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CRITICAL EXPONENTS WITH THE STOCHASTIC QUANTIZATION METHOD.

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INTRODUCTION

Some years ago, Parisi and Wu proposed a new method of quantization based on a stochastic differential equation (the Langevin equation)(1).

The original reason for introducing this method is that it is possible to study gauge theories without the necessity of fixing the gauge, as it is usually done in the standard perturbative approach by the introduction of the Faddeev-Popov ghosts.

Another interesting feature of the stochastic quantization comes from the possibility of using the Langevin equation as an alternative method,with respect to the formulation of the theory on the lattice,for computer simulations.

Moreover the stochastic quantization provides a new conceptual point of view in Quantum Field Theory and it can be applied in many different cases. For instance it has been used to study the large-N reduction of the Eguchi-Kawai model; it has been shown the link between it and the supersymmetry or with the Nicolai mapping. Stochastic quantization provides also a new, non perturbative ultraviolet regulator.

This last feature of the stochastic quantization suggests a new way of doing real computations of some physical system.

The main purpose of this work is to use this new method to compute the critical exponents of a well known three dimensional system.

The system we are looking at is described by a scalar field in a three dimensional euclidean space with a four bosons auto-interaction. As it is known, the infrared divergences don't allow a computation in three dimensions in the usual perturbative approach. Up to now this system has been studied by means of the ε-expansion (2): to extract the critical exponents, one makes the computations in four dimensions and then extrapolates the results to d=3 , considering $\varepsilon=4-d=1$ as a small parameter. This procedure is then justified since the β-function exibits an infrared stable fixed point of order ε (3, 4).

Using the stochastic quantization we can avoid the ε-expansion and study the system at the physical dimension d=3; this can be done by substituting the Markovian character of the stochastic evolution with a non- Markovian one. This provides a new regulator of the theory, which, if the choice of the function representing the non - Markovian process is suitable, can be used in a minimal subtraction scheme of the renormalization procedure. For a particular value of the regulator the system is renormalizable in d=3 dimensions and so we can compute the critical exponents using the method in ref.(3, 4).

In the first section of this work the stochstic quantization is introduced and the method of computing the critical exponents is described; in the second section the first order computations are referred (5); in the third section there are the second order computations; in the last section the link with the ε-expansion is shown and there are the conclusions.

More on stochastic quantization can be found in ref.(6).

SECTION 1

1. STOCHASTIC QUANTIZATION

To be general enough, let us consider a scalar field in a d dimensional euclidean space plus a fictitious time dimension: $\phi = \phi(x,t)$, with a $\lambda\phi^4$ interaction. The classical action is:

$$S = \int d^d x \, \frac{1}{2}\phi(x,t)\left[-\Box + m^2\right]\phi + \frac{\lambda}{4!}\phi^4 \qquad (1)$$

The Langevin equation

$$\frac{\partial}{\partial t}\phi(x,t) = -\frac{\delta S}{\delta \phi(x,t)} + \xi(x,t) \qquad (2)$$

for the classical field becomes

$$\frac{\partial}{\partial t}\phi(x,t) = -\left[-\Box + m^2\right]\phi - \frac{\lambda}{3!}\phi^3 + \xi(x,t) \qquad (3)$$

$$\phi(x,0) = 0$$

$\xi(x,t)$ is a Gaussian random source; then it satisfies

$$\langle \xi(x,t)\,\xi(x',t')\rangle = 2\,\delta(x-x')\,\delta(t-t') \qquad (4)$$

$$\langle \xi(x_1,t_1)\ldots\xi(x_{2N},t_{2N})\rangle = \sum_{\substack{\text{all} \\ \text{combinations}}} \prod_{\text{pairs}} \langle \xi(x_i,t_i)\,\xi(x_j,t_j)\rangle \qquad (4')$$

Let us denote the solution of the Langevin equation by $\phi_\xi(x,t)$. We are interested in the t infinit limit of the average over of the product of $\phi(x_i,t)$:

$$\langle \phi_\xi(x_1,t)\ldots\phi_\xi(x_n,t)\rangle = \int \mathcal{D}\phi\; \phi(x_1)\cdots\phi(x_n)\, W(\phi,t) \qquad (5)$$

where $W(\phi, t)$ is a probability distribution satisfying the Fokker-Planck equation :

$$\frac{\partial W(\phi,t)}{\partial t} = \frac{\delta^2}{\delta\phi^2} W(\phi,t) + \frac{\delta}{\delta\phi} \frac{\delta S}{\delta\phi} W(\phi,t) \qquad (6)$$

The t infinit limit of W (the equilibrium distribution) is the usual weight $\exp(-S(\phi))$ of the Feynman path integral formulation (7). So the quantum Green functions are given by:

$$G^{(n)}[\phi(x_1)\ldots\phi(x_n)] = \lim_{t\to\infty} \langle \phi(x_1,t)\ldots\phi(x_n,t)\rangle \qquad (7)$$

Moreover it can be proved that the equilibrium is not affected by the choice of the initial condition (7).

The solution of the Langevin equation can be given in terms of an integral equation:

$$\phi(x,t) = \int G(x-x', t-t')\left[\xi(x',t') - \frac{\lambda}{3!}\phi^3(x',t')\right] dx'dt' \qquad (8)$$

where G(x-x';t-t') is the stochastic propagator, satisfying:

$$(\partial_t - \Box_x + m^2)\, G(x-x'; t-t') = \delta(x-x')\,\delta(t-t') \qquad (9)$$

In p-space: $\tilde{G}(q,t)=\theta(t)\exp(-(q^2+m^2)t)$ (10)

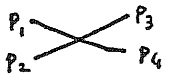
The form of eq.(8) is suitable for a perturbative expansion:

$$\phi = G*\xi - \frac{\lambda}{3!} G*(G*\xi)^3 + 3\left(-\frac{\lambda}{3!}\right)^2 G*(G*\xi)^2(G*\xi)^3 +$$
$$+3^2\left(-\frac{\lambda}{3!}\right)^3 G*(G*\xi)^2(G*\xi)^2(G*\xi)^3 + 3\left(-\frac{\lambda}{3!}\right)^3 G*(G*\xi)\left[(G*\xi)^3\right]^2 + o(\lambda^4) \qquad (11)$$

The perturbative expansion has a diagrammatic interpretation in terms of the Feynman rules:

a) Every propagator $G(q,T-t)$: a line T ---$\overset{q}{\rightarrow}$------ t

b) End lines $G^* \xi$: a line T ----------x

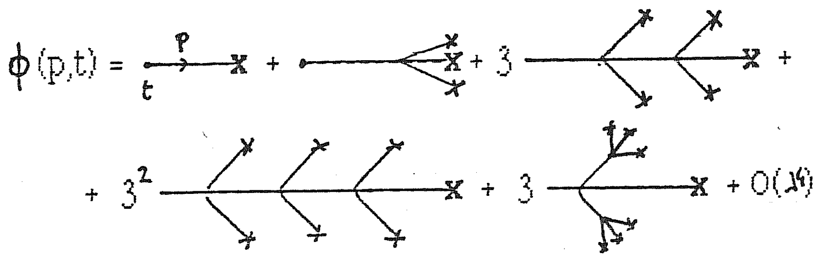
c) Every vertex  $= (2\pi)^d \delta(\Sigma_i^4 p_i)(-\frac{\lambda}{3!})$

d) Integration over momenta and times for every vertex and cross

e) Contracted random sources: if $\langle \xi(q,T)\, \xi(p,t) \rangle = \delta(p+q) g(T,t) (2\pi)^d$

then T-------x-------t gives $2g(T,t)dT\, dt\, \delta(p+q)(2\pi)^d$

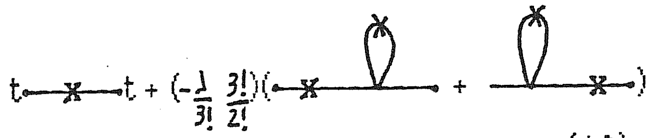
With these rules we can interpret eq.(11) as a sum of tree diagrams:



(12)

In computing the correlation functions of eq.(7) , the contractions between the ξ's produce the loops of the usual quantum theory and therefore the ultraviolet divergences The contraction of equivalent branches of the tree diagrams produce some combinatorial factors. To calculate them one must count the number of equivalent contractions in a given graph, starting from the tree diagrams, but we saw that the following rule works :

f) the total combinatorial factor in front of a graph is given by the number of independent choices for the external legs, times a 3! for every vertex,divided by the factorial of the number of the topologically equivalent lines inside the graph. As an example it is shown the first order correction to the ϕ - ϕ correlation :

$\langle \phi(p,t)\phi(q,t) \rangle =$ 

(13)

It is easy to check that in the t infinit limit this reproduces the usual first order correction to the mass in the $\lambda\phi^4$theory.

2. STOCHASTIC ANALYTIC REGULARIZATION (8).

The Markovian character of the stochastic evolution is given by the locality of the noise -noise correlation, eq.(4). An alternative way to regularize the theory, instead of using dimensional or cut-off regularizations, comes from the possibility of substituting the Markovian process with a non-Markovian one. A nice feature of this regularization is that all the symmetries of the Langevin equation have been preserved. It is sufficient to substitute the δ - function of eq.(4) with a more spreaded function; the width of this distribution is essentially the regulator if in the limit for it going to zero the δ-function is reproduced:

$$\langle \xi(x,T)\, \xi(y,t) \rangle = 2\, \delta(x-y)\, g_\sigma(T-t) \qquad \lim_{\sigma \to 0} g_\sigma(T-t) = \delta(T-t) \quad (14)$$

$$\langle \xi(p,T)\, \xi(q,t) \rangle = 2\, \delta(p+q)\, g_\sigma(T-t) (2\pi)^d \quad (14')$$

According to ref.(6) , we will use for $g_\sigma(T-t)$:

$$g_\sigma(T-t) = \frac{\sigma}{2} |T-t|^{\sigma-1} \quad (15)$$

The advantage in doing so is that , with this regulator, the divergences will appear as poles in σ , and it will be natural to use a minimal subtraction scheme for the renormalization of the theory. In fact, the function of eq. (15) has a zero of arbitrary order at $T \to t$ for sufficiently large σ , and therefore it regularizes any divergence; the physical limit occurs only at $\sigma=0$ and so only logarithmic divergencies contribute to it. In this way we see that this regularization shows the same features of analytic or dimensional regularization, but it does not change the Langevin eq.. The regularization enters the theory only at the level of the loops in the Green functions computations.

3. RENORMALIZATION.

Dimensional analysis: from eq.(14) we find that $\dim \xi = d/2+1-\sigma$. From the Langevin eq. we find $\dim \phi = d/2-1-\sigma$

$$\dim \lambda = 2\sigma+4-d = 2\sigma+\varepsilon \qquad \varepsilon=4-d \tag{16}$$

Eq.(16) suggests that the theory is renormalizable when the condition $2\sigma+\varepsilon=0$ is satisfied; in d=3 this gives $\sigma=-1/2$. This will be confirmed by a power counting analysis.

Power counting: from the perturbative expansion of eq. (12) we can see that every tree graph has one uncrossed external line E_o ; $2V+1=E_c$ crossed external lines E_c , where V is the number of vertices of the graph ; I=V-1 internal lines I . Since Green functions are given by the contraction of tree diagrams , in general a graph will have E_o external uncrossed legs, E_c crossed legs , L loops , M internal crossed lines , I internal uncrossed lines and V vertices , with the relations:

$$2M+E_c = 2V+E_o \qquad V= L-1+(E_o+E_c)/2 \tag{17}$$

Since the integration over the momenta is Gaussian, it can be done at once for every graph; then one must perform the integration over the intermediate times. It easy to see that the divergence cames when all the intermediate times go to the largest time of the graph (because in this limit there is no more the exponential damping in the momenta integration). In general we can parametrize the difference of the times t_i from the largest time T in some way, for example in polar coordinates :

$$T-t_i = r \cdot \text{times(angular variables)} \qquad (18)$$

The overall divergence now appears as a divergence for $r \to 0$. The variable r has a power $-d/2$ for every loop L; a power $(\sigma-1)$ for every crossed internal line M; a power N-1 from the differential of the N internal times:

$$r^{-\frac{d}{2}L + M(\sigma-1)} r^{N-1} dr = r^{-\frac{D}{2}-1} dr \qquad (19)$$

Since N= 2M +V -1 , combining eq.(19) with eq.(17), we get

$$D/2 = (d/2-2-\sigma)L + 3 + \sigma -(3/2 + \sigma) E_o - E_c/2 \qquad (20)$$

From eq.(20), we see that the theory is renormalizable if the degree of divergence of a given Green function (E_o and E_c given) is independent of the order of the perturbation expansion L, and therefore when

$$\sigma = \sigma^* = d/2-2 \qquad \text{or} \qquad 2\sigma^* + \varepsilon = 0 \qquad (21)$$

Eq. 21 gives us the way out for avoiding the usual ε - expansion. In fact , if one doesn' t use the stochastic regularization, eq. 21 is simply $\varepsilon=0$, so the only possibility of studying this model would be by means of the ε - expansion around d=4. Thanks to the stochastic regularization, we see that it is possible to study the model directly at d=3, since for $\sigma=-1/2$ the theory is still renormalizable. Moreover if we interpret σ and ε as 2 independent regulators of the theory, we see that there exsists a straight line in the 2-dimensional space of σ and ε , given by eq.(21), on which the theory is renormalizable. It is possible to choose any point on this line to perform the computations; ($\sigma=0$; $\varepsilon=0$) gives the ε-expansion; ($\sigma=-1/2$; $\varepsilon=1$) gives the purely stochastic case we will study; but perhaps some intermediate choice for σ and ε can be an improvement of the method.(Fig.1)

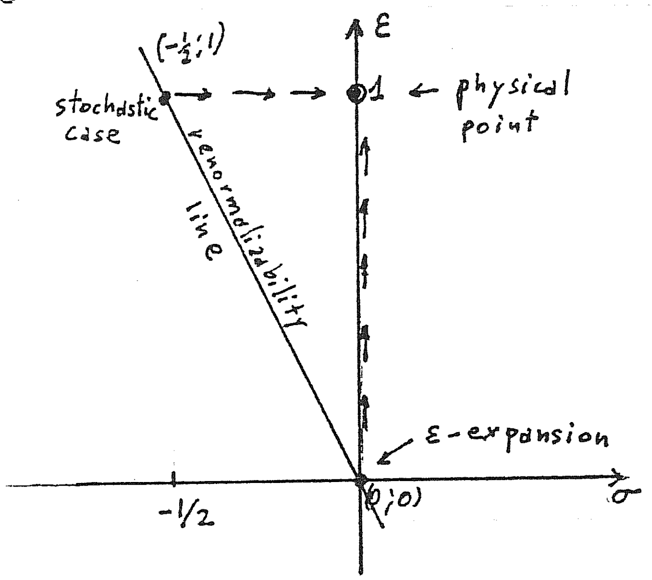


Fig. 1

Once eq.(21) has been satisfied, eq.(20) gives the superficial degree of divergence of the graphs with a fixed number of crossed and uncrossed external legs. The divergenc es appear for $D > 0$, and the logarithmic divergences appear as poles for $D=0$.

Stochastic case ($\sigma=-1/2$; $\varepsilon=1$):

$E_o=1$ $E_c=1$ $D=2$ gives a quadratic divergence (mass renormalization) and 2 logarithmic divergences as it will be seen later (wave function renormalizations).

$E_o=2$ $E_c=0$ $D=1$ gives no divergence , since $D=1$ cannot give any pole. This fact can be understood in terms of counterterms:to cancel a divergence of this kind , one should put a counterterm in eq.(14) , but this would be non-local in time, while the divergenc es appear as local terms in timeand space.

$E_o=1$ $E_c=3$ $D=0$ gives a logarithmic divergence (renormalization of the coupling constant).

ε -expansion case ($\sigma=0$; $\varepsilon=0$):

There are the same divergenc es for $E_o=1$ $E_c=1$, $E_o=1$ $E_c=3$, but now also the case $E_o=2$ $E_c=0$ gives a logarithmic divergence as now eq.(14) is local in time too (renormalization of the random source ξ). We have investigated also this case becouse we want to use the stochastic method also for $\sigma=0$ $\varepsilon=0$ in order to have a check of the whole computation: the results in this last case must be the same of that of the usual perturbative ε - expansion we can find on some reference(for instance on ref.(3)).

The divergenc es have to be reabsorbed by counterterms, so it is convenient write the Langevin equation in the renormalized form:

$$Z_t \partial_t \phi_R + Z_\phi(-\Box + m^2 + \delta m^2)\phi_R = -\frac{\lambda_R}{3!}\mu^{2\sigma+\epsilon} Z_V \phi_R^3 + Z_\xi \xi \quad (22)$$

where μ is the momentum scale at which renormalization is done. The wave function renormalization constant Z can be identified after the rescaling:

$$t = \alpha \bar{t} \qquad \xi = \alpha^{\frac{\sigma-1}{2}} \bar{\xi} \qquad \alpha = Z_t / Z_\phi \quad (23)$$

which leaves eq.(14) unchanged. Eq.(22) becames

$$Z_t^{\frac{1-\sigma}{2}} Z_\phi^{\frac{\sigma+1}{2}} Z_\xi^{-1}(\partial_t - \Box + m^2 + \delta m^2)\phi_R = -\frac{\lambda}{3!}\mu^{2\sigma+\epsilon} Z_V Z_t^{\frac{1-\sigma}{2}} Z_\phi^{\frac{\sigma-1}{2}} \phi_R^3 + \xi \quad (24)$$

$$Z^{\frac{1}{2}} = Z_t^{\frac{1-\sigma}{2}} Z_\phi^{\frac{\sigma+1}{2}} Z_\xi^{-1} \qquad \phi = Z^{\frac{1}{2}}\phi_R \quad 25)$$

As usual, to renormalize the theory , we will look at the 1 Particle Irreducible graphs; from a graph with one uncrossed external leg and n corossed external legs we can get the 1PI graph $\Gamma^{(n+1)}(x,T;y_i,t_i)$; since the t_i range in different intervals due to the $\theta(t_i-t_j)$ functions in the stochastic propagators, it is more convenient to take the Laplace transform of the 1PI: (9)

$$\Gamma^{(n+1)}(x,T;y_i,s_i) = \int \prod dt_i \, \Gamma^{(n+1)}(x,T;y_i,t_i)\exp(-\sum_i s_i t_i) \quad (26)$$

The logarithmic divergences can be computed at $s_i=0$, while the quadratic divergences will give a pole in the derivative with respect to the s_i (and also a contribution in the derivative with respect to the external momenta p_i). The procedure will be: compute only the logarithmic divergent part of the 1PI graphs and reabsorb it into the appropriate counterterms(Minimal Subtraction Scheme).

In the case $\sigma=-1/2$ $\varepsilon=1$, as we saw, there is no divergence associated to the noise-noise correlation function, and so $Z_\xi=1$; moreover, since the renormalization of the mass is given by divergences appearing as poles times m^2, in the massless case we are interested in, there is no mass renormalization:

$$Z_\xi=1 \quad ; \text{ for } m^2=0 \quad \delta m^2=0 \qquad (27)$$

Z_t and Z_ϕ will begin at order $O(\lambda^2)$ since there are no other graphs but those shown in eq.(13) at order $O(\lambda)$, and they give no contribution .

4. THE MASS INSERTION.

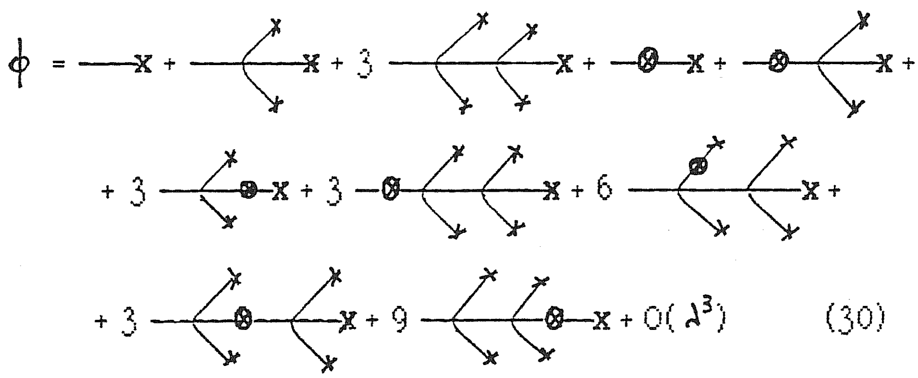
One of the critical exponents we are going to compute is related to the anomalous dimension of the mass insertion operator : $A\,\phi^2(x,t)$. It is useful , at this point, to show how to renormalize it. We introduce the mass insertion as an external operator, defining a total action:

$$S_{Tot} = S_0 + \int A\,\phi^2(x,t)\,dx \qquad (28)$$

The renormalization of the operator can be performed by studying some Green function of the external operator; we choose

$$\Gamma^{(A,2)}(x,t;y,t) = \frac{\delta}{\delta A} \langle \phi(x,t)\,\phi(y,t)\rangle_{S_{tot}} \Big|_{A=0} \qquad (29)$$

In the stochastic formalism this can be done considering the extra term $2A\,\phi(x,t)$ in the Langevin equation as a new vertex : T----⊗----t = G(p;T-t) 2A . Because of the form of the Green function we are interested in, only the linear terms in A will be involved. The perturbative expansion of ϕ is now:



(30)

Eq.(30) gives the ϕ field to be used to compute the Green function of eq.(29). As usual loops will appear and the divergenc es must be reabsorbed in the mass inserction counterterm: the renormalized Langevin equation now reads:

$$(Z_t \partial_t - Z_\phi \Box)\phi_R = -\frac{\lambda_R}{3!} Z_v \mu^{2\sigma+\varepsilon} \phi_R^3 + Z_A\, 2\, A_R \phi_R + \xi \qquad (31)$$

The Renormalization Group equation for the $\Gamma^{(A,2)}$, involve a constant defined by:

$$A_B\, \phi_B^2 = Z_{\phi^2} A_R\, \phi_R^2 \qquad (32)$$

Z_{ϕ^2} is related to the stochastic constant Z_A we are computing:

$$Z_{\phi^2} = Z_A Z_t^{1-\sigma} Z_\phi^{\sigma} \qquad (33)$$

Z_{ϕ^2} gives the anomalous dimension of the $\Gamma^{(A,2)}$:

$$\gamma_{\phi^2} = \mu \frac{\partial}{\partial \mu} \ln Z_{\phi^2} \Big|_{Bare.} \qquad (34)$$

5. THE CRITICAL EXPONENTS .

Our goal is the computation of the critical exponents γ and η, in order to compare them with the values taken from the ε - expansion and from the high temperature expansion, that we can find on some reference(see(3)pag.236). In statistical phisycs, γ is the exponent describing the scaling properties of the susceptibility in the critical region ; η is related to the scaling properties of the correlation function : (3)(4)

$$\chi \sim |T - T_c|^{-\gamma} \qquad G(r) \sim r^{-d+2-\eta} \tag{35}$$

In the field theoretical approach to statistical physics, one can see that η is essentially the anomalous dimension of the 2-points Green function ; γ is related to the anomalous dimension of the mass insertion operator γ_{ϕ^2}. From ref.(3), pag. 235, we got

$$\gamma = \left(1 + \frac{\gamma_{\phi^2}}{2-\eta} \right)^{-1} \tag{36}$$

where γ_{ϕ^2} has been defined in eq.(34). η is defined as

$$\eta = \gamma_\phi = \mu \frac{\partial}{\partial \mu} \ln Z \Big|_B \tag{37}$$

Since we identify the critical exponents from the scaling properties of the Green functions , and the Green functions begin to scale near the fixed points of the β - function, we must , first of all, study the β - function. We must look for the zeros of the β - function, verify that there exsists an InfraRed stable fixed point, and then compute the anomalous dimensions at the value of this fixed point; extrapolating to the physical value $\sigma=0$, we will get the critical exponents.

A technical remark: in computing the Renormalization Group Structure functions β , γ_{ϕ} , γ_{ϕ^2}, we must ,first of all , verify that they are finite quantities . This is a non trivial step of the computations, and also a check for them, since the renormalization constants , in general, have not only the simple poles, but also poles of arbitrary order. To get finite quantities there must be an exact cancellation of these not simple poles.

SECTION 2

FIRST ORDER COMPUTATIONS.

1. THE β-FUNCTION.

The renormalization of the coupling constant , in general, involves the computation of the wave function and of the vertex renormalization constants:

$$\lambda_B = \mu^{\omega} Z_V \, Z_t^{\sigma-1} \, Z_\phi^{-(\sigma+2)} \, \lambda_R \qquad (38)$$

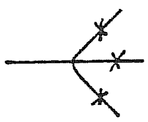
where

$$\omega = 2\,(\sigma - \sigma^*) \qquad (39)$$

Since we saw that the first contributions to Z_t and Z_ϕ are of order λ^2, at first order it is sufficient to compute only Z_V:

$$Z_V = 1 + \lambda\, \delta_1 Z_V + O(\lambda^2) \qquad (40)$$

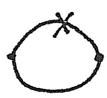
There is only one contribution to the 4-points function :

$\Gamma^{(4)}$ =  $(-\lambda Z_V) + (-\lambda/3!)^2 (3!)^2 3$  $+O(\lambda^3)$

(41)

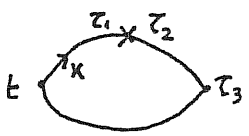
The result is finite if:

$$-\lambda^2 \delta_1 Z_V + 3\lambda^2 \frac{R}{\omega} = 0$$

with $\frac{R}{\omega}$ = divergent part of the graph  (42)

We get $\delta_1 Z_v = 3\frac{R}{\omega}$ $\quad Z_v = 1 + \lambda 3 \frac{R}{\omega} + O(\lambda^2)$ (43)

R is the residual of the single pole ; since the divergence is logarithmic the divergent part of the graph can be obtained at zero external momenta. Using the Feynman rules we have:



$$= \int \frac{d^dK}{(2\pi)^d} e^{-K^2(t-\tau_3)} \theta(t-\tau_3)\, e^{-K^2(t-\tau_1)} \theta(t-\tau_1)$$

$$\cdot\, e^{-K^2(\tau_3-\tau_2)} \theta(\tau_3-\tau_2)\, \sigma |\tau_1-\tau_2|^{\sigma-1} d\tau_1 d\tau_2 d\tau_3 \qquad (44)$$

The integration over K is Gaussian and can be performed at once

$$\int \frac{d^dK}{(2\pi)^d} e^{-AK^2} = \frac{\pi^{\frac{d}{2}}}{(2\pi)^d} A^{-\frac{d}{2}} \qquad A = 2t-\tau_1-\tau_2 \qquad (45)$$

The integral over the times can be breaken into three integrals, according to the time orderings:

a) $t \geq \tau_3 \geq \tau_2 \geq \tau_1$ b) $t \geq \tau_3 \geq \tau_1 \geq \tau_2$ c) $t \geq \tau_1 \geq \tau_3 \geq \tau_2$

The first two are equivalent , so we must do two integrals

$$I_a = \sigma \int_0^t d\tau_3 \int_0^{\tau_3} d\tau_2 \int_0^{\tau_2} d\tau_1 \frac{\pi^{\frac{d}{2}}}{(2\pi)^d} (2t-\tau_2-\tau_1)^{-\frac{d}{2}} (\tau_2-\tau_1)^{\sigma-1}$$

$$I_c = \sigma \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_3 \int_0^{\tau_3} d\tau_2 \frac{\pi^{\frac{d}{2}}}{(2\pi)^d} (2t-\tau_2-\tau_1)^{-\frac{d}{2}} (\tau_1-\tau_2)^{\sigma-1} \qquad (46)$$

Let us consider the first one. It is convenient to do the following change of variables: when there are n cronologically ordered times

$$T \geq t_n \geq t_{n-1} \geq \dots \geq t_2 \geq t_1 \qquad \text{we can put}$$

$$t_1 = T(1-\alpha_1) \qquad t_2 = T(1-\alpha_1\alpha_2)$$

$$t_n = T(1-\alpha_1\alpha_2 \cdot \alpha_n) \qquad \text{and the integral becomes}$$

$$\int_0^T dt_n \int_0^{t_n} dt_{n-1} \dots \int_0^{t_3} dt_2 \int_0^{t_2} dt_1 = \int_0^1 \prod_{i=1}^{n} d\alpha_i \, T^n \alpha_1^{n-1} \alpha_2^{n-2} \dots \alpha_{n-1} \tag{47}$$

The integral I_a now reads:

$$I_a = \sigma \frac{\pi^{d/2}}{(2\pi)^d} \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \, t^3 \alpha_1^2 \alpha_2 \, t^{-\frac{d}{2}} (1+\alpha_2)^{-\frac{d}{2}} \alpha_1^{-\frac{d}{2}+\sigma-1} \left[(1-\alpha_2)t\right]^{\sigma-1} \tag{48}$$

The integration over α_3 is trivial; the integration over α_1 produces a factor $(\sigma + 2 - d/2)^{-1}$ and this is the pole at d=3 for $\sigma = -1/2$; the integration over α_2 gives the residual of the pole and could be done taking d=3 and $\sigma = -1/2$, but one must be careful since at $\alpha_2 \rightarrow 1$ the integrand behaves as $(1-\alpha_2)^{-3/2}$ and this is not integrable . To give a sense to the integral an analytic continuation in σ is needed:

$$I_a = \left(\frac{1}{2\sqrt{\pi}}\right)^d \frac{\sigma}{\sigma+2-\frac{d}{2}} \int_0^1 d\alpha_2 \, \alpha_2 (1+\alpha_2)^{-\frac{d}{2}} (1-\alpha_2)^{\sigma-1} = \tag{49}$$

$$\underset{\substack{d\to 3\\ \sigma\to -\frac{1}{2}}}{=\!\!\longrightarrow} \left(\frac{1}{2\sqrt{\pi}}\right)^3 \frac{1}{\sigma+\frac{1}{2}} \left(-\frac{1}{2}\right)(-1)$$

The second integral can be done in the same way and we get:

$$I_c = \left(\frac{1}{2\sqrt{\pi}}\right)^3 \frac{1}{\sigma+\frac{1}{2}} \left(-\frac{1}{2}\right)(+1) \qquad (49')$$

Finally we get:

Div.Part of  $= R/\omega = I_a + I_b + I_c = \left(2\sqrt{\pi}\right)^{-3} \frac{1}{\sigma+\frac{1}{2}} \left(-\frac{1}{2}\right)(-1-1+1) =$

$$= \left(\frac{1}{2\sqrt{\pi}}\right)^3 \frac{1}{\omega} \qquad R = \left(\frac{1}{2\sqrt{\pi}}\right)^3 \qquad (50).$$

A comment on the result for the Z_v : the factor R comes from the integration over the momenta; we aspect such a factor to appear elevated to a power equal to the loops number and so it could be reabsorbed redefining the coupling constant

$$k = \lambda R \qquad (51)$$

The factor 3 in $\delta_1 Z_v$ has a purely combinatorial origin: it is the number of the channels contributing to the $\Gamma^{(4)}$ at this order. Therefore we don't expect relevant differences between the stochastic case and the ε - expansion at this order. In fact, let us compute the β - function:

$$\beta(\lambda) = \mu \frac{\partial \lambda_R}{\partial \mu}\bigg|_B = \mu \frac{\partial}{\partial \mu} \mu^{-\omega} Z_v^{-1} \lambda_B \bigg|_B = \mu \frac{\partial}{\partial \mu}\left[\mu^{-\omega} - \lambda \mu^{-\omega} \frac{3R}{\omega}\right]\lambda_B = \qquad (52)$$

$$= -\omega\lambda + 3R\lambda^2 + O(\lambda^3)$$

The β - function has a zero at

$$\lambda^* = \frac{\omega}{3R} \qquad (53)$$

It is important , at this point , to notice that since the slope of the β - function in the fixed point is positive, the fixed point is an InfraRed stable point and, since λ^* is O (ω) a perturbative approach to the infrared region is justified.

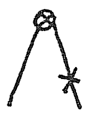
2. THE MASS INSERTION .

First of all, we want to compute γ . As we saw, since there is no contribution of order λ to the wave function renormalization, η =0 and the equations (33) and (36) semplify in

$$Z_{\phi^2} = Z_A \qquad \gamma = 1 - \gamma_{\phi^2}/2 \qquad (54)$$

From the expansion of eq(30) of ϕ , we see that the Green function for the mass insertion is:

$$\Gamma^{(A;2)} = \text{[diagram]}\, Z_A + \left(-\frac{\lambda}{3!}\right) 3! \,\text{[diagram]} + O(\lambda^2) \qquad (55)$$



Putting $Z_A = 1 + \lambda \delta_1 Z_A$ we get

$$\lambda \delta_1 Z_A - \lambda \,\text{Div. Part of}\, \text{[diagram]} = 0 \quad \text{and then} \quad \delta_1 Z_A = R/\omega \qquad (56)$$

From eq.(34) we get

$$\gamma_{\phi^2} = \mu \frac{\partial}{\partial \mu} \lambda \delta_1 Z_A = \frac{R}{\omega}\left(-\omega\lambda + O(\lambda^2)\right) = -R\lambda + O(\lambda^2) \qquad (57)$$

At the fixed point of eq.(53) and extrapolating to the physical value $\sigma = 0$ ($\omega = 1$) we obtain the critical exponent γ:

$$\gamma = 1 - \frac{1}{2}\left(-R\lambda\right)\Big|_{\lambda^*} = 1 + \frac{1}{2} R \frac{\omega}{3R} \xrightarrow[\omega \to 1]{} 1 + \frac{1}{6} \qquad (58)$$

As expected we got the same result of the ε - expansion; this is an indication that the followed procedure is correct and so we can go to the more interesting case of the second order computations , where differences are expected to arise.

3. THE WAVE FUNCTION RENORMALIZATION.

Before going to the second order , let us sketch how the renormalization of the wave function can be done(4).

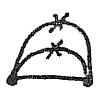
For the renormalized version of the Langevin equation (eq.(22)), we can define a renormalized stochastic propagator:

$$G_R(t) = \theta(t) \frac{1}{Z_t} e^{-p^2 \frac{Z_\phi}{Z_t} t} \qquad (59)$$

The Laplace transform of this , with respect to t, inverted is:

$$G^{-1}(s; p^2) = Z_t s + Z_\phi p^2 \qquad (60)$$

Taking the power of λ^2, the inverse of the propagator is

$$\Gamma^{(2)} = Z_t s + Z_\phi p^2 - \text{Div.Part of } (-\lambda/3!)^2 (3!)^2/2!$$  (61)

The graph we are looking at has two different divergences: one in the derivative with respect to s, giving the counterterm Z_t, and one in the derivative with respect to p^2, giving the counterterm Z_ϕ. The computation of the divergences can be done as shown in the previous paragraph, but now the integral giving the residual is not so simple, and has to be done numerically. To do the integral with the analytical continuation numerically is not straightforward, and some trick must be invented as we will show in the next section. The results are:

$$Z_t = 1 - (\lambda R)^2 \frac{1}{2} \frac{R_t}{\omega} \qquad Z_\phi = 1 - (\lambda R)^2 \frac{1}{2} \frac{R_\phi}{\omega}$$

with

$$R_t = 0.264 \pm 0.004 \qquad R_\phi = 0.196 \pm 0.003 \qquad (62)$$

From eq.(25) we get the wave function renormalization Z, and from eq.(37) the critical exponent η can be computed at the fixed point we have found. The result is:

$$\eta_\sigma = 0.055 \pm 0.001 \qquad (63)$$

to be compared with

$\eta_\varepsilon = 0.019$ of the ε - expansion and

$\eta_T = 0.04 \pm 0.01$ of the high temperature expansion. (10)

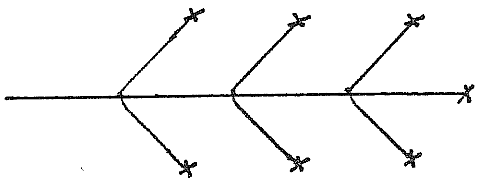
From these data it is clear that it will be very interesting to see which will be the second order corrections for both γ and η.

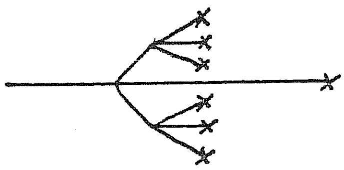
SECTION 3

SECOND ORDER COMPUTATIONS.

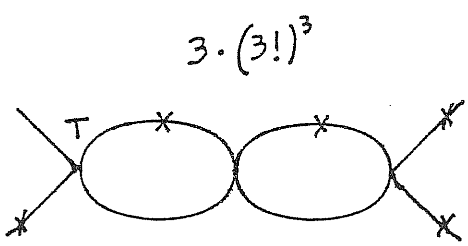
1. THE GRAPHS FOR THE β-FUNCTION.

The graphs contributing to the β- function can be constructed starting from the expansion in tree diagrams of the field ϕ, given by eq.(12). Since we are looking at the 1PI Green function $\Gamma^{(4)}$, we neglect all the 1P Reducible and all the non local in time graphs. Therefore the graphs we are looking for can be constructed starting from the tree diagrams:

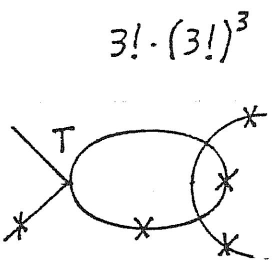
a)  (64a)

b)  (64b)

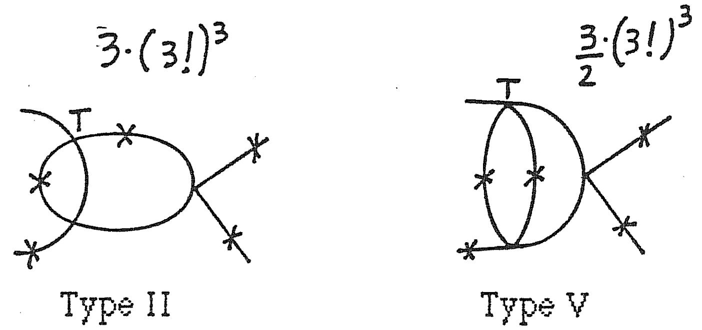
Contracting pairs of crosses we get the following graphs: from the tree diagram a)



Type 0

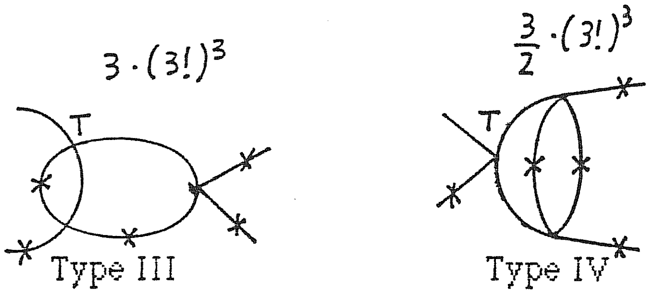


Type I



(65a)

and from the tree diagram b):



(65b)

where the combinatorial factors have been explicitly shown. The IV and V types have a subloop with a cross on each line of the loop; we know that this kind of subloop is not divergent and so these graphs have a divergence coming from the larger loop : a simple pole divergence. On the other hand , the other graphs have divergent subloops and so we aspect them to show a double pole plus a simple pole structure. For example the type 0 is essentially contributing just with a double pole: its diverging part is the square of that of the simple loop we studied in section 2, as can be easily seen:

Divergent part of type 0 = $(R/\omega)^2$ (66)

This gives rise to the problem of the cancellation of the double poles: in fact in the β- function the double pole can be , at best, multiplied by ω and so there must be a cancellation , in order to get a finite β- function.

2. THE DOUBLE POLES CANCELLATION.

Let us compute the β - function at order λ^3. We write:

$$\lambda_B = \mu^{\omega} Z_V Z_S \lambda_R \qquad Z_S = Z_t^{\sigma-1} \; Z_\phi^{-(\sigma+2)} \tag{67}$$

and

$Z_V = 1 + \lambda\, 3R/\omega + \lambda^2 \delta_2 Z_V \quad Z_S = 1 + \lambda^2 \delta_2 Z_S$ with

$$\delta_2 Z_S = (\sigma - 1)\, \delta_2 Z_t - (\sigma + 2)\, \delta_2 Z_\phi \tag{68}$$

where $\delta_2 Z_t$ and $\delta_2 Z_\phi$ are given by eq.(62).

$$\beta = \mu \frac{\partial}{\partial\mu} \left(\mu^{-\omega} Z_V^{-1} \; Z_S^{-1} \lambda_B \right)\Big|_B =$$

$$= - \omega\lambda + 3R\lambda^2 + 2\lambda^3\omega\, \delta_2 Z_S - 2\lambda^3 \omega\, 9R^2/\omega^2 + 2\omega\lambda^3 \delta_2 Z_V + O(\lambda^4) \tag{69}$$

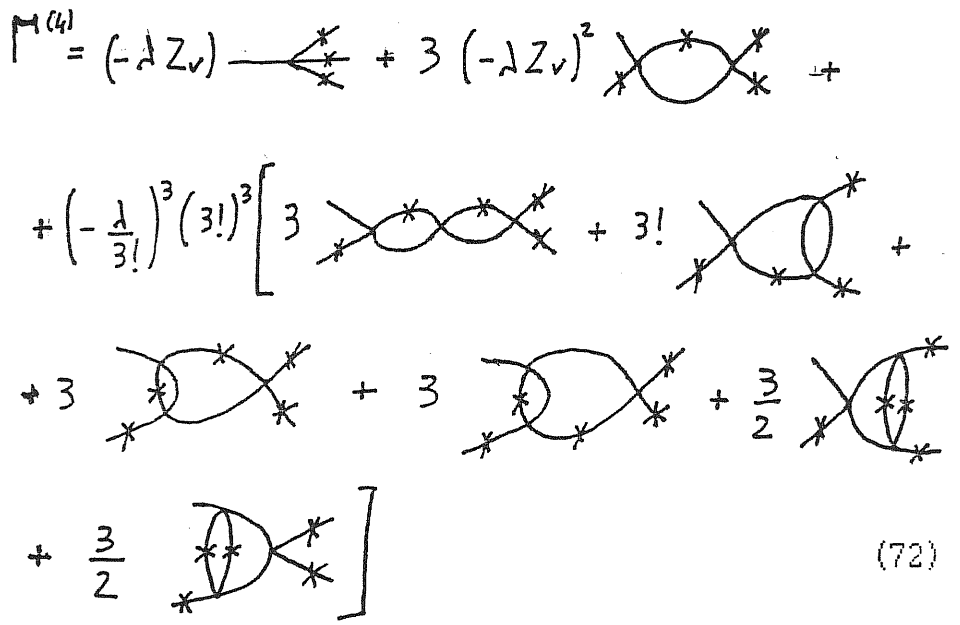
Since $\delta_2 Z_S$ has only single poles, the divergent term must be cancelled by $\delta_2 Z_V$:

$$2\lambda^3 9 \frac{R^2}{\omega} = 2\omega\lambda^3\, \delta_2 Z_V \tag{70}$$

The conclusion is that, in order to get a finite β - function, $\delta_2 Z_v$ must exibit a double pole part with the form:

$$\delta_2 Z_v = 9 \frac{R^2}{\omega^2} + \text{single pole part} \qquad (71)$$

This is not only important from a theoretical point of view, but it provides also a check of the combinatorial factors we have computed. It will be shown later that the double pole of the type I, II, III appears with a coefficient 1/2 in front of it (in general it appears with a factor one over the loop number); so let us compute the double pole contribution to $\delta_2 Z_v$:



(72)

Taking the λ^3 terms and only the double poles with their coefficients we get the following relation for the double pole part of $\delta_2 Z_v$:

$$-\lambda \lambda^2 \delta_2 Z_v + 3\frac{R}{\omega}\lambda^2 2\lambda \quad 3\frac{R}{\omega} \; -\lambda^3 3 \frac{R^2}{\omega^2}\left(1+\frac{2}{2}+\frac{1}{2}+\frac{1}{2}\right) = 0 \quad (73)$$

and then $$\delta_2 Z_v \Big|_{Double\ pole} = 9\frac{R^2}{\omega^2} \quad (74)$$

The cancellation has been proved and we will not worry any more of the double poles in Z_v. In the same way we can prove the cancellation of the double poles in the mass insertion and in the wave function renormalization. For the mass insertion, expanding the logarithm of eq.(34) in powers of λ and decomposing Z_{ϕ^2} in the standard way, we get:

$$\gamma_{\phi^2} = \mu\frac{\partial\lambda}{\partial\mu}\frac{\partial}{\partial\lambda}\ln Z_{\phi^2} = \beta(\lambda)\left[\frac{R}{\omega} + 2\lambda\,\delta_2 Z_{\phi^2} - \lambda\left(\frac{R}{\omega}\right)^2 + 0(\lambda^2)\right] =$$

$$= -\lambda R - \omega 2\lambda^2 \delta_2 Z_{\phi^2} + \omega\lambda^2\left(\frac{R}{\omega}\right)^2 + 3\frac{R^2}{\omega}\lambda^2 + O(\lambda^3)$$

therefore $$\delta_2 Z_{\phi^2}\Big|_{Double\ pole} = 2\left(\frac{R}{\omega}\right)^2 \quad (75)$$

For the wave function, writing

$$Z = 1 + \lambda^2\frac{r}{\omega} + \lambda^3\delta_3 Z \quad (76)$$

and expanding again the logarithm of eq.(37), wet get:

$$\eta = \beta(\lambda)\left[2\lambda \frac{\kappa}{\omega} + 3\lambda^2 \delta_3 Z + o(\lambda^3)\right] =$$

$$= -2\omega \lambda^2 \frac{\kappa}{\omega} + 6R\lambda^3 \frac{\kappa}{\omega} - 3\omega \lambda^3 \delta_3 Z + o(\lambda^4)$$

therefore $$\delta_3 Z\Big|_{\text{Double pole}} = 2\frac{R\kappa}{\omega^2} \qquad (77)$$

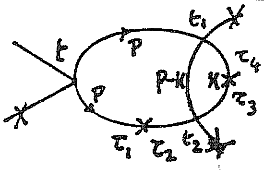
The form of the double pole parts of Z_{ϕ^2} and Z will be proved in the corresponding paragraphs.

3. THE CALCULATION OF THE GRAPHS.

Once we know how to handle the double poles, we are ready for going on with the calculation of the graphs. The program is: separate the double pole part and verify their form; then we can forget of them and look at the single pole part; we must extract explicitly the divergence and then we are left with a multiple integral, giving the residual we have to compute. This integral will be done numerically, with some trick in order to make the analytical continuation we discussed before. Two different methods will be used for the graphs of type I, II, III, and graphs of type IV, V, since they are two different kinds of graphs. Since the technique is the same for any member of the two groups, we will show only an example for case. Let us begin with the type I; using the Feynman rules we can write it and make the integration over the momenta at once:

$$\int e^{-Ap^2 - BK^2 + 2p\cdot K C} \frac{d^d p\, d^d K}{(2\pi)^d (2\pi)^d} = \frac{\pi^d}{(2\pi)^{2d}} \left(AB - C^2\right)^{-\frac{d}{2}} \tag{78}$$

The integral now reads:



$$= \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} d\tau_3 \int_0^{t_1} d\tau_4 \int_0^{t} d\tau_1 \int_0^{t} d\tau_2 \, \theta(t_2 - \tau_2) .$$

$$\cdot \frac{\pi^d}{(2\pi)^{2d}} \sigma^2 |\tau_1 - \tau_2|^{\sigma-1} |\tau_3 - \tau_4|^{\sigma-1} \left(AB - C^2\right)^{-\frac{d}{2}}$$

with $A = 2t - \tau_1 - \tau_2$ $B = 2t_1 - \tau_3 - \tau_4$ $C = t_1 - t_2$ (79)

Now we break the time intervals in a Subloop integration for the times t_2, τ_3, τ_4, with t_1 external time; Next integration for the times t_1, τ_1, τ_2, with t external time; then we break the S and N integrations in all the possible time orderings (nine cases, but really less due to symmetry in exchanging times). Let us consider one of this cases: $t \geq t_1 \geq \tau_2 \geq \tau_1$ $t_1 \geq t_2 \geq \tau_3 \geq \tau_4$. We can do two changes of variables of the kind of eq.(47) and we get:

$$I = \sigma^2 R^2 \int \prod_{i=1}^{3} d\alpha_i \, d\beta_i \; t^{2\sigma+4-d} \alpha_1^{\sigma+1-\frac{d}{2}} \beta_1^{\sigma+1} \alpha_2 \beta_2 (1-\beta_2)^{\sigma-1} (1-\alpha_2)^{\sigma-1}$$

$$b^{\sigma+2-\frac{d}{2}} \left\{\beta_1 c - \alpha_1 a^2 b\right\}^{-\frac{d}{2}} \theta(\beta_1 Q - \alpha_1 a b).$$

$$c = (1+\alpha_2)(1+\beta_2) \quad a = \alpha_2 \alpha_3 \quad Q = \beta_2 (1-\beta_3)$$

$$b = 1 - \beta_1 \beta_2 \beta_3 \tag{80}$$

This is the form of all these integrals: the first line in eq.(80) is common, the second one is different case by case, in the sense that the various Q, C , a, b, are given by different combinations of the α_i's and β_i 's. It has to be noted that, but for b, the variables defined in eq .(80) depend on $\alpha_2 \alpha_3 \beta_2 \beta_3$, while the divergence comes from the α_1 and β_1 integrations. First of all we can add and subtract the integral with the second line evaluated at $\alpha_1 = 0$; let us call this integral $\bar{I}$:

$$\bar{I} = \sigma^2 R^2 \int_0^1 \prod_{i=1}^{3} d\alpha_i d\beta_i \; \alpha_1^{\sigma+1-\frac{d}{2}} \beta_1^{\sigma+1-\frac{d}{2}} C^{-\frac{d}{2}} \alpha_2 \beta_2 (1-\alpha_2)^{\sigma-1} \cdot$$

$$\cdot (1-\beta_2)^{\sigma-1} b^{\sigma+2-\frac{d}{2}} t^{\sigma+2-\frac{d}{2}} =$$

$$= \frac{\sigma^2 R^2}{(\sigma+2-\frac{d}{2})^2} \int \prod_2^3 d\alpha_i d\beta_i \; \alpha_2 \beta_2 (1-\alpha_2)^{\sigma-1} (1-\beta_2)^{\sigma-1} C^{-\frac{d}{2}} \tag{81}$$

This gives a double pole; now we make the substitution of variables in the integrand of $I - \bar{I}$: $\alpha_1 = x\beta_1$

$$I - \bar{I} = \sigma^2 R^2 \int_0^1 \cdots \int_0^1 d\beta_1 \, \beta_1^{2\sigma+3-d} \int_0^{1/\beta_1} dx \; x^{\sigma+1-\frac{d}{2}} \cdot$$

$$\cdot \left\{ \frac{\theta(Q - x a b)}{[C - x a^2 b]^{d/2}} - \frac{1}{C^{d/2}} \right\} \tag{82}$$

Writing $1/C^{d/2} = (\Theta(1-x) + \Theta(x-1))/C^{d/2}$, we can make at once the integral for $x > 1$:

$$\int_0^1 d\beta_1\, \beta_1^{2\sigma+3-d} \int_1^{\frac{1}{\beta_1}} dx\, x^{\sigma+1-\frac{d}{2}} \left(-\frac{1}{C^{d/2}}\right) =$$

$$= -\frac{1}{2}\, C^{-\frac{d}{2}} \left(\frac{1}{\sigma+2-\frac{d}{2}}\right)^2 \tag{83}$$

This is again a double pole; adding this with the double pole of eq.(81), we get finally a double pole multiplied by 1/2, as we had to prove. What is left is the single pole; the integration in β_1 can be done, giving the divergence and the residual can be computed:

$$I - \bar{I}\Big|_{\substack{\text{single}\\\text{pole}}} = \frac{\sigma^2 R^2}{2\left(\sigma+2-\frac{d}{2}\right)} \int_0^1 \cdots \int_0^1 \frac{dz}{z} \left\{\frac{1}{(C - z a Q)^{\frac{d}{2}}} - \frac{1}{C^{\frac{d}{2}}}\right\} +$$

$$+ C^{-\frac{d}{2}} \ln\frac{Q}{a} \tag{84}$$

where we made some useful changes of variables (generating also the logarithm). The expression for the residual of the single pole is:

$$S = \sigma^2 R^2 \int_0^1 d\alpha_2\, d\alpha_3\, d\beta_2\, d\beta_3\, dz\; \alpha_2 \beta_2 (1-\alpha_2)^{\sigma-1} (1-\beta_2)^{\sigma-1} \cdot$$

$$\cdot \left\{ \frac{1}{z}\left[(C - z a Q)^{-\frac{d}{2}} - C^{-\frac{d}{2}}\right] + C^{-\frac{d}{2}} \ln\frac{Q}{a} \right\} \tag{85}$$

To do this integral is not straightforward, since analytical continuation in σ is needed. We used the generalization to more variables of the following trick: consider the integral

$$I(\sigma) = \int_0^1 d\alpha \; (1-\alpha)^{\sigma-1} F(\alpha)$$

If $\lim_{\alpha \to 1} F(\alpha) \neq 0$, the integral is not defined for $\sigma \to -1/2$.

Now add and subtract $F(1)$:

$$\int d\alpha \, (1-\alpha)^{\sigma-1} F(1) + \int d\alpha (1-\alpha)^{\sigma-1} \left[F(\alpha) - F(1) \right] =$$

$= \; 1/\sigma \; F(1) +$ an integral defined at $\sigma = -1/2$ since now $(1-\alpha)^{\sigma-1}(F(\alpha) - F(1)) \sim (1-\alpha)^{-\frac{1}{2}}$ and the integrand is integrable.

Therefore

$$\lim_{\sigma \to -\frac{1}{2}} \int d\alpha \, (1-\alpha)^{\sigma-1} F(\alpha) = -2\,F(1) + \int d\alpha \; (1-\alpha)^{\sigma-1} (F(\alpha) - F(1)) \qquad (86)$$

The programs performing multidimensional integrals have been tested using this procedure on well known integrals and the results have been quite good.

Because of the form of the graphs of type IV and V, this method was not suitable and we used another technique. As before we broke the time intervals in order to transform the absolute values of the crossed internal lines into simple differences; then we use the variable transformation of eq.(47); in this way the divergence arises when some of the new variables go to zero together. Since we are looking for the divergent part only, we can restrict ourselves to the region of integration where $\sum_i \alpha_i \leq 1$ of the hypercube of size 1 of the diverging variables α_i. This can be done multiplying the integral by

$$\int_0^1 d\lambda\, \delta(\lambda - \sum_i \alpha_i) = \theta(1 - \sum_i \alpha_i) \tag{87}$$

Rescaling then the diverging variables, the integration over λ gives the divergence; we can set $\lambda = 0$ in the residual integral and we can do it again using the analytical continuation technique.

The numerical integrals have been done using programs whose kernel was a CERN program performing multidimensional integrals with the Monte Carlo procedure. All these programs have been tested on test functions in order to optimize the ratio time and accuracy. Another important test of these programs has been given by the ε- expansion computations we will discuss in the next section. We have computed twenty-four five-dimensional integrals with an accuracy of about 10^{-3}. The results are:

$$\text{Type I} \qquad S_I = R^2 \frac{1}{4}(0.00 \pm 0.08)$$
$$\text{Type II} \qquad S_{II} = R^2 \frac{1}{4}(0.00 \pm 0.04)$$
$$\text{Type III} \qquad S_{III} = R^2 \frac{1}{4}(2.07 \pm 0.03)$$
$$\text{Type IV} \qquad S_{IV} = R^2 \frac{1}{4}(7.38 \pm 0.1)$$
$$\text{Type V} \qquad S_V = R^2 \frac{1}{4}(1.03 \pm 0.01) \qquad (88)$$

where S_i is the residual of ω^{-1}. Taking into account all the combinatorial factors the sum of the two loops contributions is:

$$-3! \frac{\lambda^3}{4} \frac{U}{\omega} = -\frac{3!}{4} \frac{\lambda^3}{\omega} \left(S_I + \frac{1}{2} S_{II} + \frac{1}{2} S_{III} + \frac{1}{4} S_{IV} + \frac{1}{4} S_V \right)$$

$$U = 3.2 \pm 0.1 \qquad (89)$$

Writing $\lambda R = k$, from eq.(69) we find that the β - function is:

$$\frac{\beta(K)}{K} = -\omega + 3K - aK^2 + O(K^3)$$

where $$a = -\frac{3}{2}\left[R_t + R_\phi - 2U \right] = 9.0 \pm 0.4 \qquad (90)$$

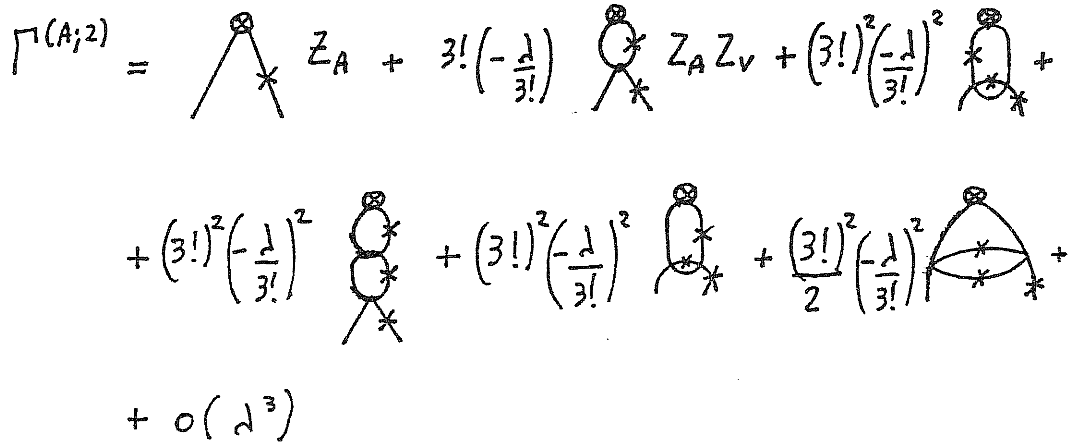
The fixed point now is

$$K^* = \frac{\omega}{3} + \frac{a}{27}\omega^2 + O(\omega^3) \qquad (91)$$

We are now in the position for computing rather easily the correction to the anomalous dimension of the mass insertion. In fact the graphs contributing to the $\Gamma^{(4)}$ enter also in the mass insertion computation.

4. THE CRITICAL EXPONENT γ .

From the expansion of the field ϕ (eq.(30)), we get the two loops contributions to the $\Gamma^{(A;2)}$:



(92)

We see that are involved the previous studied graphs of type 0, II, III, V. The second order term in Z_A is :

$$\delta_2 Z_A = 2\left(\frac{R}{\omega}\right)^2 - \frac{U_A}{\omega}\frac{R^2}{4}$$

with $U_A = S_{II} + S_{III} + \frac{1}{2} S_V = 2.62 \pm 0.07$ (93)

The double pole part of $\delta_2 Z_A$ is that we aspected (eq.(75)) in order to get γ_{ϕ^2} finite. Expanding in powers of λ eq.(33), we get $\delta_2 Z_{\phi^2}$ from $\delta_2 Z_A$:

$$\delta_2 Z_{\phi^2} = 2\left(\frac{R}{\omega}\right)^2 - \frac{1}{\omega}\frac{R^2}{4}\left[U_A + 3R_t - R_\phi\right] \tag{94}$$

From eq.(75) we get γ_{ϕ^2}:

$$\gamma_{\phi^2} = -R\lambda + 4\frac{R^2\lambda^2}{\omega} - 2\lambda^2\omega\, \delta_2 Z_{\phi^2} =$$

$$= -R\lambda + \lambda^2 R^2 \frac{1}{2} (U_A + 3R_t - R_\phi) = -k + k^2 c$$

where $c = 1.61 \pm 0.04$ (95)

We then compute $\gamma^*_{\phi^2}$ at the fixed point :

$$\gamma^*_{\phi^2} = -\frac{\omega}{3} + \omega^2 (3c + a) \frac{1}{27} + O(\omega^3) \qquad (96)$$

The critical exponent γ is now obtained expanding eq.(36) in powers of ω :

$$\gamma = 1 + \frac{\omega}{6} + \frac{1}{4} \left(\frac{2a - 6c + 3}{27} \right) \omega^2 + O(\omega^3) \qquad (97)$$

Extrapolating to $\omega = 1$, this gives

$$\gamma = 1.27 \pm 0.01 \qquad (98)$$

This is our result; it is really a good one, because we remember that from the ε - expansion we get $\gamma_\varepsilon = 1.24$ and the value of the high temperature expansion is $\gamma_T = 1.250 \pm 0.003$. (10)

What is left now is only the calculation of the critical exponent η. The strategy to follow is clear, and we are going to do the computations; up to now we have been able to extract the double pole of the wave function renormalization and proved the validity of the cancellation (see eq.(77)) (see Appendix A). The calculation of the residue of the single pole part will be our next goal; this is highly non trivial, since we must analyze three loops graphs with an external momentum different from zero.

Let us mention that the results we got, are given by an asymptotic expansion in ω andit could be necessary to use some resummation technique, as it occurs in the ε - expansion (11).

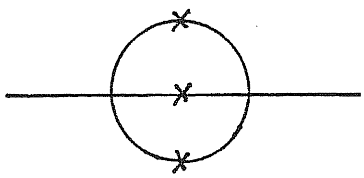
SECTION 4.

1. THE ε- EXPANSION CHECK.

We want now to use the same technique of the stochastic graphs shown in the other sections, to compute the ε- expansion. This provides a total check of the whole method; in fact the results can be in agreement with those, for instance, of ref.(3), only if we had the right graphs, combinatorial factors, there was no mistake in the integrals and in the renormalization procedure and so on. The only thing not needed now is the analytical continuation in the integrals , but this has been tested before. The ε-expansion simply corresponds to taking the limit for $\sigma=0$ and d=4. The graphs are the same as before, with the same combinatorial factors; the only substitution in the graphs is:

$$\lim_{\sigma\to 0} \frac{\sigma}{2}\left|t-t'\right|^{\sigma-1} = \delta(t-t') \qquad (99)$$

The integrals are now simpler than in the stochastic case: less variables of integration, no analytical continuation, no breaking in time intervals is needed. The technique for computing the two loops integrals is exactly the same of the stochastic case, so we check it too. An important difference is that now there is another diverging graph:



giving a renormalization to the noise-noise correlation function: a Z_ξ is needed, and one has to take it into account in computing the renormalization constants. Calling $k = \lambda R'$ where $R' = (2\sqrt{\pi})^{-4}$, we can compare our results with thoseof ref.(3), pag.248-254.

STOCHASTIC ε- EXP.	USUAL ε- EXP.
$R_\phi = -0.08332$	$R_\phi = -0.0833333 = -1/12$
$R_t = -0.14384$	$R_t = -0.1428 \quad = -1/7$
$R_t - 2R_\xi = 0.000 \pm 4 \cdot 10^{-4}$	$R_t - 2R_\xi = 0$
$\beta/K = -\varepsilon + 3k - k^2 a_\varepsilon$	$\beta/K = -\varepsilon + 3k - k^2 17/3$
$a_\varepsilon = 5.665 \pm 0.004$	$17/3 = 5.6666$
$Z_A = 1 + k/\varepsilon + 2k^2/\varepsilon^2 - k^2 U_A/\varepsilon$	
$U_A = 0.5000 \pm 0.0002$	$U_A = 1/2$
$Z_V = 1 + 3k/\varepsilon + k^2(9/\varepsilon^2 - V/\varepsilon)$	
$V = 2.999 \pm 0.002$	$V = 3$
$\gamma = 1.24379 \pm 5 \cdot 10^{-5}$	$\gamma = 1.24383$

The agreement between the values computed with stochastic methods and that of the literature is perfect, and is the best proof that all we have done is correct.

2. CONCLUSIONS.

We have shown that stochastic quantization with stochastic regularization provides a good and well defined way of doing renormalization computations; it also provides a new way of studying statistical systems in the field theoretical framework, in alternative to the ε - expansion. The results we got for the critical exponents with this new method, are as good as those obtained by other methods, and we find indications that new informations can be obtained with respect to the ε - expansion. Moreover there is the possibility of improving the method using a mixed stochastic and ε - expansion. Stochastic quantization is not only a new point of view in quantum physics, but it becomes so a new powerful practical method for studying renormalization of quantum systems.

3. ACKNOWLEDGMENTS.

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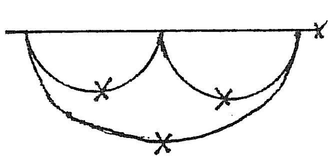
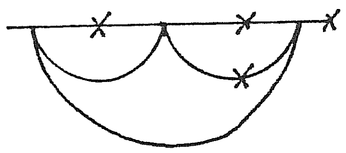
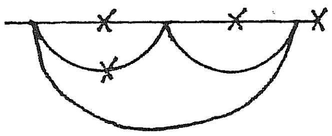
APPENDIX A.

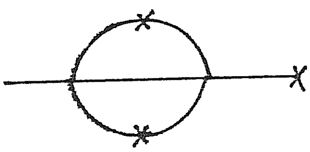
In this appendix we show the proof that the double pole part of the wave function renormalization is exactly that given by eq.(77). At order three loops there are three graphs contributing to the $\langle \phi_R \xi \rangle$ corralation function, constructed by contraction of the λ^3 tree diagrams in the expansion of the field ϕ :

$-\lambda^3$  $-\frac{\lambda^3}{2}$  $+$

$-\frac{\lambda^3}{2}$ 

(A.1)

One more contribution at order λ^3 comes from the counterterm:



$$\times(+\lambda^2)\frac{Z_v^2}{2} = \frac{\lambda^2}{2}\frac{1}{\omega}G\{t/\phi\} + \quad \text{(A.2)}$$

$$+\lambda^3\frac{3R}{\omega}\frac{1}{\omega}G\{t/\phi\}$$

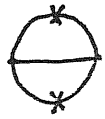
where G_t and G_ϕ are the residues of the poles in the derivatives with respect to s and p^2 respectively. At order λ^2 we got:

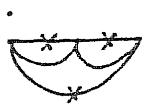
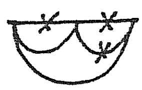
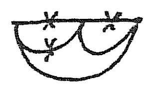
$$\delta_2 Z_t = \frac{1}{2}\frac{G_t}{\omega} \qquad \delta_2 Z_\phi = \frac{1}{2}\frac{G_\phi}{\omega} \tag{A.3}$$

The constant r defined in eq.(76) is now:

$$r = \frac{3}{4} G_t + \frac{1}{4} G_\phi \tag{A.4}$$

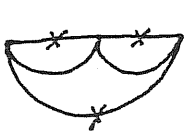
The $\Gamma^{(2)}$ is :

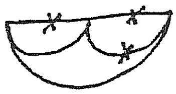
$\Gamma^{(2)} = G^{-1} -$  $\frac{\lambda^2}{2} - \lambda^3 \Big\{ \frac{3R}{\omega^2} G_t s + \frac{3R}{\omega^2} G_\phi p^2 +$

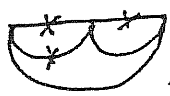
$-$  $- \frac{1}{2}$  $- \frac{1}{2}$  $\Big\}$

(A.5)

The double pole part of the three loops graphs is:

 $\Big|_{\text{Double Pole}} = \frac{2}{3}\frac{2}{\omega^2} R\, G_{\{t,\phi\}}$

 $\Big|_{D.P.} = \frac{1}{3}\frac{2}{\omega^2} R\, G_{\{t,\phi\}}$

 $\Big|_{D.P.} = \frac{1}{3}\frac{2}{\omega^2} R\, G_{\{t,\phi\}}$ (A.6)

Summing the λ^3 contribution we get:

$$\frac{R}{\epsilon^2} G_{\{t \atop \phi\}} \left[-\frac{4}{3} - \frac{1}{3} - \frac{1}{3} + 3 \right] = R\, G_{\{t \atop \phi\}} \frac{1}{\epsilon^2} \tag{A.7}$$

The constants $\delta_3 Z_t$ and $\delta_3 Z_\phi$ are

$$\delta_3 Z_t \Big|_{D.P.} = \frac{R G_t}{\epsilon^2} \qquad \delta_3 Z_\phi \Big|_{D.P.} = \frac{R G_\phi}{\epsilon^2} \tag{A.8}$$

Taking eq.(A.4) into account, $\delta_3 Z$ is:

$$\delta_3 Z \Big|_{D.P.} = \frac{3}{2} \frac{R G_t}{\epsilon^2} + \frac{1}{2} \frac{R G_\phi}{\epsilon^2} = \frac{2 R \tau}{\epsilon^2} \tag{A.9}$$

as stated by eq.(77). Eq.(A.6) can by proved writing the graphs, going to polar coordinates, and identifying the double pole part as the product of the divergences of the subloops times a coefficient. This coefficient turns out to be nothing but the number of diverging subloops divided by the loop number of the graph.