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SELF-ADJOINT OPERATORS CORRESPONDING TO HAMILTONIAN WITH n-POINT DELTA FUNCTION IN ONE DIMENSION

L. DABROWSKI

Thesis for the title "Magister Philosophiae in fisica"

Supervisors: Prof. P. Budinich
Prof. H. Grosse

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1. Introduction.

This thesis is devoted to study a class of models in Quantum Mechanics with interactions supported on a discrete set M of points. Such models are known as "point" or "contact" interactions, or zero-range potentials and have been considered in various areas of physics: since the thirties in nuclear and atomic physics in works of Thomas, Fermi, Bethe, Peierls and others; in solid state physics (e.g. Kronig- Penney model of onedimensional crystal) and since the fifties in many body physics in works of Huang, Lee, Yang, Wu and others.

The Schrodinger equation for the case of one interaction center is formally given by

$$i \frac{d}{dt} \psi = (\Delta + \delta) \psi$$

where Δ is the Laplacian and δ is the Dirac delta function. As it stands this expression does not make any sense since the multiplication of ψ by δ gives the zero wave function if $\psi(0)=0$, or a non-normalizable wave function if $\psi(0)\neq 0$. It is however possible to give a rigorous meaning to operators of such form. The methods which have been used are some limiting procedures like the scaling limit or the momentum cut-off potentials, the nonstandard analysis, the quadratic forms and the method of self-adjoint extensions of hermitian operators

Recently much work has been done on solving the spectrum exactly, finding the eigenstates and determining the the scattering parameters of such models

The case when the interaction is localized in a finite number of centers has been discussed too and a n-parameter family of possible Schrodinger operators has been described. However, from the theory of self-adjoint extensions it is known that there should be a n^2-parameter family of such operators, since we are dealing with deficiency indices (n,n). The aim of this thesis is to find all theese operators, study them and give a physical interpretation. For simplicity we shall consider in full detail only the one-dimensional case (Quantum Mechanics on the line) but all the results can be extended to D=2 and 3.

The results which we present in this thesis can in principle be also applied to more complicated situations. These include the presence of additional (smooth) potentials (for example the Culomb potential), the Dirac operator instead of the Laplacian or the derivative of delta function. The case of infinite number of centers and the delta-shell or similar interactions with a support on curves or surfaces has been discussed in the literature in some extend; this yields an infinite-parameter family of operators.

The plan of this thesis is as follows. In section 2 we shall recall the theory of self-adjoint extensions of closed symmetric operators. In section 3 we shall apply it to the Schrodinger operator with point interaction in n centers. In section 4 we shall show how to obtain these operators as the limit of the momentum cut-off potentials. In section 5 as an example we solve the case of two centers (n=2). In section 6 we shall give the outline how the delta-shell interactions should be described in our framework. Section 7 contains conclusions and final remarks.

2. Extensions of closed symmetric operators.

In this section we briefly recall the basic facts from the theory of self-adjoint extensions of closed symmetric operators. In this theory one makes use of the relation between symmetric and isometric operators which is given by the Cayley transform as follows. Assume T is a closed symmetric operator and z a complex number with nonzero imaginary part. Then the Cayley transform of T is defined on the domain

$$D(V) = \mathrm{Ran}\,(T-\bar{z}) \tag{2.1}$$

by

$$Vf = -(T-z)(T-\bar{z})^{-1}f \ , \quad f \in D(V) . \tag{2.2}$$

The operator V is isometric. The inverse transformation is given by

$$Th = (z+\bar{z}V)(1+V)^{-1}h \ , \quad h \in D(T) . \tag{2.3}$$

The defect indices (n_1, n_2) of T are equal to the dimension of the defect spaces N_{z_1} and N_{z_2} respectively, where

$$N_z = \mathrm{Ran}\,(T-z)^{\perp} \tag{2.4}$$

and $\mathrm{Im}\, z_1 < 0$, $\mathrm{Im}\, z_2 > 0$. Here $\mathcal{H}_o^{\perp}$ denotes the orthogonal complement of $\mathcal{H}_o$ in the Hilbert space $\mathcal{H}$. The defect indices of V are similarly defined with $|z_1| < 1$ and $|z_2| > 1$. These defect indices do not depend on the choice of z in the respective half planes are and are identical for T and V.

Any isometric extension $\tilde{V}$ of V maps a subspace of $D(V)^{\perp} = \mathcal{H} \ominus D(V)$ onto a subspace of $(\mathrm{Ran}\,V)^{\perp} = \mathcal{H} \ominus \mathrm{Ran}\,V$ with the same dimension. Thus V is maximal (i.e. has no strictly larger extensions) if $n_1=0$ or/and $n_2=0$ and, in particular, V is unitary if $n_1=n_2=0$. If $n_1=n_2>0$, V has (at least one) extension $\widetilde{V}$ which is of the form $\widetilde{V}=V+U$ for some unitary operator U from

$D(V)^{\perp}$ to $(\mathrm{Ran}\,V)^{\perp}$. Then all the self-adjoint extensions $\widetilde{T}$ of T can be obtained from the inverse Cayley transformation.

We see that all self-adjoint extensions are parametrized by a n^2-parameter family of unitary operators U. The domain of a self-adjoint extension T^U of T consists of vectors of the form

$$f = f_0 + g_z + U g_z \quad , \quad f_0 \in D(T) \; , \; g_z \in N_{\bar{z}} \; , \; \mathrm{Im}\, z > 0 \tag{2.5}$$

where U is the unitary map from $N_{\bar z}$ onto N_z. The action of T^U is given by

$$T^U f = T f_0 + z g_z + \bar{z} U g_z \tag{2.6}$$

The whole infromation about T^U is contained in the resolvent $R_z^U = (T^U - z)^{-1}$. Let T^U and T^W be two self-adjoint extensions of T, and R_z^U and R_z^W the corresponding resolvents. Denote by g_z^k, $k=1,\dots,n$ a basis of $N_{\bar z}$. The difference of the two resolvents satisfies the Krein formula

$$R_z^U - R_z^W = \sum_{k,l}^{n} g_z^k M_{kl}(z) \langle g_{\bar z}^l , \cdot \rangle \tag{2.7}$$

where the $n\times n$ matrix M(z) obeys the equation

$$M(z) - M(z_0) = (z - z_0) M(z) S(\bar{z}, z_0) M(z_0) \tag{2.8}$$

and $S(\bar z, z)$ is the matrix of scalar products

$$S(\bar z, z_0)_{k,l} = \langle g^k_{\bar z}, g^l_{z_0} \rangle \quad , \quad k,l = 1,\dots,n . \tag{2.9}$$

The basis g_z^k can be chosen to be an analytic function of z as follows

$$g_z^k = \left[1 + (z - z_0) R_z^W\right] g_{z_0}^k \tag{2.10}$$

where $g_{z_0}^k$ is some fixed basis and T^W is some fixed extension.

3. Construction of the general family of n-center point intereactions.

In this section we shall apply the above abstract construction to our problem. We are intersted in a Hamilton operator which looks like the free Laplacian

$$-\Delta = -\frac{d^2}{dx^2} \tag{3.1}$$

outside the set $M=\{x_j, j=1,\ldots,n\}$. The basic idea is to restrict first the self-adjoint Laplacian defined on the natural dense domain

$$\{\psi \in \mathcal{H} : \frac{d\psi}{dx} \text{ absolutely continuous}, \Delta\psi \in \mathcal{H}\} \tag{3.2}$$

in $\mathcal{H} = L^2(R, dx)$ to some appropriate domain D and then find all the self-adjoint extensions (which equal $-\Delta$ on D). The relevant dense domain D is the set of functions which vanish at all the points x_j, for $j=1,\ldots,n$ (this is meaningful since we deal with absolutely continuous functions). In this way the perturbation is felt only by the wave-functions which do not vanish on M. This is why we can write formally

$$-\frac{d^2}{dx^2} + \delta_M \tag{3.3}$$

where δ_M is some generalized potential supported on M (delta-like distribution).

To see that we can apply the method of previous chapter to the restriction of $-\Delta$ to D we pass via the Fourier transformation to the momentum representation. Then we have a multiplicative operator $H\psi(p)=p^2\psi(p)$ defined on the domain

$$D(H) = \{\psi \in L^2(R, dp) : \int |p^2\psi(p)|^2 dp < \infty, \int e^{ipx_j}\psi(p)\,dp = 0, 1\leq j\leq n\} \tag{3.4}$$

This is obviously a symmetric operator. To show that it is closed consider a sequence $\psi_m \in D(H)$, $\psi_m \to \psi$, $H\psi_m \to \phi$ with ψ and ϕ belonging to $\mathcal{H}$. We have to proove that $\psi \in D(H)$ and $H\psi = \phi$. The first statement follows from the chain of inequalities

$$\left|\int e^{ipx_j}\psi(p)\,dp\right| = \left|\int e^{ipx_j}[\psi(p)-\psi_m(p)]\,dp\right| = \lim_{N\to\infty}\left|\int e^{ipx_j}\,\chi_N(\psi-\psi_m)(p)\,dp\right|$$

$$\leq \lim_{N\to\infty}\int\left|\chi_N(\psi-\psi_m)(p)\right|dp \leq C N^{1/2}\,\|\psi-\psi_m\| \qquad (3.5)$$

$$\leq C N^{1/2} N^{-1} = C N^{-1/2} \xrightarrow[N\to\infty]{} 0$$

and

$$\|p^2\psi(p)\|^2 = \lim_{N\to\infty}\int \chi_N(p)\,p^4|\psi(p)|^2 \leq \|\chi_N p^2(\psi-\psi \qquad (3.5a)$$

$$\leq \|\chi_N p^2(\psi-\psi_m)\| + \|\chi_N(p^2\psi_m-\phi)\| + \|\chi_N\phi\|$$

$$\xrightarrow[N\to\infty]{} \|\phi\| < \infty$$

where $\chi_N(p)$ is the characteristic funcion of the set (-N,N).
The second statement is proven by

$$\|p^2\psi-\phi\| \leq \|p^2(\psi-\psi_m)\| + \|p^2\psi_m-\phi\| \xrightarrow[N\to\infty]{} 0 \qquad (3.6)$$

It is straightforward to see that the linearly independent basis of the defect space N_{-i} is

$$g_i^k(p) = (2\pi)^{-1/2}\,\frac{e^{-ipx_k}}{p^2-i}, \qquad k = 1,\dots,n. \qquad (3.7)$$

Next we observe that the self-adjoint free Laplacian in the momentum representation provides us one particular extension with the resolvent

$$R_z(p) = \frac{1}{p^2 - z} \quad . \tag{3.8}$$

We can use it as a reference extension in the formula 2.10 obtaining

$$g_z^k(p) = (2\pi)^{-1/2} \frac{e^{-ipx_k}}{p^2 - z} \quad . \tag{3.9}$$

The domain of the self-adjoint extension H^u is now given by vectors

$$\psi(p) = \phi(p) + \sum \alpha_k (g_i^k + U g_i^k) \quad , \alpha_k \in \mathbb{C} , \phi \in D(H) , \tag{3.10}$$

where the unitary operator can be written inthe basis g_i^k as

$$U g_i^k = \sum_l \mathcal{U}_{k,l} \, g_{-i}^l \quad . \tag{3.11}$$

Due to the unitarity the n×n matrix $\mathcal{U}_{k,l}$ satisfies

$$\overline{\mathcal{U}} \, S(-i,-i) \, \mathcal{U}^T = S(i,i) = S(-i,-i) \tag{3.12}$$

where $S(i,i)$ is the particular case of the matrix function 2.9

$$S_{k,l}(\bar{z}, z_0) = \int (2\pi)^{-1/2} \frac{e^{ip(x_k - x_l)}}{(p^2 - z)(p^2 - z_0)} \, dp \quad . \tag{3.13}$$

Then the action of H^u on ψ reads

$$(H^u \psi)(p) = p^2 \phi(p) + \sum_k^n \alpha_k \left(i g_i^k(p) - i \sum_{l=1}^n \mathcal{U}_{k,l} \, g_{-i}^l(p) \right) \quad , \phi \in D(H) . \tag{3.14}$$

It is easily seen that the extension 3.8 corresponds to the matrix $\mathcal{U}^{0} = -\delta_{\ell m}$.

To be able to use the Krein formula we have to determine $M(z)$ in eq. 2.7. To this aim we observe that we know the resolvents of H^{u} and H^{0} for $z=i$, thus

$$2i\left(R^{u}_{-i} - R_{-i}\right) = V \oplus U - V \oplus U_{0} = \tag{3.15}$$

$$= \sum_{j,k,\ell} \left(\mathcal{U}_{jk} + \delta_{jk}\right) S^{-1}_{j\ell}(i,i)\, g^{k}_{-i} \langle g^{\ell}_{i}, \cdot \rangle ,$$

where V denotes the Cayley transform of H. This gives

$$M(-i) = \frac{1}{2i}\left(U^{T} + 1\right) S^{-1}(i,i), \tag{3.16}$$

which by using 2.8 yields

$$M^{-1}(z) = 2i\, S(i,i)\left(U^{T} + 1\right)^{-1} - (z+i)\, S(\bar{z},-i) \tag{3.17}$$

for $\det(U^{T}+1) \neq 0$. Since the spectrum of the free Laplacian is absolutely continous and positive, the last formula together with the Krein formula imply that the spectrum of H^{u} is a positive halfaxis plus the pure point spectrum consisting of those z for which

$$\det M^{-1}(z) = 0 . \tag{3.18}$$

We define the matrix function G with the matrix elements

$$G_{k,l}(z) = \int (2\pi)^{-1/2} \frac{e^{ip(x_k - x_l)}}{p^2 - z} \, dp \quad . \tag{3.19}$$

The eq. 3.17 can be written in the form

$$M^{-1}(z) = [G(i) - G(-i)](U^T + 1)^{-1} + G(-i) - G(z) \quad . \tag{3.20}$$

We now give the physical interpretation of the operators H^u. Let $\partial\psi$ denote the difference between the derivatives of ψ from the right and from the left

$$\partial\psi(x) = \lim_{\varepsilon \to 0} \left[\frac{d}{dx}\psi(x+\varepsilon) - \frac{d}{dx}\psi(x-\varepsilon) \right] \quad . \tag{3.21}$$

In the position representation we have

$$\partial \tilde{g}_i^k (x_j) = -\delta_{ij} \quad . \tag{3.22}$$

By a short computation we get for a general vector in the domain of H^u

$$\partial\psi(x_k) = \sum_{j=1}^{n} \Lambda_{k,j} \psi(x_j) \, . \tag{3.23}$$

where the matrix Λ is related to $\mathcal{U}$ by

$$\Lambda \left(G(i) + G(-i) U^T \right) = 1 + U^T \, , \qquad \overline{\Lambda}^T = \Lambda \, . \tag{3.24}$$

This shows that we can iterpret the operators H^u in terms of the boundary conditions. If the matrix Λ is nondiagonal we have nonlocal boundary conditions, namely the jump of the first derivative at x_j will depend on the value of the function at x_l. This shows that the matrix Λ can be interpreted as the matrix of the relative strenght of the delta potential at points x_j.

4. Example : point interaction with two centers

We take n=2 and chose two points x_1 and x_2. Let

$$\Lambda = \frac{2}{|x_1 - x_2|} \begin{bmatrix} a , \gamma \\ \bar{\gamma} , b \end{bmatrix} ; \quad a, b \in \mathbb{R} , \ \gamma \in \mathbb{C} . \tag{4.1}$$

The pure point spectrum can be obtained from the equation

$$\det \begin{bmatrix} b\tau + i(ab - \gamma\bar{\gamma}) , & -\gamma\tau + i(ab + \gamma\bar{\gamma}) e^{i\tau} \\ -\bar{\gamma}\tau + i(ab - \gamma\bar{\gamma}) e^{i\tau} , & a\tau + i(ab - \gamma\bar{\gamma}) \end{bmatrix} = 0 \tag{4.2}$$

where $\tau = |x_1 - x_2| \sqrt{z}$. The bound states correspond to solutions τ on the positive imaginary axis, the virtual states to τ on the negative imaginary axis and resonances to τ not on the real axis. Sice the local case $a=b$, $\gamma=0$ has been discussed in [4] we consider the purely nonlocal case, i. e. $a=b=0, \gamma \in \mathbb{R}$. From 4.2 we get

$$\gamma e^{i\tau} - i\tau = \pm \gamma . \tag{4.3}$$

The quation for a bound state reads :

$$\gamma e^{-t} + t = \pm \gamma \quad , \quad t > 0 . \tag{4.4}$$

Thus for $0 < \gamma < 1$ there is no bound state; for $\gamma < 0$ and $\gamma > 1$ there is exactly one bound state. As far as resonances are concerned we have to solve the system

$$\begin{cases} s - \gamma e^{-t} \sin(s) = 0 \\ t + \gamma e^{-t} \cos(s) = \pm \gamma \end{cases} , \tag{4.5}$$

where $\tau = s + it$, which leads to

$$s\, ct(s) + \ln\left[\frac{\gamma}{s}\sin(s)\right] = \pm\gamma. \tag{4.6}$$

By studying this transcedental equation we get the following pattern of resonances: there are no resonances in the regions

$$2k\pi \leq s \leq (2k+1)\pi \qquad \text{if} \qquad \gamma < 0 \quad ,$$

$$(2k-1)\pi \leq s \leq 2k\pi \qquad \text{if} \qquad \gamma > 0 \quad ,$$

$$k = 1, 2, \ldots \quad ;$$

there is one resonance in the region $0 < s < \pi$ and there are two resonances in the remaining regions.

5. The operator H^u as limit of momentum cut-off potentials.

In this section we show that H^u (all the self-adjoint extensions described in the previous chapter) can be odtained as a limit $N \to \infty$ of momentum cut-off potentials H_N. Let H_N be the operator in $L^2(R, dp)$ given by

$$H_N \psi(p) = p^2 \psi(p) + \sum_{k,l}^{n} \Lambda_{k,l} \chi_N(p) E_k(p) \langle \chi_N E_l, \psi \rangle , \tag{5.1}$$

where

$$E_k(p) = (2\pi)^{-1} e^{ipx_k} \tag{5.2}$$

Since this operator has bounded potential term we can write down directly its resolvent

$$R_z^N = R_z + \sum_{k,l}^{n} R_z \chi_N E_k M_{kl}^N(z) \langle R_{\bar z} \chi_N E_l, \cdot \rangle \tag{5.3}$$

where

$$- M^N(z)^{-1}_{kl} = \Lambda^{-1}_{k,l} + \langle \chi_N E_k, R_z \chi_N E_l \rangle . \tag{5.4}$$

The convergence of H_N to H^u is now easy consequence of

$$\| R_z (\chi_N E_k - E_k) \| \xrightarrow[N \to \infty]{} 0 . \tag{5.5}$$

6. Brief outline of delta-shell interactions.

The direct application of the method of self adjoint extensions of closed operators to delta-shell or similar interactions with a support on curves or surfaces leads to infinite deficiency indices. This creates some problems. Intuitively it is clear that there should be an ∞^2-parameter family of extensions but we do not attempt the rigorous construction of all of them. We neglect technical difficulties and proceed to briefly outline the main points along the lines of section 2 and 3.

Assume that on R^3 (minus a zero-measure set which does not meet M) we have adapted coordinates (y,t) with the range $Y \times T$ such that

i). M is given by the equation t=r (thus y are internal coordinates on M and t are transversal coordinates ; $\dim Y = d$, $\dim T = 3-d$; $d = 1$ or 2)

ii) the Laplacian D separates in $L^2(Y,dy) \otimes L^2(T,dt)$ as

$$\Delta_y \otimes P_t + 1 \otimes \Delta_t \tag{6.1}$$

where Δ_y and Δ_t are differential operators in the variables y and t respectively, and P_t is a multiplicative operator in t.

In addition assume that we can solve for a complete orthonormal system of (generalized) eigenvectors X_λ of Δ_y with eigenvalues $F(\lambda)$, i.e.

$$\begin{aligned} \Delta_y X_\lambda &= F(\lambda) X_\lambda \\ \int X_\lambda(y) X_{\lambda'}(y)\, dy &= \delta(\lambda,\lambda') \\ \int X_\lambda(y) X_\lambda(y')\, d\lambda &= \delta(y,y') \end{aligned} \tag{6.2}$$

,

where $d\lambda$ is the spectral measure on Λ. Let ϑ_σ be an analogous system in $L^2(T, dt)$ with eigenvalues $E(\sigma)$ for the operator

$$\Delta_t + F(\lambda) P_t \tag{6.3}$$

Here λ is a parameter, ϑ_σ and $E(\sigma)$ depend on λ , and the measure on the labeling set Σ is $d\sigma$. Next we perform a unitary 'Fourier' transformation onto $\mathcal{H} = L^2(\Lambda \times \Sigma, d\lambda d\sigma)$ given by

$$\tilde{f}(\lambda,\sigma) = \int \vartheta_\sigma(t) \chi_\lambda(y) f(y,t)\, dy\, dt\, . \tag{6.4}$$

The Laplacian becomes now $\tilde{\Delta}$, the multiplicative operator by $E(\sigma)$ defined on a dense domain $\{ \tilde{f} \in \mathcal{H} : \int | E(\sigma) \tilde{f}(\lambda,\sigma) |^2 d\lambda\, d\sigma < \infty \}$.

The restricted domain is just

$$\tilde{D} = \left\{ \tilde{f} \in \mathcal{H} : \int \overline{\vartheta_\sigma(\tau)}\, \overline{\chi_\lambda(y)}\, \tilde{f}(\lambda,\sigma)\, d\lambda\, d\sigma = 0 \quad a.e. \right\}.$$

The restriction of $\tilde{\Delta}$ to $\tilde{D}$ is a closed symmetric operator T with deficiency indices (∞, ∞). The (generalized) basis of the deficiency space

$$N_z = Ran(T-z)^{\perp} = Ker(T^{+}-z) \quad , \quad Im\, z \neq 0 \, ,$$

can be taken in the form

$$g_z^y(\lambda,\sigma) = \frac{\vartheta_\sigma(\tau)\, \chi_\lambda(y)}{E(\sigma) - \bar{z}} \tag{6.5}$$

where $y \in Y$ is just a parameter. Denote $N_\pm$ the deficiency spaces and $g_\pm^y$ the basis for $z = \pm i$. We have $\overline{g_\pm^y} = g_\pm^{-y}$. The domain of a general self-adjoint extension H^u of T is

$$\tilde{D}(H^u) = \{\tilde{f} = h + g + Ug : h \in \tilde{D},\ g \in N_-\} \tag{6.6}$$

and the action of H^u reads

$$H^u(h + g + Ug) = E(\sigma)h + ig - iUg . \tag{6.7}$$

Here U is a bijective isometry from N_- to N_+. If we write

$$Ug_-^y = \int \mathcal{U}(y,y')\, g_+^{y'}\, dy' \tag{6.8}$$

then the integral kernel $\mathcal{U}$ satisfies the relation

$$\iint \overline{\mathcal{U}}(y,y')\, S_+(y',y'')\, \mathcal{U}(y'',y'')\, dy'dy'' = S_-(y,y''), \tag{6.9}$$

where $S_\pm(y',y'')$ is the particular case $z = \pm i$ of the (generalized) scalar product matrix

$$S_z(y,y') = \int \bar{g}_z^y(\lambda,\sigma)\, g_z^{y'}(\lambda,\sigma)\, d\lambda\, d\sigma . \tag{6.10}$$

In the position representation, H^u becomes the differential operator with the domain specified by the following boundary conditions. We conjecture that formally the (average) value of the derivative at (y,r) should be related to the value of wave-function at (y',r) as

$$\int \sum_{j=1}^{3-d} n_j \frac{\partial}{\partial t_j}\Big|_{t=r} f(y,t)\, dn = \int \Lambda(y,y') f(y',r)\, dy' , \tag{6.11}$$

where d(n) is the volume element on $S^{2-d} = \{n : \sum_{j=1}^{3-d} n_j^2 = 1\}$.

For d=2 the left hand side of this expression becomes just the jump of the first normal derivative at t=r (the result similar to point interactions). However the precise relation of the integral kernel $\Lambda(y,y')$ to the kernel $\mathcal{U}(y,y')$ has to be find out.

The work to be done in the future is computation of quantities introduced above for simplest symmetric manifolds M, for instance, if dimM=2, for: R^2 (plane), $R \times S^1$ (cylinder), $S^1 \times S^1$ (torus), S^2 (sphere), elipsoid, and, if dimM=2, for R (line), S^1 (circle) and elipse. Next, the quantities like resolvent, spectrum, eigenfunctions, resonances and scattering parameters should be explicitly determined.

7. Conclusions and final remarks.

In this thesis we have constructed the most general n^2- parameter class of Hamilton operators with poitnt interactions in n centers by using the method of self-adjoint extensions . We have interpreted these operators in terms of the (possibly nonlocal) boundary condition, and parameters as overall and relative strengths of the delta potential at different points. We have found a limiting procedure to obtain them as a norm resolvent limits of momentum cut-off potentials. As an example we have analysed the spectrum of the case with two interaction centers. Finally we have outlined the procedure to be applied for delta-shell interactions.

Some comments are in order.

Which one of the many-parameter family is the 'right' hamiltonian is a question of additional requirements. These may be for instnce the symmetry properties with respect to some involutions in the Hilbert space or transformation groups acting in R , the locality principle (it is interesting to note that nonlocal boundary conditions for the Dirac operator are usedin the original papers on the Atiyah-Singer index) or other physical requirements.

The method can be applied also to study operators of the form $\Delta + \delta_M + V$, where V is a usual potential - typically the Culomb one (c.f [9]). Some other potentials to be discussed are the magnetic monopole with the delta at the origin or some nonabelian gauge potentials in the case of spinors.

In addition to obtaining the general point interactions as norm resolvent limits of momentum cut-off potentials one should also do limits of scaled potentials or other approximations.

Infinite number of centers is the case relatively less known. Our method requires the full understanding of self adjoint extensions of closed operators with infinite deficiency indices, the method which has not been described yet (to the best of author's knowledge). In this respect this case is a bridge between M discrete and continuous. When the method will be worked out as a special case we should recover various known solutions like Kronig-Penney or n $\to \infty$ limit of finite number of centers

The motivation to study the interactions supported on a continous hypersurface M is obtaining a class of exactly solvable interesting models which have even a richer structure that the point interactions. Obtaining the general class will be useful in n-particle contact interactions or to explain the apparent discrepancy between the Thirring and Glaser solutions of the Thirring model (see e.g [14, 15, 16].

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