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QCD AT FINITE TEMPERATURE

CANDIDATO:

Dott. Zhao Wanyun

RELATORE:

Prof. R. Jengo

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QCD AT FINITE TEMPERATURE

ZHAO WANYUN

Institute of Theoretical Physics, Academia Sinica,

Beijing, China

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ABSTRACT

QCD at finite temperature is discussed in this thesis. We have especially studied high temperature inter-quark potential within the framework of perturbation theory. The result is that quarks are deconfined at high temperature.

CONTENTS

CHAPTER I - A SHORT REVIEW OF QCD AT FINITE TEMPERATURE

1.1 Introduction

It has been widely believed that strong interactions are described by a non-Abelian $SU(3)_c$ gauge theory of coloured quarks and gluons which are confined in color singlet hadronic bound states [1.1]. The theory possesses the feature of the behaviour of quasi-free quarks at short distances. The theory called QCD seems to be a good theory for strong interactions.

QCD at zero temperature also appears to be consistent with much of the successful phenomenology of strong interactions, for example, the chiral symmetry, the bag and string models of hadrons, and the notion of confinement of color and an understanding of its dynamical basis using lattice approximations or semiclassical approximations (instanton).

Since we possess a good approximate theory, QCD, for the strong interactions at zero temperature, it is natural to ask what happens when a system of elementary particles described by QCD is heated and to explore the properties of hadronic matter in wider environments, in particular at high temperature or high baryon density. There are many reasons for such an investigation, and they are both practical and purely theoretical. Firstly, we might hope to find or to produce in nature such extreme conditions and thereby test the theory in a new domain. There are places where one may look for the effects of high temperature or large baryon density on the structure of hadronic matter. For example, the interior of neutron stars has very large baryon density, $\rho \gtrsim 10^{15}$ g/cm^3, ($\rho \sim 10^{11} - 10^{14}$ g/cm^3 for nuclei, $\rho \sim 10^{15}$ g/cm^3 for neutron), and the early stages of the universe evolution, i.e. in about 10^{-5} sec. after the big bang when the universe as a whole was at high temperature comparable to hadron rest energy, $T \gtrsim 200$ MeV (i.e. $T \gtrsim 10^{12}$ °K) [1.2]. Another is that the collision of heavy ions at very high energy, $E \sim 10$-50 times that of nuclear matter, might produce the states of high temperature and density [1.3]. As the primary cosmic ray beam provides us with heavy nuclei exceeding 10^6 GeV primary energy, cosmic ray experiments can possibly provide us with a first glimpse of the states of matter. Secondly, there are purely theoretical reasons for exploring the thermodynamics of hadrons. One hopes to study the properties of hadronic matter in a set of conditions as wide as possible and then one could try to extend the theory to explain phenomena which are far removed from the observations that originally motivated the theory. The studies can test the consistency and reasonableness of QCD, and increase confidence in it.

Soon after the Salam-Weinberg model of electroweak interactions was presented, Kirzhnits, Linde, Weinberg, Dolan and Jackiw [1.4] have studied the issue of the spontaneous symmetry breaking at finite temperature. They show that for the class of theories of symmetries explicitly broken by scalar fields, symmetry is restored above some critical temperature T_c, when the temperature is low the symmetry is spontaneously broken, and for these symmetries broken dynamically, they remain broken at high temperature. The critical temperature T_c can be computed in terms of the perturbative methods developed by Bernard, Weinberg, and Dolan and Jackiw, for example, in terms of a functional-diagrammatic evaluation of the effective potential and the effective mass.

For QCD, we want to ask some questions:

1) Does confinement persist at high temperature?
2) If there is a phase transition to a non-confined phase, what is the nature of the high-temperature phase?
3) How is the critical temperature of the transition from a confinement phase to a non-confinement phase calculated?
4) What collective excitations exist in the high-temperature phase?
5) Is there a mass gap?
6) How does one evaluate the quark interaction potential at finite temperature?
7) Could the equation of state of hadronic matter at finite temperature be calculated?

One of the first attempts to explore the properties of hadrons at high temperature was carried out by Hagedorn in 1965 [1.5]. He found that the density of hadronic states increases exponentially with energy, and argued that this implies the existence of a limiting temperature, above which hadronic matter cannot exist. His results were based on a statistical boot strap hypothesis. Later, Cabbibo and Parisi [1.6] argued, in the framework of quark models of hadrons where quarks are permanently confined in hadrons, that the exponentially increasing density of state simply means that above some critical temperature quarks are liberated.

Another argument for a quark-liberating phase transition is that perturbative expansion can be used for the phase. Using renormalization group equation, Collins and Perry [1.2] showed that at sufficiently high density or temperature, physical observables could be expanded in powers of an effective coupling constant which becomes very small. The calculation of high-temperature interquark potential also implies, within the framework of the perturbation theory, $V(R) \sim \frac{\beta}{R^2} e^{-2m_{el}R}$ as $R \rightarrow \infty$ [1.7], then the quarks are deconfinement. The (color) electric screening length $(m_{el})^{-1}$ which is determined by the timelike component of the gluon propagator, is perturbatively calculable and is of order $(gT)^{-1}$, where g is the running coupling constant ($g^2 \sim (\ln \frac{T}{\mu})^{-1}$).

Therefore, there is a phase transition from quark-confinement to quark-liberation as the temperature increases. The nature of the high-temperature phase is that in this phase QCD definitely loses confinement and thermal excitations produce a plasma of quarks and gluons which screens all (color) electric flux, so that it is like a gas of quasi-free particles. This phase has a perturbatively computable (color) electric mass, m_{el}, and a perturbatively uncomputable mass gap (i.e. (color) magnetic mass of the gluon) of order (g^2T), which is developed by the spatial gauge field $A_i(x)$.

This critical temperature of the transition from a phase of hadronic matter to a phase of plasma of quarks and gluons cannot be computed by a perturbation theory. However, the critical temperature, T_c, can be calculated with use of Monte Carlo simulation methods of lattice gauge theory, and there are indications that $T_c \approx 200$ MeV and that for an SU(3) gauge theory the transition is of first order [1.8].

The ultraviolet divergences encountered in QCD at finite temperature are the same as those at zero temperature, no new temperature or density dependent infinities appear, and zero-temperature renormalization prescriptions suffice to eliminate all ultraviolet divergences. In the one-loop case, this has been explicitly shown by Weinberg [1.4] and for two loops, by Morley and Kisslinger [1.9]. A more general treatment has been presented by J.G. Taylor [1.10].

The infra-red behaviour of QCD at high temperature is controlled by the spacelike components of a gluon propagator which develops a (color) magnetic mass of order g^2T [1.11]. Since this smaller magnetic mass of order g^2T, higher order calculations of perturbation theory give increasingly infrared singularities. Beyond some order, perturbation theory at high-temperature QCD is found to be unreliable, for example, the interquark potential at high-temperature can be evaluated by means of perturbation theory only up to the order g^{10}. For high temperature, this infra-red behaviour of the theory is equivalent to that of a three-dimensional pure gauge theory with coupling g^2T [1.12]. Therefore, a reliable calculation for the magnetic mass requires a complete solution of the three-dimensional pure gauge theory.

Lattice gauge theories provide a model of quark-confinement. Polyakov and Susskind have investigated the lattice gauge theory at finite temperature, they [1.13] have given a convincing argument that the theory undergoes a phase transition to an unconfined phase as increasing the temperature. This important result illustrates how a confining theory can lose confinement. Mclerran and Svetitsky [1.14] have also obtained the same result of the quark-unconfinement at high-temperature by the Monte Carlo study of lattice gauge theory. The behaviour of the high-temperature interquark potential obtained by us in Chapter II and III implies that the quarks are liberated at high temperature. The result is consistent with the Monte Carlo study of lattice gauge theory at high-temperature.

In Sec. 1.2, we have given the Feynman rules for gauge theories at finite temperature, in order to study quantitatively the thermodynamic properties of these theories at finite temperature.

In Sec. 1.3, we have discussed the problem of the renormalization of the finite-temperature gauge theories, and the calculations have been explicitly performed through two loops.

Both phase transitions, confinement-deconfinement transition and chiral symmetry restoration, are discussed in Sec. 1.4. The values of the temperatures of these phase transitions, T_c and T_{ch}, for SU(2) and SU(3) gauge theories are given. We have also discussed the orders of these transitions and the relation between T_c and T_{ch}.

In Sec. 1.5, we have reviewed the perturbation theory in high-temperature QCD and given the results of perturbative expansions of some physical quantities such as free energy and interquark potential.

In the last section, we have simply reviewed the still open question of the infra-red behaviour encountered at high-temperature gauge field theories. There is a detailed discussion in Chapter IV.

1.2 The formalism of the finite-temperature gauge field theory

In this section we give the formalism of the finite-temperature gauge field theory. To study quantitatively the thermodynamic properties of a field theory at finite temperature, such as thermodynamic potential, critical temperature of phase transition, interquark potential at high temperature and so on, we need to know the Feynman rules for a field theory at finite temperature. Below we shall discuss the derivations of the Feynman rules for gauge theories at finite temperature. Before discussing the gauge theories, we first derive the Feynman rules for a non-gauge theory, just for the sake of simplicity.

1.2.1 Derivation of the functional formalism

We consider a non-gauge field theory described by a Hamiltonian density $\mathcal{H}(\pi, \varphi)$, where $\varphi(\vec{x}, t)$ is the Heisenberg-picture field operator and $\pi(\vec{x}, t)$ is its conjugate momentum. Then $\varphi(\vec{x}, 0)$ is the Schrödinger-picture field operator. Let $|\varphi_0\rangle$ and $|\varphi_1\rangle$ be eigenstates of $\varphi(\vec{x}, 0)$ with eigenvalues $\varphi_0(\vec{x})$ and $\varphi_1(\vec{x})$ respectively. Thus

$$\begin{aligned} \varphi(\vec{x},0)\ |\varphi_0\rangle &= \varphi_0(\vec{x})\,|\varphi_0\rangle\ , \\ \varphi(\vec{x},0)\ |\varphi_1\rangle &= \varphi_1(\vec{x})\,|\varphi_1\rangle\ . \end{aligned} \tag{1.2.1}$$

Then the Feynman functional formula in Hamiltonian form [1.15] gives the transition amplitude for going from $|\varphi_0\rangle$ at t = 0 to $|\varphi_1\rangle$ at $t = t_1$:

$$\langle\varphi_1|\,e^{-iHt_1}|\varphi_0\rangle = N\int[d\pi][d\varphi]\,exp\left\{i\int_0^{t_1}dt\int d^3x\left[\pi\dot{\varphi}-\mathcal{H}(\pi,\varphi)\right]\right\}\ , \tag{1.2.2}$$

where the integral over classical fields, $\int[d\varphi]$, runs over all possible configurations that start at $\varphi_0(\vec{x})$ at t = 0 and go to $\varphi_1(\vec{x})$ at t = t_1. The integral over momenta, $\int[d\pi]$, is unrestricted. $\dot{\varphi}(\vec{x}, t)$ is defined to be $\partial\varphi(\vec{x}, t)/\partial t$. N is a constant normalizing factor. The momentum integration is done before the field integration.

Now we introduce the temperature, let $\beta = it_1$, where β is the inverse of the temperature. In the integrand of the exponent of eq. (1.2.2) we make the variable change, it = τ. Note that $\dot{\varphi} = i\dfrac{\partial\varphi}{\partial\tau}$. We have

$$\langle\varphi_1|e^{-\beta H}|\varphi_0\rangle = N\int[d\pi][d\varphi]\,exp\left\{\int_0^\beta d\tau\int d^3\vec{x}\left[i\pi\dot{\varphi} - \mathcal{H}(\pi,\varphi)\right]\right\}, \tag{1.2.3}$$

where $\dot{\varphi}$ is now taken to mean $\dfrac{\partial\varphi}{\partial\tau}$.

In order to find the partition function Z = $Tre^{-\beta H}$, we just allow the $\int[d\varphi]$ to go over all periodic paths, i.e., those that have the same classical field at $\tau = \beta$ as at $\tau = 0$. Symbolically

$$Tre^{-\beta H} = \sum_\varphi\langle\varphi|e^{-\beta H}|\varphi\rangle =$$

$$= N\int[d\pi]\int_{periodic}[d\varphi]\,exp\left\{\int_0^\beta d\tau\int d^3\vec{x}\left[i\pi\dot{\varphi} - \mathcal{H}(\pi,\varphi)\right]\right\}. \tag{1.2.4}$$

Note that only the field integration, and not the momentum integration, is restricted to periodic orbits.

In the usual cases where $\mathcal{H}$ is no more than a quadratic function of π's, we can do the π integration immediately by completing the square. This merely replaces π in the integrand by its value at the stationary point of the integrand and adds a ghost term if the quadratic term in π is φ-dependent [1.16]. The stationary point is given by

$$i\dot{\varphi} = \frac{\partial\mathcal{H}(\pi,\varphi)}{\partial\pi}. \tag{1.2.5}$$

This, together with the addition of a possible ghost term, is just the prescription for going from the Hamiltonian to the usual effective Lagrangian, $\mathcal{L}_{eff}$, with the additional stipulation that all τ derivatives in $\mathcal{L}_{eff}$ are multiplied by i. Thus we have

$$\mathrm{Tr}\, e^{-\beta H} = N'(\beta)\int_{periodic} [d\varphi]\, \exp\left\{\int_0^\beta d\tau \int d^3\vec{x}\, \mathcal{L}_{eff}(\varphi, i\dot{\varphi})\right\} , \qquad (1.2.6)$$

where $N'(\beta)$ is a new (infinite) normalizing factor which will be determined later. The β dependence of N' comes from the evaluation of $\int [d\pi]$. The same kind of thing happens at zero temperature, but there we are interested in the $t \rightarrow \infty$ limit and so N' is an unimportant infinite constant which is usually ignored.

For a non-gauge theory, we can use eq. (1.2.6) to derive the Feynman rules in the ordinary way. The quadratic part of $\mathcal{L}_{eff}$ determines the propagators and the non-quadratic part determines the vertices. There are only two basic differences from the zero-temperature rules:

1. The presence of the $i\dot{\varphi}$ and the absence of an i multiplying $\mathcal{L}_{eff}$ just accomplish a "Wick rotation" of the rules into Euclidean space. Therefore, there is no need for an $i\epsilon$ in the propagator denominators.

2. The requirement of periodicity changes all energy integrations into energy sums for the Feynman rules. For bosons this means replacing the energy k_0 by $\frac{2\pi n}{\beta}$ and summing over integer n instead of integrating over k_0. For fermions, owing to the integral of anti-commuting C numbers, they are the same as the boson rules except for the usual minus sign for loops and the sum over odd, rather than even, multiples of $\frac{\pi}{\beta}$.

As an example of above techniques, we work out the Feynman rules of simple scalar field theory described by

$$\mathcal{L}(\varphi,\dot{\varphi}) = \mathcal{L}_{eff}(\varphi,\dot{\varphi}) = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \lambda\varphi^4 \qquad (1.2.7)$$

where the metric of space-time is taken to be

$$\eta_{\mu\nu} = diag(+1,-1,-1,-1) \quad . \tag{1.2.8}$$

We have from (1.2.6)

$$Tr\, e^{-\beta H} = N'(\beta)\int[d\varphi]\cdot exp\left\{\int_0^\beta d\tau\int d^3\vec{x}\left[-\frac{1}{2}\left((\partial_0\varphi)^2 + (\partial_i\varphi)(\partial_i\varphi) + + m^2\varphi^2\right) - \lambda\varphi^4\right]\right\} , \tag{1.2.9}$$

note that a "Wick rotation" to Euclidean space has been done.

Define the quadratic part of the action by

$$S \equiv -\frac{1}{2}\int_0^\beta d\tau\int d^3\vec{x}\left[(\partial_0\varphi)(\partial_0\varphi) + (\partial_i\varphi)(\partial_i\varphi) + m^2\varphi^2\right] \quad . \tag{1.2.10}$$

The Fourier transform of $\varphi(\vec{x},\tau)$ is

$$\varphi(\vec{x},\tau) = \frac{1}{\beta}\sum_n\int\frac{d^3\vec{k}}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\cdot e^{i\frac{2\pi}{\beta}n\tau}\,\varphi_n(\vec{k}) \quad , \tag{1.2.11}$$

$$\varphi_n(\vec{k}) = \int d^3\vec{x}\int_0^\beta d\tau\; e^{-i\vec{k}\cdot\vec{x}}\; e^{-i\frac{2\pi}{\beta}n\tau}\;\varphi(\vec{x},\tau) \quad , \tag{1.2.12}$$

because $\varphi(\vec{x},\tau)$ is periodic in the time interval $0 \leqslant \tau \leqslant \beta$, $\varphi(\vec{x},0) = = \varphi(\vec{x},\beta)$. Using the identity

$$\int_0^\beta d\tau\; e^{i\frac{2\pi}{\beta}(n-n')\tau} = \beta\,\delta_{n,n'} \quad , \tag{1.2.13}$$

we obtain the expression of S_0 in the momentum space:

$$S_0 = -\frac{1}{2}\left(\frac{1}{\beta}\right)\sum_n \int \frac{d^3\vec{k}}{(2\pi)^3}\left[\vec{k}^2 + \left(\frac{2\pi n}{\beta}\right)^2 + m^2\right] \varphi_n(\vec{k})\, \varphi_{-n}(-\vec{k}) \qquad (1.2.14)$$

.

Thus, when we write $S_0 = -\frac{1}{2}(\varphi, D\varphi)$, where the brackets denote the scalar product on our function space, we have

$$D = \vec{k}^2 + \left(\frac{2\pi n}{\beta}\right)^2 + m^2 \qquad (1.2.15)$$

,

in momentum space. The Feynman propagator, Δ_F, is then just D^{-1}. Thus in momentum space

$$\Delta_F(\omega_n, \vec{k}) = \frac{1}{\omega_n^2 + \vec{k}^2 + m^2} \qquad (1.2.16)$$

,

and in position space

$$\Delta_F(\vec{x}, \tau) = \frac{1}{\beta}\sum_n \int \frac{d^3\vec{k}}{(2\pi)^3} \cdot \frac{e^{i(\vec{k}\cdot\vec{x} + \omega_n \tau)}}{\omega_n^2 + \vec{k}^2 + m^2} \qquad (1.2.17)$$

,

where $\omega_n = \frac{2\pi n}{\beta}$.

At each vertex, except for the usual conservation of momentum, we have also the conservation of discrete energy and a factor of β due to eq. (1.2.13). Therefore, we conclude that the Feynman rules at finite temperature are the same as those at zero temperature, only with the replacements

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \frac{i}{\beta}\sum_n \int \frac{d^3\vec{k}}{(2\pi)^3} ,$$

$$k_0 \longrightarrow i\,\omega_n ,$$

(1.2.18)

$$(2\pi)^4\,\delta^4(k_1+k_2+\cdots) \longrightarrow \frac{1}{i}\,\beta\,\delta_{\omega_{n_1}+\omega_{n_2}+\cdots}\cdot(2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2+\cdots)\ ,$$

where the factors of "i" just come from the "Wick rotation" to Euclidean space.

We note that there is the β -dependent factor, $N'(\beta)$, in eq. (1.2.9), but we can also calculate finite-temperature Green functions which are statistical averages of τ -ordered fields:

$$\left\langle T\left[\varphi(\vec{x}_1,\tau_1)\cdot\varphi(\vec{x}_2,\tau_2)\cdots\right]\right\rangle \equiv \frac{Tr\left\{e^{-\beta H}\,T\left[\varphi(\vec{x}_1,\tau_1)\cdot\varphi(\vec{x}_2,\tau_2)\cdots\right]\right\}}{Tr\,e^{-\beta H}}\ , \tag{1.2.19}$$

where T is the τ -ordering symbol. Then $N'(\beta)$ is cancelled by the denominator of (1.2.19), just as in the zero-temperature S matrix.

1.2.2. Gauge Theories

We can now apply the functional formalism to gauge field theories. Before starting, however, let us show that there is a non-trivial problem involved here. Consider free electro-dynamics described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\ , \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ . \tag{1.2.20}$$

If we work, for example, in the Coulomb gauge, $\vec{\nabla}\cdot\vec{A} = 0$, or the axial gauge, $A_3 = 0$, calculate H, and take $Tre^{-\beta H}$, we come to the usual conclusion that Z describes a massless Bose gas with two degrees of polarization. However, if we work in the Feynman gauge (described by the Lagrangian $\mathcal{L} = -\frac{1}{2}\,\partial_\mu A_\nu\cdot\partial^\mu A^\nu$), calculate H, and take $Tre^{-\beta H}$, we get an obvious nonsensical result: a Bose gas with three positive - and one negative - metric states. That is,

$$\ln\left(\mathrm{Tr}\, e^{-\beta H}\right)_{\text{Feynman gauge}} = 3\int\frac{d^3\vec{k}}{(2\pi)^3}\left[\frac{-\beta\omega_k}{2} - \ln\left(1-e^{-\beta\omega_k}\right)\right] +$$

$$+\int\frac{d^3\vec{k}}{(2\pi)^3}\left[\frac{-\beta\omega_k}{2} - \ln\left(1+e^{-\beta\omega_k}\right)\right] , \qquad (1.2.21)$$

where $\omega_k = |\vec{k}|$.

The reason for the nonsense of (1.2.21) is not hard to find. Since the Hamiltonian in the Coulomb or the axial gauge has just two independent degrees of freedom, whereas in the Feynman gauge there are two extra degrees of freedom: longitudinal and time-like photons, therefore, when we take $\mathrm{Tr}e^{-\beta H}$ in the Feynman gauge we include these specious states. It can be seen that $\mathrm{Tr}e^{-\beta H}$ is not a physically meaningful quantity in all gauges, in some gauges, the spurious particles which could never come to equilibrium with a physical heat bath, are wrongly counted as physical degrees of freedom. Thus, we define a physically meaningful partition function Z which is gauge-invariant and only equals to $\mathrm{Tr}e^{-\beta H}$ in certain physical gauges. The procedure to do this is the following.

We shall be to start with (1.2.4) in a gauge which is physical. For example, consider a pure Yang-Mills theory (fermions or scalars would be trivial to add) described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu}\cdot F^{a\,\mu\nu} , \qquad (1.2.22)$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, C_{abc} A^b_\mu A^c_\nu$ and C_{abc} are the structure constants of the group.

We see [1.16] that the Hamiltonian for this theory in the axial gauge ($A_3^a = 0$) has only two degrees of freedom for each vector-meson field: A_1^a, and A_2^a. Then, in the axial gauge, (1.2.4) becomes

$$\left(\mathrm{Tr}\, e^{-\beta H}\right)_{\text{axial gauge}} = N \int \prod_a [dP_1^a][dP_2^a] \int_{\text{periodic}} [dA_1^a][dA_2^a] \cdot$$

$$\cdot \exp\left\{ \int_0^\beta d\tau \int d^3\vec{x} \left[i P_j^a \dot{A}_j^a - \mathcal{H}(A_j^a, P_j^a) \right] \right\} , \tag{1.2.23}$$

where $P_{1,2}^a$ are the conjugate momenta to $A_{1,2}^a$ and $j = 1,2$. After integrating (1.2.23) over P's, we have obtained

$$\left(\mathrm{Tr}\, e^{-\beta H}\right)_{\text{axial gauge}} = [N'(\beta)]^{2n} \int_{\text{periodic}} [dA] \cdot \prod_a \delta(A_3^a) \cdot$$

$$\cdot \exp\left[\int_0^\beta d\tau \int d^3\vec{x}\, \mathcal{L}(A, i\dot{A}) \right] , \tag{1.2.24}$$

where there is one factor of $N'(\beta)$ for each P integration (a runs from 1 to n), and $[dA]$ means integral over all the vector-meson fields.

In general, fixing a gauge corresponds to picking out a surface in the functional space of the fields which is expressed as $\prod_b \delta(F^b)$, then the right-side of eq. (1.2.24) becomes

$$[N'(\beta)]^m \int_{\text{periodic}} [dA][d\varphi]\, \exp\left[\int_0^\beta d\tau \int d^3\vec{x}\, \mathcal{L}(A, \varphi, i\dot{A}, i\dot{\varphi}) \right] \det\left(\frac{\partial F^b}{\partial \omega^c}\right) \prod_b \delta(F^b) , \tag{1.2.25}$$

where m is the total number of physical particles and polarization states in the theory. The integral in (1.2.25) is over all periodic fields (A stands for the vector-meson fields, φ for any other fields in the theory), and the determinant, $\det(\frac{\partial F^b}{\partial \omega^c})$, is due to the Faddeev-Popov trick. (The determinant is lacking in (1.2.24) because, for the axial gauge, it is $\det(\partial/\partial x_3)$ [1.16], which is a constant independent of fields and of β.)

Since the Lagrangian in (1.2.25) is gauge-invariant, the integral in (1.2.25) must be independent of how we pick our gauge surface. Therefore, (1.2.25) is a gauge-invariant quantity, thus, we define it to be the true physical partition function Z. So, we do not have to stick to a physical gauge in eq. (1.2.25); we can calculate the physical partition function even if the equations $F^b(x) = 0$ correspond to a nonphysical gauge.

From the above statements, we can see that the situation, the specification of the gauge surface by δ function and the boundary condition on the fields is essentially the same for both the finite-temperature and the zero-temperature, the only difference being the boundary condition on the fields, for the zero temperature, the fields vanish at spatial and temporal infinity, and for the finite temperature, the fields vanish at spatial infinity and are periodic at the time interval from $\tau = 0$ to $\tau = \beta$.

Therefore, we have

$$Z = [N'(\beta)]^m \int_{periodic} [dA]\,[d\varphi]\, \exp\left[\int_0^\beta d\tau \int d^3\vec{x}\, \mathcal{L}(A,\varphi,i\dot{A},i\dot{\varphi})\right] \cdot \det\left(\frac{\partial F^b}{\partial \omega^c}\right) \prod_b \delta(F^b) \quad (1.2.26)$$

Thus, we can change the surface of integration in (1.2.26) just as at zero temperature, and can write the left-hand side in any of the usual gauges (Feynman gauge, Landau gauge, the R_ξ gauges for a spontaneously broken theory, etc.). The Feynman rules at finite temperature in any of these gauges will be the zero-temperature rules with the replacements given by (1.2.18). The important point is that when we use these rules in a given gauge we calculate the true Z, which is not necessarily $\mathrm{Tr}\, e^{-\beta H}$ in that gauge.

In more complicated cases (such as non-Abelian gauge theories) we write the determinant as an integral over ghost fields, which are unphysical spin-zero particles that have a fermion-like minus for loops. The Feynman rules of these ghost fields will involve a sum over even multiples of $\frac{\pi}{\beta}$, as for bosons.

Finally, we conclude that $\mathrm{Tr}e^{-\beta H}$ is a gauge-dependent quantity which is only physically meaningful in a gauge where there are no unphysical particles. The true partition function Z is then defined to be $\mathrm{Tr}e^{-\beta H}$, where H is evaluated in one of these physical gauges. Using functional-integral methods and the Faddeev-Popov trick, the Feynman rules for Z can be derived in any convenient gauge, when we use these rules we calculate Z, which is not, in general, $\mathrm{Tr}e^{-\beta H}$ in that gauge. The Feynman rules for a given gauge are just the zero-temperature rules for that gauge, with the following substitutions:

$$k_0 \longrightarrow i\,\omega_n\,,$$

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow \frac{i}{\beta} \sum_n \int \frac{d^3\vec{k}}{(2\pi)^3}\,,$$

$$(2\pi)^4\,\delta^4(k_1+k_2+\cdots) \longrightarrow \frac{1}{i}\,(2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2+\cdots)\cdot\beta\,\delta_{\omega_{n_1}+\omega_{n_2}+\cdots}\,,$$

where $\omega_n = \frac{2\pi n}{\beta}$ for bosons and Faddeev-Popov ghosts, and $\omega_n = \frac{(2n+1)\pi}{\beta}$ for fermions.

1.3. Renormalization of the finite-temperature gauge field theory

Many authors have stressed the fact that the renormalization mechanism of the finite-temperature field theories is the same as that of the zero-temperature field theories. The zero-temperature renormalization prescriptions suffice to eliminate all ultraviolet divergences, i.e. no new temperature or density dependent infinities appear. Below, we give a justification for the above statements at two-loop order, which follows Ref. [1.9] by Kislinger and Morley.

From the last section 1.2., we have learned that the Feynman rules in finite-temperature field theory (FTF) are just the zero-temperature rules with the following replacements:

$$\int\frac{d^4K}{(2\pi)^4} \longrightarrow \frac{i}{\beta}\sum_N\int\frac{d^3K}{(2\pi)^3}, \qquad \beta=\frac{1}{T}, \qquad \text{T is temperature,}$$

$$K^0=\omega_N, \quad \omega_N=\frac{2\pi N i}{\beta} \text{ for bosons,} \qquad \omega_N=\frac{(2N+1)i\pi}{\beta} \text{ for fermions,}$$

$$N = 0, \quad 1, \quad 2, \ldots \tag{1.3.1}$$

A Feynman diagram in FTF has the general form

$$\left[\prod_\alpha \sum_N \frac{i}{\beta}\int\frac{d^3K_\alpha}{(2\pi)^3}\right]\frac{\mathcal{P}}{\left[\prod_\beta\left(K_\beta^2-m_\beta^2\right)\right]\left\{\prod_\gamma\left[\left(a_{\gamma\delta}K_\delta+b_{\gamma\delta}P_\delta\right)^2-m_\gamma^2\right]\right\}}, \tag{1.3.2}$$

where $\mathcal{P}$ involves γ matrices and polynomials in momenta, the P_δ are external momenta with

$$P_\delta^0=\omega_\delta', \quad \omega_\delta'=\frac{2\pi i\delta}{\beta} \text{ for bosons,} \qquad \omega_\delta'=\frac{(2\delta+1)i\pi}{\beta} \text{ for fermions,}$$

and m_β^2 are the squared masses.

We can convert the energy sums to contour integrals. For bosons,

$$\frac{i}{\beta}\sum_{N=-\infty}^{\infty} f(\omega_N) = \frac{1}{2\pi}\oint_C \frac{dK^0\, f(K^0)}{e^{\beta K^0}-1} \quad , \tag{1.3.3}$$

where contour C is given in Fig. 1.1.

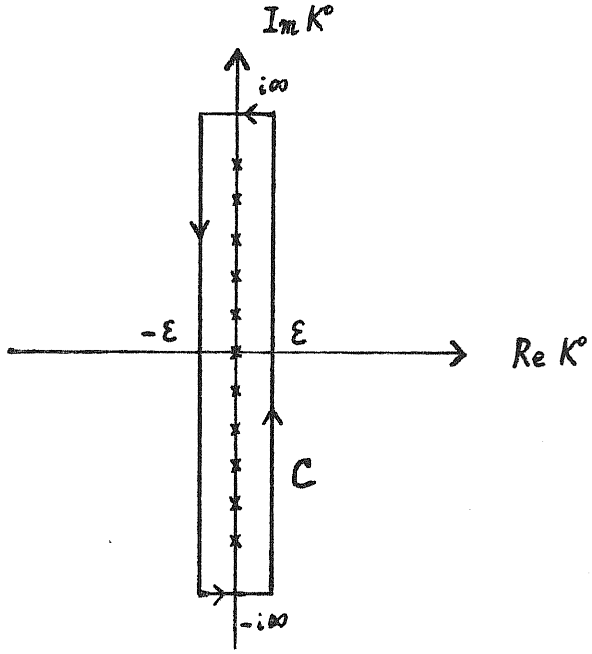


Fig. 1.1 C contour in complex K° plane, crosses are poles at $K^0 = \frac{2\pi i N}{\beta}$ for bosons and $K^0 = \frac{(2N+1)i\pi}{\beta}$ for fermions.

Thus

$$\frac{i}{\beta}\sum_{N=-\infty}^{\infty} f\left(\omega_N = \frac{2\pi i N}{\beta}\right) = \frac{1}{2\pi}\oint_C dK^0 f(K^0) \frac{e^{-\frac{\beta K^0}{2}}}{e^{\frac{\beta K^0}{2}} - e^{-\frac{\beta K^0}{2}}} =$$

$$= \frac{1}{4\pi}\oint_C dK^0 f(K^0) \frac{e^{\frac{\beta K^0}{2}} + e^{-\frac{\beta K^0}{2}}}{e^{\frac{\beta K^0}{2}} - e^{-\frac{\beta K^0}{2}}} = \frac{1}{4\pi}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dK^0 f(K^0) \frac{e^{\frac{\beta K^0}{2}} + e^{-\frac{\beta K^0}{2}}}{e^{\frac{\beta K^0}{2}} - e^{-\frac{\beta K^0}{2}}} +$$

$$+ \frac{1}{4\pi}\int_{i\infty+\epsilon}^{i\infty-\epsilon} dK^0 f(K^0) \frac{e^{\frac{\beta K^0}{2}} + e^{-\frac{\beta K^0}{2}}}{e^{\frac{\beta K^0}{2}} - e^{-\frac{\beta K^0}{2}}} + \frac{1}{4\pi}\int_{i\infty-\epsilon}^{-i\infty-\epsilon} dK^0 f(K^0) \frac{e^{\frac{\beta K^0}{2}} + e^{-\frac{\beta K^0}{2}}}{e^{\frac{\beta K^0}{2}} - e^{-\frac{\beta K^0}{2}}} +$$

$$+\frac{1}{4\pi}\int_{-i\infty-\epsilon}^{-i\infty+\epsilon} dK^0 f(K^0)\,\frac{e^{\frac{\beta K^0}{2}}+e^{-\frac{\beta K^0}{2}}}{e^{\frac{\beta K^0}{2}}-e^{-\frac{\beta K^0}{2}}}=\frac{1}{2\pi}\int_{-i\infty}^{i\infty} dK^0 f(K^0)+\frac{1}{2\pi}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dK^0\,\frac{e^{-\frac{\beta K^0}{2}}}{e^{\frac{\beta K^0}{2}}-e^{-\frac{\beta K^0}{2}}}\Big[f(K^0)+$$

$$+f(-K^0)\Big]=\frac{1}{2\pi}\int_{-i\infty}^{i\infty} dK^0 f(K^0)+\frac{1}{2\pi}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dK^0\,\frac{f(K^0)+f(-K^0)}{e^{\beta K^0}-1}\;. \tag{1.3.4}$$

For fermions

$$\frac{i}{\beta}\sum_{N=-\infty}^{\infty} f(\omega_N)=-\frac{1}{2\pi}\oint_C dK^0 f(K^0)\,\frac{e^{-\frac{\beta K^0}{2}}}{e^{\frac{\beta K^0}{2}}+e^{-\frac{\beta K^0}{2}}}=\frac{1}{2\pi}\int_{-i\infty}^{i\infty} dK^0 f(K^0)-\frac{1}{2\pi}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dK^0\,\frac{f(K^0)+f(-K^0)}{e^{\beta K^0}+1}\;. \tag{1.3.5}$$

When $T\to 0$, only first terms in eqs.(1.3.4) and (1.3.5) survive, and the second terms tend to zero, thus

$$\frac{i}{\beta}\sum_N\int\frac{d^3K}{(2\pi)^3}\;\xrightarrow[T\to 0]{}\;\int_{-i\infty}^{i\infty}\frac{dK}{2\pi}\int\frac{d^3K}{(2\pi)^3}\;,$$

which is an integral over Euclidean momenta.

Now we follow the demonstration by Kislinger and Morley. The basic step in their renormalization program is to show that all ultraviolet infinities occur in subdiagrams in which the integrals over loop momenta are temperature-independent. That is, only first terms of eqs. (1.3.4) and (1.3.5) involve ultraviolet divergences. Then the infinities are just those which would be there if we take $T\to 0$. This requires demonstrating that integrals over loop momenta associated with these second terms in eqs. (1.3.4) and (1.3.5) do not give rise to infinities. The demonstration relies on the contour closing. We now illustrate the contour-closing method. Consider eq.(1.3.2) and let us treat

$$\frac{i}{\beta}\sum_{N_1}\int\frac{d^3K_1}{(2\pi)^3} \quad ,$$

first, assuming K_1 is a fermion (we find that it is easier to evaluate all fermion loop momenta before evaluating boson loop momenta). There are two terms in eq. (1.3.5). For the second term,

$$-\frac{1}{2\pi}\int_{-i\infty+\epsilon}^{i\infty+\epsilon}dK^0\,\frac{f(K^0)+f(-K^0)}{e^{\beta K^0}+1} = -\frac{1}{2\pi}\int_{-i\infty+\epsilon}^{i\infty+\epsilon}dK^0\,\frac{f(K^0)}{e^{\beta K^0}+1} - \frac{1}{2\pi}\int_{-i\infty-\epsilon}^{i\infty-\epsilon}dK^0\,\frac{f(K^0)}{e^{-\beta K^0}+1} \quad , \tag{1.3.6}$$

we close the contours of the two integrals of the right side in eq.(1.3.6) which are drawn in Fig. 1.2.

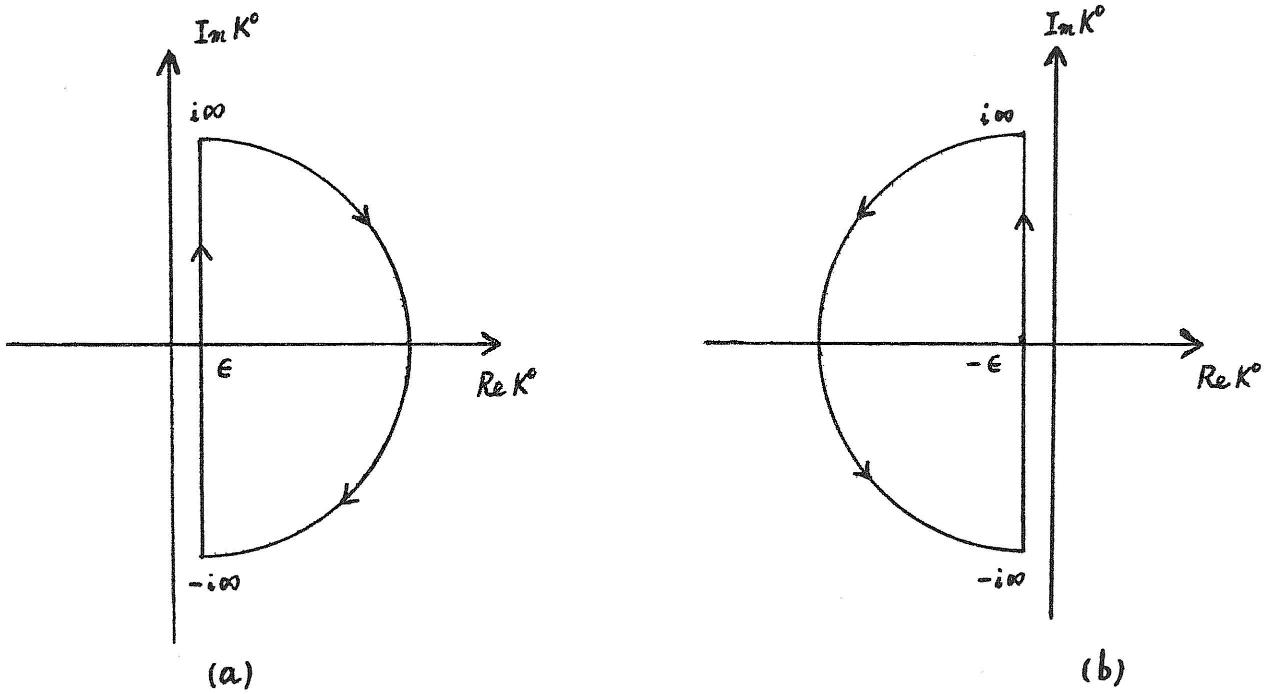


Fig.1.2. Closed integration contours

Then, the first term of (1.3.5) is left alone. Thus we go on to the next loop momentum K_2^o. When we close the K_1^o contours, keeping all other variables fixed, pick up the residues of poles in K_1^o. For example, a typical denominator containing K_1^o might be

$$\frac{1}{(K_1-K_j)^2-m^2}=\frac{1}{\left\{(K_1^o-K_j^o)+\left[(\vec{K}_1-\vec{K}_j)^2+m^2\right]^{\frac{1}{2}}\right\}\cdot\left\{(K_1^o-K_j^o)-\left[(\vec{K}_1-\vec{K}_j)^2+m^2\right]^{\frac{1}{2}}\right\}}$$

with $K_j^o=\dfrac{2N_j\pi i}{\beta}$.

Upon evaluating the contour integral we get in eq.(1.3.2) that

$$\frac{i}{\beta}\sum_{N_1}\int\frac{d^3\vec{K}_1}{(2\pi)^3}\cdot\frac{1}{(K_1-K_j)^2-m^2}\cdots\longrightarrow i\int\frac{d^3\vec{K}}{(2\pi)^3}\cdot\frac{1}{2\left[(\vec{K}_1-\vec{K}_j)^2+m^2\right]^{\frac{1}{2}}\left\{e^{\beta[(\vec{K}_1-\vec{K}_j)^2+m^2]^{\frac{1}{2}}}+1\right\}}\cdots+$$

$$+i\int\frac{d^3\vec{K}_1}{(2\pi)^3}\cdot\frac{1}{2\left[(\vec{K}_1-\vec{K}_j)^2+m^2\right]^{\frac{1}{2}}\left\{e^{\beta[(\vec{K}_1-\vec{K}_j)^2+m^2]^{\frac{1}{2}}}+1\right\}}\cdots+\frac{1}{(2\pi)^4}\int_{-i\infty}^{i\infty}dK_1^o\int d^3\vec{K}\,\frac{1}{(K_1-K_j)^2+m^2}\cdots+$$

+ other pole terms (1.3.7)

where we have used $e^{\beta K_j^o}=1$, since $K_j^o=\dfrac{2\pi i N_j}{\beta}$. In expression (1.3.7) the dots stand for the remaining factors in eq.(1.3.2) and we have suppressed the summation and integration symbols for the other variables. The other pole terms not shown come from other propagator denominators containing K_1^o. it is easier to change variables in (1.3.7), $\vec{K}_1-\vec{K}_j=\vec{K}_1'$ *and* $\int d^3\vec{K}_1\longrightarrow\int d^3\vec{K}_1'$. Now we see that closing the contours has resulted in various terms which have absolutely convergent three-momentum integrals over $\vec{K}_1'$ (i.e. the first two terms in (1.3.7), in spite of the dots). Ultraviolet infinities due to large K_1 can occur only in the term independent of T (i.e. the third term in (1.3.7) which is the term containing the T-independent Euclidean space integral). We shall refer to the first two terms in (1.3.7) as "finite K_1" ; the third term will be called "temperature-independent K_1" or TIK_1 for short.

Next, we go on to the loop momentum K_2 . Closing contours proceeds as for K_1, that is, we close the temperature-dependent K_2 contours and leave the TIK_2 contours alone. There are now some new complications which we must deal with. These occur in the terms involving the TIK_1 and the temperature-dependent K_2 contours (TDK_2). A possible factor in these terms is a denominator of the form $\left[(K_2 - K_1 - K_\ell)^2 - m^2\right]$ and closing the TDK_2 contour around the poles of this denominator (let us assume that K_2 is a boson momentum) will give rise to factors such as

$$\int_{-i\infty}^{i\infty} dK_1^0 \int d^3\vec{K}_1 \int \frac{d^3\vec{K}_2}{(2\pi)^3} \cdot \frac{1}{2\left[(\vec{K}_2-\vec{K}_1-\vec{K}_\ell)^2+m^2\right]^{\frac{1}{2}}} \cdot \frac{1}{e^{\beta\{[(\vec{K}_2-\vec{K}_1-\vec{K}_\ell)^2+m^2]^{\frac{1}{2}}+K_1^0\}}-1} \cdots \tag{1.3.8}$$

(K_ℓ^0 is actually irrelevant to the discussion). Since we are dealing with $\int_{-i\infty}^{i\infty} dK_1^0$, the K_1^0 in the argument of the exponential complicates things. We first redefine $K_2 - K_1 - K_\ell = K_2'$ with $K_2^{0\prime} = \left[(\vec{K}_2 - \vec{K}_1 - \vec{K}_\ell)^2 + m^2\right]^{1/2}$, so we have terms involving

$$\int_{-i\infty}^{i\infty} dK_1^0 \int d^3\vec{K}_1 \int \frac{d^3\vec{K}_2'}{(2\pi)^3} \frac{1}{2\left[(\vec{K}_2')^2+m^2\right]^{\frac{1}{2}}} \cdot \frac{1}{e^{\beta\{[(\vec{K}_2')^2+m^2]^{\frac{1}{2}}+K_1^0\}}-1} \cdots \tag{1.3.9}$$

Now we must close the K_1^0 contour to the right, thus avoiding any poles due to the vanishing of $\exp\left\{\beta\left[(\vec{K}_2'^2+m^2)^{\frac{1}{2}}+K_1^0\right]\right\}-1$. In doing so we will pick up poles from denominator factors such as

$$(K_1^2 - m^2), \qquad (K_1+K_j)^2 - m^2\Big|_{K_j \neq K_2}, \qquad \text{and} \quad (K_2'+K_1-K_m)^2 - m^2 .$$

In each case the resulting integration is a term involving absolutely convergent integrals over both K_1 and K_2.

Proceeding in this way we have got these integrals in eq.(1.3.2) associated with temperature-dependent are all absolutely convergent, the remaining loop momenta are associated with temperature-independent Euclidean momentum integrals. Above statements are only a crude, not rigorous demonstration. In the following, we will consider ϕ^4 theory to the two-loop level as an example.

The unrenormalized Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2} m_R^2 \phi^2 \quad ,$$

$$\mathcal{L}_I = -\frac{1}{2}\left(m_B^2 - m_R^2\right)\phi^2 - g_B \frac{\phi^4}{4!} \quad .$$

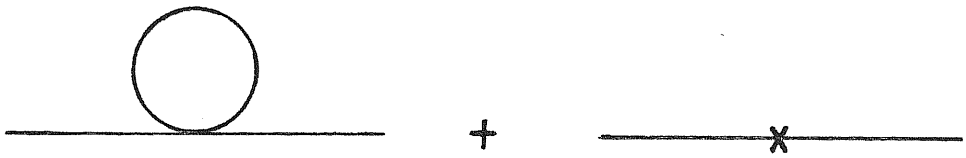
The free propagator is $i(P^2 - m_R^2)^{-1}$. The one-loop self-energy diagram and counter-term diagrams are shown in Figs. 1.3(a) and 1.3(b). From Fig.1.3(a) we get (using the above contour closing method)

$$-\frac{i g_R}{2}\mu^{4-N}\int\frac{d^N K}{(2\pi)^N}\frac{i}{K^2 - m_R^2} = -\mu^{4-N}\frac{i g_R}{2(2\pi)^N}\pi^{\frac{N}{2}}\Gamma\left(1-\tfrac{N}{2}\right)\cdot\left(m_R^2\right)^{\frac{N}{2}-1}\cdot$$

$$\cdot\frac{-i g_R}{2}\mu^{4-N}\int\frac{d^3\vec{K}}{(2\pi)^3}\frac{1}{\left(\vec{K}^2+m_R^2\right)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta\left(\vec{K}^2+m_R^2\right)^{\frac{1}{2}}}-1} = \tag{1.3.10}$$

$$= -\frac{i g_R m_R^2}{(4\pi)^2 (N-4)} - \frac{i g_R \Gamma_{finite}\left(1-\frac{N}{2}\right)\cdot m_R^2}{2\cdot(4\pi)^2} - \frac{i g_R m_R^2}{32\cdot\pi^2}\ln\left(\frac{m_R^2}{4\pi\mu^2}\right) -$$

$$-\frac{i g_R}{2}\int\frac{d^3\vec{K}}{(2\pi)^3}\frac{1}{\left(\vec{K}^2+m_R^2\right)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta\left(\vec{K}^2+m_R^2\right)^{\frac{1}{2}}}-1} \quad .$$



(a) (b)

Fig. 1.3. (a) One-loop self-energy in ϕ^4.

(b) Counterterm diagram for one loop.

From Fig. 1.3(b), we have

$$-i\left(m_B^2-m_R^2\right)=\frac{i g_R\, m_R^2}{(4\pi)^2(N-4)} \qquad (1.3.11)$$

For eq. (1.3.10),

$$-\frac{i g_R}{2}\mu^{4-N}\int\frac{d^N K}{(2\pi)^N}\frac{i}{K^2-m_R^2}=-\frac{i g_R}{2}\mu^{4-N}\frac{i}{\beta}\sum_n\int\frac{d^3\vec{K}}{(2\pi)^3}\frac{i}{K^2-m_R^2}=-\frac{i g_R}{2}\mu^{4-N}\left\{\int_{-i\infty+\epsilon}^{i\infty+\epsilon}\frac{dK^0}{2\pi}\int\frac{d^3\vec{K}}{(2\pi)^3}\cdot\frac{i}{K^2-m_R^2}\cdot\frac{1}{e^{\beta K^0}-1}+\int_{-i\infty-\epsilon}^{i\infty-\epsilon}\frac{dK^0}{2\pi}\int\frac{d^3\vec{K}}{(2\pi)^3}\cdot\frac{i}{K^2-m_R^2}\cdot\frac{1}{e^{-\beta K^0}-1}+\int_{-i\infty}^{i\infty}\frac{dK^0}{2\pi}\int\frac{d^3\vec{K}}{(2\pi)^3}\cdot\frac{i}{K^2-m_R^2}\right\}=-\frac{i g_R}{2}\mu^{4-N}\cdot\left\{\int\frac{d^3\vec{K}}{(2\pi)^3}\cdot\frac{1}{(\vec{K}^2+m_R^2)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}-1}+\int\frac{d^N\vec{K}}{(2\pi)}\cdot\frac{i}{\vec{K}^2+m_R^2}\right\}=-\frac{i g_R}{2}\mu^{4-N}\int\frac{d^3\vec{K}}{(2\pi)^3}\cdot\frac{1}{(\vec{K}^2+m_R^2)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}-1}-\mu^{4-N}\cdot\frac{i g_R}{2(2\pi)^N}\cdot\pi^{\frac{N}{2}}\,\Gamma\!\left(1-\frac{N}{2}\right)\left(m_R^2\right)^{\frac{N}{2}-1}.$$

Since

$$\Gamma(1-\tfrac{N}{2}) = \Gamma(1-\tfrac{4-\epsilon}{2}) = -\frac{2}{\epsilon} - \Gamma'(1) - 1 \quad , \qquad N = 4-\epsilon \ ,$$

$$(m_R^2)^{\frac{N}{2}-1} = (m_R^2)^{\frac{4-\epsilon}{2}-1} = m_R^2 - \frac{\epsilon}{2} m_R^2 \cdot \ln m_R^2 \quad ,$$

$$\mu^{4-N} = 1 + \epsilon \cdot \ln \mu \quad ,$$

$$2^{-N} = 2^{-4}(1+\epsilon \cdot \ln 2) \quad ,$$

$$\pi^{-\frac{N}{2}} = \pi^{-2}\left(1+\frac{\epsilon}{2}\ln\pi\right) \quad ,$$

we have

$$-\mu^{4-N} \frac{i g_R}{2(2\pi)^N} \pi^{\frac{N}{2}} \Gamma(1-\tfrac{N}{2}) (m_R^2)^{\frac{N}{2}-1} = -\frac{i g_R m_R^2}{2^5 \pi^2} \left[1 - \frac{\epsilon}{2} \ln\left(\frac{m_R^2}{4\pi\mu^2}\right)\right] \cdot$$

$$\cdot \left[-\frac{2}{\epsilon} - \Gamma'(1) - 1\right] = \frac{i g_R m_R^2}{(4\pi)^2 \cdot \epsilon} + \frac{i g_R m_R^2}{32\pi^2}(\Gamma'(1)+1) - \frac{i g_R m_R^2}{32 \cdot \pi^2} \ln\left(\frac{m_R^2}{4\pi\mu^2}\right) =$$

$$= -\frac{i g_R m_R^2}{(4\pi)^2 \cdot (N-4)} - \frac{i g_R m_R^2}{32\pi^2} \Gamma_{finite}(1-\tfrac{N}{2}) - \frac{i g_R m_R^2}{32 \cdot \pi^2} \cdot \ln\left(\frac{m_R^2}{4\pi\mu^2}\right) .$$

Since $$-\frac{1}{\epsilon} = \frac{1}{N-4} \quad \text{and} \quad -\Gamma'(1) - 1 = \Gamma_{finite}(1-\tfrac{N}{2}) \quad ,$$

we have

$$-i\frac{g_R}{2}\mu^{4-N}\int\frac{d^4K}{(2\pi)^4}\;\frac{i}{K^2-m_R^2} = -\frac{i\,g_R}{2}\int\frac{d^3K}{(2\pi)^3}\,\frac{1}{(\vec{K}^2+m_R^2)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}-1} -$$

$$-\frac{i\,g_R\,m_R^2}{(4\pi)^2\,(N-4)} + \frac{i\,g_R\,m_R^2}{32\,\pi^2}\left[\Gamma'(1)+1\right] - \frac{i\,g_R\,m_R^2}{32\,\pi^2}\cdot\ln\left(\frac{m_R^2}{4\pi\mu^2}\right) \quad . \tag{1.3.12}$$

From eqs.(1.3.12) and (1.3.11) we see that these temperature-independent infinities are cancelled, and the integral of three-dimensional loop-momentum, $\int d^3\vec{K}\,\ldots$, is absolutely convergent. So, there are no temperature-dependent infinities at one-loop approximation. Note that it can be seen from (1.3.12) that at finite temperature we have a finite-temperature -dependent radiative correction to (mass)^2 as

$$g_R\int\frac{d^3\vec{K}}{(2\pi)^3}\;\frac{1}{2\left(\vec{K}^2+m_R^2\right)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}-1} \quad . \tag{1.3.13}$$

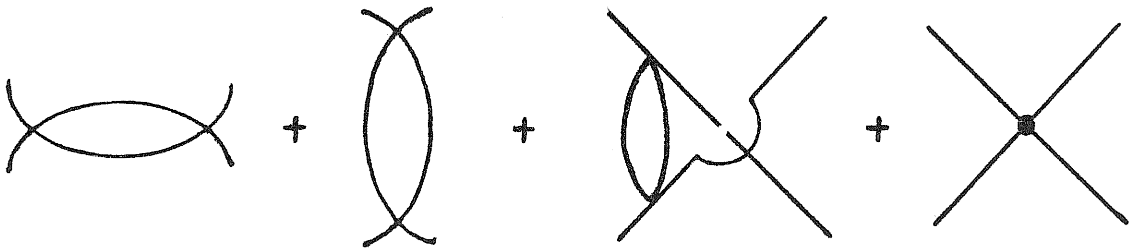
This completes the one-loop calculation.

Going to two loops we will encounter temperature-dependent infinities. We have vertex corrections from Fig. 1.4. The infinities are all the same, so we may do just Fig.1.4(a),

$$\frac{(-i\,g_R)^2}{2}\int\frac{d^NK}{(2\pi)^N}\;\frac{i}{K^2-m_R^2}\cdot\frac{i}{(P-K)^2-m_R^2} = \frac{g_R^2}{2}\cdot\frac{i\,\pi^2}{(2\pi)^4}\cdot\Gamma(2-\tfrac{1}{2}N) + \text{finite terms} \quad , \tag{1.3.14}$$

so

$$g_B = \mu^{4-N}\left[g_R - \frac{3\,g_R^2}{16\,\pi^2(N-4)}\right] \quad . \tag{1.3.15}$$



(a) (b) (c) (d)

Fig. 1.4. Vertex corrections in two loops.

The two-loop contributions to the propagator and their counterterms are shown in Fig. 1.5. Fig. 1.5(a) gives

$$-\frac{i}{2}\mu^{4-N}\left[\frac{-3g_R^2}{16\pi^2(N-4)}\right]\int\frac{d^NK}{(2\pi)^N}\,\frac{i}{K^2-m_R^2}=\frac{3}{2}\cdot\frac{\mu^{4-N}\cdot g_R^2}{16\pi^2(N-4)}\left\{\frac{i\pi^{\frac{N}{2}}}{(m_R^2)^{1-\frac{N}{2}}}\cdot\right.$$

(1.3.16)

$$\left.\cdot\frac{\Gamma(1-\frac{N}{2})}{(2\pi)^N}+i\int\frac{d^3K}{(2\pi)^3}\,\frac{1}{(\vec{K}^2+m_R^2)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}-1}\right\}\;.$$

Fig. 1.5(b) gives

$$\frac{i\,g_R\cdot\mu^{4-N}\cdot m_R^2}{(4\pi)^2\,(N-4)}\left(\frac{-i\,g}{2}\right)\int\frac{d^NK}{(2\pi)^N}\left(\frac{i}{K^2-m_R^2}\right)^2=-\,\frac{g_R^2\,m_R^2\,\mu^{4-N}}{2\,(4\pi)^2\,(N-4)}\left\{\right.$$

$$\left\{\frac{i\,\pi^{\frac{N}{2}}}{(m_R^2)^{2-\frac{N}{2}}}\cdot\frac{\Gamma(2-\frac{N}{2})}{2(2\pi)^N}-i\int\frac{d^3K}{(2\pi)^3}\left(\frac{\beta\,e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}}{\left[e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}-1\right]^2}\cdot\frac{1}{2(\vec{K}^2+m_R^2)}+\right.\right.$$

$$\left.\left.+\frac{1}{e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}-1}\cdot\frac{1}{2(\vec{K}^2+m_R^2)^{\frac{3}{2}}}\right)\right\}\,. \qquad (1.3.17)$$

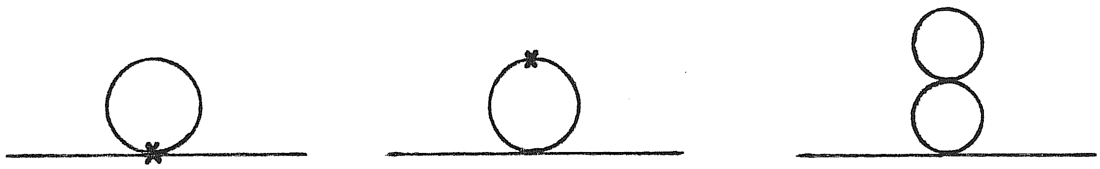
Fig. 1.5(c) gives

$$\mu^{8-2N}\cdot\frac{(-i\,g_R)^2}{4}\int\frac{d^NK}{(2\pi)^N}\left(\frac{i}{K^2-m_R^2}\right)^2\int\frac{d^N\ell}{(2\pi)^N}\cdot\frac{i}{\ell^2-m_R^2}=$$

$$=i\,\mu^{8-2N}\cdot\frac{g_R^2}{4}\left[\frac{i\,\pi^{\frac{N}{2}}\,\Gamma(2-\frac{N}{2})}{(m_R^2)^{2-\frac{N}{2}}\cdot(2\pi)^N}-i\int\frac{d^3K}{(2\pi)^3}\left(\frac{\beta\,e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}}{\left[e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}-1\right]^2}\cdot\right.\right. \qquad (1.3.18)$$

$$\left.\left.\cdot\frac{1}{2(\vec{K}^2+m_R^2)}+\frac{1}{2\left[e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}-1\right]}\cdot\frac{1}{(\vec{K}^2+m_R^2)^{\frac{3}{2}}}\right)\right]\cdot\left\{-i\,\pi^{\frac{N}{2}}\frac{\Gamma(1-\frac{N}{2})\,(m_R^2)^{\frac{N}{2}-1}}{(2\pi)^N}-\right.$$

$$\left.-i\int\frac{d^3K}{(2\pi)^3}\cdot\frac{1}{(\vec{K}^2+m_R^2)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}^2+m_R^2)^{\frac{1}{2}}}-1}\right\}\,.$$



(a) (b) (c)

Fig. 1.5. Two-loop corrections and counterterms to the propagator.

The overlapping divergence in two-loop order is the last diagram to evaluate, Fig. 1.6. This gives

$$
\begin{aligned}
&(-ig)^2\,\frac{i^3}{6}\int\frac{d^NK_2}{(2\pi)^N}\int\frac{d^NK_1}{(2\pi)^N}\,\frac{1}{K_2^2-m_R^2}\,\frac{1}{K_1^2-m_R^2}\cdot\frac{1}{(K_2-K_1-P)^2-m_R^2}= \\
&=(-ig)^2\,\frac{i^3}{6}\left[\frac{i}{\beta}\sum_{N_2}\int\frac{d^3K_2}{(2\pi)^3}\right]\left\{-i\int\frac{d^3K_1}{(2\pi)^3}\cdot\frac{1}{2(\vec{K}_1^2+m_R^2)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}_1^2+m_R^2)^{\frac{1}{2}}}-1}\cdot\right. \\
&\frac{1}{K_2^2-m_R^2}\cdot\frac{1}{(K_2-K_1-P)^2-m_R^2}\Bigg|_{K_1^0=\pm(\vec{K}_1^2+m_R^2)^{\frac{1}{2}}}-i\int\frac{d^3K_1'}{(2\pi)^3}\cdot\frac{1}{2(\vec{K}_1'^2+m_R^2)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}_1'^2+m_R^2)^{\frac{1}{2}}}-1}\cdot \\
&\frac{1}{K_2^2-m_R^2}\cdot\frac{1}{(K_1'+K_2-P)^2-m_R^2}\Bigg|_{K_1^{0\prime}=\pm(\vec{K}_1'^2+m_R^2)^{\frac{1}{2}}}+\frac{1}{2\pi}\int_{-i\infty}^{i\infty}dK_1^0\int\frac{d^3\vec{K}_1}{(2\pi)^3}\,\frac{1}{K_1^2-m_R^2}\,\frac{1}{K_2^2-m_R^2}\cdot \\
&\left.\frac{1}{(K_2-K_1-P)^2-m_R^2}\right\}.
\end{aligned}
\tag{1.3.19}
$$

We use eq.(1.3.4) to transform

$$\frac{i}{\beta}\sum_{N_2}\int\frac{d^3\vec{K}_2}{(2\pi)^3}$$

into contour integrals. We see that there are temperature-dependent infinities coming from each term on the right-hand side of eq.(1.3.19). The first two terms have temperature-dependent infinities arising from the $\int_{-i\infty}^{i\infty} dK_2^0$ integration, and contribute

$$-\frac{i\,g_R^2}{48\,\pi^2(N-4)}\int\frac{d^3\vec{K}_1}{(2\pi)^3}\cdot\frac{1}{2(\vec{K}_1^2+m_R^2)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}_1^2+m_R^2)^{\frac{1}{2}}}-1} \tag{1.3.20}$$

The last term in eq.(1.3.19) has an infinity when we take the TDK_2 contour integrals and close them over the pole from $(K_2^2 - m_R^2)^{-1}$; then we evaluate the $\int_{-i\infty}^{i\infty} dK_1^0$ integral for the resulting temperature-dependent infinity, and we get

$$-\frac{i\,g_R^2}{24\,\pi^2(N-4)}\int\frac{d^3\vec{K}_1}{(2\pi)^3}\cdot\frac{1}{2(\vec{K}_1^2+m_R^2)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}_1^2+m_R^2)^{\frac{1}{2}}}-1} \tag{1.3.21}$$

Adding these two infinities (1.3.20) and (1.3.21) together we find that Fig. 1.6 contains the temperature-dependent infinity

$$-\frac{i\,g_R^2}{16\,\pi^2(N-4)}\int\frac{d^3\vec{K}_1}{(2\pi)^3}\cdot\frac{1}{(\vec{K}_1^2+m_R^2)^{\frac{1}{2}}}\cdot\frac{1}{e^{\beta(\vec{K}_1^2+m_R^2)^{\frac{1}{2}}}-1} \tag{1.3.22}$$

Adding the temperature-dependent infinities in eqs.(1.3.16), (1.3.17), (1.3.18) and (1.3.22) we see that they cancel. What remains is the usual temperature-independent infinities arising at this order from

$$(-ig)^2 \frac{i^3}{6} \int_{-i\infty}^{i\infty} dK_2^0 \int \frac{d^3\vec{K}_2}{(2\pi)^3} \int_{-i\infty}^{i\infty} dK_1^0 \int \frac{d^3\vec{K}_1}{(2\pi)^3} \frac{1}{(K_2^2 - m_R^2)} \cdot \frac{1}{(K_1^2 - m_R^2)} \cdot$$

$$\cdot \frac{1}{\left[(K_2 - K_1 - P)^2 - m_R^2\right]} \quad . \qquad (1.3.23)$$

This completes the example. The example leads to the conclusion that all ultraviolet infinities in finite-temperature field theories are the same as those at zero-temperature, no new temperature-dependent infinity appears, and the zero-temperature renormalization prescriptions suffice to eliminate all ultraviolet divergences. In Chapter II and III, when we discuss the high-temperature interquark potential, we also give an explanation for the above conclusion.

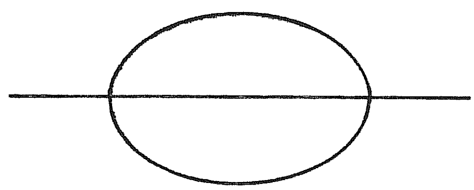


Fig. 1.6. Overlapping two-loop contribution to the propagator

1.4. The Phase Transition

In this section we would like to explore the nature of the phase transition from a confined phase in which are the usual hadronic matters to an unconfined phase in which is the quark-gluon plasma, by increasing the temperature. In the region of high-temperature phase one can use perturbation theory due to very small coupling constant $g(T)$ ($g(T) \rightarrow 0$ when $T \rightarrow \infty$). Many authors, such as A.M. Polyakov, L. Susskind [1.13] and others have rigorously proved the absence of confinement in QCD at $T \rightarrow \infty$, thus a quark-gluon-plasma must exist when the temperature is large enough. In Chapter II and III, we have also proved the unconfined nature of high-temperature phase in QCD by the example of interquark potential. In the region of low-temperature phase one has investigated the confined nature of QCD in terms of strong coupling lattice gauge theory expansions presented by Wilson [1.17] . In both limits, the low-temperature phase is confinement and the high-temperature phase is unconfinement, the problem is clear, but in the middle region, the problem is rather complicated. The mechanism of quark-gluon plasma condensation into hadron matter, which must take place at densities near the nuclear density and temperature near 200 MeV is not yet clear in detail, and there exist only qualitative considerations explaining such a phenomenon.

When the temperature (or density) increases, the average distance between the constituents of hadrons shortens. Therefore, a perturbation expansion in powers of the effective coupling may be possible in the limit of high-temperature or density, due to the asymptotic freedom of QCD at very short distances (or high momenta). Thus, sufficiently hot or dense hadron matter should become a gas of non-interacting quarks and gluons.

Strong (on the lattice, strong coupling expansion or in the continuum, semiclassical instanton method) and weak (perturbation theory) coupling approaches have evidently specified regions of applicability and thus basically give one-phase descriptions. Nevertheless, they already provide hints for a phase transition near the boundary of their regions of validity. In the perturbative treatment of the quark-gluon gas, the pressure becomes negative at some temperature value, and this has been interpreted as the onset of confinement. This approach will be shown in the subsection 1.4.1. In the strong coupling expansion, there are indications for a phase transition due to Debye screening of colour gluons, and this is the subject of the subsection 1.4.2. At high temperature, the exponentially suppression of instantons (especially, large scale instantons, $\rho \gg \frac{1}{T}$) may lead to deconfinement. The discussion concerning instantons is recommended to be read in Ref. [1.18] .

However, the above three limiting approaches cannot give us the unified whole-range (temperatures or densities) description we would like to obtain from a basic theory. Now the Monte Carlo evaluation of finite temperature QCD on the lattice provides us with a unified description. The evaluation method itself was derived and first applied to the study of the confinement problem at zero-temperature [1.19] . But its application to finite temperature statistical mechanics is quite straightforward, perhaps it is even more natural here, where the real physical temperature plays the role of the Euclidean time in the confinement problem. We shall discuss it in the subsection 1.4.3. The relations between these two phase transitions (confinement-deconfinement and chiral symmetry restoration) and their order are discussed in subsection 1.4.4.

1.4.1. Phase transition in the quark-gluon plasma

In 1979, K. Kalashnikov and V. Klimov [1.20] have given the arguments in favour of the existence of a phase transition in the quark-gluon plasma by means of perturbation theory. Their conclusions are based on investigations of the thermodynamical potential for this system.

The partition function which defines the thermodynamical properties of the quark-gluon plasma in QCD is presented by the functional integral of the usual form;

$$Z = N \int d\bar{\psi}\, d\psi\, dA\; e^{S} \quad , \tag{1.4.1}$$

where S is the Euclidean QCD action,

$$S = \int d^3x \int_0^{\beta} dx_4\, \mathcal{L}(\psi, \bar{\psi}; A) \quad , \tag{1.4.2}$$

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \frac{1}{2\alpha}\left(\partial_\mu A^a_\mu\right)^2 + \bar{C}^a\left(\partial_\mu \delta^{ab} + g f^{adb} A^d_\mu\right)\partial_\mu C^b +$$

$$+ \sum_{f=1}^{N_f} \bar{\psi}\left[\gamma_\mu\left(\partial_\mu - i g \frac{\lambda^d}{2} A^d_\mu\right) + m_f\right]\psi - \mu \sum_{f=1}^{N_f} \bar{\psi}\gamma_4\psi \quad , \tag{1.4.3}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \quad ;$$

N_f is the number of flavours, μ is the chemical potential for the baryon charge density and $\frac{\lambda^d}{2}$ are the generators of SU(3).

For the thermodynamical potential

$$\Omega = -T \ln Z \quad . \tag{1.4.4}$$

Using perturbative method the first non-trivial order (two-loop) for Ω is described by the following graphs:

$$\frac{\Omega^{(2)}}{V} = \frac{1}{12}$$  $$+\frac{1}{8}$$  $$-\frac{1}{2}$$  $$-\frac{1}{2}$$ 

where V is the volume. For the first three of above graphs the Yang-Mills fields are responsible, while the last one takes into account the contribution of quarks.

It is found that the result of the calculation of these graphs is gauge invariant, and when the quark masses, m_f, are neglected, it has the following form :

$$\frac{\Omega}{V} = -\frac{8}{45}\pi^2 T^4 - 6N_f T\int\frac{d^3k}{(2\pi)^3}\ln\left\{\left[1+e^{-\beta(k+\mu)}\right]\left[1+e^{-\beta(k-\mu)}\right]\right\} + \frac{1}{6}g^2T^4 +$$

$$+16N_f g^2\left\{\frac{1}{12}T^2\int\frac{d^3k}{(2\pi)^3}\cdot\frac{n_k}{2k} + \frac{1}{2}\left[\int\frac{d^3k}{(2\pi)^3}\cdot\frac{n_k}{2k}\right]^2\right\} \qquad (1.4.5)$$,

where $$n_k = \left\{1+e^{\beta(k-\mu)}\right\}^{-1} + \left\{1+e^{\beta(k+\mu)}\right\}^{-1} .$$

The expression (1.4.5) can be specified here both in the high-temperature and in the high-density region. For the high-temperature region, $\frac{\mu}{T} \ll 1$, then eq.(1.4.5) can be written as

$$\frac{\Omega}{V} = -\frac{8}{45}\pi^2T^4 - N_f\left[\frac{1}{2}T^2\mu^2 + \frac{7}{60}\pi^2T^4\right] +$$

$$+ g^2\left[\frac{1}{6}T^4 + N_f\left(\frac{5}{72}T^4 + \frac{T^2\mu^2}{4\pi^2}\right)\right] . \qquad (1.4.6)$$

According to the renormalization group equation, the coupling constant g should be replaced by an effective coupling constant $g_{eff}(T)$,

$$\frac{g^2}{4\pi} \longrightarrow \frac{g^2_{eff}(T)}{4\pi} = \frac{1}{1 + b_o \ln \frac{T}{S_T}} \qquad (1.4.7)$$

where $b_o = \left(\frac{11}{2} - \frac{1}{3} N_f\right) \cdot \pi^{-1}$ is due to the usual β -function, and S_T is a normalization point. Thus, using (1.4.7), eq.(1.4.6) becomes

$$-p = \frac{\Omega}{V} = -\frac{8}{45}\pi^2 T^4 - N_f\left[\frac{1}{2}T^2\mu^2 + \frac{7}{60}\pi^2 T^4\right] + + 3\pi^2 \alpha_{eff}(T)\left[\frac{1}{6}T^4 + N_f\left(\frac{5}{72}T^4 + \frac{T^2\mu^2}{4\pi^2}\right)\right] , \qquad (1.4.8)$$

where p is the pressure, and for $SU(N)_C$,

$$\alpha_{eff}(T) = \frac{g^2_{eff}(T)}{4\pi} \cdot \frac{1}{\pi} \cdot \frac{N^2-1}{2N} . \qquad (1.4.9)$$

Note that eq.(1.4.8) is valid only for $\frac{\mu}{T} \ll 1$. Thus, for $\mu \ll 1$, when the temperature decreases, the pressure becomes zero at some $T = T_o$, then

$$\alpha_{eff}(T_o) = \frac{4}{3}\left(\frac{4}{15} + \frac{7}{40}N_f\right) \cdot \left(1 + \frac{5}{12}N_f\right)^{-1} . \qquad (1.4.10)$$

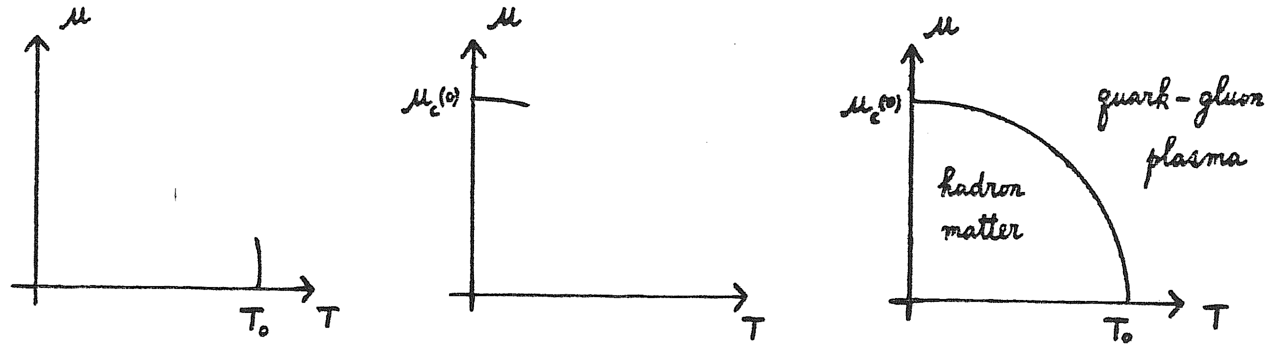
Therefore, in the vicinity of the point T_o , the pressure is zero on the curve $\mu_c = \mu_c(T)$ which is determined by the following expression

$$\frac{\mu_c^2}{T} = \frac{15\pi^2}{N_f} \cdot \frac{\left(1 + \frac{5}{12}N_f\right)^2}{7 + N_f}\left[\alpha_{eff}(T) - \alpha_{eff}(T_o)\right] . \qquad (1.4.11)$$

As the temperature decreases further, the pressure becomes negative. This means that the quark-gluon plasma becomes unstable under such conditions, thus a phase transition to the hadron matter must take place. The curve $\mu_c = \mu_c(T)$ in the vicinity of the point T_0 is drawn in Fig. 1.7(a).

An analogous analysis arises in the high-density region. The curve $\mu_c = \mu_c(T)$ in the vicinity of the point $\mu_c(0)$ is drawn in Fig. 1.7(b). In the region of intermediate temperatures and densities, as the temperature increases the phase transition occurs at lower baryon densities, therefore, the curve for $\mu_c = \mu_c(T)$ in the whole region of T and μ will probably look as shown in Fig. 1.7(c).

The above constructive arguments by perturbative treatment make one believe the existence of a phase transition from the quark-gluon plasma to the hadron matter, though they are not rigorous quantitatively. In the next subsection, we shall give a more convincing argument for the existence of a phase transition of confinement-deconfinement by means of lattice model.



(a) (b) (c)

Fig. 1.7. Phase diagrams

1.4.2. Lattice models

In this subsection we follow mainly the two papers [1.13] by A. Polyakov and L. Susskind. First, we come to the Polyakov's proof of the absence of confinement in the temperature $T \to \infty$ limit. Its main idea is that in such a limit the quantum fluctuations of the gauge fields can be neglected and that they may be treated as classical fields. This statement is motivated from the fact that, for example, the frequencies $\omega_n = 2\pi nT \to \infty$ with T and that only terms with n = 0 are essential. Zero frequency means no dependence on the time τ , so one can integrate over the period $\frac{1}{T}$ and come to the classical expression for the statistical sum of partition function as the phase space integral:

$$Z = \int \mathcal{D}x \, \mathcal{D}p \; e^{-\frac{H}{T}} \quad , \qquad (1.4.12)$$

where H is the Hamiltonian of the system. In the $A^a_o = 0$ gauge, eq.(1.4.12) can be written as

$$Z = \int \mathcal{D} A^a_n \, \mathcal{D} E^a_n \; \delta(\nabla_n E_n) \, \delta(\partial_n A^a_n) \, \det[\partial_n \nabla_n] \cdot e^{-\frac{H}{T}} \quad . \qquad (1.4.13)$$

Then one may introduce the scalar potential

$$\delta(\nabla_n E_n) = \int d\varphi \; \exp\left[\frac{i}{T} \int d^3x \cdot \varphi^a \nabla_n E^a_n\right] \quad , \qquad (1.4.14)$$

and write (1.4.13) in the following way:

$$Z = \int \mathcal{D}\varphi^a \mathcal{D}A^a_n \cdot \delta(\partial_n A^a_n) \cdot \det(\partial_n \nabla_n) \cdot \exp\left\{-\frac{1}{2g^2 T} \int d^3x \left[(\nabla_m \varphi^a)^2 + (H^a_m)^2 \right]\right\} \quad . \qquad (1.4.15)$$

Next we introduce some external current $J^a_\mu(x)$, which adds a new factor $\exp\left[i \int A^a_\mu J^a_\mu \cdot d^4x\right]$ in the statistical sum. Taking $J^a_\mu(x)$ to be the two heavy quarks at points $\vec{R}_1$, $\vec{R}_2$, the expression for the free energy, F_{12}, associated with these quarks can be written as

$$e^{-\frac{F_{12}}{T}} = \left\langle Tr\left\{exp\left[-\frac{i}{2T}\varphi^a(\vec{R}_1)\cdot\lambda^a\right]\right\}\cdot Tr\left\{exp\left[\frac{i}{2T}\varphi^b(\vec{R}_2)\cdot\lambda^b\right]\right\}\right\rangle , \tag{1.4.16}$$

where the right side of eq.(1.4.16) is the expectation of the product of these two Wilson loops. The integral over φ^a is Gaussian. The result is

$$e^{-\frac{F_{12}}{T}} = \int \mathcal{D}A_n^a \cdot \delta(\partial_n A_n^a)\cdot det(\partial_n \nabla_n)\cdot det^{-\frac{1}{2}}(\nabla^2)\cdot$$
$$\cdot exp\left[-\frac{1}{2Tg^2}\int d^3x\cdot(H_m^a)^2\right]\cdot exp\left[-\frac{g^2}{8T}\lambda^a G_{ab}(\vec{R}_1-\vec{R}_2)\cdot\lambda^b\right] , \tag{1.4.17}$$

where G_{ab} is the Green function of the equation

$$\nabla^2 G_{ab}(\vec{R}_1-\vec{R}_2) = \delta_{ab}\cdot\delta^3(\vec{R}_1-\vec{R}_2) , \tag{1.4.18}$$

and

$$\nabla_n E_n(x) \equiv \partial_n E_n(x) + [A_n, E_n] . \tag{1.4.19}$$

Now, for all decent $A_n(x)$, we have a well defined $G_{ab}(\vec{R}_1 - \vec{R}_2)$ with the property

$$G_{ab}(\vec{R}_1-\vec{R}_2) \xrightarrow[|\vec{R}_1-\vec{R}_2|\to\infty]{} 0 . \tag{1.4.20}$$

Hence

$$F_{12} \xrightarrow[|\vec{R}_1-\vec{R}_2|\to\infty]{} \text{constant.} \tag{1.4.21}$$

This means that there is no confinement at $T \longrightarrow \infty$. Since there was confinement at $T = 0$, this shows that we must have a phase transition at some $T = T_C$.

Below, we discuss the lattice model of quark confinement. Here we follow the paper by L. Susskind [1.13].

Here, we shall only discuss a simple Abelian strong-coupling model, in which confinement is rigorous at zero temperature. We suppose that the model is defined on a three-dimensional lattice. The vector field $A_{x,n}$ is defined on its sites X_n. The Hamiltonian is

$$H = \frac{g^2}{2} \sum_{x,m} (E_{x,m})^2 \quad ; \qquad E_{x,m} = \frac{1}{i} \frac{\partial}{\partial A_{x,m}} \quad . \tag{1.4.22}$$

The field is assumed to be periodic $-\pi < A_{x,n} < \pi$ and only configurations satisfying the following constraint

$$\sum_m \left(E_{x,m} - E_{x-a_m,m} \right) = 0 \tag{1.4.23}$$

are discussed, where a_m is the unit vector of the lattice. In the continuous limit eq.(1.4.23) is the usual $\partial_n E_n = 0$ condition. Such a sum is denoted by a prime index, for example the statistical sum is written as

$$Z = \sum{}' \exp\left[-\frac{g^2}{2T} \sum_{m,x} (E_{x,m})^2 \right] . \tag{1.4.24}$$

Introducing the potential φ as in (1.4.14) in order to account for (1.4.23), one has

$$Z = \prod_x \int_{-\pi}^{\pi} d\varphi_x \, \widetilde{\exp}\left[-\frac{T}{2g^2} (\varphi_x - \varphi_{x+a})^2 \right] , \tag{1.4.25}$$

where $\widetilde{\exp}(-x^2)$ is the so-called periodic Gaussian function [1.13] :

$$\widetilde{\exp}(-x^2) \equiv \sum_{m=-\infty}^{\infty} \exp\left[-(x+2\pi m)^2\right] = \sum_{n=-\infty}^{\infty} \exp\left[-\frac{n^2}{4} + i n x\right] \quad . \tag{1.4.26}$$

At T $\gg$ 1 (high temperature), one has

$$Z \underset{T\to\infty}{=\!=} \int \mathcal{D}\varphi \cdot \exp\left[-\frac{T}{2g^2}(\partial_m \varphi)^2\right] \quad . \tag{1.4.27}$$

The free energy connected with two heavy quarks at $\vec{R}_1$ and $\vec{R}_2$ may be computed to be

$$F_{12} \equiv -T \cdot \ln \left\langle e^{i\varphi(\vec{R}_1)} \cdot e^{-i\varphi(\vec{R}_2)} \right\rangle = -\frac{g^2}{|\vec{R}_1 - \vec{R}_2|} + \text{constant} \tag{1.4.28}$$

Thus, at high temperature we have usual Coulomb forces and no confinement. For non-abelian case, see Susskind's paper [1.13] . The result of non-abelian study shows that at high-temperature, thermal fluctuations eventually cause the colored gluons to form a plasma which Debye screens the quark [1.9] .

1.4.3. Monte Carlo evaluation

The Monte Carlo evaluations for finite-temperature QCD on lattice allow us to study QCD thermodynamics over the whole temperature range. Here we only give some literatures and main results.

Firstly consider the case of pure Yang-Mills theory, since the inclusion of quarks is much more complicated. The calculations based on color SU(2) [1.8, 1.21] have given the critical temperature of confinement-deconfinement,

$$T_c \approx 180 - 200 \text{ MeV} .$$

For $SU(3)_c$ [1.22 , 1.23] ,

$$T_c \approx 160 - 180 \text{ MeV} .$$

Thus, the dependence of T_c on the colour group is little.

Next consider the case including quarks, this brings in a basically new feature - the question of chiral symmetry restoration at high temperature. The lattice formulation encounters as a result the problem of species doubling, and in addition the Monte Carlo evaluation becomes considerably more complex. Nevertheless, first results both on the full QCD energy density [1.24, 1.23] and on chiral symmetry restoration [1.24, 1.25] have now appeared. In Fig. 1.8 [1.24] H. Satz and others show the overall energy density ε/T^4 , obtained by combining the pure Yang-Mills results. One concludes that full QCD with fermions indeed appears to lead to the deconfinement behaviour observed in the study of Yang-Mills systems alone. In particular, one notes that at temperatures $T \gtrsim 2T_c$ essentially all constituent degrees of freedom have been "thawed". In the following subsection we discuss the chiral symmetry restoration.

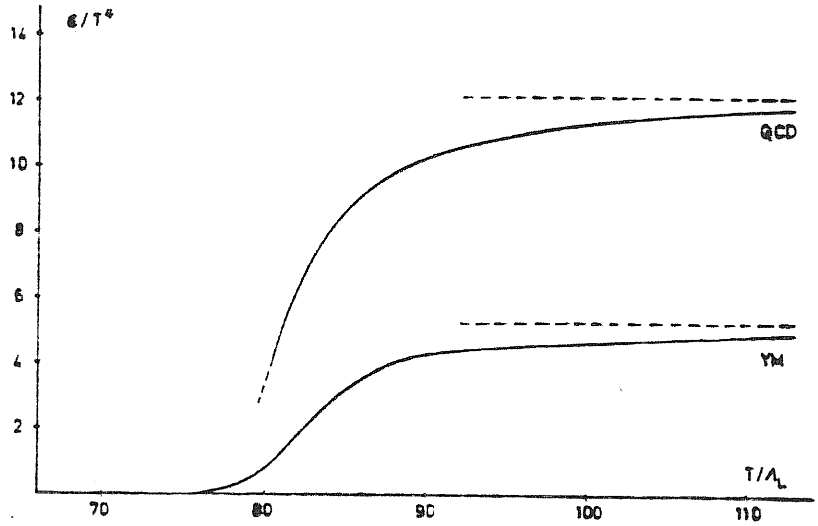


Fig. 1.8. Comparison of the energy density of full QCD with that of the SU(3) Yang-Mills theory.

1.4.4. Deconfinement and chiral symmetry restoration

In QCD thermodynamics, there are two phase transitions: one is the phase transition from confinement to deconfinement, another is the phase transition from chiral broken to chiral restored symmetry. The two phase transitions are characterized by two critical temperatures, T_c and T_{ch}. Above T_c, the temperature is high enough to render confinement unimportant: hadrons dissolve into quarks and gluons. Above T_{ch}, chiral symmetry is restored, so that quarks must be massless. For T below both T_c and T_{ch}, we have a matter of massive hadrons; for T above both T_c and T_{ch}, we have a plasma of massless quarks and gluons. The simplest conceptual situation would be $T_c = T_{ch}$; the possibility $T_c > T_{ch}$ appears rather unlikely, because it ($T_c > T > T_{ch}$) would mean that the massless quarks were bounded into hadrons.

On the other hand, $T_c < T_{ch}$ would correspond to a region of unbound massive "constituent" quarks.

In Ref. [1.24] , it is shown that chiral symmetry restoration slightly above deconfinement, with

$$T_{ch}/T_c \approx 1.3 \quad .$$

In other Refs. [1.8, 1.25]

$$T_{ch}/T_c = 1.6 \pm 0.2 \quad .$$

There also is [1.21] $T_{ch}/T_c = 1.0 \pm 0.1$.

Therefore the question of whether or not $T_c = T_{ch}$ remains an open one.

Recently, it is shown in Ref.[1.8] that these two phase transitions are second order for $SU(2)_c$, and are first order for $SU(3)_c$.

1.5. Perturbation theory in high-temperature QCD

In this section we shall study QCD at high temperature by examining the perturbative behaviour of the theory. Since the running coupling g(T) vanishes as $T \longrightarrow \infty$, we may hope that perturbation theory would be reliable at sufficiently high temperature.

1.5.1. Running coupling constant g(T)

According to general renormalization group arguments, we have

$$\int_{g}^{g(t)} \frac{dx}{A x^3} = t \quad , \tag{1.5.1}$$

where $A = -\frac{11 C_2}{48 \pi^2}$ is from β (g(t)) function for the pure Yang-Mills field without additional matter field, C_2 is the eigenvalue of the quadratic Casimir operator of the gauge group SU(N), $t = \ln \frac{\xi}{\mu}$, ξ may be large momentum transfer, high temperature or high density.

From eq.(1.5.1), we obtain

$$g^2(T,\mu) = \frac{1}{g^{-2} - 2A \ln \frac{T}{\mu}} \tag{1.5.2}$$

where μ is the renormalization mass scale.

Therefore, when $T \longrightarrow \infty$, $g^{-2}(T,\mu) \gg g^{-2}$, we can obtain that the effective running coupling constant is

$$g^2(T,\mu) = \frac{24\,\pi^2}{11 \cdot N \cdot \ln \frac{T}{\mu}} \qquad \text{for } SU(N), \qquad (1.5.3)$$

Thus, it can be seen that the running coupling constant $g(T,\mu)$ vanishes as the temperature $T \longrightarrow \infty$.

1.5.2. Free energy

The finite-temperature behaviour of any theory is specified by the partition function

$$Z = \mathrm{Tr}(e^{-\beta H})$$

and the thermal expectation of physical observables,

$$\langle Q \rangle = \frac{1}{Z} \mathrm{Tr}(e^{-\beta H} Q)$$

In the standard fashion we may derive functional integral representations for these quantities. For gauge theories,we find

$$Z = \int \mathcal{D}A_\mu \cdot \mathcal{D}\bar{\psi} \cdot \mathcal{D}\psi \; \exp\left[-\frac{1}{g^2} S(A, \bar{\psi}, \psi)\right] , \qquad (1.5.4)$$

where

$$S = \int_0^\beta dt \left(\mathcal{L}_g + \mathcal{L}_m\right) ,$$

$$\mathcal{L}_g = -\frac{1}{2} \int d^3x \; \mathrm{tr}\left(F_{\mu\nu} F_{\mu\nu}\right) ,$$

and

$$\mathcal{L}_m = \sum_{i=1}^{N_f} \int d^3x \; \bar{\psi}_i \left(\not{D} + m_i\right) \psi_i .$$

The functional integral is restricted to fields satisfying the periodic conditions,

$$A_\mu(\beta, \vec{x}) = A_\mu(0, \vec{x}) \;, \quad \psi(\beta, \vec{x}) = -\psi(0, \vec{x}) , \quad \bar{\psi}(\beta, \vec{x}) = -\bar{\psi}(0, \vec{x}) .$$

We apply standard perturbative expansions to the functional integral (1.5.4). We find that the leading contribution to the free energy density, $\mathcal{F} = -\frac{\ln Z}{\beta V}$, is given by

$$\mathcal{F}_0 = -\frac{\pi^2 T^4}{45}\left(N^2-1+\frac{7}{4}N\cdot N_f\right) ,$$

(1.5.5)

where, for SU(N), N_f is the fermion flavour number. This is simply the free energy of a gas of noninteracting massless particles [1.26] .

Kapusta [1.26] has evaluated the second and third perturbative corrections to the result (1.5.5). He found

$$\mathcal{F} = \mathcal{F}_0 + \frac{g^2 T^4}{16}\cdot\frac{1}{9}(N^2-1)\left(N+\frac{5}{4}N_f\right) - \frac{g^3 T^4}{12\pi}(N^2-1)\left[\frac{1}{3}\left(N+\frac{N_f}{2}\right)\right]^{\frac{3}{2}} . \quad (1.5.6)$$

The first $O(g^2)$ correction comes from the two-loop graphs shown in Fig. 1.9. The $O(g^3)$ term is the leading contribution from the sum of ring diagrams shown in Fig. 1.10.

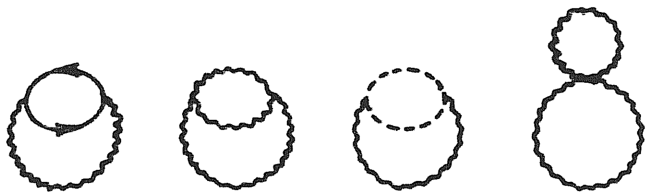


Fig. 1.9. $O(g^2)$ contributions to the free energy

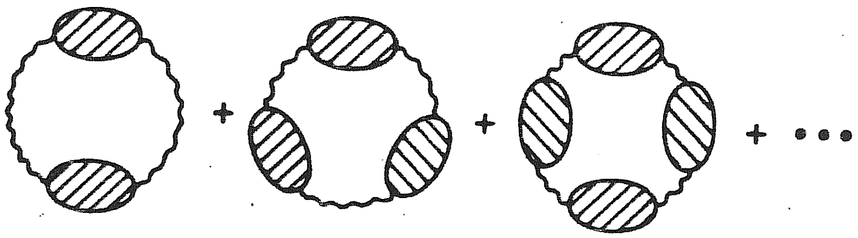


Fig. 1.10. $O(g^3)$ contribution to the free energy

1.5.3. Gluon self-energy

The full gluon propagator $\mathcal{D}^{ab}_{\mu\nu}(x) = \langle T[A^a_\mu(x)\, A^b_\nu(0)] \rangle$ may be expressed in terms of the one-particle irreducible self-energy, $\Pi^{ab}_{\mu\nu}(\omega,\vec{k})$,

$$\mathcal{D}^{ab}_{\mu\nu}(\omega_n,\vec{k}) = \left[(\omega_n^2 + \vec{k}^2)\, \delta^{ab}\delta_{\mu\nu} + \Pi^{ab}_{\mu\nu}(\omega_n,\vec{k}) \right]^{-1} . \tag{1.5.7}$$

We may construct four independent, symmetric, O(3) covariant tensors depending on a single vector k . For example,

$$A_{\mu\nu} = \delta_{\mu i}\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right)\delta_{j\nu} \quad ,$$

$$B_{\mu\nu} = \left(\delta_{\mu 0} - \frac{k_\mu k_0}{k^2}\right)\frac{k^2}{\vec{k}^2}\left(\delta_{\nu 0} - \frac{k_\nu k_0}{k^2}\right) \quad ,$$

$$C_{\mu\nu} = \frac{1}{\sqrt{2}}\left(\delta_{\mu\nu} - \frac{k_\mu k_0}{k^2}\right)\frac{k_\nu}{|\vec{k}|} + \frac{1}{\sqrt{2}}\cdot\frac{k_\mu}{|\vec{k}|}\left(\delta_{\nu 0} - \frac{k_\nu k_0}{k^2}\right),$$

$$D_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} \quad .$$

Note that

$$A_{\mu\nu} + B_{\mu\nu} = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \quad .$$

Thus the self-energy, which is always diagonal in color, may be decomposed as

$$\Pi^{ab}_{\mu\nu} = \delta^{ab}\,\Pi_{\mu\nu} = \left(\alpha A_{\mu\nu} + \beta B_{\mu\nu} + \gamma C_{\mu\nu} + \delta D_{\mu\nu}\right)\delta^{ab} \quad .$$

$\mathcal{D}_{\mu\nu}$ satisfies a Ward identity, which implies that $k_\mu k_\nu \mathcal{D}_{\mu\nu} = 1$. At zero temperature this condition plus Euclidean invariance implies that $k_\mu \Pi_{\mu\nu} = 0$, or

$$\Pi_{\mu\nu} = \frac{1}{3}\,\Pi_{\alpha\alpha}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \quad .$$

However, at finite temperature this merely provides one relation among the above coefficients, namely,

$$\delta = \frac{\gamma^2}{2(k^2+\beta)} \quad .$$

To $O(g^2)$, the self-energy is given by the one-loop diagrams in Fig. 1.11. Explicitely, these yield

$$\Pi^{ab}_{\mu\nu}(k) = \frac{g^2}{2}\,\delta^{ab} N_f \int_- \frac{d^4q}{(2\pi)^4}\, tr\left[\gamma_\mu (\not{k} + \not{q})\gamma_\nu \not{q}\right] / q^2 (k+q)^2 \; +$$

$$+ \frac{g^2}{2}\,\delta^{ab} N \int_+ \frac{d^4q}{(2\pi)^4} \Big\{ 2k_\mu k_\nu - 4(k+q)_\mu q_\nu - 4q_\mu (k+q)_\nu + 2\delta_{\mu\nu}\left[(k+q)^2 + q^2 - \right. \tag{1.5.8}$$

$$\left. \left. - 2k^2\right]\right\} / q^2 (k+q)^2 \quad .$$

Note that to one-loop order, $\Pi_{\mu\nu}$ is transverse for any frequency, and

$$\int_+ \frac{d^4p}{(2\pi)^4} = \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{1}{\beta} \cdot \sum_{\substack{n \\ \omega_n^+ = \frac{2\pi n}{\beta}}} \quad \text{and} \quad \int_- \frac{d^4p}{(2\pi)^4} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\beta} \sum_{\substack{n \\ \omega_n^- = \frac{(2n+1)\pi}{\beta}}} .$$

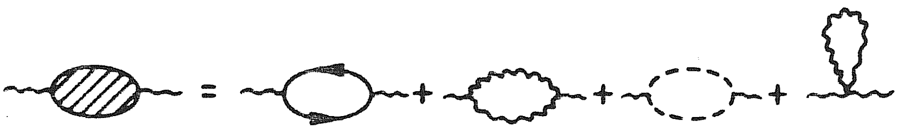


Fig. 1.11. $O(g^2)$ gluon self-energy

To study the screening caused by thermal fluctuations we should examine $\Pi_{\mu\nu}(\omega,\vec{k})$ for low spatial momentum, $\vec{k}\sim 0$. In Feynman gauge, eq.(1.5.8) can be written as

$$\Pi_{\mu\nu}(p) = g^2 N_f \int_- \frac{d^4q}{(2\pi)^4} \cdot \frac{I_{\mu\nu}(p,q)}{q^2(p+q)^2} - g^2 N \int_+ \frac{d^4q}{(2\pi)^4} \frac{I_{\mu\nu}(p,q)+J_{\mu\nu}(p)}{q^2(p+q)^2} , \qquad (1.5.9)$$

where

$$I_{\mu\nu}(p,q) = 2(p+q)_\mu q_\nu + 2q_\mu(p+q)_\nu - \delta_{\mu\nu}\left[(p+q)^2 + q^2 - p^2\right] ,$$

$$J_{\mu\nu}(p) = \delta_{\mu\nu} p^2 - p_\mu p_\nu .$$

At zero temperature one may evaluate $\Pi_{\mu\nu}$ using standard techniques (dimensional regularization). One finds

$$\Pi_{\mu\nu}(p)\Big|_{T=0} = \frac{1}{3} \frac{g^2}{(4\pi)^2}(5N - 2N_f)\left(\delta_{\mu\nu}p^2 - p_\mu p_\nu\right)\cdot \ln\left(\frac{p^2}{\mu^2}\right) ,$$

where μ is the renormalization point.

At finite temperature, $\Pi_{\mu\nu}(p)$ has in general three independent components (due to Ward identities, four independent components become as three). However, since the one-loop expression, eq.(1.5.9), is transverse (due to $p_\mu \Pi_{\mu\nu}(p) = 0$) for any energy, this reduces the number of components to two (for example, Π_{oo} and Π_{ii}). Furthermore, we shall restrict our evaluation of $\Pi_{\mu\nu}(p_o, \vec{p})$ to the limit of low spatial momentum, $\vec{p} \sim 0$. In this limit we evaluate the on-loop self-energy. When $p_o \neq 0$, the denominator of (1.5.9) may be expanded in powers of $\vec{p}$ without causing any infrared divergence. Consequently, $\Pi_{\mu\nu}(p_o \neq o, \vec{p})$ has a finite asymptotic expansion in powers of $\vec{p}$. The transversality of $\Pi_{\mu\nu}$ then implies

$$\Pi_{\mu\nu}(p_o \neq 0, \vec{p}) \sim \frac{1}{3}\left[\Pi_{\alpha\alpha}(p_o, \vec{p}=0)\right] \delta_{\mu i}\delta_{\nu i} + O(\vec{p}) .$$

On the other hand, at zero external energy, $\Pi_{\mu\nu}$ may be decomposed as

$$\Pi_{\mu\nu}(p_o=0, \vec{p}) = \left[\Pi_{oo}(p_o=0, \vec{p})\right] \delta_{\mu o}\delta_{\nu o} + \frac{1}{2}\left[\Pi_{ii}(p_o=0, \vec{p})\right] \delta_{\mu i}\left(\delta_{ij} - \frac{p_i p_j}{\vec{p}^2}\right) \delta_{j\nu} .$$

$\Pi_{ii}(p_o, \vec{p} = 0)$ can be nonvanishing only if $\Pi_{ij}(p_o = 0, \vec{p})$ has a directional singularity at $\vec{p} = 0$. However, the only part of (1.5.9) which cannot be expanded in powers of $\vec{p}$ when $p_o = 0$ is the contribution from the region where $q_o = 0$ and $\vec{q} \sim O(\vec{p})$. This contribution is of order $|\vec{p}|^3 \; \vec{p}^2/\vec{p}^4 \sim O(\vec{p})$ and hence vanishes at zero momentum. Consequently, $\Pi_{ii}(p = 0) = 0$ so that

$$\Pi_{\mu\nu}(p_o=0, \vec{p}) \sim \left[\Pi_{\alpha\alpha}(p_o=0, \vec{p}=0)\right] \delta_{\mu o}\delta_{\nu o} + O(p) .$$

Thus, we need only to compute the trace of the one-loop self-energy at zero spatial momentum. Eq.(1.5.5) yields

$$\Pi_{\mu\mu}(p) = -2g^2 N_f \int_- \frac{d^4q}{(2\pi)^4} \frac{[(p+q)^2+q^2-p^2]}{q^2\,(p+q)^2} + 2g^2 N \int_+ \frac{d^4q}{(2\pi)^4} \cdot \frac{[(p+q)^2+q^2-\frac{5}{2}p^2]}{q^2\,(p+q)^2} .$$

Eqs. (1.3.3) and (1.3.4) may now be used to extract the temperature-dependent part,

$$\delta\,\Pi_{\mu\mu}(p) = \Pi_{\mu\mu}(p) - \Pi_{\mu\mu}(p)\Big|_{T=0} .$$

we find

$$\delta\,\Pi_{\mu\mu}(p_0,\vec{p}=0) = 2g^2 \int \frac{d^4q}{(2\pi)^4} \cdot \frac{4}{q^2}\left[\frac{N}{e^{-i\beta q_0+\epsilon}-1} + \frac{N_f}{e^{-i\beta q_0+\epsilon}+1}\right] -$$

$$-g^2 \int \frac{d^4q}{(2\pi)^4} \frac{p^2}{q^2}\left[\frac{1}{(p+q)^2} + \frac{1}{(p-q)^2}\right] \cdot \left[\frac{5N}{e^{i\beta q_0+\epsilon}-1} + \frac{2N_f}{e^{-i\beta q_0+\epsilon}+1}\right] =$$

$$= 4g^2 \int \frac{d^3q}{(2\pi)^3} \cdot \frac{1}{|\vec{q}|}\left[\frac{N}{e^{\beta|\vec{q}|}-1} + \frac{N_f}{e^{\beta|\vec{q}|}+1}\right] - 2g^2 \cdot \mathrm{Re} \int \frac{d^3q}{(2\pi)^3} \cdot \frac{p_0}{|\vec{q}|(p_0+2i|\vec{q}|)} \cdot$$

$$\cdot \left[\frac{5N}{e^{\beta|\vec{q}|}-1} + \frac{2N_f}{e^{\beta|\vec{q}|}+1}\right] = \frac{2g^2}{\pi^2} \int_0^\infty dq \cdot q \left[\frac{N}{e^{\beta q}-1} + \frac{N_f}{e^{\beta q}+1}\right] -$$

$$-\frac{g^2 \cdot p_0^2}{4\pi^2} \int_0^\infty \frac{dq \cdot q}{q^2+\frac{p_0^2}{4}} \left[\frac{5N}{e^{\beta q}-1} + \frac{2N_f}{e^{\beta q}+1}\right] = \frac{1}{3} g^2 T^2 \left(N + \frac{N_f}{2}\right) -$$

$$-\frac{g^2p_o^2}{8\pi^2}(5N+2N_f)\left[\frac{\ln(\beta p_o)}{4\pi}-\psi\left(\frac{\beta p_o}{4\pi}\right)-\frac{2\pi}{\beta p_o}\right]+\frac{g^2p_o^2}{4\pi^2}(2N_f)\left[\ln\left(\frac{\beta p_o}{2\pi}\right)-\psi\left(\frac{\beta p_o}{2\pi}\right)-\right.$$

$$\left.-\frac{\pi}{\beta p_o}\right]=\frac{1}{3}g^2T^2\left(N+\frac{N_f}{2}\right)-\frac{g^2p_o^2}{(4\pi)^2}(5N-2N_f)\cdot\ln\left(\frac{\beta p_o}{4\pi}\right)^2+\frac{g^2p_o^2}{8\pi^2}\left\{5N\left[\frac{1}{n}+\psi\left(\frac{n}{2}\right)\right]-\right.$$

$$\left.-2N_f\left[2\psi(n)-\psi\left(\frac{n}{2}\right)-\ln 4\right]\right\},$$

where $\quad p_o=\frac{2\pi n}{\beta}\quad,\quad \psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}\quad.$

As $p_o\to 0$,

$$\delta\,\Pi_{\mu\mu}(p_o,\vec{p}=0)\sim\frac{1}{3}g^2T^2\left(N+\frac{N_f}{2}\right).$$

Therefore, we have

$$\Pi_{\mu\nu}(\omega_n=0,\vec{k}\to 0)\sim\frac{1}{3}g^2T^2\left(N+\frac{N_f}{2}\right)\delta_{\mu o}\,\delta_{\nu o}\,. \qquad (1.5.10)$$

The result (1.5.10) shows that A_o develops a one-loop mass due to the thermal fluctuations. This mass, which we shall call the "electric" mass m_{el}, is given by

$$m_{el}^2=\Pi_{oo}(\omega_n=0,\vec{k}=0)=\frac{1}{3}g^2T^2\left(N+\frac{N_f}{2}\right).$$

Note that both quark and gluon fluctuations contribute to the mass. The possibility for this electric mass is a direct consequence of the fact that at finite temperature the only way to approach zero (**four**) momentum, k=0, is to first set $k_o=\omega_n=0$ and then let $\vec{k}\to 0$ (since $k_o=\omega_n=\frac{2\pi n}{\beta}$, $T\neq 0$, thus $\omega_n=0$

only when n = 0). Hence $\Pi_{oo}(\omega_n = 0, \vec{k})$ is unconstrained by the transversality of the self-energy and so need not vanish at $\vec{k} = 0$ (the transversality of the self-energy means $p_\mu \Pi_{\mu\nu} = 0$, now $p_o = \omega_n = 0$, thus $\Pi_{oo}(\omega_n = 0, \vec{k}\to 0)$ need not vanish). But, the static spatial self-energy $\Pi_{ij}(\omega_n = 0,\vec{k})$ must be transverse, $k_i \Pi_{ij}(\omega_n = 0, \vec{k}) = 0$, then

$$\Pi_{ij}(\omega_n=0, \vec{k}) = \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) \Pi_{kk}(\omega_n=0, \vec{k}) . \tag{1.5.11}$$

If $\Pi_{ii}(k = 0)$ is nonzero, then the two transverse components of $\vec{A}$ will have developed a "magnetic" mass, $m^2_{mag} = \frac{1}{2}\Pi_{ii}(\omega_n=0, \vec{k}=0)$, but, as mentioned above, this Π_{ii} is of order $|\vec{k}|^3 \vec{k}^2/\vec{k}^4$ when $\vec{k} \longrightarrow 0$, $\Pi_{ii}(\omega_n=0, \vec{k}\to 0) \longrightarrow 0$. Therefore, to one-loop order the spatial components of the gauge field $\vec{A}$ remain massless.

This electric mass m_{el} implies that

$$\langle A_o(x) A_o(y)\rangle \sim e^{-m_{el}|\vec{x}-\vec{y}|} ,$$

as $|\vec{x} - \vec{y}| \longrightarrow \infty$. The nonzero frequency correlations fall off much more rapidly, as $e^{-\omega_n|\vec{x}-\vec{y}|}$. Thus A_o acquires a finite correlation length of $(m_{el})^{-1}$. Provided that higher-order corrections remain unimportant, this result implies that

$$\langle \mathrm{tr}\,\Omega(\vec{x})\, \mathrm{tr}\,\Omega^\dagger(0)\rangle \sim 1 + g^4 O(e^{-2m_{el}|\vec{x}|}) . \tag{1.5.12}$$

Note that the one gluon exchange term, of order $g^2 e^{-m_{el}|\vec{x}|}$, vanishes due to the separate traces in (1.5.12). Consequently, the heavy quark potential behaves as

$$V(R) \sim g^4 O(e^{-2m_{el}R}) , \qquad \text{as } R\to\infty . \tag{1.5.13}$$

Eq. (1.5.13) shows that heavy quarks are unconfined at high temperatures. This lack of confinement is caused by the screening of the (color) charge of the heavy quarks due to the thermal fluctuations.

At one-loop order, since $\Pi_{00}(\omega=0,\vec{k}\to 0)=m_{el}^2$ and $\Pi_{ij}(\omega=0,\vec{k}\to 0)\to 0$, eq.(1.5.7) becomes

$$\mathcal{D}^{ab}_{\mu\nu}(\omega_n=0,\vec{k}\to 0)=\left\{\left[\vec{k}^2\cdot I+\Pi(\omega_n=0,\vec{k}\to 0)\right]^{-1}\right\}^{ab}_{\mu\nu} , \tag{1.5.14}$$

where I is the unity matrix and $\Pi(\omega_n=0,\vec{k}\to 0)$ is the matrix whose matrix elements are $\Pi_{\mu\nu}$. Since $\mathcal{D}^{ab}_{\mu\nu}$ is diagonal for a,b, we have

$$\mathcal{D}^{ab}_{\mu\nu}(\omega_n=0,\vec{k}\to 0)=\delta^{ab}\begin{pmatrix}\frac{1}{\vec{k}^2+m_{el}^2} & 0 & 0 & 0\\ 0 & \frac{1}{\vec{k}^2} & 0 & 0\\ 0 & 0 & \frac{1}{\vec{k}^2} & 0\\ 0 & 0 & 0 & \frac{1}{\vec{k}^2}\end{pmatrix}_{\mu\nu}$$

Therefore

$$\mathcal{D}^{ab}_{\mu\nu}(\omega_n=0,\vec{k}\to 0)=\delta^{ab}\left\{\frac{\delta_{\mu 0}\,\delta_{\nu 0}}{\vec{k}^2+m_{el}^2}+\frac{\delta_{\mu i}\,\delta_{\nu i}}{\vec{k}^2}\right\} . \tag{1.5.15}$$

Since for a static electric field

$$\langle \vec{E}(\vec{x})\cdot\vec{E}(\vec{x}')\rangle = \langle \vec{\partial}A_0(\vec{x})\cdot\vec{\partial}A_0(\vec{x}')\rangle \quad + \text{(higher orders)},$$

eq.(1.5.15) may be interpreted as showing that a static external electric field is screened by the plasma of thermal excitations. $(m_{el})^{-1}$ is the electric length. However, for a static magnetic field

$$\langle \vec{B}(\vec{x})\,\vec{B}(\vec{x}')\rangle = \langle \vec{\partial}\times\vec{A}(\vec{x})\;\vec{\partial}\times\vec{A}(\vec{x}')\rangle + (h.o)\;.$$

Consequently, eq.(1.5.15) shows that a static external magnetic field is unscreened and so penetrates the plasma.

1.6. Infrared problem

In this section, we would like to point out some difficulties, encountered in the thermodynamics of hadronic matter at high temperatures. The source of these difficulties is the infrared problem in the thermodynamics of the massless Yang-Mills gas.

To clarify this point, let us consider a contribution to the thermodynamic potential $\Omega(T)$ from the diagrams of the order g^{2N}, containing N four-gluon vertices [1.27 , 1.28]. After a summation over the Lorentz and isotopic indices, this contribution can be represented as

$$\Omega_N(T) \sim (2\pi T)^{N+1}\cdot g^{2N}\int d^3\vec{p}_1\, d^3\vec{p}_2 \cdots d^3\vec{p}_{N+1}\cdot \sum_{n_i=-\infty}^{\infty}\prod_{k=1}^{2N}\left[(2\pi r_k T)^2 + \vec{q}_k^{\,2} + m^2(T)\right]^{-1}, \qquad (1.6.1)$$

where $\vec{q}_k$ is a uniform linear combination of $\vec{p}_i$, r_k is the corresponding combination of integers n_i, i = 1, 2,..., N+1, k = 1,2,... 2N, and m(T) is an infrared cutoff, which may appear due to high temperature effects. We remind that the summation over n_i in quantum statistics at $T \neq 0$ of eq.(1.6.1) plays the same role as the integration over $(p_o)_i$ in quantum field theory. At $m(T) \rightarrow 0$ the leading term in the sum over n_i is the term with all $n_i = 0$ (and, consequently $r_k = 0$). Therefore, at $m(T) \rightarrow 0$

$$\Omega_N(T) \sim (2\pi T)^{N+1} g^{2N} \int d^3\vec{p}_1 \cdot d^3\vec{p}_2 \cdots d^3\vec{p}_{N+1} \cdot \prod_{k=1}^{2N} \left(\vec{q}_k^{\,2} + m^2(T)\right)^{-1} \sim g^6 T^4 \left(\frac{g^2 T}{m(T)}\right)^{N-3} \tag{1.6.2}$$

From eq.(1.6.2) it is clear that, in the absence of the infrared cutoff (i.e. $m(T) = 0$), the contributions to $\Omega(T)$ of $N = 4,5,6$ etc. are of the order g^8, g^{10}, g^{12} etc. and contain power (rather than usual logarithmic) infrared divergences. From eq.(1.6.2) we see a general rule, that is, the leading infrared divergences in the quantum statistics of interacting bosons at a finite temperature are just these of the three-dimensional quantum field theory [1.12]. At $m(T) < g^2 T$, the dangerous terms in eq.(1.6.2) become finite, but due to these terms, higher orders of perturbation theory for $\Omega(T)$ become greater than the lowest ones [1.27]. Therefore it is possible to compute the thermodynamic potential of the Yang-Mills gas by means of perturbation theory only if a sufficiently large infrared cutoff appears in the theory at finite temperature. But, the infrared cutoff mass is at most of order $g^2 T$, therefore, beyond some order, perturbative expansions are meaningless. Some non-pertubative methods have to be incorporated. These problems will be discussed in detail in chapter IV.

CHAPTER II - RENORMALIZATION OF THE TWO-GLUON EXCHANGE CONTRIBUTIONS TO HIGH-TEMPERATURE INTERQUARK POTENTIAL

2.1 Introduction

The QCD possesses the feature of the almost non-interacting behaviour of quarks at short distances, known as asymptotic freedom. After the discovery of asymptotic freedom in the non-Abelian gauge theories, one has been interested in the thermodynamics of the Yang-Mills gas. One could study properties of strongly interacting matter at high temperature, $T \gtrsim 150$ MeV, or at high density, $\rho \gtrsim 10^{15}$g. There are many reasons for the investigation of the properties of hadronic matter at high temperature or high density. Firstly, there are the places where one might look for the effects of high temperature or large baryon density on the structure of hadronic matter. For example, the interior of neutron stars and the early stages of the universe evolution. Secondly, there are purely theoretical reasons for deep understanding of hadronic matter. According to general renormalization group arguments, the effective coupling constant g(T) between quarks and gluons vanishes as the temperature $T \longrightarrow \infty$, therefore we can study the properties of the Yang-Mills gas by means of perturbation theory.

In sect. 2 of this chapter we have discussed renormalization of the two-gluon exchange contributions to high-temperature interquark potential using perturbative theory.

In sect. 3 we would like to point out some difficulties which come from the infra-red problem encountered in the thermodynamics of the massless Yang-Mills gas.

2.2 Interquark potential

One can consider the thermal Wislon loop $\langle \mathrm{Tr}\,\Omega\,(\vec{x})\rangle$ as an order parameter associated to a phase transition from a confined phase to an unconfined one, as the temperature increases (the physical meaning of Wilson loop, see Appendix A). The thermal Wilson loop is defined to be

$$\langle \mathrm{Tr}\,\Omega(\vec{x})\rangle = \left\langle \mathrm{Tr}\, T\, \exp\left[ig\int_0^\beta d\tau\, \lambda^a A_0^a(\tau,\vec{x})\right]\right\rangle , \tag{2.2.1}$$

where it is assumed that the system is placed in a heat bath at a physical temperature, $1/\beta = T$. The expectation value is taken over the gauge fields $A_\mu^a(\tau, \vec{x})$, whose dynamics is here assumed to be described by a pure Yang-Mills field Lagrangian without additional matter fields. The integral over the Euclidean time τ in eq. (2.2.1) is actually an integral over a closed path due to the periodicity conditions $A_\mu^a(\vec{x}, 0) = A_\mu^a(\vec{x}, \beta)$. T represents the time ordering, and λ^a are the matrices representing the generators of a non-Abelian gauge group. We define the symbol Tr = trace/d(R) in such a way that TrI = 1, I being the identity and d(R) the dimensionality of the representation.

The thermal Wilson loop is relative to the free energy of an isolated quark (relative to the volume) [1.14, 2.1] (see Appendix B), i.e.

$$\exp[-\beta F_q] = \langle \mathrm{Tr}\,\Omega(\vec{x})\rangle , \tag{2.2.2}$$

where F_q is the free energy of the isolated quark. But the two-point function of the thermal Wilson loop is relative to the free energy, $F_{q\bar{q}}$, of a $q\bar{q}$ pair, i.e.

$$\exp[-\beta F_{q\bar{q}}] = \langle \mathrm{Tr}\,\Omega^\dagger(\vec{x})\, \mathrm{Tr}\,\Omega(\vec{y})\rangle , \tag{2.2.3}$$

where we refer to $F_{q\bar{q}} = V(\vec{x} - \vec{y})$ as the interaction potential between q and $\bar{q}$. Note that here a quark is considered as a static external particle, i.e. a heavy quark.

We are following Gross and others [1.11, 1.18], according to general renormalization group arguments, the running coupling constant should be $g^2(T,\mu)/8\pi^2 = (3/11N)(\ln(T/\mu))^{-1}$ for the group SU(N). Thus it can be seen that the running coupling constant $g(T,\mu)$ vanishes as the temperature $T \to \infty$, so that we may hope that perturbation theory is reliable at sufficiently high temperatures. We calculate below the two-point function of the thermal Wilson loop of eq. (2.2.3) using a perturbative expansion in powers of $g(T,\mu)$.

We write the two-point function of the thermal Wilson loop, $\langle \mathrm{Tr}\,\Omega^{+}(\vec{x})\,\mathrm{Tr}\,\Omega(\vec{y})\rangle$, as

$$\langle \mathrm{Tr}\,\Omega^{+}(\vec{x})\,\mathrm{Tr}\,\Omega(\vec{y})\rangle =$$

$$= \left\langle \mathrm{Tr}\left\{ \bar{T}\, exp\left[-ig\int_0^\beta d\tau\, \lambda^a A_0^a(\tau,\vec{x})\right]\right\} \mathrm{Tr}\left\{ T\, exp\left[ig\int_0^\beta d\tau'\, \lambda^{a'} A_0^{a'}(\tau',\vec{y})\right]\right\}\right\rangle, \quad (2.2.4)$$

where $\bar{T}$ means anti-time-ordering.

$$\langle \mathrm{Tr}\,\Omega^{+}(\vec{x})\,\mathrm{Tr}\,\Omega(\vec{y})\rangle = \langle \mathrm{Tr}\,\Omega^{+}(\vec{x})\,\mathrm{Tr}\,\Omega(\vec{y})\rangle_c + \langle \mathrm{Tr}\,\Omega^{+}(\vec{x})\rangle\langle \mathrm{Tr}\,\Omega(\vec{y})\rangle, \quad (2.2.5)$$

where the subscript c represents the connected two-point function of the thermal Wilson loop.

Thus

$$\langle \mathrm{Tr}\,\Omega^{+}(\vec{x})\,\mathrm{Tr}\,\Omega(\vec{y})\rangle_c = \frac{g^4}{4}\,\mathrm{Tr}(\lambda^a\lambda^b)\,\mathrm{Tr}(\lambda^{a'}\lambda^{b'})\cdot$$

$$\cdot\left\langle \bar{T}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\, A_0^a(\tau_1,\vec{x})\,A_0^b(\tau_2,\vec{x})\cdot T\int_0^\beta d\tau_1'\int_0^\beta d\tau_2'\, A_0^{a'}(\tau_1',\vec{y})\,A_0^{b'}(\tau_2',\vec{y})\right\rangle_c -$$

$$-\frac{g^6}{2\cdot 4!}\,\mathrm{Tr}(\lambda^a\lambda^b)\,\mathrm{Tr}(\lambda^{a'}\lambda^{b'}\lambda^{c'}\lambda^{d'})\left\langle \bar{T}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\, A_0^a(\tau_1,\vec{x})\,A_0^b(\tau_2,\vec{x})\,\cdot\right.$$

$$\left.\cdot\, T\int_0^\beta d\tau_1'\int_0^\beta d\tau_2'\int_0^\beta d\tau_3'\int_0^\beta d\tau_4'\, A_0^{a'}(\tau_1',\vec{y})\,A_0^{b'}(\tau_2',\vec{y})\,A_0^{c'}(\tau_3',\vec{y})\,A_0^{d'}(\tau_4',\vec{y})\right\rangle_c -$$

$$-\frac{g^6}{2\cdot 4!}\,Tr(\lambda^a\lambda^b\lambda^c\lambda^d)\,Tr(\lambda^{a'}\lambda^{b'})\,\cdot$$

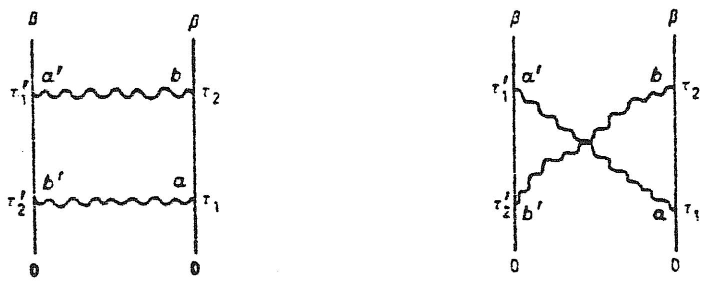
$$\cdot\Big\langle \bar{T}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\int_0^\beta d\tau_3\int_0^\beta d\tau_4\; A_0^a(\tau_1,\vec{x})\,A_0^b(\tau_2,\vec{x})\,A_0^c(\tau_3,\vec{x})\,A_0^d(\tau_4,\vec{x})\,\cdot$$

$$\cdot\, T\int_0^\beta d\tau_1'\int_0^\beta d\tau_2'\; A_0^{a'}(\tau_1',\vec{y})\,A_0^{b'}(\tau_2',\vec{y})\Big\rangle_c + \frac{g^6}{3!\,3!}\,Tr(\lambda^a\lambda^b\lambda^c)\,Tr(\lambda^{a'}\lambda^{b'}\lambda^{c'})\,\cdot$$

$$\cdot\Big\langle \bar{T}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\int_0^\beta d\tau_3\; A_0^a(\tau_1,\vec{x})\,A_0^b(\tau_2,\vec{x})\,A_0^c(\tau_3,\vec{x})\,\cdot$$

$$\cdot\, T\int_0^\beta d\tau_1'\int_0^\beta d\tau_2'\int_0^\beta d\tau_3'\; A_0^{a'}(\tau_1',\vec{y})\,A_0^{b'}(\tau_2',\vec{y})\,A_0^{c'}(\tau_3',\vec{y})\Big\rangle_c + O(g^7)\,. \qquad (2.2.6)$$

The Feynman diagrams corresponding to the connected two-point function of the thermal Wilson loop given by eq. (2.2.6) are the following (up to $O(g^6)$): Fig. 2.1 - 2.4 .



(a) (a*)

Fig. 2.1 - The connected two-point function of the thermal Wilson loop of $O(g^4)$. The wavy line represents the internal gluon line, and the solid line represents the time axis.

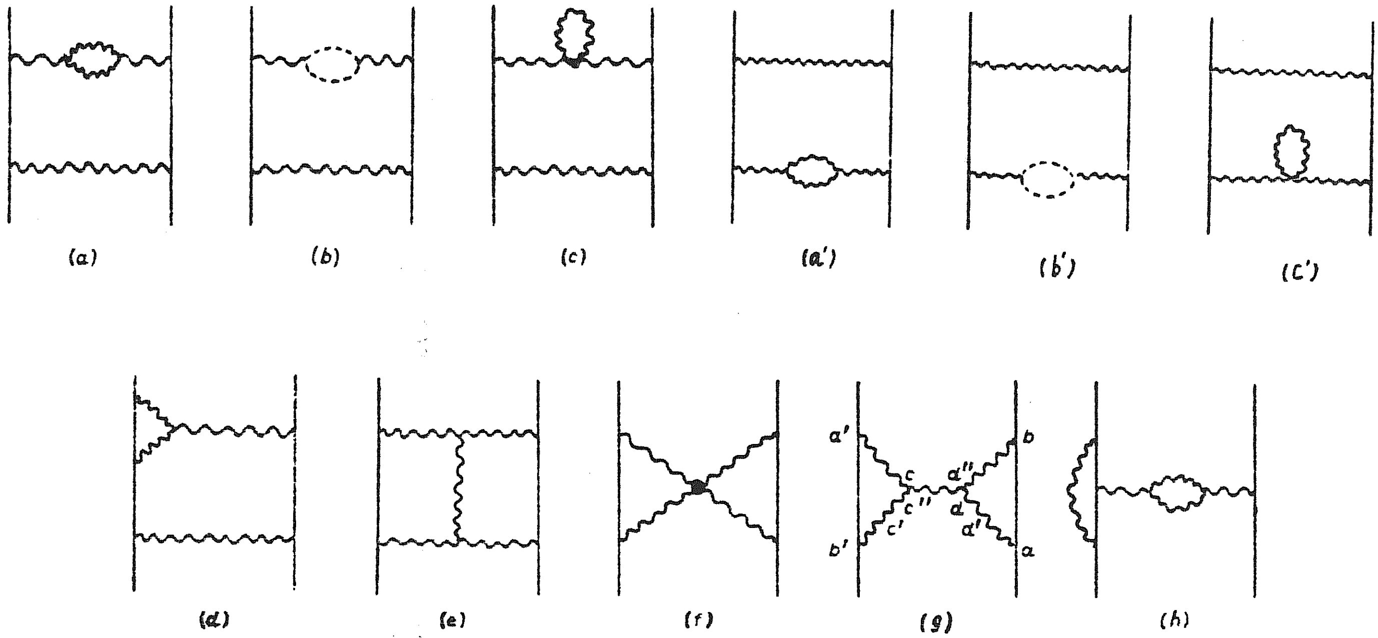


Fig. 2.2 - The connected two-point function of the thermal Wilson loop of $O(g^6)$, where the broken line represents the ghost line and there is a cross-diagram corresponding to every diagram given above.

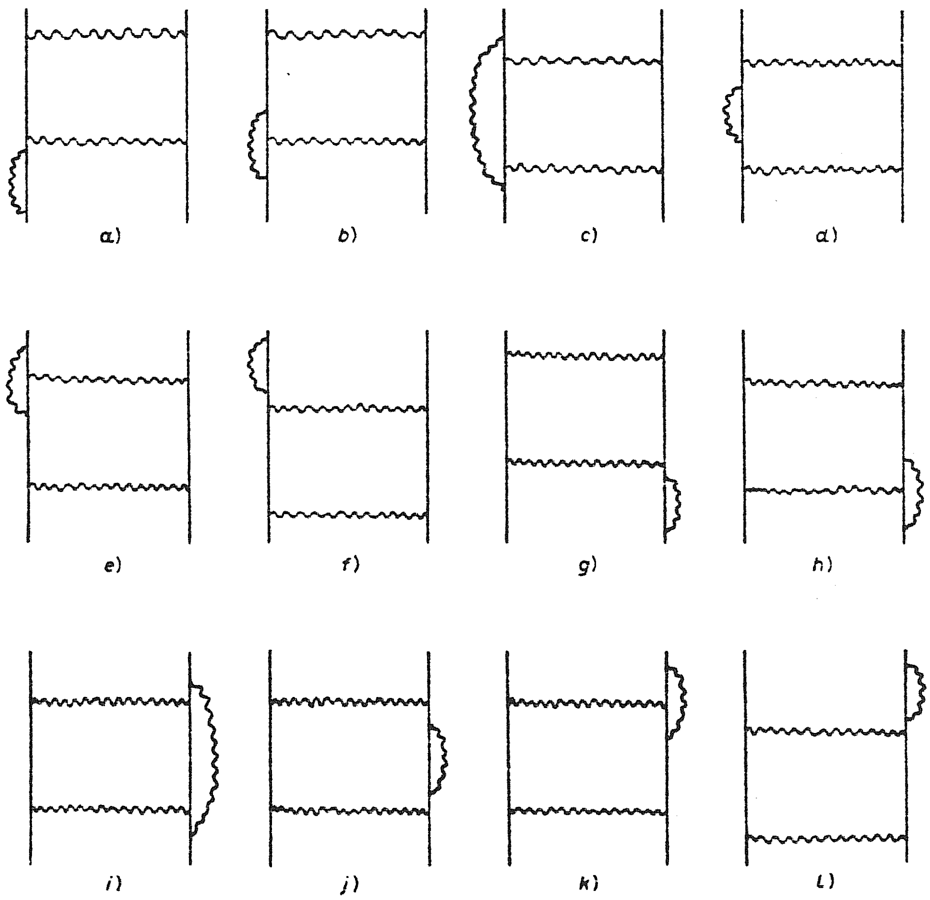


Fig. 2.3 - The connected two-point function of the thermal Wilson loop of $O(g^6)$. There is a cross-diagram corresponding to every diagram shown above.

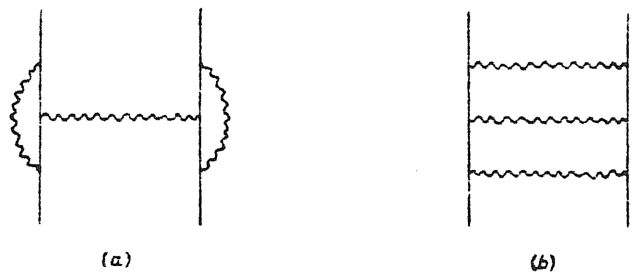


Fig. 2.4 - The connected two-point function of the thermal Wilson loop of $O(g^6)$. There are five crossed diagrams corresponding to diagram (b).

Since the thermal Wilson loop is gauge invariant [2.1], our calculations can be done in a specific gauge. We choose to work in Feynman gauge ($\alpha = 1$); furthermore, we note that only the time component, A_0^a, of the gauge fields A_μ^a appears in the two-point function of the thermal Wilson loop. It then follows that the contributions of the diagrams (d) and (f) in Fig. 2.2 to $\langle \mathrm{Tr}\,\Omega^\dagger(\vec{x})\mathrm{Tr}\,\Omega(\vec{y})\rangle_c$ vanish. The contributions of fig. 2.2(h) and fig. 2.4(a) to $\langle \mathrm{Tr}\,\Omega^\dagger(\vec{x})\mathrm{Tr}\,\Omega(\vec{y})\rangle_c$ vanish, due to $\mathrm{Tr}\,\lambda^a = 0$ and $\mathrm{Tr}(\lambda^a\lambda^b\lambda^c)\,\delta_{ac} = 0$. The contribution of fig. 2.2(g) to the two-point function of the thermal Wilson loop is also zero, due to the relation

$$\mathrm{Tr}(\lambda^{a'}\lambda^{b'})\, C_{cc'c''}\,\delta_{a'c}\,\delta_{b'c'} = 0 \quad ,$$

where the group structure constant $C_{cc'c''}$ comes from the three-gluon vertex.

Figure 2.4(b) accounting for three-gluon exchange is not considered in this chapter.

We now compute in detail the contribution $G_1(\vec{x}-\vec{y})$ of the diagrams (a) and (a*) in fig. 2.1 to the $\langle \mathrm{Tr}\,\Omega^\dagger(x)\mathrm{Tr}\,\Omega(y)\rangle_c$:

$$G_1(\vec{x}-\vec{y}) = g^4\,\mathrm{Tr}(\lambda^a\lambda^b)\,\mathrm{Tr}(\lambda^{a'}\lambda^{b'})\int_0^\beta d\tau_1'\int_0^{\tau_1'} d\tau_2'\int_\beta^0 d\tau_1\int_\beta^{\tau_1} d\tau_2 \cdot$$

$$\cdot\left[\Delta_{a'b}(\tau_1'-\tau_2,\vec{y}-\vec{x})\,\Delta_{b'a}(\tau_2'-\tau_1,\vec{y}-\vec{x})+\Delta_{a'a}(\tau_1'-\tau_1,\vec{y}-\vec{x})\,\Delta_{b'b}(\tau_2'-\tau_2,\vec{y}-\vec{x})\right], \tag{2.2.7}$$

where

$$\Delta_{a'a}(\tau_1'-\tau_1,\vec{y}-\vec{x})=\frac{1}{\beta(2\pi)^3}\sum_{n=-\infty}^{\infty}\int d^3\vec{k}\,\frac{\exp[i(2\pi/\beta)n(\tau_1'-\tau_1)+i\vec{k}\cdot(\vec{y}-\vec{x})]}{(2\pi n/\beta)^2+\vec{k}^2}\,\delta_{a'a} ,$$

choosing the Feynman gauge, and n is integer.

Since the integral

$$\int_0^\beta d\tau_1'\int_0^{\tau_1'} d\tau_2'\int_\beta^0 d\tau_1\int_\beta^{\tau_1} d\tau_2 \cdot$$

$$\cdot\left\{\exp\left[i\frac{2\pi}{\beta}(n\tau_1'-n\tau_2+m\tau_2'-m\tau_1)\right]+\exp\left[i\frac{2\pi}{\beta}(n\tau_1'-n\tau_1+m\tau_2'-m\tau_2)\right]\right\}=$$

$$=\begin{cases}\beta^4/2 , & \text{only when } n=0,\ m=0 ,\\ 0 , & \text{in other cases} ,\end{cases} \tag{2.2.8}$$

we get

$$G_1(\vec{x}-\vec{y})=\frac{g^4Q^2(N^2-1)\beta^2}{2^5\pi^2}\cdot\frac{1}{(\vec{x}-\vec{y})^2} , \tag{2.2.9}$$

where $Tr(\lambda^a\lambda^b) = Q\,\delta_{ab}$, and Q = 1/2N in the fundamental representation of the group SU(N), and we refer to the gauge field $A^a_\mu(\tau , x)$ as a $SU(N)_c$ gauge field. We have used the dimensional regularization procedure.

Now calculate the contributions of the divergent parts of the diagrams (a), (b) and (c) in fig. 2.2 and all diagrams in fig. 2.3 to the two-point function of the thermal Wilson loop.

The contributions of the diagrams (a), (b), (c) and their crossed partners in fig. 2.2 are $G_{2(a,b,c)}(\vec{x}-\vec{y})$,

$$G_{2(a,b,c)}(\vec{x}-\vec{y}) = g^4 Q^2 \int_0^\beta d\tau_1' \int_0^{\tau_1'} d\tau_2' \int_\beta^0 d\tau_1 \int_\beta^{\tau_1} d\tau_2 \Big[(Z_3-1)\,\Delta_{a'b}(\tau_1'-\tau_2,\vec{x}-\vec{y})\cdot\Delta_{b'a}(\tau_2'-\tau_1,\vec{x}-\vec{y})+$$

$$+(Z_3-1)\,\Delta_{a'a}(\tau_1'-\tau_1,\vec{x}-\vec{y})\cdot\Delta_{b'b}(\tau_2'-\tau_2,\vec{x}-\vec{y})\Big]\,\delta_{ab}\,\delta_{a'b'} = \frac{(Z_3-1)\,g^4 Q^2 (N^2-1)\,\beta^2}{2^5\,\pi^2(\vec{x}-\vec{y})^2} =$$

$$= \frac{10\,g^2 C_2}{48\,\pi^2 \epsilon}\cdot\frac{g^4 Q^2 (N^2-1)\,\beta^2}{2^5\,\pi^2(\vec{x}-\vec{y})^2}\quad, \tag{2.2.10}$$

where Z_3 is the renormalization constant of the gluon wave function, and the pole $1/\epsilon$ is coming from the dimensional regularization scheme, $\Gamma\,(3-d) = 1/\epsilon$ is the gamma function, and $d = 3-\epsilon$ is the number of dimensions in the integral, and $C_2 = N$ is the eigenvalue of the quadratic Casimir operator in the adjoint representation of the group SU(N).

The contributions of the diagram (e) and its crossed partner in fig. 2.3 to the $\langle \mathrm{Tr}\,\Omega^{+}(\vec{x})\,\mathrm{Tr}\,\Omega(\vec{y})\rangle_c$ are $G_{3(e)}(\vec{p})$ in momentum space,

$$G_{3(e)}(\vec{p}) = g^6\, Q\, \mathrm{Tr}(\lambda^a\lambda^b\lambda^a\lambda^b)\left[\frac{1}{(2\pi)^3\beta}\right]^3 \cdot \tag{2.2.11}$$

$$\cdot\frac{\beta^6\,\pi^d\,\Gamma(1-d/2)\,\Gamma(2-d/2)\left[\Gamma(d/2-1)\right]^2\,\zeta(4-d)}{4\pi^2(2\pi/\beta)^{2-d}\,(|\vec{p}|)^{4-d}\,\Gamma(d-2)}\quad,$$

where $\zeta\,(4-d)$ is the Riemann zeta function. The Riemann zeta function is defined to be $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$, Re $S > 1$, however it can be analytically continued to the whole complex plane without the point $S = 1$.

$$\mathrm{Tr}(\lambda^a\lambda^b\lambda^a\lambda^b) = \mathrm{Tr}(\lambda^a\lambda^a\lambda^b\lambda^b) + \mathrm{Tr}(\lambda^a[\lambda^b,\lambda^a]_-\lambda^b) =$$

$$= \mathrm{Tr}(\lambda^a\lambda^a\lambda^b\lambda^b) + i f_{bac}\,\mathrm{Tr}(\lambda^a\lambda^c\lambda^b)\,. \tag{2.2.12}$$

The first part $\mathrm{Tr}(\lambda^a\lambda^a\lambda^b\lambda^b)$, in eq. (2.2.12) is Abelian which is cancelled by diagram (f) in the fig. 2.3. The second part $if_{bac}\,\mathrm{Tr}(\lambda^a\lambda^c\lambda^b)$ in eq. (2.2.12) is non-Abelian, where f_{bac} are the structure constants of the gauge group:

$$i f_{bac}\,\mathrm{Tr}(\lambda^a\lambda^c\lambda^b) = -\frac{1}{4}(N^2-1)\,.$$

Due to

$$\zeta(s) = 2^s\pi^{s-1}\sin(\tfrac{1}{2}\pi s)\,\Gamma(1-s)\,\zeta(1-s) \qquad \text{and} \quad d=3-\epsilon\,,$$

we have

$$\zeta(4-d) = \zeta(1+\epsilon) \underset{\epsilon\to 0}{=} 2^{1+\epsilon}\pi^{\epsilon}\sin\left[\tfrac{1}{2}\pi(1+\epsilon)\right]\Gamma(-\epsilon)\,\zeta(-\epsilon) = \frac{1}{\epsilon}\,. \tag{2.2.13}$$

Using eq.(2.2.13) we obtain the non-Abelian part of $G_{3(e)}(\vec{p})$ as

$$\frac{g^6 Q^2 C_2 (N^2-1)\beta^2}{2^{10}\pi^5|\vec{p}|\epsilon}\,. \tag{2.2.14}$$

In co-ordinate space, the non-Abelian part of $G_{3(e)}(\vec{x}-\vec{y})$ as

$$\frac{g^6 Q^2 C_2 (N^2-1)\beta^2}{2^8\pi^4\epsilon(\vec{x}-\vec{y})^2}\,.$$

After computing various diagrams in fig. 2.3, we discover that the contributions of the diagrams (a), (d), (f), (g), (j) and (l) to $\langle \mathrm{Tr}\Omega^\dagger(\vec{x})\mathrm{Tr}\Omega(\vec{y})\rangle_c$ are the same. The contributions of diagrams (b), (e), (h) and (k) in fig. 2.3 are also the same, and the contributions of the diagrams (c) and (i) in fig. 2.3 are equal and give minus the contributions of the diagrams (a) and (d) in fig. 2.3. Thus the total contribution of all diagrams in the fig. 2.3 is equal to $G_3(\vec{x}-\vec{y})$

$$G_3(\vec{x}-\vec{y}) = \frac{g^6 Q^2 C_2 (N^2-1)\beta^2}{2^6 \pi^4 \epsilon (\vec{x}-\vec{y})^2} \quad . \tag{2.2.15}$$

The total contribution of the two diagrams of fig. 2.1, the diagrams (a), (b) and (c) in fig. 2.2 (note that also included the diagrams (a'), (b') and (c') in fig. 2.2) and all diagrams of the fig. 2.3 to the connected two-point function of the thermal Wilson loop $\langle \mathrm{Tr}\Omega^{+}(\vec{x})\mathrm{Tr}\Omega(\vec{y})\rangle$ is

$$G_1(\vec{x}-\vec{y}) + G_{2(a,b,c)}(\vec{x}-\vec{y}) + G_3(\vec{x}-\vec{y}) =$$

$$= \frac{g^4 Q^2 (N^2-1)\beta^2}{2^5 \pi^2 (\vec{x}-\vec{y})^2}\left[1 + \frac{g^2}{16\pi^2}\frac{1}{\epsilon}\frac{20}{3}C_2 + \frac{g^2 C_2}{2\pi^2\epsilon}\right] =$$

$$= \frac{g^4 Q^2 (N^2-1)\beta^2}{2^5 \pi^2 (\vec{x}-\vec{y})^2}\left[1 + \frac{11 g^2 C_2}{12\pi^2\epsilon}\right] = \frac{\bar{g}^4 Q^2 (N^2-1)\beta^2}{2^5 \pi^2 (\vec{x}-\vec{y})^2} \quad , \tag{2.2.16}$$

where

$$\bar{g}^4 = g^4\left(1 + \frac{11 g^2 C_2}{12\pi^2 \epsilon}\right) \tag{2.2.17}$$

is the renormalized coupling constant, which is consistent with the well-known charge renormalization in QCD [1.1].

From the above calculations (fig. 2.1, 2.3 and diagrams (a), (b) and (c) in fig. 2.2) we can see that in the non-Abelian case all ultra-violet divergence appearing in $\langle \mathrm{Tr}\Omega^{+}(\vec{x})\mathrm{Tr}\Omega(\vec{y})\rangle_c$ are logarithmic divergences up to $O(g^6)$ (due to using the dimensional regularization) which are absorbed by the coupling constant renormalization. The renormalization of the coupling constant in eq. (2.2.17) involves two parts, one is the gluon Green's function renormalization, and another is the renormalization of the vertex which describes gluon radiation by a static quark.

Now, we consider the contribution of the diagram (e) in fig. 2.2. An infra-red cut-off mass m(T) will be needed in the gluon propagators due to high-temperature effects [1.18]. The time component $A_0^a(\tau, \vec{x})$ of the Yang-Mills field develops an "electric" mass $M_{el}(T)$ which equal $\sqrt{(N/3)}gT$ for one-loop correction of zero-zero component of the gluon propagator in SU(N), whereas the space components $A_i^a(\tau, \vec{x})(i = 1,2,3)$ give a "magnetic" mass M_{mag}. The "electric" mass $M_{el}(T) = \sqrt{(N/3)}gT \neq 0$ at finite temperature represents the Debye screening effect [1.11, 1.18]. After taking into account the screening effect, we have for fig. 2.1

$$G_1(\vec{p}) = \frac{g^4 Q^2 (N^2-1)\beta^2}{2^7 \pi^6} \int \frac{d^3\vec{k}}{(\vec{k}^2+M_{el}^2)\left[(\vec{p}-\vec{k})^2+M_{el}^2\right]} = \tag{2.2.18}$$
$$= \frac{g^4 Q^2 (N^2-1)\beta^2}{2^6 \pi^4 |\vec{p}|} \arcsin \frac{|\vec{p}|}{\sqrt{\vec{p}^2+4M_{el}^2}} \quad ,$$

where $G_1(\vec{p})$ is the Fourier transformation of $G_1(\vec{x} - \vec{y})$.

Therefore, we get

$$G_1(\vec{x}-\vec{y}) = \frac{g^4 Q^2 (N^2-1)\beta^2}{2^6 \pi^4} \int \frac{1}{|\vec{p}|} \arcsin \frac{|\vec{p}|}{\sqrt{\vec{p}^2+4M_{el}^2}} \cdot \exp\left[i\vec{p}\cdot(\vec{x}-\vec{y})\right] d^3\vec{p} = \tag{2.2.19}$$
$$= \frac{g^4 Q^2 (N^2-1)\beta^2}{2^5 \pi^2 (\vec{x}-\vec{y})^2} \cdot \exp\left[-2M_{el}|\vec{x}-\vec{y}|\right] \quad .$$

Similarly, we can get for the divergent (ultraviolet divergent) parts of the diagrams (a), (b) and (c) in fig. 2.2 and all diagrams in fig. 2.3

$$G_{2(a,b,c)}(\vec{x}-\vec{y}) = \frac{g^2}{16\pi^2}\,\frac{1}{\epsilon}\,\frac{10}{3}\,C_2\,\frac{g^4 Q^2 (N^2-1)\beta^2}{2^5\,\pi^2}\,\frac{1}{(\vec{x}-\vec{y})^2}\exp\left[-2M_{el}|\vec{x}-\vec{y}|\right], \tag{2.2.20}$$

$$G_3(\vec{x}-\vec{y}) = \frac{g^6 Q^3 C_2 (N^2-1)\beta^2}{2^6\,\pi^4\,\epsilon}\,\frac{1}{(\vec{x}-\vec{y})^2}\exp\left[-2M_{el}|\vec{x}-\vec{y}|\right]. \tag{2.2.21}$$

Therefore, we have got

$$G_1(\vec{x}-\vec{y}) + G_{2(a,b,c)}(\vec{x}-\vec{y}) + G_3(\vec{x}-\vec{y}) = \frac{\bar{g}^4 Q^2 (N^2-1)\beta^2}{2^5\,\pi^2 (\vec{x}-\vec{y})^2}\exp\left[-2M_{el}|\vec{x}-\vec{y}|\right]. \tag{2.2.22}$$

The contribution of the diagram (e) in fig. 2.2 to the two-point function of the thermal Wilson loop is

$$\frac{g^6 Q^2 N(N^2-1)\beta}{2(2\pi)^9}\cdot$$

$$\cdot\int\frac{d^3\vec{k}_1\cdot d^3\vec{k}_2\,(\vec{k}_2-\vec{k}_1)\cdot(\vec{k}_2+2\vec{p}-\vec{k}_1)}{(\vec{k}_1^{\,2}+M_{el}^2)\,(\vec{k}_2^{\,2}+M_{el}^2)\left[(\vec{p}-\vec{k}_1)^2+M_{el}^2\right]\left[(\vec{p}+\vec{k}_2)^2+M_{el}^2\right](\vec{k}_1+\vec{k}_2)^2}, \tag{2.2.23}$$

where we have taken into account the infra-red cut-off mass M_{el}. The infra-red behaviour of eq. (2.2.23), i.e. the behaviour of eq. (2.2.23) in the domain of small external momentum $\vec{p} \ll 1$ (for further discussion of the infra-red problem, see the next section) has been found. The infra-red singular part of eq. (2.2.23) is

$$G_{2(e)}(\vec{x}-\vec{y}) = \frac{g^6 Q^2 N(N^2-1)\beta}{2^8\pi^3 M_{el}(\vec{x}-\vec{y})^2} \exp\left[-2M_{el}|\vec{x}-\vec{y}|\right] =$$

$$= \frac{g^5 Q^2 \sqrt{3N}\,(N^2-1)\beta^2}{2^8\pi^3(\vec{x}-\vec{y})^2} \exp\left[-2M_{el}|\vec{x}-\vec{y}|\right] \quad , \qquad (2.2.24)$$

where $|\vec{x}-\vec{y}| \gg 1$, because of $|\vec{x}-\vec{y}|\cdot|\vec{p}| \sim 1$, the corresponding momentum $|\vec{p}| \ll 1$.

From eqs. (2.2.22) and (2.2.24) we can see that the two-point function of the thermal Wilson loop $\langle \mathrm{Tr}\Omega^{+}(\vec{x})\mathrm{Tr}\Omega(\vec{y})\rangle_c$ decreases when the $|\vec{x}-\vec{y}|$ distance between quark and antiquark increases, and $\langle \mathrm{Tr}\Omega^{+}(\vec{x})\mathrm{Tr}\Omega(\vec{y})\rangle_c \to 0$, in $|\vec{x}-\vec{y}| \to \infty$, up to $O(g^6)$. From eqs. (2.2.2), (2.2.3) and (2.2.5) we get

$$F_{q\bar{q}} = 2F_q \quad , \qquad \text{when } |\vec{x}-\vec{y}| \to \infty \, . \qquad (2.2.25)$$

In ref. [2.2], it has been shown that at high temperature, when the running coupling constant $g^2 \sim (\ln(T/\mu))^{-1} \ll 1$, $F_q \to 0$. Therefore $F_{q\bar{q}} \to 0$ at high temperature, when $|\vec{x}-\vec{y}| \to \infty$. It can be seen that the quarks would be liberated, i.e. unconfined at high temperature, which is consistent with the Monte Carlo study of lattice gauge theory at high temperature [1.14]. According to the paper [1.18] the lack of confinement is caused by the screening of the (colour) "electric" charge of the heavy quarks due to the high-temperature effect. The range of the potential between quark and antiquark is $(2M_{el})^{-1}$ in our case (due to the two-gluon exchange; for one gluon exchange, the range of the potential is $(M_{el})^{-1}$). This $(M_{el})^{-1}$ is called the "electric" screening length.

2.3. Infra-red problem

In this section, we will discuss some difficulties encountered in the investigation of the interquark potential at high temperature by means of perturbation theory. These difficulties are due to the infra-red problem in the thermodynamics of the massless Yang-Mills gas.

To explain this point, we consider the contributions, G_0, G_2, G_4, ..., G_{2N}... to the interquark potential from these diagrams in fig. 2.5. Our interest is the infra-red behaviour of fig. 2.5, thus we consider the contribution G_{2N} that arises from the region where all spatial momenta of gluons are of order $|\vec{p}|$ ($|\vec{p}|$ is small) and all the energies of gluons vanish.

Therefore, each new loop in the diagram (a_{2N}) contributes a factor of g^2 from new vertices, a factor of $T|\vec{p}|^3$ from phase space, and a factor of $|\vec{p}|^{-4}$ from the new vertices and propagators. Since the original loop is G_0, the contribution G_{2N} will behave as

$$G_0\left(g^2 T|\vec{p}|^3 \cdot |\vec{p}|^{-4}\right)^N = G_0\left(\frac{g^2 T}{|\vec{p}|}\right)^N . \qquad (2.3.1)$$

Thus, a sum of the contributions of these diagrams in fig. 2.5 to the $\langle \mathrm{Tr}\,\Omega^+(\vec{x})\,\mathrm{Tr}\,\Omega(\vec{y})\rangle_c$ behaves as

$$\begin{aligned} &G_0(\vec{p}) + G_2(\vec{p}) + \cdots + G_{2N}(\vec{p}) + \cdots \sim \\ &\sim \frac{g^4}{T^2|\vec{p}|}\left[1 + \frac{g^2 T}{|\vec{p}|} + \left(\frac{g^2 T}{|\vec{p}|}\right)^2 + \cdots + \left(\frac{g^2 T}{|\vec{p}|}\right)^N + \cdots\right] . \end{aligned} \qquad (2.3.2)$$

This shows that in the $|\vec{p}| \lesssim g^2T$ region, this series is divergent, therefore the calculations of the two-point function of the thermal Wilson loop by means of perturbation theory are unreliable at $|\vec{p}| \lesssim g^2T$. (Equation (2.3.2) can also be obtained by the simple dimensional analysis of the loop-momentum integrals of these diagrams in fig. 2.5). From eq. (2.3.1) it can be seen that these infra-red singularities are due to the self-interactions of the massless gluons, and the higher-order corrections to the G_0 are increasingly divergent.

However, the gluon could get a mass due to high-temperature effect. The one-loop correction for the time component of gluon propagator $G_{00}(k_0 = 0, \vec{k} \to 0)$ gives an "electric" mass $M_{el} = gT$, for SU(3), and the two-loop (or above two-loop) correction for the space component of gluon propagator gives a "magnetic" mass M_{mag} of order g^2T [1.11,1.18]. Thus, if we express the perturbative expansion in terms of the full propagators which have the effective mass $M(T)(M(T) = M_{mag}$ or $M_{el})$ instead of bare massless propagators, the contribution G_{2N} of the low-momentum region, $|\vec{p}| \leqslant M(T)$, from fig. 2.5 will be of order $(g^2T/m(T))^N$, which is finite. Therefore, the infra-red divergences can be removed by such a M(T). Thus the interquark potential is computable by means of perturbation theory order by order.

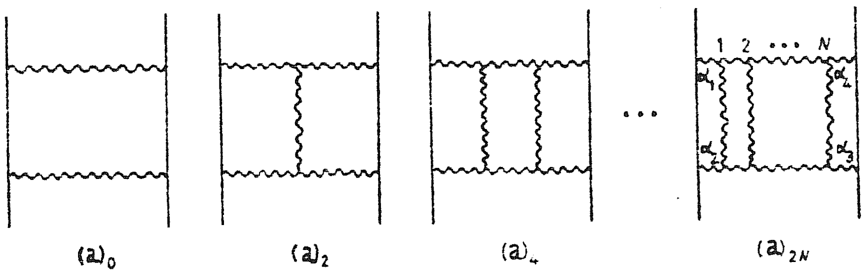


Fig. 2.5 - The diagrams of order g^{4+2N} containing 2N three-gluon vertices (N = 0,1,2, ...).

In the diagram (a_{2N}) of fig. 2.5, for α_i (i = 1,2,3,4) propagators, the Green's functions are $G^{ab}_{00}(k_0 = 0, \vec{k})$ (zero-zero component) due to the time component $A^a_0(x)$ of Yang-Mills field, but, for all other propagators, there are the contributions of G^{ab}_{ij} (space-space component). The G^{ab}_{00} gives a M_{el}, and the G^{ij}_{ab} gives a smaller magnetic mass M_{mag}. From eq. (2.2.23), we can see that in the denominator of the integrand of eq. (2.2.23), there are four propagators with the "electric" mass M_{el}, therefore, in the loop-momentum integral of eq. (2.2.23) the power of the small momenta is 8 in the numerator, and is 2 in the denominator, therefore eq. (2.2.23) is not infra-red divergent due to the cut-off mass M_{el}. But, increasing a new loop, there is a contribution factor $g^2T/|\vec{p}| \sim g^2T/M_{mag}$, thus, up to seven-loop, the $G_{2N}(N = 0,1,2,3,4,5,6)$ can be computed by means of perturbation theory, from eight-loop on, the $G_{2N}(N = 7,8, \ldots)$ are uncomputable by means of perturbation theory.

Compared with ref. [1.11], the thermodynamic potential $\Omega(T)$ in ref [1.11] is computable by means of perturbation theory only up to the order g^6, but the interquark potential in this paper is computable by means of perturbation theory up to the order g^{10}.

2.4 Conclusions

We have studied renormalization of the two-gluon exchange contributions to high-temperature interquark potential up to the order g^6 within the framework of perturbation theory. The conclusion we get is that the contributions to the interquark potential which behave as $(1/R^2)\exp\left[-2M_{el}R\right]$ fall off when the distance between the quarks increases, and the $(M_{el})^{-1}$ is the "electric" screening length. This "electric" screening effect due to high temperature generates the unconfinement of quarks.

The infra-red singularities which have arisen from the self-interactions of the gluons could be removed by an infra-red cut-off $M(T)(M_{el}$ or $M_{mag})$ due to high-temperature effect. The "electric" mass M_{el} appears in the Green's function G^{ab}_{00} $(k_0 = 0, \vec{k} \to 0)$, but a "magnetic" mass M_{mag} appears in the Green's function G^{ab}_{ij} $(k_0 = 0, \vec{k} \to 0)$, $i,j = 1,2,3$. At one-loop correction level, for the zero-zero component of the gluon propagator at finite temperature, the M_{el} equals gT, but the M_{mag} equals zero. At two-loop (or above two-loop) correction, the $M_{mag} \sim g^2T$. The removal of the infra-red divergence by such a cut-off mass $M(T)$ does not mean that the interquark potential is computable by means of perturbation theory to any power of g. Due to the smaller infra-red cut-off mass $M_{mag} \sim g^2T$, in our case, the interquark potential could be computed by means of perturbation theory only up to the order g^{10}.

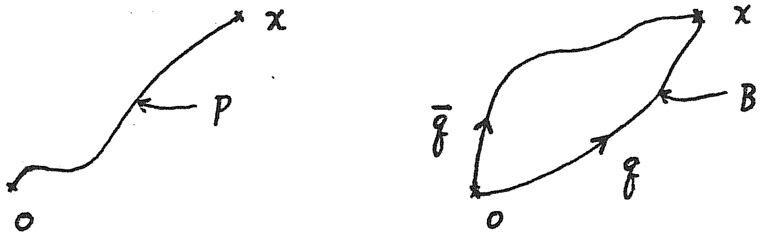
APPENDIX A

The physical meaning of Wilson loop

We would like to discuss shortly the physical meaning of the Wilson loop. When a quark goes through a path in the gauge fields from the point 0 to the point x in the four-dimensional space-time, as shown in the Fig. A.1(a), its wave function will encounter a phase change, i.e.

$$\psi \longrightarrow e^{ig\int_P A_\mu(y)\, dS^\mu}\, \psi \,, \tag{A.1}$$

where P represents the path, g is the coupling constant and $A_\mu(y)$ are the gauge fields.



(a) (b)

Fig. A.1

On the other hand, it is shown in Fig. A.1(b) that a pair quark-antiquark is created at the point 0, and annihilated at the point x. The Fig. A.1 can be considered as a quark passing on a closed path from the point 0 back to the original point 0, thus, the phase change of the quark wave function is

$$\psi \longrightarrow e^{ig\oint_B A_\mu(y)\, dS^\mu}\, \psi \,, \tag{A.2}$$

where B represents the closed path shown in Fig. A.1(b).

In the strong coupled ($g \gg 1$) lattice gauge theory, the gauge-field average of the phase factor $exp\left(ig\oint_B A_\mu(y)\, dS^\mu\right)$, i.e. the Wilson loop $\left\langle P\, exp\left(ig\oint_B A_\mu(y)\, dS^\mu\right)\right\rangle$ (in non-Abelian gauge theory, it is $\left\langle T_r P\, exp\left(ig\oint_B \lambda^a A^a_\mu(y) dS^\mu\right)\right\rangle$, the P represents a path ordering operator) behaves as $(g^2)^{-\frac{A}{a^2}} \left(=exp\left[-A\cdot\frac{\ln g^2}{a^2}\right]\right)$, where A is the minimal area enclosed by the boundary B, a being the lattice spacing.

Therefore, the $exp\left(-A\cdot\frac{\ln g^2}{a^2}\right)$ goes to zero (i.e. the Wilson loop $\left\langle P\, exp\left(ig\oint_B A_\mu(y) dS^\mu\right)\right\rangle$ goes to zero) when A is large (i.e. the loop B is large). Let m be the mass of the bound state of a pair quark-antiquark, m^{-1} can then be considered as the wave length of such a state, which is about x. As $A \sim B\cdot B$, and we can take a $B \sim (x^2)^{1/2}$ and the other as $B \sim m x^2$, we end up with $A \sim m(x^2)^{3/2}$. By keeping x^2 fixed, the mass of the bound state, m, will go to infinite as A tends to infinite; but this is impossible, therefore it is impossible for large loops. This shows that the quark path will not separate macroscopically, and there will be no quarks among the final-state particles, and there will only be the bound states.

In the weak coupling ($g \ll 1$) theory, the Wilson loop $\left\langle P\, exp\left[ig\oint_B A_\mu(y) dS^\mu\right]\right\rangle$ goes to 1 [1.17], thus the quarks will be unconfined. Thus it can be seen that the Wilson loop can be considered as an order parameter associated to a phase transition from a confined phase to an unconfined one.

When we take the Euclidean time τ as the reciprocal of the temperature, and considering the periodicity conditions $A^a_\mu(\vec{x},0) = A^a_\mu(\vec{x},\beta)$ ($\beta = \frac{1}{T}$, T is temperature) for the gauge fields, the Wilson loop becomes the Thermal Wilson loop,

$$\left\langle T_r \Omega(\vec{x}) \right\rangle = \left\langle T_r\, T\, e^{ig\int_0^\beta d\tau\, \lambda^a A^a_0(\tau,\vec{x})} \right\rangle, \qquad (A.3)$$

where T represents a time ordering operator and λ^a are the matrices representing the generators of a non-Abelian gauge group. The eq.(A.3) is still gauge invariant due to the periodicity conditions.

APPENDIX B

The derivation of the relation between Thermal Wilson line and the free energy of quark

We consider the free energy of a configuration of N_q quarks and $N_{\bar{q}}$ antiquarks, $F(\vec{r}_1,\vec{r}_2,\ldots\vec{r}_{N_q}, \vec{r}'_1,\vec{r}'_2, \ldots\vec{r}'_{N_{\bar{q}}})$. By definition,

$$\exp\left[-\beta F(\vec{r}_1,\vec{r}_2,\cdots\vec{r}_{N_q},\vec{r}'_1,\vec{r}'_2\cdots\vec{r}'_{N_{\bar{q}}})\right]=\frac{1}{N^{(N_q+N_{\bar{q}})}}\cdot\sum\langle S|e^{-\beta H}|S\rangle\;; \tag{B.1}$$

in this state, $|S\rangle$, there are N_q heavy quarks at $\vec{r}_1,\vec{r}_2, \ldots\vec{r}_{N_q}$ and $N_{\bar{q}}$ heavy anti-quarks at $\vec{r}'_1,\vec{r}'_2\ldots\vec{r}'_{N_q}$, and

$$N=\sum_{|S\rangle}\langle S|S\rangle \quad . \tag{B.2}$$

Since

$$\sum_{|S\rangle}\langle S|e^{-\beta H}|S\rangle=\sum_{|S'\rangle}\langle S'|\Big\{\sum_{(a,b)}\psi_{a_1}(\vec{r}_1,0)\,\psi_{a_2}(\vec{r}_2,0)\cdots\,\psi_{a_{N_q}}(\vec{r}_{N_q},0)\cdot$$

$$\cdot\psi^c_{b_1}(\vec{r}'_1,0)\,\psi^c_{b_2}(\vec{r}'_2,0)\cdots\,\psi^c_{b_{N_{\bar{q}}}}(\vec{r}'_{N_{\bar{q}}},0)\;e^{-\beta H}\;\psi^+_{a_1}(\vec{r}_1,0)\,\psi^+_{a_2}(\vec{r}_2,0)\cdots\psi^+_{a_{N_q}}(\vec{r}_{N_q},0)\cdot$$

$$\cdot\left(\psi^+_{b_1}(\vec{r}'_1,0)\right)^c\cdot\left(\psi^+_{b_2}(\vec{r}'_2,0)\right)^c\cdots\left(\psi^+_{b_{N_{\bar{q}}}}(\vec{r}'_{N_{\bar{q}}},0)\right)^c\Big\}\,|S'\rangle\;, \tag{B.3}$$

where these quarks and anti-quarks are drawn from $|S\rangle$, and in this $|S'\rangle$

there is no heavy quark and antiquark, and $\psi_{a_1}(\vec{r}_1,0)$ and $\psi^c_{b_1}(\vec{r}'_1,0)$ annihilated operators of quark (quantum number a_1) and anti-quark (quantum number b_1), and $\psi^+_{a_1}(\vec{r}_1,0)$ and $(\psi^+_{b_1}(\vec{r}'_1,0))^c$ are created operators of quark and anti-quark.

For any operator, $O(t)$, we have

$$e^{\beta H}\, O(t)\, e^{-\beta H} = O(t+\beta)\,. \tag{B.4}$$

Then, eq. (B.3) becomes

$$\sum_{|s\rangle}\langle s|e^{-\beta H}|s\rangle = \sum_{|s'\rangle}\langle s'|\Big\{\sum_{(a,b)} e^{-\beta H}\psi_{a_1}(\vec{r}_1,\beta)\,\psi_{a_2}(\vec{r}_2,\beta)\cdots\psi_{a_{N_q}}(\vec{r}_{N_q},\beta)\cdot$$

$$\cdot\,\psi^c_{b_1}(\vec{r}'_1,\beta)\,\psi^c_{b_2}(\vec{r}'_2,\beta)\cdots\psi^c_{b_{N_{\bar q}}}(\vec{r}'_{N_{\bar q}},\beta)\cdot\psi^+_{a_1}(\vec{r}_1,0)\,\psi^+_{a_2}(\vec{r}_2,0)\cdots\psi^+_{a_{N_q}}(\vec{r}_{N_q},0)\cdot$$

$$\cdot\big(\psi^+_{b_1}(\vec{r}'_1,0)\big)^c\cdot\big(\psi^+_{b_2}(\vec{r}'_2,0)\big)^c\cdots\big(\psi^+_{b_{N_{\bar q}}}(\vec{r}'_{N_{\bar q}},0)\big)^c\Big\}|s'\rangle =$$

$$= (-1)^{\frac{1}{2}(N_q+N_{\bar q})\cdot(N_q+N_{\bar q}-1)}\cdot\sum_{|s'\rangle}\langle s'|\Big\{\sum_{(a,b)} e^{-\beta H}\psi_{a_1}(\vec{r}_1,\beta)\,\psi^+_{a_1}(\vec{r}_1,0)\cdot$$

$$\cdot\,\psi_{a_2}(\vec{r}_2,\beta)\,\psi^+_{a_2}(\vec{r}_2,0)\cdots\psi_{a_{N_q}}(\vec{r}_{N_q},\beta)\,\psi^+_{a_{N_q}}(\vec{r}_{N_q},0)\cdot\ \psi^c_{b_1}(\vec{r}'_1,\beta)\cdot$$

$$\cdot\big(\psi^+_{b_1}(\vec{r}'_1,0)\big)^c\cdots\ \psi^c_{b_{N_{\bar q}}}(\vec{r}'_{N_{\bar q}},\beta)\cdot\big(\psi^+_{b_{N_{\bar q}}}(\vec{r}'_{N_{\bar q}},0)\big)^c\,|s'\rangle\,. \tag{B.5}$$

Using

$$\left[\psi_a(\vec{r}_i,t),\ \psi_b^{+}(\vec{r}_j,t)\right]_{+} = \delta_{ab}\,\delta_{ij}\ , \tag{B.6}$$

$$\psi_{a_i}(\vec{r}_i,t) = T\, e^{i\int_0^t d\tau\,\lambda^\alpha A_0^\alpha(\vec{r}_i,\tau)}\ \psi_{a_i}(\vec{r}_i,0)\ , \tag{B.7}$$

and, in spite of the factor $(-1)^{\frac{1}{2}(N_q+N_{\bar{q}})(N_q+N_{\bar{q}}-1)}$, eq. (B.5) becomes

$$\sum_{|s\rangle}\langle s|e^{-\beta H}|s\rangle = \sum_{|s'\rangle}\langle s'|\Big\{\sum_{(a,b)} e^{-\beta H}\ T e^{i\int_0^\beta d\tau\,\lambda^\alpha A_0^\alpha(\vec{r}_1,\tau)}\ \psi_{a_1}(\vec{r}_1,0)\,\psi_{a_1}^{+}(\vec{r}_1,0)\cdot$$

$$T e^{i\int_0^\beta d\tau\,\lambda^\alpha A_0^\alpha(\vec{r}_2,\tau)}\ \psi_{a_2}(\vec{r}_2,0)\,\psi_{a_2}^{+}(\vec{r}_2,0)\cdots T e^{i\int_0^\beta d\tau\,\lambda^\alpha A_0^\alpha(\vec{r}_{N_q},\tau)}\ \psi_{a_{N_q}}(\vec{r}_{N_q},0)\,\psi_{a_{N_q}}^{+}(\vec{r}_{N_q},0)\cdot$$

$$\psi_{b_1}^{c}(\vec{r}_1{}',0)\left(T e^{i\int_0^\beta d\tau\,\lambda^\alpha A_0^\alpha(\vec{r}_1{}',\tau)}\right)^{+}\left(\psi_{b_1}^{+}(\vec{r}_1{}',0)\right)^{c}\cdots\psi_{b_{N_{\bar{q}}}}^{c}(\vec{r}_{N_{\bar{q}}}{}',0)\cdot$$

$$\cdot\left(T e^{i\int_0^\beta d\tau\,\lambda^\alpha A_0^\alpha(\vec{r}_{N_{\bar{q}}}{}',\tau)}\right)^{+}\left(\psi_{b_{N_{\bar{q}}}}^{+}(\vec{r}_{N_{\bar{q}}}{}',0)\right)^{c}\Big\}|s'\rangle = \sum_{|s'\rangle}\langle s'|e^{-\beta H}\Big\{\sum_{(a_1)}\cdot$$

$$\cdot T e^{i\int_0^\beta d\tau\,\lambda^\alpha A_0^\alpha(\vec{r}_1,\tau)}\ \psi_{a_1}(\vec{r}_1,0)\,\psi_{a_1}^{+}(\vec{r}_1,0)\sum_{(a_2)} T e^{i\int_0^\beta d\tau\,\lambda^\alpha A_0^\alpha(\vec{r}_2,\tau)}\ \psi_{a_2}(\vec{r}_2,0)\,\psi_{a_2}^{+}(\vec{r}_2,0)\cdot$$

$$\cdots\sum_{(a_{N_q})} T e^{i\int_0^\beta d\tau\,\lambda^\alpha A_0^\alpha(\vec{r}_{N_q},\tau)}\ \psi_{a_{N_q}}(\vec{r}_{N_q},0)\,\psi_{a_{N_q}}^{+}(\vec{r}_{N_q},0)\sum_{(b_1)}\psi_{b_1}^{c}(\vec{r}_1{}',0)\cdot$$

$$\cdot\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_1',\tau)}\right)^+ \left(\psi_{b_1}^+(\vec{r}_1',0)\right)^c \cdots \sum_{(b_{N_{\bar q}})} \psi_{b_{N_{\bar q}}}^c(\vec{r}_{N_{\bar q}}',0)\cdot$$

$$\cdot\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_{N_{\bar q}},\tau)}\right)^+ \cdot \left(\psi_{b_{N_{\bar q}}}^+(\vec{r}_{N_{\bar q}}',0)\right)^c |s'\rangle \ . \tag{B.8}$$

Since

$$\sum_{(a_1)} \langle 0| T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_1,\tau)} \psi_{a_1}(\vec{r}_1,0)\, \psi_{a_1}(\vec{r}_1,0)\, |0\rangle =$$

$$= \sum_{(a_1)} \langle a_1, \vec{r}_1, 0| T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_1,\tau)} | a_1, \vec{r}_1, 0\rangle =$$

$$= \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \cdot \lambda^\alpha A_0^\alpha(\vec{r}_1,\tau)}\right) , \tag{B.9}$$

eq. (B.8) can be written as

$$\sum_{|s\rangle} \langle s| e^{-\beta H} |s\rangle = \sum_{|s'\rangle} \langle s'| e^{-\beta H}\, \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_1,\tau)}\right)\cdot$$

$$\cdot \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \cdot \lambda^\alpha \cdot A_0^\alpha(\vec{r}_2,\tau)}\right) \cdots \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_{N_q},\tau)}\right) \cdot \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_1',\tau)}\right)^+ \cdot$$

$$\mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_2{}',\tau)}\right)^+ \cdots \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_{N_{\bar{q}}}{}',\tau)}\right)^+ |s'\rangle =$$

$$= \mathrm{Tr}\left\{ e^{-\beta H}\, \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_1,\tau)}\right) \cdots \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_{N_q},\tau)}\right) \cdot \right.$$

$$\left. \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_1{}',\tau)}\right)^+ \cdots \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r}_{N_{\bar{q}}}{}',\tau)}\right)^+ \right\} . \tag{B.10}$$

We put

$$L(\vec{r}) \equiv \frac{1}{N}\, \mathrm{Tr}\left(T e^{i\int_0^\beta d\tau \lambda^\alpha A_0^\alpha(\vec{r},\tau)}\right) . \tag{B.11}$$

Thus, from eq. (B.1), we get

$$e^{-\beta F_{N_q,N_{\bar{q}}}} = \mathrm{Tr}\left\{ e^{-\beta H} L(\vec{r}_1)\, L(\vec{r}_2) \cdots L(\vec{r}_{N_q}) \cdot \right.$$

$$\left. \cdot L^+(\vec{r}_1{}') \cdot L^+(\vec{r}_2{}') \cdots L^+(\vec{r}_{N_{\bar{q}}}{}') \right\} , \tag{B.12}$$

where the trace is from $\sum\limits_{|s'\rangle}$, this summation being performed only over the states of pure gauge field, not including quark fields.

For one quark, eq. (B.12) becomes

$$e^{-\beta F_{1,0}} = Tr\left[e^{-\beta H} L(\vec{r})\right] \quad , \tag{B.13}$$

but,

$$e^{-\beta F_{0,0}} = Tr\, e^{-\beta H} \quad , \tag{B.14}$$

then,

$$e^{-\beta(F_{1,0}-F_{0,0})} = \frac{Tr\left[e^{-\beta H} L(\vec{r})\right]}{Tr\, e^{-\beta H}} \quad , \tag{B.15}$$

where $F_{0,0}$ is the free energy of pure gauge fields (gluons), and $F_{1,0}$ is the free energy of one quark and gluons.

Since

$$Tr\left[e^{-\beta H} L(\vec{r})\right] = \sum_{|s'\rangle} \langle s'| e^{-\beta H} L(\vec{r}) |s'\rangle \quad , \tag{B.16}$$

$$Tr\, e^{-\beta H} = \sum_{|s'\rangle} \langle s'| e^{-\beta H} |s'\rangle \quad , \tag{B.17}$$

$|s'\rangle$ is the state without quark and

$$H\,|s'\rangle = E_{s'}|s'\rangle \quad , \tag{B.18}$$

then,

$$\frac{Tr\left[e^{-\beta H} L(\vec{r})\right]}{Tr\, e^{-\beta H}} = \frac{\sum_{|s'\rangle}\left[e^{-\beta E_{s'}} \cdot \langle s'| L(\vec{r}) |s'\rangle\right]}{\sum_{|s'\rangle}\left[\langle s'|s'\rangle \cdot e^{-\beta E_{s'}}\right]} \quad . \tag{B.19}$$

We can scale $E_{s'}$ α times, then,

$$\frac{Tr\left[e^{-\beta H} L(\vec{r})\right]}{Tr\, e^{-\beta H}} = \frac{\sum_{|s'\rangle} \left[e^{-\alpha\beta E_{s'}} \langle s'| L(\vec{r}) |s'\rangle\right]}{\sum_{|s'\rangle} \left[\langle s'|s'\rangle\, e^{-\alpha\beta E_{s'}}\right]} . \tag{B.20}$$

We can continue the quantity, $\alpha\beta = \delta$, to the complex plane, because eq. (B.20) is analytic. We put

$$\alpha = \rho e^{i\theta} \quad , \quad \theta = -\epsilon . \tag{B.21}$$

When

$$\begin{aligned} &Re\,\alpha \longrightarrow \infty \\ &Im\,\alpha < 0 \quad , \end{aligned} \tag{B.22}$$

eq. (B.20) becomes

$$\frac{Tr\left[e^{-\beta H} L(\vec{r})\right]}{Tr\, e^{-\beta H}} = \frac{\langle 0| L(\vec{r}) |0\rangle}{\langle 0|0\rangle} . \tag{B.23}$$

(Note that $-\alpha\beta \rightarrow -\infty$, due to the fact that β is a positive pure imaginary number), where $|\,0\rangle$ is vacuum and we take $\langle 0\,|\,0\rangle = 1$. Finally, we get

$$\frac{Tr\left[e^{-\beta H} L(\vec{r})\right]}{Tr\, e^{-\beta H}} = \langle 0| L(\vec{r}) |0\rangle . \tag{B.24}$$

Therefore

$$e^{-\beta F_q} \equiv e^{-\beta(F_{1,0}-F_{0,0})} = \frac{Tr\left[e^{-\beta H} L(\vec{r})\right]}{Tr\, e^{-\beta H}} = \langle 0 | L(\vec{r}) | 0 \rangle \quad , \tag{B.25}$$

where F_q is the free energy of an isolated quark (relative to the volume), including no gluons.

This completes the derivation.

CHAPTER III - HIGH TEMPERATURE INTERQUARK POTENTIAL

3.1 INTRODUCTION

In the past few years, many authors [1.9 , 1.18] have been much interested in the issue of QCD at finite temperature and density. There is a phase transition from the usual hadronic matter, in which the quarks are confined, to the quark-gluon plasma, in which the quarks as well as the gluons are quasi-free particles, for increasing temperatures and densities [1.13] . Although confinement has not yet been proved, one is led to believe it exists; an absolute confinement is however completely excluded and quarks and gluons behave like free particles at extremely high temperatures and densities. We hope that in future experiments it may be possible to observe directly the quark-gluon plasma [1.26].

The motivation for the study of QCD at finite temperature has discussed in a previous paper [1.7]. The issue of the interquark potential is an important topic in the framework of QCD at finite temperature. The effective quark-gluon coupling constant at high temperature can be seen to be

$$g^2(T,\mu) \underset{T\to\infty}{=\!=} \frac{24\,\pi^2}{11\cdot N\,\ln\left(\frac{T}{\mu}\right)} \qquad \text{for } SU(N)_c \,,$$

as a result of the renormalization group equations.

The decrease of the effective charge at high temperature leads to the existence of a weakly interacting gas of quarks and gluons which makes it possible to use perturbation theory. In the previous paper [1.7], we have studied the renormalization of the two-gluon exchange contribution to the high temperature interquark potential, up to the order g^6 . The present paper shall give explicit analytical expressions for the finite parts of the interquark potential at high temperature through the order g^6 . Our results confirm the arguments for the non-confinement of the quarks at high temperatures. The interquark potential decreases as an exponential and a power when the distance between a quark and an antiquark increases in the high-temperature limit.

The outline of this work is as follows. In Sec.3.2 we have given the explicit analytical expressions for the finite parts of the interquark potential in the high-temperature limit through the order g^6, using perturbation theory. In Sec. 3.3 we have briefly discussed the (color) magnetic mass of the gluon and its relevance to the validity of perturbative expansions for the interquark potential at high temperature. In Sec. 3.4, we have crudely estimated the critical temperature of a phase transition from confinement to deconfinement.

3.2 INTERQUARK POTENTIAL

We know that the calculation of the interaction potential, $V(\vec{x}-\vec{y})$, between a quark q and an antiquark $\bar{q}$ can be reduced to the evaluation of the two-point function of the Thermal Wilson loop due to [1.14, 2.1]:

$$e^{-\beta V(\vec{x}-\vec{y})} = \langle \mathrm{Tr}\,\Omega^{\dagger}(\vec{x}) \cdot \mathrm{Tr}\,\Omega(\vec{y}) \rangle \qquad (3.2.1)$$

The Thermal Wilson loop $\langle \mathrm{Tr}\,\Omega(\vec{x}) \rangle$ can be considered as an order parameter associated to the phase transition from a confined phase to an unconfined one. The Thermal Wilson loop is defined to be

$$\langle \mathrm{Tr}\,\Omega(\vec{x}) \rangle = \left\langle \mathrm{Tr}\; T \exp\left[i g \int_0^{\beta} d\tau\, \lambda^a A_0^a(\tau,\vec{x}) \right] \right\rangle , \qquad (3.2.2)$$

where it is assumed that the system is placed in a heat bath at a physical temperature $\frac{1}{\beta} = T$. The expectation value is taken over the gauge fields $A_\mu^a(\tau,\vec{x})$, whose dynamics is here assumed to be described by a pure Yang-Mills field Lagrangian without additional matter fields. The integral over the Euclidean time τ in eq. (3.2.2) is actually an integral over a closed path due to the periodicity conditions $A_\mu^a(0,\vec{x}) = A_\mu^a(\beta,\vec{x})$ which keep the gauge invariance of eq. (3.2.2). Here q and $\bar{q}$ are considered as static external particles, i.e. heavy quarks. T represents the time ordering, and λ^a are the generators of non-Abelian gauge group, and g is coupling constant of quark-gluon.

We define the symbol $\mathrm{Tr} = \frac{\mathrm{trace}}{d(R)}$ in such a way that Tr I = 1, I being the identity and d(R) the dimensionality of the given representation of the gauge group.

According to general renormalization group arguments [3.1], running coupling constant in the high-temperature limit should be

$$g^2(T,\mu) = \frac{24\pi^2}{11 \cdot N \cdot \ln\left(\frac{T}{\mu}\right)} \qquad \text{for } SU(N)_c , \qquad (3.2.3)$$

where μ is the renormalization mass scale. Thus it can be seen that the $g(T,\mu)$ vanishes as the temperature $T \to \infty$, so that we may hope that perturbation theory is reliable at sufficiently high temperature. We calculate below the finite parts of the two-point function $\langle Tr\Omega^+(\vec{x}) \cdot Tr\Omega(\vec{y})\rangle$ of the Thermal Wilson loop of eq.(3.2.1), using a pertubative expansion in powers of $g(T,\mu)$.

We write the $\langle Tr\Omega^+(\vec{x}) \cdot Tr\Omega(\vec{y})\rangle$ as

$$\langle Tr\Omega^+(\vec{x}) \cdot Tr\Omega(\vec{y})\rangle =$$

$$= \left\langle Tr\left\{\bar{T} \exp\left[-ig\int_0^\beta d\tau\, \lambda^a A_0^a(\tau,\vec{x})\right]\right\} \cdot Tr\left\{T \exp\left[ig\int_0^\beta d\tau'\, \lambda^{a'} A_0^{a'}(\tau',\vec{y})\right]\right\}\right\rangle , \qquad (3.2.4)$$

where $\bar{T}$ means anti-time ordering.

$$\langle Tr\Omega^+(\vec{x})\, Tr\Omega(\vec{y})\rangle = \langle Tr\Omega^+(\vec{x})\, Tr\Omega(\vec{y})\rangle_c + \langle Tr\Omega^+(\vec{x})\rangle \cdot \langle Tr\Omega(\vec{y})\rangle , \qquad (3.2.5)$$

where the subscript c means connected two-point function of the Thermal Wilson loop. Thus,

$$\langle Tr\Omega^+(\vec{x}) \cdot Tr\Omega(\vec{y})\rangle_c = \frac{g^4}{4}\, Tr(\lambda^a \lambda^b)\, Tr(\lambda^{a'} \lambda^{b'}) \cdot$$

$$\cdot \left\langle \bar{T}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2\, A_0^a(\tau_1,\vec{x}) A_0^b(\tau_2,\vec{x})\; T\int_0^\beta d\tau_1' \int_0^\beta d\tau_2'\, A_0^{a'}(\tau_1',\vec{y}) A_0^{b'}(\tau_2',\vec{y})\right\rangle_c - \frac{g^6}{2\cdot 4!}\, Tr(\lambda^a \lambda^b) \cdot$$

$$\cdot\, Tr(\lambda^{a'} \lambda^{b'} \lambda^{c'} \lambda^{d'}) \left\langle \bar{T}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2\, A_0^a(\tau_1,\vec{x}) A_0^b(\tau_2,\vec{x})\; T\int_0^\beta d\tau_1' \int_0^\beta d\tau_2' \int_0^\beta d\tau_3' \int_0^\beta d\tau_4'\, A_0^{a'}(\tau_1',\vec{y}) A_0^{b'}(\tau_2',\vec{y}) \cdot\right.$$

$$A_0^{c'}(\tau_3',\vec{y})\, A_0^{d'}(\tau_4',\vec{y})\rangle_c - \frac{g^6}{2\cdot 4!} Tr(\lambda^a\lambda^b\lambda^c\lambda^d)\, Tr(\lambda^{a'}\lambda^{b'}) \langle \mp \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3 \int_0^\beta d\tau_4 \cdot$$

$$\cdot A_0^a(\tau_1,\vec{x})\, A_0^b(\tau_2,\vec{x})\, A_0^c(\tau_3,\vec{x})\, A_0^d(\tau_4,\vec{x}) \cdot T\int_0^\beta d\tau_1' \int_0^\beta d\tau_2'\, A_0^{a'}(\tau_1',\vec{y})\, A_0^{b'}(\tau_2',\vec{y})\rangle_c +$$

$$+ \frac{g^6}{3!\cdot 3!} Tr(\lambda^a\lambda^b\lambda^c)\cdot Tr(\lambda^{a'}\lambda^{b'}\lambda^{c'}) \cdot \langle \mp \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \int_0^\beta d\tau_3\, A_0^a(\tau_1,\vec{x})\, A_0^b(\tau_2,\vec{x})\, A_0^c(\tau_3,\vec{x}) \cdot$$

$$\cdot T\int_0^\beta d\tau_1' \int_0^\beta d\tau_2' \int_0^\beta d\tau_3'\, A_0^{a'}(\tau_1',\vec{y})\, A_0^{b'}(\tau_2',\vec{y})\, A_0^{c'}(\tau_3',\vec{y})\rangle_c + O(g^7) \quad .$$

(3.2.6)

The Feynman diagrams corresponding to the $\langle Tr\Omega^\dagger(\vec{x})\, Tr\Omega(\vec{y})\rangle_c$ given by eq. (3.2.6) are Fig. 2.1 – 2.4 (up to $O(g^6)$).

Since the thermal Wilson loop is gauge invariant, our calculations can be done in a specific gauge. We choose to work in Feynman gauge. Furthermore, we note that the contributions of the diagrams (d),(f),(g) and (h) in Fig. 2.2 and the diagram (a) in Fig. 2.4 to the $\langle Tr\Omega^\dagger(\vec{x})\, Tr\Omega(\vec{y})\rangle_c$ are zero [1.7] .

Using the dimensional regularization scheme, we have calculated the contribution $G_1(\vec{x}-\vec{y})$ of the diagrams (a) and (a*) in Fig. 2.1 to the $\langle Tr\Omega^\dagger(\vec{x})\, Tr\Omega(\vec{y})\rangle_c$ [1.7]

$$G_1(\vec{x}-\vec{y}) = \frac{g^4 Q^2 \beta^2 (N^2-1)\left[1+\epsilon(\ln 2+\gamma)\right]}{2^5\, \pi^2\, (\vec{x}-\vec{y})^{2(1-\epsilon)}} \quad , \tag{3.2.7}$$

where we refer to the gauge field $A_\mu^a(\tau,\vec{x})$ as an $SU(N)_c$ gauge field and $Tr(\lambda^a\lambda^b)=Q\,\delta_{ab}$ with $Q = \frac{1}{2N}$ in the fundamental representation and γ is the Euler constant, and the is coming from the dimensional regularization scheme.

In momentum space, the ultraviolet divergent parts of the contributions to $\langle \mathrm{Tr}\Omega^{+}(\vec{x})\cdot\mathrm{Tr}\Omega(\vec{y})\rangle_c$ coming from the graphs (a),(a'),(b),(b'),(c)(c') and their crossed partners in Fig.22 have been computed to be

$$G^{UD}_{2(a,b,c)}(\vec{p}) = \frac{g^4 Q^2 (N^2-1)\beta^2}{2^7\,\pi^3\,|\vec{p}|}\cdot\frac{g^2}{16\pi^2\epsilon}\cdot\frac{20}{3}\,C_2 \qquad . \tag{3.2.8}$$

Its Fourier transform

$$G^{UD}_{2(a,b,c)}(\vec{x}-\vec{y}) = \frac{g^4 Q^2 (N^2-1)\beta^2}{2^5\,\pi^2\,(\vec{x}-\vec{y})^2}\cdot\frac{g^2}{16\pi^2\epsilon}\cdot\frac{20}{3}\,C_2 \qquad , \tag{3.2.9}$$

where $C_2 = N$ is the eigenvalue of the quadratic Casimir operator in the adjoint representation of the group SU(N).

After having compared eq(3.2.9)with eq.(3.2.7), we discover that the ultraviolet divergent part $G^{UD}_{2(a,b,c)}(\vec{x}-\vec{y})$ at finite temperature is the same as that at zero temperature. This example explained the inference that has been explicitly shown by the authors of Refs.[1.4, 1.10] : "all ultraviolet divergences at finite temperature are independent of the temperature, no new temperature dependent ultraviolet infinities appear, and zero-temperature renormalization prescriptions suffice to eliminate all ultraviolet divergences at finite temperature" .

The finite parts of the Feynman diagrams (a),(a'),(b),(b')(c),(c') and their crossed partners in Fig.22 have been calculated to be

$$G^{F}_{2(a,b,c)}(\vec{x}-\vec{y}) \underset{T\to\infty}{\approx} \frac{g_R^6 Q^2 N(N^2-1)\beta^2}{2^9\,\pi^4\,(\vec{x}-\vec{y})^2}\cdot\frac{20}{3}\left\{\frac{3}{2}\gamma-\frac{1}{5}-\right.$$

$$\left.-\frac{1}{2}\ln\pi+\ln(\mu\beta)+\ln\left[\mu^2(\vec{x}-\vec{y})^2\right]\right\} \qquad , \tag{3.2.10}$$

at high temperature approximation, $T \to \infty$, and up to the order β^2 , where g_R is the renormalized coupling constant that is dimensionless in $3-\epsilon$ dimensions and $g_R^4 = g^4\left(1 + \frac{11 g^2 C_2}{12\pi^2 \epsilon}\right)$.

In the previous paper [1.7], we have given the total contributions of all diagrams in Fig. 2.3

$$G_3(\vec{p}) = g^6 Q\left[\frac{1}{(2\pi)^3\beta}\right]^3 \cdot \left[-\frac{1}{4}(N^2-1)\right] \cdot \frac{\beta^6 \pi^d \Gamma(1-\frac{d}{2})\Gamma(2-\frac{d}{2})\left[\Gamma(\frac{d}{2}-1)\right]^2 \zeta(4-d)}{\pi^2 \left(\frac{2\pi}{\beta}\right)^{2-d} (p^2)^{2-\frac{d}{2}}\ \Gamma(d-2)} \quad , \tag{3.2.11}$$

where $d = 3 - \epsilon$ and $\zeta(4-d)$ is the Riemann zeta function. Thus, from eq.(3.2.11) we can obtain its ultraviolet divergent $G_3^{UD}(\vec{x}-\vec{y})$ and finite parts $G_3^F(\vec{x}-\vec{y})$, respectively.

$$G_3^{UD}(\vec{x}-\vec{y}) = \frac{g^4 Q^2 (N^2-1)\beta^2}{2^5 \pi^2 (\vec{x}-\vec{y})^2} \cdot \frac{g^2 C_2}{2\pi^2 \epsilon} \quad , \tag{3.2.12}$$

$$G_3^F(\vec{x}-\vec{y}) = \frac{g_R^6 Q^2 (N^2-1)\beta^2 C_2}{2^6 \pi^4 (\vec{x}-\vec{y})^2}\left\{1 - \ln 2 - \frac{3}{2}\ln\pi + \frac{3}{2}\gamma + \ln(\mu\beta) + \ln\left[\mu^2(\vec{x}-\vec{y})^2\right]\right\} \quad . \tag{3.2.13}$$

Below, we give the contribution $G_{2(e)}(\vec{x}-\vec{y})$ of the diagram (e) in Fig.2.2 to $\langle Tr\Omega^+(\vec{x})\, Tr\Omega(\vec{y})\rangle_c$. In momentum space $G_{2(e)}(\vec{p})$ can be expressed as

$$G_{2(e)}(\vec{p}) = \frac{g^6 Q^2 N(N^2-1)\beta}{2\cdot(2\pi)^3}\int \frac{d\vec{k}_1\, d\vec{k}_2}{(2\pi)^6} \cdot \frac{(\vec{k}_2-\vec{k}_1)\cdot(\vec{k}_2+2\vec{p}-\vec{k}_1)}{\vec{k}_1^2 \cdot \vec{k}_2^2\ (\vec{p}-\vec{k}_1)^2 (\vec{p}+\vec{k}_2)^2 (\vec{k}_1+\vec{k}_2)^2} \quad . \tag{3.2.14}$$

By the Gegenbauer polynomials techniques [3.2] , we have got $G_{2(e)}(\vec{p})$ and its Fourier transform $G_{2(e)}(\vec{x}-\vec{y})$ as

$$G_{2(e)}(\vec{p}) = \frac{g^6 Q^2 N(N^2-1)\beta}{2^6 \pi^3 \, p^2}\left(\frac{1}{16} - \frac{1}{\pi^2}\right) \quad , \tag{3.2.15}$$

$$G_{2(e)}(\vec{x}-\vec{y}) = \frac{g^6 Q^2 N(N^2-1)\beta}{2^5 \, \pi \, |\vec{x}-\vec{y}|}\left(\frac{1}{16} - \frac{1}{\pi^2}\right) \quad . \tag{3.2.16}$$

Comparing eqs.(3.2.16)and(3.2.7), we notice that the contribution of Fig.2.2(e) is larger than that of Fig. 2.1 when $|\vec{x}-\vec{y}| > \frac{1}{g^2 T}$. It can be seen that the calculation of $\langle \mathrm{Tr}\Omega^{\dagger}(\vec{x}) \mathrm{Tr}\Omega(\vec{y})\rangle_c$ through perturbation theory is unreliable in the region $|\vec{x}-\vec{y}| > \frac{1}{g^2 T}$. Therefore we must take into account the infrared cutoff mass m(T) (for further discussion see next section).

If we substitute the full propagators of gluons for their bare massless propagators in Fig. 2.2(e) , we have

$$G_{2(e)}(\vec{p}) = \frac{g^6 Q^2 N(N^2-1)\beta}{2\cdot(2\pi)^3}\int\frac{d\vec{k}_1 d\vec{k}_2}{(2\pi)^6}\;\frac{(\vec{k}_2-\vec{k}_1)\cdot(\vec{k}_2+2\vec{p}-\vec{k}_1)}{(\vec{k}_1^2+m^2)(\vec{k}_2^2+m^2)[(\vec{p}-\vec{k}_1)^2+m^2][(\vec{p}+\vec{k}_2)^2+m^2](\vec{k}_1+\vec{k}_2)^2} \quad , \tag{3.2.17}$$

where $m=m_{el}$ is the gluon "electric" mass due to the temperature effect, $m_{el}^2 = \frac{N}{3}g^2T^2$ for SU(N) and one-loop correction of the time-component of gluon propagator. Among the five gluon propagators of Fig.2.2(e), four are of the kind $\langle T[A_0^a(\tau,\vec{x}) A_0^b(\tau',\vec{x}')]\rangle$, thus they are substituted by the full propagators with "electric" masses. One of them is still without mass, as the "magnetic" mass equals zero for one-loop correction of the space-component of the gluon propagator[1 18].

The n-dimensional Fourier transform of a propagator is given by

$$\int d^n k \, \frac{e^{-ik\cdot x}}{k^2+m^2} = 2\,\pi^{\frac{n}{2}} (m^2)^{\frac{n}{2}-1} \, k_{\frac{n}{2}-1}(mx) \quad , \tag{3.2.18}$$

where $k_{\frac{n}{2}-1}(mx)$ differs by a factor $\left(\frac{mx}{2}\right)^{1-\frac{n}{2}}$ from the well-known function $K_{\frac{n}{2}-1}(mx)$ (a class of Bessel functions).

By means of these k-functions we can perform the integral of eq.(3.2.17). For high temperature and large distance between the quark q and the antiquark $\bar{q}$, i.e. $T\cdot|\vec{x}-\vec{y}|\to\infty$, we get that the Fourier transform of eq.(3.2.17)

$$G_{2(e)}(\vec{x}-\vec{y}) \cong \frac{9\, g^6 Q^2 N(N^2-1)\beta}{2^{10}\,\pi^3\, m_{el}\,(\vec{x}-\vec{y})^2}\cdot e^{-2m_{el}|\vec{x}-\vec{y}|} \qquad (3.2.19)$$

Finally, we shall calculate the Feynman diagram (b) and five crossed partners in Fig.2.4 with three-gluon exchange. Their Green function $G_4(\vec{x}-\vec{y})$ can be written as

$$G_4(\vec{x}-\vec{y}) = g^6 \beta^{-3} \sum_{\ell_1,\ell_2,\ell_3=-\infty}^{\infty} \int \frac{d^3\vec{k}_1\, d^3\vec{k}_2\, d^3\vec{k}_3}{(2\pi)^9} \frac{e^{-i(\vec{k}_1+\vec{k}_2+\vec{k}_3)\cdot(\vec{x}-\vec{y})}}{(\vec{k}_1^2+\omega_1^2)(\vec{k}_2^2+\omega_2^2)(\vec{k}_3^2+\omega_3^2)}\cdot$$

$$\cdot\int_0^\beta d\tau_1'\int_0^{\tau_1'} d\tau_2'\int_0^{\tau_2'} d\tau_3'\int_\beta^0 d\tau_1\int_\beta^{\tau_1} d\tau_2\int_\beta^{\tau_2} d\tau_3 \left\{ Tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})\cdot Tr(\lambda^{a_3}\lambda^{a_2}\lambda^{a_1})\cdot e^{i\frac{2\pi}{\beta}[\ell_1(\tau_1'-\tau_1)+\ell_2(\tau_2'-\tau_2)+\ell_3(\tau_3'-\tau_3)]} + \right.$$

$$\left. + \text{ crossed graphs}\right\}, \qquad (3.2.20)$$

where $\omega_i = \frac{2\pi\ell_i}{\beta}$, i = 1,2,3.

Using the representation of k-function and the following Fourier transform

$$\frac{1}{k^{2\alpha}} = \frac{\Gamma(\lambda+1-\alpha)}{2^{2\alpha}\,\pi^{\lambda+1}\,\Gamma(\alpha)}\int d^n x\, \frac{e^{ik\cdot x}}{(x^2)^{\lambda+1-\alpha}} \qquad (3.2.21)$$

where $\lambda=\frac{1}{2}(n-2)$, we obtain the Green function

$$G_4(\vec{x}-\vec{y}) = \frac{g^6\beta^3}{2^6\pi^3|\vec{x}-\vec{y}|^3}\left\{ I(0,0,0) + \sum_{\substack{\ell_1,\ell_2,\ell_3=-\infty \\ \text{except all} \\ \ell_1,\ell_2,\ell_3=0}}^{\infty} e^{-\frac{2\pi}{\beta}(|\ell_1|+|\ell_2|+|\ell_3|)|\vec{x}-\vec{y}|}\cdot I(\ell_1,\ell_2,\ell_3)\right\}, \qquad (3.2.22)$$

where

$$I(\ell_1,\ell_2,\ell_3) = \frac{1}{\beta^6}\int_0^\beta d\tau_1'\int_0^{\tau_1'} d\tau_2'\int_0^{\tau_2'} d\tau_3'\int_\beta^0 d\tau_1\int_\beta^{\tau_1} d\tau_2\int_\beta^{\tau_2} d\tau_3 \left\{ Tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})\cdot Tr(\lambda^{a_3}\lambda^{a_2}\lambda^{a_1})\cdot\right.$$

$$\cdot e^{i\frac{2\pi}{\beta}[\ell_1(\tau_1'-\tau_3)+\ell_2(\tau_2'-\tau_2)+\ell_3(\tau_3'-\tau_1)]} + \text{crossed graphs} \Big\} , \qquad (3.2.23)$$

$$I(0,0,0) = I(\ell_1=0,\ell_2=0,\ell_3=0) = -\frac{1}{12}\,Tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3})\cdot\Big[Tr(\lambda^{a_1}\lambda^{a_2}\lambda^{a_3}) +$$

$$+ Tr(\lambda^{a_1}\lambda^{a_3}\lambda^{a_2})\Big] = -\frac{(N^2-1)\left(N-\frac{4}{N}\right)}{96\,N^2} \qquad (3.2.24)$$

For $G_4(\vec{x}-\vec{y})$, we consider only the leading term in eq.(3.2.22), i.e. I(o,o,o) term. Thus we have

$$G_4(\vec{x}-\vec{y}) \cong -\frac{g^6\beta^3(N^2-1)\left(N-\frac{4}{N}\right)}{3\;2^{11}\;\pi^3\;|\vec{x}-\vec{y}|^3} \qquad (3.2.25)$$

It can be seen from eq.(3.2.5) that $G_4(\vec{x}-\vec{y})$ falls off as the third power of $|\vec{x}-\vec{y}|$ when the distance between a quark and an antiquark increases. This differs from that of the two-gluon exchange where its power is two.

Next, we would like to discuss the case of n-gluon exchange between a pair of quarks, i.e. the following Feynman Fig. 3.1.

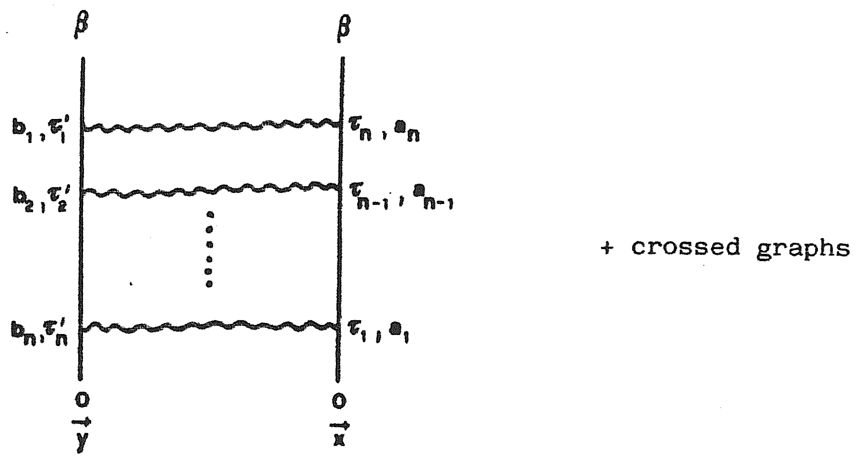


Fig 3.1 - The connected two-point function of the Thermal Wilson loop corresponding to n-gluon exchange between a quark and an antiquark. There are n!-1 crossed graphs.

By means of the representation of k-function we have got the Green function corresponding to all diagrams in Fig. 3.1.

$$G_{(n)}(\vec{x}-\vec{y}) = \frac{g^{2n}\beta^n}{2^{2n}\pi^n|\vec{x}-\vec{y}|^n}\left\{ I(0,0,\cdots,0) + \right.$$

$$\left. + \sum_{\substack{l_1,l_2,\cdots l_n=-\infty \\ \text{except all} \\ l_1,l_2,\cdots l_n=0}}^{\infty} e^{-\frac{2\pi}{\beta}(|l_1|+|l_2|+\cdots+|l_n|)\cdot|\vec{x}-\vec{y}|} \cdot I(l_1,l_2,\cdots l_n) \right\}, \qquad (3.2.26)$$

where

$$I(l_1,l_2,\cdots l_n) = \frac{1}{\beta^{2n}}\int_0^\beta d\tau_1' \int_0^{\tau_1'} d\tau_2' \cdots \int_0^{\tau_{n-1}'} d\tau_n' \int_\beta^0 d\tau_1 \int_\beta^{\tau_1} d\tau_2 \cdots \int_\beta^{\tau_{n-1}} d\tau_n \left\{ Tr(\lambda^{a_1}\lambda^{a_2}\cdots\lambda^{a_n})\cdot Tr(\lambda^{a_n}\lambda^{a_{n-1}}\cdots\lambda^{a_1})\cdot \right.$$

$$\left. \cdot e^{i\frac{2\pi}{\beta}\left[l_1(\tau_1'-\tau_n)+l_2(\tau_2'-\tau_{n-1})+\cdots+l_n(\tau_n'-\tau_1)\right]} + \text{crossed graphs} \right\}, \qquad (3.2.27)$$

$$I(0,0,\cdots,0) = \frac{(-1)^n}{(n!)^2} Tr(\lambda^{a_1}\lambda^{a_2}\cdots\lambda^{a_n})\cdot\left[\sum_{\{a_1,a_2,\cdots a_n\}} Tr(\lambda^{a_n}\lambda^{a_{n-1}}\cdots\lambda^{a_1})\right], \qquad (3.2.28)$$

The $\sum_{\{a_1,a_2,\cdots a_n\}}$ means a sum of all possible permutations over the set $\{a_1,a_2,\cdots a_n\}$.

The first term in eq.(3.2.26) is the leading term of $G_{(n)}(\vec{x}-\vec{y})$. This shows that the contributions of Fig. 3.1 of n-gluon exchange fall off as the n-th power of $|\vec{x}-\vec{y}|$ when the distance between q and $\bar{q}$ increases, if only the leading term of eq.(3.2.26) is taken into account.

Concerning the calculation of eq.(3.2.28), we can perform it by the application of the gluon projection operator:

$$\sum_a (\lambda^a)_{\alpha\beta}(\lambda^a)_{\gamma\delta} = \frac{1}{2}\left(\delta_{\alpha\delta}\,\delta_{\beta\gamma} - \frac{1}{N}\,\delta_{\alpha\beta}\,\delta_{\gamma\delta}\right), \qquad (3.2.29)$$

where the λ^a's are in the fundamental representation of SU(N).

For example, for the n = 4 case,

$$I(0,0,0,0) = \frac{1}{4608\cdot N^2}\left(N^4 - 5N^2 + 13 - \frac{9}{N^2}\right). \qquad (3.2.30)$$

By summing up the results of all Feynman diagrams in Fig. 2.1 — 2.4 , we have got the connected two-point function of the Thermal Wilson loop up to $O(g^6)$

$$\left\langle \mathrm{Tr}\Omega^{+}(\vec{x})\cdot \mathrm{Tr}\Omega(\vec{y})\right\rangle_c = G_1(\vec{x}-\vec{y}) + G_{2(a,b,c)}(\vec{x}-\vec{y}) + G_{2(e)}(\vec{x}-\vec{y}) + G_3(\vec{x}-\vec{y}) + G_4(\vec{x}-\vec{y}) \cong$$

$$\cong \frac{g_R^4 Q^2 (N^2-1)\beta^2}{2^5 \pi^2 (\vec{x}-\vec{y})^2} + \frac{9 g_R^6 Q^2 N(N^2-1)\beta}{2^{10}\pi^3 m_{el} (\vec{x}-\vec{y})^2} e^{-2m_{el}|\vec{x}-\vec{y}|} + \frac{g_R^6 Q^2 N(N^2-1)\beta^2}{2^5 \pi^4 (\vec{x}-\vec{y})^2}\Big[\frac{5}{12} +$$

$$+\frac{11}{24}\gamma - \frac{23}{24}\ln\pi - \frac{17}{12}\ln 2 + \frac{11}{12}\ln(\mu\beta)\Big] - \frac{g_R^6 (N^2-1)(N-\frac{4}{N})\beta^3}{3\cdot 2^{11}\pi^3 |\vec{x}-\vec{y}|^3} , \qquad (3.2.31)$$

at the approximations of high temperature $T\to\infty$ and large distance $|\vec{x}-\vec{y}|\mapsto\infty$.

There are three scale variables in our problem: the temperature T, the distance between q and $\bar{q}$, $|\vec{x}-\vec{y}|$, and the renormalization mass scale μ. We denote the right-side of eq.(3.2.31) by $G(|\vec{x}-\vec{y}|, \beta; \mu, g_R)$. The connected two-point Green function $G(|\vec{x}-\vec{y}|,\beta;\mu,g_R)$ of the Thermal Wilson loop satisfies the following renormalization group equation:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R}\right) G(|\vec{x}-\vec{y}|, \beta; \mu, g_R) = 0 . \qquad (3.2.32)$$

As the Thermal Wilson loop is gauge invariant, there are only two terms in the above renormalization group equation (the gauge and the anomalous dimension terms can be dropped out).

Since there is the term $\ln(\mu\beta)$ in eq.(3.2.31),it must be possible that $\left\langle \mathrm{Tr}\Omega^{+}(\vec{x})\cdot \mathrm{Tr}\Omega(\vec{y})\right\rangle_c$ can be written in terms of the running coupling constant $\bar{g}(\beta, \mu)$. Because

$$\bar{g}^2(\beta,\mu) = \frac{g_R^2}{1 - \frac{11 C_2}{24\pi^2}\ln(\mu\beta)\, g_R^2} , \qquad \bar{g}(\beta=\tfrac{1}{\mu},\mu) = g_R ; \qquad (3.2.33)$$

therefore,

$$g_R^2 = \bar{g}^2(\beta,\mu) - \frac{11 C_2}{24\pi^2}\ln(\mu\beta)\cdot\bar{g}^4(\beta,\mu) + \cdots\cdots$$

Up to $O(g^6)$,

$$g_R^4 = \bar{g}^4(\beta,\mu)\Big[1 - \frac{11 C_2}{12\pi^2}\ln(\beta\mu)\cdot\bar{g}^2(\beta,\mu) + \cdots\cdots\Big] ,$$

$$g_R^6 = \bar{g}^6(\beta,\mu)\Big[1 + \cdots\cdots\Big] .$$

Thus we get

$$\left\langle \mathrm{Tr}\,\Omega^{+}(\vec{x})\cdot \mathrm{Tr}\,\Omega(\vec{y})\right\rangle_c = \frac{\bar{g}^4 Q^2 (N^2-1)\beta^2}{2^5\,\pi^2\,(\vec{x}-\vec{y})^2} + \frac{9\,\bar{g}^6 Q^2 N(N^2-1)\beta}{2^{10}\,\pi^3\, m_{el}(\vec{x}-\vec{y})^2}\, e^{-2m_{el}|\vec{x}-\vec{y}|} +$$

$$+ \frac{\bar{g}^6 Q^2 N(N^2-1)\beta^2}{2^5\,\pi^4\,(\vec{x}-\vec{y})^2}\left[\frac{5}{12}+\frac{11}{24}\gamma-\frac{23}{24}\ln\pi-\frac{17}{12}\ln 2\right]-\frac{\bar{g}^6 (N^2-1)(N-\frac{4}{N})\beta^3}{3\cdot 2^{11}\,\pi^3\,|\vec{x}-\vec{y}|^3} \quad . \tag{3.2.34}$$

As mentioned above, a (color) electric (or magnetic) mass m(T) will be needed in the gluon propagators due to finite temperature effects. The time component $A_0^a(\tau,\vec{x})$ of the Yang-Mills field develops an "electric" mass $m_{el}(T)$ which equals $\sqrt{\frac{N}{3}}\,gT$ for one-loop correction of zero-zero component of the gluon propagator in the gauge group SU(N), whereas the space components $A_i^a(\tau,\vec{x})$ give a "magnetic" mass m_{mag} which equals zero for one-loop correction of space components of the gluon propagator and has order g^2T for two-loop (or above two-loop) correction.

This substitution of the time-component of gluon propagator (in Feynman gauge, $\alpha = 1$) ,

$$\frac{1}{k^2} \longrightarrow \frac{1}{k^2+\Pi_{00}} \quad , \tag{3.2.35}$$

implies the summation of the following diagrams:

+ . . . . . .

Fig. 3.2-The radiative corrections to the time component of the gluon propagator.

where the Π_{00} is the time component of the gluon self-energy tensor, which equals $\sqrt{\frac{N}{3}}\,gT$ to one-loop order in the case of SU(N).

the electric mass $m_{el}(T) \neq 0$ at finite temperature represents the Debye screening effect [1.11, 1.18]. After taking into account the screening effect, i.e.

making the substitution of eq.(3.2.35) in the propagators of the exchanged gluons, we have for Fig. 2.1.

$$G_1(\vec{x}-\vec{y}) = \frac{g^4 Q^2 (N^2-1)\beta^2}{2^7 \cdot \pi^6} \int \frac{d^3k\, d\vec{p}\;\; e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}}{(\vec{k}^2+m_{el}^2)\left[(\vec{p}-\vec{k})^2+m_{el}^2\right]} = \frac{g^4 Q^2 (N^2-1)\beta^2}{2^5\, \pi^2 (\vec{x}-\vec{y})^2} \cdot e^{-2m_{el}|\vec{x}-\vec{y}|} \tag{3.2.36}$$

Similarly, we can get for the diagram (b) (including the crossed diagrams) in Fig. 2.4

$$G_4(\vec{x}-\vec{y}) = \frac{g^6 \beta^3}{2^6 \pi^3 |\vec{x}-\vec{y}|^3} \left\{ \sum_{\substack{l_1,l_2,l_3 \\ =0,\pm 1,\cdots \pm\infty}} e^{-\left[\sum_{i=1}^{3} \sqrt{\left(\frac{2\pi l_i}{\beta}\right)^2 + m_{el}}\,\right]|\vec{x}-\vec{y}|} \cdot I(l_1,l_2,l_3) \right\} \tag{3.2.37}$$

For the leading term $G_4^L(\vec{x}-\vec{y})$ of eq.(3.2.37),

$$G_4^L(\vec{x}-\vec{y}) = -\frac{g^6 \beta^3 (N^2-1)(N-\frac{4}{N})}{3\cdot 2^{11}\cdot \pi^3 |\vec{x}-\vec{y}|^3}\, e^{-3m_{el}|\vec{x}-\vec{y}|} \tag{3.2.38}$$

For the leading term $G_{(n)}^L(\vec{x}-\vec{y})$ of Fig.3.1 of n-gluon exchange,

$$G_{(n)}^L(\vec{x}-\vec{y}) = \frac{g^{2n}\cdot \beta^n}{2^{2n}\, \pi^n |\vec{x}-\vec{y}|^n}\, I(0,0,\cdots,0)\cdot e^{-n\cdot m_{el}|\vec{x}-\vec{y}|} \tag{3.2.39}$$

We can also obtain the results of the diagrams (a),(a'),(b),(b'),(c) and (c') in Fig. 2.2 and all diagrams in Fig. 2.3.

By summing up the results above, we can finally get the expressions of $\langle Tr\Omega^+(\vec{x})\cdot Tr\Omega(\vec{y})\rangle_c$ up to the order $g^6(\beta,\mu)$

$$\begin{aligned}\langle Tr\Omega^+(\vec{x})\cdot Tr\Omega(\vec{y})\rangle_c = {} & \frac{\bar{g}^4 Q^2 (N^2-1)\beta^2}{2^5\cdot \pi^2 (\vec{x}-\vec{y})^2}\, e^{-2m_{el}|\vec{x}-\vec{y}|} + \frac{9\bar{g}^6 Q^2 N(N^2-1)\beta}{2^{10}\pi^3 m_{el}(\vec{x}-\vec{y})^2}\, e^{-2m_{el}|\vec{x}-\vec{y}|} + \\ & + \frac{\bar{g}^6 Q^2 N(N^2-1)\beta^2}{2^5\, \pi^4 (\vec{x}-\vec{y})^2}\left[\frac{5}{12}+\frac{11}{24}\gamma - \frac{23}{24}\ln\pi - \frac{17}{12}\ln 2\right] e^{-2m_{el}|\vec{x}-\vec{y}|} - \\ & - \frac{\bar{g}^6 (N^2-1)(N-\frac{4}{N})\beta^3}{3\cdot 2^{11}\cdot \pi^3 |\vec{x}-\vec{y}|^3}\cdot e^{-3m_{el}|\vec{x}-\vec{y}|} + O(\bar{g}^7)\end{aligned} \tag{3.2.40}$$

with the approximations of high temperature $T \to \infty$ and large distance $|\vec{x}-\vec{y}| \to \infty$.

In Ref.[2.2], it has been shown that $\langle Tr\Omega(\vec{x})\rangle \to 1$ at the high-temperature limit. Therefore, from eqs.(3.2.1) and (3.2.5) we get the high-temperature interquark potential

$$V(\vec{x}-\vec{y}) = -\frac{1}{\beta}\langle Tr\Omega^{+}(\vec{x})\cdot Tr\Omega(\vec{y})\rangle_c \quad . \qquad (3.2.41)$$

It can be seen from eqs.(3.2.40) and (3.2.41) that $V(\vec{x}-\vec{y})$ falls off as an exponential and a power when $|\vec{x}-\vec{y}|$ increases, that is, $V(\vec{x}-\vec{y}) \to 0$ as $|\vec{x}-\vec{y}| \to \infty$. Thus, the quarks would be unconfined in the high temperature limit, which is consistent with the Monte Carlo study of lattice gauge theory at high temperature [1.14]. The lack of confinement is caused by the screening of the "electric" charge of the heavy quarks due to the high temperature effect. The range of the potential between q and $\bar{q}$ is $(2m_{el})^{-1}$.

3.3 THE MAGNETIC MASS OF GLUON AND THE VALIDITY OF PERTURBATION THEORY

In this section, we shall discuss some difficulties encountered in the higher-order computations of the interquark potential at high temperature by m ans of perturbation theory. They are due to the infrared problem in the thermodynamics of a massless Yang-Mills gas. The pertubative calculability of the interquark potential is lost beyond some order of g, because the magnetic mass of the gluon is at most of order g^2T [1.18].

To explain this point, we have seen from the calculation, eq.(3.2.15), of Fig.2.2(e) that perturbative expansions for the interquark potential are unreliable in the region of small momenta, $|\vec{p}| \leqslant g^2T$. Thus, we must consider the gluon mass m(T), then, the perturbative calculation of Fig.2.2(e) is valid for all momenta. Does this mean that one can perform the perturbative expansion of the interquark potential $V(\vec{x}-\vec{y})$ to any order of g? No, since the magnetic mass of gluon is zero up to the order $g^{\frac{3}{2}}T$ [3.3], and the result of higher calculations is at most of $O(g^2T)$, the contributions of higher order graphs will be of the same order in g. We shall explain this below by means of Fig. 2.5.

We consider the contributions $G_0, G_2, G_4, \ldots, G_{2N} \ldots$ of the diagrams $(a_0), (a_2), (a_4), \ldots (a_{2N}), \ldots$ in Fig.2.5 to the interquark potential. Our interest is the infrared behaviour of Fig.2.5, thus we consider the contributions G_{2N} ($N = 0,1,2,\ldots$) that arise from the region where all spatial momenta of gluons are of order $|\vec{p}|$ ($|\vec{p}|$ is small) and all energies vanish. Therefore, each new loop in Fig.2.5 (a_{2N}) contributes a factor of g^2 from two new vertices, a factor of $T|\vec{p}|^3$ from phase space, a factor of $|\vec{p}|^{-4}$ from two new vertices and three gluon propagators. Since the original loop is G_0, the contribution G_{2N} will behave as

$$G_{2N} \sim G_0 \left(g^2 T |\vec{p}|^3 |\vec{p}|^{-4}\right)^N = G_0 \left(\frac{g^2 T}{|\vec{p}|}\right)^N \quad (3.3.1)$$

Therefore, a sum of the contributions of these diagrams $(a_0), (a_2), \ldots\ldots, (a_{2N}), \ldots\ldots$ in Fig. 2.5 to $\langle \mathrm{Tr}\Omega^\dagger(\vec{x}) \cdot \mathrm{Tr}\Omega(\vec{y})\rangle_c$ behaves as

$$G_0(\vec{p}) + G_2(\vec{p}) + \cdots + G_{2N}(\vec{p}) + \cdots \sim \frac{g^4 \cdot T^{-2}}{|\vec{p}|} \left[1 + \frac{g^2 T}{|\vec{p}|} + \left(\frac{g^2 T}{|\vec{p}|}\right)^N + \cdots \right] . \quad (3.3.2)$$

Eq.(3.3.2) shows that the series is divergent in the $|\vec{p}| \lesssim g^2 T$ region. From eq.(3.3.1) it can be seen that these infrared singularities are due to the self-interactions of the massless gluons, and the higher-order corrections to the G_0 are increasingly divergent, so we must consider an infrared cut-off mass

$m(T)$ of the gluon. After taking into account the mass $m(T)$ ($m_{el}(T)$ or $m_{mag}(T)$) and writing in configuration space, eq.(3.3.2)becomes

$$G_0(\vec{x}-\vec{y}) + G_2(\vec{x}-\vec{y}) + \cdots + G_{2N}(\vec{x}-\vec{y}) + \cdots \sim \frac{g^4 T^{-2}}{(\vec{x}-\vec{y})^2} \cdot e^{-2m_{el}|\vec{x}-\vec{y}|} \cdot \Big[1 + \frac{g^2 T}{m(T)} + \left(\frac{g^2 T}{m(T)}\right)^2 + \cdots + \left(\frac{g^2 T}{m(T)}\right)^N + \cdots \Big] \qquad (3.3.3)$$

In Fig.2.5(a_{2N}), for the four gluon propagators α_1, α_2, α_3, α_4, their corresponding Green functions are $G^{ab}_{oo}(k_o = 0,\vec{k})$, but, for all other propagators, there are the contributions from G^{ab}_{ij}. G^{ab}_{oo} develops an "electric" mass $m_{el} \sim gT$, and G^{ab}_{ij} develops a smaller "magnetic" mass $m_{mag} \sim g^2 T$. From eq(3.2.17),we can see that in the denominator of the integrand of eq(3.2.17) there are four gluon propagators with the electric mass m_{el}, consequently in the loop-momentum integral of eq.(3.2.17), power of smaller momenta ($\leqslant g^2 T$) is eight in the numerator and is two in the denominator, therefore the eq(3.2.17)is not infrared divergent due to the cutoff mass m_{el}. But, increasing a new loop, one momentum $|\vec{p}|$ is added to the denominator, thus power counting momenta brings the conclusion that up to the seven-loop graph (a_{12}) in Fig.2.5 the G_{2N} ($N = 0,1,\ldots 6$) can be calculated by means of perturbation theory; from the eight-loop Fig.2.5 (a_{14}) on, the G_{2N} ($N = 7,8,\ldots$) are uncomputable perturbatively, i.e. we have

$$G_{2N}(\vec{x}-\vec{y}) \sim \begin{cases} \dfrac{g^4 T^{-2}}{(\vec{x}-\vec{y})^2} \cdot \left(\dfrac{g^2 T}{m_{el}}\right)^N \cdot e^{-2m_{el}|\vec{x}-\vec{y}|}, & N=0,1,2,\cdots 6 \\[2ex] \dfrac{g^{10} T^{-2}}{(\vec{x}-\vec{y})^2} \cdot \left(\dfrac{g^2 T}{m_{mag}}\right)^{N-6} \cdot e^{-2m_{el}|\vec{x}-\vec{y}|}, & N \geqslant 7 \end{cases} \qquad (3.3.4)$$

It can be seen from eq(3.3.4) that if the integer $N \geqslant 7$, G_{2N} will be of the same order $g^{10} T^{-2}$ as G_{12}. Therefore $V(\vec{x}-\vec{y})$ can be computed by means of perturbation theory only up to the order $g^{10} T^{-2}$. We compare it with Ref. [1.11], where the thermodynamic potential $\Omega(T)$ is computable by means of perturbation theory only up to the order $g^6 T^4$. So, some non-perturbative methods must be incorporated to the calculations of the high-temperature interquark potential.

3.4. PHASE TRANSITION - THE CRITICAL TEMPERATURE

Expressions (3.2.40) and (3.2.41) yield that terms of lower order $\bar{g}(T,\mu)$ are larger than ~~hi~~gherorder $\bar{g}(T,\mu)$. However when temperature decreases the high order in $\bar{g}$ increases faster than the low order. Therefore, decreasing temperature to some point $T = T_c$, the value of the term of $O(\bar{g}^{6})$ is the same as that of the term $O(\bar{g}^{4})$. This means, in our opinion, that a phase transition of confinement-deconfinement occurs. In this case, we take the running coupling constant $\alpha(T_c) = \frac{\bar{g}^2(T_c)}{4\pi} \approx 1$ (see Fig. 3.3) [3.4], and the distance between a quark and an antiquark, as $|\vec{x}-\vec{y}| \approx 1$ fm. Thus, from eqs.(3.2.40) and (3.2.41), we have qualitatively estimated the critical temperature of phase transition, $T_c \approx 250$ MeV. This result is close to that of the Monte Carlo evaluation, which is $T_c \approx 200$ MeV for SU(3).

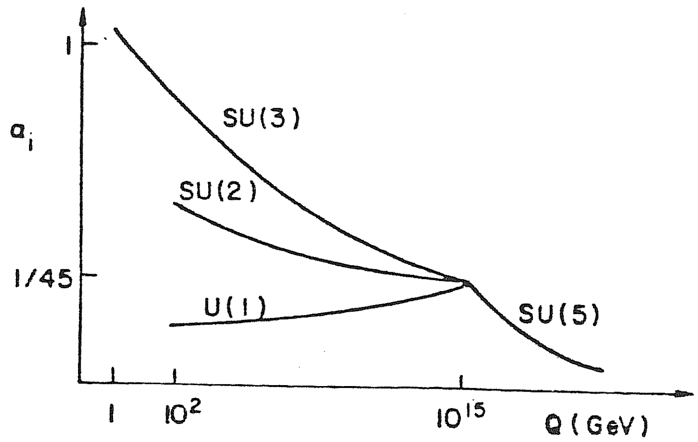


Fig. 3.3 The relation between running coupling constants and energy scale.

3.5. CONCLUSIONS

We have studied the interquark potential at high temperature through order g^6 within the framework of perturbation theory. The conclusion we get is that the interquark potential falls off as an exponential and a power when the distance between q and $\bar{q}$ increases, in the high-temperature limit and up to $O(g^6)$. It means that quarks are unconfined at high temperature, which is consistent with the Monte Carlo study of lattice gauge theory at high temperature. Since the magnetic mass of the gluon is at most of order g^2T, the high-temperature interquark potential can be calculated by means of perturbation theory only up to the order g^{10} . beyond the order g^{10} , perturbative calculations for interquark potential are meaningless. Therefore, some non-pertubative methods must be incorporated to the high-temperature interquark potential.

CHAPTER IV - INFRARED PROBLEM IN QCD AT FINITE TEMPERATURE

It can be seen, from the calculations of the interquark potential in Chapters II and III, that the loop graphs including the gluon self-interaction vertices are increasingly infrared divergent. Though the gluon masses, due to temperature effects, might cure the infrared divergences, we lose perturbative calculability because of the smaller gluon magnetic mass. In the following two subsections, we shall discuss the infrared behaviour of the temperature Green functions and the (color) gluon magnetic mass respectively.

4.1. Infrared behaviour of the temperature Green functions

In QCD, the full gluon propagator $\mathcal{D}^{ab}_{\mu\nu}(x) = \langle A^a_\mu(x) A^b_\nu(0) \rangle$ may be expressed in terms of the one-particle irreducible self-energy, $\Pi^{ab}_{\mu\nu}(\omega_n, \vec{k})$,

$$\mathcal{D}^{ab}_{\mu\nu}(\omega_n, \vec{k}) = \left\{ \left[(\omega_n^2 + \vec{k}^2) \cdot I + \Pi(\omega_n, \vec{k}) \right]^{-1} \right\}^{ab}_{\mu\nu} , \tag{4.1.1}$$

where $\Pi(\omega_n, \vec{k})$ is the gluon self-energy tensor matrix. In general, we may construct four independent, symmetric, O(3) covariant tensors depending on a single vector k. For example,

$$A_{\mu\nu} = \delta_{\mu i}\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right)\delta_{j\nu} ,$$

$$B_{\mu\nu} = \left(\delta_{\mu 0} - \frac{k_\mu k_0}{k^2}\right)\frac{k^2}{\vec{k}^2}\left(\delta_{\nu 0} - \frac{k_\nu k_0}{k^2}\right) , \tag{4.1.2}$$

$$C_{\mu\nu} = \frac{1}{\sqrt{2}}\left(\delta_{\mu 0} - \frac{k_\mu k_0}{k^2}\right)\frac{k_\nu}{|\vec{k}|} + \frac{1}{\sqrt{2}}\frac{k_\mu}{|\vec{k}|}\left(\delta_{\nu 0} - \frac{k_\nu k_0}{k^2}\right) ,$$

$$D_{\mu\nu} = \frac{k_\mu k_\nu}{k^2} .$$

Note that

$$A_{\mu\nu} + B_{\mu\nu} = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \quad . \tag{4.1.3}$$

Thus the self-energy which is always diagonal in color may be decomposed as

$$\Pi^{ab}_{\mu\nu} = \delta^{ab}\, \Pi_{\mu\nu} = \left(\alpha A_{\mu\nu} + \beta B_{\mu\nu} + \gamma C_{\mu\nu} + \delta D_{\mu\nu}\right) \delta^{ab} \quad . \tag{4.1.4}$$

Since $D_{\mu\nu}$ satisfies the Ward identity $k_\mu k_\nu D_{\mu\nu} = \xi \equiv 1$, this provides one relation among the above coefficients of eq. (4.1.4),

$$\delta = \frac{\gamma^2}{2(k^2+\beta)} \quad . \tag{4.1.5}$$

Note that from the beginning of this subsection 4.2., we have already chosen the Feynman gauge ξ = 1, and at zero frequency, $\omega_n = 0$, γ must vanish due to (Euclidean) time reversal invariance. Consequently, the static self-energy $\Pi_{\mu\nu}(\omega_n=0,\vec{k})$ is always transverse. To one-loop order, we have calculated the self-energy

$$\Pi_{00}(\omega_n=0, \vec{k}\sim 0) = \frac{N}{3} g^2 T^2 \qquad \text{for SU(N)} \quad , \tag{4.1.6}$$

$$\Pi_{ij}(\omega_n=0, \vec{k}\sim 0) = \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) \Pi_{kk}(\omega_n=0, \vec{k}\sim 0) = 0 \quad . \tag{4.1.7}$$

Therefore, we have

$$\mathcal{D}_{\mu\nu}(k_0=0,\vec{k}\sim 0) = \frac{\delta_{\mu 0}\,\delta_{\nu 0}}{\vec{k}^2+\Pi_{00}} + \frac{\delta_{\mu i}\,\delta_{\nu i}}{\vec{k}^2} \qquad (4.1.8)$$

Thus, we have seen that the electric mass is $\sqrt{\Pi_{00}}$, the magnetic mass is zero to one-loop order.

Note that

$$\mathcal{D}_{\mu\nu}(\vec{k}=0,k_0\underset{\sim}{\neq}0) = \frac{\delta_{\mu i}\,\delta_{\nu i}}{k_0^2+\frac{1}{3}\Pi_{00}} + \frac{\delta_{\mu 0}\,\delta_{\nu 0}}{k_0^2} \qquad (4.1.9)$$

These two limits, $\vec{k}\to 0$ and $k_0\to 0$, are not commutative, the different results are obtained. It can be seen from the eqs. (4.1.8) and (4.1.9). However, the infrared cut-off is relative with the limit $k_0=0,\vec{k}\to 0$, but not with the limit $\vec{k}=0,\ k_0\to 0$. The problem is that $\mathcal{D}_{\mu\nu}(k)$ is not analytic at $\vec{k}=0$.

In the following, we should like to see the infrared behaviour of the Green function of gluon, $\mathcal{D}_{\mu\nu}(k_0=0,\vec{k}\to 0)$, in the Coulomb gauge $\partial_i A_i=0$. For QED, the properties of $\mathcal{D}_{\mu\nu}(k_0=0,\vec{k}\to 0)$ in the Coulomb gauge are well known. The Green function of photon $\mathcal{D}_{\mu\nu}(k)$ at $k_0=0$ has the following structure [1.29] :

$$\begin{aligned}
\mathcal{D}_{00}(\vec{k}) &= \frac{1}{\vec{k}^2+\Pi_{00}(\vec{k})} , \\
\mathcal{D}_{0i}(\vec{k}) &= \mathcal{D}_{i0}(\vec{k}) = 0 , \\
\mathcal{D}_{ij}(\vec{k}) &= \left(\delta_{ij}-\frac{k_i k_j}{\vec{k}^2}\right)\frac{1}{\vec{k}^2+A(\vec{k})} .
\end{aligned} \qquad (4.1.10)$$

The $\Pi_{00}(\vec{k})$ is equal to the corresponding component of the polarization operator $\Pi_{\mu\nu}(k_0=0,\vec{k})$. In the high-temperature limit,

$$\Pi_{00}(\vec{k}=0) = \frac{e^2 T^2}{3} \quad . \tag{4.1.11}$$

Thus, $\mathcal{D}_{00}(k_0=0,\vec{k}\to 0) = \Pi_{00}^{-1}(\vec{k}=0) \neq \infty$ and in this sense the component $\mathcal{D}_{00}(k)$ at $k_0=0,\ \vec{k}\to 0$ behaves as the Green function of a massive field with the mass squared $m^2 = \Pi_{00}(0)$.

On the other hand, it can be shown that for all order of perturbation theory, $A(\vec{k}) \sim \vec{k}^2$ at $\vec{k} \to 0$.

The physical meaning of such a difference between the low-momentum behaviour of $G_{00}(k)$ and $G_{ij}(k)$ is very simple. In the Coulomb gauge, at $k_0=0$, the Green function of the electric field $\langle E_i E_j \rangle$ is proportional to G_{00}, and the Green function of the magnetic field $\langle H_i H_j \rangle \sim G_{ij}$. Thus, the electro-static forces in a gas of charged particles become short-range due to the Debye screening ($G_{00}(k=0) = \Pi_{00}^{-1}(0) = \lambda_D^2$, where λ_D is the Debye length). If there are no magnetic charges in the theory, the magnetic forces cannot be screened and remain long-range, in accordance with the above results, $G_{ij}(\vec{k}\sim 0) \sim \frac{1}{\vec{k}^2}$.

In QCD, in the Coulomb gauge the Green function of gluon has the same structure as the Green function for photons, except for the extra factor δ^{ab}, where a, b are color indices.

$$\begin{aligned} \mathcal{D}_{00}^{ab}(k_0=0,\vec{k}\sim 0) &= \frac{\delta^{ab}}{\vec{k}^2 + \Pi_{00}(\vec{k})} \quad , \\ \mathcal{D}_{i0}^{ab} &= \mathcal{D}_{0i}^{ab} = 0 \quad , \\ \mathcal{D}_{ij}^{ab}(k_0=0,\vec{k}\sim 0) &= \delta^{ab}\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right) G(\vec{k}) . \end{aligned} \tag{4.1.12}$$

In high temperature limit, $\Pi_{00}(\vec{k}=0)=\frac{N}{3}g^2T^2$ for the leading term and SU(N). For $G(\vec{k})$, this case of QCD is different with QED, since there are the self--interactions of gluons in QCD, the $G^{-1}(\vec{k})$ is much more singular than QED.

We consider the contribution to $G^{-1}(\vec{q})$ of an arbitrary n-loop diagram consisting of spatial gluons with zero energy in perturbation expansion. The contribution arises from the region where all spatial momenta are of order q (q is small). Each new loop contributes a factor of g^2, the phase space contributes a factor of $T\,q^3$, and the new vertices and propagators contribute a factor of q^{-4}. Thus the calculation to $G^{-1}(|\vec{q}|\sim q)$ will behave as

$$\left(g^2T\cdot q^3\cdot q^{-4}\right)^{n-1}=\left(\frac{g^2T}{q}\right)^{n-1}. \qquad (4.1.13)$$

Since to one-loop order $G(q)=\frac{1}{q^2}$, to n-loop order $G^{-1}(q)\sim q^2\left(\frac{g^2T}{q}\right)^{n-1}$. Thus

$$G^{-1}(q)\sim q^2+g^2Tq+g^4T^2+\frac{g^6T^3}{q}+\frac{g^8T^4}{q^2}+\cdots \qquad (4.1.14)$$

This series, eq. (4.1.14), is divergent for $q\lesssim g^2T$. Therefore, the perturbative calculations for G(q) are unreliable at $q\lesssim g^2T$. Analogously it can be shown that the series for $\Pi_{00}(q)$ is also divergent at $q<g^2T$. Thus the perturbation theory can give us no information about the behaviour of the Green function $\mathcal{D}^{ab}_{\mu\nu}(k_0=0,\vec{k})$ at $k<g^2T$. But, we can show only that the cut-off $m_{mag}(T)$ appearing in $G^{ab}_{ij}(k_0=0,\vec{k}\sim 0)$ would be determined by the term $\sim g^4T^2$ in eq. (4.1.14), and can not be greater than $O(g^2T)$. Therefore, some non-perturbative methods are needed for quantitative calculations of the m_{mag}.

Finally, we would like to emphasize the two facts that firstly, the magnetic properties of the Green function $\mathcal{D}_{\mu\nu}(k)$ of the gluon (or photon) at $T \neq 0$ are much more complicated than at $T = 0$. $G_{\mu\nu}(k)$ at $T \neq 0$ is not analytic about $k_\mu = 0$, it depends on the sequence of the limits $k_0 \to 0$ and $\vec{k} \to 0$, i.e. the results of first $k_0 = 0$, second $\vec{k} \to 0$ and first $\vec{k} = 0$, second $k_0 \to 0$ are different. For first $k_0 = 0$, second $\vec{k} \to 0$, we find, to one-loop order

$$\mathcal{D}_{\mu\nu}(k_0 = 0, \vec{k} \sim 0) = \frac{\delta_{\mu 0}\,\delta_{\nu 0}}{\vec{k}^2 + m_{el}^2} + \frac{\delta_{\mu i}\,\delta_{\nu i}}{\vec{k}^2} \quad . \tag{4.1.15}$$

For first $\vec{k} = 0$, second $k_0 \to 0$, to one-loop order

$$\mathcal{D}_{\mu\nu}(\vec{k} = 0, k_0 \to 0) \sim \frac{\delta_{\mu i}\,\delta_{\nu i}}{k_0^2 + \frac{1}{3} m_{el}^2} + \frac{\delta_{\mu 0}\,\delta_{\nu 0}}{k_0^2} \quad . \tag{4.1.16}$$

It can be seen that the two results, (4.1.15) and (4.1.16) are different. In 1976, Kislinger and Morley claimed that the high temperature effects lead to the appearance of a pole of the gluon Green function $\mathcal{D}_{\mu\nu}$ in (4.1.16) (we drop the color indices) at $k_0^2 = -\frac{1}{3} m_{el}^2$, $\vec{k} = 0$. The value of k_0 at this pole was interpreted as the value of the infrared cutoff, $m(T) = \frac{1}{\sqrt{3}} m_{el}$, which would solve the infrared problem in the thermodynamics of the Yang-Mills gas. All transverse gluons have acquired a mass $\frac{m_{el}}{\sqrt{3}}$, and all color fluctuations are screened, and no long-range forces exist at finite temperature. However, the results asserted by Kislinger and Morley are wrong, since the leading infrared divergences in quantum statistics of gauge fields are connected with the behaviour of the Green function $G_{\mu\nu}$ not in the limit $\vec{k} = 0$, $k_0 \to 0$, but in the static limit $k_0 = 0$, $\vec{k} \to 0$ (the latter limit just corresponds to taking $T_k = 0$ in eq. (1.6.1)).

Secondly, for QED, the electrostatic forces at $T \neq 0$ are screening due to $D_{00}(0) = \Pi_{00}^{-1}(0) = \lambda_D^2 = 3/e^2 T^2$ (λ_D is the Debye length), and become short-range. But the magnetic forces at $T \neq 0$ cannot be screened and remain long-range due to no magnetic mass for QED at $T \neq 0$. However, for QCD at $T \neq 0$, (color) static electric and magnetic fields are all screening due to $m_{el} \sim gT$ and $m_{mag} \sim g^2 T$.

4.2. The Magnetic Mass of gluon

Up to now, the problem of the gluon magnetic mass remains an open question. Below, we would like to analyze qualitatively the order of the magnetic mass of gluon.

From last subsection 4.1, we know that the infrared divergences in a four-dimensional gauge theory arise from regions where all internal energies vanish. Because

$$\int \frac{d^4k}{(2\pi)^4} \longrightarrow T \sum_n \int \frac{d^3\vec{k}}{(2\pi)^3} \xrightarrow[n=0]{} T \int \frac{d^3\vec{k}}{(2\pi)^3} ,$$

the singularities are the same as those arising in a three-dimensional gauge theory. In general the infrared behaviour at high temperature of a d-dimensional theory is given by an equivalent $(d - 1)$ - dimensional theory. Now the equivalent theory is a three-dimensional gauge theory, whose loop-expansion coupling constant is g^2T. The dimension of the coupling g^2T is mass. Except for the renormalization mass scale, the only mass which appears in the three--dimensional pure gauge theory is the coupling g^2T. Consequently, the mass gap (i.e. m_{mag}) in the three-dimensional theory can only be a pure number times g^2T. Since the magnetic mass of the gluon is of order g^2T, perturbation theory is obviously useless for computing the infrared properties of the three-dimensional theory. Thus this shows perturbative incomputability of three-dimensional non-Abelian gauge theories.

Since the gluon magnetic mass in the high temperature QCD plasma is a crucial quantity to test the validity of perturbative expansions in finite temperature QCD, the quantitative calculations of the magnetic mass of gluon have interested many physicists. If the magnetic mass-squared were exactly zero, then the perturbative expansions of the physical quantities (for example, thermodynamic potential, interquark potential, free energy and so on) would diverge. If the magnetic mass-squared were of order g^4T^2 , then the perturbation expansions are valid up to only some order, being g^6 order for the thermodynamic potential, being g^{10} order for the interquark potential. If the magnetic mass-squared were of order g^3T^2 , then the perturbation expansion is meaningfull to any order as an asymptotic series.

In 1982, Kajantie and Kapusta [4.1] reported that by using the temporal axial gauge the gluons acquire a magnetic mass-squared of order g^3T^2 and in the electric mass-squared there exists after the leading term ($=\frac{g^2T^2N}{3}$, for SU(N)) a correction of order g^3T^2 . Soon after that, Toimela [4.2] claimed the magnetic mass was vanishing up to order $g^{\frac{3}{2}}T$, but the electric mass of $O(g^{\frac{3}{2}}T)$ was gauge-parameter dependent in a covariant gauge. In both works, however, the gauge covariance in the calculation was not manifestly shown. Recently, T. Furusawa and K. Kikkawa [3.3] reported that both electric and magnetic gluon masses in the high temperature QCD plasma were evaluated up to order $g^{\frac{3}{2}}T$. The result is that the magnetic mass is shown to vanish up to order $g^{\frac{3}{2}}T$. Gauge covariance is always kept at each step of the calculations by taking into account the Ward-Takahashi identities.

Several authors have pointed out the paper by Kajantie and Kapusta were not correct, moreover, their results are not in agreement with the above qualitative analysis for the order of the magnetic mass of gluon, which is of order g^2T .

In the following, we shall introduce these two works by Toimela and by Furusawa and Kikkawa.

Toimela has calculated the values of $\Pi_{00}(0,\vec{p}\to 0)$ and $\Pi_{ii}(0,\vec{p}\to 0)$ of polarization tensor of gluon by using the Schwinger-Dyson equation truncated to one-loop level in the general covariant gauge. Up to order $g^{\frac{3}{2}}T$, he has obtained

$$\Pi_{00}(0,\vec{p}\to 0) = \frac{g^2 N}{3} T^2 + \frac{3}{4\pi}\xi\left(\frac{g^2 N}{3}\right)^{\frac{3}{2}} T^2 , \tag{4.2.1}$$

$$\Pi_{ii}(0,\vec{p}\to 0) = 0 ,$$

where ξ is the gauge-parameter.
Note that the electric mass term of order $g^{\frac{3}{2}}T$ in eq.(4.2.1) is dependent on the gauge- parameter ξ.

In 1983, Furusawa and Kikkawa have evaluated both electric and magnetic gluon masses in the high temperature QCD plasma up to order of $g^{\frac{3}{2}}$. They always have kept the gauge covariance in each step of calculations by taking into account of the Ward-Takahashi identities. The result is that the electric and magnetic masses

$$m_{el}^2 = \left[\frac{g^2 N}{3} + \frac{3}{2\pi}\left(\frac{g^2 N}{3}\right)^{\frac{3}{2}}\right] T^2 ,$$

$$m_{mag}^2 = 0 , \tag{4.2.2}$$

which are both gauge-parameter independent. In their calculations, the gauge is chosen as the axial gauge.

The results by Furusawa and Kikkawa do not agree with the preceding two reports by Kajantie, Kapusta and Toimela. In the covariant gauge (Toimela's report), the W - T identity is not simple and non-trivial vertex corrections will be needed by the gauge invariance. Since Toimela did not include the vertex correction, the result can be gauge dependent. In Kajantie and

Kapusta's temporal axial gauge, the vertex correction is also necessary to keep the gauge invariance. Taking into account this vertex correction, the same results as (4.2.2) are recovered.

Now it is clear that the leading term of the electric mass of the gluon is $\sqrt{\frac{N}{3}}\, gT$ (SU(N)). But what equals the leading term of the magnetic mass of gluon is not clear. Though the above qualitative analysis shows that the magnetic mass is of order g^2T, the coefficient of the magnetic mass $m_{mag} = C g^2 T$ was predicted to be uncalculable [1.18] in the normal sense. The case C = 0 is not excluded, but it is also possible that the coefficient C is very large, as an infinite number of diagrams were expected to contribute. Therefore, it is also possible that the summed contributions give the result $C g^2 T \sim C' g^{\frac{3}{2}} T$.

4.3. The non-perturbative approaches to the infrared problem in QCD at finite temperature

Despite this pessimism, a number of non-perturbative approaches were used to try to get at the infrared limit. We wish to shortly summarize them here.

1) Based on a truncated Schwinger-Dyson equation, an infrared cut-off of order g^2T was determined in an arbitrary covariant gauge [4.3]. However, since the coefficient depended upon the gauge fixing parameter ξ, and could even vanish for $\xi = -1$, the significance of this result is not clear.

2) At least in some gauges, it was suggested that one could study the static infrared sector of the full four-dimensional theory by considering only the n = o modes and in three dimensions [1.12, 4.4, 4.5]. The relationship of that approach to one based on a truncated Schwinger-Dyson equation is not clear. Certainly there would appear problems in trying to do that in the temporal axial gauge. Anyway, it has not yet been proven the validity of the calculation of the magnetic mass in this manner.

3) An estimate was made of the possible instanton contribution to the magnetic mass, but it was found to be quite negligible [4.6].

4) The first real evidence that the magnetic infrared cut-off really exists was provided by Monte-Carlo simulations on the SU(2) lattice [4.7]. Although these results were shown to be consistent with a g^2T behaviour, we would like to show that they are also consistent with a $g^{\frac{3}{2}}$ T behaviour. In the paper by A. Billoire, G. Lazarides and Q. Shafi, they employed Monte Carlo methods to probe the infrared structure of SU(2) Yang-Mills theory at high temperatures. In particular, they studied the free energy of the gauge invariant non-Abelian magnetic flux. These Monte Carlo data support the conjecture that the magnetic flux is screened. This inverse magnetic screening length provides a non-perturbative infrared cut-off for the quantum statistics of the SU(2) gauge theory and is estimated to be about o.24 g^2T.

5) We note that D'Hoker has been studying the infrared limit in the pure gauge theory in two space and one time dimension [4.8].

Finally, we come to the conclusion that the overall picture of the high-temperature phase of QCD is that of an electrically screening phase with a correlation length $(m_{el})^{-1} \sim \frac{1}{gT}$ and a magnetic mass gap $m_{mag} \sim g^2 T$. Owing to Debye screening, heavy quarks are not confined. Although the electric mass may be reliably calculated in lowest-order perturbation theory, a genuine calculation of the magnetic mass appears to require the complete solution of the three-dimensional pure gauge theory.

Conclusions and Prospects

We have discussed in this thesis many aspects of QCD at finite temperature. The main conclusions following from our study are:

1) There are two phases in the strongly interacting matters: the hadronic matter at low temperature (or density) and the quark-gluon plasma at high temperature (or density). The calculations of interquark potential by means of perturbative expansions in the coupling g up to g^6 order lead to the conclusion that the quarks are deconfinement in high temperature region $T \gtrsim 200$ MeV. The Monte Carlo simulation study of lattice approximation theory at high temperature gave also the same result. The quark-confinement at low temperature has been well shown by lattice gauge theory.

2) There are two possible phase transitions characterized by two critical temperatures, T_c (from confinement of quarks to deconfinement) and T_{ch} (the chiral symmetry restoration). In the lattice evaluation of QCD thermodynamics,

$$T_c = \begin{cases} [(170 - 210) \pm 30] \text{ Mev} & SU(2) \\ [(150 - 170) \pm 50] \text{ Mev} & SU(3) \end{cases},$$

thus this means little or no dependence of T_c on the color group. However, the finite temperature deconfinement transition should be of second order for SU(2), but of first order for SU(3) gauge theory. The question of whether or not $T_c = T_{ch}$ appears to remain open.

3) We ourselves have researched on the problem of the high-temperature interquark potential and its explicit analytical expressions up to order g^6 have been obtained using perturbation theory. Its ultraviolet divergents to two-loop level are the same as those at zero temperature. This confirms that all ultraviolet divergences at finite temperature gauge field theory are independent of the temperature, no new temperature dependent ultraviolet infinities appear,

and zero-temperature renormalization prescriptions suffice to eliminate all ultraviolet divergences at finite temperature.

The explicit analytical expressions of interquark potential indicates that it falls off as an exponential and a power when the distance between a quark and an anti-quark increases, in the high-temperature limit and up to $O(g^6)$.

We have roughly estimated the critical temperature for a phase transition from confinement to unconfinement through the expressions of perturbative expansions of interquark potential.

4) The overall picture of the high-temperature phase of QCD is that of an electrically screening phase with a correlation length $(m_{el})^{-1} \sim (gT)^{-1}$ and a mass gap (i.e. magnetic mass) $m_{mag} \sim g^2 T$. Owing to Debye screening, heavy quarks are not confined. The electric mass may be reliably calculated in lowest-order perturbation theory, but the magnetic mass is uncomputable by perturbation theory and the evaluation of the magnetic mass appears to require the complete solution of the three-dimensional pure gauge theory, and this means that some non-perturbative methods may be necessary. The lattice approximate calculations gave the result of $m_{mag} \cong 0.24\, g^2 T$.

5) Owing to the smaller $m_{mag} \sim g^2 T$ of the gluon the perturbative calculations for interquark potential are reliable only up to the order g^{10} and being the order g^6 for thermodynamics potential.

6) All the above discussions hold for quarkless QCD. The inclusion of quarks makes these calculations much more complicated. Although one thinks that the main results would not change or would have little changes by including quark, the truth is that genuine calculational methods for quarks are totally lacking today.

In view of the above statement, we think that the following problems should be studied further.

1) Calculation of interquark potential at high temperature through three loops and possibly, higher orders.

2) Calculation of magnetic mass of gluon is pressingly to be solved. Its leading term is whether $O(g^2T)$ or $O(g^{\frac{3}{2}}T)$.

3) It should be cleared whether $T_c = T_{ch}$ or $T_c > T_{ch}$ or $T_c < T_{ch}$.

4) Some calculational methods for quark-containing QCD must be developed.

5) The infrared behaviour of QCD in four-dimension (QCD_4) is reduced to the study for three-dimensional QCD (QCD_3). The solution of the problem for QCD_3 is not only helpful for the determination of the magnetic mass of gluon, but is also associated with the confinement of quarks. One knows QCD_2 gives confinement; unlike QCD_2 , the problem of confinement for QCD_3 remains still difficult and unsolved.

6) The (color) magnetic mass of gluon for SU(2) has been computed by means of the lattice gauge theory, but for SU(3), not yet, and now we suggest to compute it.

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