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TOPOLOGY OF MODULI SPACES OF FRAMED SHEAVES

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Contents

Abstract

This thesis is devoted to the study of some topological properties of moduli spaces of sheaves framed on an irreducible divisor.

As the moduli spaces of framed torsion-free sheaves on projective surfaces are in general not compact, they are in fact quasi-projective, we are interested in studying their homotopy type with respect to a compact proper subvariety.

We first perform this study for framed sheaves on the projective plane. We show that the moduli space $M(r, n)$ of framed torsion-free sheaves on $\mathbb{P}^2$ admits a Białynicki-Birula decomposition determined by the torus action. Using this decomposition we prove that $M(r, n)$ is homotopy equivalent to a compact irreducible invariant proper subvariety having the same fixed points set.

A generalization to framed sheaves on a nonsingular projective toric surface S is provided where we assume that there exists a projective morphism of toric surfaces $p : S \to \mathbb{P}^2$ of degree 1 and consider the framing sheaf to be supported on a divisor.

As a result, the moduli spaces M of framed torsion-free sheaves on a nonsingular projective toric surface has the homotopy type of a compact proper subvariety provided that we have a projective morphism from M onto the moduli space of ideal instantons $M_0(r, n)$ on S^4 which is equivariant with respect to the torus action.

Chapter 1

Introduction

This thesis is devoted to the study of some topological properties of moduli spaces of framed sheaves. We first perform this study for framed sheaves on the projective plane. Then a generalization to framed sheaves on a toric variety S is provided where we assume that there exists a projective morphism of toric varieties $p : S \to \mathbb{P}^2$ of degree 1. We will consider the framing sheaf to be supported on a divisor. As these moduli spaces are in general not compact, they are in fact quasi-projective varieties, we are interested in studying their homotopy type with respect to a compact proper subvariety.

Instantons and framed sheaves. The main motivation for the study of framed sheaves and their moduli spaces comes from physics and precisely from gauge theory when considering *framed instantons*. These are self-dual connections on a principle bundle $p : P \to X$ together with a *frame* that is a point in the fiber P_x on a fixed point $x \in X$.

A result by Donaldson [13] shows that there is a one-to-one correspondence between anti-self-dual instantons on the sphere S^4 framed at $\{\infty\} \in S^4$ and holomorphic bundles on the projective plane $\mathbb{P}^2$ framed at the line $\ell_\infty \subset \mathbb{P}^2$. Indeed instantons with instanton charge k on S^4 correspond to framed holomorphic vector bundles on $\mathbb{P}^2$ of rank r and second Chern class k. This means that the moduli space of instantons is isomorphic to the space $M_0^{\mathrm{reg}}(r,n)$ parametrizing locally free sheaves on $\mathbb{P}^2$ with rank r and second Chern class k.

The moduli space $M_0^{\mathrm{reg}}(r,n)$ is not compact and one can introduce the Uhlenbeck-Donaldson partial compactification $M_0(r,n)$ of $M_0^{\mathrm{reg}}(r,n)$. $M_0(r,n)$ is singular and we have a resolution of singularities $\pi : M(r,n) \to M_0(r,n)$ where $M(r,n)$ is the moduli space of framed torsion-free sheaves on $\mathbb{P}^2$ of rank r and second Chern class c_2. The latter moduli space contains $M_0^{\mathrm{reg}}(r,n)$ as an open subspace.

In [1] Atiyah, Hitchin, Drinfel'd and Manin described the moduli space of instantons in terms of linear data, the so called *ADHM data*. The moduli spaces above admit descriptions in ADHM data. This was generalized for framed bundles on $\mathbb{P}^2$ blown up at a point by King in [26], and for framed bundles on $\mathbb{P}^2$ blown-up along a finite set of points by Buchdahl in [9].

In [20], Henni gave an ADHM description to the moduli spaces of framed torsion-free sheaves on the projective plane $\mathbb{P}^2$ blown-up along a finite set of points. A similar result was established for framed torsion-free sheaves on Hirzebruch surfaces by Bartocci, Bruzzo and Rava in [3].

Moduli of framed sheaves. In [22, 23] Huybrechts and Lehn constructed the moduli space of framed sheaves on projectives curves and surfaces. More precisely, they gave a notion of stability of framed sheaves and proved the existence of a fine moduli space of stable framed sheaves on curves and on surfaces.

In [8] Bruzzo and Markushevich proved that there exists a fine moduli space of framed sheaves on projective surfaces with "good framing" and they proved this moduli space is a quasi-projective scheme.

In [7] Bruzzo, Markushevich and Tikhomirov constructed the Uhlenbeck-Donaldson compactification for the moduli space of μ-stable framed bundles. Furthermore, They showed the existence of a projective morphism from the moduli space of S-equivalence classes of semistable framed sheaves to this compactification.

Thesis results. In this thesis we will assume that the framing sheaf is trivial and is supported on an irreducible divisor. We first investigate the case of the moduli space of framed sheaves $M(r,n)$ on $\mathbb{P}^2$. In [34, Theorem 3.5(2)], Nakajima and Yoshioka stated that $M(r,n)$ is homotopy equivalent to a proper subvariety $\pi^{-1}(n[0])$ but the proof is rather obscure since it refers to papers which seem not to contain the claimed arguments. We then find interesting to give in this thesis a detailed proof of the homotopy equivalence. We first prove that both $M(r,n)$ and $\pi^{-1}(n[0])$ admit Białynicki-Birula decompositions, see Theorem 5.3.2 and Theorem 5.3.3. Using these results we show that the inclusion of $\pi^{-1}(n[0])$ in $M(r,n)$ induces isomorphisms of homology groups with integer coefficients, see Theorem 5.4.2. We then prove that both $M(r,n)$ and $\pi^{-1}(n[0])$ are simply connected, see Lemma 5.4.3 and Lemma 5.4.5. Now the homotopy equivalence between $M(r,n)$ and $\pi^{-1}(n[0])$ follows from the above results, see Theorem 5.4.6.

Furthermore, we generalize our investigation to the moduli space of framed sheaves $M(S)$ on toric surfaces S assuming that there exists a projective morphism of toric surfaces $p : S \to \mathbb{P}^2$ of degree 1. In Theorem 6.3.5 we construct a projective morphism from $M(S)$ to $M_0(r,n)$ and define a compact subvariety N of $M(S)$. Afterwards we proceed as in the previous case, namely, we prove that both $M(S)$ and N admit Białynicki-Birula decompositions, see Theorem 6.4.3 and Theorem 6.4.4. Using this results we show that the inclusion of N in $M(S)$ induces isomorphisms of homology groups with integer coefficients, see Theorem 6.4.6. We then prove that both $M(S)$ and N are simply connected, see Lemma 6.4.7. Hence the homotopy equivalence between $M(S)$ and N follows from the above results, see Theorem 6.4.9.

Thesis outline. This thesis is organized as follows.

In chapter 2, we present the basic background on framed sheaves namely the notion of (semi)stability, μ-(semi)stability of coherent sheaves and their Jordan-Hölder filtrations. Then we introduce framed sheaves, the different notions of stability and the Jordan-Hölder filtrations.

In chapter 3 we give a summary on the constructions of moduli spaces of framed sheaves done by Huybrechts and Lehn [22, 23]. Then we review the Uhlenbeck-Donaldson compactification constructed by Bruzzo, Markushevich and Tikhomirov [7] for framed sheaves on projective surfaces.

In chapter 4 we introduce the Białynicki-Birula decompositions of an algebraic scheme X. Then we restrict ourselves to algebraic varieties with a finite set of fixed points and study their decompositions.

In chapter 5 we review some results on the moduli space $M(r,n)$ studied in [33, 34, 35]. Then following the results of chapter 4, we show that $M(r,n)$ admits a Białynicki-Birula plus-decomposition and the union of the minus-cells builds up $\pi^{-1}(n[0])$ that is a compact subvariety of $M(r,n)$. Moreover, we show that these decompositions are filtrable. Afterwards, we show that the inclusion of $\pi^{-1}(n[0])$ into $M(r,n)$ induces isomorphisms of homology groups with integer coefficients and a homotopy equivalence between $M(r,n)$ and $\pi^{-1}(n[0])$.

In chapter 6 we generalize this study to the moduli space of framed sheaves on a toric surface S where we assume that there exists a projective morphism of toric surfaces $p : S \to \mathbb{P}^2$ of degree 1.

Chapter 2

Framed sheaves on projective varieties

In this chapter we present the basic background on framed sheaves. In section 2.1 we introduce the notion of (semi)stability, μ-(semi)stability of coherent sheaves and their Jordan-Hölder filtrations. In section 2.2 we introduce framed sheaves and their homomorphisms. We then present the different notions of stability for framed sheaves in section 2.2.2, and the Jordan-Hölder filtrations in section 2.2.2.

2.1 Preliminaries

In this section we collect all the definitions we need to introduce the necessary background for the following sections. The main reference is the book of Huybrechts and Lehn [24].

Let X be a Noetherian scheme.

Definition 2.1.1. The *dimension* of coherent sheaf $\mathscr{E}$ on X is the dimension of its support

$$\mathrm{Supp}(\mathscr{E}) = \{x \in X \mid \mathscr{E}_x \neq 0\}.$$

Definition 2.1.2. A coherent sheaf $\mathscr{E}$ on X is *torsion-free* if every stalk $\mathscr{E}_x$ is a torsion free $\mathscr{O}_{X,x}$-module; that is $fa = 0$ for $f \in \mathscr{O}_{X,x}$, $a \in \mathscr{E}_x$ always implies $a = 0$ or $f = 0$.

Definition 2.1.3. A coherent sheaf $\mathscr{E}$ on X is said to be *pure of dimension* d if for all nontrivial subsheaves $\mathscr{F} \subset \mathscr{E}$ the dimension of $\mathscr{F}$ is equal to d.

Definition 2.1.4. For a coherent sheaf $\mathscr{E}$ of dimension d, the unique filtration

$$0 \subset T_0(\mathscr{E}) \subset T_1(\mathscr{E}) \subset \cdots \subset T_{d-1}(\mathscr{E}) \subset T_d(\mathscr{E}) = \mathscr{E}$$

is called the *torsion filtration*, where $T_i(\mathscr{E}) \subset \mathscr{E}$ is the maximal subsheaf of dimension $\leq i$.

Remark 2.1.5. Note that $\mathscr{E}$ is pure of dimension d if and only if $T_{d-1}(\mathscr{E}) = 0$. The definition 2.1.2 of a torsion-free sheaf $\mathscr{E}$ means that for every point $x \in X$ and a nonzero section $f \in \mathscr{O}_{X,x}$, the multiplication gives rise to an injective homomorphism $\mathscr{E}_x \to \mathscr{E}_x$. Using the torsion filtration we conclude that for a torsion-free sheaf $T_{d-1}(\mathscr{E}) = 0$. Hence the notion of pure sheaf generalizes the notion of torsion-free sheaf.

Definition 2.1.6. Let m be an integer. A coherent sheaf $\mathscr{F}$ is said to be *m-regular*, if $H^i(X, \mathscr{F}(m-i)) = 0$ for each $i > 0$.

Let X be a projective scheme over field k. We define the *Euler characteristic* of a coherent sheaf $\mathscr{E}$ as follows

$$\chi(\mathscr{E}) := (-1)^i h^i(X, \mathscr{E}),$$

where $h^i(X, \mathscr{E}) = \dim_k H^i(X, \mathscr{E})$.

Now fix an ample line bundle $\mathscr{O}(1)$ on X, we define the *Hilbert polynomial* $P_{\mathscr{E}}$ of $\mathscr{E}$ by

$$m \mapsto \chi(\mathscr{E} \otimes \mathscr{O}(m)).$$

$P_{\mathscr{E}}$ can be uniquely written in the form

$$P_{\mathscr{E}}(m) = \sum_{i=0}^{\dim \mathscr{E}} \alpha_i(\mathscr{E}) \frac{m^i}{i!},$$

where $\alpha_i(\mathscr{E})$ are rational coefficients. For $d = \dim \mathscr{E}$ the coefficient $\alpha_d(\mathscr{E})$ is positive and is called the *multiplicity* of $\mathscr{E}$

Definition 2.1.7. For a coherent sheaf $\mathscr{E}$ of dimension d we define its *rank* as follows

$$\mathrm{rk}(\mathscr{E}) := \frac{\alpha_d(\mathscr{E})}{\alpha_i(\mathscr{O}_X)}.$$

Definition 2.1.8. For a coherent sheaf $\mathscr{E}$ of dimension d the *reduced Hilbert polynomial* is defined by

$$p(\mathscr{E}, m) := \frac{P(\mathscr{E}, m)}{\alpha_d(\mathscr{E})}.$$

Definition 2.1.9. A coherent sheaf $\mathscr{E}$ of dimension d is *semistable* if $\mathscr{E}$ is pure and for any proper subsheaf $\mathscr{F} \subset \mathscr{E}$ one has $p(\mathscr{F}) \leq p(\mathscr{E})$. It is said to be *stable* if the inequality is strict.

Proposition 2.1.10 (Proposition 1.2.7 in [24]). *Let $\mathscr{F}$ and $\mathscr{G}$ be semistable purely d-dimensional coherent sheaves. If $p(\mathscr{F}) > p(\mathscr{G})$, then $\mathrm{Hom}(\mathscr{F},\mathscr{G}) = 0$. If $p(\mathscr{F}) = p(\mathscr{G})$ and $f : \mathscr{F} \to \mathscr{G}$ is nontrivial, then f is injective if $\mathscr{F}$ is stable and surjective if $\mathscr{G}$ is stable. If $p(\mathscr{F}) = p(\mathscr{G})$ and $\alpha_d(\mathscr{F}) = \alpha_d(\mathscr{G})$ then any nontrivial homomorphism $f : \mathscr{F} \to \mathscr{G}$ is an isomorphism provided $\mathscr{F}$ or $\mathscr{G}$ is stable.*

Proof. Let $f : \mathscr{F} \to \mathscr{G}$ be a nontrivial homomorphism of semistable sheaves with $p(\mathscr{F}) \geq p(\mathscr{G})$. Let $\mathscr{E}$ be the image of f. Then $p(\mathscr{F}) \leq p(\mathscr{E}) \leq p(\mathscr{G})$. This contradicts the assumption $p(\mathscr{F}) > p(\mathscr{G})$. If $p(\mathscr{F}) = p(\mathscr{G})$ it contradicts the assumption that $\mathscr{F}$ is stable unless $\mathscr{F}$ is isomorphic to $\mathscr{E}$, and the assumption that $\mathscr{G}$ is stable unless $\mathscr{E}$ is isomorphic to $\mathscr{G}$. If $\mathscr{F}$ and $\mathscr{G}$ have the same Hilbert polynomial $\alpha_d(\mathscr{F})\cdot p(\mathscr{F}) = \alpha_d(\mathscr{G})\cdot p(\mathscr{G})$, then any homomorphism $f : \mathscr{F} \to \mathscr{G}$ is an isomorphism if and only if f is injective or surjective. □

Definition 2.1.11. Let $\mathscr{E}$ be a semistable sheaf of dimension d. A *Jordan-Hölder filtration* $\mathscr{E}_\bullet$ of $\mathscr{E}$ is a filtration

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_l = \mathscr{E},$$

such that $gr_i(\mathscr{E}) := \mathscr{E}_i/\mathscr{E}_{i-1}$ is stable with reduced Hilbert polynomial $p(\mathscr{E})$ for all i.

The subsheaves $\mathscr{E}_i$, $i \neq 0$ are also semistable with the same reduced Hilbert polynomial. This filtration is not unique in general.

Proposition 2.1.12 (Proposition 1.5.2 in [24]). *Jordan-Hölder filtrations always exist. Up to isomorphism, the sheaf $gr(\mathscr{E}) := \bigoplus_i gr_i(\mathscr{E})$ does not depend on the choice of the Jordan-Hölder filtration.*

Proof. Any filtration of $\mathscr{E}$ by semistable sheaves with reduced Hilbert polynomial $p(\mathscr{E})$ has a maximal refinement, whose factors are necessarily stable, i.e., $\mathscr{E}$ has a filtration by stable sheaves. Now, suppose that $\mathscr{E}_\bullet$ and $\mathscr{E}'_\bullet$ are two Jordan-Hölder filtrations of length l and l' respectively, and assume that the uniqueness of $gr(\mathscr{F})$ has been proved for all $\mathscr{F}$ with $\alpha_d(\mathscr{F}) < \alpha_d(\mathscr{E})$, where d is the dimension of $\mathscr{E}$ and α_d is its multiplicity. Let i be minimal with $\mathscr{E}_1 \subset \mathscr{E}'_i$. Then the composition of maps $\mathscr{E}_1 \to \mathscr{E}' \to \mathscr{E}'/\mathscr{E}'_{i-1}$ is nontrivial, hence by Proposition 2.1.10 it is an isomorphism when both $\mathscr{E}_1$ and $\mathscr{E}'/\mathscr{E}'_{i-1}$ are stable and $p(\mathscr{E}_1) = p(\mathscr{E}'/\mathscr{E}'_{i-1})$. Therefore $\mathscr{E}'_i \cong \mathscr{E}'_{i-1} \oplus \mathscr{E}_1$, and we have a short exact sequence

$$0 \to \mathscr{E}'_{i-1} \to \mathscr{E}/\mathscr{E}_1 \to \mathscr{E}/\mathscr{E}'_i \to 0,$$

The sheaf $\mathscr{F} = \mathscr{E}/\mathscr{E}_1$ inherits two Jordan-Hölder filtrations: the first one is defined as follows. Let $\mathscr{F}_j = \mathscr{E}_{j+1}/\mathscr{E}_1$ for $j = 0, \dots, l-1$. And the second one is given as follows. Let $\mathscr{F}'_j = \mathscr{E}'_j$ for $j = 0, \dots, i-1$ and let $\mathscr{F}'j$ be the preimage of $\mathscr{E}'_{j+1}/\mathscr{E}'_i$ for $j = i, \dots, l'-1$.

The induction hypothesis applied to $\mathscr{F}$ yields the equality $l = l'$ and the isomorphism

$$\bigoplus_{j\neq 1} \mathscr{E}_j/\mathscr{E}_{j-1} \cong \bigoplus_{j\neq i} \mathscr{E}'_j/\mathscr{E}'_{j-1}.$$

Since $\mathscr{E}_1 \cong \mathscr{E}'_i/\mathscr{E}'_{i-1}$, we get the assertion. □

This proposition motivates the following definition.

Definition 2.1.13. Two semistable sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$ with the same reduced Hilbert polynomial are called *S-equivalent* if $gr(\mathscr{E}_1) \cong gr(\mathscr{E}_2)$.

Definition 2.1.14. Let $\mathscr{E}$ be a coherent sheaf on X of dimension $d = \dim X$. We define the *degree* of $\mathscr{E}$ by

$$\deg(\mathscr{E}) := \alpha_{d-1}(\mathscr{E}) - \mathrm{rk}(E)\alpha_{d-1}(\mathscr{O}_X).$$

We define the slope of $\mathscr{E}$ by

$$\mu(\mathscr{E}) := \frac{\deg(\mathscr{E})}{\mathrm{rk}(\mathscr{E})}.$$

We are now ready to define the notion of μ-(semi)stability.

Definition 2.1.15. A coherent sheaf $\mathscr{E}$ on X of dimension $d = \dim X$ is said to be *μ-semistable* if $T_{d-1}(\mathscr{E}) = T_{d-2}(\mathscr{E})$ and $\mu(\mathscr{E}') \leq \mu(\mathscr{E})$ for all subsheaves $\mathscr{E}' \subset \mathscr{E}$ of rank $0 < \mathrm{rk}(\mathscr{E}') < \mathrm{rk}(\mathscr{E})$. $\mathscr{E}$ is said to be *μ-stable* if the inequality is strict.

Definition 2.1.16. A semistable sheaf $\mathscr{E}$ is called *polystable* if $\mathscr{E}$ is the direct sum of stable sheaves.

2.2 Framed sheaves

In this section we will give an overview on framed sheaves and their moduli. The main reference is [22, 23].

Let X be a n-dimensional nonsingular projective variety over k endowed with a very ample line bundle $\mathscr{O}_X$ and let $\mathscr{F}$ be an $\mathscr{O}_X$-module.

Definition 2.2.1. A *framed sheaf* on X is a pair $(\mathscr{E}, \alpha)$ consisting of a torsion-free sheaf $\mathscr{E}$ on X and a morphism $\alpha : \mathscr{E} \longrightarrow \mathscr{F}$ called the *framing morphism.*

The *Hilbert polynomial* of the framed sheaf $(\mathscr{E}, \alpha)$ is given by

$$P(\mathscr{E}, \alpha) := P(\mathscr{E}) - \epsilon(\alpha)\delta,$$

where δ is a rational polynomial with positive leading coefficient and $\epsilon(\alpha) := \begin{cases} 1, & \alpha \neq 0; \\ 0, & \alpha = 0. \end{cases}$

We define the *reduced Hilbert polynomial* of $(\mathscr{E}, \alpha)$ by

$$p(\mathscr{E}, \alpha) := \frac{P(\mathscr{E}, \alpha)}{\mathrm{rk}(\mathscr{E}, \alpha)},$$

where $\mathrm{rk}(\mathscr{E}, \alpha) := \mathrm{rk}(\mathscr{E})$ is the rank of the framed sheaf.

Similarly we define *the degree of the framed sheaf* $(\mathscr{E}, \alpha)$ as follows

$$\deg(\mathscr{E}, \alpha) := \deg(\mathscr{E}) - \epsilon(\alpha)\delta.$$

For a subsheaf $\mathscr{E}' \subset \mathscr{E}$ the framing homomorphism α induces a framing homomorphism $\alpha' = \alpha_{|\mathscr{E}'}$ on $\mathscr{E}'$. α also induces a framing α'' on the quotient sheaf $\mathscr{E}'' = \mathscr{E}/\mathscr{E}'$. We take the following convention: if $\alpha \neq 0$ then $\alpha'' = 0$, and if $\alpha = 0$ then α'' is the induced framing on $\mathscr{E}''$. This insures the additivity of the Hilbert polynomials

$$P(\mathscr{E}, \alpha) = P(\mathscr{E}', \alpha') + P(\mathscr{E}'', \alpha''),$$

and of the degrees

$$\deg(\mathscr{E}, \alpha) = \deg(\mathscr{E}', \alpha') + \deg(\mathscr{E}'', \alpha'').$$

Definition 2.2.2. A framed sheaf $(\mathscr{E}, \alpha)$ is *torsion-free* if $\mathscr{E}$ is torsion-free. It is *locally free* if $\mathscr{E}$ is locally free.

Definition 2.2.3. *A homomorphism* of framed sheaves $\phi : (\mathscr{E}, \alpha) \longrightarrow (\mathscr{E}', \alpha')$ is a homomorphism of sheaves $f : \mathscr{E} \to \mathscr{E}'$ for which there is a complex number $\lambda \in k$ such that $\alpha' \circ \phi = \lambda\alpha$.

2.2.1 (Semi)stable framed sheaves

Let δ be a polynomial with rational coefficients and positive leading term δ_1.

Definition 2.2.4. A framed sheaf $(\mathscr{E}, \alpha)$ is said to be *semistable* with respect to δ if it satisfies the following conditions

1. $p(\mathscr{E}') \leq p(\mathscr{E}, \alpha)$ for all nontrivial subsheaves $\mathscr{E}' \subset \ker(\alpha)$,
2. $p(\mathscr{E}', \alpha') \leq p(\mathscr{E}, \alpha)$ for all nontrivial subsheaves $\mathscr{E}' \subset \mathscr{E}$.

$(\mathscr{E}, \alpha)$ is *stable* if the inequalities are strict.

Lemma 2.2.5 (Lemma 1.2 [22]). *The kernel of a semistable framed sheaf $(\mathscr{E}, \alpha)$ is torsion free.*

Proof. Let T be the torsion part of the kernel of α, then $\mathrm{rk}(T) = 0$. Since $(\mathscr{E}, \alpha)$ is semistable, using the first inequality in the above definition we get $rP(T) \leq 0$, this implies $T = 0$. □

Lemma 2.2.6 (Lemma 1.7 [22]). *If $\deg(\delta) \geq n$ then for any semistable framed sheaf $(\mathscr{E}, \alpha)$ the framing homomorphism α is injective or zero. Conversely, if α is the inclusion homomorphism of a subsheaf $\mathscr{E}$ in $\mathscr{F}$ of positive rank, then $(\mathscr{E}, \alpha)$ is stable.*

Proof. We assume that $\alpha \neq 0$. Let $\mathscr{E}'$ be a nontrivial subsheaf of the kernel of α with $\mathrm{rk}\,\mathscr{E}' = r'$. Then the first inequality of the semistability definition gives

$$rP(\mathscr{E}') - r'P(\mathscr{E}) \leq -r'\delta.$$

The polynomials $P(\mathscr{E})$ and $P(\mathscr{E}')$ have the same degree n and the same leading term. This inequality yields a contradiction if the degree of δ is greater that n. This means that α is injective if $\deg(\delta) \geq n$.

Similarly, if α is injective the inequality is strictly satisfied since becase of the dominance of δ. □

Lemma 2.2.6 tells us that the study of semistable framed sheaves reduces to the study of stable subsheaves of the framing sheaf $\mathscr{F}$. Hence for $\deg(\delta) \geq n$, this study is covered by the theory of Grothendieck's Hilbert schemes and quot schemes. This reason yields the assumption $\deg(\delta) < n$. The polynomial δ can be written in the following form

$$\delta(m) = \delta_1 \frac{m^{n-1}}{(n-1)!} + \delta_2 \frac{m^{n-2}}{(n-2)!} + \cdot + \delta_n,$$

where the leading coefficient δ_1 is positive.

We now define another notion of (semi)stability with respect to δ_1, the leading term of the polynomial δ. For a framed sheaf $(\mathscr{E}, \alpha)$ we define its *slope* as follows

$$\mu(\mathscr{E}, \alpha) := \frac{\deg(\mathscr{E}, \alpha)}{\mathrm{rk}(E)}.$$

Definition 2.2.7. A framed sheaf $(\mathscr{E}, \alpha)$ is said to be *μ-semistable* if it satisfies the following two conditions:

1. $\mathrm{rk}(\mathscr{E})\deg(\mathscr{E}') \leq \mathrm{rk}(\mathscr{E}')(\deg(\mathscr{E} - \delta_1))$ for all nontrivial subsheaves $\mathscr{E}' \subset \ker(\alpha)$,

2. $\mathrm{rk}(\mathscr{E})(\deg(\mathscr{E}') - \delta_1) \leq \mathrm{rk}(\mathscr{E}')(\deg(\mathscr{E} - \delta_1))$ for all nontrivial subsheaves $\mathscr{E}' \subset \mathscr{E}$ with $\mathrm{rk}(\mathscr{E}') \leq \mathrm{rk}(\mathscr{E})$.

A framed sheaf $(\mathscr{E}, \alpha)$ is said to be *μ-stable* if the inequalities are strict.

We have the following implications between different notions of stability of framed sheaves

$$\mu\text{-stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu\text{-semistable}. \tag{2.1}$$

2.2.2 Jordan-Hölder filtrations

Definition 2.2.8. For a semistable framed sheaf $(\mathscr{E}, \alpha)$ with reduced Hilbert polynomial p we define its *Jordan-Hölder filtration* as follows

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = \mathscr{E},$$

such that $(gr_i(\mathscr{E}), \alpha_i)$ are stable with respect to δ with reduced Hilbert polynomial p, where $gr_i(\mathscr{E}) := \mathscr{E}_i/\mathscr{E}_{i-1}$, and α_i are the induced framing morphisms.

Proposition 2.2.9 (Propposition 1.13 in [22]). *Jordan-Hölder filtrations always exist. The framed sheaf $gr(\mathscr{E}, \alpha) := \bigoplus_{i=1}^n (\mathscr{E}_i/\mathscr{E}_{i-1}, \alpha_i)$ does not depend on the choice of the Jordan-Hölder filtration.*

Proof. The proof of Proposition 2.1.12 can be easily adapted to this case. □

As in the nonframed case, the proposition above motivates the following definition.

Definition 2.2.10. Two semistable framed sheaves $(\mathscr{E}_1, \alpha_1)$ and $(\mathscr{E}_2, \alpha_2)$ with the same reduced Hilbert polynomial are said to be *S-equivalent* if their associated Graded objects $gr(\mathscr{E}_1, \alpha_1)$ and $gr(\mathscr{E}_2, \alpha_2)$ are isomorphic.

Definition 2.2.11. For a μ-semistable framed sheaf $(\mathscr{E}, \alpha)$ of positive rank we define its *μ-Jordan-Hölder filtration* as follows

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_s = \mathscr{E},$$

such that $(gr_i(\mathscr{E}), \alpha_i)$ are μ-stable, where $gr_i(\mathscr{E}) := \mathscr{E}_i/\mathscr{E}_{i-1}$, and α_i are the induced framing morphisms.

Theorem 2.2.12 (Theorem 66 [38]). *μ-Jordan-Hölder filtrations always exist. The framed sheaf $gr^{\mu}(\mathscr{E},\alpha) := \bigoplus_{i=1}^{n}(\mathscr{E}_i/\mathscr{E}_{i-1},\alpha_i)$ does not depend on the choice of the μ-Jordan-Hölder filtration.*

Definition 2.2.13. A framed sheaf $(\mathscr{E},\alpha)$ is said to be *polystable* if $\mathscr{E}$ has a filtration $\mathscr{E}_\bullet :\ 0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_m = \mathscr{E}$, such that

1. $\mathscr{E}$ is isomorphic to the graded object $gr^{\mu}\mathscr{E} = \bigoplus_{i=1}^{n}\mathscr{E}_i/\mathscr{E}_{i-1}$.

2. The filtration $(\mathscr{E},\alpha)_\bullet :\ \cdots \subset (\mathscr{E}_i, \alpha|_{\mathscr{E}_i}) \subset (\mathscr{E}_{i+1}, \alpha|_{\mathscr{E}_{i+1}}) \subset \cdots$, is a μ-Jordan-Hölder filtration.

Chapter 3

Moduli of framed sheaves on projective surfaces

In this chapter we give a summary on the constructions of moduli spaces of framed sheaves done by Huybrechts and Lehn [22, 23]. Then we review the Uhlenbeck-Donaldson compactification constructed by Bruzzo, Markushevich and Tikhomirov [7] for framed sheaves on projective surfaces.

3.1 Construction of the moduli space

In this section we recall the construction by Huybrechts and Lehn [22] of the moduli space of framed sheaves on projective surfaces. A more detailed study is given in [23].

Let P be a polynomial, $r > 0$ the rank of a framed sheaf with Hilbert polynomial P and μ_P its slope.

Definition 3.1.1. A *family* of framed sheaves on X parametrized by a noetherian scheme T consists of a sheaf $\mathcal{E}$ flat over T and a homomorphism of sheaves $\alpha : \mathcal{E} \to p_X^* \mathcal{E}_0$. Here $\mathcal{E}_0$ is a sheaf on X and $p_X : X \times T \longrightarrow X$ is the projection on X.

Theorem 3.1.2 (Theorem 2.1 [22])**.** *There is an integer m_0 such that the following properties of a framed sheaf $(\mathscr{E}, \alpha)$ with Hilbert polynomial P and torsion-free kernel are equivalent for all $m \geq m_0$:*

1. *$(\mathscr{E}, \alpha)$ is semistable (respectively stable).*
2. *$P(m) \leq h^0((\mathscr{E}, \alpha)(m))$ and $h^0((\mathscr{E}', \alpha')(m)) \leq r' \frac{P'(m)}{r}$ (respectively $h^0((\mathscr{E}', \alpha')(m)) < r' \frac{P'(m)}{r}$) for all framed subsheaves $(\mathscr{E}', \alpha')$ of rank r', $0 \neq \mathscr{E}' \neq \mathscr{E}$.*

3. *$h^0((\mathscr{E}'',\alpha'')(m)) \geq r''\frac{P(m)}{r}$ (respectively $h^0((\mathscr{E}'',\alpha'')(m)) > r''\frac{P(m)}{r}$) for all quotients $(\mathscr{E}'',\alpha'')$ of rank r'', $0 \neq \mathscr{E}'' \neq \mathscr{E}$.*

Moreover, for any framed sheaf satisfying these conditions, $\mathscr{E}$ is m-regular.

Let P_0 be a polynomial of degree n, $P = P_0 - \delta$ and μ_P be the slope of sheaves with Hilbert polynomial P. Choose an integer $m \geq m_0$ and let V be a vector space of dimension $P_0(m)$. Let $\mathcal{H}$ denote the Hilbert schemes $\mathrm{Hilb}(V \otimes \mathscr{O}_X, P_0)$ then we have the following closed immersions

$$\mathcal{H} \longrightarrow \mathrm{Grass}(V \otimes H^0(\mathscr{O}_X(l-m)), P_0(l)) \longrightarrow \mathbb{P}(\Lambda^{P_0(l)}(V \otimes H^0(\mathscr{O}_X(l-m)))),$$

where l is sufficiently large.

Let $\mathcal{L}$ be the corresponding very ample line bundle on $\mathcal{H}$, $\mathcal{P} := \mathbb{P}(\mathrm{Hom}(V, H^0(\mathscr{F}(m)))\check{})$ and $Z' \subset \mathcal{H} \times \mathcal{P}$ be the closed subscheme of points

$$([q : V \otimes \mathscr{O}_X(-m) \to \mathscr{E}], [a : V \to H^0(\mathscr{F}(m))])$$

such that the homomorphism $a : \mathscr{E} \to \mathscr{F}$ factors through q and induces a framing $\alpha : \mathscr{E} \to \mathscr{F}$.

The group $\mathrm{SL}(V)$ act diagonally on Z' and we have natural $\mathrm{SL}(V)$-linearizations on the line bundle

$$\mathscr{O}_{Z'(n_1,n_2)} := p_{\mathcal{H}}^* \mathcal{L}^{\otimes n_1} \otimes p_{\mathcal{P}}^* \mathscr{O}_{\mathcal{P}}(n_2).$$

Make the following choice

$$\frac{n_2}{n_1} = P(l)\frac{\delta(m)}{P(m)} - \delta(l),$$

where l is large enough to insure the positivity of this term. Under these assumptions we have the following result

Proposition 3.1.3 (Proposition 3.2 in [22]). *A point $([q],[a]) \in Z'$ is (semi)stable with respect to the $\mathrm{SL}(V)$-action if and only if the corresponding framed sheaf $(\mathscr{E},\alpha)$ is (semi)stable and q induces an isomorphism $V \to H^0(\mathscr{E}(m))$.*

Now let $Z^s \subset Z^{ss} \subset Z$ be the open subschemes of stable and semistable points of Z.

Proposition 3.1.4 (Proposition 3.3 [22]). *There exists a projective scheme M^{ss} and a morphism $\pi : Z^{ss} \to M^{ss}$ which is a good quotient for the action of $\mathrm{SL}(V)$ on Z^{ss}. Moreover there is an open subscheme $M^s \subset M^{ss}$ such that $Z^s = \pi^{-1}(M^s)$ and $\pi : Z^s \to M^s$ is a geometric quotient. Two points $([q],[a])$ and $([q'],[a'])$ are mapped to the same point in M^{ss} if and only if the corresponding framed sheaves are S-equivalent.*

3.2 Uhlenbeck-Donaldson compactification

In this section we briefly recall the Uhlenbeck-Donaldson compactification of framed sheaves on projective surfaces studied by Bruzzo, Markushevich and Tikhomirov in [7].

We assume that the framing sheaf $\mathscr{F}$ is of dimension $d-1$ and is supported on an effective divisor $D \subset X$. We will consider the framed torsion-free sheaves so we assume $\deg P(m) = d$. Fix a polynomial $\delta \in \mathbb{Q}[m]$ of degree $d-1$ and positive leading coefficient δ_{d-1}.

Let us denote by $\mathcal{S}^{ss}(c, \delta)$ the family of sheaves of class c stable with respect to δ and by $\mathcal{S}^{\mu ss}(c, \delta_{d-1})$ the family of μ-semistable sheaves with respect to δ_{d-1}. By (2.1) we have the following inclusion

$$\mathcal{S}^{ss}(c, \delta) \subset \mathcal{S}^{\mu ss}(c, \delta_{d-1}). \tag{3.1}$$

Proposition 3.2.1 (Proposition 3.2 in [7]). *The family $S^{\mu ss}(c, \delta_{d-1})$ is bounded.*

Let V be a vector space of of dimension $P(m)$ for some $m \gg 0$, let $\mathcal{H} = V \otimes \mathscr{O}_X(-m)$ and let $\mathrm{Quot}(\mathcal{H}, P)$ be the Quot scheme parametrizing the coherent quotients of $\mathcal{H}$ of Hilbert polynomial P.

Let $\mathbb{P} = \mathbb{P}(\mathrm{Hom}(V, H^0(X, \mathscr{F}(m)))\check{})$ and let $Y := \mathrm{Quot}(\mathcal{H}, P, \mathscr{F})$ be the closed subscheme of $\mathrm{Quot}(\mathcal{H}, P) \times \mathbb{P}$ consisting of pairs $([g], [a])$ such that there is a morphism $\phi : \mathscr{G} \to \mathscr{F}$ for which the following diagram commutes

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{g} & \mathscr{G} \\ & \searrow^{a} & \downarrow{\scriptstyle \phi} \\ & & \mathscr{F} \end{array}$$

ϕ is uniquely determined by a.

Now fix a sufficiently large number m such that for each framed sheaf $(\mathscr{E}, \alpha) \in \mathcal{S}^{\mu ss}(c, \delta)$ is m-regular. Define $\widetilde{R}^{\mu ss}(c, \delta)$ the locally closed subscheme of $Y = \mathbb{Q}uot(\mathcal{H}, P, \mathscr{F})$ consisting of pairs $([g : \mathcal{H} \to \mathscr{E}], [a : \mathcal{H} \to \mathscr{F}])$ such that $(\mathscr{E}, \alpha) \in \mathcal{S}^{ss}(c, \delta)$ is μ-stable with respect to δ_{d-1}. The framing α is defined by $a = \alpha \circ g$ and g induces an isomorphism $V \to H^0(\mathscr{E}(m))$.

From (3.1) $\widetilde{R}^{\mu ss}(c, \delta)$ contains a subset $R^{ss}(c, \delta)$ consisting of semistable pairs. $R^{ss}(c, \delta)$ is open in $\widetilde{R}^{\mu ss}(c, \delta)$. We consider its closure $R^{\mu ss}(c, \delta)$ in $\widetilde{R}^{\mu ss}(c, \delta)$.

From now on assume that $\deg P(m) = 2$ and $\delta(m) = \delta_1 m + \delta_0$. Under these assumptions and in the same spirit of the last section; Bruzzo, Markushevich and Tikhomirov define a line bundle $\mathcal{L}(n_1, n_2)$ on $R^{\mu ss}(c, \delta)$ and prove the following result.

Proposition 3.2.2 (Proposition 3.5 in [7])**.** *For $\nu \gg 0$ the line bundle $\mathcal{L}(n_1, n_2)$ on $R^{\mu ss}(c, \delta)$ is generated by its $\mathrm{SL}(V)$-invariant sections.*

From this proposition it follows that there exists a finite dimensional subspace $W \subset W_\nu := H^0(R^{\mu ss}, \mathcal{L}(n_1, n_2)^\nu)^{\mathrm{SL}(V)}$ which generates $\mathcal{L}(n_1, n_2)^\nu$.

Let $\phi_W : R^{\mu ss}(c, \delta) \to \mathbb{P}(W)$ be the induced $\mathrm{SL}(P(m))$-invariant morphism.

Proposition 3.2.3 (Proposition 4.1 in [8])**.** *$M_W := \phi_W(R^{\mu ss}(c, \delta))$ is a projective scheme.*

Proposition 3.2.4 (Proposition 4.4 in [7])**.** *There is an integer $N > 0$ such that $\bigoplus_{l \geq 0} W_{lN}$ is a finitely generated graded ring.*

The previous two propositions allow us to define the *Uhlenbeck-Donaldson compactification.*

Definition 3.2.5. Let N be a positive integer as in proposition 3.2.4. Then $M^{\mu ss} = M^{\mu ss}(c, \delta)$ is defined by

$$M^{\mu ss} = \mathrm{Proj}\left(\bigoplus_{k \geq 0} H^0(R^{\mu ss}(c, \delta), \mathcal{L}(n_1, n_2)^{kN})^{\mathrm{SL}(P(m))} \right).$$

It is equipped with a natural morphism $\pi : R^{\mu ss}(c, \delta) \to M^{\mu ss}$ and is called the *moduli space* of μ-semistable framed sheaves.

Now define the following spaces

$$\mathcal{S}^{\mu ss}(c, \delta)^* = \{(\mathscr{E}, \alpha) \in \mathcal{S}^{\mu ss}(c, \delta) \mid \mathscr{E} \text{ is locally free at all points of } D \text{ and } \alpha \text{ induces an isomorphism } \mathscr{E}|_D \cong \mathscr{F}\},$$

$$R^{\mu ss}(c, \delta)^* = \{([g : \mathcal{H} \to \mathscr{E}], [\alpha \circ g]) \in R^{\mu ss}(c, \delta) \mid (\mathscr{E}, \alpha) \in \mathcal{S}^{\mu ss}(c, \delta)^*\},$$

$$M^{\mu ss}(c, \delta)^* = \pi(R^{\mu ss}(c, \delta)^*)$$

Theorem 3.2.6. *Assume that $\delta_1 < r \deg D$. Two framed sheaves $(\mathscr{E}_1, \alpha_1)$ and $(\mathscr{E}_2, \alpha_2)$ in $\mathcal{S}^{\mu ss}(c, \delta)^* \cap S^{ss}(c, \delta)$ define the same closed point in $M^{\mu ss}(c, \delta)^*$ if and only if $(\mathrm{gr}^\mu(\mathscr{E}_1, \alpha_1))\tilde{} = (\mathrm{gr}^\mu(\mathscr{E}_2, \alpha_2))\tilde{}$ and $l_{\mathscr{E}_1} = l_{\mathscr{E}_2}$.*

This theorem defines a set-theoretic stratification of the Uhlenbeck-Donaldson compactification.

Corollary 3.2.7. *Let $c = (r, \xi, c_2)$ be a numerical K-theory class and let*

$M^{\mu\text{-poly}}(r, \xi, c_2, \delta)^* \subset M^{\mu ss}(c, \delta)^*$ be the moduli space of μ-polystable locally free sheaves. Assume that $\delta_1 < r \deg D$. One has the following set-theoretic stratification:

$$M^{\mu ss}(c,\delta)^* = \coprod_{l\geq 0} M^{\mu\text{-}poly}(r,\xi,c_2-l,\delta)^* \times \mathrm{Sym}^l(X\setminus D).$$

Chapter 4

Białynicki-Birula decompositions

In this chapter, we introduce the Białynicki-Birula decompositions of an algebraic scheme X. These decompositions are determined by an action of a torus on X. Białynicki-Birula introduced these decompositions in [4], and studied their properties for smooth projective algebraic varieties in [5].

In [27], Konarski studied Białynicki-Birula's decompositions for normal varieties. In [28] and using a theorem due to Sumihiro [39], he gave a more elementary proof to Białynicki-Birula's theorem.

In the next sections we will give the statement of Białynicki-Birula's theorem in its general setting, then restrict ourselves to the case of algebraic varieties and study their decompositions in two cases. In the first case, when the variety is complete, the decompositions exist according to Białynicki-Birula's theorem. We will also see few examples.

In the second case when a variety is not complete, one needs to assume that the limits toward fixed points exist. We will show that in this case Białynicki-Birula's decompositions exist.

4.1 Białynicki-Birula's theorem

In this section we assume that k is an algebraically closed field and G is an algebraic one-dimensional torus, i.e., $G = \mathbb{G}_m$.

Assume X is a nonsingular reduced algebraic scheme over k. Let η be a fixed action of G on X. We will also assume that X satisfies the following condition

$$X \text{ can be covered by } G\text{-invariant quasi-affine open subschemes.} \tag{4.1}$$

Remark 4.1.1. Condition (4.1) is satisfied in the case X is normal see [39, Corollary 2]. More precisely, Corollary 2 in [39] states that X can be covered by G-invariant affine open subschemes.

We shall follow the notation in [4]. For a G-module V, we denote by V^0 the G-submodule consisting of all $v \in V$ such that $G(k) \cdot v = v$. We also denote by V^+ (respectively V^-) the G-submodule spanned by all $v \in V$ such that for $\lambda \in G(k) \cong k^*$, $\lambda \cdot v = \lambda^m v$ where $m > 0$ (respectively $m < 0$). This means that V can be written as $V = V^0 \oplus V^+ \oplus V^-$.

Under the assumptions above Białynicki-Birula proved the following theorem.

Theorem 4.1.2 ([4, Theorem 4.1]). *Let $X^G = \bigcup_{i=1}^r (X^G)_i$ be the decomposition of X^G into connected components. Then, for any $i = 1, \dots, r$, there exists a (unique) locally closed nonsingular and G-invariant subscheme X_i^+ (respectively X_i^-) of X and a (unique) morphism $\gamma^+ : X_i^+ \to (X^G)_i$ (respectively $\gamma^- : X_i^- \to (X^G)_i$) such that:*

1. *$(X^G)_i$ is a closed subscheme of X_i^+ (respectively X_i^-) and $\gamma_i^+|(X^G)_i$ (respectively $\gamma_i^-|(X^G)_i$) is the identity.*

2. *X_i^+ (respectively X_i^-) with the action of G (induced by the action of G on X) and with γ_i^+ (respectively γ_i^-) is a G-fibration over $(X^G)_i$.*

3. *For any closed $a \in (X^G)_i$, $T_a(X_i^+) = T_a(X)^0 \oplus T_a(X)^+$ (respectively $T_a(X_i^-) = T_a(X)^0 \oplus T_a(X)^-$). The dimension of the fibration defined above equals $\dim T_a(X)^+$ (respectively $\dim T_a(X)^-$), for any closed $a \in (X^G)_i$.*

The following result follows from Theorem 4.1.2.

Theorem 4.1.3 ([4, Theorem 4.3]). *Let the scheme X be complete and let $X^G = \bigcup_{i=1}^r (X^G)_i$ be the decomposition of X^G into connected components. Then there exists a unique locally closed G-invariant decomposition of X, Let $X = \bigcup_{i=1}^r (X^+)_i$ (respectively $X = \bigcup_{i=1}^r (X^-)_i$) and morphisms $\gamma^+ : X_i^+ \to (X^G)_i$ (respectively $\gamma^- : X_i^- \to (X^G)_i$) for $i = 1, \dots, r$*

1. *$(X_i^+)^G = (X^G)_i$ (respectively $(X_i^-)^G = (X^G)_i$) for $i = 1, \dots, r$.*

2. *X_i^+ with γ_i^+ (respectively X_i^- with γ_i^-) is a G-fibration over $(X^G)_i$ for $i = 1, \dots, r$.*

3. *For any closed $a \in (X^G)_i$, $T_a(X_i^+) = T_a((X^G)_i) \oplus T_a(X)^+$ (respectively $T_a(X_i^-) = T_a((X^G)_i) \oplus T_a(X)^-$) for $i = 1, \dots, r$.*

In the next section we shall see examples of these decompositions.

4.2 Decompositions of varieties with finite set of fixed points

Let X be a nonsingular algebraic variety (not necessarily complete) over $\mathbb{C}$ and let $\mathbb{C}^*$ be the multiplicative group. Assume $\mathbb{C}^*$ acts algebraically on X with a nonempty fixed points set $X^{\mathbb{C}^*}$. Let $F_1, \ldots, F_s$ be the connected components of $X^{\mathbb{C}^*}$. For any $x \in X$, one has the orbit morphism

$$\begin{aligned} \phi_x \colon \quad & \mathbb{C}^* \to X \\ & t \ \mapsto t \cdot x. \end{aligned} \tag{4.2}$$

4.2.1 Case of complete varieties

When X is complete the morphism (4.2) extends to

$$\begin{aligned} \overline{\phi}_x \colon \quad & \mathbb{P}^1 \to X \\ & t \ \mapsto t \cdot x, \end{aligned}$$

defining the limits $\overline{\phi}_x(0) = \lim_{t\to 0} t \cdot x$ and $\overline{\phi}_x(\infty) = \lim_{t\to\infty} t \cdot x$, see [5].

Let us define the following subsets of X

$$X_i^+ := \{x \in X \mid \lim_{t\to 0} t \cdot x \in F_i\},\ i = 1, \ldots, s.$$

$$X_i^- := \{x \in X \mid \lim_{t\to \infty} t \cdot x \in F_i\},\ i = 1, \ldots, s.$$

These form a decomposition of X into subspaces $X = \bigcup_i X_i^+ = \bigcup_i X_i^-$, the so called *plus* and *minus-decompositions* [5]. The subsets X_i^+ and X_i^- are locally closed by 4.1.2.

Now assume the fixed points set is finite. Then the Białynicki-Birula theorem 4.1.3 can be stated as follows

Theorem 4.2.1. *Let X be a complete variety and let the fixed points set be $X^{\mathbb{C}^*} = \{x_1, \ldots, x_s\}$. For any $i = 1, \ldots, s$ there exists a unique $\mathbb{C}^*$-invariant decomposition of X, $X = \bigcup_{i=1}^s X_i^+$ (resp. $X = \bigcup_{i=1}^s X_i^-$) such that*

1. *$x_i \in X_i^+$(resp. $x_i \in X_i^-$),*

2. *X_i^+(resp. X_i^-) is isomorphic to an affine scheme,*

3. *for any x_i, $T_{x_i}(X_i^+) = T_{x_i}(X)^+$(resp. $T_{x_i}(X_i^-) = T_{x_i}(X)^-$).*

Definition 4.2.2. A decomposition $\{X_i^+\}$ (resp. $\{X_i^-\}$) is said to be *filtrable* if there is a decreasing sequence of closed subvarieties

$$X = X_1 \supset X_2 \supset \cdots \supset X_s \supset X_{s+1} = \emptyset,$$

such that the *cells* of the decomposition are $X_i^+ = X_i \setminus X_{i+1}$ (resp. $X_i^- = X_i \setminus X_{i+1}$) for $i = 1, \ldots, s$.

Remark 4.2.3. 1. $X_i^+ = X_i \setminus X_{i+1} \subset X_i$ and X_i is closed in X so the closure $\overline{X_i^+}$ of X_i^+ lies in $X_i = \cup_{j \geq i} X_j^+$. Hence a filtrable decomposition implies that for each i

$$\overline{X_i^+} \subset \bigcup_{j \geq i} X_j^+. \tag{4.3}$$

Notice that (4.3) also holds for the minus decomposition.

2. A filtrable decomposition yields the existence of a unique cell of maximal dimension X_1^+ (resp. X_1^-) since its complement X_2 is closed in X.

3. Białynicki-Birula's decompositions of projective varieties are filtrable [5, Theorem 3].

Example 4.2.4. An elementary example of Białynicki-Birula's decompositions is the projective space. Consider $\mathbb{P}^l$ with the following $\mathbb{C}^*$-action

$$\begin{aligned} \omega \colon \qquad \mathbb{C}^* \times \mathbb{P}^l &\longrightarrow \mathbb{P}^l \\ (t, (x_0 : x_1 : \cdots : x_l)) &\longmapsto \omega(t^l x_0 : t^{l-1} x_1 : \cdots : x_l), \end{aligned}$$

The fixed points set of this action is given by

$$(\mathbb{P}^l)^{\mathbb{C}^*} = \{x_0 = (1 : 0 : \cdots : 0), \ldots, x_l = (0 : 0 : \cdots : 1)\},$$

and the corresponding plus-cells are $X_0^+ = \{x_0\}, X_1^+ = \mathbb{A}^1, \ldots, X_l^+ = \mathbb{A}^l$. Then the Białynicki-Birula plus-decomposition for $\mathbb{P}^l$ determined by the action ω is $\mathbb{P}^l = \{(1 : 0 : \cdots : 0)\} \cup \mathbb{A}^1 \cup \cdots \cup \mathbb{A}^l$. This decomposition is filtrable. Indeed $\mathbb{P}^l$ has a filtration $\mathbb{P}^l = X_l \supset X_{l-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$ where $X_i = X_i^+ \cup X_{i-1}^+$.

Similarly, one can define the Białynicki-Birula minus-decomposition of $\mathbb{P}^l$.

Example 4.2.5. A less elementary example is the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{P}^2)$ of d points in $\mathbb{P}^2$. From Example 4.2.4 the fixed point set of $\mathbb{P}^2$ is

$$(\mathbb{P}^2)^{\mathbb{C}^*} = \{x_0 = (1 : 0 : 0), x_1 = (0 : 1 : 0), x_2 = (0 : 0 : 1)\}$$

and its Białynicki-Birula decomposition is given by the union of affine spaces $\mathbb{P}^2 = \bigcup_i X_i$ where $i = 0, 1, 2$.

The $\mathbb{C}^*$ action on $\mathbb{P}^2$ naturally induces an action on $\mathrm{Hilb}^d(\mathbb{P}^2)$. Fixed points in $\mathrm{Hilb}^d(\mathbb{P}^2)$ correspond to subschemes $\bar{Z} \subset \mathbb{P}^2$ such that the support of $\bar{Z}$ is contained in the fixed points set of $\mathbb{P}^2$. This allows us to write $\bar{Z} \in \mathrm{Hilb}^d(\mathbb{P}^2)$ as a disjoint union $\bar{Z}_0 \cup \bar{Z}_1 \cup \bar{Z}_2$ where $\bar{Z}_i$ is supported on the fixed point x_i for $i = 0, 1, 2$. For each i, $\bar{Z}_i$ corresponds to a fixed point in $\mathrm{Hilb}^{d_i}(\mathbb{P}^2)$ where d_i is the length of $\mathscr{O}_{\bar{Z}_i}$.

Any subscheme $Z \subset \mathbb{P}^2$ can be written as the disjoint union $Z_0 \cup Z_1 \cup Z_2$ where the support of Z_i is contained in X_i for $i = 0, 1, 2$.

Let (d_0, d_1, d_2) be a triple of nonnegative integers such that $d_0 + d_1 + d_2 = d$. Let $W(d_0, d_1, d_2) \subset \mathrm{Hilb}^d(\mathbb{P}^2)$ be the locally closed subset corresponding to the subscheme Z, where $d_i = \mathrm{length}(\mathscr{O}_{Z_i})$ for $i = 0, 1, 2$.

Since $Z = Z_0 \cup Z_1 \cup Z_2$, the subset $W(d_0, d_1, d_2)$ can be written as follows.

$$W(d_0, d_1, d_2) \cong W(d_0, 0, 0) \times W(0, d_1, 0) \times W(0, 0, d_2).$$

Then the Hilbert scheme decomposes into cells

$$\mathrm{Hilb}^d(\mathbb{P}^2) = \bigcup_{d_0+d_1+d_2=d} W(d_0, 0, 0) \times W(0, d_1, 0) \times W(0, 0, d_2).$$

These cells are described in [15]. Their number in each dimension corresponds to the Betti numbers of the Hilbert scheme. More about this can be found in [14].

4.2.2 Case of noncomplete varieties

Suppose X is not complete and $\lim_{t\to 0} t \cdot x$ exists for every $x \in X$. In this case, the morphism (4.2) extends to $\mathbb{C}$

$$\begin{aligned} \phi'_x \colon \quad & \mathbb{C} \to X \\ & t \;\mapsto t \cdot x, \end{aligned}$$

where $\phi'_x(0) := \lim_{t\to 0} t \cdot x$.
For $i = 1, \ldots, s$ one can define the subsets $X_i^+ := \{x \in X \mid \lim_{t\to 0} t \cdot x \in F_i\}$ which form a decomposition of X into subspaces $X = \bigcup_i X_i^+$ [5]. The cells X_i^+ are locally closed by 4.1.2. Since the variety is nonsingular the cells of the decomposition are nonsingular as well.

Now suppose X as above is quasi-projective and the fixed points set is finite. The cells of the decomposition are isomorphic to affine spaces by Theorem 4.2.1. Under these assumptions the following theorem holds.

Theorem 4.2.6. *The Białynicki-Birula plus-decomposition of X is filtrable.*

Proof. According to [39, Theorem 1] there exists an equivariant projective embedding $X \to \mathbb{P}^l$ for some l.
Consider the action of $\mathbb{C}^*$ on $\mathbb{P}^l$ and let $(\mathbb{P}^l)^{\mathbb{C}^*} = \{x_1, \ldots, x_q\}$ be the finite set of fixed points. Let $\{P_j\}$ be the Białynicki-Birula plus-decomposition of $\mathbb{P}^l$. Then the fixed points set of X is the intersection

$$X^{\mathbb{C}^*} = (\mathbb{P}^l)^{\mathbb{C}^*} \cap X = \{x_{\sigma(1)}, \ldots, x_{\sigma(q)}\},$$

and the cells of the decomposition are $X^+_{\sigma(j)} = X \cap P_j$.
Since the decomposition $\{P_j\}$ is filtrable we consider the following filtration

$$\mathbb{P}^l = Y_1 \supset Y_2 \supset \cdots \supset Y_q \supset Y_{q+1} = \emptyset,$$

where $P_j = Y_j \setminus Y_{j+1}$ for $j = 1, \ldots, q$. Then

$$X^+_{\sigma(j)} = X \cap P_j = X \cap (Y_j \setminus Y_{j+1}) = (X \cap Y_j) \setminus (X \cap Y_{j+1}) = X_{\sigma(j)} \setminus X_{\sigma(j+1)},$$

where $X_{\sigma(j)} := (X \cap Y_j)$ for $j = 1, \ldots, q$ and $X_{\sigma(j+1)} \subset X_{\sigma(j)}$. Notice that for each j, $X_{\sigma(j)} = X \cap Y_j \subset X$ is a closed subvariety of X since Y_j is a closed subvariety of $\mathbb{P}^l$. It follows that the plus-decomposition of X is filtrable. □

Remark 4.2.7. Theorem 4.2.6 is also true in the case the fixed points set is not finite. The proof is basically the same.
The same holds if instead of considering the existence of the limit $\lim_{t \to 0} t \cdot x$, one considers the existence of the limit $\lim_{t \to \infty} t \cdot x$.

4.3 Decompositions determined by an n-torus

We recall that an *n-dimensional torus T* is an affine variety which is isomorphic to $(\mathbb{C}^*)^n$. It inherits a group structure from this isomorphism.

More generally, if we have an action of an n-dimensional torus the Białynicki-Birula decompositions enjoy the same properties as in the previous sections. This is due to the following lemma.

Lemma 4.3.1. *Suppose X is a normal algebraic variety endowed with a linear action of a torus T. Suppose the T-action gives rise to a nontrivial set of fixed points X^T. Then there exists a one-parameter subgroup λ of T such that $X^\lambda = X^T$.*

Proof. By [39, Corollary 2] X can be covered by a finite number of affine T-invariant open subsets. So we may assume that X is affine hence a closed subvariety of $\mathbb{A}^l$ for some l. Since the torus T acts linearly on $\mathbb{A}^l$ we will assume $X = \mathbb{A}^l$. Call the weights of T in $\mathbb{A}^l$ by $\chi_1, \ldots, \chi_l$, then it is enough to choose a one-parameter subgroup λ such that $< \chi_i, \lambda > \neq 0$ for all i. □

Remark 4.3.2. A one-parameter subgroup as in Lemma 4.3.1 is called *regular*.

Chapter 5

Moduli space on the projective plane

In this chapter we review some results on the moduli space $M(r,n)$ studied in [33, 34, 35]. This moduli space admits a projective morphism to the moduli space $M_0(r,n)$ of ideal instantons on S^4. Both $M(r,n)$ and $M_0(r,n)$ admit a torus action under which the projective morphism from $M(r,n)$ to $M_0(r,n)$ is equivariant. It is possible then to define a subvariety $\pi^{-1}(n[0])$ of $M(r,n)$ which is invariant under the torus action. We prove this subvariety is irreducible using the fact that it is isomorphic to the punctual quot scheme.

Following the results of chapter 4, we show in section 5.3 that $M(r,n)$ admits a Białynicki-Birula plus-decomposition and the union of the minus-cells builds up $\pi^{-1}(n[0])$. Moreover, these decompositions are filtrable.

In section 5.4 we show that the inclusion of $\pi^{-1}(n[0])$ into $M(r,n)$ induces isomorphisms of homology groups with integer coefficients. Furthermore, we show that this inclusion induces a homotopy equivalence between $M(r,n)$ and $\pi^{-1}(n[0])$.

5.1 Generalities on the moduli space

Most of the material in this section can be found in [33, 34, 35]. Let $M(r,n)$ be the moduli space of framed torsion free sheaves on $\mathbb{P}^2$ with rank r and second Chern class n parametrizing isomorphism classes of $(\mathscr{E},\phi)$ such that

1. $\mathscr{E}$ is a torsion free sheaf on $\mathbb{P}^2$ of rank r and second Chern class n. $\mathscr{E}$ is locally free in a neighborhood of ℓ_∞.

2. $\phi\colon \mathscr{E}|_{\ell_\infty} \xrightarrow{\cong} \mathscr{O}_{\ell_\infty}^{\oplus r}$ *framing at infinity*,

where $\ell_\infty = \{[0 : z_1 : z_2] \in \mathbb{P}^2\}$ is the line at infinity. We say two framed sheaves $(\mathscr{E}, \phi), (\mathscr{E}', \phi')$ are *isomorphic* if there exists an isomorphism $\psi : \mathscr{E} \to \mathscr{E}'$ such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{E}|_{\ell_\infty} & \xrightarrow{\psi|_{\ell_\infty}} & \mathscr{E}'|_{\ell_\infty} \\ & {}_{\phi}\searrow \quad \swarrow_{\phi'} & \\ & \mathscr{O}_{\ell_\infty}^{\oplus r} & \end{array}$$

We have the following vanishing theorem.

Lemma 5.1.1. $\mathrm{Hom}(\mathscr{E}, \mathscr{E}(-\ell_\infty)) = \mathrm{Ext}^2(\mathscr{E}, \mathscr{E}(-\ell_\infty)) = 0$.

Proof. $\mathrm{Ext}^2(\mathscr{E}, \mathscr{E}(-\ell_\infty))$ is the dual of $\mathrm{Hom}(\mathscr{E}, \mathscr{E}(-2\ell_\infty))$ by the theorem of Serre-Grothendieck. Applying the $\mathrm{Hom}(\mathscr{E}, -)$ to the following short exact sequence

$$0 \longrightarrow \mathscr{E}(-(k+1)\ell_\infty) \longrightarrow \mathscr{E}(-k\ell_\infty) \longrightarrow \mathscr{E}(-\ell_\infty) \otimes \mathscr{O}_{\ell_\infty} \longrightarrow 0,$$

one gets

$$0 \to \mathrm{Hom}(\mathscr{E}, \mathscr{E}(-(k+1)\ell_\infty)) \to \mathrm{Hom}(\mathscr{E}, \mathscr{E}(-k\ell_\infty)) \to \mathrm{Hom}(\mathscr{E}, \mathscr{E}(-\ell_\infty) \otimes \mathscr{O}_{\ell_\infty}),$$

where k is a positive integer. The last term in the exact sequence vanishes since $\mathscr{E}|_{\ell_\infty}$ is trivial. We then have

$$\mathrm{Hom}(\mathscr{E}, \mathscr{E}(-(k+1)\ell_\infty)) \cong \mathrm{Hom}(\mathscr{E}, \mathscr{E}(-k\ell_\infty)) \cong \cdots \cong \mathrm{Hom}(\mathscr{E}, \mathscr{E}(-\ell_\infty)).$$

On the other hand $\mathrm{Hom}(\mathscr{E}, \mathscr{E}(-k\ell_\infty)) \cong \mathrm{Ext}^2(\mathscr{E}, \mathscr{E}((k-3)\ell_\infty))^\vee$. By Serre's vanishing theorem $\mathrm{Ext}^2(\mathscr{E}, \mathscr{E}((k-3)\ell_\infty))^\vee = 0$ for k large. This concludes the proof. $\square$

Theorem 5.1.2. *$M(r, n)$ is a nonsingular quasi-projective variety of dimension $2nr$.*

Proof. Using the Riemann-Roch formula, this theorem follows from Lemma 5.1.1. $\square$

Let $M_0^{\mathrm{reg}}(r, n) \subset M(r, n)$ be the open subset of locally free sheaves. We define the Uhlenbeck (partial) compactification of M_0^{reg} set theoretically as follows

$$M_0(r, n) := \bigsqcup_{k=0}^{n} M_0^{\mathrm{reg}}(r, n-k) \times \mathrm{Sym}^k \mathbb{C}^2,$$

where $\operatorname{Sym}^k \mathbb{C}^2$ is the k-th symmetric product of $\mathbb{C}^2$.

The moduli space $M(r,n)$ has a description in linear data as the next theorem shows. The proof is quite long, we omit it here since it can be found in details in [33, Chapter 2].

Theorem 5.1.3. *There exists a bijection between $M(r,n)$ and the quotient of linear data (B_1, B_2, i, j) by the action of $\mathrm{GL}_n(\mathbb{C})$ such that*

$$(i)\ [B_1, B_2] + ij = 0, \tag{5.1}$$

$$(ii)\ \text{there exists no subspace } S \subsetneq \mathbb{C}^n \text{ such that } B_\alpha(S) \subset S\ (\alpha = 1,2) \text{ and } \operatorname{im} i \subset S, \tag{5.2}$$

where $B_1, B_2 \in \operatorname{End}(\mathbb{C}^n)$, $i \in \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, $j \in \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ and for $g \in \mathrm{GL}_n(\mathbb{C})$ the action is given by

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}).$$

Remark 5.1.4. The description given in Theorem 5.1.3 is often referred to as the ADHM description named after the authors of [1].

In terms of linear data $M_0(r,n)$ can be identified with the following GIT quotient.

$$M_0(r,n) \cong \{(B_1, B_2, i, j) \mid [B_1, B_2] + ij = 0\} /\!\!/ \mathrm{GL}_n(\mathbb{C}). \tag{5.3}$$

The open locus $M_0^{\mathrm{reg}}(r,n) \in M_0(r,n)$ consists of the closed orbits such that the stabilizer is trivial.

Consider the following projective morphism

$$\begin{aligned} \pi\colon \quad & M(r,n) \to M_0(r,n) \\ & (\mathscr{E}, \phi) \ \mapsto ((\mathscr{E}^{\vee\vee}, \phi), \operatorname{Supp}(\mathscr{E}^{\vee\vee}/\mathscr{E})) \in M_0^{\mathrm{reg}}(r, n-k) \times \operatorname{Sym}^k \mathbb{C}^2, \end{aligned} \tag{5.4}$$

where $\mathscr{E}^{\vee\vee}$ is the double dual of $\mathscr{E}$ and $\operatorname{Supp}(\mathscr{E}^{\vee\vee}/\mathscr{E})$ is the topological support of $\mathscr{E}^{\vee\vee}/\mathscr{E}$ counted with multiplicities.

For $k = n$, $M_0^{\mathrm{reg}}(r,0) \times \operatorname{Sym}^n \mathbb{C}^2 \simeq \operatorname{Sym}^n \mathbb{C}^2$ since $M_0^{\mathrm{reg}}(r,0)$ consists of one point that is the isomorphism class of the pair $\mathscr{O}_{\mathbb{P}^2}^{\oplus r}$ together with the trivial framing.

Consider the point $n[0] \in \operatorname{Sym}^n \mathbb{C}^2$ that is the point $0 \in \mathbb{C}^2$ counted n times. The inverse image of $n[0]$ by π is given by

$$\pi^{-1}(n[0]) = \{(\mathscr{E}, \phi) \mid \mathscr{E}^{\vee\vee} \simeq \mathscr{O}_{\mathbb{P}^2}^{\oplus r},\ \operatorname{Supp}(\mathscr{O}_{\mathbb{P}^2}^{\oplus r}/\mathscr{E}) = n[0]\}/\cong .$$

Theorem 5.1.5. *$\pi^{-1}(n[0])$ is a compact subvariety of $M(r,n)$.*

Proof. From the projective morphism (5.4), one has

$$\begin{array}{ccc} M(r,n) & \xrightarrow{\pi} & M_0(r,n) \\ \cup & & \Cup \\ \pi^{-1}(n[0]) & \longrightarrow & n[0] \end{array}$$

where the fibre $\pi^{-1}(n[0]) \cong M(r,n)_{n[0]} = M(r,n) \times_{n[0]} \operatorname{Spec} k(n[0])$ is a scheme over $k(n[0]) = k$ [18, page 89]. To prove $\pi^{-1}(n[0])$ is a variety, it is enough to prove it is a separated k-scheme of finite type [18, page 105]. $M(r,n)$ is an algebraic variety, then there exists a morphism $\phi : M(r,n) \to \operatorname{Spec} k$ that is separated and of finite type. Consider the following composition

$$\pi^{-1}(n[0]) \cong M(r,n)_{n[0]} \xhookrightarrow{i} M(r,n) \xrightarrow{\phi} \operatorname{Spec} k, \qquad \psi = \phi \circ i$$

where $i\colon M(r,n)_{n[0]} \hookrightarrow M(r,n)$ is a closed embedding, then it is separated [18, Corollary 4.6(a)] and of finite type [18, Exercise 3.13(a)]. Hence $\pi^{-1}(n[0])$ is an algebraic variety since the composition ψ is separated and of finite type. $\pi^{-1}(n[0])$ is closed in $M(r,n)$ since it is the inverse image of a closed point by a proper map, it is compact by the same argument. Hence $\pi^{-1}(n[0])$ is a compact subvariety of $M(r,n)$. □

Theorem 5.1.6 ([34, Theorem 3.5 (1)])**.** *$\pi^{-1}(n[0])$ is isomorphic to the punctual quot-scheme* $\operatorname{Quot}(r,n)$ *parameterizing zero dimensional quotients $\mathscr{O}_{\mathbb{P}^2}^{\oplus r} \to \mathscr{Q}$ with* $\operatorname{Supp}(\mathscr{Q}) = n[0]$*.*

Proof. Given a point in

$$\pi^{-1}(n[0]) = \left\{(\mathscr{E},\phi) \;\middle|\; \mathscr{E}^{\sim\sim} \simeq \mathscr{O}_{\mathbb{P}^2}^{\oplus r},\ \operatorname{Supp}(\mathscr{O}_{\mathbb{P}^2}^{\oplus r}/\mathscr{E}) = n[0]\right\}/\cong,$$

one has the exact sequence

$$0 \to \mathscr{E} \to \mathscr{O}_{\mathbb{P}^2}^{\oplus r} \to \mathscr{O}_{\mathbb{P}^2}^{\oplus r}/\mathscr{E} \to 0.$$

Thus the quotient $\mathscr{O}_{\mathbb{P}^2}^{\oplus r} \twoheadrightarrow \mathscr{O}_{\mathbb{P}^2}^{\oplus r}/\mathscr{E}$ is a point in the punctual quot-scheme defined by

$$\operatorname{Quot}(r,n) := \left\{\mathscr{O}_{\mathbb{P}^2}^{\oplus r} \twoheadrightarrow \mathscr{Q} \,|\operatorname{Supp} \mathscr{Q} = n[0]\right\}/\cong\ .$$

Conversely, given a point in $\operatorname{Quot}(r,n)$, let $\mathscr{K} := \ker(\mathscr{O}_{\mathbb{P}^2}^{\oplus r} \twoheadrightarrow \mathscr{Q}$ in $\operatorname{Quot}(r,n))$. We have the exact sequence

$$0 \to \mathscr{K} \to \mathscr{O}_{\mathbb{P}^2}^{\oplus r} \to \mathscr{Q} \to 0.$$

$\mathscr{K}$ is torsion free since it injects into a locally free sheaf. By using Lemma B.0.1, it follows that $\mathscr{K}^{\sim} \simeq \mathscr{O}_{\mathbb{P}^2}^{\oplus r}$. Notice that $\mathscr{Q}$ is supported on the point $0 \in \mathbb{P}^2$ so it vanishes on $\mathbb{P}^2 \setminus \{0\}$. In particular, $\mathscr{Q}|_{\ell_\infty}$ vanishes and the exact sequence reduces to

$$0 \longrightarrow \mathscr{E}|_{\ell_\infty} \xrightarrow{\cong} \mathscr{O}_{\ell_\infty}^{\oplus r} \longrightarrow 0.$$

This isomorphism defines the framing ϕ. The pair $(\mathscr{E}, \phi)$ is a point in $\pi^{-1}(n[0])$. □

Theorem 5.1.7. $\pi^{-1}(n[0])$ *is irreducible projective of dimension* $n(r+1)$.

Proof. The quot scheme is projective [24, Theorem 2.2.4] and irreducible of dimension $n(r+1)$ [24, Theorem 6.A.1]. Hence by Theorem 5.1.6, $\pi^{-1}(n[0])$ is an irreducible projective variety of dimension $n(r+1)$ as well. □

5.2 Torus action and fixed points

We will follow the same notation as in [34]. Let $\widetilde{T} := \mathbb{C}^* \times \mathbb{C}^* \times T$ where T is the maximal torus in $\mathrm{GL}(r, \mathbb{C})$. The action of $\widetilde{T}$ on $M(r,n)$ is defined as follows.
For $(t_1, t_2) \in \mathbb{C}^* \times \mathbb{C}^*$ let F_{t_1,t_2} be the automorphism of $\mathbb{P}^2$ defined by

$$F_{t_1,t_2}([z_0 : z_1 : z_2]) := [z_0 : t_1 z_1 : t_2 z_2]$$

and for $(e_1, \dots, e_r) \in T$ let $G_{e_1,\dots,e_r}$ be the isomorphism of $\mathscr{O}_{\ell_\infty}^{\oplus r}$ defined by

$$G_{e_1,\dots,e_r}(s_1, \dots, s_r) := (e_1 s_1, \dots, e_r s_r).$$

Then the action of $\widetilde{T}$ on a pair $(E, \Phi) \in M(r,n)$ is defined by

$$(t_1, t_2, e_1, \dots, e_r) \cdot (E, \Phi) := \left((F_{t_1,t_2}^{-1})^* E, \Phi'\right)$$

where Φ' is given by:

$$(F_{t_1,t_2}^{-1})^* E|_{\ell_\infty} \xrightarrow{(F_{t_1,t_2}^{-1})^* \Phi} (F_{t_1,t_2}^{-1})^* \mathscr{O}_{\ell_\infty}^{\oplus r} \longrightarrow \mathscr{O}_{\ell_\infty}^{\oplus r} \xrightarrow{G_{e_1,\dots,e_r}} \mathscr{O}_{\ell_\infty}^{\oplus r}$$

The $\widetilde{T}$ action is defined in a similar way on $M_0(r,n)$ and the map π given in (5.4) is equivariant.

One can define the torus action on $M(r,n)$ and $M_0(r,n)$ using the description by the linear data defined in section 5.1. The torus action can be identified with the action on the linear data given as follows

$$(B_1, B_2, i, j) \mapsto (t_1 B_1, t_2 B_2, i e^{-1}, t_1 t_2 e j)$$

where $t_1, t_2 \in \mathbb{C}^*$ and $e = (e_1, e_2, \dots, e_r) \in \mathbb{C}^r$. Note that this action preserves the conditions (5.1), (5.2), and commutes with the action of $\mathrm{GL}(r, \mathbb{C})$.

Theorem 5.2.1 ([35, Proposition 2.9])**.**

1. *The fixed points set $M(r,n)^T$ consists of finitely many points.*
2. *The fixed points set $M_0(r,n)^T$ consists of a single point $n[0] \in Sym^n\mathbb{C}^2$.*

Proof. 1. A framed sheaf $(\mathscr{E},\phi)$ is fixed if it can be written in the form $(\mathscr{E},\phi) = (\mathscr{E}_1,\phi_1) \oplus \cdots \oplus (\mathscr{E}_r,,\phi_r)$ such that $\mathscr{E}_i = \mathscr{I}_i$ is the ideal sheaf of a zero dimensional subscheme Z_i in $\mathbb{P}^2 \setminus \ell_\infty \cong \mathbb{C}^2$, and ϕ_i is an isomorphism $\mathscr{E}_i|_{\ell_\infty} \xrightarrow{\cong} \mathscr{O}_{\ell_\infty}$.
The ideal sheaves $\mathscr{I}_i$ are fixed if they are generated by monomials in the coordinate ring of $\mathbb{C}^2$ [14]. These monomials are finite and hence the ideal sheaves $\mathscr{I}_i$ form a finite family. As a result the fixed points set of $M(r,n)$ is finite.

2. Regarding $M_0^{reg}(r,n)$ as an open subset of $M(r,n)$, the set of its fixed points is finite. A framed sheaf $(\mathscr{E},\phi) \in M_0^{reg}(r,n-k)$ is fixed if is it given as above, hence $\mathscr{E} = \bigoplus_{i=1}^r \mathscr{E}_i = \bigoplus_{i=1}^r (\mathscr{I}_i)$. Since $\mathscr{E}$ is locally free then so is $\mathscr{I}_i$ for each i. This means that $\mathscr{I}_i \cong \mathscr{I}_i^{\vee\vee} \cong \mathscr{O}_{\ell_\infty}$, for each i. Hence $\mathscr{E} = \bigoplus_{i=1}^r \mathscr{O}_{\ell_\infty}$. This is a fixed point of $M_0^{reg}(r,n-k)$ if and only if $k=n$. As already noticed in the previous section, for $k=n$, $M_0^{\mathrm{reg}}(r,0) \times \mathrm{Sym}^n\,\mathbb{C}^2 \simeq \mathrm{Sym}^n\,\mathbb{C}^2$ since $M_0^{\mathrm{reg}}(r,0)$ consists of one point that is the isomorphism class of the pair $\mathscr{O}_{\mathbb{P}^2}^{\oplus r}$ together with the trivial framing. It is indeed the only fixed point of $M_0(r,n)$. □

Remark 5.2.2. From Theorem 5.2.1 it follows that $\pi^{-1}(n[0])$ contains all the fixed points of $M(r,n)$ since π is equivariant.

5.3 Białynicki-Birula decompositions

In this section, following the results of section 4 we show that both $M(r,n)$ and $\pi^{-1}(n[0])$ admit a filtrable Białynicki-Birula decomposition. It is enough to consider the action of a regular one-parameter subgroup by Lemma 4.3.1. From now on we will consider the action of $\mathbb{C}^*$ on $M(r,n)$ and on $\pi^{-1}(n[0])$.

Proposition 5.3.1. *For every element $x \in M(r,n)$, the limit $\lim_{t\to 0} t\cdot x$ exists and lies in $M(r,n)^{\mathbb{C}^*}$.*

Proof. Using the description (5.3) of $M_0(r,n)$, there exists a one-parameter subgroup of $\widetilde{T}$ such that for all x in $M_0(r,n)$, the limit $\lim_{t\to 0} t\cdot x$ exists and is $(B_1,B_2,i,j) = (0,0,0,0)$. The point $(B_1,B_2,i,j) = (0,0,0,0)$ is identified by the description (5.3) with the point $n[0] \in S^n\mathbb{C}^2$ which is the only fixed point of $M_0(r,n)$. Since π is a projective morphism, for all x in $M(r,n)$ the limit $\lim_{t\to 0} t\cdot x$ exists and lies in $\pi^{-1}(n[0])$. In particular, it is a fixed point. □

Theorem 5.3.2. *$M(r,n)$ admits a Białynicki-Birula plus-decomposition into affine spaces. Moreover, this decomposition is filtrable.*

Proof. By Proposition 5.3.1, the limit $\lim_{t\to 0} t\cdot x$ exists and is a fixed point. It follows that the orbit morphism

$$\begin{aligned} \phi_x\colon \quad \mathbb{C}^* &\to M(r,n) \\ t &\mapsto t\cdot x, \end{aligned}$$

extends to

$$\begin{aligned} \phi'_x\colon \quad \mathbb{C} &\to M(r,n) \\ t &\mapsto t\cdot x, \end{aligned}$$

where $\phi'_x(0) := \lim_{t\to 0} t\cdot x$. Hence $M(r,n)$ admits a Białynicki-Birula decomposition into affine spaces by Theorem 4.2.1. This decomposition is filtrable by Theorem 4.2.6 since $M(r,n)$ is quasi-projective. □

Let $M(r,n)^{\mathbb{C}^*} = \{x_i \mid i = 1,\dots,m\}$ and denote the cells of the decomposition by

$$M_i^+ := M(r,n)_i^+ = \{x \in M(r,n) \mid \lim_{t\to 0} t\cdot x = x_i\}.$$

The limit $\lim_{t\to\infty} t\cdot x$ does not exist for all $x \in M(r,n)$. We will only consider the points of $M(r,n)$ such that the limit exists. These points define the subspaces

$$M_i^- := M(r,n)_i^- = \{x \in M(r,n) \mid \lim_{t\to\infty} t\cdot x \text{ exists and equals } x_i\}.$$

Theorem 5.3.3. *The subvariety $\pi^{-1}(n[0])$ is the union $\pi^{-1}(n[0]) = \bigcup_i M_i^-$. Moreover, this is a filtrable decomposition of $\pi^{-1}(n[0])$ into affine spaces.*

Proof. The proof of the first claim can be found in [34, Theorem 3.5(3)]. In fact, for any $x \neq n[0]$ in $M_0(r,n)$ the limit $\lim_{t\to\infty} t\cdot x$ does not exist. Hence by the projective morphism (5.4) we deduce $\pi^{-1}(n[0]) = \bigcup_i M_i^-$. To show this is a filtrable decomposition we apply the proof of Theorem 4.2.6 to the equivariant embedding $\pi^{-1}(n[0]) \hookrightarrow \mathbb{P}^l$ obtained by composing the equivariant embeddings $\pi^{-1}(n[0]) \hookrightarrow M(r,n)$ and $M(r,n) \hookrightarrow \mathbb{P}^l$. Finally, regarding the M_i^-'s as subspaces of $M(r,n)$ and using [4, Theorem 4.1(b)], we conclude they are isomorphic to affine spaces. □

Remark 5.3.4. The subvariety $\pi^{-1}(n[0])$ being isomorphic to the punctual quot scheme is not smooth. Still the Białynicki-Birula decompositions for $\pi^{-1}(n[0])$ exist. From [39, Theorem 1] together with [39, Lemma 8] it follows that for any $x \in M(r,n)$ there exists an equivariant embedding of some neighborhood of x into $\mathbb{P}^l$. Hence for any $x \in \pi^{-1}(n[0])$ there exists an equivariant embedding of some neighborhood of x into $\mathbb{P}^l$ by composition since $\pi^{-1}(n[0])$ has an equivariant embedding into $M(r,n)$. Thus the results of [27, §1] hold for $\pi^{-1}(n[0])$.

5.4 Topological properties

Let $J = \{1, 2, \dots, m\}$ be the set of indices so that the fixed points set $X^{\mathbb{C}^*} = \{x_j \mid j \in J\}$ be ordered to yield the following filtrations

$$M(r,n) = M_1 \supset M_2 \supset \cdots \supset M_m \supset M_{m+1} = \emptyset, \tag{5.5}$$

$$\emptyset = \pi_1 \subset \pi_2 \subset \cdots \subset \pi_m \subset \pi_{m+1} = \pi^{-1}(n[0]), \tag{5.6}$$

where $M_i \backslash M_{i+1} = M_i^+$, $\pi_{i+1} \backslash \pi_i = M_i^-$ and the decomposition of $M(r,n)$ (resp. $\pi^{-1}(n[0])$) is given by $M(r,n) = \bigcup_{j \in J} M_j^+$ (resp. $\pi^{-1}(n[0]) = \bigcup_{j \in J} M_j^-$).

Define the subsets $M_{\le j}^+ := \bigcup_{i \le j} M_i^+$ and $M_{\le j}^- := \bigcup_{i \le j} M_i^-$. Then the following holds

Lemma 5.4.1. *For each j, there is an inclusion $M_{\le j}^- \hookrightarrow M_{\le j}^+$.*

Proof. We prove the assertion by using the filtration (5.5). The intersection $M_j^- \cap M_j^+$ is the fixed point $\{x_j\}$, so $M_j^- \cap (M_j \setminus M_{j+1}) = \{x_j\}$. This means $M_j^- \cap M_{j+1} = \emptyset$ since $x_j \in M_j$, $x_j \notin M_{j+1}$ and $M_{j+1} \subset M_j$. Note that $M_{j+1} = M_{\ge j+1}^+$ so $M_j^- \subset M_{\le j}^+$ for all j. Hence we get the inclusion $M_{\le j}^- \subset M_{\le j}^+$ for all j. □

Theorem 5.4.2. *The inclusion $M_{\le j}^- \hookrightarrow M_{\le j}^+$ induces isomorphisms of homology groups with integer coefficients $H_k(M_{\le j}^-) \xrightarrow{\cong} H_k(M_{\le j}^+)$ for all j and all k. In particular the inclusion $\pi^{-1}(n[0]) \hookrightarrow M(r,n)$ induces isomorphisms $H_k(\pi^{-1}(n[0])) \xrightarrow{\cong} H_k(M(r,n))$ for all k.*

Proof. We prove the theorem by induction on j.

For $j = 1$ the inclusion $M_1^- \hookrightarrow M_1^+$ induces an isomorphism of homology groups $H_k(M_1^-) \xrightarrow{\cong} H_k(M_1^+)$ for all k since M_1^- and M_1^+ are isomorphic to affine spaces.

Suppose the inclusion $M_{\le j}^- \hookrightarrow M_{\le j}^+$ induces an isomorphism of homology groups $H_k(M_{\le j}^-) \xrightarrow{\cong} H_k(M_{\le j}^+)$ for all k and consider the homology long exact sequences of the pairs $(M_{\le j+1}^-, M_{\le j}^-)$ and $(M_{\le j+1}^+, M_{\le j}^+)$ respectively. We have the following diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H_{k+1}(M^-_{\leq j+1}, M^-_{\leq j}) & \longrightarrow & H_k(M^-_{\leq j}) & \longrightarrow & H_k(M^-_{\leq j+1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow \cong & & \downarrow & & \\
\cdots & \longrightarrow & H_{k+1}(M^+_{\leq j+1}, M^+_{\leq j}) & \longrightarrow & H_k(M^+_{\leq j}) & \longrightarrow & H_k(M^+_{\leq j+1}) & \longrightarrow & \cdots
\end{array}$$

where the arrows are induced by inclusions. The middle arrow is an isomorphism by the induction hypothesis.

$M^-_{\leq j}$ is closed in $M^-_{\leq j+1}$. This follows from (4.3) by reversing the order of the inequality to meet that of the filtration (5.6). From [19, Proposition 2.22]

$$H_k(M^-_{\leq j+1}, M^-_{\leq j}) \cong H_k(M^-_{\leq j+1}/M^-_{\leq j}, p),$$

where p is the point at infinity. The quotient space $M^-_{\leq j+1}/M^-_{\leq j}$ is isomorphic to the one-point compactification of M^-_{j+1} that is homeomorphic to the Thom space $T(M^-_{j+1})$ of M^-_{j+1} [12, Ex. 138]. It follows that

$$H_k(M^-_{\leq j+1}, M^-_{\leq j}) \cong H_k(T(M^-_{j+1}), p).$$

M^-_{j+1} is isomorphic to an affine space so it is isomorphic to $N_{|x_{j+1}}$, the normal bundle to x_{j+1} in M^-_{j+1}. Hence

$$H_k(M^-_{\leq j+1}, M^-_{\leq j}) \cong H_k(T(M^-_{j+1}), p) \cong H_k(T(N_{|x_{j+1}}), p).$$

Let us denote by $V(M^+_{j+1})$ the tubular neighborhood of M^+_{j+1} in $M^+_{\leq j+1}$. By excision we get the following isomorphism

$$H_k(M^+_{\leq j+1}, M^+_{\leq j}) = H_k(M^+_{\leq j} \cup V(M^+_{j+1}), M^+_{\leq j}) \cong H_k(V(M^+_{j+1}), \partial V(M^+_{j+1})),$$

where $\partial V(M^+_{j+1}) = V(M^+_{j+1}) \setminus M^+_{j+1}$.
Denote by N the normal bundle of M^+_{j+1}. From the tubular neighborhood theorem $V(M^+_{j+1})$ is homeomorphic to N and M^+_{j+1} is homeomorphic to the zero section of N, see *e.g.* [6]. Thus

$$H_k(V(M^+_{j+1}), \partial V(M^+_{j+1})) \cong H_k(N, N_0) \cong H_k(T(N), p),$$

where $N_0 \subset N$ is the complement of the zero section in N and $T(N)$ is the Thom space of N.

Since the point x_{j+1} is the deformation retract of M^+_{j+1} then N deformation retracts to $N_{x_{j+1}}$, the fibre of N at x_{j+1}. Moreover, $N_{x_{j+1}}$ is a deformation retract of $N_{|x_{j+1}}$. It follows that $H_k(T(N_{|x_{j+1}}), p) \cong H_k(T(N_{x_{j+1}}), p)$ and hence $H_k(M^-_{\leq j+1}, M^-_{\leq j}) \cong H_k(M^+_{\leq j+1}, M^+_{\leq j})$ for all j and all k. By the five lemma we conclude the proof. □

Lemma 5.4.3. *$M(r,n)$ is simply connected.*

Proof. There exists a unique cell M_1^+ of maximal dimension that is open in $M(r,n)$ (see Remark 4.2.3). Since $M_1^+ \subset M(r,n)$ is isomorphic to an affine space then $\pi_1(M_1^+) = 0$. By [11, Theorem 12.1.5] the inclusion $M_1^+ \hookrightarrow M(r,n)$ induces a surjective map $\pi_1(M_1^+) \to \pi_1(M(r,n))$. Hence $\pi_1(M(r,n)) = 0$. □

Remark 5.4.4. Note that $M(r,n)$ is irreducible since it is nonsingular and connected.

Lemma 5.4.5. *$\pi^{-1}(n[0])$ is simply connected.*

Proof. Let $M_m^- \hookrightarrow \pi^{-1}(n[0])$ be the inclusion of the cell of maximal dimension into $\pi^{-1}(n[0])$. By [10, Theorem 3(a)] the induced map $\pi_1(M_m^-) \to \pi_1(\pi^{-1}(n[0]))$ is an isomorphism. $\pi_1(M_m^-) = 0$ since M_m^- is isomorphic to an affine space. Hence $\pi^{-1}(n[0])$ is simply connected. □

Theorem 5.4.6 ([34, Theorem 3.5 (2)])**.** *$\pi^{-1}(n[0])$ is homotopy equivalent to $M(r,n)$.*

Proof. The inclusion $\pi^{-1}(n[0]) \hookrightarrow M(r,n)$ induces morphisms of homotopy groups and we have the following diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_{k+1}(M(r,n),\pi^{-1}(n[0])) & \longrightarrow & \pi_k(\pi^{-1}(n[0])) & \longrightarrow & \pi_k(M(r,n)) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & H_{k+1}(M(r,n),\pi^{-1}(n[0])) & \longrightarrow & H_k(\pi^{-1}(n[0])) & \xrightarrow{\cong} & H_k(M(r,n)) & \longrightarrow & \cdots \end{array}$$

where the isomorphism is from Theorem 5.4.2 and $H_k(M(r,n),\pi^{-1}(n[0])) = 0$ for all k.

From Lemma 5.4.3 and Lemma 5.4.5, both $M(r,n)$ and $\pi^{-1}(n[0])$ are 1-connected, then using Hurewicz's theorem we get

$$\pi_2(M(r,n)) \cong H_2(M(r,n)),$$

$$\pi_2(\pi^{-1}(n[0])) \cong H_2(\pi^{-1}(n[0])).$$

This yields the following diagram

$$\begin{array}{ccccccc} \pi_2(\pi^{-1}(n[0])) & \longrightarrow & \pi_2(M(r,n)) & \longrightarrow & \pi_2(M(r,n),\pi^{-1}(n[0])) & \longrightarrow & 0 \\ \downarrow{\cong} & & \downarrow{\cong} & & \downarrow & & \\ H_2(\pi^{-1}(n[0])) & \xrightarrow{\cong} & H_2(M(r,n)) & \longrightarrow & 0 & & \end{array}$$

Hence we get $\pi_2(M(r,n)) \cong \pi_2(\pi^{-1}(n[0]))$ and $\pi_2(M(r,n), \pi^{-1}(n[0])) = 0$; then the pair $(M(r,n), \pi^{-1}(n[0]))$ is 2-connected. By the relative Hurewicz theorem we have

$$\pi_3(M(r,n), \pi^{-1}(n[0])) \cong H_3(M(r,n), \pi^{-1}(n[0])) = 0.$$

Iterating this process it follows that for every k,

$$\pi_k(M(r,n), \pi^{-1}(n[0])) \cong H_k(M(r,n), \pi^{-1}(n[0])) = 0.$$

Hence the long exact sequence of homotopy groups reduces to

$$\pi_k(\pi^{-1}(n[0])) \xrightarrow{\cong} \pi_k(M(r,n))$$

for all k. Finally Whitehead's theorem (see *e.g.* [37, page 370]) concludes the proof. □

Remark 5.4.7. Another proof for theorem 5.4.6 can be done by using the ADHM data defined in Theorem 5.1.3. This proof can be found in Appendix A, it is elementary and relies essentially on the ADHM description. Hence it is possible to generalize it only to moduli spaces having such a description.

Chapter 6

A generalization

In this chapter, we will generalize the study of the moduli space of framed sheaves on $\mathbb{P}^2$ performed in the previous chapter to the moduli space on a nonsingular projective toric surface S where we assume that there exists a projective morphism of toric surfaces $p : S \to \mathbb{P}^2$ of degree 1 and consider the framing sheaf to be supported on a divisor.

First we study the fixed point locus of the moduli space on toric varieties. Then we restrict ourselves to the moduli spaces $M(S)$ on the toric surface S and construct a projective morphism from $M(S)$ to $M_0(r, n)$, the moduli space of ideal instantons introduced in section 5.1. Using this projective morphism we define a compact subvariety $\widetilde{N}$ of $M(S)$ and study some topological properties, namely singular homology and homotopy equivalence between $M(S)$ and $\widetilde{N}$.

6.1 Moduli on toric surfaces

Let S be a nonsingular projective toric surface, and let $M(S)$ be the moduli space of framed torsion-free sheaves on S. Then this moduli space is a quasi-projective variety [8].

We denote by $M^{\mu ss}(S)$ the moduli space of semistable framed sheaves on S, and by $M^{\mu\text{-poly}}(S)$ the moduli space of polystable framed sheaves on S.

As shown in [7], there is a projective morphism γ from $M(S)$ onto $M^{\mu ss}(S)$ given as follows

$$\begin{aligned} \gamma : \qquad M &\longrightarrow M^{\mu ss}(S) = \coprod_{k \geq 0} M^{\mu\text{-poly}}(r, \xi, c_2 - k, \delta) \times \mathrm{Sym}^k(S \setminus D) \\ (\mathscr{E}, \alpha) &\longmapsto \left(\big(gr^{\mu}(\mathscr{E}, \alpha)\big)^{\sim}, \mathrm{Supp}\, \frac{(gr^{\mu}\mathscr{E})^{\sim}}{gr^{\mu}\mathscr{E}} \right), \end{aligned}$$

where $\mathrm{Supp}\,\mathscr{E}$ is the support of $\mathscr{E}$ counted with multiplicities.

Note that the double dual of a μ-semistable framed torsion free sheaf is a μ-polystable framed locally free sheaf, and the support of $\big((gr^\mu \mathscr{E})^{\sim}/gr^\mu \mathscr{E}\big)$ is a point in $\mathrm{Sym}^k(S \setminus D)$ where $k = c_2(gr^\mu \mathscr{E}) - c_2\big((gr^\mu \mathscr{E})^{\sim}\big)$.

6.2 Torus action and fixed points

In this section we will construct an action of the torus T on $M(S)$ following the paper of Nakajima and Yoshioka [35]. We will show that this action gives rise to a finite set of fixed points $(M(S))^{\mathbb{C}^*}$and $(M^{\mu ss}(S))^T$.

Let S be a nonsingular projective toric surface. with an action of a 2-dimensional algebraic torus T^2 and a finite set of isolated fixed points $S^{T^2} = \{x_1, \ldots, x_n\}$. Assume that the framing divisor D is toric, i.e., stable under the action of T^2, and let $\mathscr{F}$ be a locally free sheaf on D. Suppose we have an action of an r-dimensional torus T^r on the framing sheaf $\mathscr{F}$. Then this induces an action of an $(r+2)$-dimensional torus T on $M(S)$ and on $M^{\mu ss}(S)$.

Let us consider the action of $T^2 \cong \mathbb{C}^* \times \mathbb{C}^*$ on S. Then for any element (t_1, t_2) of T^2 one has an automorphism h_{t_1,t_2} of S.

$$h_{t_1,t_2} : S \longrightarrow S$$

The action of T^2 on the sheaf $\mathscr{E}$ is defined by by taking the inverse image via the automorphism h: $\mathscr{E} \mapsto \mathscr{E}' = (h^{-1}_{t_1,t_2})^* \mathscr{E}$.

To define the action on the framing α let us consider the torus $T^r \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (r-times) that acts on the framing sheaf as follows. For an element $(f_1, \ldots, f_r) \in T^r$ let $F_{f_1,\ldots,f_r}$ be the isomorphism of $\mathscr{F}$

$$F_{f_1,\ldots,f_r} : \mathscr{F} \longrightarrow \mathscr{F}$$

Then the action of $T \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ ($(r+2)$-times) on a pair $(\mathscr{E}, \alpha) \in M(S)$ is defined by

$$(t_1, t_2, f_1, \ldots, f_r) \cdot (\mathscr{E}, \alpha) := \big((h^{-1}_{t_1,t_2})^* \mathscr{E}, \alpha'\big),$$

where α' is given by the composition of the following maps:

$$\mathscr{E}'|_D = (h^{-1}_{t_1,t_2})^* \mathscr{E}|_D \xrightarrow{(h^{-1}_{t_1,t_2})^* \alpha} (h^{-1}_{t_1,t_2})^* \mathscr{F} \longrightarrow \mathscr{F} \xrightarrow{F_{f_1,\ldots,f_r}} \mathscr{F},$$

where the middle arrow is given by the action of T^2 under which $\mathscr{F}$ is stable.

The T action is defined in a similar way on $M^{\mu ss}(S)$, namely the torus action on $M^{\mu\text{-poly}}(S)$ is the same as the action on $M(S)$ and the action of T^2 on $\mathrm{Sym}^k(S \setminus D)$ is induced from that on S since D is T^2-invariant. It is possible to check that the map γ is equivariant, i.e., γ commutes with the torus action.

Indeed for a framed shead $(\mathscr{E},\alpha) \in M(S)$, the following two compositions agree:

$$(\mathscr{E},\alpha) \overset{\gamma}{\longmapsto} \left(\big(gr^\mu(\mathscr{E},\alpha)\big)^{\smallsmile\smallsmile}, \operatorname{Supp} \frac{(gr^\mu \mathscr{E})^{\smallsmile\smallsmile}}{gr^\mu \mathscr{E}} \right) \xmapsto{\text{torus action}}$$
$$\left(\big((h^{-1}_{t_1,t_2})^*(gr^\mu \mathscr{E})^{\smallsmile\smallsmile}, (gr^\mu \alpha)'\big), h_{t_1,t_2} \operatorname{Supp} \frac{(gr^\mu \mathscr{E})^{\smallsmile\smallsmile}}{gr^\mu \mathscr{E}} \right),$$

$$(\mathscr{E},\alpha) \xmapsto{\text{torus action}} \big((h^{-1}_{t_1,t_2})^* \mathscr{E}, \alpha'\big) \overset{\gamma}{\longmapsto}$$
$$\left(\Big(gr^\mu\big((h^{-1}_{t_1,t_2})^*\mathscr{E},\alpha'\big)\Big)^{\smallsmile\smallsmile}, \operatorname{Supp} \frac{\Big(gr^\mu\big((h^{-1}_{t_1,t_2})^*\mathscr{E}\big)\Big)^{\smallsmile\smallsmile}}{gr^\mu\big((h^{-1}_{t_1,t_2})^*\mathscr{E}\big)} \right).$$

This is because the torus action is compatible with the Jordan-Hölder filtration and with the double dual. The torus action on the support of a sheaf gives the support of a sheaf acted on by the torus action.

Lemma 6.2.1. *The fixed points set $(M(S))^T$ consists of finitely many points. The same holds for $(M^{\mu ss}(S))^T$.*

Proof. A framed sheaf $(\mathscr{E},\alpha)$ is fixed if it can be written in the form $(\mathscr{E},\alpha) = (\mathscr{E}_1,\alpha_1) \oplus \cdots \oplus (\mathscr{E}_r,,\alpha_r)$ such that $\mathscr{E}_i = \mathscr{I}_i \otimes \mathscr{O}(C_i)$. Here C_i is a T^2-invariant divisor that does not intersect D, $\mathscr{I}_i$ is the ideal sheaf of a zero dimensional subscheme Z_i in $S \setminus D$, and α_i is an isomorphism $\mathscr{E}_i|_D \xrightarrow{\cong} \mathscr{F}_i$, where $\mathscr{F}_i$ are rank one locally free subsheaves of $\mathscr{F}$ supported on D such that the direct sum $\bigoplus_{i=1}^r \mathscr{F}_i = \mathscr{F}$. Note that the sheaf $\mathscr{F}$ decomposes into such direct sum since it is locally free on a toric divisor in S that is a a smooth curve.
The ideal sheaves $\mathscr{I}_i$ are fixed if they are generated by monomials in the homogeneous coordinate ring (Cox ring) of S. These monomials are finite and hence the ideal sheaves $\mathscr{I}_i$ form a finite family. Moreover, the Picard group of a compact projective variety is generated by a finite number of divisors [36, Corollary 2.5]. Hence $\mathscr{E}_i = \mathscr{I}_i \otimes \mathscr{O}(C_i)$ form a finite family. As a result the fixed points set of $M(S)$ is finite.

Now regarding $M^{\mu\text{-poly}}(r,\xi,c_2-k,\delta)$ as an open subset of $M(S)$ with the corresponding invariants, the set of its fixed points is finite. A framed sheaf $(\mathscr{E},\alpha) \in M^{\mu\text{-poly}}(r,\xi,c_2-k,\delta)$ is fixed if is it given as above, hence $\mathscr{E} = \bigoplus_{i=1}^r \mathscr{E}_i = \bigoplus_{i=1}^r (\mathscr{I}_i \otimes \mathscr{O}_S(C_i))$. Since $\mathscr{E}$ is locally free then so is $\mathscr{I}_i$ for each i. We have $\mathscr{I}_i \cong \mathscr{I}_i^{\smallsmile\smallsmile} \cong \mathscr{O}_S$, hence $\mathscr{E} = \bigoplus_{i=1}^r \mathscr{O}_S(C_i)$.

The fixed points set of $\operatorname{Sym}^k(S \setminus D)$ is finite since S^T is. It follows that $(M^{\mu ss}(S))^T$ is finite. □

Remark 6.2.2. From Lemma 6.2.1 it follows that $N := \gamma^{-1}\big((M^{\mu ss}(S))^T\big)$ contains all the fixed points of $M(S)$ since γ is equivariant. But we will not need this fact in what follows.

6.3 Constructing a projective morphism

In this section we will restrict ourselves to the case where the toric surface S admits a projective morphism of toric surfaces $p : S \to \mathbb{P}^2$ of degree 1. An example of this class of surfaces is the iterated toric blowup of $\mathbb{P}^2$, i.e, the iterated blowup along a set of points that are fixed under the torus action. We will also assume that the framing sheaf is free $\mathscr{F} = \mathscr{O}_D^{\oplus r}$.

We will construct a projective morphism from the moduli space $M(S)$ of framed torsion-free sheaves on S to $M_0(r,n)$, the moduli space defined in section 5.1. To this end we will follow the construction in [34, Appendix F].

Note that under the assumptions above, the restriction of $p : S \to \mathbb{P}^2$ to the divisor D gives an isomorphism $p_{|D} : D \xrightarrow{\cong} l_\infty$, where l_∞ is the line $\{[0 : z_1 : z_2] \in \mathbb{P}^2\}$ in $\mathbb{P}^2$. The direct image of the framing sheaf on D is isomorphic to the framing sheaf on l_∞.

Next we will show that there exists a morphism between $M(S)$ and $M_0(r,n)$ and prove it is projective.

Let $\mathcal{M}(r, c_1, n)$ be the moduli space of H-stable sheaves $\mathscr{E}$ on $\mathbb{P}^2$ with rank $r := \operatorname{rk}\mathscr{E}$, first Chern class $c_1 := c_1(\mathscr{E})$ and discriminant $n := c_2(\mathscr{E}) - \frac{r-1}{2r}c_1(\mathscr{E})^2$. Assuming that $\gcd(r, < c_1, H >) = 1$ the moduli space consists of μ-stable sheaves. We define $\mathcal{M}_{loc}(r, c_1, n)$ the subscheme of $\mathcal{M}(r, c_1, n)$ consisting of μ-stable locally free sheaves.

Let $\widetilde{\mathcal{M}}(r, p^*c_1 + kC, n)$ be the moduli space of $(H - \epsilon C)$-stable sheaves $\mathscr{E}$ on S of rank r, first Chern class $c_1(\mathscr{E}) = p^*c_1 + kC$, and discriminant $n := c_2(\mathscr{E}) - \frac{r-1}{2r}c_1(\mathscr{E})^2$. Here C is a (reducible) divisor on S that does not intersect D, k is an integer and ϵ is sufficiently small.

6.3.1 Uhlenbeck compactification of $\mathcal{M}_{loc}(r, c_1, n)$

We define the Uhlenbeck compactification of the the moduli space of locally-free sheaves $\mathcal{M}_0(r, c_1, n)$ as follows.

$$\mathcal{M}_0(r, c_1, n) := \bigsqcup_l \mathcal{M}_{loc}(r, c_1, n-l) \times \operatorname{Sym}^l(\mathbb{P}^2).$$

In [29, 30], Li proved the following theorem in the general setting of moduli spaces on projective surfaces. Here is Li's theorem for our moduli spaces.

Theorem 6.3.1. *1. $\mathcal{M}_0(r, c_1, n)$ is a projective scheme.*

2. There is a projective morphism

$$\begin{aligned} \pi\colon \quad \mathcal{M}(r, c_1, n) &\longrightarrow \mathcal{M}_0(r, c_1, n) \\ \mathscr{E} &\longmapsto (\mathscr{E}^{\sim}, \operatorname{Supp}(\mathscr{E}^{\sim}/\mathscr{E})). \end{aligned} \tag{6.1}$$

When the first Chern class is zero, the morphism (6.1) reduces to the morphism (5.4) defined in section 5.1. This morphism will allow us define a new morphism from the moduli space of μ-stable sheaves on S to $\mathcal{M}_0(r, c_1, n)$.

6.3.2 Defining a morphism $\widetilde{\pi}$

In this section we will assume that the first Chern class $c_1 = k{\cdot}C$ and $0 \leq k < r$. For a sufficiently large l and assuming that $l = k$ modulo r, we define the following morphism

$$\begin{aligned} \beta : \quad \widetilde{\mathcal{M}}(r, k, \tilde{n}) &\longrightarrow \mathcal{M}(r, n) \\ \mathscr{E} \quad &\longmapsto p_*\mathscr{E}(-lC). \end{aligned}$$

The composition of β with the morphism π defined in (6.1) gives the following morphism

$$\begin{aligned} \widetilde{\pi}\colon \quad \widetilde{\mathcal{M}}(r, k, \tilde{n}) &\to \mathcal{M}_0(r, n) \\ \mathscr{E} &\mapsto \left((p_*\mathscr{E}(-lC))^{\sim}, \operatorname{Supp} \frac{(p_*\mathscr{E}(-lC))^{\sim}}{p_*\mathscr{E}(-lC)} \right). \end{aligned} \tag{6.2}$$

In the next section we will show how the morphism (6.2) restricts to a morphism between moduli spaces of framed sheaves and show it is projective.

Remark 6.3.2. Note that the definition of the morphism $\widetilde{\pi}$ relies on Lemma B.0.2. Using this lemma together with the Grothendieck-Riemann-Roch theorem, one can compute n in terms of k and $\tilde{n}$. Moreover, by the same arguments one can show that the morphism $\widetilde{\pi}$ does not depend on the choice of l. This is true because we are assuming that l is equal to k modulo r.

6.3.3 $\widetilde{\pi}$ for framed sheaves

Before getting to the definition of the morphism $\widetilde{\pi}$ for framed sheaves, we will need few results.

Lemma 6.3.3 (Lemma F.19 [34])**.** *Denote by δ_1 the leading coefficient of the polynomial δ. Assume that $\delta_1 \ll 1$, then the following holds*

1. *For a semistable framed sheaf $(\mathscr{E}, \alpha)$, $\mathscr{E}$ is torsion-free.*

2. *All torsion-free μ-semistable sheaves are μ stable.*

This lemma yields Lemma F.20 in [34]. This works in our case too. Hence using Lemma 6.3.3 we have the following result.

Lemma 6.3.4. *Assume that $\delta_1 \ll 1$. If $\epsilon > 0$ depending on δ_1 and $\mathrm{ch}(E)$ is sufficiently small, then $(\mathscr{E}, \alpha)$ is semistable with respect to $H - \epsilon C$ and δ if and only if $(p_*\mathscr{E}(-lC), \alpha)$ is semistable with respect to H and δ. In particular, the moduli space of framed torsion-free sheaves on S is contained in the moduli space of semistable pairs on S.*

This lemma states that β sends semistable framed sheaves on S to semistable framed sheaves on $\mathbb{P}^2$. Hence it extends to a morphism between moduli spaces of semistable sheaves on S and on $\mathbb{P}^2$. Since these moduli spaces are projective, β is a projective morphism. In other words, we have a projective morphism, that we denote β for simplicity, from the moduli space of framed sheaves on S to the moduli space of framed sheaves on $\mathbb{P}^2$.

$$\begin{aligned} \beta : \quad M(S) &\longrightarrow M(r,n) \\ (\mathscr{E}, \alpha) &\longmapsto \big(p_*\mathscr{E}(-lC), \phi\big), \end{aligned} \tag{6.3}$$

where the framing ϕ is given by

$$\phi : p_*\mathscr{E}(-lC)|\ell_\infty \xrightarrow{\cong} p_*\mathscr{O}_S^{\oplus r}(-lC)|\ell_\infty \cong \mathscr{O}_{\ell_\infty}^{\oplus r}, \tag{6.4}$$

since $p_*\mathscr{O}_S(-lC)$ is isomorphic to $\mathscr{O}_{\mathbb{P}^2}$.

On the other hand, the morphism π defined in (5.4) is projective, so the composition $\pi \circ \beta$ is projective. Hence we have proved the following theorem.

Theorem 6.3.5. *There is a projective morphism*

$$\begin{aligned} \widetilde{\pi} : \quad M(S) &\to M_0(r,n) \\ (\mathscr{E}, \alpha) &\mapsto \left(\big((p_*\mathscr{E}(-lC))^{\smile}, \phi\big), \mathrm{Supp}\, \frac{(p_*\mathscr{E}(-lC))^{\smile}}{p_*\mathscr{E}(-lC)} \right), \end{aligned} \tag{6.5}$$

where the framing ϕ is given by (6.4).

Remark 6.3.6. The map $\widetilde{\pi}$ is equivariant under the torus action since β is.

In the next section, using this morphism we will study some topological properties of the moduli space of framed torsion-free sheaves on S.

6.4 Some topological properties

In this section, we define the subvariety $\widetilde{N}$ of $M(S)$ that is the inverse image by $\widetilde{\pi}$ of the fixed point of $M_0(r,n)$. We show that both $M(S)$ and $\widetilde{N}$ admit a filtrable Białynicki-Birula decomposition. It is enough to consider the action of a regular one-parameter subgroup $\lambda \subset T$ as in Section 5.3. Therefore, we will consider the action of $\mathbb{C}^*$ on $M(S)$ and on $\widetilde{N}$.

Having established the projective morphism $\widetilde{\pi}$, we define the inverse image of the fixed point $n[0]$ of $M_0(r,n)$.

$$\widetilde{N} := \widetilde{\pi}^{-1}(n[0])$$

Theorem 6.4.1. *$\widetilde{N}$ is a compact subvariety of $M(S)$.*

Proof. The proof goes through the same lines as that of Theorem 5.1.5. □

Proposition 6.4.2. *For every element $x \in M(S)$, the limit $\lim_{t\to 0} t \cdot x$ exists and lies in $M(S)^{\mathbb{C}^*}$.*

Proof. The proof of this proposition go through the same lines as that of Proposition 5.3.1. □

Theorem 6.4.3. *$M(S)$ admits a Białynicki-Birula plus-decomposition into affine spaces. Moreover, this decomposition is filtrable.*

Proof. Having Proposition 6.4.2, the proof goes through the same lines as in 5.3.2. □

Let $M(S)^{\mathbb{C}^*} = \{x_i \mid i = 1, \dots, m\}$ and denote the cells of the decomposition by

$$M(S)_i^+ := M(S)_i^+ = \{x \in M(S) \mid \lim_{t\to 0} t \cdot x = x_i\}.$$

The limit $\lim_{t\to\infty} t \cdot x$ does not exist for all $x \in M(S)$. We will only consider the points of $M(S)$ such that the limit exists. These points define the subspaces

$$M(S)_i^- := M(S)_i^- = \{x \in M(S) \mid \lim_{t\to\infty} t \cdot x \text{ exists and equals } x_i\}.$$

Theorem 6.4.4. *The subvariety $\widetilde{N}$ is the union $\widetilde{N} = \bigcup_i M(S)_i^-$. Moreover, this is a filtrable decomposition of $\widetilde{N}$ into affine spaces.*

Proof. From [34, Theorem 3.5(3)] any $x \neq n[0]$ in $M_0(r,n)$ the limit $\lim_{t\to\infty} t \cdot x$ does not exist. Hence by the projective morphism $\tilde{\pi}$ we deduce that $\widetilde{N} = \bigcup_i M(S)_i^-$. To show this is a filtrable decomposition, the proof goes through the same lines as that of 5.3.3. □

Let $J = \{1, 2, \dots, m\}$ be the set of indices so that the fixed points set $X^{\mathbb{C}^*} = \{x_j \mid j \in J\}$ be ordered to yield the following filtrations

$$M(S) = M(S)_1 \supset M(S)_2 \supset \cdots \supset M(S)_m \supset M(S)_{m+1} = \emptyset,$$

$$\emptyset = \gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_m \subset \gamma_{m+1} = \widetilde{N},$$

where $M(S)_i \setminus M(S)_{i+1} = M(S)_i^+$, $\gamma_{i+1} \setminus \gamma_i = M(S)_i^-$ and the decomposition of $M(S)$ $\big($resp. $\widetilde{N}\big)$ is given by $M(S) = \bigcup_{j \in J} M(S)_j^+$ $\big($resp. $\widetilde{N} = \bigcup_{j \in J} M(S)_j^-\big)$.

Define the subsets $M(S)_{\leq j}^+ := \bigcup_{i \leq j} M(S)_i^+$ and $M(S)_{\leq j}^- := \bigcup_{i \leq j} M(S)_i^-$. Then all the results in Section 5.4 hold, the proofs are basically the same.

Lemma 6.4.5. *For each j, there is an inclusion $M(S)_{\leq j}^- \hookrightarrow M(S)_{\leq j}^+$.*

Theorem 6.4.6. *The inclusion $M(S)_{\leq j}^- \hookrightarrow M(S)_{\leq j}^+$ induces isomorphisms of homology groups with integer coefficients $H_k(M(S)_{\leq j}^-) \xrightarrow{\cong} H_k(M(S)_{\leq j}^+)$ for all j and all k. In particular the inclusion $\widetilde{N} \hookrightarrow M(S)$ induces isomorphisms $H_k(\widetilde{N}) \xrightarrow{\cong} H_k(M(S))$ for all k.*

Proof. Using Lemma 6.4.5 the proof is the same as that of Theorem 5.4.2. □

Lemma 6.4.7. *Both $M(S)$ and $\widetilde{N}$ are simply connected.*

Remark 6.4.8. Note that $M(S)$ is irreducible since it is nonsingular and connected.

Theorem 6.4.9. *$\widetilde{N}$ is homotopy equivalent to $M(S)$.*

Chapter 7

Outlook

An interesting outlook would be the generalization of the study performed in this thesis to a toric surface S without assuming the existence of a projective morphism $S \to \mathbb{P}^2$ of degree 1.

The results of chapter 5 and 6 hold for all the moduli spaces on nonsingular projective toric surfaces having a projective morphism onto $M_0(r,n)$ which is equivariant under the torus action. Indeed, to generalize these results it is enough to construct a projective morphism

$$\widetilde{\pi} \colon M(S) \to M_0(r,n).$$

However, this may not be an easy task.

As we have seen in the last chapter, to construct such a morphism we have considered a morphism

$$\begin{aligned} \beta : \quad M(S) &\longrightarrow M(r,n) \\ \mathscr{E} \quad &\longmapsto p_*\mathscr{E}(-lC), \end{aligned} \tag{7.1}$$

for a sufficiently large l.

It is to notice that when the morphism p has a greater degree, the direct image of the structure sheaf $p_*\mathscr{O}_S(-lC) \ncong \mathscr{O}_{\mathbb{P}^2}$. In this case a morphism defined as in (7.1) does not do the job since we end up in a moduli space different from $M(r,n)$.

The reason for constructing a morphism $\widetilde{\pi}$ lies in the fact that the moduli space $M_0(r,n)$ has a unique fixed point $n[0]$ and the fiber over it defines a compact subvariety containing all the fixed points of the moduli space we are considering. Moreover $M_0(r,n)$ has a description into ADHM data that was intensively used in the proofs of the results in the last two chapters.

In chapter 6 we have considered toric varieties to guarantee that the moduli space has finitely many fixed points under the torus action. This allows us to use results of chapter 4. It would be interesting to generalize this study to the moduli spaces of framed sheaves on projective surfaces having a $\mathbb{C}^*$-action. In this case, the set of fixed points under the torus action might not be finite. Another issue is the construction of the projective morphism onto $M_0(r,n)$ as discussed above.

Appendix A

Proof using ADHM data

A.1 Moduli space on the projective plane

As we havee seen in chapter 5, there is an isomorphism

$$M(r,n) \cong \left\{ (B_1, B_2, i, j) \;\middle|\; \begin{array}{l} [B_1, B_2] + ij = 0, \\ \nexists\, S \subsetneq \mathbb{C}^n \text{ such that } B_\alpha(S) \subset S \\ (\alpha = 1, 2) \text{ and } \operatorname{im} i \subset S \end{array} \right\} \Big/ \operatorname{GL}_n(\mathbb{C})$$

$$:= H / \operatorname{GL}_n(\mathbb{C}),$$

where $B_1, B_2 \in \operatorname{End}(\mathbb{C}^n)$, $i \in \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^n)$, $j \in \operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^r)$ and for $g \in \operatorname{GL}_n(\mathbb{C})$:

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1}).$$

The fibre $\pi^{-1}(n[0]) \subset M(r,n)$ is isomorphic to the quot scheme $\operatorname{Quot}(r,n)$ by Theorem 5.1.6. From [2, Lemma 2.2] we have the following description in terms of ADHM data.

$$\pi^{-1}(n[0]) \cong \left\{ (B_1, B_2, i) \;\middle|\; \begin{array}{l} [B_1, B_2] = 0 \\ \exists\, \alpha, \beta \in \mathbb{N} : {B_1}^\alpha = {B_2}^\beta = 0 \\ \nexists\, S \subsetneq \mathbb{C}^n \text{ such that } B_\alpha(S) \subset S \\ (\alpha = 1, 2) \text{ and } \operatorname{im} i \subset S \end{array} \right\} \Big/ \operatorname{GL}_n(\mathbb{C}),$$

where $B_1, B_2 \in \operatorname{End}(\mathbb{C}^n)$ and $i \in \operatorname{Hom}(\mathbb{C}^r, \mathbb{C}^n)$.

The one parameter subgroup $\mathbb{C}^*$ acts on H as follows:

$$(B_1, B_2, i, j) \mapsto (tB_1, tB_2, i, t^2 j) \text{ for } t \in \mathbb{C}^*.$$

Let us define the map:

$$\begin{aligned} \rho: \quad M(r,n) &\longrightarrow \mathbb{R} \\ (B_1, B_2, i, j) &\longmapsto \sum_i |\mu_i| + \sum_i |\lambda_i| =: \rho_{(B_1,B_2,i,j)}, \end{aligned}$$

where μ_i and λ_i are the eigenvalues of B_1 and B_2 respectively.

Consider the open neighborhood U_1 of $\pi^{-1}(n[0])$ defined by

$$U_1 := \left\{(B_1, B_2, i, j) \in M(r,n) \;\middle|\; \rho_{(B_1,B_2,i,j)} < 1\right\},$$

and let

$$U_t := t \cdot U_1 = \{t \cdot (B_1, B_2, i, j) \mid (B_1, B_2, i, j) \in U_1\},$$

be a family of open neighborhoods of $\pi^{-1}(n[0])$ parametrized by $t \in \mathbb{C}^*$.

Since $\pi^{-1}(n[0])$ is described by nilpotent matrices, i.e, matrices for which all the eigenvalues vanish, then $\pi^{-1}(n[0])$ is contained in $\subset U_t$ for all t.

A.2 Base of open neighborhoods

$M(r,n)$ and $\pi^{-1}(n[0])$ are algebraic varieties and in particular, they are locally compact topological spaces. $M(r,n)$ contains $\pi^{-1}(n[0])$ as a compact subvariety so for every neighborhood V of $\pi^{-1}(n[0])$ there exists a closed compact neighborhood W of $\pi^{-1}(n[0])$ such that $W \subset V$, [25, Theorem 18].

Let ∂W be the boundary of W and consider the following restricted map:

$$\begin{aligned} \rho_{|\partial W}: \quad \partial W &\longrightarrow \mathbb{R} \\ (B_1, B_2, i, j) &\longmapsto \rho_{(B_1,B_2,i,j)} > 0. \end{aligned}$$

This map is a real valued function on a compact space ∂W, hence it attains a minimum (maximum). Let $\epsilon = \min \rho_{|\partial W}$ then:

$$\forall (B_1, B_2, i, j) \in \partial W : \rho_{(B_1,B_2,i,j)} \geq \epsilon.$$

In the case $t = \epsilon$, we have

$$U_{t=\epsilon} \cap \partial W = \left\{(B_1, B_2, i) \in M(r,n) \;\middle|\; \rho_{(B_1,B_2,i,j)} < |\epsilon|\right\} \cap \partial W = \emptyset.$$

Hence $U_{t=\epsilon}$ is contained in $W \setminus \partial W$. It results that for each neighborhood $V \supset \pi^{-1}(n[0])$, there exists a $t \in \mathbb{C}^*$ such that U_t is contained in V. Therefore the family $\{U_t, t \in \mathbb{C}^*\}$ form a base of open neighborhoods of $\pi^{-1}(n[0])$.

A.3 U_t is homotopy equivalent to $M(r,n)$

In this section, we show that there is a homotopy between $M(r,n)$ and U_t for each t. For each $t \in \mathbb{C}^*$, we define the following isomorphism

$$\begin{array}{rccc} f_t: & U_1 & \longrightarrow & U_t \\ & (B_1,B_2,i,j) & \longmapsto & (t\cdot B_1, t\cdot B_2, i, t^2 j). \end{array}$$

In particular, U_1 is homotopy equivalent to U_t. It is enough then to show that there is a homotopy between $M(r,n)$ and U_1. Let us define the maps:

$$M(r,n) \xrightarrow{\quad f \quad} U_1 \xrightarrow{\quad g \quad} M(r,n),$$

where

$$(B_1,B_2,i,j) \mapsto f(B_1,B_2,i,j) := \left(\frac{B_1}{\rho_{(B_1,B_2,i,j)}+1}, \frac{B_2}{\rho_{(B_1,B_2,i,j)}+1}, i, j\right),$$

and

$$(B_1,B_2,i,j) \mapsto g(B_1,B_2,i,j) := \left(\frac{B_1}{\rho_{(B_1,B_2,i,j)}+1}, \frac{B_2}{\rho_{(B_1,B_2,i,j)}+1}, i, j\right).$$

Then we have a homotopy $F : [0,1] \times M(r,n) \longrightarrow M(r,n)$ defined by

$$F(\alpha,(B_1,B_2,i,j)) = \left(\frac{B_1}{(1-\alpha)\rho_{(B_1,B_2,i,j)}+1}, \frac{B_2}{(1-\alpha)\rho_{(B_1,B_2,i,j)}+1}, i, j\right).$$

Similarly, we have a homotopy $G : [0,1] \times U_1 \longrightarrow U_1$ defined by

$$G(\alpha,(B_1,B_2,i,j)) = \left(\frac{B_1}{(1-\alpha)\rho_{(B_1,B_2,i,j)}+1}, \frac{B_2}{(1-\alpha)\rho_{(B_1,B_2,i,j)}+1}, i, j\right).$$

We conclude that U_t is homotopy equivalent to $M(r,n)$ for all $t \in C^*$.

A.4 $\pi^{-1}(n[0])$ is homotopy equivalent to $M(r,n)$

$M(r,n)$ and $\pi^{-1}(n[0])$ are algebraic varieties, in particular they are algebraic sets that admit triangulations [21, 31]. Moreover, since $M(r,n)$ contains $\pi^{-1}(n[0])$ as a compact subvariety, the pair $(M(r,n), \pi^{-1}(n[0]))$ can be triangulated in a compatible way. Then there exists a basis of closed neighborhoods $\{V_\alpha \supset \pi^{-1}(n[0])\}$ homotopy equivalent to $\pi^{-1}(n[0])$, see [32, Lemma 70.1].

Since $\{U_t\}$ and $\{V_\alpha\}$ are neighborhood bases of $\pi^{-1}(n[0])$, the following inclusions $U_{t_1} \subset V_{\alpha_1} \subset U_{t_2} \subset V_{\alpha_2}$ induce homomorphisms of homotopy groups for all n

$$\pi_n(U_{t_1}) \xrightarrow{i} \pi_n(V_{\alpha_1}) \xrightarrow{j} \pi_n(U_{t_2}) \xrightarrow{k} \pi_n(V_{\alpha_2}).$$

It follows that $\pi_n(V_{\alpha_1}) \xrightarrow{\sim} \pi_n(U_{t_2})$ is an isomorphism for each n since $j \circ i$ and $k \circ j$ are isomorphisms. This implies that $\pi_n(\pi^{-1}(n[0])) \cong \pi_n(M(r,n))$.

Note that both $M(r,n)$ and $\pi^{-1}(n[0])$ are connected, by Whitehead theorem (See e.g [37, page 370]) we conclude that $\pi^{-1}(n[0])$ is homotopy equivalent to $M(r,n)$.

Remark

The maps ρ, f, and g defined above do not depend on the representatives.

Appendix B

Some useful statements

Lemma B.0.1. *Let X be a smooth projective surface and let*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}' \longrightarrow \mathscr{Q} \longrightarrow 0,$$

be an exact sequence of sheaves of $\mathscr{O}_X$-modules where $\mathscr{F}$ is torsion free, $\mathscr{F}'$ is reflexive (equivalently locally free since every reflexive sheaf on a surface is locally free) of the same rank and $\mathscr{Q}$ is zero dimensional; then $\mathscr{F}' \cong \mathscr{F}^{\check{}\check{}}$.

Proof. Dualizing the exact sequence $0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}' \longrightarrow \mathscr{Q} \longrightarrow 0$, i.e., applying the functor $\mathscr{H}om(\cdot, \mathscr{O}_X)$, we get

$$0 \longrightarrow \mathscr{H}om(\mathscr{F}', \mathscr{O}_X) \longrightarrow \mathscr{H}om(\mathscr{F}, \mathscr{O}_X) \longrightarrow \mathscr{E}xt^1(\mathscr{Q}, \mathscr{O}_X) \longrightarrow 0.$$

Note that $\mathscr{H}om(\mathscr{Q}, \mathscr{O}_X) = 0$, and $\mathscr{E}xt^1(\mathscr{F}, \mathscr{G}) = 0$ for any coherent sheaf $\mathscr{G}$ if $\mathscr{F}$ is locally free [18, Chap. III, Ex. 6.5(a)], in particular $\mathscr{E}xt^1(\mathscr{F}', \mathscr{O}_X)$ vanishes. We get

$$0 \longrightarrow \mathscr{F}'^{\check{}} \longrightarrow \mathscr{F}^{\check{}} \longrightarrow \mathscr{Q}' \longrightarrow 0,$$

where $\mathscr{Q}' := \mathscr{E}xt^1(\mathscr{Q}, \mathscr{O}_X)$ is a zero dimensional sheaf. Call S the support of $\mathscr{Q}'$, then $S \subset X$ is a closed subset of codimension 2.
Since $\mathscr{F}^{\check{}}$ and $\mathscr{F}'^{\check{}}$ are reflexive, so by [17, Prop.1.6] they are normal. Hence for every open $U \subset X$, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}'^{\check{}}(U) & \longrightarrow & \mathscr{F}^{\check{}}(U) & \longrightarrow & \mathscr{Q}'(U) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & & & \\ 0 & \longrightarrow & \mathscr{F}'^{\check{}}(U \setminus S) & \longrightarrow & \mathscr{F}^{\check{}}(U \setminus S) & \longrightarrow & \mathscr{Q}'(U \setminus S) & \longrightarrow & 0 \end{array}$$

Note that $\mathscr{Q}'(U \setminus S) = 0$ since $\mathscr{Q}'$ is supported on S. Thus the injective morphism $\mathscr{F}'^{\check{}}(U \setminus S) \to \mathscr{F}^{\check{}}(U \setminus S)$ is an isomorphism. This yields an isomorphism

$\mathscr{F}'^{\vee}(U) \xrightarrow{\cong} \mathscr{F}^{\vee}(U)$ and we get $\mathscr{Q}'(U) = \mathscr{E}xt^1(\mathscr{Q}, \mathscr{O}_X)(U) = 0$ for every open subset $U \subset X$. Hence we find $\mathscr{Q}' = \mathscr{E}xt^1(\mathscr{Q}, \mathscr{O}_X) = 0$ and $\mathscr{F}'^{\vee} \xrightarrow{\cong} \mathscr{F}^{\vee}$. Dualizing again one gets $\mathscr{F}'^{\vee\vee} \cong \mathscr{F}^{\vee\vee}$. Since $\mathscr{F}'$ is reflexive it follows that $\mathscr{F}' \cong \mathscr{F}^{\vee\vee}$. □

Lemma B.0.2. *There is an integer l_0 such that for any $\mathscr{E} \in \widetilde{M}(r,k,n)$ and any $l \geq l_0$, we have:*

1. $R^1 p_* \mathscr{E}(-lC) = 0$
2. *There is a canonical inclusion* $\mathscr{E}(-lC) \hookrightarrow p^*\big(p_*\mathscr{E}(-lC)\big)^{\vee\vee}$.
3. $\big(p_*\mathscr{E}(-lC)\big)^{\vee\vee} \cong (p_*\mathscr{E})^{\vee\vee}$.

Proof. Note that $\mathscr{E}(-lC) = \mathscr{E}(l)$ since the ideal sheaf of the exceptional divisor C on S is given by $\mathscr{O}_S(-C) = \mathscr{O}_S(1)$ see [16, IV, Lemma 4.1.(b)].

1. The proof can be found in [18, III. Theorem 8.8.(c)]
2. From [18, III. Theorem 8.8.(a)] the natural map $p^*p_*\mathscr{E}(-lC) \longrightarrow \mathscr{E}(-lC)$ is surjective, hence one gets an exact sequence:

$$0 \longrightarrow \mathscr{T} \longrightarrow p^*p_*\mathscr{E}(-lC) \longrightarrow \mathscr{E}(-lC) \longrightarrow 0.$$

Since $p^*p_*\mathscr{E}(-lC)$ and $\mathscr{E}(-lC)$ have the same rank, it follows that $\mathscr{T}$ is a torsion sheaf, but $p^*p_*\mathscr{E}(-lC)$ is torsion free, then $\mathscr{T} = 0$ and $p^*p_*\mathscr{E}(-lC) \longrightarrow \mathscr{E}(-lC)$ is an isomorphism.

On the other hand, there is a canonical inclusion $p^*(p_*\mathscr{E}(-lC)) \hookrightarrow p^*\big(p_*\mathscr{E}(-lC)\big)^{\vee\vee}$, so we have the following diagram

$$\begin{array}{ccc} p^*p_*\mathscr{E}(-lC) & \hookrightarrow & p^*\big(p_*\mathscr{E}(-lC)\big)^{\vee\vee} \\ \downarrow \wr & \nearrow & \\ \mathscr{E}(-lC) & & \end{array}$$

Hence the dotted arrow is injective and we get the inclusion $\mathscr{E}(-lC) \hookrightarrow p^*\big(p_*\mathscr{E}(-lC)\big)^{\vee\vee}$.

3. Using the inclusion above, and taking the cokernel, one gets the exact sequence

$$0 \longrightarrow \mathscr{E}(-lC) \longrightarrow p^*\big(p_*\mathscr{E}(-lC)\big)^{\vee\vee} \longrightarrow \mathscr{Q} \longrightarrow 0.$$

Tensoring by $\mathscr{O}_S(lC)$ and applying the direct image, the above exact sequence gives

$$0 \longrightarrow p_*\mathscr{E} \longrightarrow p_*\big(p^*\big(p_*\mathscr{E}(-lC)\big)^{\smile} \otimes \mathscr{O}_S(lC)\big) \longrightarrow \cdots$$

On the other hand, $p_*\mathscr{O}_S(lC) = p_*\mathscr{O}_S(-l) = \mathscr{O}_S$ see [9, page 76]. Hence, substituting and taking the quotient in the above exact sequence, one gets the short exact sequence

$$0 \longrightarrow p_*\mathscr{E} \longrightarrow \big(p_*\mathscr{E}(-lC)\big)^{\smile} \longrightarrow \mathscr{Q}' \longrightarrow 0.$$

Notice that $p_*\mathscr{E}$ is torsion free and $\big(p_*\mathscr{E}(-lC)\big)^{\smile}$ is locally free of the same rank, $\mathscr{Q}'$ is supported on points, then using Lemma B.0.1, we get the isomorphism $\big(p_*\mathscr{E}(-lC)\big)^{\smile} \cong (p_*\mathscr{E})^{\smile}$.

□

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