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On Holographic RG Flows

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Abstract

A study of holographic renormalization group (RG) flows is discussed in the framework of supergravity and string theories. Some RG flow solutions are found in $N = 4$ Chern-Simons gauged supergravity in three dimensions. The resulting solutions describe vev flows, driven by a vacuum expectation value (vev) of a relevant operator or a vev of a marginal operator, in a dual two dimensional field theory. In the minimal six dimensional supergravity, RG flow solutions, describing two dimensional vev flows driven by a vev of a marginal operator, in the presence of Yang-Mills instantons are found. The solutions describe RG flows between $N = (4, 0)$ SCFTs and have an interpretation in term of transitions between Yang-Mills vacua with different winding numbers. We also propose an interpretation in term of D1/D5 branes in type I string theory. The flow with a single instanton can be thought of as an uplifted three dimensional solution. The corresponding reduction ansatz of the (1,0) six dimensional supergravity on the $SU(2)$ group manifold giving rise to the $N = 4$ Yang-Mills gauged supergravity in three dimensions with $SU(2) \times G$ gauge group is given. Additionally, the equivalence between the reduced theory and $N = 4$ Chern-Simons $(SU(2) \ltimes \mathbf{T}^3) \times (G \times \mathbf{T}^{\dim G})$ gauged supergravity is explicitly shown. The solution with multi-instanton back ground is generalized to the case in which the Yang-Mills instantons are turned on on an asymptotically locally Euclidean (ALE) space. The corresponding solution then involves both Yang-Mills and gravitational instantons and describes RG flows from $N = 2$ two dimensional CFT in the UV to $N = 4$ CFT in the IR. Furthermore, we extend the analysis to RG flows in four dimensional field theories by studying flow solutions in the framework of type IIB and type I$'$ string theories on the ALE space. In these solutions, the UV theory is an $N = 2$ quiver gauge theory. Field theory considerations of Higgsing the corresponding UV quiver gauge theory in the UV to the IR gauge theory, which can possibly be another quiver theory or $N = 2, 4$ SYM, are discussed along with their geometric interpretation.

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Contents

Chapter 1

Introduction

As a promising candidate for a theory of quantum gravity, superstring theory, see [1, 2, 3, 4] for standard references on string theory, has been explored in various aspects, and many of its consequences, implications and applications have been realized in several contexts. One important result in string theory is the AdS/CFT correspondence. Over the past ten years, a lot of researches have shown its validity as well as its generalization. It has been proposed in [5], and the more refined definitions have been given in [6] and [7]. There are many good reviews on the subject, see for example [8] and [9] and references therein. Before proceeding further, we will give a comment on the terminology. Throughout this thesis, the word string theory means one of the five supersymmetric string theories according to which theory we are considering. Although the original AdS/CFT correspondence has been generalized to the case of non conformal field theories with non AdS gravity backgrounds and should be called, as in some references, gauge/gravity or gauge/string correspondences, we still use the terminology "AdS/CFT correspondence" in this thesis. Furthermore, the number of literatures involved in this research area is so large that it is impossible to make a complete list here. We will only give some important ideas along with relevant references to bring the readers to the works presented in the thesis.

The AdS/CFT correspondence is a duality between string theory on a certain background and a dual field theory living on the boundary of this background. One example, which is the most well studied, in the original proposal in [5] is type IIB string theory on $AdS_5 \times S^5$ with n unit of the 5-form flux. According the correspondence, this is dual to $N = 4$ supersymmetric Yang-Mills (SYM) gauge theory with gauge group $SU(n)$ living on the boundary of AdS_5. This duality is a kind of strong-weak duality in which a weakly coupled gravity theory is dual to a strongly coupled field theory or vice versa. The correspondence has been made quantitative in [6] and [7] in which the description for computing correlation functions of the dual CFT from a bulk gravity theory in the AdS space has been given. This correspondence realizes the holographic principle [10] in which the duality links the information in the bulk and boundary spaces. The mapping between bulk fields and operators of the boundary field theory has been studied in [11] for the case of type IIB on $AdS_5 \times S^5$ and $N = 4$ SYM. Most of the results have been studied in the large n or large 't Hooft coupling limit in which type IIB supergravity is a good approximation on the AdS side.

The correspondence then has been generalized and extended to the case of non conformal field theories. The corresponding gravity backgrounds are of course no longer anti-de Sister spaces since the isometry of the AdS space gives rise to conformal symmetry of the boundary field theory. There are also some attempts to obtain the correspondence in non-supersymmetric field theories. Up to now, many applications of the AdS/CFT correspondence including the study of condensed matter systems using holographic methods have been studied. One of the consequences of the AdS/CFT correspondence which is of interest in this thesis is the study of holographic renormalization group or RG flows. The backgrounds of interest in this case are the asymptotically AdS spaces. These approach AdS spaces in some limits. The interpretation in the dual field theory is that of a perturbed CFT which undergoes a renormalization group flow. In the AdS/CFT correspondence, the radial coordinate of the AdS space has an interpretation in term of the energy scale in the dual field theory, therefore a dependence of a bulk field on the radial coordinate represents a change along the energy scale corresponding to an RG flow. The advantage of the AdS/CFT correspondence is that it allows us to study perturbations and RG flows of a strongly coupled field theory using a weakly coupled gravity theory.

On the field theory side, perturbations can be given by adding a source term to the CFT Lagrangian or giving a vacuum expectation value (vev) to a certain operator. These two types of perturbations correspond to, on the gravity side, the non-normalizable and normalizable modes of the bulk fields, respectively. In the former case, conformal symmetry is explicitly broken while in the latter case, conformal symmetry is spontaneously broken. The corresponding RG flows are sometimes called operator flows and vev flows, respectively. It can happen that after an RG flow, the UV CFT approaches another CFT in the IR. This is the case if there exists a conformal fixed point in the IR. The flow solution in this case can be interpreted as an RG flow between two conformal fixed points of the dual field theory. These are the flows we are interested in. The gravity solutions are those approaching AdS spaces in two limits, sometimes called the boundary of one AdS space and the deep interior of another AdS space.

In $\text{AdS}_5/\text{CFT}_4$ correspondence, a lot of works have been done to study RG flow solutions in the dual field theories in four dimensions, see [12, 13, 14] for example. These solutions describe perturbations of $N = 4$ SYM to another CFT with lower supersymmetries. In general, flow solutions can be obtained by solving equations of motion of type IIB supergravity or working in the $N = 8$ gauged supergravity in five dimensions and use consistent reduction ansatz of type IIB on S^5 to obtain ten dimensional solutions. The latter is simpler in many aspects, and the solutions mentioned above have been found in this way. The procedure is even more simpler in the case of supersymmetric RG flows in which we can find flow solutions by solving first order differential (BPS) equations coming from supersymmetry transformations of fermions rather than solving second order field equations. Furthermore, things are more controllable in the supersymmetric solutions in both gravity and field theory sides.

In this thesis, we are mainly interested in finding holographic RG flow solutions in the dual two dimensional field theories. We also restrict ourselves to the case of supersymmetric RG flows. Moreover, since working in lower dimensional spacetime is much simpler than working directly in the ten dimensional string theory, we will follow this route and work with lower dimensional supergravity theories instead of the ten dimensional supergravities which are low energy effective theories of string theory. This is similar to the case of $\mathrm{AdS}_5/\mathrm{CFT}_4$ where we can work with five dimensional gauged supergravity and use the reduction ansatz of type IIB supergravity on S^5 to eventually obtain $AdS_5 \times S^5$ background. In $\mathrm{AdS}_3/\mathrm{CFT}_2$, the natural bulk gravity theories to begin with are of course three dimensional gauged supergravity theories. The reason for working with gauged supergravities rather than the ungauged versions is the possibilities to obtain AdS critical points which can be identified with RG fixed points in the dual field theory. According to the AdS/CFT correspondence, each critical point is interpreted as a conformal phase of the dual field theory.

Unlike in four dimensions, there are only a few literatures discussing holographic RG flows in dual two dimensional field theories [15], [16]. One reason could be that two dimensional field theories can be solved exactly in many cases by other means. Particularly, CFT_2 is very well understood in various aspects without using holographic methods. In this point of view, there does not seem to be necessary to study gravity dual of these theories since the field theories themselves are solvable even at strong coupling. On the other hand, gravity in three dimensions is also much simpler than its analogue in higher dimensions. However, the fact that both sides of the correspondence are controllable makes it possible to understand $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence in much more detail than those in other dimensions. And, hopefully, this study will give us some insights to understand how the AdS/CFT really works. This will eventually help to understand more realistic models in $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. Furthermore, $\mathrm{AdS}_3/\mathrm{CFT}_2$ is also interesting in its own right in the sense that it is useful in the study of black hole entropies, see [17] and references therein.

Gravity theory in three dimensions has been studied for a long time [18, 19, 20, 21, 22]. It has been used to study various aspects of quantum gravity and as a toy model toward quantizing gravity in four dimensional spacetime. In [23], it has been shown that three dimensional Einstein gravity can be written as a Chern-Simons theory which makes quantization more traceable. For more detail on three dimensional gravity, the reader is referred to the book [24]. Pure gravity in three dimensions is topological since there is no propagating degree of freedom. In a sense, there is no local dynamical degree of freedom. This is a result of constraints from Einstein equations, the mix spatial and time components of Einstein equations, together with coordinate transformations. However, there are finite number of global degrees of freedom unless the spacetime is topologically trivial [24].

As stated above, our interest lies in the theory of supergravity in three dimensions. Like the non-supersymmetric counterpart, pure supergravity in three dimensions is also topological and admits arbitrary number of supersymmetries. Nevertheless, supergravity coupled to matter fields is not a topological theory but is in the form of non-linear sigma model coupled to supergravity. In three dimensions, vector fields are dual to scalars, so the bosonic matter fields can be given purely in terms of scalars. This makes the matter coupled supergravity in the form of non-linear sigma model coupled to supergravity. It is in the matter coupled theory that there is an upper bound on the number of supersymmtries namely $N \leq 16$ or equivalently 32 supercharges. The $N = 16$ and $N = 8$ theories with the corresponding scalar target spaces $\frac{E_{8(8)}}{SO(16)}$ and $\frac{SO(8,k)}{SO(8)\times SO(k)}$, with k being number of supermultiplets, have been constructed in [25]. The classification of ungauged supergravity, with all admissible values of N, $1 \leq N \leq 16$, in three dimensions has then been carried out in [26] together with their scalar target spaces, which are symmetric spaces determined by supersymmetry, for $N > 4$. For $N \leq 4$, supersymmetry is not powerful enough to determine the scalar manifolds, and target spaces which are not symmetric spaces are allowed.

The first gauged supergravity in three dimensions has been constructed in [27] using the notion of embedding tensor which provides a G covariant formulation of the gauged supergravity theories. G is the global symmetry of the theory which, in the case of symmetric target spaces, is the same as the global symmetry group, or isometry group, of the coset space G/H. Various gaugings with both compact and non-compact gauge groups have been classified in [28]. As in the higher dimensional analogues, there exist gaugings with non-semisimple gauge groups of the form $G_0 \ltimes \mathbf{T}^n$, with $\mathbf{T}^n$ being abelian translational symmetries in n dimensions, and furthermore, complex gauge groups are possible as well [29]. The similar construction has been used to construct the half maximal $N = 8$ gauged supergravity in three dimensions in [30]. Eventually, three dimensional gauged supergravities with all possible values of N have been completed in [31]. In this work, the ungauged theories discovered in [26] have been generalized to implement the local gauge symmetry coming from promoting some subgroup of the isometries of the target manifold that can be extended to a symmetry of the full matter coupled Lagrangian.

Using the duality between vectors and scalars in three dimensions, the only propagating degrees of freedom in the ungauged theories are scalar fields. In this way, the global symmetry of the theory is realized at the level of the Lagrangian. Unlike in higher dimensions, it seems not possible to gauge any global symmetries of the theory since there are no vector fields to act as gauge fields. However, vector fields can enter the Lagrangian via the Chern-Simons term rather than the usual

Yang-Mills kinetic term. The presence of the Chern-Simons term which does not introduce propagating degrees of freedom is in turn required by supersymmetry. The resulting gauged theory still have equal number of bosonic and fermionic degrees of freedom. The formulation stated above gives rise to Chern-Simons gauged supergravities. Note that in this formulation, there is no restriction on the number of vector fields or equivalently, on the dimension of the gauge group, therefore, we can gauge various choices of gauge groups provided that they can be embedded in the global symmetry group and are allowed by supersymmetry. This is the reason for the huge amount of possible gauge groups in three dimensional gauged supergravity.

The embedding tensor makes it possible to classify admissible gauge groups, that can be gauged consistently with supersymmetry, by using a group theoretical method for the case of symmetric target manifolds. The embedding tensor is also useful in the construction of higher dimensional gauged supergravities [32, 33, 34, 35, 36], for a good review see [37]. However, apart from the three dimensional ones which play an important role several parts of the thesis, we will not review these gauged supergravities in this thesis since we will not explicitly use them here.

In the dimensional reduction scenario, lower dimensional gauged supergravities can be obtained from dimensional reductions of higher dimensional theories. The theories obtained form dimensional reduction always have gauge fields with Yang-Mills kinetic terms. This implies that all of the three dimensional gauged supergravities mentioned above cannot be obtained from any known dimensional reductions. Nevertheless, it has been discovered in [38] that a Chern-Simons gauged supergravity with a non-semisimple gauged group $G \ltimes \mathbf{T}^{\dim G}$, with $\mathbf{T}^{\dim G}$ transforming in the adjoint representation of G, is on-shell equivalent to a Yang-Mills gauged supergravity with a semisimple gauge group G. This equivalence has been applied to the $N = 8$ theory with gauge group $SO(4) \ltimes \mathbf{T}^{\infty}$ in [39] to describe the Kaluza-Klein spectrum of the (2,0) six-dimensional supergravity on $AdS_3 \times S^3$. Remarkably, all massive vector fields arising from the KK modes can be incorporated in this theory within a single scalar manifold $\frac{SO(8,\infty)}{SO(8)\times SO(\infty)}$.

Recall that the background of interest in the AdS$_3$/CFT$_2$ correspondence is AdS_3 which according to the AdS/CFT correspondence, will have a dual two dimensional CFT living on its boundary. In AdS$_{d+1}$/CFT$_d$, the conformal group of the CFT$_d$ is identified with the isometry group of AdS_{d+1}, $SO(2, d)$. For $d = 2$, the isometry of AdS_3 is $SO(2, 2) \sim SO(2, 1) \times SO(2, 1) \sim SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ which is the global conformal symmetry of the two dimensional CFTs. On the other hand, the local conformal transformations correspond to an infinite dimensional group generated by the Virasoro algebra. It has been shown long ago before the AdS/CFT correspondence in [40] that the asymptotic symmetry of AdS_3 is the infinite dimensional conformal group with central charge $c = \frac{3L}{2G}$ where L and G are AdS_3 radius and Newton constant, respectively. The extension to the case of three dimensional supergravity giving rise to the asymptotic symmetry in the form of the superconformal group on the boundary has been studied in [41] and [42]. The dual CFT for pure gravity in three dimensions has been studied in [43] with the dual CFT identified with the monster theory of Frenkel, Lepowsky and Meurman. Recently, it has been shown that an asymptotic symmetry of the higher spin AdS gravity in three dimensions is given by a two dimensional W algebra [44], [45], and a gravity dual of the two dimensional W_N minimal models in the large N limit has been proposed in [46].

As noted before, to embedding the lower dimensional solutions to string theory, it is necessary to have a consistent reduction ansatz from ten to three dimensions. It can happen that a lower dimensional geometry is singular but the corresponding ten dimensional geometry obtained by uplifting the lower dimensional one, is completely smooth. This is crucially relies on the existence of the consistent reduction ansatz. Therefore, we have to start with three dimensional gauged supergravity with non-semisimple gauge groups as these are the only ones up to now we know how to obtain their higher dimensional origin. Currently, there are only a few explicit reduction ansatze of higher dimensional supergravities reduced to three dimensions. One example is given in [47] and [48] in which the reduction of the minimal $N = (1, 0)$ pure supergravity in six dimensions on $SU(2)$ group manifold has been studied. The resulting theory is the Yang-Mills $N = 4$ three dimensional gauged supergravity with $SU(2)$ gauge group coupled to three massive vector fields coming from the reduction of the self-dual two form field. And more recently, the embedding of $N = 2$ three dimensional gauged supergravity in eleven dimensions has been proposed in [49].

In this thesis, we will give a report on new RG flows in a dual two dimensional field theory in the framework of three dimensional gauged supergravities as well as in the higher dimensional point of view. Hopefully, this work partially fills the gap in the literatures on two dimensional holographic RG flows. We also find a new reduction ansatz of (1,0) six dimensional supergravity on $SU(2)$ group manifold. The resulting theory is the $N = 4$ gauged supergravity in three dimensions with a gauge group $SU(2) \times G$ without massive vector fields as opposed to previous works in this direction. Moreover, we explore RG flow solutions in the presence of a multi-instanton background along with its effects on RG flows and the values of central charges at the fixed points. The flows can also be interpreted as transitions between different instanton vacua. We further generalize the solutions by studying flow solutions with Yang-Mills instantons turned on on the ALE background. We finally extend the study to more interesting RG flows in four dimensional field theories by considering flow solutions in type IIB and type I$'$ string theories. We now discuss the outline of the thesis which is essentially based on the works done in [50, 51, 52, 53, 54, 55].

In chapter 2, we review the construction of three dimensional gauged supergravities using the $SO(N)$ covariant formulation of [31]. For a good review on supergravity theories, we refer the reader to [56] and [57] for general discussions. We will then introduce the notion of the embedding tensor characterizing the embedding of the gauge group in the global symmetry group of the ungauged theory. We give a detailed discussion of the target space geometry in general and then specialize to the case of symmetric target spaces which can be written as coset spaces G/H. We also give the full Lagrangian as well as the gauge and supersymmetry transformations of all fields. Finally, we review the on-shell equivalence between non-semisimple Chern-Simons gauged supergravity and semisimple Yang-Mills gauged supergravity. This is a peculia feature of supergravity theories in three dimensions. This chapter forms the base of many chapters in the thesis since the formulation and many formulae given here will be extensively used in later chapters.

In chapter 3, we review the basic idea of the AdS/CFT correspondence. We then introduce the notion of holographic renormalization and holographic RG flows. We will also discuss the holographic c-theorem and review its proof. We then move to study supersymmetric AdS_3 vacua of $N = 4$ gauged supergravity in three dimensions with various amount of preserved supersymmetries. The scalar target space is $\frac{SO(4,4)}{SO(4)\times SO(4)} \times \frac{SO(4,4)}{SO(4)\times SO(4)}$. We gauge the $SO(4) \ltimes \mathbf{T}^6$ non-semisimple subgroup of the global symmetry $SO(4,4)$. The motivation for gauging a non-semisimple gauge group is that, by the equivalence between Chern-Simons and Yang-Mills gauged supergravities as mentioned above, this gauging allows us to relate the resulting theory to the Yang-Mills gauged supergravity obtained from dimensional reduction of some higher dimensional theory. This mechanism has been employed, for example, in [39] for $N = 8$, where it has been shown that a gauging by $SO(4) \ltimes \mathbf{T}^6$ indeed reproduces, at the $N = 8$ point in the scalar manifold, the Kaluza-Klein spectrum of the six-dimensional (2,0) supergravity on $AdS_3 \times S^3$[58]. The latter is the background one obtains by taking the near horizon geometry of a D1-D5 system of type IIB theory on $K3$ or T^4, corresponding to a SCFT_2 with (4,4) supersymmetry. We will also study the flow between different vacua with different cosmological constants but the same amount of supersymmetry. Quite remarkably, we will be able to find an analytic flow solution between vacua with (3,1) supersymmetry involving two active scalar fields. For the case of a flow between (2,0) vacua which involves three active scalars, we will discuss a numerical flow solution. The flows turn out to be vev flows driven by vacuum expectation values of some relevant operators in the UV. Examples of vev flows are known in four dimensional super-conformal field theories, in particular in $N = 2$ SCFT, where they have been studied using Seiberg-Witten solution in connection with the Argyres-Douglas fixed points[59, 60, 61]. To the best of our knowledge, the flows given here are the first examples of vev flows between two AdS vacua in a gauged supergravity context.

Chapter 4 is devoted to the $SU(2)$ reduction of the (1,0) six dimensional supergravity constructed in [62] and [63]. We will begin with a discussion of a group manifold reduction and specialize to the reduction on the $SU(2)$ group manifold. We then review the minimal six dimensional supergravity coupled to an antisymmetric tensor and Yang-Mills multiplets. The equations of motion for the bosonic fields and all supersymmetry transformations are given together with a Lagrangian of some specific cases. As is well known, there does not exist an invariant Lagrangian without introducing the auxiliary fields for the case where the number of tensor multiplets is different from one due to the (anti) self duality of the three-form field strength of the antisymmetric tensor fields in the tensor and gravity multiplets. On the other hand, coupled to one tensor multiplet, the theory does admit a Lagrangian formulation in some cases as we will see in this chapter. We will perform the $SU(2)$ group manifold reduction of (1,0) six dimensional supergravity coupled to an anti-symmetric tensor and G Yang-Mills multiplets and obtain $SU(2) \times G$ Yang-Mills gauged supergravity in three dimensions. The resulting theory contains $4(1 + \dim G)$ bosons and $4(1 + \dim G)$ fermions with the scalar manifold being $\mathbf{R} \times \frac{SO(3, \dim G)}{SO(3)\times SO(\dim G)}$. While it is known that the most general $SU(2)$ reduction, including massive vector fields, is consistent, the novel feature this chapter is that we make a further truncation by removing the massive vector fields and show that it is consistent. We then construct an $N = 4$ Chern-Simons $(SO(3) \ltimes \mathbf{T}^3) \times (G \times \mathbf{T}^{\text{dim}G})$ gauged supergravity with a scalar manifold $\frac{SO(4,1+\text{dim}G)}{SO(4)\times SO(1+\text{dim}G)}$ and show that it is indeed equivalent to $SO(3) \times G$ Yang-Mills gauged supergravity with scalar manifold $\mathbf{R} \times \frac{SO(3,\text{dim}G)}{SO(3)\times SO(\text{dim}G)}$ after removing $3 + \text{dim}G$ scalars corresponding to the translational symmetries. The result completely agrees with the discovery in [38].

In chapter 5, we will study RG flow solutions in the dual two dimensional field theories with Yang-Mills instantons in the supergravity background by looking for supersymmetric solutions of the six dimensional supergravity theory studied in the previous chapter. We first discuss a six dimensional flow solution which, in fact, is the uplift to six dimensions of an RG flow of the $N = 4$ three dimensional gauged supergravity and preserves half of the supersymmetries. This solution involves an $SU(2)$ instanton on $\mathbb{R} \times S^3$, $\mathbb{R}$ parametrized by the radial coordinate, with topological charge equal to 1, which in the three dimensional setting is seen as a scalar's background. The instanton interpolates between the $|0\rangle$ Yang-Mills vacuum with winding number 0 in the IR and $|1\rangle$ vacuum with winding number 1 in the UV. We then move to study solutions involving multi-instanton gauge fields of an arbitrary semisimple gauge group G. In this case, the solution is genuinely six dimensional in the sense that it cannot be obtained as an uplifted solution of a three dimensional theory, roughly, because it involves higher modes on S^3. The instanton interpolates between $|N\rangle$ vacuum in the UV and $|0\rangle$ vacuum in the IR. The solution has been studied long ago in [64, 65] but in different contexts. We will look at it from another point of view by regarding it as an RG flow solution interpolating between the UV and IR CFT's corresponding to two AdS_3 limits. The central charge at the two fixed points of course respects the c-theorem and admits an interpretation in terms of the dynamics of the $D1/D5$ dual system giving rise to a $(4, 0)$ SCFT in the decoupling limit[66, 67, 68].

In chapter 6, we study RG flows in both two- and four- dimensional contexts. We still work in the framework of (1,0) six dimensional supergravity, but now we replace the transverse $\mathbb{R}^4$ with an ALE manifold of A_{N-1} type. We will adopt on it the well-known Gibbons-Hawking multi-center metric [69]. Furthermore, we also study a flow solution involving Yang-Mills instantons turned on on the ALE space, thereby generalizing the solution discussed in chapter 5. Explicit instanton solutions on an ALE space can be written down for the $SU(2)$ gauge group [70], [71], [72] and we will then restrict ourselves to these solutions. The resulting supergravity solutions describe RG flows in two dimensional dual field theories and have asymptotic geometries $AdS_3 \times S^3/\mathbb{Z}_N$ in the UV and $AdS_3 \times S^3$ in the IR. The former arises from the limit where one goes to the boundary of the ALE, the latter when one zooms near one of the smooth ALE centers. Notice that in this case the solution describes the flow from a (2,0) UV CFT to a (4,0) IR CFT, contrary to the case of chapter 5, where both fixed points were (4,0) CFT's. Indeed, in the UV we have $\mathbb{Z}_N$ projection, due to asymptotic topology of the ALE space.

We will then move to study more interesting and more realistic RG flow solutions in the dual four dimensional field theory in the context of ten dimensional type IIB and type I′ theories (by the latter we mean IIB on $T^2/(-1)^{F_L}\Omega I_2$, the double T-dual of type I on T^2 [73]) on an ALE background. These solutions describe RG flows of four dimensional UV CFT's with $N = 2$ supersymmetry. In the type IIB case our solution is a variation on the theme discussed in [74, 75, 76] for the ALE space of the form $\mathbb{C}^3/\mathbb{Z}_3$ and for the conifold, respectively, which describe flows from $N = 1$ to $N = 4$ CFT's. Our flows interpolate between $N = 2$ quiver gauge theories with product gauge group in the UV and the $N = 4$ $SU(n)$ supersymmetric Yang-Mills theory in the IR. The corresponding asymptotic geometries are $AdS_5 \times S^5/\mathbb{Z}_N$ and $AdS_5 \times S^5$.

The discussion becomes more interesting in type I′ theory: in this case we find that the critical points are described by the geometries $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ in the UV and $AdS_5 \times S^5/\mathbb{Z}_2$ in the IR. The $\mathbb{Z}_2$ is identified with $(-1)^{F_L}\Omega I_2$. The UV gauge groups are more complicated than those of type IIB case, and are among the (unoriented) quiver gauge groups discussed in [77]. The quiver diagrams have different structures depending on whether N is even or odd, and for N even there are in addition two possible projections, resulting in two different quiver structures. This is what will make the discussion of RG flows richer and more interesting. We will in fact verify the agreement between the geometric picture emerging from the supergravity solutions and the corresponding field theory description, where the flows are related to the Higgsing of the gauge group, i.e. they are driven by vacuum expectation values of scalar fields belonging to the hypermultiplets of the $N = 2$ theories.

We also consider more general RG flows, in which not all the UV gauge group is broken to a single diagonal IR subgroup. In other words, the IR theory can be another, smaller, quiver gauge theory. The associated flows are the flows between two $N = 2$ quiver gauge theories, and the corresponding geometries are given by $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ and $AdS_5 \times S^5/(\mathbb{Z}_M \times \mathbb{Z}_2)$ with $M < N$. We will find that field theory considerations do not allow all possible flows with arbitrary values of

M and N and some symmetry breaking patterns are forbidden. Actually, we will see that these features are reproduced by the geometry, having to do with the fact that the Ω projection does not allow arbitrary ALE geometry since it projects out or identify the geometric moduli, as was already observed in a different context in [78]. In fact, we will obtain a very satisfactory agreement between with the field theory and the supergravity pictures.

Chapter 7 gives a summary of the results presented in this thesis. Some comments on the results and open problems as well as directions for future researches are discussed.

Additionally, there are two appendices. One of them presents new results, and the other gives a review on the relevant information used in the main text. Appendix A presents other vacua of three dimensional gauged supergravities. The theories considered in this appendix are $N = 8, 9, 10$ theories. The vacua of $N = 4$ theory which are not involved in the flows studied in chapter 3 are also given here. The study of critical points of gauged supergravities is useful in the AdS/CFT correspondence. In the original AdS_5/CFT_4 correspondence, critical points of $N = 8$ five dimensional gauged supergravity found in [79] describe various phases of $N = 4$ SYM. Recently, the interest in the AdS/CFT correspondence has been extended to AdS_4/CFT_3 which might give some insight to condensed matter systems, for example, superconductors. Some critical points of the corresponding four dimensional gauged supergravity have been studied in [80, 81] soon after its construction [82], and recently, some new vacua of this theory have been identified in [83, 84]. In three dimensions, the analogous study appears in [15, 85, 86, 87] for $N = 8$ and $N = 16$ theories. In this appendix, we will give some results on critical points of $N = 4, 8, 9, 10$ three dimensional gauged supergravities with various gauge groups.

The last appendix deals with gravitational and Yang-Mills instantons which play an important role in several places in the thesis. The aim of this appendix is to review the relevant formulae used in the main text along with some derivations. This section is by no means intentionally a complete review on the corresponding subject.

Chapter 2

Three Dimensional Gauged Supergravity

In this chapter, we review the construction of gauged supergravity in three dimensions. This chapter is mainly based on [31] and [26]. We begin with a discussion of ungauged three dimensional supergravity coupled to a non-linear sigma model describing bosonic matter fields in three dimensions. The Lagrangian and supersymmetry transformations of this theory along with its symmetries are discussed. The symmetries consist of the R-symmetry and isometries of the target space parametrized by scalar fields. We focus our discussion on symmetric target spaces which can be written in term of a coset space of the form G/H. We then move to gaugings some of these symmetries by introducing a notion of an embedding tensor along with consistency conditions imposed by supersymmetry. We end this chapter by reviewing the equivalence between semisimple Yang-Mills and non-semisimple Chern-Simons gauged supergravities in three dimensions [38].

2.1 Ungauged supergravity in three dimensions

Although pure supergravity in three dimensions is topological and allows any number N of supersymmetry, the theory coupled to matter fields exists only for $N \leq 16$ [26]. In three dimensions, bosonic matter fields are scalars since vector fields can be dualized to scalars, so the coupled theory is in the form of non-linear sigma model coupled to N extended supergravity. The existence of the bound $N \leq 16$ is consistent with higher dimensional theories, but at the same time seems surprising. This is because there is no physical restriction on the number of supercharges in three dimensions.

In four dimensions, the requirement for the absence of massless particles with helicity larger than two imposes the condition $N \leq 8$. This bound is carried over to theories in dimension higher than four since compactifications of these theories that lead to sensible four dimensional theories must respect this bound. So, in dimension larger than four, the maximum number of supercharges is also implied by the physical requirement in four dimensions. This is, however, not the case in three dimensions because there is no helicity in three dimensions. So, we cannot use the physical reason to impose a restriction on the number of supercharges. On the other hand, the reduction of $N/2$ extended supergravities in four dimensions with even N gives rise to supergravities with N supersymmetries in three dimensions. This suggests the bound $N \leq 16$ for at least theories obtained from dimensional reductions. However, there is a constraint on the target space of the non-linear sigma model as shown in [26]. For a theory with $N > 4$, the target spaces must be symmetric, and beyond $N = 16$, there is no known symmetric space. This is related to the fact that there is no exceptional group beyond E_8 to act as an isometry group of the scalar target manifolds [26]. So, in three dimensions, the restriction on the maximum number of supercharges namely 32 is purely mathematical.

The sigma model is parametrized by scalar fields ϕ^i whose superpartners are χ^i. The coupling to N extended supersymmetries requires the existence of $N-1$ almost complex structures $f^{Pi}{}_j$ with $P = 2\ldots N$, $i, j = 1\ldots, d$, and d is the dimension of the target space. These $f^{Pi}{}_j$ are hermitean and generate Clifford algebra

$$\begin{aligned} g_{ij}f^{Pj}{}_k + g_{kj}f^{Pj}{}_i &= 0, \\ f^{Pi}{}_k f^{Qk}{}_j + f^{Qi}{}_k f^{Pk}{}_j &= -2\delta^{PQ}\delta^i_j\,. \end{aligned} \qquad (2.1)$$

We can construct f^{IJ}_{ij} tensors which generate $SO(N)$ R-symmetry by

$$f^{1P} = -f^{P1} = f^P, \qquad f^{PQ} = f^{[P}f^{Q]}\,. \qquad (2.2)$$

These f^{IJ}_{ij} satisfy the following identities

$$\begin{aligned} f^{IJ}_{ij} &= -f^{JI}_{ij} = -f^{IJ}_{ji}\,, \\ f^{IJ}f^{KL} &= f^{[IJ}f^{KL]} - 4\delta^{I[K}f^{L]J} - 2\delta^{I[K}\delta^{L]J}\mathbf{1}, \\ f^{IJij}f^{KL}{}_{ij} &= 2d\delta^{I[K}\delta^{L]J} - \delta_{N,4}\epsilon^{IJKL}\mathrm{Tr}J, \end{aligned} \qquad (2.3)$$

where $I, J = 1, \ldots N$. The $J^i{}_j$ is relevant only for $N = 4$ and defined by

$$J = \frac{1}{6}\epsilon_{PQR}f^Pf^Qf^R = \frac{1}{24}\epsilon^{IJKL}f^{IJ}f^{KL}\,. \qquad (2.4)$$

It commute with the almost complex structures and satisfy

$$\begin{aligned} Jf^P &= f^PJ, \qquad J^2 = \mathbf{1}, \qquad J_{ij} = J_{ji}, \\ f^Pf^Q &= -\delta^{PQ}\mathbf{1} - \epsilon^{PQR}Jf^R\,. \end{aligned} \qquad (2.5)$$

J has eigenvalues ± 1 and is covariantly constant. The target space for $N = 4$ is a product of two Riemannian spaces with dimension $d_\pm$, $d_- + d_+ = d$. It turns out that both of the two subspaces are quaternionic manifold, so $d_\pm$ are multiple of 4. The product structure is due to the fact that there are two inequivalent multiplets for $N = 4$. Each subspace corresponds to each multiplet. In fact, inequivalent multiplets exist for value of $N = 4$ mod 4, but the requirement that the local symmetry H containing $SO(N)$ act irreducibly on the target space rules out all but $N = 4$ cases. This is because the $SO(4)$ R-symmetry itself factors into two $SO(3)$'s. The two $SO(3)$ factors act separately on the two subspaces. So, the $N = 4$ case is special, and in particular, we can write (2.3) as

$$f^{IJij} f^{KL}{}_{ij} = 4(d_+ \mathbb{P}_+^{IJKL} + d_- \mathbb{P}_-^{IJKL}) \tag{2.6}$$

by using the projectors

$$\mathbb{P}_\pm^{IJKL} = \frac{1}{2}\delta^{I[K}\delta^{L]J} \mp \frac{1}{4}\epsilon^{IJKL} \,. \tag{2.7}$$

We now move to the geometry of the target space and its implications. Since some results for the case $N = 1, 2$ are different from $N > 2$, and in this work, we are interested in theory with $N > 2$ mainly $N = 4$, we will only review the $N > 2$ case and refer the reader to [31] for the detailed discussion of $N = 1, 2$. For $N > 2$, f^P are only almost complex structures. As shown in [26], the Nijenhuis tensors are given by

$$N_{Pij}^k = f_{Pi}^l D(\Gamma)_{[j} f_{Pl]}^k - f_{Pj}^l D_{[i} f_{Pl]}^k, \qquad \text{no sum on } P \,. \tag{2.8}$$

These satisfy $N_{Pji}^j = 0$ but vanish only for $N = 2$ in which f_{ij}^P is covariantly constant with respect to the Christoffel connection Γ_{ij}^k according to [26]

$$D_k(\Gamma) f_{Pij} + Q_{PQ} f_{ij}^Q + Q_k^Q (f_{[P} f_{Q]})_{ij} = 0 \,. \tag{2.9}$$

For $N = 2$, we simply have $Q_{PQ} = 0$, $P, Q = 2$.

The coupling between supergravity and non-linear sigma model involves the $SO(N)$ connections Q_i^{IJ}, formed by combining Q^P and Q^{PQ}, on the target space. These connections are non-trivial in the sense that

$$R_{ij}^{IJ} = \partial_i Q_j^{IJ} - \partial_j Q_i^{IJ} + 2Q_i^{K[I} Q_j^{J]K} = \frac{1}{2} f_{ij}^{IJ} \,. \tag{2.10}$$

The f^{IJ} tensors are covariantly constant with respect to Christoffel connection Γ_{ij}^k and Q_i^{IJ}

$$D_i(\Gamma, Q^{IJ}) f_{jk}^{IJ} = \partial_i f_{jk}^{IJ} - 2\Gamma_{i[k}^l f_{j]l}^{IJ} + 2Q_i^{K[I} f_{jk}^{J]K} = 0 \,. \tag{2.11}$$

The integrability condition for (2.11) gives

$$R_{ijmk} f^{IJm}{}_l - R_{ijml} f^{IJm}{}_k = -f_{ij}^{K[I} f_{kl}^{J]K} \tag{2.12}$$

where R_{ijkl} is the target space Riemann tensor. Contracting with f^{MNkl} and g^{jl} gives, respectively,

$$
\begin{aligned}
R_{ijkl} f^{IJkl} &= \frac{1}{4} d f_{ij}^{IJ} , && (2.13)\\
R_{ij} = g^{kl} R_{ikjl} &= \left(N - 2 + \frac{1}{8} d \right) g_{ij} \, . && (2.14)
\end{aligned}
$$

From these results, the target space is an Einstein space with non-trivial $SO(N)$ holonomy. For the $N = 4$ case with the target space being a product of two quaternionic spaces, both the two subspaces are Einstein, and equations (2.13) and (2.14) become

$$
\begin{aligned}
R_{ijkl} f^{IJkl} &= \frac{1}{2} (d_+ \mathbb{P}_+^{IJ,KL} + d_- \mathbb{P}_-^{IJ,KL}) f_{ij}^{KL} , && (2.15)\\
R_{ij} &= g^{kl} R_{ikjl} = \left(2 + \frac{1}{8} d \right) g_{ij} + \frac{1}{8} (d_+ - d_-) J_{ij} \, . && (2.16)
\end{aligned}
$$

We can extract the f^{IJ} part of the Riemann tensor by decomposing the Riemann tensor as

$$
R_{ijkl} = \hat{R}_{ijkl} + \frac{1}{8} f_{ij}^{IJ} f_{kl}^{IJ} \, . \qquad (2.17)
$$

We then write the integrability condition (2.12) as

$$
\hat{R}_{ijmk} f^{IJm}{}_{l} - \hat{R}_{ijml} f^{IJm}{}_{k} = 0 \, . \qquad (2.18)
$$

We now introduce a set of antisymmetric tensors h_{ij}^α that commute with $SO(N)$ generators f^{IJ}

$$
h_{ik}^\alpha f^{IJk}{}_{j} - h_{jk}^\alpha f^{IJk}{}_{i} = 0 \, . \qquad (2.19)
$$

h^α generate a subgroup H' of $SO(d)$ that commutes with $SO(N)$. The algebra of H' is given by

$$
h^\alpha h^\beta - h^\beta h^\alpha = f^{\alpha\beta}{}_{\gamma} h^\gamma \qquad (2.20)
$$

with structure constants $f^{\alpha\beta}{}_{\gamma}$. h^α are covariantly constant with respect to the Christoffel and $\Omega_i^{\alpha\beta}$ connections

$$
D(\Gamma)_i h_{jk}^\alpha - \Omega_i^{\alpha}{}_{\beta} h_{jk}^\beta = 0 \, . \qquad (2.21)
$$

The Riemann tensor can now be written as

$$
R_{ijkl} = \frac{1}{8} (f_{ij}^{IJ} f_{kl}^{IJ} + C_{\alpha\beta} h_{ij}^\alpha h_{kl}^\beta) \qquad (2.22)
$$

with a symmetric tensor $C_{\alpha\beta}$. The holonomy group is then contained in $SO(N) \times H' \subset SO(d)$ and acts irreducibly on the target space. We can also normalize h^α as in [26] by

$$
h_{ij}^\alpha h^{\beta ij} = 2 d_N \delta^{\alpha\beta} \qquad (2.23)
$$

where d_N is the number of bosonic states in the supermultiplet of N extended supersymmetries, see [26] for the value of d_N for all values of N. $\delta^{\alpha\beta}$ is an invariant tensor of H' with $f^{\alpha\beta\gamma} = \delta^{\delta\gamma} f^{\alpha\beta}_{\ \ \delta}$ being the totally antisymmetric structure constants. The H' curvature is given by

$$R^{\alpha}{}_{\beta ij} = 2(\partial_{[i}\Omega_{j]}{}^{\alpha}{}_{\beta} - \Omega_{[i}{}^{\gamma}{}_{\alpha}\Omega_{j]}{}^{\beta}{}_{\gamma}) = \frac{1}{8} f^{\alpha\gamma}{}_{\beta} C_{\gamma\delta} h^{\delta}_{ij}\,. \tag{2.24}$$

We can also restrict $\Omega_i^{\alpha\beta}$ to the form $\Omega_i^{\alpha\beta} \propto f^{\alpha\beta}{}_{\gamma} Q_i^{\gamma}$.

There exist only theories with $N = 1,\ldots,6$, 8, 9, 10, 12 and 16. The supergravity fields consist of the vielbein e^a_μ and gravitini ψ^I_μ. Following [31], we will use the $SO(N)$ covariant formulation in which the almost complex structures will not appear explicitly. We first define a new basis for the spin $\frac{1}{2}$ fields

$$\chi^{iI} = (\chi^i, f^{Pi}{}_j \chi^j) \tag{2.25}$$

with the $SO(N)$ covariant constraint

$$\chi^{iI} = \mathbb{P}^{Ii}_{Jj}\chi^{jJ} = \frac{1}{N}(\delta^{IJ}\delta^i_j - f^{IJi}{}_j)\chi^{jJ}\,. \tag{2.26}$$

The total number of fermions is still d as can be seen from the trace of the projector $\mathbb{P}^{Ii}_{Ii} = d$. The Lagrangian for the ungauged theory is given by [31]

$$\begin{aligned}\mathcal{L}_0 \;=\;& -\frac{1}{2} i\epsilon^{\mu\nu\rho}(e^a_\mu R_{\nu\rho a} + \bar{\psi}^I_\mu D_\nu \psi^I_\rho) - \frac{1}{2} e g_{ij}\left(g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j + \frac{1}{N}\bar{\chi}^{iI} \not{D}\chi^{jI}\right)\\ &+\frac{1}{4} e g_{ij}\bar{\chi}^{iI}\gamma^\mu\gamma^\nu\psi^I_\mu(\partial_\nu\phi^j + \hat{\partial}_\nu\phi^j) - \frac{1}{24N^2} e R_{ijkl}\bar{\chi}^{iI}\gamma_a\chi^{jI}\bar{\chi}^{kJ}\gamma^a\chi^{lJ}\\ &+\frac{1}{48N^2} e\left[3(g_{ij}\bar{\chi}^{iI}\chi^{jI})^2 - 2(N-2)(g_{ij}\bar{\chi}^{iI}\gamma^a\chi^{jJ})^2\right]\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}D_\mu\psi^I_\nu &= \left(\partial_\mu + \frac{1}{2}\omega^a{}_\mu\gamma_a\right)\psi^I_\nu + \partial_\mu\phi^i Q_i^{IJ}\psi^J_\nu,\\ D_\mu\chi^{iI} &= \left(\partial_\mu + \frac{1}{2}\omega^a{}_\mu\gamma_a\right)\chi^{iI} + \partial_\mu\phi^j(\Gamma^i_{jk}\chi^{kI} + Q_j^{IJ}\chi^{iJ}).\end{aligned} \tag{2.28}$$

$\omega^a{}_\mu$ is the usual spacetime spin connection contracted with ϵ_{abc} as followed from the $\gamma_a\gamma_b$ identity given below, and e is the determinant of e^a_μ defined by

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}. \tag{2.29}$$

The Lagrangian is invariant under the following supersymmetry transformations

$$\begin{aligned}\delta e^a_\mu &= \frac{1}{2}\bar{\epsilon}^I\gamma^a\psi^I_\mu,\\ \delta\psi^I_\mu &= D_\mu\epsilon^I - \frac{1}{8} g_{ij}\bar{\chi}^{iI}\gamma^\nu\chi^{jJ}\gamma_{\mu\nu}\epsilon^J - \delta\phi^i Q_i^{IJ}\psi^J_\mu,\\ \delta\phi^i &= \frac{1}{2}\bar{\epsilon}^I\chi^{iI},\\ \delta\chi^{iI} &= \frac{1}{2}(\delta^{IJ}\mathbf{1} - f^{IJ})^i_j\hat{\not{\partial}}\phi^j\epsilon^J - \delta\phi^j(\Gamma^i_{jk}\chi^{kI} + Q_j^{IJ}\chi^{iJ}),\end{aligned} \tag{2.30}$$

where the supercovariant derivative and the covariant derivative are defined by

$$\begin{aligned}
\hat{\not{\partial}}_\mu \phi^i &= \partial\phi^i - \frac{1}{2}\bar{\psi}^I_\mu \chi^{iI}, \\
D_\mu \epsilon^I &= \left(\partial_\mu + \frac{1}{2}\omega^a{}_\mu \gamma_a\right)\epsilon^I + \partial_\mu \phi^i Q_i^{IJ}\epsilon^J, \qquad (2.31)
\end{aligned}$$

respectively. We also use the same conventions as [31] and [26] namely γ^a are hermitean and satisfy

$$\gamma_a\gamma_b = \delta_{ab} + i\epsilon_{abc}\gamma_c\,. \qquad (2.32)$$

We can change to $(-++)$ metric by multiply conjugate spinors and ϵ_{abc} by i and ϵ^{abc} by $-i$. We will do these in the actual computation in later chapters. The field dependent $SO(N)$ R-symmetry transformations

$$\begin{aligned}
\delta\psi^I_\mu &= \Lambda^{IJ}(\phi)\psi^J_\mu, \qquad \delta\chi^{iI} = \Lambda^{IJ}(\phi)\chi^{iJ}, \\
\delta Q_i^{IJ} &= -D_i\Lambda^{IJ}(\phi), \qquad \delta f^{IJ} = 2\Lambda^{K[I}(\phi)f^{J]K} \qquad (2.33)
\end{aligned}$$

and target space diffeomorphisms are not, in general, an invariance of the Lagrangian but are rather reparametrizations within certain equivalence classes. The invariance of the Lagrangian consists of target space isometries including appropriate $SO(N)$ R-symmetry rotations. The isometries are generated by Killing vectors $X^i(\phi)$. Accompanied with $SO(N)$ rotations $\mathcal{S}^{IJ}(X,\phi)$, some of these isometries can be extended to the invariance of the Lagrangian. The target space metric g_{ij} is invariant under X^i as X^i is the Killing vectors. We also require the invariance of Q_i^{IJ} and f_{ij}^{IJ} up to $SO(N)$ transformations. Using (2.33), we can write down the following conditions for our requirements

$$\begin{aligned}
\mathcal{L}_X g_{ij} &= 0, \qquad \mathcal{L}_X Q_i^{IJ} + D_i\mathcal{S}^{IJ} = 0, \\
\mathcal{L}_X f_{ij}^{IJ} &- 2\mathcal{S}^{K[I} f_{ij}^{J]K} = 0 \qquad (2.34)
\end{aligned}$$

where $\mathcal{L}_X$ denotes the Lie derivative along X^i. The Lagrangian (2.27) is now invariant under the transformations

$$\begin{aligned}
\delta\phi^i &= X^i, \qquad (2.35)\\
\delta\psi^I_\mu &= \mathcal{S}^{IJ}\psi^J_\mu = \mathcal{V}^{IJ}\psi^J_\mu - \delta\phi^i Q_i^{IJ}\psi^J_\mu, \qquad (2.36)\\
\delta\chi^{iI} &= \chi^{jI}\partial_j X^i + \mathcal{S}^{IJ}\chi^{i,J} = D_j X^i \chi^{jI} + \mathcal{V}^{IJ}\chi^{iJ} \\
&\quad -\delta\phi^j(\Gamma^i_{jk}\chi^{kI} + Q_j^{IJ}\chi^{iJ}) \qquad (2.37)
\end{aligned}$$

where $\mathcal{V}^{IJ} = X^j Q_j^{IJ} + \mathcal{S}^{IJ}$. Equations (2.10) and (2.11) imply that the second equation of (2.34) can be written as

$$D_i\mathcal{V}^{IJ} = \frac{1}{2}f_{ij}^{IJ}X^j \qquad (2.38)$$

and the third equation of (2.34) becomes the integrability condition for (2.38)

$$f^{IJk}{}_{[i}D_{j]}X_k = f^{K[I}_{ij}\mathcal{V}^{J]K}\,. \tag{2.39}$$

Contracting with f^{MNij}, we find

$$f^{IJij}D_iX_j = \begin{cases} \frac{1}{2}d\mathcal{V}^{IJ}, & \text{for} N \neq 4 \\ (d_+\mathbb{P}_+^{IJ,KL} + d_-\mathbb{P}_-^{IJ,KL})\mathcal{V}^{KL}, & \text{for} N = 4 \end{cases}. \tag{2.40}$$

We then find that

$$D^iD_i\mathcal{V}^{IJ} = \begin{cases} \frac{1}{4}d\mathcal{V}^{IJ}, & \text{for} N \neq 4 \\ \frac{1}{2}(d_+\mathbb{P}_+^{IJ,KL} + d_-\mathbb{P}_-^{IJ,KL})\mathcal{V}^{KL}, & \text{for} N = 4 \end{cases}. \tag{2.41}$$

This result shows that there is no restriction to extend an isometry to a symmetry of the Lagrangian for $N > 2$. The symmetry of the Lagrangian coming from the extension of some isometries is generated by an algebra $\mathfrak{g}$ with generators $X^{\mathcal{M}}$, $\mathcal{M} = 1, \ldots \dim\mathfrak{g}$, satisfying

$$X^{\mathcal{M}i}\partial_iX^{\mathcal{N}} - X^{\mathcal{N}i}\partial_iX^{\mathcal{M}} = f^{\mathcal{MN}}{}_{\mathcal{K}}X^{\mathcal{K}}\,. \tag{2.42}$$

$f^{\mathcal{MN}}{}_{\mathcal{K}}$ are structure constants of this algebra. Closure condition of the algebra implies the condition for $SO(N)$ transformations

$$\begin{aligned} \left[\mathcal{S}^{\mathcal{M}}, \mathcal{S}^{\mathcal{N}}\right]^{IJ} &= \mathcal{S}^{\mathcal{M}IK}\mathcal{S}^{\mathcal{N}KJ} - \mathcal{S}^{\mathcal{N}IK}\mathcal{S}^{\mathcal{M}KJ} \\ &= -f^{\mathcal{MN}}{}_{\mathcal{K}}\mathcal{S}^{\mathcal{K}IJ} + (X^{\mathcal{M}i}\partial_i\mathcal{S}^{\mathcal{N}IJ} - X^{\mathcal{N}i}\partial_i\mathcal{S}^{\mathcal{M}IJ})\,. \end{aligned} \tag{2.43}$$

where $\mathcal{S}^{\mathcal{M}IJ} = \mathcal{S}^{IJ}(\phi, X^{\mathcal{M}})$. We can also write this equation as

$$\left[\mathcal{V}^{\mathcal{M}}, \mathcal{V}^{\mathcal{N}}\right]^{IJ} = -f^{\mathcal{MN}}{}_{\mathcal{K}}\mathcal{V}^{\mathcal{K}IJ} + \frac{1}{2}f^{IJ}_{ij}X^{\mathcal{M}i}X^{\mathcal{N}j} \tag{2.44}$$

where (2.42) and the second equation of (2.34) have been used, and $\mathcal{V}^{\mathcal{M}IJ} = \mathcal{V}^{IJ}(\phi, X^{\mathcal{M}})$. The integrability condition (2.39) implies that $D_iX_j - \frac{1}{4}f^{IJ}_{ij}\mathcal{V}^{IJ}$ commutes with the almost complex structures and can be expanded in terms of h^{α}_{ij}

$$D_iX_j - \frac{1}{4}f^{IJ}_{ij}\mathcal{V}^{IJ} = h^{\alpha}_{ij}\mathcal{V}^{\mathcal{M}}{}_{\alpha}\,. \tag{2.45}$$

Using the notation $\mathcal{V}^{\mathcal{M}i} = X^{\mathcal{M}i}$ and $D_iD_jX_k = R_{ijkl}X^l$, the following differential equations can be derived [31]

$$\begin{aligned} D_i\mathcal{V}^{\mathcal{M}IJ} &= \frac{1}{2}f^{IJ}_{ij}\mathcal{V}^{\mathcal{M}j}, \\ D_i\mathcal{V}^{\mathcal{M}}{}_j &= \frac{1}{4}f^{IJ}_{ij}\mathcal{V}^{\mathcal{M}IJ} + h^{\alpha}_{ij}\mathcal{V}^{\mathcal{M}}{}_{\alpha}, \\ D_i\mathcal{V}^{\mathcal{M}}{}_{\alpha} &= \frac{1}{8}C_{\alpha\beta}h^{\beta}_{ij}\mathcal{V}^{\mathcal{M}j} \end{aligned} \tag{2.46}$$

where the derivatives now contain Γ and $SO(N) \times H'$ connections. It has been shown in [31] that these identities and (2.42) can be used to derive

$$
\begin{aligned}
f^{\mathcal{MN}}{}_{\mathcal{K}}\mathcal{V}^{\mathcal{K}}_{\ i} &= \frac{1}{4}f^{IJ}_{ij}(\mathcal{V}^{\mathcal{M}IJ}\mathcal{V}^{\mathcal{N}j} - \mathcal{V}^{\mathcal{N}IJ}\mathcal{V}^{\mathcal{M}j}) + h^{\alpha}_{ij}(\mathcal{V}^{\mathcal{M}}_{\ \alpha}\mathcal{V}^{\mathcal{N}j} - \mathcal{V}^{\mathcal{N}}_{\ \alpha}\mathcal{V}^{\mathcal{M}j}), \\
f^{\mathcal{MN}}{}_{\mathcal{K}}\mathcal{V}^{\mathcal{K}}_{\ \alpha} &= f^{\beta\gamma}{}_{\alpha}\mathcal{V}^{\mathcal{M}}_{\ \beta}\mathcal{V}^{\mathcal{N}}_{\ \gamma} + \frac{1}{8}C_{\alpha\beta}h^{\beta}_{ij}\mathcal{V}^{\mathcal{M}i}\mathcal{V}^{\mathcal{N}j} \,.
\end{aligned} \tag{2.47}
$$

Equations (2.44) and (2.47) can be used to reveal the algebraic structure of the target space symmetries. To do this, we define an algebra $\mathfrak{a}$ which is an image of $\mathfrak{g}$ under the homomorphism $\mathcal{V}$

$$
\begin{aligned}
\mathcal{V} &: \mathfrak{g} \to \mathfrak{a}, \\
\mathcal{V}(X^{\mathcal{M}}) &= \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}} = \frac{1}{2}\mathcal{V}^{\mathcal{M}}{}_{IJ}t^{IJ} + \mathcal{V}^{\mathcal{M}}{}_{\alpha}t^{\alpha} + \mathcal{V}^{\mathcal{M}}{}_{i}t^{i} \,.
\end{aligned} \tag{2.48}
$$

The algebra $\mathfrak{a}$ is an extension of $\mathfrak{so}(N) \oplus \mathfrak{h}'$ and generated by $\{t^{\mathcal{A}}\} = \{t^{IJ}, t^{\alpha}, t^{i}\}$ satisfying

$$
\begin{aligned}
[t^{IJ}, t^{KL}] &= -4\delta^{[I[K}t^{L]J]}, \qquad [t^{IJ}, t^{A}] = -\frac{1}{2}f^{IJ,AB}t_{B}, \qquad [t^{\alpha}, t^{\beta}] = f^{\alpha\beta}{}_{\gamma}t^{\gamma}, \\
\left[t^{A}, t^{B}\right] &= \frac{1}{4}f^{AB}_{IJ}t^{IJ} + \frac{1}{8}C_{\alpha\beta}h^{\beta AB}t^{\alpha}, \qquad [t^{\alpha}, t^{A}] = h^{\alpha}{}_{B}{}^{A}t^{B} \,.
\end{aligned} \tag{2.49}
$$

Using the fact that $\mathcal{V}$ is a homomorphism

$$
\mathcal{V}(\left[X^{\mathcal{M}}, X^{\mathcal{N}}\right]) = \left[\mathcal{V}(X^{\mathcal{M}}), \mathcal{V}(X^{\mathcal{M}})\right] = f^{\mathcal{MN}}{}_{\mathcal{K}}\mathcal{V}(X^{\mathcal{K}}) \tag{2.50}
$$

and (2.48), we readily recover (2.44) and (2.47) after matching the coefficients of t^{IJ}, t^{α} and t^{i} on both sides of (2.50). In addition, equation (2.38) can be written as

$$
D_i\mathcal{V}(X^{\mathcal{M}}) = \left[g_{ij}t^{j}, \mathcal{V}(X^{\mathcal{M}})\right] \,. \tag{2.51}
$$

One of the results of [26] is that for theory with $N > 4$, the target space is a symmetric space of the form G/H. For $N \leq 4$, the target space is not necessarily symmetric. However, in this work, we are interested only in the theories with symmetric target spaces, so we restrict ourselves to symmetric target spaces. A detailed discussion of symmetric spaces can be found in, for example, [57]. We now review the structure of symmetric spaces and describe the formulation given above in the case of symmetric target spaces. In this case, the symmetry of the Lagrangian which consists of some isometries and R-symmetry is given by the action of the group G. The scalar fields are described by a G value matrix L on which the global G and the local H symmetries act as multiplications from the left and right, respectively. The H symmetry can be used to eliminate $\dim(H)$ spurious degrees of freedom in L. After this "gauge fixing", we are left with a coset representative L. The target space is then $d = \dim(G/H) = \dim(G) - \dim(H)$ dimensional and hence paremetrized by d scalars ϕ^i. The isometries act on $L(\phi)$ from the left with the compensating H action on the right to maintain the coset representative in a particular gauge

$$gL(\phi^i) = L(\phi'^i)h(\phi^i)\,. \tag{2.52}$$

Consider infinitesimal transformation $\phi'^i = \phi^i + X^{\mathcal{M}i}\lambda_{\mathcal{M}}$ together with $g = 1 + t^{\mathcal{M}}\lambda_{\mathcal{M}}$ and $h = 1 + \frac{1}{2}\mathcal{S}^{\mathcal{M}IJ}X^{IJ}\lambda_{\mathcal{M}} + \mathcal{S}^{\mathcal{M}\alpha}X^{\alpha}\lambda_{\mathcal{M}}$, we then find

$$X^{\mathcal{M}i}\partial_i L = t^{\mathcal{M}}L - \frac{1}{2}\mathcal{S}^{\mathcal{M}IJ}LX^{IJ} - \mathcal{S}^{\mathcal{M}\alpha}LX^{\alpha}\,. \tag{2.53}$$

The index $\mathcal{M}$ is now G adjoint indices. In this case, the $H = SO(N)\times H'$, and G generators $t^{\mathcal{M}}$ decompose into $\{X^{IJ}, X^{\alpha}, Y^A\}$ where X^{IJ}, X^{α} and Y^A are $SO(N)$, H' and coset generators, respectively. The non-compact generators transform in a spinor representation of $SO(N)$. The target space metric is given by

$$g_{ij} = e_i^A e_j^B \delta_{AB} \tag{2.54}$$

where the vielbein e_i^A as well as the $H = SO(N)\times H'$ composite connections are obtained from the decomposition

$$L^{-1}\partial_i L = \frac{1}{2}Q_i^{IJ}X^{IJ} + Q_i^{\alpha}X^{\alpha} + e_i^A Y^A\,. \tag{2.55}$$

Indices A, B can be thought of as "flat" target space indices, and e_i^A together with its inverse e_A^i can be used to converse indices i, j to A, B and vice versa.

The map $\mathcal{V}$ in (2.48) is now an isomorphism and takes the form

$$L^{-1}t^{\mathcal{M}}L = \mathcal{V}^{\mathcal{M}}_{\ \mathcal{A}} = \frac{1}{2}\mathcal{V}^{\mathcal{M}}_{\ IJ}X^{IJ} + \mathcal{V}^{\mathcal{M}}_{\ \alpha}X^{\alpha} + \mathcal{V}^{\mathcal{M}}_{\ A}Y^A\,. \tag{2.56}$$

The algebra (2.49) is then isomorphic to the G algebra characterized by [26]

$$\begin{aligned}
\left[X^{IJ}, X^{KL}\right] &= -4\delta^{[I[K}t^{L]J]}, \qquad \left[X^{\alpha}, X^{\beta}\right] = f^{\alpha\beta}_{\ \ \gamma}X^{\gamma}, \qquad \left[X^{IJ}, X^{\alpha}\right] = 0,\\
\left[X^{IJ}, Y^A\right] &= -\frac{1}{2}\Gamma^{IJ}_{AB}Y^B, \qquad \left[X^{\alpha}, Y^A\right] = -h^{\alpha}_{AB}Y^B,\\
\left[Y^A, Y^B\right] &= \frac{1}{4}\Gamma^{IJ}_{AB}X^{IJ} + \frac{1}{8}C_{\alpha\beta}h^{\alpha}_{AB}X^{\beta}\,.
\end{aligned} \tag{2.57}$$

From this algebra, the f^{IJ} tensors are then identified with Γ^{IJ} constructed from $SO(N)$ gamma matrices $\Gamma^I_{A\dot{A}}$. The precise relation is given by

$$f^{IJ}_{ij} = -\frac{1}{2}(\Gamma^I\Gamma^J - \Gamma^J\Gamma^I)_{AB}e_i^A e_j^B = -\Gamma^{IJ}_{AB}e_i^A e_j^B\,. \tag{2.58}$$

The H' generators h^{α} satisfy

$$\begin{aligned}
h^{\alpha}_{AC}\Gamma^I_{C\dot{B}} + h^{\alpha}_{\dot{B}\dot{C}}\Gamma^I_{A\dot{C}} &= 0\\
h^{\alpha}_{AC}h^{\beta}_{CB} - h^{\beta}_{AC}h^{\alpha}_{CB} &= f^{\alpha\beta}_{\ \ \gamma}h^{\gamma}_{AB}
\end{aligned} \tag{2.59}$$

and the $C_{\alpha\beta}$ tensor coincides with the one defined previously. The integrability condition for (2.55) gives rise to [26]

$$\begin{aligned} D_{[i}e^A_{j]} &= \partial_{[i}e^A_{j]} + \left(\frac{1}{4}Q^{IJ}_{[i}\Gamma^{IJ}_{AB} + Q^\alpha_{[i}h^\alpha_{AB}\right)e^B_{j]} = 0, && (2.60)\\ R^{IJ}_{ij} &= -\frac{1}{2}e^A_i e^B_j \Gamma^{IJ}_{AB}, && (2.61)\\ R^\alpha_{ij} &= -\frac{1}{8}e^A_i e^B_j C_{\alpha\beta}h^\beta_{AB} && (2.62)\end{aligned}$$

with R^{IJ}_{ij} defined in (2.10). The R^α_{ij} are defined as

$$R^\alpha_{ij} = \partial_i Q^\alpha_j - \partial_j Q^\alpha_i + f^\alpha{}_{\beta\gamma}Q^\beta_i Q^\gamma_j \,. \tag{2.63}$$

Recall that $\Omega^{\alpha\beta}_i \propto f^{\alpha\beta}{}_\gamma Q^\gamma_i$, we see that this is related to $R^\alpha{}_{\beta ij}$ defined in (2.24) by $R^\alpha{}_{\beta ij} \propto f^{\alpha\beta}{}_\gamma R^\gamma_{ij}$. The Riemann tensor is then given by

$$\begin{aligned} R_{ijkl} &= -e^A_k e^B_l \left(\frac{1}{4}R^{IJ}_{ij}\Gamma^{IJ}_{AB} + R^\alpha_{ij}h^\alpha_{ij}\right)\\ \text{or} \qquad R^{ABCD} &= \frac{1}{8}(\Gamma^{IJ}_{AB}\Gamma^{IJ}_{CD} + C_{\alpha\beta}h^\alpha_{AB}h^\beta_{CD})\,. && (2.64)\end{aligned}$$

It can be shown that the differential relations (2.38) follow from the above information [57]. For example, from the second equation in (2.34), the transformations of Q^{IJ}_i and Q^α_i under G are given by

$$\partial_i X^{\mathcal{M}j}Q_j + X^{\mathcal{M}j}\partial_j Q^{IJ}_i + \partial_i\mathcal{S}^{\mathcal{M}} + \left[Q_i, \mathcal{S}^{\mathcal{M}}\right] = 0 \tag{2.65}$$

where

$$\begin{aligned} Q_i &= \frac{1}{2}Q^{IJ}_i X^{IJ} + Q^\alpha_i X^\alpha\\ \mathcal{S}^{\mathcal{M}} &= \frac{1}{2}\mathcal{S}^{\mathcal{M}IJ}X^{IJ} + \mathcal{S}^{\mathcal{M}\alpha}X^\alpha\,. && (2.66)\end{aligned}$$

Multiplying (2.53) by L^{-1} from the left and using (2.55) and (2.56), we find

$$\mathcal{V}^{\mathcal{M}} \equiv \frac{1}{2}\mathcal{V}^{\mathcal{M}}{}_{IJ}X^{IJ} + \mathcal{V}^{\mathcal{M}}{}_\alpha X^\alpha = \mathcal{S}^{\mathcal{M}} + X^{\mathcal{M}i}Q_i\,. \tag{2.67}$$

as well as $X^{\mathcal{M}i} = g^{ij}e^A_j\mathcal{V}^{\mathcal{M}}_A$. Introducing $\hat{R}_{ij} = \frac{1}{2}R^{IJ}_{ij}X^{IJ} + R^\alpha_{ij}X^\alpha$ and using the above results, we can write (2.65) as

$$D_i\mathcal{V}^{\mathcal{M}} = \hat{R}_{ij}X^{\mathcal{M}j}\,. \tag{2.68}$$

By using (2.61) and (2.62), we readily obtain the same result as (2.46).

Finally, the spinor fields χ^{iI} are also redefined to

$$\chi^{\dot{A}} = \frac{1}{N}e^A_i\Gamma^I_{A\dot{A}}\chi^i I \tag{2.69}$$

which transform in the conjugate spinor representation of $SO(N)$.

2.2 Gauged supergravity in three dimensions

In this section, we will gauge some symmetry of the ungauged Lagrangian given in the previous section. As our main interest is in the theory with symmetric target space, the gauge group G_0 is then a subgroup of the symmetry group G. We follow the covariant formulation by introducing the notion of embedding tensor introduced in [27]. This has been first applied to the maximal $N = 16$ gauged supergravity in three dimensions. This method has also been used in many cases including the theory in other space-time dimensions with different numbers of supersymmetries.

The embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ acts as a projection operator on the group G to the gauge group G_0. The gauge generators are given by

$$X^i = g\Theta_{\mathcal{M}\mathcal{N}}\Lambda^{\mathcal{M}}(x)X^{\mathcal{N}i} \tag{2.70}$$

where $\Lambda^{\mathcal{M}}(x)$ and g are space-time dependent parameters of gauge transformations and the gauge coupling, respectively. The dimension of the gauge group is given by the rank of the embedding tensor. The embedding tensor is gauge invariant and symmetric in its two indices. In three dimensions, it is an element in the symmetric tensor product of the two adjoint representations of G as indicated by the indices $\mathcal{M}$ and $\mathcal{N}$. The requirement that gauge generators form close algebra implies the following constraint on the embedding tensor

$$\Theta_{\mathcal{M}\mathcal{P}}\Theta_{\mathcal{N}\mathcal{Q}}f^{\mathcal{P}\mathcal{Q}}{}_{\mathcal{R}} = \hat{f}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\Theta_{\mathcal{P}\mathcal{R}} \tag{2.71}$$

with structure constants of the gauge group $\hat{f}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$. This can be easily shown by noting that the generators of the gauge group are given by

$$J_{\mathcal{M}} = \Theta_{\mathcal{M}\mathcal{N}}t^{\mathcal{N}}\,. \tag{2.72}$$

The requirement that these generators form an algebra gives

$$[J_{\mathcal{M}}, J_{\mathcal{N}}] = \hat{f}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}J_{\mathcal{P}}\,. \tag{2.73}$$

Using the G algebra $\left[t^{\mathcal{M}}, t^{\mathcal{N}}\right] = f^{\mathcal{M}\mathcal{N}}{}_{\mathcal{R}}t^{\mathcal{R}}$, we can write (2.73) as

$$\Theta_{\mathcal{M}\mathcal{P}}\Theta_{\mathcal{N}\mathcal{Q}}f^{\mathcal{P}\mathcal{Q}}{}_{\mathcal{R}} = \hat{f}_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\Theta_{\mathcal{P}\mathcal{R}}\,. \tag{2.74}$$

Gauge invariance of the embedding tensor can be then written as

$$\hat{f}_{\mathcal{M}\mathcal{P}}{}^{\mathcal{Q}}\Theta_{\mathcal{Q}\mathcal{N}} + \hat{f}_{\mathcal{N}\mathcal{P}}{}^{\mathcal{Q}}\Theta_{\mathcal{Q}\mathcal{M}} = \Theta_{\mathcal{P}\mathcal{L}}(f^{\mathcal{K}\mathcal{L}}{}_{\mathcal{M}}\Theta_{\mathcal{N}\mathcal{K}} + \hat{f}^{\mathcal{K}\mathcal{L}}{}_{\mathcal{N}}\Theta_{\mathcal{M}\mathcal{K}}) = 0\,. \tag{2.75}$$

This constraint is the so-called quadratic constraint.

In the ungauged theory, all bosonic degrees of freedom are carried by scalars, so there seem to be no vector fields to act as gauge fields. However, we can introduce gauge fields to the Lagrangian via the so-called Chern-Simons term

$$\mathcal{L}_{\text{CS}} = \frac{i}{4} g \epsilon^{\mu\nu\rho} A_\mu^{\mathcal{M}} \Theta_{\mathcal{M}\mathcal{N}} \left(\partial_\nu A_\rho^{\mathcal{N}} - \frac{1}{3} g \hat{f}_{\mathcal{P}\mathcal{Q}}{}^{\mathcal{N}} A_\nu^{\mathcal{P}} A_\rho^{\mathcal{Q}} \right). \tag{2.76}$$

In this form, vector fields do not carry propagating degrees of freedom. So, the number of bosonic degrees of freedom remains the same as required by supersymmetry. This term is also required by supersymmetry as we will see below. We now introduce the gauge covariant derivatives

$$\begin{aligned}
\mathcal{D}_\mu \phi^i &= \partial_\mu \phi^i + g \Theta_{\mathcal{M}\mathcal{N}} A_\mu^{\mathcal{M}} X^{\mathcal{N} i}, \\
\mathcal{D}_\mu \psi_\nu^I &= \left(\partial_\mu + \frac{1}{2} \omega_\mu^a \gamma_a \right) \psi_\nu^I + \partial_m u \phi^i Q_i^{IJ} \psi_\nu^J + g \Theta_{\mathcal{M}\mathcal{N}} A_\mu^{\mathcal{M}} \mathcal{V}^{\mathcal{N} IJ} \psi_\nu^J, \\
\mathcal{D}_\mu \chi^{iI} &= \left(\partial_\mu + \frac{1}{2} \omega_\mu^a \gamma_a \right) \chi^{iI} + \partial_\mu \phi^j (\Gamma^i_{jk} \chi^{kI} + Q_j^{IJ} \chi^{iJ}) \\
&\quad g \Theta_{\mathcal{M}\mathcal{N}} A_\mu^{\mathcal{M}} (\delta^i_j \mathcal{V}^{\mathcal{N} IJ} - \delta^{IJ} g^{ik} D_k \mathcal{V}^{\mathcal{N}}_j \chi^{jI}).
\end{aligned} \tag{2.77}$$

Under gauge transformations, the gauge fields transform as

$$\Theta_{\mathcal{M}\mathcal{N}} \delta A_\mu^{\mathcal{M}} = \Theta_{\mathcal{M}\mathcal{N}} (-\partial_\mu \Lambda^{\mathcal{M}} + g \hat{f}_{\mathcal{P}\mathcal{Q}}{}^{\mathcal{M}} A_\mu^{\mathcal{P}} \Lambda^{\mathcal{Q}}). \tag{2.78}$$

The corresponding filed strength is given by

$$\Theta_{\mathcal{M}\mathcal{N}} F_{\mu\nu}^{\mathcal{M}} = \Theta_{\mathcal{M}\mathcal{N}} (\partial_\mu A_\nu^{\mathcal{M}} - \partial_\nu A_\mu^{\mathcal{M}} - g \hat{f}_{\mathcal{P}\mathcal{Q}}{}^{\mathcal{M}} A_\mu^{\mathcal{P}} A_\nu^{\mathcal{Q}}). \tag{2.79}$$

The covariant derivative on the supersymmetry transformation parameter is also modified to

$$\mathcal{D}_\mu \epsilon^I = \left(\partial_\mu + \frac{1}{2} \omega_\mu^a \gamma_a \right) \epsilon^I + \partial_\mu \phi^i Q_i^{IJ} \epsilon^J + g \Theta_{\mathcal{M}\mathcal{N}} A_\mu^{\mathcal{M}} \mathcal{V}^{\mathcal{N} IJ} \epsilon^J . \tag{2.80}$$

It is not unexpected that the introduction of the above covariant derivatives makes the Lagrangian not invariant under supersymmetry transformations given in the previous section. It is necessary to modify the Lagrangian with extra terms to restore supersymmetry. We simply review the main results and refer the reader to [31] for more detailed discussions. The commutators of two covariant derivatives give rise to the terms involving gauge field strengths for example,

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] \phi^i = g \Theta_{\mathcal{M}\mathcal{N}} F_{\mu\nu}^{\mathcal{M}} X^{\mathcal{N} i} . \tag{2.81}$$

These supersymmetry violating terms are of the form

$$-\frac{1}{2} i g \Theta_{\mathcal{M}\mathcal{N}} F_{\nu\rho}^{\mathcal{N}} \epsilon^{\mu\nu\rho} \left(\mathcal{V}^{\mathcal{N} IJ} \bar{\psi}_\mu^I \epsilon^J + \frac{1}{2} \mathcal{V}^{\mathcal{N}}{}_i \bar{\chi}^{iI} \gamma_\mu \epsilon^I \right) \tag{2.82}$$

which can be canceled by adding $\mathcal{L}_{\text{CS}}$ to the Lagrangian together with the supersymmetry transformation of the gauge fields given below. The supersymmetry transformation of the vector fields in the covariant derivatives leads to the terms at order g

$$-eg\Theta_{\mathcal{MN}}(2\mathcal{V}^{\mathcal{M}IJ}\bar{\psi}^I_\mu\epsilon^J+\mathcal{V}^{\mathcal{M}}_{\ i}\bar{\chi}^{iI}\gamma_\mu\epsilon^I)\mathcal{V}^{\mathcal{N}}_{\ j}\mathcal{D}^\mu\phi^j\,. \tag{2.83}$$

To cancel these terms, we then introduce the fermion mass-like terms to the Lagrangian together with the modified supersymmetry transformations of fermions at order g. These extra terms take the form

$$\begin{aligned}\mathcal{L}_{\text{mass-like}} &= ge\left(\frac{1}{2}A_1^{IJ}\bar{\psi}^I_\mu\gamma^{\mu\nu}\psi^J_\nu+A_{2i}^{IJ}\bar{\psi}^I_\mu\gamma^\mu\chi^{jJ}+\frac{1}{2}A_{3ij}^{IJ}\bar{\chi}^{iI}\chi^{jJ}\right)\\ \delta\psi^{I(g)}_\mu &= gA_1^{IJ}\epsilon^J,\\ \delta\chi^{iI(g)} &= -gNA_2^{JIi}\epsilon^J\,.\end{aligned} \tag{2.84}$$

Before completing the gauged Lagrangian, we define the T-tensor as follows

$$T_{\mathcal{AB}}=\mathcal{V}^{\mathcal{M}}_{\ \ \mathcal{A}}\Theta_{\mathcal{MN}}\mathcal{V}^{\mathcal{N}}_{\ \ \mathcal{B}}\,. \tag{2.85}$$

The T-tensor is the image of the embedding tensor under the map $\mathcal{V}$ introduced before. In components, this can be written

$$\begin{aligned}T^{IJ,KL} &= \mathcal{V}^{\mathcal{M}IJ}\Theta_{\mathcal{MN}}\mathcal{V}^{\mathcal{N}KL}, \qquad T^{IJi}=\mathcal{V}^{\mathcal{M}IJ}\Theta_{\mathcal{MN}}\mathcal{V}^{\mathcal{N}i},\\ T^{ij} &= \mathcal{V}^{\mathcal{M}i}\Theta_{\mathcal{MN}}\mathcal{V}^{\mathcal{N}j}, \qquad T^i_\alpha=\mathcal{V}^{\mathcal{M}i}\Theta_{\mathcal{MN}}\mathcal{V}^{\mathcal{N}}_{\ \ \alpha},\\ T_{\alpha\beta} &= \mathcal{V}^{\mathcal{M}}_{\ \ \alpha}\Theta_{\mathcal{MN}}\mathcal{V}^{\mathcal{N}}_{\ \ \beta}, \qquad T^{IJ}_{\ \ \ \alpha}=\mathcal{V}^{\mathcal{M}IJ}\Theta_{\mathcal{MN}}\mathcal{V}^{\mathcal{N}}_{\ \ \alpha}\,.\end{aligned} \tag{2.86}$$

The tensors A_1 and A_3 are evidently symmetric. Further more, A_2 and A_3 must satisfy the projection condition similar to (2.26)

$$\begin{aligned}A_1^{IJ} &= A_1^{JI}, \qquad A_{3ij}^{IJ}=A_{3ji}^{JI},\\ \mathbb{P}^{Jj}_{Ii}A_{2j}^{KJ} &= A_{2i}^{KI}, \qquad \mathbb{P}^{Jj}_{Ii}A_{3jk}^{JK}=A_{3ik}^{IK}\,.\end{aligned} \tag{2.87}$$

The cancelation is achieved by various identities for the T-tensor. For the full list of these identities, we refer to [31]. We emphasize again that we will restrict ourselves to theories with $N>2$. So, only the results relevant to these theories will be given. The tensors A_1, A_2 and A_3 that satisfy the constraints (2.87) are determined in terms of the T-tensors as follows

$$A_1^{IJ} = -\frac{4}{N-2}T^{IM,JM}+\frac{2}{(N-1)(N-2)}\delta^{IJ}T^{MN,MN}, \tag{2.88}$$

$$A_{2j}^{IJ} = \frac{4}{N(N-2)}f^{M(Im}_{\ \ \ \ \ j}T^{J)M}_{\ \ \ \ \ m}+\frac{2}{N(N-1)(N-2)}\delta^{IJ}f^{KL\ \ m}_{\ \ \ \ j}T^{KL}_{\ \ \ \ m}+\frac{2}{N}T^{IJ}_{\ \ \ j}, \tag{2.89}$$

$$A_{3ij}^{IJ} = \frac{1}{N^2}\Big[-2D_{(i}D_{j)}A_1^{IJ}+g_{ij}A_1^{IJ}+A_1^{K[I}f_{ij}^{J]K}+2T_{ij}\delta^{IJ}-4D_{[i}T^{IJ}_{\ \ \ j]}-2T_{k[i}f^{IJk}_{\ \ \ \ j]}\Big]. \tag{2.90}$$

In order to cancel the extra supersymmetry transformation at order g^2, we need to include the scalar potential to the Lagrangian

$$V = -\frac{4g^2}{N}\left(A_1^{IJ}A_1^{IJ} - \frac{N}{2}g^{ij}A_{2i}^{IJ}A_{2j}^{IJ}\right). \tag{2.91}$$

The cancelation of the supersymmetry transformations of the potential requires the quadratic identities

$$\begin{aligned} 2A_1^{IK}A_1^{KJ} - NA_2^{IKi}A_{2i}^{JK} &= \frac{1}{N}\delta^{IJ}(2A_1^{KL}A_1^{KL} - NA_2^{KLi}A_{2i}^{KL}), \\ 3A_1^{IK}A_{2j}^{KJ} + Ng^{kl}A_{2k}^{IK}A_{3lj}^{KJ} &= \mathbb{P}_{Jj}^{Ii}(3A_1^{KL}A_{2i}^{KL} + Ng^{kl}A_{2k}^{LK}A_{3lj}^{KL}). \end{aligned} \tag{2.92}$$

These conditions arise from the quadratic constraint on the embedding tensor (2.75). This has been explicitly shown in [28] for $N = 16$ theory.

All the above results can be summarized in a single statement that consistency of the gaugings implied by supersymmetry is given by a constraint on the $T^{IJ,KL}$

$$T^{IJ,KL} = T^{[IJ,KL]} - \frac{4}{N-2}\delta^{[I|[K}T^{L]M,M|J]} - \frac{2}{(N-1)(N-2)}\delta^{I[K}\delta^{L]J}T^{MN,MN}, \tag{2.93}$$

or equivalently,

$$\mathbb{P}_{\boxplus}T^{IJ,KL} = 0\,. \tag{2.94}$$

The symbol $\boxplus$ denotes the $\boxplus$ representation of $SO(N)$. For symmetric target spaces, this constraint can be lifted to the constraint on $\Theta_{\mathcal{MN}}$. The final result is that, see [31] for details, the embedding tensor must satisfy a projection condition analogous to (2.94)

$$\mathbb{P}_{R_0}\Theta_{\mathcal{MN}} = 0\,. \tag{2.95}$$

The representation R_0 appears in the decomposition of the symmetric tensor product of the adjoint representations of G under G and contains the representation $\boxplus$ of $SO(N)$ when decomposed under $SO(N)$. This condition is more useful than (2.94) because we can work with $\Theta_{\mathcal{MN}}$ which is a constant tensor rather than the field dependent T-tensors.

We now collect the full gauged Lagrangian and the corresponding supersymmetry transformations

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_0 + ge\left(\frac{1}{2}A_1^{IJ}\bar{\psi}_\mu^I\gamma^{\mu\nu}\psi_\nu^J + A_{2i}^{IJ}\bar{\psi}_\mu^I\gamma^\mu\chi^{jJ} + \frac{1}{2}A_{3ij}^{IJ}\bar{\chi}^{iI}\chi^{jJ}\right) \\ &\quad + \frac{4eg^2}{N}\left(A_1^{IJ}A_1^{IJ} - \frac{N}{2}g^{ij}A_{2i}^{IJ}A_{2j}^{IJ}\right) \end{aligned} \tag{2.96}$$

and

$$\begin{aligned}
\delta e^a_\mu &= \frac{1}{2}\bar{\epsilon}^I\gamma^a\psi^I_\mu \\
\delta A^{\mathcal{M}}_\mu &= 2\mathcal{V}^{\mathcal{M}IJ}\bar{\psi}^I_\mu\epsilon^J + \mathcal{V}^{\mathcal{M}}_i\bar{\chi}^{iI}\gamma_\mu\epsilon^I, \\
\delta\phi^i &= \frac{1}{2}\bar{\epsilon}^I\chi^{iI}, \\
\delta\psi^I_\mu &= \mathcal{D}_\mu\epsilon^I - \frac{1}{8}g_{ij}\bar{\chi}^{iI}\gamma^\nu\chi^{jJ}\gamma_{\mu\nu}\epsilon^J - \delta\phi^i Q^{IJ}_i\psi^J_\mu + gA^{IJ}_1\epsilon^J, \\
\delta\chi^{iI} &= \frac{1}{2}(\delta^{IJ}\mathbf{1} - f^{IJ})^i{}_j\hat{\not{\mathcal{D}}}\phi^j\epsilon^J - \delta\phi^j(\Gamma^i_{jk}\chi^{kI} + Q^{IJ}_j\chi^{iJ}) - gNA^{JIi}_2\epsilon^J \qquad (2.97)
\end{aligned}$$

with $\mathcal{D}_\mu\epsilon^I$ defined, previously. For completeness, we also give gauge transformations of various fields

$$\begin{aligned}
\delta\phi^i &= g\Theta_{\mathcal{MN}}\Lambda^{\mathcal{M}}X^{\mathcal{N}i}, \\
\delta\psi^I_\mu &= g\Theta_{\mathcal{MN}}\Lambda^{\mathcal{M}}\mathcal{V}^{\mathcal{N}IJ}\psi^J_\mu - \delta\phi^i Q^{IJ}_i\psi^J_\mu, \\
\delta\chi^{iI} &= g\Theta_{\mathcal{MN}}\Lambda^{\mathcal{M}}(\chi^{jI}D_j\mathcal{V}^{\mathcal{N}i} + \mathcal{V}^{\mathcal{N}IJ}\chi^{iJ}) - \delta\phi^j(\Gamma^i_{jk}\chi^{kI} + Q^{IJ}_j\chi^{iJ}), \\
\Theta_{\mathcal{MN}}\delta A^{\mathcal{M}}_\mu &= \Theta_{\mathcal{MN}}(-\partial_\mu\Lambda^{\mathcal{M}} + g\hat{f}_{\mathcal{PQ}}{}^{\mathcal{M}}A^{\mathcal{P}}_\mu\Lambda^{\mathcal{Q}}). \qquad (2.98)
\end{aligned}$$

We end this section by giving some information about critical points of the scalar potential. This will be used extensively in later chapters. The critical (stationary) points are the points satisfying the condition

$$3A^{IK}_1A^{KJ}_{2j} + Ng^{kl}A^{IK}_{2k}A^{KJ}_{3lj} = 0\,. \qquad (2.99)$$

This relation follows from the variation of the scalar potential in (2.91). For supersymmetric critical points, unbroken supersymmetries are characterized by the two equivalent conditions

$$\begin{aligned}
A^{JI}_{2i}\epsilon^J &= 0 \\
\text{and} \qquad A^{IK}_1A^{KJ}_1\epsilon^J &= -\frac{V_0}{4}\epsilon^I = \frac{1}{N}(A^{IJ}_1A^{IJ}_1 - \frac{1}{2}Ng^{ij}A^{IJ}_{2i}A^{IJ}_{2i})\epsilon^I, \qquad (2.100)
\end{aligned}$$

where V_0 is the value of the potential at the critical point. The equivalence between these relations follows from the quadratic constraint (2.92). Obviously, A^{IJ}_{2i} are entirely zero at the maximal supersymmetric points. From (2.100), unbroken supersymmetries ϵ^I are eigenvectors of A^{IJ}_1 with the corresponding eigenvalues $\pm\sqrt{\frac{-V_0}{4g^2}}$. We will explain this point in more detail in the next chapter.

2.3 Chern-Simons and Yang-Mills gauged supergravities

The formulation given in the previous section is referred to as Chern-Simons gauged supergravity in which the gauge fields appear in the Lagrangian via Chern-Simons terms. On the other hand, in three dimensional gauged supergravities obtained from dimensional reductions of higher dimensional theories, gauge fields normally come with the usual Yang-Mills kinetic terms. In general, the two formulations are not equivalent. This fact a priori implies that the theories discussed in the previous section cannot be obtained from higher dimensional theories via any known mechanisms. It turns out that there is an exception. It has been shown in [38] that for a class of gaugings with non-semisimple gauge groups, the Chern-Simons gauged theories are indeed on-shell equivalent to certain Yang-Mills gauged theories with semisimple gauge groups. As discussed in the previous section, all propagating bosonic degrees of freedom are carried by scalars, but with Yang-Mills kinetic terms, gauge fields do carry propagating degrees of freedom. So, in the process of converting a Chern-Simons gauged theory to a Yang-Mills one, some scalars must disappear, and the corresponding degrees of freedom will be shifted to vector fields. We will see this in this section. The material given here closely follows [38]. We will not include the coupling to massive vector fields here, because this is not directly relevant to the works presented in this thesis. Most of the notations are also parallel to those in [38] except for the $(-++)$ metric.

For conveniences, we also repeat the relevant term in the Lagrangian in the $(-++)$ metric

$$\mathcal{L} = \frac{1}{4}eR + g\mathcal{L}_{\text{CS}} - \frac{1}{4}\mathcal{P}^A_\mu \mathcal{P}^{A\mu} - V\,. \tag{2.101}$$

We can derive the equation of motion for vector fields

$$\Theta_{\mathcal{MN}} F^{\mathcal{M}}_{\mu\nu} = \epsilon_{\mu\nu\rho}\Theta_{\mathcal{MN}}\mathcal{V}^{\mathcal{N}}{}_A \mathcal{P}^{\rho A} \tag{2.102}$$

where the fermionic terms have been omitted because they are not relevant for our discussion here. In this section, as in [38], we will use the notations $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ defined by

$$\mathcal{Q}_\mu + \mathcal{P}_\mu = L^{-1}(\partial_\mu + g\Theta_{\mathcal{MN}} A^{\mathcal{M}}_\mu t^{\mathcal{N}})L\,. \tag{2.103}$$

Notice that in this equation the derivative is a space time derivative obtained by pulling back $\partial_i L$ in (2.55) $\partial_\mu L = \partial_\mu \phi^i \partial_i L$. Similarly, we can identify

$$\mathcal{Q}_\mu = \partial_\mu \phi^i \left(\frac{1}{2}Q^{IJ}_i X^{IJ} + Q^\alpha_i X^\alpha\right), \qquad \mathcal{P}_\mu = \partial_\mu \phi^i e^A_i Y^A\,. \tag{2.104}$$

We begin with the description of the gauge group and its embedding tensor. We take the non-semisimple gauge group of the form $G_0 \ltimes \mathcal{T}_n$ with $n = \text{dim}G_0$. $\mathcal{T}_n$ contains n commuting translational generators which transform as the adjoint representation of the semisimple group G_0. The corresponding Lie algebras are denoted by $\mathfrak{g}_0 = \{T^m\}$ and $\mathfrak{t}_n = \{T^{\underline{m}}\}$, respectively, with the indices $m, \underline{m} = 1, \dots n$. The $G_0 \ltimes \mathcal{T}_n$ algebra is given by the following commutation relations

$$[T^m, T^n] = f^{mn}{}_p T^p, \qquad [T^m, T^{\underline{n}}] = f^{mn}{}_p T^{\underline{p}}, \qquad [T^{\underline{m}}, T^{\underline{n}}] = 0, \tag{2.105}$$

where $f^{mn}{}_p$'s are structure constants of G_0. A consistent gauging is described by the embedding tensor with non-zero components

$$g\Theta_{m\underline{n}} = g\Theta_{\underline{m}n} = g_1\eta_{mn}, \qquad g\Theta_{\underline{mn}} = g_2\eta_{mn} \tag{2.106}$$

where η_{mn} is the $\mathfrak{g}_0$ Cartan-Killing form. The coupling constants g_1 and g_2 are not independent but related to each other by the relation which describes the embedding of $G_0 \ltimes \mathcal{T}_n$ in G. The gauge fields of G_0 and $\mathcal{T}_n$ are denoted by $C^m_\mu \equiv A^m_\mu$ and $B^m_\mu \equiv A^{\underline{m}}_\mu$, respectively. Under gauge transformations with the above embedding tensor, these transform as

$$\begin{aligned} \delta B^m_\mu &= \partial_\mu \Lambda^{\underline{m}} + g_1 f^m{}_{kl} B^k_\mu \Lambda^{\underline{l}}, \qquad f^m{}_{kl} = \eta_{nk} f^{mn}{}_l, \\ \delta C^m_\mu &= \partial_\mu \Lambda^m + g_1 f^m{}_{kl} C^k_\mu \Lambda^{\underline{l}} + f^m{}_{kl} B^k_\mu (g_1 \Lambda^l + g_2 \Lambda^{\underline{l}}). \end{aligned} \tag{2.107}$$

The associated field strengths are given by

$$\begin{aligned} B^m_{\mu\nu} &= 2\partial_{[\mu} B^m_{\nu]} + g_1 f^m{}_{kl} B^k_\mu B^l_\nu, \\ C^m_{\mu\nu} &= 2\partial_{[\mu} C^m_{\nu]} + 2g_1 f^m{}_{kl} C^k_{[\mu} B^l_{\nu]} + g_2 f^m{}_{kl} B^k_\mu B^l_\nu\,. \end{aligned} \tag{2.108}$$

We now separate n scalars, ϕ_m, corresponding to the translations $T^{\underline{m}}$'s from the coset representative $L(\phi^i) = L(\phi, \tilde{\phi})$. The remaining scalars $\tilde{\phi}$ parametrize the reduced coset G'/H' which will be identified with the Yang-Mills coset. So, we write

$$L(\phi, \tilde{\phi}) = e^{\phi_m T^{\underline{m}}} \tilde{L}(\tilde{\phi}) \tag{2.109}$$

and also define

$$\tilde{C}^m_{\mu\nu} = C^m_{\mu\nu} - f^{mn}{}_l \phi_n B^l_{\mu\nu}\,. \tag{2.110}$$

To derive other results given below, it is useful to note a formula

$$e^{-\phi_m T^{\underline{m}}} T^n e^{\phi_m T^{\underline{m}}} = T^n + \phi_m f^{nm}{}_l T^{\underline{l}}\,. \tag{2.111}$$

This is easily shown by remembering $\left[[T^n, T^{\underline{m}}], T^{\underline{l}}\right] = 0$. To describe gauge transformations of these fields, we recall the gauge transformation of the scalars ϕ^i given in the previous section. Using $L = L(\phi^i)$ and the fact that the isometry acts on L as a left multiplication by G, we can write the gauge transformation of L as

$$\delta L = -g\Theta_{\mathcal{MN}} \Lambda^{\mathcal{M}} t^{\mathcal{N}} L\,. \tag{2.112}$$

Together with the gauge transformations of vector fields, we find

$$\begin{aligned} \delta B^m_{\mu\nu} &= g_1 f^m{}_{kl} B^k_{\mu\nu} \Lambda^{\underline{l}}, \qquad \delta\phi_m = -g_1 \eta_{mn} \Lambda^n - (g_2 \eta_{mn} + g_1 f^l{}_{mn} \phi_l) \Lambda^{\underline{n}}, \\ \delta \tilde{C}^m_{\mu\nu} &= g_1 f^m{}_{kl} \tilde{C}^k_{\mu\nu} \Lambda^{\underline{l}}, \qquad \delta \tilde{L} = -g_1 \eta_{mn} \Lambda^{\underline{m}} T^n \tilde{L}\,. \end{aligned} \tag{2.113}$$

We can use Λ^m gauge transformations to gauge ϕ_m away since ϕ's are shifted under these transformations. We then define the following quantities for the reduced coset $\tilde{L}$

$$\begin{aligned}\tilde{\mathcal{V}}^{\mathcal{M}}{}_{\mathcal{A}} t^{\mathcal{A}} &= \tilde{L}^{-1} t^{\mathcal{M}} \tilde{L}, \\ \tilde{\mathcal{Q}}_\mu + \tilde{\mathcal{P}}_\mu &= \tilde{L}^{-1}(\partial_\mu + g_1 \eta_{mn} B^m_\mu T^n)\tilde{L}\,.\end{aligned} \tag{2.114}$$

Using (2.109) and (2.111), we can show that these quantities are related to those of L by

$$\tilde{\mathcal{V}}^{\underline{m}}{}_{\mathcal{A}} = \mathcal{V}^{\underline{m}}{}_{\mathcal{A}}, \qquad \tilde{\mathcal{V}}^{m}{}_{\mathcal{A}} = \mathcal{V}^{m}{}_{\mathcal{A}} - f^{mn}{}_k \phi_n \mathcal{V}^{\underline{k}}{}_{\mathcal{A}}\,. \tag{2.115}$$

With (2.106) and $\Theta_{\mathcal{M}\mathcal{N}}$ being symmetric, it is easily verified that

$$\tilde{T}_{\mathcal{A}\mathcal{B}} = \Theta_{\mathcal{M}\mathcal{N}} \tilde{\mathcal{V}}^{\mathcal{M}}{}_{\mathcal{A}} \tilde{\mathcal{V}}^{\mathcal{N}}{}_{\mathcal{B}} = \Theta_{\mathcal{M}\mathcal{N}} \mathcal{V}^{\mathcal{M}}{}_{\mathcal{A}} \mathcal{V}^{\mathcal{N}}{}_{\mathcal{B}} = T_{\mathcal{A}\mathcal{B}} \tag{2.116}$$

which is a manifestation of the fact that the T-tensor is gauge invariant. Similarly, we find the relation

$$\mathcal{Q}_\mu + \mathcal{P}_\mu = \tilde{\mathcal{Q}}_\mu + \tilde{\mathcal{P}}_\mu + D_\mu \phi_m \tilde{\mathcal{V}}^{\underline{m}}{}_{\mathcal{A}} t^{\mathcal{A}} \tag{2.117}$$

where the covariant derivative is defined by

$$D_\mu \phi_m = \partial_\mu \phi_m + \eta_{mn}(g_1 C^n_\mu + g_2 B^n_\mu) + g_1 f^k{}_{mn} \phi_k\,. \tag{2.118}$$

We then return to the duality equation (2.102) which gives

$$\begin{aligned}B^m_{\mu\nu} &= e\epsilon_{\mu\nu\rho} \tilde{\mathcal{V}}^{\underline{m}}{}_A (\tilde{\mathcal{P}}^{A\rho} + \tilde{\mathcal{V}}^{\underline{n}}{}_A D^\rho \phi_n), \\ \tilde{C}^m_{\mu\nu} &= e\epsilon_{\mu\nu\rho} \tilde{\mathcal{V}}^{m}{}_A (\tilde{\mathcal{P}}^{A\rho} + \tilde{\mathcal{V}}^{\underline{n}}{}_A D^\rho \phi_n).\end{aligned} \tag{2.119}$$

Everything is now invariant under Λ^m. We can eliminate ϕ_m from the equations of motion by solving (2.119)

$$\begin{aligned}e\epsilon_{\mu\nu\rho} D^\rho \phi_m &= M_{mn} B^n_{\mu\nu} - e\epsilon_{\mu\nu\rho} M_{mn} \tilde{\mathcal{V}}^{\underline{n}}{}_A \tilde{\mathcal{P}}^{A\rho}, \\ \tilde{C}^m_{\mu\nu} &= e\epsilon_{\mu\nu\rho}(\tilde{\mathcal{V}}^{m}{}_A - \tilde{\mathcal{V}}^{m}{}_B \tilde{\mathcal{V}}^{\underline{k}}{}_B M_{kl} \tilde{\mathcal{V}}^{\underline{l}}{}_A)\tilde{\mathcal{P}}^{A\rho} + \tilde{\mathcal{V}}^{m}{}_A \tilde{\mathcal{V}}^{\underline{k}}{}_A M_{kn} B^n_{\mu\nu}\,.\end{aligned} \tag{2.120}$$

We have defined $M^{mn} \equiv \tilde{\mathcal{V}}^{\underline{m}}{}_A \tilde{\mathcal{V}}^{\underline{n}}{}_A$ and assumed the existence of its inverse M_{mn}. As shown in [38], by using $[D_\mu, D_\nu]\,\phi_m = \eta_{mn}(g_1 \tilde{C}^n_{\mu\nu} + g_2 B^n_{\mu\nu})$, we find the integrability condition for the first equation in (2.120)

$$\begin{aligned}D^\nu(M_{mn} B^n_{\mu\nu}) &= e\epsilon_{\mu\nu\rho} D^\nu (M_{mn} \tilde{\mathcal{V}}^{\underline{n}}{}_A \tilde{\mathcal{P}}^{A\rho}) + g_1 \eta_{mn} \tilde{\mathcal{V}}^{n}{}_A (\delta_{AB} - \tilde{\mathcal{V}}^{\underline{k}}{}_A M_{kl} \tilde{\mathcal{V}}^{\underline{l}}{}_B) \tilde{\mathcal{P}}^B_\mu \\ &\quad + \frac{1}{2} e\epsilon_{\mu\nu\rho}(g_2 \eta_{mn} + g_1 \eta_{mk} \tilde{\mathcal{V}}^{k}{}_A \tilde{\mathcal{V}}^{\underline{l}}{}_A M_{ln}) B^{n\nu\rho}\,.\end{aligned} \tag{2.121}$$

which is the Yang-Mills equation for $B^n_{\mu\nu}$. This equation can be derived from the Lagrangian

$$\begin{aligned}\tilde{\mathcal{L}} &= \frac{1}{4} eR - g_2 \tilde{\mathcal{L}}_{\text{CS}}(B) - \frac{1}{8} e M_{mn} B^{m\mu\nu} B^n_{\mu\nu} - \frac{1}{4} e G_{AB} \tilde{\mathcal{P}}^A_\mu \tilde{\mathcal{P}}^{B\mu} \\ &\quad + \frac{1}{4} \epsilon^{\mu\nu\rho} M_{mn} \tilde{\mathcal{V}}^{\underline{n}}{}_A B^m_{\mu\nu} \tilde{\mathcal{P}}^A_\rho - eV\end{aligned} \tag{2.122}$$

where

$$
\begin{aligned}
G_{AB} &= \delta_{AB} - \tilde{\mathcal{V}}^{\underline{m}}{}_A M_{mn} \tilde{\mathcal{V}}^{\underline{n}}{}_B, \qquad M_{mn} = (\tilde{\mathcal{V}}^{\underline{m}}{}_A \tilde{\mathcal{V}}^{\underline{n}}{}_A)^{-1}, \\
\tilde{\mathcal{L}}_{\text{CS}}(B) &= \frac{1}{4}\epsilon^{\mu\nu\rho} B^m_\mu \eta_{mn} \left(\partial_\nu A^n_\rho + \frac{1}{3} g_1 f^n{}_{kl} B^k_\nu B^l_\rho\right). \qquad (2.123)
\end{aligned}
$$

This is a Yang-Mills gauged supergravity with $d-n$ scalar fields parametrizing the coset space G'/H' with gauge group G_0. The corresponding gauge transformations are given by

$$
\delta \tilde{L} = -g_1 \eta_{mn} \Lambda^{\underline{m}} T^n \tilde{L}, \qquad \delta B^m_\mu = \partial_\mu \Lambda^{\underline{m}} + g_1 f^m{}_{kl} B^k_\mu \Lambda^{\underline{l}}. \qquad (2.124)
$$

It is also possible to eliminate ϕ_m using (2.120) in the fermionic terms in the Lagrangian and supersymmetry transformations. This equivalence has also been shown in [31] in reverse direction namely by writing a Yang-Mills Lagrangian in Chern-Simons form. This has been done by introducing extra gauge fields and compensating scalars so that we obtain the original Yang-Mills Lagrangian after gauge fixing. In the next chapter, we will study RG flow solutions by finding supersymmetric domain wall solutions of $N = 4$ gauged supergravity in the formulation studied in this chapter. The theories constructed here will also be used in several parts of this thesis.

Chapter 3

RG flows from $N = 4$ three dimensional gauged supergravity

In this chapter, we will study holographic RG flow solutions in $N = 4$ three dimensional gauged supergravity. The RG flow solution is a domain wall of the gauged supergravity and interpolates between two AdS_3 critical or fixed points of the scalar potential. These critical points correspond to the fixed points of the beta function in the dual boundary field theory. In the dual field theory, we start with a CFT in the UV with some perturbations which are source terms in the Lagrangian or vacuum expectation values of some operators. The theory is then driven by these perturbations to undergo an RG flow to another fixed point at which the theory becomes a CFT again. In some cases, it can happen that the UV CFT flows to non-conformal field theory because there is no IR fixed point. In this work, we are interested only in the RG flows with drive some UV CFT's to another CFT's in the IR. In term of the gravity dual, this means that we are looking for supergravity solutions that have asymptotic AdS_3 limits at both ends. We start with the notion of holographic renormalization and RG flows based on the general principle of the AdS/CFT correspondence. To find the RG flow solutions, we study the scalar potentials of $N = 4$ gauged supergravity and identify some of their critical points. Our main interest is supersymmetric RG flows, so we look for supersymmetric domain wall solutions which approach AdS_3 at the end points. This is achieved by studying the solutions of the BPS equations obtained by setting supersymmetry transformations of fermionic fields to zero. We end this chapter by identifying the dimension of the operator that drives the flow and computing the mass spectrum of scalar fields.

3.1 Holographic renormalization and RG flows

According to the AdS/CFT correspondence, a strongly coupled quantum field theory living on d dimensional space, M_d, can be described by a weakly coupled gravity theory living in a $d+1$ dimensional bulk space whose boundary is M_d. In particular, in the case of AdS_{d+1} bulk space, the dual field theory is a d dimensional conformal field theory. We are interested in an asymptotically AdS bulk geometry which approaches an AdS space for a certain limit. If the geometry admits two AdS regions namely at the boundary and deep interior, this background describes an RG flow of a disturbed UV CFT to another CFT in the IR. The rest of the discussion in this section closely follows [88, 89, 90, 9]. For further details, we refer the reader to these references and also [91, 92, 93, 94]. We mainly focus on relevant results we will use in this thesis. The full detail of the derivation can be found in the above mentioned references.

As first introduced in [7], correlation functions of the boundary field theory can be obtained from the AdS/CFT correspondence via

$$Z_{\text{string}}[\Phi_0]|_{M\times\tilde{M}} = \langle e^{-\int_{\partial M}\Phi_0\mathcal{O}}\rangle_{\text{QFT}}\,. \tag{3.1}$$

This states that the partition function of string theory on $M\times\tilde{M}$ with a compact manifold $\tilde{M}$ can be identified with the generating functional of the correlation functions of the field theory operator $\mathcal{O}$ with the value of the bulk field Φ on the boundary ∂M, Φ_0, being a source of $\mathcal{O}$. In many cases of interest, M is AdS or asymptotically AdS spaces. Quantization of string theory on a non-trivial background and, in many cases, with non-trivial Ramon-Ramon fields turns out to be difficult, and currently, this has not been achieved. However, in some limits, we can work with its low energy limit rather than the full string theory. For example, in the original AdS/CFT correspondence, type IIB theory on $AdS_5\times S^5$ is dual to $N=4$ $SU(N)$ supersymmetric Yang-Mills (SYM) theory. In the large N limit corresponding to strongly coupled gauge theory with large 't Hooft coupling $\lambda = g^2_{\text{PYM}}N$, we can approximate string theory with its low energy effective theory namely type IIB supergravity. Furthermore, if a consistent truncation to lower dimensions can be performed, we can work with the lower dimensional supergravity theory. This is the strategy we follow in this chapter where we will study RG flow solutions in three dimensional gauged supergravity. After the solutions have been found, we can uplift them to ten dimensions in the case where the reduction ansatz exists.

The bulk fields of interest to us are the metric $g_{\mu\nu}$, vector fields A_μ and scalars ϕ. In the dual field theory, the boundary values of these fields can be identified with sources coupled to stress tensor T_{ij}, currents J_i and scalar operator $\mathcal{O}$, respectively. We can then write (3.1) as

$$\begin{aligned} Z_{\text{SUGRA}}[\phi_{(0)}, g_{(0)}, A_{(0)}] &= \int_{\{\phi,g,A\}\sim\{\phi_{(0)},g_{(0)},A_{(0)}\}} D\phi DgDAe^{-S[\phi,g,A]} \\ &= \langle e^{-\int d^4x\sqrt{g_{(0)}}\left(\mathcal{O}(x)\phi_{(0)}(x)+A_{i(0)}J^i+\frac{1}{2}g_{(0)ij}T^{ij}\right)}\rangle\,. \end{aligned} \tag{3.2}$$

In this equation, $\phi_{(0)}$, $A_{(0)}$ and $g_{(0)}$ are the values of the corresponding bulk fields at the boundary. In the limit where all stringy corrections can be neglected, the bulk path integral is dominated by the leading term which is the on-shell supergravity action

$$\int_{\{\phi,g,A\}\sim\{\phi_{(0)},g_{(0)},A_{(0)}\}} D\phi DgDAe^{-S[\phi,g,A]} = e^{-S_{\text{on-shell}}[\phi_{(0)},g_{(0)}.A_{(0)}]}\,. \tag{3.3}$$

So, to leading order, we can identify the on-shell bulk action with the QFT generating functional for connected diagrams.

$$S_{\text{on-shell}}[\phi_{(0)}, g_{(0)}, A_{(0)}] = -W_{\text{QFT}}[\phi_{(0)}, g_{(0)}, A_{(0)}]. \tag{3.4}$$

We can obtain correlation functions of the dual operators by taking functional derivatives on the on-shell action. For example, the one-point functions are given by

$$\begin{aligned} \langle \mathcal{O}(x)\rangle &= \frac{1}{\sqrt{g_{(0)}(x)}}\frac{\delta S}{\delta\phi_{(0)}(x)}, \\ \langle J_i(x)\rangle &= \frac{1}{\sqrt{g_{(0)}(x)}}\frac{\delta S}{\delta A_{(0)_i}(x)}, \\ \langle T_{ij}(x)\rangle &= \frac{1}{\sqrt{g_{(0)}(x)}}\frac{\delta S}{\delta g_{(0)_{ij}}(x)}\,. \end{aligned} \tag{3.5}$$

The higher point functions can be obtained by further differentiations. It is well known that the QFT correlation functions need to be renormalized because of their divergences. In the holographic context, this is also the case for the on-shell classical action which is also divergent due to the infinite volume of the bulk spacetime. We need to regularize and subtract the infinities by appropriate counter terms to make it finite. This process is called holographic renormalization. It is customary to work with the metric on the asymptotically AdS_{d+1} space with radial coordinate ρ

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(\rho,x)dx^i dx^j, \qquad i,j = 1,\dots,d\,. \tag{3.6}$$

In this coordinate system, the boundary is at $\rho = 0$. In the holographic renormalization, the cutoff at $\rho = \epsilon > 0$ is introduced. Then, the divergent terms is removed by adding some counter terms. The general procedure is to study the bulk field equations in the limit near the AdS boundary. We expand the bulk fields in the series in ρ. For example, $g_{\mu\nu}$ and ϕ can be expanded as

$$\begin{aligned} g(\rho,x)_{ij} &= g_{(0)ij}(x) + g_{(2)ij}(x)\rho + \dots \\ \phi(\rho,x) &= \rho^{\frac{d-\Delta}{2}}(\phi_{(0)}(x) + \rho\phi_{(2)}(x) + \rho^2\phi_{(4)}(x) + \dots). \end{aligned} \tag{3.7}$$

For even d, there are logarithmic terms in both expansions. For the metric, this term arises at order $\frac{d}{2}$ while for a scalar field, this occurs at order $\frac{\Delta}{2}$ where Δ is the conformal dimension of the operator dual to this scalar, see the references for details.

The necessity for the logarithmic terms is due to the termination of the recurrence relation for determining $g_{(d)}$ and $\phi_{(2n)}$ for $n = \Delta - \frac{d}{2}$. So, the logarithmic terms are needed to avoid this problem. After solving the field equations, we substitute the solution back to the action with the cutoff $\rho = \epsilon$. We finally subtract the divergent terms in the limit $\epsilon \to 0$ and end up with the renormalized on-shell action S_{ren}

$$S_{\text{ren}} = \lim_{\epsilon \to 0}(S_{\text{reg}} + S_{\text{ct}}) \tag{3.8}$$

where S_{reg} and S_{ct} are the regularized action and the corresponding counter terms.

By definition, the variation of S_{ren} give one-point functions

$$\delta S_{\text{ren}} = \int d^4x \sqrt{g_{(0)}} \left(\frac{1}{2} \langle T_{ij} \rangle \delta g^{ij}_{(0)} + \langle J_i \rangle \delta A^i_{(0)} + \langle \mathcal{O} \rangle \delta \phi_{(0)} \right). \tag{3.9}$$

This directly gives the expectation values $\langle T_{ij} \rangle$, $\langle J_i \rangle$ and $\langle \mathcal{O} \rangle$. Since we will not explicitly compute the correlation functions in this work, we will not carry out the whole holographic renormalization program. However, we will come back to this procedure in chapter 5 in order to compute the central charge of the dual CFT as well as the level of the current algrbras. In what follow, we are interested only in scalar operator $\mathcal{O}$ which is relevant for our discussion of RG flows.

We now move to holographic RG flows. These are backgrounds in the forms of kinks or domain walls in the corresponding (super) gravity bulk theory. Since the radial coordinate of the AdS space can be interpreted as an energy scale in the dual field theory at the boundary, RG flow solutions describing RG flows in the Poincare invariant field theories will have only a non trivial radial dependence. In this work, we are interested in the supersymmetric flow solutions, so our starting point is the supergravity theory. Furthermore, since the dual operators are scalar operators, the corresponding flow solutions will involve radial dependent scalar fields. Scalar fields depending non trivially on AdS radial coordinate are called active scalars while for those which do not are call inert scalars.

The $(d+1)$ dimensional metric ansatz is given by

$$ds^2 = e^{2A(r)} \gamma_{ij} dx^i dx^j + dr^2, \qquad i, j = 1, \ldots, d. \tag{3.10}$$

Throughout this thesis, we always work with $(-++\ldots+)$ metric. The coordinate r is related to the ρ defined above by $\rho = e^{-2r}$. This form is manifestly in variant under d dimensional diffeomorphism. We are particularly interested in the case in which $\gamma_{ij} = \eta_{ij}$, so from now on, we restrict ourselves to this case. The d dimensional part of the metric is obviously Poincare invariant. We require that the solution interpolates between two AdS_{d+1} regions, one as $r \to \infty$ and the other one as $r \to -\infty$. At both ends, the metric is AdS_{d+1} in which $A(r) = \frac{r}{L}$ with AdS radius L. This feature has a natural interpretation as an RG flow of the perturbed CFT in the UV to another CFT in the IR. This background is sometime called asymptotically AdS.

We now present the holographic c-theorem which is one of the important consequences of holographic RG flows. In a two dimensional field theory, it has been proved in [95] that in an RG flow, there exist a c-function which decreases monotonically along the flow from the UV to the IR, and this function has the same values as the central charges at the fixed points. In higher dimensional field theories, there is no complete proof of this c-theorem. On the other hand, for holographic RG flows, there does exist a proof of the "holographic c-theorem" for any d dimensional dual field theories. One result obtained in [12] is based on Einstein equations and Weaker energy condition in the bulk. Another result can be found in [96]. The latter proof is based on the geometric consideration and Raychadhuri theorem. This result also shows a close relationship between the irreversibility of the RG flow in the dual field theory and the singularity theorem in the bulk gravity. We will review only the former result because it is directly related to the works presented here.

We study Einstein equations with the metric (3.10). We find the Ricci tensor components

$$\begin{aligned} R_{ij} &= -e^{2A}[A'' + (D-1)(A')^2]\eta_{ij}, \\ R_{rr} &= -(D-1)[A'' + (A')^2] \end{aligned} \tag{3.11}$$

where $D = d+1$. Using Einstein equation $G_{\mu\nu} = \kappa_D^2 T_{\mu\nu}$, it is easily shown that

$$(D-2)A'' = \kappa_D^2 (T^t_{\ t} - T^r_{\ r}). \tag{3.12}$$

For our metric ansatz, the corresponding stress tensor is diagonal

$$T^\mu_{\ \nu} = \text{diag}(-\rho, p_1, p_2, \ldots, p_r). \tag{3.13}$$

Recall the weaker energy condition $T_{\mu\nu}\zeta^\mu\zeta^\nu \geq 0$ for an arbitrary null vector ζ^μ, we find $\rho + p_i \geq 0$. If there is no matter fields apart from the cosmological constant, we will recover the AdS$_D$ geometry, and $\rho + p_i = 0$ for all $i = 1, \ldots D-2, r$. In this case, there is no RG flow, and the corresponding field theory is the conformal field theory. We then conclude that if the matter fields in the bulk satisfy the weaker energy condition, $A'' \leq 0$ from (3.12). This result implies the monotonicity of A'. We now follow [12] and define the holographic c-function

$$C(r) = \frac{C_0}{A'^{D-2}}\,. \tag{3.14}$$

This function has been introduced already in [14] in another but equivalent form. The constant C_0 is universal for a particular supergravity theory in the sense that its value is the same for all CFTs dual to AdS vacua of this supergravity. At the fixed points of the scalar potential, $C(r)$ will give values of the corresponding central charges.

Although this result is based on the weaker energy condition, for RG flow solutions involving only the metric and scalars, it is generally true. These are flow solutions we will consider throughout this thesis. To see this, we consider a gravity theory coupled to scalar fields with scalar potential V

$$I = \frac{1}{2\kappa_D^2}\int d^D x\sqrt{-g}\left(R - \frac{1}{2}\mathcal{M}_{IJ}P^{I\mu}P^J_\mu - V\right). \tag{3.15}$$

Notice that this is of the same form as the action of three dimensional gauged supergravity in the previous chapter. In the RG flow solution, the scalars depend only on r. It can be shown by a straightforward calculation, that

$$T^t{}_t - T^r{}_r = -\mathcal{M}_{IJ}P^I_r P^J_r \tag{3.16}$$

which is negative for a positive definite $\mathcal{M}_{IJ}$. It has also been shown in [14] that the c-theorem is a consequence of equations of motion for the flows involving only scalars and the metric.

Once an RG flow background has been found, the dimension of the operator driving the flow and the correlation functions of the dual operator can be obtained. The latter can be found by studying the fluctuations around the background solution and using the procedure described above. We now review how to extract the dimension of the perturbing operator. Near the UV fixed point which can be identified with $r \to \infty$, the metric becomes AdS_{d+1}, and the scalar behaves as

$$\phi(r) = \tilde{\phi} + e^{\frac{(\Delta_1 - d)r}{L}}\phi_{(0)} + e^{\frac{\Delta_1 r}{L}}\phi_{(2\Delta - d)} + \dots \tag{3.17}$$

where $\tilde{\phi}$ is the scalar background given by the flow solution. We have included only the leading terms in the expansion. From this equation, $\phi_{(0)}$ and $\phi_{(2\Delta-d)}$ are interpreted as the source and the vacuum expectation value (vev) of the dual operator $\mathcal{O}$. Notice that for $\Delta_1 > \frac{d}{2}$, the source term dominates while for $\Delta_1 < \frac{d}{2}$ the vev term dominates. In the first case, the flow is driven by a relevant operator of dimension Δ_1. This corresponds to adding an extra term to the UV CFT Lagrangian, and conformal symmetry is explicitly broken. For the latter case, conformal symmetry is spontaneously broken by a vev of a relevant operator of dimension Δ_1. In some cases, the flow can be driven by a vev of a marginal operator of dimension d as we will see in later chapters. The case for $\Delta = \frac{d}{2}$ is special since the asymptotic behavior of the scalar field is given by

$$\phi(r) = \tilde{\phi} + e^{-\frac{dr}{2L}}\left(\frac{r}{L}\phi_{(0)} + \phi_{(2\Delta - d)}\right) + \dots. \tag{3.18}$$

Note that the source term comes with an additional factor of r corresponding to the $\log\rho$ term in the (ρ, x^i) coordinates.

Since, we are interested in the case in which the UV CFT flows to another CFT in the IR, the flow will approach the IR fixed point at the end of the flow.

The driving operator then becomes irrelevant ($\Delta > d$) at the IR fixed point. The fluctuation near the IR point is given by

$$\phi(r) = \tilde{\phi} + \bar{\phi} e^{\frac{(\Delta_2 - d)r}{L}} + \ldots, \qquad r \to -\infty, \tag{3.19}$$

and the corresponding operator has dimension Δ_2 in the IR.

We end this section with the relation between mass and conformal dimension of various bulk fields.

- scalar fields [7]:

$$\Delta_\pm = \frac{1}{2}(d \pm \sqrt{d^2 + 4m^2}) \qquad \text{or} \qquad m^2 L^2 = \Delta(\Delta - d)$$

- spinor fields [97]:

$$\Delta = \frac{1}{2}(d + 2|m|)$$

- vector fields:

$$\Delta_\pm = \frac{1}{2}(d \pm \sqrt{(d-2)^2 + 4m^2})$$

- p-form fields [98]:

$$\Delta = \frac{1}{2}(d \pm \sqrt{(d-2p)^2 + 4m^2})$$

- first order $\frac{d}{2}$-form fields (d even):

$$\Delta = \frac{1}{2}(d + 2|m|)$$

- spin $\frac{3}{2}$ fields [99, 100]:

$$\Delta = \frac{1}{2}(d + 2|m|)$$

- spin 2 fields [101]:

$$\Delta = \frac{1}{2}(d + \sqrt{d^2 + 4m^2})$$

3.2 $N = 4$ three dimensional gauged supergravity

We are now in the position to give our RG flow solutions. These are supersymmetric solutions of $N = 4$ gauged supergravity in three dimensions. The scalar fields parametrize the coset space $\frac{SO(4,4)}{SO(4)\times SO(4)} \times \frac{SO(4,4)}{SO(4)\times SO(4)}$. The gauge group is of semi-simple type, $SO(4)\ltimes \mathbf{T}^6$. The choice of the gauge group is motivated by the fact that three dimensional gauged supergravity with a semi-simple gauge group is equivalent to a Yang-Mills gauged supergravity as shown in chapter 2. In the present case, the theory we consider is equivalent to $N = 4$ $SO(4)$ Yang-Mills gauged supergravity. It is possible to uplift this theory and its solutions to higher dimensions if the reduction ansatz exists. We will give some progress in this direction in the next chapter.

We begin with the supersymmetry transformations of fermionic fields given in chapter 2 with zero fermions

$$\begin{aligned} \delta\psi^I_\mu &= \mathcal{D}_\mu\epsilon^I + A_1^{IJ}\gamma_\mu\epsilon^J, \\ \delta\chi^{iI} &= \frac{1}{2}(\delta^{IJ}\mathbf{1} - f^{IJ})^i{}_j \not{\mathcal{D}}\phi^j\epsilon^J - NA_2^{JIi}\epsilon^J\,. \end{aligned} \tag{3.20}$$

We also recall the conditions for supersymmetric vacua (2.100). These conditions mean that the preserved supersymmetries correspond to the eigenvalues of A_1^{IJ} which equal $\pm\sqrt{\frac{-V_0}{4}}$, since in our normalization $-V_0 = L^{-2}$, where L is the radius of AdS_3. More in detail, let us choose AdS_3 coordinates r, x_0, x_1, and metric $ds^2 = dr^2 + e^{2r/L}(-dx_0^2 + dx_1^2)$. From the previous remarks, it follows that for each eigenvector $v^I_\pm$ of A_1^{IJ}, with eigenvalue $\pm\sqrt{\frac{-V_0}{4}}$, if we form the spinor $\epsilon^I_\pm = \epsilon_\pm \otimes v^I_\pm$, then the BPS condition for the gravitino variation (3.20) becomes identical to the Killing spinor equation for $\epsilon_\pm$ on AdS_3 i.e. $\mathcal{D}_\mu\epsilon_\pm = \pm\frac{1}{2L}\gamma_\mu\epsilon_\pm$. Using the explicit expression for the spin connection for the above metric, one can see that one solution to this equation is an x_0, x_1-independent spinor obeying $\gamma^r\epsilon_\pm = \pm\epsilon_\pm$, where γ^r is the flat gamma matrix. This corresponds to a left (right) Poincare' supersymmetry in the boundary CFT. The other solution gives rise to the superconformal charge in the boundary CFT, has a non-trivial x_0, x_1 dependence and is constructed with a constant spinor obeying the opposite γ^r projection condition.

Therefore, it is convenient to classify the critical points by presenting their preserved supersymmetries in the form of (n_-, n_+) corresponding to the n_+ and n_- positive and negative eigenvalues of A_1^{IJ} whose modulus equals $\sqrt{\frac{-V_0}{4}}$. These coincide with the number of left-(right-) moving Poincare' supersymmetries of the dual SCFT_2. Of course, the total number of supersymmetries is doubled by the inclusion of the superconformal ones.

To summarize, the procedure of finding supersymmetric vacua is the following. From (2.100), we look for the Killing spinors ϵ^I which are annihilated by some of the A_{2i}^{JI}. At the same time, ϵ^I must also be the eigenvector of A_1^{IJ}. Clearly, maximal supersymmetric vacua are annihilated by all of the components of A_{2i}^{JI}, and ϵ^I is an eigenvector of A_1^{IJ} for all directions I. The ϵ^I characterizing partially supersymmetric vacua will be an eigenvector of A_1^{IJ} for certain directions labeled by some values of I, and will be annihilated only by the A_{2i}^{JI} in the corresponding directions.

3.2.1 Vacua of the $N = 4$ Theory

The target space in our case is the product of two quaternionic manifolds, that we take to be $SO(4,4)/SO(4) \times SO(4)$. A convenient (redundant) parametrization of cosets is given by the following SO(4,4) group element

$$L_i = \frac{1}{2} \begin{pmatrix} X_i + e_i^t & Y_i + e_i^t \\ -X_i + e_i^t & e_i^t - Y_i \end{pmatrix}, \tag{3.21}$$

where $i = 1, 2$ refers to the two spaces. e_i is a 4×4 matrix in $GL(4, \mathbb{R})$, $X_i = E_i + B_i e_i^t$, $Y_i = -E_i + B_i e_i^t$. B_i is an antisymmetric 4×4 matrix, and $E_i = e_i^{-1}$. The inverse of L_i is

$$L_i^{-1} = \frac{1}{2} \begin{pmatrix} X_i^t + e_i & X_i^t - e_i \\ -Y_i^t - e_i & e_i - Y_i^t \end{pmatrix}. \tag{3.22}$$

One can eliminate 6 of the 22 parameters in L by using the right action of the diagonal $SO(4)$ action, for example by bringing e_i into an upper triangular form. The following Lie algebra elements,

$$t^{\mathcal{A}} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \qquad t^{\mathcal{B}} = \begin{pmatrix} b & b \\ -b & -b \end{pmatrix} \tag{3.23}$$

where all entries are 4×4 antisymmetric blocks, together with an identical copy for the second space, will be gauged. In other words, the semisimple part of the gauge group will be the diagonal $SO(4)_D$ in the $(SO(4))^4$ of the product $(\frac{SO(4,4)}{SO(4)\times SO(4)})^2$, corresponding to generators $t^{\mathcal{A}}$. On the other hand, the nilpotent generators, $t^{\mathcal{B}}$, generate diagonal shift symmetries $B_{1,2} \to B_{1,2} + 2b$. Also, it is clear that the $\mathcal{B}$-generators transform in the adjoint representation with respect to the diagonal $SO(4)$. For a and b, we can take a basis of antisymmetric matrices given by $J^{IJ} = \varepsilon^{IJ} - \varepsilon^{JI}$, with $(\varepsilon^{IJ})_{KL} = \delta_{IK}\delta_{JL}$. Similarly, we can use the following basis for the 16 non-compact generators of $SO(4,4)$:

$$Y^{ab} = \begin{pmatrix} 0 & \varepsilon^{ab} \\ (\varepsilon^t)^{ab} & 0 \end{pmatrix}. \tag{3.24}$$

Since in the present case both the R symmetry group and the gauge group are $SO(4)$, it is convenient to split the corresponding Lie algebras generators into self-dual and anti-self-dual components J_+ and J_- respectively:

$$J_+^{IJ} = J^{IJ} + \frac{1}{2}\epsilon^{IJKL}J^{KL} \qquad \text{and} \qquad J_-^{IJ} = J^{IJ} - \frac{1}{2}\epsilon^{IJKL}J^{KL} \tag{3.25}$$

which are $SU(2)_+$ and $SU(2)_-$ generators in the $SO(4) = SU(2)_+ \oplus SU(2)_-$ Lie algebra decomposition. We will adopt this decomposition both for $\mathcal{A}$- and $\mathcal{B}$-type generators. Correspondingly, the two-forms tensors f^{IJ} introduced in the previous section have, say, self-dual components on the first quaternionic space and anti-self-dual components on the second. In our formalism and in a flat basis, they can be expressed as:

$$f^{IJ}_{\pm\ ab,cd} = \mathrm{Tr}((\varepsilon^t)^{ab} J^{IJ}_{\pm} \varepsilon^{cd}). \tag{3.26}$$

At this stage, we can proceed to construct the supergravity theory with the gauging of $SO(4) \ltimes \mathbf{T}^6$ and in particular, verify its consistency, along the lines reviewed in the previous section. As explained there, the main ingredients are given by the tensors A_1 and A_2, which determine the scalar potential and the supersymmetry variations of the fermionic fields. They are constructed through the T-tensors, which in turn are obtained by uplifting the embedding tensor $\Theta_{\mathcal{M}\mathcal{N}}$ into G by using $\mathcal{V}^{\mathcal{M}}_{\mathcal{P}}$, with $\mathcal{P}$ running over the generators of G corresponding to the R-symmetries $\mathcal{P} = IJ$, and the non-compact coset directions $\mathcal{P} = ab$ in the first and second space.

Gauge invariance restricts the Θ tensors to have components, $\Theta_{\mathcal{A}\mathcal{B}}$ and $\Theta_{\mathcal{B}\mathcal{B}}$, which are proportional to the $SO(4)$ Killing form, schematically $\delta_{\mathcal{A}\mathcal{B}}$ and $\delta_{\mathcal{B}\mathcal{B}}$, respectively. The proportionality constants are gauge couplings, and, of course, we should specify here to which of the four $SU(2)$'s the $\mathcal{A}$, $\mathcal{B}$ indices belong. Therefore, a priori we expect four couplings g_{1s}, g_{1a}, g_{2s}, and g_{2a}. The a and s labels indicate the self-dual and anti-self-dual $SU(2)$, respectively, and 1 refers to the $\mathcal{A}\mathcal{B}$ couplings whereas 2 refers to the $\mathcal{B}\mathcal{B}$ ones.

We now give all the ingredients to find critical points of the scalar potential. These include all components of $\mathcal{V}$. Indices referring to each target space coordinates, $i, j, k, \ldots$, will be traded by a pair of indices of the type $a, b, c, \ldots$ from 1 to 4. Antisymmetric pairs of capital letters $I, J, K, \ldots$ label $SO(4)$ adjoint indices.

$$\begin{aligned}
\mathcal{V}^{LJ,MK}_{\pm \mathrm{a}} &= -\frac{1}{4}\mathrm{Tr}[(e_1^t J^{LJ}_+ X_1^t + X_1 J^{LJ}_+ e_1) J^{MK}_{\pm} + (e_2^t J^{LJ}_- X_2^t + X_2 J^{LJ}_- e_2) J^{MK}_{\pm}],\\
\mathcal{V}_{\pm 1,2\mathrm{a}}{}^{MK}_{ab} &= \mathrm{Tr}[(e^t_{1,2}\varepsilon_{ab} X^t_{1,2} + Y_{1,2}\varepsilon_{ab} e_{1,2}) J^{MK}_{\pm}],\\
\mathcal{V}^{LJ,MK}_{\pm \mathrm{b}} &= -\frac{1}{4}\mathrm{Tr}[(e_1^t J^{LJ}_+ e_1^t + e_2^t J^{LJ}_- e_2^t) J^{MK}_{\pm}],\\
\mathcal{V}_{\pm 1,2\mathrm{b}}{}^{MK}_{ab} &= \mathrm{Tr}(e^t_{1,2}\varepsilon_{ab} e_{1,2} J^{MK}_{\pm}).
\end{aligned} \tag{3.27}$$

The string of indices $\pm 1, 2\mathrm{a}$ ($\pm 1, 2\mathrm{b}$) indicates $\mathcal{A}$ ($\mathcal{B}$)-type gauging in the first (second) space with (anti-)self-dual $SU(2)$.

The T-tensors turn out to be

$$\begin{aligned}
T^{LJ,MK} &= g_{1s}(\mathcal{V}_{+\mathrm{a}}^{LJ,PQ}\mathcal{V}_{+\mathrm{b}}^{MK,PQ}+\mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+\mathrm{a}}^{MK,PQ})+g_{1a}(\mathcal{V}_{-\mathrm{a}}^{LJ,PQ}\mathcal{V}_{-\mathrm{b}}^{MK,PQ}\\
&\quad +\mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-\mathrm{a}}^{MK,PQ})+g_{2s}\mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+\mathrm{b}}^{MK,PQ}+g_{2a}\mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-\mathrm{b}}^{MK,PQ},\\
T_{1ab}^{\ LJ} &= g_{1s}(\mathcal{V}_{+\mathrm{a}}^{LJ,PQ}\mathcal{V}_{+1\mathrm{b}ab}^{\ \ PQ}+\mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+1\mathrm{a}ab}^{\ \ PQ})+g_{1a}(\mathcal{V}_{-\mathrm{a}}^{LJ,PQ}\mathcal{V}_{-1\mathrm{b}ab}^{\ \ PQ}\\
&\quad +\mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-1\mathrm{a}ab}^{\ \ PQ})+g_{2s}\mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+1\mathrm{b}ab}^{\ \ PQ}+g_{2a}\mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-1\mathrm{b}ab}^{\ \ PQ},\\
T_{2ab}^{\ LJ} &= g_{1s}(\mathcal{V}_{+\mathrm{a}}^{LJ,PQ}\mathcal{V}_{+2\mathrm{b}ab}^{\ \ PQ}+\mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+2\mathrm{a}ab}^{\ \ PQ})+g_{1a}(\mathcal{V}_{-\mathrm{a}}^{LJ,PQ}\mathcal{V}_{-2\mathrm{b}ab}^{\ \ PQ}\\
&\quad +\mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-2\mathrm{a}ab}^{\ \ PQ})+g_{2s}\mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+2\mathrm{b}ab}^{\ \ PQ}+g_{2a}\mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-2\mathrm{b}ab}^{\ \ PQ}. \qquad (3.28)
\end{aligned}$$

It turns out that the consistency requirement on $T^{IJ,KL}$, discussed in the previous section, requires $g_{2a} = -g_{2s}$. Moreover, we find it is convenient for the subsequent analysis to redefine the couplings from g_{1s}, g_{1a} to g_n, g_p as follows:

$$g_{1s} = g_p + g_n \qquad \text{and} \qquad g_{1a} = g_p - g_n. \qquad (3.29)$$

Now, we study various vacua of this theory. We begin by choosing an ansatz for the coset L. We have two spaces. We set $B_1 = B_2 = 0$ and choose diagonal e_i's:

$$e_1 = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix} \qquad \text{and} \qquad e_2 = \begin{pmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{pmatrix}. \qquad (3.30)$$

Notice that the shift gauge symmetry would allow us to set one of the two B's to zero and the left $SO(4)$ gauge symmetry can be used to diagonalize one of the two e's, so the ansatz above is indeed a truncation of the full twenty-dimensional moduli space. We have checked the consistency of this truncation explicitly. That is, we have verified that the remaining fields appear at least quadratically in the action, and therefore setting them to zero solves their equations of motion. We then proceed to analyze the BPS conditions $\delta\psi^I_\mu = 0$ and $\delta\chi^{iI} = 0$ using (3.20), within this eight-dimensional subspace.

We give below the vacuum expectation values of e_1, e_2, the A_1^{IJ} eigenvalue (A_1) satisfying $|A_1|^2 = -V_0/4$ and the corresponding preserved supersymmetries (n_-, n_+) for the AdS_3 vacuum that are relevant to the flow solutions we will show in the next subsection. Other vacua are shown in Appendix A. We will use the notation $\mathbf{I}_n$ for the $n \times n$ identity matrix throughout this thesis.

(1,3) vacua

- I.

$$
\begin{aligned}
e_1 &= \sqrt{\frac{-2(g_n+g_p)}{g_{2s}}}\mathbf{I}_4 \\
e_2 &= \sqrt{\frac{-2(g_n+g_p)}{g_{2s}}}(-1,1,1,1) \\
A_1 &= \frac{32(g_n+g_p)^2}{g_{2s}} \qquad \text{and} \qquad V_0 = \frac{-4096(g_n+g_p)^4}{g_{2s}^2}.
\end{aligned} \tag{3.31}
$$

- II.

$$
\begin{aligned}
e_1 &= \sqrt{\frac{2(g_p-g_n)}{g_{2s}}}(1,-1,-1,-1) \\
e_2 &= -\sqrt{\frac{2(g_p-g_n)}{g_{2s}}}\mathbf{I}_4 \\
A_1 &= \frac{-32(g_n-g_p)^2}{g_{2s}} \qquad \text{and} \qquad V_0 = \frac{-4096(g_n-g_p)^4}{g_{2s}^2}.
\end{aligned} \tag{3.32}
$$

- III.

$$
\begin{aligned}
e_1 &= \sqrt{\frac{g_n(g_p^2-g_n^2)}{g_{2s}g_n^2}}\Big(\frac{g_n}{g_p},-1,-1,-1\Big) \\
e_2 &= -\sqrt{\frac{g_n(g_p^2-g_n^2)}{g_{2s}g_n^2}}\Big(\frac{g_n}{g_p},1,1,1\Big) \\
A_1 &= \frac{-8(g_n^2-g_p^2)^2}{g_{2s}g_ng_p} \qquad \text{and} \qquad V_0 = \frac{-256(g_n^2-g_p^2)^4}{g_{2s}^2g_n^2g_p^2}.
\end{aligned} \tag{3.33}
$$

(2,0) vacua

- IV.

$$
e_1 = (-a_1,a_1,a_2,a_2) \qquad e_2 = (b_1,b_1,b_2,b_2) \tag{3.34}
$$

$$
\begin{aligned}
a_1 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(g_p - g_n + \sqrt{5g_n^2 + 2g_pg_n + g_p^2})}} \\
a_2 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(g_n - g_p + \sqrt{5g_p^2 + 2g_pg_n + g_n^2})}} \\
b_1 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(3g_n + g_p + \sqrt{5g_n^2 + 2g_pg_n + g_p^2})}} \\
b_2 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(\sqrt{g_n^2 + 2g_ng_p + 5g_p^2} + g_n + 3g_p)}} \qquad (3.35)
\end{aligned}
$$

$$
A_1 = \frac{-32(g_n - g_p)^2}{g_{2s}} \quad \text{and} \quad V_0 = -\frac{4096(g_n - g_p)^4}{g_{2s}^2}. \qquad (3.36)
$$

- V.

$$
e_1 = (a_1, a_2, a_3, a_3) \qquad e_2 = (b_1, b_1, b_2, b_2) \qquad (3.37)
$$

$$
\begin{aligned}
a_1 &= -\frac{1+t}{1-t+\sqrt{1+t^2}}\sqrt{\frac{2g_p(1-t+\sqrt{1+t^2})}{g_{2s}t(1+t)\sqrt{1+t^2}}} \times \\
&\quad \sqrt{(t-1)\left\{t^3 - t^2 + t - 1 + (t - t^2 - 1)\sqrt{1+t^2}\right\}} \\
a_2 &= \sqrt{\frac{2tg_p(t-1)^2(1+t)\sqrt{1+t^2}}{g_{2s}(1-t+\sqrt{1+t^2})}} \times \\
&\quad \frac{1}{\sqrt{(t-1-t^2)(t-1)\sqrt{1+t^2} - t^2 + (1-t+t^2)^2}} \\
a_3 &= \sqrt{\frac{2g_p(1-t^2)}{g_{2s}(t-1+\sqrt{1+t^2})}} \\
b_1 &= \sqrt{\frac{2g_p(1-t^2)}{g_{2s}(1+t+\sqrt{1+t^2})}} \\
b_2 &= \sqrt{\frac{2g_p(1-t^2)}{g_{2s}(1+t+\sqrt{1+t^2})}} \\
A_1 &= \frac{-8(g_n^2 - g_p^2)^2}{g_{2s}g_ng_p} \quad \text{and} \quad V_0 = -\frac{256(g_n^2 - g_p^2)^4}{(g_{2s}g_ng_p)^2}, \qquad (3.38)
\end{aligned}
$$

where we have introduced $t = \frac{g_n}{g_p}$.

Out of all vacua, there are only three possibilities in connecting two vacua. That means we will have only three RG flows in the dual field theories. All these three flows are the flows between I and III, II and III, and between IV and V. The last flow is the only possible flow among V and other (2,0) points. This is because we cannot find any values of g_n, g_p and g_{2s} so that both e_1 and e_2 of the two end points of the flow are real apart from the IV and V pair. There are three possibilities in order to make IV and V real at the same time. These are given by

$$\begin{array}{ll} & t<-1, g_p<0, g_{2s}<0 \\ \text{or} & t<1, g_p>0, g_{2s}>0 \\ \text{or} & t>1, g_p>0, g_{2s}<0. \end{array} \tag{3.39}$$

For definiteness, we choose the last range and further choose $t = 2$, $g_p = 1$ and $g_{2s} = -1$ in our numerical solution. For all the critical points given above, we have checked that there exist at least one possible set of g_p, g_n and g_{2s} such that all the square roots in any critical points are real, although any two different critical points may not be made real with the same values of g_p, g_n and g_{2s}.

There might be more possibilities apart from these three flows. However, we could not find any interpolating solutions both analytically and numerically apart from those three mentioned above. Remarkably, we find only the flows between critical points which have the same supersymmetries. In the next section, we will give these solutions explicitly.

3.2.2 The Flow Between $(1,3)$ Vacua

In this subsection, we study a supersymmetric flow between two of the AdS_3 vacua with the same, $(1,3)$, amount of supersymmetries but with different cosmological constants, found in the previous subsection. We recall the three dimensional domain wall metric

$$ds^2 = e^{2A(r)}(-dt^2 + dx^2) + dr^2. \tag{3.40}$$

This becomes AdS_3 of radius L for $A(r) = \frac{r}{L}$. This is related to the vacuum energy V_0 as $L^2 = -\frac{1}{V_0}$, since in our normalization Einstein's equations read $L_{\mu\nu} = -2V_0 g_{\mu\nu}$.

We start by giving an ansatz for the scalars with non-trivial r-dependence,

$$e_1 = \begin{pmatrix} b(r) & 0 & 0 & 0 \\ 0 & a(r) & 0 & 0 \\ 0 & 0 & a(r) & 0 \\ 0 & 0 & 0 & a(r) \end{pmatrix}, \qquad e_2 = \begin{pmatrix} -b(r) & 0 & 0 & 0 \\ 0 & a(r) & 0 & 0 \\ 0 & 0 & a(r) & 0 \\ 0 & 0 & 0 & a(r) \end{pmatrix}. \tag{3.41}$$

Since now we are going to allow the scalars to have r dependence, we need to worry about possible contributions of the intrinsic connection Q^{IJ}_μ and the gauge fields $A^{\mathcal{M}}_\mu$ to the BPS equations (3.20). In addition, of course, the Yang-Mills equations of motion may be non-trivial. Indeed, r-dependent scalars may a priori source the gauge fields in case they give rise to a non-trivial gauge current $J^{\mathcal{M}}_\mu$. From the scalar kinetic term given in (2.27), we have

$$\mathcal{L}_{\rm kin} = -\frac{1}{2}eg_{ij}g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j = -\frac{1}{2}eg^{\mu\nu}e^A_\mu e^B_\nu\delta_{AB} \tag{3.42}$$

with $e^A_\mu = \partial_\mu\phi^i e^A_i$ being the pullback of the vielbein on the target space. Using the formula (2.55), we then find

$$\begin{aligned}\mathcal{L}_{\rm kin} &= \frac{1}{2}\sqrt{g}[\mathrm{Tr}(L^{-1}\partial_\mu L L^{-1}\partial^\mu L) + 2\Theta_{\mathcal{MN}}A^{\mathcal{M}\mu}\mathrm{Tr}(L^{-1}t^{\mathcal{N}}\partial_\mu L)\\ &\quad +\Theta_{\mathcal{MN}}\Theta_{\mathcal{KL}}A^{\mathcal{M}\mu}A^{\mathcal{K}}_\mu\mathrm{Tr}(L^{-1}t^{\mathcal{N}}t^{\mathcal{L}}L)].\end{aligned} \tag{3.43}$$

From (3.43), we see that the gauge fields couple to the scalar fields via a current

$$J^{\mathcal{N}}_\mu = \sqrt{g}\mathrm{Tr}(L^{-1}t^{\mathcal{N}}\partial_\mu L). \tag{3.44}$$

For diagonal e_1 and e_2, the current is zero, so we can consistently satisfy the equation of motion for the gauge fields by setting $A^{\mathcal{M}}_\mu = 0$. As promised, our flows involve only scalars and the metric. So, the holographically proved c-theorem mentioned before is guaranteed in our flow ansatz. Furthermore, all of the composite connections Q's are also zero in this diagonal ansatz. The BPS equations can be obtained by using (3.20). The $\delta\chi^{iI} = 0$ conditions give

$$\frac{db}{dr} = 24g_n ab^2 + 16g_p b^3 - 8a^3(g_n - g_{2s}b^2) \tag{3.45}$$

$$\frac{da}{dr} = 16g_p a^3 + 8g_n a^2 b + \frac{8a^4(g_n + g_{2s}b^2)}{b}. \tag{3.46}$$

This ansatz preserves $(1,3)$ supersymmetry, so we have $(1,3)$ supersymmetry throughout the flow. We proceed by taking one of the scalars as an independent variable. Changing the variables to $b(r) = z$ and $a(r) = a(z)$, we can write (3.45) and (3.46) as a single equation

$$\frac{da}{dz} = \frac{a^2(g_n z^2 + 2g_p za + (g_n + g_{2s}z^2)a^2)}{2g_p z^4 + 3g_n z^3 a + (g_{2s}z^3 - g_n z)a^3}. \tag{3.47}$$

We solve this by writing $a(z) = zf(z)$. Then, (3.47) becomes

$$z\frac{df}{dz} = -\frac{2f(g_p + g_n f)(f^2-1)}{(g_n - g_{2s}z^2)f^3 - 2g_p - 3g_n f}. \tag{3.48}$$

This equation can be solved for z as a function of f. We find

$$z = \pm\sqrt{\frac{g_n(f^2-1)}{g_{2s}f^2 + (g_n^2 f^3 + g_n g_p f^2)c_1}}. \tag{3.49}$$

We then obtain

$$b = \pm\sqrt{\frac{g_n(f^2-1)}{g_{2s}f^2+(g_n^2 f^3+g_n g_p f^2)c_1}}, \tag{3.50}$$

$$a = fb, \tag{3.51}$$

and (3.45) and (3.46) lead to the same equation for f

$$\frac{df}{dr} = \frac{16g_n(g_p+g_n f)(f^2-1)^2}{f(g_{2s}+(g_n g_p+g_n^2 f)c_1)}. \tag{3.52}$$

We can solve for r in term of f and find

$$\begin{aligned} r &= c_2+\frac{1}{64g_n}\left[\frac{2(-fg_{2s}g_n+g_{2s}g_p+g_n(g_p^2-g_n^2)c_1)}{(f^2-1)(g_n^2-g_p^2)}-\frac{g_{2s}g_n\ln(1-f)}{(g_n+g_p)^2}\right.\\ &\left.+\frac{g_{2s}g_n\ln(1+f)}{(g_n-g_p)^2}-\frac{4g_{2s}g_n^2 g_p\ln(fg_n+g_p)}{(g_n^2-g_p^2)^2}\right]. \end{aligned} \tag{3.53}$$

The constant c_2 is irrelevant and can be set to zero by shifting the coordinate r. So, from now on, we will use $c_2 = 0$ and choose a definite sign, + sign, for z.

We now move to the gravitino variation $\delta\psi_\mu^I$. The BPS condition gives an equation for the warp factor $A(r)$:

$$\begin{aligned} \frac{dA}{dr} &= -\frac{1}{f^2(g_{2s}+(g_n g_p-g_n^2 f)c_1)^2}[16g_n(f^2-1)(3f^2(c_1 g_n(g_n^2+g_p^2)+g_{2s}g_p)\\ &-2g_n f^3(2c_1 g_n g_p+g_{2s})-2g_n f(2c_1 g_n g_p+g_{2s})+c_1 g_n^3 f^4\\ &+g_p(c_1 g_n g_p+g_{2s}))]. \end{aligned} \tag{3.54}$$

Changing the variable from r to f, we find

$$\frac{dA}{df} = \frac{1}{fg_n+g_p}\left[\frac{g_p+f(3fg_p+g_n(3+f^2))}{f(f^2-1)}-\frac{g_{2s}g_n}{g_{2s}+g_n(fg_n+g_p)c_1}\right]. \tag{3.55}$$

This can be solved and give

$$A = c_3+\ln f-2\ln(1-f^2)+\ln(g_p+fg_n)+\ln(g_{2s}+g_n(g_p+g_n f)c_1). \tag{3.56}$$

The constant c_3 can be set to zero by rescaling coordinates x^0 and x^1. We require that A_1 must not change sign along the flow, so these are the only two possible flows namely the flow between I and III critical points and between II and III points. We choose the value of c_1 in such a way that the solution goes to one critical point at one end and to the other critical point at the other end. In order to identify the UV point with $r = \infty$ and the IR point with $r = -\infty$, we choose $g_{2s} < 0$ in the followings.

In the flow between I and III critical points, we chose $c_1 = -\frac{g_{2s}}{g_n(g_n+g_p)}$, $g_n g_p < 0$ and obtain

$$\begin{aligned}
b &= \sqrt{-\frac{(g_n+g_p)(1+f)}{g_{2s}f^2}} \\
a &= \sqrt{-\frac{(g_n+g_p)(1+f)}{g_{2s}}} \\
r &= \frac{1}{64}\left[-\frac{2g_{2s}}{(1+f)(g_n^2-g_p^2)} - \frac{g_{2s}\ln(1-f)}{(g_n+g_p)^2}\right. \\
&\quad \left.+\frac{g_{2s}\ln(1+f)}{(g_n-g_p)^2} - \frac{4g_{2s}g_ng_p\ln(fg_n+g_p)}{(g_n^2-g_p^2)^2}\right] \\
A &= \ln f - \ln(1-f) - 2\ln(1+f) + \ln(g_p+fg_n)
\end{aligned} \tag{3.57}$$

where we have absorbed all the constants in c_3 for the last equation. We see that $A \to \infty$ at $f = 1$ and $A \to -\infty$ at $f = -\frac{g_p}{g_n}$. In the dual CFT, the I point corresponds to the UV fixed point while the III point corresponds to the IR point. The ratio of the central charges is given by

$$\frac{c_{UV}}{c_{IR}} = -\frac{(g_n-g_p)^2}{4g_ng_p}. \tag{3.58}$$

It is easy to show that this is always greater than 1 as it should.

The flow between II and III are given by $c_1 = \frac{g_{2s}}{g_n(g_n-g_p)}$, and $g_ng_p > 0$. We find that

$$\begin{aligned}
b &= \sqrt{\frac{(g_n-g_p)(f-1)}{g_{2s}f^2}} \\
a &= \sqrt{\frac{(g_n-g_p)(f-1)}{g_{2s}}} \\
r &= \frac{1}{64}\left[\frac{2g_{2s}}{(1-f)(g_n^2-g_p^2)} - \frac{g_{2s}\ln(1-f)}{(g_n+g_p)^2}\right. \\
&\quad \left.+\frac{g_{2s}\ln(1+f)}{(g_n-g_p)^2} - \frac{4g_{2s}g_ng_p\ln(fg_n+g_p)}{(g_n^2-g_p^2)^2}\right] \\
A &= \ln f - 2\ln(1-f) - \ln(1+f) + \ln(g_p+fg_n).
\end{aligned} \tag{3.59}$$

In this case, we see that $A \to \infty$ at $f = -1$ and $A \to -\infty$ at $f = -\frac{g_p}{g_n}$. In the dual CFT, the II point corresponds to the UV fixed point while the III point corresponds to the IR point. The ratio of the central charges is given by

$$\frac{c_{UV}}{c_{IR}} = \frac{(g_n+g_p)^2}{4g_ng_p}. \tag{3.60}$$

Again,this agrees with the c-theorem.

We next compute the scalar mass spectrum for the eight scalars. We parametrize the eight scalars as follow:

$$\begin{aligned} a_1(r) &= a_{10}e^{s_1(r)} \qquad a_2(r) = a_{20}e^{s_2(r)} \\ a_3(r) &= a_{30}e^{s_3(r)} \qquad a_4(r) = a_{40}e^{s_4(r)} \\ b_1(r) &= a_{50}e^{s_5(r)} \qquad b_2(r) = a_{60}e^{s_6(r)} \\ b_3(r) &= a_{70}e^{s_7(r)} \qquad b_4(r) = a_{80}e^{s_8(r)} \end{aligned} \tag{3.61}$$

where all the s_i, $i = 1, \ldots 8$ are canonically normalized scalars. From the scalar mass matrix M^2, we can find the conformal dimensions (Δ) of the operators in the dual CFT by using the relation

$$\Delta(\Delta - 2) = m^2 L^2. \tag{3.62}$$

We find the following mass matrices.

- $f = 1$:

$$M^2 = \frac{2048(g_n + g_p)^4}{g_{2s}^2} \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{3.63}$$

  The eigenvalues of M^2L^2 are (3, -1, -1, -1, 0, 0, 0, 0) corresponding to $\Delta = (3, 1, 2)$. All the eight eigenvectors are given by

$$\begin{aligned} v_1 &= (1,1,1,1,1,1,1,1) \qquad v_2 = (-1,0,0,1,-1,0,0,1) \\ v_3 &= (-1,0,1,0,-1,0,1,0) \qquad v_4 = (-1,1,0,0,-1,1,0,0) \\ v_5 &= (0,0,0,-1,0,0,0,1) \qquad v_6 = (0,0,-1,0,0,0,1,0) \\ v_7 &= (0,-1,0,0,0,1,0,0) \qquad v_8 = (-1,0,0,0,1,0,0,0). \end{aligned} \tag{3.64}$$

  Our flow corresponds to the combination $v_2 + v_3 + v_4$ which has eigenvalue -1, $\Delta = 1$. This is consistent with the fact that the flow is driven by a relevant operator.

- $f = -1$:

$$M^2 = \frac{2048(g_n - g_p)^4}{g_{2s}^2} \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{3.65}$$

The eigenvalues of M^2L^2 are (3, -1, -1, -1, 0, 0, 0, 0) corresponding to $\Delta = (3, 1, 2)$. All the eight eigenvectors are given by

$$
\begin{aligned}
u_1 &= (1,1,1,1,1,1,1,1) \qquad u_2 = (-1,0,0,1,-1,0,0,1) \\
u_3 &= (-1,0,1,0,-1,0,1,0) \qquad u_4 = (-1,1,0,0,-1,1,0,0) \\
u_5 &= (0,0,0,-1,0,0,0,1) \qquad u_6 = (0,0,-1,0,0,0,1,0) \\
u_7 &= (0,-1,0,0,0,1,0,0) \qquad u_8 = (-1,0,0,0,1,0,0,0).
\end{aligned} \tag{3.66}
$$

As in the previous case, the flow ansatz is the combination $u_2 + u_3 + u_4$ which has eigenvalue -1, $\Delta = 1$ and corresponds to a relevant operator.

- $f = -\frac{g_p}{g_n}$:

$$
M^2 = \frac{256(g_n^2 - g_p^2)^4}{(g_{2s}g_ng_p)^2} \begin{pmatrix}
\frac{3}{2} & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 \\
0 & \frac{3}{2} & 0 & 0 & 0 & -\frac{1}{2} & 1 & 1 \\
0 & 0 & \frac{3}{2} & 0 & 0 & 1 & -\frac{1}{2} & 1 \\
0 & 0 & 0 & \frac{3}{2} & 0 & 1 & 1 & -\frac{1}{2} \\
\frac{3}{2} & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 \\
0 & -\frac{1}{2} & 1 & 1 & 0 & \frac{3}{2} & 0 & 0 \\
0 & 1 & -\frac{1}{2} & 1 & 0 & 0 & \frac{3}{2} & 0 \\
0 & 1 & 1 & -\frac{1}{2} & 0 & 0 & 0 & \frac{3}{2}
\end{pmatrix}. \tag{3.67}
$$

The eigenvalues of M^2L^2 are (3, 3, 3, 3, 0, 0, 0, 0) corresponding to $\Delta = (3, 2)$. All the eight eigenvectors are given by

$$
\begin{aligned}
w_1 &= (0, \frac{2}{3}, \frac{2}{3}, -\frac{1}{3}, 0,0,0,1) \qquad w_2 = (0, \frac{2}{3}, -\frac{1}{3}, \frac{2}{3}, 0,0,1,0) \\
w_3 &= (0, -\frac{1}{3}, \frac{2}{3}, \frac{2}{3}, 0,1,0,0) \qquad w_4 = (1,0,0,0,1,0,0,0) \\
w_5 &= (0, -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}, 0,0,0,1) \qquad w_6 = (0, -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}, 0,0,1,0) \\
w_7 &= (0, \frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}, 0,1,0,0) \qquad w_8 = (-1,0,0,0,1,0,0,0).
\end{aligned} \tag{3.68}
$$

Our flow corresponds to the combination $w_1 + w_2 + w_3 - 3w_4$ which has eigenvalue 3, $\Delta = 3$. This is consistent with the fact that at the IR, the operator must be irrelevant.

We also compute the mass spectrum for the full scalar manifold. Using gauge transformation, we are left with twenty scalars. At the UV points $f = \pm 1$, six of the extra twelve scalars have $M^2L^2 = -\frac{1}{4}$, and the other six are massless. At the IR point $f = -\frac{g_p}{g_n}$, out of the extra twelve scalars, there are six massless scalars and six scalars with $M^2L^2 = \frac{3}{4}$.

The behavior of the scalars at large r is given by the linearized equations

$$
\begin{aligned}
\frac{da}{dr} &= \frac{8a_0}{b_0}[2a(r)(2a_0^2(b_0^2g_{2s}+g_n)+3a_0b_0g_p+b_0^2g_n)+b(r)(a_0^2(b_0^2g_{2s}-g_n)+b_0^2g_n) \\
&\quad +a_0^2b_0^2g_{2s}+a_0^2g_n+2a_0b_0g_p+b_0^2g_n] \\
\frac{db}{dr} &= \frac{8}{b_0}[3a(r)(a_0^3(b_0^2g_{2s}-g_n)+a_0b_0^2g_n)+2b_0^2b(r)(a_0^3g_{2s}+3a_0g_n+3b_0g_p) \\
&\quad +a_0^3b_0^2g_{2s}-a_0^3g_n+3a_0b_0^2g_n+2b_0^3g_p]
\end{aligned} \tag{3.69}
$$

where a_0 and b_0 are the values of $a(r)$ and $b(r)$ at the critical point. For the UV ($r \to \infty$) point, $f = 1$ and $f = -1$, we find

$$a(r) \sim e^{-r/L}, \qquad b(r) \sim e^{-r/L}. \tag{3.70}$$

For the IR point ($r \to -\infty$), we find

$$a(r) \sim e^{r/L}, \qquad b(r) \sim e^{r/L}. \tag{3.71}$$

According to the discussion in the previous section, we find that in our flow, the first term of the scalar fluctuation given in (3.18) is absent, so there is no source term. The flow is therefore of the so-called vev flow, corresponding to the deformation of the UV theory by an expectation value of an operator of dimension one. Near the IR point, the scalar behaves as $e^{(\Delta-2)r/L}$. We then find that, in the IR, the corresponding operator is irrelevant with dimension 3.

3.2.3 The Flow Between $(2,0)$ Vacua

Now, we consider the flow between IV and V critical points.

We begin by giving the ansatz for e_1 and e_2,

$$
\begin{aligned}
e_1 &= \sqrt{\frac{2(g_p-g_n)}{g_{2s}}}\begin{pmatrix} x(r) & 0 & 0 & 0 \\ 0 & q(r) & 0 & 0 \\ 0 & 0 & z(r) & 0 \\ 0 & 0 & 0 & z(r) \end{pmatrix} \\
e_2 &= \sqrt{\frac{2(g_p-g_n)}{g_{2s}}}\begin{pmatrix} y(r) & 0 & 0 & 0 \\ 0 & y(r) & 0 & 0 \\ 0 & 0 & w(r) & 0 \\ 0 & 0 & 0 & w(r) \end{pmatrix}.
\end{aligned} \tag{3.72}
$$

Consistency condition for the BPS equations requires

$$x = -\frac{(g_n+g_p)y^2}{q(g_n+g_p-2g_ny^2)} \tag{3.73}$$

$$w = \sqrt{\frac{g_n+g_p}{g_n+g_p+2g_pz^2}}\,z. \tag{3.74}$$

The $\delta\chi^{iI}$ equations give

$$\frac{dz}{dr} = \frac{1}{g_{2s}(g_n+g_p)q^2y^2(g_n+g_p-2g_ny^2)}\{8(g_n+g_p)z^3(2q^2y^2(2(g_n^3-g_ng_p^2)y^4 +(2g_p^3+6g_ng_p^2-4g_n^3)y^2+(g_n-2g_p)(g_n+g_p)^2)+g_nq^4(g_n+g_p-2g_ny^2)^2 +g_n(g_n+g_p)^2y^4)\} \quad (3.75)$$

$$\frac{dy}{dr} = \frac{8y(g_n+g_p-2g_ny^2)}{g_{2s}(g_n+g_p)}\Bigg\{-\frac{2(g_n+g_p)y^2}{g_n+g_p}\Bigg((g_n-g_p)^2z^2-2g_n^2+3g_ng_p-g_p^2 +\frac{(g_n-g_p)^2(g_n+g_p)z^2}{g_n+g_p+2g_pz^2}\Bigg)+\frac{g_p(g_p-g_n)(g_n+g_p)^2y^4}{q^2(g_n+g_p-2g_ny^2)^2}+g_p(g_p-g_n)q^2\Bigg\} \quad (3.76)$$

$$\frac{dq}{dr} = -\frac{8(g_n-g_p)q(g_n+g_p-2g_ny^2)}{g_{2s}(g_n+g_p)y^2}\Bigg\{\frac{(g_n+g_p)^2y^4}{q^2(g_n+g_p-2g_ny^2)}\Bigg(g_nz^2-g_py^2 +\frac{g_n(g_n+g_p)z^2}{g_n+g_p+2g_pz^2}\Bigg)+q^2\Bigg(\frac{2(g_n+g_p)^2y^4(g_p-2g_n+(g_n-g_p)z^2)}{q^2(g_n+g_p-2g_ny^2)^2} +\frac{(g_n+g_p)z^2}{g_n+g_p+2g_pz^2}\Big(\frac{2(g_n-g_p)(g_n+g_p)^2y^4}{q^2(g_n+g_p-2g_ny^2)}-g_n\Big)-g_nz^2+g_py^2\Bigg) +\frac{2g_p(g_n+g_p)q^2y^2}{g_n+g_p-2g_ny^2}\Bigg\}. \quad (3.77)$$

This flow ansatz preserves (2,0) supersymmetry along the entire flow. We now change the variables to z_1, h, and p

$$y = \sqrt{\frac{g_n+g_p}{2g_n(1+z_1)}} \quad (3.78)$$

$$z = \sqrt{\frac{g_n+g_p}{2g_ph}} \quad (3.79)$$

$$q = \sqrt{-\frac{(g_n+g_p)\sqrt{p^2-4}}{g_nz_1(p^2-4+p\sqrt{p^2-4})}} \quad (3.80)$$

and rescale r to $r\frac{8(g_n^2-g_p^2)}{g_{2s}g_ng_p}$. The final forms of (3.75), (3.76), and (3.77) are

$$\frac{dz_1}{dr} = \frac{(g_n^2-g_p^2-h(g_p^2p-2g_n(g_n-2g_p)+g_ph(4g_n-2g_p+g_pp)))}{h(h+1)} \quad (3.81)$$

$$\frac{dh}{dr} = \frac{g_n^2-g_p^2+z_1(g_n^2p(1+z_1)-2(g_p(g_p-2g_n)+g_n(g_n-2g_p)z_1))}{z_1(1+z_1)} \quad (3.82)$$

$$\frac{dp}{dr} = -(p^2-4)\left[g_n^2\left(\frac{1}{h}+\frac{1}{1+h}\right)-\frac{g_p^2}{z_1}-\frac{g_p^2}{1+z_1}\right]. \quad (3.83)$$

We proceed by taking p as an independent variable and obtain

$$\frac{dz_1}{dp} = \frac{(g_p^2 - g_n^2 + (g_p^2 p + 4g_n g_p - 2g_n^2)h + g_p(4g_n + g_p(p-2))h^2)z_1(1+z_1)}{(p^2-4)(g_n^2(1+2h)z_1(1+z_1) - g_p^2 h(1+h)(1+2z_1))} \tag{3.84}$$

$$\frac{dh}{dp} = \frac{h(1+h)(g_p^2 - g_n^2 + 2g_p(g_p - 2g_n)z_1 + 2g_n(g_n - 2g_p)z_1^2 - g_n^2 p z_1(1+z_1))}{(p^2-4)(g_n^2(1+2h)z_1(1+z_1) - g_p^2 h(1+h)(1+2z_1))}$$
$$. \tag{3.85}$$

Recall that $g_n = tg_p$, we find that the two critical points are given by

- IV:

$$p = -2, \qquad h = \frac{1}{4}(t - 1 + \sqrt{5 + 2t + t^2}),$$
$$\text{and} \qquad z_1 = \frac{1 - t + \sqrt{1 + 2t + 5t^2}}{4t}, \tag{3.86}$$

  and

- V:

$$p = 2 - \frac{2}{t} - 2t, \qquad h = \frac{1}{2}(t - 1 + \sqrt{1+t^2}),$$
$$\text{and} \qquad z_1 = \frac{1 - t + \sqrt{1+t^2}}{2t}. \tag{3.87}$$

We now give the numerical solution. Choosing $t = 2$, we find the numerical values for the critical points

$$\begin{array}{llll} \text{IV}: & p = -2.000, & h = 1.151, & z_1 = 0.500 \\ \text{V}: & p = -3.000, & h = 1.618, & z_1 = 0.309. \end{array} \tag{3.88}$$

The numerical solutions for the flow are given in Fig.3.1 and Fig.3.2.

The gravitino variation gives an equation for $A(p)$, with $t = 2$,

$$\begin{aligned} \frac{dA}{dp} = & -\frac{8g_p^2[(p^2-2)\sqrt{p^2-4} + p^3 - 4p]}{\sqrt{p^2-4}(p\sqrt{p^2-4} + p^2 - 4)^2} \times \\ & [(p+6)h(p)^2(2z_1(p)+1) + ph(p)(1 - 2z_1(p)(4z_1(p)+3)) \\ & -2z_1(p)(2pz_1(p) + 2p + 3) - 3]/[-4(2h(p)+1)z_1(p)^2 \\ & +2((h(p)-3)h(p) - 2)z_1(p) + h(p)(h(p)+1)]. \end{aligned} \tag{3.89}$$

Choosing $g_{2s} = -1$ and $g_p = 1$, we find the numerical solution for A as shown in Fig.3.3.

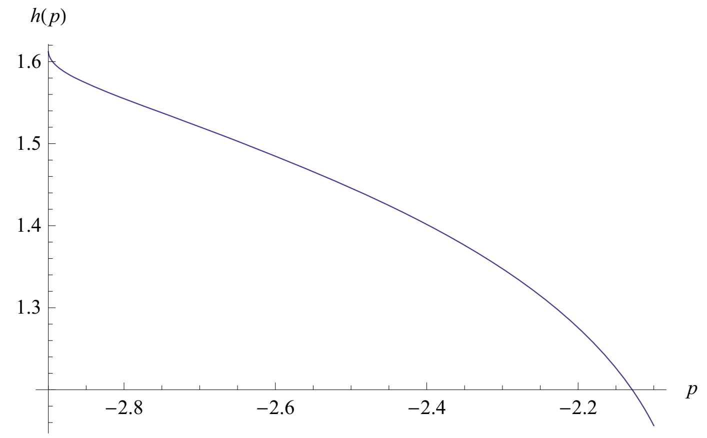

Figure 3.1: $h(p)$ solution.

In this flow, the point IV is the UV fixed point, and V is the IR. The ratio of the central charges is

$$\frac{c_{UV}}{c_{IR}} = \frac{(g_n + g_p)^2}{4g_n g_p}. \tag{3.90}$$

This ratio is greater than 1 in consistent with the c-theorem. We also compute the scalar mass matrices at both critical points, but the form of the matrices is too complicated to be written here. We give only the numerical values of the eigenvalues in our choice of $g_p = 1$, $g_n = 2$ and $g_{2s} = -1$.

- IV: Eigenvalues of M^2L^2 are $(3.70, -1.00, -1.00, -0.97, 0.36, 0.36, 0.00, 0.00)$ with eigenvectors

$$\begin{aligned}
U_1 &= (-0.47, -0.47, -0.44, -0.44, -0.16, -0.16, -0.24, -0.24) \\
U_2 &= (0.33, -0.33, 0.44, -0.44, 0.00, 0.00, 0.44, -0.44) \\
U_3 &= (0.63, -0.63, -0.23, 0.23, 0.00, 0.00, -0.23, 0.23) \\
U_4 &= (0.47, 0.47, -0.44, -0.44, 0.16, 0.16, -0.24, -0.24) \\
U_5 &= (0.00, 0.00, -0.49, 0.49, -0.14, 0.14, 0.49, -0.49) \\
U_6 &= (0.00, 0.00, -0.10, 0.10, 0.69, -0.69, 0.10, -0.10) \\
U_7 &= (0.22, 0.22, -0.06, -0.06, -0.66, -0.66, 0.11, 0.11) \\
U_8 &= (-0.04, -0.04, -0.33, -0.33, 0.12, 0.12, 0.61, 0.61).
\end{aligned} \tag{3.91}$$

Our flow ansatz corresponds to U_4 with $\Delta = 1.168$ which is dual to a relevant operator. Note also that, our ansatz does not correspond to the one which

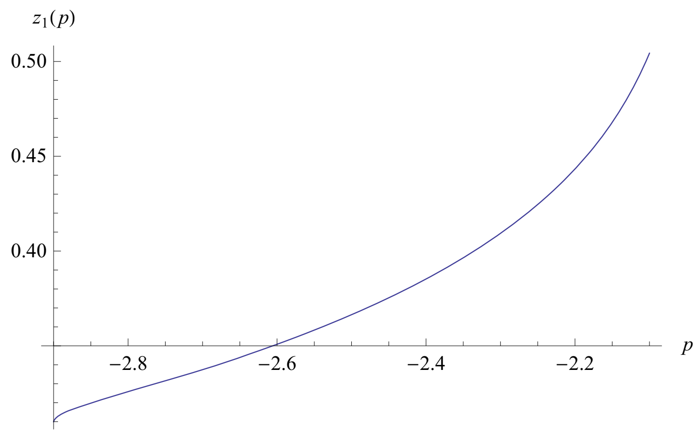

Figure 3.2: $z_1(p)$ solution.

saturates the bound $M^2L^2 = -1$. This means the dual operator is not the most relevant one.

- V: Eigenvalues of M^2L^2 are $(4.17, 3.33, 3.33, 3.33, 0.84, 0.84, 0.84, 0.00)$ with eigenvectors

$$\begin{aligned}
V_1 &= (-0.211, -0.894, -0.211, -0.211, -0.130, -0.130, -0.130, -0.130) \\
V_2 &= (0.201, -0.031, 0.063, -0.159, -0.609, 0.001, 0.090, 0.742) \\
V_3 &= (0.390, -0.398, 0.523, 0.432, 0.400, 0.011, -0.103, 0.241) \\
V_4 &= (-0.293, 0.159, 0.015, -0.258, 0.359, -0.004, -0.801, 0.227) \\
V_5 &= (0.255, 0.002, -0.712, 0.572, -0.039, 0.086, -0.300, 0.046) \\
V_6 &= (-0.146, 0.011, 0.387, 0.287, -0.526, 0.391, -0.384, -0.411) \\
V_7 &= (0.757, 0.004, -0.047, -0.499, 0.007, 0.156, -0.207, -0.328) \\
V_8 &= (0.130, 0.130, 0.130, 0.130, -0.211, -0.893, -0.211, -0.211). \quad (3.92)
\end{aligned}$$

  Our flow ansatz corresponds to V_1 with $\Delta = 3.275$ which is dual to an irrelevant operator.

The behavior near $r \to \infty$ can be obtained as in the previous case. With $g_p = 1$, $g_n = 2$, and $g_{2s} = -1$, we find that

$$p(r) \sim e^{-2r/L}, \qquad z_1(r),\ h(r) \sim e^{-1.168r/L}. \quad (3.93)$$

At the IR point, we find

$$z_1(r),\ h(r),\ p(r) \sim e^{1.275r/L}. \quad (3.94)$$

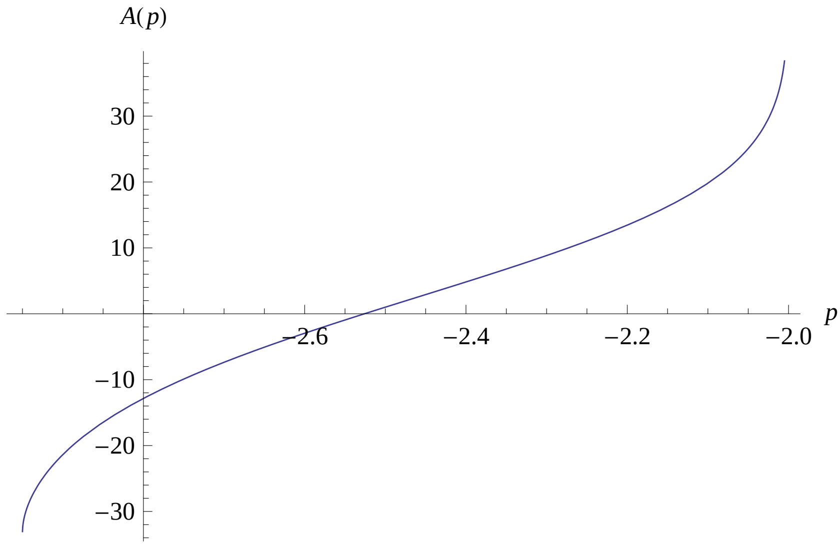

Figure 3.3: $A(p)$ solution.

From the dominant term near the UV fixed point, we see that the flow solution describes the deformation of the UV theory by a vacuum expectation value of an operator of dimension 1.168. We find that this flow is also a vev flow. The corresponding operator in the IR theory is an irrelevant operator of dimension 3.275.

In this chapter, we have studied $N = 4$ three dimensional gauged supergravities and their AdS_3 supersymmetric vacua. We have found analytic solutions interpolating between two $(1, 3)$ vacua. These solutions describe Renormalization Group flows between two fixed points of the dual boundary field theory. We have checked that the flows agree with the c-theorem, in particular the central charges of UV fixed points are strictly greater than those of the IR ones. We have also found a numerical solution describing the flow between (2,0) vacua with similar qualitative features. In both cases, we found vev flows, i.e. flows driven by vacuum expectation values of relevant operators with dimensions $\Delta = 1$ and $\Delta = 1.168$, respectively, as opposed to the most common case where the flow is driven by a perturbing relevant operator. It is possible to uplift the solutions given here to higher dimensions. All that remains is to find a consistent reduction ansatz. In the next chapter, we will give a first step to proceed in this direction. As we will see, a class of $N = 4$ Chern-Simons gauged supergravity with target space $\frac{SO(4,n)}{SO(4)\times SO(n)}$ can be obtained from the $SU(2)$ group manifold reduction of (1,0) six dimensional supergravity.

Chapter 4

$N = 4$ three dimensional gauged supergravity from $SU(2)$ reduction

In this chapter, we will perform a dimensional reduction of (1,0) six dimensional supergravity on the $SU(2)$ group manifold. This results in the $N = 4$ gauged supergravity in three dimensions whose flows solutions have been studied in the previous chapter. However, the theory obtained in this chapter is not exactly the same as the one studied in chapter 3. The reduced theory has $SU(2) \times SU(2) \sim SO(4)$ gauge group but only one quaternionic target space as opposed to the product of two quaternionic spaces in chapter 3. However, this result is a good starting point toward the full theory with two quaternionic target spaces. The resulting theory is also interesting in the sense that we can successfully decouple massive vector fields usually arise from the reduction of two-form fields in six dimensions. So, our theory is described by pure $N = 4$ gauged supergravity in three dimensions.

We begin with a notion of group manifold reduction particularly on $SU(2)$ group manifold. Then, we introduce our starting point, the (1,0) six dimensional supergravity coupled to an antisymmetric tensor multiplet as well as Yang-Mills multiplets of an arbitrary gauge group G. Since this theory will be used extensively in the next two chapters, we review it in sufficient detail. We will proceed by performing the $SU(2)$ reduction, writing down the three dimensional action and giving diagonalized fermion kinetic terms in three dimensions. We end this chapter by constructing $N = 4$ Chern-Simons gauged supergravity whose construction has been reviewed in chapter 2 and finally show that the theory obtained from dimensional reduction is indeed equivalent to the Chern-Simons theory as mentioned at the end of chapter 2.

4.1 $SU(2)$ reduction and (1,0) six dimensional supergravity

In dimensional reductions, the higher dimensional spacetime is decomposed into a product of the lower dimensional spacetime and an internal space. Our main aim is to obtain $N = 4$ gauged supergravity in three dimensions from higher dimensional theories. Recall that the relevant gauge group in our case is $SO(4)$ not the Chern-Simons gauge group $SO(4) \ltimes \mathbf{T}^6$ since the theory obtained from a dimensional reduction is always in the form of Yang-Mills gauged theory. Accordingly, we would expect to obtain this theory by a dimensional reduction in which the internal space contains an S^3 factor since we can identify the gauge group $SO(4)$ as the isometry of the S^3. The simplest possibility of the internal space is the S^3 itself. This leads us to consider the reduction of the minimal six dimensional supergravity on S^3. Minimal supersymmetry, $N = (1,0)$, in six dimensions has eight supercharges which precisely gives $N = 4$ supersymmetry in three dimensions. For the case in which the gauge group $SO(4)$ is identified with the isometry of the S^3, all six gauge fields of $SO(4)$ are kept in the reduction process. In the language of [102], this is called consistent sphere reduction. However, it has been argued in [47] that the consistent sphere reduction of the minimal supergravity in six dimensions is not possible to obtain a reduced theory admitting AdS_3 vacua. The resulting three dimensional theory is clearly not what we need since AdS_3 vacua are the important ingredient in our study. The resolution to this problem is to perform instead the reduction on an $SU(2)$ group manifold which is topologically S^3. We will call this reduction $SU(2)$ reduction from now on. The consistent reduction can be obtained by keeping only fields which are singlet under left action of $SU(2)$. This means that we can consistently keep only $SU(2)$ gauge fields not the full $SO(4)$ gauge fields. To obtain $SO(4)$ gauged supergravity, our strategy is to begin with a supergravity theory coupled to $SU(2)$ gauge fields in six dimensions. The resulting three dimensional theory will become $SU(2) \times SU(2) \sim SO(4)$ gauged supergravity.

We now review the two main ingredients we need in this chapter, the group manifold reduction and the minimal six dimensional supergravity coupled to an antisymmetric tensor and Yang-Mills gauge fields of an arbitrary gauge group G. It turns out that it is possible to obtain a larger class of three dimensional gauged supergravities with gauge group $SU(2) \times G$. So, we consider (1,0) six dimensional supergravity coupled to G Yang-Mills multiplets and then set $G = SU(2)$ to obtain the $SU(2) \times SU(2)$ gauged supergravity mentioned above.

4.1.1 Group manifold reductions

In the usual Kaluza-Klein dimensional reduction, the fields do not depend on the coordinates of the internal space. There is a generalization of this reduction in which we allow the fields to depend on the internal coordinates in such a way that these dependencies cancel in the lower dimensional field equations as well as in the lower dimensional action. This is sometimes called generalized dimensional reduction [103]. One of the new features in this generalized dimensional reduction of higher dimensional supergravities is that we obtain gauged supergravity theories in lower dimensions. The group structure arises from the coordinate transformations in the internal space. The internal space is then in the form of a group manifold, and the reduction is called group manifold reduction or Sherk Schwarz reduction or DeWitt reduction.

We now briefly review the reduction studied in [103]. We also refer the reader to [103, 104, 105] for more details. The D dimensional coordinates z^M are decomposed into (x^μ, y^α) describing the lower dimensional spacetime and the internal space. We choose the D dimensional diffeomorphism parameters ξ^M to be

$$\xi^\mu(x,y) = \xi^\mu(x), \qquad \xi^\alpha(x,y) = (U^{-1}(y))^\alpha_\beta \xi^\beta(x)\,. \tag{4.1}$$

In the usual reductions, the matrix U becomes an identity matrix. The y dependence is encoded entirely in the this matrix. We now show how the group structure of the internal manifold arises from the internal diffeomorphism. A commutator of two transformations with parameters ξ_1 and ξ_2 gives a new transformation with parameter

$$\xi^M = \xi_2^N \partial_N \xi_1^M - \xi_1^N \partial_N \xi_2^M\,. \tag{4.2}$$

The commutator of the transformations with parameters ξ^μ and ξ^ν will give the lower dimensional diffeomorphism as usual while the commutator of ξ^μ and ξ^α transformations gives another internal transformation. The new feature arises in considering the commutator of ξ^α and ξ^β transformations. We find

$$\begin{aligned}
\xi^\alpha &= \xi_2^M(x,y)\partial_M \xi_1^\alpha(x,y) - \xi_1^M(x,y)\partial_M \xi_2^\alpha(x,y) \\
&= (U^{-1})^{\alpha'}_\beta (U^{-1})^{\beta'}_\gamma (\partial_{\beta'} U^\alpha_{\alpha'} - \partial_{\alpha'} U^\alpha_{\beta'})\xi_1^\beta(x)\xi_2^\gamma(x) \\
&= f^\alpha{}_{\beta\gamma}\xi_1^\beta(x)\xi_2^\gamma(x)\,.
\end{aligned} \tag{4.3}$$

The consistency of the reduction requires that the y dependence of the matrix U cancel to give the constants $f^\alpha{}_{\beta\gamma}$. These constants will be identified with the group structure constants. The group generators can be constructed by

$$L_\alpha = (U^{-1})^\beta_\alpha \partial_\beta \tag{4.4}$$

which satisfy the algebra

$$[L_\alpha, L_\beta] = f^\gamma{}_{\alpha\beta} L_\gamma\,. \tag{4.5}$$

All the fields carrying internal indices will get a y dependence through U. For example, the metric g_{MN} is decomposed into $g_{\mu\nu}$, $g_{\mu\alpha} \sim A_{\alpha\mu}$ and $g_{\alpha\beta} \sim H_{\alpha\beta}$. We now have a reduction ansatz

$$\begin{aligned}
g_{\mu\nu}(x,y) &= g_{\mu\nu}(x), \\
A^\alpha_\mu(x,y) &= (U^{-1})^\alpha_\beta A^\beta_\mu(x), \\
g_{\alpha\beta}(x,y) &= U^{\alpha'}_\alpha U^{\beta'}_\beta H_{\alpha'\beta'}(x)\,.
\end{aligned} \tag{4.6}$$

This prescription is extended to other fields as well, see [103]. The whole process can be further generalized by introducing the y dependence through the tangent space transformations of internal space, internal tangent space rotations. All fields with internal tangent space indices for example the vielbein will pick up a y dependence. We will not proceed in this direction further since the above discussion is sufficient for our proposes, see [105] for more detail.

On a G group manifold of dimension $\dim G$, the isometries on the manifold are given by the left and right actions of the group G. These actions correspond to right and left translations, respectively. The full isometry can be written as $G_L \times G_R$. We can also define left (L) and right (R) invariant one-forms given by

$$L = \sigma^\alpha T_\alpha = g^{-1}dg, \qquad R = \rho^\alpha T_\alpha = dgg^{-1} \tag{4.7}$$

where $g \in G$, $\alpha = 1, \ldots, \dim G$, and T_α are generators of G. In what follow, we will focus our attention on the left invariant one-form. The left invariant one-forms satisfy

$$d\sigma^\alpha = -\frac{1}{2} f^\alpha_{\ \beta\gamma} \sigma^\beta \wedge \sigma^\gamma \tag{4.8}$$

where $f^\alpha_{\ \beta\gamma}$ are G structure constants. In the group manifold reduction, it is consistent to keep only fields which are invariant under G_L. This consistency is due to the fact that the G_L singlet fields cannot form non singlet representations of G, so they cannot act as sources for the non singlet fields. The equations of motion for the non singlet fields are then automatically satisfied with non singlet fields being zero. In the next section, we will make a group manifold reduction with $G = SU(2) \sim S^3$. Notice that if we work with the right invariant one-form, equation (4.8) will be replaced by

$$d\rho^\alpha = \frac{1}{2} f^\alpha_{\ \beta\gamma} \rho^\beta \wedge \rho^\gamma \, . \tag{4.9}$$

We can also express the σ^α in the coordinate basis as $\sigma^\alpha = U^\alpha_\beta dy^\beta$.

4.1.2 Minimal (1,0) six dimensional supergravity

Six dimensional gauged supergravity coupled to an antisymmetric tensor and Yang-Mills multiplets has been constructed in [62]. The theory has been generalized, to coupled to n_T tensor multiplets, n_V vector multiplets and n_H hypermultiplets in [63], see also [106, 107]. The complete minimal supergravity in six dimensions including all possible couplings to all (1,0) supermultiplets has been worked out in [108] including quartic fermion couplings. The theory of interest to us is a truncation of this theory namely the ungauged (1,0) six dimensional supergravity coupled to an antisymmetric tensor multiplet and Yang-Mills multiplets of gauge group G. We then choose $n_T = 1$, $n_V = \dim G$ and $n_H = 0$. In this case, the theory can admit a Lagrangian formulation. For $n_T \neq 1$, it is not possible to have a covariant

Lagrangian without including auxiliary fields since the two form field in the supergravity multiplet has selfdual field strength while the antisymmetric tensor field in the tensor multiplet comes with anti-selfdual field strength. Only for $n_T = 1$, we can write down an invariant kinetic term for the two form field. The (1,0) six dimensional supergravity is also interesting in the study of string theory and has been obtained from various compactifications of string and M theories see for example [109], [110]. These compactifications have been used to study many aspects of string dualities in six dimensions [111], [112].

We follow the notation of [63] with the metric signature $(-+++++)$. The theory considered here contains $N = 1$ supergravity multiplet, one antisymmetric tensor multiplet and $\dim G$ Yang-Mills multiplets of an arbitrary gauge group G. We also assume that the group G commutes with the $SU(2) \sim Sp(1)$ R-symmetry group. The field content in this case is given by the graviton $e_M^{\hat{M}}$, gravitino ψ_M^A, third rank anti-symmetric tensor G_{3MNP} which is the field strength of the 2-form field b_{MN}, scalar θ, spin $\frac{1}{2}$ fermion χ, G gauge fields A_M^I with $I = 1, 2, \ldots, \dim G$ and the G gauginos λ^I. The six dimensional spacetime indices are $M, N = 0, \ldots 5$ with the tangent space indices $\hat{M}, \hat{N} = 0, \ldots 5$ while $A, B = 1, 2$ are $Sp(1)$ R-symmetry indices. The scalar θ parametrizes the coset space $SO(1,1)$ which is a special case of $\frac{SO(n_T,1)}{SO(n_T)}$ for n_T tensor multiplets. In the presence of gauge fields, anomaly cancelation requires the modifications of the Bianchi identity for G_3 as well as the G_3 equation of motion. These modifications are characterized by two types of parameters v^z and $\tilde{v}^z$. The index z label various components of the gauge group which can be a product of simple groups. In this thesis, we need only bosonic field equations and all supersymmetry transformations. The bosonic field equations are given by [63]

$$\begin{aligned} R_{MN} - \frac{1}{2}g_{MN}R - \frac{1}{3}e^{2\theta}\left(3G_{3MPQ}G_{3N}{}^{PQ} - \frac{1}{2}g_{MN}G_{3PQR}G_3{}^{PQR}\right) &- \partial_M\theta\partial^M\theta \\ +\frac{1}{2}g_{MN}\partial_P\theta\partial^P\theta - C^z\mathrm{Tr}_z\left(2F_M^PF_{NP} - \frac{1}{2}g_{MN}F_{PQ}F^{PQ}\right) &= 0, \qquad (4.10) \\ e^{-1}\partial_M(eg^{MN}\partial_N\theta) - \frac{1}{2}(v^ze^{\theta} - \tilde{v}^ze^{-\theta})\mathrm{Tr}_z(F_{MN}F^{MN}) & \\ -\frac{1}{3}e^{2\theta}G_{3MNP}G_3{}^{MNP} &= 0, \qquad (4.11) \\ D_N(ee^{\theta}F^{IMN}) + e(v^ze^{2\theta}G^{MNP}F^I_{NP} - \tilde{v}^z(*G_3)^{MNP}F_{NP}) &= 0, \qquad (4.12) \\ D_M(ee^{2\theta}G_3{}^{MNP}) + \frac{1}{4}\tilde{v}^z\epsilon^{NPQRML}\mathrm{Tr}_z(F_{QR}F_{ML}) &= 0 \qquad (4.13) \end{aligned}$$

where $C^z = v^ze^{\theta} + \tilde{v}^ze^{-\theta}$. We have set hypermultiplet scalars to zero, and by our assumption on the gauge group G, the term C^{AB} disappears as it should for the ungauged theory. The modified Bianchi identity is given by

$$DG_3 = v^z\mathrm{Tr}_z(F \wedge F). \qquad (4.14)$$

We can also write (4.13) as

$$D(e^{2\theta} * G_3) = -\tilde{v}^z \mathrm{Tr}_z(F \wedge F). \qquad (4.15)$$

In these equations, the role of v^z and $\tilde{v}^z$ looks more symmetric. Supersymmetry transformations, to leading order in fermionic fields, are given by

$$\begin{aligned}
\delta e_M^{\hat{M}} &= \bar{\epsilon}\Gamma^{\hat{M}}\psi_M, \\
\delta\psi_M &= D_M\epsilon + \frac{1}{24}e^{\theta}\Gamma^{NPQ}\Gamma_M G_{3NPQ}\epsilon - \frac{1}{16}\Gamma_M\chi\bar{\epsilon}\chi - \frac{3}{16}\Gamma^N\chi\bar{\epsilon}\Gamma_{MN}\chi \\
&\quad + \frac{1}{32}\Gamma_{MNP}\chi\bar{\epsilon}\Gamma^{NP}\chi - \frac{1}{16}C_z\mathrm{Tr}_z(18\lambda\bar{\epsilon}\Gamma_M\lambda - 2\Gamma_{MN}\lambda\bar{\epsilon}\Gamma^N\lambda \\
&\quad + \Gamma_{NP}\lambda\bar{\epsilon}\Gamma_M{}^{NP}\lambda), \\
\delta b_{MN} &= 2v^z\mathrm{Tr}_z(A_{[M}\delta A_{N]}) - e^{-\theta}\bar{\epsilon}\Gamma_{[M}\psi_{N]} + \frac{1}{2}e^{-\theta}\bar{\epsilon}\Gamma_{MN}\chi, \\
\delta\theta &= \bar{\epsilon}\chi, \\
\delta\chi &= \frac{1}{2}\Gamma^M\partial_M\theta\epsilon - \frac{1}{12}e^{\theta}\Gamma^{MNP}G_{3MNP}\epsilon + \frac{1}{2}(v^z e^{\theta} - \tilde{v}^z e^{-\theta})\mathrm{Tr}_z[\Gamma^M\lambda(\bar{\epsilon}\Gamma_M\lambda)], \\
\delta A_M &= -\bar{\epsilon}\Gamma_M\lambda, \\
\delta\lambda_A &= \frac{1}{4}\Gamma^{MN}F_{MN}\epsilon_A - C_z^{-1}(v^z e^{\theta} - \tilde{v}^z e^{-\theta})\bar{\chi}_{(A}\lambda_{B)}\epsilon^B. \qquad (4.16)
\end{aligned}$$

For both v^z and $\tilde{v}^z$ non zero, there is no invariant action which can be written down only for $v^z\tilde{v}^z = 0$. We choose to work with a special case $v^z = 1$ and $\tilde{v}^z = 0$ in performing the $SU(2)$ reduction. In this case, the Lagrangian is given by [63]

$$\begin{aligned}
e^{-1}\mathcal{L} &= \frac{1}{4}R - \frac{1}{12}e^{2\theta}G_{3MNP}G_3{}^{MNP} - \frac{1}{4}\partial_M\theta\partial^M\theta - \frac{1}{2}\bar{\psi}_M\Gamma^{MNP}D_N\psi_P \\
&\quad - \frac{1}{2}\bar{\chi}\Gamma^M D_M\chi - \frac{1}{4}e^{\theta}F^I_{MN}F^{IMN} - e^{\theta}\bar{\lambda}^I\Gamma^M D_M\lambda^I \\
&\quad + \frac{1}{2}e^{\theta}\bar{\chi}\Gamma^{MN}\lambda^I F^I_{MN} + \frac{1}{2}\bar{\psi}_M\Gamma^N\Gamma^M\chi\partial_N\theta - \frac{1}{2}e^{\theta}\bar{\psi}_M\Gamma^{NP}\Gamma^M\lambda^I F^I_{NP} \\
&\quad - \frac{1}{24}e^{\theta}G_{3MNP}\big[\bar{\psi}^L\Gamma_{[L}\Gamma^{MNP}\Gamma_{Q]}\psi^Q - 2\bar{\psi}_L\Gamma^{MNP}\Gamma^L\chi - \bar{\chi}\Gamma^{MNP}\chi \\
&\quad + 2e^{\theta}\bar{\lambda}^I\Gamma^{MNP}\lambda^I\big] \qquad (4.17)
\end{aligned}$$

where $e = \sqrt{-g}$. We have also used the normalization of the G gauge generators T^I such that $\mathrm{Tr}(T^I T^J) = \delta^{IJ}$, and $A_M = A^I_M T^I$, $\lambda = \lambda^I T^I$. The three form field strength in differential form language can be written as

$$G_3 = db + F^I \wedge A^I - \frac{1}{6}g_2 f_{IJK}A^I \wedge A^J \wedge A^K \qquad (4.18)$$

where g_2 and f_{IJK} are coupling and structure constants of the gauge group G, respectively. The equations of motion for various fermions can be found in [63]. We will not repeat them here because they will not be needed in our discussion. In the next section, we will give the reduction ansatz and perform the $SU(2)$ reduction of this (1,0) six dimensional supergravity.

4.1.3 Reduction ansatz on $SU(2)$ group manifold

We now give our reduction ansatz. We will put a hat on all the six dimensional fields from now on to distinguish them from the three dimensional fields. We use the following reduction ansatz:

$$\begin{aligned} d\hat{s}^2 &= e^{2f} ds^2 + e^{2g} h_{\alpha\beta} \nu^\alpha \nu^\beta \,, \\ \hat{A}^I &= A^I + A^I_\alpha \nu^\alpha, \qquad \nu^\alpha = \sigma^\alpha - g_1 A^\alpha \,, \\ \hat{F}^I &= d\hat{A}^I + \frac{1}{2} g_2 f_{IJK} \hat{A}^J \wedge \hat{A}^K \\ &= F^I - g_1 A^I_\alpha F^\alpha + \mathcal{D} A^I_\alpha \wedge \nu^\alpha + \frac{1}{2}(g_2 A^J_\alpha A^K_\beta f_{IJK} - \epsilon_{\alpha\beta\gamma} A^I_\gamma) \nu^\alpha \wedge \nu^\beta \end{aligned} \quad (4.19)$$

where the $SU(2) \times G$ covariant derivative is given by

$$\mathcal{D} A^I_\alpha = dA^I_\alpha + g_1 \epsilon_{\alpha\beta\gamma} A^\beta A^I_\gamma + g_2 f_{IJK} A^J A^K_\alpha \,. \quad (4.20)$$

The three dimensional field strength $F^I = dA^I + \frac{1}{2} g_2 f_{IJK} A^J \wedge A^K$. From the metric, we can read off the vielbein components

$$\hat{e}^a = e^f e^a, \; \hat{e}^i = e^g L^i_\alpha \nu^\alpha \text{ with } h_{\alpha\beta} = L^i_\alpha L^i_\beta. \quad (4.21)$$

The left-invariant $SU(2)$ 1-forms σ^α satisfy

$$d\sigma^\alpha = -\frac{1}{2} \epsilon_{\alpha\beta\gamma} \sigma^\beta \wedge \sigma^\gamma \,. \quad (4.22)$$

The $\epsilon_{\alpha\beta\gamma}$ and f_{IJK} are the $SU(2)$ and G structure constants, respectively. We can also write down the σ^α in terms of the Euler angles (ψ, θ, ϕ) on S^3

$$\begin{aligned} \sigma_1 &= \cos\psi d\theta + \sin\theta \sin\psi d\phi, \\ \sigma_2 &= \cos\psi \sin\theta d\psi - \sin\psi d\theta, \\ \sigma_3 &= d\psi + \cos\theta d\phi \,. \end{aligned} \quad (4.23)$$

The metric $h_{\alpha\beta}$ and a (3×3) matrix L^i_α are unimodular. The spin connections are given by [47]

$$\begin{aligned} \hat{\omega}_{ab} &= \omega_{ab} + e^{-f}(\partial_b f \eta_{ac} - \partial_a f \eta_{bc}) \hat{e}^c + \frac{1}{2} g_1 e^{g-2f} F^i_{ab} \hat{e}^i, \\ \hat{\omega}_{ai} &= -e^{-f} P_{aij} \hat{e}^j - e^{-f} \partial_a g \hat{e}^i + e^{g-2f} F^i_{ab} \hat{e}^b, \\ \hat{\omega}_{ij} &= e^{-f} Q_{aij} \hat{e}^a + \frac{1}{2} e^{-g}(T^{kl} \epsilon_{ijl} + T^{jl} \epsilon_{ikl} - T^{il} \epsilon_{jkl}) \hat{e}^k \end{aligned} \quad (4.24)$$

where

$$\begin{aligned} P_{aij} &= \frac{1}{2}\big[(L^{-1})^\alpha_i D_a L^j_\alpha + (L^{-1})^\alpha_j D_a L^i_\alpha\big] = \frac{1}{2}(L^{-1})^\alpha_i (L^{-1})^\beta_j D_a h_{\alpha\beta}, \\ Q_{aij} &= \frac{1}{2}\big[(L^{-1})^\alpha_i D_a L^j_\alpha - (L^{-1})^\alpha_j D_a L^i_\alpha\big], \\ F^i &= L^i_\alpha F^\alpha, \qquad T^{ij} = L^i_\alpha L^j_\alpha, \qquad DL^i_\alpha = dL^i_\alpha - g_1 \epsilon_{\alpha\beta\gamma} A^\gamma L^i_\beta. \end{aligned} \quad (4.25)$$

We use the same conventions as in [47] namely

$$
\begin{aligned}
F^\alpha &= dA^\alpha + \frac{1}{2} g_1 \epsilon_{\alpha\beta\gamma} A^\beta \wedge A^\gamma, \\
DF^\alpha &= dF^\alpha + g_1 \epsilon_{\alpha\beta\gamma} A^\beta \wedge A^\gamma = 0, \\
D\nu^\alpha &= d\nu^\alpha + g_1 \epsilon_{\alpha\beta\gamma} A^\beta \wedge \nu^\gamma = -g_1 F^\alpha - \frac{1}{2} \epsilon_{\alpha\beta\gamma} \nu^\beta \wedge \nu^\gamma. \qquad (4.26)
\end{aligned}
$$

The indices $(M, \hat{M})$ reduce to (μ, a) in three dimensions while the S^3 part is described by indices (α, i). The ansatz for $\hat{G}_3$ is

$$
\begin{aligned}
\hat{G}_3 &= h\varepsilon_3 + a\epsilon_{\alpha\beta\gamma} \nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma + \epsilon_{\alpha\beta\gamma} C^\alpha \wedge \nu^\beta \wedge \nu^\gamma + H^\alpha \wedge \nu^\alpha \\
&\quad + \hat{F}^I \wedge \hat{A}^I - \frac{1}{6} g_2 f_{IJK} \hat{A}^I \wedge \hat{A}^J \wedge \hat{A}^K. \qquad (4.27)
\end{aligned}
$$

The first line in (4.27) is the $d\hat{b}$ which must be closed. This requires that

$$
H^\alpha = 2DB^\alpha - 6ag_1 F^\alpha. \qquad (4.28)
$$

We also choose the one form $C_\alpha = \frac{1}{2} A^I_\alpha A^I$ to further simplify the ansatz and truncate the vector field C^α out. Putting all together, we end up with the following $\hat{G}_3$ ansatz

$$
\begin{aligned}
\hat{G}_3 &= \tilde{h}\varepsilon_3 + \bar{F}^\alpha \wedge \nu^\alpha + \frac{1}{2} K_{\alpha\beta} \wedge \nu^\alpha \wedge \nu^\beta \\
&\quad + \frac{1}{6} \tilde{a} \epsilon_{\alpha\beta\gamma} \nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma \qquad (4.29)
\end{aligned}
$$

where $\tilde{h} = h\varepsilon_3 + \tilde{F}^I \wedge A^I - \frac{1}{6} g_2 A^I \wedge A^J \wedge A^K f_{IJK}$. We have defined the following quantities

$$
\begin{aligned}
\bar{F}^\alpha &= A^I_\alpha (\tilde{F}^I + F^I) - 6ag_1 F^\alpha, \qquad \tilde{F}^I = F^I - g_1 A^I_\alpha F^\alpha, \\
K_{\alpha\beta} &= A^I_\beta \mathcal{D} A^I_\alpha - A^I_\alpha \mathcal{D} A^I_\beta, \\
\tilde{a} &= 6a - A^I_\alpha A^I_\alpha + \frac{1}{3} g_2 A^3, \qquad A^3 \equiv A^I_\alpha A^J_\beta A^K_\gamma f_{IJK} \epsilon_{\alpha\beta\gamma}, \qquad (4.30)
\end{aligned}
$$

and a is a constant. The ansatz for the Yang-Mills fields can be rewritten as

$$
\hat{F}^I = \tilde{F}^I + \mathcal{D} A^I_\alpha \wedge \nu^\alpha + \frac{1}{2} \mathcal{F}^I_{\alpha\beta} \nu^\alpha \wedge \nu^\beta \qquad (4.31)
$$

where $\mathcal{F}^I_{\alpha\beta} = g_2 A^J_\alpha A^K_\beta f_{IJK} - A^I_\gamma \epsilon_{\alpha\beta\gamma}$.

The volume form in three dimensions is defined by

$$
\varepsilon_3 = \frac{1}{6} e^{3f} \epsilon_{abc} e^a \wedge e^b \wedge e^c \equiv e^{3f} \omega_3\,. \qquad (4.32)
$$

The six dimensional gamma matrices decompose as [47]

$$\begin{aligned} \Gamma^{\hat{A}} &= (\Gamma^a, \Gamma^i), \qquad \Gamma^a = \gamma^a \otimes \mathbf{I}_2 \otimes \sigma_1, \\ \Gamma^i &= \mathbf{I}_2 \otimes \gamma^i \otimes \sigma_2, \qquad \Gamma_7 = \mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \sigma_3 \\ \gamma^{abc} &= \epsilon^{abc}, \qquad \gamma^{ijk} = i\epsilon^{ijk}, \qquad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \ \{\gamma_i, \gamma_j\} = 2\delta_{ij}. \end{aligned} \tag{4.33}$$

The conventions are $\eta_{AB} = (- + + + + +)$, $\eta_{ab} = (- + +)$ and $\epsilon^{012} = \epsilon^{345} = 1$. We further choose

$$\gamma^0 = i\tilde{\sigma}^2, \ \gamma^1 = \tilde{\sigma}^1, \ \gamma^2 = \tilde{\sigma}^3, \qquad \gamma^i = \tau^i \tag{4.34}$$

where $\tilde{\sigma}^i$, τ^i, $i = 1, 2, 3$ are the usual Pauli matrices. Also the chirality condition $\Gamma_7 \epsilon^A = \epsilon^A$ becomes $\mathbf{I}_2 \otimes \mathbf{I}_2 \otimes \sigma_3 \epsilon^A = \epsilon^A$.

Before proceeding further, let us count the number of degrees of freedom. Table I shows all three dimensional fields arising from the six dimensional ones.

6D fields	3D fields	3D number of degrees of freedom
$\hat{g}_{MN}$	$g_{\mu\nu}$	non propagating
	A^α_μ	3
	$h_{\alpha\beta}$	5
	g	1
$\hat{b}_{MN}$	$b_{\mu\nu}$	non propagating
	$b_{\mu\alpha}$	3
	$b_{\alpha\beta}$	3
$\hat{\theta}$	θ	1
$\hat{A}^I_M$	A^I_μ	dim$G$
	A^I_α	3 dim$G$
$\hat{\psi}_M$	ψ_μ	non propagating
	ψ_i	12
$\hat{\lambda}^I$	λ^I	4 dim$G$
$\hat{\chi}$	χ	4

Table I: Three dimensional fields and the associated number of degrees of freedom.

From table I, there are 16+4dim$G$ bosonic and 16+4dim$G$ fermionic degrees of freedom in the full reduced theory. In this counting, each six dimensional fermion gives rise to 4 three dimensional fermions. In the reduction of the six dimensional theory, the component $\hat{b}_{\mu\alpha}$ will give rise to massive vector fields in three dimensions. Our goal is to truncate this theory to obtain a three dimensional $N = 4$ gauged supergravity involving only gravity, scalars and gauge fields without massive vector fields. The resulting theory will have 4(1+dim G) bosonic and 4(1+dim G) fermionic propagating degrees of freedom. To achieve this, we need to truncate 12 degrees of freedom out. From the $\hat{G}_3$ ansatz expressed entirely in terms of gauge fields, scalars coming from the gauge fields in six dimensions and constants, we see that all the fields coming from $\hat{b}_{MN}$ have been truncated out. This accounts for 6 degrees of freedom. We will see below that $h_{\alpha\beta}$ and θ, comprising 6 degrees of freedom, will be truncated, too.

In the fermionic sector, we find that the truncation is given by

$$\hat{\psi}_i - \frac{1}{2}\Gamma_i\hat{\chi} - 2e^{\theta-g}A^I_\alpha(L^{-1})_i^\alpha\hat{\lambda}^I = 0. \qquad (4.35)$$

Indeed, this removes 12 fermionic degrees of freedom. In order to check that this truncation is compatible with supersymmetry to leading order in fermions, we start by putting our ansatz to the $\delta\hat{\psi}_i$, $\delta\hat{\chi}$ and $\delta\hat{\lambda}^I$ given in (4.16). The result is

$$\begin{aligned}
\delta\hat{\psi}_i &= \frac{1}{8}g_1 e^{g-2f}\not{F}^i(\mathbf{I}_2\otimes\mathbf{I}_2\otimes\mathbf{I}_2)\epsilon - \frac{i}{2}e^{-f}(\not{P}_{ij}-\not{\partial}g\delta_{ij})(\mathbf{I}_2\otimes\gamma^j\otimes\sigma_3)\epsilon \\
&+\frac{i}{2}e^{-g}\left(T_{ij}-\frac{1}{2}T\delta_{ij}\right)(\mathbf{I}_2\otimes\gamma^j\otimes\mathbf{I}_2)\epsilon + e^\theta\left[\frac{1}{8}e^{-2f-g}(L^{-1})_j^\alpha\not{\bar{F}}^\alpha(\mathbf{I}_2\otimes\gamma^j\otimes\sigma_2)\right. \\
&+\frac{1}{4}\tilde{h}(1\mathbf{I}_2\otimes\mathbf{I}_2\otimes\sigma_1) + \frac{i}{4}e^{-f-2g}(L^{-1})_l^\beta(L^{-1})_j^\gamma\epsilon_{ljk}A^I_\gamma\not{\mathcal{D}}A^I_\beta(\mathbf{I}_2\otimes\gamma^k\otimes\sigma_1) \\
&\left.+\frac{i}{4}\tilde{a}e^{-3g}(\mathbf{I}_2\otimes\mathbf{I}_2\otimes\sigma_2)\right](\mathbf{I}_2\otimes\gamma^i\otimes\sigma_2)\epsilon \qquad (4.36)\\
\delta\hat{\chi} &= \frac{1}{2}\not{\partial}\theta\epsilon - e^\theta\left[\frac{1}{2}\tilde{h}(\mathbf{I}_2\otimes\mathbf{I}_2\otimes\sigma_1) + \frac{1}{4}e^{-2f-g}(L^{-1})_i^\alpha\not{\bar{F}}^\alpha(\mathbf{I}_2\otimes\gamma^i\otimes\sigma_2)\right. \\
&+\frac{i}{2}e^{-f-2g}(L^{-1})_i^\beta(L^{-1})_j^\gamma\epsilon_{ijk}A^I_\gamma\not{\mathcal{D}}A^I_\beta(\mathbf{I}_2\otimes\gamma^k\otimes\sigma_1) \\
&\left.+\frac{i}{2}\tilde{a}e^{-3g}(\mathbf{I}_2\otimes\mathbf{I}_2\otimes\sigma_2)\right]\epsilon \qquad (4.37)\\
\delta\hat{\lambda}^I &= \frac{1}{4}[e^{-2f}\not{\tilde{F}}^I(\mathbf{I}_2\otimes\mathbf{I}_2\otimes\mathbf{I}_2) + 2ie^{-f-g}(L^{-1})_i^\alpha\not{\mathcal{D}}A^I_\alpha(\mathbf{I}_2\otimes\gamma^i\otimes\sigma_3) \\
&+ie^{-2g}(L^{-1})_i^\alpha(L^{-1})_j^\beta\epsilon_{ijk}\mathcal{F}^I_{\alpha\beta}(\mathbf{I}_2\otimes\gamma^k\otimes\mathbf{I}_2)]\epsilon \qquad (4.38)
\end{aligned}$$

We have used the notations $\not{\hat{F}}^I = \hat{F}^I_{MN}\Gamma^{MN}$ etc. From these equations and $\mathbf{I}_2\otimes\mathbf{I}_2\otimes\sigma_3\epsilon^A = \epsilon^A$, we find, to leading order in fermions, that

$$\delta\hat{\psi}_i - \frac{1}{2}\Gamma_i\delta\hat{\chi} - 2e^{\theta-g}A^I_\alpha(L^{-1})_i^\alpha\delta\hat{\lambda}^I = 0 \qquad (4.39)$$

provided that

$$h_{\alpha\beta} = e^{\theta-2g}(12a\delta_{\alpha\beta} - 2A^I_\alpha A^I_\beta) \equiv e^{\theta-2g}N_{\alpha\beta}. \qquad (4.40)$$

This is the truncation in the bosonic sector. From (4.40), it follows that

$$\theta = 2g - \frac{1}{3}\ln N \qquad (4.41)$$

where $N \equiv \det(N_{\alpha\beta})$. In proving the above result, the following relations are useful

$$\begin{aligned} L^i_\alpha L^j_\beta T_{ij} &= e^{2\theta-4g} N_{\alpha\gamma} N_{\beta\gamma} \\ T &= T_{ii} = e^{\theta-2g} N_{\alpha\alpha} \\ L^i_\alpha L^j_\beta P_{aij} &= \frac{1}{2} D_a (e^{\theta-2g} N_{\alpha\beta}). \end{aligned} \tag{4.42}$$

Furthermore, from (4.40), it can be easily checked that

$$\delta[h_{\alpha\beta} - e^{\theta-2g}(12a\delta_{\alpha\beta} - 2A^I_\alpha A^I_\beta)] = 0 \tag{4.43}$$

to leading order in fermions by using (4.35). To check (4.43), we start by noting that

$$\hat{g}_{\alpha\beta} = e^{2g} h_{\alpha\beta} = e^{\theta}(12a\delta_{\alpha\beta} - 2A^I_\alpha A^I_\beta). \tag{4.44}$$

It follows that, with $\hat{g}_{\alpha\beta} = \frac{1}{2}(\Gamma_\alpha\Gamma_\beta + \Gamma_\beta\Gamma_\alpha)$,

$$\begin{aligned} \delta\hat{g}_{\alpha\beta} &= \delta\theta\hat{g}_{\alpha\beta} - 2e^{\theta}(A^I_\alpha \delta A^I_\beta + \delta A^I_\alpha A^I_\beta), \text{ or} \\ \bar{\epsilon}(\Gamma_\alpha\psi_\beta + \Gamma_\beta\psi_\alpha) &= \delta\theta\hat{g}_{\alpha\beta} - 2e^{\theta}(A^I_\alpha \delta A^I_\beta + A^I_\beta \delta A^I_\alpha) \\ &= \bar{\epsilon}\frac{1}{2}(\Gamma_\alpha\Gamma_\beta + \Gamma_\beta\Gamma_\alpha)\chi + 2e^{\theta}\bar{\epsilon}(\Gamma_\beta\lambda^I A^I_\alpha + \Gamma_\alpha\lambda^I A^I_\beta), \\ \text{or} \qquad \bar{\epsilon}\Gamma_\alpha\left(\psi_\beta - \frac{1}{2}\Gamma_\beta\chi - 2e^{\theta}A^I_\beta\lambda^I\right) &+ (\alpha \leftrightarrow \beta) = 0 \end{aligned} \tag{4.45}$$

where we have temporarily dropped the hats on the fermions in order to simplify the expressions. So, the relation (4.40) is compatible with supersymmetry. Equations (4.40) and (4.41) give another truncation in the bosonic sector and remove 6 degrees of freedom. The bosonic degree of freedoms are then given by $1 + 3\dim(\mathrm{G})$ scalars, g and A^I_α, and $3 + \dim(\mathrm{G})$ vectors, A^α and A^I. So, the reduced theory contains $4(1 + \dim G)$ propagating degrees of freedom and involves only gravity, scalars and vector gauge fields.

We now check the consistency of the six dimensional field equations. It is convenient to rewrite equations (4.11), (4.12) and (4.13) in differential forms. We find that these equations can be written as

$$\hat{\mathcal{D}}(e^{2\hat{\theta}} \hat{*} \hat{G}_3) = 0, \tag{4.46}$$

$$\hat{\mathcal{D}}(e^{\hat{\theta}} \hat{*} \hat{F}^I) - 2e^{2\hat{\theta}} \hat{*} \hat{G}_3 \wedge \hat{F}^I = 0, \tag{4.47}$$

$$\hat{d}\hat{*}\hat{d}\hat{\theta} + e^{\hat{\theta}} \hat{*} \hat{F}^I \wedge \hat{F}^I + 2e^{2\hat{\theta}} \hat{*} \hat{G}_3 \wedge \hat{G}_3 = 0\,. \tag{4.48}$$

In order to obtain the canonical Einstein-Hilbert term in three dimensions, we choose $f = -3g$ from now on. Before giving equations of motion, we give here the Hodge dual of $\hat{F}^I$ and $\hat{G}_3$

$$
\begin{aligned}
\hat{*}\hat{F}^I &= \frac{1}{3!}e^{6g} * \tilde{F}^I \epsilon_{\alpha\beta\gamma}\nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma + \frac{1}{2}e^{-2g}h^{\alpha\delta}\epsilon_{\beta\gamma\delta} * \mathcal{D}A^I_\alpha \wedge \nu^\beta \wedge \nu^\gamma \\
&\quad + \frac{1}{2}e^{-10g}\mathcal{F}^I_{\alpha\beta}h_{\gamma\delta}\epsilon_{\alpha\beta\delta}\omega_3 \wedge \nu^\gamma , \qquad (4.49)\\
\hat{*}\hat{G}_3 &= -\frac{1}{3!}e^{3g}\tilde{h}\epsilon_{\alpha\beta\gamma}\nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma + \frac{1}{2}e^{4g}h^{\alpha\delta}\epsilon_{\beta\gamma\delta} * \bar{F}^\alpha \wedge \nu^\beta \wedge \nu^\gamma \\
&\quad -\frac{1}{2}e^{-4g}\epsilon_{\alpha\beta\gamma}h_{\gamma\delta} * K_{\alpha\beta} \wedge \nu^\delta + \tilde{a}e^{-12g}\omega_3 \ . \qquad (4.50)
\end{aligned}
$$

The $\hat{*}$ and $*$ are Hodge dualities in six and three dimensions, respectively. After using our ansatz in (4.46), (4.47) and (4.48), we find the following set of equations

$$
\begin{aligned}
&\mathcal{D}(e^{2\theta+3g}\tilde{h}) = 0 \, , \qquad (4.51)\\
&\mathcal{D}(e^{\theta+6g}N^{\alpha\beta} * \bar{F}^\beta) + g_1c_1F^\alpha + \frac{1}{2}\epsilon_{\alpha\beta\gamma}N^{\alpha'\beta}N^{\beta'\gamma} * K_{\alpha'\beta'} = 0 \, , \qquad (4.52)\\
&\mathcal{D}(N^{\alpha\gamma}N^{\beta\delta} * K_{\alpha\beta}) - g_1e^{\theta+6g}(N^{\alpha\gamma} * \bar{F}^\alpha \wedge F^\delta - N^{\alpha\delta} * \bar{F}^\alpha \wedge F^\gamma) = 0 \, , \qquad (4.53)\\
&\mathcal{D}(e^{\theta+6g} * \tilde{F}^I) + 2c_1\tilde{F}^I - 2e^{\theta+6g}N^{\alpha\alpha'} * \bar{F}^{\alpha'} \wedge \mathcal{D}A^I_\alpha \\
&\qquad + N^{\alpha\alpha'}N^{\beta\beta'}\mathcal{F}^I_{\alpha\beta} * K_{\alpha'\beta'} + g_2N^{\alpha\delta}f_{IJK}A^J_\delta * \mathcal{D}A^K_\alpha = 0 \, , \qquad (4.54)\\
&\mathcal{D}(N^{\alpha\beta} * \mathcal{D}A^I_\beta) + g_1e^{\theta+6g} * \tilde{F}^I \wedge F^\alpha - 2e^{\theta+6g}N^{\alpha\beta} * \bar{F}^\beta \wedge \tilde{F}^I \\
&\qquad + 2N^{\alpha\alpha'}N^{\beta\beta'} * K_{\alpha'\beta'} \wedge \mathcal{D}A^I_\beta + \frac{1}{2}e^{-\theta-6g}N^{\alpha'\beta}N^{\beta'\gamma}\mathcal{F}^I_{\alpha'\beta'}\epsilon_{\alpha\beta\gamma}\omega_3 \\
&\qquad - \tilde{a}e^{2\theta-12g}\epsilon_{\alpha\beta\gamma}\mathcal{F}^I_{\beta\gamma}\omega_3 + g_2f_{IJK}e^{-\theta-6g}A^J_\beta\mathcal{F}^K_{\alpha'\beta'}N^{\alpha'\beta}N^{\alpha\beta'}\omega_3 = 0 \, , \qquad (4.55)\\
&2d * dg - \frac{1}{3}d\ln N + e^{\theta+6g} * \tilde{F}^I \wedge \tilde{F}^I + N^{\alpha\alpha'} * \mathcal{D}A^I_{\alpha'} \wedge \mathcal{D}A^I_\alpha \\
&\qquad + \frac{1}{2}e^{\theta+6g}N^{\alpha\alpha'} * \bar{F}^{\alpha'} \wedge \bar{F}^\alpha + \frac{1}{2}N^{\alpha\alpha'}N^{\beta\beta'} * K_{\alpha'\beta'} \wedge K_{\alpha\beta} + c_1^2e^{-2\theta-12g}\omega_3 \\
&\qquad + \frac{1}{2}e^{-\theta-6g}N^{\alpha\alpha'}N^{\beta\beta'}\mathcal{F}^I_{\alpha\beta}\mathcal{F}^I_{\alpha'\beta'}\omega_3 + \tilde{a}^2e^{-12g}\omega_3 = 0 \, , \qquad (4.56)
\end{aligned}
$$

where we have used the summation convention on $\alpha, \beta, \ldots$ regardless their upper or lower positions, and $N^{\alpha\beta} \equiv (N^{-1})_{\alpha\beta}$. We have also used the solution for equation (4.51) namely

$$\tilde{h}e^{2\theta+3g} = c_1 \qquad (4.57)$$

with a constant c_1 in other equations. Equation (4.53) can be obtained by multiplying (4.55) by $A^I_{\beta'}N^{\beta\beta'}$ and antisymmetrizing in α and β. By using the explicit forms of the Ricci tensors given in [47], the scalar equation (4.55), after multiplied by $e^{8g-\theta}N^{\beta\beta'}A^I_{\beta'}$ and symmetrized in α and β, is the same as component ij of the Einstein equation with the trace part of Einstein equation taking care of equation (4.56). Component ab the Einstein equation will give three dimensional Einstein equation which we will not give the explicit form here. Equation (4.52) gives Yang-Mills equations for A^α. The combination [(4.54) $+2A^I_\alpha$(4.52)] gives Yang-Mills equations for A^I

$$\mathcal{D}[e^{\theta+6g}[(\delta_{IJ}+4A^I_\alpha A^J_\beta N^{\alpha\beta})F^J - 24g_1 a A^I_\alpha N^{\alpha\beta}F^\beta]] + 2c_1 F^I + g_2 f_{IJK}N^{\alpha\beta}A^J_\beta * \mathcal{D}A^K_\alpha + g_2 f_{IJK}N^{\alpha\alpha'}N^{\beta\beta'}A^J_\alpha A^K_\beta * K_{\alpha'\beta'} = 0\,. \quad (4.58)$$

We have checked that the equation for F^α is the same as component ai of the Einstein equation. So, there are two Yang-Mills equations for F^α and F^I, one equation for g and one equation for A^I_α. All six dimensional field equations are satisfied by our ansatz.

4.1.4 Three dimensional gauged supergravity Lagrangian

All three dimensional equations of motion obtained in the previous subsection can be obtained from the following Lagrangian, with $\hat{e} = ee^{3f+3g} = ee^{-6g}$,

$$\begin{aligned}\mathcal{L} &= \frac{1}{4}R*\mathbf{1} - \frac{1}{2}N^{-\frac{1}{3}}e^{8g}\big[(\delta_{IJ}+4A^I_\alpha A^J_\beta N^{\alpha\beta})*F^I\wedge F^J - 48ag_1 N^{\alpha\beta}A^I_\beta * F^\alpha\wedge F^I \\ &\quad +6ag_1^2(24aN^{\alpha\beta}-\delta_{\alpha\beta})*F^\alpha\wedge F^\beta\big] - *d\big(2g-\frac{1}{12}\ln N\big)\wedge d\big(2g-\frac{1}{12}\ln N\big) \\ &\quad -\frac{1}{2}N^{\alpha\beta}*\mathcal{D}A^I_\alpha\wedge\mathcal{D}A^I_\beta - N^{\alpha\alpha'}N^{\beta\beta'}A^I_\beta A^J_{\beta'}*\mathcal{D}A^I_\alpha\wedge\mathcal{D}A^J_{\alpha'} - V*\mathbf{1} + \mathcal{L}_{\rm CS}\end{aligned} \quad (4.59)$$

which is the same as the dimensional reduction of the Lagrangian

$$\mathcal{L}_B = \frac{1}{4}\hat{R}\hat{*}\mathbf{1} - \frac{1}{4}\hat{*}d\hat{\theta}\wedge d\hat{\theta} - \frac{1}{2}e^{2\hat{\theta}}\hat{*}\hat{G}_3\wedge\hat{G}_3 - \frac{1}{2}e^{\hat{\theta}}\hat{*}\hat{F}^I\wedge\hat{F}^I \quad (4.60)$$

together with the Chern-Simons terms. The scalar potential and the Chern-Simons Lagrangian are given by

$$\begin{aligned} V &= \frac{1}{4}\big[N^{-\frac{2}{3}}(N_{\alpha\beta}N_{\alpha\beta}-\frac{1}{2}N_{\alpha\alpha}N_{\beta\beta}) + 2N^{-\frac{2}{3}}e^{-8g}\tilde{a}^2 \\ &\quad +N^{\frac{1}{3}}e^{-8g}N^{\alpha\alpha'}N^{\beta\beta'}\mathcal{F}^I_{\alpha\beta}\mathcal{F}^I_{\alpha'\beta'} - 2c_1^2 N^{\frac{2}{3}}e^{-16g}\big], \quad (4.61)\\ \mathcal{L}_{\rm CS} &= 2c_1\big(F^I\wedge A^I - \frac{1}{6}g_2 f_{IJK}A^I\wedge A^J\wedge A^K\big) \\ &\quad -12ag_1^2c_1\big(F^\alpha\wedge A^\alpha - \frac{1}{6}g_1\epsilon_{\alpha\beta\gamma}A^\alpha\wedge A^\beta\wedge A^\gamma\big)\,. \quad (4.62)\end{aligned}$$

In order to make formulae simpler and the symmetries of the scalar manifold more transparent, we make the following rescalings. We first restore the coupling g_1 in the appropriate places by setting

$$a = \frac{\bar{a}}{g_1^2}. \quad (4.63)$$

We can then remove the constant a by setting

$$
\begin{aligned}
c_1 &= \frac{\bar{c}_1}{6\bar{a}}, \qquad e^g = \frac{e^{\bar{g}}}{g_1^{\frac{1}{4}}}, \qquad A^I_\alpha = \frac{\sqrt{6\bar{a}}}{g_1}\bar{A}^I_\alpha, \\
g_2 &= \frac{\bar{g}_2}{\sqrt{6\bar{a}}}, \qquad N_{\alpha\beta} = \frac{6\bar{a}}{g_1^2}\bar{N}_{\alpha\beta}, \qquad e^\theta = \frac{g_1^{\frac{3}{2}}}{6\bar{a}}e^{\bar{\theta}} \\
&\text{and} \qquad A^I = \sqrt{6\bar{a}}\bar{A}^I\,.
\end{aligned}
\tag{4.64}
$$

After removing all the bars, we obtain the Lagrangian

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{4}R * \mathbf{1} - \frac{1}{2}e^{2\sqrt{2}\Phi}[(\delta_{IJ} + 4N^{\alpha\beta}A^I_\alpha A^J_\beta) * F^I \wedge F^J - 8N^{\alpha\beta}A^I_\beta * F^\alpha \wedge F^I \\
&\quad + (4N^{\alpha\beta} - \delta_{\alpha\beta}) * F^\alpha \wedge F^\beta] - \frac{1}{2} * d\Phi \wedge d\Phi - \frac{1}{2}N^{\alpha\beta} * \mathcal{D}A^I_\alpha \wedge \mathcal{D}A^I_\beta \\
&\quad - N^{\alpha\alpha'}N^{\beta\beta'}A^I_\beta A^J_{\beta'} * \mathcal{D}A^I_\alpha \wedge \mathcal{D}A^J_{\alpha'} - V + \mathcal{L}_{\text{CS}}
\end{aligned}
\tag{4.65}
$$

where we have introduced the canonically normalized scalar for the gauge singlet combination

$$
\Phi = 2\sqrt{2}g - \frac{\sqrt{2}}{12}\ln N\,. \tag{4.66}
$$

The scalar potential and Chern-Simons terms are now

$$
\begin{aligned}
V &= \frac{1}{4}\big[g_1^2 N^{-1}e^{-2\sqrt{2}\Phi}(N_{\alpha\beta}N_{\alpha\beta} - \frac{1}{2}N_{\alpha\alpha}N_{\beta\beta}) + 2N^{-1}e^{-2\sqrt{2}\Phi}\tilde{a}^2 - 2c_1^2 e^{-4\sqrt{2}\Phi} \\
&\quad + e^{-2\sqrt{2}\Phi}N^{\alpha\alpha'}N^{\beta\beta'}\mathcal{F}^I_{\alpha\beta}\mathcal{F}^I_{\alpha'\beta'}\big], \qquad (4.67) \\
\mathcal{L}_{\text{CS}} &= 2c_1\Bigg[F^I \wedge A^I - \frac{1}{6}g_2 f_{IJK}A^I \wedge A^J \wedge A^K \\
&\quad - \big(F^\alpha \wedge A^\alpha \wedge - \frac{1}{6}g_1\epsilon_{\alpha\beta\gamma}A^\alpha \wedge A^\beta \wedge A^\gamma\big)\Bigg] \qquad (4.68)
\end{aligned}
$$

with

$$
\begin{aligned}
\tilde{a} &= g_1(1 - A^I_\alpha A^I_\alpha) + \frac{1}{3}g_2 A^3\,, \\
\mathcal{F}^I_{\alpha\beta} &= g_2 A^J A^K f_{IJK} - g_1\epsilon_{\alpha\beta\gamma}A^I_\gamma\,.
\end{aligned}
\tag{4.69}
$$

We first look at the scalar matrix appearing in the gauge kinetic terms

$$
\mathbf{M} = \begin{pmatrix} \mathcal{M}_{\alpha\beta} & \mathcal{M}_{\alpha J} \\ \mathcal{M}_{I\beta} & \mathcal{M}_{IJ} \end{pmatrix} = e^{2\sqrt{2}\Phi}\begin{pmatrix} 4N^{\alpha\beta} - \delta_{\alpha\beta} & -4N^{\alpha\beta}A^J_\beta \\ -4N^{\alpha\beta}A^I_\alpha & \delta_{IJ} + 4N^{\alpha\beta}A^I_\alpha A^J_\beta \end{pmatrix}. \tag{4.70}
$$

Introducing the matrix notation for ${A^I}_\alpha \equiv \mathbf{A}$ which is an $n \times 3$, $n = \dim G$, we find

$$
N_{\alpha\beta} \equiv \mathbf{N} = 2\Big(\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}\Big), \qquad N^{\alpha\beta} \equiv \mathbf{N}^{-1} = \frac{1}{2\Big(\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}\Big)}, \tag{4.71}
$$

$$
\mathbf{M} = e^{2\sqrt{2}\Phi}\begin{pmatrix} \frac{\mathbf{I}_3 + \mathbf{A}^t\mathbf{A}}{\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}} & -2\frac{1}{\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}}\mathbf{A}^t \\ -2\frac{1}{\mathbf{I}_n - \mathbf{A}\mathbf{A}^t}\mathbf{A} & \frac{\mathbf{I}_n + \mathbf{A}^t\mathbf{A}}{\mathbf{I}_n - \mathbf{A}^t\mathbf{A}} \end{pmatrix}. \tag{4.72}
$$

It follows that

$$\mathbf{M}^{-1} = e^{-2\sqrt{2}\Phi} \begin{pmatrix} \frac{\mathbf{I}_3+\mathbf{A}^t\mathbf{A}}{\mathbf{I}_3-\mathbf{A}^t\mathbf{A}} & 2\frac{1}{\mathbf{I}_3-\mathbf{A}^t\mathbf{A}}\mathbf{A}^t \\ 2\frac{1}{\mathbf{I}_n-\mathbf{A}\mathbf{A}^t}\mathbf{A} & \frac{\mathbf{I}_n+\mathbf{A}^t\mathbf{A}}{\mathbf{I}_n-\mathbf{A}^t\mathbf{A}} \end{pmatrix}. \tag{4.73}$$

The scalars form the coset space $\mathbf{R} \times \frac{SO(3,n)}{SO(3)\times SO(n)}$ with the factor $\mathbf{R}$ corresponding to Φ. The scalar kinetic terms give rise to the metric on $\mathbf{R} \times \frac{SO(3,n)}{SO(3)\times SO(n)}$

$$\begin{aligned} & -\frac{1}{2} * d\Phi \wedge d\Phi - \frac{1}{2} N^{\alpha\beta} * \mathcal{D}A^I_\alpha \wedge \mathcal{D}A^I_\beta - N^{\alpha\alpha'} N^{\beta\beta'} A^I_\beta A^J_{\beta'} * \mathcal{D}A^I_\alpha \wedge \mathcal{D}A^J_{\alpha'} \\ & = -\frac{1}{2} * d\Phi \wedge d\Phi - \frac{1}{4}\mathrm{Tr}\Big(\frac{1}{\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}} * \mathcal{D}\mathbf{A}^t \wedge \frac{1}{\mathbf{I}_n - \mathbf{A}\mathbf{A}^t}\mathcal{D}\mathbf{A}\Big). \end{aligned} \tag{4.74}$$

With all these results, the Lagrangian can be simply written as

$$\begin{aligned} \mathcal{L} \; = \; & \frac{1}{4} R * \mathbf{1} - \frac{1}{2} * d\Phi \wedge d\Phi - \frac{1}{4}\mathrm{Tr}\Big(\frac{1}{\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}} * \mathcal{D}\mathbf{A}^t \wedge \frac{1}{\mathbf{I}_n - \mathbf{A}\mathbf{A}^t}\mathcal{D}\mathbf{A}\Big) \\ & -\frac{1}{2} e^{2\sqrt{2}\Phi} \mathcal{M}_{\mathcal{A}\mathcal{B}} * F^{\mathcal{A}} \wedge F^{\mathcal{B}} - V + \mathcal{L}_{\rm CS} \end{aligned} \tag{4.75}$$

where $\mathcal{A}, \mathcal{B} = (\alpha, I)$.

We now come to supersymmetries of our truncated theory. We will show that this truncation is indeed compatible with supersymmetry namely supersymmetry transformations of various components of $\hat{b}_{MN}$ must be consistent with our specific choices of $C^\alpha = \frac{1}{2} A^I_\alpha A^I$. This ensures that all the truncated fields will not be generated via supersymmetry. For the filed $\hat{b}_{\mu\nu}$, we have eliminated it by using the equation of motion for $\hat{G}_3$ in (4.51). Because of its non propagating nature, we do not need to worry about it. We now check the supersymmetry transformations $\delta\hat{G}_{3\mu\alpha\beta}$ and $\delta\hat{G}_{3\mu\nu\alpha}$. It is more convenient to work with the transformation of the filed strength $\hat{G}_3$. With equation (4.35), the component $\delta\hat{b}_{\alpha\beta}$ vanishes identically. The $\delta\hat{G}_{3\mu\alpha\beta}$ gives the condition

$$\delta\hat{b}_{\mu\alpha} = \delta(A^I_\alpha A^I_\mu - 6ag_1 A^\alpha_\mu)\,. \tag{4.76}$$

Using $A^I_\mu = \hat{A}^I_\mu + g_1 A^I_\alpha A^\alpha_\mu$ and $\delta\hat{b}_{\mu\alpha}$ from (4.16), we find that

$$\begin{aligned} \delta\hat{b}_{\mu\alpha} - \delta(A^I_\alpha A^I_\mu - 6ag_1 A^\alpha_\mu) \;&=\; 2A^I_\alpha \bar{\epsilon}\Gamma_\mu \hat{\lambda}^I - e^{-\theta}\bar{\epsilon}\Gamma_\mu \hat{\psi}_\alpha + \frac{1}{2} e^{-\theta}\bar{\epsilon}(\Gamma_{\mu\alpha} + \hat{g}_{\mu\alpha})\hat{\chi} \\ &=\; -e^{-\theta}\bar{\epsilon}\Gamma_\mu\big(\hat{\psi}_\alpha - \frac{1}{2}\Gamma_\alpha\hat{\chi} - 2e^\theta A^I_\alpha \hat{\lambda}^I\big) = 0 \end{aligned} \tag{4.77}$$

where we have used

$$\hat{g}_{\mu\alpha} = \hat{e}^i_\mu \hat{e}^i_\alpha = -g_1 h_{\alpha\beta} e^{2g} A^\beta_\mu = -g_1 e^\theta N_{\alpha\beta} A^\beta_\mu\,. \tag{4.78}$$

Note that

$$\hat{\psi}_i - \frac{1}{2}\Gamma_i\hat{\chi} - 2e^{\theta-g} A^I_\alpha (L^{-1})^\alpha_i \hat{\lambda}^I = e^{-g}(L^{-1})^\alpha_i\big(\hat{\psi}_\alpha - \frac{1}{2}\Gamma_\alpha\hat{\chi} - 2e^\theta A^I_\alpha\hat{\lambda}^I\big). \tag{4.79}$$

$\delta\hat{G}_{3\mu\nu\alpha}$ is simply the derivative of the previous result namely

$$\delta\hat{G}_{3\mu\nu\alpha} = 2\partial_{[\mu}\delta\hat{b}_{\nu]\alpha} = 2\partial_{[\mu}\delta(A^I_{\nu]}A^I_\alpha - 6ag_1A^\alpha_{\nu]}). \qquad (4.80)$$

We have then verified that our truncated theory is a supersymmetric theory. We will also give a confirmation to this claim in the next section in which we will show that this theory is on-shell equivalent to a manifestly supersymmetric $(SO(3) \ltimes \mathbf{R}^3) \times (G \ltimes \mathbf{R}^n)$ Chern-Simons gauged supergravity.

The final issue we should add here is the diagonalization of the fermion kinetic terms. Applying the result of [113], we find that our fermion kinetic Lagrangian can be written as

$$e^{-1}\mathcal{L}_{\text{Fkinetic}} = -\frac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho - \frac{1}{2}\bar{\chi}\Gamma^\mu D_\mu\chi - \frac{1}{2}(\delta^{IJ} + 4N^{\alpha\beta}A^I_\alpha A^I_\beta)\bar{\lambda}^I\Gamma^\mu\mathcal{D}_\mu\lambda^I \quad (4.81)$$

where the three dimensional fields are given by

$$\begin{aligned}
\psi_a &= e^{-\frac{3g}{2}}(\hat{\psi}_a + \Gamma_a\Gamma^i\hat{\psi}_i), \\
\psi_i &= e^{-\frac{3g}{2}}\left(\hat{\psi}_i - \frac{1}{2}\Gamma_i\hat{\chi}\right) = 2e^{-\frac{3g}{2}+\theta}A^I_i\hat{\lambda}^I,\ A^I_i = A^I_\alpha e^{-g}(L^{-1})^\alpha_i, \\
\chi &= e^{-\frac{3g}{2}}\left(\Gamma^i\hat{\psi}_i + \frac{1}{2}\hat{\chi}\right), \\
\lambda^I &= e^{\frac{\theta}{2}-\frac{3g}{2}}\hat{\lambda}^I.
\end{aligned} \qquad (4.82)$$

4.2 Chern-Simons and Yang-Mills gaugings in three dimensions

In this section, we show the on-shell equivalence between non-semisimple Chern-Simons and semisimple Yang-Mills gaugings in three dimensions [38]. We will construct Chern-Simons gauged supergravity with gauge groups $(SO(3)\ltimes\mathbf{T}^3)\times(G\ltimes\mathbf{T}^n)$, $n = \text{dim}G$, and show that the gauging is consistent according to the criterion given in [31]. We then show that this theory is on-shell equivalent to the $SU(2)\times G$ gauged supergravity obtained from $SU(2)$ reduction in the previous section.

For conveniences, we recall some necessary equations, we will use throughout this section, from chapter 2 with appropriate changes for our present case. The Yang-Mills Lagrangian is given by

$$\begin{aligned}
e^{-1}\tilde{\mathcal{L}} &= \frac{1}{4}R + e^{-1}h_1\tilde{\mathcal{L}}_{\text{CS}} - \frac{1}{8}\mathbf{M}_{mn}F^{m\mu\nu}F^n_{\mu\nu} - \frac{1}{4}G_{AB}\tilde{\mathcal{P}}^A_\mu\tilde{\mathcal{P}}^{B\mu} \\
&\quad + \frac{1}{4}e^{-1}\epsilon^{\mu\nu\rho}\mathbf{M}_{mn}\tilde{\mathcal{V}}^{\underline{n}}{}_A F^m_{\mu\nu}\tilde{\mathcal{P}}^A_\rho - V\,.
\end{aligned} \qquad (4.83)$$

We also repeat here the quantities appearing in (4.83)

$$
\begin{aligned}
G_{AB} &= \delta_{AB} - \tilde{\mathcal{V}}^{m}_{\ A}\mathbf{M}_{mn}\tilde{\mathcal{V}}^{n}_{\ B}, \qquad \mathbf{M}_{mn} = (\tilde{\mathcal{V}}^{m}_{\ A}\tilde{\mathcal{V}}^{n}_{\ A})^{-1},\\
\tilde{\mathcal{L}}_{\rm CS} &= \frac{1}{4}\epsilon^{\mu\nu\rho}A^{m}_{1\mu}\eta_{mn}\left(\partial_\nu A^{n}_{1\rho} + \frac{1}{3}g_1 f^{n}_{\ kl}A^{k}_{1\nu}A^{l}_{1\rho}\right)\\
&\quad + \frac{1}{4}\epsilon^{\mu\nu\rho}A^{m}_{2\mu}\eta_{mn}\left(\partial_\nu A^{n}_{2\rho} + \frac{1}{3}g_2 f^{n}_{\ kl}A^{k}_{2\nu}A^{l}_{2\rho}\right),\\
\tilde{\mathcal{V}}^{\mathcal{M}}_{\ \mathcal{A}}t^{\mathcal{A}} &= \tilde{L}^{-1}t^{\mathcal{M}}\tilde{L},\\
\tilde{\mathcal{Q}}_\mu + \tilde{\mathcal{P}}_\mu &= \tilde{L}^{-1}(\partial_\mu + g_1\eta_{1mn}A^{m}_{1\mu}t^{n}_{1} + g_2\eta_{2mn}A^{m}_{2\mu}t^{n}_{2})\tilde{L}\,. \qquad (4.84)
\end{aligned}
$$

The field strengths of A_1 and A_2, F_1 and F_2, are included in the $F^{m\mu\nu}$. A_1 and A_2 are gauge fields of $SO(3)$ and G, respectively.

We are now in a position to construct a consistent Chern-Simons gauged supergravity with gauge groups $(SO(3) \ltimes \mathbf{T}^3) \times (G \ltimes \mathbf{T}^n)$. We proceed as in [50] using the formulation of [31].

The $4(1+n)$ scalar fields are described by a coset space $\frac{SO(4,1+n)}{SO(4)\times SO(n+1)}$. We parametrize the coset by

$$
L = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix} \qquad (4.85)
$$

where A is a symmetric 4×4 matrix, B is a $4\times(n+1)$ matrix, and C is a symmetric $(n+1)\times(n+1)$ matrix. These matrices satisfy the relations

$$
\begin{aligned}
A^2 - BB^t &= \mathbf{I}_4,\\
AB - BC &= 0,\\
C^2 - B^tB &= \mathbf{I}_{n+1}\,. \qquad (4.86)
\end{aligned}
$$

The gauging is characterized by the embedding tensor

$$
\Theta_{\mathcal{M}\mathcal{N}} = g_1\delta_{a_1b_1} + g_2\delta_{a_2b_2} + h_1\delta_{b_1b_1} + h_2\delta_{b_2b_2}. \qquad (4.87)
$$

The ranges of the indices are $a_1, b_1 = 1, 2, 3$ and $a_2, b_2 = 1, \ldots, n$. We denote the $(5+n)\times(5+n)$ matrix in the block form

$$
\left(\begin{array}{c|c} 4\times 4 & 4\times(n+1) \\ \hline (n+1)\times 4 & (n+1)\times(n+1) \end{array}\right). \qquad (4.88)
$$

With this form, the generators of $SO(4,1+n)$ can be shown as

$$
\left(\begin{array}{c|c} J_{SO(4)} & Y \\ \hline Y^t & J_{SO(n+1)} \end{array}\right) \qquad (4.89)
$$

with Y being non-compact and given by $e_{a\hat{I}} + e_{\hat{I}a}$. We further divide each block by separating its last row and last column from the rest and use the following ranges of indices:

$$
\alpha,\, \beta = 1, 2, 3, \qquad I,\, J = 1, \ldots, n, \qquad \hat{I} = 5, \ldots, n+5, \text{ and } a,\, b = 1, \ldots, 4.
$$

Various gauge groups are described by the following generators:

$$\begin{aligned}
SO(3) &: \; J^{\alpha}_{a_1} = \epsilon_{\alpha\beta\gamma} e_{\beta\gamma} , \\
G &: \; J^{I}_{a_2} = f^{I}{}_{JK} e_{JK} , \\
\mathbf{T}^3 &: \; J^{\alpha}_{b_1} = e_{\alpha,n+5} + e_{n+5,\alpha} + e_{4\alpha} - e_{\alpha 4} , \\
\mathbf{T}^n &: \; J^{I}_{b_2} = e_{4,I+4} + e_{I+4,4} + e_{n+5,I+4} - e_{I+4,n+5} \\
& \quad \text{with } \; (e_{ab})_{cd} = \delta_{ac}\delta_{bd}, \text{ etc} .
\end{aligned} \tag{4.90}$$

Schematically, these gauge generators are embedded in the $(5+n) \times (5+n)$ matrix as

$$\left(\begin{array}{c|c|c|c}
J_{a_1}(3 \times 3) & \begin{array}{c} -b_1 \\ (3 \times 1) \end{array} & & \begin{array}{c} b_1 \\ (3 \times 1) \end{array} \\ \hline
b_1^t(1 \times 3) & & b_2^t(1 \times n) & \\ \hline
 & \begin{array}{c} b_2 \\ (n \times 1) \end{array} & \begin{array}{c} J_{a_2} \\ (n \times n) \end{array} & \begin{array}{c} -b_2 \\ (n \times 1) \end{array} \\ \hline
b_1^t(1 \times 3) & & b_2^t(1 \times n) &
\end{array} \right) \tag{4.91}$$

where each b_1 and b_2 correspond to various e's factors in J_{b_1} and J_{b_2} in (4.90). Notice that the shift generators have components in both $SO(4) \times SO(n+1)$ and Y parts. Furthermore, J_{b_1} and J_{b_2} transform as adjoint representations of the gauge groups $SO(3)$ and G, respectively.

From this information, we can construct T-tensors and check the consistency of the gauging according to the criterion $\mathbb{P}_{\boxplus} T^{IJ,KL} = 0$. The consistency requires that

$$h_2 = -h_1 . \tag{4.92}$$

The $4(1+n)$ scalars correspond to the non-compact generators Y. After using the shift symmetries to remove some of the shifted scalars and gauge fields, we are left with $1+3n$ scalars embedded in $(5+n) \times (5+n)$ matrix as

$$\tilde{L} = \left(\begin{array}{c|c|c|c}
\frac{1}{\sqrt{\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}}} & & \frac{1}{\sqrt{\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}}}\mathbf{A}^t & \\ \hline
 & \cosh\sqrt{2}\Phi & & \sinh\sqrt{2}\Phi \\ \hline
\frac{1}{\sqrt{\mathbf{I}_n - \mathbf{A}\mathbf{A}^t}}\mathbf{A} & & \frac{1}{\sqrt{\mathbf{I}_n - \mathbf{A}\mathbf{A}^t}} & \\ \hline
 & \sinh\sqrt{2}\Phi & & \cosh\sqrt{2}\Phi
\end{array} \right) . \tag{4.93}$$

Note that in (4.93), we have chosen a specific form of A, B and C. $\mathbf{A}$ is an $n \times 3$ matrix to be identified with A^I_α in the previous section. The resulting coset space is readily recognized as $\mathbf{R} \times \frac{SO(3,n)}{SO(3)\times SO(n)}$ in which Φ corresponds to the $\mathbf{R} \sim SO(1,1)$

part.

We now use equations (2.55) and (2.56) together with the definition of A_1 and A_2 tensors given in chapter 2 to compute all the $\mathcal{V}$'s, various components of the T-tensor as well as A_1 and A_2 tensors. To avoid confusion with the G adjoint indices $I, J, K, \ldots$, we temporarily label R-symmetry indices with $\bar{I}, \bar{J}, \bar{K}, \ldots$. The coordinate index on the target space i will be denoted by a pair of indices specifying the entries of the L. In order to simplify the equations, we introduce a symbolic notation R for the R-symmetry generators including their indices. We start by giving all the $\mathcal{V}^{\mathcal{M}}_{\bar{I}\bar{J}}$'s.

$$\begin{aligned}
\mathcal{V}^R_{a_\alpha} &= \frac{1}{4}\epsilon_{\alpha\beta\gamma}(ARA)_{\gamma\beta}, \qquad a_\alpha = \epsilon_{\alpha\beta\gamma}e_{\beta\gamma}, \\
\mathcal{V}^R_{a_I} &= -\frac{1}{4}f^I{}_{JK}(B^tRB)_{JK}, \qquad \mathcal{V}^R_{b_\alpha} = \frac{1}{2}\mathcal{H}(AR)_{\alpha 4}, \;\; \mathcal{H} = A_{44} - B_{4,n+1}, \\
\mathcal{V}^R_{b_I} &= \frac{1}{2}\mathcal{H}(B^tR)_{I4}\,.
\end{aligned} \tag{4.94}$$

The $\mathcal{V}^{\mathcal{M}}_i$'s are given by

$$\begin{aligned}
\mathcal{V}^{a_\alpha}_{\delta L} &= \epsilon_{\alpha\beta\gamma}B_{\gamma L}, \qquad \mathcal{V}^{a_I}_{\delta M} = -f_{IJK}B_{\delta J}C_{KM}, \qquad \mathcal{V}^{b_\alpha}_{\delta,n+1} = \mathcal{H}A_{\delta\alpha}, \\
\mathcal{V}^{b_\alpha}_{4L} &= \mathcal{H}B_{\alpha L}, \qquad \mathcal{V}^{b_I}_{4L} = \mathcal{H}C_{IL}, \qquad \mathcal{V}^{b_I}_{\delta,n+1} = \mathcal{H}B_{\delta I}.
\end{aligned} \tag{4.95}$$

The T-tensors are given by

$$\begin{aligned}
T^{RR'} &= \frac{1}{16}\left[-8g_1R_{\alpha 4}R'_{\alpha 4}\mathrm{det}A + 8g_2\mathcal{H}\frac{B^3}{6} + 4h_1\mathcal{H}^2R_{\gamma 4}R'_{\gamma 4}\right], \\
T^R_{\delta,n+1} &= \frac{1}{4}\left[-2g_1\mathcal{H}\mathrm{det}AR_{\delta 4} + 2g_2\mathcal{H}\frac{B^3}{6}R_{\delta 4} + 2h_1\mathcal{H}^2R_{\delta 4}\right], \\
T^R_{4L} &= \frac{1}{4}\big[-g_1\mathcal{H}\epsilon_{\alpha\beta\gamma}B_{\alpha L}(ARA)_{\beta\gamma} + g_2\mathcal{H}f^I{}_{JK}C_{IL}(B^tRB)_{JK}\big], \\
T^R_{\delta L} &= \frac{1}{4}\big[2g_1\mathcal{H}\epsilon_{\alpha\beta\gamma}A_{\delta\beta}B_{\gamma L}(AR)_{\alpha 4} - 2g_2\mathcal{H}f_{IJK}B_{\delta J}C_{KL}(B^tR)_{I4}\big]
\end{aligned} \tag{4.96}$$

where $B^3 = \epsilon_{\alpha\beta\gamma}f_{IJK}B_{\alpha I}B_{\beta J}B_{\gamma K}$. Before moving on, we note the useful relations

$$\begin{aligned}
R_{\alpha\beta} &= \epsilon_{\alpha\beta\gamma}R_{\gamma 4}, \qquad (R^{\bar{K}(\bar{I}}R^{\bar{J})\bar{K}})_{a4} = 3\delta^{\bar{I}\bar{J}}\delta_{a4}, \\
R^{\bar{K}(\bar{I}}_{i4}R^{\bar{J})\bar{K}}_{j4} &= -\delta_{i\alpha}\delta_{j\alpha}\delta^{\bar{I}\bar{J}}, \qquad R^{\bar{K}(\bar{I}}_{[i|l|}R^{\bar{J})\bar{K}}_{j]4} = \delta_{\alpha l}\epsilon_{\alpha ij}\delta^{\bar{I}\bar{J}}, \\
R^{\bar{K}(\bar{I}}_{\alpha 4}R^{\bar{J})\bar{K}}_{\beta 4} &= -\delta_{\alpha\beta}\delta^{\bar{I}\bar{J}}\,.
\end{aligned} \tag{4.97}$$

The following combination is useful in computing $A^{\bar{I}\bar{J}}_{2i}$

$$\begin{aligned}
f^{\bar{K}(\bar{I}\;\;j}_{\;\;\;\;4,n+1}T^{\bar{J})\bar{K}}_j &= \frac{3}{2}\delta^{\bar{I}\bar{J}}\Big(-g_1\mathcal{H}\mathrm{det}A + \frac{1}{6}g_2\mathcal{H}B^3 + h_1\mathcal{H}^2\Big), \\
f^{\bar{K}(\bar{I}\;\;j}_{\;\;\;\;\delta L}T^{\bar{J})\bar{K}}_j &= -\frac{3}{4}\delta^{\bar{I}\bar{J}}\mathcal{H}(g_1\epsilon_{\alpha\beta\gamma}\epsilon_{\delta\beta'\gamma'}A_{\beta\beta'}A_{\gamma\gamma'}B_{\alpha L} \\
&\quad -g_2f_{IJK}B_{\beta J}B_{\gamma K}\epsilon_{\delta\beta\gamma}C_{IL}).
\end{aligned} \tag{4.98}$$

We then find A_1 and A_2 tensors

$$\begin{aligned} A_1^{\bar{I}\bar{J}} &= -2T\delta^{\bar{I}\bar{J}}, \\ A_{2i}^{\bar{I}\bar{J}} &= \frac{1}{2}T_i^{\bar{I}\bar{J}} + \frac{1}{6}X_i\delta^{\bar{I}\bar{J}} \end{aligned} \tag{4.99}$$

where we have defined the following quantities

$$\begin{aligned} T &= 2\Big(- g_1\mathcal{H}\text{det}A + \frac{1}{6}g_2\mathcal{H}B^3 + \frac{1}{2}h_1\mathcal{H}^2\Big), \\ X_{4,n+1} &= \frac{3}{2}\Big(- g_1\mathcal{H}\text{det}A + \frac{1}{6}g_2\mathcal{H}B^3 + h_1\mathcal{H}^2\Big), \\ X_{\delta L} &= -\frac{3}{4}\mathcal{H}(g_1\epsilon_{\alpha\beta\gamma}\epsilon_{\delta\beta'\gamma'}A_{\beta\beta'}A_{\gamma\gamma'}B_{\alpha L} \\ &\quad -g_2 f_{IJK}B_{\beta J}B_{\gamma K}\epsilon_{\delta\beta\gamma}C_{IL}). \end{aligned} \tag{4.100}$$

By using $\mathcal{V}$'s given above and computing $\tilde{\mathcal{P}}^A$ from (4.84), we find that

$$\tilde{\mathcal{V}}^{\underline{n}}{}_A\tilde{\mathcal{P}}^A_\mu = 0 \tag{4.101}$$

So, there is no coupling term between scalars and gauge field strength in (4.83). Another consequence of this is that the scalar metric G_{AB} in (4.83) is effectively δ_{AB}. We can also compute the scalar manifold metric which is given by the general expression

$$ds^2 = \frac{1}{8}\text{Tr}(\tilde{L}^{-1}d\tilde{L}|_Y\tilde{L}^{-1}d\tilde{L}|_Y) \tag{4.102}$$

where $|_Y$ means that we take the coset component of the corresponding one-form. Using the relation $\mathbf{A}^t\frac{1}{\sqrt{\mathbf{I}_n-\mathbf{A}\mathbf{A}^t}} = \frac{1}{\sqrt{\mathbf{I}_3-\mathbf{A}^t\mathbf{A}}}\mathbf{A}^t$, we find, after a straightforward calculation,

$$\tilde{L}^{-1}d\tilde{L}|_Y = \left(\begin{array}{c|c|c|c} 0 & & \frac{1}{\sqrt{\mathbf{I}_3-\mathbf{A}^t\mathbf{A}}}d\mathbf{A}^t\frac{1}{\sqrt{\mathbf{I}_n-\mathbf{A}\mathbf{A}^t}} & \\ \hline & 0 & & \sqrt{2}d\Phi \\ \hline \frac{1}{\sqrt{\mathbf{I}_n-\mathbf{A}\mathbf{A}^t}}d\mathbf{A}\frac{1}{\sqrt{\mathbf{I}_3-\mathbf{A}^t\mathbf{A}}} & & 0 & \\ \hline & \sqrt{2}d\Phi & & 0 \end{array}\right) \tag{4.103}$$

where we have given only the coset components to simplify the equation. The scalar metric is then given by

$$ds^2 = \frac{1}{2}d\Phi d\Phi + \frac{1}{4}\text{Tr}\Big(\frac{1}{\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}}d\mathbf{A}^t\frac{1}{\mathbf{I}_n - \mathbf{A}\mathbf{A}^t}d\mathbf{A}\Big). \tag{4.104}$$

This is exactly the same scalar metric appearing in the scalar kinetic terms in (4.74). The scalar matrix appearing in the gauge field kinetic terms can be computed as follows. From (4.83) and (4.84), we can write

$$\mathbf{M}_{mn} = \bar{\mathbf{M}}^{-1}_{mn} \qquad \text{where} \qquad \bar{\mathbf{M}}_{mn} = \tilde{\mathcal{V}}^{\underline{m}}{}_A \tilde{\mathcal{V}}^{\underline{n}}{}_A\,. \tag{4.105}$$

In our case, the indices $\underline{m}, \underline{n} = b_1, b_2$, and $m, n = a_1, a_2$. With properly normalized coset generators Y^A, we find that

$$\bar{\mathbf{M}} = \begin{pmatrix} \bar{\mathbf{M}}_{a_1a_1} & \bar{\mathbf{M}}_{a_1a_2} \\ \bar{\mathbf{M}}_{a_2a_1} & \bar{\mathbf{M}}_{a_2a_2} \end{pmatrix}, \qquad \bar{\mathbf{M}}_{a_{\hat{i}}a_{\hat{j}}} = \tilde{\mathcal{V}}^{b_{\hat{i}}}{}_A \tilde{\mathcal{V}}^{b_{\hat{j}}}{}_A$$
$$\text{where } \hat{i},\, \hat{j} = 1, 2 \text{ and } \tilde{\mathcal{V}}^{b_{\hat{i}}}{}_A = \mathrm{Tr}(\tilde{L}^{-1} J_{b_{\hat{i}}} \tilde{L} Y^A). \tag{4.106}$$

After some algebra, we find that the matrix $\mathbf{M}_{a_{\hat{i}}a_{\hat{j}}}$ is the same as $\mathcal{M}_{\mathcal{AB}}$ in (4.75). So, the reduced scalar coset from the Chern-Simons gauged theory is the same as that in the Yang-Mills gauged theory obtained form the $SU(2)$ reduction.

Finally, we have to check the scalar potential. From the embedding tensor, we can compute the potential by using the formula

$$V = A_1^{\bar{I}\bar{J}} A_1^{\bar{I}\bar{J}} - 2g^{ij} A_{2i}^{\bar{I}\bar{J}} A_{2j}^{\bar{I}\bar{J}}. \tag{4.107}$$

Using (4.107) and the above expressions, we can compute the potential

$$V = 16T^2 - 2\Big(\frac{1}{4} T_i^{\bar{I}\bar{J}} T_i^{\bar{I}\bar{J}} + \frac{1}{9} X_i X_i\Big). \tag{4.108}$$

After some manipulations, we can show that the resulting potential is the same as (4.68) with the following identifications

$$\mathcal{H} \to N^{\frac{1}{6}} e^{-4g} = e^{-\sqrt{2}\Phi}, \qquad A \to \left(\begin{array}{c|c} \frac{1}{\sqrt{\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}}} & 0 \\ \hline 0 & \cosh\sqrt{2}\Phi \end{array}\right),$$
$$B \to \left(\begin{array}{c|c} \frac{1}{\sqrt{\mathbf{I}_3 - \mathbf{A}^t\mathbf{A}}}\mathbf{A}^t & 0 \\ \hline 0 & \sinh\sqrt{2}\Phi \end{array}\right), \qquad C \to \left(\begin{array}{c|c} \frac{1}{\sqrt{\mathbf{I}_n - \mathbf{A}\mathbf{A}^t}} & 0 \\ \hline 0 & \cosh\sqrt{2}\Phi \end{array}\right). \tag{4.109}$$

We have now completely shown that the Chern-Simons gauged theory constructed in this section is the same as the Yang-Mills gauged theory obtained from the $SU(2)$ reduction in the previous section.

In the case where $G = SU(2)$, the $SU(2) \times SU(2)$ Yang-Mills gauged theory is the same as $(SU(2) \ltimes \mathbf{T}^3)^2$ Chern-Simons gauged theory with scalar manifold $\frac{SO(4,4)}{SO(4)\times SO(4)}$. Such quaternionic space has been already considered in the previous chapter, however the $(SU(2) \ltimes \mathbf{T}^3)^2$ gauging appearing there is different from the one in this chapter. The two gauged $SU(2)$'s in that case are the diagonal subgroups of the two $SU(2)_L$ and the two $SU(2)_R$ respectively of the $SO(4) \times SO(4)$. We can again construct the $N = 4$ theory using the parametrization of the target space in terms of e and B matrices as in the previous chapter. For completeness, we give the necessary ingredients here. All notations are the same as in chapter 3 except that there is only one target space. The $\mathcal{V}$'s are given by

$$\begin{aligned}
\mathcal{V}_{\pm\mathrm{a}}^{LJ,MK} &= -\frac{1}{4}\mathrm{Tr}[(e^t J_+^{LJ} X^t + X J_+^{LJ} e) J_\pm^{MK}], \\
\mathcal{V}_{\pm\mathrm{a}}{}_{ab}^{MK} &= \mathrm{Tr}[(e^t \varepsilon_{ab} X^t + Y \varepsilon_{ab} e) J_\pm^{MK}], \\
\mathcal{V}_{\pm\mathrm{b}}^{LJ,MK} &= -\frac{1}{4}\mathrm{Tr}[e^t J_+^{LJ} e^t J_\pm^{MK}], \\
\mathcal{V}_{\pm\mathrm{b}}{}_{ab}^{MK} &= \mathrm{Tr}(e^t \varepsilon_{ab} e J_\pm^{MK}).
\end{aligned} \tag{4.110}$$

It is then straightforward to compute the T-sensors with the embedding tensor as in the previous chapter. These are given by

$$\begin{aligned}
T^{LJ,MK} &= g_{1s}(\mathcal{V}_{+\mathrm{a}}^{LJ,PQ}\mathcal{V}_{+\mathrm{b}}^{MK,PQ} + \mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+\mathrm{a}}^{MK,PQ}) + g_{1a}(\mathcal{V}_{-\mathrm{a}}^{LJ,PQ}\mathcal{V}_{-\mathrm{b}}^{MK,PQ} \\
&\quad + \mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-\mathrm{a}}^{MK,PQ}) + h_{1s}\mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+\mathrm{b}}^{MK,PQ} + h_{1a}\mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-\mathrm{b}}^{MK,PQ}, \\
T_{ab}^{LJ} &= g_{1s}(\mathcal{V}_{+\mathrm{a}}^{LJ,PQ}\mathcal{V}_{+\mathrm{b}}{}_{ab}^{PQ} + \mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+\mathrm{a}}{}_{ab}^{PQ}) + g_{1a}(\mathcal{V}_{-\mathrm{a}}^{LJ,PQ}\mathcal{V}_{-\mathrm{b}}{}_{ab}^{PQ} \\
&\quad + \mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-\mathrm{a}}{}_{ab}^{PQ}) + h_{1s}\mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+\mathrm{b}}{}_{ab}^{PQ} + h_{1a}\mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-\mathrm{b}}{}_{ab}^{PQ}.
\end{aligned} \tag{4.111}$$

The consistency condition from supersymmetry requires that

$$h_{1a} = -h_{1s}. \tag{4.112}$$

Together with the f^{IJ} in chapter 3, we can compute A_1 and A_2 tensors as well as the scalar potential. The action of shift symmetry generators is to shift B. We can simply set $B = 0$ in this parametrization to obtain the Yang-Mills coset. Although the identification of (A_α^I, Φ) and e is more complicated than the previous case, with the help of *Mathematica*, it can be shown that the two theories are indeed equivalent.

In this chapter, we have obtained Yang-Mills $SU(2)\times G$ gauged supergravity in three dimensions from $SU(2)$ group manifold reduction of six dimensional (1,0) supergravity coupled to an anti-symmetric tensor and G Yang-Mills multiplets. We have also given consistent truncations in both bosonic and fermionic fields from which the resulting consistent reduction ansatz followed. The truncation, which removes three dimensional massive vector fields, results in an $N = 4$ supergravity theory describing $4(1 + \dim G)$ bosonic propagating degrees of freedom, $1 + 3\dim G$ scalars and $3 + \dim G$ gauge fields, together with $4(1 + \dim G)$ fermions. The scalar fields are coordinates in the coset space $\mathbf{R} \times \frac{SO(3,\dim G)}{SO(3)\times SO(\dim G)}$.

Furthermore, we have explicitly constructed the $N = 4$ Chern-Simons $(SO(3)\ltimes \mathbf{T}^3) \times (G \ltimes \mathbf{T}^{\dim G})$ gauged supergravity in three dimensions, following the general procedure detailed in [31]. The scalar manifold $\frac{SO(4,1+\dim G)}{SO(4)\times SO(1+\dim G)}$ becomes $\mathbf{R} \times \frac{SO(3,\dim G)}{SO(3)\times SO(\dim G)}$ after removing the scalars corresponding to the translations or shift symmetries. We have shown the agreement between the resulting Lagrangian and the Lagrangian obtained from dimensional reduction i.e. the gauge field kinetic terms, the scalar manifold metrics and scalar potentials.

We have not given the supersymmetry transformations of the three dimensional fields here. These can, in principle, be obtained by direct computations or using the results in [113] with our truncations. Although supersymmetry transformations of fermions are essential, for example for finding BPS solutions, it is more convenient to work with the equivalent Chern-Simons gauged theory as the latter turns out to be simpler than the equivalent Yang-Mills theory, see [38] for a discussion. In particular, the consistency of the Chern-Simons gauging is encoded in a single algebraic condition on the embedding tensor [27, 28, 29, 31].

Chapter 5

Two dimensional RG flows and Yang-Mills instantons

In this chapter, we will study RG flow solutions in (1,0) six dimensional supergravity coupled to an anti-symmetric tensor and Yang-Mills multiplets corresponding to a semisimple group G. We turn on G instanton gauge fields, with instanton number N, in the conformally flat part of the 6D metric. The solution interpolates between two (4,0) supersymmetric $AdS_3 \times S^3$ backgrounds and describes an RG flow in the dual two dimensional SCFT. For the single instanton case and $G = SU(2)$, there exist a consistent reduction ansatz to three dimensions as shown in the previous chapter, and the solution in this case can be interpreted as an uplifted three dimensional solution. Correspondingly, we present the solution in the framework of $N = 4$ $(SU(2) \ltimes \mathbf{T}^3)^2$ three dimensional gauged supergravity. We will also give an interpretation of the supergravity solution in terms of the D1/D5 system in type I string theory on K3, whose effective field theory is expected to flow to a (4,0) SCFT in the infrared.

5.1 An RG flow solution from six dimensional supergravity on $SU(2)$ group manifold

In the ansatz for the flow solution, we set $A^I_\mu = A^\alpha_\mu = 0$, $A^I_\alpha = \delta^I_\alpha A$ and $L^i_\alpha = \delta^i_\alpha$ in the reduction ansatz of the previous chapter. After various scalings together with the standard domain wall ansatz for the metric in three dimensions, we end up with

$$\begin{aligned} ds^2 &= e^{2w(r)}(-dx_0^2+dx_1^2)+dr^2+\frac{e^{2q(r)}}{4g_1^2}\delta_{\alpha\beta}\sigma^\alpha\sigma^\beta, \\ \hat{A}^I &= A^I_\alpha\sigma^\alpha = A\delta^I_\alpha\sigma^\alpha = A\sigma^I, \\ \hat{F}^I &= \frac{1}{g_1}dA\wedge\sigma^I+\frac{1}{2g_1^2}(g_2A^2-g_1A)\epsilon_{IJK}\sigma^J\wedge\sigma^K, \\ \hat{G}_3 &= h\varepsilon_3+\frac{1}{6g_1^3}\left(g_1+2g_2A^3-3g_1A^2\right)\epsilon_{IJK}\sigma^I\wedge\sigma^J\wedge\sigma^K\,. \end{aligned} \quad (5.1)$$

The scalings have been performed to restore the g_1 and g_2 in the appropriate positions in the solution. This makes the comparison with the three dimensional solution given in section 5.2 more evident. Notice the particular ansatz for A^I_α which gives $K_{\alpha\beta}=0$. This is the reason for the consistency of the truncation of the three dimensional gauge fields, $A^I_\mu=0$ and $A^\alpha_\mu=0$. It can be easily checked that all the three dimensional field equations given in chapter 4 are satisfied by our ansatz. The S^3 part of the metric and that in the previous chapter are related by

$$\frac{e^{2q}}{4g_1^2}\delta_{\alpha\beta}=e^{2g}h_{\alpha\beta}=2e^\theta(1-A^2)\delta_{\alpha\beta} \quad (5.2)$$

where we have used the bosonic truncation relation (4.40) after scalings. We will see later that our solution satisfies this relation and is indeed a solution of the theory obtained in the previous chapter. The supersymmetric flow solution can be found by considering the Killing spinor equations coming from the supersymmetry transformation of fermions. From the metric, we can read off the vielbeins

$$\hat{e}^a=e^w dx^\mu,\;\; \hat{e}^{\hat{r}}=dr,\;\;\text{and}\;\; \hat{e}^i=\frac{e^q}{2g_1}\sigma^i \quad (5.3)$$

We can compute the following spin connections

$$\begin{aligned} \hat{\omega}^{a\hat{r}} &= w'\hat{e}^a, \\ \hat{\omega}^{\hat{r}i} &= -q'\hat{e}^i, \\ \hat{\omega}^{ij} &= -g_1e^{-q}\epsilon_{ijk}\hat{e}^k\,. \end{aligned} \quad (5.4)$$

The index a is the tangent space index for $\mu=0,1$, and $'$ means $\frac{d}{dr}$. For conveniences, we also repeat here the decompositions of the six dimensional gamma matrices from [47]

$$\begin{aligned} \Gamma^{\hat{A}} &= (\Gamma^a,\Gamma^i), \qquad \Gamma^a=\gamma^a\otimes\mathbf{I}_2\otimes\sigma_1, \\ \Gamma^i &= \mathbf{I}_2\otimes\gamma^i\otimes\sigma_2, \qquad \Gamma_7=\mathbf{I}_2\otimes\mathbf{I}_2\otimes\sigma_3, \\ \gamma^{abc} &= \epsilon^{abc}, \qquad \gamma^{ijk}=i\epsilon^{ijk}, \qquad \{\gamma_a,\gamma_b\}=2\eta_{ab},\;\;\{\gamma_i,\gamma_j\}=2\delta_{ij} \end{aligned} \quad (5.5)$$

with the same conventions given in [47]. We further specify the three dimensional gamma matrices by the following choice

$$\gamma^0 = i\tilde{\sigma}_2, \qquad \gamma_1 = \tilde{\sigma}_1, \qquad \gamma^2 = \tilde{\sigma}_3\,. \tag{5.6}$$

Using (4.36), (4.37) and (4.38), we find

$$\begin{aligned}
\delta\lambda^I = 0 \quad &: \quad A' = -2(g_2A^2 - g_1A)e^{-q}, \\
\delta\chi = 0 \quad &: \quad \theta' = e^{\theta}\big[h - 8e^{-3q}\big(g_1 + 2g_2A^3 - 3g_1A^2\big)\big], \\
\delta\psi_i = 0 \quad &: \quad q' = -g_1e^{-q} + \frac{1}{2}\big[e^{\theta}\big[h + 8e^{-3q}\big(g_1 + 2g_2A^3 - 3g_1A^2\big)\big]\big], \\
\delta\psi_a = 0 \quad &: \quad w' = -\frac{1}{2}e^{\theta}[h + 8e^{-3q}(g_1 + 2g_2A^3 - 3g_1A^2)]
\end{aligned} \tag{5.7}$$

where we have used $\tilde{\sigma}_3 \otimes \mathbf{I}_2 \otimes \mathbf{I}_2\epsilon = \epsilon$. So, the solution preserves half of the (1,0) supersymmetry in six dimensions. We fix h by using the equation of motion for $\hat{G}_3$

$$\hat{D}(e^{2\hat{\theta}}\hat{*}\hat{G}_3) = 0\,. \tag{5.8}$$

This gives $he^{3q+2\theta} = c_1$ with a constant c_1. Using this result and changing the coordinate r to $\tilde{r}$ given by $\frac{d\tilde{r}}{dr} = e^{-q}$, we find that the above equations can be rewritten as

$$\theta' = e^{\theta-2q}(c_1e^{-2\theta} - 8\tilde{a})\,, \tag{5.9}$$

$$q' = -g_1 + \frac{1}{2}e^{\theta-2q}(e^{-2\theta}c_1 + 8\tilde{a})\,, \tag{5.10}$$

$$w' = -\frac{1}{2}e^{\theta-2q}(c_1e^{-2\theta} + 8\tilde{a})\,, \tag{5.11}$$

$$A' = -2(g_2A^2 - g_1A), \tag{5.12}$$

where $\tilde{a} = g_1 + 2g_2A^3 - 3g_1A^2$. The $'$ is now $\frac{d}{d\tilde{r}}$. Before solving these equations, let us look at the fixed points given by the conditions $\theta' = q' = A' = 0$. There are two fixed points:

- I:

$$\begin{aligned}
A &= 0, \qquad \theta = \frac{1}{2}\ln\frac{c_1}{8g_1}, \\
q &= \frac{1}{4}\ln\frac{8c_1}{g_1},
\end{aligned} \tag{5.13}$$

- II:

$$\begin{aligned}
A &= \frac{g_1}{g_2}, \qquad \theta = \frac{1}{2}\ln\frac{c_1g_2^2}{8g_1(g_2^2 - g_1^2)}, \\
q &= \frac{1}{4}\ln\frac{8c_1(g_2^2 - g_1^2)}{g_1g_2^2}.
\end{aligned} \tag{5.14}$$

Equation (5.12) can be solved and gives

$$A = \frac{g_1}{g_2 - e^{g_1 C_2 - 2g_1\tilde{r}}}. \tag{5.15}$$

Taking the combination (5.9)+ 2 (5.10), we find

$$z' = 2e^{-z}c_1 - 2g_1 \tag{5.16}$$

where $z = \theta + 2q$. From (5.16), we find the solution for z is

$$z = \ln \frac{c_1 - e^{-2g_1\tilde{r}+C_3}}{g_1}. \tag{5.17}$$

From (5.15), we see that the fixed point I is at $\tilde{r} \to -\infty$ while the II point is at $\tilde{r} \to \infty$. Regularity of A requires that $-e^{g_1 C_2}$ must have the same sign as g_2. For convenience, we choose

$$C_2 = \frac{1}{g_1}\ln(-g_2).$$

From (5.17), z blows up as $\tilde{r} \to -\infty$, so the solution breaks down at the I point. To overcome this problem, we choose z to be constant in such a way that (5.16) is satisfied identically. This can be achieved by setting

$$z = \ln \frac{c_1}{g_1}. \tag{5.18}$$

This means $\theta = \ln \frac{c_1}{g_1} - 2q$. We can see that this condition is satisfied at both fixed points, and equations (5.10) and (5.9) collapse to a single equation namely

$$q' = 4e^{-4q}\frac{c_1}{g_1}\left(g_1 - \frac{e^{4g_1\tilde{r}}(3 + e^{2g_1\tilde{r}})g_1^3}{g_2^2(1 + e^{2g_1\tilde{r}})^3}\right) - \frac{g_1}{2}. \tag{5.19}$$

This equation can be solved, and we find

$$q = \frac{1}{4}\ln\left[8e^{-2g_1\tilde{r}}\left(c_1 e^{2g_1\tilde{r}}\left(\frac{1}{g_1} - \frac{g_1}{g_2^2}\right) - \frac{c_1 g_1(2 + 3e^{2g_1\tilde{r}})}{g_2^2(1 + e^{2g_1\tilde{r}})^2} + \frac{54C_4}{g_2^2}\right)\right]. \tag{5.20}$$

In order to make the solution for q interpolates between the two values at both fixed points, we need to choose

$$C_4 = \frac{c_1 g_1}{27}. \tag{5.21}$$

We finally find

$$A = \frac{g_1}{g_2(1 + e^{-2g_1\tilde{r}})} \tag{5.22}$$

$$q = \frac{1}{4}\ln\frac{8c_1(g_2^2 + 2g_2^2 e^{2g_1\tilde{r}} + (g_2^2 - g_1^2)e^{4g_1\tilde{r}})}{g_1 g_2^2(1 + e^{2g_1\tilde{r}})^2} \tag{5.23}$$

$$w = -q - g_1\tilde{r} \tag{5.24}$$

We neglect all additive constants to w because they can be absorbed in the rescaling of x_0 and x_1. The solution for q approaches the fixed point I and II as $\tilde{r} \to \mp\infty$, respectively.

At the fixed points, the six dimensional metric is given by

$$ds^2 = e^{-2q_0 - 2g_1\tilde{r}} dx_{1,1}^2 + e^{2q_0} d\tilde{r}^2 + \frac{e^{2q_0}}{4g_1^2} \delta_{\alpha\beta}\sigma^\alpha\sigma^\beta \tag{5.25}$$

where q_0 is the value of q at the fixed points. By rescaling the x^μ and $\tilde{r}$ by a factor of e^{-q_0} and $-e^{q_0}$, respectively, we can write (5.25) as

$$ds^2 = e^{\frac{2\bar{r}}{L}} dx_{1,1}^2 + d\bar{r}^2 + \frac{R^2}{4} \delta_{\alpha\beta}\sigma^\alpha\sigma^\beta \tag{5.26}$$

which is the $AdS_3 \times S^3$ metric. The radii of AdS_3 and S^3 are given by $L = \frac{e^{q_0}}{g_1}$ and $R = \frac{e^{q_0}}{g_1}$, respectively. The central charge in the dual CFT is given by [40]

$$c = \frac{3L}{2G_N^{(3)}} \sim e^{4q_0}$$

where we have used the relation between Newton constants in three and six dimensions $G_N^{(3)} = \frac{G_N^{(6)}}{\text{Vol}(S^3)}$. We find the ratio of the central charges

$$\frac{c_{\text{I}}}{c_{\text{II}}} = \frac{e^{4q_0}|_{\text{I}}}{e^{4q_0}|_{\text{II}}} = \frac{1}{1 - \frac{g_1^2}{g_2^2}} > 1. \tag{5.27}$$

From this equation, we find that the flow respects the c-theorem as it should, and point I is the UV point while point II is the IR point. Note that $d\bar{r} = -e^q d\tilde{r}$, so the UV and IR points correspond to $\bar{r} \to \pm\infty$. We can interpret $\bar{r}$ as an RG scale in the dual two dimensional field theory. From the solutions for q, θ and A, we can check that the relation (5.2) is satisfied. So, the solution is indeed a solution of the theory considered in chapter 4 and can be obtained from three dimensional gauged supergravity. We will give this solution in the three dimensional framework in the next section.

We briefly look at the behavior of the scalar fields near the UV point I. From (5.22) and (5.23), we find that

$$A \sim e^{2g_1\tilde{r}} \sim e^{-\frac{2\bar{r}}{L}} \text{ and } e^q \sim e^{-\frac{2\bar{r}}{L}}. \tag{5.28}$$

We can see that the flow is driven by a vacuum expectation value of a marginal operator of dimension two. Although this is not expected, we will confirm this fact in section 5.2 in which we will reobtain this solution in the three dimensional gauged supergravity. So, this flow is a vev flow driven by a vacuum expectation value of a marginal operator. Notice that gauge-field background of (5.1) that we have found corresponds to a single $SU(2)$ instanton on the four-space (r, S^3), interpolating between winding number 0 for $\bar{r} \rightarrow -\infty$ and winding number 1 for $\bar{r} \rightarrow +\infty$. In the next section we will generalize this result to a multi-instanton configuration for semisimple G gauge fields, which therefore will not admit a three dimensional interpretation.

Throughout this section, we have mainly studied the flow solution in the context of the $SU(2)$ reduction to three dimensions. This leads to the form of the solution given above. Before discussing the multi-instanton case, we would like to change the form of the solution to make contact with what we will find in the next section. First of all, we can change the coordinates in (5.1) to R given by

$$\frac{dR}{dr} = -g_1 R e^{-q}\,. \tag{5.29}$$

We have put a minus sign in order to identify the UV point with $R \rightarrow \infty$ and the IR with $R \rightarrow 0$. We then find that the metric is given by

$$ds^2 = e^{2w}(-dx_0^2 + dx_1^2) + \frac{e^{2q}}{g_1 R^2} dy^i dy^i \tag{5.30}$$

where $dy^i dy^i = dR^2 + \frac{R^2}{4}\sigma^\alpha\sigma^\alpha$ is the flat metric of the four dimensional space. This is the form of the metric we will see in section 5.3 in which the 4-dimensional part is conformally flat. The second point is the solution for A in (5.15). Recall that the relation between R and $\tilde{r}$ is $\frac{dR}{d\tilde{r}} = -g_1 R$, we can write

$$A = \frac{\lambda^2}{g_2 R(\lambda^2 + R^2)} \tag{5.31}$$

where we have chosen $C_2 = \frac{1}{g_1}\ln\left(-\frac{g_2}{\lambda^2}\right)$. This is a single instanton solution at the origin $R = 0$ in the polar coordinates. Notice that this is the instanton solution in the singular gauge in which the winding number come from the contribution near $R = 0$. In section 5.3, we will study a flow solution with N instantons but in the regular gauge.

5.2 Flow solution from $N = 4$ three dimensional gauged supergravity

In this section, we study a flow solution in $N = 4$ three dimensional $(SO(3) \ltimes \mathbf{T}^3) \times (G \ltimes \mathbf{T}^{\dim G})$ Chern-Simons gauged supergravity. As shown in chapter 4, this theory is equivalent to $SO(3) \times G$ Yang-Mills gauged theory obtained from $SU(2)$ reduction of six dimensional supergravity whose flow solution has been studied in the previous section. We are interested in the case $G = SO(3)$. We will see that the solution we are going to find is the same as that in section 5.1 but now in another framework.

5.2.1 $(SO(3) \ltimes \mathbf{T}^3) \times (SO(3) \ltimes \mathbf{T}^3)$ gauged supergravity

We now construct three dimensional gauged supergravity with gauge groups $(SO(3)\ltimes \mathbf{R}^3) \times (SO(3) \ltimes \mathbf{R}^3)$. This can be obtained from the theory constructed in chapter 4 by setting $G = SO(3)$. In the following, we will focus on this spacial case. The scalar fields are described by $\frac{SO(4,4)}{SO(4)\times SO(4)}$ coset manifold. We parametrize the coset by

$$L = \begin{pmatrix} \frac{1}{1-A(r)^2}\mathbf{I}_{3\times 3} & 0 & \frac{A(r)}{1-A(r)^2}\mathbf{I}_{3\times 3} & 0 \\ 0 & \cosh h(r) & 0 & \sinh h(r) \\ \frac{A(r)}{1-A(r)^2}\mathbf{I}_{3\times 3} & 0 & \frac{1}{1-A(r)^2}\mathbf{I}_{3\times 3} & 0 \\ 0 & \sinh h(r) & 0 & \cosh h(r) \end{pmatrix}. \tag{5.32}$$

The $SO(3)$ generators are given by

$$J^A_{a_1} = \epsilon_{ABC}e_{BC}, \qquad J^A_{a_2} = \epsilon_{ABC}e_{B+4,C+4}, \;\; A, B, C = 1, 2, 3 \tag{5.33}$$

where $(e_{AB})_{CD} = \delta_{AC}\delta_{BD}$ are 8×8 matrices. The translational symmetries are generated by

$$\begin{aligned} J^A_{b_1} &= -e_{A4} + e_{A8} + e_{4A} + e_{8A} \\ J^A_{b_2} &= -e_{4,4+A} + e_{A+4,4} - e_{A+4,8} + e_{8,A+4}, \;\; A = 1, 2, 3. \end{aligned} \tag{5.34}$$

$SO(4)$ R-symmetry generators are given by

$$J^{IJ}_{\pm} = J^{IJ} \pm \frac{1}{2}\epsilon_{IJKL}J^{KL}, \qquad J^{IJ} = e_{IJ} - e_{JI}, \;\; I, J, \ldots = 1, \ldots 4. \tag{5.35}$$

In the $N = 4$ theory, the $SO(4)$ R-symmetry decomposes to $SO(3)_+$ and $SO(3)_-$, and each factors acts separately on the two scalar target spaces. In our case called the degenerate case, there is only one target space, so we have only one $SO(3)$ which we will denote by $SO(3)_+$. Non compact generators of $SO(4, 4)$ are

$$Y_{ab} = e_{a,b+4} + e_{b+4,a}, \;\; a, b = 1, 2, 3. \tag{5.36}$$

We now proceed by computing the T-tensors which are given by

$$\begin{aligned} T^{IJ,KL} &= g_1(\mathcal{V}^{AIJ}_{\mathrm{a}_1}\mathcal{V}^{AKL}_{\mathrm{b}_1} + \mathcal{V}^{AIJ}_{\mathrm{b}_1}\mathcal{V}^{AKL}_{\mathrm{a}_1}) + g_2(\mathcal{V}^{AIJ}_{\mathrm{a}_2}\mathcal{V}^{AKL}_{\mathrm{b}_2} + \mathcal{V}^{AIJ}_{\mathrm{b}_2}\mathcal{V}^{AKL}_{\mathrm{a}_2}) \\ &\quad +h_1\mathcal{V}^{AIJ}_{\mathrm{b}_1}\mathcal{V}^{AKL}_{\mathrm{b}_1} + h_2\mathcal{V}^{AIJ}_{\mathrm{b}_2}\mathcal{V}^{AKL}_{\mathrm{b}_2}, \end{aligned} \tag{5.37}$$

$$\begin{aligned} T^{IJ}_{ab} &= g_1(\mathcal{V}^{AIJ}_{\mathrm{a}_1}\mathcal{V}_{\mathrm{b}_1}{}^{A}_{ab} + \mathcal{V}^{AIJ}_{\mathrm{b}_1}\mathcal{V}_{\mathrm{a}_1}{}^{A}_{ab}) + g_2(\mathcal{V}^{AIJ}_{\mathrm{a}_2}\mathcal{V}_{\mathrm{b}_2}{}^{A}_{ab} + \mathcal{V}^{AIJ}_{\mathrm{b}_2}\mathcal{V}_{\mathrm{a}_2}{}^{A}_{ab}) \\ &\quad +h_1\mathcal{V}^{AIJ}_{\mathrm{b}_1}\mathcal{V}_{\mathrm{b}_1}{}^{A}_{ab} + h_2\mathcal{V}^{AIJ}_{\mathrm{b}_2}\mathcal{V}_{\mathrm{b}_2}{}^{A}_{ab}. \end{aligned} \tag{5.38}$$

With the coset representative L, we can compute all the needed quantities using

$$\begin{aligned} L^{-1}D_\mu L &= \frac{1}{2}Q^{IJ}_\mu X^{IJ} + Q^\alpha_\mu X^\alpha + e^A_\mu Y^A\,, \\ L^{-1}t^{\mathcal{M}}L &= \frac{1}{2}\mathcal{V}^{\mathcal{M}IJ}X^{IJ} + \mathcal{V}^{\mathcal{M}}_{\ \alpha}X^\alpha + \mathcal{V}^{\mathcal{M}}_{\ A}Y^A\,. \end{aligned} \tag{5.39}$$

The first equation is simply equation (2.55) pulled back to the spacetime, and the second equation is repeated for reader's convenience. The consistency condition from supersymmetry, $\mathbb{P}_{\boxplus}T^{IJ,KL}=0$, requires that $h_2=-h_1$. The resulting $\mathcal{V}_{\mathcal{A}}^{\ \mathcal{M}}$ are given by

$$\begin{aligned}
\mathcal{V}_{\text{a}_{1,2}}^{AKL} &= -\frac{1}{2}\text{Tr}[L^{-1}J^{A}_{\text{a}_{1,2}}LJ^{KL}_{+}],\\
\mathcal{V}_{\text{b}_{1,2}}^{AKL} &= -\frac{1}{2}\text{Tr}[L^{-1}J^{A}_{\text{b}_{1,2}}LJ^{KL}_{+}],\\
\mathcal{V}_{\text{a}_{1,2}}^{Aab} &= \frac{1}{2}\text{Tr}[L^{-1}J^{A}_{\text{a}_{1,2}}L(e_{a,b+4}+e_{b+4,a})],\\
\mathcal{V}_{\text{b}_{1,2}}^{Aab} &= \frac{1}{2}\text{Tr}[L^{-1}J^{A}_{\text{b}_{1,2}}L(e_{a,b+4}+e_{b+4,a})]
\end{aligned} \tag{5.40}$$

where $A,B,\ldots=1,2,3$ label $SO(3)$ gauge generators, and a pair of indices $a,b,\ldots=1,2,3$ labels target space coordinates. With an appropriate normalization, the tensor f^{IJ} is given by

$$f^{IJ}_{ab,cd}=2\text{Tr}(e_{ba}J^{IJ}_{+}e_{cd}). \tag{5.41}$$

With all these ingredients, we can now compute A_1 and A_2 tensors which give the scalar potential via equations (2.88), (2.89) and (2.91). The potential for these two scalars is given by

$$V=e^{-4h}\left[h_1^2+\frac{2e^{2h}}{(A^2-1)^3}(g_1^2+A^2(g_2A(4g_1+g_2A(A^2-3))-3g_1^2))\right]. \tag{5.42}$$

This simple looking potential admits five different critical points. We can identify supersymmetric critical points by using the procedures explained in chapter 3. All non trivial critical points are given in table II.

Critical points	A_0	h_0	V_0	Preserved supersymmetries
I	0	$\ln\left(-\frac{h_1}{g_1}\right)$	$-\frac{g_1^4}{h_1^2}$	non supersymmetric
II	0	$\ln\left(\frac{h_1}{g_1}\right)$	$-\frac{g_1^4}{h_1^2}$	(4,0)
III	$\frac{g_1}{g_2}$	$\ln\left(\frac{\sqrt{-h_1^2(g_1^2-g_2^2)}}{g_1g_2}\right)$	$-\frac{g_1^4g_2^4}{(g_1^2-g_2^2)^2h_1^2}$	(4,0)
IV	$\frac{g_1}{g_2}$	$\ln\left(-\frac{\sqrt{-h_1^2(g_1^2-g_2^2)}}{g_1g_2}\right)$	$-\frac{g_1^4g_2^4}{(g_1^2-g_2^2)^2h_1^2}$	non supersymmetric
V	$\frac{g_2}{g_1}$	$\ln\frac{h_1\sqrt{g_1^2-g_2^2}}{\sqrt{g_1^4-g_1^2g_2^2+g_2^4}}$	$-\frac{g_1^4-g_1^2g_2^2+g_2^4}{(g_1^2-g_2^2)^2h_1^2}$	non supersymmetric

Table II: Critical points of the potential (5.42). A_0 and h_0 are vacuum expectation values at the critical point of A and h, respectively.

5.2.2 An RG flow solution

Supersymmetric flow equations can be obtained from supersymmetry transformations of fermions, $\delta\psi^I_\mu = 0$ and $\delta\chi^{iI} = 0$. The metric ansatz is chosen to be

$$ds^2 = e^{2f} dx^2_{1,1} + dr^2 \,. \tag{5.43}$$

Recall equation (3.20) together with the definition of A_1 and A_2 tensors, we find, from $\delta\chi^{Ii}$,

$$\frac{dA}{dr} = \frac{2e^{-h}A(g_2A - g_1)}{\sqrt{1-A^2}} \tag{5.44}$$

$$\frac{dh}{dr} = \frac{2e^{-2h}}{(1-A^2)^{\frac{3}{2}}}[e^h(g_1 - g_2A^3) + h_1(A^2-1)\sqrt{1-A^2}]. \tag{5.45}$$

We can easily check that (5.44) and (5.45) have two critical points which are exactly the same as II and III points in table II. With a new function g and new coordinate $\tilde{r}$ given by

$$g = h + \ln\sqrt{1-A^2} \qquad \text{and} \qquad d\tilde{r} = e^{-g}dr, \tag{5.46}$$

we can write (5.44) as

$$A' = \frac{dA}{d\tilde{r}} = 2A(g_2A - g_1). \tag{5.47}$$

The solution for A is

$$A = -\frac{g_1}{e^{2g_1\tilde{r}+g_1C_1} - g_2} \,. \tag{5.48}$$

As in section 5.1, we choose $C_1 = \frac{1}{g_1}\ln(-g_2)$ and end up with

$$A = \frac{g_1}{(e^{2g_1\tilde{r}}+1)g_2} \,. \tag{5.49}$$

With (5.46) and (5.49), equation (5.45) can be rewritten as

$$g' = \frac{dg}{d\tilde{r}} = -2\Bigg[g_1\Bigg(-\frac{g_1h_1e^{-g}}{g_2^2(e^{2g_1\tilde{r}}+1)^2} + \frac{g_1+g_2}{g_2e^{2g_1\tilde{r}}+g_1+g_2} + \frac{1}{\frac{g_2e^{2g_1\tilde{r}}}{g_2-g_1}+1} - 2\Bigg) + h_1e^{-g} + g_1\tanh(g_1\tilde{r})\Bigg]. \tag{5.50}$$

This can be solved, and the solution is

$$g = \ln\left[\frac{(h_1 + 16C_2g_1e^{2g_1\tilde{r}})(g_2^2e^{4g_1\tilde{r}} + 2g_2^2e^{2g_1\tilde{r}} - g_1^2 + g_2^2)}{g_1g_2^2(1+e^{g_1\tilde{r}})^2}\right]. \tag{5.51}$$

This solution interpolates between II and III fixed points provided that we choose $C_2 = 0$. We now move to $\delta\psi^I_\mu$. With the solutions for A and g, the gravitino variation gives

$$\frac{df}{dr} = -\frac{g_1^2g_2^2(e^{2g_1r}+1)(g_1^2(e^{2g_1r}-1) + g_2^2(e^{2g_1r}+1)^3)}{h_1(g_2e^{2g_1r}-g_1+g_2)^2(g_2e^{2g_1r}+g_1+g_2)^2} \,. \tag{5.52}$$

After going to $\tilde{r}$ coordinate, we find the solution

$$f = g_1\tilde{r} - \ln[2(1 + e^{2g_1\tilde{r}})] + \frac{1}{2}\ln[2(g_1^2 - g_2^2(1 + e^{2g_1\tilde{r}})^2)] \tag{5.53}$$

where, as usual, we have ignored all additive constants because they can be absorbed by rescaling x^0 and x^1. The AdS_3 radius is given by

$$L = \frac{e^{g_0}}{g_1} = \frac{1}{\sqrt{V_0}}. \tag{5.54}$$

We can compute the ratio of the central charges between the two fixed points

$$\frac{c_{\rm II}}{c_{\rm III}} = \frac{1}{1 - \frac{g_1^2}{g_2^2}} > 1. \tag{5.55}$$

By the c-theorem, we see that point II and III correspond to the UV and IR CFTs, respectively. This is in agreement with the solution found in section 5.1. So, the solutions from both theories are the same. This is the result, at the level of solutions, of the fact that the two theories are equivalent as shown in chapter 4. Near the UV point, the scalars behave as

$$\delta A \sim e^{-2r/L}, \qquad \delta h \sim e^{-4r/L}, \quad L = \frac{h_1}{g_1^2}\,. \tag{5.56}$$

We see that the flow is driven by a marginal operator dual to A of dimension 2. h is dual to an irrelevant operator of dimension 4. Up to quadratic order in the scalars, the potential (5.42) at the UV point is given by

$$V = -\frac{g_1^4}{h_1^2} + \frac{4g_1^4}{h_1^2}h^2\,. \tag{5.57}$$

We find that the scalar A is massless at this point and dual to a marginal operator. The scalar kinetic terms are

$$\mathcal{L}_{\text{scalar kinetic}} = \frac{1}{2}\left(\frac{3A'^2}{(A^2-1)^2} + h'^2\right). \tag{5.58}$$

At the UV point, $A = 0$, all the kinetic terms are canonically normalized, and we can read off the values of mass squared directly from the potential. In the unit of $\frac{1}{L^2}$, h has $m^2L^2 = 8$ which gives exactly $\Delta = 4$ in agreement with the asymptotic behavior. At the IR point, A becomes massive with positive mass squared as can be seen from the expansion of the potential

$$V = -\frac{g_1^4}{h_1^2(1 - \frac{g_1^2}{g_2^2})} + \frac{12g_1^4g_2^8}{(g_1^2 - g_2^2)^4h_1^2}A^2 + \frac{4g_1^4g_2^4}{(g_1^2 - g_2^2)^2h_1^2}h^2\,. \tag{5.59}$$

The positive mass squared means that the potential has a minimum at the IR point, and the dual operator is irrelevant as it should. To compute the mass squared at the IR point, we redefine A to $A = \tanh\phi$. Then, (5.58) becomes

$$\frac{1}{2}(3\phi'^2 + h'^2). \tag{5.60}$$

The potential near the IR point is, to quadratic terms in scalars, given by

$$\frac{V}{V_{0\mathrm{IR}}} = 1 - 4h^2 - \frac{12g_2^4}{(g_1^2 - g_2^2)^2}\phi^2. \tag{5.61}$$

At this fixed point, h and ϕ have $m^2L^2 = 8$ and $m^2L^2 = \frac{24}{1-\frac{g_1^2}{g_2^2}}$, respectively. We find that

$$\Delta_h = 4 \qquad \text{and} \qquad \Delta_\phi = 1 + \sqrt{1 + \frac{24g_2^4}{(g_1^2 - g_2^2)^2}} > 2\,. \tag{5.62}$$

As expected, the operator dual to ϕ is irrelevant with dimension greater than 2.

5.3 RG flow solutions and multi-instantons

In this section, we generalize the solution obtained in section 5.1 by considering the gauge field configuration describing N instantons. We will further make an extension to gauge fields of an arbitrary gauge group G. The solution we will study is very similar to the solution given in [64] and further studied in [65]. In this paper, we give an interpretation of this solution in the context of an RG flow in the dual two dimensional field theory. We start by reobtaining this solution and then discuss its implication in term of the RG flow.

5.3.1 Flow solutions

Since we are going to use the full six-dimensional theory, we will now turn on both v^z and $\tilde{v}^z$. Throughout this section, we also assume that both v^z and $\tilde{v}^z$ are positive. If this is not the case, the phase transition discussed in [64] is unavoidable. To leading order in fermionic fields, the supersymmetry transformations of fermions are the same as the ones previously used in section 5.1. On the other hand, the bosonic field equations are

$$\begin{aligned} \hat{D}(e^{2\hat{\theta}}\hat{*}\hat{G}_3) + \tilde{v}^z\hat{F}^I \wedge \hat{F}^I &= 0, & (5.63)\\ \hat{D}[(v^z e^{\hat{\theta}} + \tilde{v}^z e^{-\hat{\theta}})\hat{*}\hat{F}^I] - 2v^z e^{2\hat{\theta}}\hat{*}\hat{G}_3 \wedge \hat{F}^I + 2\tilde{v}^z\hat{*}\hat{G}_3 \wedge \hat{F}^I &= 0, & (5.64)\\ \hat{d}\hat{*}\hat{d}\hat{\theta} + (v^z e^{\hat{\theta}} + \tilde{v}^z e^{-\hat{\theta}})\hat{*}\hat{F}^I \wedge \hat{F}^I + 2e^{2\hat{\theta}}\hat{*}\hat{G}_3 \wedge \hat{G}_3 &= 0\,. & (5.65) \end{aligned}$$

It is easy to see that if we set $v^z = 1$, $\tilde{v}^z = 0$ and take a spherically symmetric single instanton configuration (i.e. an instanton at the origin of $\mathbb{R}^4$) for the gauge field A^I, then the above equations reduce to the ones discussed in section 5.1. The Bianchi identity is

$$\hat{D}\hat{G}_3 = v^z \hat{F}^I \wedge \hat{F}^I . \tag{5.66}$$

We take an ansatz for the metric as

$$ds_6^2 = e^{2f}(-dx_0^2 + dx_1^2) + ds_4^2 \tag{5.67}$$

where f only depends on the coordinates z^α, $\alpha = 2, \ldots 5$ of the four dimensional metric $ds_4^2 = g_{\alpha\beta} dz^\alpha dz^\beta$. We first look at the $\delta\lambda^I = 0$ equation. We can satisfy this condition by choosing $F^I_{\alpha\beta}$ to be self dual because of the anti-self duality of the $\Gamma_{\alpha\beta}$, $\alpha, \beta = 2, \ldots 5$. The anti-self duality of $\Gamma_{\alpha\beta}$ is implied by the condition $\Gamma_7\epsilon = \epsilon$ and the two-dimensional chirality $\Gamma_{01}\epsilon = \epsilon$ chosen in $\delta\psi_\mu = 0$ below. The indices $I, J \ldots = 1, 2, \ldots, \dim G$ are now G adjoint indices. The gauge fields and three form field strength are

$$\begin{aligned} \hat{A}^I &= A^I, \qquad \hat{F}^I = F^I, \\ \hat{G}_3 &= G + dx_0 \wedge dx_1 \wedge d\Lambda . \end{aligned} \tag{5.68}$$

The hatted fields are six dimensional ones while the unhatted fields represented by differential forms without indices have only components along ds_4^2. The x_0 and x_1 components will be shown explicitly. The three form field satisfies the Bianchi identity (5.66) which gives $D\,G = v^z F^I \wedge F^I$. The dual of $\hat{G}_3$ is

$$\hat{*}\hat{G}_3 = e^{-2f} * d\Lambda - e^{2f} dx_0 \wedge dx_1 \wedge *G \tag{5.69}$$

where $\hat{*}$ and $*$ are Hodge duals in six and four dimensions, respectively. We have used the same convention as [63] namely $\epsilon^{012345} = 1$. Using equation (5.63), we find

$$D(e^{2\theta-2f} * d\Lambda) = \tilde{v}^z F^I \wedge F^I , \tag{5.70}$$

$$D(e^{2\theta+2f} * G) = 0 \Rightarrow *G = e^{-2\theta-2f} d\tilde{\Lambda} . \tag{5.71}$$

We take F^I to be self dual with respect to the four dimensional $*$. This corresponds to an instanton configuration. The dual of $\hat{F}^I$ is given by

$$\hat{*}\hat{F}^I = -e^{2f} dx_0 \wedge dx_1 \wedge *F^I . \tag{5.72}$$

We now come to supersymmetry transformations. Using our ansatz and the results given above, we find the Killing spinor equations

$$\begin{aligned} \delta\chi &= \frac{1}{2}\partial\!\!\!/\theta\epsilon - \frac{1}{12}e^\theta \hat{G}\!\!\!/_3\epsilon \\ &= \frac{1}{2}\partial\!\!\!/\theta + \frac{1}{2}e^{\theta-2f}\partial\!\!\!/\Lambda - \frac{1}{2}e^{-\theta-2f}\partial\!\!\!/\tilde{\Lambda} = 0 \end{aligned} \tag{5.73}$$

$$\begin{aligned} \delta\psi_\mu &= D_\mu\epsilon + \frac{1}{24}e^\theta \hat{G}\!\!\!/_3\Gamma_\mu\epsilon, \ \mu = 0, 1 \\ &= \frac{1}{2}\Gamma_\mu\partial\!\!\!/ f - \frac{1}{4}e^{\theta-2f}\Gamma_\mu\partial\!\!\!/\Lambda - \frac{1}{4}e^{-\theta-2f}\Gamma_\mu\partial\!\!\!/\tilde{\Lambda} = 0 . \end{aligned} \tag{5.74}$$

We have used the notation $\hat{\not{G}}_3 \equiv \Gamma^{MNP}\hat{G}_{3MNP}$ and a projector $\Gamma_{2345}\epsilon = \epsilon$ which is also equivalent to $\Gamma_{01}\epsilon = \epsilon$. This implies that the solution preserves half of the six dimensional supersymmetry. Taking the combination [(5.73) − 2(5.74)], we find

$$2d\Lambda = -e^{-\theta+2f}d(\theta - 2f). \tag{5.75}$$

The solution is easily found to be

$$\Lambda = \frac{1}{2}e^{-\theta+2f} + C_1 \tag{5.76}$$

with a constant C_1. Similarly, the combination [(5.73) + 2(5.74)] gives

$$de^{\theta+2f} - 2d\tilde{\Lambda} = 0 \Rightarrow \tilde{\Lambda} = \frac{1}{2}e^{\theta+2f} + C_2 \tag{5.77}$$

with a constant C_2. The equation from $\delta\psi_\alpha = 0$ gives

$$D_\alpha\epsilon - \frac{1}{2}\not{\partial} f\Gamma_\alpha\epsilon = 0\,. \tag{5.78}$$

We now make the following ansatz for the 4-dimensional metric

$$ds_4^2 = e^{2g}dy^i dy^i. \tag{5.79}$$

With the supersymmetry transformation parameter of the form $\epsilon = e^{\frac{f}{2}}\tilde{\epsilon}$, we can write equation (5.78) as

$$\partial_i\tilde{\epsilon} - \frac{1}{2}\Gamma_{ji}\partial^j(f+g)\tilde{\epsilon} = 0\,. \tag{5.80}$$

To satisfy this equation, we simply choose $g = -f$ and find that $\tilde{\epsilon}$ is a constant spinor. So, we have solved all the Killing spinor equations.

We can easily check that equation (5.64) is identically satisfied with our explicit forms of Λ and $\tilde{\Lambda}$. We now solve equations (5.70) and the Bianchi identity (5.66). We start with the $SU(2)$ instanton configuration from [114]

$$A_i = \frac{i}{2}\bar{\sigma}_{ij}\partial_j \ln\rho, \qquad \rho = 1 + \sum_{a=1}^{n}\frac{\lambda_a^2}{(y-y_a)^2}\,. \tag{5.81}$$

Notice that, we have rescaled the A_i form [114] by a factor of $\frac{1}{2}$. This can be done without loosing generalities, see for example, [115] for a discussion. The $\bar{\sigma}_{ij}$ matrices are anti-self dual and can be written in terms of Pauli matrices σ_x as [114]

$$\bar{\sigma}_{xy} = \frac{1}{4i}[\sigma_x,\sigma_y], \qquad \bar{\sigma}_{x4} = -\frac{1}{2}\sigma_x, \;\; x,y = 1,2,3\,. \tag{5.82}$$

This solution can be generalized to any semi-simple group G to obtain the solution given in [116]. We can write the above $\bar{\sigma}_{ij} = \bar{\eta}^I_{ij}t^I$ where $\bar{\eta}^I_{ij}$ and t^I are 't Hooft's tensor generating $SU(2)$ subgroup of $SO(4)$ and $SU(2)$ generators, respectively. For any group G, the solution is [116]

$$A^I_i = G^I_a \bar{\eta}^a_{ij} \partial_j \ln \rho, \quad a = 1, 2, 3\,. \tag{5.83}$$

For $G = SU(2)$, we simply have $G^I_a = \delta^I_a$. The solution for any group G can be obtained by embedding $SU(2)$ in G [116]. In order to solve (5.70) and (5.66), we need to compute $F^I_{ij} F^{Iij}$. For $SU(2)$ instanton, we have [114]

$$F^I_{ij} F^{Iij} = -\Box\Box \ln \rho\,. \tag{5.84}$$

For G instanton, the result is the same up to some numerical factors, from [116],

$$F^I_{ij} F^{Iij} = -\frac{2}{3} c(G) d(G) \Box\Box \ln \rho\,. \tag{5.85}$$

We now come to the Bianchi identity for G, $DG = v^z F^I \wedge F^I$, which gives

$$\Box e^{-(\theta+2f)} = -v^z F^I_{ij} F^{Iij} = v^z \frac{2}{3} c(G) d(G) \Box\Box \ln \rho \tag{5.86}$$

where we have used [114]

$$*(F^I \wedge F^I) = *(*F^I \wedge F^I) = \frac{1}{2} F^I_{ij} F^{Iij} = -\frac{1}{3} c(G) d(G) \Box\Box \ln \rho\,. \tag{5.87}$$

We can solve (5.86) and obtain

$$\begin{aligned} e^{-(\theta+2f)} &= \frac{d}{r^2} + v^z \left(\frac{2}{3} c(G) d(G) \Box \ln \rho + \sum_{a=1}^{n} \frac{4}{(y-y_a)^2} \right) \\ &\equiv \frac{d}{r^2} + v^z \frac{2}{3} c(G) d(G) \Box \ln \tilde{\rho}\,. \end{aligned} \tag{5.88}$$

We have removed the singularities in the solution by defining $\tilde{\rho} = \rho \prod_{a=1}^{n} (y - y_a)^2$. Inserting Λ from (5.76) in (5.70), with $*d\Lambda$ replaced by $e^{-2f} * d\Lambda$, gives

$$\Box e^{\theta-2f} = -\tilde{v}^z F^I_{ij} F^{Iij} = \tilde{v}^z \frac{2}{3} c(G) d(G) \Box\Box \ln \rho\,. \tag{5.89}$$

The solution is similar to the previous equation

$$e^{\theta-2f} = \frac{c}{r^2} + \tilde{v}^z \frac{2}{3} c(G) d(G) \Box \ln \tilde{\rho} \tag{5.90}$$

where c is an integration constant. The two integration constants c and d are proportional, respectively, to the fluxes of $\hat{*}\hat{G}_3$ and $\hat{G}_3$ through the S^3. Therefore, they represent, respectively, the number of $D1$ and $D5$ branes. We can directly see this by considering for example, the $\hat{G}_3$ flux near $r = 0$

$$Q_1 = \frac{1}{8\pi^2}\int_{S^3} e^{2\theta}\hat{*}\hat{G}_3 = \frac{c}{4} \tag{5.91}$$

$$Q_5 = \frac{1}{8\pi^2}\int_{S^3} \hat{G}_3 = \frac{d}{4}\,. \tag{5.92}$$

We have used the same normalization of Q_1 and Q_5 as in [65]. Indeed, we can regard our six dimensional theory as a subsector of type I theory compactified on $K3$. In this solution, we have $D5$ branes wrapped on $K3$ and $D1$ branes transverse to it. The solution we give here is the same as the gauge dyonic string studied in [112] and [65].

The behaviors of e^{-4f} near $r \to \infty$ and $r \to 0$ are given by

$$r \to \infty : \qquad e^{-4f} \sim \frac{(c + 4\tilde{v}^z N)(d + 4v^z N)}{r^4} = \frac{16(Q_1 + \tilde{v}^z N)(Q_5 + v^z N)}{r^4}, \tag{5.93}$$

$$r \to 0 : \qquad e^{-4f} \sim \frac{cd}{r^4} = \frac{16 Q_1 Q_5}{r^4}\,. \tag{5.94}$$

We have introduced the instanton number N given by

$$N = \frac{1}{32\pi^2}\int d^4y (*F)_{ij}F^{ij} = -\frac{1}{48\pi^2}c(G)d(G)\int d^4y \Box\Box \ln\tilde{\rho}\,. \tag{5.95}$$

At the fixed points, the metric is given by

$$ds_6^2 = \frac{r^2}{L^2}(-dx_0^2 + dx_1^2) + \frac{L^2}{r^2}dr^2 + L^2 ds^2(S^3). \tag{5.96}$$

where we have rewritten the four dimensional flat metric in the polar coordinates

$$dy^i dy^i = dr^2 + r^2 ds^2(S^3). \tag{5.97}$$

The metric (5.96) is readily seen to be $AdS_3 \times S^3$ metric with the AdS_3 and S^3 having the same radius L. The AdS radii at the fixed points near $r \sim \infty$ and $r \sim 0$ are $L = [(c + 4\tilde{v}^z N)(d + 4v^z N)]^{\frac{1}{4}}$ and $L = (cd)^{\frac{1}{4}}$, respectively. In the dual two dimensional conformal field theory, this solutions describes an RG flow from the CFT UV to the CFT IR with the ratio of the central charges

$$\frac{c|_0}{c|_\infty} = \frac{e^{-4f}|_0}{e^{-4f}|_\infty} = \frac{cd}{(c + 4\tilde{v}^z N)(d + 4v^z N)} < 1 \tag{5.98}$$

where we have used the relation between the central charge and AdS radius $c \sim \frac{L}{G_N^{(3)}} \sim \frac{L\mathrm{Vol}(S^3)}{G_N^{(6)}} \sim e^{-4f}$. The UV point corresponds to $r = \infty$ while the IR point is at $r = 0$.

To extract the dimension of the operator driving the flow, we need to consider the behavior of the fluctuation of the metric around $AdS_3 \times S^3$ near the UV point. To simplify the manipulation, we first consider here a single instanton at the origin $y^i_a = 0$. With this simplification, the solution, up to group theory factors which are not relevant for this discussion, is given by

$$e^{-4f} = \left(\frac{c}{r^2} + \tilde{v}^z \Box \ln(r^2 + \lambda^2)\right)\left(\frac{d}{r^2} + v^z \Box \ln(r^2 + \lambda^2)\right). \tag{5.99}$$

As $r \to \infty$, the solution behaves

$$\begin{aligned} e^{-4f} &\sim \frac{(c + 4\tilde{v}^z)(d + 4v^z)}{r^4}\left(1 + \frac{8\lambda^2[c + d + 4(\tilde{v}^z + v^z)]}{r^2(c + 4\tilde{v}^z)(d + 4v^z)} + \ldots\right) \\ \text{or} \qquad e^{2g} &= e^{-2f} \\ &\sim \frac{\sqrt{(c + 4\tilde{v}^z)(d + 4v^z)}}{r^2}\left(1 + \frac{4\lambda^2[c + d + 4(\tilde{v}^z + v^z)]}{r^2(c + 4\tilde{v}^z)(d + 4v^z)} + \ldots\right) \end{aligned} \tag{5.100}$$

From this equation, we find the fluctuation

$$\delta e^g \sim \frac{2\lambda^2[c + d + 4(\tilde{v}^z + v^z)]}{r^2(c + 4\tilde{v}^z)(d + 4v^z)} \tag{5.101}$$

which give $\Delta = 2$ in agreement with the result of the previous section. We can also see this in the coordinate $\tilde{r} = [(c + 4\tilde{v}^z)(d + 4v^z)]^{\frac{1}{4}} \ln r$ in which

$$ds_6^2 = e^{\frac{2\tilde{r}}{L}}(-dx_0^2 + dx_1^2) + d\tilde{r}^2 + R^2 ds(S^3)^2 \tag{5.102}$$

and $\delta e^g \sim e^{\frac{-2\tilde{r}}{L}}$. We have identified the AdS_3 and S^3 radii $L = R = [(c + 4\tilde{v}^z)(d + 4v^z)]^{\frac{1}{4}}$. In the general case with N instantons, it can be checked through a more complicated algebra that the fluctuation of the metric behaves as $\sim r^{-2}$ near the UV point. This can be seen as follows. $\tilde{\rho}$ have an expansion in powers of $r^{2n} + r^{2n-2}$ with $r^2 = y_i y_i$. From this, we find that $\Box \ln \tilde{\rho} \sim \frac{1}{r^{4n}}(r^2 + r^4 + \ldots r^{4n-2})$ from which we see that r^{-2} is the leading term we have found in (5.93) while the subleading r^{-4} gives $\Delta = 2$ as in the single instanton case. So, our flow is a vev flow driven by a vacuum expectation value of a marginal operator.

We end this subsection by giving a comment on the anti-instanton gauge field configuration. We need to choose the three dimensional chirality $\Gamma_{01}\epsilon = -\epsilon$ which implies the self dual $\Gamma_{\alpha\beta}$ from $\Gamma_7\epsilon = -\Gamma_{2345}\epsilon = \epsilon$. So, the condition $\delta\lambda^I = 0$ is still satisfied. The BPS equations (5.73) and (5.74) are modified by some sign changes. We find the following equations

$$\frac{1}{2}\partial\!\!\!/\theta - \frac{1}{2}e^{\theta - 2f}\partial\!\!\!/\Lambda + \frac{1}{2}e^{-\theta - 2f}\partial\!\!\!/\tilde{\Lambda} = 0 \tag{5.103}$$

$$\frac{1}{2}\partial\!\!\!/ f + \frac{1}{4}e^{\theta - 2f}\partial\!\!\!/\Lambda + \frac{1}{4}e^{-\theta - 2f}\partial\!\!\!/\tilde{\Lambda} = 0\,. \tag{5.104}$$

This change results in an extra minus sign in $\Lambda = -\frac{1}{2}e^{-\theta+2f} + C_1$. The field strength $*F^I = -F^I$ gives an extra minus sign in equation $DG = v^z F^I \wedge F^I$. The final result is

$$e^{-(\theta+2f)} = \frac{d}{r^2} - v^z \frac{2}{3} c(G)d(G)\Box \ln \tilde{\rho} \tag{5.105}$$

$$e^{\theta-2f} = \frac{c}{r^2} - \tilde{v}^z \frac{2}{3} c(G)d(G)\Box \ln \tilde{\rho} \tag{5.106}$$

with the behavior near the fixed points

$$r \to \infty : \quad e^{-4f} \sim \frac{(c - 4\tilde{v}^z N)(d - 4v^z N)}{r^4}, \tag{5.107}$$

$$r \to 0 : \quad e^{-4f} \sim \frac{cd}{r^4}. \tag{5.108}$$

In this case, N is now negative.

5.3.2 Central charges of the dual CFT

We now give some comments on the central charge of the dual (4,0) CFT. We have mentioned that solutions to the six-dimensional supergravity given in the previous sections can be interpreted as a D1/D5 brane system in type I string theory on K3. As type I and heterotic theories are S-dual to each other [117], this D1/D5 system is dual to the F1/NS5 brane system in the heterotic theory. We choose to work with heterotic string theory on K3 with the string frame effective action given by [118]

$$\begin{aligned} I_6 &= \frac{(2\pi)^3}{\alpha'^2} \int d^6x \sqrt{-g} e^{2\theta} \left[R_6 + 4\partial_M \theta \partial^M \theta - \frac{1}{12} G_{MNP} G^{MNP} \right] \\ &\quad + \int_{M_6} \frac{1}{4(2\pi)^3 \alpha'} B \wedge \sum_\alpha \tilde{v}^\alpha \mathrm{tr} F_\alpha \wedge F_\alpha \end{aligned} \tag{5.109}$$

where we have given only the relevant terms for our discussion. All the notations are the same as those in [118] including the modified three-form field strength

$$G = dB - \frac{\alpha'}{4} \sum_\alpha v^\alpha \Omega(F^\alpha) \tag{5.110}$$

where $\Omega(F^\alpha)$ is the Chern-Simons term of the gauge field A^α.

To compute the central charge, we need to know the coefficient of the Einstein-Hilbert term. The central charge is then given by [40]

$$c = \frac{3\ell}{2G_N^{(3)}} \tag{5.111}$$

where ℓ is the AdS_3 radius. Note that the central charge can be written as $c = 24\pi\alpha$ with α being the coefficient of the Einstein-Hilbert term.

This result has been computed in many references with different approaches, see for example [119, 120]. However, we will give a derivation of this result by using the computation of chiral correlators involving T_{zz} ($\bar{T}_{\bar{z}\bar{z}}$) in the spirit of [7]. This, to the best of our knowledge, has not appeared in the literatures. The two point function will directly give the value of the central charge. We first start with the gravitational action in three dimensions of the form

$$I = \alpha \left[\int_M d^3x(\sqrt{G}R_G + 2\Lambda) - \int_{\partial M} d^2x\sqrt{g}2K \right] . \qquad (5.112)$$

The second term is the Gibbons-Hawking term with the induce metric g and extrinsic curvature of the boundary ∂M, K. The coefficient α is dimensionless provided that we use the unit AdS_3 in the measure $d^3x\sqrt{g}$.

We now carry out some calculation in the holographic renormalization reviewed in chapter 3. We adopt Euclidean signature and take the metric [93]

$$ds^2 = G_{\mu\nu}dx^\mu dx^\nu = \ell^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{g_{ij}(x,\rho)}{\rho} dx^i dx^j \right) . \qquad (5.113)$$

In these coordinates, the extrinsic curvature is given by $K_{ij} = -\frac{2\rho}{\ell}\partial_\rho g_{ij}$, and the boundary is at $\rho = 0$. We look for the solution of g_{ij} of the form

$$g(x,\rho) = g_{(0)} + \rho g_{(2)} + h_{(2)}\rho \ln \rho + \dots . \qquad (5.114)$$

The Einstein equations give [93]

$$\begin{aligned} \rho[2g'' - 2g'g^{-1}g' + \mathrm{Tr}(g^{-1}g')g'] + R_g - \mathrm{Tr}(g^{-1}g')g &= 0 \qquad (5.115) \\ \nabla_i \mathrm{Tr}(g^{-1}g') - \nabla^j g'_{ij} &= 0 \qquad (5.116) \\ \mathrm{Tr}(g^{-1}g'') - \frac{1}{2}\mathrm{Tr}(g^{-1}g'g^{-1}g') &= 0 . \qquad (5.117) \end{aligned}$$

Using the expansion (5.114), we find the relevant equations

$$\begin{aligned} \nabla^j g^{(2)}_{ij} &= \frac{1}{2}\nabla_i R_g \qquad (5.118) \\ \mathrm{Tr} g_{(2)} &= = \frac{1}{2} R_g . \qquad (5.119) \end{aligned}$$

We now expand the background metric $g_{(0)}$ about the flat metric. We use the complex coordinates with the convention of [2] and write the metric $g_{(0)}$ as

$$g_{(0)} = \begin{pmatrix} h_{zz} & \frac{1}{2} + h_{z\bar{z}} \\ \frac{1}{2} + h_{z\bar{z}} & h_{\bar{z}\bar{z}} \end{pmatrix} . \qquad (5.120)$$

To simplify the equations, we will keep only the $h_{\bar{z}\bar{z}}$ component non zero. This is enough to find the $T_{zz}T_{zz}$ correlation functions. In the complex coordinates, equation (5.118) takes the form

$$\nabla_z g^{(2)}_{\bar{z}\bar{z}} + \nabla_{\bar{z}} g^{(2)}_{zz} - 2h_{\bar{z}\bar{z}}\nabla_z g^{(2)}_{zz} = \frac{1}{4}\partial_z R_g . \qquad (5.121)$$

Equation (5.119) gives

$$g^{(2)}_{z\bar{z}} - h_{\bar{z}\bar{z}} g^{(2)}_{zz} = \frac{1}{8} R_g \,. \tag{5.122}$$

This can be used to eliminate $g^{(2)}_{z\bar{z}}$ in (5.121). We finally find

$$\partial_{\bar{z}} g^{(2)}_{zz} - h_{\bar{z}\bar{z}} \partial_z g^{(2)}_{zz} - 2\partial_z h_{\bar{z}\bar{z}} g^{(2)}_{zz} = \frac{1}{8} \partial_z R_g \,. \tag{5.123}$$

This equation has precisely the structure of Virasoro's Ward identity for the generating function of connected correlators of T_{zz}. A different holographic derivation of it has been discussed in [121].

The Ricci scalar R_g is given by

$$R_g = 2 g^{z\bar{z}}_{(0)} R_{z\bar{z}} = 4 \partial_z^2 h_{\bar{z}\bar{z}} \,. \tag{5.124}$$

We can now solve (5.123) for $g^{(2)}$ order by order. The first order equation is simply given by

$$\partial_{\bar{z}} g^{(2)}_{zz} = \frac{1}{2} \partial_z^3 h_{\bar{z}\bar{z}} \,. \tag{5.125}$$

This is easily solved by recalling $\partial_{\bar{z}} \frac{1}{z} = 2\pi \delta(z)$ and taking

$$g^{(2)}_{zz} = -\frac{3}{2\pi} \int d^2 w \frac{1}{(z-w)^4} h_{\bar{w}\bar{w}}(w) \,. \tag{5.126}$$

Back to our action (5.112), we can evaluate this action on the solution (5.113) with the expansion (5.114). This gives [120]

$$\delta I = \alpha \int d^2 x \sqrt{g} (g^{(2)ij} - g^{kl}_{(0)} g^{(2)}_{kl} g^{(0)ij}) \delta g^{(0)}_{i,j} \,. \tag{5.127}$$

Although, in our coordinates, the boundary is at the lower limit of the ρ integration, $\rho = 0$, in contrast to [120] in which the boundary is at the upper limit of the η integration, $\eta = \infty$, we find δI with the same sign as that in [120]. This is because of the extra minus sign in the extrinsic curvature K_{ij}.

In the complex coordinates and with only $h_{\bar{z}\bar{z}} \neq 0$, we find

$$\delta I = 2\alpha i \int d^2 z g^{(2)}_{zz} \delta g^{(0)}_{\bar{z}\bar{z}} \,. \tag{5.128}$$

This gives the one point function for the stress-energy tensor. Using the solution (5.126), we can find the two point function by differentiate one more. The result is

$$\langle T(z) T(w) \rangle = (-2\pi)^2 \frac{\delta^2}{\delta h_{\bar{z}\bar{z}}(z) \delta h_{\bar{w}\bar{w}}(w)} e^{iS}|_{h_{\bar{z}\bar{z}}=0} = \frac{12\pi\alpha}{(z-w)^4} \,. \tag{5.129}$$

We have used our normalization factor of -2π in the definition of the stress-energy tensor. This normalization has been determined by computing the three point function $\langle T(z_1)T(z_2)T(z_3)\rangle$ which in turn can be obtained by solving (5.123) to the second order. After matching this three point function with the CFT $\langle T(z_1)T(z_2)T(z_3)\rangle$, we find the normalization factor. We then compare (5.129) with the OPE $T(z)T(w) \sim \frac{c_L}{2(z-w)^4} + \ldots$, we obtain

$$c_L = 24\pi\alpha \,. \tag{5.130}$$

A similar analysis can be done for the $\langle \bar{T}\bar{T}\rangle$ with non zero h_{zz}. The right moving central charge is then given by

$$c_R = 24\pi\alpha \,. \tag{5.131}$$

In principle, we can use (5.123) to find any n point function of the CFT's stress-energy tensor. However, the above analysis only involves ether $h_{\bar{z}\bar{z}}$ or h_{zz}. With all h_{zz}, $h_{z\bar{z}}$ and $h_{\bar{z}\bar{z}}$ non zero, we have also checked, to leading order, that there is no $T\bar{T}$ correlation function, but there is a coupling between $h_{z\bar{z}}$ and h_{zz} and between $h_{z\bar{z}}$ and $h_{\bar{z}\bar{z}}$. These couplings can be removed by adding some local counter terms to the two dimensional action. Beyond leading order, it is not clear what we can learn from a very complicated equation coming from (5.123).

We end this section by briefly discussing the contribution to the Kac-Moody level from the gauge Chern-Simons term. Following [122], the gauge field can be expanded as

$$A = A^{(0)} + \rho A^{(1)} + \ldots \,. \tag{5.132}$$

The Lagrangian for the gauge field including the Chern-Simons term is

$$I = -\frac{1}{2}\int *F^a \wedge F^a - \frac{\beta}{2}\int \left(A^a \wedge dA^a + \frac{2}{3} f_{abc} A^a \wedge A^b \wedge A^c \right) . \tag{5.133}$$

We will suppress the gauge group index on A from now on to make the expression compact. From this action, it is straightforward to find the equation of motion and find, in the $A_\rho = 0$ gauge,

$$\partial_i A_j^{(0)} - \partial_j A_i^{(0)} = 0 \,. \tag{5.134}$$

As discussed in [122], it is necessary to add a boundary term in order to obtain only the left moving current. This boundary term is given by

$$I_b = \frac{\beta}{2}\int d^2x \sqrt{g} g^{ij} A_i^{(0)} A_j^{(0)} \,. \tag{5.135}$$

Notice the sign change as oppose to the result in [122]. We can solve (5.134) in complex coordinates by taking

$$A_z(z) = -\frac{1}{2\pi}\int d^2w \frac{1}{(z-w)^2} A_{\bar{w}}(w) \,. \tag{5.136}$$

Inserting this into I_b, we obtain

$$I_b = -i\frac{\beta}{2\pi}\int d^2z d^2w A_{\bar{w}}(w)\frac{1}{(z-w)^2}A_{\bar{z}}(z)\,. \tag{5.137}$$

We can now find

$$\langle J(z)J(w)\rangle = (-2\pi)^2\frac{\delta^2}{\delta A_{\bar{z}}(z)\delta A_{\bar{w}}(w)}e^{iI_b}|_{A_{\bar{z}}=0} = \frac{4\pi\beta}{(z-w)^2} \tag{5.138}$$

which gives $k = 8\pi\beta$ by using the OPE $J(z)J(w) \sim \frac{k}{2(z-w)^2} + \ldots$. The factor -2π is again due to our normalization of the current. The central charge is $c = 6k = 48\pi\beta$.

We now come back to the action (5.109) and take the metric ansatz to be

$$ds_6^2 = \ell^2\left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}dx_{1,1}^2\right) + \ell^2 ds^2(S^3). \tag{5.139}$$

We have used the AdS_3 metric in the (ρ, x^0, x^1) coordinates. The three-form field is

$$G = \ell^2 h\epsilon_3 + \ell^2 a\omega_3 \tag{5.140}$$

where ϵ_3 and ω_3 are the volume form on the unit AdS_3 and S^3, respectively. The radii of AdS_3 and S^3 are the same by the equation of motion for θ with constant θ. The G equations of motion and Bianchi identity require h and a to be constant. Einstein equation determines the value of $h = a = 2$. The fluxes of G and $*G$ are given by

$$Q_1 = \frac{(2\pi)^3}{\alpha'^2}\int_{S^3} e^{2\theta} * G = \frac{(2\pi)^5\ell^2 e^{2\theta}}{\alpha'^2}, \tag{5.141}$$

$$Q_5 = \frac{(2\pi)^3}{\alpha'^2}\int_{S^3} G = \frac{(2\pi)^5\ell^2}{\alpha'^2}\,. \tag{5.142}$$

In order to relate Q_1 and Q_5 to the number of F1 strings and NS5 branes, N_1 and N_5, we match our ansatz to the F1/NS5 solution in six dimensions given in [123]. The three-form field strength is

$$G = 2\alpha' N_5(\mathrm{Vol}_{AdS_3} + \mathrm{Vol}_{S^3})\,. \tag{5.143}$$

We find that, after matching the flux of this solution with that of our ansatz,

$$N_1 = 2\pi\alpha' Q_1, \qquad N_5 = \frac{\alpha' Q_5}{(2\pi)^5}\,. \tag{5.144}$$

We make a reduction of (5.109) on S^3 and obtain

$$\begin{aligned} I_3 &= \frac{\alpha'^2 Q_1 Q_5}{2(2\pi)^5}\int d^3x\sqrt{g}R + \frac{\alpha' Q_1}{4}\int_{M3} v\Omega(F^\alpha) + \frac{\alpha' Q_5}{4(2\pi)^6}\int_{M3}\tilde{v}\Omega(F^\alpha) \\ &\quad -\frac{\alpha' Q_1 Q_5}{8(2\pi)^5}\int_{M3}\Omega(F^I) \\ &= \frac{N_1 N_5}{4\pi}\int d^3x\sqrt{g}R + \frac{N_1}{8\pi}\int_{M3} v\Omega(F^\alpha) + \frac{N_5}{8\pi}\int_{M3}\tilde{v}\Omega(F^\alpha) \\ &\quad -\frac{N_1 N_5}{2(8\pi)}\int_{M3}\Omega(F^I) \end{aligned} \tag{5.145}$$

where we have given only the Einstein-Hilbert and Chern-Simons terms which are relevant for the present discussion. The $SU(2)$ Chern-Simons term $\Omega(F^I)$ cannot be determined by the dimensional reduction of the action (5.109). As in the previous chapter, its presence in the effective action is implied by the equation of motion for F^I.

We finally find the central charges

$$c_L = 6N_1N_5, \qquad c_R = 6N_1N_5. \tag{5.146}$$

The Kac-Moody levels of the $SU(2)$ and gauge group G_α can be computed from the Chern-Simons terms of the $SU(2)$ and G^α gauge fields. The result is given by

$$\text{SU(2) level} \ : \ k_{SU(2)} = 8\pi\beta = N_1N_5, \tag{5.147}$$

$$\text{G}_\alpha \text{ level} \ : \ k_\alpha = 2(v^\alpha N_1 + \tilde{v}^\alpha N_5). \tag{5.148}$$

We now summarize what we have studied in this chapter. We have found three analytic RG flow solutions in six and three dimensional supergravities. In six dimensional supergravity, we have found the solution in which the internal components, outside the AdS_3 part, of the gauge fields, describe a configuration with N instantons. We have discussed separately the case $N = 1$. This is interesting in the sense that the solution can be obtained from uplifting the three dimensional solution. We have also given the corresponding solution in the Chern-Simons gauged supergravity. With the reduction given in chapter 4, the solution can be lifted to six dimensions and easily seen that it is indeed the same as the six dimensional solution.

The flows describe a deformation of the UV CFT by a vacuum expectation value of a marginal operator. Interestingly, these RG flows have an interpretation in terms of Yang-Mills instantons tunnelling between $|N\rangle$ Yang-Mills vacuum in the UV and $|0\rangle$ in the IR, and this fact is in turn related to the different values of the central charge at the two fixed points. In the general N instanton solution, there is a subtlety of phase transitions occurring whenever v and $\tilde{v}$ change sign. We have avoided this issue by assuming the positivity of both v and $\tilde{v}$. We do not have a clear interpretation of this phase transition in the dual CFT, so it would be interesting to study this issue in more detail.

Chapter 6

Gravitational and Yang-Mills instantons in holographic RG flows

In this chapter, we continue our study of RG flow solutions. We will generalize the solutions by including a gravitational instanton background in the form of asymptotically locally Euclidean (ALE) spaces of A_{N-1} type. A two-dimensional RG flow from a UV (2,0) CFT to a (4,0) CFT in the IR is found in the context of (1,0) six dimensional supergravity as in the previous chapter. The solution interpolates between $AdS_3 \times S^3/\mathbb{Z}_N$ and $AdS_3 \times S^3$ geometries. We will also find solutions involving non trivial gauge fields in the form of $SU(2)$ Yang-Mills instantons on ALE spaces. Both flows are vev flows driven by the vacuum expectation value of a marginal operator.

We then move on to study RG flows in four dimensional field theories in type IIB and type I′ string theories. In type IIB theory, we will find the flow interpolating between $AdS_5 \times S^5/\mathbb{Z}_N$ and $AdS_5 \times S^5$ geometries. The field theory interpretation is that of an $N = 2$, $SU(n)^N$ quiver gauge theory flowing to $N = 4$, $SU(n)$ gauge theory. In type I′ theory, the solution describes an RG flow from $N = 2$ quiver gauge theory with a product gauge group to $N = 2$ gauge theory in the IR, with gauge group $USp(n)$. The corresponding geometries are $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ and $AdS_5 \times S^5/\mathbb{Z}_2$, respectively. We also explore more general RG flows, in which both the UV and IR CFTs are $N = 2$ quiver gauge theories and the corresponding geometries are $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ and $AdS_5 \times S^5/(\mathbb{Z}_M \times \mathbb{Z}_2)$. Finally, we discuss the matching between the geometric and field theoretic pictures of the flows.

6.1 RG flows in six dimensional supergravity

In this section, we will find flows solution in (1,0) six dimensional supergravity as in the previous chapter. We begin with a review of relevant formulae we will use throughout this section. We proceed by studying an RG flow solution on the ALE background and compute the ratio of the central charges of the UV and IR fixed points. We then include $SU(2)$ instantons on the ALE background. This is also a generalization of the solution studied in the previous chapter in which the flow involves only Yang-Mills instantons. We will see that the result is a combined effect of gravitational instantons studied here and $SU(2)$ instantons studied in the previous chapter. Finally, we will discuss the left and right central charges with a subleading correction including curvature squared terms on the gravity side.

6.1.1 An RG flow with graviational instantons

We now study a supersymmetric RG flow solution in (1,0) six dimensional supergravity. Most of the formulae used here are the same as those in section 5.3. The difference is the flat $\mathbb{R}^4$ part of the metric being replaced by the ALE space. The metric ansatz, in this case, is then given by

$$ds_6^2 = e^{2f}(-dx_0^2 + dx_1^2) + e^{2g}ds_4^2 \,. \tag{6.1}$$

The four dimensional metric ds_4^2 will be chosen to be the gravitational multi-instantons of [69]. This is an asymptotically locally Euclidean space (ALE) with the metric

$$ds_4^2 = V^{-1}(d\tau + \vec{\omega}.d\vec{x})^2 + V d\vec{x}.d\vec{x}. \tag{6.2}$$

The function V is given by

$$V = \sum_{i=1}^{N} \frac{1}{|\vec{x} - \vec{x}_i|} \,. \tag{6.3}$$

The function $\vec{\omega}$ is related to V via

$$\vec{\nabla} \times \vec{\omega} = \vec{\nabla} V, \tag{6.4}$$

and the τ has period 4π. We also choose the gauge

$$\vec{\omega}.d\vec{x} = \sum_{i=1}^{N} \cos\theta_i d\phi_i \tag{6.5}$$

as in [124]. The point $\vec{x}_i$ is the origin of the spherical coordinates (r_i, θ_i, ϕ_i) with $r_i = |\vec{x} - \vec{x}_i|$. The ansatz for three-form field $\hat{G}_3$ and gauge field $\hat{F}^I$ are the same as in section 5.3. Although $\hat{A}^I = 0$ in the present case, it is more convenient to work with non-zero $\hat{A}^I$ since equations with non-zero $\hat{A}^I$ will be used later in the next subsection.

We now recall the result from the previous chapter with some differences emphasized. Solving BPS equations and equations of motion gives

$$\Lambda = \frac{1}{2}e^{-\theta+2f} + C_1, \qquad \tilde{\Lambda} = \frac{1}{2}e^{\theta+2f} + C_2 \tag{6.6}$$

with constants of integration C_1 and C_2. Defining $\epsilon = e^{\frac{f}{2}}\tilde{\epsilon}$ and taking $g = -f$, we find the Killing spinor equation on the ALE space, from equation (5.80) with ∂_α replaced by D_α,

$$D_\alpha \tilde{\epsilon} = 0 \tag{6.7}$$

which requires that $\tilde{\epsilon}$ is a Killing spinor on the ALE space. D_α is a covariant derivative on the ALE space. The ALE space has $SU(2)$ holonomy and admits two Killing spinors out of the four spinors. Therefore, the flow solution entirely preserves $\frac{1}{4}$ of the eight supercharges, or $N = 2$ in two dimensional langauge, along the flow.

In this subsection, we study only the effect of gravitational instantons, so we choose $A^I = 0$ from now on. Using (6.6), we can write (5.66) and (5.70) as

$$\Box\, e^{-\theta-2f} = 0 \qquad \text{and} \qquad \Box\, e^{\theta-2f} = 0\,. \tag{6.8}$$

The $\Box$ in these equations is the covariant scalar Laplacian on the ALE space

$$\Box = \frac{1}{V}[V^2\partial_\tau^2 + (\vec{\nabla} - \vec{\omega}\partial_\tau).(\vec{\nabla} - \vec{\omega}\partial_\tau)]. \tag{6.9}$$

Our flow is described by a simple ansatz as follows. We first choose $\theta = 0$. It is straightforward to check that all equations of motion as well as BPS equations are satisfied. We then have only a single equation to be solved

$$\Box\, e^{-2f} = 0\,. \tag{6.10}$$

We now choose f to be τ independent of the form

$$e^{-2f} = \frac{c}{|\vec{x} - \vec{x}_1|} \tag{6.11}$$

where c is a constant. This is clearly a solution of (6.10) since for τ independent functions, the $\Box$ reduce to the standard three dimensional Laplacian $\vec{\nabla}.\vec{\nabla}$. We will now show that this solution describes an RG flow between two fixed points given by $|\vec{x}| \to \infty$ and $\vec{x} \to \vec{x}_1$. We emphasize that the point $\vec{x}_1$ is purely conventional since any point x_i with $i = 1 \ldots N$ will work in the same way. Notice that for general τ dependent solution, the solution to the harmonic function will be given by the Green function on ALE spaces. The explicit form of this Green function will be given in the next subsection. Furthermore, with τ dependent solution, the IR fixed point of the flow can also be given by $\vec{x} \to \vec{y}$ where $\vec{y}$ is a regular point on the ALE space rather than one of the ALE canter $\vec{x}_i$. The crucial point in our discussion is the behavior of the Green function near the fixed points such that the geometry contains AdS_3. However, for the present case, we restrict ourselves to the ansatz (6.11).

When $|\vec{x}| \to \infty$, we have

$$\begin{aligned} e^{-2f} &= \frac{c}{|\vec{x} - \vec{x}_1|} \to \frac{c}{\zeta}, \qquad \zeta \equiv |\vec{x}|, \\ V &= \sum_{i=1}^{N} \frac{1}{|\vec{x} - \vec{x}_i|} \to \frac{N}{\zeta}\,. \end{aligned} \tag{6.12}$$

In this limit, the ALE metric becomes

$$ds_4^2 = \frac{\zeta}{N}(d\tau + N\cos\theta d\phi)^2 + \frac{N}{\zeta}(d\zeta^2 + \zeta^2 d\Omega_2^2) \tag{6.13}$$

where we have written the flat three dimensional metric $d\vec{x}.d\vec{x}$ in spherical coordinates with the S^2 metric $d\Omega_2^2$. The factor $N\cos\theta d\phi$ arises from $\sum_{i=1}^N \cos\theta_i d\phi_i$ since in the limit $|\vec{x}| \to \infty$ all (θ_i, ϕ_i) are the same to leading order. By changing the coordinate ζ to r defined by $\zeta = \frac{r^2}{4N}$, we obtain

$$ds_4^2 = dr^2 + \frac{r^2}{4}\left[\left(\frac{d\tau}{N} + \cos\theta d\phi\right)^2 + d\Omega_2^2\right]. \tag{6.14}$$

The full six-dimensional metric is then given by

$$ds_6^2 = \frac{r^2}{4Nc}dx_{1,1}^2 + \frac{4Nc}{r^2}dr^2 + 4Nc\left[\left(\frac{d\tau}{N} + \cos\theta d\phi\right)^2 + d\Omega_2^2\right]. \tag{6.15}$$

The expression in the bracket is the metric on $S^3/\mathbb{Z}_N$. So, the six dimensional geometry is $AdS_3 \times S^3/\mathbb{Z}_N$ with the radii of AdS_3 and $S^3/\mathbb{Z}_N$ being $L_\infty = 2\sqrt{Nc}$.

When $\vec{x} \to \vec{x}_1$, we find

$$\begin{aligned} e^{-2f} &= \frac{c}{\xi}, \qquad \xi \equiv |\vec{x} - \vec{x}_1|, \\ V &= \sum_{i=1}^N \frac{1}{|\vec{x} - \vec{x}_i|} = \frac{1}{\xi}. \end{aligned} \tag{6.16}$$

The ALE metric becomes

$$ds_4^2 = \xi(d\tau + \cos\theta_1 d\phi_1)^2 + \frac{1}{\xi}(d\xi^2 + \xi^2 d\Omega_2^2). \tag{6.17}$$

In this limit, $\sum_{i=1}^N \cos\theta_i d\phi_i \sim \cos\theta_1 d\phi_1$ to leading order. Writing $\xi = \frac{r^2}{4}$, we obtain

$$ds_4^2 = dr^2 + \frac{r^2}{4}[(d\tau - \cos\theta_1 d\phi_1)^2 + d\Omega_2^2] \tag{6.18}$$

which is the metric on $\mathbb{R}^4$. The six-dimensional metric now takes the form

$$ds_6^2 = \frac{r^2}{4c}dx_{1,1}^2 + \frac{4c}{r^2}dr^2 + 4c d\Omega_3^2 \tag{6.19}$$

where $d\Omega_3^2$ is the metric on S^3. This geometry is $AdS_3 \times S^3$ with AdS_3 and S^3 having the same radius $2\sqrt{c}$. With the central charge of the dual CFT given by

$$c = \frac{3L}{2G_N^{(3)}}, \tag{6.20}$$

we find the ratio of the central charges

$$\begin{aligned}\frac{c_1}{c_\infty} &= \frac{L_1 G^{(3)}_{N\infty}}{L_\infty G^{(3)}_{N1}} = \frac{L_1 \text{Vol}(S^3)}{L_\infty \text{Vol}(S^3/\mathbb{Z}_N)} \\ &= N\left(\frac{L_1}{L_\infty}\right)^4 = \frac{1}{N}\end{aligned} \tag{6.21}$$

where we have used $G^{(3)}_N = \frac{G^{(6)}_N}{\text{Vol}(M)}$ for six-dimensional theory compactified on a compact space M. The flow interpolates between $AdS_3 \times S^3/\mathbb{Z}_N$ in the UV to $AdS_3 \times S^3$ in the IR. The UV CFT has (2,0) supersymmetry because of the $\mathbb{Z}_N$ projection, so our flow describes an RG flow from the (2,0) CFT in the UV to the (4,0) CFT in the IR.

We now consider the central charge on the gravity side including the curvature squared terms. The bulk gravity is three dimensional, and the Riemann tensor can be written in terms of the Ricci tensor and Ricci scalar. To study the effect of higher derivative terms, we add the $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term to the (1,0) six dimensional action. The supersymmetrization of this term has been studied in [125]. We temporarily drop the hat to simplify the expressions. The Lagrangian with the auxiliary fields integrated out is given by [126]

$$\begin{aligned}\mathcal{L} &= \sqrt{-g}e^{-2\theta}\left[R + 4\partial_\mu\theta\partial^\mu\theta - \frac{1}{12}G_3^{\mu\nu\rho}G_{3\mu\nu\rho}\right] + \frac{1}{4}\alpha\sqrt{-g}\tilde{R}^{\mu\nu\rho\sigma}\tilde{R}_{\mu\nu\rho\sigma} \\ &\quad + \frac{1}{16}\beta\epsilon^{\mu\nu\rho\sigma\tau\lambda}\tilde{R}^{\alpha\beta}{}_{\mu\nu}\tilde{R}_{\alpha\beta\rho\sigma}b_{\tau\lambda}\end{aligned} \tag{6.22}$$

where $\tilde{R}_{\mu\nu\rho\sigma}$ is computed with the modified connection $\tilde{\Gamma}^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\mu\nu} - \frac{1}{2}G^\rho_{3\mu\nu}$. The $b_{\lambda\tau}$ is the two-form field whose field strength is G_3. Reducing (6.22) on S^3 with $G_3 = 2S\epsilon_3 + 2m\omega_3$ where ϵ_3 and ω_3 are volume forms on ds^2_3 and S^3, respectively gives [126]

$$\begin{aligned}e^{-1}\mathcal{L} &= e^{-2\theta}(R + 4\partial_\mu\theta\partial^\mu\theta + 4m^2 + 2S^2) + 4mS \\ &\quad -2\beta m\left[RS + 2S^3 - \frac{1}{4}\epsilon^{\mu\nu\rho}\left(R^{ab}{}_{\mu\nu}\omega_{\rho ab} + \frac{2}{3}\omega_\mu{}^a{}_b\omega_\nu{}^b{}_c\omega_\rho{}^c{}_a\right)\right] \\ &\quad +\frac{1}{4}\alpha(4R^{\mu\nu}R_{\mu\nu} - R^2 - 8\partial_\mu S\partial^\mu S + 12S^4 + 4RS^2).\end{aligned} \tag{6.23}$$

As shown in [126], $S = -m$ on the AdS_3 background, and m is related to the AdS radius via $m = \frac{1}{L}$. The left and right moving central charges can be computed as in [120, 127]. The result is [126]

$$c_L = \frac{3L}{2G^{(3)}_N}\left(1 + \frac{4\beta}{L^2}\right), \qquad c_R = \frac{3L}{2G^{(3)}_N}\,. \tag{6.24}$$

We find that

$$\text{UV}: \qquad c_L = \frac{48\pi^2 c^2 N}{G_N^{(6)}}\left(1+\frac{\beta}{cN}\right), \qquad c_R = \frac{48\pi^2 c^2 N}{G_N^{(6)}}, \tag{6.25}$$

$$\text{IR}: \qquad c_L = \frac{48\pi^2 c^2}{G_N^{(6)}}\left(1+\frac{\beta}{c}\right), \qquad c_R = \frac{48\pi^2 c^2}{G_N^{(6)}}. \tag{6.26}$$

We end this subsection by finding the dimension of the dual operator driving the flow. This is achieved by expanding the metric around the UV fixed point, $|\vec{x}| \to \infty$ in our solution. e^{-2f} and V can be expanded as

$$\begin{aligned} e^{-2f} &= \frac{c}{|\vec{x}-\vec{x}_1|} \sim \frac{1}{\zeta}\left(1+\frac{a_1\cos\varphi_1}{\zeta}-\frac{a_1^2}{2\zeta^2}\right)+\ldots, \\ V &= \sum_{i=1}^{N}\frac{1}{|\vec{x}-\vec{x}_i|} \sim \frac{N}{\zeta}+\sum_{i=1}^{N}\left(\frac{a_i\cos\varphi_i}{\zeta^2}-\frac{a_i^2}{2\zeta^3}\right)+\ldots \end{aligned} \tag{6.27}$$

where φ_i are angles between $\vec{x}$ and $\vec{x}_i$. We have also defined $\zeta \equiv |\vec{x}|$ and $a_i \equiv |\vec{x}_i|$. By substituting (6.27) in (6.1), it is then straightforward to obtain the behavior of the metric fluctuation which is of order $\mathcal{O}(r^{-2})$. This gives $\Delta = 2$ indicating that the flow is driven by a vacuum expectation value of a marginal operator.

6.1.2 An RG flow with gravitational and $SU(2)$ Yang-Mills instantons

We now add Yang-Mills instantons to the solution given in the previous subsection. This involves constructing instantons on ALE spaces. Some explicit instantons solutions on an ALE space are given in [70]. We are interested in $SU(2)$ instantons whose explicit solutions can be written down. The solution can be expressed in the form [70]

$$A^I_\alpha dx^\alpha = -\eta^I_{ab} e^a E^b \ln H\,. \tag{6.28}$$

The vielbein e^a_α and its inverse E^α_a for the metric (6.2) are given by

$$e^0 = V^{-\frac{1}{2}}(d\tau+\vec{\omega}.d\vec{x}), \qquad e^l = V^{\frac{1}{2}}dx^l, \tag{6.29}$$

$$E_0 = V^{\frac{1}{2}}\frac{\partial}{\partial\tau}, \qquad E_l = V^{-\frac{1}{2}}\left(\frac{\partial}{\partial x^l}-\omega_l\frac{\partial}{\partial\tau}\right). \tag{6.30}$$

The η^I_{ab}'s are the usual 't Hooft tensors and $l = 1,2,3$. This form resembles the $SU(2)$ instantons on the flat space $\mathbb{R}^4$. Self duality of F^I requires that H satisfies the harmonic equation on the ALE space

$$\nabla_a\nabla^a H = 0\,. \tag{6.31}$$

The solution is given by

$$H = H_0 + \sum_{j=1}^{n} \lambda_j G(x, y_j) \tag{6.32}$$

where H_0 and λ_j are constants, and $G(x, y_j)$ is the Green's function on the ALE space given in [124] with $x = (\tau, \vec{x})$. Its explicit form is

$$G(x, x') = \frac{\sinh U}{16\pi^2 |\vec{x} - \vec{x}'|(\cosh U - \cos T)} \tag{6.33}$$

where

$$\begin{aligned} U(x, x') &= \frac{1}{2}\sum_{i=1}^{N} \ln\left(\frac{r_i + r_i' + |\vec{x} - \vec{x}'|}{r_i + r_i' - |\vec{x} - \vec{x}'|}\right), \qquad r_i = |\vec{x} - \vec{x}_i|, \\ T(x, x') &= \frac{1}{2}(\tau - \tau') + \sum_{i=1}^{N} \tan^{-1}\left[\tan\left[\frac{\phi_i - \phi_i'}{2}\right] \frac{\cos\frac{\theta_i + \theta_i'}{2}}{\cos\frac{\theta_i - \theta_i'}{2}}\right]. \end{aligned} \tag{6.34}$$

This solution is obviously τ dependent and can be thought of as a generalization of the τ independent solution of [71]. The latter is subject to the constraint $n \leq N$ since the finite action requires that the instantons must be put at the ALE centers. We emphasize here that the $\vec{y}_j$ inside the y_j in (6.32) needs not necessarily coincide with the ALE center $\vec{x}_i$. Therefore, $\vec{y}_j$ could be any point, ALE center or regular point, on the ALE space. However, in our flow solution given below, we will choose one of the $\vec{y}_j$'s to coincide with one of the ALE centers $\vec{x}_i$'s which is, by our convention, chosen to be $\vec{x}_1$.

As in the flat space case, we can write

$$F^I_{ab} F^{Iab} = -4\Box\Box \ln H \tag{6.35}$$

which can be shown by using the properties of η^I_{ab} given in [115] and the fact that H is a harmonic function on the ALE space as well as the Ricci flatness of the ALE space. Using this relation, we obtain

$$*(F^I \wedge F^I) = *(*F^I \wedge F^I) = \frac{1}{2} F^I_{ij} F^{Iij} = -2\Box\Box \ln H\,. \tag{6.36}$$

Equations (5.66) and (5.70) become

$$\begin{aligned} \Box\, e^{-\theta - 2f} &= 4v\Box\Box \ln H, \qquad (6.37) \\ \Box\, e^{\theta - 2f} &= 4\tilde{v}\Box\Box \ln H\,. \qquad (6.38) \end{aligned}$$

The solutions to these equations are of the form

$$\begin{aligned} e^{-\theta - 2f} &= f_1 + 4v\Box \ln H, \\ e^{\theta - 2f} &= f_2 + 4\tilde{v}\Box \ln H \end{aligned} \tag{6.39}$$

where f_1 and f_2 are solutions to the homogeneous equations. The Green function $G(x,x')$ in (6.33) is singular when $x \sim x'$. The behavior of $G(x,x')$ in this limit is [124]

$$G(x,x') = \frac{1}{4\pi^2 |x-x'|^2} \tag{6.40}$$

where

$$|x-x'|^2 = V|\vec{x}-\vec{x}'|^2 + V^{-1}[\tau - \tau' + \vec{\omega}.(\vec{x}-\vec{x}')]^2. \tag{6.41}$$

We remove this singularity, in our case $x' \sim y_j$, from our solution by adding $G(x,y_j)$, with appropriate coefficients, to (6.39). We also choose f_1 and f_2 to be $\frac{c}{|\vec{x}-\vec{x}_1|}$ and $\frac{d}{|\vec{x}-\vec{x}_1|}$, respectively. This choice is analogous to the solution in the previous subsection with c and d being constants. Collecting all these, we find

$$\begin{aligned} e^{-\theta-2f} &= \frac{c}{|\vec{x}-\vec{x}_1|} \\ &\quad +4v\left[\Box \ln\left(H_0 + \sum_{j=1}^{n} \lambda_j G(x,y_j)\right) + 16\pi^2 \sum_{j=1}^{n} G(x,y_j)\right], \quad (6.42) \\ e^{\theta-2f} &= \frac{d}{|\vec{x}-\vec{x}_1|} \\ &\quad +4\tilde{v}\left[\Box \ln\left(H_0 + \sum_{j=1}^{n} \lambda_j G(x,y_j)\right) + 16\pi^2 \sum_{j=1}^{n} G(x,y_j)\right]. \quad (6.43) \end{aligned}$$

The metric warp factor e^{-2f} can be obtained by multiplying (6.42) and (6.43). We now study the behavior of this function in the limits $\vec{x} \to \vec{x}_1$ and $|\vec{x}| \to \infty$.

As $\vec{x} \to \vec{x}_1$, the terms involving $G(x,x_1)$ in the square bracket in (6.42) and (6.43) do not contribute since the poles of the two terms cancel each other. The other terms involving $G(x,y_j)$, $\vec{y}_j \neq \vec{x}_1$, are subleading compared to f_1 and f_2. We find

$$e^{-\theta-2f} = \frac{d}{|\vec{x}-\vec{x}_1|}, \qquad e^{\theta-2f} = \frac{c}{|\vec{x}-\vec{x}_1|} \tag{6.44}$$

or

$$e^{-2f} = \frac{\sqrt{cd}}{|\vec{x}-\vec{x}_1|}. \tag{6.45}$$

By using the coordinate changing as in the previous subsection $|\vec{x}-\vec{x}_1| = \frac{r^2}{4}$, it can be shown that the metric is of the form of $AdS_3 \times S^3$

$$ds_6^2 = \frac{r^2}{4\sqrt{cd}} dx_{1,1}^2 + \frac{4\sqrt{cd}}{r^2} dr^2 + 4\sqrt{cd}\, d\Omega_3^2. \tag{6.46}$$

As $|\vec{x}| \to \infty$, the Green function (6.33) becomes

$$G(x,x') = \frac{1}{16\pi^2 |\vec{x}-\vec{x}'|} \tag{6.47}$$

because U defined in (6.34) becomes infinite. We find

$$e^{-\theta-2f} = \frac{c}{|\vec{x}-\vec{x}_1|} + 4v\sum_{i=1}^{n}\frac{1}{|\vec{x}-\vec{y}_i|} \sim \frac{c+4vn}{|\vec{x}|},$$
$$e^{\theta-2f} = \frac{d}{|\vec{x}-\vec{x}_1|} + 4\tilde{v}\sum_{i=1}^{n}\frac{1}{|\vec{x}-\vec{y}_i|} \sim \frac{d+4\tilde{v}n}{|\vec{x}|}. \tag{6.48}$$

The warp factor is now given by

$$e^{-2f} = \frac{\sqrt{(c+4nv)(d+4n\tilde{v})}}{|\vec{x}|}. \tag{6.49}$$

The six-dimensional metric becomes $AdS_3 \times S^3/\mathbb{Z}_N$, with $|\vec{x}| = \frac{r^2}{4N}$,

$$ds_6^2 = \frac{r^2}{\ell^2}dx_{1,1}^2 + \frac{\ell^2}{r^2}dr^2 + \ell^2\left[\left(\frac{d\tau}{N} + \cos\theta d\phi\right)^2 + d\Omega_2^2\right] \tag{6.50}$$

where the AdS_3 radius is given by

$$\ell = 2\sqrt{N}[(c+4nv)(d+4n\tilde{v})]^{\frac{1}{4}}. \tag{6.51}$$

The ratio of the central charges can be found in the same way as that in the previous subsection and is given by

$$\frac{c_1}{c_\infty} = N\left(\frac{L_1}{L_\infty}\right)^4 = \frac{cd}{N(c+4nv)(d+4n\tilde{v})}. \tag{6.52}$$

For $N = 1$, the ALE space becomes a flat $\mathbb{R}^4$, and we obtain the result of the previous chapter. As in the previous subsection, the solution describes an RG flow from a (2,0) CFT to a (4,0) CFT in the IR. The central charges to curvature squared terms are given by

$$\text{UV}: c_L = \frac{48\pi^2(c+4nv)(d+4n\tilde{v})N}{G_N^{(6)}}\left(1 + \frac{\beta}{N\sqrt{(c+4nv)(d+4n\tilde{v})}}\right),$$
$$c_R = \frac{48\pi^2(c+4nv)(d+4n\tilde{v})N}{G_N^{(6)}}, \tag{6.53}$$
$$\text{IR}: c_L = \frac{48\pi^2 cd}{G_N^{(6)}}\left(1+\frac{\beta}{cd}\right), \qquad c_R = \frac{48\pi^2 cd}{G_N^{(6)}}. \tag{6.54}$$

As in the previous subsection, it can be shown that this is also a vev flow driven by a vev of a marginal operator of dimension two.

6.2 RG flows in type IIB and type I$'$ theories

In this section, we study an RG flow solution in type IIB theory on an ALE background. Since there is no gauge field in type IIB theory, the corresponding flow solution only involves gravitational instantons. We also consider a solution in type I$'$ theory which is a T-dual of the usual type I theory on T^2 and can also be obtained from type IIB theory on $T^2/(-1)^{F_L}\Omega I_2$. As we will see, in type I$'$ theory, there are more possibilities of the gauge groups for the quiver gauge theory in the UV and, as a result, more possible RG flows.

6.2.1 RG flows in type IIB theory

We now study a supersymmetric flow solution in type IIB theory. We begin with supersymmetry transformations of the gravitino ψ_M and the dilatino χ. These can be found in various places, see for example [128, 129], and are given by

$$\begin{aligned}
\delta\chi &= iP_M\Gamma^M\epsilon^* - \frac{i}{24}F_{M_1M_2M_3}\Gamma^{M_1M_2M_3}\epsilon, \\
\delta\psi_M &= \nabla_M\epsilon - \frac{i}{1920}F^{(5)}_{M_1M_2M_3M_4M_5}\Gamma^{M_1M_2M_3M_4M_5}\Gamma_M\epsilon \\
&\quad + \frac{1}{96}F_{M_1M_2M_3}({\Gamma_M}^{M_1M_2M_3} - 9\delta^{M_1}_M\Gamma^{M_2M_3})\epsilon
\end{aligned} \tag{6.55}$$

where

$$\begin{aligned}
P_M &= \frac{1}{2}(\partial_M\phi + ie^{\phi}\partial_M C_0), \\
F_{M_1M_2M_3} &= e^{-\frac{\phi}{2}}H_{M_1M_2M_3} + ie^{\frac{\phi}{2}}F_{M_1M_2M_3}\,.
\end{aligned} \tag{6.56}$$

In our ansatz, we choose $\phi = 0$, $C_0 = 0$ and $F_{M_1M_2M_3} = 0$, so $\delta\chi = 0$ is automatically satisfied. The ten dimensional metric is given by

$$ds^2 = e^{2f}dx^2_{1,3} + e^{2g}ds^2_4 + e^{2h}(dr^2 + r^2d\theta^2). \tag{6.57}$$

The metric ds^2_4 is the ALE metric in (6.2), and the functions f, g and h depend only on ALE coordinates y^a and r. We will use indices $\mu, \nu = 0, \ldots, 3$, $a, b = 4, \ldots, 8$. The ansatz for the self-dual five-form field strength is

$$F^{(5)} = \tilde{F} + \hat{*}\tilde{F} \tag{6.58}$$

where $\hat{*}$ is the ten dimensional Hodge duality. We choose $\tilde{F}$ to be

$$\begin{aligned}
\tilde{F} &= dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge(U^{(1)} + Kdr) + rG^{(3)}\wedge dr\wedge d\theta + r\tilde{G}^{(4)}\wedge d\theta, \\
\hat{*}\tilde{F} &= e^{-4f}(e^{2(g+h)} * U^{(1)} + e^{4g} * Krd\theta) + e^{4f-2(g+h)}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge *G^{(3)} \\
&\quad + e^{4(f-g)} * \tilde{G}^{(4)}dx^0\wedge dx^1\wedge dx^2\wedge dx^3\wedge dr
\end{aligned} \tag{6.59}$$

with all functions depending only on y^a and r. We have used the convention $\epsilon_{01234567r\theta} = 1$. The notation $X^{(n)}$ means the n-form X on the four dimensional space whose metric is ds_4^2. The Bianchi identity $DF^{(5)} = 0$ and self duality condition impose the conditions

$$
\begin{aligned}
dU^{(1)} &= 0, \qquad dK = \partial_r U^{(1)} \Rightarrow U^{(1)} = d\Lambda, \qquad K = \partial_r\Lambda + c_1, \\
*G^{(3)} &= e^{-4f+2(g+h)} d\Lambda, \qquad *\tilde{G}^{(4)} = e^{-4(f-g)} K\,.
\end{aligned}
\tag{6.60}
$$

The $*$ and d are the Hodge dual and exterior derivative on ds_4^2, and c_1 is a constant.

From (6.57), we can read off the vielbein components

$$
e^{\hat{\mu}} = e^f dx^\mu, \qquad e^{\hat{a}} = e^g \bar{e}^{\hat{a}}, \qquad e^{\hat{r}} = e^h dr, \qquad e^{\hat{\theta}} = e^h r d\theta\,. \tag{6.61}
$$

The $\bar{e}^{\hat{a}}$ is the vielbein on the ALE space. The spin connections are given by

$$
\begin{aligned}
\omega^{\hat{\theta}}{}_{\hat{r}} &= e^{-h}\left(\frac{1}{r} + h'\right) e^{\hat{\theta}}, \qquad \omega^{\hat{\theta}}{}_{\hat{a}} = e^{-g}\partial_a h e^{\hat{\theta}}, \qquad \omega^{\hat{r}}{}_{\hat{a}} = e^{-g}\partial_a h e^{\hat{r}} - e^{-h} g' e^{\hat{a}}, \\
\omega^{\hat{a}}{}_{\hat{b}} &= e^{-g}(\partial_b g \delta^a_c - \partial^a g \delta^b_c) e^{\hat{c}} + e^{-g}\bar{\omega}^{\hat{a}}{}_{\hat{b}}, \qquad \omega^{\hat{\mu}}{}_{\hat{a}} = e^{-g}\partial_a f e^{\hat{\mu}}, \\
\omega^{\hat{\mu}}{}_{\hat{r}} &= e^{-h} f' e^{\hat{\mu}}
\end{aligned}
\tag{6.62}
$$

where $\bar{\omega}^{\hat{a}}{}_{\hat{b}}$ are spin connections on the ALE space. We also use the following ten dimensional gamma matrices

$$
\begin{aligned}
\Gamma_{\hat{\mu}} &= \gamma_{\hat{\mu}} \otimes \mathbf{I}_4 \otimes \mathbf{I}_2, \qquad \Gamma_{\hat{a}} = \tilde{\gamma}_5 \otimes \gamma_{\hat{a}} \otimes \mathbf{I}_2, \\
\Gamma_{\hat{r}} &= \tilde{\gamma}_5 \otimes \hat{\gamma}_5 \otimes \sigma_1, \qquad \Gamma_{\hat{\theta}} = \tilde{\gamma}_5 \otimes \hat{\gamma}_5 \otimes \sigma_2\,.
\end{aligned}
\tag{6.63}
$$

The chirality condition on ϵ is $\Gamma_{11}\epsilon = \tilde{\gamma}_5 \otimes \hat{\gamma}_5 \otimes \sigma_3 \epsilon = \epsilon$. $\tilde{\gamma}_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ and $\hat{\gamma}_5 = \gamma_4\gamma_5\gamma_6\gamma_7$ are chirality matrices in x^μ and y^a spaces, respectively. With only 5-form turned on, the relevant BPS equations come from

$$
\delta\psi_M = \nabla_M \epsilon - \frac{i}{1920} \not{F}^{(5)} \Gamma_M \epsilon \tag{6.64}
$$

where $\not{F}^{(5)} = F^{(5)}_{M_1 M_2 M_3 M_4 M_5} \Gamma^{M_1 M_2 M_3 M_4 M_5}$. It is now straightforward to show that all the BPS equations are satisfied provided that we choose

$$
h = g = -f, \qquad \Lambda = 2e^{4f}, \qquad \epsilon = e^{\frac{1}{2}f + \frac{i}{2}\sigma_3\theta}\hat{\epsilon} \tag{6.65}
$$

with $\hat{\epsilon}$ being the Killing spinor on the ALE space and satisfying the condition

$$
\bar{\nabla}_a \hat{\epsilon} = 0\,. \tag{6.66}
$$

Furthermore, $\hat{\epsilon}$ satisfies a projection condition $\gamma_5\hat{\epsilon} = \hat{\epsilon}$. So, the solution is again $\frac{1}{4}$ supersymmetric along the flow. With these conditions inserted in (6.58), we obtain the 5-form field

$$
F^{(5)} = 2dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge \hat{d}\Lambda + 2e^{-8f} \hat{*}\hat{d}\Lambda \tag{6.67}
$$

where now $\hat{*}$ and $\hat{d}$ are those on the six dimensional space ALE$\times\mathbb{R}^2$ with coordinates (y^a, r, θ). Equation $DF^{(5)} = 0$ then gives

$$\hat{d}(e^{-8f}\hat{*}\hat{d}\Lambda) = \hat{d}\hat{*}\hat{d}e^{-4f} = 0\,. \tag{6.68}$$

So, the function e^{-4f} satisfies a harmonic equation on ALE$\times\mathbb{R}^2$.

It turns out to be difficult to find the explicit form of this harmonic function. This function can be constructed from the Green's function whose existence has been shown in [130], see also [131]. We now consider the behavior of this funciton at the two fixed points. The six dimensional metric is given by

$$d\tilde{s}^2 = V^{-1}(d\tau + \vec{\omega}.d\vec{x})^2 + Vd\vec{x}.d\vec{x} + dr^2 + r^2 d\theta^2\,. \tag{6.69}$$

As $|\vec{x}| \to \infty$, with the coordinate changing given in the previous section, the ALE metric become $\mathbb{R}^4/\mathbb{Z}_N$. So, the metric of the whole six dimensional space can be written as

$$d\tilde{s}^2 = dR^2 + R^2 ds^2(S^5/\mathbb{Z}_N) \tag{6.70}$$

where $R^2 = 4N|\vec{x}| + r^2$.

Similarly, we can show that as $\vec{x} \to \vec{x}_1$, the metric becomes the flat $\mathbb{R}^6$ metric

$$d\tilde{s}^2 = d\tilde{R}^2 + \tilde{R}^2 d\Omega_5^2 \tag{6.71}$$

where $\tilde{R}^2 = 4|\vec{x} - \vec{x}_1| + r^2$.

So, in order to interpolate between two conformal fixed points, this function must satisfy the boundary condition

$$e^{-4f} \sim \frac{1}{R^4}\,. \tag{6.72}$$

at both ends. There is also a relative factor of N between the two end points. This is due to the fact that the integral of the harmonic equation (6.68) on $d\tilde{s}^2$ must vanish, and this integral is in turn reduced to the integral of the gradient of the Green's function over S^5 and $S^5/\mathbb{Z}_N$ at the two end points. So, with all these requirements, the required harmonic function has boundary conditions

$$\begin{aligned} \vec{x} \to \vec{x}_1 \quad &: \qquad e^{-4f} = \frac{C}{R^4}, \\ |\vec{x}| \to \infty \quad &: \qquad e^{-4f} = \frac{CN}{R^4}\,. \end{aligned} \tag{6.73}$$

The full metrics at both end points take the form

$$\begin{aligned} \vec{x} \to \vec{x}_1 \quad &: \qquad ds_{10}^2 = \frac{R^2}{\sqrt{C}} dx_{1,3}^2 + \frac{\sqrt{C}}{R^2} dR^2 + \sqrt{C} d\Omega_5^2 \\ |\vec{x}| \to \infty \quad &: \qquad ds_{10}^2 = \frac{\tilde{R}^2}{\sqrt{NC}} dx_{1,3}^2 + \frac{\sqrt{NC}}{\tilde{R}^2} d\tilde{R}^2 + \sqrt{NC} ds^2(S^5/\mathbb{Z}_N). \end{aligned} \tag{6.74}$$

We obtain the two AdS_5 radii $L_1 = C^{\frac{1}{4}}$ and $L_\infty = (CN)^{\frac{1}{4}}$. The central charge is given by [132]

$$a = c = \frac{\pi L^3}{8G_N^{(5)}} \,. \tag{6.75}$$

The ratio of the central charges is given by

$$\frac{a_1}{a_\infty} = \frac{c_1}{c_\infty} = \frac{L_1^8 \mathrm{Vol}(S^5)}{L_\infty^8 \mathrm{Vol}(S^5/\mathbb{Z}_N)} = N\left(\frac{L_1}{L_\infty}\right)^8 = \frac{1}{N}. \tag{6.76}$$

The flow describes the deformation of $N=2$ quiver $SU(n)^N$ gauge theory in the UV to $N=4$ $SU(n)$ SYM in the IR in which the gauge group $SU(n)$ is the diagonal subgroup of $SU(n)^N$.

We now compute the central charges to curvature squared terms. Higher derivative corrections to the central charges in four dimensional CFTs have been considered in many references, see for example [132, 133, 134]. The five dimensional gravity Lagrangian with higher derivative terms can be written as

$$\mathcal{L} = \frac{\sqrt{-g}}{2\kappa_5^2}(R + \Lambda + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}). \tag{6.77}$$

Λ is the cosmological constant. The central charges a and c appear in the trace anomaly

$$\begin{aligned}\langle T^\mu{}_\mu \rangle &= \frac{c}{16\pi^2}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\right) \\ &\quad - \frac{a}{16\pi^2}(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2).\end{aligned} \tag{6.78}$$

Compare this with the holographic Weyl anomaly gives

$$\begin{aligned} a &= \frac{\pi L^3}{8G_N^{(5)}}\left[1 - \frac{4}{L^4}(10\hat{\alpha} + 2\hat{\beta} + \hat{\gamma})\right], \\ c &= \frac{\pi L^3}{8G_N^{(5)}}\left[1 - \frac{4}{L^4}(10\hat{\alpha} + 2\hat{\beta} - \hat{\gamma})\right] \end{aligned} \tag{6.79}$$

where we have separated the AdS_5 radius out of $\alpha = \frac{\hat{\alpha}}{L^2}$, $\beta = \frac{\hat{\beta}}{L^2}$ and $\gamma = \frac{\hat{\gamma}}{L^2}$. Only γ can be determined from string theory calculation. Furthermore, there is an ambiguity in α and β due to field redefinitions.

For $N=4$ SYM with gauge group $SU(n)$ in the IR, there is no correction from $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ term. To this order, the central charges are then given by

$$a_{IR} = c_{IR} = \frac{\pi^4 L^8}{8G_N^{(10)}} = \frac{\pi^4 C^2}{8G_N^{(10)}} \,. \tag{6.80}$$

On the other hand, in the UV, we have $N=2$, $SU(n)^N$ quiver gauge theory. The central charges are

$$\begin{aligned} a_{UV} &= \frac{\pi^4 N C^2}{8G_N^{(10)}}\left[1-\frac{4}{NC}(10\hat{\alpha}+2\hat{\beta}+\hat{\gamma})\right], \\ c_{UV} &= \frac{\pi^4 N C^2}{8G_N^{(10)}}\left[1-\frac{4}{NC}(10\hat{\alpha}+2\hat{\beta}-\hat{\gamma})\right]. \end{aligned} \tag{6.81}$$

The constant C in our solution is related to the number of D3-branes, N_3. The leading term in a and c is of order C^2 while the subleading one is of order C as expected. The analysis of the metric fluctuation can be carried out as in the six dimensional case and gives $\Delta = 2$. The flow is a vev flow driven by a vacuum expectation value of a relevant operator of dimension two.

Before discussing the RG flow on the dual field theory, let us recall that ALE gravitational instantons admit a hyperkahler quotient construction, which can be understood nicely in terms of the moduli space of a transverse (regular) D-brane probe moving off the orbifold fixed point in $\mathbb{R}^4/\mathbb{Z}_N$ [77], [78]. Starting with $U(N)$ valued fields X, $\bar{X}$ on which one performs the $\mathbb{Z}_N$ projection, one denotes the invariant (one-dimensional) components by $X_{i,i+1}$, $\bar{X}_{i+1,i}$, for $i=0,\ldots,N-2$, $X_{N-1,0}$, $\bar{X}_{0,N-1}$, which are the links of the quiver diagram corresponding to the A_{N-1} extended Dynkin diagram. The resulting gauge group is $U(1)^N$, with a trivially acting center of mass $U(1)$. It is convenient to introduce the doublet fields Φ_r

$$\Phi_r = \begin{pmatrix} X_{r-1,r} \\ \bar{X}^\dagger_{r,r-1} \end{pmatrix} \tag{6.82}$$

for $r=1,\ldots,N-1$, and

$$\Phi_0 = \begin{pmatrix} X_{N-1,0} \\ \bar{X}^\dagger_{0,N-1} \end{pmatrix} \tag{6.83}$$

After removing the trivial center of mass $U(1)$, the gauge group is $U(1)^{N-1}$, and the Φ's have definite charges with respect to it. After introducing Fayet-Iliopoulos (FI) terms $\vec{D}_r$, $r=0,\ldots,N-1$, with $\sum_r \vec{D}_r = 0$, corresponding to closed string, blowing-up mouduli, one gets the following potential:

$$U = \sum_{r=0}^{N-1}\left(\Phi_r^\dagger\vec{\sigma}\Phi_r - \Phi_{r+1}^\dagger\vec{\sigma}\Phi_{r+1} + \vec{D}_r\right)^2. \tag{6.84}$$

and the $N-1$ independent D-flatness conditions, are then given by:

$$\Phi_{r+1}^\dagger\vec{\sigma}\Phi_{r+1} - \Phi_r^\dagger\vec{\sigma}\Phi_r = \vec{D}_r\,. \tag{6.85}$$

The ALE metric (6.2) can be obtained after defining the ALE coordinate and centers

$$\vec{x} = \Phi_0^\dagger \vec{\sigma} \Phi_0, \qquad \vec{x}_i = \sum_{r=0}^{i-1} \vec{D}_r, \tag{6.86}$$

respectively, and computing the gauge invariant kinetic term on the Φ's, subject to the D-terms constraints [78].

This procedure can be generalized to the case of n regular D3-branes transverse to the ALE space. Starting with $U(nN)$ valued Chan-Paton factors, the resulting theory after projection, is the $N = 2$ $SU(n)^N$ gauge theory, with hypermultiplets formed by the fields X_{ij} and $\bar{X}_{ij}$ related to the links of the quiver diagram as above, but now in the $(n, \bar{n})$, $(\bar{n}, n)$ representations of the $SU(n)$'s at the vertices of the quiver diagram. In addition, there are adjoint scalars W_i in the adjoint of $SU(n)$, belonging to the vector multiplets. The theory is conformally invariant and describes the dual $N = 2$ SCFT at the UV point.

In order to match with the RG flow from the UV to IR described previously on the gravity side, which gives an $N = 4$ theory in the IR, we consider the Higgs branch of the $N = 2$ UV theory discussed above. Therefore, we set $\langle W_i \rangle = 0$ and give vev's to the hypermultiplets X_{ij}, $\bar{X}_{ij}$. The equations governing the vacua of the theory are then the obvious matrix generalization of (6.85), with $N-1$ independent triplets of FI terms for the $N-1$ $U(1)$'s, in an $SU(2)_R$ invariant formulation or can be written in and $N = 1$ fashion directly in terms of X_{ij}, $\bar{X}_{ij}$ and their hermitean conjugates. In any case, it is clear that by giving digonal vev's to X 's ($\bar{X}$'s)

$$\langle X_{ij} \rangle = x_{ij} \mathbf{I}_n, \qquad \text{for all}\, i, j \tag{6.87}$$

compatible with the D-flatness conditions, we can break $SU(n)^N$ down to the diagonal $SU(n)$, with a massless spectrum coinciding with that of $N = 4$ SYM theory for $SU(n)$ gauge group. A similar flow, from $N = 1$ to $N = 4$, has been studied in [74] and [76] in the case of the ALE space $\mathbb{C}_3/\mathbb{Z}_3$.

Notice that we can have intermediate possibilities for the IR point. In terms of the geometry, this can happen when some of the ALE centers x_i coincide with each other. Recalling the ALE metric, we have already seen that in the UV

$$V \sim \frac{N}{|\vec{x}|}, \qquad |\vec{x}| \to \infty\,.$$

In the IR, if we let M centers, $M < N$ to coincide with $\vec{x}_1$ say, and zoom near $\vec{x}_1$, we have

$$V \sim \frac{M}{|\vec{x} - \vec{x}_1|}, \qquad \vec{x} \to \vec{x}_1\,. \tag{6.88}$$

The ALE geometry then develops a $\mathbb{Z}_M$ singularity in the IR. Therefore, all possibilities with any values of N and M should be allowed as long as $M < N$. We can also compute the ratio of the central charges by repeating the same procedure as in the previous section and end up with the result

$$\frac{a_{IR}}{a_{UV}} = \frac{c_{IR}}{c_{UV}} = \frac{M}{N} < 1\,. \tag{6.89}$$

On the other hand, on the field theory side, we can partially Higgs the gauge group $SU(n)^N$ down to $SU(n)^M$, for any $M < N$. That is we have flows between the corresponding quiver diagrams.

6.2.2 RG flows in Type I′ string theory

As we mentioned, type I′ is obtained from type I theory by two T-duality transformations along the two cycles of T^2. In this process, D9-branes will become D7-branes and the $SO(32)$ gauge group is broken to $SO(8)^4$, corresponding to the four fixed points of T^2. It has been shown in [73], that the resulting theory is dual to type IIB theory on $T^2/(-1)^{F_L}\Omega I_2$. One then considers a stack of D3-branes near one of the fixed points and in the near horizon geometry one gets $AdS_5 \times S^5/\mathbb{Z}_2$. This corresponds to a dual $N = 2$ CFT, with $USp(2n)$ gauge group and $SO(8)$ global flavor symmetry [135, 136], with matter hypermultiplets in the antisymmetric representation of $USp(2n)$ and also in the (real) $(2n, 8)$ of $USp(2n) \times SO(8)$. In our case, we are replacing $\mathbb{R}^4$ with ALE space, or in the orbifold limit, with $\mathbb{R}^4/\mathbb{Z}_N$. Similar to the type IIB case, the UV field theory will be obtained by performing the orbifold $\mathbb{Z}_N$ projection of the above field content, which in turn will be recovered at the IR point after Higgsing.

On the supergravity side, we will restrict our analysis to the two-derivative terms in the affective action. Therefore, the ansatz of the previous subsection can be carried over to this case. In particular, the Bianchi identity for the 5-form will be unchanged. Otherwise, one would have to switch on also D7-brane instantons on the ALE space in order to compensate for the $R \wedge R$ term present on the right-hand side of the Bianchi identity at order $\mathcal{O}(\alpha')$. The analysis in this case is closely similar to the previous case apart from the facts that we start with 16 supercharges in ten dimensions rather than 32, and the final equation for e^{-4f} is the same as before. Following similar analysis as in the previous subsection, we can show that the solution interpolates between $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ in the UV and $AdS_5 \times S^5/\mathbb{Z}_2$ in the IR, with $\mathbb{Z}_2$ being the orientation reversal operator Ω.

As mentioned above, the field theory interpretation will involve flows between $N = 2$ quiver gauge theories with different gauge groups. The gauge group in the UV will be obtained by considering orbifolding/orientifolding a system of D3/D7 branes, whereas the IR group will be obtained by Higgsing, like in type IIB case. According to [77], the choice of gauge groups depends on the values of N as well as on the choice of a $\mathbb{Z}_N$ phase relating Ω and $\mathbb{Z}_N$ projections. In our case, the D3-brane worldvolume gauge theory descends from the theory on D5-brane for which $\gamma(\Omega)^t = -\gamma(\Omega)$. In what follows, we will use the notations of [77] and also refer the reader to this reference for more detail on the quiver gauge theory. We first review the consistency conditions for the $\mathbb{Z}_N \times \mathbb{Z}_2$ actions [77]

$$\begin{aligned} \Omega^2 = 1 &\quad : \quad && \gamma(\Omega) = \chi(\Omega)\gamma(\Omega)^t, \\ \Omega g = g\Omega &\quad : \quad && \gamma(g)\gamma(\Omega)\gamma(g)^t = \chi(g,\Omega)\gamma(\Omega), \\ g^N = 1 &\quad : \quad && \gamma(g)^N = \chi(g)1 \end{aligned} \tag{6.90}$$

where $g \in \mathbb{Z}_N$ and $\chi(\Omega)$, $\chi(g)$ and $\chi(g,\Omega)$ are phases. As shown in [77], we can set $\chi(g) = 1$. Furthermore, we are interested in the case of $\chi(\Omega) = -1$ on the D3-branes and $\chi(\Omega) = +1$ on the D7's. In type I theory, there are five cases to consider, but only three of them are relevant for us. These are N odd, $\chi(g,\Omega) = 1$, N even, $\chi(g,\Omega) = 1$ and N even, $\chi(g,\Omega) = \xi$ with $\xi = e^{\frac{2\pi i}{N}}$. We now consider RG flows in these cases.

$\chi(g,\Omega) = 1$, N odd

In this case, the gauge group is given by

$$\begin{aligned} G_1 &= USp(v_0) \times [U(v_1) \times U(v_2) \times \ldots \times U(v_{\frac{N-1}{2}})] \\ &= \{U_0, U_1, \ldots, U_{N-1} | U_i U^t_{N-i} = 1, 1 \leq i \leq N-1\}. \end{aligned} \tag{6.91}$$

Our convention is that $USp(2n)$ has rank n. The full configuration involves also the quiver theory on D9-branes which give rise to D7-branes in our case. Our main aim here is to study the symmetry breaking of the gauge group on D3-branes. The presence of D7-branes is necessary to make the whole system conformal. For the UV quiver gauge theory to be conformal, we choose $v_0 = v_1 = \ldots = v_{\frac{N-1}{2}} = n$ with an appropriate number of D7-branes such that the field theory beta function vanishes. Using the notation of [77], we denote the vector spaces associated to the nodes of the inner quiver, the D3-branes, by V_i and those of the outer quiver on D7-branes by W_i. There is also an identification of the nodes $V_i = V_{N-i}$ and similarly for W_i's, see [77]. This condition gives rise to the relation between the gauge groups of different nodes as shown in (6.91).

The gauge theory on the D7-branes is described by similar gauge group structure but with $USp(2n)$ replaced by $O(2n)$ due to the opposite sign of $\chi(\Omega)$. In addition to the vector multiplets, there are hypermultiplets, X, $\bar{X}$, associated to the links connecting the V_i's and I, J related to the links connecting V_i's and W_i's, in bifundamental representations of the respective gauge groups. The vanishing of the beta function can be achieved by setting $w_0 = 4$, $w_{\frac{N-1}{2}} = w_{\frac{N+1}{2}} = 2$ and the other w_i's zero. Notice that the non trivial gauge groups on the outer quiver are associated with two types of the nodes of the inner quiver. The first type consists of the nodes with USp gauge groups while the second type contains nodes connected to each other by antisymmetric scalars. The corresponding outer gauge groups for these two types are $SO(4)$ and $U(2)$, respectively.

It is easy to see the reason for this pattern of inner/outer gauge groups: the point is that for the inner nodes with $U(2n)$ gauge groups and connected by $U(2n)$ bifundamental scalars, the corresponding part of the quiver diagram is essentially the same as the quiver diagram arising from type IIB theory in which all the gauge groups are unitary. It is well-known that this quiver gauge theory is supeconformal without any extra field contents.

As observed in [77, 137], the above construction matches with the ADHM construction of $SO(n)$ instantons on ALE spaces: for example, the assignement of D7-brane gauge group given above means that, at the boundary of the ALE space, which has fundamental group $\pi_1 = \mathbb{Z}_N$, the $SO(8)$ flat connection has holonomy which breaks $SO(8)$ down to $SO(4) \times U(2) \sim SO(4) \times SU(2) \times U(1)$. On the other hand, G_1 is the ADHM gauge group, related to the number of instantons (D3-branes).

We now consider a Higgsing of this theory, and we need to be more precise about the representations of the matter fields. The nodes are connected to each other by the bifundamental scalars X and $\bar{X}$. These scalars are subject to some constraints given by

$$\begin{aligned} X_{01} &= -(X_{N-1,0}\omega_{2n})^t, \qquad X_{\frac{N-1}{2},\frac{N+1}{2}} = -(X_{\frac{N-1}{2},\frac{N+1}{2}})^t, \\ X_{i,i+1} &= (X_{N-i-1,N-i})^t, \qquad 1 \le i \le \frac{N-3}{2}, \\ \bar{X}_{10} &= (\omega_{2n}\bar{X}_{0,N-1})^t, \qquad \bar{X}_{\frac{N+1}{2},\frac{N-1}{2}} = -(\bar{X}_{\frac{N+1}{2},\frac{N-1}{2}})^t, \\ \bar{X}_{i+1,i} &= (\bar{X}_{N-i,N-i-1})^t, \qquad 1 \le i \le \frac{N-3}{2} \end{aligned} \tag{6.92}$$

where ω_{2n} represents the symplectic form of dimension $2n$. We will show that after the RG flow, the theory will flow to $USp(2n)$, $N = 2$ gauge theory in which the gauge group $USp(2n)$ is the diagonal subgroup of the $USp(2n)$ and $USp(2n)$ subgroups of all the $U(2n)$'s. We first illustrate this with the simple case of $N = 5$. The corresponding quiver diagram is shown in Figure 6.1. In the figure, the outer quiver and the inner one are connected to each other by scalar fields I_i, J_i (we have omitted the $\bar{X}$'s on the diagram). Notice that the gauge groups in the outer quiver are orthogonal and unitary groups due to the opposite sign of $\chi(\Omega)$. We will be interested in the Higgs branch, i.e. we set the vev's of the scalars in the vector multiplets to zero. Furthermore, we will also set $\langle I \rangle = \langle J \rangle = 0$ in all the cases we will discuss in the following. The D-flatness conditions are then obtained from those of the type IIB case by suitable projections/identifications on the X's and $\bar{X}$'s. The main difference, compared to the type IIB case, comes from the gauge group, which involve an $USp(2n)$ factor at the 0-th vertex and has $U(2n)$ factors which are related in the way indicated in Figure 6.1: as a result, the corresponding FI terms obey $\vec{D}_0 = 0$, $\vec{D}_1 = -\vec{D}_4$, $\vec{D}_2 = -\vec{D}_3$, with similar relations for higher odd N.

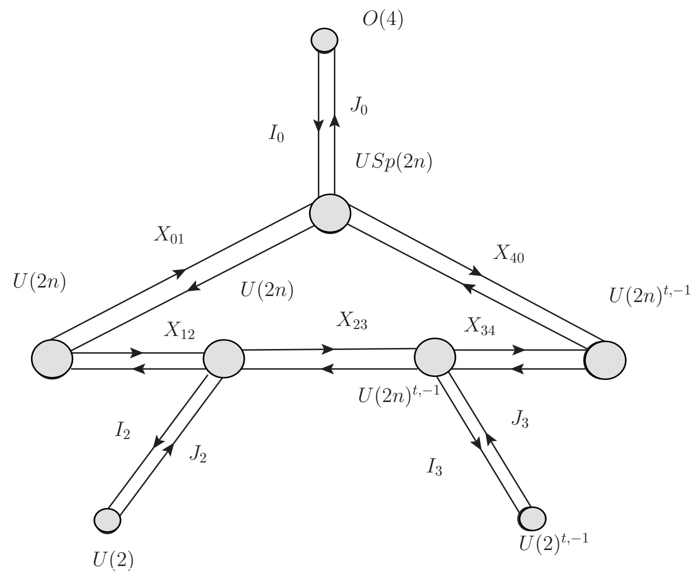

Figure 6.1: Quiver diagram for $\chi(\Omega) = -1$, $\chi(g, \Omega) = 1$ and $N = 5$.

We will give only the flows in which X and $\bar{X}$ acquire vev's. The above conditions then give

$$X_{01} = -X_{40}^t, \qquad X_{12} = X_{34}^t, \qquad X_{23} = -X_{23}^t \tag{6.93}$$

and similarly for $\bar{X}$. We choose the vev's as follows

$$\langle X_{01} \rangle = a\mathbf{I}_{2n}, \qquad \langle X_{12} \rangle = b\mathbf{I}_{2n}, \qquad \langle X_{23} \rangle = c\omega_{2n} \tag{6.94}$$

where a, b and c are constants. The vev's for $\bar{X}$ are similar but with different parameters $\bar{a}$, $\bar{b}$ and $\bar{c}$. Notice also that we only need to give vev's to the independent fields since the vev's of other fields can be obtained from (6.93). From now on, we will explicitly analyze only the X's. The analysis for $\bar{X}$'s follows immediately.

The field X_{ij} transforms as $g_i X_{ij} g_j^{-1}$ where g_i and g_j are elements of the two gauge groups, G_i and G_j, connected by X_{ij}. The unbroken gauge group is the subgroup of $USp(2n) \times U(2n) \times \ldots \times U(2n)$ that leaves all these vev's invariant. The invariance of X_{01} requires that g_1 is a symplectic subgroup of $U(2n)$ and $g_1 = g_0$. The invariance of X_{12} imposes the condition $g_1 = g_2 = g_0$ and so on. In the end, we find that the gauge group in the IR is $USp(2n)_{\text{diag}}$. For any odd N, the whole process works in the same way apart from the fact that there are more nodes similar to X_{12}. These nodes can be given vev's proportional to the identity. Taking this into account, we end up with scalar vev's

$$\begin{aligned}\langle X_{01}\rangle &= a_{01}\mathbf{I}_{2n}, \qquad \langle X_{\frac{N-1}{2},\frac{N+1}{2}}\rangle = a_{\frac{N-1}{2},\frac{N+1}{2}}\omega_{2n}, \\ \langle X_{i,i+1}\rangle &= a_{i,i+1}\mathbf{I}_{2n}, \qquad 1 \leq i \leq \frac{N-3}{2},\end{aligned} \tag{6.95}$$

and the unbroken gauge group is $USp(2n)_{\mathrm{diag}}$. In addition, one can verify that the masless spectrum is precisely that of the superconformal $USp(2n)$ theory with $SO(8)$ global symmetry described at the beginning of this section.

$\chi(g,\Omega) = 1$**,** N **even**

In this case, we have the gauge group

$$\begin{aligned}G_2 &= USp(v_0) \times [U(v_1) \times \ldots \times U(v_{\frac{N}{2}-1})] \times USp(v_{\frac{N}{2}}) \\ &= \left\{U_0, \ldots, U_{N-1} | U_i U_{N-i}^t = 1, 1 \leq i \leq N-1, i \neq \frac{N}{2}\right\}.\end{aligned} \tag{6.96}$$

Compared to the previous case, there is an additional $USp(v_{\frac{N}{2}})$ gauge group at the $\frac{N}{2}^{\mathrm{th}}$ node. As before, we choose $v_0 = v_1 = \ldots = v_{\frac{N}{2}} = 2n$ and $w_0 = w_{\frac{N}{2}} = 4$ with other w_i's being zero, corresponding to the breaking of the D7-brane gauge group from $SO(8)$ down to $SO(4) \times SO(4)$. The scalars are subject to the constraints

$$\begin{aligned}X_{01} &= \omega_{2n}(X_{N-1,o})^t, \qquad X_{\frac{N}{2},\frac{N+2}{2}} = -\omega_{2n}(X_{\frac{N-2}{2},\frac{N}{2}})^t, \\ X_{i,i+1} &= (X_{N-i-1,N-i})^t, \qquad 1 \leq i < \frac{N-2}{2}.\end{aligned} \tag{6.97}$$

The corresponding quiver diagram for $N = 4$ is shown in Figure 6.2.

As for the FI terms in this case, clearly $\vec{D}_0 = \vec{D}_2 = 0$ and $\vec{D}_1 = -\vec{D}_3$, with the obvious generalization for higher even N. We can choose the following vev's to Higgs the theory

$$\begin{aligned}\langle X_{01}\rangle &= x_{01}\mathbf{I}_{2n}, \qquad \langle X_{\frac{N-2}{2},\frac{N}{2}}\rangle = x_{\frac{N-2}{2},\frac{N}{2}}\mathbf{I}_{2n}, \\ \langle X_{i,i+1}\rangle &= x_{i,i+1}\mathbf{I}_{2n}, \qquad 1 \leq i < \frac{N-2}{2}.\end{aligned} \tag{6.98}$$

The symmetry breaking is the same as in the previous case. These vev's are invariant under the unbroken gauge group $USp(2n)_{\mathrm{diag}}$, and one can verify that massless hypermultiplets fill the spectrum of the $N = 2$ theory discussed in the previous case.

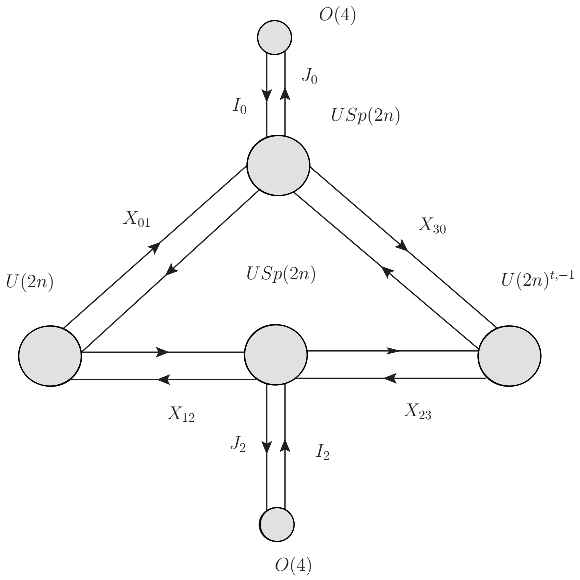

Figure 6.2: Quiver diagram for $\chi(\Omega) = -1$, $\chi(g, \Omega) = 1$ and $N = 4$.

$\chi(g, \Omega) = \xi$, N even

It is possible to choose $\chi(g, \Omega) = \xi$ for N even as shown in [77], and this is our last case. We adopt the range of the index i from 1 to N in this case. The relevant gauge group is given by

$$\begin{aligned} G_3 &= U(v_1) \times U(v_2) \times \ldots \times U(v_{\frac{N}{2}}) \\ &= \{U_1, \ldots, U_N | U_i U^t_{N-i+1} = 1, 1 \leq i \leq N\}. \end{aligned} \tag{6.99}$$

We are interested in the case $v_1 = v_2 = \ldots = v_{\frac{N}{2}} = 2n$ and $w_1 = w_{\frac{N}{2}} = 2$ with other w_i's being zero, i.e. the D7 gauge group is now broken down to $U(2) \times U(2)$. The conditions on the scalar fields are

$$\begin{aligned} X_{N1} &= -X^t_{N1}, \qquad X_{\frac{N}{2}, \frac{N+2}{2}} = -(X_{\frac{N}{2}, \frac{N+2}{2}})^t, \\ X_{i,i+1} &= (X_{N-i,N-i+1})^t, \qquad 1 \leq i \leq \frac{N-2}{2}\,. \end{aligned} \tag{6.100}$$

The quiver diagram for $N = 4$ and $\xi = i$ is shown in Figure 6.3. Notice the relations $\vec{D}_1 = -\vec{D}_4$, $\vec{D}_2 = -\vec{D}_3$ and so on for higher even N. There are two possibilities for Higgsing this theory. The first one involves only the vev's

$$\langle X_{i,i+1} \rangle = b_{i,i+1} \mathbf{I}_{2n}, \qquad 1 \leq i \leq \frac{N-2}{2}\,. \tag{6.101}$$

The unbroken gauge group is the diagonal subgroup of $U(v_1)\times\ldots\times U(v_{\frac{N}{2}})$, $U(2n)_{\rm diag}$. The second possibility is to give vev's to all scalars including the antisymmetric ones

$$\begin{aligned}\langle X_{i,i+1}\rangle &= b_{i,i+1}\mathbf{I}_{2n}, \qquad 1\le i\le \frac{N-2}{2}, \\ \langle X_{N1}\rangle &= b_{N1}\omega_{2n}, \qquad \langle X_{\frac{N}{2},\frac{N+2}{2}}\rangle = b_{\frac{N}{2},\frac{N+2}{2}}\omega_{2n}\,. \end{aligned} \tag{6.102}$$

In this case, the resulting gauge group is further broken down to $USp(2n)_{\rm diag}$.

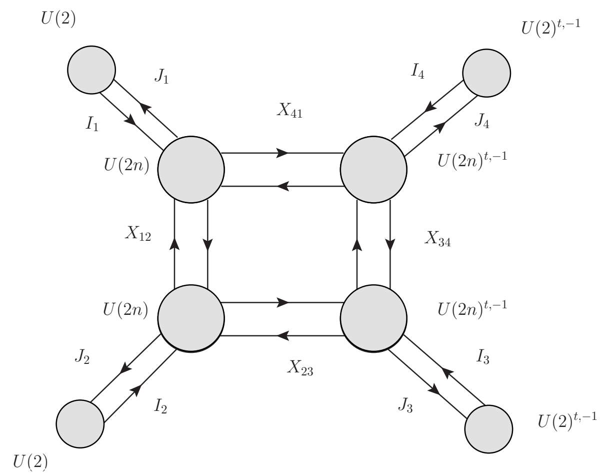

Figure 6.3: Quiver diagram for $\chi(\Omega)=-1$, $\chi(g,\Omega)=i$ and $N=4$.

6.3 Symmetry breaking and geometric interpretations

In this section, like in the type IIB case, we consider more general symmetry breaking patterns in the field theory and match them with the possible flows emerging from the supergravity solution. This involves the cases in which the gauge groups in the quiver gauge theory are not completely broken down to a single diagonal subgroup. After symmetry breaking, the IR CFT is again a quiver gauge theory with a reduced number of gauge groups, and of course, the number of nodes is smaller. We will show that some symmetry breaking patterns are not possible on the field theory side, at least by giving simple vev's to scalar fields.

We now consider the possibility of RG flows from a UV CFT which is a quiver gauge theory with the corresponding geometry $AdS_5\times S^5/(\mathbb{Z}_N\times\mathbb{Z}_2)$ to an IR CFT which is associated to the geometry $AdS_5\times S^5/(\mathbb{Z}_M\times\mathbb{Z}_2)$ and $M<N$. We saw that in the type IIB case this was always possible, and geometrically it was related to geometries developing a $\mathbb{Z}_M$ orbifold singularity obtained by bringing M

centers together in the smooth ALE metric.

Let us start from the field theory side. It is easy to see that it is not always possible to have a flow from one quiver diagram to the other. For example, we consider a flow from the diagram in Figure 6.1, $N = 5$, to Figure 6.2, $N = 4$. This can be done by giving a vev to X_{23} and $\bar{X}_{32}$ which transform in the antisymmetric tensor representation of $U(2n)$. The gauge group $U(2n)$ at the node v_2 and v_3 will be broken to $USp(2n)$. The resulting IR theory is then described by the quiver diagram in Figure 6.2. Continuing the process by Higgsing Figure 6.2 to the diagram with $N = 3$, we find that it is not possible to completely break the $USp(2n)$ gauge group at v_2 with the remaining scalars transforming in the antisymmetric tensor representation of $U(2n)$. It might be achieved by giving a vev to complicated composite operators, but we have not found any of these operators. Note also that this is the case only for reducing the value of N by one unit. If we Higgs the $N = 6$ to $N = 4$ or in general N to $N - 2$, this flow can always be achieved by giving vev's to X_{12} and $\bar{X}_{21}$. The gauge groups $U(2n)$ at v_1 and v_2 as well as at their images v_{N-1} and v_{N-2} will be broken to $U(2n)_{\text{diag}}$. The resulting quiver diagram is the same type as the original one with two nodes lower. What we are interested in is the problematic cases in which the flow connects two types of diagrams and lowers N by one unit.

We now begin with a diagram of the type shown in Figure 6.3. As mentioned in the previous section, this type is only possible for even N. It is easily seen that giving a vev to the $U(2n)$ antisymmetric scalars X_{1N} and $\bar{X}_{N1}$ reduces the diagram to the $N-1$ diagram of the type shown in Figure 6.1. Furthermore, a diagram with $N - 2$ nodes of the type in Figure 6.2 can be obtained by giving an additional vev to $X_{\frac{N}{2},\frac{N+2}{2}}$ and $\bar{X}_{\frac{N+2}{2},\frac{N}{2}}$. Now, the problem arises in deriving this diagram from the odd N diagram. As before, the $USp(2n)$ gauge groups at v_0 must be completely broken leaving only scalars in the $U(2n)$ antisymmetric tensor. Actually, it seems to be impossible to obtain this type of quiver diagrams from any of the other two types by Higgsing in a single or multiple steps since the process involves the disappearance of the USp gauge group.

We now discuss how the above field theory facts match with the geometry on the supergravity side. We will follow the approach in [78], where some peculiarities of type I string theory on $\mathbb{Z}_N$ orientifolds where clarified. The idea is to use a regular D1-brane (in type I theory), to probe the background geometry, following the same logic explained for the type IIB case in the previous subsection. In that case, we saw that one could reproduce the full smooth ALE geometry by switching on FI terms, which are background values of closed string moduli. The $\mathbb{Z}_N$ projection has generically the effect of reducing unitary groups down to SO/USp subgroups and/or of identifying pairs of unitary groups, in a way which depends on the details of the projection. We can indeed consider a probe D1-brane in the present orientifold context and derive its effective field theory for the three cases discussed in the previous section by assuming an orthogonal projection $\chi(\Omega) = 1$. In the following, the diagrams in Figures 6.1, 6.2 and 6.3 will be referred to as type I, II and III quivers, respectively.

For the case 1, $\chi(g,\Omega) = 1$, $N = 2m+1$ odd, we will have m pairs of conjugate $U(1)$'s as gauge groups, (with an $O(1)$ "gauge group" at the 0-th vertex of the inner quiver diagram of Figure 6.1), with appropriate identifications of the scalars X, $\bar{X}$. For example for $N = 3$, we have $X_{01} = X_{20}$ plus X_{12}, similarly for $\bar{X}$ fields. Consequently, for the FI terms, we will have $D_0 = 0$ and $D_i = -D_{N-i}$, $i = 1, \ldots, m$. Translating these data to the ALE centers $\vec{x}_i$ via (6.86) as in the previous subsection, we see that for $N = 2m+1$ there are m $\mathbb{Z}_2$ singularities. There is in addition a simple pole in the function V, which is however a smooth point in the geometry as long as it is kept distinct from the other poles. The function V in the ALE metric (6.2) is then given by

$$V = \frac{1}{|\vec{x} - \vec{x}_1|} + \sum_{i=2}^{m+1} \frac{2}{|\vec{x} - \vec{x}_i|} \,. \tag{6.103}$$

If we choose the IR point by setting $\vec{x} \to \vec{x}_1$, we end up with the flow from $N = 2$ quiver gauge theory of type I to the $N = 2$, $USp(2n)_{\text{diag}}$ gauge theory. The flow from type I quiver with $N = 2m+1$ to type I quiver with $N = 2m-1$ can be obtained by choosing $\vec{x} \to \vec{x}_1$ with $\vec{x}_i = \vec{x}_1$ for $i = 2, \ldots, m-1$. Finally, the flow to type II quiver in the case 2 can be achieved by setting $\vec{x}_i = \vec{x}_2$ for $i = 3, \ldots, m-1$ and $\vec{x} \to \vec{x}_2$.

For the case 2, $\chi(g,\Omega) = 1$, $N = 2m$ even, we will have $O(1)$ at the nodes 0 and m, and the remaining $U(1)$'s are pairwise conjugate, and there are obvious identifications for the X and $\bar{X}$ fields. Consequently, $\vec{D}_0 = \vec{D}_m = 0$ and $\vec{D}_i = -\vec{D}_{N-i}$, $i = 1, \ldots, m-1$. In terms of the ALE metric, we see that there are m $\mathbb{Z}_2$ singularities. The corresponding V function is

$$V = \sum_{i=1}^{m} \frac{2}{|\vec{x} - \vec{x}_i|} \,. \tag{6.104}$$

The possible flows are the following. First of all, to obtain the $N = 2$, $USp(2n)_{\text{diag}}$ gauge theory in the IR, we choose $\vec{x} \to \vec{x}'$ where $\vec{x}'$ is any regular point. The full Green function $G(x, x')$ will behave in the same way as $\vec{x} \sim \vec{x}_i$. In this case, the IR geometry is a smooth space. Another possible flows are given by Higgsing type II diagram with $N = 2m$ to the same type with $N = 2m-2$. This is achieved by setting $\vec{x}_i = \vec{x}_1$ for $i = 2, \ldots, m-1$ and $\vec{x} \to \vec{x}_1$.

Finally, for the case 3, $\chi(g,\Omega) = \xi$, therefore $N = 2m$ even, we have m pairs of conjugate $U(1)$ factors and consequently $\vec{D}_i = -\vec{D}_{N-1-i}$, $i = 0, \ldots, m$, which implies $m-1$ $\mathbb{Z}_2$ singularities plus two smooth points in the geometry. The function V is given by

$$V = \frac{1}{|\vec{x} - \vec{x}_1|} + \frac{1}{|\vec{x} - \vec{x}_{2m}|} + \sum_{i=2}^{m} \frac{2}{|\vec{x} - \vec{x}_i|} \,. \tag{6.105}$$

The flow from type III quiver to $N = 2$, $USp(2n)_{\rm diag}$ gauge theory is given by $\vec{x} \to \vec{x}_1$ or $\vec{x} \to \vec{x}_{2m}$. If we choose $\vec{x} \to \vec{x}_1$ and $\vec{x}_i = \vec{x}_1$ for $i = 2, \ldots, 2m-1$, we obtain the flow from type III quiver with $N = 2m$ to type I quiver with $N = 2m-1$. On the other hand, if we choose $\vec{x} \to \vec{x}_2$ and $\vec{x}_i = \vec{x}_2$ for $i = 3, \ldots, 2m-1$, we find a flow from type III quiver with $N = 2m$ to type II quiver with $N = 2m-2$. The flow from type III quiver with $N = 2m$ to type III quiver with $N = 2m-2$ is given by setting $\vec{x} \to \vec{x}_1$ and $\vec{x}_i = \vec{x}_{2m} = \vec{x}_1$ for $i = 2, \ldots, 2m-2$.

Notice that the V in (6.105) cannot be obtained from either (6.103) or (6.104) since both of them have none or only one single singularities while V in (6.105) has two. Furthermore, the flow from type II quiver to type I quiver is not allowed because there is no single singularity in (6.104), but there is one in (6.103). All the flows given above exactly agree with those obtained from the field theory side. So, we see that the effect of the Ω projection is to remove some of the blowing up, closed string, moduli and therefore the geometry cannot be completely smoothed out. Generically there remain $\mathbb{Z}_2$ singularities. Of course higher singularities can be obtained by bringing together the centers surviving the Ω projection. We summarize all possible flows in table III. The UV geometry is always $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ with $\vec{x}_{\rm UV} \to \infty$. The $V_{\rm UV}$ is given by that of (6.2) while $V_{\rm IR}$'s can be obtained by the $\vec{x}_{\rm IR}$ given in the table via (6.103), (6.104) and (6.105). In the Flow column, the notation I$(2m+1)\to$ II$(2m)$ means the flow from type I quiver with $N = 2m+1$ to type II quiver with $N = 2m$ etc. The $N = 2$, $USp(2n)_{\rm diag}$ gauge theory is denoted by I(1). The ALE centers are labeled in the same ordering as in equations (6.103), (6.104) and (6.105). Finally, $\vec{x}_{\rm IR}$'s are the IR points with the notation $\vec{x}'$ denoting any regular point away from the ALE center $\vec{x}_i$'s.

Flow	$\vec{x}_{\rm IR}$
I$(2m+1) \to$ I(1)	$\vec{x}_1$
II$(2m) \to$ I(1)	$\vec{x}'$
III$(2m) \to$ I(1)	$\vec{x}_1$
I$(2m+1) \to$ I$(2n+1, n < m)$	$\vec{x}_i = \vec{x}_1$, $i = 2, \ldots, n$
I$(2m+1) \to$ II$(2n, n \leq m)$	$\vec{x}_i = \vec{x}_2$, $i = 3, \ldots, n+1$
II$(2m) \to$ II$(2n, n < m)$	$\vec{x}_i = \vec{x}_1$, $i = 2, \ldots, n$
III$(2m) \to$ III$(2n, n < m)$	$\vec{x}_i = \vec{x}_1 = \vec{x}_{2m}$, $i = 2, \ldots, n$
III$(2m) \to$ II$(2n, n \leq m-1)$	$\vec{x}_i = \vec{x}_2$, $i = 2, \ldots, n$
III$(2m) \to$ I$(2n+1, n \leq m-1)$	$\vec{x}_i = \vec{x}_1$, $i = 2, \ldots, n$

Table III: All possible RG flows of the $N = 2$ quiver gauge theories arising in type I$'$ theory.

We now summarize what we have studied in this chapter. We have studied RG flow solutions in the four and two dimensional field theories on the background of the A_N ALE space. The flows in two dimensions are similar to the solution given in chapter 5 with the flat four dimensional space replaced by the ALE space. The flows are vev flows driven by a vacuum expectation value of a marginal operator. The dual field theory description is that of the (2,0) UV CFT flows to the (4,0) theory in the IR. The corresponding geometries are $AdS_3 \times S^3/\mathbb{Z}_N$ and $AdS_3 \times S^3$. We have computed the central charges in both the UV and IR to curvature squared terms in the bulk. The ratio of the central charges to the leading order contains a factor of N as expected from the ratio of the volumes of the S^3 and $S^3/\mathbb{Z}_N$ on which the six dimensional supergravity is reduced.

In type IIB theory, we have studied a flow solution describing an RG flow in four dimensional field theory. It involves the Green's function on ALE $\times\, \mathbb{R}^2$, which we were unable to find explicitely, but whose existence is guaranteed. The solution interpolates between $AdS_5 \times S^5/\mathbb{Z}_N$ and $AdS_5 \times S^5$. The flow is again a vev flow driven by a vacuum expectation value of a relevant operator of dimension two. The flow drives the $N = 2$ quiver gauge theory with the gauge group $SU(n)^N$ in the UV to the $N = 4$ $SU(n)_{\text{diag}}$ supersymmetric Yang-Mills theory in the IR. The hypermultiplets acquire vacuum expectation values proportional to the identity matrix and break $SU(n)^N$ to its diagonal subgroup $SU(n)_{\text{diag}}$ in the IR. The central charges a and c have also been computed to the curvature squared terms.

Moreover, we have studied a flow solution in type I$'$ theory. The flow solution interpolates between $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ and $AdS_5 \times S^5/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ is $(-1)^{F_L}\Omega I_2$. The flow is again driven by a vacuum expectation value of a relevant operator of dimension two. In contrast to the type IIB case, the field theory description is more complicated and more interesting. There are three cases to be considered. For N odd and $\chi(g, \Omega) = 1$, the flow drives the $N = 2$ quiver gauge theory with the gauge group $USp(2n) \times U(2n) \times \ldots \times U(2n)$ to the $N = 2$, $USp(2n)_{\text{diag}}$ gauge theory. For N even and $\chi(g, \Omega) = 1$, the flow describes an RG flow from $N = 2$ quiver $USp(2n) \times U(2n) \times \ldots \times U(2n) \times USp(2n)$ gauge theory to $N = 2$, $USp(2n)_{\text{diag}}$ gauge theory. Finally, for N even and $\chi(g, \Omega) = e^{\frac{2\pi i}{N}}$, we find the flow from $N = 2$ quiver $U(2n) \times \ldots \times U(2n)$ gauge theory to $N = 2$, $U(2n)_{\text{diag}}$ gauge theory for vanishing expectation values of the antisymmetric bifundamental scalars. With non-zero antisymmetric scalar expectation values, the gauge group in the IR is reduced to $USp(2n)_{\text{diag}}$.

We have also generalized the previous discussion to RG flows between two $N = 2$ quiver gauge theories in both type IIB and type I$'$ theories. The gravity solution interpolates between $AdS_5 \times S^5/(\mathbb{Z}_N \times \mathbb{Z}_2)$ and $AdS_5 \times S^5/(\mathbb{Z}_M \times \mathbb{Z}_2)$ geometries. In type IIB theory, the flows work properly as expected from the field theory side in a strightforward way. In type I$'$ theory, field theory considerations forbid some symmetry breaking patterns. However, this is in agreement with the geometrical picture, after one takes into account the restrictions put on the geometry by the orientifolding procedure.

Chapter 7

Conclusions

Throughout this thesis, we have studied a number of holographic RG flow solutions in the framework of supergravity theories. All solutions found in this thesis are vev flows driven by vacuum expectation values of relevant or marginal operators. In some cases, we have given an interpretation of the resulting solution in term of some D-branes configurations in string theory. A reduction ansatz for (1,0) supergravity coupled to an antisymmetric tensor and Yang-Mills multiplets in six dimensions on $SU(2)$ group manifold giving rise to $N = 4$ $SU(2) \times G$ Yang-Mills gauged supergravity in three dimensions has been found. Furthermore, the reduced theory has been shown to be on-shell equivalent to the $N = 4$ Chern-Simons gauged supergravity with the corresponding gauge group $(SU(2) \ltimes \mathbf{T}^3) \times (G \ltimes \mathbf{T}^{\text{dim}G})$ as discovered in [38]. The role of Yang-Mills and gravitational instantons in RG flows has been emphasized, and their effects on the value of the central charge at the fixed points has been worked out. The corresponding RG flows with Yang-Mills instantons turned on on $\mathbb{R}^4$ can be interpreted as transitions between different Yang-Mills vacua. We have also studied RG flows in $N = 2$ quiver gauge theories in four dimensions in the framework of type IIB and type I$'$ string theories. We have given a geometric interpretation of various symmetry breaking patterns. This completely agrees with the field theory consideration. We now end the thesis with some comments on the results together with the remaining open problems which will give us some directions for future works.

The gaugings considered in chapter 3 are of non semi-simple Chern-Simons type, giving rise to semi-simple Yang-Mills theories upon integrating out scalar fields corresponding to translational symmetries. In the $N = 8$ theory, the $(4, 4)$ point is related to the Kaluza-Klein reduction of type IIB theory on $AdS_3 \times S^3 \times S^3 \times S^1$, and it would be interesting to identify the marginal deformations which take the theory to other less supersymmetric vacua, i.e. to generalize the discussion of [138], where the marginal deformation from $(4, 4)$ to $(3, 3)$ vacua has been worked out in detail, to the (k, k) vacua with $k < 3$. From the higher dimensional perspective, the $N = 8$ case is related to the brane configuration in type IIB theory whose near horizon geometry is $AdS_3 \times S^3 \times S^3 \times S^1$ [139], dual to a SCFT$_2$ with "large" $(4, 4)$

superconformal algebra[138, 140]. The $N = 4$ case seems to be related, via a $\mathbb{Z}_2$ projection, to the $N = 8$ theory, and it would be interesting to see how this is acting on the corresponding type IIB theory background. This would presumably help us in understanding the nature of the dual SCFT_2.

For the reduction ansatz given in chapter 4, a natural open problem is how to obtain three dimensional $N = 4$ gauged supergravity with two quaternionic scalar manifolds, for example to recover the theory studied in chapter 3. Presumably we would need to add hypermultiplets to the six dimensional theory, whose scalars themselves live on a quaternionic manifold, or perhaps, we may even need to start with extended supersymmetry in six dimensions. As discussed in [31], $N = 4$ gauged supergravity in three dimensions is related to $N = 2$ supergravity in four dimensions. The two quaternionic target spaces correspond to scalar fields in the vector and hyper multiplets, respectively. It could be that the reductions giving rise to four dimensional $N = 2$ supergravity including both vector and hyper multiplets scalars will give some hints to obtain $N = 4$ gauged supergravity in three dimensions with two scalar target manifolds. This issue needs further investigations.

In chapter 5, we have found RG flow solutions in the presence of Yang-Mills instantons. The flows describe a deformation of the UV CFT by a vacuum expectation value of a marginal operator. Interestingly, these RG flows have an interpretation in terms of Yang-Mills instantons tunnelling between $|N\rangle$ Yang-Mills vacuum in the UV and $|0\rangle$ in the IR, and this fact is in turn related to the different values of the central charge at the two fixed points. In the general N instanton solution, there is a subtlety of phase transitions occurring whenever v and $\tilde{v}$ change sign. We have avoided this issue by assuming the positivity of both v and $\tilde{v}$. We do not have a clear interpretation of this phase transition in the dual CFT, so it would be interesting to study this issue in more detail.

In chapter 6, we have studied many RG flows in both two and four dimensional field theories. Two dimensional RG flows have been found in the framework of (1,0) six dimensional supergravity while the four dimensional solutions have been studied in type IIB and type I$'$ string theories. Here, we will give a few comments regarding the type I$'$ case. If we include higher order terms in the effective action, we need, among other things, to switch on the $F \wedge F$ to ensure the Bianchi identity for the 5-form

$$d\tilde{F}^{(5)} = \frac{\alpha'}{4}(\text{Tr}R \wedge R - \text{Tr}F \wedge F)\delta^{(2)}(\vec{z}) \tag{7.1}$$

F being the field strength of the $SO(8)$ gauge group and $\vec{z}$ a coordinate on the transverse $\mathbb{R}^2$. In particular, we need to include $SO(8)$ instantons on the ALE spaces (with the standard metric, the warp factor being irrelevant due to conformal invariance). It would be interesting to relate ALE's instanton configurations to the pattern of symmetry breaking of the global $SO(8)$ group involved in the various flows discussed in chapter 6. As already mentioned, the UV group is determined by the holonomy of the flat connection at the ALE's boundary, which is in turn part of the ADHM data. It would be interesting to understand the IR group in a similar way.

It is certainly true that a lot of works remain to be done in order to understand RG flows in both two and four dimensional field theories although many works have already appeared. As we have seen, holography and the AdS/CFT correspondence provide a very useful tool to explore various aspects of strongly coupled field theories. Hopefully, future investigations will clarify and give some insights to the questions mentioned above as well as other open problems stated elsewhere.

Appendix A

Vacua of three dimensional gauged supergravities

In this appendix, we give some vacua of three dimensional gauged supergravities. The strategy to find critical points of the scalar potential has already been discussed in the main text chapter 3. The theories studied here are $N = 4, 8, 9, 10$ gauged supergravity theories whose vacua are given below.

A.1 Vacua of $N = 4$ theory

The vacua of $N = 4$ theory that do not involve in the flow solutions are given by:

A.1.1 (0,4) vacuum

- VI.

$$
\begin{aligned}
e_1 &= \sqrt{\frac{-2(g_n+g_p)}{g_{2s}}}\mathbf{I}_4, \qquad e_2 = \sqrt{\frac{2(g_p-g_n)}{g_{2s}}}\mathbf{I}_4, \\
A_1 &= \frac{32 g_n g_p}{g_{2s}}, \qquad \text{and} \qquad V_0 = -\frac{4096 g_n^2 g_p^2}{g_{2s}^2}.
\end{aligned} \tag{A.1}
$$

A.1.2 (3,0) vacua

- VII.

$$
\begin{aligned}
e_1 &= a\Big(1, -\frac{g_m+g_p}{g_n+g_p+g_{2s}a^2}, 1, 1\Big), \qquad e_2 = \sqrt{\frac{(g_p^2-g_n^2)}{g_p^2-g_n^2+g_{2s}g_p a}}a\mathbf{I}_4, \\
a &= \sqrt{\frac{g_n^3-g_n^2 g_p - g_n g_p^2 + g_p^3 + \sqrt{g_n^6 - g_n^4 g_p^2 - g_n^2 g_p^4 + g_p^6}}{g_n g_p g_{2s}}} \\
A_1 &= -\frac{8(g_n^2-g_p^2)^2}{g_{2s} g_n g_p}, \qquad \text{and} \qquad V_0 = \frac{-256(g_n^2-g_p^2)^4}{(g_{2s}^2 g_n g_p)^2}
\end{aligned} \tag{A.2}
$$

- VIII.

$$
\begin{aligned}
e_1 &= a\Big(1, -\frac{g_m + g_p}{g_n + g_p + g_{2s}a^2}, 1, 1\Big), \qquad e_2 = \sqrt{\frac{(g_p^2 - g_n^2)}{g_p^2 - g_n^2 + g_{2s}g_p a}}\, a\mathbf{I}_4, \\
a &= \sqrt{\frac{g_n^3 - g_n^2 g_p - g_n g_p^2 + g_p^3 - \sqrt{g_n^6 - g_n^4 g_p^2 - g_n^2 g_p^4 + g_p^6}}{g_n g_p g_{2s}}} \\
A_1 &= -\frac{8(g_n^2 - g_p^2)^2}{g_{2s} g_n g_p}, \qquad \text{and} \qquad V_0 = \frac{-256(g_n^2 - g_p^2)^4}{(g_{2s}^2 g_n g_p)^2} \qquad \text{(A.3)}
\end{aligned}
$$

A.1.3 (2,0) vacua

- IX.

$$
e_1 = -(a_1, a_1, b_1, b_2) \qquad e_2 = (b_1, b_1, b_2, b_2) \qquad \text{(A.4)}
$$

$$
\begin{aligned}
a_1 &= 2\sqrt{\frac{g_n^2 - g_p^2}{g_{2s}(g_n - g_p + \sqrt{5g_n^2 + 2g_n g_p + g_p^2})}} \\
a_2 &= 2\sqrt{\frac{g_n^2 - g_p^2}{g_{2s}(g_p - g_n + \sqrt{5g_p^2 + 2g_n g_p + g_n^2})}} \\
b_1 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(3g_n + g_p - \sqrt{5g_n^2 + 2g_p g_n + g_p^2})}} \\
b_2 &= 2\sqrt{\frac{g_n^2 - g_p^2}{g_{2s}(\sqrt{g_n^2 + 2g_n g_p + 5g_p^2} - g_n - 3g_p)}} \qquad \text{(A.5)}
\end{aligned}
$$

$$
A_1 = \frac{-32(g_n - g_p)^2}{g_{2s}} \qquad \text{and} \qquad V_0 = -\frac{4096(g_n - g_p)^4}{g_{2s}^2}. \qquad \text{(A.6)}
$$

- X.

$$
e_1 = (-a_1, a_1, a_2, a_2) \qquad e_2 = (b_1, b_1, b_2, b_2) \qquad \text{(A.7)}
$$

$$
\begin{aligned}
a_1 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(g_p - g_n + \sqrt{5g_n^2 + 2g_p g_n + g_p^2})}} \\
a_2 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(g_n - g_p - \sqrt{5g_p^2 + 2g_p g_n + g_n^2})}} \\
b_1 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(3g_n + g_p + \sqrt{5g_n^2 + 2g_p g_n + g_p^2})}} \\
b_2 &= 2\sqrt{\frac{g_n^2 - g_p^2}{g_{2s}(\sqrt{g_n^2 + 2g_n g_p + 5g_p^2} - g_n - 3g_p)}} \qquad \text{(A.8)}
\end{aligned}
$$

$$A_1 = \frac{-32(g_n - g_p)^2}{g_{2s}} \quad \text{and} \quad V_0 = -\frac{4096(g_n - g_p)^4}{g_{2s}^2}. \tag{A.9}$$

- XI.

$$e_1 = (-a_1, a_1, a_2, a_2) \qquad e_2 = (b_1, b_1, b_2, b_2) \tag{A.10}$$

$$\begin{aligned}
a_1 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(g_p - g_n - \sqrt{5g_n^2 + 2g_p g_n + g_p^2})}} \\
a_2 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(g_n - g_p + \sqrt{5g_p^2 + 2g_p g_n + g_n^2})}} \\
b_1 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(3g_n + g_p - \sqrt{5g_n^2 + 2g_p g_n + g_p^2})}} \\
b_2 &= 2\sqrt{\frac{g_p^2 - g_n^2}{g_{2s}(\sqrt{g_n^2 + 2g_n g_p + 5g_p^2} + g_n + 3g_p)}}
\end{aligned} \tag{A.11}$$

$$A_1 = \frac{-32(g_n - g_p)^2}{g_{2s}} \quad \text{and} \quad V_0 = -\frac{4096(g_n - g_p)^4}{g_{2s}^2}. \tag{A.12}$$

- XII.

$$e_1 = (-a_1, a_1, a_2, a_2) \qquad e_2 = (b_1, b_1, b_2, b_2) \tag{A.13}$$

$$\begin{aligned}
a_1 &= \sqrt{\frac{g_p - g_n}{g_{2s}}}\sqrt{\frac{g_n - g_p + \sqrt{5g_n^2 + 2g_n g_p + g_p^2}}{g_n}} \\
a_2 &= \sqrt{\frac{g_p - g_n}{g_{2s}}}\sqrt{-\frac{g_n - g_p + \sqrt{g_n^2 + 2g_n g_p + 5g_p^2}}{g_p}} \\
b_1 &= \sqrt{\frac{g_p - g_n}{g_{2s}}}\sqrt{\frac{3g_n + g_p - \sqrt{5g_n^2 + 2g_n g_p + g_p^2}}{g_n}} \\
b_2 &= \sqrt{\frac{g_p^2 - g_n^2}{g_{2s} g_p}}\sqrt{-\frac{g_n - g_p + \sqrt{g_n^2 + 2g_n g_p + 5g_p^2}}{2g_p - \sqrt{g_n^2 + 2g_n g_p + 5g_p^2}}}
\end{aligned} \tag{A.14}$$

$$A_1 = \frac{-32(g_n - g_p)^2}{g_{2s}} \quad \text{and} \quad V_0 = -\frac{4096(g_n - g_p)^4}{g_{2s}^2}. \tag{A.15}$$

A.2 Vacua of $N = 8$ theory

In this section, we study some vacua of $(SO(4) \ltimes \mathbf{T}^6) \times (SO(4) \ltimes \mathbf{T}^6)$, $N = 8$ gauged supergravity in three dimensions. We restrict our discussion to the target space $\frac{SO(8,8)}{SO(8)\times SO(8)}$.

A.2.1 $N = 8$ three dimensional gauged supergravity

We parametrize the coset elements L as in the $N = 4$ case, but now obviously e is an element of $GL(8, \mathbb{R})$ and B is an antisymmetric 8×8 matrix. The resulting L depends on 92 parameters, but, again using the right action of a diagonal $SO(8)$, one can bring e to an upper triangular form, thereby reducing the number of parameters to 64. As for the non compact generators, the Y^{ab} introduced before carry over in the obvious way to the present case, with $a, b = 1, \ldots, 8$.

We are going to gauge the subgroup $(SO(4) \ltimes \mathbf{T}^6)^2$. Accordingly, we introduce gauge group generators:

$$t^{\mathcal{A}} = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix}, \qquad t^{\mathcal{B}} = \begin{pmatrix} b_1 & 0 & b_1 & 0 \\ 0 & b_2 & 0 & b_2 \\ -b_1 & 0 & -b_1 & 0 \\ 0 & -b_2 & 0 & -b_2 \end{pmatrix}. \tag{A.16}$$

Here all entries are 4×4 matrices, a_1 (a_2) are generators of the first (second) $SO(4)$, b_1 and b_2 are antisymmetric and correspond to independent shifts of B. More precisely, the upper and lower 4×4 diagonal blocks of B will be shifted by $2b_1$ and $2b_2$, respectively, and therefore could be set to zero. Generators carrying index 1 commute with those carrying index 2, and one checks the structure of the gauge group stated above. The f-tensors are constructed as follows: we choose a basis of symmetric, real $SO(8)$ gamma matrices with 8×8 off-diagonal blocks Γ^I, so that:

$$f^{IJ}_{ab,cd} = -\frac{1}{2}\mathrm{Tr}(\varepsilon^{ba}[\Gamma^I, \Gamma^J]\varepsilon^{cd}). \tag{A.17}$$

As for the embedding tensor Θ, the structure discussed in the $N = 4$ case extends naturally to the present case, and now we expect a priori 8 couplings corresponding to the 8 $SU(2)$'s (including the $\mathcal{B}$ generators). We then proceed by first computing the $\mathcal{V}$'s which are given by

$$\begin{aligned} \mathcal{V}_{\pm\mathrm{a}}{}^{LJ,MK} &= \frac{1}{4\sqrt{2}}\mathrm{Tr}[\Gamma^{JL}(eJ_{\pm}^{MK}X + X^t J_{\pm}^{MK}e^t)], \\ \mathcal{V}_{\pm\mathrm{b}}{}^{LJ,MK} &= \frac{1}{2\sqrt{2}}\mathrm{Tr}[J_{\pm}^{JL}e\Gamma^{MK}e^t], \\ \mathcal{V}_{\pm\mathrm{a}}{}_{ab}^{MK} &= \frac{1}{\sqrt{2}}\mathrm{Tr}[\varepsilon_{ab}(X^t J_{\pm}^{MK}e^t + eJ_{\pm}^{MK}Y)], \\ \mathcal{V}_{\pm\,\mathrm{b}}{}_{ab}^{MK} &= \frac{2}{\sqrt{2}}\mathrm{Tr}[\varepsilon_{ab}eJ_{\pm}^{MK}e^t]. \end{aligned} \tag{A.18}$$

Here $\Gamma^{JL} = -[\Gamma^J, \Gamma^L]/2$ and all indices run from 1 to 8 and $J_{\pm}^{MK}$ are the (anti-) self-dual $SU(2)$ generators in $SO(4) \times SO(4) \subset SO(8)$, corresponding to the first (second) $SO(4)$ for $M, K = 1, \ldots, 4$ ($M, K = 5, \ldots, 8$), respectively.

The T tensors are

$$\begin{aligned}
T^{LJ,MK} &= g_{1s}(\mathcal{V}_{+\mathrm{a}}^{LJ,PQ}\mathcal{V}_{+\mathrm{b}}^{MK,PQ} + \mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+\mathrm{a}}^{MK,PQ}) + g_{1a}(\mathcal{V}_{-\mathrm{a}}^{LJ,PQ}\mathcal{V}_{-\mathrm{b}}^{MK,PQ} \\
&\quad + \mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-\mathrm{a}}^{MK,PQ}) + g_{2s}(\mathcal{V}_{+\mathrm{a}}^{LJ,P'Q'}\mathcal{V}_{+\mathrm{b}}^{MK,P'Q'} + \mathcal{V}_{+\mathrm{b}}^{LJ,P'Q'}\mathcal{V}_{+\mathrm{a}}^{MK,P'Q'}) \\
&\quad + g_{2a}(\mathcal{V}_{-\mathrm{a}}^{LJ,P'Q'}\mathcal{V}_{-\mathrm{b}}^{MK,P'Q'} + \mathcal{V}_{-\mathrm{b}}^{LJ,P'Q'}\mathcal{V}_{-\mathrm{a}}^{MK,P'Q'}) + h_{1s}\mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+\mathrm{b}}^{MK,PQ} \\
&\quad + h_{1a}\mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-\mathrm{b}}^{MK,PQ} + h_{2s}\mathcal{V}_{+\mathrm{b}}^{LJ,P'Q'}\mathcal{V}_{+\mathrm{b}}^{MK,P'Q'} + h_{2a}\mathcal{V}_{-\mathrm{b}}^{LJ,P'Q'}\mathcal{V}_{-\mathrm{b}}^{MK,P'Q'}, \\
T_{ab}^{LJ} &= g_{1s}(\mathcal{V}_{+\mathrm{a}}^{LJ,PQ}\mathcal{V}_{+\mathrm{b}ab}{}^{PQ} + \mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+\mathrm{a}ab}{}^{PQ}) + g_{1a}(\mathcal{V}_{-\mathrm{a}}^{LJ,PQ}\mathcal{V}_{-\mathrm{b}ab}{}^{PQ} \\
&\quad + \mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-\mathrm{a}ab}{}^{PQ}) + g_{2s}(\mathcal{V}_{+\mathrm{a}}^{LJ,P'Q'}\mathcal{V}_{+\mathrm{b}ab}{}^{P'Q'} + \mathcal{V}_{+\mathrm{b}}^{LJ,P'Q'}\mathcal{V}_{+\mathrm{a}ab}{}^{P'Q'}) \\
&\quad + g_{2a}(\mathcal{V}_{-\mathrm{a}}^{LJ,P'Q'}\mathcal{V}_{-\mathrm{b}ab}{}^{P'Q'} + \mathcal{V}_{-\mathrm{b}}^{LJ,P'Q'}\mathcal{V}_{-\mathrm{a}ab}{}^{P'Q'}) + h_{1s}\mathcal{V}_{+\mathrm{b}}^{LJ,PQ}\mathcal{V}_{+\mathrm{b}ab}{}^{PQ} \\
&\quad + h_{1a}\mathcal{V}_{-\mathrm{b}}^{LJ,PQ}\mathcal{V}_{-\mathrm{b}ab}{}^{PQ} + h_{2s}\mathcal{V}_{+\mathrm{b}}^{LJ,P'Q'}\mathcal{V}_{+\mathrm{b}ab}{}^{P'Q'} + h_{2a}\mathcal{V}_{-\mathrm{b}}^{LJ,P'Q'}\mathcal{V}_{-\mathrm{b}ab}{}^{P'Q'}, \quad \text{(A.19)}
\end{aligned}$$

where $P, Q, \ldots = 1, \ldots, 4$ and $P', Q', \ldots = 5, \ldots, 8$. Here L, J, M, K are $SO(8)$ R-symmetry indices, and $a, b = 1, \ldots, 8$ label the 64 non-compact generators in $SO(8,8)$. $P, Q = 1, \ldots, 4$ and $P', Q' = 5, \ldots, 8$ label the first and second $SO(4)$, respectively. We have included also the 8 coupling constants, but actually, consistency imposes relations among them:

$$\begin{aligned}
g_{1a} &= -g_{1s}, \qquad g_{2a} = -g_{2s} \\
h_{1a} &= -h_{1s}, \qquad \text{and} \qquad h_{2a} = -h_{2s}.
\end{aligned} \qquad \text{(A.20)}$$

Notice that if we set the type-2 couplings to zero i.e. $g_{2s} = g_{2a} = h_{2s} = h_{2a} = 0$, we decouple the second $SO(4)$ and therefore we recover a truncation of the single $SO(4)$ gauging studied in [39] as the supergravity dual of the D1-D5 system in IIB theory on $K3$ or T^4. It can be obtained by reducing (2,0) six-dimensional supergravity on $AdS_3 \times S^3$.

A.2.2 Vacua of $N = 8$ gauged supergravity

A simple class of supersymmetric AdS vacua can be obtained as follows. We parameterize e and B as:

$$e = \begin{pmatrix}
a_1 & 0 & 0 & 0 & e_{15} & e_{16} & e_{17} & e_{18} \\
0 & a_2 & 0 & 0 & e_{25} & e_{26} & e_{27} & e_{28} \\
0 & 0 & a_3 & 0 & e_{35} & e_{36} & e_{37} & e_{38} \\
0 & 0 & 0 & a_4 & e_{45} & e_{46} & e_{47} & e_{48} \\
0 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_8
\end{pmatrix}, \qquad \text{(A.21)}$$

$$
B = \begin{pmatrix} 0 & 0 & 0 & 0 & b_{15} & b_{16} & b_{17} & b_{18} \\ 0 & 0 & 0 & 0 & b_{25} & b_{26} & b_{27} & b_{28} \\ 0 & 0 & 0 & 0 & b_{35} & b_{36} & b_{37} & b_{38} \\ 0 & 0 & 0 & 0 & b_{45} & b_{46} & b_{47} & b_{48} \\ -b_{15} & -b_{25} & -b_{35} & -b_{45} & 0 & 0 & 0 & 0 \\ -b_{16} & -b_{26} & -b_{36} & -b_{46} & 0 & 0 & 0 & 0 \\ -b_{17} & -b_{27} & -b_{37} & -b_{47} & 0 & 0 & 0 & 0 \\ -b_{18} & -b_{28} & -b_{38} & -b_{48} & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{A.22}
$$

We have used the shift symmetry to set to zero the diagonal 4×4 blocks of B and the $SO(4) \times SO(4)$ left action to diagonalize the diagonal blocks of e. For diagonal $e = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ and $B = 0$, we cannot find any interesting solutions apart from the trivial one with (4,4) supersymmetry. All the truncations below have been checked to be consistent, in the sense that there are no tadpoles for the remaining scalars.

We find a class of solutions by setting:

$$
\begin{aligned}
a_2 &= a_3 = a_4 = a_1, \qquad a_6 = a_7 = a_8 = a_5, \\
b_{15} &= \frac{1}{4}(c_{15} - c_{26} - c_{37} + c_{48}), \qquad b_{16} = \frac{1}{4}(-c_{16} - c_{25} - c_{38} - c_{47}), \\
b_{17} &= \frac{1}{4}(c_{18} + c_{27} - c_{36} - c_{45}), \qquad b_{18} = \frac{1}{4}(c_{17} - c_{28} + c_{35} - c_{46}), \\
b_{25} &= \frac{1}{4}(-c_{16} - c_{25} + c_{38} + c_{47}), \qquad b_{26} = \frac{1}{4}(-c_{15} + c_{26} - c_{37} + c_{48}), \\
b_{27} &= \frac{1}{4}(c_{17} - c_{28} - c_{35} + c_{46}), \qquad b_{28} = \frac{1}{4}(-c_{18} - c_{27} - c_{36} - c_{45}), \\
b_{35} &= \frac{1}{4}(c_{18} - c_{27} + c_{36} - c_{45}), \qquad b_{36} = \frac{1}{4}(-c_{17} - c_{28} + c_{35} + c_{46}), \\
b_{37} &= \frac{1}{4}(-c_{15} - c_{26} - c_{37} - c_{48}), \qquad b_{38} = \frac{1}{4}(-c_{16} + c_{25} + c_{38} - c_{47}), \\
b_{45} &= \frac{1}{4}(-c_{17} - c_{28} - c_{35} - c_{46}), \qquad b_{46} = \frac{1}{4}(-c_{18} + c_{27} + c_{36} - c_{45}), \\
b_{47} &= \frac{1}{4}(-c_{16} + c_{25} - c_{38} + c_{47}), \qquad b_{48} = \frac{1}{4}(c_{15} + c_{26} - c_{37} - c_{48}),
\end{aligned} \tag{A.23}
$$

and all other parameters are zero. We can choose

$$
\begin{aligned}
c_{16} &= c_{17} = c_{18} = c_{25} = c_{27} = c_{28} = 0, \\
c_{35} &= c_{36} = c_{38} = c_{45} = c_{46} = c_{47} = 0.
\end{aligned} \tag{A.24}
$$

Supersymmetric vacua require

$$
g_{1s} = -a_1^2 h_{1s}, \qquad g_{2s} = -a_5^2 h_{2s}, \qquad h_{2s} = \frac{a_1^4}{a_5^4} h_{1s}. \tag{A.25}
$$

- **(1,1) critical point**
  This point is given by $c_{15} = 0$,

$$A_1 = (-\frac{16g_{1s}^2}{h_{1s}}, \frac{16g_{1s}^2}{h_{1s}}, -\frac{8g_{1s}^2}{h_{1s}}\sqrt{4 + a_1^2 a_5^2 c_{26}^2}, \frac{8g_{1s}^2}{h_{1s}}\sqrt{4 + a_1^2 a_5^2 c_{26}^2}, -\frac{8g_{1s}^2}{h_{1s}}\sqrt{4 + a_1^2 a_5^2 c_{37}^2}, \frac{8g_{1s}^2}{h_{1s}}\sqrt{4 + a_1^2 a_5^2 c_{37}^2}, -\frac{8g_{1s}^2}{h_{1s}}\sqrt{4 + a_1^2 a_5^2 c_{48}^2}, \frac{8g_{1s}^2}{h_{1s}}\sqrt{4 + a_1^2 a_5^2 c_{48}^2}) \quad \text{(A.26)}$$

  and $V_0 = -\frac{1024 g_{1s}^4}{h_{1s}^2}$.

- **(2,2) critical point**
  This point is given by $c_{15} = 0$ and $c_{26} = 0$.

- **(3,3) critical point**
  This point is given by $c_{15} = 0$,$c_{26} = 0$ and $c_{37} = 0$.

- **(4,4) critical point**
  This point is given by $c_{15} = 0$, $c_{26} = 0$, $c_{37} = 0$ and $c_{48} = 0$.

All of them have the same cosmological constant. A_1 for the last three points is given by setting some of the appropriate values of c's to zero in (A.26).

We also find other solutions with non zero parameters

$$a_2 = a_3 = a_4 = a_1, \qquad a_6 = a_7 = a_8 = a_5,$$
$$e_{15} = e_{26} = e_{37} = e_{48} = e,$$
$$b_{16} = -b_{25}, \qquad b_{38} = -b_{47} \quad \text{(A.27)}$$

subject to these relations $a_5^2 + e^2 = -\frac{g_{2s}}{h_{2s}}$, $a_1^2 = -\frac{g_{1s}}{h_{1s}}$ and $\frac{g_{1s}^2}{h_{1s}} = \frac{g_{2s}^2}{h_{2s}}$. Note that in this case, we also turn on some off-diagonal elements of e. The solutions are given by:

- **(2,2) critical point**
  This solution has $A_1 = \frac{16g_{1s}^2}{h_{1s}}$ giving the same cosmological constant as in the previous case.

- **(2,3) critical point**
  This can be obtained from the previous case by setting $b_{25} = b_{47}$ or $b_{25} = -b_{47}$.

There is no possible flow solution between all these critical points since all critical points have the same value of the cosmological constant.

A.3 Vacua of $N = 9$ theory

In this section, we study some vacua of $N = 9$ theory. The construction of the theory is the same as $N = 4$ and $N = 8$ theories studied previously. For $N > 8$, the scalar target space is unique because there is only one supermultiplet in these cases. In the present case, the scalar manifold is given by the exceptional coset $\frac{F_{4(-20)}}{SO(9)}$. We will study some vacua of this theory with gauge groups $SO(p) \times SO(9-p)$ for $p = 0, 1, 2, 3, 4$, $G_{2(-14)} \times SL(2)$ and $Sp(1, 2) \times SU(2)$. All these gauge groups have been shown to be consistent gaugings in [31].

A.3.1 $N = 9$ three dimensional gauged supergravity

We begin with the $\frac{F_{4(-20)}}{SO(9)}$ coset. The 52 generators of the compact F_4 have been explicitly constructed by realizing F_4 as an automorphism group of the Jordan algebra J_3 in [141]. There are 16 non-compact and 36 compact generators in $F_{4(-20)}$. Under $SO(9)$, the 52 generators decompose as

$$\mathbf{52} \rightarrow \mathbf{36} + \mathbf{16}$$

where **36** and **16** are adjoint and spinor representations of $SO(9)$, respectively. The non-compact $F_{4(-20)}$ can be obtained from the compact F_4 by using "Weyl unitarity trick", see [142] for an example with G_2. This is achieved by introducing a factor of i to each generator corresponding to the non-compact generators. From [141], the compact subgroup $SO(9)$ is generated by, in the notation of [141], $c_1, \ldots, c_{21}$, $c_{30}, \ldots, c_{36}$, $c_{45}, \ldots, c_{52}$. We have chosen the same $SO(9)$ subgroup as in [141] among the three possibilities, see [141] for a discussion. The remaining 16 generators are our non-compact ones which we will define by

$$Y^A = \begin{cases} ic_{A+21} & \text{for } A = 1, \ldots, 8 \\ ic_{A+28} & \text{for } A = 9, \ldots, 16 \end{cases}. \tag{A.28}$$

Note that the $SO(9)$ generators c_i in [141] are labeled by the F_4 adjoint index. In order to apply the $SO(9)$ covariant formulation of $N = 9$ theory, we need to relabel them by using the $SO(9)$ antisymmetric tensor indices i.e. X^{IJ}. To do this, we first repeat the relevant algebra given in chapter 2, for conveniences,

$$[t^{IJ}, t^{KL}] = -4\delta^{[I[K}t^{L]J]}, \qquad [t^{IJ}, t^A] = -\frac{1}{2}f^{IJ,AB}t_B, \qquad [t^A, t^B] = \frac{1}{4}f^{AB}_{IJ}t^{IJ} \tag{A.29}$$

where we have used the flat target space indices in f^{IJ}_{AB} and the non-compact generators, t^A. Using the first commutator in (A.29), we can map all c_i's forming $SO(9)$

to the desired form X^{IJ}. We find the following mapping between c_i and X^{IJ}

$$
\begin{aligned}
X^{12} &= c_1,\ X^{13}=-c_2,\ X^{23}=c_3,\ X^{34}=c_6,\ X^{14}=c_4,\ X^{24}=-c_5,\\
X^{15} &= c_7,\ X^{25}=-c_8,\ X^{35}=c_9,\ X^{45}=-c_{10},\ X^{56}=-c_{15},\ X^{16}=c_{11},\\
X^{26} &= -c_{12},\ X^{46}=-c_{14},\ X^{36}=c_{13},\ X^{17}=c_{16},\ X^{27}=-c_{17},\ X^{47}=-c_{19},\\
X^{37} &= c_{18},\ X^{67}=-c_{21},\ X^{57}=-c_{20},\ X^{78}=-c_{36},\ X^{18}=c_{30},\ X^{28}=-c_{31},\\
X^{48} &= -c_{33},\ X^{38}=c_{32},\ X^{68}=-c_{35},\ X^{58}=-c_{34},\ X^{29}=-c_{46},\ X^{19}=c_{45},\\
X^{49} &= -c_{48},\ X^{39}=c_{47},\ X^{69}=-c_{50},\ X^{59}=-c_{49},\\
X^{89} &= -c_{52},\ X^{79}=-c_{51}\,. \qquad \text{(A.30)}
\end{aligned}
$$

The next step is to find the f^{IJ}. In order to be compatible with the F_4 algebra given in [141], we need to use the second and the third commutators in (A.29) to extract the component of f^{IJ}_{AB}. There are eight independent f^{IJ} from which all other components follow. On the other hand, the components of f^{IJ} are essentially the structure constants of the F_4 algebra given in [141]. From both methods, it is not difficult to obtain these f^{IJ}_{AB} by using the *Mathematica* program given in [141]. However, we will not give them here due to their complicated form.

We now come to various gaugings characterized by the embedding tensors Θ. The embedding tensors for the compact gaugings with gauge groups $SO(p)\times SO(9-p)$, $p=0,\ldots,4$ are given by [31]

$$\Theta_{IJ,KL}=\theta\delta^{KL}_{IJ}+\delta_{[I[K}\Xi_{L]J]} \qquad \text{(A.31)}$$

where

$$\Xi_{IJ}=\begin{cases} 2\big(1-\frac{p}{9}\big)\delta_{IJ} & \text{for } I\leq p \\ -2\frac{p}{9}\delta_{IJ} & \text{for } I>p \end{cases}, \qquad \theta=\frac{2p-9}{9}\,. \qquad \text{(A.32)}$$

There is only one independent coupling constant, g. The gauge generators can be easily obtained from $SO(9)$ generators X^{IJ} by choosing appropriate values for the indices I,J. For example, in the case of $SO(2)\times SO(7)$ gauging, we have the following gauge generators

$$
\begin{aligned}
SO(7) &: T^{ab}_1=X^{ab},\ a,b=1,\ldots 7,\\
SO(2) &: T_2=X^{89}\,. \qquad \text{(A.33)}
\end{aligned}
$$

We then move to non-compact gaugings with gauge groups $G_{2(-14)}\times SL(2)$ and $Sp(1,2)\times SU(2)$. We find the following embedding tensors

$$G_{2(-14)}\times SL(2)\ :\ \Theta_{\mathcal{MN}}=\eta^{G_2}_{\mathcal{MN}}-\frac{1}{6}\eta^{SL(2)}_{\mathcal{MN}}\,, \qquad \text{(A.34)}$$

$$Sp(1,2)\times SU(2)\ :\ \Theta_{\mathcal{MN}}=\eta^{Sp(1,2)}_{\mathcal{MN}}-12\eta^{SU(2)}_{\mathcal{MN}} \qquad \text{(A.35)}$$

where η^{G_0} is the Cartan Killing form of the gauge group G_0. We now identify the gauge generators in these two gaugings. In $G_{2(-14)}\times SL(2)$ gauging, we can find the corresponding generators as follows. The generators of $G_{2(-14)}$ are obtained by using the embedding of $G_{2(-14)}$ in $SO(7)$ generated by X^{IJ}, $I,J=1,\ldots,7$. The adjoint representation of $SO(7)$ decomposes under $G_{2(-14)}$ as

$$\mathbf{21}\rightarrow\mathbf{14}+\mathbf{7}\,. \tag{A.36}$$

The generators of $G_{2(-14)}$ can be explicitly found by combinations of $SO(7)$ generators [143]

$$\begin{aligned}
T_1 &= \frac{1}{\sqrt{2}}(X^{36}+X^{41}),\ \ T_2=\frac{1}{\sqrt{2}}(X^{31}-X^{46}),\\
T_3 &= \frac{1}{\sqrt{2}}(X^{43}-X^{16}),\ \ T_4=\frac{1}{\sqrt{2}}(X^{73}-X^{24}),\\
T_5 &= -\frac{1}{\sqrt{2}}(X^{23}+X^{47}),\ \ T_6=-\frac{1}{\sqrt{2}}(X^{26}+X^{71}),\\
T_7 &= \frac{1}{\sqrt{2}}(X^{76}-X^{21}),\ \ T_8=\frac{1}{\sqrt{6}}(X^{16}+X^{43}-2X^{72}),\\
T_9 &= -\frac{1}{\sqrt{6}}(X^{41}-X^{36}+2X^{25}),\ \ T_{10}=-\frac{1}{\sqrt{6}}(X^{31}+X^{46}-2X^{57}),\\
T_{11} &= \frac{1}{\sqrt{6}}(X^{73}+X^{24}+2X^{15}),\ \ T_{12}=-\frac{1}{\sqrt{6}}(X^{74}-X^{23}+2X^{65}),\\
T_{13} &= \frac{1}{\sqrt{6}}(X^{26}-X^{71}+2X^{35}),\ \ T_{14}=\frac{1}{\sqrt{6}}(X^{21}+X^{76}-2X^{45}).
\end{aligned} \tag{A.37}$$

We have verified that these generators satisfy G_2 algebra given in [144]. The $SL(2)$ generators are

$$J_1=i\sqrt{2}(c_{22}+c_{27}),\ \ J_2=i\sqrt{2}(c_{37}+c_{42}),\ \ J_3=2c_{52} \tag{A.38}$$

which can be easily checked that they commute with all T's and form $SL(2)$ algebra.

The generators of non-compact $Sp(1,2)$ can be constructed by first finding its compact subgroup generators $Sp(1)\times Sp(2)\sim SO(3)\times SO(5)$. The latter can be obtained by taking $SO(8)$ with generators X^{IJ}, $I,J=1,\ldots,8$. We then identify the $SO(3)$ generators with X^{IJ} for $I,J=1,\ldots,3$ and $SO(5)$ with X^{IJ} for $I,J=4,\ldots,8$. The eight non-compact generators of $Sp(1,2)$ can be obtained by taking combinations of Y^A's which commute with the $SU(2)$ gauge group. The latter has three generators obtained by looking for the combinations of $SO(9)$ generators that commute with $SO(3)\times SO(5)$ mentioned above. We find the following gauge generators:

- Sp(1,2):

$$
\begin{aligned}
Q_1 &= \sqrt{2}c_1,\;\; Q_2 = -\sqrt{2}c_2,\;\; Q_3 = \sqrt{2}c_3,\;\; Q_4 = \sqrt{2}c_4,\;\; Q_5 = -\sqrt{2}c_5,\\
Q_6 &= \sqrt{2}c_6,\;\; Q_7 = \sqrt{2}c_7,\;\; Q_8 = -\sqrt{2}c_8,\;\; Q_9 = \sqrt{2}c_9,\;\; Q_{10} = -\sqrt{2}c_{10},\\
Q_{11} &= -c_{21} - c_{52},\;\; Q_{12} = c_{51} - c_{35},\;\; Q_{13} = c_{50} + c_{36},\\
Q_{14} &= Y_1 + Y_{10},\;\; Q_{15} = Y_2 - Y_9,\;\; Q_{16} = Y_3 + Y_{13},\\
Q_{17} &= Y_4 + Y_{16},\;\; Q_{18} = Y_5 - Y_{11},\;\; Q_{19} = Y_6 - Y_{15},\\
Q_{20} &= Y_7 + Y_{14},\;\; Q_{21} = Y_8 - Y_{12}\,. \qquad \text{(A.39)}
\end{aligned}
$$

- SU(2):

$$
K_1 = \frac{1}{2}(c_{52} - c_{21}),\;\; K_2 = -\frac{1}{2}(c_{35} + c_{51}),\;\; K_3 = \frac{1}{2}(c_{36} - c_{50}). \qquad \text{(A.40)}
$$

Using the above embedding tensors and equation (2.56), we can find all the $\mathcal{V}$'s and T-tensors. The generators are normalized by

$$
\mathrm{Tr}(c_i c_j) = -6\delta_{ij}. \qquad \text{(A.41)}
$$

With this normalization, we find that

$$
\begin{aligned}
\mathcal{V}^{\alpha IJ} &= -\frac{1}{6}\mathrm{Tr}(L^{-1}T^{\alpha}_{G}LX^{IJ}) \qquad \text{(A.42)}\\
\mathcal{V}^{\alpha A} &= \frac{1}{6}\mathrm{Tr}(L^{-1}T^{\alpha}_{G}LY^{A}) \qquad \text{(A.43)}
\end{aligned}
$$

where we have introduced the symbol T^{α}_{G} for gauge group generators. T^{α}_{G} will be replaced by some appropriate generators of the gauge group being considered in each gauging.

With the above generators together with (A.42) and (A.43), we can compute the T-tensors

$$
\begin{aligned}
T^{IJ,KL} &= \mathcal{V}^{IJ,\alpha}\mathcal{V}^{KL,\beta}\delta^{SO(p)}_{\alpha\beta} - \mathcal{V}^{IJ,\alpha}\mathcal{V}^{KL,\beta}\delta^{SO(9-p)}_{\alpha\beta}, \qquad \text{(A.44)}\\
T^{IJ,A} &= \mathcal{V}^{IJ,\alpha}\mathcal{V}^{A,\beta}\delta^{SO(p)}_{\alpha\beta} - \mathcal{V}^{IJ,\alpha}\mathcal{V}^{A,\beta}\delta^{SO(9-p)}_{\alpha\beta} \qquad \text{(A.45)}
\end{aligned}
$$

for compact gaugings and

$$
\begin{aligned}
T^{IJ,KL} &= \mathcal{V}^{IJ,\alpha}\mathcal{V}^{KL,\beta}\eta^{G_1}_{\alpha\beta} - K\mathcal{V}^{IJ,\alpha}\mathcal{V}^{KL,\beta}\eta^{G_2}_{\alpha\beta}, \qquad \text{(A.46)}\\
T^{IJ,A} &= \mathcal{V}^{IJ,\alpha}\mathcal{V}^{A,\beta}\eta^{G_1}_{\alpha\beta} - K\mathcal{V}^{IJ,\alpha}\mathcal{V}^{A,\beta}\eta^{G_2}_{\alpha\beta} \qquad \text{(A.47)}
\end{aligned}
$$

for non-compact gaugings with K being $\frac{1}{6}$ and 12 for $G_1 \times G_2 = G_{2(-14)} \times SL(2)$ and $Sp(1,2) \times SU(2)$, respectively. We have used summation convention over gauge indices α, β with the notation δ^{G_0} and η^{G_0} meaning that the summation is restricted to the G_0 generators.

It is now straightforward to compute the A_1, A_2 tensors and the scalar potential in each gauging. In the next subsection, we will give the scalar potentials for all gaugings mentioned above along with some of their critical points.

A.3.2 Vacua of $N = 9$ gauged supergravity

In this subsection, we give some vacua of the $N = 9$ gauged theory with the gaugings mentioned in the previous subsection. We will discuss the isometry groups of the background with maximal supersymmetries at $L = \mathbf{I}$. This is a supersymmetric extension of the $SO(2,2) \sim SO(1,2) \times SO(1,2)$ isometry group of AdS_3. The superconformal group can be identified by finding its bosonic subgroup and representations of supercharges under this group. A similar study has been done in [86] for models with $N = 16$ supersymmetry. The full list of superconformal groups in two dimensions can be found in [145]. We first start with compact gaugings.

Vacua of compact gaugings

Since the scalar manifold involves 16 scalars and the coset representative L is a 26×26 matrix, it is extremely difficult to compute the scalar potential for the full scalar manifold. However, we can study the scalar potential on some part of the full scalar manifold. It has been shown in [80] that the critical points obtained from the potential restricted on a scalar manifold which is invariant under some subgroups of the gauge group are critical points of the full potential. This invariant manifold is parametrized by all scalars which are singlets under the chosen symmetry. To make things more manageable, we will not study the scalar potential with more than four scalars. We choose to parametrize the scalars by using the coset representative

$$L = e^{a_1 Y_1} e^{a_2 Y_2} e^{a_3 Y_{15}} e^{a_4 Y_{16}} \,. \tag{A.48}$$

For any invariant manifold with a certain residual symmetry, our choice for L in (A.48) certainly does not cover the whole invariant manifold. Therefore, the critical points on this submanifold may not be critical points of the potential on the whole scalar manifold. Nevertheless, we can use the argument of [80] as a guideline to find critical points. After identifying the critical points, we then use the stationarity condition (2.99) to check whether our critical points are truly critical points of the scalar potential.

Let us identify some residual symmetries of (A.48). In $SO(9)$ gauging, with only $a_1 \neq 0$, L has $SO(7)$ symmetry. For $a_1, a_2 \neq 0$, L preserves $SO(6)$ symmetry. With $a_1, a_2, a_3 \neq 0$ and $a_1, a_2, a_3, a_4 \neq 0$, L preserves $SU(3)$ and $SU(2)$, respectively. In other gauge groups, L will have different residual symmetry. We will discuss the residual symmetry of each critical point, separately. We find that in all cases, non trivial supersymmetric critical points arise with at most two non zero scalars. With all four scalar fields turned on, the conditions $A_{2i}^{JI} \epsilon^J = 0$ are satisfied if and only if two of the scalars vanish. So, we give below only potentials with two scalars.

In (A.48), we have used the basis elements of Y's to parametrize each scalar field. We also find that, in this parametrization, all the sixteen scalars are on equal footing in the sense that any four of the Y's among sixteen of them give the same structure of the potential. As a consequence, any two non zero scalars in (A.48) give rise to the same critical points with the same location and cosmological constant. Notice that this is not the case if we use different parametrization of L. For example, by using linear combinations of Y_i's as basis for the four scalars in (A.48), different choices of Y_i's in each basis may give rise to different structures of the scalar potential.

We recall our notation namely V_0 is the cosmological constant and (n_-, n_+) refers to the number of supersymmetries in the dual two dimensional filed theory. As usual, the n_+ (n_-) corresponds to the number of positive (negative) eigenvalues of A_1^{IJ}. For definiteness, we will keep a_1 and a_2 non zero. Furthermore, we give the values of scalar fields up to a trivial sign change.

- $SO(9)$ gauging:
  The scalar potential is

$$\begin{aligned} V &= \frac{1}{32}g^2(-1390 - 232\cosh(2a_1) + 6\cosh(4a_1) + 4\cosh[2(a_1 - 2a_2)] \\ &\quad +4\cosh(4a_1 - 2a_2) - 112\cosh[2(a_1 - a_2)] + \cosh[4(a_1 - a_2)] \\ &\quad -232\cosh(2a_2) + \cosh[4(a_1 + a_2)] - 112\cosh[2(a_1 + a_2)] \\ &\quad +6\cosh(4a_2) + 4\cosh[2(2a_1 + a_2)] + 4\cosh[2(a_1 + 2a_2)]). \end{aligned} \tag{A.49}$$

This is the case in which the full R-symmetry group $SO(9)$ is gauged. There is no non trivial critical point with two scalars. For $a_2 = 0$, there are two critical points, but only the $L = \mathbf{I}$ solution has any supersymmetry.

Critical points	a_1	V_0	Preserved supersymmetry
1	0	$-64g^2$	(9,0)
2	$\cosh^{-1} 2$	$-100g^2$	-

The corresponding A_1 tensor at the supersymmetric point is

$$A_1^{(1)} = \text{diag}(-4, -4, -4, -4, -4, -4, -4, -4, -4)\,. \tag{A.50}$$

The notation $A_1^{(1)}$ means that this is the value of the A_1 tensor evaluated at the critical point number 1 in the table. For $L = \mathbf{I}$, the background isometry is given by $Osp(9|2, \mathbb{R}) \times SO(1, 2)$. The non-supersymmetric critical point has unbroken $SO(7)$ gauge symmetry. This point is closely related to the non-supersymmetric $SO(7) \times SO(7)$ point found in $N = 16$ $SO(8) \times SO(8)$ gauged supergravity [86]. Both the location and the value of the cosmological constants compared to the $L = \mathbf{I}$ point are very similar to that in [86].

- $SO(8)$ gauging:
  The potential is

$$V = -\frac{1}{16}g^2[(26+2\cosh(2a_1)+\cosh[2(a_1-a_2)]+2\cosh(2a_2) + \cosh[2(a_1+a_2)])^2 - 32(\cosh^2 a_2 \sinh^2(2a_1) + \cosh^4 a_1 \sinh^2(2a_2))]. \qquad (A.51)$$

This case is very similar to the $SO(9)$ gauging. There are two critical points with a single scalar.

Critical points	a_1	V_0	Preserved supersymmetry
1	0	$-64g^2$	(8,1)
2	$\cosh^{-1} 2$	$-100g^2$	-

The A_1 tensor is

$$A_1^{(1)} = \mathrm{diag}(-4,-4,-4,-4,-4,-4,-4,-4,4)\,. \qquad (A.52)$$

For $L = \mathbf{I}$, the background isometry is given by $Osp(8|2,\mathbb{R}) \times Osp(1|2,\mathbb{R})$. The critical point 2 is invariant under G_2 subgroup of $SO(8)$. Apart from the splitting of supercharges and residual gauge symmetry, the critical points in this gauging are the same as the $SO(9)$ gauging.

- $SO(7) \times SO(2)$ gauging:
  In this gauging, the potential is

$$V = -\frac{1}{36864}g^2[9(342+40\cosh a_1+18\cosh(2a_1)-4\cosh(a_1-2a_2) +16\cosh(a_1-a_2)+3\cosh[2(a_1-a_2)]+12\cosh(2a_1-a_2) +8\cosh a_2+50\cosh(2a_2)+16\cosh(a_1+a_2)+3\cosh[2(a_1+a_2)] +12\cosh(2a_1+a_2)-4\cosh(a_1+2a_2))^2+8(-576\cosh^2\frac{a_2}{2}(-3 +\cosh a_2-3\cosh a_1(1+\cosh a_2))^2\sinh^2 a_1-9(-1 -8\cosh a_1(-1+\cosh a_2)+47\cosh a_2+3\cosh(2a_1)(1+\cosh a_2) +6\cosh^2 a_1(1+\cosh a_2))^2\sinh^2 a_2)]. \qquad (A.53)$$

We find one supersymmetric critical point with

$$V_0 = -144g^2, \qquad a_1 = \cosh^{-1}\frac{5}{3}, \qquad a_2 = \cosh^{-1} 2 \qquad (A.54)$$

with the value of the A_1 tensor

$$A_1 = \begin{pmatrix} -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{14}{3} & 0 & 0 & 0 & 0 & 0 & -\frac{8\sqrt{2}}{3} \\ 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & -\frac{8\sqrt{2}}{3} & 0 & 0 & 0 & 0 & 0 & \frac{14}{3} \end{pmatrix}. \tag{A.55}$$

After diagonalization, we find

$$A_1 = \text{diag}(-10, -10, -6, -10, -10, -10, -10, 6, 6). \tag{A.56}$$

This is a (1,2) point with $SU(2)$ symmetry. With $a_2 = 0$, we find the following critical points

Critical points	a_1	V_0	Preserved supersymmetry
1	0	$-64g^2$	(7,2)
2	$\cosh^{-1}\frac{7}{3}$	$-\frac{1024}{9}g^2$	(0,1)

The corresponding values of the A_1 tensor are

$$\begin{aligned} A_1^{(1)} &= \text{diag}\,(-4, -4, -4, -4, -4, -4, -4, 4, 4) \\ \text{and } A_1^{(2)} &= \text{diag}\left(-4, -4, -4, -4, -4, -4, -4, \frac{16}{3}, 8\right). \end{aligned} \tag{A.57}$$

For $L = \mathbf{I}$, the background isometry is given by $Osp(7|2, \mathbb{R}) \times Osp(2|2, \mathbb{R})$. The critical point 2 preserves $SU(3)$ symmetry. The location and value of the cosmological constant relative to the $L = \mathbf{I}$ point are similar to the $G_2 \times G_2$ point in $SO(8) \times SO(8)$ gauged $N = 16$ supergravity. In our result, the residual gauge symmetry is the $SU(3)$ subgroup of G_2 which is in turn a subgroup of $SO(7)$.

- $SO(6) \times SO(3)$ gauging:
  We find the potential

$$\begin{aligned} V = \; & \frac{1}{128}g^2(-3886 - 424\cosh(2a_1) + 6\cosh(4a_1) + 4\cosh[2(a_1 - 2a_2)] \\ & +4\cosh(4a_1 - 2a_2) - 1536\cosh(a_1 - a_2) - 208\cosh[2(a_1 - a_2)] \\ & + \cosh[4(a_1 - a_2)] - 424\cosh(2a_2) + 6\cosh(4a_2) \\ & -1536\cosh(a_1 + a_2) - 208\cosh[2(a_1 + a_2)] + \cosh[4(a_1 + a_2)] \\ & +4\cosh[2(2a_1 + a_2)] + 4\cosh[2(a_1 + 2a_2)]). \end{aligned} \tag{A.58}$$

One supersymmetric critical point is

$$V_0 = -256g^2, \qquad a_1 = \cosh^{-1} 2, \qquad a_2 = \cosh^{-1} 3 \,. \tag{A.59}$$

with the value of the A_1 tensor

$$A_1 = \begin{pmatrix} -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -16 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & -4\sqrt{3} \\ 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & -4\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} . \tag{A.60}$$

This can be diagonalized to

$$A_1 = \mathrm{diag}\,(-16, -16, -8, -16, -16, -16, 8, 8, 8) \,. \tag{A.61}$$

This is a (1,3) point and has $SO(3) \subset SO(6)$ symmetry. With $a_2 = 0$, we find the following critical points

Critical points	a_1	V_0	Preserved supersymmetry
1	0	$-64g^2$	(6,3)
2	$\cosh^{-1} 3$	$-144g^2$	(0,2)

The corresponding values of the A_1 tensor are

$$\begin{aligned} A_1^{(1)} &= \mathrm{diag}\,(-4, -4, -4, -4, -4, -4, 4, 4, 4) \\ \text{and } A_1^{(2)} &= \mathrm{diag}\,(-10, -10, -10, -10, -10, -10, 6, 6, 10) \,. \end{aligned} \tag{A.62}$$

For $L = \mathbf{I}$, the background isometry is given by $Osp(6|2, \mathbb{R}) \times Osp(3|2, \mathbb{R})$. The critical point 2 is also invariant under $SO(3)$ subgroup of $SO(6)$.

- $SO(5) \times SO(4)$ gauging:
  The potential for this gauging is

$$\begin{aligned} V &= \frac{1}{32} g^2 (3 + \cosh a_1 \cosh a_2)^2 (-86 + 2\cosh(2a_1) - 24\cosh(a_1 - a_2) \\ &\quad + \cosh[2(a_1 - a_2)] + 2\cosh(2a_2) - 24\cosh(a_1 + a_2) \\ &\quad + \cosh[2(a_1 + a_2)]). \end{aligned} \tag{A.63}$$

There is no critical point with two non zero scalars. With $a_2 = 0$, we find the following critical points:

Critical points	a_1	V_0	Preserved supersymmetry
1	0	$-64g^2$	(5,4)
2	$\cosh^{-1} 5$	$-256g^2$	(0,3)

.

The corresponding values of the A_1 tensor are

$$\begin{aligned} A_1^{(1)} &= \text{diag}\,(-4,-4,-4,-4,-4,4,4,4,4) \\ \text{and}\ \ A_1^{(2)} &= \text{diag}\,(-16,-16,-16,-16,-16,8,8,8,16)\,. \end{aligned} \quad \text{(A.64)}$$

For $L = \mathbf{I}$, the background isometry is given by $Osp(5|2,\mathbb{R}) \times Osp(4|2,\mathbb{R})$. The critical point 2 preserves $SO(4)_{\text{diag}}$ symmetry which is the diagonal subgroup of $SO(4) \times SO(4)$ with the first $SO(4)$ being a subgroup of $SO(5)$.

Vacua of non-compact gaugings

We now give some critical points of the non-compact gaugings. The isometry group of the background with $L = \mathbf{I}$ consists of the maximal compact subgroup of the gauge group and $SO(2,2)$ as the bosonic subgroup. Using the generators given in the appendix, we can compute the scalar potentials for these two gaugings. Notice that in the non-compact gaugings, all sixteen scalars are not equivalent. At the maximally symmetric vacua, the gauge group is broken down to its maximal compact subgroup, and some of the scalars become Goldstone bosons making some of the vector fields massive. This "Higgs-mechanism" results in the propagating n_{ng} massive vector fields where n_{ng} denotes the number of non compact generators which are broken at the critical point. The total number of degrees of freedom remains the same because of the disappearance of the n_{ng} scalars, Goldstone bosons. For further detail, see [86] in the context of $N = 16$ models.

- $G_{2(-14)} \times SL(2)$ gauging:
  The coset representative is chosen to be

$$L = e^{a_1 Y_3} e^{a_2 Y_{13}}\,. \quad \text{(A.65)}$$

This parametrization has residual gauge symmetry $SU(2)$ which is a subgroup of $G_{2(-14)}$. With one of the scalars vanishing, L has $SU(3)$ symmetry. The potential with two scalars is given by

$$\begin{aligned} V &= \frac{1}{4608} g^2 [-23406 - 2520\cosh(2a_1) + 70\cosh(4a_1) + 8\cosh(4a_1 - 3a_2) \\ &\quad +28\cosh[2(a_1 - 2a_2)] + 28\cosh(4a_1 - 2a_2) - 560\cosh[2(a_1 - a_2)] \\ &\quad +\cosh[4(a_1 - a_2)] - 1792\cosh(2a_1 - a_2) + 56\cosh(4a_1 - a_2) \\ &\quad +3472\cosh(a_2) - 6104\cosh(2a_2) - 16\cosh(3a_2) + 198\cosh(4a_2) \\ &\quad -560\cosh[2(a_1 + a_2)] + \cosh[4(a_1 + a_2)] - 1792\cosh(2a_1 + a_2) \\ &\quad +28\cosh[2(2a_1 + a_2)] + 56\cosh(4a_1 + a_2) + 28\cosh[2(a_1 + 2a_2)] \\ &\quad +8\cosh(4a_1 + 3a_2)]. \end{aligned} \quad \text{(A.66)}$$

We find the following critical points:

critical point	a_1	a_2	V_0	preserved supersymmetries
1	0	0	$-\frac{64}{9}g^2$	(7,2)
2	0	$\cosh^{-1}\frac{1}{2}\sqrt{\frac{11+\sqrt{57}}{2}}$	$-\frac{551+21\sqrt{57}}{72}g^2$	-
3	$\cosh^{-1}2$	0	$-\frac{100}{9}g^2$	(0,1)
4	$\cosh^{-1}\frac{3}{2}$	$\cosh^{-1}\frac{2}{\sqrt{3}}$	$-\frac{1024}{81}g^2$	(1,2)

The corresponding values of the A_1 tensor are

$$\begin{aligned} A_1^{(1)} &= \text{diag}\left(-\frac{4}{3},-\frac{4}{3},-\frac{4}{3},-\frac{4}{3},-\frac{4}{3},-\frac{4}{3},-\frac{4}{3},\frac{4}{3},\frac{4}{3}\right), \\ A_1^{(3)} &= \begin{pmatrix} -\frac{7}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & -\frac{7}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{7}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{7}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{7}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{3} \end{pmatrix} \end{aligned} \tag{A.67}$$

and

$$A_1^{(4)} = \begin{pmatrix} -\frac{28}{9} & 0 & 0 & 0 & 0 & -\frac{4}{9} & 0 & 0 & 0 \\ 0 & -\frac{28}{9} & 0 & 0 & 0 & 0 & \frac{4}{9} & 0 & 0 \\ 0 & 0 & -\frac{8}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{14}{9} & 0 & 0 & 0 & 0 & -\frac{2}{3}\sqrt{\frac{5}{3}} \\ 0 & 0 & 0 & 0 & -\frac{8}{3} & 0 & 0 & 0 & 0 \\ -\frac{4}{9} & 0 & 0 & 0 & 0 & -\frac{28}{9} & 0 & 0 & 0 \\ 0 & \frac{4}{9} & 0 & 0 & 0 & 0 & -\frac{28}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{16}{9} & 0 \\ 0 & 0 & 0 & -\frac{2}{3}\sqrt{\frac{5}{3}} & 0 & 0 & 0 & 0 & \frac{14}{9} \end{pmatrix}. \tag{A.68}$$

$A_1^{(3)}$ and $A_1^{(4)}$ can be diagonalized to

$$\begin{aligned} A_1^{(3)} &= \text{diag}\left(-\frac{7}{3},-\frac{11}{3},-\frac{7}{3},-\frac{7}{3},-\frac{7}{3},-\frac{7}{3},-\frac{7}{3},\frac{5}{3},\frac{7}{3}\right), \\ A_1^{(4)} &= \text{diag}\left(-\frac{32}{9},-\frac{32}{9},-\frac{8}{3},-\frac{16}{9},-\frac{8}{3},-\frac{8}{3},-\frac{8}{3},\frac{16}{9},\frac{16}{9}\right). \end{aligned} \tag{A.69}$$

For $L = \mathbf{I}$, the gauge group is broken down to its compact subgroup $G_2 \times SO(2)$. The background isometry is given by $G(3) \times Osp(2|2,\mathbb{R})$. There are two $SU(3)$ points with completely broken supersymmetry (point 2) and (0,1) supersymmetry (point 3). Point 4 has $SU(2)$ symmetry.

- $Sp(1,2)\times SU(2)$ gauging:
We choose the coset representative

$$L = e^{a_1(Y_1-Y_{10})}e^{a_2(Y_2+Y_9)}\,. \tag{A.70}$$

This has symmetry $SO(3)\times SO(3)$ if any one of the scalars vanishes. This is the case in which our critical points lie. This symmetry is a subgroup of the $SO(5)\times SO(3)$ compact subgroup of $Sp(1,2)$ with the first $SO(3)$ being a subgroup of $SO(5)$. We find the potential

$$\begin{aligned} V &= \frac{1}{32}g^2[-1390-232\cosh(2\sqrt{2}a_1)+6\cosh(4\sqrt{2}a_1)\\ &\quad +4\cosh[2\sqrt{2}(a_1-2a_2)]-112\cosh[2\sqrt{2}(a_1-a_2)]\\ &\quad +\cosh[4\sqrt{2}(a_1-a_2)]+4\cosh[2\sqrt{2}(2a_1-a_2)]-232\cosh(2\sqrt{2}a_2)\\ &\quad +6\cosh(4\sqrt{2}a_2)-112\cosh[2\sqrt{2}(a_1+a_2)]+\cosh[4\sqrt{2}(a_1+a_2)]\\ &\quad +4\cosh[2\sqrt{2}(2a_1+a_2)]+4\cosh[2\sqrt{2}(a_1+2a_2)]]. \end{aligned} \tag{A.71}$$

Some of the critical points are given by

critical point	a_1	a_2	V_0	preserved supersymmetries
1	0	0	$-64g^2$	(5,4)
2	0	$\frac{\cosh^{-1}2}{\sqrt{2}}$	$-100g^2$	-
3	$\frac{\ln(2-\sqrt{3})}{\sqrt{2}}$	0	$-100g^2$	-
4	$\frac{\ln(2+\sqrt{3})}{\sqrt{2}}$	0	$-100g^2$	-

with the corresponding A_1 tensor

$$A_1^{(1)} = \text{diag}\,(-4,-4,-4,-4,-4,4,4,4,4) \tag{A.72}$$

for the critical point 1. For $L=\mathbf{I}$, the gauge group is broken down to its compact subgroup $Sp(1)\times Sp(2)\times SU(2)\sim SU(2)\times SO(5)\times SU(2)$. The two $SU(2)$'s factors combine to $SO(4)$ under which the right handed supercharges transform as **4**. So, the background isometry is given by $Osp(5|2,\mathbb{R})\times Osp(4|2,\mathbb{R})$. Point 2, 3, and 4 are $SO(3)\times SO(3)$ points with completely broken supersymmetry.

We have checked that all critical points given above are truely critical points of the corresponding potential. There exist many supersymmetric flow solutions interpolating between some of the supersymmetric vacua. However, we will not give them here since they are not directly related to the thesis mainline. We refer the reader to [54] for the discussions on these solutions. We end this section by summarizing all critical points found in the $N = 9$ gauged supergravity in table IV.

Critical point	Gauge group	V_0	Unbroken SUSY	Unbroken gauge symmetry
1	$SO(9)$	$-64g^2$	$(9,0)$	$SO(9)$
2	$SO(9)$	$-100g^2$	-	$SO(7)$
3	$SO(8)$	$-64g^2$	$(8,1)$	$SO(8)$
4	$SO(8)$	$-100g^2$	-	G_2
5	$SO(7)\times SO(2)$	$-64g^2$	$(7,2)$	$SO(7)\times SO(2)$
6	$SO(7)\times SO(2)$	$-\frac{1024}{9}g^2$	$(0,1)$	$SU(3)$
7	$SO(7)\times SO(2)$	$-144g^2$	$(1,2)$	$SU(2)$
8	$SO(6)\times SO(3)$	$-64g^2$	$(6,3)$	$SO(6)\times SO(3)$
9	$SO(6)\times SO(3)$	$-144g^2$	$(0,2)$	$SO(3)$
10	$SO(6)\times SO(3)$	$-256g^2$	$(1,3)$	$SO(3)$
11	$SO(5)\times SO(4)$	$-64g^2$	$(5,4)$	$SO(5)\times SO(4)$
12	$SO(5)\times SO(4)$	$-256g^2$	$(0,3)$	$SO(4)_{\text{diag}}$
13	$G_{2(-14)}\times SL(2)$	$-\frac{64}{9}g^2$	$(7,2)$	$G_{2(-14)}\times SO(2)$
14	$G_{2(-14)}\times SL(2)$	$-\frac{551+21\sqrt{57}}{72}g^2$	$-$	$SU(3)$
15	$G_{2(-14)}\times SL(2)$	$-\frac{100}{9}g^2$	$(0,1)$	$SU(3)$
16	$G_{2(-14)}\times SL(2)$	$-\frac{1024}{81}g^2$	$(1,2)$	$SU(2)$
17	$Sp(1,2)\times SU(2)$	$-64g^2$	$(5,4)$	$SO(5)\times SO(4)$
18	$Sp(1,2)\times SU(2)$	$-100g^2$	$-$	$SO(3)\times SO(3)$

Table IV: Some critical points of $N = 9$ gauged supergravity in three dimensions.

A.4 Vacua of $N = 10$ theory

In this section, we continue with identifying critical points of gauged supergravity in three dimensions. This section deals with $N = 10$ gauged supergravity which contains 32 scalar fields parametrizing the exceptional coset space $\frac{E_{6(-14)}}{SO(10)\times U(1)}$. The procedure is much similar to that of the previous section. Particularly, the generators of E_6 are obtained by extending the F_4 generators used in the previous section. The admissible gauge groups considered here involve both compact and non-compact gauge groups which are maximal subgroups of $SO(10) \times U(1)$ and $E_{6(-14)}$, respectively. They are given by $SO(p)\times SO(10-p)\times U(1)$ for $p = 6, \ldots 10$, $SO(5) \times SO(5)$, $SU(4, 2) \times SU(2)$, $G_{2(-14)} \times SU(2, 1)$ and $F_{4(-20)}$. We employ the technique introduced in [80] to find critical points of the scalar potentials.

A.4.1 $N = 10$ three dimensional gauged supergravity

In this subsection, we review the structure of $N = 10$ three dimensional gauged supergravity. We start by describing the scalar target space manifold and the necessary ingredients. The scalar manifold of $N = 10$ theory is a 32 dimensional symmetric space $\frac{E_{6(-14)}}{SO(10)\times U(1)}$. We will use the E_6 generators constructed in [146]. Notice that there is an additional factor $H' = U(1)$, in the compact subgroup $H = SO(N) \times H'$, in this theory in contrast to $N = 9$ and $N = 16$ theories studied in the previous section and [86]. The 78 generators of E_6 are given in [141] for the first 52 generators and in [146] for the remaining 26. We can construct the non-compact form $E_{6(-14)}$ by making 32 generators non-compact using "Weyl unitarity". These transform as a spinor representation of $SO(10)$ and are given by

$$Y^A = \left\{ \begin{array}{ll} ic_{A+21} & \text{for } A = 1, \dots, 8 \\ ic_{A+28} & \text{for } A = 9, \dots, 16 \\ ic_{A+37} & \text{for } A = 17, \dots, 32 \end{array} \right. . \tag{A.73}$$

The 46 compact generators are the generators of $SO(10) \times U(1)$ and are given by

$$\begin{aligned} X^{1,10} &= -c_{71}, \; X^{2,10} = c_{72}, \; X^{3,10} = -c_{73}, \; X^{4,10} = c_{74}, \; X^{5,10} = c_{75}, \\ X^{6,10} &= c_{76}, \; X^{7,10} = c_{77}, \; X^{8,10} = c_{78}, \; X^{9,10} = \tilde{c}_{53} \end{aligned} \tag{A.74}$$

together with the $SO(9)$ generators given in the previous section. The $U(1)$ subgroup is generated by $X = 2\tilde{c}_{70}$. The $\tilde{c}_{53}$ and $\tilde{c}_{70}$ are defined by [146]

$$\tilde{c}_{53} = \frac{1}{2}c_{53} + \frac{\sqrt{3}}{2}c_{70} \qquad \text{and} \qquad \tilde{c}_{70} = -\frac{\sqrt{3}}{2}c_{53} + \frac{1}{2}c_{70} \, . \tag{A.75}$$

All the f^{IJ}'s components can be obtained from the structure constants of the $[X^{IJ}, Y^A]$ given in [141] and [146].

The embedding tensors for the compact gaugings with gauge groups $SO(p) \times SO(10-p) \times U(1)$, $p = 6, \dots, 10$ and $SO(5) \times SO(5)$ are given by [31]

$$\Theta_{IJ,KL} = \theta \delta^{KL}_{IJ} + \delta_{[I[K}\Xi_{L]J]} + \frac{1}{3}(5-p)\Theta_{U(1)} \tag{A.76}$$

where

$$\Xi_{IJ} = \left\{ \begin{array}{rl} 2\big(1 - \frac{p}{10}\big)\delta_{IJ} & \text{for } I \leq p \\ -\frac{p}{5}\delta_{IJ} & \text{for } I > p \end{array} \right. , \qquad \theta = \frac{p-5}{5} \, . \tag{A.77}$$

For $p = 5$, the gauge group is $SO(5) \times SO(5)$ which lies entirely in $SO(10)$. This is the case in which the $U(1)$ is not gauged. The generators for these gauge groups can be obtained by choosing appropriate generators of $SO(10)$

$$\begin{aligned} T_1^{IJ} &= X^{IJ}, \qquad I, J = 1, \dots p, \\ T_2^{IJ} &= X^{IJ}, \qquad I, J = p+1, \dots 10 \end{aligned} \tag{A.78}$$

and $U(1)$ generator $2\tilde{c}_{70}$.

Non-compact gaugings considered in this work are those given in [31]. The gauge groups are $SU(4,2)\times SU(2)$, $G_{2(-14)}\times SU(2,1)$ and $F_{4(-20)}$. We find the following embedding tensors

$$\begin{aligned} G_{2(-14)}\times SU(2,1) &: \Theta_{\mathcal{MN}}=\eta^{G_2}_{\mathcal{MN}}-\frac{2}{3}\eta^{SU(2,1)}_{\mathcal{MN}} && \text{(A.79)}\\ SU(4,2)\times SU(2) &: \Theta_{\mathcal{MN}}=\eta^{SU(4,2)}_{\mathcal{MN}}-6\eta^{SU(2)}_{\mathcal{MN}} && \text{(A.80)}\\ F_{4(-20)} &: \Theta_{\mathcal{MN}}=\eta^{F_{4(-20)}}_{\mathcal{MN}} && \text{(A.81)} \end{aligned}$$

where η^{G_0} is the Cartan Killing form of the gauge group G_0. The corresponding gauge generators of these three gaugings are given as follows. The $G_{2(-14)}$ generators are the same as those given in the previous section. The $SU(2,1)$ generators are given by

$$\begin{aligned} J_1 &= -c_{52}, \qquad J_2=-\tilde{c}_{53}, \qquad J_3=-c_{78}, \qquad J_4=\tilde{c}_{70},\\ J_5 &= \frac{1}{\sqrt{2}}(Y_1+Y_6), \qquad J_6=\frac{1}{\sqrt{2}}(Y_9+Y_{14}),\\ J_7 &= \frac{1}{\sqrt{2}}(Y_{21}+Y_{24}), \qquad J_8=\frac{1}{\sqrt{2}}(Y_{25}+Y_{30})\,. \end{aligned} \quad \text{(A.82)}$$

We have normalized these generators according to the embedding tensor given above.

In $SU(4,2)\times SU(2)$ gauging, the relevant generators are given by

- $SU(4,2)$:

$$\begin{aligned} Q_i &= c_i, \qquad i=1,\ldots,15,\\ Q_{16} &= \frac{1}{\sqrt{2}}(c_{52}+c_{77}),\ Q_{17}=\frac{1}{\sqrt{2}}(c_{51}-c_{78}),\ Q_{18}=\frac{1}{\sqrt{2}}(\tilde{c}_{53}-c_{36}),\\ Q_{19} &= \tilde{c}_{70},\ Q_{20}=\frac{1}{\sqrt{2}}(Y_1+Y_{23}),\ Q_{21}=\frac{1}{\sqrt{2}}(Y_2-Y_{22}),\\ Q_{22} &= \frac{1}{\sqrt{2}}(Y_3+Y_{24}),\ Q_{23}=\frac{1}{\sqrt{2}}(Y_4-Y_{21}),\ Q_{24}=\frac{1}{\sqrt{2}}(Y_5+Y_{20}),\\ Q_{25} &= \frac{1}{\sqrt{2}}(Y_6+Y_{18}),\ Q_{26}=\frac{1}{\sqrt{2}}(Y_7-Y_{17}),\ Q_{27}=\frac{1}{\sqrt{2}}(Y_8-Y_{19}),\\ Q_{28} &= \frac{1}{\sqrt{2}}(Y_9+Y_{27}),\ Q_{29}=\frac{1}{\sqrt{2}}(Y_{10}-Y_{29}),\ Q_{30}=\frac{1}{\sqrt{2}}(Y_{11}-Y_{25}),\\ Q_{31} &= \frac{1}{\sqrt{2}}(Y_{12}+Y_{30}),\ Q_{32}=\frac{1}{\sqrt{2}}(Y_{13}+Y_{26}),\ Q_{33}=\frac{1}{\sqrt{2}}(Y_{14}-Y_{28}),\\ Q_{34} &= \frac{1}{\sqrt{2}}(Y_{15}-Y_{32}),\ Q_{35}=\frac{1}{\sqrt{2}}(Y_{16}+Y_{31})\,. \end{aligned} \quad \text{(A.83)}$$

- $SU(2)$:

$$K_1=\frac{1}{2}(c_{51}+c_{78}),\ \ K_2=-\frac{1}{2}(c_{52}-c_{77}),\ \ K_3=\frac{1}{2}(c_{36}+\tilde{c}_{53}). \quad \text{(A.84)}$$

To find the above generators, we first look at the generators of the compact subgroup $SU(4) \times SU(2) \times U(1)$ of the $SU(4,2)$. Using the fact that $SU(4) \sim SO(6)$ and $SU(2) \times SU(2) \sim SO(4)$, we can identify $SU(4) \times SU(2) \times SU(2)$ with $SO(6) \times SO(4) \subset SO(10)$. The $U(1)$ generator is simply $\tilde{c}_{70}$. The final non-compact gauge group is $F_{4(-20)}$ whose generators can be easily identified by $c_1, \ldots, c_{52}$ in the construction of the E_6 given in [146].

With the same conventions as in the previous section, various components of $\mathcal{V}$'s are given by

$$\mathcal{V}^{\alpha IJ} = -\frac{1}{6}\text{Tr}(L^{-1}T^{\alpha}_{G}LX^{IJ}) \tag{A.85}$$

$$\mathcal{V}^{\alpha A} = \frac{1}{6}\text{Tr}(L^{-1}T^{\alpha}_{G}LY^{A}) \tag{A.86}$$

$$\mathcal{V}^{IJ}_{U(1)} = -\frac{1}{6}\text{Tr}(L^{-1}XLX^{IJ}) \tag{A.87}$$

$$\mathcal{V}^{A}_{U(1)} = \frac{1}{6}\text{Tr}(L^{-1}XLY^{A}) \tag{A.88}$$

with gauge generators T_G. Using the above embedding tensor, we find the following T-tensors

$$T^{IJ,KL} = \mathcal{V}^{IJ,\alpha}\mathcal{V}^{KL,\beta}\delta^{SO(p)}_{\alpha\beta} - \mathcal{V}^{IJ,\alpha}\mathcal{V}^{KL,\beta}\delta^{SO(10-p)}_{\alpha\beta} + \frac{1}{3}(5-p)\mathcal{V}^{IJ}_{U(1)}\mathcal{V}^{KL}_{U(1)}, \tag{A.89}$$

$$T^{IJ,A} = \mathcal{V}^{IJ,\alpha}\mathcal{V}^{A,\beta}\delta^{SO(p)}_{\alpha\beta} - \mathcal{V}^{IJ,\alpha}\mathcal{V}^{A,\beta}\delta^{SO(10-p)}_{\alpha\beta} + \frac{1}{3}(5-p)\mathcal{V}^{IJ}_{U(1)}\mathcal{V}^{A}_{U(1)} \tag{A.90}$$

for compact gaugings and

$$T^{IJ,KL} = \mathcal{V}^{IJ,\alpha}\mathcal{V}^{KL,\beta}\eta^{G_1}_{\alpha\beta} - K\mathcal{V}^{IJ,\alpha}\mathcal{V}^{KL,\beta}\eta^{G_2}_{\alpha\beta}, \tag{A.91}$$

$$T^{IJ,A} = \mathcal{V}^{IJ,\alpha}\mathcal{V}^{A,\beta}\eta^{G_1}_{\alpha\beta} - K\mathcal{V}^{IJ,\alpha}\mathcal{V}^{A,\beta}\eta^{G_2}_{\alpha\beta} \tag{A.92}$$

for non-compact gaugings with K being $\frac{2}{3}$ and 6 for $G_1 \times G_2$ being $G_{2(-14)} \times SU(2,1)$ and $SU(4,2) \times SU(2)$, respectively. For $F_{4(-20)}$ gauging, we have the simpler expressions for the T-tensors namely

$$\begin{aligned} T^{IJ,KL} &= \mathcal{V}^{IJ,\alpha}\mathcal{V}^{KL,\beta}\eta^{F_{4(-20)}}_{\alpha\beta}, \\ T^{IJ,A} &= \mathcal{V}^{IJ,\alpha}\mathcal{V}^{A,\beta}\eta^{F_{4(-20)}}_{\alpha\beta}. \end{aligned} \tag{A.93}$$

We are now in a position to compute the scalar potential for all the gauge groups mentioned above. The potentials and some of their critical points will be given in the next subsection.

A.4.2 Vacua of $N = 10$ gauged supergravity

In this subsection, we give some vacua of the $N = 10$ gauged theory with the gaugings described in the previous section. We will also discuss the isometry groups of the background with maximal supersymmetry at $L = \mathbf{I}$. This is a supersymmetric extension of the $SO(2,2) \sim SO(1,2)\times SO(1,2)$ isometry group of AdS$_3$. As a general strategy, we give the trivial critical point in which all scalars are zero, $L = \mathbf{I}$, as the first critical point. It is also useful to compare the cosmological constants of other critical points with the trivial one. According to the AdS/CFT correspondence, the cosmological constant V_0 is related to the central charge in the dual CFT as $c \sim \frac{1}{\sqrt{-V_0}}$, so we will give the ratio of the central charges for each non trivial critical point with respect to the trivial critical point at $L = \mathbf{I}$.

A.4.3 Vacua of compact gaugings

The compact gauging includes gauge groups $SO(p) \times SO(10-p) \times U(1)$ for $p = 6, \ldots, 10$ and $SO(5) \times SO(5)$. We give the scalar potential in $SO(p) \times SO(10-p) \times U(1)$ for $p = 7, \ldots, 10$ gaugings in the G_2 invariant scalar sector. For $SO(6) \times SO(4) \times U(1)$ gauging, we study the potential in $SO(4)_{\text{diag}}$ and $SU(3)$ sectors. Finally, for $SO(5) \times SO(5)$ gauging, we study the potential in $SO(5)_{\text{diag}}$, $SO(4)_{\text{diag}}$ and $SO(3)_{\text{diag}}$ sectors.

$SO(10) \times U(1)$ gauging

We will study the potential in the G_2 invariant scalar manifold. From 32 scalars, there are four singlets under $G_2 \subset SO(p)$, $p = 7, \ldots, 10$. These four scalars correspond to non-compact directions of $SU(2,1)$. We use the same parametrization as in [86], namely using three compact generators of the $SU(2)$ subgroup and one non-compact generator. With this parametrization, the coset representative takes the form

$$L = e^{a_1 c_{78}} e^{a_2 \tilde{c}_{53}} e^{a_3 c_{52}} e^{b_1 (Y_1 + Y_6)} e^{-a_3 c_{52}} e^{-a_2 \tilde{c}_{53}} e^{-a_1 c_{78}} \,. \tag{A.94}$$

This choice of L will also be used in the next three gauge groups. In this $SO(10) \times U(1)$ gauging, the potential is given by

$$V = \frac{1}{2} g^2 [-101 - 28 \cosh(2\sqrt{2} b_1) + \cosh(4\sqrt{2} b_1)]. \tag{A.95}$$

The potential does not depend on a_1, a_2 and a_3.

The first critical point is the trivial one in which all scalars are zero. We find

$$V_0 = -64 g^2, \qquad A_1 = -4 \mathbf{I}_{10} \,. \tag{A.96}$$

This is the critical point with (10,0) supersymmetry according to our convention. The corresponding background isometry is $Osp(10|2, \mathbb{R}) \times SO(2,1)$.

The second critical point is at $b_1 = \frac{\cosh^{-1} 2}{\sqrt{2}}$ with cosmological constant $V_0 =$

$-100g^2$. This is a non-supersymmetric point. The ratio of the central charges between this point and the maximally supersymmetric point is

$$\frac{c_{(0)}}{c_{(1)}} = \sqrt{\frac{V_0^{(1)}}{V_0^{(0)}}} = \frac{5}{4}\,. \tag{A.97}$$

Here and from now on, the notations $c_{(0)}$ and $c_{(i)}$ mean the central charges of the trivial and i^{th} non trivial critical points, respectively.

For $a_1 = a_3 = 0$, the coset representative (A.94) has a larger symmetry $SO(7)$. This $SO(7)$ is embedded in $SO(8)$ in such a way that it stabilizes one component of the $SO(8)$ spinor. In [86], this $SO(7)$ has been called $SO(7)^{\pm}$ according to a component of $\mathbf{8}_s$ or $\mathbf{8}_c$ is stabilized. Our critical point is parametrized only by b_1, so has $SO(7)$ symmetry. Notice that this point is very similar to the non-supersymmetric $SO(7)\times SO(7)$ critical point of the $SO(8)\times SO(8)$ gauged $N=16$ theory given in [86] and the $SO(7)$ point in $SO(9)$ gauged $N=9$ theory studied in [54]. The similarity mentioned here and in the following means that the location and the value of the cosmological constant relative to the trivial point are similar for these points. We do not know whether this is only an accident or there is a precise relation (to be specified if exists) between these critical points.

$SO(9)\times U(1)$ gauging

The potential in this gauging is much more complicated than the previous gauge group and depends on all four scalars. We will use the local $H = SO(10)\times U(1)$ symmetry to remove the $e^{-a_3 c_{52}} e^{-a_2 \tilde{c}_{53}} e^{-a_1 c_{78}}$ factor in (A.94) to simplify the computation and reduce the calculation time. The potential is given by

$$
\begin{aligned}
V &= -\frac{1}{327680}g^2\Bigg[-2\left(64\cos(2a_1)(1+3\cos(2a_3))\cosh\left(\frac{b_1}{\sqrt{2}}\right)\sinh^3\left(\frac{b_1}{\sqrt{2}}\right)\right. \\
&\quad +16\left(-4\cosh\left(\frac{b_1}{\sqrt{2}}\right)\left(\cos(2a_3)-4\cos^2a_1\cos(2a_2)\sin^2a_3\right.\right. \\
&\quad \left.+4\sin(2a_1)\sin a_2\sin(2a_3)\right)\sinh^3\left(\frac{b_1}{\sqrt{2}}\right)+\left(3+29\cosh\left(\sqrt{2}b_1\right)\right)\times \\
&\quad \left.\left.\sinh\left(\sqrt{2}b_1\right)\right)\right)^2+5120\left(4\cos^2a_2\cos(2a_3)+2\cos(2a_1)\left(-2\cos^2a_2\right.\right. \\
&\quad \left.\left.+(-3+\cos(2a_2))\cos(2a_3)\right)+8\sin(2a_1)\sin a_2\sin(2a_3)\right)^2\sinh^8\left(\frac{b_1}{\sqrt{2}}\right) \\
&\quad -2621440\cos^2a_1\cos^2a_2(\cos a_3\sin a_1+\cos a_1\sin a_2\sin a_3)^2\sinh^6\left(\frac{b_1}{\sqrt{2}}\right) \\
&\quad -384\left(64\sinh^6\left(\frac{b_1}{\sqrt{2}}\right)\right)\left(4\cos(2a_3)\sin(2a_1)\sin a_2+\left(-1+3\cos(2a_1)\right.\right. \\
&\quad \left.\left.-2\cos^2a_1\cos(2a_2)\right)\sin(2a_3)\right)^2-96\left(32\left(\cosh\left(\frac{b_1}{\sqrt{2}}\right)\left(4\cos^2a_1\cos^2a_3\right.\right.\right. \\
&\quad \left.+\left(3+\cos(2a_2)-2\cos(2a_1)\sin^2a_2\right)\sin^2a_3-2\sin(2a_1)\sin a_2\sin(2a_3)\right)\times \\
&\quad \left.\left.\sinh^3\left(\frac{b_1}{\sqrt{2}}\right)+\left(1+3\cosh\left(\sqrt{2}b_1\right)\right)\sinh\left(\sqrt{2}b_1\right)\right)\right)^2-4\left(4\left(8\cosh\left(\frac{b_1}{\sqrt{2}}\right)\right.\right. \\
&\quad \times\left(\cos(2a_2)-2\cos^2a_2\cos(2a_3)+\cos(2a_1)\left(2\cos^2a_2-(-3+\cos(2a_2))\right.\right. \\
&\quad \left.\left.\times\cos(2a_3)\right)-4\sin(2a_1)\sin a_2\sin(2a_3)\right)\sinh^3\left(\frac{b_1}{\sqrt{2}}\right)+6\sinh\left(\sqrt{2}b_1\right) \\
&\quad \left.\left.+29\sinh\left(2\sqrt{2}b_1\right)\right)\right)^2-1024\sinh^6\left(\frac{b_1}{\sqrt{2}}\right)\left(12\cos(2a_1)\sin(2a_3)\right. \\
&\quad \left.+16\cos(2a_3)\sin(2a_1)\sin(a_2)-4\left(1+2\cos^2a_1\cos(2a_2)\right)\sin(2a_3)\right)^2 \\
&\quad -\left(-8\left(8\cosh\left(\frac{b_1}{\sqrt{2}}\right)\left(-\cos(2a_3)+\cos(2a_1)(1+3\cos(2a_3))+4\cos^2a_1\times\right.\right.\right. \\
&\quad \left.\cos(2a_2)\sin^2a_3-4\sin(2a_1)\sin a_2\sin(2a_3)\right)\sinh^3\left(\frac{b_1}{\sqrt{2}}\right)+6\sinh\left(\sqrt{2}b_1\right) \\
&\quad \left.\left.\left.+29\sinh\left(2\sqrt{2}b_1\right)\right)\right)^2\Bigg]
\end{aligned}
\tag{A.98}
$$

Although we do not have a systematic way of finding critical points of this complicated potential, we find some critical points, numerically.

The first critical point is the maximally supersymmetric (9,1) point

$$
\begin{aligned}
a_1 &= a_2=a_3=b_1=0, \qquad V_0=-64g^2, \\
A_1 &= \mathrm{diag}\left(-4,-4,-4,-4,-4,-4,-4,-4,-4,4\right).
\end{aligned}
\tag{A.99}
$$

The background isometry is given by $Osp(9|2,\mathbb{R})\times Osp(1|2,\mathbb{R})$.

The second critical point is given by

$$\begin{aligned} b_1 &= \frac{1}{\sqrt{2}}\cosh^{-1}\frac{7}{3}, \qquad a_1=\pi,\ a_2=\frac{3\pi}{2},\ a_3=\frac{\pi}{2},\ V_0=-\frac{1024}{9}g^2, \\ A_1 &= \text{diag}\left(-8,-8,-8,-8,-8,-8,-8,\frac{16}{3},-\frac{16}{3},-\frac{16}{3}\right). \end{aligned} \tag{A.100}$$

This G_2 critical point has (2,1) supersymmetry with

$$\frac{c_{(0)}}{c_{(1)}}=\frac{4}{3}. \tag{A.101}$$

This critical point should be compared with the (1,1) $G_2\times G_2$ point in the $SO(8)\times SO(8)$ gauged $N=16$ theory. The two points have similar locations and values of the cosmological constant relative to the trivial point.

The third critical point in this gauging is given by

$$\begin{aligned} b_1 &= \frac{1}{\sqrt{2}}\cosh^{-1}2, \qquad a_1=a_3=\frac{\pi}{2}, \qquad a_2=\text{arbitrary}, \qquad V_0=-100g^2, \\ A_1 &= \text{diag}\left(-7,-7,-7,-7,-7,-7,-7,-7,7,-5\right). \end{aligned} \tag{A.102}$$

This is a (1,0) point with G_2 symmetry and

$$\frac{c_{(0)}}{c_{(2)}}=\frac{5}{4}. \tag{A.103}$$

$SO(8)\times SO(2)\times U(1)$ **gauging**

The potential in the G_2 sector is given by

$$\begin{aligned} V &= \frac{1}{2048}g^2\Big[-88549-21112\cosh(\sqrt{2}b_1)-22148\cosh(2\sqrt{2}b_1)+56\cosh(3\sqrt{2}b_1) \\ &\quad +681\cosh(4\sqrt{2}b_1)+256\Big[4\cos^2 a_1\cos(2a_3)[43+13\cosh(\sqrt{2}b_1)]+\cos(2a_1)\times \\ &\quad [85+27\cosh(\sqrt{2}b_1)]\Big]\sinh^6\left(\frac{b_1}{\sqrt{2}}\right)+128\,[3\cos(4a_1) \\ &\quad +16\cos^2 a_1\cos(2a_1)\cos(2a_3)+8\cos^4 a_1\cos(4a_3)]\sinh^8\left(\frac{b_1}{\sqrt{2}}\right)\Big]. \end{aligned} \tag{A.104}$$

The potential does not depend on a_2. We find the following critical points.

First of all, when $a_1=a_2=a_3=b_1=0$, we find the maximally supersymmetric critical points. At this point, we find

$$\begin{aligned} V_0 &= -64g^2, \\ A_1 &= \text{diag}\left(-4,-4,-4,-4,-4,-4,-4,-4,4,4\right). \end{aligned} \tag{A.105}$$

This point has (8,2) supersymmetry and $Osp(8|2,\mathbb{R})\times Osp(2|2,\mathbb{R})$ as the background isometry group.

The next point is given by

$$b_1 = \cosh^{-1} 2, \qquad a_1 = a_3 = 0, \qquad V_0 = -100g^2 \,. \tag{A.106}$$

This is an $SO(7)$ non-supersymmetric point with

$$\frac{c_{(0)}}{c_{(1)}} = \frac{5}{4}. \tag{A.107}$$

This point is very similar to the non-supersymmetric $SO(7)\times SO(7)$ point of the $SO(8)\times SO(8)$ gauged $N=16$ theory studied in [86].

The third critical point is given by

$$\begin{aligned}
b_1 &= \frac{1}{\sqrt{2}}\cosh^{-1}\frac{7}{3}, \qquad a_1 = 0, \qquad a_3 = \frac{\pi}{2}, \qquad V_0 = -\frac{1024}{9}g^2,\\
A_1 &= \begin{pmatrix}
-8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -8 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{16}{3} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & x_3
\end{pmatrix}
\end{aligned} \tag{A.108}$$

where

$$\begin{aligned}
x_1 &= -\frac{4}{3}[-5+\cos(2a_2)], \qquad x_2 = \frac{4}{3}\sin(2a_2),\\
x_3 &= \frac{4}{3}[5+\cos(2a_2)].
\end{aligned} \tag{A.109}$$

We find that this is the (1,1) point with G_2 symmetry, and the diagonalized A_1 tensor is given by

$$A_1 = \mathrm{diag}\left(-8,-8,-8,-8,-8,-8,-8,8,-\frac{16}{3},\frac{16}{3}\right). \tag{A.110}$$

The ratio of the central charges is

$$\frac{c_{(0)}}{c_{(2)}} = \frac{4}{3}\,. \tag{A.111}$$

This point is similar to the $G_2\times G_2$ point with (1,1) supersymmetry in $SO(8)\times SO(8)$ gauged $N=16$ theory.

$SO(7)\times SO(3)\times U(1)$ **gauging**

In this gauging, we still work with the G_2 invariant scalar sector. The potential is given by

$$V=-\frac{1}{32}g^2[1301+448\cosh(\sqrt{2}b_1)+308\cosh(2\sqrt{2}b_1)-9\cosh(4\sqrt{2}b_1)]. \quad \text{(A.112)}$$

This case is very similar to the $SO(10)\times U(1)$ gauging in the sense that the potential dose not depend on a_1, a_2 and a_3 and admits two critical points.

The first critical point is as usual at $L=\mathbf{I}$. This point is a (7,3) point with

$$\begin{aligned} V_0 &= -64g^2 \\ A_1 &= \text{diag}(-4,-4,-4,-4,-4,-4,-4,4,4,4). \end{aligned} \quad \text{(A.113)}$$

The background isometry is $Osp(7|2,\mathbb{R})\times Osp(3|2,\mathbb{R})$.

The second critical point is given by

$$b_1=\frac{1}{\sqrt{2}}\cosh^{-1}\frac{7}{3}, \qquad V_0=-\frac{1024}{9}g^2\,. \quad \text{(A.114)}$$

The A_1 tensor is very complicated, so we refer the reader to [55] for its explicit form. Remarkably, a complicated A_1 can be diagonalized to

$$A_1=\text{diag}\left(-8,-8,-8,-8,-8,-8,-8,8,8,\frac{16}{3}\right). \quad \text{(A.115)}$$

So, this critical point has (0,1) supersymmetry with

$$\frac{c_{(0)}}{c_{(1)}}=\frac{4}{3}\,. \quad \text{(A.116)}$$

Notice that this point has G_2 symmetry although it is characterized only by b_1. This is because the $SO(7)$ in the gauge group is not the same as $SO(7)^{\pm}$, and b_1 is not invariant under this $SO(7)$. The $SO(7)$ in the gauge group is embedded in $SO(8)$ as $\mathbf{8}_v\rightarrow\mathbf{7}+\mathbf{1}$. This point is similar to the (1,1) $G_2\times G_2$ point in [86].

$SO(6)\times SO(4)\times U(1)$ **gauging**

We first study the potential in the $SO(4)_{\text{diag}}$ scalar sector. There are four singlets in this sector corresponding the non-compact directions of $SO(2,2)\sim SO(2,1)\times SO(2,1)$. We parametrize the coset representative by

$$L=e^{a_1[V_1,V_2]}e^{b_1V_1}e^{-a_1[V_1,V_2]}e^{a_2[V_3,V_4]}e^{b_2V_1}e^{-a_2[V_3,V_4]}, \quad \text{(A.117)}$$

where

$$\begin{aligned} V_1 &= j_1 + j_2, \\ V_2 &= j_3 - j_4, \\ V_3 &= j_3 + j_4, \\ V_4 &= j_1 - j_2, \end{aligned} \tag{A.118}$$

and

$$\begin{aligned} j_1 &= Y_1 + Y_5 - Y_9 + Y_{13} - Y_{17} - Y_{21} + Y_{30} + Y_{32}, \\ j_2 &= Y_2 + Y_{10} - Y_{11} + Y_{18} + Y_{19} - Y_{28} + Y_{31} + Y_3, \\ j_3 &= Y_4 + Y_7 + Y_{12} - Y_{15} + Y_{20} + Y_{23} + Y_{26} - Y_{27}, \\ j_4 &= Y_6 - Y_8 + Y_{14} + Y_{16} - Y_{22} + Y_{24} + Y_{25} - Y_{29}. \end{aligned} \tag{A.119}$$

We find the potential

$$\begin{aligned} V &= -4g^2[6 + 4\cosh(4\sqrt{2}b_1) + \cosh[4\sqrt{2}(b_1 - b_2)] + 4\cosh(4\sqrt{2}b_2) \\ &\quad + \cosh[4\sqrt{2}(b_1 + b_2)]]. \end{aligned} \tag{A.120}$$

There is no non-trivial critical point in this potential. So, there is no critical point with $SO(4)_{\text{diag}}$ symmetry.

Next, we will consider the $SU(3)$ invariant sector. The $SU(3)$ is a subgroup of $SO(6) \sim SU(4)$. There are eight singlets in this sector. The coset representative is parametrized by

$$L = e^{a_1 c_{36}} e^{a_2 c_{51}} e^{a_3 c_{52}} e^{a_4 \tilde{c}_{53}} e^{a_5 c_{77}} e^{a_6 c_{78}} e^{b_1 Y_1} e^{b_2 Y_3} \tag{A.121}$$

in which the eight scalars correspond to non-compact directions of $SU(2,2)$. As usual, we have used the local H symmetry to simplify the parametrization of L. The potential has a very complicated form, so we will not repeat it here. Its explicit form is given in [55]. We find two critical points.

The trivial (6,4) critical point at $L = \mathbf{I}$ is given by

$$\begin{aligned} V_0 &= -64g^2, \\ A_1 &= \text{diag}(-4, -4, -4, -4, -4, -4, 4, 4, 4, 4). \end{aligned} \tag{A.122}$$

The background isometry is $Osp(6|2, \mathbb{R}) \times Osp(4|2, \mathbb{R})$.

The non trivial critical point is given by

$$\begin{aligned} a_i &= \frac{\pi}{2}, \qquad i = 1, \ldots, 6, \\ b_1 &= b_2 = \cosh^{-1}\sqrt{3}, \qquad V_0 = -144g^2, \\ A_1 &= \text{diag}(-10, -10, -10, -10, -10, -10, 6, 6, 10, 10). \end{aligned} \tag{A.123}$$

This point preserves (0,2) supersymmetry and $SU(3)$ symmetry. The ratio of the central charges is

$$\frac{c_{(0)}}{c_{(1)}} = \frac{3}{2}. \tag{A.124}$$

$SO(5) \times SO(5)$ **gauging**

We start with the potential in the $SO(5)_{\text{diag}}$ scalar sector. There are two singlets in this sector corresponding to the non-compact directions of $SL(2)$. We parametrize the coset representative by

$$L = e^{a_1 V} e^{b_1 U} e^{-a_1 V} \tag{A.125}$$

where the compact and non-compact generators of $SL(2)$ are given by

$$V = \frac{1}{\sqrt{2}}\left(c_{11} - c_{17} + c_{32} - c_{48} + c_{75} + \frac{\sqrt{3}}{2}\tilde{c}_{70}\right), \tag{A.126}$$

$$U = Y_3 - Y_5 - Y_{12} + Y_{16} + Y_{17} - Y_{18} + Y_{27} + Y_{29}\,. \tag{A.127}$$

The potential is given by

$$V = -8g^2(5 + 3\cosh(4b_1)) \tag{A.128}$$

which does not have any non-trivial critical points.

We then move to smaller unbroken gauge symmetry namely $SO(4)_{\text{diag}}$. The parametrization of L is the same as in (A.117). The potential turns out to be the same as that of $SO(6) \times SO(4) \times U(1)$ gauging, and, of course, does not have any non trivial critical points.

To proceed further, we need to reduce the residual symmetry to a smaller group. The next sector we will consider is $SO(3)_{\text{diag}}$. There are eight singlets in this sector. These are non-compact directions of $SO(4,2) \sim SU(2,2)$. We parametrize the coset representative in this sector by

$$L = e^{a_1 c_{10}} e^{a_2 c_{14}} e^{a_3 c_{15}} e^{a_4 c_{19}} e^{a_5 c_{20}} e^{a_6 c_{21}} e^{b_1 Z_1} e^{b_2 Z_2} \tag{A.129}$$

where

$$Z_1 = Y_1 + Y_{11} - Y_{20} - Y_{29}, \qquad Z_2 = Y_2 + Y_{13} - Y_{24} + Y_{27}\,. \tag{A.130}$$

The potential depends on all eight scalars. Its explicit form is given in [55].

The trivial (5,5) critical point at $L = \mathbf{I}$ is characterized by

$$V_0 = -64g^2, \qquad A_1 = \text{diag}\,(-4,-4,-4,-4,-4,4,4,4,4,4)\,. \tag{A.131}$$

The corresponding background isometry group is $Osp(5|2,\mathbb{R}) \times Osp(5|2,\mathbb{R})$.

We find a non trivial critical point given by

$$a_i = \frac{\pi}{2}, \quad i = 1,\ldots,6, \qquad b_2 = 0,$$
$$b_1 = \frac{\cosh^{-1} 5}{2}, \qquad V_0 = -256g^2,$$
$$A_1 = \text{diag}\,(-8,-8,-8,16,16,-16,-16,16,16,16)\,. \tag{A.132}$$

This critical point has (3,0) supersymmetry with the ratio of the central charges

$$\frac{c_{(0)}}{c_{(1)}} = 2\,. \tag{A.133}$$

A.4.4 Vacua of non-compact gaugings

We now consider non-compact gaugings with gauge groups $SU(4,2)\times SU(2)$, $G_{2(-14)}\times SU(2,1)$ and $F_{4(-20)}$. At $L=\mathbf{I}$, the gauge group is broken down to its maximal compact subgroup, and the bosonic part of the background isometry is formed by this subgroup and $SO(2,2)$. These three gauge groups contain $SU(3)$ subgroup, so we study the potential in the $SU(3)$ scalar sector in all non-compact gaugings. For $G_{2(-14)}\times SU(2,1)$ and $F_{4(-20)}$ gaugings, the $SU(3)\subset G_2$ sector consists of eight scalars which is twice the number of scalars in the G_2 sector. The $SU(3)$ is embedded in G_2 as $\mathbf{7}\rightarrow\mathbf{3}+\bar{\mathbf{3}}+\mathbf{1}$. The eight scalars correspond to non-compact directions of the $SO(4,2)\sim SU(2,2)\subset E_{6(-14)}$. For $SU(4,2)\times SU(2)$ gauging, the $SU(3)$ is embedded in $SU(4)\subset SU(4,2)$ as $\mathbf{4}\rightarrow\mathbf{3}+\mathbf{1}$. Similarly, the eight scalars are described by non-compact directions of $SU(2,2)$. This sector is essentially the same as that used in $SO(6)\times SO(4)\times U(1)$ gauging.

Fortunately, we do not need to deal with all eight scalars. In these three gaugings, four of the eight $SU(3)$ singlets lie along the gauge group, so only four directions orthogonal to the gauge group are relevant. This is because the singlets which are parts of the gauge group will drop out from the potential and correspond to flat directions of the potential. The relevant four singlets are contained in the $SU(2,1)$ sub group of $SU(2,2)$. We also study the potentials in other sectors specific to each gauging. The details of these sectors will be explained below.

$G_{2(-14)}\times SU(2,1)$ gauging

If we study the potential in the G_2 sector in this gauging, we will find the constant potential. This is because all scalars in the G_2 sector are parts of the gauge group and will drop out from the potential. We then start with $SU(3)\subset G_2$ sector. As discussed above, this sector contains four relevant scalars parametrized by

$$L=e^{a_1c_{52}}e^{a_2c_{78}}e^{a_3\tilde{c}_{53}}e^{b_1(Y_1-Y_6)}e^{-a_3\tilde{c}_{53}}e^{-a_2c_{78}}e^{-a_1c_{52}}\,. \tag{A.134}$$

The potential is given by

$$V=\frac{1}{18}g^2[-101-28\cosh(2\sqrt{2}b_1)+\cosh(4\sqrt{2}b_1)]. \tag{A.135}$$

There are two critical points. The first one is the trivial critical point given by $L=\mathbf{I}$ and

$$\begin{aligned} V_0 &= -\frac{64}{9}g^2,\\ A_1 &= \text{diag}\left(-\frac{4}{3},-\frac{4}{3},-\frac{4}{3},-\frac{4}{3},-\frac{4}{3},-\frac{4}{3},-\frac{4}{3},\frac{4}{3},\frac{4}{3},\frac{4}{3}\right). \end{aligned} \tag{A.136}$$

We find that this point has (7,3) supersymmetry. The symmetry of this point is given by the maximal compact subgroup $G_2\times SU(2)\times U(1)$ of $G_{2(-14)}\times SU(2,1)$. The left handed supercharges transform as **7** under G_2 while the right handed supercharges transform as **3** under the $SU(2) \sim SO(3)$. So, the background isometry is given by $G(3) \times Osp(3|2, \mathbb{R})$.

The second critical point is characterized by

$$\begin{aligned}
b_1 &= \frac{\cosh^{-1} 2}{\sqrt{2}}, \qquad V_0 = -\frac{100}{9} g^2, \\
A_1 &= \begin{pmatrix}
-\frac{7}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{7}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{7}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{7}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{11}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{7}{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{7}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & y_1 & y_4 & y_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & y_4 & y_2 & y_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & y_5 & y_6 & y_3
\end{pmatrix}
\end{aligned} \tag{A.137}$$

where

$$\begin{aligned}
y_1 &= \frac{1}{6}[13 - \cos(2a_1) - 2\cos^2 a_1 \cos(2a_2)], \\
y_2 &= \frac{1}{6}[13 + \cos(2a_1) - 2\cos(2a_2)\sin^2 a_1], \\
y_3 &= \frac{1}{3}(6 + \cos(2a_2)), \qquad y_4 = \frac{1}{3}\cos^2 a_2 \sin(2a_1), \\
y_5 &= -\frac{1}{3}\cos a_1 \sin(2a_2), \qquad y_6 = \frac{1}{3}\sin a_1 \sin(2a_2)\,.
\end{aligned} \tag{A.138}$$

We can diagonalize A_1 to

$$A_1 = \text{diag}\left(-\frac{11}{3}, -\frac{7}{3}, -\frac{7}{3}, -\frac{7}{3}, -\frac{7}{3}, -\frac{7}{3}, -\frac{7}{3}, \frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right) \tag{A.139}$$

from which we find that this is a (0,1) supersymmetric critical point. The ratio of the central charges relative to the $L = \mathbf{I}$ point is

$$\frac{c_{(0)}}{c_{(1)}} = \frac{5}{4}\,. \tag{A.140}$$

This $SU(3)$ point is similar to the (0,1) $SU(3)$ point in $G_{2(-14)} \times SL(2)$ gauged $N = 9$ theory.

We now study the potential in different sector, $SU(2)_{\text{diag}}$ sector. From the $SU(3)$ sector discussed above, the next symmetry to consider could be the $SU(2) \subset SU(3)$. In general, we expect more scalars than those appearing in the $SU(3)$ sector. This will make the calculation takes much longer time. We then consider $SU(2)_{\text{diag}}$

sector in which $SU(2)_{\rm diag} \subset SU(2) \times SU(2)$. The first and second $SU(2)$'s are subgroups of $SU(3) \subset G_{2(-14)}$ and $SU(2,1)$, respectively. There are four singlets in this sector corresponding to the non-compact directions of $SO(4,1) \sim Sp(1,1)$. We choose to parametrize the coset representative by applying three $SO(3) \subset SO(4) \sim SO(3) \times SO(3)$ rotations as follow

$$L = e^{a_1 c_8} e^{a_2 c_{17}} e^{a_3 c_{20}} e^{b_1 (Y_2 - Y_{16} + Y_{19} + Y_{29})} e^{-a_3 c_{20}} e^{-a_2 c_{17}} e^{-a_1 c_8} \,. \tag{A.141}$$

The potential is

$$V = \frac{1}{72} g^2 [-269 - 192\cosh(2b_1) - 52\cosh(4b_1) + \cosh(8b_1)] \,. \tag{A.142}$$

There is one non trivial critical points given by

$$b_1 = \cosh^{-1}\sqrt{2}, \qquad V_0 = -16g^2 \,. \tag{A.143}$$

This is a supersymmetric point with the associated A_1 tensor given in [55]. After diagonalization, we find

$$A_1 = \text{diag}\left(-4,-4,-4,-4,-\frac{10}{3},-2,-2,2,2,2\right) \tag{A.144}$$

which gives (2,3) supersymmetry. The ratio of the central charges is

$$\frac{c_{(0)}}{c_{(2)}} = \frac{3}{2} \,. \tag{A.145}$$

This critical point has $SU(2)_{\rm diag} \times U(1)$ symmetry.

$F_{4(-20)}$ gauging

In this gauging with simple gauge group, we study the potential in the G_2 and $SU(3)$ scalar sectors. We start with the G_2 sector. Two of the four scalars are parts of the gauge group, so we only need to parametrize the coset representative with the other two scalars. These two scalars correspond to the non-compact directions of $SL(2)$. The L is then parametrized by

$$L = e^{a_1 c_{52}} e^{b_1 (Y_{25} + Y_{30})} e^{-a_1 c_{52}} \,. \tag{A.146}$$

The potential is

$$V = \frac{g^2}{8}[-101 - 28\cosh(2\sqrt{2}b_1) + \cosh(4\sqrt{2}b_1)]. \tag{A.147}$$

There are two critical points. The first one is trivial and given by

$$\begin{aligned} L &= \mathbf{I}, \qquad V_0 = -16g^2, \\ A_1 &= \text{diag}\,(-2,-2,-2,-2,-2,-2,-2,-2,-2,2) \,. \end{aligned} \tag{A.148}$$

This is the maximally supersymmetric point with (9,1) supersymmetry. The gauge symmetry is broken down to its maximal compact subgroup $SO(9)$, and the background isometry is $Osp(9|2,\mathbb{R}) \times Osp(1|2,\mathbb{R})$.

The second critical point is given by

$$\begin{aligned} b_1 &= \frac{\cosh^{-1} 2}{\sqrt{2}}, \qquad V_0 = -25g^2, \\ A_1 &= \begin{pmatrix} -\frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{7}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{7}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{7}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_1 & w_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_3 & w_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{11}{2} \end{pmatrix} \end{aligned} \tag{A.149}$$

where

$$w_1 = -3 - \frac{1}{2}\cos(2a_1), \;\; w_2 = \frac{1}{2}[-6 + \cos(2a_1)], \;\; w_3 = \cos a_1 \sin a_1 \,. \tag{A.150}$$

The A_1 tensor can be diagonalized to

$$A_1 = \text{diag}\left(\frac{11}{2}, -\frac{7}{2}, -\frac{7}{2}, -\frac{7}{2}, -\frac{7}{2}, -\frac{7}{2}, -\frac{7}{2}, -\frac{7}{2}, -\frac{7}{2}, -\frac{5}{2}\right). \tag{A.151}$$

This critical point is a (1,0) point with

$$\frac{c_{(0)}}{c_{(1)}} = \frac{5}{4} \tag{A.152}$$

and preserves $SO(7) \subset SO(9) \subset F_{4(-20)}$ symmetry.

In the $SU(3)$ sector, there are eight singlets, but four of them are parts of the $F_{4(-20)}$. So, there are four singlets orthogonal to the gauge group. These are non-compact directions of $SU(2,1)$, and L can be parametrized by

$$L = e^{a_1 c_{34}} e^{a_2 c_{49}} e^{a_3 c_{52}} e^{b_1 Y_{21}} e^{-a_3 c_{52}} e^{-a_2 c_{49}} e^{-a_1 c_{34}} \,. \tag{A.153}$$

The potential is given by

$$V = \frac{g^2}{8}[-101 - 28\cosh(2\sqrt{2}b_1) + \cosh(4\sqrt{2}b_1)] \tag{A.154}$$

which is the same as the potential in the G_2 sector. The non-trivial critical point is at the same position and cosmological constant, $b_1 = \cosh^{-1} 2$, $V_0 = -25g^2$. The residual symmetry is $SO(7)$ as in the previous critical point. Although the A_1 tensor in this case is more complicated, it is the same as (A.151) after diagonalization. The explicit form of A_1 can be found in [55].

$SU(4,2) \times SU(2)$ gauging

This gauging is the most difficult one to find a suitable scalar sector in order to reveal non trivial critical points and still have a manageable number of scalars. We start with the $SO(4)_{\text{diag}}$ scalar sector. The $SO(4)_{\text{diag}}$ is formed by taking the subgroup $SU(2)\times SU(2)\times SU(2)\times SU(2)$ of $SU(4,2)\times SU(2)$. The first two $SU(2)$'s are subgroups of $SU(4) \subset SU(4,2)$, the third $SU(2)$ is the $SU(2) \subset SU(4,2)$. Our $SO(4)_{\text{diag}}$ is the diagonal subgroup of $(SU(2) \times SU(2)) \times (SU(2) \times SU(2)) \sim SO(4) \times SO(4)$. There are two singlets in this sector. These are non-compact directions of $SL(2)$, and L can be parametrized by

$$\begin{aligned} L &= e^{a_1 c_{15}} e^{b_1 \tilde{Y}} e^{-a_1 c_{15}}, \\ \tilde{Y} &= Y_1 + Y_2 - Y_6 - Y_7 - Y_9 + Y_{10} - Y_{14} + Y_{15} \\ &\quad + Y_{17} - Y_{18} - Y_{22} + Y_{23} - Y_{27} + Y_{28} - Y_{29} - Y_{32} \end{aligned} \tag{A.155}$$

which, unfortunately, gives a constant potential $V = -16g^2$. So, we move to a smaller residual symmetry to obtain a non trivial structure of the potential.

We now study the potential in the scalar sector parametrizing the $SU(3)$ invariant manifold. This $SU(3)$ is a subgroup of $SU(4) \subset SU(4,2)$. The eight singlet scalars in this sector are the non-compact directions of $SO(4,2) \sim SU(2,2)$. The four directions which are orthogonal to the gauge group are non-compact directions of $SU(2,1) \subset SU(2,2)$. The coset representative is given by

$$\begin{aligned} L &= e^{a_1(c_{51}+c_{78})} e^{a_2(c_{36}+\tilde{c}_{53})} e^{a_3(c_{77}-c_{52})} e^{b_1(Y_1-Y_{23})} \\ &\quad e^{-a_3(c_{77}-c_{52})} e^{-a_2(c_{36}+\tilde{c}_{53})} e^{-a_1(c_{51}+c_{78})} . \end{aligned} \tag{A.156}$$

We find the potential

$$V = -2g^2(5 + 3\cosh(2b_1)) \tag{A.157}$$

which, again, does not admit any non trivial critical points.

The next sector we will study is $SU(2)_{\text{diag}}$. This symmetry is a diagonal subgroup of $SU(2) \times SU(2)$ in which the first $SU(2)$ is a subgroup of $SU(4) \subset SU(4,2)$, and the second $SU(2)$ is the $SU(2)$ factor in the gauge group. There are four scalars in this sector. These scalars are non-compact directions of $SU(2,1)$, and L can be parametrized by

$$L = e^{a_1 c_{10}} e^{a_2 c_{14}} e^{a_3 c_{15}} e^{b_1 Y} e^{-a_3 c_{15}} e^{-a_2 c_{14}} e^{-a_1 c_{10}} \tag{A.158}$$

where

$$Y = Y_7 - Y_6 - Y_{12} - Y_{16} + Y_{17} + Y_{18} + Y_{30} + Y_{31} . \tag{A.159}$$

The corresponding potential is

$$V = \frac{g^2}{8}[-101 - 28\cosh(4\sqrt{2}b_1) + \cosh(8\sqrt{2}b_1)] . \tag{A.160}$$

We now discuss its trivial critical point at $L = \mathbf{I}$. This point is characterized by

$$V_0 = -16g^2, \qquad A_1 = \mathrm{diag}\left(-2, -2, -2, -2, -2, -2, 2, 2, 2, 2\right). \tag{A.161}$$

The critical point has (6,4) supersymmetry. The gauge group is broken down to its maximal compact subgroup $SU(4) \times SU(2) \times U(1) \times SU(2)$. The left handed supercharges transform as **6** under $SU(4) \sim SO(6)$ while the right handed supercharges transform as **4** under $SU(2) \times SU(2) \sim SO(4)$. So, the background isometry is given by $Osp(6|2, \mathbb{R}) \times Osp(4|2, \mathbb{R})$.

The non trivial critical point with $SU(2)_{\mathrm{diag}} \times SU(2) \times SU(2) \times U(1)$ symmetry is given by

$$b_1 = \frac{1}{\sqrt{2}} \cosh^{-1} \sqrt{\frac{3}{2}}, \qquad V_0 = -25g^2\,. \tag{A.162}$$

The associated A_1 tensor can be found in [55]. It can be diagonalized to

$$A_1 = \mathrm{diag}\left(\frac{11}{2}, \frac{11}{2}, \frac{11}{2}, \frac{11}{2}, -\frac{7}{2}, -\frac{7}{2}, -\frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}, -\frac{5}{2}\right). \tag{A.163}$$

So, this is a (4,0) point with

$$\frac{c_{(0)}}{c_{(1)}} = \frac{5}{4}\,. \tag{A.164}$$

In this section, we have studied critical points of $N = 10$ three dimensional gauged supergravity with both compact and non-compact gauge groups. Remarkably, all critical points found in this paper are AdS critical points. This is in contrast to the results of [86] in which some Minkowski and dS vacua have been found. All critical points identified in this section together with the value of the cosmological constant, unbroken gauge symmetry and residual supersymmetry are listed in Table V.

Critical point	Gauge group	V_0	Unbroken SUSY	Unbroken gauge symmetry
1	$SO(10) \times U(1)$	$-64g^2$	$(10,0)$	$SO(10) \times U(1)$
2	$SO(10) \times U(1)$	$-100g^2$	-	$SO(7)$
3	$SO(9) \times U(1)$	$-64g^2$	$(9,1)$	$SO(9) \times U(1)$
4	$SO(9) \times U(1)$	$-\frac{1024}{9}g^2$	$(2,1)$	G_2
5	$SO(9) \times U(1)$	$-100g^2$	$(1,0)$	G_2
6	$SO(8) \times SO(2)$ $\times U(1)$	$-64g^2$	$(8,2)$	$SO(8) \times SO(2)$ $\times U(1)$
7	$SO(8) \times SO(2)$ $\times U(1)$	$-100g^2$	-	$SO(7)$
8	$SO(8) \times SO(2)$ $\times U(1)$	$-\frac{1024}{9}g^2$	$(1,1)$	G_2
9	$SO(7) \times SO(3)$ $\times U(1)$	$-64g^2$	$(7,3)$	$SO(7) \times SO(3)$ $\times U(1)$
10	$SO(7) \times SO(3)$ $\times U(1)$	$-\frac{1024}{9}g^2$	$(0,1)$	G_2
11	$SO(6) \times SO(4)$ $\times U(1)$	$-64g^2$	$(6,4)$	$SO(6) \times SO(4)$ $\times U(1)$
12	$SO(6) \times SO(4)$ $\times U(1)$	$-144g^2$	$(0,2)$	$SU(3)$
13	$SO(5) \times SO(5)$	$-64g^2$	$(5,5)$	$SO(5) \times SO(5)$
14	$SO(5) \times SO(5)$	$-256g^2$	$(3,0)$	$SO(3)_{\text{diag}}$
15	$G_{2(-14)} \times SU(2,1)$	$-\frac{64}{9}g^2$	$(7,3)$	$G_{2(-14)} \times SU(2)$ $\times U(1)$
16	$G_{2(-14)} \times SU(2,1)$	$-\frac{100}{9}g^2$	$(0,1)$	$SU(3)$
17	$G_{2(-14)} \times SU(2,1)$	$-16g^2$	$(2,3)$	$SU(2)_{\text{diag}} \times U(1)$
18	$F_{4(-20)}$	$-16g^2$	$(9,1)$	$SO(9)$
19	$F_{4(-20)}$	$-25g^2$	$(1,0)$	$SO(7)$
20	$SU(4,2) \times SU(2)$	$-16g^2$	$(6,4)$	$SU(4) \times SU(2)$ $\times SU(2) \times U(1)$
21	$SU(4,2) \times SU(2)$	$-25g^2$	$(4,0)$	$SU(2)_{\text{diag}} \times SU(2)$ $\times SU(2) \times U(1)$

Table V: Some critical points of $N = 10$ gauged supergravity in three dimensions.

Appendix B

On instantons

In this appendix, we review some relevant information about Yang-Mills and gravitational instantons which play an important role in various places throughout this thesis. This chapter follows the discussions given in [115], [114] and [116]. We will not aim to give a complete review on this vast subject but only provide the results which are useful for finding RG flow solutions discussed in the main text and make the thesis sufficiently self-contained. The main propose of this appendix is solely to focus on the explicit solutions and useful formulae. The full detail can be found in the literatures.

B.1 Yang-Mills instantons

Instantons are solutions of field equations in Euclidean space with finite action. We restrict ourselves to the case of instantons on a four dimensional Euclidean space although instantons can be defined on higher dimensional spaces [147]. These solutions are described by a self-dual or antiself-dual field strength. The real instanton solutions are possible only in Euclidean spaces since the squared of the Hodge duality is minus one, $*^2 = -1$, in Minkowski spaces rather than plus one as in the Euclidean cases. Since we only need the explicit instanton solutions in this thesis, we will not review the ADHM formalism [148] which is very useful in constructing multi-instanton solutions. The interested reader should consult [149] and [150] for good reviews.

We begin with the Euclidean Yang-Mills action with gauge group G in four dimensions

$$I = \frac{1}{4g^2}\int d^4x F^a_{\mu\nu}F^{a\mu\nu}, \qquad a = 1, \ldots, \text{dim}G\,. \tag{B.1}$$

The G generators T^a satisfy the algebra $\left[T^a, T^b\right] = f^{ab}{}_c T^c$ with structure constants $f^{ab}{}_c$. The field strength is defined by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{abc}A^b_\mu A^c_\nu\,. \tag{B.2}$$

We will restrict ourselves to semisimple gauge groups so that we can work with the totally antisymmetric structure constants f_{abc}. Since the Cartan-Killing metric is non degenerate in this case, we can work with f_{abc} or f^{abc}. The Yang-Mills equation is straightforwardly obtained from (B.1)

$$D_\mu F^{a\mu\nu} = 0 \tag{B.3}$$

where the covariant derivative is defined by

$$D_\mu F^a_{\nu\rho} = \partial_\mu F^a_{\nu\rho} + f_{abc} A^b_\mu F^c_{\nu\rho}\,. \tag{B.4}$$

The requirement for finite action imposes the condition that the field strength is asymptotically zero faster than $\frac{1}{r^2}$, $r^2 = |x|^2$ where x^μ, $\mu = 1, \ldots, 4$, are coordinates on $\mathbb{R}^4$. This implies that the gauge fields are pure gauge asymptotically

$$A^a_\mu = U^{-1}\partial_\mu U, \qquad r \to \infty \text{ and } U \in G\,. \tag{B.5}$$

The gauge field subject to this boundary condition can be classified by the winding number, or topological charge, defined by

$$N = \frac{1}{32\pi^2}\int d^4x F^a_{\mu\nu}(*F^a)^{\mu\nu} = \frac{1}{32\pi^2}\int d^4x F^a_{\mu\nu}F^{a\mu\nu} \tag{B.6}$$

where the Hodge duality is given by $(*F^a)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{a\rho\sigma}$ with $\epsilon_{1234} = 1$. The winding number is clearly gauge invariant. Instantons are described by self-dual or antiself-dual field configurations

$$F^a_{\mu\nu} = \pm(*F^a)_{\mu\nu} = \pm\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{a\rho\sigma}\,. \tag{B.7}$$

The field equation is trivially satisfied due to the Bianchi identity $D_{[\mu}F^a_{\nu\rho]} = 0$. The action is minimized by these configurations as can be seen by

$$I = \frac{1}{4g^2}\int d^4x F^2 = \frac{1}{8g^2}\int d^4x(F \mp *F)^2 \mp \frac{1}{4g^2}\int d^4x F * F \geq \pm\frac{8\pi^2}{g^2}N \tag{B.8}$$

where we have used the definition of the winding number and omitted explicit contractions of indices.

We begin with the $SU(2)$ instantons. The ansatz is given by

$$A_\mu = \alpha\sigma_{\mu\nu}\partial^\nu \ln\rho, \qquad A_\mu = A^a_\mu T^a\,. \tag{B.9}$$

At this point, we discuss all the notations involved throughout this section. The $\sigma^{\mu\nu}$ are defined by

$$\sigma^{\mu\nu} = \frac{1}{2}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \tag{B.10}$$

where

$$\sigma^\mu = (\sigma^a, i\mathbf{I}), \qquad \bar\sigma^\mu = (\sigma^a, -i\mathbf{I}). \tag{B.11}$$

The σ^a, $a = 1,2,3$ are the usual Pauli matrices. We similarly define

$$\bar\sigma^{\mu\nu} = \frac{1}{2}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu). \tag{B.12}$$

The Euclidean Lorentz generators on the two spinor representations are given by

$$M^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} \qquad \text{and} \qquad M^{\mu\nu} = \frac{1}{2}\bar\sigma^{\mu\nu} \tag{B.13}$$

which satisfy

$$[M^{\mu\nu}, M^{\rho\sigma}] = \delta^{\nu\rho}M^{\mu\sigma} - \delta^{\nu\sigma}M^{\mu\rho} - \delta^{\mu\rho}M^{\nu\sigma} - \delta^{\mu\sigma}M^{\nu\rho}\,. \tag{B.14}$$

Furthermore, the $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ matrices are antiself-dual and self-dual, respectively,

$$\sigma_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma}, \qquad \bar\sigma_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\bar\sigma^{\rho\sigma}\,. \tag{B.15}$$

We can also expand these matrices in terms of Pauli matrices using the 't Hooft tensors $\eta^a_{\mu\nu}$ and $\bar\eta^a_{\mu\nu}$ as

$$\sigma_{\mu\nu} = i\bar\eta^a_{\mu\nu}\sigma^a, \qquad \bar\sigma_{\mu\nu} = i\eta^a_{\mu\nu}\sigma^a\,. \tag{B.16}$$

We immediately deduce that $\eta^a_{\mu\nu}$ and $\bar\eta^a_{\mu\nu}$ are, respectively, self-dual and antiself-dual. In components, $\eta^a_{\mu\nu}$ and $\bar\eta^a_{\mu\nu}$ are given by

$$\begin{aligned} \eta^a_{ij} &= \epsilon^a{}_{ij}, \qquad \eta^a_{i4} = -\eta^a_{4i} = \delta^a_i, \\ \bar\eta^a_{ij} &= \epsilon^a{}_{ij}, \qquad \bar\eta^a_{i4} = -\bar\eta^a_{4i} = -\delta^a_i \end{aligned} \tag{B.17}$$

for $i = 1,2,3$. Before solving for the instanton solution, we note the following identities

$$\begin{aligned} [\sigma_{\mu\nu}, \sigma_{\rho\sigma}] &= 2\delta_{\nu\rho}\sigma_{\mu\sigma} - 2\delta_{\nu\sigma}\sigma_{\mu\rho} - 2\delta_{\mu\rho}\sigma_{\nu\sigma} - 2\delta_{\mu\sigma}\sigma_{\nu\rho}, \\ \{\sigma_{\mu\nu}, \sigma_{\rho\sigma}\} &= 2(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) + 2\epsilon_{\mu\nu\rho\sigma}, \\ [\bar\sigma_{\mu\nu}, \bar\sigma_{\rho\sigma}] &= 2\delta_{\nu\rho}\bar\sigma_{\mu\sigma} - 2\delta_{\nu\sigma}\bar\sigma_{\mu\rho} - 2\delta_{\mu\rho}\bar\sigma_{\nu\sigma} - 2\delta_{\mu\sigma}\bar\sigma_{\nu\rho}, \\ \{\bar\sigma_{\mu\nu}, \bar\sigma_{\rho\sigma}\} &= 2(\delta_{\mu\sigma}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\sigma}) - 2\epsilon_{\mu\nu\rho\sigma}, \\ \epsilon_{\mu\nu\rho\sigma}\sigma_{\sigma\tau} &= \delta_{\mu\tau}\sigma_{\nu\rho} - \delta_{\nu\tau}\sigma_{\mu\rho} + \delta_{\rho\tau}\sigma_{\mu\nu}, \\ \epsilon_{\mu\nu\rho\sigma}\bar\sigma_{\sigma\tau} &= -\delta_{\mu\tau}\bar\sigma_{\nu\rho} + \delta_{\nu\tau}\bar\sigma_{\mu\rho} - \delta_{\rho\tau}\bar\sigma_{\mu\nu}\,. \end{aligned} \tag{B.18}$$

Using (B.9) and the identities (B.18), we can directly compute the field strength

$$\begin{aligned} F_{\mu\nu} = {} & [\alpha\sigma_{\nu\sigma}\partial_\mu\partial_\sigma \ln\rho + 2\alpha^2\sigma_{\mu\sigma}\partial_\nu\ln\rho\partial_\sigma\ln\rho - (\mu\leftrightarrow\nu)] \\ & -2\alpha^2\sigma_{\mu\nu}\partial_\sigma\ln\rho\partial^\sigma\ln\rho\,. \end{aligned} \tag{B.19}$$

Since we work with the flat Euclidean space, we will not be careful with the upper or lower indices. The dual of $F_{\mu\nu}$ is given by, with the help of (B.18),

$$\begin{aligned}(*F)_{\mu\nu} &= \sigma_{\nu\sigma}(\alpha\partial_\sigma\partial_\mu \ln\rho - 2\alpha^2\partial_\sigma \ln\rho\partial_\mu \ln\rho) - (\mu \leftrightarrow \nu) \\ &\quad +\sigma_{\mu\nu}\alpha\partial_\sigma \ln\rho\partial^\sigma \ln\rho \,. \end{aligned} \tag{B.20}$$

The self-duality condition yields

$$\partial_\mu\partial^\mu \ln\rho + 2\alpha\partial_\mu \ln\rho\partial^\mu \ln\rho = 0\,. \tag{B.21}$$

We can rescale the ρ to $\rho^{\frac{1}{2\alpha}}$, which is effectively the same as setting $\alpha = \frac{1}{2}$, to obtain

$$\frac{\Box\rho}{\rho} = \frac{\partial_\mu\partial^\mu\rho}{\rho} = 0\,. \tag{B.22}$$

A solution for ρ is given by

$$\rho = 1 + \sum_{i=1}^{n} \frac{\lambda_i}{(x-x_i)^2} \tag{B.23}$$

which gives

$$A_\mu = \frac{1}{2}\sigma_{\mu\nu}\partial^\nu \ln\left[1 + \sum_{i=1}^{n} \frac{\lambda_i}{(x-x_i)^2}\right]. \tag{B.24}$$

This solution describes the field configuration with n instantons. The parameters λ_i and x_i can be identified with the scale and position of the instantons. Notice that this solution is singular at $x = x_i$. The solution is called the instanton solution in a singular gauge. We can make a singular gauge transformation to obtain a solution which is regular at $x = x_i$. The latter solution is in the so-called regular gauge. Although it is regular at $x = x_i$, it is indeed singular at $x = \infty$. We now discuss one simple example namely a single instanton solution at $x = a$

$$A_\mu = \frac{1}{2}\sigma_{\mu\nu}\partial^\nu \ln\left[1 + \frac{\lambda}{(x-a)^2}\right] = \frac{\sigma_{\mu\nu}\lambda^2(x-a)^\nu}{(x-a)^2((x-a)^2+\lambda^2)}. \tag{B.25}$$

With a gauge transformation $U = \frac{i\bar{\sigma}_\mu x^\mu}{\sqrt{x^2}}$, we find a regular gauge instanton solution

$$\tilde{A}_\mu = -\frac{\bar{\sigma}_{\mu\nu}(x-a)^\nu}{(x-a)^2+\lambda^2}\,. \tag{B.26}$$

As discussed in [114], we can also remove the singularities at $x = x_i$ by adding a harmonic function $\Box\ln(x - x_i)$ to the ρ solution for each i. This is precisely what we have done in chapter 5. The antiself-dual instanton solution can be found by repeating the above procedure and replacing $\sigma_{\mu\nu}$ by $\bar{\sigma}_{\mu\nu}$. We will not go through the detail here, see [115].

Instantons are characterized by the scales λ_i and positions x_i as well as gauge gauge orientations. For $SU(2)$ instantons, there are three gauge parameters corresponding three $SU(2)$ transformations. These transformations are global symmetries which are left after fixing gauge symmetries. The actions of these symmetries are given by

$$A_\mu \to U^{-1}(SU(2))A_\mu U(SU(2)). \tag{B.27}$$

We can embed the $SU(2)$ in a larger gauge group to obtain instantons of the new gauge group. We begin with $SU(N)$, $N > 2$. We can embed $SU(2)$ in the 2×2 submatrix of the $N \times N$ defining representation of $SU(N)$. This is clearly not the only possible embedding. Instead of embedding the 2×2 fundamental representation of $SU(2)$ in the fundamental representation of $SU(N)$, we can embed a spin j representation of $SU(2)$ inside $SU(N)$ provided that $2j+1 \leq N$. After embedding, we can act on the resulting solution with an $SU(N)$ transformation as in (B.27), with $U(SU(2))$ being replaced by $U(SU(N))$, and find a new solution. Of course, not all transformations give a new solution. To embed $SU(2)$, we break $SU(N)$ down to $SU(N-2)\times U(1)\times SU(2)$, so there is a stability group $SU(N-2)\times U(1)$ commuting with the embedded $SU(2)$. We must then remove this redundancies. So, the $SU(N)$ instanton is described by $4N-5+5 = 4N$ parameters which are sometimes called collective coordinates. The $4N-5$ is the dimension of the coset space $\frac{SU(N)}{SU(N-2)\times U(1)}$, and we have added the scale and position of the instanton which constitute 5 parameters.

Recall that

$$A_\mu = \frac{1}{2}\sigma_{\mu\nu}\partial^\nu \ln\rho = \bar{\eta}^a_{\mu\nu}\left(\frac{i\sigma_a}{2}\right)\partial^\nu \ln\rho, \tag{B.28}$$

we can recognize the appearance of $SU(2)$ generators $\frac{i\sigma^a}{2}$. The $SU(N)$ solution is readily given by

$$A_\mu = \bar{\eta}^a_{\mu\nu}T_a\partial^\nu \ln\rho \tag{B.29}$$

where T_a are generators of $SU(2)$ embedded in $SU(N)$ with a certain $SU(2)$ representation.

For $SO(N)$ instantons, we can embed $SU(2)$ instanton in one of the two $SU(2)$'s inside the $SO(4) \sim SU(2)\times SU(2) \subset SO(N)$ with the same procedure as described above. The solution is characterized by $4N-8$ parameters since the stability group in this case is $SO(N-4)\times SU(2)$ with the $SU(2)$ being another $SU(2)$ factor in $SO(4)$. Note also that the two $SU(2)$'s are generated by $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$.

For gauge group $USp(2N)$, we can embed $USp(2) \sim SU(2)$ instanton in $USp(2N)$. The stability group is $USp(2N-2)$, so the number of collective coordinates is given by $4(N+1)$. For the n instantons solution, the number of collective coordinates is given by the above mentioned result multiplied by n.

We now review an instanton solution obtained in [116] and extensively used in chapter 5. The solution describes instantons of an arbitrary semisimple gauge group G. The solution we will rederive covers all the cases mentioned above. We begin with Yang-Mills equation

$$D_\mu F^{a\mu\nu} = 0 \qquad \text{or} \qquad \partial_\mu F^{a\mu\nu} = -f^{abc} A^b_\mu F^{c\mu\nu} \,. \tag{B.30}$$

Choosing the Fock-Schwinger gauge

$$x_\mu A^{a\mu} = 0, \tag{B.31}$$

we can write the gauge potential as

$$A^a_\mu = -\int_0^1 d\alpha \alpha F^a_{\mu\nu}(\alpha x) x^\nu \,. \tag{B.32}$$

The field strength computed from (B.32) must be the same as $F^{a\mu\nu}$, so we impose the condition

$$F^a_{\mu\nu}[A(F)] = F^a_{\mu\nu} \,. \tag{B.33}$$

The Yang-Mills equation becomes

$$\partial_\mu F^{a\mu\nu} = f^{abc} F^{c\mu\nu} \int_0^1 d\alpha \alpha F^b_{\mu\lambda}(\alpha x) x^\lambda \,. \tag{B.34}$$

Follow [116], we take an ansatz for $F^a_{\mu\nu}$ to be

$$F^a_{\mu\nu} = G^a_{\mu\nu} \psi(x^2) \,. \tag{B.35}$$

$G^a_{\mu\nu}$ is a constant antisymmetric in μ and ν. It is also self-dual or antiself-dual and can be expanded in terms of the 't Hooft tensors as

$$G^a_{\mu\nu} = G^a_I \eta^I_{\mu\nu} + \bar{G}^a_I \bar{\eta}^I_{\mu\nu} \,. \tag{B.36}$$

We now note the important information and identities of the 't Hooft tensors before proceed further. As mentioned above, they generate $SU(2) \times SU(2) \sim SO(4)$ which is the Euclidean Lorentz group in four dimensions. We can write

$$\bar{\eta}^a = \frac{1}{2}(J_a + K_a), \qquad \eta^a = \frac{1}{2}(J_a - K_a) \tag{B.37}$$

where J_a and K_a correspond to rotation and boost generators, respectively. These generators can be identified with the Lorentz generators $L_{\mu\nu}$ as $J_a = \frac{1}{2}\epsilon_{abc} L_{bc}$ and $K_a = L_{a4}$, $a, b, c = 1, 2, 3$. Furthermore, they satisfy

$$[J_a, J_b] = -\epsilon_{abc} J_c, \qquad [K_a, K_b] = -\epsilon_{abc} J_c, \qquad [J_a, K_b] = -\epsilon_{abc} K_c \,. \tag{B.38}$$

Apart from being self-dual and antiself-dual, the $\eta^a_{\mu\nu}$ and $\bar{\eta}^a_{\mu\nu}$ also satisfy a number of identities which are useful for our discussion later. These identities are given by

$$\begin{aligned} \eta^a_{\mu\nu}\bar{\eta}^{b\mu\nu} &= 0, \qquad [\eta_a,\eta_b] = -\epsilon_{abc}\eta_c, \qquad [\bar{\eta}_a,\bar{\eta}_b] = -\epsilon_{abc}\bar{\eta}_c, \\ \{\eta_a,\eta_b\} &= -2\delta_{ab}, \qquad \{\bar{\eta}_a,\bar{\eta}_b\} = -2\delta_{ab}, \qquad [\eta_a,\bar{\eta}_b] = 0\,. \end{aligned} \tag{B.39}$$

The identities involving various contractions of η^a and $\bar{\eta}^a$ are given by

$$\begin{aligned} \epsilon_{abc}\eta_{b\mu\nu}\eta_{c\rho\sigma} &= \delta_{\mu\rho}\eta_{a\nu\sigma} + \delta_{\nu\sigma}\eta_{a\mu\rho} - \delta_{\mu\sigma}\eta_{a\nu\rho} - \delta_{\nu\rho}\eta_{a\mu\sigma}, \\ \eta_{a\mu\nu}\eta_{a\rho\sigma} &= \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \epsilon_{\mu\nu\rho\sigma}, \\ \eta_{a\mu\rho}\eta^{\mu}_{b\sigma} &= \delta_{ab}\delta_{\rho\sigma} + \epsilon_{abc}\eta_{c\rho\sigma}, \\ \epsilon_{\mu\nu\rho\tau}\eta_{a\sigma}{}^{\tau} &= \delta_{\sigma\mu}\eta_{a\nu\rho} + \delta_{\sigma\rho}\eta_{a\mu\nu} - \delta_{\sigma\nu}\eta_{a\mu\rho}, \\ \eta_{a\mu\nu}\eta^{\mu\nu}_{a} &= 12, \qquad \eta_{a\mu\nu}\eta^{\mu\nu}_{b} = 4\delta_{ab}, \qquad \eta_{a\mu\rho}\eta^{\mu}_{a\sigma} = 3\delta_{\rho\sigma}, \end{aligned} \tag{B.40}$$

together with another copy for $\bar{\eta}^a$ except the ones involving $\epsilon_{\mu\nu\rho\sigma}$

$$\begin{aligned} \bar{\eta}_{a\mu\nu}\bar{\eta}_{b\rho\sigma} &= \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma}, \\ \epsilon_{\mu\nu\rho\sigma}\bar{\eta}^{\sigma}_{a\tau} &= -\delta_{\sigma\mu}\bar{\eta}_{a\nu\rho} - \delta_{\sigma\rho}\bar{\eta}_{a\mu\nu} + \delta_{\sigma\nu}\bar{\eta}_{a\mu\rho}\,. \end{aligned} \tag{B.41}$$

We now come back to the instanton solution. Inserting the ansatz (B.35) into the gauge field (B.32), we find

$$A^a_\mu = -G^a_{\mu\nu}x^\nu\Phi(x^2) \tag{B.42}$$

where $\Phi(x^2) \equiv \frac{1}{2x^2}\int_0^{x^2} du\psi(u)$. In the last step, we have changed the integration variable from α to $u \equiv \alpha^2 x^2$. The Yang-Mills equation becomes

$$f^{abc}G^c_{\mu\nu}G^b_{\nu\sigma}x^\sigma\Phi(x^2)\psi(x^2) - 2G^a_{\mu\nu}x^\nu\psi'(x^2) = 0 \tag{B.43}$$

where $'$ denotes the derivative with respect to x^2. As shown in [116], the above equation is satisfied provided that we choose

$$f^{abc}G^c_{\mu\lambda}G^{b\lambda}{}_{\nu} = 2G^a_{\mu\nu}, \tag{B.44}$$

$$\psi'(x^2) = \Phi(x^2)\psi(x^2)\,. \tag{B.45}$$

The constraint (B.33) can be written as

$$\begin{aligned} G^a_{\mu\nu}\psi(x^2) &= 2\Phi'(x^2)(G^a_{\mu\lambda}x_\nu - G^a_{\nu\lambda}x_\mu)x^\lambda + 2G^a_{\mu\nu}\Phi(x^2) \\ &\quad + f^{abc}G^b_{\mu\lambda}G^c_{\nu\rho}x^\lambda x^\rho\Phi(x^2)^2\,. \end{aligned} \tag{B.46}$$

Using the identities for η^a, we can rewrite equation (B.44) as

$$G^a_I = \frac{1}{2}\epsilon_{IJK}f^{abc}G^b_J G^c_K \tag{B.47}$$

where we have specified to the case of self-dual instantons by setting $\bar{G}^a_I = 0$. We then separate equation (B.46) into self-dual and antiself-dual parts

$$G^a_I(2\Phi + x^2\Phi') + \frac{1}{4}\epsilon_{IJK} f^{abc} G^b_J G^c_K x^2 \Phi^2 \;=\; G^a_I \psi, \tag{B.48}$$

$$x^\nu x^\lambda \bar{\eta}^I_{\nu\mu} \eta^K_{\mu\lambda} \left(G^a_K \Phi' - \frac{1}{4}\epsilon_{KML} f^{abc} G^b_M G^c_L \Phi^2 \right) \;=\; 0\,. \tag{B.49}$$

With the help of (B.47), these two equations can be reduced to

$$x^2\Phi^2 + 2\Phi + \frac{1}{2}x^2\Phi^2 \;=\; \psi, \tag{B.50}$$

$$\Phi' - \frac{1}{2}\Phi^2 \;=\; 0\,. \tag{B.51}$$

The corresponding solutions are given by

$$\Phi \;=\; -\frac{2}{x^2+\lambda^2}, \tag{B.52}$$

$$\psi \;=\; -\frac{4\lambda^2}{(x^2+\lambda^2)^2} \tag{B.53}$$

with a constant λ. Introducing G_I defined by $G_I = G^a_I T^a$ and using $\left[T^a, T^b\right] = f^{abc} T^c$, we can write (B.47) as

$$[G_I, G_J] = \epsilon_{IJK} G_K \tag{B.54}$$

which is the $SU(2)$ algebra. Collecting everything together, we find

$$A^a_\mu = G^a_{\mu\nu} \frac{2x^\nu}{x^2+\lambda^2}, \qquad F^a_{\mu\nu} = -G^a_{\mu\nu} \frac{4\lambda}{(x^2+\lambda^2)^2}\,. \tag{B.55}$$

Notice that A^a_μ can be written as $A^a_\mu = G^a_{\mu\nu}\partial^\nu \ln(x^2+\lambda^2)$. This is precisely the form of a single instanton at the origin in the regular gauge. From the $SU(2)$ instanton solution discussed before, we learn that the corresponding solution in the singular gauge is given by

$$A^a_\mu = G^a_{\mu\nu}\partial^\nu \ln\left(1 + \frac{\lambda^2}{x^2}\right). \tag{B.56}$$

We can generalize this solution by replacing the the harmonic function by the multi-center function. The multi-instanton solution in the singular gauge is then given by

$$A^a_\mu = G^a_{\mu\nu}\partial^\nu \ln\left[1 + \sum_{i=1}^{n} \frac{\lambda_i^2}{(x-x_i)^2}\right]. \tag{B.57}$$

It is now straightforward to compute the field strength and the winding number. One useful relation is given by [114]

$$F^a_{\mu\nu} F^{a\mu\nu} = -\Box\Box \ln\rho \tag{B.58}$$

for $SU(2)$ instantons with $A^a_\mu = \frac{1}{2}\eta^a_{\mu\nu}\partial^\nu \ln\rho$, $\rho = 1 + \sum_{i=1}^n \frac{\lambda_i}{(x-x_i)^2}$. We will derive this relation in the next section in the case of instantons on ALE spaces. In order to avoid the repeatition, we simply state the result, which is a special case for instantons on flat spaces, and give the derivation only once in the next section. The winding number for the $SU(2)$ instantons is given by [114]

$$N = -\frac{1}{16\pi^2}\int d^4x \Box\Box \ln\rho = n\,. \tag{B.59}$$

For the present G instanton solution, there is an additional group theory factor coming from G^a_I. The winding number in this case is given by [116]

$$N = \frac{1}{2}G^a_I G^a_I n = \frac{2}{3}c(G)d(G)n\,. \tag{B.60}$$

$c(G)$ is the value of the $SU(2)$ quadratic Casimir operator of G_I which generate $SU(2)$ as shown above. As the representation of $SU(2)$ is characterized by the value of "spin", j, we find that $c(G) = j(j+1)$ if G_I generate the $SU(2)$ algebra in the spin j representation, and $d(G)$ is the trace of the unit matrix in the spin j representation of $SU(2)$. We briefly comment on the value of $d(G)$ for one example in which $G = SU(N)$. We can have instantons provided that $2j+1 \leq N$, but the $d(G)$ does not need to be the same as N. If $d(G) = N$, the instanton is called the maximal spin instanton. The corresponding winding number is $\frac{1}{6}N(N^2-1)n$. We end this section by giving the relation analogous to (B.58) for G instanton

$$F^a_{\mu\nu}F^{a\mu\nu} = -\frac{2}{3}c(G)d(G)\Box\Box \ln\rho\,. \tag{B.61}$$

B.2 Gravitational instantons

Gravitational instantons play an important role in the path integral approach to quantum gravity [151]. This is similar to the Yang-Mills instantons in gauge theories in the sense that they are classical non singular solutions of Einstein equations with finite action, and some of them have self-dual curvature tensors which is the analogue of self-dual gauge field strengthes. There are a number of gravitational instantons discovered until now. However, we will not discuss all of them since not all of these solutions are used or have direct contact with the works presented in this thesis. The review in this appendix is not aimed to be complete. For more detailed information, the reader is referred to [152, 69, 153] on which this appendix is based. We begin with a discussion of the ALE spaces of type A_k and then move to the Yang-Mills instanton solutions on ALE spaces. The metric of the A_k ALE space has been discovered long ago in [69]. For the D and E types, it is more difficult to find the explicit form of the corresponding metric. Nevertheless, recently, the metric of type D_k gravitational instantons has been found in [154].

Gravitational instantons are complete non-singular positive definite Einstein manifolds. They correspond to non-singular stationary points of the action functional used to compute the path integral. In this definition, gravitational instantons are not necessarily described by (anti) self-dual Riemann tensor. For those with non self-duality, the interpretation in term of tunneling geometries is possible in much the same way as Yang-Mills instantons can be interpreted as tunneling between Yang-Mills vacua. These gravitational instantons are said to have a Lorentzian section. As noted above, four dimensional Lorentzian spaces do not admit second rank (anti) self-dual tensors due to the Hodge duality operation being squared to minus one. Therefore, gravitational instantons with (anti) self-dual Riemann tensor do not have a Lorentzian section and have nothing to do with tunneling geometries. The latter is the class of solutions we are interested in.

The positive action conjecture does not allow non-trivial asymptotically Euclidean spaces but does allow asymptotically locally Euclidean (ALE) spaces as gravitational instantons. These ALE spaces have self-dual Riemann tensor as well as self-dual spin connections. Since a four dimensional hyper-Kahler manifold has self dual or anti-self dual Riemann tensor, a gravitational instanton can be described by a hyper-Kahler manifold. The ALE gravitational instanton can be constructed from a higher dimensional hyper-Kahler manifold by hyper-Kahler quotient [155]. However, we will not review this construction here. The interested reader is referred to [155] or [70]. (Anti) Self-duality of the Riemann tensor, ${R^\rho}_{\sigma\mu\nu} = \pm\frac{1}{2}{\epsilon^{\alpha\beta}}_{\mu\nu}{R^\rho}_{\sigma\alpha\beta}$, implies the vanishing of the Ricci tensor

$$\begin{aligned} R_{\sigma\nu} &= {R^\rho}_{\sigma\rho\nu} = \pm\frac{1}{2}{\epsilon^{\alpha\beta}}_{\rho\nu}{R^\rho}_{\sigma\alpha\beta} \\ &= \mp\frac{1}{2}{\epsilon^{\rho\alpha\beta}}_{\nu}R_{\sigma\rho\alpha\beta} = 0 \end{aligned} \tag{B.62}$$

where we have used the property of the Riemann tensor $R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}$ and $R_{\mu[\nu\rho\sigma]} = 0$. So, ALE spaces are Ricci flat spaces and are solutions of the vacuum Einstien equation.

Furthermore, in four dimensions, self-duality implies that the space has $SU(2)$ holonomy and admitting Killing spinors satisfying $\nabla_a\eta = 0$. These are given by one chiral spinors which do not see any curvature since their integrability condition is automatically satisfied $[\nabla_a, \nabla_b]\,\eta = \frac{1}{4}R_{abcd}\Gamma^{cd}\eta$. This makes ALE spaces an interesting supersymmetric background, and they are used in studying dualities in string theory, see for example [156]. More information on gravitational instantons can be found in [157] and [158].

We now review the ALE spaces of type A_k whose metric is given by [69]

$$ds^2 = V^{-1}(d\tau + \vec{\omega}.d\vec{x})^2 + V d\vec{x}.d\vec{x} \tag{B.63}$$

where

$$\vec{\nabla}\times\vec{\omega} = \vec{\nabla}V, \qquad \text{and} \qquad V = V_0 + \sum_{i=1}^{N}\frac{1}{|\vec{x}-\vec{x}_i|}\,. \tag{B.64}$$

The ALE instantons are described by $V_0 = 0$. For $V_0 \neq 0$, this metric describes the multi-center Taub-NUT solution. This solution is also a blown up of the orbifold $\mathbb{R}^4/\mathbb{Z}_N$. The solution has topological invariant quantities $\chi = N$ and $\tau = N - 1$. The Euler number χ and the signature or the index τ are defined by [153]

$$\chi = \frac{1}{128\pi^2}\int_M \epsilon^{\mu\nu}{}_{\rho\sigma} R_{\mu\nu\alpha\beta} R^{\rho\sigma\lambda\kappa} \epsilon^{\alpha\beta}{}_{\lambda\kappa}\sqrt{g}d^4x, \tag{B.65}$$

$$\tau = \frac{1}{96\pi^2}\int_M R_{\mu\nu\rho\sigma} R^{\mu\nu}{}_{\alpha\beta}\epsilon^{\rho\sigma\alpha\beta}\,. \tag{B.66}$$

As discussed in [153], the additional boundary terms are necessary for non-compact manifolds. We refer the reader to [153] for their explicit forms. As shown in the main text, at $|\vec{x}| \to \infty$ and near one of the singularity $\vec{x} \to \vec{x}_i$, the geometry is given by $\mathbb{R}^4/\mathbb{Z}_N$ and $\mathbb{R}^4$, respectively.

The next issue we will address here is the Yang-Mills instanton solutions on the ALE spaces. The construction analogous to the ADHM construction on $\mathbb{R}^4$ has been given in [159] and sometimes called KN construction. We have not used the full construction here, so we only discuss some explicit solutions in special cases. It is enough for our propose to review the $SU(2)$ instanton solution on ALE spaces given in [71], see also [70]. Before going to the detailed discussion of the instanton solution, we repeat the vielbein and its inverse of the metric (B.63) for later conveniences

$$e^0 = V^{-\frac{1}{2}}(d\tau + \vec{\omega}.d\vec{x}), \qquad e^l = V^{\frac{1}{2}}dx^l, \tag{B.67}$$

$$E_0 = V^{\frac{1}{2}}\frac{\partial}{\partial\tau}, \qquad E_l = V^{-\frac{1}{2}}\left(\frac{\partial}{\partial x^l} - \omega_l\frac{\partial}{\partial\tau}\right). \tag{B.68}$$

The ansatz for the gauge fields is given by

$$A_0 = \frac{1}{2}\vec{G}.\vec{\sigma}, \qquad \vec{A} = \frac{1}{2}[\vec{G}.\vec{\sigma}\vec{\omega} - V(\vec{G}\times\vec{\sigma})]. \tag{B.69}$$

As shown in [71], the self-duality condition can be satisfied by imposing the conditions

$$\vec{\nabla}\times(V\vec{G}) = 0, \qquad \vec{\nabla}.(V\vec{G}) - (V\vec{G}).(V\vec{G}) = 0\,. \tag{B.70}$$

The solution for $\vec{G}$ is of the form

$$\vec{G} = -\frac{1}{V}\vec{\nabla}\ln H \tag{B.71}$$

with a harmonic function H satisfying $\vec{\nabla}^2 H = 0$. For $H = V$, the gauge fields are the same as the spin connections of the metric (B.63) which are given by

$$\omega^{\hat{\tau}}{}_{\hat{i}} = -\frac{1}{2}V^{-\frac{3}{2}}[\partial_i V e^{\hat{\tau}} + (\partial_j\omega_i - \partial_i\omega_j)e^{\hat{j}}], \tag{B.72}$$

$$\omega^{\hat{i}}{}_{\hat{j}} = \frac{1}{2}V^{-\frac{3}{2}}[(\partial_j\omega_i - \partial_i\omega_j)e^{\hat{\tau}} + (\delta^i_k\partial_j V - \delta^j_k\partial_i V)e^{\hat{k}}]. \tag{B.73}$$

The spin connections are (anti) self dual as can be seen from the relation $\vec{\nabla} V = \pm \vec{V} \times \vec{\omega}$.

The solution for H is given by

$$H = H_0 + \sum_{j=1}^{n} \frac{\lambda_j}{|\vec{x} - \vec{y}_j|} \,. \tag{B.74}$$

At this stage, the $\vec{y}_j$'s do not need to coincide with the ALE centers, $\vec{x}_i$ in V. It has been argued in [71] that the finite action condition requires that the centers of H which do not coincide with those of V have to be omitted since these give rise to infinite values of the Yang-Mills action. This argument implies that $n \leq N$.

In the ansatz given above, the function H is τ independent. We can generalize this ansatz to the form [70]

$$A^I_\alpha dx^\alpha = -\eta^I_{ab} e^a E^b \ln H \tag{B.75}$$

in which H is now a function of both $\vec{x}$ and τ. By using the identities for η^a's given in the previous section, we can straightforwardly compute the field strength

$$\begin{aligned} F^I_{\mu\nu} &= \nabla_\mu A^I_\nu - \nabla_\nu A^I_\mu + \epsilon^{IJK} A^J_\mu A^K_\nu \\ &= \eta^I_{ab} E^{b\lambda} (e^a_\mu \nabla_\nu \nabla_\lambda \ln H - e^a_\mu \partial_\lambda \ln H \partial_\nu \ln H) - (\mu \leftrightarrow \nu) \,. \end{aligned} \tag{B.76}$$

Using $\nabla_a \nabla_b H = \nabla_b \nabla_a H$, we can show that the self-duality condition requires

$$\nabla_a \nabla^a H = 0 \,. \tag{B.77}$$

We emphasize here that ∇ is the ALE covariant derivative while $\vec{\nabla}$ is the gradient operator in $\mathbb{R}^3$ with the metric $d\vec{x}.d\vec{x}$. The solution for H is of the form $H = H_0 + \sum_{i=1}^{n} G(x, x_i)$ where $G(x, x_i)$ is the scalar Green function on ALE spaces given in [124]. Since the explicit form of $G(x, x_i)$ is essentially the same as that given in chapter 6, we will not repeat it here. We also point out that in this generalized solution with τ dependence, the $\vec{x}_i$'s need not coincide with the ALE centers, so the constraint $n \leq N$ can be relaxed.

We now end this section with the derivation of the identity $F^I_{\mu\nu} F^{I\mu\nu} = -4\Box\Box \ln H$. From (B.76) and $\eta^I_{ab}\eta^I_{cd}$ identity given in the previous section, we find

$$\begin{aligned} F^a_{\mu\nu} F^{a\mu\nu} &= 2\eta^I_{ab}\eta^I_{cd} E^{b\lambda} E^{d\rho} e^a_\mu e^{c\mu} (\nabla_\nu \nabla_\lambda \ln H - \partial_\lambda \ln H \partial_\nu \ln H) \times \\ &\quad (\nabla^\nu \nabla_\rho \ln H - \partial_\rho \ln H \partial^\nu \ln H) - 2\eta^I_{ab}\eta^I_{cd} E^{b\lambda} E^{d\rho} e^a_\mu e^{c\nu} \times \\ &\quad (\nabla_\nu \nabla_\rho \ln H - \partial_\lambda \ln H \partial_\nu \ln H)(\nabla^\mu \nabla_\rho \ln H - \partial_\rho \ln H \partial^\mu \ln H) \\ &= -4\Box\Box \ln H \end{aligned} \tag{B.78}$$

where

$$\Box\Box \ln H = -2\nabla_\mu \nabla_\nu \ln H \nabla^\mu \nabla^\nu \ln H + 4\partial_\mu \ln H \partial_\nu \ln H \nabla^\mu \nabla^\nu \ln H \,. \tag{B.79}$$

In order to show the above identity, we recall that H is harmonic on ALE spaces, $\Box H = \nabla_\mu \nabla^\mu H = 0$, and ALE spaces are Ricci flat. The latter is necessary in order to interchange the covariant derivatives in $\nabla_a \nabla_b \nabla^b \ln H = \nabla_b \nabla_a \nabla^b \ln H$. Notice that for the flat space $\mathbb{R}^4$, $e^a_\mu = \delta^a_\mu$, $E^\mu_a = \delta^\mu_a$ and $\nabla \to \partial$, we recover the result of [114] which has also been used in chapter 5. Notice that there is a factor of 4 in (B.78) compared to the analogous relation in chapter 5. This is because we have not put $\frac{1}{2}$ in the ansatz (B.75).

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