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THE DECOMPOSABLE SETS OF MEASURABLE FUNCTIONS : SOME PROPERTIES AND APPLICATIONS TO MULTIVALUED MAPS

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INTRODUCTION

The present work is devoted to discuss some recent results involving a particular kind of sets of measurable functions, i.e. the decomposable sets. The definition of decomposability will be given, together with some properties of decomposable sets, in Chapter 1.

The main part of this thesis is concerned with multivalued maps having decomposable values: two selection theorems are proven in Chapter 3. As a technical tool for them, Chapter 2 contains a discussion of Liapunov's Convexity Theorem, together with some generalizations.

Most of the results collected in this work have a surprising property: every statement can be obtained, with possibly few changes and remarks, from famous statements concerning convex sets, just substituting the word "convex" by the word "decomposable". This characteristic has been pointed out by C. Olech [22]. As an example of these results we mention here the fact that every decomposable and closed set in a separable L^1- space is an absolute retract. The reason of this analogy lies on the fact that the convexity is "transferred" from the set of functions into the measure space which is their domain. In fact, the decomposability property - which essentially is the possibility of perturbing a function by cutting its graph on a measurable set and piecing on that set the corresponding part of the graph of another function -, together with the nonatomicity of a measure on the domain of the function, provides an interpolation method which plays the role of convex combinations.

1. THE DECOMPOSABILITY

1.1 INTRODUCTION

A notion of decomposability seems to have appeared first in a paper by Rockafellar [23] , although it is used, for example, in a two years earlier article by Olech [21] concerning optimal control theory, and in Pontrjagin's Maximum Principle. In the Rockafellar work, the definition is given for a vector space of measurable functions, hence it is added, and not substituted, to a convexity assumption. The definition we give in Section 1 can be found in a paper by Hiai and Umegaki [14] , in which it used to define a multivalued conditional expectation. In the multifunction framework, the decomposability has been implicitely used first by Antosiewicz and Cellina [1] , later by Bressan [3] , Cellina and Marchi [8], and then appears in its full generality in a paper by Fryszkowski [11] .

In Section 1 we report from [14] an interesting characterization of the closed and decomposable subsets of L^p. In Section 2 some properties of decomposable sets which make them to look like convex sets are listed. They are due to Olech [22] and to Bressan and Colombo [4] .

1.2 THE BASIC DEFINITION

Throughout this thesis, $(T, \mathfrak{F}, \mu)$ denotes a measure space with a σ-algebra $\mathfrak{F}$ of subsets of T and a positive measure μ. Given a μ-integrable function $f: T \to \mathbb{R}$, we write $f \cdot \mu$ for the measure having density f with respect to μ. We denote by

$\sigma\{A_\lambda : \lambda \in \Lambda\}$ the σ-algebra generated by a family of measurable sets $A_\lambda \in \mathcal{F}$. If E is a Banach space with norm $||\cdot||_E$, M denotes the vector space of the functions $u : T \to E$, which are measurable with respect to $\mathcal{F}$ and to the Borel subsets of E, while $L^p(T;E)$, $1 \leq p \lneq \infty$, is the Banach space of the functions $u \in M$ such that $||u||_E \in L^p(T;\mathbb{R})$, with norm $||u||_p = (\int_T ||u||_E^p \, d\mu)^{1/p}$ (see [24,pag.132]). The vector space of the multivalued maps F from T into the subsets of E which are (weakly) measurable (i.e. for any open set $U \subseteq E$ we have that $F^-(U) = \{ t \in T : F(t) \cap U \neq \emptyset \} \in \mathcal{F}$) is indicated by $\mathfrak{M}$. We recall here the basic property of these multifunctions :

THEOREM 1.1 . Let F be a multivalued function from T into the closed nonempty subsets of E and assume thet E is separable. Then the following statements are equivalent :

(i) $F \in \mathfrak{M}$;

(ii) (Castaing) there exists a sequence $(f_n)_n$ in M such that

$$F(t) = cl_E \{f_n(t) : n \in N\} \quad \text{for all } t \in T .$$

For the proof see, for instance, [16] and [15,Thm. 5.6] .

We indicate by S_F^p the set of all the L^p selections from F, i.e.

$$S_F^p = \{u \in L^p(T;E) : u(t) \in F(t) \text{ a.e. in } T\} .$$

Given two metric spaces X,Y with distances d_X, d_Y respectively, the distance on their product is $d_{X\times Y} = d_X + d_Y$. The open ε-neighborhood of a set $S \subseteq X$ is $B(S,\varepsilon) = \{x \in X;\ d(x,S) < \varepsilon\}$. The diameter of S is $\operatorname{diam}(S) = \sup \{ d(x,x') : x,x' \in S \}$. The set-theoretic difference between two sets A,B is written $A \setminus B$; their symmetric difference is $A \Delta B = (A \setminus B) \cup (B \setminus A)$. $\# A$ stands for the cardinality of the set A, while χ_A is the characteristic function of A.

Following [14], we now introduce the main concept discussed in this thesis.

DEFINITION 1.2. A set $K \subseteq \mathcal{M}$ is <u>decomposable</u> if

$$u \cdot \chi_A + v \cdot \chi_{T \setminus A} \in K \qquad \text{whenever } u, v \in K,\ A \in \mathcal{F}.$$

The collection of all nonempty decomposable subsets of a subspace L of M is denoted by $\mathbf{D}(L)$. For any set $H \subseteq L$, the <u>decomposable hull</u> of H in **L** is

$$\mathrm{dec}_L[H] = \bigcap \{K \in D(L) \; ; \; H \subseteq K\}.$$

Clearly, $\mathrm{dec}_L[H]$ represents the smallest decomposable subset of L which contains H.

The following result provides a characterization of the closed and decomposable subsets of $L^p(T;E)$ and gives easy examples of such sets. It is due to Hiai and Umegaki [14, Thm. 3.1].

<u>THEOREM 1.3</u>. Let μ be σ-finite and E be separable. Then a nonempty closed subset K of $L^p(T;E)$, $1 \leq p < \infty$, is decomposable if and only if there exists $F \in \mathcal{M}$, with F(t) closed for every $t \in T$, such that $K = S_F^p$.

The following Lemma, which is essentially a L^p version of the Representation Theorem 1.1, will be needed in the proof.

<u>LEMMA 1.4.</u> Let $F \in \mathcal{M}$ have closed values and let $1 \leq p < \infty$. If S_F^p is nonempty, then there exists a sequence $(f_n)_n$ contained in S_F^p such that $F(t) = cl_E\{u_n(t) : n \in \mathbb{N}\}$ for all $t \in T$. Furthermore, for each $u \in S_F^p$ and $\varepsilon > 0$, there exists a finite measurable partition $\{A_1, \ldots, A_n\}$ of T such that

$$\| u - \sum_{i=1}^{n} u_i \cdot \chi_{A_i} \|_p < \varepsilon.$$

For the proof, see [14, Lemma 1.1, 1.3].

<u>Proof of Theorem 1.3.</u> The decomposability of S_F^p is obvious. Conversely, let $K \in D(L^p(T;E))$ be closed and let $(u_i)_i \subseteq L^p$ be a sequence such that $\{u_i(t) : i \in \mathbb{N}\}$, by Lemma 1.4, is dense in E for all t. Set, for each i, $\alpha_i = \inf \{ \|u_i - v\|_p : v \in K\}$ and choose a sequence $\{v_{ij} : j \geq 1\} \subseteq K$ such that $\| u_i - v_{ij} \|_p \to \alpha_i$. Define the function $F \in \mathfrak{M}$ by setting $F(t) = cl_E \{ v_{ij}(t) : i,j \geq 1 \}$; we now prove that $K = S_F^p$.

To see that $S_F^p \subseteq K$, let $u \in S_F^p$ and $\varepsilon > 0$: by Lemma 1.4 we can take a finite measurable partition $\{ A_1,\dots,A_n \}$ of T and a subset $\{ w_1,\dots,w_n \} \subseteq \{v_{ij}\}$ such that

$$\| u - \sum_{k=1}^{n} w_k \cdot \chi_{A_k} \|_p < \varepsilon .$$

Since $\sum_{k=1}^{n} w_k \cdot \chi_{A_k} \in K$ and K is closed, we have that $u \in K$.

Conversely, let us assume by contradiction that $K \supsetneq S_F^p$. Then we can take $u \in K$, $A \in \mathcal{F}$ with $\mu(A) > 0$ and $\delta > 0$ such that

$$\inf_{i,j \geq 1} \| u(t) - v_{ij}(t) \|_E \geq \delta \qquad \text{for all } t \in A.$$

Fix an index i such that the set

$$B = A \cap \{t \in T : \| u(t) - u_i(t) \|_E < \delta/3 \}$$

has positive measure and define the sequence $(v'_j)_j \subseteq K$ by setting

$$v'_j = u \cdot \chi_B + v_{ij} \cdot \chi_{T \setminus B} \qquad ,j \geq 1 .$$

Then, since for all $t \in B$

$$\|u_i(t) - v_{ij}(t)\|_E \geq \|u(t) - v_{ij}(t)\|_E - \|u(t) - u_i(t)\|_E > \frac{2\delta}{3} ,$$

we have that, for every $j \geq 1$,

$$\begin{aligned} \| u_i - v_{ij} \|_p^p - \alpha_i^p &\geq \\ &\geq \| u_i - v_{ij} \|_p^p - \| u_i - v'_j \|_p^p \\ &\geq \int_B (\|u_i(t) - v_{ij}(t)\|_E^p - \|u_i(t) - u(t)\|_E^p) \, d\mu \\ &\geq \left[\left(\frac{2}{3}\delta\right)^p - \left(\frac{\delta}{3}\right)^p \right] \cdot \mu(B) > 0 , \end{aligned}$$

which is a contradiction. ∎

1.3 SOME PROPERTIES OF DECOMPOSABLE SUBSETS OF L^1.

Throughout this section, μ denotes a finite, positive and nonatomic measure, i.e. there exists no $E \in \mathcal{M}$ such that $\mu(E) > 0$ and for all measurable $F \subseteq E$ either $\mu(F) = 0$ or $\mu(F) = \mu(E)$.

In this case, a number of analogies beetween decomposable and convex subsets of $L^1(T;R)$ - or of $L^1(T;E)$ - can be found. They were first noticed by C. Olech in [22] . These properties depend on the following fact, that will be discussed in the next chapter : there exists a totally ordered subset $\{ A_\alpha : \alpha \in [0,1] \}$ of $\mathcal{M}$ such that $\mu(A_\alpha) = \alpha \mu(T)$ for every $\alpha \in [0,1]$; as a consequence, any two functions u,v in K can be joined by a continuous path contained in K. We collect here three theorems about the geometrical and the topological structure of decomposable sets in L^1. They correspond to the Krein-Milman's and to the Carathéodory's theorems on the extremal points of a convex set and to the Dugundji's extension theorem, which we state here for reference.

<u>THEOREM 1.A</u> (Krein-Milman, Carathéodory). Suppose X is a topological vector space on which X* separates points. If D is a compact and convex subset of X, then D is the closed convex hull of the set of its extreme points (we say that $e \in D$ is extreme for D if the equality $e = \lambda a + (1 - \lambda)b$ with $0 < \lambda < 1$ and $a,b \in D$ implies $e = a = b$). When $X = R^n$, for any $x \in D$ there exist at most n + 1 extreme points $e_0, e_1, \ldots, e_n$ of D such that $x \in co \{e_0, e_1, \ldots , e_n\}$.

<u>THEOREM 1.B</u> (Dugundji). Let A be a closed subset of a metric space X and let D be a convex subset of a Banach space Y. Then every continuous map $f : A \to D$ has a continuous extension $\tilde{f} : X \to D$.

The following definition allows us to state Theorem 1.A for decomposable sets.

DEFINITION 1.5 (Olech). A point w of a decomposable subset K of $L^1(T;\mathbb{R}^n)$ is an extreme point of K if there exists a convex cone $C \subseteq \mathbb{R}^n$, with the property that $C \cap (-C) = \{0\}$ and $C \cup (-C) = \mathbb{R}^n$, such that for each $u \in K$, $w(t) - u(t) \in C$ for μ-a.e. $t \in T$.

We recall that, for a convex set $D \subseteq \mathbb{R}^n$, e is extreme if and only if there exists a convex cone C, with the property that $C \cap (-C) = \{0\}$, such that $e - d \notin C$ for each $d \in D$. Hence, the above notion of extremality for decomposable sets can be seen as an analogue to the (pointwise) one for convex sets in L^1.

THEOREM 1.6 (Olech). Let $K \in D(L^1(T;\mathbb{R}^n))$ be closed and integrably bounded (i.e. there exists $m \in L^1(T;\mathbb{R})$ such that $|u(t)| \leq m(t)$ μ-a.e. in T for every $u \in K$). Then $w \in K$ is an extreme point of K if and only if $\int_T w\, d\mu$ is an extreme point of $\int K = \{\int_T u\, d\mu : u \in K\}$. Moreover, $\int K$ is a convex and compact subset of R^n, hence the set of the extreme points of K is nonempty.

THEOREM 1.7 (Olech). Let $K \in D(L^1(T;\mathbb{R}^n))$ be closed and integrably bounded. Then for each $u \in \overline{co}\, K$ there exist at most n + 1 extreme points of K, $w_o,\dots,w_n$, and a measurable partition of T, $A_o,\dots,A_n$, such that

$$\int_T \sum_{i=0}^{n} w_i \cdot \chi_{A_i}\, d\mu = \int_T u\, d\mu \quad .$$

These two results were stated by Olech [21, Theorems 7.1,7.2] in a different framework and later reformulated in this language [22, Theorems 1,2].

The last result of this section has its origin in a paper by A. Cellina [5], in which it was proven that a particular decomposable set K has the compact fixed point property, that is every continuous map $f : K \to K$ with relatively compact image has a fixed point. This theorem has been later generalized to all the decomposable sets in L^1 by A. Fryszkowski [12] and represents the decomposable counterpart of the Schauder's Fixed Point Theorem. We state it as a corollary of the announced analogue of Dugundji's theorem.

THEOREM 1.8. Let A be a closed subset of a metric space X. If either X or $L^1(T;E)$ is separable, then every continuous map $f:A\to L^1(T,E)$ has a continuous extension $\tilde{f}:A\to L^1(T;E)$ such that $\tilde{f}(X)\subseteq \mathrm{dec}\,[\,f(A)]$.

COROLLARY 1.9 If $L^1(T;E)$ is separable, then every closed decomposable subset $K\subseteq L^1(T;E)$ is a retract of the whole space.

Theorem 1.8 yields a general fixed point theorem, which is valid for L^1 spaces over any abstract measure space $(T,\mathcal{F},\mu)$ with a non-atomic probability measure μ .

COROLLARY 1.10. Every closed decomposable set $K\subseteq L^1(T;E)$ has the compact fixed point property.

Indeed, if $L^1(T;E)$ is separable, then Corollary 1.10 is an immediate consequence of Corollary 1.9. To cover the case where $L^1(T;E)$ is not separable, let $f:K\to K$ be a continuous map whose image is relatively compact, and let X be the closure of the convex hull of f(K). Since X is compact, it is obviously separable. Using Theorem 1.8, extend the identity map ι on $X\cap K$ to a continuous map $\tilde{\iota}:X\to K$. The composition $f\circ\tilde{\iota}$ maps X into $X\cap K$. By Schauder's theorem, it has a fixed point $\bar{x}\in X\cap K$, which is then a fixed point of f.

PROOF OF THEOREM 1.8.

We assume first that $L^1(T;E)$ is separable. For each $x\in X\setminus A$, take an open ball $B(x,r_x)$ with radius $r_x<\frac{1}{2}d(x,A)$. The family $\{B(x,r_x)\,;\,x\in X\setminus A\}$ is an open covering of the paracompact space $X\setminus A$, hence it admits an open nbd-finite refinement $\{V_i\,;\,i\in I\}$. Here I is a possibly uncountable set of indexes. For each i, choose two points $x_i\in V_i$ and $y_i\in A$ such that

$d(x_i,y_i) < 2\, d(x_i,A)$. Using the separability assumption, select a countable subset $D = \{ u_n \; ; \; n \geq 1 \}$ of $f(A)$ which is dense on $f(A)$. Define the sequence $(g_k)_{k \geq 0}$ in $L^1(T;\mathbb{R})$ by setting

$$g_k(t) = \| u_m(t) - u_n(t) \|_E \qquad \text{whenever } k = 2^m \cdot 3^n \text{ for some } m,n \geq 1;$$

$$g_k(t) = 1 \qquad \text{otherwise.}$$

Applying Theorem 2.6 below to this sequence, we obtain a family $\{ \Phi(\tau,\lambda) \}$ of measurable subsets of T with the properties a) $\div$ c). For each $i \in I$, choose $u_{\nu(i)} \in D$ such that $\| u_{\nu(i)} - f(y_i) \| < d(x_i,y_i)$. Let $\{ p_i(\cdot) \; ; \; i \in I \}$ be a continuous partition of unity subordinated to the covering $\{V_i\}$. For every $n \geq 1$, define the open set $W_n = \bigcup \{ V_i \; ; \; \nu(i) = n \}$ and let $q_n(x) = \sum_{\nu(i)=n} p_i(x)$.
Clearly, $\{ q_n(\cdot) \; ; \; n \geq 1 \}$ is a continuous partition of unity subordinated to the locally finite open covering $\{W_n\}$. Construct a sequence of continuous functions $(h_n)_{n \geq 1}$ such that $h_n \equiv 1$ on supp (q_n) and supp $(h_n) \subseteq W_n$.

For every $x \in X \setminus A$, define $\lambda_n(x) = \sum_{m \leq n} q_m(x)$, $n \geq 0$, and consider the function

$$\tau(x) = \sum_{m,n \geq 1} h_m(x) \cdot h_n(x) \cdot 2^m \cdot 3^n .$$

Notice that τ is continuous on $X \setminus A$ and that

$$\tau(x) \geq 2^m \cdot 3^n \qquad \forall\, x \in \mathrm{supp}(q_m) \cap \mathrm{supp}(q_n) . \tag{1.1}$$

We can now extend the map f to the whole space X by setting

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A \\ \sum_{n\geq 1} u_n \cdot \chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))} & \text{if } x \in X \setminus A . \end{cases}$$

It is clear that $\tilde{f}$ maps X into dec[f(A)] . Moreover, $\tilde{f}$ is continuous on $X \setminus A$, because the functions $\tau(\cdot)$ and $\lambda_n(\cdot)$ $(n \geq 0)$ are continuous, the characteristic function of the set $\Phi(\tau,\lambda)$ varies continuously in $L^1(T;\mathbb{R})$ w.r.t. the parameters τ and λ, and because the summation defining $\tilde{f}$ is locally finite. To prove that $\tilde{f}$ is continuous on A, let $a \in A$ and $\varepsilon > 0$ be given. Choose $\delta > 0$ such that $\delta < \varepsilon/12$ and $\| f(y) - f(a) \|_1 < \varepsilon/2$ whenever $y \in A$, $d(y,a) < 12\delta$.

If $d(x,a) < \delta$ and $x \in V_i$ for some $i \in I$, then $\operatorname{diam}(V_i) < 2\delta$, $d(x_i,A) < 3\delta$ and $d(x_i,y_i) < 6\delta$. Therefore, $p_i(x) \neq 0$ implies that $d(y_i,a) < 9\delta$,

$\| f(y_i) - f(a) \|_1 < \varepsilon/2$ and $\| u_{\nu(i)} - f(a) \|_1 < \varepsilon$.

From the last inequality , it follows that

$$\| u_n - f(a) \|_1 < \varepsilon \qquad \forall n \text{ such that } q_n(x) \neq 0 . \qquad (1.2)$$

For any $x \in X \setminus A$ with $d(x,a) < \delta$, fix an integer j for which $q_j(x) \neq 0$.

Using (1.1) , (1.2) and the property c) of the sets $\Phi(\tau,\lambda)$, we obtain the estimate

$$\| f(a) - \tilde{f}(x) \|_1 \leq \| f(a) - u_j \|_1 + \| u_j - \tilde{f}(x) \|_1$$

$$\leq \varepsilon + \sum_{n=1}^{\infty} \int_T \| u_j - u_n \|_E \cdot \chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))} \, d\mu$$

$$= \varepsilon + \sum_{n=1}^{\infty} \int_T g_{2^j 3^n} \cdot \chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))} \, d\mu$$

$$= \varepsilon + \sum_{n=1}^{\infty} q_n(x) \cdot \int_T g_{2^j 3^n} \, d\mu = \varepsilon + \sum_{n=1}^{\infty} q_n(x) \, \| u_j - u_n \|_1 \leq 3\varepsilon .$$

Since ε was arbitrary, this completes the proof in the case where $L^1(T;E)$ is separable.

When X is separable, only minor modifications of the above arguments are needed. Consider again the open covering $\{B(x,r_x) \; ; \; x \in X \setminus A\}$ and a locally finite refinement $\{V_i \; ; \; i \in I\}$. Notice that in this case the set I is necessarily countable, since $X \setminus A$ is separable. For each i, choose $x_i \in V_i$, $y_i \in A$ such that $d(x_i,y_i) < 2\, d(x_i,A)$. It now suffices to define the countable set $D = \{f(y_i) \; ; \; i \in I\} \subset L^1(T;E)$ and arrange its elements into a sequence, say $D = \{u_n \; ; \; n \geqq 1\}$. From this point on, the proof runs exactly as in the previous case. ∎

2. SOME TECHNICAL RESULTS ON MEASURE THEORY.

2.1 INTRODUCTION

In the analysis of decomposable sets, one of the most important tools is a method to interpolate among points with continuity, which is not a convex combination. More precisely, let X be a topological space and L be a topological vector space and suppose given a finite open covering $\mathcal{U} = (U_i)_{i=1,\dots,p}$ of X and a corresponding family of points $Y = (y_i)_{i=1,\dots,p}$ in L. The usual way to "extract" a continuous function f from the multifunction $x \to \{ y_i : x \in U_i \}$ is to take a continuous partition of unity $(\pi_i(\cdot))_{i=1,\dots,p}$ subordinate to $\mathcal{U}$ and to define

$$f(x) = \sum_{i=1}^{p} \pi_i(x) \cdot y_i \quad . \tag{2.1}$$

If K is any convex set containing Y, then $f(X) \subseteq K$, while if K is not convex, the property $f(X) \subseteq K$ may be lost. To maintain the relation $f(X) \subseteq K$ in the case when L is $L^1(T;E)$, a nonatomic measure μ over $\mathcal{F}$ is given and K is decomposable, Antosiewicz and Cellina [1] used the following substitute of (2.1) :

$$f(x) = \sum_{i=1}^{p} \chi_i(x) \cdot y_i \quad , \tag{2.2}$$

where $\chi_i : X \to L^1(T;\{0,1\})$ are continuous functions such that $\sum_{i=1}^{p} \chi_i(x) \equiv 1$ for every $x \in X$. The functions χ_i can be constructed by taking a family $(A_\alpha)_{\alpha \in [0,1]}$ of measurable subsets of T such that

$$\begin{cases} A_\alpha \subseteq A_\beta & \text{if } \alpha \leq \beta \quad , \\ \mu(A_\alpha) = \alpha \cdot \mu(T) & \text{for all } \alpha \in [0,1] \quad , \end{cases} \tag{2.3}$$

and by defining $\alpha_i(x) = \sum_{j=1}^{i} \pi_j(x)$, $\alpha_o(x) \equiv 0$ and $\chi_i(x) = \chi_{A_{\alpha_i(x)} \setminus A_{\alpha_{i-1}(x)}}$ (see Fig.1). The existence of a family of sets satisfying (2.3) is equivalent to the nonatomicity of μ , as it will be shown in the Corollary 2.4 below. The property

$f(X) \subseteq K$ holds by the definition of decomposability, and f is simply seen to be continuous (see Proposition 2.8 below).

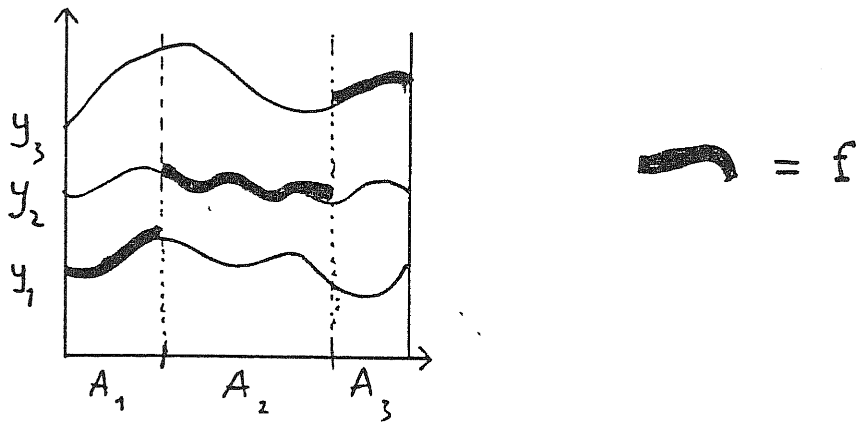

Fig.1

Together with these analogies with convex combinations there is, however, a major difference.

In Banach spaces, the metric and the algebraic structures are linked together by the fact that balls are convex . On the other hand, balls in $L^1(T;E)$ are not decomposable. The failure of this basic property is a primary source of technical difficulties. If $\bar{y} \in L^1$ and $\| y_i - \bar{y} \|_1 \leq \rho$ for all $i \in \{1, \ldots , p\}$, without additional assumptions on the sets A_λ the only available estimate for (2.2) is

$$\| f(x) - \bar{y} \|_1 \leq p \cdot \rho \quad .$$

This bound can be improved if the sets A_λ are more carefully chosen. In [11] the author defines the measures μ_i by setting

$$\mu_i(A) = \int_A \| y_i - \bar{y} \|_E \, d\mu \qquad (A \in \mathcal{F}).$$

By Corollary 2.4 below , one can then choose a family of sets A_λ satisfying the additional conditions

$$\mu_i(A_\lambda) = \lambda \mu_i(T) \quad , \quad (\lambda \in [0,1] \; , \; i = 1,\ldots,p) \qquad (2.4)$$

If these special sets are used in (2.2), the stronger estimate $\| f(x) - \bar{y} \|_1 \leq \rho$ holds.

This chapter is devoted to construct families $(A_\alpha)_\alpha$ for which (2.4) holds. For a finite set Y of functions, hence for finitely many measures, the existence of such a family is equivalent to the Liapunov's Convexity Theorem, as it will be shown in the next section. In general, (2.4) is not true for an infinite family of measures $(\mu_i)_{i \in I}$, as Liapunov's counterexample 2.5 shows. Thus, in order to apply this interpolation method to a space X without assuming its compactness, a kind of infinite dimensional extension of Corollary 2.4 is needed. We give in the last section two different generalizations: the first one, Theorem 2.6, provides the desired interpolation, while the second one, Theorem 2.9, although a more precise analogue to Corollary 2.4, is not applicable here* .

2.2 THE CASE OF FINITELY MANY MEASURES

We state first the famous Liapunov's Convexity Theorem, which is the foundation of every result presented here.

<u>THEOREM 2.1</u> (Liapunov). The range of every nonatomic finitely dimensional vector measure , i.e. of every (bounded) countably additive function $\nu : \mathcal{A} \to \mathbb{R}^n$ whose components have no atoms, is compact and convex.

Many proofs of this theorem can be found in the literature; we report here a simplified rewriting of the convexity part in the paper by Halmos [13] , by whose arguments some of the results stated in this chapter were inspired.

Before the proof we give some definitions which simplify the

* The last chapter of the monography "Vector Measures" by J.Diestel and J.J.Uhl (1977,A.M.S. Providence Rhode Island) contains a deep discussion of many theorems on the range of vector measures.

notations. We recall first that the lenght of a finitely dimensional vector measure ν , $|\nu|$, is the numerical measure which is the sum of the total variation of each component of ν ; a measure ν' over $\mathcal{H}$ is absolutely continuous with respect to ν -and we write $\nu' << \nu$ - if $|\nu'| << |\nu|$.

<u>DEFINITION 2.2</u> ([7]). A family $(A_\alpha)_{\alpha\in[0,1]}$, $A_\alpha \in \mathcal{H}$, is called <u>increasing</u> if

$$A_\alpha \subseteq A_\beta \quad \text{when} \quad \alpha \leq \beta \quad .$$

An increasing family is called <u>refining</u> $A \in \mathcal{H}$ with respect to the measure ν if $A_0 = \emptyset$, $A_1 = A$ and $\nu(A_\alpha) = \alpha \cdot \nu(A)$ for every $\alpha \in [0,1]$.

<u>DEFINITION 2.3</u> ([13]). A vector measure ν is called <u>convex</u> if for every A there exists a family $(A_\alpha)_\alpha$ refining A withrespect to ν ; is called <u>semi-convex</u> if for every $A \in \mathcal{H}$ thre exists a $B \subseteq A$ such that $\nu(B) = \nu(A)/2$.

<u>REMARK</u>. The property of convexity says that the range of ν is star-shaped and that the segment $[0,\nu(A)]$ is covered by the measures of an increasing family contained in A (see Fig. 2) .

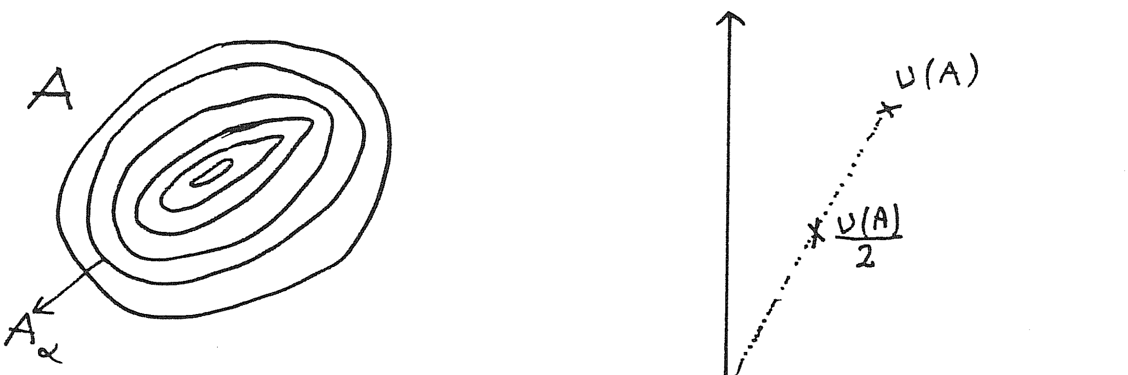

Fig. 2

PROOF OF THEOREM 2.1 (CONVEXITY PART).

We suppose, for simplicity, that $\nu = (\nu_1, \nu_2)$ is 2-dimensional and has positive components.

The proof will be carried out in a few steps:

a) a semi-convex measure is convex;

b) the range of a convex measure is convex;

c) a nonatomic measure is semi-convex (hence it has convex range).

Proof of a). (Fryszkowski [12,Prop. 1.1]). Let $A \in \mathcal{F}$. By definition there exist measurable sets $A_{1/2}$, $A_{1/4} \subseteq A_{1/2}$ and $A' \subseteq A \setminus A_{1/2}$ such that $\nu(A_{1/2}) = \frac{1}{2}\nu(A)$ and $\nu(A_{1/4}) = \frac{1}{4}\nu(A) = \nu(A')$; define $A_{3/4} = A_{1/2} \cup A'$: then $\nu(A_{3/4}) = \frac{3}{4}\nu(A)$. Using this method we can construct an increasing family of subsets of A, A_α, with $\alpha = k/2^n$ ($n \in \mathbb{N}$ and $0 \leq k \leq 2^n$) such that $\nu(A_\alpha) = \alpha\nu(A)$. Define, for every $\alpha \in [0,1]$, $A_\alpha = \bigcup_{k/2^n \leq \alpha} A_{k/2^n}$. Then the family $(A_\alpha)_\alpha$ is obviously increasing, while it also refines A with respect to ν, because

$$\nu_i(A) = \sup_{k/2^n \leq \alpha} \nu_i(A_{k/2^n}) = \alpha\nu_i(A),$$

for $i = 1,2$. □

Proof of b). This claim will be proven if we prove the following statement, which is itself of interest :

b_1) let ν be a convex measure and let $A,B \in \mathcal{F}$; then for every $\lambda \in [0,1]$ there exist a measurable set $C(\lambda)$ such that :

1) $C(0) = A$; $C(1) = B$;

2) $\nu(C(\lambda)) = (1 - \lambda)\,\nu(A) + \lambda\nu(B)$.

To prove b_1), let $(A_\alpha)_\alpha$ (resp. $(B_\alpha)_\alpha$) be a family refining $A \setminus B$ (resp. $B \setminus A$) with respect to ν . Define

$$C(\lambda) = (A \cap B) \dot{\cup} A_{1-\lambda} \dot{\cup} B_\lambda \quad ;$$

then, 1) is obviuos, while 2) follows from

$$\begin{aligned}(C(\lambda)) &= \nu(A \cap B) + (1 - \lambda)\,\nu(A \setminus B) + \lambda(B \setminus A)\\ &= (1 - \lambda)[\nu(A \cap B) + \nu(A \setminus B)] + \lambda[\nu(A \cap B) + \nu(B \setminus A)]\\ &= (1 - \lambda)\,\nu(A) + \lambda\nu(B) \quad .\end{aligned}$$

□

Proof of c). The proof will be carried by induction. First we show the following proposition :

c_1) the range of a nonatomic numerical measure ν is the closed interval $[0,\nu(T)]$.

We begin by showing that every $A \in \mathfrak{F}$ such that $\nu(A) > 0$ contains measurable subsets of arbitrarily small positive measure. In fact, by definition A contains some $B \in \mathfrak{F}$ such that $0 < \nu(B) < \nu(A)$; among the sets B and $A \setminus B$, call A_1 the set whose measure ν is not bigger than $\nu(A)/2$. Similarly we can construct a set $A_2 \subseteq A_1$ such that $0 < \nu(A_2) < \nu(A_1)/2$, and so on by induction.

Let now $\alpha \in]0, (T)[$: we can find an $A_1 \in \mathfrak{F}$ such that $0 < \nu(A_1) \leq \alpha$. If the equality holds, we have finished; otherwise, suppose to have constructed a family of pairwise disjoint measurable sets $(A_i)_{i \in I}$ such that $\sum_{i \in I} \nu(A_i) < \alpha$ ($\Rightarrow \# I \leq \aleph_0$) : we can find a measurable $A_I \subseteq T \setminus \bigcup_{i \in I} A_i$ such that

$$0 < \nu(A_I) \leq \alpha - \sum_{i \in I} \nu(A_I) ,$$

and so on by transfinite induction. In this way we obtain a countable family of pairwise disjoint sets the union of which has measure α . □

Let now $\nu = (\nu_1, \nu_2)$ be two-dimensional. We can suppose that ν_2 is absolutely continuous with respect to ν_1 : in fact we can study the measure $\nu' = (\nu_1 + \nu_2, \nu_2)$, which has the preceeding property, and note that

$$\nu(B) = \tfrac{1}{2} \nu(A) \Longleftrightarrow \nu'(B) = \tfrac{1}{2} \nu'(A).$$

Our next step is therefore the following :

c_2) let ν' be a measure which is absolutely continuous with respect to ν and let $(A_\alpha)_\alpha$ be a family refining $A \in \mathfrak{F}$ with respect to ν . Then the function $\alpha \to \nu'(A_\alpha)$ is continuous.

In fact, let $\varepsilon > 0$ and choose $\delta > 0$ such that $\forall B \in \mathfrak{F}$

$$|\nu(B)| < \delta \Rightarrow |\nu'(B)| < \varepsilon \quad .$$

If $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ and $\alpha_2 - \alpha_1 < \delta/|\nu(A)|$, then

$$|\nu(A_{\alpha_2} \setminus A_{\alpha_1})| = (\alpha_2 - \alpha_1)|\nu(A)| < \delta$$

and therefore $|\nu'(A_{\alpha_2}) - \nu'(A_{\alpha_1})| = |\nu'(A_{\alpha_2} \setminus A_{\alpha_1})| < \varepsilon$. □

End of the proof of c). Let $A \in \mathfrak{F}$; then, by c_1) there exists a measurable $B \subseteq A$ such that $\nu_1(B) = \frac{1}{2} \nu_1(A)$. If $\nu_2(B) = \frac{1}{2} \nu_2(A)$, there is nothing to prove. Otherwise, we can assume that $\nu_2(B) < \frac{1}{2} \nu_2(A)$ and $\nu_2(A \setminus B) > \frac{1}{2} \nu_2(A)$. Applying proposition b_1) to the measure ν and to the sets B, $A \setminus B$, we obtain a family $C(\lambda)$ of subsets of A such that $\nu_1(C(\lambda)) \equiv \frac{1}{2} \nu_1(A)$, $\nu_2(C(0)) < \frac{1}{2} \nu_2(A)$ and $\nu_2(C(1)) > \frac{1}{2} \nu_2(A)$. Since $\nu_2 << \nu_1$, the continuity property shown in b_2) yields a $\bar{\lambda}$ such that $\nu_2(C(\bar{\lambda})) = \frac{1}{2} \nu_2(A)$ and the proof is concluded. ■

<u>COROLLARY 2.4</u> (Fryszkowski [11,Prop.1.1]). Let μ be a positive and finite measure over $\mathfrak{F}$ and let $\nu = (\nu_1, \dots, \nu_n)$ be a measure absolutely continuous with respect to μ . Then if (and only if) μ is nonatomic there exists a family $(A_\alpha)_{\alpha \in [0,1]}$ refining T with respect to $(\nu_1, \dots, \nu_n, \mu)$.

<u>Proof</u>. Apply parts a) and c) of the proof of Theorem 2.1 to the measure $(\nu_1, \dots, \nu_n, \mu)$. ■

2.3 THE CASE OF INFINITELY MANY MEASURES.

<u>EXAMPLE 2.5</u> (Liapunov [17]). A ℓ_2-valued (hence with infinitely many components) nonatomic measure of bounded variation whose range is not convex.

Let $\mathfrak{F}$ be the Borel subsets of $[0,2\pi]$ and m be the Lebesgue measure. Take a complete orthogonal system $(w_n)_{n \in \mathbb{N}}$ in $L_2([0,2\pi];\mathbb{R})$ such that every w_n

assumes only the values ± 1 and such that $w_o = \chi_{[0,2\pi]}$, while $\int_0^{2\pi} w_n \, dm = 0$ for $n \geq 1$ (e.g. the Walsh functions). For each n, define ν_n on $\mathfrak{F}$ by setting

$$\nu_n(E) = 1/2^n \int_E [1 + w_n(t)]/2 \, dm \qquad , E \in \mathfrak{F} .$$

Define $\nu: \mathfrak{F} \to \ell_2$ as

$$\nu(E) = (\nu_o(E), \nu_1(E), \ldots) \quad .$$

Then $\| \nu(E) \|_2 \leq 2 \, m(E)$ for every $E \in \mathfrak{F}$, therefore ν is a vector measure of bounded variation which is clearly nonatomic. We will prove that the range of ν is not convex by showing that there does not exist a set $\bar{E} \in \mathfrak{F}$ such that $\nu(\bar{E}) = \nu([0,2\pi])/2 = (\pi, \pi/4, \pi/8, \ldots)$. Notice first that $\nu_o(E) = m(E)$ and $\nu_n(E) = \frac{1}{2^n} m(E \cap U_n)$, where $U_n = \{ s \in [0,2\pi] : w_n(s) = +1 \}$, for every $E \in \mathfrak{F}$. Suppose that $\bar{E}$ exists : then we have that $\pi = m(\bar{E})$ and $\pi/2^{n+1} = m(\bar{E} \cap U_n)/2^n$; since $m(U_n) = m(\bar{E}) = \pi$, it follows that

$$m(\bar{E} \cap U_n) = \pi/2 = m(\bar{E} \setminus U_n) = m(U_n \setminus \bar{E}) = m([0,2\pi] \setminus (\bar{E} \cup U_n))$$

for $n \geq 1$. Define $f = \chi_{\bar{E}} - \chi_{[0,2\pi] \setminus \bar{E}}$: we have that $\int_0^{2\pi} f \, w_o \, dm = \pi - \pi = 0$ and, for $n \geq 1$, that

$$\int_0^{2\pi} f \, w_n \, dm = m(U_n \cap \bar{E}) + m([0,2\pi] \setminus (\bar{E} \cup U_n))$$
$$= m(U_n \cap \bar{E}) - m(\bar{E} \setminus U_n) - m(U_n \setminus \bar{E}) + m([0,2\pi] \setminus (\bar{E} \cup U_n)) = 0.$$

Since $f \in L_2([0,2\pi])$ and $f \neq 0$, this contradicts the completeness of the system $(w_n)_n$. ∎

This example shows that Corollary 2.4 is false when ν takes values in an infinitely dimensional vector space. The following theorems extend Corollary 2.4 to the infinite case by allowing the family $(A_\alpha)_\alpha$ to be non costant and to continuously depend on the components ν_i of ν. The meaning of "continuous" will be precised in each statement. Up to now, a fully constructive method to obtain a refining family $(A_\alpha^{\nu_i})_\alpha$ from a given measure ν_i does not seem to exist. Hence, the continuity of $(A_\alpha^{\nu_i})_\alpha$ with respect to ν_i, which is necessary to provide a continuous interpolation method, is the core of the infinitely dimensional case. In Theorem 2.6 this problem is solved, for countably many measures, by adding a new real parameter τ and by a more precise choice of the $(A_\alpha^{\nu_i})_\alpha$. In Theorem 2.9 a slightly different kind of continuity follows from a complicated interpolation method among measurable sets which is originated by Halmos' proof of Liapunov's Convexity Theorem.

The two following results can be found in [4,Lemma 4.1,4.2] .

THEOREM 2.6. Let $(T,\mathcal{F},\mu)$ be a measure space with a σ-algebra $\mathcal{F}$ of subsets of T and a non-atomic probability measure μ on $\mathcal{F}$. Let $(g_n)_{n\geq 0}$ be a sequence of non-negative functions in $L^1(T;\mathbb{R})$ with $g_0 \equiv 1$. Then there exists a map $\Phi: \mathbb{R}^+\times[0,1] \to \mathcal{F}$ with the following properties:

a) $\Phi(\tau,\lambda_1) \subseteq \Phi(\tau,\lambda_2)$ if $\lambda_1 \leq \lambda_2$,

b) $\mu(\Phi(\tau_1,\lambda_1) \,\Delta\, \Phi(\tau_2,\lambda_2)) \leq |\lambda_1 - \lambda_2| + 2|\tau_1 - \tau_2|$,

c) $\int_{\Phi(\tau,\lambda)} g_n \, d\mu = \lambda \cdot \int_T g_n \, d\mu$, $\forall n \leq \tau$,

for all $\lambda, \lambda_1, \lambda_2 \in [0,1]$, $\tau, \tau_1, \tau_2 \geq 0$.

Proof. The Theorem will be proved first in a special case, assuming that

$$\int_T g_n \, d\mu = 1 \qquad \forall n \geq 0 . \tag{2.5}$$

By induction on n, we shall define a sequence of families of measurable sets $\{A_\lambda^n ; \lambda \in [0,1]\}$, $n \geq 0$, and a decreasing sequence of σ-algebras $\mathcal{F}^n \subseteq \mathcal{F}$ with the following properties:

i) $\mu(A_\lambda^n) = \lambda$,

ii) $A_\lambda^n \in \mathcal{F}^{n-1}$,

iii) $A^n_{\lambda_1} \subseteq A^n_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$,

iv) $\mathscr{F}^n = \sigma\{A^n_\lambda \; ; \; \lambda \in [0,1]\}$,

v) $\mu(A) = \int_A g_i \, d\mu$ whenever $A \in \mathscr{F}^n$, $i \leq n$.

To do this, using Liapunov's theorem 2.1 , construct a family of sets $\{A^0_\lambda\}$ such that i) and iii) hold for $n = 0$, and let $\mathscr{F}^0$ be the σ-algebra generated by the sets A^0_λ . Let now A^m_λ be defined for all $\lambda \in [0,1]$ and all $m \leq n-1$, so that the properties i) ÷ v) hold. Apply Liapunov's theorem to the two non-atomic measures μ and $\mu_n = g_n \cdot \mu$, on the measurable space $(T, \mathscr{F}^{n-1})$. This yields a family of sets $\{ A^n_\lambda \; ; \; \lambda \in [0,1] \}$ such that i), ii) and iii) hold for n. Define $\mathscr{F}^n$ by iv). We then have

$$\mu(A) = \int_A g_n \, d\mu \qquad (A \in \mathscr{F}^n) \quad ,$$

because the equality holds whenever $A = A^n_\lambda$ for some λ , and the family of sets $\{A^n_\lambda \; ; \; \lambda \in [0,1]\}$ is increasing and, by definition, generates $\mathscr{F}^n$.

If $i < n$, then $A \in \mathscr{F}^n$ implies $A \in \mathscr{F}^i \supseteq \mathscr{F}^n$, hence v) is a consequence of the inductive hypothesis .

We now define the sets $\Phi(\tau,\lambda)$ as follows.

If τ is an integer, $\Phi(\tau,\lambda) = A^\tau_\lambda$.

If $\tau = n + \delta$ with n integer, $0 < \delta < 1$, we consider two cases: when $\lambda \leq \delta$ we set $\Phi(\tau,\lambda) = A^{n+1}_\lambda$; when $\lambda > \delta$ we set $\Phi(\tau,\lambda) = A^{n+1}_\delta \cup A^n_\xi$, where ξ is the smallest number in $[0,1]$ for which the equality $\mu(A^{n+1}_\delta \cup A^n_\xi) = \lambda$ holds.

Notice that for any n, δ the function

$$\xi \to \psi(\xi) = \mu(A^{n+1}_\delta \cup A^n_\xi)$$

is Lipschitz-continuous and non-decreasing, with $\psi(0) = \delta$, $\psi(1) = 1$. In the case $\delta < \lambda \leq 1$, the set $\{ \xi \in [0,1] ; \psi(\xi) = \lambda \}$ is non-empty, closed and connected, hence it contains a minimal element. The map Φ is thus well defined.

The verification of a) is elementary. By construction, we also have

$$\mu(\Phi (\tau,\lambda)) = \lambda \qquad \forall \tau \geq 0 , \quad \lambda \in [0,1] \quad . \tag{2.6}$$

Observe that on $\mathscr{F}^n$ the measures $g_1 \cdot \mu , \dots , g_n \cdot \mu$ all coincide with $\mu = g_0 \cdot \mu$, because of v). Since $\Phi(\tau,\lambda) \in \mathscr{F}^n$ whenever $\tau \geq n$, (2.6) implies c). To prove b) notice that a) and (2.6) together yield

$$\mu(\Phi (\tau, \lambda_1) \; \Delta \; \Phi (\tau, \lambda_2)) = |\lambda_1 - \lambda_2| \qquad \forall \tau, \lambda_1, \lambda_2 \quad .$$

Therefore, to establish b), it suffices to prove the inequality

$$\mu(\Phi(\tau_1,\lambda) \; \Delta \; \Phi (\tau_2,\lambda)) \leq 2|\tau_1 - \tau_2| . \tag{2.7}$$

Moreover, we can assume that $\tau_1 < \tau_2$ and that τ_1, τ_2 both belong to the same interval $[n,n+1]$. For i = 1,2 , set $\delta_i = \tau_i - n$ and, if $\lambda > \delta_i$, let $\Phi(\tau_i,\lambda) = A^{n+1}_{\delta_i} \cup A^n_{\xi_i}$. Three cases must be considered.

1) If $\lambda \leq \delta_1 < \delta_2$, then $\Phi(\tau_1,\lambda) = \Phi(\tau_2,\lambda) = A^{n+1}_\lambda$ and (2.7) holds trivially.

2) If $\delta_1 \leq \lambda \leq \delta_2$, then $\mu(\Phi(\tau_1,\lambda) \; \Delta \; \Phi(\tau_2,\lambda)) \leq \mu((A^{n+1}_{\delta_1} \cup A^n_{\xi_1}) \; \Delta \; A^{n+1}_{\delta_1}) + \mu(A^{n+1}_{\delta_1} \; \Delta \; A^{n+1}_{\delta_2}) = (\lambda - \delta_1) + (\delta_2 - \delta_1) \leq 2 (\delta_2 - \delta_1) = 2|\tau_1 - \tau_2|$.

3) If $\delta_1 < \delta_2 \leq \lambda$, observe that $A^{n+1}_{\delta_1} \subseteq A^{n+1}_{\delta_2}$ and $A^n_{\xi_1} \supseteq A^n_{\xi_2}$. Using these relations, we obtain

$$\begin{aligned}
&\mu(\Phi(\tau_2,\lambda) \setminus \Phi(\tau_1,\lambda)) + \mu(\Phi(\tau_1,\lambda) \setminus \Phi(\tau_2,\lambda)) \\
&\leq \mu(A^{n+1}_{\delta_2} \setminus A^{n+1}_{\delta_1}) + \mu(\Phi(\tau_1,\lambda) \setminus (A^{n+1}_{\delta_1} \cup A^n_{\xi_2})) \\
&= (\delta_2 - \delta_1) + \mu(\Phi(\tau_1,\lambda)) - \mu(A^{n+1}_{\delta_1} \cup A^n_{\xi_2}) \\
&\leq (\delta_2 - \delta_1) + \lambda - [\lambda - (\delta_2 - \delta_1)] = 2|\tau_1 - \tau_2| .
\end{aligned}$$

Here the last inequality is deduced from the inclusion

$$(A_{\delta_2}^{n+1} \cup A_{\xi_2}^{n}) \setminus (A_{\delta_2}^{n+1} \setminus A_{\delta_1}^{n+1}) \subseteq (A_{\delta_1}^{n+1} \cup A_{\xi_2}^{n}) .$$

The above estimates complete the proof of Theorem 2.6 under the additional assumption (2.5).

To treat the general case, for each $n \geq 0$, set $\hat{g}_n \equiv 1$ if $g_n = 0$ μ-almost everywhere; otherwise define $\hat{g}_n = [\int_T g_n \, d\mu]^{-1} \cdot g_n$. If $\{\Phi(\tau,\lambda)\}$ is a family of sets which satisfy a) ÷ c) for the sequence $(\hat{g}_n)$, one can easily check that these same sets satisfy a) ÷ c) for the sequence (g_n) as well. ■

COROLLARY 2.7. Let X be a separable metric space, and let $\phi_n : X \to L^1(T;\mathbb{R})$, $h_n : X \to [0,1]$ $(n \geq 1)$, be two sequences of continuous functions, with $\phi_n(x)(t) \geq 0$ $\forall x \in X$, $\forall t \in T$, and such that $\{ \text{supp}(h_n) ; n \geq 1\}$ is a locally finite (closed) covering of X . Then, for every $\varepsilon > 0$ and every continuous, strictly positive function $\ell : X \to \mathbb{R}^+$, there exist a continuous function $\tau : X \to \mathbb{R}^+$ and a map $\Phi : \mathbb{R}^+ \times [0,1] \to \mathcal{F}$ which satisfy conditions a), b) in THEOREM 2.6 together with

c') for all $x \in X$, $\lambda \in [0,1]$ and $n \geq 1$, if $h_n(x) = 1$ then

$$\left| \int_{\Phi(\tau(x),\lambda)} \phi_n(x) \, d\mu - \lambda \cdot \int_T \phi_n(x) \, d\mu \right| < \varepsilon / 4\ell(x) .$$

Proof. Let $\varepsilon > 0$ and be given. For every $x \in X$, choose an open neighborhood U_x of x which intersects the supports of finitely many functions h_n , so that the set of indexes $I_x = \{ n ; U_x \cap \text{supp}(h_n) \neq \emptyset \}$ is finite. Set $\psi_n(x) = h_n(x) \cdot \phi_n(x) \in L^1(T;R)$ and define

$$V_x = \{ x' \in U_x ; \|\psi_n(x') - \psi_n(x)\|_1 < \varepsilon / 8\ell(x) , \forall n \in I_x \} . \tag{2.8}$$

The family $\{ V_x ; x \in X \}$ is an open covering of the paracompact separable space X. Hence, there exists a sequence of functions $k_m : X \to [0,1]$ such that the family $\{ \operatorname{supp}(k_m) ; m \geq 1 \}$ is a countable nbd-finite refinement of $\{V_x\}$ and the sets $W_m = \{ x \in X ; k_m(x) = 1 \}$ still cover X. For all $m \geq 1$, select x_m such that $W_m \subseteq V_{x_m}$. Define the sequence $(g_j)_{j \geq 0}$ in $L^1(T;\mathbb{R})$ by setting : $g_j = \psi_n(x_m)$ if $j = 2^m \cdot 3^n$ for some integers $m,n \geq 1$; $g_j \equiv 1$ otherwise. Moreover, set

$$\tau(x) = \sum_{m,n \geq 1} k_m(x) \cdot h_n(x) \cdot 2^m \cdot 3^n \quad . \tag{2.9}$$

The function τ is continuous, because the summation in (2.9) is locally finite.

Using Theorem 2.6, construct a map Φ which satisfies a) ÷ c) for the sequence $(g_j)_{j \geq 0}$. We claim that c') holds as well. To see this, fix $x \in X$, $n \geq 1$ and $\lambda \in [0,1]$. For some index m, $x \in W_m$.

If $h_n(x) = 1$, then

$$\left| \int_{\Phi(\tau(x),\lambda)} \phi_n(x)\, d\mu - \lambda \int_T \phi_n(x)\, d\mu \right|$$

$$\leq \int_{\Phi(\tau(x),\lambda)} | \psi_n(x) - \psi_n(x_m) |\, d\mu + \left| \int_{\Phi(\tau(x),\lambda)} \psi_n(x_m)\, d\mu - \lambda \int_T \psi_n(x_m) d\mu \right| +$$

$$+ \lambda \int_T | \psi_n(x_m) - \psi_n(x) |\, d\mu$$

$$\leq 2 \| \psi_n(x) - \psi_n(x_m) \|_1 + \left| \int_{\Phi(\tau(x),\lambda)} g_{2^m \cdot 3^n}\, d\mu - \lambda \int_T g_{2^m \cdot 3^n}\, d\mu \right| .$$

By (2.8), since $x \in V_{x_m}$, the first term of this last expression is less than $\varepsilon/4\, \ell(x)$, while the second term vanishes because $\tau(x) \geq 2^m \cdot 3^n$, by (2.9). ■

PROPOSITION 2.8. Let $\Phi : \mathbb{R}^+ \times [0,1] \to \mathcal{F}$ be a map satisfying a) and b) of Theorem 2.6 and such that $\mu(\Phi(\tau,\lambda)) = \lambda$ for each τ. Let moreover X be a topological space and $\tau : X \to \mathbb{R}^+$, $\lambda : X \to [0,1]$, $\phi : X \to L^1(T;E)$ be continuous maps. Then the map $\psi : X \to L^1(T;E)$ defined by $\psi(x) = \phi(x)\chi_{\Phi(\tau(x),\lambda(x))}$ is continuous.

Proof. The characteristic function of the sets $\Phi(\tau,\lambda)$ varies continuously in $L^1(T;\mathbb{R})$, since

$$\| \chi_{\Phi(\tau,\lambda)} - \chi_{\Phi(\tau_o,\lambda_o)} \|_1 = \mu(\Phi(\tau,\lambda) \,\Delta\, \Phi(\tau_o,\lambda_o)) ,$$

by a) of Theorem 2.6. ■

The last result of the chapter is due to Cellina, Colombo and Fonda [7] .

THEOREM 2.9. Let $\mathfrak{M}$ be the set of positive finite measures ν on $\mathcal{F}$ which are absolutely continuous with respect to the nonatomic measure μ , with the metric induced by the norm $\|\nu\|$ given by the total variation of ν, and let X be a compact metric space. Let moreover $x \to \nu_x$ be a continuous map from X into $\mathfrak{M}$. Then, for every $x \in X$ there exists an increasing family $(A^x_\alpha)_\alpha$ of measurable subsets of T satisfying

$$\nu_x(A^x_\alpha) = \alpha\nu_x(T) \qquad \text{for all } \alpha\in[0,1]$$

and such that the map $x \to (A^x_\alpha)_\alpha$ is continuous, in the sense that for every $x_o \in X$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x, x' \text{ and } x'' \text{ in } B(x_o,\delta) \text{ implies } \sup_{\alpha\in[0,1]} \nu_x(A^{x'}_\alpha \,\Delta\, A^{x''}_\alpha) < \varepsilon .$$

3. TWO SELECTION THEOREMS.

3.1 INTRODUCTION.

The results to which this chapter is devoted have their origin in a paper by Antosiewicz and Cellina [1] , where the existence of solutions to a Cauchy Problem for differential inclusions without convexity of the right-hand side is proven. Their approach is based on a continuous selection theorem for the map

$$G : K \to 2^{L^1(T;\mathbb{R}^n)}$$

$$x \to \{ u \in L^1(T;\mathbb{R}^n) : u(t) \in F(t,x(t)) \text{ a.e. on } I \} \quad ,$$

where I is a compact interval and K is a compact set of Lipschitzean functions from I into $\mathbb{R}^n$. This map is easily seen to have decomposable values. In the proof, an interpolation formula of the type (2.2) is used.

More recently [11] , Fryszkowski introduced explicitely the decomposability in the multivalued maps framework and, by using Liapunov's Convexity Theorem, stated an abstract version of the Antosiewicz and Cellina selection theorem, valid for lower semicontinuous maps. The use of Liapunov's theorem provides almost for free the interpolation formula; hence the complicated arguments which had to be employed in different versions of Antosiewicz-Cellina theorem (see [3] and [8]) are avoided. The Friszkowski's abstract setting can be used to prove the existence of solutions to differential inclusions (see [9]). The compactness hypothesis on the domain of the function, which is essential in Friszkowski's paper, can be removed by using Thorem2.6. Hence, in the next section we prove a continuous selection theorem with only a separability assumption on the domain.

A parallel theory for upper semicontinuous maps with decomposable values hs been also developed. As it is well known, upper semicontinuous maps admit, in general, only approximate selections. The analogue of Friszkowski's theorem was proven by Cellina, Colombo and Fonda [6] . In the last section we prove a theorem, with no compactness assumptions, which is the counterpart of that one in Section 2. Both Sections 2 and 3 are contained in a paper by Bressan and Colombo [4] .

These two results are, again, almost precisely analogous to two theorems which are valid for maps with convex values. We state them here for reference.

<u>THEOREM 3.A</u> (Michael's Selection Theorem [19]). Let X be a paracompact space and Y be a Banach space and let F from X into the closed convex subsets of Y be lower semicontinuous. Then there exists $f : X \to Y$, a continuous selection from F.

<u>THEOREM 3.B</u> (Cellina's Approximate Selection Theorem [2,p. 84]). Let X be a metric space, Y a Banach space and F a Hausdorff-upper semicontinuous map from X into the convex subsets of Y. Then, for every $\varepsilon > 0$ there exists a locally Lipschitzean map $f_\varepsilon : X \to Y$ such that

$$\text{graph } (f_\varepsilon) \subseteq \text{graph } (F) + \varepsilon B \quad ,$$

i.e. f_ε is an ε-approximate selection of F, and $f_\varepsilon(X) \subseteq \text{co } F(X)$.

We recall that a multifunction $F : X \to 2^Y$ is lower semicontinuous (l.s.c.) iff the set $F^+(C) = \{ x \in X : F(x) \subseteq C \}$ is closed for every closed set $C \subseteq Y$. A map $F : X \to 2^Y$ is Hausdorff-upper semicontinuous (H-u.s.c.) iff, for every $x_0 \in X$ and every $\varepsilon > 0$, there exists a neighbourhood V of x_0 such that $F(x) \subseteq B(F(x_0),\varepsilon)$ for all $x \in V$.

3.2. THE LOWER SEMICONTINUOUS CASE.

THEOREM 3.1. Let X be a separable metric space and let $F : X \to D(L^1(T;E))$ be a l.s.c. multifunction with closed, decomposable values. Then F has a continuous selection.

PROOF OF THEOREM 3.1.

In what follows, the main arguments are taken from [11] . We list first some preliminary results.

PROPOSITION 3.2. For every family $\mathcal{H}$ of non-negative measurable functions $u : T \to \mathbb{R}^+$, there exists a measurable function $v : T \to \mathbb{R}^+$ such that

i) $v \leq u$ μ-a.e. for all $u \in \mathcal{H}$,

ii) if w is a measurable function such that $w \leq u$ μ-a.e. for all $u \in \mathcal{H}$, then $w \leq v$ a.e. .

Furthermore, there exists a sequence (u_n) in $\mathcal{H}$ such that

$$v(t) = \inf \{ u_n(t) \ ; \ n \geq 1 \} \quad \text{for a.e. } t \text{ in } T.$$

If the family $\mathcal{H}$ is directed downwards (i.e., if for every $u, u' \in \mathcal{H}$ there exists $w \in \mathcal{H}$ such that $w \leq u$ and $w \leq u'$ μ-a.e.), then the sequence (u_n) can be chosen to be decreasing.

For the proof, see Neveu [20, p. 121] .

By ii), the function v is unique up to μ-equivalence. It represents the greatest lower bound of $\mathcal{K}$ in the sense of μ-a.e. inequality, and is denoted by ess inf $\{u \; ; \; u \in \mathcal{K}\}$.

PROPOSITION 3.3. Let K be a nonempty, closed decomposable subset of $L^1(T;E)$ and let $\psi(t) = \text{ess inf}\{ \|u(t)\|_E \; ; \; u \in K\}$. Then, for every $v_\circ \in L^1(T;\mathbb{R})$ such that $v_\circ(t) > \psi(t)$ a.e. , there exists an element $u_\circ \in K$ such that

$$\| u_\circ(t)\|_E < v_\circ(t) \qquad \mu\text{-a.e.} \qquad . \tag{3.1}$$

Proof. Notice that the set $\mathcal{K} = \{\|u(\cdot)\|_E \; ; \; u \in K\}$ is a decomposable subset of $L^1(T;\mathbb{R})$. Therefore, it is directed downwards. Using Proposition 3.2, take a sequence $(u_n)_{n\geq 1}$ in K such that

$$\| u_m(t)\|_E \geq \|u_n(t)\|_E \qquad \forall \, m < n \; , \; t \in T \; ,$$

$$\psi(t) = \lim_{n\to\infty} \|u_n(t)\|_E \qquad \mu\text{-a.e.} \qquad .$$

Let now $v_\circ$ be given, with $v_\circ(t) > \psi(t)$ a.e. , and define the increasing sequence of sets : $T_\circ = \emptyset$, $T_n = \{ t \in T \; ; \; \|u_n(t)\|_E < v_\circ(t) \}$, $n \geq 1$.

Observe that $\mu(T \setminus \bigcup_{n\geq 0} T_n) = 0$. Define the sequence (w_n) by setting

$$w_n(t) = \begin{cases} u_k(t) & \text{if } t \in T_k \setminus T_{k-1} \; , \; k = 1, \ldots, n-1 \; , \\ u_n(t) & \text{if } t \in T \setminus \bigcup_{k<n} T_k \; . \end{cases}$$

Since K is decomposable, each w_n belongs to K . Moreover, the sequence $w_n(t)$ is eventually constant for a.e. $t \in T$, and $\|w_n(t)\|_E \leq \|u_1(t)\|_E$ μ-a.e. ;

hence, by the Dominated Convergence Theorem, w_n converges in $L^1(T;E)$ to some function $u_\circ$. Clearly, $u_\circ \in K$ because K is closed. Finally, if $t \in T_n \setminus T_{n-1}$ for some n, then $\|u_\circ(t)\|_E = \|u_n(t)\|_E < v_\circ(t)$. Therefore, $u_\circ$ satisfies (3.1). ■

PROPOSITION 3.4. Let X be a metric space and let $F : X \to D(L^1(T;E))$ be a l.s.c. map with closed decomposable values. For all $x \in X$, set $\psi_x(t) =$ ess inf $\{\|u(t)\|_E \; ; \; u \in F(x)\}$. Then the multivalued map $P : X \to L^1(T;\mathbb{R})$ defined as

$$P(x) = \{ v \in L^1(T;\mathbb{R}) \; ; \; v(t) > \psi_x(t) \quad \mu\text{-a.e.} \} \tag{3.2}$$

is lower semicontinuous.

Proof. Let C be an arbitrary closed subset of $L^1(T;\mathbb{R})$. It suffices to show that, if $P(x_n) \subseteq C$ for some sequence $(x_n)_{n\geq 1}$ converging to $x_\circ$, then also $P(x_\circ) \subseteq C$. To this purpose, fix any $v_\circ \in P(x_\circ)$ and take, by Proposition 3.3, a function $u_\circ \in F(x_\circ)$ such that $\|u_\circ(t)\|_E < v_\circ(t)$ μ-a.e.. Because of the lower semicontinuity of F, there exists a sequence $u_n \in F(x_n)$ such that $u_n \to u_\circ$ in $L^1(T;E)$. Then, for every $n \geq 1$, the function $v_n = \|u_n\|_E + v_\circ - \|u_\circ\|_E$ belongs to $P(x_n)$ which is contained in C. Since the sequence (v_n) converges to $v_\circ$ in the norm of $L^1(T;\mathbb{R})$ and C is closed, this implies $v_\circ \in C$. ■

PROPOSITION 3.5. Let X be a metric space and let $G : X \to D(L^1(T;E))$ be a l.s.c. map with closed decomposable values. Assume that $g : X \to L^1(T;E)$ and

$\phi : X \to L^1(T;\mathbb{R})$ are continuous functions such that, for every $x \in X$, the set

$$H(x) = \{\, u \in G(x) \ ; \ \| u(t) - g(x)(t)\|_E < \phi(x)(t) \quad \mu\text{-a.e.} \}$$

is nonempty. Then the map $H : X \to D(L^1(T;E))$ is l.s.c. with decomposable values.

Proof. For every $x \in X$, $H(x)$ is the intersection of two decomposable sets, hence it is decomposable. To check the lower semicontinuity of H, let C be any closed subset of $L^1(T;E)$. It suffices to show that, for any sequence (x_n) in X converging to a point $x_\circ$, if $H(x_n) \subseteq C$ for all $n \geq 1$, then $H(x_\circ) \subseteq C$. To this purpose, fix any $u_\circ \in H(x_\circ)$. Because of the lower semicontinuity of G, there exists a sequence $u_n \in G(x_n)$ such that $u_n \to u_\circ$ in $L^1(T;E)$. By possibly taking a subsequence, we can assume that $u_n(t)$, $g(x_n)(t)$, $\phi(x_n)(t)$ converge to $u_\circ(t)$, $g(x_\circ)(t)$, $\phi(x_\circ)(t)$ respectively, μ-a.e. in T. Applying Egorov's theorem to these sequences w.r.t. the measure $\phi(x_\circ)\cdot\mu$, for each $i \geq 1$ we obtain a measurable set $T_i \subseteq T$ such that u_n, $g(x_n)$ and $\phi(x_n)$ converge uniformly on T_i and $\int_{T \setminus T_i} \phi(x_\circ)\, d\mu < 1/i$. For each $k \geq 1$, consider the sets

$$T_i^k = \{\, t \in T_i \ ; \ \|u_\circ(t) - g(x_\circ)(t)\|_E < \phi(x_\circ)(t) - 1/k \,\} .$$

Notice that $\bigcup_{k\geq 1} T_i^k = T_i$ and $T_i^k \subseteq T_i^{k+1}$. Hence, for every $i \geq 1$, there exists a k(i) such that

$$\int_{T_i \setminus T_i^{k(i)}} \phi(x_\circ)\, d\mu < 1/i \ .$$

Define $T_i' = T_i^{k(i)}$. The sets T_i' have the following properties :

$$\int_{T \setminus T_i'} \phi(x_\circ)\, d\mu \ < \ 2/i \ , \tag{3.3}$$

$$\| u_\circ(t) - g(x_\circ)(t) \|_E < \phi(x_\circ) - 1/k(i) \qquad , \quad \forall\, t \in T_i' \ . \tag{3.4}$$

By (3.4) and by the uniform convergence on T_i' , for all $i \geq 1$ there exists some n_i such that

$$\| u_n(t) - g(x_n)(t) \|_E < \phi(x_n)(t) \qquad \forall\, t \in T_i' \ , \ n \geq n_i . \tag{3.5}$$

We can also assume that the sequence $(n_i)_{i \geq 1}$ is strictly increasing. For each n, choose an arbitrary $w_n \in H(x_n)$ and set, for $n_i \leq n < n_{i+1}$, $v_n = u_n \cdot \chi_{T_i'} + w_n \cdot \chi_{T \setminus T_i'}$. Since $H(x_n)$ is decomposable, $v_n \in H(x_n)$. We claim that $v_n \to u_\circ$ in $L^1(T;E)$, which implies $u_\circ \in C$. Indeed, for $n_i \leq n < n_{i+1}$, (3.3) and (3.5) yield

$$\begin{aligned} \| v_n - u_\circ \|_1 \leq & \int_{T \setminus T_i'} \| w_n - g(x_n) \|_E \, d\mu + \int_{T \setminus T_i'} \| g(x_n) - g(x_\circ) \|_E \, d\mu \ + \\ & + \int_{T \setminus T_i'} \| g(x_\circ) - u_\circ \|_E \, d\mu + \int_{T_i'} \| u_n - u_\circ \|_E \, d\mu \\ \leq & \int_{T \setminus T_i'} \phi(x_n)\, d\mu + \| g(x_n) - g(x_\circ) \|_1 + \int_{T \setminus T_i'} \phi(x_\circ)\, d\mu + \| u_n - u_\circ \|_1 \\ \leq & \left[2/i + \| \phi(x_n) - \phi(x_\circ) \|_1 \right] + \| g(x_n) - g(x_\circ) \|_1 + 2/i + \| u_n - u_\circ \|_1 \ . \end{aligned}$$

As $n \to +\infty$, we also have $i \to +\infty$, hence our claim is proved.

■

The next result, concerning the existence of approximate selections, is the core of the whole proof of Theorem 3.1.

PROPOSITION 3.6 Let X be a separable metric space and let $G : X \to D(L^1(T;E))$ be a l.s.c. map with closed decomposable values. Then, for every $\varepsilon > 0$, there exist continuous maps $f_\varepsilon : X \to L^1(T;E)$ and $\phi_\varepsilon : X \to L^1(T;\mathbb{R})$ such that f_ε is an ε-approximate selection of G, in the sense that, for each $x \in X$, the set

$$G_\varepsilon(x) = \{ u \in G(x) \ ; \ \| u(t) - f_\varepsilon(x)(t) \|_E < \phi_\varepsilon(x)(t) \quad \mu\text{-a.e.} \} \quad (3.6)$$

is non-empty, and $\| \phi_\varepsilon(x) \|_1 < \varepsilon$. Moreover, the map $x \to G_\varepsilon(x)$ is l.s.c. with decomposable values.

Proof. Fix $\varepsilon > 0$. For every $\bar{x} \in X$ and $\bar{u} \in G(\bar{x})$, the multivalued map Q defined as

$$Q(x) = \{ v \in L^1(T;\mathbb{R}) \ ; \ v(t) \geq \text{ess inf}\{ \|u(t) - \bar{u}(t)\|_E ; \ u \in G(x) \} \text{ for a.e. } t \in T \} \quad (3.7)$$

is l.s.c. with closed convex values. To see this, define

$F(x) = \{ u - \bar{u} \ ; \ u \in G(x) \}$. Then the map F is also l.s.c. with closed decomposable values. By Proposition 3.4, the multivalued map P defined in (3.2) is l.s.c. . Hence Q is also l.s.c. , because Q(x) is the closure of P(x), for all $x \in X$. It is therefore possible to apply Michael's theorem to Q and obtain a continuous selection $\phi_{\bar{x},\bar{u}}$ such that $\phi_{\bar{x},\bar{u}}(x) \in Q(x)$ for all $x \in X$ and $\phi_{\bar{x},\bar{u}}(\bar{x}) \equiv 0$. The family of sets

$$\{ \{ x \in X \; ; \|\phi_{\bar{x},\bar{u}}(x)\|_1 < \varepsilon/4\}; \quad \bar{x} \in X \, , \; \bar{u} \in G(\bar{x}) \}$$

is an open covering of the separable metric space X, therefore it has a countable nbd-finite open refinement $\{ V_n \; ; \; n \geq 1 \}$. Let $\{p_n(\cdot)\}$ be a continuous partition of unity subordinated to the covering $\{ V_n \}$ and let $\{h_n(\cdot)\}$ be a family of continuous functions from X into $[0,1]$ such that $h_n \equiv 1$ on $supp(p_n)$ and $supp(h_n) \subset V_n$. For every $n \geq 1$, choose x_n, u_n such that $V_n \subseteq \{x \; ; \|\phi_{x_n,u_n}(x)\|_1 < \varepsilon/4 \}$ and set $\phi_n = \phi_{x_n,u_n}$. The functions ϕ_n have the following properties :

$$\phi_n(x)(t) \geq \text{ess inf} \{ \| u(t) - u_n(t) \|_E \; ; \; u \in G(x) \} \quad , \tag{3.8}$$

$$p_n(x) \cdot \| \phi_n(x) \|_1 \leq p_n(x) \cdot \varepsilon/4 \qquad (x \in X, \; n \geq 1) \; . \tag{3.9}$$

Corollary 2.7, applied to the sequences $\{\phi_n\}$ and $\{h_n\}$, and to the function ℓ :

$\ell(x) = \sum_{n\geq 1} h_n(x)$, yields a continuous function $\tau \colon X \to \mathbb{R}^+$ and a family $\{\Phi(\tau,\lambda)\}$ of measurable subsets of T satisfiying a), b) and c').

It is now possible to construct the functions f_ε and ϕ_ε . Set $\lambda_0 \equiv 0$, $\lambda_n(x) = \sum_{m\leq n} p_m(x)$, and define

$$f_\varepsilon(x) = \sum_{n\geq 1} u_n \chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))} \quad ,$$

$$\phi_\varepsilon(x) = \varepsilon/4 + \sum_{n\geq 1} \phi_n(x) \cdot \chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))} \quad .$$

Clearly, f_ε and ϕ_ε are continuous, because the above summations are locally finite. Let G_ε be defined by (3.6). To check that the values of G_ε are non-empty, fix any $x \in X$. For every $n \geq 1$, use Proposition 3.3 and select $u_x^n \in G(x)$ such that

$$\|u_x^n(t) - u_n(t)\|_E < \varepsilon/4 + \text{ess inf}\{\|u(t) - u_n(t)\|_E \,;\, u \in G(x)\} \qquad (3.10)$$

μ-a.e. in T. Then

$$u_x = \sum_{n\geq 1} u_x^n \cdot \chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))}$$

lies in $G(x)$, because $G(x)$ is decomposable. We claim that $u_x \in G_\varepsilon(x)$. Indeed, (3.8) and (3.10) yield

$$\|u_x(t) - f_\varepsilon(x)(t)\|_E \leq \sum_{n\geq 1} \|u_x^n(t) - u_n(t)\|_E \cdot \chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))}(t)$$

$$< \phi_\varepsilon(x)(t) \qquad \mu\text{-a.e. in } T.$$

Hence $G_\varepsilon(x) \neq \emptyset$. Being the intersection of two decomposable sets, $G_\varepsilon(x)$ is also decomposable. The lower semicontinuity of G_ε follows from Proposition 3.5. To conclude the proof of Proposition 3.6, it now suffices to show that $\|\phi_\varepsilon(x)\|_1 < \varepsilon$ for every x. Set $I(x) = \{ n\geq 1 \,;\, p_n(x) > 0 \}$ and notice that $1 \leq \# I(x) \leq \ell(x)$. From c') in Corollary 2.7 and (3.9) we deduce

$$\|\phi_\varepsilon(x)\|_1 = \varepsilon/4 + \sum_{n\geq 1} \int_T \phi_n(x)\chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))}\, d\mu$$

$$< \varepsilon/4 + \sum_{n\in I(x)} \left[p_n(x)\cdot\|\phi_n(x)\|_1 + \varepsilon/2\cdot\ell(x)\right] \leq \varepsilon/4 + \left[\varepsilon/4 + \frac{\# I(x)\cdot\varepsilon}{2\,\ell(x)}\right] \leq \varepsilon .$$

■

At this stage, everything is ready for the completion of the proof of

Theorem 3.1.

Let the function F be given. Construct two sequences of continuous maps $f_n : X \to L^1(T;E)$ and $\phi_n : X \to L^1(T;\mathbb{R})$, and a sequence of l.s.c. multifunctions G_n with decomposable values , such that, for all $x \in X$ and $n \geq 1$,

i) $G_n(x) = \{u \in F(x) \ ; \ \|u(t) - f_n(x)(t)\|_E < \phi_n(x)(t) \ \mu\text{-a.e.}\} \neq \emptyset$,

ii) $\|f_n(x)(t) - f_{n-1}(x)(t)\|_E \leq \phi_n(x)(t) + \phi_{n-1}(x)(t)$ μ-a.e. in T $(n \geq 2)$,

iii) $\|\phi_n(x)\|_1 < 2^{-n}$.

To do this, define f_1 and ϕ_1 by applying Proposition 3.6 with G = F , $\varepsilon = 1/2$. Let now f_m , ϕ_m and G_m be defined so that i) ÷ iii) hold for all m = 1, ... , n-1. To construct f_n and ϕ_n , apply again Proposition 3.6 with $\varepsilon = 2^{-n}$, defining G(x) to be the closure of $G_{n-1}(x)$, for all x. By induction, the maps f_n , ϕ_n and G_n can be defined for all $n \geq 1$. By ii), the sequence $(f_n)_{n \geq 1}$ is Cauchy in the L^1-norm, hence it converges uniformly to some continuous function $f : X \to L^1(T;E)$. By i) and iii) , $d_{L^1}(f_n(x), F(x)) < 2^{-n}$. Since F(x) is closed, this implies that $f(x) \in F(x)$ for all $x \in X$, hence f is a selection of F.

∎

3.3. THE UPPER SEMICONTINUOUS CASE.

THEOREM 3.7. Let X be a metric space and let $F: X \to D(L^1(T;E))$ be a H-u.s.c. multifunction with decomposable values. If either X or $L^1(T;E)$ is separable, then for every $\varepsilon > 0$ there exists a continuous map $f_\varepsilon : X \to L^1(T;E)$ such that

$$\text{graph}\ \{f_\varepsilon\} \subseteq B\ (\text{graph}\ \{F\}\ , \varepsilon).$$

Moreover, $f_\varepsilon(X) \subseteq \text{dec}_{L^1}[F(X)]$.

Proof. We state first a technical lemma concerning paracompact spaces.

LEMMA 3.8. Let X be a paracompact topological space. For every $x \in X$, let U_x be an open neighborhood of x and let $M(x)$ be an integer number. Then there exists a continuous function $\tau : X \to \mathbb{R}$ such that $\tau(x) \geq \min \{ M(x') ; x \in U_{x'} \}$ for every $x \in X$.

Proof. Let $\{ V_i ; i \in I \}$ be an open nbd-finite refinement of the covering $\{U_x\}$, and let $\{ p_i(\cdot) ; i \in I \}$ be a continuous partition of unity subordinated to $\{V_i\}$. For each i, select a point x_i such that $V_i \subseteq U_{x_i}$. Define

$\tau(x) = \sum_{i \in I} p_i(x)\, M(x_i)$. Clearly, τ is continuous. Moreover,

$\tau(x) \geq \min \{ M(x_i) ; p_i(x) \neq 0 \} \geq \min \{ M(x_i) ; x \in U_{x_i} \}$

$\geq \min \{ M(x') ; x \in U_{x'} \}$.

PROOF OF THEOREM 3.7.

The following proof is an adaptation of the arguments given in [6] .

Assume first that $L^1(T;E)$ is separable. Fix $\varepsilon > 0$. For every $x \in X$, choose a number $\delta(x) \in]0,\varepsilon/6[$ such that $F(x') \subseteq B(F(x),\varepsilon/6)$ whenever $x' \in B(x,\delta(x))$. Let $\{ V_i ; i \in I\}$ be an open nbd-finite refinement of the covering $\{ B(x,\delta(x)/2) ; x \in X \}$ of X. For each i, choose $x_i \in X$ such that $V_i \subseteq B(x_i,\delta(x_i)/2)$ and select $u_i \in F(x_i)$. For $i,j \in I$, choose also $v_{i,j} \in F(x_j)$ such that

$$\| u_i - v_{i,j} \|_1 \leq \varepsilon/6 + \inf\{\|u_i - v\|_1 ; v \in F(x_j)\} = \varepsilon/6 + d_{L^1}(u_i, F(x_j)). \qquad (3.11)$$

Let $D = \{y_n ; n \geq 1\}$ be a countable dense subset of $F(X)$. For every $i \in I$ select a $y_{\nu(i)} \in D$ for which $\|u_i - y_{\nu(i)}\|_1 < \varepsilon/6$. The set D' of all functions $g \in L^1(T;R)$ of the form $g(t) = \|y_m(t) - y_n(t)\|$, $m,n \geq 1$, is countable. Arrange its elements into a sequence, say, $D' = \{g_k ; k \geq 1\}$. Let $\{ p_i(\cdot) ; i \in I\}$ be a continuous partition of unity subordinated to the covering $\{V_i\}$. For every $n \geq 1$, define the open set $W_n = \bigcup \{V_i ; \nu(i) = n\}$ and let $q_n(x) = \sum_{\nu(i)=n} p_i(x)$. Clearly, $\{q_n(\cdot) ; n \geq 1\}$ is a continuous partition of unity, subordinated to the nbd-finite open covering $\{W_n\}$. Define

$$\lambda_n(x) = \sum_{m \leq n} q_m(x) \quad , \quad (n \geq 0, x \in X) . \qquad (3.12)$$

For every $x \in X$, take an open neighborhood U_x of x which intersects finitely many sets V_i. Setting $I(U_x) = \{ i \in I : U_x \cap V_i \neq \emptyset \}$, this of course means that $N(x) = \# I(U_x)$ is a finite integer. For every couple of indexes $i,j \in I(U_x)$, choose a $y_{\nu(i,j,x)} \in D$ such that

$$\| y_{\nu(i,j,x)} - v_{i,j} \|_1 < \varepsilon / 6N(x). \qquad (3.13)$$

Let $M(x)$ be an integer so large that the set $\{g_k \; ; \; 1 \le k \le M(x)\}$ contains the finite set of functions $\{\|y_{\nu(i)} - y_{\nu(i,j,x)}\|_E \; ; \; i,j \in I(U_x)\} \subseteq D'$.

Applying Lemma 3.8 to the collection of neighborhoods $\{U_x \; ; \; x \in X\}$ and integers $M(x)$, we get the existence of a continuous function $\tau : X \to \mathbb{R}^+$ such that

$$\tau(x) \ge \min\{M(x') \; ; \; x \in U_{x'}\}. \tag{3.14}$$

Recalling (3.12), the map $f_\varepsilon : X \to L^1(T;\Xi)$ can now be defined by setting

$$f_\varepsilon(x) = \sum_{n\ge 1} y_n \cdot \chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))} \; . \tag{3.15}$$

Here $\{\Phi(\tau,\lambda)\}$ is the family of sets constructed in Theorem 2.6, relative to the sequence $(g_k)_{k\ge 1}$ of the elements of D'. It is easily checked that f_ε is continuous and takes values inside dec $\overline{F(X)}$. To show that f_ε is an ε-approximate selection, fix $x \in X$ and define $I(x) = \{i \in I \; ; \; p_i(x) \ne 0\}$, $J(x) = \{n \ge 1 \; ; \; q_n(x) \ne 0\}$. Notice that $\# J(x) \le \# I(x) < +\infty$. Since $I(x)$ is finite, there exists an $\hat{\imath} \in I(x)$ such that $\hat{\delta} = \delta(x_{\hat{\imath}}) = \max\{\delta(x_i) \; ; \; i \in I(x)\}$. For every $i \in I(x)$ we have that $x_i \in B(x_{\hat{\imath}}, \hat{\delta})$, hence

$$F(x_i) \subseteq B(F(x_{\hat{\imath}}), \varepsilon/6) \; . \tag{3.16}$$

Take a point $z \in X$ such that $x \in U_z$ and $M(z) = \min\{M(x') \; ; \; x \in U_{x'}\}$. For every $n \in J(x)$, select an index $i_n \in I(x) \subseteq I(U_z)$ such that $\nu(i_n) = n$. Define

$$w = \sum_{n\ge 1} y_{\nu(i_n,\hat{\imath},z)} \cdot \chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))} \; ,$$

$$w' = \sum_{n\ge 1} v_{i_n,\hat{\imath}} \cdot \chi_{\Phi(\tau(x),\lambda_n(x)) \setminus \Phi(\tau(x),\lambda_{n-1}(x))} \; .$$

Notice that $w' \in F(x_{\hat{i}})$. For every $n \in J(x)$, using (3.11), (3.13) and (3.16) we obtain

$$\|y_n - y_{\nu(i_n,i,z)}\|_1 \le \|y_n - u_{i_n}\|_1 + \|u_{i_n} - v_{i_n,i}\|_1 + \|v_{i_n,i} - y_{\nu(i_n,i,z)}\|_1$$

$$\le \varepsilon/6 + [\varepsilon/6 + d_{L^1}(u_{i_n}, F(x_{\hat{i}}))] + \varepsilon/(6\, N(z)) \le \tfrac{2}{3}\varepsilon . \tag{3.17}$$

Relying on the properties of the sets $\Phi(\tau,\lambda)$ and recalling that by (3.14) $\tau(x) \ge M(z)$, from (3.17) we deduce the estimates

$$\| f_\varepsilon(x) - w\|_1 = \sum_{n\ge 1} \int_T \|y_n - y_{\nu(i_n,\hat{i},z)}\|_E \cdot \chi_{\Phi(\tau(x),\lambda_n(x))\setminus\Phi(\tau(x),\lambda_{n-1}(x))} d\mu$$

$$= \sum_{n\ge 1} q_n(x)\cdot \|y_n - y_{\nu(i_n,\hat{i},z)}\|_1 \le \tfrac{2}{3}\varepsilon , \tag{3.18}$$

$$\|w - w'\|_1 = \sum_{n\ge 1} \int_T \|y_{\nu(i_n,\hat{i},z)} - v_{i_n,\hat{i}}\|_E \cdot \chi_{\Phi(\tau(x),\lambda_n(x))\setminus\Phi(\tau(x),\lambda_{n-1}(x))} d\mu$$

$$\le \sum_{n\in J(x)} \|y_{\nu(i_n,\hat{i},z)} - v_{i_n,\hat{i}}\|_1 \le \frac{\# J(x)\cdot\varepsilon}{6\, N(z)} \le \frac{\# I(U_z)\cdot\varepsilon}{6\, N(z)} = \varepsilon/6. \tag{3.19}$$

Putting together (3.18) and (3.19), one has

$$d_{X\times L^1}((x,f_\varepsilon(x)),(x_{\hat{i}},w'))$$

$$\le d_X(x,x_{\hat{i}}) + \|f_\varepsilon(x) - w\|_1 + \|w - w'\|_1 < \varepsilon/6 + 2\varepsilon/3 + \varepsilon/6 = \varepsilon .$$

Hence $(x,f_\varepsilon(x)) \in B(\mathrm{graph}(F),\varepsilon)$. This completes the proof in the case where $L^1(T;E)$ is separable .

When X is separable, a slight modification of the above arguments is needed. The nbd-finite open covering $\{V_i \,;\, i \in I\}$ of X is now countable, because of the separability assumption. It is therefore possible to define the countable set $D = \{u_i \,;\, i \in I\} \cup \{v_{i,j} \,;\, i,j \in I\}$ and arrange it into a sequence, say, $D = \{y_n \,;\, n \ge 1\}$. After this choice of the set D, the rest of the proof goes exactly as in the previous case. ∎

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