y_2^n(t))$$

and so in particular the end points $\gamma_1(t_1)$, $\gamma_2(t_1)$ have the same x coordinate. Therefore they can be connected to each other by a straight line segment of the form $\lambda(t) = \gamma_1(t_1) + vt$ which lies inside the plane $\mathcal{Y}$ that passes through $(t_1, 0, ..., 0)$. Here v is the unit vector in the direction $(0, y_2^1(t_1) - y_1^1(t_1), ..., y_2^n(t_1) - y_1^n(t_1))$. Let $\ell(\cdot)$ denote the length. We will show that $\ell(\lambda) > 0$ leads to a contradiction thus proving the proposition.

We first prove a lemma which gives sufficient conditions, in terms of the existence of a family of differential forms, to ensure $\ell(\lambda) = 0$. Then in the following sections we will show that such a family can be constructed.

Lemma 3.3.2. *Assume $\{\alpha^k\}_k$ is a sequence of C^1 differential forms defined on some domain $\tilde{V} \subset V$ containing the curves $\gamma_1, \gamma_2, \lambda$, a constant $c > 0$ such that for*

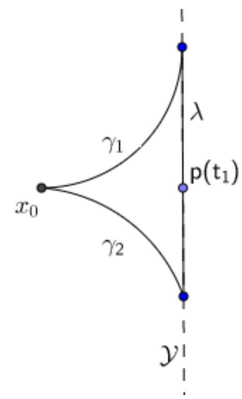

Figure 3.2:

every k:

$$1.\ \alpha^k(X^k) = 0 \tag{3.3.1}$$

$$2.\ d\alpha^k = 0 \tag{3.3.2}$$

$$3.\ \min_t \alpha^k(\dot{\lambda}(t)) \geq c \tag{3.3.3}$$

$$4.\ \lim_{k\to\infty} |\alpha^k(X^k - X)|_\infty = 0 \tag{3.3.4}$$

Then $\ell(\lambda) = 0$*.*

Proof. Let S be a surface in $\tilde{V}$ bounded by the curves $\gamma_1, \gamma_2, \lambda$ (whose union forms a simple, closed, piecewise smooth curve, see Figure 3.2). By Stoke's Theorem we have

$$\int_{\gamma_1} \alpha^k + \int_\lambda \alpha^k - \int_{\gamma_2} \alpha^k = \int_S d\alpha^k$$

and so

$$|\int_\lambda \alpha^k| \leq |\int_{\gamma_2} \alpha^k - \int_{\gamma_1} \alpha^k + \int_S d\alpha^k|.$$

Since $|\int_\lambda \alpha^k| \geq \min_t \alpha^k(\dot{\lambda}(t))\ell(\lambda)$ we can write this as

$$\min_t \alpha^k(\dot{\lambda}(t))\ell(\lambda) \leq |\int_{\gamma_2} \alpha^k| + |\int_{\gamma_1} \alpha^k| + |\int_S d\alpha^k| \tag{3.3.5}$$

Using Equations (3.3.2), (3.3.3) and (3.3.5) we have

$$\ell(\lambda) \leq \frac{1}{c}(|\int_{\gamma_2} \alpha^k| + |\int_{\gamma_1} \alpha^k|) \tag{3.3.6}$$

and using Equation (3.3.1) and $\dot{\gamma}_1(t) = X(\gamma_1(t))$ we get

$$|\int_{\gamma_1} \alpha^k| = |\int_0^{t_1} \alpha^k(X)(\gamma_1(s))ds| = |\int_0^{t_1} \alpha^k(X^k - X)(\gamma_1(s))ds|$$

which implies

$$|\int_{\gamma_1} \alpha^k| \leq 2t_1|\alpha^k(X^k - X)|_\infty. \tag{3.3.7}$$

The same applies to γ_2. Then plugging Equation (3.3.7) into (3.3.6), we get that for all k

$$\ell(\lambda) \leq \frac{1}{c}|\alpha^k(X^k - X)|_\infty$$

which, due to (3.3.4), goes to 0 as k goes to ∞. □

3.3.2 Definition of α^k

To construct α^k satisfying conditions of Lemma 3.3.2, we are going to define a change of coordinates that straightens flow of each X^k. Since the image of the flow is a straight lines of the form $\frac{\partial}{\partial t}$, then it will also be nullified by constant differential forms of the form dz^j. Pulling back these constant differential forms will give us the required differential forms α^k(see Figure 3.5). Now we make this more precise.

Let $\epsilon_1 > 0$ be small enough such that the box $U_1 = (-\epsilon_1, \epsilon_1)^{n+1}$ centered at 0 is in V. Let $\epsilon_2 < \epsilon_1$ be small enough so that for $U_2 = (-\epsilon_2, \epsilon_2)^{n+1}$, $e^{tX^k}(U_2) \subset U_1$ for all $|t| \leq \epsilon_2$. We decrease t_1 if necessary so that γ_ℓ for $\ell = 1, 2$ are in U_2 and $t_1 < \epsilon_2$ and denote the line $p(t) = (t, 0, 0, ..., 0)$. Given some $t \leq \epsilon_2$, we denote the subspaces:

$$P_t = \{x = t\} \cap U_2 \simeq [\epsilon_2, \epsilon_2]^n \quad \text{and} \quad D^k_{\epsilon_2} = \bigcup_{0 \leq t < \epsilon_2} e^{tX^k}(P_{t_1}).$$

The domains $D^k_{\epsilon_2}$ will be the domains on which we will define the forms α^k. The assumptions of of Lemma 3.3.2 require that they should contain the curves $\gamma_1, \gamma_2, \lambda$. This is proven in the next lemma.

Lemma 3.3.3. *There exists an open subset $\tilde{V} \subset \cap_{k=1}^{\infty} D^k_{\epsilon_2}$ such that for t_1 small enough, it contains the curves γ_1, γ_2 and λ.*

Proof. Fix $\delta \leq \epsilon_2/2$. Choose t_1 small enough so that for $\ell = 1, 2$,

$$d(\gamma_\ell(t), p(t)) = |(0, y^1_\ell(t), ..., y^n_\ell(t))| \leq \delta$$

for all $0 \leq t \leq t_1$ and that $e^{sX^k}(P_{t_1}) \cap V_2$ contains a box $[-\frac{\epsilon_2}{2}, \frac{\epsilon_2}{2}]^n \subset P_{t_1+s}$ centered at $p(t_1 + s)$ for all $|s| \leq t_1$. These conditions are possible to obtain since the X^k are uniformly bounded in norm which guarantees that λ and $\gamma_\ell([0, t_1])$ are in $D^k_{\epsilon_2}$ and that $D^k_{\epsilon_2}$ all contain a uniformly sized box centered at the axis $y = 0$ (see Figure 3.3). □

Note that each P_t is a codimension one subspace of U_2. Since P_{t_1} is transverse to X^k, $D^k_{\epsilon_2}$ is an $n+1$ dimensional open subset of U_1. Let $\phi : [-\epsilon_2, \epsilon_2]^n \to P_{t_1}$ be a

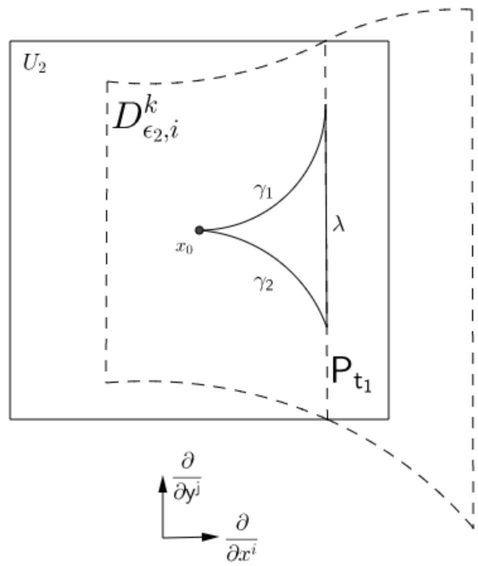

Figure 3.3: The domains

parametrization of P_{t_1} with coordinate representation $\phi(z^1, ..., z^n)$. We can assume that $D\phi\frac{\partial}{\partial z^i} = \frac{\partial}{\partial y^i}$ for all $i = 1, ..., n$. Then we define the change of coordinates :

$$\psi_k : [-\epsilon_2, \epsilon_2]^{n+1} \to D^k_{\epsilon_2} \subset U_1$$

by

$$\psi_k(t, z^1, ..., z^m) = e^{tX^k}(\phi(z^1, ..., z^m)).$$

This simply takes points of P_{t_1} and flows them by an amount equal to t.

Lemma 3.3.4. *The maps ψ_k are diffeomorphisms onto their image.*

Proof. We will show that these maps are diffeomorphisms by showing that they are local diffeomorphisms and that they are injective. To show that it is a local diffeomorphism, it is enough to show that the columns of $D\psi_k$ are everywhere linearly independent. One of the columns is:

$$\frac{\partial \psi_k}{\partial t} = X^k$$

while the others are of the form

$$\frac{\partial \psi_k}{\partial \tilde{y}^i} = De^{tX^k} \frac{\partial}{\partial y^i}.$$

By Lemma 3.2.6, for t small enough, De^{tX^k} preserves $\mathcal{Y}$ and $\mathcal{Y}$ is transverse to X^k. Therefore $D\psi^k$ is invertible at every point and therefore is a local diffeomorphism. So it remains to show it is injective. If it is not injective then there exists two integral curves ξ_1, ξ_2 that start at P_{t_1} and intersect at their final point. By uniqueness of solutions, this means that $\xi_1 \circ \xi_2^{-1}$ is an integral curve of X^k that starts at P_{t_1} and comes back to P_{t_1} (see Figure 3.4). This either means that the x component of $\xi_1 \circ \xi_2^{-1}$ first increases and then decreases or first decreases then increases. Neither is possible due to the form of the vector fields $X^k = \frac{\partial}{\partial x} + ...$ which implies that the x component is monotone. $\square$

Now we are ready to define α_k. First, for every $j = 1, ..., n$, let

$$\alpha_j^k = (\psi_k^{-1})^* dz^j.$$

In the next subsection, we will show that for some fixed i_0, the differential forms $\{\alpha_{i_0}^k\}$ are the required differential forms.

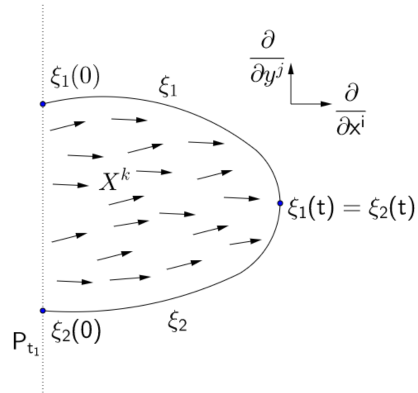

Figure 3.4: Injectivity

3.3.3 Choosing $\alpha^k_{i_0}$

Lemma 3.3.5. *For some choice of i_0, the differential 1-forms $\alpha^k_{i_0}$ satisfy the conditions given in Lemma 3.3.2.*

Proof. First of all

$$\alpha^k_j(X^k) = (\psi_k^{-1})^* dz^j(X^k) = dz^j(D\psi_k^{-1}X^k) = dz^j(\frac{\partial}{\partial t}) = 0$$

and

$$d\alpha^k_j = d(\psi_k^{-1})^* dz^j = (\psi_k^{-1})^* ddz^j = 0$$

which prove that conditions (3.3.1) and (3.3.2) hold for all j and k.

To check the remaining conditions, we need to calculate the inverse of ψ_k explicitly. One can by direct calculation check that the inverse of ψ_k is given by

$$\psi_k^{-1}(x^i, y^1, ..., y^m) = (x - t_1, \phi^{-1} \circ e^{-(x-t_1)X^k}(x, y^1, ..., y^n))$$

straight.png

Figure 3.5:

Therefore $\psi_k^{-1}(x, y^1, ..., y^m)$ is like flowing the y coordinates of $(x, y^1, ..., y^n)$ by the amount $-(x - t_1)$ and replacing the first coordinate by the amount of time required to get there from P_{t_1}. For simplicity we denote $T = \phi^{-1} : V_i \to [-\epsilon_2, \epsilon_2]^n$, $s = -(x - t_1)$ and $(x, y^1, ..., y^n) = (x, y)$.

To prove (3.3.3), note that $(\psi_k^{-1})^*$ in coordinates is the $(n+1) \times (n+1)$ matrix which transpose of the differential $D(\psi_k^{-1})$. One has

$$D(\psi_k^{-1})|_{(x,y)} = \begin{pmatrix} 0 & \cdot & & \cdot & 0 \\ & & & & \\ & & & & \\ & [DT \circ & De^{sX^k}(x,y)]_{n\times n+1} & & \\ & & & & \end{pmatrix} + \begin{pmatrix} 1 & 0 & \cdot & \cdot & 0 \\ \ulcorner\urcorner & 0 & 0 & \cdot & \cdot \\ & 0 & 0 & 0 & \cdot \\ -DT(X^k) & \cdot & \cdot & \cdot & \cdot \\ \llcorner\lrcorner & 0 & \cdot & \cdot & 0 \end{pmatrix} \tag{3.3.8}$$

For $x = t_1$ we have $s = 0$ and by the property $De^{0X^k} = Id$, we get

$$(\psi_k^{-1})^*|_{x=t_1}(dz^j) = dy^j - [DT(X^k)]_j dx^j. \tag{3.3.9}$$

We will now show that for some i_0 all $\alpha_{i_0}^k$ satisfy Equation (3.3.3) (at least up to changing some orientations), i.e we will show that for some i_0, we have $\alpha_{i_0}^k(\dot{\lambda})(\lambda(s)) > c > 0$ for all s. To show this, tote that the curve λ is of the form $\lambda(s) = sv$ for a fixed unit vector v that lies inside $\mathcal{Y}$ and therefore since $\dot{\lambda}(s) = v = \sum_{j=1}^n v^j \frac{\partial}{\partial y^j}$ and $\lambda \subset P_{t_1}$, we have by Equation (3.3.9) for all k

$$\alpha_j^k(\dot{\lambda}(s)) = (\psi_k^{-1})^*|_{x=t_1}(dy^j)(\dot{\lambda}(s)) = v^j.$$

Since $|v| = 1$, there exists a constant $c > 0$ and i_0 such that $|v^{i_0}| > c$. By reversing the orientation of the loop formed by $\gamma_1, \gamma_2, \lambda$ if necessary and therefore reversing the direction of λ, we can assume v^{i_0} is positive, that is

$$\alpha_{i_0}^k(\dot{\lambda}(s)) > c$$

for all k, which proves (3.3.3).

Finally to prove (3.3.4), we will relate the quantity $|\alpha^k(X^k - X)|_\infty$ to the quantity given in the definition of exterior regularity (see Definition 3.1.17), which goes to 0 by assumption. Note that $\alpha^k = (\psi_k^{-1})^* dz^{i_0}$. Therefore

$$\alpha^k(X^k - X) = dz^{i_0}(D\psi_k^{-1}(X^k - X))$$

But $X^k - X = \sum_{i=1}^n (b^{i,k} - b^i)\frac{\partial}{\partial y^i} \in \mathcal{Y}$ and looking at the form of $D\psi_k^{-1}$ given in equation (3.3.8) we see that

$$\|D\psi_k^{-1}|_{(x,y)}(X^k - X)\| = \|De^{sX^k}(X^k - X)\|$$

where $s = -(x - t_1)$. By Proposition 3.2.2, denoting $y = e^{sX^k}(x)$ we have

$$\begin{aligned} |De_x^{sX^k}(X^k - X)|_\infty &\leq \sup_{s,r,j \in \{1,\dots,s_k\}} |B^{k,s}(X^k - X)|_\infty ||B^{-k,r}||_\infty e^{m\epsilon_1 M_B^{k,j}} \\ &\leq \sup_{s,r,j \in \{1,\dots,s_k\}} \|B^{k,s}|_E\| \|B^{-k,r}\|_\infty e^{m\epsilon_1 M_B^{k,j}} \end{aligned}$$

where the last inequality is given by the fact that X^k annihilates $B^{k,i}$. This quantity goes to 0 by the exterior regularity condition which gives (3.3.4). □

3.4 Applications

In this section we will prove Theorems 5 to 9. Theorems 8 and 9 are direct applications of our Main Theorem and Theorem 8 implies Theorem 7 which implies Theorem 6 which implies Theorem 5.

3.4.1 Proof of Theorem 9

We first start by recalling some standard properties of dominated splittings, for details one can consult the book [49] and the article [38].

Let E^0 be a C^1 subbundle transverse to F, then by the domination of the splitting the sequence of subbundles $E^k := \phi_*^{-k} E^0 = D\phi^{-k} E^0$ converges in angle to E as $k \to \infty$. Now fix any $p \in M$ and a neighborhood U of p, we suppose that coordinates systems $(x^1, ..., x^m, y^1, ..., y^n)$ are defined in U and all the other notations in Section 3.1.5 are also adapted to this present Section relatively to the sequence of distributions $\{E^k, k \geq 1\}$ and its limit E.

Let $\{V_j\}_{j=1}^N$ be a cover of M by open balls such that for each $j \in \{1, ..., N\}$, $\mathcal{A}^1(E^0)$ admits an orthonormal frame C^j. Notice that for each $k > 1$, $\{\phi^{-k}(V_j)\}_{j=1}^l$ is an open cover of M and $\{C^{k,j} = (\phi^k)^* C^j\}_{j=1}^l$ is a frame of $\mathcal{A}^1(E^k)$ such that for $j = 1, ..., l$, $C^{k,j}$ is defined in $\phi^{-k}(V_j)$. Let $\{U^{k,i}\}_{i=1}^{n_k}$ be the open cover of U given by the connected components of $U \cap \phi^{-k}(V_j)$. Notice that for each $U^{k,i}$ there is a frame $A^{k,i}$ of $\mathcal{A}^1(E^k)$ which is the restriction of the relevant $C^{k,j}$.

We are going to check that the open cover $\{U^{k,i}\}_{i=1}^{n_k}$ and the corresponding sections $\{A^{k,i}\}_{i=1}^{n_k}$ satisfy asymptotic involutivity and exterior regularity. From the definition of the $C^{k,j}$'s we have that

$$A_p^{k,i} = C_{\phi^k(p)}^{\ell_i} \circ D\phi_p^k.$$

To check the compatibility of the cover we first observe that, by standard estimates for dominated splittings, we have that $\phi_p^k \mathcal{Y}_p$ is converging to F and so in particular

$\phi_p^k \mathcal{Y}_p$ is transverse to E^0. Then we can write

$$(A_p^{k,i}|_{\mathcal{Y}_p})^{-1} = (D\phi_p^k)^{-1} \circ (C^{\ell_i}_{\phi^k(p)}|_{\phi_*^k \mathcal{Y}_p})^{-1}$$

which implies

$$\|A_p^{k,i} \circ (A_p^{k,j}|_{\mathcal{Y}_p})^{-1}\| = \|C^{\ell_i}_{\phi^k(p)} \circ (C^{\ell_j}_{\phi^k(p)}|_{\phi_*^k \mathcal{Y}_p})^{-1}\|.$$

The compatibility follows from the fact that $C_1 := C^{\ell_i}$ and $C_2 := C^{\ell_j}|_{\phi_*^k \mathcal{Y}}$ are orthonormal and therefore $C_1, C_2 : (E^0)^\perp \to \mathbb{R}^n$ are isometries. Let $v \in \mathbb{R}^n$ and $u = C_2^{-1}(v)$ and we write $u = u_1 + u_2$ with $u_1 \in E^0$ and $u_2 \in (E^0)^\perp$. Then we have $C_1(u) = C_1(u_2)$ and $C_2(u_2) = v$ and using that $C_1|_{(E^0)^\perp}$ and $C_2|_{(E^0)^\perp}$ are isometries we have

$$\|C_1 \circ C_2^{-1}(v)\| = \|C_1(u)\| = \|C_1(u_2)\| = \|u_2\| = \|C_2(u_2)\| = \|v\|$$

which gives that $\|C_1 \circ C_2^{-1}\| = 1$ which then implies the compatibility.

Therefore it remains to prove asymptotic involutivity and exterior regularity. For both cases we need to estimate

$$\|(A_p^{k,i}|_{\mathcal{Y}_p})^{-1}\| = \frac{1}{\|A_p^{k,i}|_{\mathcal{Y}_p}\|}.$$

Since $\mathcal{Y}_p$ is transverse to E_p, again by standard estimates for dominated splittings, there exists a constant $C > 0$ such that for all $p \in M$ and for any $v \in \mathcal{Y}_p$ with $|v| = 1$

$$|D\phi_p^k v| \geq C m(D\phi^k|_{F_p}).$$

Therefore $||A_p^{k,i}|_{\mathcal{Y}_p}|| \geq Cm(D\phi^k|_{F_p})$ and so

$$\|(A_p^{k,i}|_{\mathcal{Y}_p})^{-1}\| \leq \frac{1}{Cm(D\phi^k|_{F_p})}. \tag{3.4.1}$$

Another common term for both asymptotic involutivity and exterior regularity is $M^{k,i}$ which we estimate as follows. For $X \in \mathbb{R}^n$, $Y \in E^k$ we have

$$|dA^{k,i}(A^{-k,i}X, Y)| = |dC^{\ell_i}(D\phi^k \circ D\phi^{-k} \circ C^{-1,i}X, D\phi^k Y)| = |dC^{\ell_i}(C^{-1,i}X, D\phi^k Y)|.$$

Notice that $\|dC^{l_i}\|$ is uniformly bounded then we can estimate the right hand side by the product of the norms of the two vectors. Since $\|C^{-1,i}X\|$ is also uniformly bounded we have

$$M^{k,i} = \sup_{X \in \mathbb{R}^n, Y \in E^k} |dA^{k,i}(A^{-k,i}X, Y)| \leq C\|D\phi^k|_{E^k}\|.$$

To estimate the last remaining term for the asymptotic involutivity, notice that if $X, Y \in E^k$ then

$$|dA^{k,i}(X, Y)| = |dC^{\ell_i}(D\phi^k X, D\phi^k Y)| \leq C\|D\phi^k|_{E^k}\|^2,$$

which implies that

$$\|dA^{k,i}|_{E^k}\| \leq C\|D\phi^k|_E\|^2.$$

So using these last three estimates we have

$$\|dA^{k,i}|_{E^k}\|_\infty \|A^{-k,j}\|_\infty e^{\epsilon M^{k,i}} \leq \frac{\|D\phi^k|_{E^k}\|^2}{m(D\phi^k|_F)} e^{\epsilon\|D\phi^k|_{E^k}\|}.$$

By the domination, there exists $r < 1$ such that for k large enough we have

$$\sup_{p \in M}\{\|D\phi_p^k|_{E_p^k}\|\} < r^k \inf_{p \in M}\{m(D\phi^k|_{F_p})\}.$$

Moreover by the definition of E^k, the linear growth assumption holds also for E^k, and therefore choosing ϵ small enough, the right hand side goes to zero as $k \to \infty$ and so the asymptotic involutivity is satisfied.

Similarly, to estimate the last remaining term of the exterior regularity condition, notice that we have

$$|A^{k,i}(X_i^k - X_i)| = |C^{\ell_i}(D\phi^k(X^k - X)| \leq \|D\phi^k X^k\| + \|D\phi^k X\| \leq 2\max\{\|D\phi^k|_{E^k}\|, \|D\phi^k|_E\|\}$$

which implies that

$$|A^{k,i}(X_i^k - X_i)|_\infty \|A^{-k,j}\| e^{kM^{k,\ell}}| \leq \frac{2\max\{\|D\phi^k|_{E^k}\|, \|D\phi^k|_E\|\}}{m(D\phi^k|_F)} e^{\epsilon\|D\Phi^k|_{E^k}\|}$$

which also goes to 0 for ϵ small enough, giving exterior regularity.

3.4.2 Proof of Theorem 8

The proof consists of checking that strong asymptotic involutivity and strong exterior regularity imply asymptotic involutivity and exterior regularity respectively, therefore Theorem 8 follows directly from the Main Theorem.

First of all, we replace the covering $\{U_i^k\}_{i=1}^{s_k}$ with the whole neighbourhood U so that $s_k \equiv 1$ and the compatibility condition is automatically satisfied. Then A^k simply becomes the matrix formed by the 1-forms $\{\eta_1^k, ..., \eta_n^k\}$ given by the strong asymptotic involutivity and B^k becomes the matrix formed by the 1-forms $\{\beta_1^k, ..., \beta_n^k\}$ given by the strong exterior regularity assumption. Moreover, by the strong version of asymptotic involutivity and exterior regularity, the sequences of 1-forms η_i^k and β_i^k converge to a basis η_i and β_i of $\mathcal{A}^1(E)$ and therefore $\|A^k\|_\infty, \|B^k\|_\infty, \|A^{-k}\|_\infty, \|B^{-k}\|$

are uniformly bounded in k. Then it is easy to check that $||dA^k||_\infty \leq C \max_i\{|d\eta_i^k|\}$, $\|dB^k\|_\infty \leq C \max_i\{|d\beta_i^k|\}$ for some constant $C > 0$ and for all k. Therefore from the definition of M^k(we omit the superscript ℓ since $s_k \equiv 1$) we have (using C as a generic constant)

$$M_A^k \leq \|dA^k\| \cdot \|A^{-k}\| \leq C \max_i\{|d\eta_i^k|\} \text{ and } M_B^k \leq \|dB^k\| \cdot \|B^{-k}\| \leq C \max_i\{|d\beta_i^k|\}.$$

Moreover, using the fact that $E^k \subset \ker(\eta_i)$ for all i, we have $\|d\eta_j^k|_{E^k}\| \leq C\|\eta_1^k \wedge ... \wedge d\eta_j^k\|$ which implies that $\|dA^k|_{E^k}\| \leq C \sup_j \|\eta_1^k \wedge ... \wedge d\eta_j^k\|$. Combining these observations, we have

$$\|dA^k|_{E^k}\| \cdot \|A^{-k}\| e^{\epsilon M_A^k} \leq C \sup_j \|\eta_1^k \wedge ... \wedge d\eta_j^k\| e^{C\epsilon \max_i\{|d\eta_i^k|\}}$$

which converges to zero by strong asymptotic involutivity.

For the exterior regularity, we observe that for all k, j we have

$$\|B^k|_E\| = \|(B^k - B)|_E\| \leq C \max_{j \in \{1,\dots,n\}} |\beta_j^k - \beta_j|_\infty$$

where the first equality is true because E annihilates B. Combining this with the bound on M_B^k, we get the exterior regularity.

and $\sup_{s \in \{1,\dots,m\}} |B^{k,i}(X_s^k - X_s)|_\infty \leq C \sup_{s \in \{1,\dots,n\}} |\beta_s^k - \beta_s|_\infty$ since the maximal angle between E^k and E is proportional to that between $\mathcal{A}^1(E^k)$ and $\mathcal{A}^1(E)$. Combining these observations one gets that conditions given in Theorem 8 imply those in the Main Theorem.

3.4.3 Proof of Theorem 7

We will prove Theorem 7 assuming Theorem 8. Recall that E has modulus of continuity $w_1(s)$ and has modulus of continuity $w_2(s)$ with respect to the variables y^ℓ, which means that it has some basis of sections $Z_i = \sum_{\ell=1}^m b_{i\ell}(x,y)\frac{\partial}{\partial x^i} + \sum_{j=1}^n c_{ij}(x,y)\frac{\partial}{\partial y^j}$ where $b_{i\ell}$ and c_{ij} have modulus of continuity $w_1(s)$ and have modulus of continuity $w_2(s)$ with respect to the variables y^ℓ. We can also find a basis of E of the form

$$X_i = \frac{\partial}{\partial x^i} + \sum_{j=1}^n a_{ij}(x,y)\frac{\partial}{\partial y^j}.$$

We claim that $a_{ij}(x,y)$ has modulus of continuity $w_1(s)$ and has modulus of continuity $w_2(s)$ with respect to the variables y^ℓ. To prove this claim, we write X_i as linear combinations of Z_i where the coefficients in the combinations are obtained from $b_{i\ell}$ and c_{ij} by summing, dividing (whenever non-zero) and multiplying. These coefficients are a_{ij} and by proposition 3.5.1 they have the same modulus of continuity properties as $b_{i\ell}$ and c_{ij}.

Now we are going to prove that each X_i is uniquely integrable. As in section 3.3, given X_i, it suffices to prove uniqueness restricted to the plane span$\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}\}$. Define the 1-forms

$$\eta_j = dy^j - a_{ij}(x,y)dx^i$$

so that $X_i \subset \cap_{j=1}^n ker(\eta_j)$. These 1-forms also have the modulus of continuity properties as above.

To prove that X_i is uniquely integrable, it is sufficient, by Theorem 8, to prove that η_j are exterior regular in this plane.

Lemma 3.4.1. *η_j are strongly exterior regular.*

Proof. Let $a^{\epsilon}_{ij}(x,y)$ be mollifications of a_{ij} (see (3.5.1) in the Appendix). Then define

$$\eta^{\epsilon}_j = dy^j - a^{\epsilon}_{ij}(x,y)dx \quad \text{and} \quad d\eta^{\epsilon}_j = -\sum_{i=1}^{n} \frac{\partial a^{\epsilon}_{ij}}{\partial y^i}(x,y)dy^i \wedge dx.$$

By proposition 3.5.2

$$|\frac{\partial a^{\epsilon}_{ij}}{\partial y^i}|_\infty \leq K \sup_{|s|\leq\epsilon} \frac{w_2(s)}{s} \quad \text{and} \quad |a^{\epsilon}_{ij} - a_{ij}|_\infty \leq K \sup_{|s|\leq\epsilon} w_1(s)$$

for all i, j and ϵ, for some $K > 0$. Therefore setting

$$\eta^k_j = dy^j - a^{\frac{1}{k}}_{ij}(x,y)dx$$

we have that

$$|\eta^k_j - \eta_j|_\infty e^{|d\eta^k_j|_\infty} \leq K \sup_{|s|\leq\epsilon} w_1(s) e^{w_2(s)/s}$$

which goes to 0, which gives the strong exterior regularity. □

This completes the proof of Theorem 7.

3.4.4 Proof of Theorem 6

Now we will prove that Theorem 7 implies Theorem 6. To apply Theorem 7, note that the problem of integrating a system of first order PDEs into a problem of integrating a bundle, i.e. solving the PDE (3.1.6) is equivalent to integrating the differential forms

$$\eta_i = dy^i - \sum_{j=1}^{n} F^{ij}(x,y)dx^j.$$

Consider the matrix $\hat{F}^I(\xi)$ (see section 3.1.2 and Theorem 6 for the definition) and its inverse $B(\xi)$. Let $E = \cap_{i=1}^{m} ker(\eta_i)$ so that $\mathcal{A}^1(E) = \text{span}\{\eta_i\}_{i=1}^{m}$. Let

$$\alpha_\ell = \sum_{j=1}^{m} B_{\ell j}(\xi)\eta_j$$

which forms a basis for $\mathcal{A}(1(E))$. Since B is the inverse of $\hat{F}^I(\xi)$ and $\hat{F}^I(\xi)$ corresponds to the columns $i_1, ..., i_n$ of the matrix $\hat{F}$, the new differential forms are of the form

$$\alpha_\ell = d\xi_{i_\ell} + \sum_{j \notin i_1,\ldots,i_n}^{n} g_{ij}(\xi)d\xi^j$$

Note that $g_{ij}(\xi)$ are $C^{w_1(s)}$ functions that have modulus of continuity $w_2(s)$ with respect to variables $\{\xi_i\}$ for $i \in \{i_1, ..., i_n\}$. Indeed the functions f_{ij} satisfy these properties and so do the entries of the matrix $\hat{F}^I(\xi)$. Therefore the matrix $B(\xi)$ whose entries are formed by taking quotients, products and sums of elements of $\hat{F}^I(\xi)$ have the same properties, by Proposition 3.5.1. Since g_{ij} are obtained by products and sums of entries from $B(\xi)$ and f_{ij} and 1's they also have the same property, again by Proposition 3.5.1. Defining the vector-fields for $\ell \notin i_1, ..., i_n$

$$X_\ell = \frac{\partial}{\partial \xi^\ell} + \sum_{j \in i_1,\ldots,i_n} g_{\ell j}(\xi)\frac{\partial}{\partial \xi^j}$$

we see that E is spanned by X_ℓ, and so is transverse to $\{\frac{\partial}{\partial \xi^j}\}$ for $j \in i_1, ..., i_n$. And in particular with respect to variables $\{\xi^j\}$ for $j \in \{i_1, ..., i_n\}$, E has modulus of continuity $w_2(s)$ and in general it has modulus of continuity $w_1(s)$ where $w_1(s), w_2(s)$ satisfy (3.1.3). And so by theorem 7, E is uniquely integrable.

This completes the proof of theorem 6.

Proof of proposition 3.1.7. Note that solving the PDE given in the proposition is equivalent to integrating the system of differential forms

$$\eta_i = dy^i - G_i(y^i) \sum_{j=1}^{m} \frac{\partial H_i}{\partial x^j}(x) dx^j.$$

We will prove that these differential forms satisfy strong asymptotic involutivity and strong exterior regularity.

Let G_i^ϵ and H_i^ϵ be mollifications of H_i and G_i. Then denote

$$\eta_i^k = dy^i - G_i^{1/k}(y^i) \sum_{j=1}^{m} \frac{\partial H_i^{1/k}}{\partial x^j}(x) dx^j.$$

Note that $|\eta_i^k - \eta_i|_\infty \to 0$. This is because by proposition 3.5.2, if a function H is C^1, and H^ϵ are its mollifications, then derivatives of H^ϵ converge to derivative of H. Denoting the differential form $\alpha_i^k = \sum_{j=1}^m \frac{\partial H_{ij}^{1/k}}{\partial x^j}(x)dx^j$, this system can be written as

$$\eta_i^k = dy^i - G_i^{1/k}(y^i)\alpha_i^k.$$

Let d_x be the exterior differentiation with respect to only x coordinates. Then $\alpha_i^k = d_x H_i^{1/k}$, so $d\alpha_i^k = d_x^2(H_i^{1/k}) = 0$. Therefore

$$d\eta_i^k = -\frac{\partial G_i^{1/k}}{\partial y^i}(y^i)dy^i \wedge \alpha_i^k.$$

Now lets show strong asymptotic involutivity:

$$\eta_1^k \wedge ... \wedge \eta_n^k \wedge d\eta_\ell^k = -\bigwedge_{i=1}^{n}(dy^i - G_i 1/k(y^i)\alpha_i^k) \wedge (\frac{\partial G_\ell^{1/k}}{\partial y^\ell}dy^\ell \wedge \alpha_\ell^k)$$

Note that $\alpha_\ell^k \wedge \alpha_\ell^k = 0$ therefore the expression above reduces to

$$= -dy^\ell \wedge \bigwedge_{i \neq \ell}(dy^i - G_i 1/k(y^i)\alpha_i^k) \wedge (\frac{\partial G_\ell^{1/k}}{\partial y^\ell}dy^\ell \wedge \alpha_\ell^k)$$

However the expression above now contains a term of the form

$$dy^\ell \wedge \frac{\partial G_\ell^{1/k}}{\partial y^\ell}dy^\ell$$

which is 0. therefore

$$\eta_1^k \wedge ... \wedge \eta_n^k \wedge d\eta_\ell^k = 0.$$

So by Theorem 8 this system is integrable. This proves Proposition 3.1.7. □

3.4.5 Proof of Theorem 5

Now we show how Theorem 6 implies Theorem 5. We want to show uniqueness of solutions for the ODE given in equation (3.1.1). We will prove Theorem 5 by showing that it is a special case of Theorem 6 with $m = 1$ (so that there is no j index for $\hat{F}$). First note that it can be written as the PDE:

$$\frac{\partial y^i}{\partial t} = F^i(t, y) \tag{3.4.2}$$

Then the matrix $\hat{F}(\xi)$ defined in Theorem 6, specialised to the case of Theorem 5, is the $n \times (n+1)$ matrix which is obtained by adjoining $n \times n$ identity matrix with the column vector formed from $F(\xi)$. We recall that $\tilde{F} = (F, 1)$. The condition $\tilde{F}^i(\xi) \neq 0$ with $i = 1, 2, ..., n+1$ in Theorem 5 is equivalent to the condition $\det(\hat{F}^I(\xi)) \neq 0$ with $I = (1, 2, ..., i-1, i+1, ..., n+1)$ in Theorem 6. And moreover the condition 3.1.7 given in Theorem 6 is equivalent to the condition 3.1.3 given in Theorem 5.

This finishes the proof of Theorem 5.

3.5 Appendix

In the appendix we prove or cite some properties of modulus of continuities and mollifications that we use in section 3.4.

3.5.1 Modulus of continuity

Proposition 3.5.1. *Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be function with modulus of continuity $w_f(s)$, $w_g(s)$. Let $K = \max\{\sup_s f(s), \sup_s g(s)\}$ and $c = \inf_s\{g(s)\}$. Then*

1. *Then $f + g$ has modulus of continuity $w_f(s) + w_g(s)$.*

2. *If* $K \leq \infty$, $f.g$ *has modulus of continuity* $w_f(s) + w_g(s)$

3. *If* $c > 0$ *then* $\frac{f}{g}$ *has modulus of continuity* $w_f(s) + w_g(s)$

4. *The function* $-xln(x)$ *has modulus of continuity* $-xln(x)$ *for* $0 < x < \frac{1}{e}$

5. *The function* x^α *for* $0 < \alpha < 1$ *has modulus of continuity* x^α.

6. *Assume* $f(x^1, ..., x^n)$ *has modulus of continuity* $w_i(s)$ *with respect to each variable. Let* $w(s)$ *be a bounded, convex function such that* $w_i(s) \leq w(s)$ *for all* s. *Then* $f(x)$ *has modulus of continuity* $w(\sqrt{2}s)$.

7. *Assume* $f = (f^1(x^1, ..., x^n), ..., f^m(x^1, ..., x^n))$ *is such that each* f^i *has modulus of continuity* w_{ij} *with respect to variable* x^j. *Let* $w_j(s)$ *be such that* $w_{ij}(s) \leq w_j(s)$ *for all* i. *Then* f *has modulus of continuity* $w_j(s)$ *with respect to variable* x^j.

Proof. For the first one

$$|f(x)+g(x)-f(y)-g(y)| \leq |f(x)-f(y)|+|g(x)-g(y)| \leq w_f(|x-y|)+w_g(|x-y|)$$

For the second one

$$|f(x)g(x) - f(y)g(y)| \leq |f(x)g(x) - f(y)g(x)| + |f(y)g(x) - f(y)g(y)|$$

$$\leq K(w_f(|x-y|) + w_g(|x-y|))$$

For the third one note that $\frac{f}{g} = f\frac{1}{g}$ and that

$$|\frac{1}{g}(x) - \frac{1}{g}(y)| = |\frac{g(y) - g(x)}{g(y)g(x)}| \leq \frac{1}{c^2}w(|x-y|)$$

For the fourth and the fifth one see [2]. For the sixth one consider the case $f = f(x^1, x^2)$,

$$|f(x^1,x^2) - f(y^1,y^2)| \leq |f(x^1,x^2) - f(y^1,x^2)| + |f(y^1,x^2) - f(y^1,y^2)|$$

$$\leq w_1(|x^1 - y^1|) + w_2(|x^2 - y^2|) \leq w(|x^1 - y^1| + |x^2 - y^2|)$$

Since $|x^1 - y^1| + |x^2 - y^2| \leq \sqrt{2}|(x^1,x^2) - (y^1,y^2)|$ we get

$$|f(x^1,x^2) - f(y^1,y^2)| \leq w(\sqrt(2)|(x^1,x^2) - (y^1,y^2)|)$$

The last one is also almost direct, indeed

$$|f(x^1,...,x^j+t,...,x^n) - f(x^1,...,x^j,...,x^n)|^2 = \sum_{i=1}^{m} |f^i(..x^j+t...) - f^i(...x^j...)|^2$$

$$\leq \sum_{i=1}^{m} K w_{ij}^2(|t|) \leq K' w_j^2(|t|).$$

□

3.5.2 Mollifications

In this section we investigate the modulus of continuity of the standard sequence of mollifications. We recall that, for a continuous function $f(x^1,...,x^n)$, the family of mollifiers of f, $\{f^\epsilon\}_{\epsilon>0}$ is defined by

$$f^\epsilon(x) = \int_{B(x,\epsilon)} \phi_\epsilon(y) f(x-y) dy \tag{3.5.1}$$

where $\phi_\epsilon(y) = \epsilon^{-n} e^{\epsilon^2/(|y-x|^2-\epsilon^2)}/I_n$ for $|y-x| < \epsilon$ and $I_n = \int_{B(x,\epsilon)} e^{\epsilon^2/(|y-x|^2-\epsilon^2)} dy$. The following properties of mollifications are similar to those used in [59] but are formulated here in terms of modulus of continuity rather than the Hölder norm.

Proposition 3.5.2. *Assume $f(x^1, ..., x^n)$ is a continuous function with modulus of continuity w and, for $j = 1, ..., n$, let w_j be the modulus of continuity of f with respect to the variable x^j. Then there exists a constants $K > 0$ such that for all $j = 1, ..., n$ and $\epsilon > 0$ we have*

$$|f^\epsilon - f|_\infty \leq \frac{K}{\epsilon^n}\int_{|s|\leq\epsilon} s^{n-1}w(s)ds \quad \text{and} \quad |\frac{\partial f^\epsilon}{\partial x^j}| \leq \frac{K}{\epsilon^{n+1}}\int_{|s|\leq\epsilon} s^{n-1}w_j(s)ds.$$

Proof. For the first inequality we have

$$|f^\epsilon(x) - f(x)| = \int_{B(0,\epsilon)} |\phi_\epsilon(x)|f(x-y) - f(x)|dy \leq |\phi_\epsilon|_\infty \int_{B(0,\epsilon)} w(|y|)dy.$$

To get the bound in the statement, we first observe that $|\phi_\epsilon| \leq \epsilon^{-n}$ and the bound of $\int_{B(0,\epsilon)} w(|y|)dy$ follows from a standard change of coordinates by passing to polar coordinates $(r, \theta_1, ..., \theta_{n-1})$ for which the volume form is $dV = r^{n-1}f(\theta)drd\theta_1...d\theta_{n-1}$.

For the second inequality, first note that for every $j = 1, ..., n$ we have

$$\int_{B(0,\epsilon)} \frac{\partial \phi_\epsilon}{\partial x^j} = 0 \tag{3.5.2}$$

and

$$\frac{\partial \phi_\epsilon}{\partial x^j}(x) = \frac{1}{\epsilon^{n+1}}\frac{\partial \phi}{\partial x^j}(\frac{x}{\epsilon}) \tag{3.5.3}$$

where $\phi(x) = e^{1/(|x|^2-1)}$ for $|x| < 1$. Moreover, letting $\hat{y}_i = (y_1, ..., y_{i-1}, 0, y_{i+1}, ..., y_n)$ we have

$$|\int_{B(0,\epsilon)} \frac{\partial \phi_\epsilon}{\partial x^j}(y)f(x - \hat{y}_i)dy| = 0 \tag{3.5.4}$$

as can be seen beywriting the integral as a multiple integral with respect to the coordinates and noticing that with respect to the i'th coordinate, the function

$f(x-\hat{y}_i)$ is a constant function. Therefore it comes out of the integral and multiplies $\int_{B(0,\epsilon)} \frac{\partial \phi_\epsilon}{\partial x^j}(y)dy$ which is equal to zero by (3.5.2). From (3.5.4), we can write

$$|\frac{\partial f^\epsilon}{\partial x^i}(x)| = |\int_{B(0,\epsilon)} \frac{\partial \phi_\epsilon}{\partial x^j}(y)f(x-y)dy| = |\int_{B(0,\epsilon)} \frac{\partial \phi_\epsilon}{\partial x^j}(y)(f(x-y) - f(x-\hat{y}_j)dy|$$

Bounding $\frac{\partial \phi_\epsilon}{\partial x^j}$ by $|D\phi_\epsilon|_\infty$ and using the modulus of continuity of f with respect to the j'th coordinate we have

$$|\frac{\partial f^\epsilon}{\partial x^i}(x)| \leq |D\phi_\epsilon|_\infty \int_{B(0,\epsilon)} w_j(|y|)dy \leq |D\phi|_\infty \frac{1}{\epsilon^{n+1}} \int_{|s|\leq\epsilon} s^{n-1} w_j(s)ds$$

where the last inequality is again achieved by passing to polar coordinates and using equation (3.5.3). This finishes the proof of the proposition. □

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