Extrinsic Geometry
And W - Gravities

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1. Introduction and Summary

The consistent quantum theory of gravity is, together with the quark confinement problem in the theory of strong interactions, one of the most important open problems in the contemporary theoretical physics. In the last few years great hopes were concentrated on the string theory, which is supposed to give complete, consistent unification of all interactions, including gravity. The string theory itself includes as an essential part the theory of two-dimensional gravity, as it is based on the requirement of diffeomorphism invariance on the worldsheet. Hence, the study of 2D quantum gravity is of two-fold interest, being unavoidable step in the development of the string theory and, on the other hand, presenting a theoretical laboratory for the realistic cosmology and gravity.

However, the conceptual and technical achievements of the recent years were most impressive in the theory of critical strings, while strings in non-critical dimensions appeared to be a rather though subject. Only recently there was a fast growth of interest in this direction. This is mainly due to the successes in the description of the non-perturabative 2D gravity based on the theory of random surfaces (or the equivalent matrix models) [1], the more deep understanding of the quantization of the Liouville theory [2, 3] and the recent construction of the physical spectrum of the non-critical strings [4].

The theory of the noncritical strings can be defined as a quantum theory of 2-D gravity interacting with matter fields [5, 6]. Which type of gravity we have to couple to the matter fields depends on our choice of the string action. The simplest area action $A = \int \sqrt{-h}\mathrm{d}^2x$ (i.e. minimal surfaces) in noncrtical dimensions $d < 26$ requires a consistent quantization of the "pure " 2-D gravity represented in conformal gauge by the Liouville action [7]. However if we define the string as a theory of surfaces immersed in $M_{1,p}$ (or more generally in $M_{p,q}$) we have to consider, following Polyakov's arguments [8], larger class of surfaces in $M_{1,p}$ which are described not only by the metric h_{ij} $(i,j = 1,2)$ but also by their second quadratic forms b^{α}_{ij} $(\alpha = 1,\cdots,p-1)$ and torsions $\nu_i^{\alpha\beta}$. In this case the string action is given by the geometric invariants characterizing the surface (i.e. certain determinats and traces of h_{ij}, b^{α}_{ij} and $\nu_i^{\alpha\beta}$). The gravitational degrees of freedom of such theories include together with the Liouville mode ($\varphi \leftrightarrow h_{ij}$) the "extrinsic modes" coming from the independent components of b^{α}_{ij} and $\nu_i^{\alpha\beta}$. Therefore the corresponding noncritical string model should be a theory of *"extended" 2-D gravity* interacting with the matter fields.

The induced action of two-dimensional gravity is well known and has the form:

$$\Gamma = \frac{d}{96\pi} \int_M \sqrt{g} \left(R \frac{1}{\Delta} R \right).$$

Here d^{-1} plays the role of a coupling constant, Δ is the Laplacian in the metric g_{ab}, R is the scalar curvature and M is the manifold in consideration. This action is naturally induced by massless particles and appears in the string functional integral [7].

The most simple form this formula takes is in the conformal gauge: $g_{ab} = e^{\phi}\delta_{ab}$, where it becomes a free field action. However, troubles appear in trying to quantize the theory in this gauge. We have to set a cut-off, such that it is competible with the general covariance. Generally, it is not known how to do this. In fact, trying to regularize the theory, we obtain a non-linear interaction term of the form:

$$\Gamma^{c}_{reg} = \frac{1}{\Lambda^2} \int \sqrt{g} R^2 \sim \frac{1}{\Lambda^2} \int (\partial\phi)^2 e^{\phi}.$$

For that reason it was proposed in [5] to quantize the theory in a light-cone gauge, defined by:

$$ds^2 = dx^+ dx^- + h_{++}(dx^+)^2.$$

A remarkable property of this gauge is the $SL(2,R)$- current algebra generated by h_{++}. This current algebra implements differential equations defining the correlation functions of the theory. These differential equations involve constant parameters wich are subject to finite renormalization. Unlike the conformal gauge, the quantization in the light-cone gauge does not spoil the general covariance. Indeed, one can add a regulator to the action:

$$\Gamma^{l.c.}_{reg} = \frac{1}{\Lambda^2} \int \sqrt{g} R^2 \sim \frac{1}{\Lambda^2} \int (\partial_-^2 h_{++})^2$$

with no non-linear terms. This term modifies the propagator of the h- field without touching the vertices and makes the theory convergent.

In two dimensions one can consider various extensions of gravity. The best known is the theory of 2D supergravity [9] which includes a spin-$\frac{3}{2}$ gravitino field as a superpartner of the graviton. There exist also higher-spin extensions of 2D gravity. These theories are based upon an algebra which is of W- type. In general, the latter are non-linear algebras: the OPE of two generators closes only on normal ordered products of the other W- generators. In other words, the closure of the algebra is achieved only in the enveloping algebra of the generators, thus, it is not a Lie algebra. Originally introduced [10] as a higher spin extension of the Virasoro algebra, it rapidly became clear that W- algebras are related to various other important structures of theoretical physics, such as cosets of affine Lie algebras [11], gauged WZW- models [12, 13], reductions of KP- hierarchy [14, 15], Toda field theories [16], integrable IRF lattice models [17] and, more recently, matrix models and random surfaces [18].

W- gravity can be thought of as the gauge theory of local W- algebra symmetries in the same sense as two-dimensional gravity can be thought as the result of gauging the Virasoro algebra. The gauge fields of such theory include the two dimensional metric g_{ij} and a (possibly infinite) number of higher-spin gauge fields $A_{ij\ldots k}$. Perhaps the most important reason to consider W- gravity is that the range of "weak gravity" is enlarged. Untill for the usual gravity it is in the region: $c \leq 1, \quad c \geq 25$, for W- gravity associated with the affine algebra $\hat{g}$ one obtains: $c \leq r, \quad c \geq r + 4hD$, where r is the rank, h- the dual Coxeter number and D- the dimension of the algebra g. Owing to this fact, one can even define the string theory in four-dimensional space time without compactification. The minimal W- gravity to treat such object corresponds to $sl(5)$ or $so(8)$. If the space-time interpretation of these current algebras is possible, the W- string can be regarded as an important candidate for realistic string theory.

Decpite recent progress in the understanding and the classification of 2D classical and quantum W- gravities [5, 6, 12, 19, 20], a deep question remains unanswered: what replaces the general covariance on the two-dimensional surface in the case of 2D gravity when it is extended to W- gravity ? In other words: what is W- geometry, or as a dynamical question: what is the invariant definition of W- gravity ? Although Witten's approach to CFT's based on 2+1 Chern-Simons actions clarified the geometry of WZW-models the question for the geometrical meaning of the higher spin currents ($s > 2$) W_{ijk} , $V_{ijkl},\ldots$, of the W-algebra models, [16, 21, 22], and its primary fields remains open. Concerning W-gravities, the Drinfeld-Sokolov Hamiltonian reduction, [12, 23, 24], provides them with natural phase space geometry and leaves unanswered the question about the origin of the new geometrical objects A_{ijk}, $B_{ijkl},\ldots$, coupled to the higher spin currents.

In this thesis we present an attempt to give an answer (presumably incomplete) to some of the above questions and discuss some recent developments on the subject. However, the full self-consistent theory of W- gravity is, at least to my knowledge, still an open problem.

This thesis is organized as follows:

In Chapter 2 we give an overview of the, already classic, results concerning two-dimensional quantum gravity. First, we discuss in some details the induced gauge theories. The reason is the analogy between both theories which is used in the proper treatment of 2D gravity. We obtain the induced action which results after the integration over the matter fields. After an appropriate parametrization of the gauge fields and currents it turns out to be equivalent to the WZW action. To prove this one uses the anomaly equation and the corresponding anomaly in the gauge transformation of the action. Then we discuss the dynamics of the gauge fields, more precisely some aspects of the renormalization of the theory. It is shown that the central charge of the corresponding current algebra does not renormalize.

The theory of 2D gravity is treated in a similar way. We choose to work in a light-cone gauge for the metric which is analogous to our choice in the gauge case. Unlike the latter, however, the anomaly equation and the anomaly transformation of the action contain third order derivatives of h_{++} which leads to different renormalization effects. The equation of motion for h_{++} is solved in terms of currents $J^{\pm,0}(x^+)$ which obey $SL(2,R)$ current algebra as a consequence of the transformation properties of h_{++}. The dynamical description of the theory implies that the total stress-energy tensor (and therefore the total central charge) should vanish. This leads to an exact formula for the renormalized central charge k of the current algebra wich in the classical limit $c \to -\infty$ reduces to $k_0 \sim \frac{c}{6}$. The underlying $SL(2,R)$ current algebra implements differential equations for the correlation functions of the theory. In the case of two-point functions these give the renormalized values of the anomalous dimensions of the primary fields interacting with gravity.

Chapter 3 is a review of the known results about W- gravity. The latter are obtained mainly by using the structure and the representations of the underlying W- algebras. Hence, we begin with brief discussion of their properties. The most natural formalism to treat this subject turns out to be the free-field (or Feigin-Fucks type) realization. The construction of the generators $W^{(k)}$ is rather complicated and we give here only the first few examples. Then we proceed with more or less standard description of the highest-weight representations, the role of the screening operators, null vectors and degenerate representations, operator product algebra and minimal models. We discuss also the possible $n \to \infty$ limit of W_n- algebras resulting in new linear algebra W_∞ or its contraction w_∞.

In the absence of covariant formulation the analysis of W- gravities is based on a natural ansatz for the gauge fields, their transformation properties and the analog of the light-cone gauge. We show that using these assumptions one can rather straightforwardly generalize the previous results for the anomaly equations, anomaly of the action and the equations of motion of the usual gravity. As a result we obtain that the hidden symmetry of W- gravity in the light-cone gauge is given by the corresponding affine Lie algebra $\hat{g}$. Again, the vanishing of the total central charge implies a general formula for the renormalized central charge of the current algebra. Using the Sugawara construction for the stress tensor we obtain also the anomalous dimensions of operators in the presence of W- gravity. At the end we discuss briefly the properties of w_∞- gravity (which can be thought of as a gauge theeory of w_∞- algebra). It is shown that in the process of cancellation of the anomalies the renormalized currents close no more w_∞ but just W_∞- algebra.

In Chapter 4 we present original results on the connection between W_n- gravities and the geometry of the affine surfaces of constant mean curvature immersed in higher-dimensional affine spaces A_n. First we give a brief introduction to the theory of affine curves in A_n and our proof that W_2 and W_3 minimal models have as a classical limit ($c \to -\infty$) the geometries of the affine curves in A_2 and A_3 respectively. It is based on the identification of the normalized "affine velocities"

$v^{\mu}_{(n)}$ with the classical limit of specific primary fields of W_n- models. Also, the classical limit of certain null vectors coincides with the affine Frenet equation for the curve imbedded in A_n. We propose also an affine geometrical derivation of the KdV equation.

The following section is devoted to the geometry of the affine surfaces immersed in A_3. It can be defined uniquely by its two fundamental forms: $\varphi = h_{ij}dx^i dx^j$ and $\psi = A_{ijk}dx^i dx^j dx^k$ satisfying the condition $h^{ij}A_{ijk} = 0$. In the last section we derive our main result, namely that W_3- gravity is equivalent to the affine geometry of the constant mean curvature affine surfaces in A_3. The (gauge-fixed) affine structure equations in light-cone gauge are invariant under W- transformations generated by h and A. As a consequence of the integrability conditions the "extended metrics" h and A satisfy the W- trace anomaly equations. Concerning W_n- gravities we conjecture that they are described by certain class of affine surfaces in A_n.

Chapter 5 presents an attempt to identify the noncritical string models represented by the geometry of surfaces of constant mean curvature and certain other restrictions with the theory of $WO(p,q)$- gravities. Our starting point are the structure equations for the moving frame fields t^{μ}_i and N^{μ}_{α}. Choosing an appropriate basis we fix part of the gauge symmetries. The remaining restricted $SO(p,q)$- gauge transformations are found by using a method proposed by Polyakov. We show that imposing the condition of constant mean curvature (and some further restrictions in the case of surfaces embedded in higher dimensional spaces) the latter close the classical extended $WO(p,q)$- algebra. The anomaly equations for the extended metrics are again a consequence of the integrability conditions. Their derivation in the case of surfaces embedded in $M_{3,3}$ is highly nontrivial and it confirms our conjecture for the general case. For $M_{2,2}$ we present also a discussion on how the self-intersection properties appear in the context of the extended $WO(2,2)$-gravity.

Appendix A can be considered as a guide to the Lie-Cartan theory of the geometric invariants which is one of the basic tools in the construction of the Klein geometries. In Appendix B we give the proof of the W_3- symmetry of the affine structure equations and the detailed derivation of the corresponding infinitesimal transformations. In Appendix C we derive, using the method of Polyakov, the restricted (field dependent) gauge transformation laws of the currents $T_{\pm}$in the case of surfaces embedded in $M_{2,2}$. We also derive the transformations of the "extended metrics" $\tilde{h}_{\pm}$. In Appendix D we derive systematically the anomaly equations from the Gauss-Codazzi equations for the specific surfaces embedded in $M_{3,3}$. It includes also the complete expressions for the transformation laws of the fields W and V of conformal spins 3 and 4 respectively.

2. Two Dimensional Quantum Gravity

In this Chapter we give a brief introduction to the the theory of two-dimensional quantum gravity. It is based mainly on the classic works of Polyakov [5] and Knizhnik, Polyakov and Zamolodchikov (KPZ) [6]. It was proposed there to quantize 2D gravity in a light-cone gauge for the metric which turns out to be much more convenient than the usual conformal gauge used in the string theory. Using a parametrization of the light-cone component h_{++} similar to the one used in the induced gauge theories, it is possible to write down a local covariant action which is the gravitational analog of the WZW- action. In particular it reveals the hidden $SL(2,R)$ current algebra symmetry of the theory. It is exploated below to give general expressions for the renormalized central charge of the current algebra and the anomalous dimensions of the primary fields in presence of gravitational interaction. Comparison with the results obtained in the random surface models shows perfect agreement between both theories.

2.1. Induced Gauge Theories

Let us consider first the induced gauge theories. The reason of doing this is that there are a lot of analogies between these and the induced gravity theory which is our goal in this chapter. One can also show, [12, 25], that it is possible to obtain the induced gravity action in chiral gauge from that of the induced gauge theory with gauge group $SL(2,R)$.

Consider the simplest example of fermions in the adjoint representation of the group G minimally coupled to external G-gauge fields:

$$\mathcal{L} = \psi_+^i(\partial_-\delta^{ij} + A_-^{[i,j]})\psi_+^j + (+ \leftrightarrow -), \tag{2.1}$$

($x^\pm$ are the coordinates of the 2D surface), or in a more compact form:

$$\mathcal{L} = \bar{\psi}\gamma_\alpha(\partial_\alpha + A_\alpha)\psi. \tag{2.2}$$

The induced action of the theory is obtained after integrating out the matter fermions and is given by the determinant of the Dirac operator in (2.2):

$$e^{i\Gamma_{ind}(A)} = \{Det[\gamma_\alpha(\partial_\alpha + A_\alpha)]\}^{1/2} = \int \mathcal{D}\psi e^{iS(\psi,A)}. \tag{2.3}$$

One can obtain the explicit form of $\Gamma_{ind}(A)$ computing the Feynman graphs and summing the perturbation series. The result is a non local non polynomial expression in A:

$$\begin{aligned}\Gamma_{ind}(A) &= \int d^2xTr\left(A\frac{\partial_+}{\partial_-}A + \frac{2}{3}A\left[\frac{1}{\partial_-}A, \frac{\partial_+}{\partial_-}A\right] + \dots\right)\\ &= \int d^2xTr\left(A\sum_{n\geq 0}\frac{1}{n+2}\left[\frac{1}{\partial_-}A, \cdot\right]^n\frac{\partial_+}{\partial_-}A\right).\end{aligned} \tag{2.4}$$

Polyakov and Wiegmann, [26], found a very elegant alternative formulation for $\Gamma(A)$. It is based on the so called anomaly equation:

$$\partial_- J_+ + \partial_+ A_- + [A_-, J_+] = 0, \tag{2.5}$$

(and analogous for A_+ and J_-), which can be easily proved by direct computation. Now, let us parametrize the gauge fields:

$$\begin{aligned}A_- &= -(\partial_- h)h^{-1}\\ A_+ &= -(\partial_+ g)g^{-1},\end{aligned} \tag{2.6}$$

where g and h are elements of the group G. For the currents J_α one obtains respectively:

$$\begin{aligned}J_+ &= (\partial_+ h)h^{-1}\\ J_- &= (\partial_- g)g^{-1},\end{aligned} \tag{2.7}$$

because the anomaly equation (2.5) states that the curvature of the gauge field with components (A_-, J_+) vanishes:

$$\nabla_- J_+ = -\partial_+ A_-. \tag{2.8}$$

Using the anomaly equation (2.5) it is easy to obtain the variation of the induced action $\Gamma_-(A_-)$ (the part of $\Gamma(A)$ depending only on A_-):

$$\delta\Gamma_-(A_-) \sim \int TrJ_+\delta A_- = -\int Tr(\nabla_- J_+)\epsilon \tag{2.9}$$

under the gauge transformation. Using (2.8) this becomes:

$$\delta\Gamma_-(A_-) \sim \int Tr(\partial_+ A_-)\epsilon.$$

In terms of the new variables (2.6) the above transformation can be written as:

$$\delta\Gamma \sim -\int Tr\left(\partial_+(\partial_- h h^{-1})\delta h h^{-1}\right),$$

where we put $h + \delta h = (\exp \epsilon) h$.

Now one can look for an action which has the same transformation property. The result is a WZW action for the group-valued field $h \in G$:

$$\Gamma(h) \sim \int d^2x Tr(\partial_+ h^{-1}\partial_- h) + \int d^3x \epsilon^{\alpha\beta\gamma} Tr(h_{,\alpha} h^{-1} h_{,\beta} h^{-1} h_{,\gamma} h^{-1}) \qquad (2.10)$$

with $d^3x = dx^3 dx^+ dx^-$ and $\epsilon^{3+-} = -1$. A similar expression can be obtained for $\Gamma_+(A_+)$ with changed sign and h replaced by g. The final form of the total action is:

$$\Gamma_{ind}(A_+, A_-) = \Gamma_+(A_+) + \Gamma_-(A_-) - 2\int Tr(A_+ A_-), \qquad (2.11)$$

where $-2\int Tr A_+ A_-$ is the local counterterm we have to add to ensure gauge invariance. This covariant action can be viewed as the induced action of a gauged WZW model [27]:

$$\Gamma(h, A_+, A_-) = \Gamma(h) + \int Tr(J_+ A_- + J_- A_+ - \frac{k}{2} A_+ A_- + \frac{k}{2} A_- h A_+ h^{-1}).$$

Parametrizing $A_\pm$ as before one can show that after integrating out matter, i.e. the fields h, the induced action of the gauged WZW model is indeed given by (2.11).

To perform now the functional integration over A we have to fix a gauge. We choose:

$$A_- = 0, \qquad (2.12)$$

which results in the following Lagrangian:

$$\mathcal{L} = \psi_-(\partial_+ + A_+)\psi_- + \psi_+ \partial_- \psi_+ + \eta \partial_- \epsilon,$$

where η and ϵ are anticommuting ghosts in the adjoint representation of the gauge group. In order to compute the central charge of the theory we have to use the constraints on the currents, namely that in any gauge theory the total current is zero. At the quantum level this reads:

$$\left.\frac{\delta Z}{\delta A_-}\right|_{A_-=0} \equiv \langle J_+^{tot} \rangle = 0. \qquad (2.13)$$

For the model considered above we then have:

$$J_+^{tot} \equiv \psi_+\psi_+ + \eta\epsilon + const A_+ = 0. \tag{2.14}$$

We see that J_+^{tot} is a sum of three contributions coming from matter, ghosts and gauge currents. Each of them satisfies separately a Kac-Moody algebra with central charge k:

$$[J^a(x), J^b(y)] = k\delta_{ab}\delta'(x-y) + f_{abc}J^c(y).$$

Then the constraint (2.14) implies that the sum of the central charges vanishes:

$$k^{tot} = k_{matter} + k_{ghost} + \mathrm{k} = 0, \tag{2.15}$$

k is the central charge of the gauge part of the currents. One can calculate:

$$k_{ghost} = 2C_v,$$

where C_v is the Casimir of the gauge group, and therefore the following relation between the central charges

$$\mathrm{k} = -(k_{matter} + 2C_v) \tag{2.16}$$

holds.

Let us now consider the problem of renormalization of the model, i.e. compute the renormalization constant Z_A for the gauge field and the renormalization of the Kac-Moody central charge k. We shall compute the 1-loop corrections using the expansion:

$$A_+ = \bar{A} + a$$

around the classical configuration $\bar{A}$ in the functional integral:

$$Z = \int \mathcal{D}A_+ e^{i\mathrm{k}\Gamma(A_+)}.$$

We have:

$$\begin{aligned} Z &= e^{i\mathrm{k}\Gamma(\bar{A})} \int \mathcal{D}a \exp i\frac{\mathrm{k}}{2} \int a.a \frac{\partial^2\Gamma}{\partial A_+ \partial A_+}\bigg|_{A_+=\bar{A}} \\ &= e^{i\mathrm{k}\Gamma(\bar{A})} Det^{-1/2} \left| \frac{\delta^2\Gamma}{\delta A_+ \delta A_+} \right|_{A_+=\bar{A}}. \end{aligned}$$

Since $J_- = \frac{\delta\Gamma}{\delta A_+}$, the 1-loop induced action will have the form:

$$\Gamma^{(1)}(\bar{A}) = -\frac{1}{2}\log Det\left|\frac{\delta J_-}{\delta A_+}\right|.$$

Using the transformation properties of A_+ and J_- and computing the corresponding determinants one gets:

$$\Gamma^{(1)}(\bar{A}) = C_v\left(\Gamma_+(A_+) - \Gamma_-(J_-)\right).$$

Now we use the identity:

$$\Gamma_{ind}(A_+, J_-) \equiv \Gamma_+(A_+) + \Gamma_-(J_-) - 2\int Tr A_+ J_- = 0, \tag{2.17}$$

which follows from the gauge invariance of $\Gamma_{ind}(A_+, J_-)$ and the vanishing field strength of the gauge field with components (A_+, J_-) due to the anomaly equation (2.5). Thus, the 1-loop correction is given by:

$$\Gamma^{(1)}(\bar{A}) = 2C_v\Gamma_+(\bar{A}) - 2C_v\int Tr\frac{\delta\Gamma_+}{\delta A}\bar{A}$$

and for the total effective action up to one loop we have:

$$\Gamma^R_{ind}(\bar{A}) = (\mathrm{k} + 2C_v)\Gamma_{ind}\left[\left(1 - \frac{2C_v}{\mathrm{k} + 2C_v}\right)\bar{A}\right]. \tag{2.18}$$

From this result we can read-off the renormalization constant for the gauge field:

$$Z_A = 1 - \frac{2C_v}{\mathrm{k} + 2C_v} = 1 + \frac{2C_v}{k_{matter}}$$

and see that the central charge k does not renormalize. Actually, since k is an integer, it cannot be renormalized also at higher loop levels.

2.2 Induced 2D Gravity

Let us now turn to the description of the two dimensional gravity theory. We shall follow an approach very similar to the one developed for the case of gauge fields using the analogies between both theories. The Lagrangian for massless

Majorana fermions on a curved world sheet is given by:

$$L = (\det e)\bar{\psi}\gamma^a e_a^\alpha \partial_\alpha \psi, \tag{2.19}$$

where e_α^a are the "zweibeins", defined as:

$$e_a^\alpha e_b^\beta \delta^{ab} = g^{\alpha\beta}.$$

It is convenient to make a change of variables:

$$\begin{aligned} &\phi_- = e_{+-}^{1/2}\psi_- \\ &h_{++} = \frac{e_{++}}{e_{+-}}, \qquad h_{+-} = e_{+-}e_{-+}, \qquad h_{--} = \frac{e_{--}}{e_{+-}}, \end{aligned} \tag{2.20}$$

under which the Lagrangian (2.19) takes a form similar to the one of gauge theory (2.1) :

$$L = \phi_-(\partial_+ - h_{++}\partial_-)\phi_- + (+ \leftrightarrow -). \tag{2.21}$$

Again, our goal is to derive the induced action, obtained after the integration over the matter fields, and to investigate its properties under renormalization. In view of the analogy with the gauge case we again expect for Γ_{ind} the following structure:

$$\Gamma_{ind}(h_{++}, h_{--}, h_{+-}) = \Gamma_+(h_{++}) + \Gamma_-(h_{--}) + \Lambda(h_{++}, h_{--}, h_{+-}), \tag{2.22}$$

where Λ is is a local counterterm needed to ensure the general covariance.

The covariant form of this action, which appears in the string functional integral, is well known and have the following non local form:

$$\Gamma = \frac{c}{96\pi}\int d^2x\sqrt{g}\left(R\frac{1}{\Delta}R\right) \tag{2.23}$$

where c is the central charge of the matter system, Δ is the Laplacian in the metric g_{ab} and R is the scalar curvature. The most simple form of this action is in a conformal gauge: $g_{ab} = e^\phi \delta_{ab}$ where it reduces to a free field action. Unfortunately in quantizing this theory one have to introduce a cut-off which would spoil the general covariance. Instead, it was proposed in [5] to consider a light-cone gauge for the metric defined by:

$$ds^2 = dx^+dx^- + h_{++}(x^+, x^-)(dx^+)^2, \tag{2.24}$$

i.e. to put (compare with the case of gauge fields) $h_{--} = 0, \qquad h_{+-} = 1$. Again, one can at this point use the parametrization:

$$\partial_+ f = h_{++}\partial_- f$$

and obtain the following local form of the action:

$$\Gamma[f] = \frac{c}{24\pi}\int d^2x \left[\frac{\partial_-^2 f\partial_-\partial_+ f}{(\partial_- f)^2} - \frac{(\partial_-^2 f)^2\partial_+ f}{(\partial_- f)^3}\right], \tag{2.25}$$

which is the gravitational analog of the WZW- action (2.10) . Instead of working directly with the action (2.25) we shall follow the approach developed in the previous section using the anomalous equations and the Ward identities for the correlation functions.

Consider the partition function:

$$Z = \int \mathcal{D}h_{++} e^{ik_0\Gamma_+(h_{++})}, \tag{2.26}$$

where

$$\Gamma_+ \sim \log\det(\partial_+ - h_{++}\partial_-)$$

and k_0 is the bare central charge which can be expressed in terms of the matter central charge c:

$$k_0 = \frac{c}{6}. \tag{2.27}$$

Here $e^{ik_0\Gamma_+(h_{++})}$ can be considered as before as the generating functional for the correlation functions of the stress-energy tensor:

$$e^{ik_0\Gamma_+(h_{++})} = \left\langle e^{\frac{i}{\pi}\int h_{++}T_{--}}\right\rangle_{matt} \tag{2.28}$$

(average is performed over the matter fields), where T_{--} satisfies the usual OPE:

$$T_{--}(x^-)T_{--}(0) = \frac{c}{2}\frac{1}{(x^-)^4} + \frac{2}{(x^-)^2}T_{--}(0) + \frac{1}{x^-}\partial_- T_{--}(0). \tag{2.29}$$

It is not difficult to derive the anomalous conservation low for the stress-energy tensor using (2.28) and (2.29) . In fact, take the derivative of the correlation function $\langle T_{--}\ldots\rangle$:

$$\begin{aligned}\partial_+\langle T_{--}(x^+,x^-)\ldots\rangle &= \frac{1}{\pi}\int d^2y h_{++}(y)\partial_+\langle T_{--}(y)T_{--}(x)\ldots\rangle_0 \\ &= \langle X(x^+,x^-)\ldots\rangle \\ X(x^+,x^-) &= \frac{c}{12}\partial_-^3 h_{++} + 2(\partial_- h_{++})T_{--} + h_{++}\partial_- T_{--},\end{aligned} \tag{2.30}$$

where $\langle\ldots\rangle_0$ is the vacuum expectation value of the conformal theory and we used the identity:

$$\partial_+ \frac{1}{x^- - y^-} = -\pi\delta^2(x-y).$$

In this way we get the following anomaly equation for T_{--}:

$$\nabla_+ T_{--} \equiv \partial_+ T_{--} - 2(\partial_- h_{++})T_{--} - h_{++}\partial_- T_{--} = \frac{c}{12}\partial_-^3 h_{++}. \tag{2.31}$$

It helps us to obtain the anomalous transformation low of the induced action $\Gamma_+(h_{++})$. For this purpose consider the following infinitesimal transformation of the coordinates:

$$\begin{aligned} x^- &\to x^- + \omega_+(x^+, x^-) \\ x^+ &\to x^+ \end{aligned} \tag{2.32}$$

under which h_{++} transforms as:

$$\delta h_{++} = \nabla_+ \omega_+ \equiv (\partial_+ - h_{++}\partial_-)\omega_+ + (\partial_- h_{++})\omega_+. \tag{2.33}$$

Then from (2.28) , (2.31) and (2.33) one obtains:

$$\delta\Gamma_{ind} = \int (\delta h_{++})T_{--} = -\int \omega_+ \nabla_+ T_{--} = -\frac{c}{12}\int (\partial_-^3 h_{++})\omega_+. \tag{2.34}$$

We need now the equation of motion for h_{++}. From (2.31) one finds that it is given by:

$$\partial_-^3 h_{++} = 0, \tag{2.35}$$

which can be solved obviously by expanding in powers of x^-:

$$h_{++}(x^+, x^-) = J^+ - 2x^- J^0 + (x^-)^2 J^- \tag{2.36}$$

the coefficients $J^{\pm,0}$ depending only of x^+.

We are now going to obtain the symmetries generated by these "currents". Consider the N- point function of fields ϕ which transform according to:

$$\delta\phi = \omega_+ \partial_- \phi + \lambda(\partial_- \omega_+)\phi. \tag{2.37}$$

Performing a change of variables in the functional integral (2.26) and using (2.37) one finds the following Ward identity:

$$\begin{aligned} &\partial_-^3 \langle h_{++}(z)\phi(x_1)\ldots\phi(x_n)\rangle \\ &\quad = \sum_j \left[\delta(z-x_i)\partial_- + \lambda\partial_-\delta(z-x_j)\right] \langle\phi(x_1)\ldots\phi(x_n)\rangle. \end{aligned}$$

This can be integrated to give:

$$\begin{aligned}
&\langle h_{++}\phi(x_1)\ldots\phi(x_n)\rangle \\
&=\sum_j\left[\frac{(z^- - x_j^-)^2}{(z^+ - x_j^+)}\frac{\partial}{\partial x_j^-}+2\lambda\frac{(z^- - x_j^-)}{(z^+ - x_j^+)}\right]\langle\phi(x_1)\ldots\phi(x_n)\rangle,
\end{aligned}\tag{2.38}$$

or in terms of the currents (2.36) :

$$\langle J^a(z)\phi(x_1)\ldots\phi(x_n)\rangle=\sum_j\frac{l_j^a(x)}{(z^+ - x_j^+)}\langle\phi(x_1)\ldots\phi(x_n)\rangle,\tag{2.39}$$

where:

$$\begin{aligned}
l_j^- &= \frac{\partial}{\partial x_j^-}\\
l_j^0 &= x_j^-\frac{\partial}{\partial x_j^-}+\lambda\\
l_j^+ &= (x_j^-)^2\frac{\partial}{\partial x_j^-}+2\lambda x_j^-.
\end{aligned}\tag{2.40}$$

It is straightforward to check that l_j^a's give a realization of the generators of $SL(2,R)$. In fact the WI (2.39) is exactly the kind of identity one gets from the WZW theory. In the same way one can study the correlation functions of h_{++} itself. The result can be expressed equivalently by the OPE:

$$h_{++}(z)h_{++}(0)=-\frac{c}{24}\frac{(z^-)^2}{(z^+)^2}-\frac{z^-}{z^+}h_{++}(0)-\frac{1}{2}\frac{(z^-)^2}{z^+}\partial_- h_{++}(0).\tag{2.41}$$

Finally, substituting the solution (2.36) , one finds that actually the J's satisfy the $SL(2,R)$ current algebra:

$$J^a(z_1)J^b(z_2)=\frac{k_0}{2}\frac{q^{ab}}{z_{12}^2}+\frac{f_c^{ab}}{z_{12}}J^c(z_2)+reg,\tag{2.42}$$

where q^{ab} is the Killing metric and f_c^{ab} - the structure constants of $SL(2,R)$. The relation (2.42) shows also that indeed k_0 plays the role of a central charge of the current algebra. At this point we see the advantages of the light-cone gauge. In fact in that gauge the density $\sqrt{g}=1$, the left-moving matter does not interact with gravity and the hidden $SL(2,R)$ Kac-Moody symmetry of the theory becomes manifest. We are now going to exploite just this symmetry in proving that, unlike the gauge case, the central charge k_0 gets renormalized.

Instead of repeating the analysis we made for the induced gauge theories we shall proceed in a different way which allows us to get the exact result. We introduce the total stress-energy tensor:

$$T^{tot}_{++} = \left.\frac{\delta\Gamma}{\delta h_{--}}\right|_{h_{--}=0} \tag{2.43}$$

and require that it is zero. In our case of fermionic matter coupled to gravity we have for example:

$$T^{tot}_{++} = \phi_+\partial_+\phi_+ + \eta_{++}\partial_+\epsilon_- + \zeta\partial_+\epsilon_+ + T^{grav}_{++} = 0, \tag{2.44}$$

where η_{++}, ϵ_- are the ghosts of spin 2, -1 respectively corresponding to the gauge choise $h_{--} = 0$ and ζ, ϵ_+ of spin 0, 1 correspond to $h_{+-} = 1$. The condition (2.44) implies that the total central charge also vanish:

$$c^{tot} = c - 26 - 2 + c^{grav} = 0, \tag{2.45}$$

c is in general the matter central charge. In order to determine c^{grav} and T^{grav}_{++} we make use of the Sugawara construction in terms of the currents (2.36) . The most general form of T^{grav}_{++} is given by:

$$T^{grav}_{++} = const(q_{ab}J^aJ^b + B\partial_+J^0). \tag{2.46}$$

Due to the inhomogeneous term it satisfies a Virasoro algebra with modified central charge depending on k_0 and B. To determine the unknown coefficient B we should fix the residual gauge symmetry presented in the theory. In fact it is easily seen that the transformations:

$$\begin{aligned} x^+ &\to x^+ + \epsilon(x^+) \\ x^- &\to x^- - (\partial_+\epsilon)x^- + \eta(x^+) \end{aligned} \tag{2.47}$$

preserve the form of the light-cone metric (2.24) . Under these transformations h changes as:

$$\begin{aligned} \delta_\epsilon h &= \epsilon\partial_+ h - x^-\epsilon'\partial_- h + 2\epsilon' h - x^-\epsilon'' \\ \delta_\eta h &= \eta\partial_- h + \eta'. \end{aligned} \tag{2.48}$$

Translating this in terms of the current coefficients (2.36) one convince himself that the generators of (2.48) are given by:

$$\begin{aligned} Q_\epsilon &= \int dx^+\epsilon(x)T^{tot}_{++} \\ Q_\eta &= \int dx^+\eta(x)J^-. \end{aligned} \tag{2.49}$$

Now, in the same way as we put T^{tot}_{++} we should require that also

$$J^{-} = 0. \tag{2.50}$$

This additional constraint determines completely T^{grav}_{++}:

$$T^{grav}_{++} = \frac{1}{k_0 + 2} q_{ab} J^a J^b + \partial_+ J^0. \tag{2.51}$$

and therefore the gravity central charge:

$$c^{grav} = \frac{3k_0}{k_0 + 2} - 6k_0. \tag{2.52}$$

Finally, from the condition (2.45) , one obtains the exact result:

$$c - 13 = \frac{6}{k_0 + 2} + 6(k_0 + 2) \tag{2.53}$$

which, compared to (2.27) , shows how k_0 gets renormalized in the presence of gravitational interaction.

We now turn to the problem of the renormalization of the anomalous dimensions of the operators. For that purpose we shall derive differential equations for their correlation functions. We are interested mainly in the so called primary fields with transformation:

$$\delta\phi_\lambda = (\epsilon\partial_- + \lambda\partial_-\epsilon)\phi_\lambda. \tag{2.54}$$

Here λ is the $SL(2,R)$ weight, therefore:

$$J^a(z)\phi_\lambda(x) = \frac{l^a(x^-)}{z - x^+}\phi_\lambda(x) + reg \tag{2.55}$$

and l^a are the $SL(2,R)$ generators introduced in (2.40) . Since only the right-moving matter interacts with gravity, we expect the following structure of the correlation functions:

$$\langle \phi(x_1)\dots\phi(x_n)\rangle \sim \sum_k \mathcal{F}_k(x_1^+, \dots x_n^+) F_k(x_1^+, x_1^-; \dots; x_n^+, x_n^-). \tag{2.56}$$

The "blocks" F_k satisfy a differential equation which can be derived in the following way. Recall the Sugawara construction for the stress-energy tensor (2.51) . One can expand T and J in Laurent series in the usual way and obtain the corresponding relation for the coefficients L_n^{grav} and J_n :

$$L_n^{grav} = \frac{1}{k_0+2} q_{ab} \sum_k : J_k^a J_{n-k}^b : -(n+1) J_n^0. \tag{2.57}$$

If we insert now (2.57) for the case $n = -1$ in the correlation functions (2.56) we obtain the desired equation:

$$\left[(k_0+2)\frac{\partial}{\partial x_i^+} - \sum_{j \neq i} \frac{q_{ab} l_i^a(x_i) l_j^b(x_j)}{x_i^+ - x_j^+} \right] F_k(x_1^+, x_1^-; \ldots; x_n^+, x_n^-) = 0, \tag{2.58}$$

where we used $L_{-1} = \partial_+$ and the current WI (2.55) . In the case of two-point function (2.56) the above equation leads to the following relation:*

$$\Delta_0 + \lambda + \frac{\lambda(\lambda+1)}{k_0+2} = 0, \tag{2.59}$$

where Δ_0 is defined by:

$$L_0^{matt} \phi_\lambda = \Delta_0 \phi_\lambda$$

and is therefore the "bare" dimension of the primary field in the absence of gravitational interaction. What is the scaling dimension of Φ_λ in the presence of gravity? The easiest way to see this is to perform a Weyl transformation of the metric:

$$ds^2 \to \Lambda ds^2 = \Lambda(dx^+ dx^- + h_{++}(dx^+)^2)$$

with a constant factor Λ. The rescaled metric then can be turned back to the gauge (2.24) by means of coordinate transformation:

$$x^- \to \Lambda^{-1} x^-,$$

which, as can be easily seen, is generated by the operator:

$$-J_0^0 = -\oint \frac{dx^+}{2\pi i} J^0(x^+).$$

Therefore the scaling dimension is given by $\Delta = -\lambda$ and the relation (2.59) becomes:

$$\Delta - \Delta_0 = \frac{\Delta(1-\Delta)}{k_0+2}. \tag{2.60}$$

The equations (2.53) and (2.60) allow one to compute critical exponents for massless field theory interacting with the induced gravity in the "weak gravity"

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* Actually, one can use directly (2.57) with $n = 0$.

regime: $c \leq 1$ or $c \geq 25$. In this region the quadratic equation (2.53) have real solutions for k_0:

$$k_0 + 2 = \frac{c - 13 + \sqrt{(1-c)(25-c)}}{12}, \tag{2.61}$$

where we choose the $(+)$–sign, compatible with the semiclassical limit $k_0 \sim c/6$.

3. Higher Spin Extensions of Two Dimensional Gravity – W - Gravities

W- gravity was introduced as a formal generalization of the ordinary 2D gravity described in Chapter 2. It contains in addition to the metric g_{ij} and the stress-energy tensor T_{ij} gauge fields and conserved currents of spin greater than two. In the same way in which 2D gravity is connected to the Virasoro algebra, W- gravity, as stated by its name, is based upon an algebra which is of W- type. The latter, as it is well known, are non-linear and not Lie algebras. This explains the great interest in such theories as a part of the more general interest in the quantum non-linear gauge theories. Another motivation to study W- gravity is their essential role in the description of the non-critical W- strings. The most important advantage of the latter is that the range of "weak gravity" is enlarged from the usual bosonic string theory. In fact, we shall see below that this region in the case of W- gravity becomes:

$$c \leq r, \qquad\qquad c \geq r + 4hD$$

where r is the rank, h - the dual Coxeter number and D - the dimension of the corresponding algebra g. Owing to this fact, if the space-time interpretation is possible, W- string can be regarded as an important candidate for more realistic (even four dimensional) string theory.

3.1. W - Algebras and Representations

The W- algebras realize an infinite dimensional symmetries which are extensions of the conformal symmetry of the two-dimensional quantum field theory. They include the Virasoro algebra as a subalgebra but have more complicated structure, including generators of spin higher than two. Another specific property is their nonlinearity, so that they are not ordinary Lie algebras.

The classical and most familiar example is the Zamolodchikov's W_3- algebra [10]. It contains in addition to the stress-energy tensor $T(z) \equiv W^{(2)}(z)$ additional current $W^{(3)}(z) \equiv W(z)$ of spin 3. The defining OPE's have the following form:

$$\begin{aligned}
T(z)T(z') &= \frac{c/2}{(z-z')^4} + \frac{2T(z')}{(z-z')^2} + \frac{\partial T(z')}{z-z'} + [\Lambda(z') + \frac{3}{10}\partial^2 T(z')] \\
T(z)W(z') &= \frac{3W(z')}{(z-z')^2} + \frac{\partial W(z')}{z-z'} \\
W(z)W(z') &= \frac{c/3}{(z-z')^6} + \frac{2T(z')}{(z-z')^4} + \frac{\partial T(z')}{(z-z')^3} \\
&\quad + \frac{3}{10}\frac{\partial^2 T(z')}{(z-z')^2} + \frac{1}{15}\frac{\partial^3 T(z')}{z-z'} + \frac{2\beta\Lambda(z')}{(z-z')^2} + \frac{\beta\partial\Lambda(z')}{z-z'}.
\end{aligned} \tag{3.1}$$

where:

$$\beta = \frac{16}{5c+22}, \qquad \Lambda(z) =: T(z)T(z) : -\frac{3}{10}\partial^2 T(z).$$

One can expand the currents in Laurent series $T(z) = \sum L_n z^{-n-2}$ and $W(z) = \sum W_n z^{-n-3}$. The above OPE's of the currents are then equivalent to the commutation relations:

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m} \\
[L_n, W_m] &= (3n-(n+m))W_{n+m} \\
[W_n, W_m] &= \frac{c}{3.5!}(n^2-4)(n^2-1)n\delta_{n+m} \\
&\quad + (n-m)\left[\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right] L_{n+m} \\
&\quad + \beta(n-m)\Lambda_{n+m}
\end{aligned} \tag{3.2}$$

where:

$$\Lambda_n = \sum_k : L_k L_{n-k} : + \frac{1}{5}x_n L_n$$

$$x_{2n} = (1+n)(1-n) \qquad\qquad x_{2n+1} = (2+n)(1-n).$$

The appearence of the composite field $\Lambda(z)$ in the RHS of (3.1) makes already evident in this most simple example the nonlinear structure of the W- algebras. At this point one has two possibilities. Either to consider $\Lambda(z)$ as a new current of spin 4 and compute its OPE's with T and W. This introduces still other currents and one has to define fields of higher and higher spin. This never stops. The other possibility is to accept the nonlinear structure of the algebra (which is thus not a Lie algebra) and keep $\Lambda(z)$ as a composite field. Closure is then obtained in the enveloping algebra of the L_n's and W_n's.

One can show that there is a close connection between the W_3 algebra defined above and the affine $\widehat{su(3)}$ algebra. In fact, this statement is true also in the general case, i.e. for any given affine Lie algebra $\hat{g}$, one can construct corresponding W- algebra by using the Miura transformation [21, 22](see also [11, 16, 24]). This gives a natural classification of the W- algebras corresponding to that of the affine algebras. For W- algebras of A-D-E type it is known that they contain currents whose spins are given by the orders of Casimir invariants of the corresponding Lie algebra.* In this Chapter we shall be concerned mainly with W- algebras which correspond to $A_{n-1} \equiv su(n)$ (called simply W_n- algebras), and give the general results for the remaining ones which can be treated in analogous way.

The W_n- algebras contain generators $L_m \equiv W_m^{(2)}$, $W_m^{(3)}$, ..., $W_m^{(n)}$ of spins 2,...,n (corresponding to the Casimirs of A_{n-1}). These are the coefficients in the Laurent expansion $W^{(k)}(z) = \sum W_k^{(n)} z^{-n-k}$ of the conserved currents $W^{(k)}(z)$ with OPEs:

$$\begin{aligned} T(z)W^{(k)}(z') &= \frac{kW^{(k)}(z')}{(z-z')^2} + \frac{\partial W^{(k)}(z')}{z-z'} \\ W^{(k)}(z)W^{(l)}(z') &= \sum_{r=0}^{k+l-1} : \frac{differential \quad polynomial(W^{(s)})(z')}{(z-z')^{k+l-r}} : . \end{aligned} \tag{3.3}$$

We shall show below that they have series of unitary representations of central charge:

$$c = (n-1)\left(1 - \frac{n(n+1)}{p(p+1)}\right), \qquad p = n+1, n+2, \ldots . \tag{3.4}$$

The first member of (3.4) is just the conformal unitary series $c = 1 - \frac{6}{p(p+1)}$.

The simplest approach to deal with the W- algebras is the free-field (or Feigin-Fuchs type [28]) construction [10, 22]. For this purpose introduce $n-1$ scalar fields ϕ^i, $i = 1, \ldots, n-1$ with two-point correlator:

$$\langle \phi^i(z_1)\phi^j(z_2)\rangle = -\delta^{ij}\log z_{12}$$

The stress-energy tensor for this system have the following general form:

$$T(z) = -\frac{1}{2}\partial\phi(z)\cdot\partial\phi(z) + 2i\alpha_0\rho\cdot\partial^2\phi(z). \tag{3.5}$$

We choose ρ to be the Weyl vector of A_{n-1} defined as half the sum of the positive

---

* The W-algebras of B- type has a little different properties. In this case there is a fermionic generator with half-integer spin

roots. It is given by:

$$\rho = \sum_{i=1}^{n-1} \lambda_i, \tag{3.6}$$

where λ_i are the fundamental weights of A_{n-1}. In (3.5) we have pulled out a constant $2i\alpha_0$ for later convenience. The stress-tensor (3.5) satisfies the Virasoro algebra with central charge given by:

$$c = (n-1) - 48\alpha_0^2\rho^2. \tag{3.7}$$

One can try to construct in a similar way also the other currents $W^{(k)}$ [11], i.e. to express them as differential polynomials in ϕ^i of dimension k and determine the unknown coefficients by computing their OPE with the stress tensor. Obviously such program is realistic only for small k. The other way is to use the hidden $su(n)$- symmetry of the algebras. Having constructed the current $W^{(3)}$ in the case of W_3- algebra, one observes an underlying structure related to the weights of $su(3)$. This motivates the following guess. Consider the differential operator of order n:

$$\begin{aligned}(2i\alpha_0)^n\mathcal{D}_n &=: \prod_{\mu=1}^{n}(2i\alpha_0\partial_z + h_\mu \cdot \partial\phi(z)) : \\ &\equiv: (2i\alpha_0\partial_z + h_n \cdot \partial\phi(z)) \dots (2i\alpha_0\partial_z + h_1 \cdot \partial\phi(z)) :,\end{aligned} \tag{3.8}$$

where h_μ are the n weights of the vector representation. It can be expanded in the following "canonical form":

$$\mathcal{D}_n = \partial_z^n + \sum_{k=1}^{n}(2i\alpha_0)^{-k}u_k(z)\partial_z^{n-k}. \tag{3.9}$$

The transformation from (3.8) to (3.9) is known as Miura transformation [22]. One finds:

$$u_1 = \sum_\mu h_\mu \cdot \partial\phi = 0$$

by the properties of the weights of the vector representation. It can be shown that there is a close connection between the remaining u_k and the currents $W^{(k)}$. In fact, the second coefficient in (3.9) reads:

$$u_2 = \sum_{\mu>\nu} : h_\mu \cdot \partial\phi h_\nu \cdot \partial\phi : +2i\alpha_0 \sum_\mu (n-\mu) h_\mu \cdot \partial^2\phi \tag{3.10}$$

and after some algebraic manipulations this becomes:

$$u_2 = -\frac{1}{2} : \partial\phi(z) \cdot \partial\phi(z) : +2i\alpha_0\rho \cdot \partial^2\phi(z), \tag{3.11}$$

i.e. exactly the stress-energy tensor in the form (3.5) . The situation is more involved for the higher-spin currents. One can show that, in general, $W^{(k)}$ can be expressed as a differential polynomial of u_i, $i \leq k$. Typical examples are given by:

$$
\begin{aligned}
W^{(3)} &= u_3 - \frac{n-2}{2}(2i\alpha_0)\partial u_2 \\
W^{(4)} &= u_4 + \beta\partial u_3 + \gamma\partial^2 u_2 + \delta : u_2^2 :
\end{aligned}
\tag{3.12}
$$

for some coefficients β, γ and δ. More details one can find in the original works [22, 29]. A nice discussion of the classical case (i.e. disregarding the normal ordering effects) is given in [15].

The main problem in studying the algebra (3.3) concerns its highest- weight irreducible representations. They are in one-to-one correspondence with the primary fields of the corresponding 2D conformal QFT models obeying such symmetry. As it is well known, the latter are represented in the free-field construction by vertex operators, i.e. normal ordered exponents of ϕ:

$$V_\alpha(z) =: e^{i\alpha\cdot\phi(z)} : . \tag{3.13}$$

This is confirmed from their OPE with the stress tensor (3.5):

$$T(z)V_\alpha(z') = \frac{\Delta_2(\alpha)V_\alpha(z')}{(z-z')^2} + \frac{\partial V_\alpha(z')}{z-z'} \tag{3.14}$$

stating that indeed V_α is a primary field of dimension:

$$\Delta_2(\alpha) = \frac{1}{2}\alpha\cdot(\alpha - 4\alpha_0\rho). \tag{3.15}$$

Of special interest in what follows are the operators of conformal dimension one, the so called screening operators. The corresponding vertex operators are defined as:

$$V_\pm^j(z) \equiv V_{\alpha_\pm e_j}(z) =: e^{i\alpha_\pm e_j\cdot\phi(z)} : \tag{3.16}$$

and (3.15) in the case $\Delta = 1$ gives a quadratic equation for $\alpha_\pm$:

$$\alpha_\pm^2 - 2\alpha_0\alpha_\pm - 1 = 0.$$

Thus, the total number of screening operators is $2(n-1)$. Such operators are particularly important since the singular terms in the OPE $W^{(k)}(z_1)V_\pm^j(z_2)$ combine to give a total derivative. This, in turn, means that the screening charge $Q_\pm^j = \oint dz V_\pm^j(z)$ is an invariant of the W- algebra (3.3) .

We are now going to discuss the highest-weight representations of (3.3) . The highest-weight state is defined in the standard way:

$$
\begin{aligned}
&W_n^{(k)}|\alpha\rangle = 0, \qquad n > 0\\
&W_0^{(k)}|\alpha\rangle = \bar{\Delta}_k(\alpha)|\alpha\rangle, \qquad k = 2,3,\ldots,n.
\end{aligned}
\tag{3.17}
$$

The other states of the representation are obtained by applying to $|\alpha\rangle$ monomials of operators $W_{-n}^{(k)}$ with $n \geq 0$. The connection to the free-field construction is given by the relation:

$$
\lim_{z\to 0} V_\alpha(z)|0\rangle = |\alpha\rangle, \tag{3.18}
$$

where $|0\rangle$ is the $SL(2,C)$- invariant vacuum of the theory. It is clear that this state is annihilated by all positive modes of the $W^{(k)}$ and thus is a highest-weight state. It is not difficult to compute the values of $\bar{\Delta}_k(\alpha)$. In fact, one first obtains the eigenvalues of the zero modes of u_k's which we call $\Delta_k(\alpha)$. Then, $\bar{\Delta}_k(\alpha)$ and $\Delta_k(\alpha)$ are related by equations of the type (3.12) , e.g. $\bar{\Delta}_3(\alpha) = \Delta_3(\alpha) + (n-2)(2i\alpha_0)\Delta_2(\alpha)$ and, of corse, $\bar{\Delta}_2(\alpha) = \Delta_2(\alpha)$. One applies the differential operator (3.8) to the highest-weight state (3.17) and uses the Miura transformation. This results in a system of linear algebraic equations with solution:

$$
\Delta_k(\alpha) = (-i)^k \sum_{\mu_1>\mu_2>\ldots>\mu_k} \prod_{m=1}^{k} \left(h_{\mu_m}\cdot\alpha + 2\alpha_0(k-m)\right) \tag{3.19}
$$

and one easily checks that this correctly reproduces $\Delta_2(\alpha)$ (3.15) .

Among the representations of (3.3) , of special interest are the so called degenerate representations. They are characterized by the existence, at some level N, of a state $|\chi_N\rangle$, called null vector, which is again highest-weight state (3.17) . It is clear that such representation cannot be irreducible. Actually, one can show that the null vector, together with its descendents, decouples from the rest of the representation. Thus one can consistently set it to zero. This has important consequences for the theory since it leads to differential equations for the correlation functions of the corresponding fields. With enough number of null vectors (completely degenerate representations) one can completely determine all the correlation functions thus leading to an integrable theory. As an illustration, and for later use, we present here the simplest example: the first level null vector of the W_3- algebra. It has the form:

$$|\chi_1\rangle = \left(2\Delta_2 W^{(3)}_{-1} - 3\bar{\Delta}_3 L_{-1}\right)|\Delta\rangle \tag{3.20}$$

provided:

$$9\bar{\Delta}_3^2 = 2\Delta_2^2\left(\frac{32}{22+5c}(\Delta_2+\frac{1}{5})-\frac{1}{5}\right). \tag{3.21}$$

The free field formalism described above provides a natural way of constructing null vectors using the properties of the screening operators [22]. Consider the state:

$$|\chi^j_\pm(\beta)\rangle = Q^j_\pm|\Delta(\beta)\rangle \equiv \oint dz : e^{i\alpha_\pm e_j\cdot\phi(z)} :: e^{i\beta\cdot\phi(0)} : |0\rangle. \tag{3.22}$$

Such expression is well-defined and not trivially zero if and only if:

$$\alpha_\pm e_j\cdot\beta = -l_j - 1, \qquad\qquad l_j = 0,1,2,\ldots. \tag{3.23}$$

We choose $\beta = 4\alpha_0\rho - \alpha^* - \alpha_\pm e_j$ so that (3.22) is clearly a descendent of the highest-weight state with $\Delta(4\alpha_0\rho-\alpha^*) = \Delta(\alpha)$. Due to the invariance properties of the screening charges $Q^j_\pm$ (3.16) , it is also a highest-weight state, so it is a null vector. In the general case one considers a state:

$$|\chi^{j,l'}_\pm\rangle = \oint\ldots\oint V^j_\pm\ldots V^j_\pm|\Delta(\beta)\rangle \tag{3.24}$$

with l'_j screening operators and imposes again analyticity condition analogous to (3.23) . It can be solved giving the following parametrization for α:

$$\alpha = \sum_{j=1}^{n-1}\left((1-l_j)\alpha_- + (1-l'_j)\alpha_+\right)\lambda_j, \tag{3.25}$$

where $l_j, \quad l'_j = 1,2,\ldots.$ Note that $\lambda = \sum_j(l_j-1)\lambda_j$ and $\lambda' = \sum_j(l'_j-1)\lambda'_j$ are highest (i.e. dominant) weights of $su(n)$. Thus, we may rewrite (3.25) as:

$$\alpha = -\alpha_-\lambda - \alpha_+\lambda'. \tag{3.26}$$

So the W_n–highest-weight is labelled by a pair (λ,λ') of highest-weights of $su(n)$. Analogous correspondence holds also in the general case of A-D-E type W- algebras. We shall use it in the next Chapter in the description of the W- gravities.

The properties of the field theoretical model having W- algebra symmetry are determined by the operator product algebra of the fields corresponding to its representations. In general this algebra contains an infinite number of fields.

However, it may happen that, due to various symmetries, it is truncated. In fact, one can show that in our case of W- algebras for the special value $\alpha_+^2 = \frac{p'}{p}$ the OPE algebra closes on a finite number of fields. Denoting it by $\mathcal{A}$, the field content of the model can be expressed simbolycally as:

$$\mathcal{A} = \oplus_{\sum l_j \leq p'-1} \oplus_{\sum l'_j \leq p-1} \Phi_{\lambda(l),\lambda'(l')}. \tag{3.27}$$

These models are called minimal models. Since $2\alpha_0 = \alpha_+ + \alpha_- = \frac{p'-p}{\sqrt{pp'}}$, the central charge and the conformal dimensions of these (p,p') minimal models are given by:

$$\begin{aligned} c^n_{pp'} &= (n-1)\left(1 - n(n+1)\frac{(p-p')^2}{pp'}\right) \\ \Delta_2(l|l') &\equiv \Delta_2(l_1 \dots l_{n-1}|l'_1 \dots l'_{n-1}) \\ &= \frac{12\left[\sum_{i=1}^{n-1}(pl_i - p'l'_i)\lambda_i\right]^2 - n(n^2-1)}{24pp'}, \end{aligned} \tag{3.28}$$

where the range of l_j and l'_j is clear from (3.27) . The remaining eigenvalues $\bar{\Delta}_k$ are also obtained by direct substitution in the corresponding expression (3.19) .

Among the minimal models, the most interesting are those obeying the unitarity condition. In this case $p' = p+1$ as can be easily seen by using the coset construction based on the coset:

$$\frac{\hat{su(n)}_k \otimes \hat{su(n)}_1}{\hat{su(n)}_{k+1}}, \qquad p = k+n. \tag{3.29}$$

All the quantum numbers characterizing the unitary models can be obtained trivially by inserting $p' = p+1$ directly in (3.28) and (3.19) .

Up to now we discussed the properties of the W_n- algebras. These are not Lie algebras for any finite $n > 2$, owing to the presence of non-linear terms in the commutation relations, which rapidly increase in complexity as n increases. However, it has been argued that the structure of these algebras should become much simpler in the limit $n \to \infty$ [30]. The process of taking this limit is not uniquely defined; one can in principle arrive at different algebras depending on how one first rescales the generators and structure constants of the finite n algebra. One particular limit that has been discussed in the literature [31] is of the form:

$$[w_m^{(i)}, w_n^{(j)}] = ((j-1)m - (i-1)n)w_{m+n}^{(i+j-2)}, \tag{3.30}$$

where $w_m^{(i)}$ is a generator of conformal spin i. This algebra, called w_∞, can be considered as equivalent to the algebra of smooth area-preserving diffeomorphisms of the cylinder $S^1 \times R^1$ [32]. Although the W_n- algebras have central terms for each conformal spin, the limit (3.30) admits a central term only in the Virasoro sector. One might suppose, therefore, that there should exist some different limiting procedure in which the central terms of the W_n- algebra are retained in the $n \to \infty$ limit. From a physical point of view such a limit is more interesting, since it would allow the existence of unitary representations with nontrivial dependence on the higher-spin generators.

Such algebra, called W_∞, was considered in [32, 33] and has the form:

$$[V^i_m, V^j_n] = \sum_{l=0}^{\infty} q^{2l} g^{ij}_{2l}(m,n) V^{i+j-2l}_{m+n} + q^{2i} c_i(m) \delta^{ij} \delta_{m+n}, \tag{3.31}$$

where the structure constants are given by:

$$g^{ij}_{2r}(m,n) = \frac{\phi^{ij}_{2r}}{2(2r+1)!} N^{ij}_{2r}(m,n) \tag{3.32}$$

with:

$$\phi^{ij}_{2r} = \sum_{k=0}^{r} \prod_{i=1}^{k} \frac{(2l-3)(2l+1)(2r-2l+3)(r-l+1)}{l(2i-2l+3)(2j-2l+3)(2i+2j-4r+2l+3)}$$

$$N^{ij}_{2r}(m,n) = \sum_{k=0}^{2r+1} (-1)^k \binom{2r+1}{k} \times (2i+2-2r)_k [2j+2-k]_{2r+1-k} [i+1+m]_{2r+1-k} [j+1+n]_k .$$

The form of the algebra is unique up to an arbitrary constant c which sets the scale of the central terms:

$$c_i = 2^{2i-3} \frac{i!(i+2)!}{(2i+1)!!(2i+3)!!} c.$$

It can be viewed as a deformation of the w_∞ algebra (3.30) , from which the latter is obtained by contraction in which we put $q=0$.

3.2. W - Gravity

The main obstacle in dealing with the W- gravity is presented by the nonlinearity of the W- algebras. This, together with the c- dependence of the structure constants, leads to non-local, non-linear Ward identities which are hard to solve.

Another difficulty is the absence of covariant formulation of the theory. One cannot write down a covariant action as in the case of the usual gravity (2.23) . The deep reason for such situation is that the geometry behind the W- gravity is not known. Some attempts for its description, from different points of view, have been done recently but the situation is still not completely clear. We shall describe some of these attempts in the next Chapters.

Despite the covariant formulation is not known, one can quite straightforwardly extend the KPZ arguments for the case of W- gravity. For this purpose, corresponding to the fact that there exist some higher spin charges, we need to introduce some additional higher spin gauge symmetries. Lets make the following natural ansatz [20]:

1. The gauge fields corresponding to the W- generators are components of a symmetric tensor generalizing the metric tensor $g_{\mu\nu}$. Denote them as $A^{(n)}_{\mu_1\ldots\mu_n}$.

2. The gauge symmetries corresponding to these fields are parametrized by $n-1$- symmetric tensors $k^{(n-1)}_{\lambda_1\ldots\lambda_{n-1}}$. The gauge transformation of A is given by:

$$\delta_k A^{(n)}_{\mu_1\ldots\mu_n} = \nabla_{(\mu_1} k^{(n-1)}_{\mu_2\ldots\mu_n)} + \ldots . \tag{3.33}$$

3. By using this gauge symmetry we can pick "light-cone gauge" for A:*

$$\begin{aligned} A^{(n)}_{zz\ldots z} &\neq 0 \\ (other \quad components) &= 0. \end{aligned} \tag{3.34}$$

4. For W- algebra associated with the affine algebra $\hat{g}$ introduce a set of connection fields $\{A^{(n)}\}_{n\in C}$, C is the set of orders of the Casimir operators of $\hat{g}$.

By using these assumptions we shall discuss below the hidden symmetry of the W- gravity and find the general expressions for the renormalized central charge and anomalous dimensions. As in the case of W- algebras the discussion will be performed for the most simple W_3- gravity and the results will be then generalized.

There are two different approaches one can follow in the description of the W_3- gravity. The first one is based on a classical Lagrangian for scalar fields which is invariant under W_3- transformations. Then one introduces gauge fields

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⋆ In this section we shall slightly change the notations: $x^+ \to z, \quad x^- \to \bar{z}$.

and determines their transformations in such a way that the resulting Lagrangian remains invariant. The quantum corrections are obtained in $\frac{1}{c}$ (loop) expansion using the usual Feynmann diagram techniques. We shall follow here a KPZ - like approach based directly on the defining W_3- algebra OPEs (3.1) . It has the advantage to lead to general results which are then confirmed by the perturbation expansion.

Again we are interested in the induced action defined as the generating functional for the current correlation functions:

$$e^{i\Gamma_{ind}(h,A)} = \left\langle e^{\frac{i}{\pi}\int(h\bar{T}+A\bar{W})}\right\rangle_{matt}, \tag{3.35}$$

where A is the new gauge field corresponding to the conserved current W. The anomalous conservation laws for the currents are derived in exactly the same way as in the case of the usual gravity (2.30) . Using the OPE (3.1) we obtain:

$$\begin{aligned}
\nabla_z\bar{T}(z,\bar{z}) \equiv& \partial\bar{T} - 2\bar{\partial}h\bar{T} - h\bar{\partial}\bar{T} - 3\bar{\partial}A\bar{W} - 2A\bar{\partial}\bar{W} \\
=& \frac{c}{12}\bar{\partial}^3 h \\
\nabla_z\bar{W}(z,\bar{z}) \equiv& \partial\bar{W} - (3\bar{\partial}h + h\bar{\partial})\bar{W} \\
& - \left(\frac{1}{3}\bar{\partial}^3 A + \frac{1}{2}\bar{\partial}^2 A\bar{\partial} + \frac{3}{10}\bar{\partial}A\bar{\partial}^2 + \frac{1}{15}A\bar{\partial}^3\right)\bar{T} \\
& - \beta(2\bar{\partial}A + A\bar{\partial})\bar{\Lambda} \\
=& \frac{c}{360}\bar{\partial}^5 A.
\end{aligned} \tag{3.36}$$

The gauge transformations of the connections can be deduced from (3.35) and (3.36) . In the classical limit $c \to -\infty$ the latter are given by:

$$\begin{aligned}
\delta_{\epsilon,k}h =& \partial\epsilon - h\bar{\partial}\epsilon + \bar{\partial}h\epsilon \\
& + \frac{1}{30}(-2A\bar{\partial}^3 + 3\bar{\partial}A\bar{\partial}^2 - 3\bar{\partial}^2 A\bar{\partial} + 2\bar{\partial}^3 A)k \\
\delta_{\epsilon,k}A =& \partial k - h\bar{\partial}k + 2k\bar{\partial}h + \epsilon\bar{\partial}A - 2\bar{\partial}\epsilon A,
\end{aligned} \tag{3.37}$$

which agree with our ansatz (3.33) . From (3.36) ,(3.37) we can evaluate the anomaly of the gauge symmetry of the action:

$$\begin{aligned}
\delta\Gamma &= \frac{1}{\pi}\int(\delta h\bar{T} + \delta A\bar{W})d^2 z \\
&= -\frac{1}{\pi}\int(\epsilon\nabla\bar{T} + k\nabla\bar{W})d^2 z \\
&= -\frac{c}{12\pi}\int\left(\epsilon\bar{\partial}^3 h + \frac{1}{30}k\bar{\partial}^5 A\right)d^2 z.
\end{aligned} \tag{3.38}$$

The equations of motion for h and A are given by:

$$\bar{\partial}^3 h = 0$$
$$\bar{\partial}^5 A = 0 \tag{3.39}$$

and are direct generalization of (2.35) . Analogously to (2.36) , their solution can be presented as an expansion in $\bar{z}$:

$$\begin{aligned} h &= I^+(z) - \bar{z} I^0(z) - \frac{1}{2}\bar{z}^2 I^-(z) \\ A &= J^{++}(z) + \bar{z} J^+(z) + \frac{1}{2}\bar{z}^2 J^0(z) + \frac{1}{2}\bar{z}^3 J^-(z) + \frac{1}{4}\bar{z}^4 J^{--}(z). \end{aligned} \tag{3.40}$$

By the investigation of Ward identities [5], it is straightforward to derive the OPEs of h and A:

$$\begin{aligned} h(z,\bar{z})h(0,0) &= -\frac{c}{24}\frac{\bar{z}^2}{z^2} - \frac{\bar{z}}{z}h - \frac{1}{2}\frac{\bar{z}^2}{z}\bar{\partial} h \\ h(z,\bar{z})A(0,0) &= -2\frac{\bar{z}}{z}A - \frac{1}{2}\frac{\bar{z}^2}{z}\bar{\partial} A \\ A(z,\bar{z})A(0,0) &= \frac{c}{96}\frac{\bar{z}^4}{z^2} + \left(\frac{1}{2}\frac{\bar{z}^3}{z} + \frac{1}{4}\frac{\bar{z}^4}{z}\bar{\partial}\right) h, \end{aligned} \tag{3.41}$$

where we have rescaled the gauge fields:

$$h \to \frac{c}{12}h, \qquad A \to \frac{ic}{12\sqrt{10}}A. \tag{3.42}$$

One can easily check that these OPEs are equivalent to $sl(3)$ current algebra in terms of the component currents I and J defined above. The central charge of the current algebra is given by:

$$k = \frac{c}{24}. \tag{3.43}$$

Actually, the decomposition of the currents follows directly the decomposition of the adjoint representation of $sl(3)$ into representations of $sl(2)$ subalgebra of I^+, I^0 and I^-: $(8) \to (3) + (5)$. This suggests the following generalization. Let $\{X^\alpha, Y^\alpha, H^\alpha\}_{\alpha=1\ldots r}$ be the Cartan-Weyl basis of g. Define the $sl(2)$ subalgebra as follows:

$$\begin{aligned} I^- &= \sum_\alpha X^\alpha \\ I^0 &= \sum_{\alpha,\beta} A^{-1}_{\alpha,\beta} H^\beta \\ I^+ &= \sum_{\alpha,\beta} A^{-1}_{\alpha,\beta} Y^\beta, \end{aligned} \tag{3.44}$$

where $A^{-1}_{\alpha\beta}$ is the inverse of the Cartan matrix. It is known that adjoint representation of g can be decomposed into representations of this $sl(2)$ subalgebra:

$$(D) = \bigoplus_{n\in C}(2n-1). \tag{3.45}$$

As we have done in the $sl(3)$ case (3.40) , we can assign the currents corresponding to the elements appearing in $(2n-1)$ to the coefficients of Taylor expansion of $A^{(n)}$ with respect to $\bar{z}$. Remark that the OPE:

$$\bar{W}^{(n)}(\bar{z})\bar{W}^{(n)}(0) \sim \frac{c}{n}\bar{z}^{-2n} + O(\bar{z}^{-2n+1})$$

implies:

$$\bar{\partial}^{2n-1}A^{(n)} = 0, \qquad (n \in C). \tag{3.46}$$

Let us now turn to the description of the dynamics of the gauge fields based on the partition function:

$$Z = \int \prod_n \mathcal{D}A^{(n)} e^{i\Gamma_{ind}[A^{(n)}]}. \tag{3.47}$$

For each gauge connection $A^{(n)}$ we have to consider the gauge fixing condition:

$$\frac{\delta\Gamma}{\delta A^{(n)}_{\bar{z}\ldots\bar{z}}} = \frac{\delta\Gamma}{\delta A^{(n)}_{\bar{z}\ldots\bar{z}z}} = \ldots = \frac{\delta\Gamma}{\delta A^{(n)}_{\bar{z}z\ldots z}} = 0. \tag{3.48}$$

If we use the ansatz (3.33) the ghost Lagrangian originated from this gauge fixing is given by:

$$\mathcal{L}_{ghost} = \sum_{n\in C}\mathcal{L}^{(n)}_{ghost} = \sum_{n\in C}\sum_{j=0}^{n-1} b^{(n)}_{j+1}\nabla c^{(n)}_{(-j)}, \tag{3.49}$$

where $c^{(n)}_{-j}$ is a ghost of spin $(-j)$ and $b^{(n)}_{j+1}$ is an antighost of spin $(j+1)$. The first condition (3.48) :

$$\frac{\delta\Gamma}{\delta h} \equiv T^{tot} = 0 \tag{3.50}$$

states again that the total stress-energy tensor is zero. The contribution of W-gravity can be calculated analogously to (2.51) :

$$T^{grav} = T^{Sug} + \partial_z I^0(z) \tag{3.51}$$

where T^{Sug} is the Sugawara type energy-momentum tensor of $\hat{g}$ and I^0 is the corresponding current introduced in (3.40) . According to the condition (3.50) , the total central charge vanishes:

$$c_{matter} + c_{gravity} + c_{ghost} = 0, \tag{3.52}$$

where now:

$$\begin{aligned} c_{ghost} &= \sum_{n \in C} \sum_{j=1}^{n} (-12j^2 + 12j - 2) \\ c_{grav} &= \frac{kD}{k+h} - 12\rho^2 k. \end{aligned} \tag{3.53}$$

Here r and D are respectively rank and dimension of g, h is the dual Coxeter number and ρ is the Weyl vector. In the case of A-D-E type algebras the ghost central charge (3.53) has the form:

$$c_{ghost} = -r - (1+h)^2 D.$$

Summarizing, (3.52) leads to the following relation between k and $c_{matter} \equiv c$:

$$-(k+h) = \frac{2hD + r - c + \sqrt{(r-c)(r+4hD-c)}}{2hD}. \tag{3.54}$$

In the classical limit this reduces to:

$$k \sim \frac{c}{hD} \tag{3.55}$$

which coincides with the previous result (3.43) . In a recent paper [25] Ooguri et all. computed the first nontrivial terms in the $\frac{1}{c}$ expansion of the effective action for W_3- gravity. Their results are in a perfect agreement with the general formula (3.54) .

Anomalous dimensions of the operators can be computed in exactly the same way as we did for the usual 2D gravity (2.58) , where now $l^{(a)}$ give a realization of the algebra g. The equivalent of the general result (2.60) in our case reads:

$$\Delta_0(l|l') = -\frac{2\lambda(\lambda-\rho)}{k+h} + 2\lambda\rho, \tag{3.56}$$

where λ are highest weights of the current algebra:

$$\lambda = \frac{1}{2}\sum_i \left(\frac{p}{q}(l_i - 1) - (l'_i - 1)\right)\lambda_i$$

and $\Delta_0(l|l')$ are the bare dimensions of the primary fields as given above (3.28) :

$$\Delta_0(l|l') = \frac{\left[\sum(pl_i - ql'_i)\lambda_i\right]^2 - \rho^2(p-q)^2}{2pq}.$$

Analogous formulas can be derived also for the other quantum numbers of the W- algebras. For that purpose we need to solve the more general constraints (3.48) for arbitrary n.

Finally, we shall discuss briefly the theory of w_∞ gravity. The starting point is a Lagrangian for a free scalar field φ that takes its values in some Lie algebra g:

$$\mathcal{L} = \frac{1}{2} tr(\partial\varphi\bar{\partial}\varphi). \tag{3.57}$$

It is invariant under the transformation:

$$\delta\varphi = k_l(\partial\varphi)^{l+1} \tag{3.58}$$

provided k_l are functions of z only. One can easily check that these transformations close to form the w_∞ algebra (3.30) .

To gauge this symmetry we first assume that k_l depends on $\bar{z}$ as well as z. Then introduce gauge fields A_l and consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2} tr(\partial\varphi\bar{\partial}\varphi) - \sum_0^\infty \frac{1}{l+2} A_l tr(\partial\varphi)^{l+2}. \tag{3.59}$$

Note that the currents:

$$w^{(l)} = tr(\partial\varphi)^{l+2} \tag{3.60}$$

are conserved as a consequence of the field equation $\partial\bar{\partial}\varphi = 0$. The Lagrangian (3.59) is invariant under (3.58) provided that the gauge fields transform as:

$$\delta A_l = \bar{\partial} k_l - \sum_j [(j+1)A_j \partial k_{l-j} - (l-j+1) k_{l-j} \partial A_j]. \tag{3.61}$$

Actually, due to the presence of additional symmetries of (3.59) , not all the gauge fields are truly independent [34]. It can be shown that putting the additional gauge fields (and the corresponding parameters) to zero and adding compensating transformations leads to truncation of w_∞ to W_r- gravity where r is the rank of g.

In trying to quantize the above theory one encounters the problem of cancellation of the anomalies. There are two kinds of anomalies that arise. The first, called universal anomalies, are given by local expressions involving the background gauge fields only. They are governed by the central charge structure of the algebra. The universal anomalies are cancelled by constructing a "critical theory" in which they are simply cancelled against the ghost contribution. The second kind are the so called matter field dependent anomalies and arise from diagrams with external matter fields. These are more difficult to deal with. For their cancellation one introduces local finite counterterms simultaneously correcting the matter and gauge field transformation rules. This is equivalent to adjusting the currents (3.60) . The first simple examples for the spin-2 and spin-3 currents (which are in fact exact) are given by:

$$\begin{aligned} V^0 &= \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}\sqrt{\hbar}\partial^2\varphi \\ V^1 &= \frac{1}{3}(\partial\varphi)^3 + \frac{1}{2}\sqrt{\hbar}\partial\varphi\partial^2\varphi + \frac{1}{12}\hbar\partial^3\varphi. \end{aligned} \tag{3.62}$$

One can proceed in this way renormalizing all the currents. It can be shown [35] that, as a result of this procedure, the modified currents will generate not the original w_∞ algebra but precisely the W_∞ algebra (3.31) !

4. Affine Geometry and W_n-Gravities

4.1. Introduction

The present Chapter is an attempt to answer the following question:*given a set of "metrics" $\{h, h^{\dagger}, A, B, \ldots\}$ which are the geometries characterized by this geometrical data* (i.e. what do the W-geometries look like) ? Our main observation is that the $(1-D)$ CFT's and the W-gravities can be described in terms of particles and string mechanics, i.e. as differential geometries of curves and surfaces immersed in certain spaces V_n with groups of motion $\tilde{G}_n = T_n \cdot G$ [36].

Our basic tool in the study of the geometries of the CFT's and W-gravities is the famous F.Klein's classification of the geometries proposed in his "Erlangen program" in 1872 and further developed by W.Blaschke [37], E.Cartan [38], M.Fubini [39], etc. (our prefered textbook is [40]). The Klein geometries can be defined as theories of the geometric invariants of transitive transformation groups* . Specific examples of such geometries are the geometries of curves and surfaces imbeded in Euclidean E_n (or Minkowski $M_{p+q=n}$), affine A_n and projective RP^n (or CP^n) spaces.

A remarkable property of the *geometric invariants* $k_i(y^{(0)}, \ldots, y^{(s)})$ (curvatures, torsions etc.), $y^{(s)}(x) = \partial_x^s y$, describing an embedding of a curve in V_n:

$$y^{\mu}(x) : R \rightarrow V_n, \qquad V_n = E_n(M_{p+q}), \quad A_n, \quad RP^n, \quad \mu = 1, \ldots, n.$$

is that they appear as generators of the infinithesimal transformations of WG_n-algebras† where $G_n = \tilde{G}/T_n$ for E_n and A_n and $G_n = SL(n+1, R)$ for RP^n. Therefore *the classification of WG_n-symmetric $(1-D)$ CFT's is (to some extend) isomorphic to the Klein classification of the geometries of curves.* We restrict

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$\star$ A brief introduction to the Lie-Cartan theory of geometric invariants is presented in App. A.

† The ortogonal case ($G_n = SO(n)$ or $SO(p,q)$) is an exeption: the number of invariants is larger than the number of the generators of WG_n (see App.A and [41]). To get WG_n-algebras we must impose specific conditions on the k_i decreasing its number. This restricts the possible curves geometries to the case of helices (see [41] for the case of surfaces).

our further discussion on the cases of affine curves (Sect.2) and affine surfaces (Sect.3,4) only (see [41] for $V_n = M_{p+q=n}$).

To complete the identification of the affine particles ($\equiv$ affine curves immersed in A_n) with W_n-CFT's models we have to find the geometric counterparts of the primary fields and null-vectors of W_n-models. As we shall show the classical limit ($c \to -\infty$) of specific null-vectors (n-th level for W_n) coincides with the affine Frenet equations (2.6) for the curve immersed in A_n. The normalized "affine velocites"

$$v^{\mu}_{(n)} = (\partial_x y^{\mu})(det(y', \ldots, y^{(n)}))^{-\frac{1}{n}}$$

(of dimension $\Delta(v^{\mu}_{(n)}) = \frac{1-n}{2}$) are the classical limits of specific primary fields of W_n-models. For example in the simplest case of plane affine curves ($n = 2$ and $W_2 \equiv$ Virasoro algebra) we have:

$$(\partial_x^2 + k)v^{\mu}(x) = 0, \qquad \mu = 1, 2 \tag{4.1}$$

($k(x)$ is the affine curvature), as an equation of motion ($\equiv$ affine Frenet equation) and the quantum counterpart of v^{μ} ($\Delta(v^{\mu}) = -\frac{1}{2}$) is the primary field $\phi_{21}(x)$ of the Virasoro minimal models ($\Delta_{21}(c) = \frac{5-c+\sqrt{(1-c)(25-c)}}{16}$). The Lagrangian of such a particle is

$$L = \int \left(\epsilon^{\mu\nu} y'_{\mu} y''_{\nu}\right)^{\frac{1}{3}} dt + \alpha \int \epsilon^{\mu\nu} y_{\mu} y'_{\nu} dt \tag{4.2}$$

The appearence of the KdV L-operator in (4.1) leads to the following *affine geometrical interpretation of KdV equation.* Consider a family of plane affine curves $y^{\mu}(x,t)$ generated by a given curve $y^{\mu}(x,t_0)$ by specific small deformations

$$y^{\mu}(x,t) = y^{\mu}(x,t_0) + (t - t_0)k\partial_x y^{\mu} + \ldots$$

(i.e. $\partial_t y = k\partial_x y$). As it is shown in Sect.2 the KdV equation

$$\partial_t k = 6k\partial_x k + \partial_x^3 k \tag{4.3}$$

describes how the affine curvature $k(x,t)$ changes under these transformations.

The main difference between the geometries of affine curves and affine surfaces in A_n, concerning their symmetries, is that in the case of surfaces the number of the geometric invariants is bigger than the number of generators of the W_n-algebras (see App.A). To obtain geometries relevant to the W_n-gravities we have to consider a specific class of affine surfaces which are described by $n-1$ differential invariants only. For example, the (classical) W_3-gravity is isomorphic to the geometry of the affine surfaces of constant affine mean curvature immersed in A_3. Such an identification is based on the following properties of the affine surface geometry (see Sect.3 and 4):

a) each affine surface in A_3 is defined uniquely by its two fundamental forms:

$$\varphi = h_{ij}dx^i dx^j, \qquad \psi = A_{ijk}dx^i dx^j dx^k, \qquad h^{ij}A_{ijk} = 0$$

b) the (gauge fixed) affine structure equations (3.1) (or (3.5)): $\partial_i g = A_i g$, in light-cone gauge are invariant under W_3-transformations generated by (see eqs.(4.60)):

$$T = -2B_{--}, \qquad W = 4(A_{---} + \frac{1}{2}\partial_- T)$$

c) as a consequence of the integrability conditions (i.e. affine Gauss-Codazzi) of the eqs.(4.42), the "metrics" h_{ij} and A_{ijk} satisfy the W_3-"trace anomaly" equations (4.63). Integrating these anomalies one can construct an effective action of W_3-induced gravity:

$$\delta S = \int T\delta\bar{h}d^2x + \int W\delta A d^2x$$

and the corresponding equations of motion are

$$\partial_-^3 \bar{h} = 0 = \partial_-^5 A. \tag{4.4}$$

The eqs.(4.4) together with the specific transformation laws of $\bar{h}$ and A (4.61) are at the origin of the $SL(3,R)$-current algebra symmetries of the affine surface geometry.

This affine geometrical description of the W_3-gravity suggests an interpretation of it as a theory of the noncritical affine string, i.e., affine curve moving in (target) affine space A_3. Its consistent quantization requires to consider not all the affine surfaces in A_3 but only the ones of constant mean curvature. A peculiar property of such a string is that we have as a space-time the affine 3-D space A_3 and as a consequence we have no gravitons in it.

One could wonder how general is the affine geometrical approach to W_n-gravities. In this Chapter we present the proof that *the quantum W_3-gravity has as a classical limit ($c \longrightarrow -\infty$) the geometry of affine surfaces of constant mean* *curvature in* A_3. Concerning the W_n-gravities our cojecture is that they indeed are described by certain class of affine surfaces in A_n. However the theory of the 2D-surfaces immersed in A_n for $n > 3$ is not developed enough and we have in support of this congecture only some preliminary results for $n = 4$, namely the construction of the induced metric h_{ij} and the proof that the affine surface in A_4 can be described by one quadratic form h_{ij}, one cubic form A_{ijk} and one quartic form B_{ijkl}.

4.2. Affine Curves and W_n-models

The affine curves $y^{\mu}(x)$ $(\mu = 1, \ldots, n)$ in A_n are completely defined by $n-1$ differential invariants $k_i(y^{(1)}, \ldots, y^{(s)})$ (see App.A). The Cartan's method for the explicit construction of k_i is based on the affine Frenet equations for the (normalized) moving frame span by

$$v_{\mu}^{k} = y_{\mu}^{(k)} \left(det(y^{(1)}, \ldots, y^{(n)})\right)^{-\frac{1}{n}}, \qquad k = 1, \ldots, n \tag{4.5}$$

i.e.

$$\partial_x g = Ag, \qquad A = (\partial_x g)g^{-1}, \qquad detg = 1 \tag{4.6}$$

where g is $(n \times n)$-frame matrix, $(g)_{\mu}^{k} = v_{\mu}^{k}$. In general A is an arbitrary matrix of $sl(n, R)$ (i.e. $TrA = 0$) and eq.(4.6) is invariant under local $SL(n, R)$ transformations. For our specific choise of the frame (4.5) , A has the particular form:

$$A = \begin{pmatrix} * & 1 & 0 & \ldots & 0 \\ * & * & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \ldots & 1 \\ * & * & * & \ldots & * \end{pmatrix}$$

which is preserved by local lower triangular $SL(n, R)$ matrix transformations:

$$\begin{aligned} \tilde{A} &= hAh^{-1} + (\partial h)h^{-1} \\ \tilde{g} &= hg, \qquad h = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ * & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & 1 \end{pmatrix}. \end{aligned} \tag{4.7}$$

We can use this residual symmetry to further fix the basis v_μ^k, choosing $(h)_k^i$ such that

$$\tilde{v}_\mu^{(2)} = \partial v_\mu^{(1)}, \qquad \tilde{v}_\mu^{(k)} = \partial \tilde{v}_\mu^{(k-1)}, \qquad k = 3, \dots, n \tag{4.8}$$

i.e.

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -k_1 & -k_2 & -k_3 & \dots & -k_{n-1} & 0 \end{pmatrix} \tag{4.9}$$

In this basis the affine Frenet eq.(4.6) reduces to*

$$\left(\partial^n + k_{n-1}\partial^{n-2} + \ldots + k_1\right) v_\mu^{(1)} = 0. \tag{4.10}$$

The invariants k_i take the simplest form when one chooses as a worldline coordinate x the invariant parameter σ (the affine arc length):

$$d\sigma = \left(det(y^{(1)}, \dots, y^{(n)})\right)^{\frac{2}{n(n+1)}} dx \tag{4.11}$$

i.e.

$$det(\dot{y}, \dots, y^{[n]}) = \pm 1, \qquad y^{[n]} = \partial_\sigma^n y(\sigma).$$

In this case as a consequence of (4.10) ($v_\mu^{(1)} = \dot{y}_\mu$), the curvatures k_i have the form:

$$\begin{aligned} k_1 &= -\det(\ddot{y}, \dots, y^{[n+1]}) \\ k_i &= -det(\dot{y}, \dots, y^{[i-1]}, y^{[m]}, y^{[i+1]}, \dots, y^{[n+1]}). \end{aligned} \tag{4.12}$$

To make all these constructions and their symmetries clearer and more explicit we consider in details two examples: $n = 2$ and $n = 3$.

* the same equations appear in the generalized Toda equations approach to W_n [16] and in the Drinfeld-Sokolov Hamiltonian reduction [23].

4.2.1. *Affine plane curves* $(n=2)$

Choosing the moving frame in the form (4.5) we can write (4.6) as follows:

$$\partial_x \begin{pmatrix} v_\mu^{(1)} \\ v_\mu^{(2)} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\frac{(I,III)}{(I,II)} & 1 \\ -\frac{(II,III)}{(I,II)} & \frac{1}{2}\frac{(I,III)}{(I,II)} \end{pmatrix} \begin{pmatrix} v_\mu^{(1)} \\ v_\mu^{(2)} \end{pmatrix} \tag{4.13}$$

where $(I,II) \equiv det(y', y'')$. The residual gauge freedom (4.7) allows to simplify eq.(4.13) . Taking

$$h = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, \qquad a = \frac{1}{2}\frac{(I,III)}{(I,II)}$$

we get $\tilde{A}$ in the form

$$\tilde{A} = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}, \qquad k = \frac{1}{2}\frac{(I,IV)}{(I,II)} + \frac{3}{2}\frac{(II,III)}{(I,II)} - \frac{3}{4}\left(\frac{(I,III)}{(I,II)}\right)^2 \tag{4.14}$$

The $(SL(2,R)\cdot T_2)$ invariant $k(x)$ is called affine curvature and it characterizes completely the affine curves in A_2 up to $SL(2,R)\cdot T_2$ (global) transformations.

A remarkable property of $k(x)$ given by (4.14) is that it transforms under x-reparametrizations $x \to \tilde{x} = f(x)$ as the $(1-D)$ stress-energy tensor ($T(z)$ of [42]):

$$\tilde{k}(x) = (f')^2 k(f) + \frac{1}{2}\left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right). \tag{4.15}$$

Under these transformations the affine velocity $v_\mu^{(1)}$ behaves as a primary field of dimension $\Delta(v) = -\frac{1}{2}$:

$$\tilde{v}_\mu^{(1)}(x) = (f')^{-\frac{1}{2}} v_\mu^{(1)}(f(x)) \tag{4.16}$$

An easy computation shows that the transformations (4.15) ,(4.16) leave invariant the corresponding $n=2$ Frenet equation:

$$(\partial^2 + k) v_\mu^{(1)} = 0. \tag{4.17}$$

As was noted by Polyakov [43] in the context of 2D-gravity the origin of such a symmetry is in the $SL(2,R)$ local gauge invariance of the non-gauge fixed Frenet equation (4.6) . The specific choise of the moving frame, as in (4.13) ,(4.14) , is equivalent to a gauge fixing of

$$A = \begin{pmatrix} A_0 & A_- \\ A_+ & -A_0 \end{pmatrix}$$

by imposing the conditions $A_- = 1$,$A_0 = 0$. Then from the infinithesimal form of (4.7)

$$\begin{aligned} \delta_\epsilon A^a &= f^{abc}\epsilon_b A_c + \partial\epsilon^a, \qquad A = T_a A^a \\ \delta_\epsilon v_\mu^{(\alpha)} &= \epsilon^a (T_a)^{\alpha\beta} v_\mu^{(\beta)}, \qquad f^{abc} = \epsilon^{abc}, (a = 0, \pm) \end{aligned} \tag{4.18}$$

and the gauge fixing conditions we obtain

$$\epsilon_0 = -\partial\epsilon_+, \qquad \epsilon_- = \frac{1}{2}\partial\epsilon_0 - \epsilon_+ k$$

Therefore the residual $SL(2,R)$ gauge symmetry

$$\begin{aligned} \delta_{\epsilon_+} k &= 2(\partial\epsilon_+)k + \epsilon_+\partial k + \frac{1}{2}\partial^3\epsilon_+ \\ \delta_{\epsilon_+} v_\mu^{(1)} &= (-\frac{1}{2}\partial\epsilon_+ + \epsilon_+\partial) v_\mu^{(1)} \end{aligned} \tag{4.19}$$

is nothing but the diffeomorphisms generated by the *affine curvature k*.

Following the analogy with the curves in E_n (or $M_{p,q}$) which have particle interpretation, it is natural to consider the affine curves as trajectories of the "affine particles". One can take as an action for such a particle the affine arc lenght (4.11) , i.e.

$$S_{n=2} = \int \left(det(y', y'')\right)^{\frac{1}{3}} dx$$

In the case of the plane affine curves ($n = 2$) one can construct one more ($SL(2,R) \cdot T_2$ and reparametrization) invariant

$$S_1 = \int det(y, y') dx$$

which has a meaning of a number of the selfintersections of the affine curve. The sum of these two integral invariants (4.2) can be taken as a total affine particle action.

The most remarkable property of the affine plane curves ($\equiv$ particles) is that they appear as a classical limit ($c \to -\infty$) of the Virasoro algebra minimal models [42, 44]. To prove this statement we have only to rephrase the Al.Zamolodchikov's arguments [44] concerning certain covariant differential operators appearing in the light-cone quantization of 2D-gravity. The first observation is that in this limit the primary field ϕ_{21} behaves as a primary field of dimension $\Delta_{21}(-\infty) = -1/2$, i.e.

$$\Delta_{21}(c) = \frac{5 - c + \sqrt{(1-c)(25-c)}}{16} \to \Delta_{21}(-\infty) = -\frac{1}{2} - \frac{9}{2c}. \tag{4.20}$$

To identify it with the affine velocity $v_\mu^{(1)}$ (4.16) we have to prove that the null-vector condition [42] :

$$\left(L_{-1}^2 - \frac{2(2\Delta_{21}+1)}{3}L_{-2}\right) \mid \phi_{21}\rangle = 0 \tag{4.21}$$

or equivalently

$$\partial_x^2\phi_{21}(x) - \frac{2(2\Delta_{21}+1)}{3} : T(x)\phi_{21}(x) := 0$$

has (4.17) as a classical limit. This is indeed the case if we apply the Zamolodchikov's limiting procedure [44]* :

$$\begin{aligned} L_m &= \frac{c}{6}l_m, \qquad m \le -2, \qquad l_{-p-2} = \frac{1}{p!}\partial_x^p k, \qquad p \ge 0 \\ L_m &= l_m, \qquad m \ge -1, \qquad l_{-1} = \partial_x \end{aligned} \tag{4.22}$$

and substitute (4.20) in (4.21) . There is one point to clarify: we have two velocities $v_1^{(1)}$ and $v_2^{(1)}$ ($\mu = 1,2$) and only one primary field ϕ_{21} in correspondence with them. One can easily remove this apparent contradiction considering the standart parametrization of the affine curves: $y^1 = x$, $y^2 = g(x)$ in which the fact that normalized velocity $v_\mu^{(1)}(x)$ has only one independent component $v_2^{(1)} = \frac{g'}{\sqrt{g''}}$ is manifest.

Reversing all these arguments we can state that the field ϕ_{21} represents the quantum velocity of the affine particle in A_2. Broadly speaking the Virasoro minimal models can be considered as a quantum geometry of the affine plane curves. However such a statement should be supported by an appropriate geometrical interpretation for the other primary fields ϕ_{nm} of the corresponding models which is missing at the moment. Our conjecture is that the Virasoro minimal models are QFT version of a system of interacting affine particles.

The richness of the affine differential geometry is not exhausted by its role in the geometry of the CFT's. It offers us one more miracle: the affine geometrical interpretation of the KdV equation. Consider a family of affine plane curves $y^\mu(\sigma,t)$ generated by the curve $y^\mu(\sigma,t_0)$ of curvature $k(\sigma,t_0)$ by small deformations in its tangent directions $\partial_\sigma y^\mu$:

$$y^\mu(\sigma,t) = y^\mu(\sigma,t_0) + (t-t_0)k\partial_\sigma y^\mu + \dots \tag{4.23}$$

---

* l_m are the generators of the classical Virasoro algebra

(σ is the invariant parameter (4.11)). As a consequence of (4.23) we have:

$$\partial_t y^\mu = k \partial_\sigma y^\mu. \tag{4.24}$$

In this parametrization the equations defining the curve $y^\mu(\sigma, t)$ take the following form:

$$\left(\partial_\sigma^2 + k(\sigma, t_0)\right) \partial_\sigma y^\mu(\sigma, t_0) = 0 \tag{4.25}$$

and

$$k(\sigma, t_0) = -\epsilon^{\mu\nu} \partial_\sigma^2 y_\mu \partial_\sigma^3 y_\nu \tag{4.26}$$

The question we adress is: how the affine curvature $k(\sigma, t)$ changes under these deformations (i.e. if y^μ changes acording to (4.24))? By simple differentiations of the eqs.(4.24) and (4.25) we get:

$$\partial_t k = 6k\partial_\sigma k + \partial_\sigma^3 k \tag{4.27}$$

or

$$\partial_t k = \partial_\sigma \theta, \qquad \theta = 3k^2 + \partial_\sigma^2 k \tag{4.28}$$

Therefore the affine curvature k of the (continuous) family of curves (4.23) must satisfy the KdV equation (4.27) .

Although we have no full understanding of the deformation (4.23) at the classical level we shall address here few questions concerning its quantum (i.e. Virasoro minimal models) counterpart:

a) while the quantum affine curves are related to the Virasoro minimal models ($\partial_\sigma y \sim \phi_{21}, k = \frac{6}{c}T$) one could wonder how the Zamolodchikov's ϕ_{21} flows [45,46] of the Virasoro minimal models fits with the geometry of the certain families of affine plane curves.

b) what is the quantum (perturbed Virasoro models ?) counterpart of the KdV? Its form (4.28) suggests to look for an answer in the perturbed conservation laws of the stress-tensor $T = \frac{c}{6}k$ and its descendents [45,46], i.e. to interprate KdV as a classical limit of these conservation laws equations.

4.2.2. *3-D space affine curves*

In the basis (4.5) the Frenet equation for a curve $y^\mu(x)$ immersed in A_3 has the form (4.6) with

$$A = \begin{pmatrix} -a & 1 & 0 \\ 0 & -a & 1 \\ b_1 & b_2 & 2a \end{pmatrix}$$

where

$$b_1 = \frac{(II,III,IV)}{(I,II,III)}, \qquad b_2 = -\frac{(I,III,IV)}{(I,II,III)}, \qquad a = 1/3\partial_x \ln(I,II,III).$$

The residual gauge symmetry preserving this form of A forms an abelian subgroup of $SL(3,R)$:

$$h_\alpha = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \alpha^2 + \alpha\prime & 2\alpha & 1 \end{pmatrix}, \qquad h_\alpha h_\beta = h_{\alpha+\beta}.$$

Choosing $\alpha = a$, i.e. keeping the tangent $v_\mu^{(1)}$ unchanged and changing the "affine normals" $v_\mu^{(2)}$,$v_\mu^{(3)}$ according to (4.8) we get

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k_1 & -k_2 & 0 \end{pmatrix} \tag{4.29}$$

with

$$\begin{aligned} k = k_2 =& 2\frac{(I,III,IV)}{(I,II,III)} + \frac{(I,II,IV)}{(I,II,III)} - \frac{4}{3}\left(\frac{(I,II,IV)}{(I,II,III)}\right)^2 \\ w \equiv k_1 - \frac{1}{2}k_2' =& -\frac{5}{3}\frac{(II,III,IV)}{(I,II,III)} - \frac{1}{6}\frac{(I,II,VI)}{(I,II,III)} - \frac{5}{6}\frac{(I,III,V)}{(I,II,III)} - \frac{20}{27}\left(\frac{(I,II,IV)}{(I,II,III)}\right)^3 + \\ &+\frac{5}{3}\frac{(I,II,IV)(I,III,IV)}{(I,II,III)^2} + \frac{5}{6}\frac{(I,II,IV)(I,II,V)}{(I,II,III)^2} \end{aligned} \tag{4.30}$$

From these explicit constructions one can easily chek that the *affine curvature* k and the *affine torsion* w are indeed $SL(3,R).T_3$ (global) invariants. They define completely each affine curve immersed in A_3.

Another simple cosequence of eqs.(4.30) are the transformation laws of the affine invariants k and w under reparametrizations $x \to \tilde{x} = f(x) = x + \epsilon(x)$:

$$\begin{aligned} \delta_\epsilon k &= 2(\partial\epsilon)k + \epsilon\partial k + 2\partial^3\epsilon \\ \delta_\epsilon w &= 3(\partial\epsilon)w + \epsilon\partial w \end{aligned} \tag{4.31}$$

(i.e. $\Delta(k) = 2, \Delta(w) = 3$). The affine velocity $v_\mu = y'_\mu(I,II,III)^{-\frac{1}{3}}$ transforms as a primary field of dimension $\Delta(v) = -1$:

$$\delta_\epsilon v_\mu = -(\partial\epsilon)v_\mu + \epsilon\partial v_\mu. \tag{4.32}$$

These tranformations do not exhaust the symmetries of the gauge fixed Frenet equation for the affine curve in A_3:

$$\left[\partial^3 + k\partial + \left(\frac{1}{2}k' + w\right)\right] v_\mu = 0. \tag{4.33}$$

Applying Polyakov's method [43] for the residual $SL(3,R)$-gauge transformations encoded in the specific form (4.29) of $\bar{A} = A^a T_a$ $(a = 1, \ldots, 8)^\star$, we have first to solve part of the eqs.(4.18) for 6 of the 8 parameters ϵ_a:

$$\epsilon_i(x) = \epsilon_i(x; \epsilon, \eta; k, w), \qquad i = 1, \ldots, 6.$$

Substituting these ϵ_i in the remaining equations for δk, δw and δv_μ we get on top of the reparametrizations (4.31) , (4.32) another set of one parameter (η) transformations which leave invariant (4.33) :

$$\begin{aligned}
\delta_\eta k &= 3(\partial\eta)w + 2\eta\partial w \\
\delta_\eta w &= -\frac{1}{6}\partial^5\eta - \frac{5}{6}(\partial^3\eta)k - \frac{5}{4}(\partial^2\eta)\partial k - \frac{3}{4}(\partial\eta)\partial^2 k - \\
&\quad - \frac{1}{6}\eta\partial^3 k - \frac{2}{3}(\partial\eta)k^2 - \frac{2}{3}\eta k\partial k \\
\delta_\eta v_\mu &= \left[\frac{2}{3}\partial^2\eta + \frac{2}{3}\eta k - (\partial\eta)\partial + \eta\partial^2\right] v_\mu.
\end{aligned} \tag{4.34}$$

The explicit form of the transformations (4.31) and (4.34) suggests an interpretation of the curvature k and the torsion w as generators of the (classical) W_3-algebra [21,22]. Accepting the standard definitions

$$\delta_\epsilon A(x) = \int dz\epsilon(z)\{k(z), A(x)\}, \qquad \delta_\eta A(x) = \int dz\eta(z)\{w(z), A(x)\}$$

we can derive from (4.31) the Virasoro algebra in terms of Poisson brackets:

$$\{k(x), k(z)\} = (k(x) + k(z))\,\partial_x\delta(x - z) + 2\partial_x^3\delta(x - z)$$

and from (4.34) the corresponding ones for the spin-3 current $w(x)$ [21,22].

Similarly to the affine plane curves the affine curves in A_3 can be considered as a classical limit $(c \to -\infty)$ of the W_3-minimal models.This identification is

$\star$ T_a are 3×3-matrices of the adjoint representation of $SL(3,R)$

based on the transformation properties (4.32) ,(4.34) of v_μ and of the Frenet equation (4.33) . Consider the field $\phi\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ of dimension and "torsion" [21] (see (3.21))

$$\begin{aligned}\Delta\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} &= \frac{14-c+\sqrt{(2-c)(98-c)}}{36}\\ w^2\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} &= \frac{2}{9}\Delta^2\left[2b^2(\Delta+\frac{1}{5})-\frac{1}{5}\right]\end{aligned}\tag{4.35}$$

where $b^2=\frac{16}{22+5c}$.

It has degeneraces at levels 1 and 2, the corresponding null-vectors being (3.21):

$$\begin{aligned}&(2\Delta W_{-1}-3wL_{-1})\mid\phi\rangle=0\\ &\Big(\Delta(5\Delta+1)W_{-2}-12wL_{-1}^2+6w(\Delta+1)L_{-2}\Big)\mid\phi\rangle=0.\end{aligned}\tag{4.36}$$

We multiply the second level null-vector by W_{-1} and commute it to the right using the W_3-algebra and the first null-vector (4.36) . This results in an effective third level null-vector:

$$\Big(c_1L_{-1}^3+c_2L_{-2}L_{-1}+c_3L_{-3}+c_4W_{-3}\Big)\mid\phi\rangle=0\tag{4.37}$$

where

$$\begin{aligned}c_1&=-\frac{288w^2}{\Delta(5\Delta+1)}\\ c_2&=-5\Delta(5\Delta+1)b^2+\frac{144(\Delta+1)w^2}{\Delta(5\Delta+1)}\\ c_3&=-5\Delta^2(5\Delta+1)b^2-\frac{45(\Delta+1)(\Delta-3)w^2}{\Delta(5\Delta+1)}\\ c_4&=18(\Delta+1)w.\end{aligned}$$

Following the Zamolodchikov's limiting procedure (4.22) and requiring

$$\begin{aligned}&W_m=\frac{c}{24}w_m, \qquad m\le -3, \qquad w_{-m-3}=\frac{1}{m!}\partial_x^m w(x), \qquad m\ge 0\\ &W_m=w_m, \qquad m\ge -2\\ &L_n=\frac{c}{24}l_n, \qquad n\le -2, \qquad l_{-p-2}=\frac{1}{p!}\partial_x^p k, \qquad p\ge 0\\ &L_n=l_n, \qquad n\ge -1, \qquad l_{-1}=\partial_x\end{aligned}$$

we find that (4.33) is the $c\to-\infty$ limit of (4.37) and that $\phi\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ is the quantum counterpart of the affine velocity v_μ, ($\mu=1,2,3$).

The origin of the classical Z_3-symmetry is in the permutation symmetry of the components of v_μ: (v_1, v_2, v_3). The Z_3-doublet of fields of equal dimensions $\phi\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}$ and $\phi\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ (and opposite Z_3-charge) is in correspondence with the two independent components of v_μ in the parametrization $y_1 = x$, $y_2 = y_2(x)$, $y_3 = y_3(x)$. The affine particle interpretation of the W_3-models is based on the followihg action:

$$S_{n=3} = \int \left(\epsilon^{\mu\nu\rho} y'_\mu y''_\nu y'''_\rho\right)^{\frac{1}{6}} dx.$$

Analogously in the general case of W_n-algebras one considers the field $\phi(1\ldots 1 \mid 2 \ldots 1)$ (see [22]) of dimension (3.28):

$$\Delta(1\ldots 1 \mid 2\ldots 1) = \frac{(n-1)(2n+1) - c + \sqrt{(c-n+1)(c-n+1-4n(n^2-1)}}{4n^2}.$$

In the classical limit this expression behaves as

$$\Delta \to \frac{1-n}{2} - \frac{(n^2-1)^2}{2c}$$

and therefore it is natural to consider the field ϕ as a quantum counterpart of the normalized affine velocities $v_\mu^{(n)} = (\partial_x y_\mu)\left(det(y^{(1)}, \ldots, y^{(n)})\right)^{-\frac{1}{n}}$.

In addition it exhibits $n-2$ degeneraces at level 1 and one at level 2.Our conjecture is that combining them in the way analogous to the case of W_3-algebra, they result in an effective n-th level null-vector.Its classical limit should be the n-dimensional affine Frenet equation (4.10) . In other words our conjecture is that W_n-minimal models ($n \geq 2$) have as a classical limit the geometry of the affine curves in A_n and that the geometric invariants k_i describing these curves appear as generators of the W_n-algebras (i.e. the higher spin currents).

Among the many open questions concerning the geometry of W_n CFT's we should mention the following two:

a) our present discussion was concentrated on the geometry of the $1-D$ ("left mouvers") W_n-models only. The consistent geometric interpretation of the 2D W_n CFT's should be related to the geometry of the affine surfaces in A_n (in conformal gauge !). Although the remaining part of this Chapter is devoted to the study of the affine surfaces we don't have a complete answer to this question.

b) the affine geometry of curves in A_n is based on the group $SL(n,R)\cdot T_n$. The (gauged) $SL(n,R)$-symmetries of the corresponding Frenet equations are in the origin of its relations with W_n CFT's. The question arises: are there other geometries based on the same group? An example of such geometries are the projective curves (and surfaces) in RP^{n-1} (with $SL(n,R)$ as a group of motion). Our preliminary results [47] confirme this possibility: W_n-minimal models can have as classical limits the geometry of projective curves in RP^{n-1} as well.

4.3. Affine Surfaces

Consider 2D differentiable manifold M_2 immersed in the affine space A_3:

$$y^{\mu}(x_i): \qquad M_2 \rightarrow A_3, \qquad \mu = 1,2,3; \quad i = 1,2$$

The affine differential geometry studies the properties of surfaces invariant under unimodular affine transformations: $\tilde{y}^{\mu} = \alpha^{\mu\nu} y_{\nu} + \beta^{\mu}$, where $(\alpha,\beta) \in SL(3,R) \cdot T_3 \equiv G$. Introducing at each point of M_2 a moving frame $\{y^{\mu}_{,i}, N^{\mu} : a = det(y_{,1}, y_{,2}, N) \neq 0\}$ we can write the surface analog of the affine Frenet equations in the form:

$$\partial_i g = A_i g, \qquad g = \begin{pmatrix} y^1_{,1} & \cdots & y^3_{,1} \\ y^1_{,2} & \cdots & y^3_{,2} \\ N^1 & \cdots & N^3 \end{pmatrix} a^{-\frac{1}{3}} \tag{4.38}$$

where $A_i = (\partial_i g) g^{-1}$. The problem is to realize the elements of A_i's in terms of covariant geometric objects characterizing the geometry (extrinsic!) of a surface immersed in A_3. According to the *Radon theorem* [48] each affine surface in A_3 is defined uniquely (modulo G-transformations) by its two fundamental forms:

a) the first one is the metric quadratic form $\varphi = h_{ij} dx^i dx^j$

b) the second one is the Fubini-Pick cubic form $\psi = A_{ijk} dx^i dx^j dx^k$ satisfying the apolarity condition

$$h^{ij} A_{ijk} = 0. \tag{4.39}$$

The induced (pseudo) Riemanian metric* $h_{ij}(x_k)$ has the following explicit form

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$\star$ often called Blaschke-Berwald metric

[48]:

$$h_{ij} = H_{ij}\left(det(H_{ij})\right)^{-\frac{1}{4}}, H_{ij} = \frac{1}{2}\epsilon^{kl}\epsilon^{\mu\nu\rho}y^{\mu}_{,ij}y^{\nu}_{,k}y^{\rho}_{,l}. \qquad (4.40)$$

In terms of the y^{μ} derivatives the Fubini-Pick form ψ is given by

$$\psi = \frac{det(y_{,1}, y_{,2}, d^3y)}{(deth_{ij})^{\frac{1}{4}}} - \frac{3}{2}d\varphi. \qquad (4.41)$$

Choosing the affine normal N^{μ} in the form

$$N^{\mu} = \frac{1}{2}\Delta(h)y^{\mu}$$

we can finaly rewrite the matrices A_i in terms of h_{ij},A_{ijk} and their first derivatives. The result are the following affine structure equations [49, 37]

$$\begin{aligned} y^{\mu}_{,ij} &= \Gamma^{k}_{ij}y^{\mu}_{,k} + A^{k}_{ij}y^{\mu}_{,k} + h_{ij}N^{\mu} \\ N^{\mu}_{,i} &= -B^{k}_{j}y^{\mu}_{,k} \end{aligned} \qquad (4.42)$$

where Γ^{k}_{ij} are the usual Christoffel simbols of h_{ij} (i.e. the Levi-Cevita connection) and $B_{ij} = h_{ik}B^{k}_{j}$ is the second quadratic form. The integrability conditions of the system of equations (4.42) are given by the affine analog of the Ricci-Mainardi-Gauss-Codazzi equations [49]:

$$\begin{aligned} R_{ijkl} &= h^{nm}\left(A_{njk}A_{mil} - A_{njl}A_{mik}\right) + \frac{1}{2}\left(h_{ik}B_{jl} - h_{il}B_{jk} - h_{jk}B_{il} + h_{jl}B_{ik}\right) \\ B_{ij;k} - B_{ik;j} &= A^{l}_{ij}B_{lk} - A^{l}_{ik}B_{lj} \\ A_{ijk;l} - A_{ijl;k} &= \frac{1}{2}\left(h_{ik}B_{jl} - h_{il}B_{jk} + h_{jk}B_{il} - h_{jl}B_{ik}\right) \end{aligned} \qquad (4.43)$$

where $B_{ij;k}$ denotes the covariant derivative of B_{ij}:

$$B_{ij;k} = \partial_k B_{ij} - \Gamma^{l}_{ik}B_{lj} - \Gamma^{l}_{jk}B_{li}$$

and R_{ijkl} is 2D Riemann tensor of h_{ij}. As is well known these equations in the matrix form (4.38) of the eq.(4.42) are nothing but the zero curvature condition for the $SL(3, R)$-connection A_i:

$$\partial_i A_j - \partial_j A_i + [A_j, A_i] = 0. \qquad (4.44)$$

We can further simplify eqs.(4.43)

$$\begin{aligned} R &= J + 2H \\ A^{s}_{jk;s} &= Hh_{jk} - B_{jk} \end{aligned} \qquad (4.45)$$

where $R = R_{1212}$ is the scalar curvature, $H = \frac{1}{2}B_{ij}h^{ij}$ is the mean curvature and $J = A_{ijk}A^{ijk}$ is the so called Pick invariant. The forms B_{ij},h_{kl} and A_{ijk}

define three independent geometric invariants (see App.A) which describe the affine surfaces in A_3: $R(h)$,$\det h^{-1}B = k_1 k_2 = K$ (affine Gauss curvature),[†] $Trh^{-1}B = k_1 + k_2 = 2H$, J satisfying the constraint (4.45) . One could find this description to contradict the Radon theorem: we consistently use one more quadratic form B_{ij} to characterize the affine surfaces. In fact we must formulate the above theorem more precisely. One indeed needs three independent forms h_{ij},B_{ij} and A_{ijk} to write the eqs.(4.42) .It is only on the solution of the integrability conditions (4.43) (or (4.45)) when we can express B_{ij} in terms of h_{ij}, A_{ijk} and their derivatives.

In this breef introduction to the theory of the affine surfaces we left unanswered two principal questions: (a) what is the geometrical meaning of the cubic form A_{ijk}? (b) how one can succeed in inducing a metric h_{ij} on the surface having no any metric properties in A_3? An elegant answer of these questions one can find in the modern version of the affine differential geometry[*] [38] of surfaces based on the Mayer-Cartan structure equations for the transformation group $G = SL(3, R) \cdot T_3$.

Consider the 1-forms ω^α,ω^β_α dual to the frame fields (y^μ, t_α), $\alpha = 1, 2, 3$, where t_α is an arbitrary (normalized, $\det(t_1, t_2, t_3) = 1$) local A_3 frame (in particular one can take $t^\mu_i = y^\mu_{;i}$ and $t^\mu_3 = N^\mu$ as such a frame). In other words we have to construct a principle frame bundle over A_3 with structure group G and ω^α,ω^α_β are the corresponding G-connections. According to these definitions we have:

$$dy^\mu = \omega^\alpha t^\mu_\alpha, \qquad dt^\mu_\alpha = \omega^\beta_\alpha t^\mu_\beta \tag{4.46}$$

and calculating $d^2 = 0$ we obtain the structure equations of G:

$$\begin{aligned} d\omega^\alpha &= \omega^\beta \wedge \omega^\alpha_\beta \\ d\omega^\beta_\alpha &= \omega^\gamma_\alpha \wedge \omega^\beta_\gamma, \qquad \sum_{\alpha=1}^{3} \omega^\alpha_\alpha = 0. \end{aligned} \tag{4.47}$$

We have to impose $\omega^3 = 0$ in order to restrict these forms to the surface M_2. However this is not an independent requirement if we choose t_1 and t_2 to be

† the common eigenvalues of the pair of quadratic forms h_{ij} and B_{ij} are called principal affine curvatures

* at this point we are closely following ref.[50].

tangent to M_2. As a cosequence of (4.47) we have $\omega^i \wedge \omega_i^3 = 0$ and therefore

$$\omega_i^3 = \sum_{i=1}^{2} H_{ij}\omega^j, \qquad H_{ij} = H_{ji}. \tag{4.48}$$

Let $\det H_{ij} \neq 0$, then the quadratic form

$$\varphi = h_{ij}\omega^i\omega^j, \qquad h_{ij} = (\det H_{ij})^{-\frac{1}{4}} H_{ij}$$

is affine invariant and it defines a (pseudo) Riemannian metric on M_2. The condition

$$\omega_3^3 = -\frac{1}{4} d \ln \det H, \qquad d\omega_3^3 = 0$$

allows to define an affine invariant normal vector $N^\mu = (\det H)^{\frac{1}{4}} t_3^{\mu}$ ‡ . Substituting this requirement in the structure equations (4.47) we get $\omega_3^i \wedge \omega_i^3 = 0$, i.e.

$$\omega_3^i = -\sum B^{ij}\omega_j^3 = -\sum B^{ij} H_{jk}\omega^k. \tag{4.49}$$

Eq.(4.49) gives rise to another affine invariant quadratic form

$$\theta = -\omega_3^i \omega_i^3 = B^{ij}\omega_i^3\omega_j^3. \tag{4.50}$$

The remaining two eqs.(4.47)

$$d\omega_i^3 = \omega_i^j \wedge \omega_j^3$$

together with (4.48) imply the introduction of the Fubini-Pick form A_{ijk}:

$$\left(dh_{ik} - h_{il}\omega_k^l - h_{kl}\omega_i^l\right) \wedge \omega^k = 0$$

and therefore:

$$Dh_{ik} \equiv dh_{ik} - h_{il}\omega_k^l - h_{kl}\omega_i^l = A_{ikj}\omega^j. \tag{4.51}$$

The l.h.s. of (4.51) is by definition the covariant derivative of h_{ik} defined with respect to the induced affine connection ω_i^j. While in general $Dh_{ik} \neq 0$ we conclude that ω_i^j does not coincide with the Levi-Cevita connection $\tilde{\omega}_i^j$ of the affine metric h_{ij}. This observation makes clear the geometrical meaning of the

‡ we choose for simplicity $\det H_{ij} = const, (\omega_3^3 = 0)$, which holds in the case of light-cone gauge for h_{ij}

cubic form: it measures the deviation of the induced affine connection ω_i^j from the Levi-Cevita connection $\bar{\omega}_i^j$ of the induced metric h_{ij}:

$$\omega_j^i - \bar{\omega}_j^i = h^{ik} A_{jkl} \omega^l. \tag{4.52}$$

The apolarity condition (4.39) simply follows from (4.51) and $\det H_{ij} = const.$

The last step is the derivation of the affine structure equations (4.42) from eq.(4.46) . Choosing local coordinates x^i such that $\omega^i = dx^i$ we have

$$\bar{\omega}_j^i = \Gamma^i_{jk} dx^k, \qquad t_i^\mu = \frac{dy^\mu}{dx_i}$$

and therefore

$$\omega_j^i = \left(\Gamma^i_{jk} + h^{il} A_{jlk}\right) dx^k. \tag{4.53}$$

It remains to substitute (4.48) ,(4.49) and (4.53) in (4.46) . As a result we get eqs.(4.42) .

Among the various interesting (and unusual) properties of the affine surfaces we should mention that the affine and reparametrization invariant area functional is given by:

$$A = \int_{M_2} \mid \det h_{ij} \mid^{\frac{1}{4}} d^2 x. \tag{4.54}$$

The appearence of a fourth root is not surprising.It reflects the fact that h_{ij} by definition (4.40) contains four derivatives (instead of two in the euclidean case $M_2 \to E_3$). One can further use (4.54) as an action for the "affine strings". Concerning the problem of writing the most general action for the affine string in A_3 one should add to (4.54) the linear combination of the other independent invariants describing the embedding of M_{1+1} in A_3, i.e. J, H^2 and $\det B = K$. Such an action will represent the affine analog of the Polyakov rigid string action [8].

4.4 W_3-Gravity as a Geometry of the Affine Surfaces

The formal definition of the W_3-gravity is as a Borel gauge fixed $SL(3, R)$-WZW model [12, 20]. Its light-cone formulation [20] is a 2D field theory of interacting spin-2 h_{ij} and spin-3 A_{ijk} fields satisfying specific "trace anomalies" equations. The basic feature of this theory is its W_3-symmetry mixed with the "hidden" $SL(3,R)$-current algebra symmetries.

From the other side many of the ingredients of the "induced W_3-gravity" can be recognized in the geometry of the affine surfaces immersed in A_3. The problem is to find the exact correspondence between them. Our main observation is that *W_3-gravity coincides with the geometry of 2D affine surfaces of constant affine mean curvature* $\left(H = \frac{1}{2}(k_1 + k_2)\right)$ *immersed in A_3*. As a byproduct this fact offers a new affine geometrical interpretation of the gauged WZW-models.

Let us first rewrite the affine structure equations (4.38) (or (4.42)) in light-cone gauge: $h_{--} = 0$, $h_{+-} = \frac{1}{2}$, $h_{++} = h$; $A_{---} = C$, $A_{+++} = D$ and the apolarity condition imposes $A_{-++} = h^2C$, $A_{--+} = hC$. Taking as a moving frame $y^\mu_\pm = \partial_\pm y^\mu$ and $N^\mu = \frac{1}{2}\Delta(h)y^\mu$ the eqs.(4.38) and (4.42) in the following matrix form:

$$\partial_\pm \begin{pmatrix} y_+ \\ y_- \\ N \end{pmatrix} = A_\pm \begin{pmatrix} y_+ \\ y_- \\ N \end{pmatrix} \tag{4.55}$$

where

$$\begin{aligned} A_- &= \begin{pmatrix} -2hC & 2C & 0 \\ \partial_- h - 2h^2 C & 2hC & \frac{1}{2} \\ 4hB_{--} - 2B_{+-} & -2B_{--} & 0 \end{pmatrix}, \\ A_+ &= \begin{pmatrix} \partial_- h - 2h^2 C & 2hC & \frac{1}{2} \\ \partial_- h^2 + \partial_+ h + 2D - 4h^3 C & 2h^2 C - \partial_- h & h \\ -2B_{++} + 4hB_{+-} & -2B_{+-} & 0 \end{pmatrix}. \end{aligned} \tag{4.56}$$

Note that the specific form of $A_\pm$ reflects our particular choise of the frame. We can further gauge fix the initial local $SL(3,R)$-gauge symmetry of eqs.(4.38) taking into account that the local $SL(2,R)$ transformations of the tangents only, i.e. $g_0(x_\pm) : t_i \to \tilde{t}_i$ which do not change our affine surface M_{1+1}. The transformations which mix the tangents t_i with the normal N map a given surface to another one and describe the flow of the invariants J, K, H, i.e. certain family of surfaces.

Using the $SL(2,R)$ gauge freedom we choose a new tangent frame $\tilde{t}_1 = y_-$,

$\tilde{t}_2 = y_+ - hy_-$ by an appropriate gauge transformation

$$\tilde{A}_\pm = g_0 A_\pm g_0^{-1} + (\partial g_0) g_0^{-1}$$
$$\tilde{t}_\alpha = g_0 t_\alpha, \qquad g_0 = \begin{pmatrix} 1 & 0 & 0 \\ -h & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.57}$$

In this basis the eqs.(4.55) , (4.56) have the following simple form:

$$\tilde{A}_- = \begin{pmatrix} 0 & 2C & 0 \\ 0 & 0 & \frac{1}{2} \\ -H & -2B_{--} & 0 \end{pmatrix}, \qquad \tilde{A}_+ = \begin{pmatrix} \partial_- h & 2hC & \frac{1}{2} \\ A & -\partial_- h & \frac{h}{2} \\ -2B_{++} + 2hB_{+-} & -2B_{+-} & 0 \end{pmatrix} \tag{4.58}$$

where $H = 2B_{+-} - 2hB_{--}$ is the afine mean curvature and $A \equiv 2(D - h^3 C) = \frac{1}{8}\frac{J}{C}$ (J is the Pick invariant).

Having fixed part of the local $SL(3,R)$ gauge symmetries of the affine structure equations the main question is: which are the remaining gauge symmetries and how the corresponding infinithesimal transformations look like? At this point we have to remember once more the Polyakov method* [43] for deriving the residual gauge transformations we already have used in the case of affine curves (see Sect.2.2). By permutations of rows and coloumns we can write $\tilde{A}_-$ in the form similar to the one of eq.(4.29) :

$$\tilde{A}'_- = \begin{pmatrix} 0 & 0 & 2C \\ -H & 0 & -2B_{--} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}. \tag{4.59}$$

To obtain the W_3-algebra we must restrict ourselves to the *subclass of affine surfaces of constant mean curvature*, i.e. *to impose the following auxiliary condition*: $H = -\frac{1}{2}$. Then substituting the gauge fixing conditions (and $H = -\frac{1}{2}$) encoded in the particular form of $\tilde{A}_-$ in the $SL(3,R)$ gauge transformation laws:

$$\delta_\epsilon A^a = f^{abc}\epsilon^b A^c + \partial\epsilon^a, \qquad A = T_a A^a, \qquad a = 1, \ldots, 8$$

we obtain the W_3 infinithesimal transformations (see App.B for the proof):

$$\begin{aligned} \delta T =& -4\partial_-^3\epsilon T + 2\partial_-\epsilon T + \epsilon\partial_- T + 3\partial_-\eta W + 2\eta\partial_- W \\ \delta W =& \frac{16}{3}\partial_-^5\eta - \frac{40}{3}\partial_-^3\eta T - 20\partial_-^2\eta\partial_- T - 12\partial_-\eta\partial_-^2 T - \frac{8}{3}\partial_-^3 T + \\ &+ \frac{16}{3}(\partial_-\epsilon)T^2 + \frac{8}{3}\eta\partial_- T^2 + 3(\partial_-\epsilon)W + \epsilon\partial_- W \end{aligned} \tag{4.60}$$

$\star$ see also [24]

where $T = -2B_{--}$, $W = 4(C - \partial_- B_{--})$.

Therefore the W_3-symmetries of the affine (gauge fixed) structure equations for the affine surfaces with constant mean curvature are generated by the $_{--}$ and $_{---}$ components of the second quadratic form B_{ij} and the Fubini-Pick form A_{ijk} respsctively.

We can easily apply the above procedure to analyze the restricted gauge transformations of $\tilde{A}_+$ given by (4.58) with $H = -\frac{1}{2}$. As it is shown in App.B this gives us the transformations of the (improved) metric $\bar{h} = h + 4\partial_- C$ and the field $A = 2(D - h^3 C)$ (i.e. improved $_{+++}$ component of A_{ijk}) coupled to the spin-3 current W:

$$\begin{aligned} \delta\bar{h} &= \left[\partial_+ - \bar{h}\partial_- + \partial_-\bar{h}\right]\epsilon + \\ &+\frac{8}{3}\left[2A\partial_-^3 - 3(\partial_- A)\partial_-^2 - 5(\partial_-^2 A)\partial_- - 2(\partial_-^3 A) + 4T\partial_- A\right]\eta \\ \delta A &= \left[\partial_+ + 2\partial_-\bar{h} - \bar{h}\partial_-\right]\eta - 2\partial_-\epsilon A + \epsilon\partial_- A. \end{aligned} \tag{4.61}$$

Comparing our results with the results of refs.[43, 12] we can conclude that the theory of the affine surfaces of constant mean curvature have the same symmetries and the same equations as the Borel gauge fixed $SL(3,R)$-WZW model.†

The next question is how the hidden $SL(3,R)$ current algebra symmetry and the specific "trace anomalies" equations (see ref.[20]) appear in the context of the affine surface theory. The answer is extremely simple: they are nothing but the integrability conditions (4.43) of the affine structure equations, i.e.

$$\partial_+\tilde{A}_- - \partial_-\tilde{A}_+ + \left[\tilde{A}_-, \tilde{A}_+\right] = 0.$$

These equations in the light-cone gauge and for $H = -\frac{1}{2}$ have the form:

$$\begin{aligned} &\partial_-^2 h = \frac{1}{2}A\left[W - 2\partial_- T\right] - \frac{1}{4} \\ &\partial_- A = -\left(B_{++} + \frac{1}{2}h^2 T + \frac{1}{2}h\right) \\ &\partial_-\left(B_{++} + \frac{1}{2}h^2 T + \frac{1}{2}h\right) = -\frac{1}{2}AT \\ &\left[\partial_+ - 2\partial_- h - h\partial_-\right]T = (\partial_- A)(W - 2\partial_- T) \\ &\left[\partial_+ - 3\partial_- h - h\partial_-\right](W - 2\partial_- T) = T. \end{aligned} \tag{4.62}$$

Eliminating B_{++} from the system (4.62) and performing some tedious manipu-

† note that in this gauge the Witten term is identicaly zero and the WZW equations coincide with (4.38) , i.e. the σ-model equations

lations on the remaining equations we find that T, W and $\tilde{h}$, A should satisfy the following "anomalies equations":

$$
\begin{aligned}
&-4\partial_-^3\tilde{h} = \left(\partial_+ - 2\partial_-\tilde{h} - \tilde{h}\partial_-\right)T - 3(\partial_-A)W - 2A\partial_-W \\
&\frac{16}{3}\partial_-^5 A = \left(\partial_+ - 3\partial_-\tilde{h} - \tilde{h}\partial_-\right)W + \\
&\qquad + \left[\frac{40}{3}\partial_-^3 A + 20(\partial_-^2 A)\partial_- + 12(\partial_- A)\partial_-^2 + \frac{8}{3}A\partial_-^3\right]T - \frac{8}{3}(2\partial_- A + A\partial_-)T^2
\end{aligned}
\tag{4.63}
$$

We can further impose $T \approx 0$ and $W \approx 0$ as classical constraints (which gives W_3) and then we obtain as equations of motion for $\tilde{h}$ and A

$$
\begin{aligned}
\partial_-^3\tilde{h} &= 0 \\
\partial_-^5 A &= 0.
\end{aligned}
\tag{4.64}
$$

The geometrical meaning of these constraints is that the Pick invariant J, the Gauss curvature $K = \det B$ and the scalar curvature $R = -\frac{1}{2}\partial_-^2 h$ ($R \neq 2K$) are constants, i.e. all invariants describing the embedding of the affine surface M_{1+1} in A_3 are constants.

Following Matsuo [20] we can obtain the eqs.(4.64) considering the following effective action:

$$
\delta S = \int T\delta\tilde{h}d^2x + \int W\delta A d^2x \sim \int \epsilon\partial_-^3 h d^2x + \alpha\int \eta\partial_-^5 A d^2x,
$$

where we have used the W_3-variations (4.61) and the anomalies equations (4.63) .

The general solutions of (4.64) can be written in the form:

$$
\begin{aligned}
&\tilde{h}(x^+,x^-) = I^+(x^+) - x^- I^0(x^+) - \frac{1}{2}(x^-)^2 I^-(x^+) \\
&A(x^+,x^-) = J^{++}(x^+) + x^- J^+(x^+) + \frac{1}{2}(x^-)^2 J^0(x^+) + \frac{1}{2}(x^-)^3 J^-(x^+) + \frac{1}{4}(x^-)^4 J^{--}(x^+).
\end{aligned}
\tag{4.65}
$$

As it has been proven by Matsuo [20], the 8 currents $I^{\pm,0}$ and $J^{\pm\pm}$, $J^{\pm,0}$ obey all the properties of the generators of (left) $SL(3,R)$ current algebra.

With this we have completed the proof of our conjecture that the (classical) induced W_3-gravity is nothing but the geometry of the affine surfaces of constant mean curvature immersed in 3D affine space A_3.

5. Extrinsic Geometry of Strings and W-Gravity

5.1. Introduction

The main statement of the present Chapter is that the *noncritical string models represented by the geometry of surfaces of constant mean curvatures (for $p+q=3,\ 4$) and certain other restrictions (for $p+q>4$) are equivalent to the $WO(p,q)$-gravities* [12, 20, 22, 40]. Vice versa the quantum $WO(p,q)$-gravities have as a classical limit ($c\to-\infty$) the extrinsic geometries of specific surfaces immersed in $M_{p,q}$. Remember that W-gravities, due to their higher symmetries, can be solved by the methods of W-algebra representations [22, 12, 20]. Therefore in this way we are singling out a class of solvable noncritical string models.

Our starting point are the structure equations for the moving frame fields $t_i^\mu = y_{,i}^\mu(x)$ and $N_\alpha^\mu(x)$ [51]:

$$\begin{aligned} y^\mu_{,ij} &= \Gamma^k_{ij} y^\mu_{,k} + b^\alpha_{ij} N^\mu_\alpha, \\ N^{\mu\alpha}_{,i} &= -b^\alpha_{ij} h^{jk} y^\mu_{,k} + \nu_i^{\alpha\beta} N^\mu_\beta, \end{aligned} \tag{5.1}$$

where $y_{,i}=\partial_i y$, h_{ij} is the induced metric: $h_{ij}=\eta_{\mu\nu}y^\mu_{,i}y^\nu_{,j}$, and Γ^k_{ij} are the Christoffel's symbols for h_{ij}. We will use the light-cone gauge: $ds^2 = dx^+dx^- + h(dx^+)^2$ throughout this Chapter. Then we define $A_\pm$ as follows

$$\partial_\pm g = A_\pm g, \qquad g \equiv \begin{pmatrix} t^\mu_- \\ t^\mu_+ \\ N^\mu_\alpha \end{pmatrix}. \tag{5.2}$$

The specific form of $A_\pm(\Gamma^k_{ij}, b^\alpha_{ij}, \nu_i^{\alpha\beta})$ reflects our choice of the local frame and surface coordinates. In general $A_\pm \in so(p,q)$ (for surfaces imbedded in flat $M_{p,q}$) and they transform as gauge fields under local $SO(p,q)$ gauge transformations. However we should distinguish two type of transformations: a) $G_0 = SO(1,1)\times SO(p-1,q-1)$–gauge transformations which do not mix tangent $\{t_i\}$ with normal $\{N_\alpha\}$ space, i.e. leave the surface invariant ; b) transformations which mix $\{t_i\}$ with $\{N_\alpha\}$ and therefore map a given surface to another one. Choosing an appropriate basis in $\{t_i\}$ and $\{N_\alpha\}$ we can gauge fix G_0-symmetries. The remaining restricted $SO(p,q)$-gauge transformations of A_- can be found by using the Polyakov's method [43]. The analysis of the A_- transformations shows that for the constant mean curvature surfaces in $M_{1,p}, p = 2,3$, the restricted gauge transformations close the (classical) extended $WO(1,p)$-algebra. For $p = 2$ this is the Virasoro algebra generated by $T = b_{--}$ and for $p = 3$ the complexified Virasoro algebra with generators $T_\pm = b^1_{--} \pm i b^2_{--}$. In the case of constant mean curvature surfaces imbeded in $M_{2,2}$ the corresponding algebra is nothing but the doubled Virasoro algebra and it coincides with WD_2-algebra with $T_\pm = b^1_{--} \pm b^2_{--}$. The A_+-transformations generate the transformation laws for the "extended metrics" $\tilde{h}_\pm$ (coupled to the extended currents $T_\pm$). For $p = 2$ we obtain the well known transformation law of the light-cone metric $\tilde{h} = h + c$:

$$\delta_\epsilon \tilde{h} = \partial_+ \epsilon - \partial_- \epsilon \tilde{h} + \epsilon \partial_- \tilde{h}, \tag{5.3}$$

For p=3 and $M_{2,2}$ we get

$$\delta_{\epsilon^+,\epsilon^-} \tilde{h}_\pm = \partial_+ \epsilon^\pm - \partial_- \epsilon^\pm \tilde{h}_\pm + \epsilon^\pm \partial_- \tilde{h}_\pm, \tag{5.4}$$

where $\tilde{h}_\pm$ are defined as

$$\tilde{h}_\pm = h + \frac{c_1 \mp e c_2}{\Lambda_1 \mp e\Lambda_2}, \tag{5.5}$$

and $e = i$ for $M_{1,3}$; $e = 1$ for $M_{2,2}$. (See Sect. 3 for more details.)

In more complicated cases of surfaces embedded in higher dimensional spaces we have to impose further restrictions to single out surface geometries corresponding to the solvable $WSO(p,q)$- gravities. The first example is represented by the theory of surfaces embedded in $M_{3,3}$ for which, as we shall show in Sec. 4, the geometrical conditions (leading to theories of $WSO(3,3)$- gravity) must be choosen as follows:

$$\begin{aligned}
&\Lambda^\alpha \Lambda_\alpha = 1, \qquad && \Lambda_1 = -1,\ \Lambda_2 = 0, \quad && \Lambda_3 + \Lambda_4 = 0, \\
&b^2_{--} = 0, && b^3_{--} + b^4_{--} = -1, && \Lambda_3 - \Lambda_4 = \frac{4}{3} b^1_{--}, \\
&\nu^{23}_- + \nu^{24}_- = 0, && \text{other } \nu^{\alpha\beta}_- = 0.
\end{aligned} \tag{5.6}$$

At the moment we do not have clear understanding for the geometrical nature of such class of surfaces. The fact is that the currents

$$T = \frac{10}{3} b^1_{--}, \qquad W = 2\nu^{23}_-, \qquad V = (b^3_{--} - b^4_{--}), \tag{5.7}$$

(of conformal spins 2, 3 and 4) generate an algebra isomorphic to WD_3-algebra [22]. The corresponding "extended metrics" are given by

$$H = h - \frac{3}{5} c_1, \qquad A = \frac{1}{2}(\nu^{23}_+ + \nu^{24}_+), \qquad B = -(c_3 + c_4), \tag{5.8}$$

and again their transformation properties are encoded in the restricted gauge transformations of A_+.

The W-symmetries of our models are an indication that the geometry of such specific surfaces imbedded in $M_{p,q}$ are related to the W-gravities. To show their exact equivalence we have to find the origin of the hidden $SO(p,q)$- current algebra symmetry and to derive the "extended trace anomaly" equations [5, 20] in the specific context of the surface geometry. The basic tool in solving these problems is the analysis of the *integrability conditions* (called Gauss-Codazzi equations) of the structure equations (5.1) (see [51]):

$$\partial_+ A_- - \partial_- A_+ + [A_-, A_+] = 0, \tag{5.9}$$

for the restricted class of surfaces we are considering. For example, in the simplest case of constant mean curvature ($\Lambda = 1$) surfaces embedded in $M_{1,2}$, one derives from (5.9) the Polyakov's trace anomaly equation [5] for $\tilde{h} = h + c$

$$(\partial_+ - \tilde{h}\partial_- - 2\partial_- \tilde{h})T = -\frac{1}{2}\partial^3_- \tilde{h}. \tag{5.10}$$

Imposing further the condition of *constant scalar curvature*

$$\partial^3_- \tilde{h} = 0, \tag{5.11}$$

we have

$$\tilde{h} = J^+_+(x^+) - 2J^0_+(x^+)x^- + J^-_+(x^+)(x^-)^2, \tag{5.12}$$

and as a consequence of (5.3), $J^{\pm,0}_+$ span the $so(2,1) \approx sl(2,R)$-current algebra. Similar arguments take place also in the case of constant mean curvature surfaces embedded in $M_{2,2}$ and $M_{1,3}$. The anomaly equations for the extended metrics $\tilde{h}_\pm$ are again a consequence of the integrability conditions (5.9) and their explicit form happens to coincide with (5.10) (see Section 3). The $so(2,2)$ (or $so(1,3)$)

-current algebras appear as hidden symmetries of these models. The next case we analize in details (see Section 4) is a specific class of surfaces in $M_{3,3}$. The derivation of the extended anomaly equations for the metrics H, A and B from the Gauss-Codazzi equations (5.9)is highly nontrivial and it confirms *our conjecture that the anomaly equations for the $WSO(p,q)$-gravities are consequence of the integrability conditions for certain class of surfaces embedded in $M_{p,q}$*. However we have no general proof of this statement (except for $p+q=3,4,6$).

An important property of the surfaces immersed in higher dimensional spaces is their richer geometrical and topological structure. Thus, for $M_{2,2}$ (or $M_{1,3}$) one have to take in consideration also the possibility of selfintersecting surfaces. In Sect. 3 we present our discussion on how the selfintersection property appears in the context of the extended $WSO(2,2)$-gravity.

We have to mention the similarity of most of our constructions and arguments to the corresponding ones in the Borel gauged $SO(p,q)$-WZW models [12, 20, 24,43]. In fact, for $p+q=3,4$ and 6 the structure equations (5.1) (or (5.2)) for a specific class of constant mean curvature surfaces in light-cone-like gauge coincide with the equations of motion of the Borel gauge fixed $SO(p,q)$- WZW models, i.e. with the $WO(p,q)$- gravities in their light-cone gauge formulation. The present Chapter can be considered as an attempt for differential geometric and string interpretation of these models.

5.2. 2D Gravity as Extrinsic Geometry of Strings in $M_{1,2}$

As it is well known the most general classical 2-D gravity can be also described as extrinsic geometry of surfaces embedded in appropriate flat Minkowski space $M_{p,q}$ (of signature (p,q)). One could wonder what is the quantum counterpart of such an equivalence. Our statement is that the Polyakov's 2-D quantum gravity [5, 6] interacting with the conformal matter is equivalent to the (quantum) geometry of surfaces of constant mean curvature embedded in $M_{1,2}$.

Consider an arbitrary 2-D surface immersed in $M_{1,2}$. According to the classical Gauss theorem [40] each surface in $M_{1,2}$ is uniquely characterized (modulo global $T_3 \cdot SO(2,1)$ transformations) by its first and second fundamental forms:

$$ds^2 = h_{ij}dx^i dx^j, \qquad H = \{h_{ij}\}$$
$$\varphi = b_{ij}dx^i dx^j, \qquad B = \{b_{ij}\}$$

The geometric invariants describing this embedding are certain det's and traces of H and B:

$$\begin{aligned} 2R = K = \det H^{-1}B = k_1 k_2 \\ \Lambda = TrH^{-1}B = h^{ij}b_{ij} = k_1 + k_2 \end{aligned} \tag{5.13}$$

where R, K and Λ are the scalar, Gauss and mean curvatures respectively. The eigenvalues k_i of the pair of forms H, B: $\det(H^{-1}B - K) = 0$ are called principal curvatures. Shortly, to define a surface in $M_{1,2}$ we need to know its two (independent) invariants, say K and Λ.

Introduce at each point of the surface $P_{1,1}$ a local orthogonal moving frame $t^\mu_\alpha(x_i)(\mu, \alpha = 1,2,3)$: $\eta_{\mu\nu}t^\mu_\alpha t^\nu_\beta = \eta_{\alpha\beta}$, $\eta_{\mu\mu} = (1,-1,-1)$. We can choose $t^\mu_1(x_i)$ and $t^\mu_2(x_i)$ to be tangent to $M_{1,1}$ and then the normal is $t^\mu_3 \equiv N^\mu = \epsilon^{\mu\nu\rho}t^\nu_1 t^\rho_2$. Let $y^\mu(x_i)$ are the functions defining the immersion, i.e. $y^\mu : P_{1,1} \to M_{1,2}$. Now we can construct $t^\mu_\alpha(x_i)$ and the induced h_{ij} and b_{ij} in terms of y^μ and its derivatives only. In light-cone gauge (l.c.g.) we have:

$$\begin{aligned} &h_{\pm\pm} = y^\mu_\pm y_{\pm\mu}, \qquad h_{+-} = y^\mu_+ y_{-\mu} = 1/2, \qquad h_{++} = h, \qquad h_{--} = 0 \\ &t^\mu_1 = \frac{y^\mu_+}{\sqrt{h}}, \qquad t^\mu_2 = -\frac{y^\mu_+}{\sqrt{h}} + 2\sqrt{h}y^\mu_-, \qquad y^\mu_\pm = \partial_\pm y^\mu \\ &b_{\pm\pm} = -y^\mu_{\pm\pm}N_\mu, \qquad b_{+-} = -y^\mu_{+-}N_\mu \end{aligned} \tag{5.14}$$

The structure equations (1.1) describe the local changes of the moving frame on $P_{1,1}$:

$$\partial_\pm g = A_\pm g \tag{5.15}$$

where in l.c.g. $A_\pm \in so(2,1)$ take the form:

$$\begin{aligned} A_- &= \begin{pmatrix} 0 & \frac{\partial_- h}{2h} & \frac{b_{+-}}{\sqrt{h}} \\ \frac{\partial_- h}{2h} & 0 & -\frac{b_{+-}}{\sqrt{h}} + 2\sqrt{h}b_{--} \\ \frac{b_{+-}}{\sqrt{h}} & \frac{b_{+-}}{\sqrt{h}} - 2\sqrt{h}b_{--} & 0 \end{pmatrix} \\ A_+ &= \begin{pmatrix} 0 & \frac{\partial_+ h}{2h} + \partial_- h & \frac{b_{+-}}{\sqrt{h}} \\ \frac{\partial_+ h}{2h} + \partial_- h & 0 & -\frac{b_{++}}{\sqrt{h}} + 2\sqrt{h}b_{+-} \\ \frac{b_{++}}{\sqrt{h}} & \frac{b_{++}}{\sqrt{h}} - 2\sqrt{h}b_{+-} & 0 \end{pmatrix} \end{aligned} \tag{5.16}$$

Using the "allowed" gauge transformations $g_0 \in G_0 = G_{tangents} \otimes G_{normals}$ (here these are the $SO(2,1)$-transformations that leave $P_{1,1}$ invariant, i.e. $G_0 =$

$SO(1,1)$) we can further fix the tangents $t^{\mu}_{\pm}$. We take g_0 in the form:

$$g_0 = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 \\ \sinh\alpha & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \alpha = -\frac{1}{2}\log h$$

and as a result of the gauge transformations

$$\begin{aligned} \tilde{A}_{\pm} &= g_0 A_{\pm} g_0^{-1} + (\partial_{\pm} g_0) g_0^{-1} \\ \tilde{g} &= g_0 g \end{aligned} \tag{5.17}$$

we obtain

$$\tilde{t}_1 = y_+ + (1-h)y_-, \qquad \tilde{t}_2 = -y_+ + (1+h)y_-$$

$$\tilde{A}_- = \begin{pmatrix} 0 & 0 & b_{--}+\Lambda \\ 0 & 0 & b_{--}-\Lambda \\ b_{--}+\Lambda & -b_{--}+\Lambda & 0 \end{pmatrix}, \qquad \tilde{A}_+ = \begin{pmatrix} 0 & \partial_- h & a_- \\ \partial_- h & 0 & -a_+ \\ a_- & a_+ & 0 \end{pmatrix} \tag{5.18}$$

where $\Lambda = b_{+-} - hb_{--}$, $a_{\pm} = b_{++} \mp (1 \pm h) b_{+-}$.

We are now interested in the symmetries of the eqs.(5.15) which remain after the partial gauge fixing of the $SO(2,1)$-symmetries. An effective method for extracting such residual gauge transformations was recently proposed by Polyakov [43]. To start with the $SO(2,1)$-transformations of $\tilde{A}_{\pm} = (\partial_{\pm}\tilde{g})\tilde{g}^{-1} = T_a J^a_{\pm}(x_+, x_-)$:

$$\delta_{\epsilon} J^a_{\pm} = \epsilon^{abc}\epsilon^b(x_{\pm}) J^c_-(x_{\pm}) + \partial\epsilon^a \tag{5.19}$$

where $J^a_{\pm}$ are the generators of the $SO(2,1)$ current algebra. Comparing the general form of A_-:

$$A_- = 2\begin{pmatrix} 0 & J^0_- & 1/2(J^+_- + J^-_-) \\ J^0_- & 0 & 1/2(J^+_- - J^-_-) \\ 1/2(J^+_- + J^-_-) & 1/2(J^-_- - J^+_-) & 0 \end{pmatrix}$$

with its gauge fixed form $\tilde{A}_-$ given by eq.(5.18) we obtain:

$$J^+_- = b_{--}, \qquad J^-_- = \Lambda, \qquad J^0_- = 0 \tag{5.20}.$$

Let us further fix $J^-_- = 1$ (i.e.*we consider surfaces of constant mean curvature* $\Lambda = 1$ *only*) and substitute (5.20) (with $\Lambda = 1$) in (5.19) .Two of the eqs.(5.19)

$$\delta J^-_- = 0 = \delta J^0_-$$

allow to realize $\epsilon_{0,+}$ in terms of ϵ_- [43]

$$\epsilon_0 = \frac{1}{2}\partial_-\epsilon_-, \qquad \epsilon_+ = (-\frac{1}{2}\partial_-^2 + b_{--})\epsilon_- \tag{5.21}$$

and the third one gives the restricted gauge transformations we are looking for ($\epsilon_- \equiv \epsilon$):

$$\delta_\epsilon b_{--} = 2(\partial_-\epsilon)b_{--} + \epsilon\partial_- b_{--} - \frac{1}{2}\partial_-^3\epsilon \tag{5.22}$$

Therefore the residual gauge symmetries of eqs.(5.15) are nothing but diffeomorphisms generated by b_{--}. This fact suggests the identification of $--$ component of b_{ij} with $--$ component of the stress-tensor of the considered model* : $b_{--} = T_{--}$.

If we remove the condition $\Lambda = 1$ and consider an arbitrary surface in $M_{1,2}$ by applying the same Polyakov's procedure we find the following complicated transformations:

$$\begin{aligned}
\delta_{\epsilon_+} J^+(x) &= \int dz\epsilon_+(z)\left\{J^-(z)J^+(x) - \partial_z\delta(z-x)\right\} \\
\delta_{\epsilon_-} J^+(x) &= -\int dz\epsilon_-(z)J^+(z)J^+(x) \\
\delta_{\epsilon_+} J^-(x) &= -\int dz\epsilon_+(z)J^-(z)J^-(x) \\
\delta_{\epsilon_-} J^-(x) &= \int dz\epsilon_-(z)\left\{J^+(z)J^-(x) - \partial_z\delta(z-x)\right\}
\end{aligned}$$

The lack of clear understanding of the nature and the properties of the corresponding $\{\delta_{\epsilon_\pm}\}$-algebra is one of the reasons to consider the geometry of the surfaces of constant mean curvature only. The similarity of the $\Lambda = const$ case to the Borel gauged $SL(2,R)$ - WZW model [52] and to Drinfeld-Sokolov Hamiltonian reduction [12,23, 24] is an indication that perhaps only the $\Lambda = const$ surfaces lead to exactly solvable model. This suggests the following strategy to the solvable geometries of surfaces in $M_{p,q}$: to look for such surfaces, whose structure equations obey $WSO(p,q)$ symmetries.

Turning back to the symmetries of eqs.(5.15) we have to examine the $\bar{A}_+$ - transformations too. The eqs.(5.18) and (5.19) together with (5.21) lead to

---

⋆ which is in fact a Grassmanian σ-model $G_{2,2} = SO(2,1)/SO(1,1)$ (see ref.[8]), constrained by $\Lambda = const$.

δ_ϵ transformation laws for b_{++}, b_{+-} and $\tilde{h} = h + c$ (c is defined below):

$$\begin{aligned}
\delta_\epsilon \tilde{h} &= -(\partial_- \epsilon)\tilde{h} + \epsilon \partial_- \tilde{h} + \partial_+ \epsilon \\
\delta_\epsilon b_{+-} &= (\partial_+ - \partial_- h)\left(\epsilon b_{--} - \frac{1}{2}\partial_-^2 \epsilon\right) + (\partial_- \epsilon) b_{+-} \\
\delta_\epsilon b_{++} &= \delta_\epsilon \tilde{h} + \delta_\epsilon (h b_{+-}).
\end{aligned} \tag{5.23}$$

Note that the first of eqs.(5.23) exactly coincides with the transformation law of the metric in the Polyakov's 2-D gravity [5, 6].

The integrability conditions for the eqs.(5.15)

$$\partial_+ \tilde{A}_- - \partial_- \tilde{A}_+ + [\tilde{A}_-, \tilde{A}_+] = 0 \tag{5.24}$$

are satisfied identicaly if h_{ij} and b_{ij} are constructed by y^μ's as in (5.14) . Considering h_{ij} and b_{ij} as independent (of y^μ) variables and the surface geometry as $G_{2,2}$ σ-model (constrained by $\Lambda = const$) the equations (5.24) impose certain restrictions on h and b:

$$\begin{aligned}
&\partial_-^2 h = 2b_{--}\left(b_{++} - h^2 b_{--} - 2h\right) - 1/2 \\
&[\partial_+ - 2\partial_- h - h\partial_-]\, b_{--} = 0 \\
&\partial_- \left(b_{++} - h^2 b_{--} - 2h\right) = 0
\end{aligned}$$

One can rewrite them in the form:

$$\begin{aligned}
&\partial_-^2 (h + c) = -K \\
&[\partial_+ - (h+c)\partial_- - 2\partial_-(h+c)]\, b_{--} = -\frac{1}{2}\partial_-^3 (h+c)
\end{aligned} \tag{5.25}$$

where $c = b_{++} - h^2 b_{--} - 2h$ and K is the Gauss curvature. The second equation (5.25) is nothing but the Polyakov's "trace anomaly" equation [5] for the induced 2-D gravity with a metric $\tilde{h} = h + c$ and the stress-tensor $T_{--} = b_{--}$.

For the surfaces of constant K (and constant mean curvature Λ) the first equation (5.25) gives:

$$\partial_-^3 \tilde{h} = 0 \tag{5.26}$$

and therefore

$$\tilde{h} = J_+^+(x^+) - 2J_+^0(x^+)x^- + J_+^-(x^+)(x^-)^2.$$

According to refs.[5, 6] the currents $J_+^{\pm,0}$ close $SL(2,R)$ current algebra and their transformation properties are a cosequence of (5.23) .

All these properties of the geometry of constant mean curvarure surfaces $P_{1,1}$ embedded in $M_{1,2}$ encoded in their structure equations lead to the conclusion that such an extrinsic geometry (having h_{ij} and b_{ij} as independent variables) is equivalent to the 2-D Polyakov's gravity [5, 6]. One could further repeat all the KPZ steps for constructing the corresponding quantum theory.

Many of the properties of the surfaces of constant mean curvature have a simple form in the conformal gauge as well. For example if we take $h_{ij} = \begin{pmatrix} 0 & 2e^{\varphi} \\ 2e^{\varphi} & 0 \end{pmatrix}$, $ds^2 = e^{-2\varphi}dzd\bar{z}$, and write the integrability conditions (5.24) we get the Liouville equation:

$$\Box\varphi = \Lambda e^{\varphi}$$

as a counterpart of eqs.(5.25) .

One could wonder what is the action for such an extrinsic geometrical model. The answer is extremely simple: the action for $G_{2,2}$ σ-model with the constraint $\Lambda = const.$ The same result can be obtained starting from the Borel gauged WZW action (i.e. with $J_-^- = 1$ constraint) and replacing the currents $J_\pm^a$ with the corresponding geometrical variables h_{ij}, b_{ij} according to our identification (see eqs.(5.20) and (5.18)). All this if we take h_{ij} and b_{ij} as independent (of y^μ's) variables. One can define equaly well the classical geometry of surfaces in $M_{1,2}$ using only the $y^\mu(x_i)$ variables (h_{ij}, b_{ij} then are given by (5.14)). The main equations determining y^μ are:

$$\Box_h y_\mu = \frac{\Lambda}{2}\epsilon_{\mu\nu\rho}y^\nu_{,i}y^\rho_{,j}\epsilon^{ij}, \qquad \Lambda = const. \tag{5.27}$$

The corresponding string action describing such "nonminimal" surfaces ($\Box y^\mu \neq 0$) is given by:

$$S = \int \sqrt{-g}d^2x + \frac{\Lambda}{6}\int \epsilon_{\mu\nu\rho}\epsilon^{ij}y^\mu y^\nu_{,i}y^\rho_{,j}d^2x \tag{5.28}$$

where $g = \det(\partial_i y^\mu \partial_j y^\mu)$. The second term has a topological meaning and it is related to the degree $(\deg f)$ of the "normal" Gauss map $f_y : P_{1,1} \to T^*P_{1,1} \sim S_{1,1}$. We should mention that in eqs.(5.27) ,(5.28) the metric h_{ij} is not gauge fixed.

In contrast with the critical strings one can not use free oscilators to quantize (5.27) . However one can further speculate that the quantization of such a string is equivalent to the KPZ-quantization of the 2-D gravity interacting with the conformal matter.

5.3. Noncritical Strings in $M_{2,2}$ and $WSO(2,2)$ - Gravity

The extrinsic geometry of constant mean curvature surfaces in 4 dimensions $(2+2$ or $3+1$, $\eta_{\mu\nu} = diag(1,-1,-1,e^2), e = \sqrt{\pm 1})$ in many aspects is a trivial generalization of the results for $d_{cl} = 2+1$, $\quad (d_q \leq 1)$ from Sect.2. The immersion $y^\mu(x_i): \quad P_{1,1} \to M_{2,2}$ $(M_{1,3})$ is characterized by h_{ij}, two second quadratic forms $b^\alpha_{ij} (\alpha = 1,2)$ and the torsion vector ν_i. These quantities can be realized in terms of the moving frame fields $T^\mu_A(x_i) : \eta_{\mu\nu} T^\mu_A T^\nu_B = \eta_{AB}$, $(A, B, \mu, \nu = 1, \dots, 4)$ as follows:

$$
\begin{aligned}
h_{ij} &= \eta_{\mu\nu} y^\mu_{,i} y^\nu_{,j}, \qquad b^\alpha_{,ij} = -y^\mu_{,ij} N^{\alpha\nu} \eta_{\mu\nu} \\
\nu_i &= -N^{\mu 1}_{,i} N^{\nu 2} \eta_{\mu\nu}
\end{aligned}
\tag{5.29}
$$

where we denote $T^\mu_3 = N^{1\mu}$, $T^\mu_4 = N^{2\mu}$. We fix the G_0 gauge freedoms:

$$
\begin{aligned}
G_0 &= SO(1,1) \times SO(2) \qquad for \qquad M_{1,3} \qquad (e = i) \\
G_0 &= SO(1,1) \times SO(1,1) \qquad for \quad M_{2,2} \qquad (e = 1)
\end{aligned}
$$

choosing a particular moving frame (in l.c.g.):

$$
\begin{aligned}
T^\mu_1 &= y^\mu_+ + (1-h) y^\mu_- \qquad N^{1\mu}_- N^{2\nu} \eta_{\mu\nu} = \nu_- = 0 \\
T^\mu_2 &= -y^\mu_+ + (1+h) y^\mu_-.
\end{aligned}
\tag{5.30}
$$

In this basis the structure equations (1.1) get simplified: $\partial_\pm g = A_\pm g$, and

$$
A_- = \begin{pmatrix} 0 & 0 & b_1 + \Lambda_1 & b_2 + \Lambda_2 \\ 0 & 0 & b_1 - \Lambda_1 & b_2 - \Lambda_2 \\ b_1 + \Lambda_1 & -(b_1 - \Lambda_1) & 0 & 0 \\ -e^2(b_2 + \Lambda_2) & e^2(b_2 - \Lambda_2) & 0 & 0 \end{pmatrix}, \qquad A_+ = \begin{pmatrix} 0 & \partial_- h & a^1_+ & a^2_+ \\ \partial_- h & 0 & a^1_- & a^2_- \\ * & * & 0 & \nu \\ * & * & e^2 \nu & 0 \end{pmatrix}
\tag{5.31}
$$

where

$$
\begin{aligned}
b^\alpha_{--} &= b_\alpha, \qquad c_\alpha = b^\alpha_{++} - h^2 b^\alpha_{--} - 2\Lambda_\alpha h, \qquad \nu = \nu_+ \\
a^\alpha_\pm &= \Lambda_\alpha + h b_\alpha \pm (c_\alpha + h\Lambda_\alpha), \qquad \Lambda_\alpha = b^\alpha_{+-} - h b^\alpha_{--}.
\end{aligned}
\tag{5.32}
$$

To find the residual $SO(2,2)$ (or $SO(3,1)$) gauge symmetries of eqs.(1.1) we apply again the Polyakov's method [43], explained in Sect.2. As it is shown in App.C for *surfaces of constant mean curvatures* (i.e. Λ_α are constants and $\Lambda_1 \pm e\Lambda_2 \neq 0$) the A_--restricted gauge transformations generate the following two "diffeomorphisms" for the fields $b_1 \pm eb_2 = T_\pm$:

$$\begin{aligned}
\delta_{\epsilon^+}T_+ &= -\frac{1}{2}\frac{1}{\Lambda_1 - e\Lambda_2}\partial_-^3\epsilon^+ + 2(\partial_-\epsilon^+)T_+ + \epsilon^+\partial_-T_+, \qquad \delta_{\epsilon^-}T_+ = 0 \\
\delta_{\epsilon^-}T_- &= -\frac{1}{2}\frac{1}{\Lambda_1 + e\Lambda_2}\partial_-^3\epsilon^- + 2(\partial_-\epsilon^-)T_- + \epsilon^-\partial_-T_-, \qquad \delta_{\epsilon^+}T_- = 0.
\end{aligned} \tag{5.33}$$

For $e = 1$ ($M_{2,2}$) the infinithesimal transformations (5.33) give rise to the $WSO(2,2)$ algebra wich happens to coincide with the WD_2 of ref.[22]. This algebra is nothing but the direct sum of two Virasoro algebras. For the surfaces ($\Lambda_\alpha = const$) in $M_{1,3}$, ($e = i$), the transformations (5.33) generate the complexified Virasoro algebra.

The doubling of the Virasoro algebra in the case of $M_{2,2}$ is a hint to look for two "metrics" $\tilde{h}_\pm$ sharing the transformation law (5.23) . This is indeed the case and the analysis of the A_+-restricted gauge transformations(see App.C) allows us to conclude that the $WSO(2,2)$ ($WSO(3,1)$)- extended metrics are given by:

$$\tilde{h}_\mp = h + \frac{c_1 \pm ec_2}{\Lambda_1 \pm e\Lambda_2}. \tag{5.34}$$

They obey the desired transformation laws:

$$\begin{aligned}
\delta_{\epsilon^-}\tilde{h}_+ &= \partial_+\epsilon^- - (\partial_-\epsilon^-)\tilde{h}_+ + \partial_-\tilde{h}_+\epsilon^-, \qquad \delta_{\epsilon^+}\tilde{h}_+ = 0 \\
\delta_{\epsilon^+}\tilde{h}_- &= \partial_+\epsilon^+ - (\partial_-\epsilon^+)\tilde{h}_- + \partial_-\tilde{h}_-\epsilon^+, \qquad \delta_{\epsilon^-}\tilde{h}_- = 0.
\end{aligned} \tag{5.35}$$

(see App.C for the proof).

The next step is to derive the "extended trace anomaly equations" from the integrability conditions (1.4) (Gauss-Codazzi eqs.) for the corresponding structure equations. It is so only when h_{ij},b_{ij} and ν in (5.1) are treated as y^μ - independent variables. Substituting (5.31) in (1.4) we obtain the following system of equations:

$$\begin{aligned}
&\partial_-^2 h = 2(b_1c_1 - e^2b_2c_2) - 2\Lambda_1^2 + 2e^2\Lambda_2^2 \equiv -K \\
&\partial_-\nu = 2(b_1c_2 - b_2c_1) \equiv K^\perp \\
&(\partial_+ - h\partial_-)\, b_\alpha = 2(\partial_-h)b_\alpha + \partial_-\Lambda_\alpha - b_\beta\nu_{\beta\alpha} \\
&(\partial_+ - h\partial_-)\, \Lambda_\alpha = \partial_-c_\alpha - \Lambda_\beta\nu_{\beta\alpha}, \qquad \nu = \nu_{12} = e^2\nu_{21}.
\end{aligned} \tag{5.36}$$

For $\Lambda_\alpha = const$ eqs.(5.36) can be written in the form:

$$\left[\partial_+ - \tilde{h}_\pm\partial_- - 2(\partial_-\tilde{h}_\pm)\right] T_\pm = -\frac{1}{2(\Lambda_1 \mp e\Lambda_2)}\partial_-^3\tilde{h}_\pm \tag{5.37}$$

and

$$\partial_-^2 \tilde{h}_\mp = 2T_\mp(c_1 \pm ec_2) - 2(\Lambda_1 + e\Lambda_2)(\Lambda_1 - e\Lambda_2). \tag{5.38}$$

Introducing the Gauss and normal* curvatures K and $K^\perp$ as in eqs.(5.36) we have:

$$\partial_-^2 \tilde{h}_\pm = -K \pm eK^\perp. \tag{5.39}$$

For *surfaces of constant K and $K^\perp$* eqs.(5.39) imply:

$$\partial_-^3 \tilde{h}_\pm = 0.$$

As a consequence $\tilde{h}_\pm$ contain the currents of the $SO(2,1) \otimes SO(2,1)$-current algebra for $e = 1$ ($M_{2,2}$) and of the $SO(3,1)$-current algebra for $e = i$ (i.e. $M_{1,3}$). Their transformation properties are encoded in the $\tilde{h}_\pm$ transformation laws (5.35)

An important property of the geometry of the constant mean curvature surfaces embedded in $M_{2,2}$ is its equivalence to the geometries of two (independent) surfaces of constant mean curvature embedded in $M_{1,2}$. It reflects the fact that the system of equations (5.37) ,(5.39) splits in two independent parts ($\{\tilde{h}_+, T_+\}$ and $\{\tilde{h}_-, T_-\}$) and each of them is nothing but the eq.(5.25) for the surfaces in $M_{1,2}$. The algebraic origin of this property is in the nonsimplicity of $SO(2,2) = SO(2,1) \otimes SO(2,1)$ and as a consequence the (constrained $\Lambda_\alpha = const$) Grassmanian σ-model $G_{2,4} = \frac{SO(2,2)}{SO(1,1)\times SO(1,1)} \equiv G_{2,2} \otimes G_{2,2} = S_{1,1} \otimes S_{1,1}$, representing the surface geometry in $M_{2,2}$ is a product of two $G_{2,2}$ σ-models.

One of the motivations to consider surfaces immersed in $M_{2,2}$ is that one expect to enlarge the class of surfaces which can be immersed in $M_{1,2}$ with the surfaces obeying new geometrical properties. The question we have to answer is: which are these new geometric characteristics and how one can compute them? The answer is that the surfaces in $M_{2,2}$, together with the old ($M_{1,2}$)-property to have certain number of handles g:

$$2(1-g) = \frac{1}{2\pi} \int_{P_2} K\sqrt{-g}d^2x = \chi(P_2) \tag{5.40}$$

obey new ($M_{2,2}$)- characteristic: the number of selfintersections q:

$$\chi^\perp(TP_2^\perp) = \frac{1}{2\pi} \int_{P_2} K^\perp\sqrt{-g}d^2x = 2q \tag{5.41}$$

$\star$ the curvature of the induced connection in the normal frame bundle i.e. the genus of the normal space $TP_2^{\perp}$. To avoid the possible confusions† due to the noncompactness of the surfaces $P_{1,1}$ in $M_{2,2}$ at this point only we consider their compactifications P_2 (immersed in E_4). According to Hoffman-Osserman [53] theorem the generalized Gauss map (see also [54]):

$$\gamma_f : \qquad P_2 \to G_{2,4} = S_2 \times S_2, \qquad f : \qquad P_2 \to E_4$$

splits into two factors $\gamma_f^{\pm}$ (which are the projections of γ_f on both spheres S_2) and the corresponding Euler characteristics are given by:

$$\chi(P_2) = \deg \gamma_f^+ + \deg \gamma_f^-$$
$$\chi^{\perp}(TP_2^{\perp}) = \deg \gamma_f^+ - \deg \gamma_f^- .$$

This fact suggests that the Lagrangian describing the constrained $G_{2,4}$ model ($\Lambda_\alpha = const$), i.e. the specific noncritical string in $M_{2,2}$, is the sum of two $M_{1,2}$ Lagrangians (5.28) : $L(\Lambda_1, \Lambda_2) = L(\Lambda_1) + L(\Lambda_2)$. However, we have no an explicit construction of $\deg \gamma_f^{\pm}$ in terms of the original $y^\mu : \quad P_{1,1} \to M_{2,2}$.

Another approach to the $G_{2,4}$ σ-model action is to take h_{ij},b^α_{ij} and ν_i as independent variables and to consider the Polyakov's extrinsic geometrical action:

$$S = \alpha_0 \int \sqrt{-g} d^2x + \alpha_1 \int P_+ \sqrt{-g} d^2x + \alpha_2 \int P_- \sqrt{-g} d^2x \qquad (5.42)$$

where$P_\pm$ are the following two invariants:

$$P_+ = h^{il} b^\alpha_{ik} b_{\alpha lj} h^{kj} = -K + 4\Lambda^\alpha \Lambda_\alpha, \qquad \Lambda^\alpha = b^\alpha_{ij} h^{ij}$$
$$P_- = \epsilon^{jk} \epsilon^{\alpha\beta} h^{lm} b^\alpha_{lj} b^\beta_{mk} = K^{\perp}.$$

For the surfaces of constant mean curvature $\Lambda^\alpha \Lambda_\alpha = const$ the Polyakov's action simplifies. The second term gives contributions proportional to the area term and the Euler characteristics. Therefore the action describing the geometry of the constant mean curvature surfaces in E_4 is:

$$S(\Lambda^2 = const) = Area(P_2) + \chi(P_2) + \chi^{\perp}(TP_2^{\perp}) \qquad (5.43)$$

This suggests the following form of the partition function $Z(A)$ for a surface of fixed area A and selfintersection number $q(A)$:

$$Z(A) = \sum_{\chi, q} A^{-1+\frac{1}{2}\chi(\gamma_{str}-2)} e^{-mA} (-1)^{q_{str}(A)}$$

According to our analysis of the symmetries of the structure equations (5.1) for $P_{1,1}$ embedded in $M_{2,2}$ (and $\Lambda_\alpha = const$) such an extrinsic geometry is equiv-

† and mainly because we do not know how to formulate these quantities for $P_{1,1}$.

alent (classicaly) to the $WSO(2,2)$ -gravity interacting with WD_2 -conformal matter. It is natural to consider the quantum $WSO(2,2)$-gravity and matter as a quantization of the noncritical string with action (5.43) . Applying the KPZ-quantization to the $WSO(2,2)$-gravity (with $WSO(2,2)$ ghosts and $WSO(2,2)$ matter (see refs.[12, 20])) from the condition of the vanishing of the total stress-tensor

$$T_{gr} + T_{matt} + T_{gh} = 0, \qquad T_{gr} = T_+ + T_-$$

one obtains:

$$d - 56 + \frac{6\kappa}{\kappa + 4} - 6\kappa = 0$$

(d_q is the number of the matter fields, $c_{gh} = -56$) and therefore

$$k + 2 = \frac{26 - d_q - \sqrt{(2 - d_q)(50 - d_q)}}{24}, \qquad \frac{\kappa}{2} = -k - 4 \qquad (5.44)$$
$$d_q = c_{matt}.$$

This leads to the restriction $d_q \leq 2 = d_{cl.}/2$. The analog of the KPZ formula for the scaling dimensions $\Delta, \tilde{\Delta}$ in terms of the conformal dimensions of the matter fields $\Delta_0 = \Delta_0^+ + \Delta_0^-$, $\tilde{\Delta}_0 = \Delta_0^+ - \Delta_0^-$ * is:

$$\Delta - \Delta_0 = \frac{\Delta(1 - \Delta) - (\tilde{\Delta})^2}{k + 2}, \qquad \tilde{\Delta} - \tilde{\Delta}_0 = \frac{\tilde{\Delta}(1 - 2\Delta)}{k + 2}. \qquad (5.45)$$

Due to the fact that $WSO(2,2) = Vir \otimes Vir$ all these results are trivial doubling of the ones of the KPZ [6]. The only nontrivial element is coming from the Z_2-odd sector ("Ramond-like" sector) of the WD_2 algebra [55]. The generator $\tilde{T} = T^+ - T^-$ is Z_2-twisted around the fields $V_{n,m}^{matt}$ of this sector and therefore the representations of $WSO(2,2)$ -algebra are characterized by only one number $\Delta_0(n,m) \neq \Delta_0^+ + \Delta_0^-$ (see ref.[55]). The KPZ formula in the Z_2-odd sector is:

$$\Delta_{n,m}^0 - \Delta = \frac{\Delta(1 - \Delta)}{k + 2}.$$

The analog of the identity I ($\Delta_0 = 0 = \tilde{\Delta}_0$) here is the field of the lowest dimension $V_{1,1}$: $\Delta_0^{1,1} = \frac{c_{matt}}{32}(c_{matt} = 2c_{matt}^{Vir})$. This analogy together with the fact that γ_{str} is the solution of (5.45) for $\Delta_0 = 0 = \tilde{\Delta}_0$ lead us to the following

* $\Delta_0^{\pm}$ are given by Kac formula for the Virasoro minimal models

conjecture: the quantum counterpart of the selfintersection number q_{str} is given by :

$$q_{str} - \frac{c_{matt}}{32} = \frac{q_{str}(1-q_{str})}{k+2}$$

and that the selfintersection properties are described by the fields from the Z_2 -odd sector of the WD_2-algebra.

For $WSO(2,2)$-gravity interacting with WD_2 minimal unitary models $d_q = c_{matt} = 2(1-6/p(p+1))$, k is quantized according to (5.44) . Therefore the central charge of the pure gravity is quantized too:

$$c_{gr} = \frac{6\kappa}{\kappa+2} - 6\kappa. \tag{5.46}$$

From the other side as a consequence of eq.(5.33) the (classical) central charge of the pure gravity is given as a function of the constant mean curvatures Λ_1, Λ_2:

$$c_{gr} = \frac{12\Lambda_1}{\Lambda_2^2 - \Lambda_1^2}. \tag{5.47}$$

One can speculate that geometricaly the quantization of c_{gr} means quantization of the mean curvatures, i.e. $\Lambda_i = \Lambda_i(k)$ and only surfaces with some discrete values of Λ_i are allowed.

The main obstacle in applying all this technology to the surfaces embedded in $M_{1,3}$ is our poor understanding of the representations of $WSO(1,3)$ algebra.

5.4. Extrinsic Geometries and $WSO(p,q)$-Gravities

In the previous sections we have shown the equivalence between the extrinsic geometry of certain surfaces embedded in $M_{1,2}$ (and $M_{2,2}$ or $M_{1,3}$) and the $WSO(1,2)$ (and $WSO(2,2)$ or $WSO(1,3)$) -gravities. Both examples are based on Virasoro- and $SL(2,R)$-current algebras. One could think that such an equivalence does not hold in the general case of surfaces embededd in $M_{p,q}$. In this section we demonstrate that the extrinsic geometry of the specific class of surfaces imbedded in $M_{3,3}$ is indeed equivalent to the $WSO(3,3)$-gravity. However we have no general proof for such a statement for the general case $p+q > 6$.

Consider a surface imbedded in flat $M_{3,3}$ with the metric $\eta_{\mu\nu} = \eta_\mu \delta_{\mu\nu}$, $\mu, \nu =$

$1, 2, \cdots, 6$, $\eta_\mu = \pm 1$ † . Let us take the tangent vectors T^μ_α as in eq.(5.14) Then the structure equations (5.1)can be written in the form

$$\partial_\pm g = A_\pm g,$$

$$A_- = \begin{pmatrix} 0 & 0 & b^\alpha_{--} + \Lambda_\alpha \\ 0 & 0 & b^\alpha_{--} - \Lambda_\alpha \\ -\eta_\alpha(b^\alpha_{--} + \Lambda_\alpha) & \eta_\alpha(b^\alpha_{--} - \Lambda_\alpha) & \nu^{\alpha\beta}_- \end{pmatrix}, \tag{5.48}$$

$$A_+ = \begin{pmatrix} 0 & \partial_- h & a^\alpha_+ \\ \partial_- h & 0 & a^\alpha_- \\ -\eta_\alpha a^\alpha_+ & \eta_\alpha a^\alpha_- & \nu^{\alpha\beta}_+ \end{pmatrix},$$

where $a^\alpha_\pm = b^\alpha_{+-} \pm (b^\alpha_{++} - h b^\alpha_{+-})$ and $\Lambda_\alpha = \frac{1}{4} b^\alpha_{ij} h^{ij} = b^\alpha_{+-} - h b^\alpha_{--}$. From here on we set $b_\alpha = b^\alpha_{--}$, $c_\alpha = b^\alpha_{++} - h b^\alpha_{+-} - 2h\Lambda_\alpha$. The $G_0 = SO(1,1) \times SO(2,2)$-gauge freedoms are partially fixed. To get $WSO(3,3)$-symmetric surface model we should do two more steps: (1) to gauge fix the $SO(2,2)$-local rotations of the normal vectors, imposing appropriate conditions on $\nu^{\alpha\beta}_-$ and (2) to find a set of geometrical constraints selecting the class of "solvable" surface geometries. From the $SL(4,R) \approx SO(3,3)$ Hamiltonian reduction [23], we know that the desired $WSO(3,3)$-algebra can be obtained if we take A_- in the form:

$$A^{(4)}_- = \begin{pmatrix} 0 & \frac{3}{10}T & \frac{1}{2}W & V \\ 1 & 0 & \frac{2}{5}T & \frac{1}{2}W \\ 0 & 1 & 0 & \frac{3}{10}T \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{5.49}$$

The corresponding $so(3,3)$-form of A_- is

$$A^{(6)}_- = \begin{pmatrix} 0 & 0 & \frac{3}{10}T - 1 & 0 & \frac{1}{2}(V - 1 + \frac{2}{5}T) & \frac{1}{2}(-V - 1 - \frac{2}{5}T) \\ 0 & 0 & \frac{3}{10}T + 1 & 0 & \frac{1}{2}(V - 1 - \frac{2}{5}T) & \frac{1}{2}(-V - 1 + \frac{2}{5}T) \\ * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & \frac{1}{2}W & -\frac{1}{2}W \\ * & * & 0 & \frac{1}{2}W & 0 & 0 \\ * & * & 0 & \frac{1}{2}W & 0 & 0 \end{pmatrix}. \tag{5.50}$$

† we take $(\eta_\mu) = (1, -1, 1, -1, 1, -1)$.

The comparison of (5.50)with (5.48)leads to the following geometrical constraints:

$$\begin{aligned}
&\Lambda_1 = -1, \quad \Lambda_2 = 0, \quad \Lambda_3 = -\Lambda_4 = \frac{1}{5}T, \\
&b_1 = \frac{3}{10}T, \quad b_2 = 0, b_3 = \frac{1}{2}(V-1), \; b_4 = -\frac{1}{2}(V+1), \\
&\nu_-^{23} = -\nu_-^{24} = \frac{1}{2}W, \text{other } \nu_-^{\alpha\beta} = 0.
\end{aligned} \tag{5.51}$$

Note that $\sum_\alpha \eta_\alpha (\Lambda_\alpha)^2 = 1$ (constant mean curvature condition). At the moment we do not have clear understanding for the geometrical nature of such class of surfaces.

Next we have to show how the $WSO(3,3)$ symmetry of the structure equations (5.1)arises from the restricted $SO(3,3)$-gauge transformations. In doing that we are following exactly the same procedure as before (which is now more complicated due to the large number of parameters) [9]. As a result of tedious calculations we find the following 3-parametric transformations of the currents (of conformal spins 2, 3 and 4)

$$T = \frac{10}{3} b^1_{--}, \quad W = \nu_-^{23} - \nu_-^{24}, \quad V = b^3_{--} - b^4_{--}, \tag{5.52}$$

$$\begin{aligned}
\delta_{\epsilon,\eta,\tilde\epsilon} T =& -5\partial_-^3 \epsilon + 2\partial_-\epsilon T + \epsilon\partial_- T \\
&+ 3\partial_-\eta W + 2\eta\partial_- W + 4\partial_-\tilde\epsilon V + 3\tilde\epsilon\partial_- V \\
\delta_{\epsilon,\eta,\tilde\epsilon} W =& \; \partial_-^5 \eta + 3\partial_-\epsilon W + \epsilon\partial_- W + (\text{ more terms}), \\
\delta_{\epsilon,\eta,\tilde\epsilon} V =& -\frac{1}{20}\partial_-^7 \tilde\epsilon + 4\partial_-\epsilon V + \epsilon\partial_- V + (\text{ more terms}).
\end{aligned} \tag{5.53}$$

The explicit expressions for δW and δV are presented in Appendix D. The transformations (5.53) close the (classical) W_4 or WD_3-algebra.

The last and most difficult step is the derivation of the extended anomaly equations from the Gauss-Codazzi equations (5.9). For surfaces embedded in $M_{3,3}$ they have the form:

$$\begin{aligned}
&\partial_-^2 h = -2\sum_\alpha \eta_\alpha [b_\alpha c_\alpha - (\Lambda_\alpha)^2] \\
&(\partial_+ - 2\partial_- h - h\partial_-) b_\alpha = \partial_- \Lambda_\alpha + \sum_\beta \left[(\Lambda_\beta + h b_\beta)\nu_-^{\beta\alpha} - b_\beta \nu_+^{\beta\alpha} \right] \\
&(\partial_+ - h\partial_-)\Lambda_\alpha = \partial_- c_\alpha + \sum_\beta \left[(c_\beta + h\Lambda_\beta)\nu_-^{\beta\alpha} - \Lambda_\beta \nu_+^{\beta\alpha} \right] \\
&\partial_+ \nu_-^{\alpha\beta} - \partial_- \nu_+^{\alpha\beta} = -2\eta_\alpha (c_\alpha b_\beta - b_\alpha c_\beta) + ([\nu_+, \nu_-])^{\alpha\beta}.
\end{aligned} \tag{5.54}$$

Introducing the extended "metrics":

$$H = h - \frac{3}{5}c_1, \quad A = \frac{1}{2}(\nu_+^{23} + \nu_+^{24}), \quad B = -(c_3 + c_4)$$

and eliminating part of the variables, we find as a consequence of (5.54)that $H,\ A,\ B$ and $T,\ W,\ V$ satisfy the following system of equations:

$$\begin{aligned}
&-5\partial_-^3 H = (\partial_+ - 2\partial_- H - H\partial_-)T - 3\partial_- AW - 2A\partial_- W - 4\partial_- BV - 3B\partial_- V,\\
&\partial_-^5 A = (\partial_+ - 3\partial_- H - H\partial_-)W + (\text{more terms}),\\
&-\frac{1}{20}\partial_-^7 B = (\partial_+ - 4\partial_- H - H\partial_-)V + (\text{more terms}),
\end{aligned} \tag{5.55}$$

A detailed analysis shows that these are indeed the expected trace anomaly equations for $WSO(3,3)$- gravity. (see Appendix D for details).

Further restrictions on the scalar curvature and certain other invariants lead to the following "equations of motion" for $H,\ A,\ B$:

$$\partial_-^3 H = \partial_-^5 A = \partial_-^7 B = 0, \tag{5.56}$$

The solutions of these equations contain 15 functions $J_+^{\mu\nu}(x^+),\ \mu,\ \nu = 1,\cdots,6$ ($J_+^{\mu\nu} = -\eta_\mu\eta_\nu J_+^{\mu\nu}$) which should be related to the generators of the $so(3,3)$-current algebra as their transformation laws (which can be derived from the transformation properties of A_+) tell us.

This completes our schematic demostration that the extrinsic geometry of the surfaces (4.1) possesses all the properties of the $WSO(3,3)$ ($\approx W_4$) -gravity. Concerning the KPZ-quantization of such theory we assume that it is given by the quantum W_4-gravity [12, 20]. According to refs. [12, 20] the vanishing of the total stress-energy tensor restricts the number of the quantum matter fields d_q to be $d_q \leq 3$.

Further generalizations to surfaces embedded in $M_{p,q}$ requires better understanding of the nature of the class of surfaces singled out by the D-S Hamitionian reduction.

6. Discussion

To summarize, in the present thesis we described the general properties of the extended W- gravities. It was shown that using a natural anzatz for the gauge fields and the defining relations of the underlying W- algebras it is possible to derive general expressions for the renormalized central charge and the anomalous dimensions of operators. It is clear that a lot remains to be done. The theories should be consistently quantized and the conformal and W- anomalies should be carefully investigated. If it turns out that one can construct anomaly free theories, this would be an encouraging step in the studying of the W- strings. The most important open problem, however, concerns the geometry underlying the W- gravities. In the last two Chapters we discussed an attempt to make connection between the latter and the extrinsic geometries of certain class of surfaces embedded in higher dimensional spaces.

The relation between W_3-gravity and the geometry of affine surfaces in A_3, described in Chapter 4, addresses the question for the uniqueness of the proposed affine geometrical interpretaion. In fact there exists one more candidate for the role of W_3-geometry. It is the projective geometry of surfaces embedded in RP^3. We have no satisfactory proof that such a geometry is a specisic classical limit of the quantum W_3-gravity [47] but there are two indications that it should be so. The first is the Fubini theorem [39] that the projective surfaces in RP^3 are described by their two fundamental forms: one quadratic h_{ij} and one cubic A_{ijk}. The second is that the corresponding (gauge fixed) projective structure equations obey specific $SL(4, R)$ residual gauge symmetries similar to the $SL(3, R)$ ones in the affine case.

Concerning the quantum W_3-gravity (i.e. the quantization of the affine surface geometry) we have to mention that the geometric quantities h_{ij}, A_{ijk} should be considered as quantum fields independent from the matter fields $y^\mu(x^+, x^-)$. Remember now the identification of the tangent v^μ to the affine line in the A_3 as a classical limit of the primary fields $\phi\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}$ and $\phi\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}$ of the W_3-minimal models (see Sect.2.3). Let us introduce a set of (affine) x^+- and x^--coordinate lines on the affine surface M_{1+1} such that in each point they are tangent to $y^{\mu}_{\pm}$. Therefore we have introduced in each point of M_{1+1} the corresponding primary fields of the W_3-model. This gives us an affine geometrical interpretation of the interacting W_3-gravity and W_3-minimal model as an interaction of the tangents of the local frame $y^{\mu}_{\pm}$ with the fluctuating affine surface geometry given by h_{++} and A_{+++}. We can have another interpretation of the same system as an affine particle (or "affine Wilson line") moving on the random affine surface and interacting with its quantum geometry. This picture can be generalized to the case of WB_n and WD_n-gravities* (see [41]) interacting with the corresponding minimal models. It suggests that the WG-gravity can interact only with the fields of the WG-minimal model.

All our results concerning the equivalence of certain noncritical strings to the W- gravities were derived in the light-cone gauge. The question arises whether this property holds in arbitrary gauge ? Although we have no satisfactory proof for the general case, the preliminary analysis shows that such an equivalence *is gauge independent.* For example, the integrability conditions (5.9) for constant mean curvature surfaces in $M_{1,2}$ in conformal gauge lead to the Liouville equation and to two Liouville equations for surfaces embedded in $M_{2,2}$. Our *conjecture* is that the integrability conditions (i.e. the Gauss-Codazzi equations) for certain class of surfaces in the conformal gauge for $h_{ij}{}^{\dagger}$ generate the generalized Toda equations (see [16]). In other words the conformal gauge counterpart of the "extended trace anomaly" equations are the Toda systems of the corresponding affine algebra $\hat{g}$.

Our interpretation of the induced 2-D gravity as extrinsic geometry of surfaces embedded in $M_{1,2}$ (i.e. $d_{cl} = 3$) might create certain confusions. As we know from the KPZ- quantization [6] the well defined quantum gravity (interacting with conformal matter) exists only for $d_q \leq 1$ (or $d_q \geq 25$). To make the things clearer we should repeat once more our main statement concerning W- geometries: *the quantum $WSO(p,q)$ gravities have as a quasiclassical limit $(c \to -\infty)$ the extrinsic geometries of a certain class of surfaces embedded in flat*

$\star$ including the Riemannian $WSO(2,1)$-gravity

$\dagger$ and certain specific "G_0-gauge fixing" of the remaining "gravitational" degrees of freedom b^{α}_{ij}, $\nu^{\alpha\beta}_{i}$.

$M_{p,q}$. The following three properties of the class of solvable extrinsic geometries:

a) their $WSO(p,q)$ - symmetries of the (gauge fixed) structure equations

b) the "hidden" $SO(p,q)$ - current algebra

c) the extended anomaly equations hidden in the Gauss-Codazzi equations lead us to the *assumption that the quantization of such geometries is given by the* $WSO(p,q)$ *- gravities interacting with conformal matter*. However, we have no explanation of the discrepancy between d_{cl} and d_q, i.e. why $3 = d_{cl} \neq d_{quant} \leq 1$ (in general $p+q \neq d_q = \frac{p+q}{2}$). We expect the *intrinsic geometrical description* of the $WSO(p,q)$ - gravities to answer this question.

Turning back to our starting point- *the Klein geometries and the Lie-Cartan theory of the geometric invariants*- we should note that the problem of their quantization remains open. Our few examples of quantum field theories which have as a classical limit ($c \to -\infty$) specific Klein geometries show that these allmost forgotten (by the physicists) geometries can play an important role in the understanding of the noncritical string models.

APPENDIX A

A transformation group acting in some subspace of R^m is defined by an r-parameter Lie group G and a family of maps $y = F(x,a), x, y \in R^m, a \in G$, provided

$$F(x,e) = x$$
$$F(F(x,a),b) = F(x,ba)$$

Let $a(t) = \alpha t + \ldots$ be a one parameter subgroup G_α of G. $y(t) = F(x,a(t))$ is the trajectory of the fixed point x under the action of the one-parameter subgroup:

$$y(t) = x + \alpha \mathcal{W}(x)t + \ldots \tag{A1}$$

where the $r \times m$ matrix $\mathcal{W}(x)$ is the Jacobian $\mathcal{W}(x) = \left(\frac{\partial F_i}{\partial a_i}\right)_{a=e}$. Define the Lie derivative of a function $f(x)$ on R^m under a one-parameter group G_α by:

$$\mathcal{L}_\alpha f = \left(-\frac{d}{dt} f\left(F(x,a(t)^{-1})\right)\right)_{t=0}$$

A simple computation shows that:

$$\mathcal{L}_\alpha f(x) = \sum_{i=1}^{m} (\alpha \mathcal{W}(x))_i \frac{\partial}{\partial x_i} f(x) \equiv \xi_\alpha grad \quad f(x).$$

The vector field $\xi_\alpha(x) \equiv \alpha \mathcal{W}(x)$ is the tangent at x to the trajectory of the one-parameter group G_α (see (A1)).

The Lie derivatives of a transformation group F form a Lie algebra under addition and commutation which is isomorphic to the Lie algebra of the group G. The corresponding transformation group is completely determined by its Lie derivatives.

The transformation group is called localy transitive at the point x if for any other point y in some neighborhood of x there exists $a \in G$ such that $y = F(x,a)$.

The central point of any geometry is the study of the invariants of the transitive transformation groups. Among them the most important are those ones which are not changed under any differentiable change of coordinates- the so called geometric invariants. These are the only invariants that have an intrinsic geometrical meaning.

The differential invariants are functions of the points x and their derivatives with respect to some parameters (independent variables). For example the function $f(x)$ is an invariant of the transformation group F if $f(x) = f(F(x,a))$. As a consequence

$$\mathcal{L}_\alpha f \equiv \alpha \mathcal{W}(x) grad \quad f = 0. \tag{A2}$$

But for the transitive transformation groups the rank r of $\mathcal{W}$ is greather than the dimension of the space: $r > m$, and the equation (A2) has only a trivial solution $f = const$. Therefore any nontrivial invariant of F must depend at least on the first (and higher) derivatives of x. The number of invariants is determined by the fundamental existence theorem for the solutions of the system of partial differential equations of type (A2) .

Let us consider a curve in R^m and and choose the coordinate x_1 as an independent variable: $x_i = x_i(x_1), i = 2, \ldots, m$. In this case we have $m-1$ "new" variables $x_i' = \frac{dx_i}{dx_1}$. The new "prolonged" Lie derivatives will act on functions of $2m-1$ variables $\psi(x_1, \ldots, x_m; x_2', \ldots, x_m')$:

$$\mathcal{L}^{(1)}\psi = \left[\sum_{i=1}^{m} \xi_i \frac{\partial}{\partial x_i} + \sum_{j=2}^{m} \eta_j \frac{\partial}{\partial x_j'}\right] \psi \tag{A3}$$

For the computation of η_j we observe that $\mathcal{L}^{(1)}$ is acting in fact in some subspace of R^{2m-1} in which

$$dx_j - x_j' dx_1 = 0 \tag{A4}$$

holds. Also, from the commutativity of the second partial derivatives we have

$$d\mathcal{L}^{(1)} = \mathcal{L}^{(1)} d. \tag{A5}$$

From (A4) and (A5) we obtain

$$\eta_j = \frac{d\xi_j}{dx_1} - x_j' \frac{d\xi_1}{dx_1}. \tag{A6}$$

Analogously, the second prolongation is

$$\mathcal{L}^{(2)} = \sum_{i=1}^{m} \xi_i \frac{\partial}{\partial x_i} + \sum_{i=2}^{m} \eta_i \frac{\partial}{\partial x_i'} + \sum_{i=2}^{m} \zeta_i \frac{\partial}{\partial x_i''}$$

and ζ_i are computed from (A5) and

$$dx_j' - x_j'' dx_1 = 0.$$

The invariant function of order k will be a solution of a system of equations of type (A2) :

$$\mathcal{L}_\alpha^{(k)} \psi \left(x_1, \ldots, x_m; \ldots; x_2^{(k)}, \ldots, x_m^{(k)}\right) = 0.$$

It is clear that we can "prolonge" the Lie derivatives to infinity and at the same time the rank r of $\mathcal{W}(x)$ is not changed. Therefore every transitive transformation group has nontrivial differential invariants. Actualy, there exists a finite set of independent invariants and all the others are functionaly dependent on them (for example if $f^{(2)}$ and $f^{(3)}$ are invariants of order 2 and 3 respectively, then $\frac{df^{(3)}}{df^{(2)}}$ is an invariant of order 4, etc.). It can be shown that if the number of the independent variables is n, then the number of the independent invariants is $< n + m$.

Consider for example a curve in two dimensional Euclidean space. The 2D Euclidean group of motions has three parameters (1 rotation, 2 translations) $r = 3$, $m = 2$, and each prolongation adds one dimension to the space in which the group is acting. So we expect two differential invariants of 2-nd and 3-th order respectively. The prolonged matrix $\mathcal{W}$ in this case is given by

$$\mathcal{W}^{(3)}(x) = \begin{pmatrix} -x_2 & x_1 & 1+(x_2')^2 & 3x_2'x_2'' & 3(x_2'')^2 - 4x_2'x_2''' \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The second order invariant is a solution of the equations

$$\begin{aligned} -x_2\frac{\partial f}{\partial x_1} + x_1\frac{\partial f}{\partial x_2} + \left(1+(x_2')^2\right)\frac{\partial f}{\partial x_2'} + 3x_2'x_2''\frac{\partial f}{\partial x_2''} &= 0 \\ \frac{\partial f}{\partial x_1} &= 0 \\ \frac{\partial f}{\partial x_2} &= 0 \end{aligned}.$$

They can be integrated directly and one obtains

$$f^{(2)} = \frac{x_2''}{(1+(x_2')^2)^{3/2}}.$$

This is actualy the curvature of the curve embedded in E_2. Analogously for the third order invariant:

$$f^{(3)} = \frac{\left(1+(x_2')^2\right)x_2''' - 3x_2'(x_2'')^2}{(1+(x_2')^2)^3}.$$

Actualy in the case of curves in Euclidean spaces one can introduce an "invariant parameter" - the arc length of the curve σ. One can show that in fact $f^{(3)} = \frac{dk}{d\sigma}$, so the independent differential invariant in this case is only $f^{(2)}$, i.e. the curvature.

Consider the general case of a curve in R^m. The Euclidean group has $\frac{m(m-1)}{2} +$ m parameters and each prolongation adds $m-1$ dimensions. Therefore, given the invariant parameter, we obtain $m-1$ independent differential invariants.

Another example is the theory of curves in affine spaces. The group of motion is $SL(m,R)\cdot T_m$ (T_m = translations in R^m). It has m^2+m-1 parameters. Each prolongation of the Lie derivatives gives $m-1$ new variables. According to the above considerations this theory has $m-1$ independent differential invariants of order $\leq m+2$ (with respect to the invariant parameter - the "affine arc length").

In the theory of surfaces there are two independent variables. The simplest case is a surface in 3D Euclidean space. It can be defined localy as a function $x_3 = x_3(x_1,x_2)$. Here, we should consider the derivatives $p = \frac{\partial x_3}{\partial x_1}$, $q = \frac{\partial x_3}{\partial x_2}$, $r = \frac{\partial_2^{x_3}}{\partial_{x_1^2}}$, $s = \frac{\partial_2^{x_3}}{\partial_{x_1}\partial_{x_2}}$, $t = \frac{\partial_2^{x_3}}{\partial_{x_2^2}}$, etc. Analogously to the case of curves the first prolongation is found to be

$$\begin{aligned}
\mathcal{L}_1^{(1)} &= -x_3\partial_2 + x_2\partial_3 + pq\frac{\partial}{\partial_p} + (1+q^2)\partial_q \\
\mathcal{L}_2^{(1)} &= -x_3\partial_1 + x_1\partial_3 + (1+p^2)\partial_p + pq\partial_q \\
\mathcal{L}_3^{(1)} &= -x_2\partial_1 + x_1\partial_2 - q\partial_p + p\partial_q
\end{aligned}$$

The group of Euclidean motions has 6 parameters. Two prolongations will add 5 dimensions. Hence, the theory of surfaces in R^3 is determined by two invariants of second order.

Another interesting example considered in Chapter 4 is a surface embedded in 3D affine space A_3. The group of unimodular affine transformations has 11 parameters. Therefore we expect the theory to depend on one independent invariant of order 3 and three independent invariants of order 4.

APPENDIX B

In this Appendix we show that after fixing the $SL(3,R)$ gauge symmetry, the remaining "currents" in (4.58) , i.e. $T = -2B_{--}$ and $W = 4(C - \partial_- B_{--})$ generate the W_3-transformations. We find also the variations of the improved metric $\tilde{h} = h + 4\partial_- A$ and the field $A = 2(D - h^3 C)$ under the above transformations.

First, for further convenience* , we make an ulterior gauge transformation (4.57) of the matrices (4.58) using

$$\bar{g}_0 = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

* in this gauge W transforms as a primary field with respect to T

$\alpha = -2B_{--}(\equiv T)$. The matrices (4.58) take the form

$$\tilde{A}_-^{(new)} = \frac{1}{2}\begin{pmatrix} 0 & W & T \\ 0 & 0 & 1 \\ 1 & T & 0 \end{pmatrix}$$

$$\tilde{A}_+^{(new)} = \begin{pmatrix} h_- + 2AT & \frac{h}{2}W - h\partial_- T - 2h_- T - 2AT^2 + \partial_+ T & \frac{1}{2} + \frac{1}{2}hT \\ 2A & -h_- - 2AT & \frac{1}{2}h \\ -2(B_{++} + \frac{1}{2}h^2 T + \frac{h}{4}) & 1/2 + \frac{1}{2}hT + 2T(B_{++} + \frac{1}{2}h^2 T + h/2) & 0 \end{pmatrix}. \tag{B1}$$

The $SL(3,R)$ gauge transformations of the fields in (B1) are given by:

$$\delta A = \partial\epsilon + [\epsilon, A], \tag{B2}$$

$A = A_a T^a$, $\epsilon = \epsilon_a T^a$, $a = 1, \ldots, 8$ and T^a are the matrix generators of $SL(3,R)$:

$$T^1 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} \quad T^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1/2 \end{pmatrix} \quad T^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$T^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T^5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad T^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad .$$
$$T^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad T^8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We want to preserve the choosen gauge (B1) . This condition combined with the transformation properties (B2) gives (algebraic) equations for the (part of) ϵ's, which are easily resolved:

$$\begin{aligned} \epsilon_4 &= -\partial\epsilon_1 + \epsilon_5 W + \epsilon_7 T \\ \epsilon_8 &= \partial\epsilon_2 + \epsilon_5 W + \epsilon_6 T \\ \epsilon_6 - \epsilon_7 &= -2\partial\epsilon_5 \\ \epsilon_1 - \epsilon_2 &= 4\partial(\epsilon_6 + \epsilon_7) \\ \epsilon_1 + \epsilon_2 &= \frac{8}{3}\partial^2\epsilon_5 - \frac{4}{3}\epsilon_5 T \\ \epsilon_3 &= \frac{8}{3}\partial^4\epsilon_5 - \frac{16}{3}\partial^2\epsilon_5 T - \frac{4}{3}\epsilon_5\partial^2 T + \epsilon_5 T^2 + 1/2(\epsilon_6 + \epsilon_7)W. \end{aligned} \tag{B3}$$

The remaining independent $\epsilon_6 + \epsilon_7 \equiv \epsilon$ and $\epsilon_5 = \eta$ parameters serve as parameters of the transformations generated by T and W.

In fact from (B2) one obtaines:

$$\begin{aligned}\delta T &= \partial(\epsilon_4+\epsilon_8)+1/4(\epsilon_1-\epsilon_2)T-1/2(\epsilon_6-\epsilon_7)W\\ \delta W &= 2\partial\epsilon_3+1/2(\epsilon_1-\epsilon_2)W+(\epsilon_4-\epsilon_8)T.\end{aligned}\tag{B4}$$

Substituting the relations (B3) in (B4) we get the final answer (4.60) . It is obvious that ϵ and η are the parameters of the transformations generated by T and W respectively.

Let us turn now to the matrix $\tilde{A}_+^{new}$ in (B1) and consider the transformations of the combinations $\tilde{h}=A_6+A_7$ and $A=A_5$:

$$\begin{aligned}\delta\tilde{h} =&\partial_+\epsilon+1/2(\frac{1}{2}\epsilon_1+\epsilon_2)h+2(\epsilon_1+\frac{1}{2}\epsilon_2)(B_{++}+\frac{1}{2}h^2T+\frac{1}{4}h)-\\ &+(\epsilon_6+\epsilon_7)(h_-+2AT)-2(\epsilon_4-\epsilon_8)A+4\eta T\partial_-A\\ \delta A =&\partial_+\eta-(\epsilon_1-\epsilon_2)A+2\eta(h_-+2AT)-2\epsilon_6(B_{++}+\frac{1}{2}h^2T+\frac{1}{2}h)-\\ &+1/2(\epsilon_6-\epsilon_7)h.\end{aligned}\tag{B5}$$

The parameters are fixed by the same equations (B3) . After substituting their solutions in (B5) we arive to the transformations (4.61) . They are in agreement with the previous results [20] and demonstrate that endeed $\tilde{h}$ and A are the fields "conjugate" to T and W, as stated in the begining.

APPENDIX C

In this appendix we derive, using the method of Polyakov [43],the restricted (field dependent) gauge transformation laws for $T_\pm$, as announced in (5.33) . We also derive the transformation laws for the extended metrics $\tilde{h}_\pm$.

The $SO(2,2)$ (or $SO(1,3)$, depending on whether $e=1$ or $e=i$)-gauge transformations

$$\delta A_\pm=[\epsilon,A_\pm]+\partial_\pm\epsilon,$$

$$\epsilon=\begin{pmatrix}0 & \epsilon_0 & \epsilon 1+\eta_1 & \epsilon_2+\eta_2\\ \epsilon_0 & 0 & \epsilon_1-\eta_1 & \epsilon_2-\eta_2\\ \epsilon_1+\eta_1 & -(\epsilon_1-\eta_1) & 0 & \eta_0\\ -e^2(\epsilon_2+\eta_2) & e^2(\epsilon_2-\eta_2) & e^2\eta_0 & 0\end{pmatrix},\tag{C1}$$

for A_- given by (3.3) and $\Lambda_{1,2}=$ consts ($\Lambda_1\neq\Lambda_2$) give 4 equations for the 6 parameters and the transformation laws for b_1 and b_2:

$$\begin{aligned}\delta b_1 &= \partial_- \epsilon_1 + \epsilon_0 b_1 - e^2 \eta_0 b_2, \\ \delta b_2 &= \partial_- \epsilon_2 + \epsilon_0 b_2 - \eta_0 b_1,\end{aligned} \tag{C2}$$

and

$$\begin{aligned}&2(\epsilon_1 \Lambda_1 - \eta_1 b_1) + 2e^2(\eta_2 b_2 - \epsilon_2 \Lambda_2) + \partial_- \epsilon_0 = 0, \\ &-(\epsilon_0 \Lambda_1 + e^2 \eta_0 \Lambda_2) + \partial_- \eta_1 = 0, \\ &-(\epsilon_0 \Lambda_2 + e^2 \eta_0 \Lambda_1) + \partial_- \eta_2 = 0, \\ &2(\epsilon_1 \Lambda_2 - \eta_1 b_2) - 2(\eta_2 b_1 - \epsilon_2 \Lambda_1) + \partial_- \eta_0 = 0,\end{aligned} \tag{C3}$$

The solutions of the system (C3)are:

$$\begin{aligned}\epsilon_0 &= \frac{1}{\Lambda_1^2 - e^2 \Lambda_2^2}(\Lambda_1 \partial_- \eta_1 - e^2 \Lambda_2 \partial_- \eta_2), \\ \eta_0 &= \frac{1}{\Lambda_1^2 - e^2 \Lambda_2^2}(\Lambda_1 \partial_- \eta_2 - \Lambda_2 \partial_- \eta_1), \\ \epsilon_1 \pm e\epsilon_2 &= \frac{1}{\Lambda_1 \mp e\Lambda_2}\Big[(b_1 \pm eb_2)(\eta_1 \mp e\eta_2) - \frac{1}{2}\frac{1}{\Lambda_1 \mp e\Lambda_2}\partial_-^2(\eta_1 \mp e\eta_2)\Big].\end{aligned} \tag{C4}$$

Setting $T_\pm = b_1 \pm b_2$, $\epsilon^\pm = \frac{\eta_1 \mp e\eta_2}{\Lambda_1 \mp e\Lambda_2}$, and substituting (C4)in (C2)we get (5.33) .

Similar analysis of the A_+-transformations leads to the following transformations

$$\begin{aligned}\delta(c_1 + h\Lambda_1) &= -[\epsilon_0(c_1 + h\Lambda_1) - e^2 \nu \eta_2 - \partial_- h \eta_1 + e^2 \eta_0 (c_2 + h\Lambda_2)] + \partial_+ \eta_1, \\ \delta(c_2 + h\Lambda_2) &= -[\epsilon_0(c_2 + h\Lambda_2) - \nu \eta_1 - \partial_- h \eta_2 + \eta_0 (c_1 + h\Lambda_1)] + \partial_+ \eta_2.\end{aligned} \tag{C5}$$

Taking into account (C4)in the tansformation laws of the new "metrics"

$$\tilde{h}_\pm = h + \frac{c_1 \mp e c_2}{\Lambda_1 \mp e\Lambda_2}, \tag{C6}$$

we find

$$\delta_{\epsilon^+, \epsilon^-} \tilde{h}_\pm = \partial_+ \epsilon^\pm - \partial_- \epsilon^\pm \tilde{h}_\pm + \epsilon^\pm \partial_- \tilde{h}_\pm. \tag{C7}$$

We note that all the above calculations can be used to derive similar formulas for the group $SO(1,2)$ by simply setting the last row and column of the 4×4 matrices to zero. In this way we obtaine again (5.21) ,(5.22) and (5.23) .

APPENDIX D

In this appendix we derive systematicaly the anomaly equations, eq.(5.55), from the Gauss-Codazzi ones, (5.54) , for the specific surfaces embedded in $M_{3,3}$. We also give the complete expressions for the transfomation laws of the fields W and V of (conformal) spins 3 and 4 respectively.

Let us write down the Gauss-Codazzi equations in components imposing the geometrical constraints (4.4). In the following we focus our attention on only a part of them:

$$\partial_-^2 h = 2\Big(1 - \frac{3}{10}c_1 T + \frac{1}{2}(c_3 - c_4) - \frac{1}{2}(c_3 + c_4)V\Big), \tag{D1}$$

$$\partial_- c_1 + \frac{1}{5}(\nu_+^{13} + \nu_+^{14})T = 0, \tag{D2}$$

$$\partial_-(c_3 + c_4) + (\nu_+^{13} + \nu_+^{14}) = 0, \tag{D3}$$

$$\frac{2}{5}(\partial_+ - h\partial_-)T = \partial_-(c_3 - c_4) + c_2 W + (\nu_+^{13} - \nu_+^{14}) + \frac{2}{5}\nu_+^{34}T, \tag{D4}$$

$$\frac{3}{10}(\partial_+ - 2\partial_- h - h\partial_-)T = -\frac{1}{2}(\nu_+^{13} - \nu_+^{14}) + \frac{1}{2}(\nu_+^{13} + \nu_+^{14})V, \tag{D5}$$

$$\nu_+^{34} = 2\partial_- h + \frac{3}{10}(\nu_+^{13} + \nu_+^{14})T, \tag{D6}$$

$$\partial_-\nu_+^{34} = -(c_3 - c_4) - (c_3 + c_4)V + \frac{1}{2}(\nu_+^{23} + \nu_+^{24})W, \tag{D7}$$

$$\partial_-(\nu_+^{23} + \nu_+^{24}) = 2c_2. \tag{D8}$$

One can easily solve (D3), (D8), (D6), and (D7)to get:

$$\begin{aligned}
&\nu_+^{13} + \nu_+^{14} = -\partial_-(c_3 + c_4),\\
&c_2 = \frac{1}{2}\partial_-(\nu_+^{23} + \nu_+^{24}),\\
&\nu_+^{34} = 2\partial_- h - \frac{3}{10}\partial_-(c_3 + c_4)T,\\
&3\partial_-^3 h = 2 + \frac{3}{10}\partial_-(\partial_-(c_3 + c_4)T) - \frac{3}{5}c_1 T - 2(c_3 + c_4)V + \frac{1}{2}(\nu_+^{23} + \nu_+^{24})W,\\
&c_3 - c_4 = -\frac{4}{3} + \frac{2}{5}c_1 T + \frac{1}{10}\partial_-(\partial_-(c_3 + c_4)T) + \frac{1}{3}(c_3 + c_4)V + \frac{1}{6}(\nu_+^{23} + \nu_+^{24})W.
\end{aligned} \tag{D9}$$

From (D4)and (D5)by eliminating $\partial_+ T$ and using the above solutions we obtain:

$$\begin{aligned}
\nu_+^{13} - \nu_+^{14} = &-\frac{2}{5}\partial_-(c_3 + c_4)V - \frac{6}{25}\partial_-(c_1 T) - \frac{3}{50}\partial_-^2(\partial_-(c_3 + c_4)T) + \frac{9}{25}\partial_- c_1 T\\
&- \frac{1}{5}\partial_-\Big((c_3 + c_4)V + \frac{1}{2}(\nu_+^{23} + \nu_+^{24})W\Big) - \frac{3}{10}\partial_-(\nu_+^{23} + \nu_+^{24})W.
\end{aligned} \tag{D10}$$

Now we substitute $\nu_+^{13} - \nu_+^{14}$ back into (D5), which, together with (D2)gives:

$$\begin{aligned}\partial_+ T =& -5\partial_-^3 h + (2\partial_- h + h\partial_-)T - \frac{3}{5}(-5\partial_-^3 c_1 + (2\partial_- c_1 + c_1\partial_-)T)\\ &+ \frac{3}{2}\partial_-(\nu_+^{23} + \nu_+^{24})W + (\nu_+^{23} + \nu_+^{24})\partial_- W - 4\partial_-(c_3 + c_4)V - 3(c_3 + c_4)\partial_- V.\end{aligned} \tag{D11}$$

Setting $H = h - \frac{3}{5}c_1$, $A = \frac{1}{2}(\nu_+^{23} + \nu_+^{24})$ and $B = -(c_3 + c_4)$ we get from (D11)the anomaly equation for the "metric" H:

$$-5\partial_-^3 H = (\partial_+ - 2\partial_- H - H\partial_-)T - (3\partial_- AW + 2A\partial_- W + 4\partial_- BV + 3B\partial_- V). \tag{D12}$$

We also present here the complete expressions for the transformation laws of the fields $W$and V:

$$\begin{aligned}\delta_{\epsilon,\eta,\tilde\epsilon} W =& \partial_-^5\eta + 3\partial_-\epsilon W + \epsilon\partial_- W + 4\partial_-\eta V + 2\eta\partial_- V - 2\partial_-^3\eta T - 3\partial_-^2\eta\partial_- T\\ &- \frac{9}{5}\partial_-\eta\partial_-^2 T - \frac{2}{5}\eta\partial_-^3 T - \frac{7}{5}\partial_-^3\tilde\epsilon W - \frac{14}{5}\partial_-^2\tilde\epsilon\partial_- W - 2\partial_-\tilde\epsilon\partial_-^2 W\\ &- \frac{1}{2}\tilde\epsilon\partial_-^3 W + \frac{26}{25}\partial_-\tilde\epsilon TW + \frac{27}{50}\tilde\epsilon\partial_- TW + \frac{17}{25}\tilde\epsilon T\partial_- W + \frac{16}{25}\partial_-(\eta T)T,\end{aligned} \tag{D13}$$

$$\begin{aligned}\delta_{\epsilon,\eta,\tilde\epsilon} V =& -\frac{1}{20}\partial_-^7\tilde\epsilon + 4\partial_-\epsilon V + \epsilon\partial_- V + \frac{7}{25}\partial_-^5\tilde\epsilon T + \frac{7}{10}\partial_-^4\tilde\epsilon\partial_- T + \frac{21}{25}\partial_-^3\tilde\epsilon\partial_-^2 T\\ &+ \frac{14}{25}\partial_-^2\tilde\epsilon\partial_-^3 T + \frac{1}{5}\partial_-\tilde\epsilon\partial_-^4 T + \frac{3}{100}\tilde\epsilon\partial_-^5 T + \frac{3}{5}\partial_-^3\tilde\epsilon V + \frac{9}{10}\partial_-^2\tilde\epsilon\partial_- V + \frac{1}{2}\partial_-\tilde\epsilon\partial_-^2 V\\ &+ \frac{1}{10}\tilde\epsilon\partial_-^3 V - \frac{49}{125}\partial_-^3\tilde\epsilon T^2 - \frac{147}{125}\partial_-^2\tilde\epsilon T\partial_- T - \frac{88}{125}\partial_-\tilde\epsilon T\partial_-^2 T - \frac{59}{100}\partial_-\tilde\epsilon(\partial_- T)^2\\ &- \frac{177}{500}\tilde\epsilon\partial_- T\partial_-^2 T - \frac{39}{250}\tilde\epsilon T\partial_-^3 T - \frac{7}{5}\partial_-^3\eta W - \frac{7}{5}\partial_-^2\eta\partial_- W - \frac{3}{5}\partial_-\eta\partial_-^2 W\\ &- \frac{1}{10}\eta\partial_-^3 W + \frac{72}{625}\partial_-\tilde\epsilon T^3 + \frac{108}{625}\tilde\epsilon T^2\partial_- T - \frac{14}{25}\partial_-\tilde\epsilon TV - \frac{7}{25}\tilde\epsilon\partial_- TV\\ &- \frac{7}{25}\tilde\epsilon T\partial_- V + \frac{26}{25}\partial_-\eta TW + \frac{1}{2}\eta\partial_- TW + \frac{9}{25}\eta T\partial_- W + \frac{3}{4}\partial_-(\eta W)W.\end{aligned} \tag{D14}$$

The corresponding anomaly equations can be obtained formaly from the above transformation laws by the folowing substitution

$$\delta_{\epsilon,\eta,\tilde\epsilon} \to \partial_+, \quad \epsilon \to H, \quad \eta \to A, \quad \tilde\epsilon \to B. \tag{D15}$$

We have proved that the anomaly equations obtained in this way can also be derived from the Gauss-Codazzi equations for the surfaces we are considering, i.e. taking A_- as in eq.(5.50).

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