ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

N-String Vertex in the Operatorial Formalism

Thesis Submitted for the Degree of
Doctor Philosophiae

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Academic Year 1987/88

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Acknowledgements

I would like to express my gratitude to Paolo Di Vecchia for his being constantly present during the different stages of my work.

I acknowledge Alberto Clarizia for his collaboration.

Lots of thanks go to M. Bonini, G. Cristofano, M. Frau, A. Lerda, R. Musto, F. Nicodemi, R. Pettorino, H. Sarmadi and S. Sciuto for helpful discussions; particular thanks go to R. Pettorino also for his encouragements during these years.

Furthermore I would also like to thank all friends and collegues of mine from SISSA with whom I have surely lived very important moments, both from a scientific and a human point of view.

Finally I am grateful to NORDITA (Copenhagen) for their very kind hospitality and to SISSA and Fondazione "Angelo Della Riccia" for financial support.

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Chapter 1

Introduction

String theories [1] are today seriously regarded as realistic candidates for a unified description of all the fundamental interactions.

Their structure is strongly based on the two-dimensional anomaly free conformal field theory [2] defined on the world-sheet of the string. It is well known that a one-dimensional string in its motion sweeps out a two-dimensional surface $x^\mu(\tau,\sigma)$ ($\mu = 1, ..., D$) imbedded in a D-dimensional *target space*. The whole information about dynamical properties of the string can be derived from the intrinsic geometry of the world-sheet: they are independent on the kind of parametrization chosen for $x^\mu(\tau,\sigma)$. In order to achieve a manifestly reparametrization invariant two-dimensional action, a metric $g_{\alpha\beta}(\tau,\sigma)$ can be defined on the surface; the parametrization invariance can be exploited to put this metric equal to the flat two-dimensional Minkowski one except for a local scale factor. This choice defines the *conformal gauge*: in this gauge the action of the theory still exhibits a residual invariance, that is the invariance under transformations of the *conformal group*. So the conformal group constitutes a gauge group in string theories.

It is worthwhile to observe that the space-time, in string theories, should have no fundamental place, but it should be a *prediction* of the theory. This fact gives the possibility of bypassing some problems connected, for example, with canonical quantization of gravity. When quantizing gravity, indeed, an operator valued field corresponds to the space-time metric which is classically regarded as a dynamical variable and that operator must have vanishing commutation relations with other observables. The operatorial character of the metric causes a lacking of a unique geometry and leads to an undeterminess for the sign of the distance between two points with a consequent ambiguity for the definitions of space- and time- like distances.

This is a good indication for claiming that string theories provide a consistent quantum mechanical framework in which gravity can be embodied in a very natural way; but there are other two observations on which this claim can rest. The first is that string theories reproduce, in a suitable limit, the Einstein equation of motion for the metric in the flat space-time; the second one concerns the problem of non-renormalizability of gravity. String theories indeed seem to be finite, order by order in perturbation theory, therefore there are no renormalizations to perform. So strings may remove the problem of non-renormalizability of gravity replacing the non-renormalizable interactions of gravity by interactions which reproduce gravity at "low" energies while avoiding unphysical infinities. Intuitively this is very probable: the unphysical infinities of quantum gravity arise in the short-distance (ultra-violet) regime, but this is precisely where the strings show their "stringy" character.

Hence a very relevant problem in string theories is to check their finitess, through the study of perturbative expansions.

String interactions result from non-trivial topology of the Riemann surface constituted by the world-sheet; this is also a very peculiar feature in string theories: for point-like particles there exists a separation between free theory and interacting theory and furthermore the nature of the interaction must be specified. On the contrary this separation is meaningless in string theories. In a Lorentz covariant formulation, the action of an interacting string coincides with the one of a free string: the interaction is simply indicated by the topology of the world-sheet swept out by the string. If external source terms are added to the free action of a string, these terms are forced by the intrinsic structure of the theory to describe just physical string states. External strings can be thought, from a geometrical point of view, as attached to the world sheet of the propagating string. So one can claim that *strings can interact consistently only with strings.* Hence if string theory is *a* fundamental theory of *anything* it must be *the* fundamental theory of *everything.*

Until now explicit expressions for scattering amplitudes at any order of perturbation theory have been calculated only for the bosonic string, but some partial results have already been obtained also for the fermionic string.

Many different approaches have been developed by now for computing scattering amplitudes in string theories. They include the functional integral technique both in the covariant [3] and light-cone gauge [4], the covariant old operator formalism where the orbital and ghost degrees of freedom circulate in the loops [5] [6] [7] [8], a more hybrid technique where both the operator formalism and the path integral are used [9], a group theoretical approach based on overlap equations [10] and a new operator formalism based on the construction of the so-called g-vacuum [11].

Among these, the covariant path-integral formalism is the most elegant and transparent from a geometrical point of view but at the moment it does not allow to go behind very first perturbative orders in getting explicit expressions for the scattering amplitudes. In this formalism the relativistic quantum theory of the string is obtained by a natural extension of the path-integral formulation of a point-like particle. Indeed the integral over the "trajectories" becomes now, according a Polyakov's proposal [12], an integral over surfaces, each characterized by its own metric $g_{\alpha\beta}$. Moreover the perturbative parameter is *identified* with the genus p, that is defined as the number of holes or handles of the surface. Therefore the perturbative series becomes a sum over all topologies of the world-sheet; for a closed string, the tree level corresponds to a sphere ($p = 0$), the one-loop level to a torus ($p = 1$), the two-loop level to a surface with two handles ($p = 2$) and so forth.

In this approach it is possible to write any string scattering amplitude as a sum over all the inequivalent topologies labelled by the number of handles g; the scattering amplitude involving N physical states is indeed given by:

$$A(1,2,...,N) = \sum_{g=0}^{\infty} \int \frac{\mathcal{D}g_{\alpha\beta}\mathcal{D}x^{\mu}}{\mathcal{N}} \prod_{i=1}^{N} d^2\xi_i \text{exp}\left\{-S\left(x, g^{\alpha\beta}\right) + (\text{Source Terms})\right\}$$

where $S\left(x, g^{\alpha\beta}\right)$ is the action of the bosonic string; $\xi \equiv (\xi_1 = \tau, \xi_2 = \sigma)$ denotes the coordinates on the Riemann surface on which the metric $g_{\alpha\beta}$ is defined, $\mathcal{N}$ is the volume of the gauge group of the theory, and the ξ_i's are the Koba-Nielsen coordinates of the punctures, i.e. the positions of the N external states on the world-sheet. The source terms depend on the external states and therefore on the coordinates ξ_i.

After having performed the functional integrations over $g_{\alpha\beta}$ and x^{μ}, that can be exactly computed, it still remains to evaluate an integral over a finite dimensional space spanned by $3g-3$ parameters (*moduli*), that describe inequivalent surfaces of genus g. The integrand is connected to the determinants that come just from having integrated over $g_{\alpha\beta}$ and x^{μ} respectively.

Hence in this approach in order to compute the multiloop amplitudes one must choose a parametrization of the moduli space [13] and compute the contribution of the measure, of the determinants and of the source terms as functions of the moduli.

For $g = 2$ [14][15] and $g = 3$ [14] explicit expressions of the integrand describing the amplitude have been obtained, by using the $\frac{1}{2}g(g+1)$ elements of the period matrix as the $3g-3$ moduli. For higher genus a rather explicit expression has been obtained only in the light cone approach of ref. [4] apart from a correction factor, that has a complicated form in terms of the first abelian integrals.

The old operatorial formalism[16], revised within the framework of the covariant approach with BRST invariance, has been instead revealed very helpful in order to analyse the various string interactions and loop corrections perturbatively.

The starting point for this purpose are the dual vertices: the three-reggeon vertex constructed by Sciuto [18] and made symmetric by Caneschi, Schwimmer and Veneziano [19] and the N-Reggeon vertex. In the dual resonance model these vertices were written in terms of oscillators: the conformal invariance underlying string theories allows to write them in terms of fields of the two-dimensional conformal theory defined on the world-sheet. So doing one can apply the methods of conformal field theory straightforwardly. Requiring the consistency with BRST invariance corresponding to the two-dimensional reparametrization invariance of the world-sheet, various proposals in this direction have been made for the three-string vertex and the N-string vertex, including the ghost contributions.

The aim of this thesis is just to illustrate one of the possible constructions of an N-string vertex in some cases and to show its main properties.

This is a very fundamental operator in string theories, defined as a bra vector in the direct product of the Fock spaces of N strings. When saturated by N arbitrary physical states, it is required to provide their corresponding tree amplitudes. Correlation functions of a general conformal field theory exhibit invariance under transformations of a subgroup of the conformal group, namely the *projective group*. In string theories this invariance is exhibited by amplitudes: it is connected to the freedom in choosing a suitable system of local coordinates for the point z_i in which the ith external string is attached to the world-sheet. Hence projective invariance is expected to play a fundamental rule in constructing an N-string vertex. This can be indeed expressed in terms of matrices belonging to unitary irreducible representations of the projective group [20] in the Fock spaces of the string states.

Besides providing the tree amplitudes, an N-string vertex can be also used in loop calculations [17]. An arbitrary multiloop amplitude can be in fact obtained simply sewing together two or more legs with the insertion of a propagator. In this sense it seems to have the same rule as Lagrangian has in an ordinary field theory.

In this thesis a definition of N-string vertex [21] is at first given in the case of the bosonic closed string and in the case of both open and closed interacting strings [22]. In other words an $(N+M)$-string vertex is constructed describing the interaction between N open and M closed strings. It has the important property of being projective and BRST invariant. In addition it reproduces the correct physical tree scattering amplitude [23] when saturated with physical states.

If $M = 0$ this vertex is identical to the N-Reggeon vertex of ref. [21]. When $N = 0$ it gives instead an M-closed string vertex, that has an $SL(2, C)$ projective invariance and it is invariant under the two independent BRST transformations of the closed string.

For $N = M = 1$, the transition between one open string state and one closed string state is obtained. It has a form that is identical to the expression constructed by Caneschi, Schwimmer and Veneziano for three open strings and generalized in ref. [24] to include the ghost degrees of freedom in order to achieve BRST invariance. However in the mixed case one set of oscillators describes the open string while the other two sets correspond to the two sectors of the closed string. In addition the zero modes of the two sectors must be identified.

This form of the open - closed string transition is very natural because a closed string can be viewed in this formalism as two open strings attached together at the two end points. For the same reason a vertex operator for the emission of a closed string state is given by the product of two vertex operators corresponding each to the emission of an open string state with half the momentum of the closed string.

Within the framework of Witten's string field theory, Shapiro and Thorn [25] have constructed a BRST invariant open - closed string transition. Since by construction the two vertices reproduce the correct open - closed string scattering amplitudes for on shell physical states, they must differ by a conformal transformation as it happens for the vertex for three open strings.

The case of the vertex for two open and one closed string is also considered and it is shown that it can be obtained by sewing together a 3-open string vertex with the open - closed string transition vertex after the insertion of a twisted propagator. This is a consequence of the general factorization property of the mixed vertex and shows that in general a closed string interacts with open strings through a direct closed - open string transition: in other words, it interacts with open strings after having become itself an open string.

In the operatorial formalism, *sewing* vertices through a twisted propagator simply means to consider a product of elements of a particular set of operators defined in the Fock space of oscillators, called *canonical forms*, to which both vertices and propagator can be shown to belong.

The definition of N-string vertex is then extended to the case of the Neveu-Schwarz string [26] In that case it was shown that a consistent N-string vertex should contain *orbital* oscillators and *spin* anti-commuting oscillators, according to a realization in the Fock space of two different unitary irreducible representations of the projective group. However no expression was given in terms of conformal fields showing explicitly the connection between the N-string vertex and the scattering amplitude for on-shell physical states. In addition the vertex was not written in a manifestly super-projective invariant form. An expression for the 3-Reggeon vertex was found [27], by integrating gauge identities that come out from demanding invariance under a transformation that belongs to the graded (super-) extension of the projective group; and also a BRST invariant generalization was formulated [28].

In this thesis two different constructions of the N-string vertex for the N.S. open string are given. The first one is only manifestly projective invariant while the second one is manifestly super-projective invariant. The details of the derivation lie on the structure of the unitary irreducible representations of the projective and super-projective group. These vertices are connected in a manifest way to the N-point amplitude in the super Koba-Nielsen form [29]. Examples of direct computation of the N-tachyons scattering-amplitude are given in both cases, as well as, for $N = 3$, generalizations of the Caneschi-Schwimmer-Veneziano vertex are easily obtained.

These applications show how the formalism of the N-string vertex not only reproduces the one that makes use of the vertex operators, but differently from the latter, it has the advantage that the external states may not be on the mass shell. It is this property that allows to construct vertices with N strings external to g loops by starting from an $N + 2g$-string vertex and sewing $2g$ legs together after the insertion of a twisted propagator [5].

This formalism seems to be sufficiently general to be presumably applied to any conformal field theory. In this case the role of the N-string vertex for string theories is played by an N-point vertex, that has the property of reproducing the N-point correlation functions involving the primary fields of the theory when it is saturated with the corresponding N highest weight states.

There are of course some differences between the case of a string theory and the one of an arbitrary conformal field theory. In the former our interest has been directed to amplitudes so that an integration over the Koba-Nielsen variables is needed; in the latter we are interested to correlation functions so such an integration is not needed. Furthermore the conformal group in an arbitrary conformal field theory is not a gauge group as in the case of string theories, hence no integration over the moduli of the Riemann surface must be performed.

These concepts are applied here explicitly to a fermionic free theory [30]; an N-point vertex for free fermions on an arbitrary Riemann surface is written: it turns out to be a function of the Szegö kernel written in terms of the Poincaré Θ series. In particular $V_{1;g}$ reproduces the g-vacuum discussed in [11].

The thesis is organized as follows.

In chapter 2 we review the basic features of the BRST invariant operatorial formalism, giving also the expressions of the scattering amplitudes in the case of bosonic open and closed strings in interaction.

In chapter 3 we discuss the properties of the N-string vertex, that is given both in the case of only closed strings and in the one of mixed strings. Projective and BRST invariance of these vertices are shown. Here it is also illustrated the technique of the *conformal cut-off*, very helpful in treating the zero modes in the orbital contribution to the N-string vertex. This technique consists in writing the position operator x^μ as a conformal field of weight $\frac{1}{2}\varepsilon$ and to perform all the calculations with ε which is supposed to be sent just at the end.

In chapter 4 it is discussed the generalization to the Neveu-Schwarz string, giving a projective and super-projective invariant expressions for the N-string vertex.

Chapter 5 is devoted to the possibility of generalizing the technique of the N-string vertex to an arbitrary conformal field theory, constructing an N-point g-loop vertex for the fermionic field which gives correlation functions on arbitrary Riemann surfaces. The sewing used for such a construction is illustrated in some detail. The same procedure is then considered for free bosons checking in this way bosonization of the free fermionic theory on an arbitrary Riemann surface.

In Appendix A we give definitions and properties of unitary irreducible representations of the projective group for a conformal field of arbitrary weight, which have a considerable role in the definition of an N-string vertex; in Appendix B we give some details about the Schottky description of the Riemann surface.

Throughout this work we use the space-time metric $\eta^{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$, a value $\alpha' = \frac{1}{2}$ for the Regge slope, and (super) Koba-Nielsen variables $Z \equiv (z, \phi)$, with $z = e^{i\tau}$, becoming real after a Wick rotation, and ϕ a Grassmann variable.

Chapter 2

BRST Operatorial Formalism

The aim of this chapter is to review the main features of the operatorial formalism with BRST invariance embodied.

The operatorial formalism was introduced in late 60's in the framework of dual models [31], when they were not interpreted in terms of string dynamics, as a way to exhibit explicitly the factorization property of the Veneziano amplitude. In fact a typical contribution to the Veneziano integrand has an expression of the form $\exp\{b_i b_j c\}$, where b_i, b_j are scalars and c is a constant; factorization of the amplitude means to split this expression into the product of two objects, one depending on b_i and another depending on b_j. This can be directly achieved by introducing creation and annihilation operators satisfying canonical commutation relations. Through factorization of tree amplitudes describing the interaction of an arbitrary number of external ground states it was possible to define spectrum, propagators and vertices of the theory.

This approach was very successful, but could not overcome a huge obstacle: the propagation of unphysical states in the loops.

Yang-Mills theories have provided the solution of this problem: it is enough to compensate the contribution of unphysical states with the one of Fadeev-Popov ghosts, by building a BRST invariant formalism, we are going now to illustrate.

2.1 Bosonic strings

A bosonic string is described by the following action:

$$S\left(x^{\mu},g^{\alpha\beta}\right)=-\frac{T}{2}\int d^{2}\xi\sqrt{g}g^{\alpha\beta}(\xi)\partial_{\alpha}x^{\mu}\partial_{\beta}x^{\nu}G_{\mu\nu}(x) \tag{2.1.1}$$

where $\xi \equiv (\xi_1 = \tau, \xi_2 = \sigma)$ defines a system of local coordinates on the two-dimensional world-sheet, on which the metric $g_{\alpha\beta}$ is defined; g is the absolute value of the determinant of $g_{\alpha\beta}$; $G_{\mu\nu}$ is the background space-time metric; the parameter T has dimensions of (lenght)$^{-2}$ or (mass)^2 and can be identified as the string tension: it is related to the universal Regge slope parameter by $T = (2\pi\alpha')^{-1}$.

The action (2.1.1) is invariant under the following local symmetry transformations:

i) reparametrizations of the world-sheet coordinates (world-sheet diffeomorphisms):

$$\begin{aligned} \delta x^{\mu}(\xi) &= \varepsilon^{\alpha}\delta_{\alpha}x^{\mu}(\xi) \\ \delta g_{\alpha\beta}(\xi) &= \varepsilon^{\gamma}\delta_{\gamma}g_{\alpha\beta}+\delta_{\alpha}\varepsilon^{\gamma}g_{\gamma\beta}+\delta_{\beta}\varepsilon^{\gamma}g_{\alpha\gamma}; \end{aligned} \tag{2.1.2}$$

ii) Weyl transformations:

$$\delta g_{\alpha\beta}=\Lambda(\xi)g_{\alpha\beta} \tag{2.1.3}$$

Furthermore (2.1.1) exhibits also global symmetries that reflect the symmetry of the background in which the string is propagating; for flat Minkowski space this is just Poincarè invariance.

Since there is no kinetic term for $g_{\alpha\beta}$ in (2.1.1), its classical equation of motion implies the vanishing of the energy-momentum tensor, which is just defined by the variational derivative of S with respect to the two-dimensional metric:

$$T_{\alpha\beta}=\partial_{\alpha}x\partial_{\beta}x-\frac{1}{2}g_{\alpha\beta}g^{\gamma\delta}\partial_{\gamma}x\,\partial_{\delta}x=0 \tag{2.1.4}$$

The local symmetries *i*) and *ii*) can be exploited to choose the three independent elements of $g_{\alpha\beta}$ so that

$$g_{\alpha\beta} = \rho(\xi)\eta_{\alpha\beta} \tag{2.1.5}$$

with $\eta_{\alpha\beta}$ ($\eta_{00} = -\eta_{11} = -1$) being the two-dimensional Minkowski metric. Eq. (2.1.5) defines the **conformal gauge**: this is the gauge where the world-sheet metric coincides with the two-dimensional flat metric except for a local scale factor.

Setting $g_{\alpha\beta}$ according this choice does not completely use up the gauge freedom. Indeed any combined reparametrization and Weyl scaling for which:

$$\delta^\alpha \xi^\beta + \delta^\beta \xi^\alpha = \Lambda \eta^{\alpha\beta} \tag{2.1.6}$$

preserves the gauge choice. These transformations which leave the conformal gauge invariant are defined as **conformal transformations**. They in turn define the *conformal group.*

The conformal gauge has the advantage that by it the action (2.1.1) simplifies to:

$$S = -\frac{1}{2\pi}\int d^2\sigma \eta^{\alpha\beta}\partial_\alpha x \partial_\beta x \tag{2.1.7}$$

where the flat metric has been assumed for the Minkowski space-time and $T = 1/\pi$.

The Eulero-Lagrange equation generated by (2.1.7) is nothing but the free two-dimensional wave equation:

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) x^\mu = 0. \tag{2.1.8}$$

In the case of open strings the stationarity of (2.1.7) is guaranteed by the vanishing of a *surface term*:

$$\int d\tau \left[\frac{\partial}{\partial\sigma} x^\mu \delta x^\mu \bigg|_{\sigma=\pi} - \frac{\partial}{\partial\sigma} x^\mu \delta x^\mu \bigg|_{\sigma=0} \right] = 0 \tag{2.1.9}$$

giving the open string boundary conditions. For closed strings the stationarity of (2.1.7) is ensured by the periodicity condition on x^μ:

$$x^\mu(\tau,0) = x^\mu(\tau,\pi). \tag{2.1.10}$$

The general solution in two dimensions of the massless wave equation (2.1.8) can be conveniently expressed in terms of complex coordinates introduced on the Euclidean world sheet:

$$z = \xi^1 + i\xi^2 \qquad \bar{z} = \xi_1 - i\xi_2. \tag{2.1.11}$$

It can be indeed written as a sum of two arbitrary functions:

$$x^\mu(z,\bar{z}) = x^\mu_L(z) + x^\mu_R(\bar{z}) \tag{2.1.12}$$

In the solution (2.1.12) two sectors appear: the sector x^μ_R describes *right-moving* modes of the string and x^μ_L describes *left-moving* modes.

For the closed string the boundary condition (2.1.10) determines the following general solution:

$$\begin{aligned} x^\mu_L(z) &= \frac{1}{2}(q^\mu - ip^\mu \log z) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n z^{-n} \\ x^\mu_R(\bar{z}) &= \frac{1}{2}(q^\mu - ip^\mu \log \bar{z}) + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}^\mu_n \bar{z}^{-n} \end{aligned} \tag{2.1.13}$$

In (2.1.13) q^μ and p^μ may be respectively considered as the center of mass position and momentum of the string.

For the open string the boundary conditions essentially identify the two sectors, in the sense that left- and right-moving components combine into standing waves and therefore the variable z can be taken real. One has the following solution:

$$x^\mu(z) = q^\mu - ip^\mu \log z + i\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n z^{-n} \tag{2.1.14}$$

Since the left and right sectors of string can be treated separately, in what follows we will often refer to one of them or, equivalently, to the open string.

With the choice (2.1.11) of the world-sheet coordinates, the conformal group can be thought as consisting of all reparametrizations:

$$z \longrightarrow \xi(z) \qquad \bar{z} \longrightarrow \bar{\xi}(\bar{z}) \tag{2.1.15}$$

where $\xi(z)$ and $\bar{\xi}(\bar{z})$ are arbitrary analytical functions. Thus one can think of the world-sheet as a complex manifold, constituting a **Riemann surface**.

$x^\mu(z)$ is meromorphic in z, except for the branch points of the logarithm in the zero mode part. From a point of view of conformal theories, it can be regarded as a primary conformal field of dimension 0, even if, as we will note in a little while, this is not a quite proper statement.

In general, a **primary conformal field** with dimension Δ is defined through its transformation law under a finite analytic reparametrization $z \longrightarrow \tilde{z}(z)$ [2] [33]:

$$\tilde{\phi}_\Delta(\tilde{z}) = \left(\frac{\partial z}{\partial \tilde{z}}\right)^\Delta \phi_\Delta(z). \tag{2.1.16}$$

A general conformal field of dimension Δ admits the following expansion at $z = 0$:

$$\phi_\Delta(z) = \sum_{n \in Z} \phi_n z^{-n-\Delta} \tag{2.1.17}$$

When ϕ_Δ is thought as a quantum field the expansion coefficients in (2.1.17) are identified with creation and annihilation operators. The distinction between creation and annihilation operators of course depends on the choice of vacuum. Annihilation [creation] operators are defined as annihilating the right [left] vacuum. The right vacuum is chosen as the state annihilated by all operators multiplying a singular z-dependence (at $z = 0$).

This implies that for $x^\mu(z)$, the annihilation operators are p^μ and α_n^μ with $n \geq 1$; so the following covariant commutation rules can be imposed:

$$[\alpha_m^\mu, \alpha_n^\nu] = m g^{\mu\nu} \delta_{m,-n}$$

$$[q^\mu, p^\nu] = i g^{\mu\nu}.$$

The oscillators α_n^μ of the field $x^\mu(z)$ are related to conventionally normalized harmonic oscillator operators by:

$$\begin{aligned} \alpha_m^\mu &= \sqrt{m} a_m^\mu, & m > 0 \\ \alpha_{-m}^\mu &= \sqrt{m} a_m^{\mu+}, & m > 0 \end{aligned}$$

According to the above general rule for the choice of vacuum, for $x^\mu(z)$ the vacuum $|p = 0, 0 >$ of the Fock space is annihilated by α_n with $n \geq 0$ ($\alpha_0^\mu \equiv p^\mu$) ; the left vacuum is defined as the state $< q = 0, 0|$ so that any operator in the expansion of $x^\mu(z)$ annihilates either the right or the left vacuum, but not both. Consequently the normal ordered product is defined as the one that has:

i) α_{-n}^μ to the left of α_n^μ for any $n \geq 1$;

ii) q^μ to the left of p^μ.

These rules give:

$$x^\mu(z) x^\nu(w) =: x^\mu(z) x^\nu(w) : -g^{\mu\nu} \ln(z - w), \qquad |z| > |w| \tag{2.1.18}$$

or, equivalently, the following contraction:

$$< x^\mu(z) x^\nu(w) >= -g^{\mu\nu} \ln(z - w) \tag{2.1.19}$$

The correlation function (2.1.19) contains implicitly a definition of time ordering on the world-sheet. Actually the variables (2.1.11) allow to consider the latter one as radial ordering: they send the cylinder (τ real, σ in the interval $[0, 2\pi]$) representing the world surface of the free closed string into the complex plane

(Riemann sphere), where the time evolution becomes a radial evolution from the origin to the infinity point [32]. For bosonic fields, like $x^\mu(z)$ and its derivatives, the radial ordered product $\mathcal{R}$ is defined as follows:

$$\begin{aligned} \mathcal{R}\left(\phi(z)\eta(w)\right) &= \phi(z)\eta(w) && \text{if } |z| > |w| \\ &= \psi(w)\eta(z) && \text{if } |z| < |w| \end{aligned} \tag{2.1.20}$$

Eq. (2.1.18) can be seen as a simple example of operator product expansion (OPE) that is in general obtained by normal ordering the time ordered product of operators in order to pick up the singularities of the latter one.

Eq. (2.1.19) demonstrates that the field x^μ does not have a well-defined conformal dimension, and so strictly speaking, it is not a proper quantum conformal field, but in general it is necessary to consider only operators involving ∂x and e^{ikx} which are genuine primary conformal fields of dimensions 1 and $p^2/2$ respectively.

The quantum generators of the conformal transformations defined on the world-sheet are the operators L_n satisfying the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{D}{12}n(n^2-1)\delta_{n,-m} \tag{2.1.21}$$

and have the following expression in terms of oscillators:

$$L_n = \frac{1}{2}\sum_{l \in Z} : \alpha_l^\mu \alpha_{n-l}^\nu : g_{\mu\nu} \tag{2.1.22}$$

For closed strings two sets of operators L_n and $\bar{L}_n$ must be introduced containing respectively α_n and $\bar{\alpha}_n$ oscillators.

L_n's are the Fourier components of the two-dimensional energy-momentum tensor.

Since the classical energy-momentum tensor is vanishing, also its Fourier components must be vanishing. In quantizing the theory, this condition on L_n becomes an additional condition which characterizes *physical states* in the Fock space generated by the orbital oscillators α_n^μ that is not positive definite because of the negative metric in the commutation relations of the time components: physical states, indeed, correspond to the subspace defined by the conditions [34]:

$$(L_n - \delta_{n,0})|\text{phys} >= 0, \quad n = 0, 1, 2..., \tag{2.1.23}$$

which is equivalent to claim that a physical state is a *highest weight vector* of the Virasoro algebra.

For a closed string the classical conditions:

$$L_n = \bar{L}_n = 0 \tag{2.1.24}$$

become in the quantized theory:

$$L_n|\text{phys} >= \bar{L}_n|\text{phys} >= 0 \qquad n > 0$$

$$(L_0 + \bar{L}_0 - 2)|\text{phys} >= (L_0 - \bar{L}_0)|\text{phys} >= 0.$$

In a general conformal field theory the energy-momentum tensor is traceless and symmetric and so it has only two independent components $T_{zz}(z)$ $[T(z)]$ and $T_{\bar{z}\bar{z}}$ $[\bar{T}(\bar{z})]$; $T(z)$, and analogously $\bar{T}(\bar{z})$, can be expanded as follows:

$$T(z) = \sum_{n \in Z} L_n z^{-n-2}. \tag{2.1.25}$$

This expansion also shows that it is a primary field with dimension $\Delta = 2$.

Eq. (2.1.25) implies that the energy-momentum tensor relative to our theory is given by:

$$T(z) = -\frac{1}{2} : \left(\frac{\partial x}{\partial z}\right)^2 (z) : . \tag{2.1.26}$$

From the contraction of the field $\partial_z x$, easily obtained from (2.1.19), one obtains from $T(z)$ the following OPE: [33]

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg. terms.} \tag{2.1.27}$$

with $|z| > |w|$.

The OPE (2.1.27) embodies the Virasoro algebra; indeed eq. (2.1.25) implies the following expression for L_n:

$$L_n = \frac{1}{2\pi i} \oint dz^{n+1} T(z) \tag{2.1.28}$$

where the contour integral encircles the origin. From this expression it is possible compute explicitly the commutator of L_n and L_m, obtaining:

$$\begin{aligned} [L_n, L_m] = {} & \frac{1}{(2\pi i)^2} \oint_{z=0} dz\, z^{n+1} \left\{ \oint_{|w|<|z|} dw\, w^{m+1}\, T(z)T(w) \right. \\ & \left. - \oint_{|w|>|z|} dw\, w^{m+1}\, T(w)T(z) \right\} \\ = {} & \frac{1}{(2\pi i)^2} \oint_{w=0} dw \oint_{z=w} dz\, z^{n+1}\, w^{m+1}\, T(z)T(w) \end{aligned} \tag{2.1.29}$$

By substituting the right-hand side of eq. (2.1.27) in (2.1.29) one gets the result of eq. (2.1.21).

Since the energy momentum tensor is the generating function of the L_n's, it provides an helpful tool to derive informations about the transformations properties of the primary conformal fields characterizing the theory. The variation of a general conformal field of dimension Δ is indeed given by:

$$[L_n, \phi_\Delta(z)] = z^n \left[(n+1)\Delta\phi_\Delta(z) + z\partial\phi_\Delta(z)\right]. \tag{2.1.30}$$

But eq. (2.1.28) implies the equivalence between this commutator and the following OPE:

$$T(z)\phi_\Delta(w) = \frac{\Delta\phi_\Delta(w)}{(z-w)^2} + \frac{\partial\phi_\Delta(z)}{z-w} + \text{terms regular as } z \to w \tag{2.1.31}$$

The anomaly of the Virasoro algebra (2.1.21) describes the lack of invariance of the quantum theory under deformations of the two dimensional metric. It vanishes for $n = 0, \pm 1$, so the generators L_{-1}, L_0 and L_1 form a subalgebra, that is the maximal closed subalgebra of the Virasoro algebra; this is the projective group $SL(2, C)$, i.e. the group of the linear fractional transformations:

$$z \rightarrow \gamma(z) = \frac{az + b}{cz + d} \tag{2.1.32}$$

with a, b, c and d complex coefficients and $ad - bc \neq 0$. In particular, the operator L_1 generates translations; L_0 generates infinitesimal dilations of the coordinate z and in the coordinate system σ, τ, it is a generator of "time" shifts, hence it plays the same role as the hamiltonian one.

The vacuum $|p = 0, 0 >$ above defined in the Fock space of the orbital oscillators is projective invariant:

$$L_n |p = 0, 0 >= 0 \qquad\qquad n = -1, 0, 1 \tag{2.1.33}$$

The same property is also exhibited by the vacuum $< p = 0, 0|$. Since the commutation relations of $SL(2, C)$ are unaffected by the anomaly of the Virasoro algebra, the correlation functions on the complex plane ($\sim$ amplitudes) will exhibit invariance under (2.1.32).

2.2 Ghosts

For $D = 26$ the anomaly of the algebra (2.1.21) is cancelled by the contribution of the Fadeev-Popov ghosts. In the covariant path-integral quantization approach these are introduced in (2.1.1) in order to deal with the gauge-fixing determinants; this is in fact represented as an integral over a conjugate pair of anticommuting *ghosts* $c, \bar{c}$ and *antighosts* $b, \bar{b}$, with different tensor structures, being c a contravariant vector field and b a covariant symmetric, traceless tensor.

Introducing ghosts in the theory amounts to add to the classical action of the string (2.1.7) a term, which in the conformal gauge reads:

$$S_{gh} = \frac{1}{\pi}\int d^2z \left[b\partial_{\bar{z}}c + \bar{b}\partial_z\bar{c}\right]. \tag{2.2.34}$$

This action implies the following equations of motion for b and c:

$$\partial_{\bar{z}}b = \partial_{\bar{z}}c = 0. \tag{2.2.35}$$

The solutions of eq. (2.2.35) are:

$$\begin{aligned} b(z) &= \sum_{n\in Z} b_n z^{-n-2} \\ c(z) &= \sum_{n\in Z} c_n z^{-n+1} \end{aligned} \tag{2.2.36}$$

with analogous expansions for $\bar{b}(\bar{z})$ and $\bar{c}(\bar{z})$.

The equations (2.2.36) show that $b(z)$ and $c(z)$ are conformal fields of weights $\Delta = 2, -1$ respectively.

The coefficients in (2.2.36) are identified, in a scheme of covariant quantization, with creation and annihilation operators. Since the annihilation operators are defined as the operators multiplying a singular z-dependence (at $z = 0$), they result to be b_n with $n > -2$ for $b(z)$ and c_n for $n > 1$ for $c(z)$. Hence the following canonical anticommutation relations must be imposed:

$$\{b_n, c_m\} = \delta_{n+m,0} \qquad \{c_n, c_m\} = \{b_n, b_m\} = 0 \tag{2.2.37}$$

with all other anticommutators vanishing and the normal ordering is defined by:

$$\begin{aligned} :c_n b_{-n}: \; &= c_n b_{-n} \qquad \text{if} \qquad n \leq 1 \\ & -b_{-n}c_n \qquad \text{if} \qquad n \gtreqless 2 \end{aligned} \tag{2.2.38}$$

with

$$b_n^+ = b_{-n} \qquad\qquad c_n^+ = c_{-n} \tag{2.2.39}$$

From the definition (2.2.38) of normal ordering it is immediately to derive, for instance, the following propagator on the plane for the b, c system:

$$< b(z)c(w) >= \frac{1}{z-w}. \tag{2.2.40}$$

The introduction of the ghosts fields in the action brings to a redefinition of the energy-momentum tensor; indeed for the b, c system it reads now:

$$T(z) = T_x(z) + T_{gh}(z) \tag{2.2.41}$$

where

$$T_{gh}(z) =: c\partial b(z) + 2(\partial c)b(z) : . \tag{2.2.42}$$

T_x is defined in (2.1.26).

It is to observe that, despite the asymmetrical tensor structures, c and b enter symmetrically in the action (2.2.35), which refers to a *flat* world-sheet, but this is not so on a curved world-sheet. Analogously they do not enter symmetrically in the energy-momentum tensor (2.2.42) which is obtained considering the variation with respect to the world-sheet metric.

The OPE $T_{gh}(z)T_{gh}(w)$ that can be derived by eqs. (2.2.40) and (2.2.42) determines the Virasoro algebra of the operators L_n^{gh}, Fourier components of T_{gh}:

$$T_{gh}(z)T_{gh}(w) = \frac{-13}{(z-w)^4} + \frac{2T_{gh}(z)}{(z-w)^2} + \frac{\partial_z T_{gh}(z)}{z-w} + \text{reg. terms} \tag{2.2.43}$$

Hence the OPE $T(z)T(w)$ relative to the energy-momentum tensor (2.2.41) of the complete theory is immediately obtained from (2.1.27) and (2.2.43):

$$T(z)T(w) = \frac{(D-26)/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg. terms} \tag{2.2.44}$$

Hence in $D = 26$ dimensions the total anomaly of the Virasoro algebra vanishes. This result allows to apply consistently the scheme of the BRST quantization in order to decouple unphysical states in the direct product of the Fock spaces of the orbital and ghost modes.

In general, for any physical system with symmetry operators G_i defining a closed Lie algebra G:

$$[G_i, G_j] = \sum_{i,j,k} f_{ijk} G_k \tag{2.2.45}$$

one defines the ghost number operator:

$$U = \sum_i c_i b_i \tag{2.2.46}$$

(where a normal ordering must be introduced in the case of infinite-dimensional Lie algebras in order to have a well-defined operator) and the BRST operator:

$$Q = \sum_i c^i G_i + \frac{1}{2} \sum_{i,j,k} f_{ijk} c_i c_j b_k \tag{2.2.47}$$

which can be shown to have the basic property to be nilpotent:

$$Q^2 = 0 \tag{2.2.48}$$

that derives by considering the commutation relations (2.2.45) and the Jacobi identity.

In the case of the string, the BRST operator associated with the residual gauge symmetry under analytic reparametrizations is defined by:

$$Q = \sum_n c_n \mathcal{L}_n + \frac{1}{2} \sum_{n,m,l} : c_n c_m b_l : (n - m) \delta_{n+m+l,0} \tag{2.2.49}$$

while the ghost number operator is:

$$j_0 = \sum_{n=-\infty}^{+\infty} : c_n b_{-n} : \tag{2.2.50}$$

Both the BRST operator and the ghost number operator can be obtained as integrals of current densities:

$$Q = \oint \frac{dz}{2\pi i} j_{BRST} = \oint \frac{dz}{2\pi i} : \left[c \left(T^{x} + \frac{1}{2} T^{gh} \right) \right] : \tag{2.2.51}$$

and

$$j_0 = \oint \frac{dz}{2\pi i} : cb : \tag{2.2.52}$$

Q acts on the fields as:

$$\begin{aligned} [Q, x(z)] &= c\partial x(z) \\ \{Q, c(z)\} &= c\partial c(z) \\ \{Q, b(z)\} &= T(z) \end{aligned} \tag{2.2.53}$$

For $D = 26$ the quantum BRST charge Q is nilpotent if the normal ordering is the one defined in (2.2.38). In order to define the vacuum state with respect to this ordering let us consider the properties of the states $|q = m>$, eigenstates of the *ghost number operator* relative to eigenvalue m.

The ghost state $|q = m>$ satisfies the following relations:

$$\begin{aligned} b_n|q = m >&= 0 \qquad\qquad n > -2 + m \\ c_n|q = m >&= 0 \qquad\qquad n > 1 - m \end{aligned} \tag{2.2.54}$$

Furthermore:

$$\begin{aligned} b_m|q = m > &= |q = m - 1 > \\ c_m|q = m - 1 > &= |q = m > . \end{aligned} \tag{2.2.55}$$

The right vacuum is then given by the state $|q = 0 >$.

The left vacuum is determined by the following considerations. The number ghost operator (2.2.50) is not antihermitian; indeed from the hermiticity properties (2.2.39) one gets:

$$j_0 + j_0^+ = 3 \tag{2.2.56}$$

The relation (2.2.56) has heavy consequences in the theory; first of all the scalar product:

$$< q = 0|q = 0 >= \frac{1}{3} < q = 0|(j_0 + j_0^+)|q = 0 > \tag{2.2.57}$$

vanishes; more generally (2.2.56) implies that the scalar product $< q = q'|q = q'' >$ between two states of definite ghost number is different from zero only if $q' + q'' = 3$; the non vanishing product is given by:

$$< q = 3|q = 0 >=< q = 0|c_{-1}c_0c_1|q = 0 >= 1.$$

Hence the (adjoint) left vacuum for ghosts is $< q = 3|$.

Let us see now how the definition of physical states can be implemented in the direct product of the Fock spaces of the orbital and ghost modes; in this space the right vacuum is given simply by:

$$|\text{vacuum} >= |p = 0; 0 > \otimes |q = 0 > . \tag{2.2.58}$$

In this case, physical states of the free string theory (or the asymptotic states of the complete theory) are defined by the condition [35]:

$$Q|phys >= 0 \tag{2.2.59}$$

From the nilpotency of Q it follows that the condition (2.2.59) is clearly satisfied by any state of the form:

$$|\text{phys} >= Q|\Lambda > . \tag{2.2.60}$$

So a physical state is not represented by a single vector but by a *cohomology class* of Q: two vectors satisfying (2.2.59) whose difference can be written in the form (2.2.60) are equivalent and represent the same physical state.

The states (2.2.60) are called *spurious states*. Since the BRST operator is self adjoint spurious states have zero norm:

$$< \Lambda|Q^\dagger Q|\Lambda >=< \Lambda|Q^2|\Lambda >= 0. \tag{2.2.61}$$

Furthermore $[Q, A]_\pm$ for any A commutes (or anticommutes) with Q and hence gives a physical zero norm state when applied to a physical state. In particular by setting $A = b_n$ one gets:

$$l_n = \{Q, b_n\} = L_n + \sum_m (n-m) : c_m b_{n-m} : \tag{2.2.62}$$

The l_n's are (in 26 dimensions) the quantum mechanical anomaly free generators of the conformal algebra.

The vacuum (2.2.58) is both BRST invariant and projective invariant:

$$Q|p_\mu = 0, 0; q = 0 >= l_{\pm 1,0}|p_\mu = 0, 0; q = 0 > \tag{2.2.63}$$

It is important for unitarity of the space-time S-matrix that spurious states decouple from physical amplitude, i.e. that physical amplitudes only depend on the BRST-cohomology classes of the external physical states.

Freeman and Olive [36] have shown that in each cohomology class it is possible to choose a representative of the following form:

$$|\text{phys} >= |\text{phys} >_x \otimes |q = 1 > \tag{2.2.64}$$

where $|q = 1 >= c_1|q = 0 >$ and $|\text{phys} >_x$ [37] satisfies (2.1.23).

The statement of the decoupling of spurious states is that any representative of the same BRST-cohomology class gives exactly the same physical amplitudes.

The presence of a nonvanishing background charge in (2.2.56) is a consequence of the Riemann-Roch theorem; the ghost number current satisfies the following OPE:

$$T_{gh}(z)j(w) = \frac{\partial j(w)}{z-w} + 2\frac{j(w)}{(z-w)^2} + \frac{Q}{(z-w)^4} \qquad (2.2.65)$$

The extra term, that makes $j(w)$ not quite a primary field, is a consequence of the anomaly appearing in the conservation equation for the ghost number current. Integrating the anomaly equation one gets the Riemann-Roch theorem (which can be proved by the "ordinary" Atiyah-Singer index theorem in two dimensions):

$$(\#\ \text{of } b\text{-zero modes}) - (\#\ \text{of } c\text{-zero modes}) = Q(g-1) \qquad (2.2.66)$$

where $b[c]$-zero modes are referred to the $b[c]$ solutions of the equation of motion (2.2.35); furthermore g is the number of the handles of the world sheet and $Q = 3$.

The anomaly in the ghost number current conservation can be put in analogy with the case of the axial anomaly in an instanton background where chiral charge is anomalously non-conserved. The ghosts are "chiral" because they are purely right moving (by the equations of motion).

c-zero modes correspond to globally defined conformal reparametrizations and are also called *conformal killing vectors*; b-zero modes are known as *moduli* and they correspond to deformations of the metric which cannot be brought about by infinitesimal reparametrizations. On the sphere, that is to say for $g = 0$, there are precisely three c-zero modes. Since the sphere has no handles one can conclude by the Riemann-Roch theorem that there are no b-zero modes on it. For $g = 1$ c has one complex zero mode and none for $g \geq 2$. Consequently the torus has two moduli while for Riemann surfaces with two or more handles there are $3(g-1)$ moduli.

2.3 Vertex operators

There exists a one-to-one correspondence between physical states and conformal fields, in the sense that it is always possible to find, given an on shell physical state

|phys > a conformal field called **vertex operator** of |phys >, which generates it at $z = 0$, i.e. at $t = -\infty$:

$$\lim_{z\to 0} V_\alpha(z)|p = 0, 0 >= |\alpha > . \qquad (2.3.67)$$

Conformal invariance dictates that vertex operators associated to physical states are conformal fields with dimensions Δ and $\bar{\Delta}$ respectively in z and $\bar{z}$ both equal to 1.

The vertex operator associated to the lowest state of the open string, the tachyonic state $|p = k, 0 >$ is:

$$V(z) =: \exp(ikx(z)) : \qquad (2.3.68)$$

In fact:

$$\lim_{z\to 0} : \exp(ikx(z)) : |p = 0, 0 >= |p = k, = 0 > \qquad (2.3.69)$$

The transformation properties under the conformal group of (2.3.68) can be deduced from the OPE with the energy-momentum tensor (2.1.26):

$$T_x(z) : \exp(ikx(\xi)) = \frac{\partial_\xi : \exp(ikx(\xi)) :}{z-\xi} + \frac{k^2/2 : \exp(ikx(\xi)) :}{(z-\xi)^2} + \text{reg. terms} \qquad (2.3.70)$$

which, compared to (2.1.31) suggests that $: \exp(ikx(\xi)) :$ has dimension $\Delta = 1$ as required by the conformal invariance only if $k^2 = 2$, that in fact corresponds to the mass shell condition for the tachyonic state.

Analogously the vertex operator associated to the massless photonic state of the string:

$$|\text{photon} >= -i\varepsilon \cdot a_1^+|p = 0, k >$$

is given by:

$$V_{ph} = \partial_z x^\mu \varepsilon_\mu \exp\left(ikx(z)\right). \tag{2.3.71}$$

So a physical state |phys > can be also interpreted as an asymptotic "in" state created by the conformal field $V_\alpha(z)$.

Furthermore the following properties hold:

$$\begin{aligned} \lim_{z\to\infty} < p = 0, 0|z^2 V_\alpha(z) &= < \text{phys}| \\ [L_n, V_\alpha(z)] &= \frac{d}{dz}\left[z^{n+1}V_\alpha(z)\right] \\ V_\alpha^+\left(\frac{1}{z}, -k\right) &= z^2(-1)^N V_\alpha(z,k) \end{aligned} \tag{2.3.72}$$

where N is the level of the state.

In $D = 26$ dimensions it is possible to write explicitly the vertex operator for an arbitrary physical state by using only the tachyon and the photon vertex operators [37]:

$$V_{N_i;\varepsilon_i}(z,\pi) = Z\prod_j \left[\oint_z dz_j x'^\mu(z_j)\varepsilon_\mu^{ij} e^{ik_j x(z_j)}\right] : e^{ipx(z)} : \tag{2.3.73}$$

The integral over the variable z_j is evaluated along a curve of the complex plane z_j containing the point z.

The singularity of the integrand function is a pole at $z_j = z$, that arises from bringing all creation operators to the left of the annihilation ones, provided that the following conditions are satisfied:

$$pk_i = -N_i \tag{2.3.74}$$

where N_i are integers. Furthermore the momentum π of the vertex is given by:

$$\pi = p - \sum_j N_j k. \tag{2.3.75}$$

The vertex (2.3.76), acting on the vacuum, reproduces the *transverse states* $|DDF>$:

$$\lim_{z\to 0} V_{N_j;\varepsilon_j}(z,\pi) = \prod_j A_{i_j;N_j}|0,p> \qquad (2.3.76)$$

where

$$A_{i,N} = \frac{1}{2\pi i}\oint dz x'_\mu(z)\varepsilon_i^\mu e^{-iNkx(z)}. \qquad (2.3.77)$$

The index i of these operators runs over $D-2$ transverse dimensions of space-time. They commute with the Virasoro operators.

The transverse states (2.3.76) form a complete and orthonormal basis in the subspace of the physical states if $D = 26$.

The vertex operator $V_{\alpha_i\beta_i}(z_i,\bar{z}_i,k_i)$ associated with the physical state $|\alpha_i,\beta_i>$ of a closed string can be written as a product of two vertices of open string, respectively associated to the left and right sector with the zero modes identified:

$$V_{\alpha_i\beta_i}(z_i,\bar{z}_i,k_i) =: V_{\alpha_i}\left(z_i,\frac{1}{2}k_i\right) V_{\beta_i}\left(\bar{z}_i,\frac{1}{2}k_i\right) : . \qquad (2.3.78)$$

The ordering explicitly indicated is relative to the zero modes – with the q operator on the left of the p operator. Eq. (2.3.78) appears to be very natural because a closed string can be viewed as two open strings attached together at the two end points. $V_{\alpha_i\beta_i}$ is a primary field with conformal weights $\Delta = \bar{\Delta} = 1$.

The physical state $|phys>$ in the Fock space of orbital and ghost oscillators can be obtained from the right vacuum (2.2.58):

by

$$|\psi> = \lim_{z\to 0} c(z)V_\alpha(z)|0> \qquad (2.3.79)$$

where $V_\alpha(z)$ is the vertex operator associated to the physical state $|\alpha>$.

2.4 Tree scattering amplitudes

A fundamental feature of string theory, which is new with respect to the ordinary field theories, is that if one adds to the free action a term describing the interaction between strings and an "external source", then this source is forced by the structure of the theory to be a string itself. So one can claim that strings can interact consistently only with strings.

Conformal invariance underlying string theories allows to visualize a typical Feynman diagram as the compactified world sheet of the propagating string with the incoming and outgoing external strings projected to points, at which the corresponding vertex operators are inserted. These diagrams can be classified by their topology, being the order of a diagram defined by its *genus*, that is to say the number of holes or handles of the surface. Therefore the perturbative series becomes a sum over all the topologies of the world sheet.

Let us here examine what is the form assumed by tree N-point amplitudes in the cases of open string states, of closed string states and of interacting open and closed strings. These amplitudes are written as integrals over a finite number of variables, z_i with $(i = 1, ..., N)$ at which the external strings are attached to the world sheet. From a geometrical point of view the variables z_i are interpreted as coordinates of *punctures*, corresponding to the insertion points of the vertex operators on the world-sheet, that so becomes a *punctured surface.* For this kind of surfaces, the location of the punctures contains geometrical properties, hence the invariance under reparametrizations is restricted to those transformations which leave the punctures fixed.

2.4.1 Open strings

For an open string the world sheet has boundaries, and emitted open string states are attached to these boundaries. This is just a consequence of the fact that two open strings interact attaching to each other at the end points, i.e. the points corresponding to $\sigma = 0, \pi$.

The world sheet of an open string can be conveniently mapped in a disk or the upper half plane with external strings appearing as punctures on the boundary. The order of the punctures is meaningful. In fact if we think to the disk with a certain chosen order of the punctures then it is easy to see that a conformal transformation can turn them only in a *cyclic* order.

Invariance under cyclic permutations is the translation in geometrical terms of the planar duality. This invariance suggests that the integral expressing the amplitude over the coordinates of the punctures is carried out only over values of them corresponding to a given cyclic order. So one can write the following scattering amplitude involving N on shell physical states of the open string $|\alpha_i>$, $(i = 1, 2, ..., N)$ is given by:

$$A(\alpha_1, ..., \alpha_N) = \int \frac{\prod_{i=1}^{N}[dz_i \theta(z_i - z_{i-1})]}{dV_{abc}} < p = 0; 0| \prod_{i=1}^{N} V_{\alpha_i}(z_i)|p = 0; 0 > \quad (2.4.80)$$

where V_{α_i} are the vertex operators associated with the physical states $|\alpha_i>$ and where the puncture are ordered along the real axis through the θ's functions.

At the end of the sect. 2.1 it has been said that in a general conformal theory the correlation functions on the complex plane are invariant under the transformations of the projective group, i.e. invariant under translations, dilations and special conformal transformations. In string theories this claim amounts to say that tree scattering amplitudes, which can be seen as correlation functions of the vertex operators, must be projective invariant. Indeed the amplitude (2.4.80) is that its integrand is invariant under the group $SL(2, R)$ of the projective transformations with real coefficients:

$$z_i \longrightarrow \tilde{z}_i = \frac{az_i + b}{cz_i + d} \quad (2.4.81)$$

with $ad - bc = 1$.

As a consequence of the projective invariance the locations of three of the punctures is made completely arbitrary: so three of the integration variables z_a,

z_b and z_c can be arbitrarily fixed and the volume element dV_{abc} must be inserted:

$$dV_{abc} = \frac{dz_a dz_b dz_c}{(z_a - z_b)(z_a - z_c)(z_b - z_c)} \tag{2.4.82}$$

The conventional choice is the one that fixes the three variables in such a way that $z_1 \to \infty$, $z_2 = 1$ and $z_N = 0$.

The projective invariance of the integrand of (2.4.80) is easily checked: dV_{abc} is left invariant under (2.4.81); the vertex operator V_{α_i} is a primary field with dimension $\Delta = 1$ and therefore its transformation is:

$$V_{\alpha_i}\left(\frac{az_i + b}{cz_i + d}; k_i\right) = (cz_i + d)^2 V_{\alpha_i}(z_i, k_i) \tag{2.4.83}$$

In addition one has:

$$d\tilde{z}_i = \frac{dz_i}{(cz_i + d)^2} \tag{2.4.84}$$

Hence the integrand in (2.4.80) is projective invariant.

Although the expression (2.4.80) is manifestly conformal invariance, it is however not manifestly BRST invariant and it contains explicitly the volume of the projective group.

A BRST invariant amplitude can be obtained replacing the vertex operators corresponding to the fixed variables with $c(z)V_\alpha(z)$ obtaining in this case a vertex operator with dimension $\Delta = 0$ and leaving unchanged the remaining $N - 3$ vertex operators.

With these modifications (2.4.80) becomes:

$$\begin{aligned} A(\alpha_1, ..., \alpha_N) &= \int \frac{\prod_{i=1}^N [dz_i \theta(z_i - z_{i-1})]}{dz_a dz_b dz_c} \\ &< p = 0, 0, q = 0| \prod_{i=1}^N \hat{V}_{\alpha_i}(z_i) |p = 0, 0, q = 0 > \end{aligned} \tag{2.4.85}$$

where $\hat{V}_{\alpha_i}(z_i) = V_{\alpha_i}(z_i)$ for $i \neq a, b, c$, $V_{\alpha_i}(z_i) = c(z_i)V_{\alpha_i}(z_i)$ for $i = a, b.c$ and $|q = 0, 0 >$ represents the BRST and projective invariant vacuum.

The expression (2.4.80) follows from (2.4.85) by means of the equation:

$$< q = 0|c(z_a)c(z_b)c(z_c)|q = 0 >= (z_a - z_b)(z_a - \ldots \tag{2.4.86}$$

that can be easily checked.

The amplitude (2.4.85) is now BRST invariant since:

$$\{Q, c(z)V_\alpha(z)\} = 0 \quad \{Q, V_\alpha(z)\} = \frac{d}{dz}\left[c(z)V_\alpha(z)\right]. \tag{2.4.87}$$

Using the relation

$$V_\alpha(z) = z^{L_0 - \Delta} V_\alpha(1) z^{-L_0} \tag{2.4.88}$$

with $\Delta = 0$ for $i = a, b, c$ and $\Delta = 1$ for the others and introducing the new variables $x_i = z_i/(z_i - 1)$ for $i = 3, ..., N-1$, it is easy to rewrite (2.4.85) in the fully factorized form:

$$A(\alpha_1, ...\alpha_N) = < q = 1; \alpha_1|c(1)V_{\alpha_2}(1)\, D \, \ldots \, V_{\alpha_{N-2}}(1)\, D\, V_{\alpha_{N-1}}|q = 1; \alpha_N > \tag{2.4.89}$$

where the choice $a = 1$, $b = 2$, $c = N$ and $V_i \equiv V_{\alpha_i}$ has been performed. In obtaining (2.4.89) the usual choice z_a, z_b, z_c must be performed and the following relations must be used:

$$\begin{aligned} \lim_{z \to 0} c(z)|q = 0 >= |q = 1 > \\ \lim_{z \to \infty} < q = 0|c(z)/z^2 = < q = 1| \end{aligned} \tag{2.4.90}$$

The propagator D is given by:

$$D = \frac{b_0}{L_0} = b_0 \int_0^1 \, dx x^{L_0 - 1}. \tag{2.4.91}$$

(2.4.89) contains vertices and propagators which are locally BRST invariant. In fact

$$\{Q, c(1)V(1)\} = 0 \quad \left\{Q, b_0 x^{L_0 - 1}\right\} = \frac{d}{dx} x^{l_0} \tag{2.4.92}$$

The extra terms appearing in the right hand side of (2.4.92) do not give any contribution and therefore the amplitude (2.4.85) is BRST invariant.

2.4.2 Closed strings

The world sheet of a closed string is topologically equivalent to a sphere, which can be mapped by stereographic projection onto the complex plane, on which the external strings are attached at specific points z_i. Differently from the open string in this case there is no natural ordering of the external strings: this means that, unlike the open string case where the emission must occur from a boundary of the world sheet, closed string emissions occur from the interior of the world sheet.

The scattering amplitude involving M physical states of the bosonic closed string can be written as follows [38]:

$$A(\alpha_1, \beta_1, k_1; ...; \alpha_M, \beta_M, k_M) = \int \frac{\prod_{i=1}^{M} d^2 z_i}{dV_{abc}} < 0|R\left(\prod_{i=1}^{M} V_{\alpha_i\beta_i}(z_i, \bar{z}_i, k_i)\right)|0 > \quad (2.4.93)$$

where $(z_i, \bar{z}_i)$ are the Koba-Nielsen variables associated with the ith external state of momentum k_i:

$$z_i = e^{2i(\tau_i+\sigma_i)}, \quad \bar{z}_i = e^{2i(\tau_i-\sigma_i)} \quad (2.4.94)$$

After a Wick rotation z_i and $\bar{z}_i$ become the complex conjugates of each other. R refers to the ordering prescription with respect to the moduli of the variables z_i, that are integrated over the whole complex plane.

The integrand in (2.4.93) is invariant under the group $SL(2, C)$ of the projective transformations with complex coefficients. As a consequence of the projective invariance, three of the integration variables z_a, z_b and z_c can be arbitrarily fixed and the volume element dV_{abc} must be inserted:

$$dV_{abc} = \frac{d^2 z_a d^2 z_b d^2 z_c}{|z_a - z_b|^2 |z_b - z_c|^2 |z_a - z_c|^2}. \quad (2.4.95)$$

This amplitude can be made BRST invariant through a procedure similar to the one considered for the open string case.

2.4.3 Mixed strings

More generally, one can calculate a scattering amplitude involving both open and closed strings [23]. For this aim one must first introduce the vertex operators corresponding to the emission of a closed string from an open one.

The form of this vertex can be guessed by the following considerations. The emission of a closed string from an open one can be considered by adding to the free action of the open string an interaction term describing the interaction of the end points of the open string with the closed string: this term is represented by an integral over the D-dimensional target space of the product between the closed string (2.3.78) vertex and a "current" generated by the open string, whose tensorial character will depend on the one of the closed string vertex. Now, the most general vertex of a closed string is written in terms of the tachyonic vertex $\exp(ikx)$ where $x(z, \bar{z})$ is the free closed string position field (2.1.13); so the coupling with the open string current makes to substitute in this vertex the free open string position field (2.1.14). This implies that the left and right movers of the vertex of the closed string must be identified and the interaction term factorizes into a product of two vertices of open strings, one depending on the variable z and the other on $\bar{z}$. Hence the following vertex for the emission of a closed string from an open string can be considered having the following form.

$$V_{\alpha\beta}(z, \bar{z}, k) = V_{\alpha}(z, \frac{1}{2}k)V_{\beta}(\bar{z}, \frac{1}{2}). \tag{2.4.96}$$

It is very important to stress that, differently from the closed string vertices, (2.4.96) contains only a normal ordering for each constituent vertices and not an overall ordering; actually, this prescription guarantees the right conformal behaviour of the vertex (2.4.96). This is, in fact, given by the following OPE with the energy-momentum tensor (2.1.26):

$$T(z)V_\alpha(w,\frac{1}{2}k)V_\beta(\bar{w},\frac{1}{2}k) = \frac{\partial_w V_\alpha(w,\frac{1}{2}k)V_\beta(\bar{w},\frac{1}{2}k)}{z-w} + \frac{\partial_{\bar{w}} V_\alpha(w,\frac{1}{2}k)V_\beta(\bar{w},\frac{1}{2}k)}{z-\bar{w}}$$
$$+ \frac{V_\alpha(w,\frac{k}{2}k)^2}{(z-w)} + \frac{V_\beta(\bar{w},\frac{1}{2}k)^2}{(z-\bar{w})} + \text{reg.terms}$$

An overall ordering prescription of (2.4.96) would introduce in (2.4.3) a term with a singularity $(z-w)(z-\bar{w})$, breaking the conformal invariance.

The vertex operators (2.4.96) allows to write the mixed scattering amplitude.

The scattering amplitude for N open and M closed physical states interacting, here denoted with $|\alpha_i, p_i>$ and $|\alpha_j, \beta_j, k_j>$ respectively, is given by:

$$A(N,M) = \int \frac{\prod_{i=1}^N [dx_i\theta(x_i - x_{i+1})] \prod_{j=1}^M d^2 z_j}{dV_{abc}}$$
$$\times <0|R^* \left(\prod_{i=1}^N \prod_{j=1}^M V_{\alpha_i}(x_i;p_i) V_{\alpha_j\beta_j}(z_j,\bar{z}_j;k_j \right) |0> \qquad (2.4.97)$$

where $V_{\alpha_i}(x_i;p_i)[V_{\alpha_j\beta_j}(z_j,\bar{z}_j;k_j)]$ is the vertex operator associated with the state $|\alpha_i;p_i>$ $[|\alpha_j,\beta_j;k_j>]$. The Koba-Nielsen variables for the closed strings are integrated over the whole complex plane, while those for the open strings are integrated along the real axis with the ordering given by the θ functions. The R^* prescription refers to the ordering of the closed string states among themselves and with respect to the open string states according to the moduli of their variables z and x.

(2.4.97) can be diagrammatically represented, for example, by the upper half of the complex plane with the punctures relative to the open strings inserted on the boundary of the world-sheet, i.e. on the real axis and with the punctures relative to the closed strings inserted in the interior of it.

Although (2.4.97) also contains closed strings its integrand is only invariant under real projective transformations as in the case of open strings. Consequently, three real variables can be fixed arbitrarily and the corresponding volume dV_{abc} must be introduced.

Chapter 3

BRST Invariant Bosonic N-String Vertex

A definition of BRST N-string vertex is given in this chapter both for bosonic open and closed strings. Starting from these vertices, a mixed $(N + M)$-string vertex is then constructed: when it is saturated with N open string and M closed string physical states it reproduces their corresponding scattering amplitudes. As a particular case we obtain a BRST invariant vertex for the open - closed string transition.

Furthermore the *conformal cut* technique is discussed: it reveals very helpful in treating orbital zero modes.

3.1 Bosonic N-string vertex

In sect. 2.4.2 scattering amplitudes involving N physical bosonic strings, both open and closed, were given in terms of vertex operators, stressing their invariance under the group of projective transformations. By exploiting this invariance, it is possible to define a bra vector in the Fock space of the string, that we will define N- **string vertex**, which is required to satisfy the following properties:

i) when applied to N physical string states, it must give the correct tree amplitude;

ii) it must be invariant under transformations of the projective group;

iii) it must be BRST invariant.

The underlying projective invariance of the tree amplitudes is extremely relevant in the construction of such a vertex. When written explicitly in terms of oscillators in its expression appear infinite matrices belonging to a suitable unitary irreducible representation of the projective group, the space basis being the Fock space generated by the orbital and ghost modes.

So we want to start by giving some properties of these representations: more details about this subject can be found in App. A.

3.1.1 Unitary irreducible representations of the projective group

To an arbitrary projective transformation of the form

$$z' = \lambda z = \frac{az+b}{cz+d} \tag{3.1.1}$$

with a, b, c, d constants and $ad - bc = G \neq 0$, one can associate the 2×2 matrix:

$$\Lambda \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

It can be very often convenient to normalize Λ such that $G = 1$.

Given an arbitrary quantum conformal field of weight Δ, a projective transformation induces in the Fock space of the relative oscillators an infinite matrix $D_{mn}(\Lambda)$, defined by the following equation:

$$\sum_{n=\Delta}^{\infty} D_{mn}^{(J)}(\Lambda) N(J,n) z^{n-\Delta} = N(J,m) \left(\frac{cz+d}{\sqrt{|G|}} \right)^{2J} \left(\frac{az+b}{cz+d} \right)^{m} \tag{3.1.2}$$

where

$$N(J,n) = \sqrt{\frac{\Gamma(n-2J)}{\Gamma(n+1)}} \tag{3.1.3}$$

with $J \neq 0 = -\Delta$.

(3.1.2) provides straightforwardly, for example, the infinite matrix $M_{nm}(V)$ for orbital oscillators, considering $J \to 0$:

$$
\begin{aligned}
n, m \geq 1 \qquad & M^{\varepsilon}_{nm}(V) = \tfrac{1}{m!}\sqrt{\tfrac{m}{n}}\,\partial_z^m \left[\left(\frac{az+b}{cz+d}\right)^n\right]_{z=0} \\
& M_{n0}(V) = \sqrt{\tfrac{1}{n}}\left(\frac{b}{d}\right)^n \\
n \geq 1 \qquad & \\
& M_{0n}(V) = \sqrt{\tfrac{1}{n}}\left(-\frac{c}{d}\right)^n \\
& M_{00}(V) = -\log\left(\frac{d}{\sqrt{ad-bc}}\right)
\end{aligned}
\qquad (3.1.4)
$$

Analogously for $J = -1$ one obtains an infinite-dimensional representation of the projective group $E_{nm}(V)$ in the Fock space defined by the oscillators of $c(z)$; it is given by:

$$
n, m \geq -1 \qquad E_{nm}(V) = \tfrac{1}{(m+1)!}\,\partial^{m+1}\left[\left(\frac{az+b}{cz+d}\right)^{m+1} \cdot \frac{1}{V'(z)}\right]_{z=0} . \qquad (3.1.5)
$$

3.1.2 Sciuto-Della Selva-Saito vertex

The basic ingredient for constructing an N-string vertex is the Sciuto-Della Selva-Saito (SDS) vertex [18].

Let us first examine the definition of SDS vertex for the open string, limitatly to the orbital part.

The scattering amplitude (2.4.80) involving three open strings can be written, according (2.4.89) as:

$$
A(\alpha_1, k_1; \alpha_2, k_2; \alpha_3, k_3) = < \alpha_1, k_1 | V_{\alpha_2}(1, k_2) | \alpha_3, k_3 > . \qquad (3.1.6)
$$

The aim is the one of defining an operator that, taken among the states $|\alpha_j, k_j>$ with $j = 1, 2, 3$, reproduces (3.1.6); one possibility is to consider an operator W_i, depending on two sets of oscillators acting on two different Fock spaces: the one of the propagating string that constitutes the set of *auxiliary oscillators* and the other one relative to the emitted string which will be denoted by an index i. The dependence on these sets of oscillators must be such that the following condition is satisfied:

$$ {}_i < x = 0, 0|W_i|\alpha_2, k_2 >_i = V_{\alpha_2}(1; k_2), \tag{3.1.7} $$

i.e. the operator W_i we are looking for, closed between the state $|\alpha_2, k_2 >_i$ defined in the Fock space of the emitted string and the left vacuum of this one, must become depending on only the set of auxiliary oscillators with such a dependence that it can reproduce exactly the vertex operator relative to the state $|\alpha_2, k_2 >$.

An operator that satisfies (3.1.7) is given by:

$$ W_i =: \exp\left\{\oint dz\left[-x(1-z)x_i'(z)\right]\right\}: \tag{3.1.8} $$

originally constructed by Sciuto and written in this form by Della Selva and Saito [40]. The contour of integration encircles the point $z = 0$ and a factor $1/2\pi i$ in front of the integral is understood to be. That this vertex reproduces (3.1.7) can be seen considering the expansion of the field $x^\mu(z)$ around $z = 0$:

$$ x^\mu(z) = q^\mu - ip^\mu \log z + i\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(a_n^\mu z^{-n} - a_n^{\mu+} z^n\right), $$

so obtaining:

$$ {}_i < x = 0, 0|W_i|\alpha, k >_i =: exp\left\{ikx(1)\right\}\exp\left\{i\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n!}a_n^{(i)}\partial^n x(1)\right\}|\alpha >_i : \tag{3.1.9} $$

where the quantity at the right hand coincides with the expression of the vertex operator, in $z = 1$, associated to the state $|\alpha >_i$ with momentum k. Hence (3.1.9) shows that, given an arbitrary state $|\alpha >$, W_i allows to compute the corresponding vertex operators V_α in the point $z = 1$ of the world-sheet. If the state $|\alpha >$ is physical, i.e. a highest weight state of the Virasoro algebra, then the vertex will be a primary field with weight $\Delta = 1$.

The SDS vertex defined in (3.1.8) does not treat the three external legs in a cyclic symmetric way, since it acts on one external state as bra and on the other two states as kets; it is clear that, in order to make it cyclic symmetric, all of them must be on the same side: for instance, the bra must become a ket like the other two states, or, in other words one has to twist the leg relative to the bra. This was performed by Caneschi, Schwimmer and Veneziano [19], who multiplied the vertex (3.1.8) with the following *twisting operator*:

$$\Omega_x^+ = (-1)^{L_0^x - \frac{1}{2}p^2} e^{L_1^x} \tag{3.1.10}$$

and introducing a third set of oscillators; in this way one obtains a cyclic symmetric vertex written as:

$$V_x = \int d^D(x)_1 < x, 0|_2 < x, 0|_3 < x, 0|$$

$$\exp\left\{-\sum_{r=1}^{3}\left[\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}p^{(r)}a_n^{(r+1)} + \sum_{n,m=1}^{\infty}(-1)^m\sqrt{\frac{m}{n}}\begin{pmatrix} n \\ m \end{pmatrix}a_m^{(r)}a_n^{(r+1)}\right]\right\}. \tag{3.1.11}$$

The SDS vertex can be made BRST invariant by adding the contribution of the ghost and antighost fields ([24]); (3.1.8) so becomes:

$${}_i < x = 0, 0, q = 3|W_i \tag{3.1.12}$$

with

$$W_i =: \exp\left\{\oint dz\left[-x(1-z)x_i'(z) - c(1-z)b_i(z) + b(1-z)c_i(z)\right]\right\} : . \tag{3.1.13}$$

This form of the SDS vertex with ghosts is guessed by observing that the integrand appearing in (3.1.8) is dimensionless because $x(z)$ has dimension 0, while the dimension of dz and $x_i'(z)$ exactly cancel. Using the same rule in the case of the ghosts it seems natural to have in the integrand a product of the ghost times the antighost coordinates.

The property of BRST invariance of (3.1.13) is guaranteed by the holding of the following equation [24]:

$${}_i < x = 0, 0, q = 3|\left[Q_i + Q, W_i\right] \tag{3.1.14}$$

where Q_i and Q are the BRST charges corresponding to the fields $x_i(z)$ and $x(z)$ in terms of which the SDS vertex is defined; they have the following expression:

$$Q = \oint dz : \left\{ -\frac{1}{2} c_i(z) \left[x'(z)\right]^2 + b(z)c(z)c'(z) \right\} : . \tag{3.1.15}$$

Let us now consider the extension of the definition of SDS vertex to the closed string.

Since the vertex operator for the emission of a closed string state is the product of the vertex operators corresponding to two open string states, also the SDS vertex for a closed string will be the product of two vertices for the open string. They will depend on the variables z and $\bar{z}$ respectively and contain the same oscillators for the zero modes. So this suggests that the SDS vertex for a closed string is given by:

$$_{i\bar{i}} < x = 0, 0| \; W_{i\bar{i}} \tag{3.1.16}$$

where

$$W_{i\bar{i}} \; = : W_i \cdot W_{\bar{i}} \; : \tag{3.1.17}$$

with

$$\begin{aligned} W_i &= \; : \exp\left\{ \oint dz[-x(1-z)x_i'(z)] \right\} : \\ W_{\bar{i}} &= \; : \exp\left\{ \oint d\bar{z}[-\bar{x}(1-\bar{z})\bar{x}_{\bar{i}}'(\bar{z})] \right\} : \end{aligned}$$

The contours of integration encircle the points $z = 0$ and $\bar{z} = 0$ respectively and also here a factor $1/2\pi i$ in front of the integrals is understood. The operator (3.1.16) depends on four sets of oscillators : two sets are labelled with the indices i and $\bar{i}$, relative to the emitted string while the others correspond to two independent sets (except for the zero modes) of *auxiliary* oscillators and are relative to the propagating string.

The state $_{i\bar{i}}< x = 0, 0|$ is the vacuum of the Fock space of the emitted string with vanishing eigenvalue of the center of mass variable $q^i \equiv q^{\bar{i}}$. It is easy to convince oneself that the vertex (3.1.16), acting on any state, will reproduce the corresponding vertex operator. In particular if the state is physical then the vertex operator will be a conformal field with weights $\Delta = \bar{\Delta} = 1$. In fact considering the following expansions for $x^\mu(z)$ and $\bar{x}^\mu(\bar{z})$:

$$
\begin{aligned}
x^\mu(z) &= \frac{1}{2}\left[q^\mu - i\frac{p^\mu}{2}\log z + i\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(a_n^\mu z^{-n} - a_n^{\mu\dagger} z^n)\right] \\
\bar{x}^\mu(\bar{z}) &= \frac{1}{2}\left[q^\mu - i\frac{p^\mu}{2}\log \bar{z} + i\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(\bar{a}_n^\mu \bar{z}^{-n} - \bar{a}_n^{\mu\dagger} \bar{z}^n)\right]
\end{aligned}
$$

one gets :

$$
\begin{aligned}
{}_{i\bar{i}}< x = 0, 0|\, W_{i\bar{i}}|\alpha_i, \beta_i, k_i >_{i\bar{i}} &= \, : V_{\alpha_i}\left(1, \frac{k_i}{2}\right) \cdot V_{\beta_i}\left(1, \frac{k_i}{2}\right) :\, = \\
&= V_{\alpha_i\beta_i}(1, 1, k_i) \qquad (3.1.18)
\end{aligned}
$$

3.1.3 BRST invariant N string vertex

Eq. (3.1.7) shows that the SDS vertex, when applied to any highest weight state, provides the corresponding vertex operator, computed in $z = 1$. On the other hand eq. (2.4.80) indicates that the tree amplitude for physical states, that we want to reproduce, is expressed in terms of vertex operators computed in a general point z of the world-sheet. But the vertex operator associated to a highest weight state is a conformal field of dimension 1, so the transformation properties of this operator under projective transformations are well-known. Hence it is very straightforward to obtain from the SDS vertex the vertex operator associated to a highest weight state in any point z: it is enough to consider the transformed operator under the projective transformation that sends $1 \to z$. For a not physical state this is not true. Nevertheless we would like to transform a vertex operator associated to an *arbitrary* state from the point 1 to z, considering the transformation property of the SDS vertex W_i, that is not a primary field, under this projective transformation. This is so because we could then collect N SDS vertices in order to obtain the general coupling of N arbitrary states, without any restriction to the only physical ones.

A possible solution is the one of considering the explicit expression of the operators $\hat{\gamma}_i$'s which in the Fock space of the auxiliary oscillators realize the projective group and compute explicitly the transformed W_i' of W_i:

$$W_i' = \hat{\gamma}_i W_i \hat{\gamma}_i^{-1} \tag{3.1.19}$$

and, in order to have the N-string vertex, one will consider the product of N such objects.

First of all we must choose a suitable projective transformation $\gamma_i(z)$ which sends $1 \to z_i$ with $i = 1, ..., N$ for every puncture of the world-sheet; the three parameters which characterize a projective transformation can be fixed in such a way that ∞ is transformed in z_{i-1}, 1 in z_i and 0 in z_{i+1} or, using Lovelace notation: [41]:

$$\gamma_i \equiv \begin{bmatrix} \infty & 1 & 0 \\ z_{i-1} & z_i & z_{i+1} \end{bmatrix} \tag{3.1.20}$$

If we introduce the projective transformation:

$$V_i(z) \equiv \gamma_i(z) = \frac{a_i z + b_i}{c_i z + d_i} \tag{3.1.21}$$

the parameters are fixed by (3.1.20) to be the following:

$$a_i = z_{i-1}(z_i - z_{i+1}) \quad b_i = z_i(z_{i+1} - z_{i-1})$$
$$c_i = z_i - z_{i+1} \qquad d_i = z_{i+1} - z_{i-1} \tag{3.1.22}$$

The variables z_{i-1} and z_{i+1} which appear in the definition of the projective transformation $\gamma_i(z)$ are relative to an ordering of the variables z_i $(i = 1, ..., N)$ chosen in advance.

From a geometrical point of view performing such a projective transformation correspond to choose a suitable system of local coordinates on the Riemann surface; in particular the projective transformation $V_i^{-1}(z)$ corresponds to a choice of local coordinates vanishing at the Koba-Nielsen point z_i.

The operator $\hat{\gamma}_i$ which in the Fock space of the auxiliary oscillators performs the projective transformation $\gamma_i(z)$ can be expressed in terms of the generators L_{-1}, L_0 and L_1 of the projective group in the following way:

$$\hat{\gamma}_i = \exp\left\{-\frac{c_i}{a_i}L_1\right\}\left[\frac{a_i^2}{a_id_i - b_ic_i}\right]^{L_0}\exp\left\{-\frac{b_i+a_i}{a_i}L_1\right\} \tag{3.1.23}$$

which allows to compute $\hat{\gamma}_i W_i \hat{\gamma}_i^{-1}$. Indeed one has:

$$\hat{\gamma}_i W_i \hat{\gamma}_i^{-1} = \exp\left\{\frac{1}{2}p_i^2 \log\frac{a_id_i - b_ic_i}{d_i^2}\right\}\exp\left\{\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(-\frac{c_i}{d_i}\right)^n p_i a_n^{(i)}\right\}$$

$$\times\exp\left\{\sum_{n=2}^{\infty}c_n^{(i)}\left[-\frac{n(n+1)}{2}\left(-\frac{c_i}{d_i}\right)^{n-1}b_{-1}^{(i)} + (n^2-1)\left(-\frac{c_i}{d_i}\right)^n b_0^{(i)} - \frac{n(n-1)}{2}\left(-\frac{c_i}{d_i}\right)^{n+1}b_1^{(i)}\right]\right.$$

$$\times : \exp\left\{\oint dz\left[-x\left(V_i(z)\right)x_i'(z) + c\left(V_i(z)\right)b^{(i)}(z)\frac{1}{V_i'(z)} + b\left(V_i(z)\right)c^{(i)}(z)V'(z)^2\right]\right\} : \tag{3.1.24}$$

If the product of N vertices (3.1.24) is taken between the projective vacuum $|p=0,0,q= \ 0>$ both on left and right hand, one obtains the following:

$$< p=0,0,q=0|\prod_{i=1}^{N}\hat{\gamma}_i W_i \hat{\gamma}_i^{-1}|p=0,0,q=0> \ =$$

$$= \cdot\exp\left\{-\sum_{\substack{i,j=1\\ i\neq j}}^{N}\left[\frac{1}{2}\sum_{\substack{n=0\\ m=0}}^{\infty}a_n^{(i)}M_{nm}(\Gamma V_i^{-1}V_j)a_m^{(j)} + \sum_{\substack{n=2\\ m=-1}}^{\infty}c_n^{(i)}E_{nm}(\Gamma V_i^{-1}V_j)b_m^{(j)}\right]\right\}$$

$$\cdot\delta^{(D)}\left(\sum_{i=1}^{N}p^{(i)}\right)\prod_{n=-1}^{1}\left\{\sum_{i=1}^{N}\sum_{m=-1}^{1}E_{nm}\left(V_i\right)b_m^{(i)}\right\}, \tag{3.1.25}$$

Here $M_{nm}(V)$ and $E_{nm}(V)$ defined in (3.1.4) and (3.1.5) respectively. Furthermore the matrix Γ is given by:

$$\Gamma \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

corresponding to the inversion:

$$\Gamma(z) = \frac{1}{z}$$

The expectation value (3.1.25) establishes a connection between the product of transformed SDS vertices and the N-string vertex written in the old days of dual models by Lovelace [41]. Furthermore it easy to check that (3.1.25) is invariant under a projective transformation acting on the variables z_i:

$$z_i \rightarrow z_i' = \frac{\alpha z_i + \beta}{\gamma z_i + \delta}$$

or

$$\Delta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Under such a transformation, the matrix V_i is transformed in ΔV_i and therefore $V_i^{-1}V_j$ is transformed in $V_i^{-1}V_j$; this implies that the arguments of the matrices D_{nm} and E_{nm} are invariant under any projective transformation D. The same properties are also valid for the fermionic d-functions appearing in (3.1.25).

A BRST and projective invariant N-string vertex can be now simply written in terms of (3.1.25) in the following way:

$$V_N = \int \frac{\prod_{i=1}^{N} dz_i}{dV_{abc}} \frac{\prod_{i=1}^{N-1} \theta(z_i - z_{i+1})}{\prod_{i=1}^{N} |\gamma_i'(1)|} \prod_{i=1}^{N} [_i < x = 0,0,q=3|]$$
$$\times < p = 0,0,q=0| \prod_{i=1}^{N} \hat{\gamma}_i W_i \hat{\gamma}_i^{-1} | p = 0,0,q=0 > \qquad (3.1.26)$$

with

$$dV_{abc} = \frac{dz_a dz_b dz_c}{|\gamma_a'(1)\gamma_b'(1)\gamma_c'(1)|} \qquad (3.1.27)$$

The scattering amplitude (2.4.85) is reproduced from V_N by saturating it with three physical states of the type $|phys > \otimes |q=1>$, obtaining a vertex operator with weight $\Delta = 0$ and with the other $N-3$ states of the type $|phys > \otimes |q=0>$, obtaining a vertex operator with conformal weight $\Delta = 1$.

The BRST invariance of V_N derives straightforwardly from the BRST invariance of the SDS vertex (3.1.14); indeed taking into account that each Q_j commutes with all the factors in the product $\prod_{i=1}^{N} \hat{\gamma}_i W_i \hat{\gamma}_i^{-1}$ except the one with the same index and that the (3.1.14) holds, then it is possible to write:

$$< p = 0,0; q=0| \prod_{i=1}^{N} [_i < x = 0,0,q=3|] \left[\sum_{j=1}^{N} Q_j, \prod_{i=1}^{n} \hat{\gamma}_i W_i \hat{\gamma}_i^{-1} \right] | p = 0,0,q=0 >=$$

$$- < p = 0,0,q=0| \prod_{i=1}^{N} {}_i < x = 0,0,q=3| \left[Q, \prod_{i=1}^{N} \hat{\gamma}_i W_i \hat{\gamma}_i^{-1} \right] | p = 0,0,q=0 >= 0 \qquad (3.1.28)$$

since

$$Q|p = 0, 0, q = 0 >= 0. \tag{3.1.29}$$

The M-closed string vertex can be defined in the following way in terms of the SDS vertex (3.1.16) $W_{i\bar{i}}$:

$$V_M = \int \frac{\prod_{i=1}^{M} d^2 z_i}{dV_{abc}} \frac{1}{\prod_{i=1}^{M} |\gamma_i'(1)|^2} \cdot$$

$$\prod_{i=1}^{M} ({}_{i\bar{i}}< x = 0, 0|) < p = 0, 0| R \left(\prod_{i=1}^{M} \hat{\gamma}_i \hat{\bar{\gamma}}_i \, W_{i\bar{i}} \, \hat{\bar{\gamma}}_i^{\,-1} \hat{\gamma}_i^{\,-1} \right) |p = 0, 0> \tag{3.1.30}$$

where $\hat{\gamma}_i$ and $\hat{\bar{\gamma}}_i$ are operators which act in the Fock space of the auxiliary oscillators, performing the projective transformations:

$$z \to \gamma_i(z) \quad \bar{z} \to \bar{\gamma}_i(\bar{z}) \; ; \qquad \gamma_i(1 - z) = V_i(z) = \frac{a_i z + b_i}{c_i z + d_i}$$

or, using Lovelace notation:

$$V_i \equiv \begin{bmatrix} \infty & 0 & 1 \\ z_{i-1} & z_i & z_{i+1} \end{bmatrix} \qquad \bar{V}_i \equiv \begin{bmatrix} \infty & 0 & 1 \\ \bar{z}_{i-1} & \bar{z}_i & \bar{z}_{i+1} \end{bmatrix} \tag{3.1.31}$$

and

$$\gamma_i \equiv \begin{bmatrix} \infty & 1 & 0 \\ z_{i-1} & z_i & z_{i+1} \end{bmatrix} \qquad \bar{\gamma}_i \equiv \begin{bmatrix} \infty & 1 & 0 \\ \bar{z}_{i-1} & \bar{z}_i & \bar{z}_{i+1} \end{bmatrix} \tag{3.1.32}$$

The parameters are given by:

$$a_i = z_{i-1}(z_i - z_{i+1}) \qquad b_i = z_i(z_{i+1} - z_{i-1})$$
$$c_i = z_i - z_{i+1} \qquad d_i = z_{i+1} - z_{i-1} \tag{3.1.33}$$

(and the complex conjugate values are taken for the parameters characterizing $\bar{V}_i$). The couples of variables, $(z_{i-1}, \bar{z}_{i-1})$ and $(z_{i+1}, \bar{z}_{i+1})$ here introduced, are relative to the external states that respectively forego and follow immediately the ith state in the various terms of the sum on the permutations prescribed by the R ordering.

The fact that the M-closed string vertex (3.1.30) reproduces exactly the amplitude (2.4.93), by acting on M physical states, easily follows from (3.1.18) and from the well-known transformation property of primary fields, that for $V_{\alpha_i \beta_i}$ reads:

$$\hat{\gamma}_i\hat{\bar{\gamma}}_i V_{\alpha_i\beta_i}(1,1,k_i)\hat{\bar{\gamma}}_i^{-1}\hat{\gamma}_i^{-1} = [\gamma_i'(1)]^{\Delta}[\bar{\gamma}'_i(1)]^{\bar{\Delta}} V_{\alpha_i\beta_i}(1,1,k_i) \tag{3.1.34}$$

with $\Delta = \bar{\Delta} = 1$.

The vacuum expectation value in (3.1.30) over the auxiliary oscillators can be explicitly performed. One gets:

$$\begin{aligned} V_M &= \int \frac{\prod_{i=1}^M d^2 z_i}{dV_{abc}} \frac{1}{\prod_{i=1}^M |\gamma_i'(1)|^2} \sum_{\{permut.\}} \prod_{i=1}^M ({}_{i\bar{i}}\!< x = 0, 0|) \cdot \\ & \times \exp\left\{ -\sum_{\substack{i,j=1 \\ i\neq j}}^M \frac{1}{2} \left[\sum_{\substack{n=0 \\ m=0}}^{\infty} a_n^i M_{nm}(\Gamma V_i^{-1} V_j)\, a_m^j + \sum_{\substack{n=0 \\ m=0}}^{\infty} \bar{a}_n^{\bar{i}} \bar{M}_{nm}(\Gamma \bar{V}_i^{-1} \bar{V}_j)\, \bar{a}_m^{\bar{j}} \right] \right\} \cdot \\ & \times \delta^D\left(\sum_{i=1}^M p^i\right) \end{aligned} \tag{3.1.35}$$

where $a_0^i = \bar{a}_0^{\bar{i}} = p^i/2$. The matrix M_{nm} $[\bar{M}_{nm}]$ is related to the infinite-dimensional representation of the projective group (see refs. [20] and [26]) with $J \to 0$, corresponding to the coordinate $x^\mu(z)$ $[\bar{x}^\mu(\bar{z})]$ defined in (3.1.4).

From (3.1.35) it is rather simple to verify the projective invariance. Indeed if one performs an $SL(2, C)$ projective transformation on the external variables such that:

$$z_i \to P(z_i) = \frac{\alpha z_i + \beta}{\gamma z_i + \delta} \tag{3.1.36}$$

then $V_i \to PV_i$ and $\bar{V}_i \to \bar{P}\bar{V}_i$. Therefore the arguments $\Gamma V_i^{-1} V_j$ and $\bar{\Gamma}\bar{V}_i^{-1}\bar{V}_j$ of the matrices M_{nm} and $\bar{M}_{nm}$ are left invariant under the transformation (3.1.36). Furthermore it is very easy to show that the integration measure in (3.1.35) is projective invariant. Hence the full integrand in (3.1.35) is projective invariant and also the requirement 2. is satisfied.

The previous vertex can be made BRST invariant by adding the contribution of the ghost and antighost fields as in the case of the open string. In this case eq. (3.1.16) becomes:

$${}_{i\bar{i}} < x = 0, 0, q = 3, \bar{q} = 3|\; W_{i\bar{i}} \tag{3.1.37}$$

where

$$W_{i\bar{i}} =: W_i W_{\bar{i}} : \tag{3.1.38}$$

with

$$\begin{aligned} W_i &= :\exp\left\{\oint dz[-x(1-z)x_i'(z) - c(1-z)b_i(z) + b(1-z)c_i(z)]\right\}: \\ W_{\bar{i}} &= :\exp\left\{\oint d\bar{z}[-\bar{x}(1-\bar{z})\bar{x}_i'(\bar{z}) - \bar{c}(1-\bar{z})\bar{b}_i(\bar{z}) + \bar{b}(1-\bar{z})\bar{c}_i(\bar{z})]\right\}: \end{aligned}$$

In terms of (3.1.37) the M-closed string vertex reads now:

$$V_M = \int \frac{\prod_{i=1}^M d^2 z_i}{dV_{abc}} \frac{1}{\prod_{i=1}^M |\gamma_i'(1)|^2} \prod_{i=1}^M ({}_{i\bar{i}}< x = 0, 0, q = 3; \bar{q} = 3|) \cdot$$

$$\cdot < p = 0, 0, q = 0, \bar{q} = 0|R\left(\prod_{i=1}^M \hat{\gamma}_i \hat{\bar{\gamma}}_i W_{i\bar{i}} \hat{\bar{\gamma}}_i^{-1} \hat{\gamma}_i^{-1}\right)|p = 0, 0, q = 0, \bar{q} = 0 > \tag{3.1.39}$$

with

$$dV_{abc} = \frac{d^2 z_a d^2 z_b d^2 z_c}{|\gamma_a'(1)\gamma_b'(1)\gamma_c'(1)|^2}. \tag{3.1.40}$$

The difference in the measure between (3.1.30) and (3.1.39) corresponds to the fact that the scattering amplitude relative to physical states is now obtained from (3.1.39) by saturating it with three physical states of the type $|q = \bar{q} = 1>$ $\otimes\ |transv.>$ (obtaining a vertex operator with conformal weights $\Delta = \bar{\Delta} = 0$) and with the other $M - 3$ states of the type $|q = \bar{q} = 0> \otimes |transv.>$ (obtaining a vertex operator $V_{\alpha\beta}(z, \bar{z})$ with conformal weights $\Delta = \bar{\Delta} = 1$). In such a way the M-closed string vertex will reproduce the scattering amplitude proposed by Friedan, Martinec and Shenker [33].

The matrix element for the auxiliary oscillators can be computed and we get the following expression for the M-closed string vertex:

$$V_M = \int \frac{\prod_{i=1}^M d^2 z_i}{dV_{abc}} \frac{1}{\prod_{i=1}^M |\gamma_i'(1)|^2} \sum_{\{permut.\}} \prod_{i=1}^M ({}_{i\bar{i}}< x = 0, 0, q = 3, \bar{q} = 3|) \cdot$$

$$\cdot \exp\left\{ -\sum_{\substack{i,j=1 \\ i \neq j}}^M \left[\frac{1}{2} \sum_{\substack{n=0 \\ m=0}}^\infty a_n^i M_{nm}(\Gamma V_i^{-1} V_j)\, a_m^j + \frac{1}{2} \sum_{\substack{n=0 \\ m=0}}^\infty \bar{a}_n^{\bar{i}} \bar{M}_{nm}(\Gamma \bar{V}_i^{-1} \bar{V}_j) \bar{a}_m^{\bar{j}} + \right.\right.$$

$$+ \sum_{\substack{n=2 \\ m=-1}}^{\infty} c_n^i E_{nm}(\Gamma V_i^{-1} V_j)\, b_m^j + \sum_{\substack{n=2 \\ m=-1}}^{\infty} \bar{c}_n^{\bar{i}} \bar{E}_{nm}(\Gamma \bar{V}_i^{-1} \bar{V}_j) \bar{b}_m^{\bar{j}} \Bigg] \Bigg\} \cdot$$

$$\cdot \delta^D \left(\textstyle\sum_{i=1}^M p^i \right) \cdot \prod_{n=-1}^{1} \left\{ \sum_{i=1}^{M} \sum_{m=-1}^{1} E_{nm}(V_i) b_m^i \right\} \cdot \prod_{n=-1}^{1} \left\{ \sum_{i=1}^{M} \sum_{m=-1}^{1} \bar{E}_{nm}(\bar{V}_i) \bar{b}_m^{\bar{i}} \right\} \tag{3.1.41}$$

where

$E_{nm}(V)$ and $\bar{E}_{nm}(\bar{V})$ provide an infinite-dimensional representation of the projective group with $J = 1$, corresponding to the ghost coordinates $c(z)$ and $\bar{c}(\bar{z})$. Besides the projective invariance (for the new terms in eq. (3.1.41) it is immediate to show they are left invariant), the M-closed string vertex constructed exhibits also BRST invariance, in fact:

$$V_M \sum_{i=1}^{M} Q_i = V_M \sum_{i=1}^{M} \bar{Q}_i = 0 \tag{3.1.42}$$

as a consequence of the BRST invariance of the constituent SDS vertices used in (3.1.39).

Q_i and $\bar{Q}_i$ are the BRST charges corresponding to the coordinates of the i-th closed string:

$$\begin{aligned} Q_i &= \oint dz : \left\{ -\frac{1}{2} c_i(z) [x_i'(z)]^2 + b_i(z) c_i(z) c_i'(z) \right\} : \\ \bar{Q}_i &= \oint d\bar{z} : \left\{ -\frac{1}{2} \bar{c}_i(\bar{z}) [\bar{x}_i'(\bar{z})]^2 + \bar{b}_i(\bar{z}) \bar{c}_i(\bar{z}) \bar{c}_i'(\bar{z}) \right\} : . \end{aligned} \tag{3.1.43}$$

Therefore also the requirement 3. is satisfied.

In conclusion we have constructed an M-closed string vertex that is projective and BRST invariant and when saturated with physical states it reproduces their corresponding scattering amplitudes.

3.2 BRST invariant mixed string vertex

In this section we want to generalize the construction of the previous sections to include also a number of open string states by constructing an M-closed and N-open string vertex.

As in the case of a vertex involving only open or closed strings the starting point is the scattering amplitude with N open and M closed physical states, that we denote with $|\alpha_i, p_i>$ and $|\alpha_j, \beta_j, k_j>$ respectively. Such an amplitude is given by [23]:

$$\begin{aligned} A(N,M) &= \int \frac{\prod_{i=1}^{N}[dx_i\, \vartheta(x_i - x_{i+1})]\prod_{j=1}^{M} d^2 z_j}{dV_{abc}} \cdot \\ &\cdot < 0|R^* \left(\prod_{i=1}^{N} \prod_{j=1}^{M} V_{\alpha_i}(x_i; p_i) V_{\alpha_j \beta_j}(z_j, \bar{z}_j; k_j) \right) |0> \end{aligned} \quad (3.2.44)$$

where $V_{\alpha_i}(x_i; p_i)$ $[V_{\alpha_j\beta_j}(z_j, \bar{z}_j; k_j)]$ is the vertex operator associated with the state $|\alpha_i; p_i>$ $[|\alpha_j, \beta_j; k_j>]$. The Koba-Nielsen variables for the closed strings are integrated over the whole complex plane, while those for the open strings are integrated along the real axis with the ordering given by the ϑ-functions. The R^* prescription refers to the ordering of the closed string states among themselves and with respect to the open string states according to the moduli of their variables z and x.

Although (3.2.44) contains also closed strings its integrand is only invariant under real projective transformations as in the case of open strings. Consequently three real variables can be fixed arbitrarily and the corresponding volume dV_{abc} must be introduced.

Still the basic ingredient for the construction of a mixed string vertex is an SDS vertex, that describes the interaction between a closed and an open string. It is given by:

$$_{i\bar{i}} < x = 0, 0|W_i \cdot W_{\bar{i}} \quad (3.2.45)$$

where

$$\begin{aligned} W_i &= : \exp\left\{ \oint dz[-x_i'(z)\, x(1-z)] \right\} : \\ W_{\bar{i}} &= : \exp\left\{ \oint d\bar{z}[-\bar{x}_{\bar{i}}'(\bar{z})\, x(1-\bar{z})] \right\} : \end{aligned}$$

with $x(1-z)$ and $x(1-\bar{z})$ containing the same set of auxiliary oscillators. The vertex (3.2.45) has the property that, acting on any closed string state $|\alpha_i, \beta_i, k_i>$, gives the corresponding vertex operator in the space of the auxiliary oscillators:

$$ {}_{i\bar{i}}< x = 0,0|W_i W_{\bar{i}}|\alpha_i, \beta_i, k_i>_{i\bar{i}} = V_{\alpha_i}\left(1, \frac{k_i}{2}\right) V_{\beta_i}\left(1, \frac{k_i}{2}\right) \equiv V_{\alpha_i\beta_i}(1, k_i). \quad (3.2.46) $$

In terms of this new SDS vertex we can construct a mixed string vertex that is given by:

$$ V_{N;M} = \int \frac{\prod_{i=1}^{N}[dx_i \vartheta(x_i - x_{i+1})]\prod_{j=1}^{M} d^2 z_j}{dV_{abc}} \frac{1}{|\prod_{i=1}^{N}\gamma_i'(1)\prod_{j=1}^{M}\gamma_j'(1)\,\gamma_{\bar{j}}'(1)|} \cdot $$

$$ \times \prod_{i=1}^{N}({}_i< x = 0,0|) \prod_{j=1}^{M}({}_{j\bar{j}}< x = 0,0|) \cdot $$

$$ \times < p = 0,0|R^* \left(\prod_{i=1}^{N} \hat{\gamma}_i W_i \hat{\gamma}_i^{-1} \prod_{j=1}^{M} \hat{\gamma}_j W_j \hat{\gamma}_j^{-1} \hat{\gamma}_{\bar{j}} W_{\bar{j}} \hat{\gamma}_{\bar{j}}^{-1}\right) |p = 0,0> \quad (3.2.47) $$

where $\hat{\gamma}_i$ (or $\hat{\gamma}_j$) is the operator performing the following projective transformation:

$$ \gamma_i \equiv \begin{bmatrix} \infty & 1 & 0 \\ u_a & u_i & u_b \end{bmatrix}. \quad (3.2.48) $$

Here the variables u_a, u_i and u_b stand for anyone of the variables x_i, z_j and $\bar{z}_j$ according to the sequence of the vertices in (3.2.47) in a generic term of the sum prescribed by the R^* ordering. By construction if we now saturate the vertex (3.2.47) with physical states we immediately get the amplitude (3.2.44). This follows from (3.2.46) and from the transformation property of the vertex operators under (3.2.48):

$$ \hat{\gamma}_i V_{\alpha_i}(1, h_i)\hat{\gamma}_i^{-1} = \gamma_i'(1)\; V_{\alpha_i}(u_i, h_i) \quad (3.2.49) $$

where V_{α_i} can be either an open string vertex operator or one of the two vertices in which $V_{\alpha_j\beta_j}$ can be split.

The matrix element over the auxiliary oscillators can be computed and we get:

$$\begin{aligned} V_{N;M} &= \int \frac{\prod_{i=1}^{N}[dx_i \vartheta(x_i - x_{i+1})] \prod_{j=1}^{M} d^2 z_j}{dV_{abc}} \frac{1}{|\prod_{i=1}^{N} \gamma_i'(1) \prod_{j=1}^{M} \gamma_j'(1)\gamma_{\bar{j}}'(1)|} \cdot \\ &\times \delta^D \left(\sum_{i=1}^{N} p^i + \sum_{j=1}^{M} k^j\right) \sum_{\{permut.\}} \prod_{i=1}^{N+2M} ({}_i< x = 0, 0|) \cdot \\ &\times \exp\left\{ - \sum_{\substack{i,j=1 \\ i \neq j}}^{N+2M} \frac{1}{2} \left[\sum_{\substack{n=0 \\ m=0}}^{\infty} a_n^i M_{nm}(\Gamma V_i^{-1} V_j)\, a_m^j \right] \right\}. \end{aligned} \tag{3.2.50}$$

From eq. (3.2.50) it can immediately be seen that the integrand of our mixed vertex is invariant under projective transformations. In fact also in this case the effect of an arbitrary projective transformation P on x_i or z_i is that $V_i \to PV_i$ and therefore the arguments $\Gamma V_i^{-1} V_j$ of the matrices M_{nm} are left invariant. This, together with the projective invariance of the integration measure, guarantees the projective invariance of the whole integrand in (3.2.47).

The BRST invariance can be implemented as in the case of the only open or closed string vertex by adding the contribution of the ghost coordinates. The vertex (3.2.45) becomes:

$$ {}_{i\bar{i}} < x = 0, 0, q = 3, \bar{q} = 3 | W_i \cdot W_{\bar{i}} \tag{3.2.51}$$

with

$$\begin{aligned} W_i &= :\exp\left\{ \oint dz \left[-x_i'(z) x(1-z) - c(1-z) b_i(z) + b(1-z) c_i(z) \right] \right\}: \\ W_{\bar{i}} &= :\exp\left\{ \oint dz \left[-\bar{x}_{\bar{i}}'(\bar{z}) x(1-\bar{z}) - c(1-\bar{z}) \bar{b}_{\bar{i}}(\bar{z}) + b(1-\bar{z}) \bar{c}_{\bar{i}}(\bar{z}) \right] \right\}: \end{aligned}$$

The complete N-open M-closed string vertex reads:

$$\begin{aligned} V_{N;M} = \int & \frac{\prod_{i=1}^{N} [dx_i \vartheta(x_i - x_{i+1})] \prod_{j=1}^{M} d^2 z_j}{dV_{abc}} \frac{1}{|\prod_{i=1}^{N} \gamma_i'(1) \prod_{j=1}^{M} \gamma_j'(1) \gamma_{\bar{j}}'(1)|} \cdot \\ & \times \prod_{i=1}^{N} ({}_i< x = 0, 0, q = 3|) \prod_{j=1}^{M} ({}_{j\bar{j}}< x = 0, 0, q = 3, \bar{q} = 3|) \cdot \\ \times < p = 0, 0, q = 0 | R^* & \left(\prod_{i=1}^{N} \hat{\gamma}_i W_i \hat{\gamma}_i^{-1} \prod_{j=1}^{M} \hat{\gamma}_j W_j \hat{\gamma}_j^{-1} \hat{\gamma}_{\bar{j}} W_{\bar{j}} \hat{\gamma}_{\bar{j}}^{-1} \right) | p = 0, 0, q = 0 > \end{aligned} \tag{3.2.52}$$

Now, if we fix for example a closed string variable z_a and an open string variable x_b, the volume dV_{abc} is given by:

$$dV_{abc} = \frac{d^2 z_a dx_b}{|\gamma'_b(1)\gamma'_a(1)\gamma'_{\bar{a}}(1)|} \tag{3.2.53}$$

The expectation value in (3.2.52) can be explicitly computed, yielding:

$$\begin{aligned} V_{N:M} &= \int \frac{\prod_{i=1}^{N}[dx_i \vartheta(x_i - x_{i+1})] \prod_{j=1}^{M} d^2 z_j}{dV_{abc}} \frac{1}{|\prod_{i=1}^{N}\gamma'_i(1)\prod_{j=1}^{M}\gamma'_j(1)\gamma'_{\bar{j}}(1)|} \\ & \sum_{\{permut.\}} \prod_{i=1}^{N+2M} ({}_i< x=0;0;q=3|) \cdot \\ & \cdot \exp\left\{ -\sum_{\substack{i,j=1 \\ i\neq j}}^{N+2M} \left[\frac{1}{2} \sum_{\substack{n=0 \\ m=0}}^{\infty} a_n^i M_{nm}(\Gamma V_i^{-1} V_j)\, a_m^j + \right.\right. \\ & \left.\left. + \sum_{\substack{n=2 \\ m=-1}}^{\infty} c_n^i E_{nm}(\Gamma V_i^{-1} V_j)\, b_m^j \right] \right\} \cdot \\ & \cdot \delta^D\left(\sum_{i=1}^{N} p^i + \sum_{j=1}^{M} k^j\right) \prod_{n=-1}^{1} \left\{ \sum_{i=1}^{N+2M} \sum_{m=-1}^{1} E_{nm}(V_i) b_m^i \right\}. \end{aligned} \tag{3.2.54}$$

The mixed vertex so defined is BRST invariant, since it satisfies the following relation:

$$V_{N:M} \sum_{i=1}^{N} Q_i = V_{N:M} \sum_{j=1}^{M} Q_j = V_{N:M} \sum_{j=1}^{M} \bar{Q}_j = 0$$

where Q_i [Q_j and $\bar{Q}_j$] are the BRST charges corresponding to the coordinates of the open [closed] strings, defined as in (3.1.43).

3.3 Vertex for the closed - open string transition

From the expression of the mixed string vertex it is straightforward to derive the vertex relative to the open - closed string transition. The final result is:

$$\begin{aligned}
V_{1;1} &= {}_1< x=0,0,q=3|\ {}_2< x=0,0,q=3|\ {}_{\bar 2}< \bar x=0,0,\bar q=3| \cdot \\
&\cdot \exp\left\{ -\left[\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left(a^1_n \cdot \frac{p}{2} + a^2_n \cdot \frac{p}{2} + \bar a^{\bar 2}_n \cdot k \right) + \right.\right. \\
&+ \sum_{\substack{n=1\\m=1}}^{\infty} (-1)^n \sqrt{\frac{m}{n}} \begin{pmatrix} n \\ m \end{pmatrix} \left(a^1_n \cdot a^2_m + a^2_n \cdot \bar a^{\bar 2}_m + \bar a^{\bar 2}_n \cdot a^1_m \right) + \\
&+ \sum_{n=2}^{\infty} \sum_{m=-1}^{1} (-1)^m \begin{pmatrix} n+1 \\ m+1 \end{pmatrix} \left(c^1_n \cdot b^2_m + c^2_n \cdot \bar b^{\bar 2}_m + \bar c^{\bar 2}_n \cdot b^1_m \right) + \\
&\left.\left. + \sum_{\substack{n=2\\m=2}}^{\infty} (-1)^n \begin{pmatrix} m-2 \\ n-2 \end{pmatrix} \left(c^1_n \cdot \bar b^{\bar 2}_m + c^2_n \cdot b^1_m + \bar c^{\bar 2}_n \cdot b^2_m \right) \right] \right\} \cdot \delta^D(p+k) \cdot \\
&\left(b^1_{-1} - b^1_0 - b^2_0 + b^2_1 + \bar b^{\bar 2}_0 \right) \cdot \left(b^2_{-1} - b^2_0 - \bar b^{\bar 2}_0 + \bar b^{\bar 2}_1 + b^1_0 \right) \cdot \\
&\left(\bar b^{\bar 2}_{-1} - \bar b^{\bar 2}_0 - b^1_0 + b^1_1 + b^2_0 \right). \qquad (3.3.55)
\end{aligned}$$

It is obtained from eq. (3.2.54) in the special case with $N = M = 1$. In this case the measure factor in (3.2.54) is equal to 1 and the terms depending on the oscillators can be easily computed if we observe that:

$$\Gamma V_i^{-1} V_{i+1} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \qquad (3.3.56)$$

and

$$\Gamma V_{i+1}^{-1} V_i = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \qquad (3.3.57)$$

The expression of the vertex (3.3.55) is exactly analogous to the BRST invariant generalization of the Caneschi-Schwimmer-Veneziano vertex where now one oscillator is relative to the open string and the others correspond to the two sectors of the closed string; this reproduces the analogy between the transition amplitude from a closed to an open physical on-shell string state and an amplitude for three open string states, already observed in ref. [23] after factorization of the mixed scattering amplitudes. In our approach this form for the open - closed string transition vertex is very natural and also it can be considered as a constituent piece in the factorization of a generic mixed string vertex, as it will be shown with an example in the following section. But first it is interesting to make use of eq. (3.3.55) for computing, for example, the amplitude between an open string photon described by the state:

$$-i\,\varepsilon \cdot a_1^{\dagger}|0> \otimes \; |q=1> \tag{3.3.58}$$

and an antisymmetric tensor state of the closed string:

$$\varepsilon^{\mu\nu}(a^{\dagger}_{1;\mu}\bar{a}^{\dagger}_{1;\nu} - a^{\dagger}_{1;\nu}\bar{a}^{\dagger}_{1;\mu})|0> \otimes \; |q=\bar{q}=1> \,. \tag{3.3.59}$$

Such an amplitude is obtained by saturating eq. (3.3.55) with the two states (3.3.58) and (5.1.29). The result is:

$$A(\text{photon} \rightarrow \text{antisymm.tensor}) = \varepsilon_{\mu}\; \varepsilon_{\rho\sigma}(p^{\rho}g^{\mu\sigma} - p^{\sigma}g^{\mu\rho})\delta^{D}(p+k) \tag{3.3.60}$$

where p denotes the photon momentum. It is also easy to see that the photon - graviton or photon - dilaton amplitudes are vanishing as expected from angular momentum conservation.

3.4 Factorization property of the mixed vertex

Another interesting example is the vertex for one closed and two open strings, that is a particular case of our general vertices (3.2.47) and (3.2.52) for $N = 2$ and $M = 1$. Its integrand will depend on a complex z_A and two real Koba-Nielsen variables z_1 and z_2 relative to the closed and open strings respectively. The projective invariance allows us to fix three of these variables. We are then left with an integral over the fourth one, which corresponds to the fact that the closed string interacts with the open strings by first becoming an open string through the open - closed string transition and then generating the two open strings.

This can be seen immediately by *sewing* the open - closed string transition vertex with the 3-open string vertex after the insertion of a twisted propagator (see fig.2), and we shall get the $N = 2$ $M = 1$ vertex. The integration variable of the propagator corresponds to the variable that cannot be fixed in this vertex as previously observed.

But before considering this sewing, we would like at first to explain what is, from the point of view of the operatorial formalism, the meaning to give to the sewing of a general N-string vertex with a propagator. For this aim we will now introduce the definition of *canonical forms*: a canonical form is just a particular form in which some operators defined in the Fock space of the oscillators can be put, with the property that the product of two canonical forms is still a canonical form. So if it was possible to define both the N-string vertex and the propagator as canonical forms, then the sewing of the latter with the former would have the simple meaning of product of two canonical forms, which can be easily obtained. This is what we are illustrating in the next subsections.

3.4.1 Canonical forms

A **canonical form** in the Fock space generated by the oscillators relative to the field $x^\mu(z)$ is defined [31] as any operator O which can be put in the following way:

$$O =: \exp\left\{\sum_{n=1}^{\infty} a_n^+ A_n\right\} : \exp\left\{\sum_{\substack{n=1\\m=1}}^{\infty} a_n^+ (C_{nm} - \delta_{nm}) a_m\right\} : \exp\left\{\sum_{n=1}^{\infty} B_n a_n\right\} \exp\{-\phi\} \quad (3.4.61)$$

or, equivalently, introducing a zero mode $a_0^\mu = a_0^{\mu+} = p_\mu$:

$$O =: \exp\{\sum_{n=0}^{\infty} a_n^+ (C_{nm} - \delta_{nm})\} : \quad (3.4.62)$$

where the matrix C_{nm} provides an infinite irreducible representation of the projective group.

Let us consider, for the sake of simplicity, the orbital part of the exponential of the N-string vertex for open strings and let us extract from it the terms relative to the leg E to which we mean to sew the propagator; it is:

$$\hat{V}_E = \exp\left\{ -\sum_{\substack{i=1\\i\neq E}}^{N}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} a_n^{(i)} M_{nm}(\Gamma V_i V_E) a_m^{(E)} - \sum_{\substack{i=1\\i\neq E}}^{N}\sum_{n=0}^{\infty} a_n^{(i)} M_{n0}(\Gamma V_i^{-1} V_E) p \right\} \quad (3.4.63)$$

This operator can be put in the following form:

$$\exp\left\{ \sum_{m=1}^{\infty} B_m a_m^{(E)} \right\} \exp\{\phi\} \quad (3.4.64)$$

with

$$\begin{aligned} B_m &= -\sum_{\substack{i=1\\i\neq E}}^{N}\sum_{n=1}^{\infty} a_n^{(i)} M_{nm}(\Gamma V_i^{-1} V_E) \\ \phi &= \sum_{\substack{i=1\\i\neq E}}^{N}\sum_{n=0}^{\infty} a_n^{(i)} M_{n0}(\Gamma V_i^{-1} V_E) p \end{aligned} \quad (3.4.65)$$

Hence it is a canonical form, according to the definition (3.4.61).

Another example of canonical form is given by the twisted propagator. The reason for introducing such an object lies in the fact that open strings have an "orientation" since their couplings to a given final state are not invariant if an anticyclic permutation of the particles in the final state is made: so they must remember in which way they are to be coupled. Since the N-string vertex we have defined is symmetric, in the sense that all the external strings have the same orientation, a twist is necessary when two of them are sewn.

A possible definition of twisted propagator [31] is given by:

$$\hat{P}(x) \equiv x^{L_0} \Omega\, (1-x)^W \quad (3.4.66)$$

with

$$\Omega = e^{L_1} (-1)^{L_0 - \frac{p^2}{2}} \qquad W = L_0 - L_1 \quad (3.4.67)$$

corresponding to an intercept $\alpha_0 = 1$. $\hat{P}(x)$ generates the following projective transformation:

$$P(x) = \begin{pmatrix} x & -x \\ x & -1 \end{pmatrix} \qquad \text{or} \qquad P(x) = \begin{bmatrix} \infty & 1 & 0 \\ 1 & 0 & x \end{bmatrix}.$$

It is a canonical form, since:

$$x^{L_0} =: \exp\left\{\sum_{\substack{n=1\\m=1}}^{\infty} a_n^+ \left(M_{nm}(\Lambda) - \delta_{nm}\right) a_m\right\} : x^{\frac{1}{2}p^2} \tag{3.4.68}$$

with

$$\Lambda = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\Omega(x) = \exp\left[\sum_{n=1}^{\infty} a_n^+ \frac{p}{\sqrt{n}}\right] : \exp\left[\sum_{\substack{n=1\\m=1}}^{\infty} a_n^+ (M_{nm}(\Xi) - \delta_{nm}) a_m\right] :$$

$$\exp\left[\sum_{n=1}^{\infty} \frac{px^n}{\sqrt{n}} a_n\right] (1-x)^{\frac{1}{2}p^2} \tag{3.4.69}$$

with

$$\Xi = \begin{pmatrix} 1 & -1 \\ x & -1 \end{pmatrix}$$

Therefore $\hat{P}(x)$ satisfies the def. (3.4.61) with

$$\begin{aligned} A_n &= p\frac{x^n}{\sqrt{n}} \\ B_n &= \frac{px^n}{\sqrt{n}} \\ C_{nm} &= M_{nm}(P(x))\phi = \frac{1}{2}p^2 \ln[x(1-x)] \end{aligned}$$

where

$$P(x) \equiv \begin{pmatrix} x & -x \\ x & -1 \end{pmatrix}.$$

3.4.2 Sewing a propagator to a vertex

This operation is nothing but a product of canonical forms.

The product of two canonical forms O_1 and O_2 given by eq. (3.4.61) is again a canonical form with

$$\begin{aligned} \phi &= \phi_1 + \phi_2 + \sum_{n=1}^{\infty} B_{1n} A_{2n} \\ A &= A_1 + C_1 A_2 \\ B &= B_2 + B_1 C_2 \\ C &= C_1 C_2 \end{aligned} \tag{3.4.70}$$

So the sewing of the twisted propagator on the leg E of the N-string vertex means to apply this definition of product to (3.4.64) and (3.4.68) and in this way it is possible to show, by using the conservation of momentum, that:

$$\exp\left\{ - \sum_{i \neq E} \sum_{\substack{n=0 \\ m=0}}^{\infty} a_n^{(i)} M_{nm}(\Gamma V_i^{-1} V_E) a_m^{(E)} \right\} P^E(x) =$$

$$\exp\left\{ - \sum_{\substack{i=1 \\ i \neq E}}^{N} \sum_{\substack{n=0 \\ m=0}}^{\infty} a_n^{(i)} M_{nm}(\Gamma V_i^{-1} V_E P^E(x)) a_m^E \right\}$$

(3.4.2) shows that sewing the propagator to a leg of an N-string vertex simply means to multiply it by the argument of the matrices M_{nm} relative to that leg.

3.4.3 Sewing two vertices

Here we want to show in which way it is possible to sew a mixed vertex with $M = N = 1$ and a 3-open string vertex. If this procedure could lead to a mixed vertex $M = 1 N = 2$, then it would be a proof of the correct factorization property of the mixed vertex.

For the sake of simplicity we show the steps of the sewing procedure limiting ourselves to the orbital part. The procedure for the ghost part follows similar lines and introduces one new fermionic δ function besides the three ones present in the general vertex (last term in (3.2.54)), yielding an equivalent form of the vertex, but we do not want here to deal with this feature (see ref. [42]).

We want to compute the following expression:

$$\int \frac{dV_{1:1}}{dV_{A\bar{A}E}} \hat{V}_{1;1}(A,\bar{A},E) \int_0^1 \frac{dx}{x^2} \hat{P}_E(x) \int \frac{dV_{3:0}}{dV_{F12}} \hat{V}^{\dagger}_{3;0}(F,1,2) \tag{3.4.71}$$

where the integration volumes stand for the usual ones, respectively of a mixed vertex with $N = M = 1$ and of a 3-Reggeon vertex, while E and F denote the two legs to be sewed together. $\hat{V}_{1;1}(A,\bar{A},E)$ coincides with the orbital part of (3.3.55).

Since $\hat{V}_{1;1}(A,\bar{A},E)$ and $\hat{V}_{3;0}(F,1,2)$ are both defined as bra vectors, as first step we have to invert the leg F and then identify it with the leg E: in other words we have to write $\hat{V}^{\dagger}_{3;0}$ with E in the place of F; it is obtained from $\hat{V}_{3;0}(F,1,2)$ through the following substitutions:

$$a_n^F \to -a_n^{\dagger E} \qquad\qquad p_\mu^F \to -p_\mu^E \tag{3.4.72}$$

$$_F< x = 0, 0| \;\to\; |x = 0, 0>_E \tag{3.4.73}$$

The twisted propagator used here is given by [31]:

$$T = \int_0^1 \frac{dx}{x^2} \hat{P}(x) \tag{3.4.74}$$

where $\hat{P}(x)$ is given by (3.4.2).

As a second step we have to identify the Koba-Nielsen variables of the two vertices as follows:

$$z_E = z_1 \quad z_F = z_A \tag{3.4.75}$$

and we choose the variable z_2 as a function of x identified with the following anharmonic ratio:

$$x \equiv (z_2, z_1, z_A, z_{\bar{A}}) \equiv \frac{(z_2 - z_1)(z_A - z_{\bar{A}})}{(z_2 - z_{\bar{A}})(z_A - z_1)} \tag{3.4.76}$$

The identification of the variables as in (3.4.75) and (3.4.76) imply the identity:

$$\hat{V}_{1;1}(A,\bar{A},E)\hat{P}_E(x)\hat{V}^{\dagger}_{3;0}(F,1,2) = \hat{V}_{2;1}(A,\bar{A},1,2) \tag{3.4.77}$$

which means that the expression (3.4.71) can be rewritten in the following form:

$$\begin{aligned}
&\int_0^1 \frac{dx}{x^2}\prod_i\prod_I ({}_i< x=0,0|\,{}_I< x=0,0|)\cdot \\
&\cdot\exp\left\{-\frac{1}{2}\sum_{n,m=0}^{\infty}\left[\sum_{i\neq j} a^i_n M_{nm}(\Gamma V_i^{-1}V_j)a^j_m + \sum_{I\neq J} a^I_n M_{nm}(\Gamma V_I^{-1}V_J)a^J_m\right]\right\}\cdot \\
&\cdot\exp\left\{-\frac{1}{2}\sum_{n,m=0}^{\infty}\sum_i\sum_I\left[a^i_n M_{nm}(\Gamma V_i^{-1}V_I)a^I_m + a^I_n M_{nm}(\Gamma V_I^{-1}V_i)a^i_m\right]\right\}\cdot \\
&\cdot\,\delta^D(k+p_1+p_2)
\end{aligned} \tag{3.4.78}$$

with $i,j = A,\bar{A}$ and $I,J = 1,2$. The projective matrices V's are however defined according to the ordering in the two component vertices and not to the one in the composite vertex. The correct ordering is obtained performing two projective transformations Q_A and Q_1, in the space of the variables A and 1 respectively, that change the old transformations in the new ones, as follows:

$$V_A^{(1;1)}Q_A = V_A^{(2;1)} \qquad V_1^{(3)}Q_1 = V_1^{(2;1)}$$

It turns out that the corresponding operators have the following form:

$$\hat{Q}_A = (z_1, z_{\bar{A}}, z_2, z_A)^{W_A} \tag{3.4.79}$$

and

$$\hat{Q}_1 = (z_2, z_A, z_1, z_{\bar{A}})^{W_1}. \tag{3.4.80}$$

Furthermore the measure in (3.4.78) must be rewritten in terms of the Koba-Nielsen variables of the composite vertex. Putting all together we can finally write (3.4.71) as follows:

$$\begin{aligned}
V_{2;1} \;=\; &\int \frac{\prod_i dz_i \vartheta(z_1-z_2)}{dV_{abc}}\prod_i\left(\frac{1}{|\gamma_i'(1)|}\right)\prod_i ({}_i< x=0,0|)\cdot \\
&\cdot\exp\left\{-\frac{1}{2}\sum_{i\neq j}\sum_{n,m=0}^{\infty} a^i_n M_{nm}(\Gamma V_i^{-1}V_j)a^j_m\right\}\cdot \\
&\cdot(z_1,z_{\bar{A}},z_2,z_A)^{-W_A+1}(z_2,z_A,z_1,z_{\bar{A}})^{-W_1+1}\,\cdot \\
&\cdot\delta^D(k+p_1+p_2)
\end{aligned}$$

with $i, j = A, \bar{A}, 1, 2$.

Therefore we obtain the vertex for one closed and two open strings, apart from the two projective transformations that do not give any contribution if we saturate that with physical states.

Since we obtained this vertex by sewing it is guaranteed to *factorize* correctly. In fact, *factorization* is simply the sewing in reverse.

Furthermore we have also shown in a particular case that a closed string interacts with open strings through the direct open - closed string transition.

The foregoing procedure can be followed step by step, as said, for the ghosts. In this case the twisted propagator to be used in sewing the two vertices has the following form:

$$T = (b_0 - b_1) \int_0^1 \frac{dx}{x(1-x)} \hat{P}(x) \tag{3.4.81}$$

corresponding to an intercept $\alpha_0 = 0$.

3.5 Conformal cut off formalism

Zero modes can give rise to complications both in the orbital and in the ghost contribution to the N-string vertex.

For the orbital part, it is possible to show that the zero modes problem can be avoided from the beginning. In fact in the previous sections it has been pointed out that the field $x^\mu(z)$, strictly speaking, is not a genuine conformal field of dimension 0, but it is a genuine conformal field of weight $\frac{1}{2}\varepsilon$, where ε is a positive number. The technique of the conformal cut-off consists in treating the position operator $x(z)$ with its true conformal weight and to send ε to zero at the end of calculations.

This formalism provides a more straightforward derivation of the N-string vertex with respect to the one given in the previous sections: in particular it gives the possibility to obtain in a much simpler the vacuum expectation value of the product of the projective transformed SDS vertices. For this aim we need introducing a particular kind of operators in the Fock space that transform like irreducible tensor operators under a transformation of the projective group. These are said *field operators.*

3.5.1 Field operators

In a Fock space generated by canonical oscillators an arbitrary element of the projective group can be expressed through exponentiation of the operators representing the generators L_1, L_0, L_1 of the projective group, which are given by:

$$L_i = \sum_{\substack{n=0\\m=0}}^{\infty} a_n^+ D^J_{nm}(\Lambda_i) a_m \qquad (3.5.82)$$

where Λ_i are the representation of these generators in terms of 2×2 matrices:

$$\Lambda_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \Lambda_+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \qquad \Lambda_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

An arbitrary element of the group can be expressed as:

$$U = \exp[\alpha_+ L_+] \exp[\alpha_0 L_0] \exp[\alpha_- L_-] \qquad (3.5.83)$$

The exponentiation of an operator L_i is simply obtained through the following [31]:

$$\exp\left[\sum_{\substack{n=0\\m=0}}^{\infty} a_n^+ D_{nm}(\Lambda) a_m\right] =: \exp\left[\sum_{\substack{n=0\\m=0}}^{\infty} a_n^+ (D_{nm}(\exp[\Lambda] - \delta_{nm}) a_m\right] : \qquad (3.5.84)$$

where the quantity at right hand of (3.5.84) has been put in a *canonical form,* according to the definition given in the previous section. It is possible to show that canonical forms satisfy the group multiplication rules.

So an arbitrary group element (3.5.83) can be written in the canonical form:

$$U(\Lambda) =: \exp[a_n^+ (D_{nm}(\Lambda) - \delta_{nm}) a_m] \qquad (3.5.85)$$

By using canonical forms it is possible to show the simple transformation laws under projective transformations of the *field operators.* The latter are operators in the Fock space that transform like irreducible tensor operators under a projective transformation.

If V is a 2×2 matrix corresponding to a projective transformation, the field operators operator-valued functions $F_n^J(V)$ and $\tilde{F}_n^J(V)$ defined in the following way:

$$F_n^J(V) = \sum_{m=0}^{\infty} a_m^+ D_{mn}^J(V)$$
$$\tilde{F}_n^J = \sum_{m=0}^{\infty} a_m D_{mn}^J(\Gamma V) \tag{3.5.86}$$

Then, writing a group element in Fock space $U(\Lambda)$ in canonical form, it follows that:

$$U(\Lambda)F_n^J(V)U^{-1}(\Lambda) = F_n^J(\Lambda V)$$
$$U(\Lambda)\tilde{F}_n^J(V)U^{-1}(\Lambda) = \tilde{F}_n^J(\Lambda V). \tag{3.5.87}$$

In the next subsection we will show how these objects, considered for the orbital oscillators, are connected to the definition of $x(z)$ and to the one of the SDS vertex.

3.5.2 N-string vertex with conformal cut off

In the Fock space of the orbital oscillators, let us introduce the following field operators:

$$F_n^\varepsilon(V) = \sum_{m=0}^{\infty} a_m^+ D_{mn}^\varepsilon(V)$$
$$\tilde{F}_n^\varepsilon(V) = \sum_{m=0}^{\infty} D_{mn}^\varepsilon(\Gamma V)\, a_m \tag{3.5.88}$$

where the definition of D_{mn}^ε is, as usual, derived from (3.1.2):

$$D_{mn}^\varepsilon(V) = \sqrt{\frac{\Gamma(m+\varepsilon)\,\Gamma(n+1)}{\Gamma(m+1)\,\Gamma(n+\varepsilon)}}\,\frac{1}{n!}\partial^n \left[\frac{1}{(cz+d)^\varepsilon}\left(\frac{az+b}{cz+d}\right)^m\right]\Bigg|_{z=0} \tag{3.5.89}$$

Let us now consider:

$$\mathcal{F}_0(V) = \tilde{F}_0^\varepsilon(V) - F_0^\varepsilon(V). \tag{3.5.90}$$

It is straightforward to show that:

$$\lim_{\varepsilon\to 0} i\sqrt{\Gamma(\varepsilon)}\mathcal{F}_0(V) = x\left[V(0)\right] \tag{3.5.91}$$

i.e. this limit reproduces the ordinary position field $x(z)$ at the point $V(0)$, with a_0 and a_0^+ related to p and q by the following equation:

$$\begin{aligned} a_0 &= \frac{p}{\sqrt{\varepsilon}} - \frac{1}{2} i\sqrt{\varepsilon} q \\ a_0^+ &= \frac{p}{\sqrt{\varepsilon}} + \frac{1}{2} i\sqrt{\varepsilon} q. \end{aligned} \tag{3.5.92}$$

But we can now go from $V(0)$ to $V(z)$ through a projective transformation, under which $\mathcal{F}_0$ has the simple transformation law (3.5.87).

Hence it is possible to give a redefinition of this field, seen as a conformal field with dimension $\frac{1}{2}\varepsilon$ for any arbitrary ε through the quantity $i\sqrt{\Gamma(\varepsilon)}\mathcal{F}_0(V)$. This turns out to be:

$$x_\varepsilon(z) = i\sum_{m=0}^{\infty}\sqrt{\frac{\Gamma(m+\varepsilon)}{\Gamma(m+1)}}\left[a_m z^{-(m+\varepsilon/2)} - a_m^+ z^{m+\varepsilon/2}\right] z^{-\varepsilon/2} \tag{3.5.93}$$

The vacuum for the Fock space generated by the oscillators a_n and a_n^+ with $n \geq 0$ is defined by:

$$a_m|0>=0 \qquad <0|a_m^+ = 0 \qquad m \geq 0;$$

with respect to this vacuum one can introduce a new ordering that we will denote by ;; which means that the oscillators $a_n^{(i)}$ must be put on the right side of the $a_n^{(i)+}$ with $n \geq 0$. So taking into account this definition one can compute, for example, the 2-point function for the field x_ε, which turns out to be:

$$< x_\varepsilon(z)x_\varepsilon(y) >= \frac{\Gamma(\varepsilon)}{(z-y)^\varepsilon} \qquad |z| > |y| \tag{3.5.94}$$

From (3.5.93) one gets the following expansion for the momentum operator:

$$x'_\varepsilon(z) = -\frac{i}{z}\left\{\varepsilon\sqrt{\Gamma}(\varepsilon)z^{-\varepsilon} + \sum_{m=1}^{\infty}\sqrt{\frac{\Gamma(m+\varepsilon)}{\Gamma(m+1)}}\left[a_m(m+\varepsilon)z^{-m-\varepsilon} + a_m^+ z^m\right]\right\} \tag{3.5.95}$$

In this formalism we can give the following definition of the SDS vertex:

$$U_i(1) =; \exp\left\{-\oint dz x_(1-z) x_i'(z)\right\}; \tag{3.5.96}$$

where the index ε has been omitted and where, as usual, the field $x(z)$ contains auxiliary oscillators.

The integral appearing in (3.5.96) can be computed easily if the limit $\varepsilon \to 0$ is performed for the non zero modes; so one has:

$$\oint dz x(1-z) x_i'(z) = -i\left\{\sqrt{\varepsilon}x(1)a_0 + \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n!}\partial^n x(1-z)|_{z=0} a_n^{(i)}\right\} \tag{3.5.97}$$

Since $\mathcal{F}_0(1-z)$ is connected to the definition of the position field in 1, one may also wonder whether there are connections between $\mathcal{F}_n(V)$ with $n \geq 1$ and the contribution deriving from the non zero modes in (3.5.97). Therefore let us consider:

$$\mathcal{F}_n(V) = \tilde{F}_n^{\varepsilon}(V) - F_n^{\varepsilon}(V) \qquad n \geq 1 \tag{3.5.98}$$

It is easy verify that in the limit $\varepsilon \to 0$ one has:

$$i\mathcal{F}_n = \frac{\sqrt{n}}{n!}\partial^n x(V(z))|_{z=0} \tag{3.5.99}$$

Hence the operators $\mathcal{F}_n$ result to be connected to the SDS vertex (3.5.96) through the relation:

$$U_i(1) =; \exp\left\{-\sum_{n=0}^{\infty} \mathcal{F}_n a_n^{(i)}\right\}; \tag{3.5.100}$$

from which, taking into account the transformation laws of the field operators, one has:

$$U_i(\gamma) \equiv \hat{\gamma} U_i(1) \hat{\gamma}^{-1} =; \exp\left\{-\oint dz x(\gamma(1-z)) x_i'(z)\right\}; \tag{3.5.101}$$

This shows that in the ε-formalism, the SDS vertex transforms under projective transformations as a conformal field of weight zero. Hence the computation of the expectation value of a product of more operators as (3.5.101) can be very easily performed taking into account the following commutation relations:

$$\left[\tilde{F}_m^{\varepsilon}(V_i(z)), F_n^{\varepsilon}(V_j(z))\right] = D_{mn}^{\varepsilon}(\Gamma V_i^{-1} V_j) \tag{3.5.102}$$

where the transposition rule for the matrices D^{ε}_{mn} has been used:

$$D^{\varepsilon}_{mn}(\Lambda) = D^{\varepsilon}_{nm}(\Gamma\Lambda\Gamma^{-1}) \tag{3.5.103}$$

So one has:

$$< 0|\prod_{i=1}^{N} U_i(\gamma)|0 >= \exp\left\{-\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\sum_{n,m=0}^{\infty} a_n^{(i)} D_{nm}(\Gamma V_i^{-1} V j) a_m^{(j)}\right\} \tag{3.5.104}$$

The same result is reproduced by the following operator which does not make use of auxiliary oscillators:

$$V_N =< 0|; \exp\left\{-\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N} \oint dz x_i(V_{ij}(z)) x_j'(z)\right\};. \tag{3.5.105}$$

So we can conclude that:

$$< \prod_{i=1}^{N} \hat{\gamma} U_i \hat{g}^{-1} >= N_\varepsilon < 0|; \exp\left\{-\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N} \oint dz x_i \left(V_{ij}(z) x_j'(z)\right)\right\}. \tag{3.5.106}$$

where

$$N_\varepsilon = (2\pi\varepsilon)^{\frac{DN}{2}}$$

The right hand quantity of eq. (3.5.106) gives the possibility of writing the integrand of the N-string vertex in a very suggestive way.

From (3.5.104) the Lovelace results are recovered by taking into account that the state $|k >_\varepsilon$ with which the N-string vertex must be saturated in this formalism, is defined, when represented in the x-basis, by the function:

$$< x|k >_\varepsilon = \frac{1}{(2\pi)}^{D/2} \exp\left(ikx - \frac{1}{4}\varepsilon x^2\right)$$

that is an eigenstate of $\sqrt{\varepsilon}$ with eigenvalue k.

Chapter 4

N-Reggeon Vertex for the Neveu-Schwarz String

Two N-string vertices for arbitrary open string states – *Reggeons* – are constructed for the Neveu-Schwarz string. One is manifestly invariant only under the projective group, while the other has the full super-projective invariance. When they are saturated with physical states they reproduce the correct physical tree scattering amplitudes.

4.1 Fermionic strings

We want to review briefly here some generalities about the fermionic string theory [33].

Basically the treatment of fermionic strings is a straightforward generalization of the case of the bosonic string. So the classical action for the fermionic string can be derived from (2.1.1) by introducing a world-sheet supersymmetry that relates the string coordinate $x^\mu(\xi)$ to a two-dimensional supersymmetric partner spin-1/2 field $\psi^\mu(\xi)$, and the world-sheet metric $g_{\alpha\beta}$ to the world-sheet gravitino (spin-$\frac{3}{2}$), χ_α. In terms of these fields the classical action, with a flat metric for the background space-time metric, is given by [43]:

$$S = \int d^2\xi\sqrt{g}\frac{1}{2}g^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu + \frac{i}{2}\psi^\mu\gamma_\alpha\nabla_\alpha\psi_\mu + \frac{i}{2}\left(\chi_\alpha\gamma^\beta\gamma^\alpha\psi^\mu\right)\left(\partial_\beta x_\mu - \frac{i}{4}\chi_\beta\psi_\mu\right) \quad (4.1.1)$$

which is obtained from (2.1.1) by requiring invariance under *local supersymmetry transformations*:

$$\begin{aligned}
\delta g_{\alpha\beta} &= i\epsilon\left(\gamma_\alpha\chi_\beta + \gamma_\beta\chi_\alpha\right)\\
\delta\chi_\alpha &= 2\nabla_\alpha\,\epsilon\\
\delta x^\mu &= i\epsilon\psi^\mu\\
\delta\psi^\mu &= \gamma^\alpha\left(\partial_\alpha x^\mu - \frac{i}{2}\chi_\alpha\psi^\mu\right)\epsilon.
\end{aligned} \quad (4.1.2)$$

Here ϵ is an infinitesimal world-sheet spinor, γ_α are the two-dimensional Dirac matrices and ∇ is the covariant derivative.

Furthermore, since in (4.1.1) there is no kinetic term for the gravitino, that like $g_{\alpha\beta}$ is a nondynamical Lagrange multiplier, one has the vanishing of both the two-dimensional energy-momentum tensor and the spinor current, which is just defined by the variational derivative of the action with respect to the gravitino field.

It is important to observe here that one of the most important differences between fields of integer and half odd integer spin is that whereas the former must be single-valued the latter are defined only up to a sign which may not be globally well-defined: this means that a spinor may change its sign under parallel transport along a closed contour. This implies that, in order the action (4.1.1) is well-defined we have to specify which **spin structure** on the world-sheet we are using.

4.1.1 The super conformal gauge

By appropriately choosing local world-sheet coordinates and supersymmetry parameter, one can define a **superconformal gauge**:

$$\begin{aligned}
g_{\alpha\beta} &= \rho(\xi)\eta_{\alpha\beta}\\
\chi_\alpha &= \gamma_a\zeta
\end{aligned} \quad (4.1.3)$$

in which (4.1.1) becomes:

$$S = \int d^2\xi \left[\frac{1}{2}\partial_\alpha x^\mu \partial^\alpha x_\mu - \frac{i}{2}\bar{\psi}\gamma^\alpha\partial_\alpha\psi\right] \tag{4.1.4}$$

Supersymmetry can be made manifest by formulating the theory in a complex **superspace** coordinate, $Z = (z, \theta)$, where θ is odd (Grassmannian) so that $\theta^2 = 0$, the super line element, $dZ = dzd\theta$, and super derivative, $D = \partial_\theta + \theta\partial_z$, satisfying $D^2 = \partial_z$ one may combine the bosonic spin 0 field, x, with the fermionic spin $\frac{1}{2}$ field, ψ, into one **superfield**, $X(Z)$, given by the expansion:

$$X^\mu(z, \bar{z}, \theta, \bar{\theta}) = x^\mu(z, \bar{z}) + \theta\psi^\mu(z, \bar{z}) + \bar{\theta}\bar{\psi}^\mu(z, \bar{z}) + \theta\bar{\theta}F^\mu(z, \bar{z}) \tag{4.1.5}$$

where $F^\mu(z, \bar{z})$ is the so-called *auxiliary field.*

In super space the gauge fixed action reads:

$$S[X] = \frac{1}{2}\int dZd\bar{Z}\bar{D}XDX \tag{4.1.6}$$

where the integration over odd variables is defined by

$$\int d\theta = 0, \qquad \int d\theta\theta = 1 \tag{4.1.7}$$

The solution of the equation of motion implied by the action (4.1.6) is given by:

$$X^\mu = X^\mu(z) + X^\mu(\bar{z}) + \theta\psi^\mu(z) + \bar{\theta}\bar{\psi}^\mu(\bar{z}), \tag{4.1.8}$$

where the fermionic coordinate admits the following expansion:

$$\psi^\mu(z) = -i\sum_r b_r^\mu z^{-r-\frac{1}{2}} \tag{4.1.9}$$

where the index r is half odd integer for the Neveu-Schwarz model (bosonic sector) and integer in the *Ramond* one (fermionic sector).

Eq. 4.1.8 shows that right moving and left moving fields on the world-sheet decouple, so one can consider the two sectors separately exactly as in the bosonic case.

4.1.2 Super conformal algebra

The super conformal conditions given by eq. (4.1.3) can be formulated in super space. The superconformal residual invariance is generated by the super energy-momentum tensor:

$$T(Z) = -\frac{1}{2} DXD^2X(Z) \tag{4.1.10}$$

This can be expanded as:

$$T(Z) = \sum_r G_r z^{-r-\frac{3}{2}} + \theta \sum_n L_n z^{-n-2} \tag{4.1.11}$$

where the summation variable r is half odd (integer) in the Neveu-Schwarz (Ramond) sector.

The expansion (4.1.11) of the energy-momentum tensor defines the generators of the super conformal algebra:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n,-m} \\ \{G_m, G_n\} &= 2L_{m+n} + \frac{1}{3} c\left(m^2 - \frac{1}{4}\right)\delta_{m,-n} \\ [L_m, G_n] &= \left(\frac{1}{2}m - n\right) G_{m+n} \end{aligned} \tag{4.1.12}$$

In the Neveu-Schwarz sector $L_1, L_0, L_1, G_{-1/2}, G_{1/2}$ generate the **super projective** subalgebra ($O\,Sp(1|2)$); we will deal with the elements of this subalgebra in sect. 4.2.

Under *finite* super conformal transformations $Z \to \tilde{Z}(Z)$, one has:

$$\tilde{\Phi}_\Delta(\tilde{Z})(d\tilde{Z})^{2\Delta} = \Phi_\Delta(Z)(dZ)^{2\Delta} \tag{4.1.13}$$

and so:

$$\tilde{\Phi}_\Delta(\tilde{(Z)}) = \Phi_\Delta(Z)(\tilde{D}\theta)^{2\Delta} \tag{4.1.14}$$

where $\tilde{D}\theta$ denotes the *super jacobian* of the transformation.

This tensor, analogously to the bosonic case, can be exploited to derive informations about the transformation properties of a primary super conformal field under a super conformal reparametrization. Indeed a general **primary super conformal field**, $\Phi_\Delta(Z)$, of dimension Δ has the expansion:

$$T(Z)\Phi_\Delta(Z') = \Delta\frac{\theta-\theta'}{(z-z')^2}\Phi_\Delta(Z')+\frac{1/2}{z-z'}D'\Phi_\Delta(Z')+\frac{\theta-\theta'}{z-z'}\partial'\Phi_\Delta(Z) \quad (4.1.15)$$

X is a primary super conformal field of dimension 0.

4.1.3 Physical states, vertex operators and scattering amplitudes in the bosonic sector

The bosonic sector of the fermionic string can be quantized by requiring together with the commutation relations for the bosonic oscillators also the following anti-commutation relations for the fermionic oscillators:

$$\{b^\mu_r, b^\nu_s\} = g^{\mu\nu}\delta_{r+s,0}. \quad (4.1.16)$$

From eq. (4.1.11), considered for the Neveu-Schwarz sector, taking again the annihilation operators to be those multiplying a singular z-dependence at $z=0$ we get immediately the definition of the vacuum for the NS string.

Two different *pictures* can be used in order to define the states of the bosonic sector of the string. These pictures, that can be shown to be equivalent, are called respectively $\mathcal{F}_1$ and $\mathcal{F}_2$.

In the $\mathcal{F}_2$ picture, the on shell physical states are characterized by the following conditions:

$$L_n|phys>= G_r|phys>= \left(L_0-\frac{1}{2}\right)|Phys>=0 \quad (4.1.17)$$

with $n, r > 0$.

There exists a correspondence between the physical states in the picture $\mathcal{F}_1$ and those in the picture $\mathcal{F}_2$, given by the following relation:

$$|\mathrm{phys}(\mathcal{F}_1)>= G_{-1/2}|\mathrm{phys}(\mathcal{F}_2)> . \quad (4.1.18)$$

A physical state $|\text{phys}(\mathcal{F}_1) >$ is not annihilated by $G_{1/2}$. In fact:

$$G_{1/2}|\text{phys}(\mathcal{F}_1) > \quad = G_{1/2}G_{-1/2}|\text{phys}(\mathcal{F}_2) >$$
$$= (2L_0 - G_{-1/2}G_{1/2})|\text{phys}(\mathcal{F}_2) = |\text{phys}(\mathcal{F}_2 > . \qquad (4.1.19)$$

$|\text{phys}(\mathcal{F}_1) >$ is annihilated by G_r with $r \geq 3/2$.

A vertex operator $V_\alpha(z, \theta; k)$ corresponds to the physical state $|\alpha >$; in the $\mathcal{F}_2$ picture the request of superconformal invariance of the theory makes this operator a superconformal primary field of dimension $\Delta = 1/2$. The following conditions are satisfied by the operator V_α:

$$\begin{aligned}
\lim_{z,\theta \to 0} V_\alpha(z, \theta, k)|0 > &= |\alpha, k > \\
\lim_{z,\theta \to 0} < 0|V_\alpha^+(z, \theta, k)|0 > &= < \alpha, k| \\
V_\alpha^+\left(\frac{1}{z}, \theta, -k\right) &= V_\alpha(z, z\theta, k)z \\
[L_n, V_\alpha^x] &= \left[z^{n+1}\partial_z + \frac{1}{2}(n+1)z^n\right] V_\alpha^x \\
\left[L_n, V_\alpha^\psi\right] &= \partial_z \left[z^{n+1} V_\alpha^\psi\right] \\
[\lambda G_r, V_\alpha^x] &= \lambda z^{r+\frac{1}{2}} V_\alpha^\psi \\
\left[\lambda G_r, V_\alpha^\psi\right] &= \lambda \partial_z \left[z^{r+\frac{1}{2}} V_\alpha^x\right]
\end{aligned} \qquad (4.1.20)$$

that generalize to the fermionic string the conditions (2.3.72).

V_α^x and V_α^ψ are the two components of V_α:

$$V_\alpha(z, \theta, k) = V_\alpha^x(z, k) + \theta V_\alpha^\psi(z, k) \qquad (4.1.21)$$

where λ is a constant Grassman parameter.

The vertex associated to the tachyon state is given, similarly to the bosonic string case, by the vertex operator

$$V(z, \theta) =: e^{ikx(z,\theta)} : .$$

A physical state $|\text{phys}(\mathcal{F}_1) >$ is associated to a vertex operator with dimension $\Delta = 1$, as in the bosonic case. This is the reason why the $\mathcal{F}_1$ picture was used to give a manifestly $SL(2,R)$ invariant formula for the tree amplitudes in which cyclic symmetry can be readily established as in the case of the bosonic string theory.

The scattering amplitude for an arbitrary state of the fermionic string is given by the super Koba-Nielsen variables formula:

$$\int \frac{\prod_{i=1}^{N}(dz_i d\theta_i)}{dV_{abc}} < 0| \prod_{i=1}^{N} V_{\alpha_i}(z_i, \theta_i; k_i)|0 > \tag{4.1.22}$$

where dV_{abc} is given by (2.4.82).

4.2 Super-projective transformations

In this section we give some definitions and derive some useful formulas about super-projective transformations, that will be used in constructing a super-projective invariant N-string vertex.

Let us first define a super-projective transformation on the variable (z, φ) in the following way [44]:

$$\begin{aligned} z &\rightarrow \frac{az + b + \alpha\varphi}{cz + d + \beta\varphi} \\ \varphi &\rightarrow \frac{\bar{\alpha}z + \bar{\beta} + \bar{A}\varphi}{cz + d + \beta\varphi} \end{aligned} \tag{4.2.23}$$

with:

$a, b, c, d \in \mathbf{R}$; $G = ad - bc \neq 0$,

α, β Grassmann parameters,

$$\bar{\alpha} = \frac{1}{\sqrt{G}}(a\beta - c\alpha) \quad ; \quad \bar{\beta} = \frac{1}{\sqrt{G}}(b\beta - d\alpha)$$

$$\bar{A} = \frac{1}{\sqrt{G}}\left(G - \tfrac{3}{2}\alpha\beta\right) \quad ; \quad \bar{B} = \frac{1}{\sqrt{G}}\left(G - \tfrac{1}{2}\alpha\beta\right) = \sqrt{G - \alpha\beta} \, .$$

This transformation can be also and nicely represented (a multivalued representation) by the following 3×3 super-matrix[1]:

[1] See for example ref. [45] or [46] for definitions of super-matrices and super-determinants.

$$V^s \equiv \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \bar{\alpha} & \bar{\beta} & \bar{A} \end{pmatrix} \tag{4.2.24}$$

acting on vectors $\xi \equiv (x_1, x_2, \varphi x_2)$, with $z = x_1/x_2$.

Its super-determinant is simply given by:

$$\text{Det}\, V^s = \bar{B} .$$

This is a particular superconformal transformation[2]. In fact (4.2.23) can be also written just as in the general superconformal case [45]:

$$\begin{aligned} z \;\; &\rightarrow \;\; V^z(z,\varphi) = \frac{az+b}{cz+d} + \varphi\, \frac{\bar{\alpha}z+\bar{\beta}}{cz+d}\, \frac{\bar{B}}{cz+d} \;=\; v(z) + \varphi\, \Phi(z) J(z) \\ \varphi \;\; &\rightarrow \;\; V^\varphi(z,\varphi) = \frac{\bar{\alpha}z+\bar{\beta}}{cz+d} + \varphi\, \frac{\bar{B}}{cz+d} \;=\; \Phi(z) + \varphi\, J(z) \end{aligned} \tag{4.2.25}$$

with v, Φ and J satisfying:

$$\partial_z v(z) = (J(z))^2 - \Phi(z)\partial_z \Phi(z).$$

The group of these transformations is the graded extension of the projective (or Möbius) group and in the previous section we have seen that in the Fock space of the string states its generators are the operators $L_{-1}, L_0, L_1, G_{-\frac{1}{2}}$ and $G_{\frac{1}{2}}$.

The super-Jacobian [47] of a superconformal transformation is simply given by:

$$\begin{aligned} sJ\frac{(z',\varphi')}{(z,\varphi)} \;&=\; \left(\frac{\partial\varphi'}{\partial\varphi}\right)^{-2} \left(\frac{\partial z'}{\partial z}\frac{\partial \varphi'}{\partial\varphi} - \frac{\partial\varphi'}{\partial z}\frac{\partial z'}{\partial\varphi}\right) \;= \\ &=\; \left(\frac{\partial}{\partial\varphi} + \varphi\,\frac{\partial}{\partial z}\right) V_\varphi(z,\varphi) \;=\; DV_\varphi(z,\varphi) \end{aligned}$$

hence for the transformations (4.2.25) one has:

$$sJ\frac{(z',\varphi')}{(z,\varphi)} \;=\; \frac{\bar{B}}{cz+d+\beta\varphi} . \tag{4.2.26}$$

[2]As in the general case there are two possibilities in choosing the sign in front of square-roots; they correspond to two disconnected parts of the group.

The behaviour of a primary superfield, $F(z,\varphi)$, under a super projective transformation, according to the general rule for superconformal transformations [48], is the following:

$$F(z,\varphi) \;\rightarrow\; \hat{V}^s\, F\, (\hat{V}^s)^{-1} = \left(\frac{\bar{B}}{cz+d+\beta\varphi}\right)^{2\Delta} F(V^z,V^\varphi) \tag{4.2.27}$$

where Δ is the superconformal weight of F. The expression of the operator $\hat{V}^s$ for an arbitrary super-projective transformation in the Fock space, corresponding to the (4.2.23), is given by[3]:

$$\hat{V}^s \;=\; e^{-(c/a)L_1}\; e^{(\bar{\alpha}/a)G_{1/2}} \left(\frac{a^2}{G-\alpha\beta}\right)^{L_0} e^{-(\alpha/a)G_{-1/2}}\; e^{(b/a)L_{-1}}. \tag{4.2.28}$$

Now we can introduce the supersymmetric extension of the Sciuto-Della Selva-Saito vertex:

$$W_i = W_{0_i} + \theta_i W_{1_i} = :\exp\left\{-\oint dZ\, DX^{(i)}(z,\varphi)\cdot X^{(0)}(1+z-\theta_i\varphi,\theta_i+\varphi)\right\}: \tag{4.2.29}$$

Our expression for the fermionic field $\psi^\mu(z)$ is:

$$\psi^\mu(z) \;=\; -i\sum_{r=1/2}^{\infty}\left(b_r^\mu z^{-r-\frac{1}{2}} + b_r^{\mu\dagger} z^{r-\frac{1}{2}}\right) \quad .$$

The superfield $X^{(0)}$ depends on auxiliary, or interaction oscillators (in the following without any superscript). The superfield $X^{(i)}$ depends on the oscillators acting in the Fock space of the i-th string. θ_i is a parameter of a super-projective transformation acting on the variables (z,φ). More explicitly we can write:

$$\begin{aligned} {}_i\langle x=0,0|W_i \;&=\; {}_i\langle x=0,0| \;:\; \exp\{ip^{(i)}\cdot x(1)\}\times \\ &\exp\left\{i\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n!}\,a_n^{(i)}\cdot\partial_z^n\, x(1+z)|_{z=0} - i\sum_{r=1/2}^{\infty}\frac{1}{(r-1/2)!}\,b_r^{(i)}\cdot\partial_z^{r-\frac{1}{2}}\psi(1+z)|_{z=0}\right\}\times \end{aligned}$$

[3]In the following, writing only $\hat{V}$, we will mean a simple projective transformation, that on the variables yields:

$$z\rightarrow v(z) \qquad \varphi\rightarrow\sqrt{v'(z)}\,\varphi$$

and the super-Jacobian is: $\quad sJ = \sqrt{v'(z)}$.

$$\exp\left\{ \theta_i \left[ip^{(i)} \cdot \psi(1) + i \sum_{n=1}^{\infty} \tfrac{\sqrt{n}}{n!}\, a_n^{(i)} \cdot \partial_z^n\, \psi(1+z)|_{z=0} + \right.\right.$$

$$\left.\left. + i \sum_{r=1/2}^{\infty} \tfrac{1}{(r-1/2)!}\, b_r^{(i)} \cdot \partial_z^{r+\frac{1}{2}} x(1+z)|_{z=0} \right] \right\} :=$$

$$= \;{}_i{<}x=0,0| \;\; : \exp\{\, ip^{(i)} \cdot X(1,\theta_i) \,\} \times$$

$$\exp\left\{ i \sum_{n=1}^{\infty} \tfrac{\sqrt{n}}{n!}\, a_n^{(i)} \cdot \partial_z^n X(1+z.\theta_i)|_{z=0} \right\} \times$$

$$\exp\left\{ -i \sum_{r=1/2}^{\infty} \tfrac{1}{(r-1/2)!}\, b_r^{(i)} \cdot \partial_z^{r-\frac{1}{2}} DX(1+z.\theta_i)|_{z=0} \right\} : \tag{4.2.30}$$

It is easy to realize that the W vertex is constructed in such a way that, applying the operator (4.2.30) to an arbitrary physical state of the i-th string, $|\alpha_i, k_i{>}_i$, we get the vertex operator[4] relative to that state, computed in $(1,\theta_i)$:

$${}_i{<}x=0,0|\, W_i\, |\alpha_i, k_i{>}_i = V_{\alpha_i}(1,\theta_i; k_i) \tag{4.2.31}$$

This operator has superconformal weight $\Delta = \frac{1}{2}$, and we know how it transforms under projective or super-projective transformations; whereas the W vertex is not even a primary superfield. Nevertheless we want to transform it from the point 1 to the point z_i and collect N of them so that we get the general coupling of N arbitrary states of the N.S. string, without any restriction to the physical ones. We have two ways to pursue our task, that we are going to explain in the sections 3 and 4 respectively.

4.3 Projective invariant N-Reggeon vertex

In this section we construct an N-Reggeon vertex using the more familiar projective transformations on the vertex (4.2.29) previously introduced. We keep

[4] See for example the second ref. of [29].

the θ's parameters for the sake of completeness: this choice will be discussed at the end of the section. We will get a manifestly projective invariant vertex. We can perform one of the possible projective transformations such that 1 is mapped into z_i (and θ_i is left only scaled). We choose the following transformation:

$$\gamma_i \equiv \begin{bmatrix} \infty & 1 & 2 \\ z_{i-1} & z_i & z_{i+1} \end{bmatrix} \qquad \gamma_i(1+z) = v_i(z) = \frac{a_i z + b_i}{c_i z + d_i} \tag{4.3.32}$$

and $v_i(z)$ has the usual parametrization:

$$\begin{aligned} a_i &= z_{i-1}(z_i - z_{i+1}) & b_i &= z_i(z_{i+1} - z_{i-1}) \\ c_i &= z_i - z_{i+1} & d_i &= z_{i+1} - z_{i-1} \end{aligned} \tag{4.3.33}$$

The expression for the corresponding operator $\hat{\gamma}_i$ is:

$$\hat{\gamma}_i = e^{-\frac{c_i}{a_i}L_1} \left(\frac{a_i^2}{G_i}\right)^{L_0} e^{\frac{b_i - a_i}{a_i}L_{-1}} .$$

So we introduce the following definition for the N-Reggeon vertex:

$$\begin{aligned} \mathrm{R}_N = & \int \frac{\prod_{i=1}^N dZ_i}{dV_{abc}} \frac{\prod_{i=1}^{N-1} \vartheta(z_i - z_{i+1})}{\prod_{i=1}^N \gamma_i'(1)} \times \\ & \prod_{i=1}^N ({}_i{<}x{=}0,0|) \ \ {<}p{=}0,0| \prod_{i=1}^N \hat{\gamma}_i W_i \hat{\gamma}_i^{-1} |0,p{=}0{>} \end{aligned} \tag{4.3.34}$$

where: $\quad Z_i \equiv (z_i, \theta_i)$,

$$dV_{abc} = \frac{dz_a\, dz_b\, dz_c}{(z_a - z_b)(z_b - z_c)(z_a - z_c)} \qquad \text{and} \qquad \gamma_i'(1) = \frac{(z_i - z_{i+1})(z_{i-1} - z_i)}{z_{i+1} - z_{i-1}}.$$

This operator, applied to N physical states, yields the tree-amplitude in the super Koba-Nielsen form, as it can be immediately shown making use of formulas (4.2.27) and (4.2.31).

It is very instructive to work out from the expression (4.3.34) a form without the auxiliary oscillators. For the actual computation of the projective transformation on W_i it turns out useful to represent the orbital part of W_i by the unitary irreducible representation of the projective group with spin $J = -\frac{\varepsilon}{2}$ (with $\varepsilon \to 0$

after having performed the whole computation). The use of this representation, called for brevity *ε-representation*, involves a reordering with respect to the *zero-modes* because the oscillators a_0 and $a_0^\dagger$ are needed instead of q and p. As regards the spin part it can be represented by the *UIR* of spin $J = -\frac{1}{2}$.

In this way we get, apart from c-factors, an operator that behaves exactly like a primary field of superconformal weight $\Delta = 0$. We want to give here its expression written in the following concise form:

$$; \exp\left\{-\oint dZ\, DX_\varepsilon^{(i)}(z,\varphi)\cdot X_\varepsilon(1+z-\theta_i\varphi,\theta_i+\varphi)\right\} ;$$

The subscript ε in the fields of this covariant vertex refers to the field x_ε that has conformal weight $\Delta = \frac{\varepsilon}{2}$ defined in the previous chapter.

Hence the projective transformation γ_i on the vertex W_i can be easily computed and also the vacuum expectation value of formula (4.3.34). We get the following expression:

$$\begin{aligned}
\mathrm{R}_N \;=\; & \int \frac{\prod_{i=1}^N dZ_i}{dV_{abc}} \frac{\prod_{i=1}^{N-1}\vartheta(z_i - z_{i+1})}{\prod_{i=1}^N \gamma_i'(1)} \prod_{i=1}^N ({}_i\langle x=0,0|)\times \\
& \exp\left\{-\sum_{\substack{i,j=1\\ i\neq j}}^N \frac{1}{2}\left[\sum_{\substack{n=0\\ m=0}}^\infty a_m^{(i)} M_{mn}(\Gamma V_i^{-1}V_j)\, a_n^{(j)} + i\sum_{\substack{r=1/2\\ s=1/2}}^\infty b_r^{(i)} D_{rs}^{1/2}(\Gamma V_i^{-1}V_j)\, b_s^{(j)}\right]\right\}\times \\
& \exp\left\{-\sum_{\substack{i,j=1\\ i<j}}^N \left[\theta_i \left(\sum_{\substack{r=1/2\\ n=0}}^\infty \sqrt{r+1/2}\, b_r^{(i)} M_{r+\frac{1}{2},n}(\Gamma V_i^{-1}V_j)\, a_n^{(j)} + \right.\right.\right. \\
& \qquad\qquad \left. -i\sum_{\substack{m=0\\ s=1/2}}^\infty \sqrt{m}\, a_m^{(i)} D_{m+\frac{1}{2},s}^{1/2}(\Gamma V_i^{-1}V_j)\, b_s^{(j)}\right) + \\
& \quad + \theta_j\left(\sum_{\substack{m=0\\ s=1/2}}^\infty a_m^{(i)} M_{m,s+\frac{1}{2}}(\Gamma V_i^{-1}V_j)\, b_s^{(j)}\sqrt{s+1/2} + \right. \\
& \qquad\qquad \left. +i\sum_{\substack{r=1/2\\ n=0}}^\infty b_r^{(i)} D_{r,n+\frac{1}{2}}^{1/2}(\Gamma V_i^{-1}V_j)\, a_n^{(j)}\sqrt{n}\right) +
\end{aligned}$$

$$- \theta_i\theta_j \left(\sum_{\substack{r=1/2 \\ s=1/2}}^{\infty} \sqrt{r+1/2}\, b_r^{(i)} M_{r+\frac{1}{2},s+\frac{1}{2}}(\Gamma V_i^{-1} V_j)\, b_s^{(j)} \sqrt{s+1/2} + \right.$$

$$\left. -i \sum_{\substack{n=0 \\ m=0}}^{\infty} \sqrt{m}\, a_m^{(i)} D^{1/2}_{m+\frac{1}{2},n+\frac{1}{2}}(\Gamma V_i^{-1} V_j)\, a_n^{(j)} \sqrt{n} \right) \Bigg] \Bigg\} \times$$

$$\times\, \delta^D \left(\textstyle\sum_{i=1}^N p^{(i)} \right) \qquad (4.3.35)$$

where $a_0^{(i)} = p^{(i)}$, and $\sqrt{0}$ has to be understood as 1.

The infinite matrix $\{M_{mn}\}$ is related to the true $(J=-\varepsilon/2)$-*representation* element $\{D^\varepsilon_{mn}\}$; that is:

$$n,m \geq 1 \qquad D^\varepsilon_{mn}(V) = \tfrac{1}{n!}\sqrt{\tfrac{n}{m}}\, \partial^n \left[\left(\frac{az+b}{cz+d} \right)^m \right]_{z=0} = M_{mn}(V)$$

$$D^\varepsilon_{m0}(V) = \sqrt{\tfrac{\varepsilon}{m}} \left(\frac{b}{d} \right)^m = \sqrt{\varepsilon}\, M_{m0}(V)$$

$$m \geq 1$$

$$D^\varepsilon_{0m}(V) = \sqrt{\tfrac{\varepsilon}{m}} \left(-\frac{c}{d} \right)^m B = \sqrt{\varepsilon}\, M_{0m}(V)$$

$$D^\varepsilon_{00}(V) = 1 - \varepsilon \log \left(\frac{d}{\sqrt{G}} \right) = 1 + \varepsilon\, M_{00}(V)$$

The matrix $\{D^{1/2}_{rs}\}$ is the $(J=-1/2)$-*representation* element, given by[5]:

$$D^{1/2}_{rs}(V) = \tfrac{1}{(s-1/2)!}\, \partial^{s-\frac{1}{2}} \left[\frac{\sqrt{G}}{cz+d} \left(\frac{az+b}{cz+d} \right)^{r-\frac{1}{2}} \right]_{z=0} .$$

It is worth to point out that the latter matrix has a transposition rule depending on the sign of G:

$$[D^{1/2}(V)]^T = \frac{G}{|G|}\, D^{1/2}(\Gamma V^{-1} \Gamma).$$

By V we have always meant the real 2×2 matrix:

[5]The following is a two-valued representation. The previous one also is two-valued because of normalization factors.

$$V \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

without restrictions on the determinant, and by Γ the matrix:

$$\Gamma \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The coefficients of the V_i and V_j matrices are given by (4.3.33); the product $\Gamma V_i^{-1} V_j$ means the composition of the two projective transformations:

$$U_i = \Gamma V_i^{-1} \leftrightarrow \begin{bmatrix} z_{i-1} & z_i & z_{i+1} \\ 0 & \infty & 1 \end{bmatrix} \qquad \text{and} \qquad V_j \leftrightarrow \begin{bmatrix} \infty & 0 & 1 \\ z_{j-1} & z_j & z_{j+1} \end{bmatrix} .$$

Therefore, like the bosonic N-Reggeon vertex, we can conclude that this vertex is projective invariant *at sight.* In fact a projective transformation on the variables Z_i of external legs:

$$z_i \;\longrightarrow\; \Lambda(z_i)$$

$$\theta_i \;\longrightarrow\; \sqrt{\Lambda'(z_i)}\,\theta_i$$

corresponds to matrix products ΛV_i and ΛV_j in the argument of the infinite matrices $\{D\}$, that consequently remain unchanged. The transformation on the θ's is compensated by a factor:

$$\left[\prod_{i=1}^{N} \sqrt{\Lambda'(z_i)}\right]^{-1}$$

that comes out from the integration measure in (4.3.35).

However the expression (4.3.35) is not super-projective invariant and we can't fix to *zero* two of the θ's. We may indeed integrate from the beginning over the θ's but this would be paid with a certain complication in computing the vacuum expectation value in (4.3.34). Alternatively a simplification can be obtained putting *all* the θ's to *zero* and using the W_{0_i} of formula (4.2.29) in the expression (4.3.34). In the subsequent expression (4.3.35) we are left with the first two lines only. This means, on the other hand, that we have to work in the old $\mathcal{F}_1$ Fock space [49]$_a$, in which the tachyon state is given by:

$$k_i \cdot b^{\dagger(i)}_{\frac{1}{2}} |0, k_i>_i \tag{4.3.36}$$

and all other physical states are related to the usual ones (that live in the $\mathcal{F}_2$ Fock space [49]$_b$) by:

$$|\,\mathrm{phys}\,(\mathcal{F}_1)> \;\; = G_{-\frac{1}{2}}\,|\,\mathrm{phys}\,(\mathcal{F}_2)> .$$

We have applied the $\mathrm{R}_N^{(\mathcal{F}_1)}$ vertex to N tachyon-states like (4.3.36); the N-pion amplitude [49]$_b$ is easily derived:

$$A(N\text{tach.})\begin{cases} = \mathrm{R}_N^{(\mathcal{F}_1)} \prod_{i=1}^N \left(k_i \cdot b^{\dagger(i)}_{\frac{1}{2}}|0,k_i>_i\right) = \\ = \int \frac{\prod_{i=1}^N dz_i}{dV_{abc}} \prod_{i=1}^{N-1} \vartheta(z_i - z_{i+1})\, \delta^D\left(\sum_{i=1}^N k_i\right) (-1)^{\frac{N}{2}} \times \\ \prod_{\substack{i,j=1 \\ i<j}}^N (z_i - z_j)^{k_i \cdot k_j} \sum_{\{i_1,\dots,i_N\}} (-1)^{\wp} \frac{k_{i_1}\cdot k_{i_2}}{z_{i_1}-z_{i_2}} \cdots \frac{k_{i_{N-1}}\cdot k_{i_N}}{z_{i_{N-1}}-z_{i_N}} & \text{N even} \\ = 0 & \text{N odd} \end{cases}$$

where the summation runs over permutations with $i_1 < i_2, \dots, i_{N-1} < i_N$; and $\wp$ is the parity of the permutation.

The 3-Reggeon vertex can be also easily derived:

$$\begin{aligned} \mathrm{R}_3^{(\mathcal{F}_1)} \;=\; & \prod_{i=1}^3 \left({}_i\!<x=0,0|\right) \delta^D\left(\sum_{i=1}^3 p^{(i)}\right) \times \\ & \exp\left\{ -\sum_{i=1}^3 \left[\sum_{n=1}^\infty \frac{1}{\sqrt{n}}\, a_n^{(i)} \cdot p^{(i+1)} + \sum_{\substack{n=1\\m=1}}^\infty (-1)^m \sqrt{\frac{m}{n}} \begin{pmatrix} n \\ m \end{pmatrix} a_n^{(i)} \cdot a_m^{(i+1)} + \right.\right. \\ & \left.\left. + \sum_{\substack{r=1/2\\s=1/2}}^\infty (-1)^{s-\frac{1}{2}} \begin{pmatrix} r-1/2 \\ s-1/2 \end{pmatrix} b_r^{(i)} \cdot b_s^{(i+1)} \right] \right\}. \end{aligned}$$

Before passing to the discussion of a super-projective invariant version of the N-Reggeon vertex we want to remark that the $R_N^{(\mathcal{F}_1)}$ vertex can be also rewritten in the following way (in the ε-representation):

$$
\begin{aligned}
\mathrm{R}_N^{(\mathcal{F}_1)} &= \int \frac{\prod_{i=1}^{N} dz_i}{dV_{abc}} \frac{\prod_{i=1}^{N-1} \vartheta(z_i - z_{i+1})}{\prod_{i=1}^{N} \gamma_i'(1)} \prod_{i=1}^{N} ({}_i\!< a_0^\dagger = 0, 0|) \times \\
&; \exp\left\{ -\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \oint dZ\, X_{V_{ij}}^{(i)} \left(v_{ij}(z), \sqrt{v_{ij}'(z)}\, \varphi \right) \cdot DX_\varepsilon^{(j)}(z,\varphi) \right\} ; \times \\
&(2\pi\varepsilon)^{DN/2}\ \delta^D \left(\textstyle\sum_{i=1}^{N} p^{(i)} \right)
\end{aligned}
\qquad (4.3.37)
$$

where different vacua are used and also different $|0, k_i >_\varepsilon$ states. The X_ε fields in (4.3.37) have superconformal weight $\Delta = \frac{\varepsilon}{2}$, so that in the case of the transformation $V_i^{-1}V_j$ we have:

$$
X_{V_{ij}}^{(i)} = X_\varepsilon^{(i)} \left(v_{ij}(z), \sqrt{v_{ij}'(z)}\, \varphi \right) \left(\sqrt{v_{ij}'(z)} \right)^{\varepsilon}
$$

where $$v_{ij}(z) = \frac{a_{ij} z + b_{ij}}{c_{ij} z + d_{ij}},$$

and a_{ij}, b_{ij}, c_{ij} and d_{ij} are elements of the matrix $V_i^{-1}V_j$.

4.4 Super-projective invariant N-Reggeon vertex

As we already pointed out all the previous expressions for the N-Reggeon vertex are merely projective invariant. We can construct an N-Reggeon vertex in an alternative and suggestive way, by transforming with a *super*-projective transformation the W_{0_i} vertex from the point $(1,0)$ to the point (z_i, θ_i) , and then collecting N such vertices as in (4.3.34); eventually we can also eliminate the auxiliary oscillators.

Let us first proceed looking directly for a form without auxiliary oscillators like (4.3.37), i.e. a *manifestly* super-projective invariant expression. It should contain, in the exponent, the field:

$$X^{(i)}_{V^S_{ij}} = X^{(i)}_{\varepsilon}(V^z_{ij}(z,\varphi), V^{\varphi}_{ij}(z,\varphi)) \left[\frac{\bar{B}_{ij}}{c_{ij}z + d_{ij} + \beta_{ij}\varphi}\right]^{\varepsilon}. \qquad (4.4.38)$$

V^s_{ij} is a super-projective transformation given by the product $(V^s_i)^{-1}V^s_j$—that can be easily computed making use of super-matrices (4.2.24). $(V^s_i)^{-1}$ corresponds to the transformation:

$$\begin{aligned} z &\rightarrow \frac{d_i z - b_i + \bar{\beta}_i\varphi}{-c_i z + a_i - \bar{\alpha}_i\varphi} \\ \varphi &\rightarrow \frac{-\beta_i z + \alpha_i + \bar{A}_i\varphi}{-c_i z + a_i - \bar{\alpha}_i\varphi} \end{aligned}$$

while V^s_j corresponds to (4.2.23). The coefficients a, b, c and d are given by (4.3.33). α and β take the following values:

$$\alpha_i = \sqrt{y_i}\,\frac{z_i\theta_{i-1} - z_{i-1}\theta_i}{z_{i-1} - z_i} \qquad \beta_i = \sqrt{y_i}\,\frac{\theta_{i-1} - \theta_i}{z_{i-1} - z_i} \qquad (4.4.39)$$

with $y_i = (z_{i-1} - z_i)(z_i - z_{i+1})(z_{i+1} - z_{i-1})$.

We will also use for the transformation V^s_{ij} a shortened notation as in (4.2.25):

$$\begin{aligned} z &\rightarrow V^z_{ij}(Z) = v_{ij}(z) + \varphi\,\Phi_{ij}(z)J_{ij}(z) \\ \varphi &\rightarrow V^{\varphi}_{ij}(Z) = \Phi_{ij}(z) + \varphi\,J_{ij}(z) \end{aligned} \qquad (4.4.40)$$

and in Lovelace extended notation:

$$V^s_{ij} \leftrightarrow \left[\begin{array}{ccc} (z_{i-1},\theta_{i-1}) & (z_i,\theta_i) & (z_{i+1},\Theta_i) \\ (\infty,0) & (0,0) & (1,0) \end{array}\right] \circ \left[\begin{array}{ccc} (\infty,0) & (0,0) & (1,0) \\ (z_{j-1},\theta_{j-1}) & (z_j,\theta_j) & (z_{j+1},\Theta_j) \end{array}\right];$$

Θ_j is obviously a combination of θ_{j-1} and θ_j.

It is rather easy to convince oneself that a super-projective transformation, Λ^s, acting on all external legs, implies:

$$V^s_i \rightarrow \Lambda^s\,V^s_i$$

so that the superfield (4.4.38) is left invariant. This fact assures that the operator part of the N-Reggeon vertex is super-projective invariant.

We also introduce the following new integration measure:

$$d\mu_N^s = \frac{\prod_{i=1}^N dZ_i}{dV_{abc}^s} \; \frac{\prod_{i=1}^{N-1} \vartheta(z_i - z_{i+1})}{\prod_{i=1}^N \left[sJ \frac{(V_i^z, V_i^\varphi)}{(z,\varphi)} \right]_{\substack{z=0 \\ \varphi=0}}} \tag{4.4.41}$$

where we use the super-projective invariant expression for dV_{abc}^s [50]:

$$dV_{abc}^s = \frac{dZ_a \, dz_b \, dZ_c}{[(Z_a - Z_b)(Z_b - Z_c)(z_a - z_b)(z_b - z_c)]^{1/2}} \; .$$

The new integration measure (4.4.41) is super-projective invariant by itself.

Finally a manifestly super-projective N-Reggeon vertex for the N.S. string is given by:

$$\begin{aligned} \mathrm{R}_N^s &= \int d\mu_N^s \prod_{i=1}^N ({}_i\!<\! a_0^\dagger \!=\! 0, 0|) \times \\ &\; ; \exp\left\{ -\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \oint dZ \, X_{V_{ij}^S}^{(i)} \left(V_{ij}^z(Z), V_{ij}^\varphi(Z) \right) \cdot DX_\varepsilon^{(j)}(z,\varphi) \right\} \; ; \times \\ &\quad (2\pi\varepsilon)^{DN/2} \; \delta^D \left(\textstyle\sum_{i=1}^N p^{(i)} \right) \; . \end{aligned} \tag{4.4.42}$$

Translating the different conventions on states of expression (4.4.42) in the usual ones, and letting ε go to zero, we obtain the following N-Reggeon vertex in oscillator form:

$$\begin{aligned} \mathrm{R}_N^s &= \int d\mu_N^s \prod_{i=1}^N ({}_i\!<\! x \!=\! 0, 0|) \; \exp\Bigg\{ -\sum_{\substack{i,j=1 \\ i<j}}^N \Bigg\{ p^{(i)} \log \left(\frac{DV_{ij}^\varphi}{V_{ij}^z} \right) \Bigg|_{\substack{z=0 \\ \varphi=0}} p^{(j)} + \\ &+ \sum_{n=1}^\infty a_n^{(i)} \tfrac{1}{\sqrt{n}} \left(\frac{1}{V_{ij}^z} \right)^n \Bigg|_{\substack{z=0 \\ \varphi=0}} p^{(j)} + \sum_{n=1}^\infty p^{(i)} \tfrac{\sqrt{n}}{n!} \, \partial^n \left[\log \left(\frac{DV_{ij}^\varphi}{V_{ij}^z} \right) \right]_{\substack{z=0 \\ \varphi=0}} a_n^{(j)} + \\ &\qquad + \sum_{\substack{n=1 \\ m=1}}^\infty a_m^{(i)} \tfrac{1}{n!} \sqrt{\tfrac{n}{m}} \, \partial^n \left[\left(\frac{1}{V_{ij}^z} \right)^m \right]_{\substack{z=0 \\ \varphi=0}} a_n^{(j)} + \\ &- \sum_{\substack{r=1/2 \\ s=1/2}}^\infty b_r^{(i)} \tfrac{1}{(s-1/2)!} \, \partial^{s-\frac{1}{2}} \left[\frac{DV_{ij}^\varphi}{V_{ij}^z} \left(\frac{1}{V_{ij}^z} \right)^{r-\frac{1}{2}} \right]_{\substack{z=0 \\ \varphi=0}} b_s^{(j)} + \end{aligned}$$

$$+\sum_{r=1/2}^{\infty} b_r^{(i)} \left[\frac{V_{ij}^{\varphi}}{V_{ij}^{z}} \left(\frac{1}{V_{ij}^{z}}\right)^{r-\frac{1}{2}}\right]_{\substack{z=0\\ \varphi=0}} p^{(j)} + \sum_{\substack{r=1/2\\ n=1}}^{\infty} b_r^{(i)} \frac{\sqrt{n}}{n!} \partial^{n} \left[\frac{\text{[illegible]}}{V_{ij}} \left(\frac{1}{V_{ij}^{z}}\right)^{r-\frac{1}{2}}\right]_{\substack{z=0\\ \varphi=0}} a_n^{(j)} +$$

$$-\sum_{s=1/2}^{\infty} p^{(i)} \frac{1}{(s-1/2)!} \partial^{s-\frac{1}{2}} \left[\frac{-\alpha_{ij}}{a_{ij}z + b_{ij}}\right]_{z=0} b_s^{(j)} +$$

$$\left.\left. -\sum_{\substack{s=1/2\\ n=1}}^{\infty} a_n^{(i)} \frac{\sqrt{n}}{(s-1/2)!} \partial^{s-\frac{1}{2}} \left[\frac{DV_{ij}^{z}}{V_{ij}^{z}} \left(\frac{1}{V_{ij}^{z}}\right)^{n}\right]_{\substack{z=0\\ \varphi=0}} b_s^{(j)} \right\}\right\} \times$$

$$\times\, \delta^D \left(\sum_{i=1}^{N} p^{(i)}\right) . \tag{4.4.43}$$

The previous expression can be written in the following compact form:

$$\mathrm{R}_N^s = \int d\mu_N^s \prod_{i=1}^{N} [{}_i< x = 0, 0|] \delta \left(\sum_{i=1}^{N} p_i\right)$$

$$\prod_{i<j} \exp\left[-\sum_{n,m=0}^{\infty} \mathcal{D}_{nm}(V_{ij}) A_n^{(i)} A_m^{(j)}\right] \tag{4.4.44}$$

with $V_{ij} = V_i^{-1} V_j$.

The indices n and m run over all non negative integer and half integer numbers and $A_n^{(i)} = a_n^{(i)}$ or $b_n^{(i)}$ depending if n is integer or half integer. The matrix $\mathcal{D}_{nm}$ is given by:

$$\begin{aligned}
\mathcal{D}_{nm} &= \frac{1}{n[m]} D^{2m} \left[\left(-n\frac{V^{\theta}(z,\theta)}{V^{z}(z,\theta)}\right)^{2(n-[n])} [V^{z}(z,\theta)]^{-[n]}\right]\Bigg|_{z=\theta=0} \qquad n \geq \frac{1}{2},\ m \geq 0 \\
\mathcal{D}_{00} &= \log\left|\frac{DV^{\theta}(0,0)}{V^{z}(0,0)}\right| \\
\mathcal{D}_{0m} &= \frac{1}{[m]!} D^{2m-1} \left[\frac{V^{z}(z,\theta)}{DV^{\theta}(z,\theta)} \partial_z \left(\frac{V^{\theta}(z,\theta)}{V^{z}(z,\theta)}\right)\right]\Bigg|
\end{aligned} \tag{4.4.45}$$

where $[m]$ is equal to m if m is integer and $(m - \frac{1}{2})$ if m is half-integer.

As in the bosonic string the form of the vertex (4.4.45) is independent from the particular choice of the superprojective transformations $V_i(Z)$, that generalize the transformations $V_i(z)$ of the bosonic string.

The expression (4.4.43) reproduces the scattering amplitude of N tachyon-states as it can be easily checked by using the following formulas:

$$\frac{\bar{B}_i}{d_i} = \sqrt{\frac{z_i - z_{i+1}}{z_{i+1} - z_{i-1}}\,(Z_{i-1} - Z_i)}$$
$$\frac{b_{ij}}{\bar{B}_{ij}} = (Z_i - Z_j)\,\frac{z_{i+1} - z_{i-1}}{\bar{B}_i}\,\frac{z_{j+1} - z_{j-1}}{\bar{B}_j}\ .$$

We immediately get:

$$\mathrm{R}^s_N \prod_{i=1}^{N} (|0, k_i>_i) = \int \frac{\prod_{i=1}^N dZ_i}{dV^s_{abc}}\ \prod_{i=1}^{N-1} \vartheta(z_i - z_{i+1}) \times$$
$$\prod_{i=1}^{N} \frac{d_i}{\bar{B}_i}\ \exp\left[-\sum_{\substack{i,j=1\\ i<j}}^{N} \log\left(\frac{\bar{B}_{ij}}{b_{ij}}\right) k_i \cdot k_j\right]\ \delta^D\left(\sum_{i=1}^N k_i\right) =$$
$$= \int \frac{\prod_{i=1}^N dZ_i}{dV^s_{abc}}\ \prod_{i=1}^{N-1} \vartheta(z_i - z_{i+1}) \prod_{\substack{i,j=1\\ i<j}}^{N} (Z_i - Z_j)^{k_i\cdot k_j}\ \delta^D\left(\sum_{i=1}^N k_i\right)\ .$$

If we restrict ourselves to the case $N = 3$ we reproduce the 3-Reggeon vertex constructed in ref. [27]. In fact in this case we are left with only one θ and it is quite easy to compute the actual coefficients of the composed transformation (4.4.40) and consequently all the terms appearing in the formula (4.4.43).

In conclusion we get:

$$\mathrm{R}^s_3 = \int d\theta \prod_{i=1}^{3} ({}_i\!<x{=}0, 0|)\ \exp\left\{-\sum_{i=1}^{3}\left[\sum_{n=1}^{\infty} \tfrac{1}{\sqrt{n}}\, a_n^{(i)} \cdot p^{(i+1)} + \right.\right.$$
$$\left.\left. + \sum_{\substack{n=1\\ m=1}}^{\infty} (-1)^n \sqrt{\tfrac{n}{m}} \begin{pmatrix} m \\ n \end{pmatrix} a_m^{(i)} \cdot a_n^{(i+1)} + \sum_{\substack{r=1/2\\ s=1/2}}^{\infty} (-1)^{s-\frac{1}{2}} \begin{pmatrix} r-1/2 \\ s-1/2 \end{pmatrix} b_r^{(i)} \cdot b_s^{(i+1)}\right]\right\} \times$$
$$\left\{1 + \theta \sum_{i=1}^{3}\left[\sum_{r=1/2}^{\infty} b_r^{(i)} \cdot p^{(i+1)} + \sum_{\substack{r=1/2\\ n=1}}^{\infty} (-1)^n \sqrt{n} \begin{pmatrix} r-1/2 \\ n \end{pmatrix} b^{(i)} \cdot a_n^{(i+1)} + \right.\right.$$
$$\left.\left. + \sum_{\substack{r=1/2\\ n=1}}^{\infty} (-1)^{r-\frac{1}{2}}\ \tfrac{r+1/2}{\sqrt{n}} \begin{pmatrix} n \\ r+1/2 \end{pmatrix} a_n^{(i)} \cdot b_r^{(i+1)}\right]\right\}\ \delta^D\left(\sum_{i=1}^3 p^{(i)}\right)\ .$$

As we said in the beginning of this section it is possible to show, after fairly long calculations, that the vertex (4.4.43) is equal to:

$$\begin{aligned} \mathrm{R}^s_N &= \int d\mu^s_N \prod_{i=1}^{N} ({}_i{<}x{=}0,0|) \times \\ & <p{=}0,0| \prod_{i=1}^{N} \hat{\gamma}^s_i \, W_{0_i} \, (\hat{\gamma}^s_i)^{-1} \, |0,p{=}0> \; . \end{aligned} \qquad (4.4.46)$$

In this expression the super-projective transformation γ^s_i is the generalization of (4.3.32):

$$\gamma^s_i \equiv \begin{bmatrix} (\infty,0) & (1,0) & (2,0) \\ (z_{i-1},\theta_{i-1}) & (z_i,\theta_i) & (z_{i+1},\Theta_i) \end{bmatrix} \qquad \begin{array}{l} \gamma^z_i(1+z,\varphi) = V^z_i(z,\varphi) \\ \gamma^\varphi_i(1+z,\varphi) = V^\varphi_i(z,\varphi) \end{array}$$

with V^z_i and V^φ_i given by (4.2.23) or (4.2.25) and the coefficients given by (4.3.33) and (4.4.39).

The operator $\hat{\gamma}^s_i$ is like (4.2.28) but with the following parametrization:

$$\hat{\gamma}^s_i = e^{-\frac{c_i}{a_i}L_1} \; e^{\frac{\bar{\alpha}_i}{a_i}G_{1/2}} \; \left(\frac{a_i^2}{G_i - \alpha_i\beta_i}\right)^{L_0} \; e^{-\frac{\alpha_i}{a_i}G_{-1/2}} \; e^{\frac{b_i - a_i}{a_i}L_{-1}} .$$

Still, after use of (4.2.27) and (4.2.31), one can immediately check that the vertex (4.4.46), applied to N arbitrary physical states, yields the general tree-amplitude.

The N-Reggeon vertex we have constructed in this section is the best candidate for the introduction of the ghost fields: the structure of the terms in formula (4.4.43) should suggest the way how to treat those; for this purpose we believe that it is very useful to introduce unitary representations of the super-projective group, as well as that the calculations needed in going from (4.4.46) to (4.4.43) may be drastically simplified by using their properties as in the case of previous section.

Chapter 5

g-Loop Vertices for Free Fermions and Bosons

This chapter is devoted to the possibility of applying the formalism of the N-string vertex to any conformal field theory. The starting point in this case is the N-point vertex, which has now the property of reproducing the N-point correlation functions involving the primary fields of the theory when it is saturated with the corresponding N highest weight states. By means of the sewing procedure one can then construct the N-point g-loop vertex, that gives the correlation functions of primary fields on a genus g Riemann surface when it is saturated with the corresponding primary fields. This program of working is developed for free fermionic theories.

The g-loop vertex for free fermions is expressed in terms of the Szegö kernel and we study its connections with the g-vacuum, that is the starting point for the computation of multiloop amplitudes in a recently developed operator formalism on arbitrary Riemann surfaces. [11]

The same construction for free bosons allows to check bosonization of the free fermionic theory on an arbitrary Riemann surface.

5.1 g-loop vertex for free fermions

The starting object is the vertex with N external legs for free fermions on the sphere [20]:

$$V_{N,0}^{F} = \prod_{i=1}^{N} {}_i < 0| \exp\left\{ -\frac{1}{2} \sum_{r,s=\frac{1}{2}}^{\infty} \sum_{\substack{i,j=1 \\ i\neq j}}^{N} b_r^{(i)} D_{rs}^{(1/2)}(U_i V_j) b_s^{(j)} \right\} \tag{5.1.1}$$

where $U_i \equiv \Gamma V_i^{-1}$ and

$$D_{rs}^{(1/2)}(V(z)) = \frac{1}{(s-1/2)!} \partial^{s-1/2} \left[V'(z)^{\frac{1}{2}} \left(V(z)\right)^{r-\frac{1}{2}} \right] \Big|_{z=0} \tag{5.1.2}$$

is an infinite dimensional representation of the projective group corresponding to a conformal weight $\Delta = \frac{1}{2}$ and $\Gamma(z) = 1/z$. $V_i^{-1}(z)$'s are projective transformations corresponding to a choice of local coordinates vanishing at the Koba-Nielsen points z_i.

Although in the past the choice of $V_i^{-1}(z)$ found by Lovelace [41] has been widely used, our results do not depend on the particular choice for $V_i(z)$.

The vertex (5.1.1) reproduces the correlation functions of the free fermionic theory. Indeed if we want for instance to compute the fermion propagator on the sphere we have to saturate $V_{2,0}^{F}$ with the fermion states obtained by the relation:

$$\lim_{z\to z_i} \left[V_i^{-1}(z)' \right]^{\frac{1}{2}} \psi \left[V_i^{-1}(z) \right] |0> = \frac{1}{[V_i'(0)]^{1/2}} b_{-\frac{1}{2}}^{(i)} |0> \tag{5.1.3}$$

corresponding to the local coordinate $V_i^{-1}(z)$ vanishing at the point $z = z_i$.

Here the following expression for the fermionic field is assumed:

$$\psi(z) = \sum_{r=1/2}^{\infty} \left(b_r z^{-r-\frac{1}{2}} + b_r^{+} z^{r-\frac{1}{2}} \right).$$

By saturating (5.1.1) with two states as in (5.1.3) we get the Green's function for free fermions on the sphere:

$$< \psi(z)\psi(y) >= \frac{1}{z-y} \tag{5.1.4}$$

In the same way one can obtain from (5.1.1) and (5.1.3) an arbitrary correlation function of the free fermionic theory on the sphere.

5.1.1 Sewing procedure

The vertex $V^F_{N,g}$ is obtained by sewing together $2g$ legs of $V^F_{N+2g,0}$ after the insertion of the following twisted propagator [31]:

$$P(x_\mu) = (-1)^{(1-B_\mu)F_\mu} x_\mu^{L_0} \Omega (1-x_\mu)^W \tag{5.1.5}$$

on the μth leg, that is sewn to the next one. The twist operator Ω and W are given by:

$$\Omega = e^{L_{-1}}(-1)^{L_0 - \frac{p^2}{2}} \quad W = L_0 - L_1. \tag{5.1.6}$$

Even if we use for convenience the twisted propagator we believe that our results are largely independent from the specific choice of the propagator as it will appear from subsequent calculations.

Furthermore B_μ in (5.1.5) can be regarded as the μth component of a g-dimensional vector, which can take the values 0 or 1, corresponding to the possible insertion in $P(x_\mu)$ of the fermion parity operator $(-1)^F$. The quantities B_μ's specify the boundary conditions fixed for the fermionic fields around the g homology cycles commonly denoted by b of a compact Riemann surface of genus g: hence the vector B, so defined, takes into account of the spin structures.

Let us examine in some detail how the sewing procedure works.

The starting point is the vertex $V^F_{N+2g,0}$ on the sphere defined by (5.1.1). We will use the following conventions about the indices:

$$i, j = 1, \ldots, N \qquad \mu, \nu = 1, \ldots, g \tag{5.1.7}$$

and we will sew the leg $N+2\mu-1$, that will be labelled only by $2\mu-1$ with the leg $N+2\mu$, that will be labelled only by 2μ.

The first step is joining the propagator (5.1.5) to the each of the legs labelled by $2\mu-1$ of the vertex $V_{N+2g,0}$, according the meaning given to this operation in sect. 3.4.2 and to connect it to the leg 2μ. Oscillators of the latter must be inverted according to the following rules:

$$b_r^{(2\mu)} \longrightarrow -b_r^{+(2\mu)} \tag{5.1.8}$$

Furthermore the vacuum of the Fock space of the string labelled by 2μ must be transformed from left vacuum to right vacuum, i.e.:

$$_{2\mu}<0| \rightarrow |0>_{2\mu} \tag{5.1.9}$$

Taking into account the transposition property of the matrices $D_{nm}^{(1/2)}$:

$$D_{nm}^{(1/2)}(U_iV_j) = -D_{mn}^{(1/2)}(U_jV_i)$$

and that the insertion of the propagator $P_{2\mu-1}$ on the $(2\mu-1)$th leg amounts to send:

$$\begin{aligned} V_{2\mu-1} &\rightarrow V_{2\mu-1}P_{2\mu-1} \equiv \tilde{V}_{2\mu-1} \\ U_{2\mu-1} &\rightarrow P_{2\mu-1}\Gamma V_{2\mu}^{-1} \equiv \tilde{U}_{2\mu-1} \end{aligned} \tag{5.1.10}$$

one obtains from (5.1.1) the following vertex:

$$\tilde{V}_{N+2g,0}^{+} = \prod_{i=1}^{N} [{}_i< 0|] \prod_{\mu=1}^{g} [{}_{2\mu-1}< 0|]$$

$$\exp\left\{ -\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N}\sum_{r,s=\frac{1}{2}}^{\infty} b_r^{(i)} D_{rs}(U_iV_j) b_s^{(j)} - \sum_{i=1}^{N}\sum_{\nu=1}^{g}\sum_{r,s=\frac{1}{2}}^{\infty} b_r^{(i)} D_{rs}(U_i\tilde{V}_{2\nu-1}) b_s^{(2\nu-1)} \right.$$

$$+\sum_{i=1}^{N}\sum_{\nu=1}^{g}\sum_{r,s=\frac{1}{2}}^{\infty} b_r^{(i)} D_{rs}(U_iV_{2\mu}) b_{-s}^{(2\mu)} + \frac{1}{2}\sum_{\substack{\mu\nu=1\\ \mu\neq\nu}}^{g}\sum_{r,s=\frac{1}{2}}^{\infty} b_r^{(2\mu-1)} D_{rs}(\tilde{U}_{2\mu-1}\tilde{V}_{2\nu-1}) b_s^{(2\nu-1)} +$$

$$\left. \sum_{\mu\nu=1}^{g}\sum_{r,s=\frac{1}{2}}^{\infty} b_{-r}^{(2\mu)} D_{rs}(U_{2\mu}\tilde{V}_{2\nu-1}) b_s^{(2\nu-1)} - \frac{1}{2}\sum_{\substack{\mu\nu=1\\ \mu\neq\nu}}^{g} D_{rs}(U_{2\mu}V_{2\nu}) b_s^{(2\nu)} \right\} \prod_{\mu=1}^{g} [|0>_{2\mu}] \tag{5.1.11}$$

where $\tilde{V}$ means that the insertion of the propagator has been performed.

Constructing a g-loop vertex sewing each leg $2\mu - 1$ to the next one 2μ, after the insertion of the propagator, means to calculate a multiple trace of the operator (5.1.11). Each trace can be conveniently computed introducing a set of *coherent states*; in the Fock space generated by oscillators b_n the latter are defined as follows:

$$|\beta > \equiv e^{-\sum_{m=\frac{1}{2}}^{\infty} \beta_m b_m} |0> \tag{5.1.12}$$

They satisfy the properties:

$$b_n|\beta> = \beta_n|\beta> \tag{5.1.13}$$

$$<\beta|b_{-n} = -<\beta|\beta_n^* \tag{5.1.14}$$

By this choice the μth loop trace of the vertex (5.1.11) results to be defined as:

$$\mathrm{Tr}_{(2\mu-1.2\mu)} \tilde{V}_{N+2g}^{+} = \int \prod_{n=\frac{1}{2}}^{\infty} d^2\beta_n^{\mu} e^{\sum_{n=\frac{1}{2}}^{\infty} \beta_n^{*\mu} \beta_n^{\mu}} {}_{2\mu-1}<\beta|\tilde{V}_{N+2g}^{+}|\beta>_{2\mu} \tag{5.1.15}$$

Hence the N-point g-loop vertex is given by:

$$V_{N.g} = \prod_{\mu=1}^{g} \mathrm{Tr}_{(2\mu-1.2\mu)} \tilde{V}_{N+2g}^{+} \tag{5.1.16}$$

The traces in (5.1.16) lead actually to gaussian integrals, which allow to write $V_{N.g}$ as follows:

$$V_{N.g} = [\det \mathcal{M}]^{\frac{1}{2}} \exp\left[-\frac{1}{2}(X|\mathcal{M}|X)\right] \tag{5.1.17}$$

where the matrix $\mathcal{M}$ is:

$$\mathcal{M} \equiv \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix}$$

being the elements $\mathcal{M}_{ij}$ themselves representation infinite matrices:

$$
\begin{aligned}
(\mathcal{M}_{11})^{\mu\mu}_{rs} &= 0 & (\mathcal{M}_{11})^{\mu\nu}_{rs} &= D_{rs}(U_{2\mu}V_{2\nu}) \qquad \mu \neq \nu \\
(\mathcal{M}_{12})^{\mu\nu}_{rs} &= \delta_{rs}\delta^{\mu\nu} + \mathcal{F}^{\mu\nu}_{rs} & \mathcal{F}^{\mu\nu}_{rs} &= (-1)^{B_\nu} D_{rs}(U_{2\mu}\tilde{V}_{2\nu-1}) \\
(\mathcal{M}_{21})^{\mu\nu}_{rs} &= -\delta_{rs}\delta^{\mu\nu} - \mathcal{E}^{\mu\nu}_{rs} & \mathcal{E}^{\mu\nu}_{rs} &= (-1)^{B_\mu} D_{rs}(\tilde{U}_{2\mu-1}V_{2\nu}) \\
(\mathcal{M}_{\epsilon\epsilon})^{\mu\mu}_{rs} &= 0 & (\mathcal{M}_{22})^{\mu\nu}_{rs} &= (-1)^{B_\mu+B_\nu+1} D_{rs}(\tilde{U}_{2\mu-1}\tilde{V}_{2\nu-1})
\end{aligned}
\tag{5.1.18}
$$

and

$$
(X|^\nu_s \equiv \left(-\sum_{i=1}^{N}\sum_{r=\frac{1}{2}}^{\infty} b^{(i)}_r D_{rs}(U_i V_{2\nu}), (-1)^{B_\nu} \sum_{i=1}^{N}\sum_{r=\frac{1}{2}}^{\infty} b^{(i)}_r D_{rs}(U_i\tilde{V}_{2\nu-1}) \right| \tag{5.1.19}
$$

The inverse of the matrix $\mathcal{M}$ can be easily computed if one writes it as:

$$
\mathcal{M} = K(1+\mathcal{H}) \tag{5.1.20}
$$

with $\mathcal{H}$ consequently defined by (5.1.18), so that:

$$
\begin{aligned}
\mathcal{M}^{-1} &= -\sum_{n=0}^{\infty}(-\mathcal{H})^n K \\
\det\mathcal{M} &= \exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}\mathcal{H}^n\right]
\end{aligned}
\tag{5.1.21}
$$

with

$$
K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} \mathcal{E} & -\mathcal{M}_{22} \\ \mathcal{M}_{11} & \mathcal{F} \end{pmatrix}
$$

The explicit computation of $\mathcal{H}^n$ gives:

$$
(\mathcal{H}^n)^{\mu\nu}_{rs} = \begin{pmatrix} (-1)^{B_\mu} D_{rs}(\tilde{U}_{2\mu-1}\Sigma^{n-1}V_{2\nu} & (-1)^{B_\mu+B_\nu} D_{rs}(\tilde{U}_{2\mu-1}\Sigma^{n-1}\tilde{V}_{2\nu-1}) \\ D_{rs}(U_{2\mu}V_{2\nu}) & (-1)^{B_\nu} D_{rs}(U_{2\mu}\tilde{V}_{2\nu-1}) \end{pmatrix} \tag{5.1.22}
$$

with

$$
\Sigma = \sum_{\mu=1}^{g}\left[S_\mu + S_\mu^{-1}\right] \tag{5.1.23}
$$

where we have put:

$$S_\mu \equiv \tilde{V}_{2\mu-1}U_{2\mu}$$
$$S_\mu^{-1} \equiv V_{2\mu}\tilde{U}_{2\mu-1} \tag{5.1.24}$$

Σ^n contains a term $S_\mu S_\mu^{-1}$ that must be equal to zero.

The calculus of $(X|\mathcal{M}|X)$ through Eq. (5.1.22) and the one of det$\mathcal{M}$ yield to the following automorphized expression for the vertex:

$$V_{N,g}^F = \prod_\alpha{}' \prod_{n=1}^{\infty}\left(1-(-1)^{N_\alpha^B}k_\alpha^{n-\frac{1}{2}}\right)\prod_{i=1}^{N}[_i<0|]$$
$$\times \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{N}\sum_{r,s=\frac{1}{2}}^{\infty} b_r^{(i)} D_{rl}^{(1/2)}(U_i)\sum_{\alpha\in\mathcal{S}}{}'(-1)^{N_\alpha^B}D_{lm}^{(1/2)}(T_\alpha)D_{ms}^{(1/2)}(V_j)b_s^{(j)}\right\} \tag{5.1.25}$$

where the sum runs over the elements T_α of the discontinuous Schottky group $\mathcal{S}$ (cfr. App. B) generated by S_μ. $\sum'$ denotes that in the terms where $i=j$ the identity must be left out and, furthermore, a sum over the indices l and m from 1/2 to ∞ is understood; N_α^B is defined as follows:

$$N_\alpha^B = \sum_{\mu=1}^{g} B_\mu N_\mu \tag{5.1.26}$$

where N_μ corresponds to the number of times the generator S_μ appears in the element T_α of the Schottky group. Furthermore $\prod_\alpha{}'$ in (5.1.25) denotes a product over all prime classes each characterized by the multiplier k_α.

Using the relation:

$$D_{rl}^{(1/2)}(U_i)\sum_{\alpha\in\mathcal{S}}{}'(-1)^{N_\alpha^B}D_{lm}^{(1/2)}(T_\alpha)D_{ms}^{(1/2)}(V_j) =$$
$$= \frac{1}{(r-1/2)!}\frac{1}{(s-1/2)!}\partial_z^{r-\frac{1}{2}}\partial_y^{s-\frac{1}{2}}\left[G(V_i(z),V_j(y)) - \frac{\delta_{ij}}{V_i(z)-V_j(y)}\right]\Bigg|_{z=y=0} \tag{5.1.27}$$

where

$$G(z,y) \equiv \sum_\alpha(-1)^{N_\alpha^B}\frac{[T_\alpha'(y)]^{1/2}}{z-T_a(y)} \tag{5.1.28}$$

it is possible to rewrite the vertex (5.1.25) in the following covariant form:

$$V^F_{N,g} = \prod_\alpha{}' \prod_{n=1}^{\infty}\left(1-(-1)^{N^B_\alpha} k_\alpha^{n-\frac{1}{2}}\right) V^F_{N,0}$$

$$\times \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{N}\sum_{r,s=\frac{1}{2}}^{\infty} b_r^{(i)} \frac{1}{(r-\frac{1}{2})!}\frac{1}{(s-\frac{1}{2})!}\partial_y^{s-\frac{1}{2}}\partial_z^{r-\frac{1}{2}}\mathcal{G}\left[V_i(z),V_j(y)\right]\Big|_{z=y=0} b_s^{(j)}\right\} \quad (5.1.29)$$

where

$$\mathcal{G}\left[V_i(z),V_j(y)\right] = G\left[V_i(z),V_j(y)\right] - \frac{1}{V_i(z)-V_j(y)}. \quad (5.1.30)$$

The g-loop vertex (5.1.29) consists of three pieces. The first one is just the g-loop partition function for a Majorana fermion:

$$Z_F^{(g)} = \prod_\alpha{}' \prod_{n=1}^{\infty}\left(1-(-1)^{N^B_\alpha} k_\alpha^{n-\frac{1}{2}}\right). \quad (5.1.31)$$

The second one is the vertex for the sphere while the last term is the contribution coming from the automorphization procedure needed to go from the sphere to an arbitrary genus g Riemann surface.

(5.1.30) represents the Green function regolarized.

5.1.2 Two-point function

If we saturate $V_{2,g}$ derived by the g-loop vertex with two states as in (5.1.3) we get the fermion correlation function on a genus g Riemann surface:

$$<\psi(z)\bar{\psi}(y)>_g = G(z,y) \quad (5.1.32)$$

with $G(z,y)$ defined in (5.1.28).

This function is called the Szegö kernel in the literature. Actually the Szegö kernel is usually defined in a different form in terms of the Θ-functions and the prime form:

$$G(z,y)=\frac{\Theta\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}\left(\tau\middle|\int_y^z\omega\right)}{E(z,y)\Theta\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}(\tau|0)} \tag{5.1.33}$$

where (α_1,α_2) define an even spin structure on the Riemann surface. In (5.1.33) τ and $E(z,y)$ denote respectively the period matrix and the prime form of a Riemann surface of genus g.

Since the two expressions (5.1.28) and (5.1.33) have the same singularities and periodicity properties around the various cycles they must be identical.

It is interesting to check the equivalence between (5.1.28) and (5.1.33) on the torus where (5.1.28) gives the following expression for the spin structures $\chi=(-+)$ and $\chi=(--)$, corresponding to $B=0$ and $B=1$ respectively:

$$G_{-\pm}(z,y)=\sum_{n=-\infty}^{+\infty}\frac{(\pm 1)^n k^{n/2}}{z-k^n y}. \tag{5.1.34}$$

while from (5.1.33) one gets:

$$G_{-\pm}(z,y)=\frac{1}{z-y}\prod_{n=1}^{\infty}\frac{(1-k^n)^2}{(1\mp k^{n-1/2})^2}\frac{\left(1\mp k^{n-1/2}\frac{y}{z}\right)\left(1\mp k^{n-1/2}\frac{z}{y}\right)}{\left(1-k^n\frac{y}{z}\right)\left(1-k^n\frac{z}{y}\right)} \tag{5.1.35}$$

In deriving (5.1.35) we have used the following explicit expression of the prime form [51]:

$$E(z,y)=(z-y)\prod_{n=1}^{\infty}\frac{(z-k^n y)(y-k^n z)}{zy(1-k^n)^2}. \tag{5.1.36}$$

It is possible to check that the two expressions are identical [52] using the identities R_{18} and R_{21} of ref. [53]. It is also easy to see that they have the same expansion around $k=0$.

It is interesting to obtain from our formalism the g-vacuum, that is the starting point for the computation of multiloop amplitudes in the new operator formalism [11]. It is given by the vertex $V_{1,g}$. By choosing the local coordinates such that $V_1(z)=z$ we get for $V_{1,g}$ the following expression:

$$V_{1.g} = Z_F^{(g)} < 0| \exp \left[-\frac{1}{2} \sum_{rs=1/2}^{\infty} b_r B_{rs} b_s \right] \tag{5.1.37}$$

with

$$B_{rs} = \frac{1}{\left(r-\frac{1}{2}\right)!} \frac{1}{\left(s-\frac{1}{2}\right)!} \partial_y^{r-1/2} \partial_z^{s-1/2} \left[G(z,y) - \frac{1}{z-y} \right] \Big|_{z=0} \tag{5.1.38}$$

that reproduces the definition of g-vacuum for fermions given in [11].

5.2 g-loop vertex for free bosons

The previous construction of $V_{N.g}^F$ can be extended to the case where we have many fermion fields.

A very interesting feature of the two-dimensional quantum conformal theories is that free bosons and free fermions are equivalent in the sense that there is a one-to-one correspondence between of their correlation function. In particular in the case of two Majorana fermions the theory is equivalent to the one with a free real scalar field. So we want to construct $V_{N.g}^B$ for this theory and check how bosonizations works in our formalism.

A scalar field $\phi(z)$ admits (at $z = 0$) the following expansion::

$$\phi(z) = q - i\alpha_0 \log z + i \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n} \tag{5.2.39}$$

with commutation relations:

$$\begin{aligned} [\alpha_n, \alpha_m] &= n\delta_{n+m.0} \\ [q, \alpha_0] &= i \end{aligned} \tag{5.2.40}$$

For this field the zero mode is compactified and is given by: is compactified and it is given by

$$\alpha_0^{(i)} = -iN_i$$

where N_i is an operator that has integer eigenvalues.

The vertex $V^B_{N,g}$ for scalar fields can be computed by means of the same sewing procedure used for fermions. It is given by:

$$\begin{aligned} V^B_{N,g} &= \prod_\alpha{}' \prod_{n=1}^{\infty} \frac{1}{(1-k_\alpha^n)} V^B_{N,0} \\ &\times \exp\left\{ \frac{1}{2} \sum_{i,j=1}^{N} \sum_{n,m=0}^{\infty} \alpha_n^{(i)} \frac{1}{n!} \frac{1}{m!} \partial_z^n \partial_y^m \log E\left[V_i(z), V_j(y)\right]\Big|_{z=y=0} \alpha_m^{(j)} \right\} \\ &\times \Theta \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \left(\tau \Big| \frac{1}{2\pi} \sum_{i=1}^{N} \sum_{n=0}^{\infty} \alpha_n^{(i)} \frac{1}{n!} \partial_z^n \int_{z_0}^{V_i(z)} \omega^\mu \right) \end{aligned} \quad (5.2.41)$$

where ω^μ's are the g first abelian differentials.

The vertex $V^B_{N,0}$ appearing in (5.2.41) is the following:

$$V_{N,0} = \prod_{i=1}^{N} \left[\sum_{n_i} {}_i < n_i, 0_\alpha | \right] \delta\left(\sum_{i=1}^{N} N_i \right) \exp\left[-\frac{1}{2} \sum_{i \neq j} \sum_{n,m=0}^{\infty} a_n^{(i)} D_{nm}(U_i V_j) a_m^{(j)} \right] \quad (5.2.42)$$

being D_{nm} the infinite matrix representation of the projective group with $J \to 0$.

Bosonizing free complex fermions means that they are expressed in the bosonic theory by:

$$\psi(z) \to e^{\phi(z)} \qquad \bar{\psi}(z) \to e^{-\phi(z)}. \quad (5.2.43)$$

In order to compute the fermion Green's function in this theory we must saturate $V^B_{2,g}$ with the states:

$$\begin{aligned} \lim_{z \to z_1} \left[V_1'^{-1}(z)\right]^{\frac{1}{2}} e^{\phi(z)} |0>_1 &= \frac{1}{[V_1'(0)]^{\frac{1}{2}}} |n_1 = 1, 0_\alpha >_1 \\ \lim_{z \to z_2} \left[V_2'^{-1}(z)\right]^{\frac{1}{2}} e^{-\phi(z)} |0>_2 &= \frac{1}{[V_2'(0)]^{\frac{1}{2}}} |n_2 = -1, 0_\alpha >_2 . \end{aligned} \quad (5.2.44)$$

So one obtains the following two-point function:

$$< \psi(z)\bar{\psi}(z) >= \frac{\Theta \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \left(\tau | \frac{1}{2\pi i} \int_{z_1}^{z_2} \omega^\mu \right)}{E(z_1, z_2) \Theta \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} (\tau | 0)} \quad (5.2.45)$$

that has been normalized dividing by the bosonic partition function:

$$Z_B^{(g)} = \prod_\alpha{}' \prod_{n=1}^{\infty} \frac{1}{(1-k_\alpha^n)} \Theta \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} (\tau|0). \tag{5.2.46}$$

Bosonization implies that:

$$\left(Z_F^{(g)}\right)^2 = Z_B^{(g)} \tag{5.2.47}$$

This equation can be explicitly checked in the case of the torus for those spin structures described by both vertices (5.1.29) and (5.2.41). In fact from (5.1.29) we get:

$$Z_F^{(1)} = \prod_{n=1}^{\infty} \left(1-(-1)^B k^{n-\frac{1}{2}}\right) \tag{5.2.48}$$

while from (5.2.41) we get:

$$Z_B^{(1)} = \prod_{n=1}^{\infty} \left(\frac{1}{1-k^n}\right) \Theta \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} (\tau|0)$$

$$= \prod_{n=1}^{\infty} \left(\frac{1}{1-k^n}\right) \sum_{n=-\infty}^{+\infty} k^{\frac{1}{2}n^2} \left(e^{2\pi i \alpha_2} k^{\alpha_1}\right)^n k^{\frac{1}{2}\alpha_1^2} e^{2\pi i \alpha_1 \alpha_2}. \tag{5.2.49}$$

So (5.2.47) is simply checked using the Jacobi identity:

$$\sum_{n=-\infty}^{+\infty} k^{\frac{1}{2}n^2} z^n = \prod_{n=1}^{\infty} (1-k^n) \left(1+k^{n-\frac{1}{2}} z\right) \left(1+k^{n-\frac{1}{2}} \frac{1}{z}\right) \tag{5.2.50}$$

with the identification $1-B=2\alpha_2$. By saturating the vertex $V_{N,g}^B$ (5.2.41) with the states $|n_i, 0>_i$ we can compute the more general correlation function:

$$< \prod_{i=1}^{N} e^{n_i \phi(z_i)} >= \prod_{i<j} [E(z_i, z_j)] \frac{\Theta \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \left(\tau | \frac{1}{2\pi i} \sum_{i=1}^{N} n_i \int_{z_0}^{z_i} \omega^\mu\right)}{\Theta \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} (\tau|0)} \tag{5.2.51}$$

that reproduces the result obtained in ref. [54] with other techniques.

In particular if the vertex (5.2.41) is saturated with M states $|n_i = 1, 0_\alpha>$ $(i=1,...,M)$ and with M states $|n_j = -1, 0_\alpha>$ $(j=1,...,M)$, then from the "addition-theorem" for Abelian functions [55], it is possible to write the corresponding correlation function (5.2.51) as follows:

$$< \prod_{i=1}^{M} \exp -[\phi(z_i)] \prod_{j=1}^{M} \exp[\phi(y_j)] >= \det \Big(< \psi(z_i) \quad (5.2.52)$$

that is equal to the correlation function of M fields $\psi(z_i)$ and $\bar{\psi}(y_j)$.

In conclusion we have seen that the sewing procedure used in the old operator formalism for free bosonic and fermionic theory provides a very powerful tool for computing correlation functions on an arbitrary Riemann surface starting from those on the sphere and the propagator (5.1.5) and for showing bosonization.

Appendix A

Unitary Irreducible Representations of the Projective Group

To every real projective transformation of the form

$$\tilde{z} = T(z) = \frac{az+b}{cz+d}, \tag{A..1}$$

with $G = ad - bc$, one associates the 2×2 matrix:

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

which can be thought to act on a vector $\xi = (\xi_1, \xi_2)$, the two components of which are called *homogeneus coordinates* of the variable z. The set of these matrices define the group $SL(2, R)$ isomorphic to $SU(1,1)$. The unitary irreducible representations of the projective group are obtained by considering the transformation properties of the monomials of the components of the spinor ξ, namely $\xi_1^a \xi_2^b$, which provides a basis for them. The powers a and b can be arranged in such a way these basic states are written as:

$$|Jkm >= N(J,k,m)(\xi_1\xi_2)^J \left(\frac{\xi_1}{\xi_2}\right)^{k+m}, \tag{A..2}$$

where $N(J,k,m)$ is a normalization coefficient. The numbers J and k label irreducible representations.

When a transformation Λ is performed on the two-component vector ξ, correspondently the basic states are transformed in:

$$|J,k,m>'= N(J,k,m)(\xi_1'\xi_2')^J \left(\frac{\xi_1'}{\xi_2'}\right)^{k+m} = \sum_{n=0}^{\infty} D^{J,k}_{mn}(\Lambda)|Jkn> . \qquad (A..3)$$

Let us consider the representations with $J = -k$. In this case eq.(A..3) may be written explicitly in the following way:

$$N(J,-J,m)(cz+d)^{2J}\left(\frac{az+b}{cz+d}\right)^m = \sum_{n=0}^{\infty} D^J_{mn}(\Lambda)N(J,-J,n)z^n \qquad (A..4)$$

where $z = \frac{\xi_1}{\xi_2}$.

The generators of $SU(1,1)$ are denoted by $L_0, L_{\pm1}$ and satisfy the commutation relations:

$$[L_0, L_{\pm1}] = \mp L_{\pm1}$$

$$[L_1, L_{-1}] = 2L_0$$

These generators are represented by the following matrices 2×2:

$$\Lambda_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Lambda_+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \Lambda_- = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

Their matrix elements in the (J,k) irreducible representation are so given by:

$$\begin{aligned} D^{(J,k)}_{mn}(\Lambda_0) &= (k+m)\delta_{mn} \\ D^{(J,k)}_{mn}(\Lambda_+) &= \frac{N(J,k,m)}{N(J,k,n)}(k+m-J)\delta_{n,m+1} \\ D^{J,k}_{mn}(\Lambda_-) &= \frac{N(J,k,m)}{N(J,k,n)}(k+m+J)\delta_{n,m-1} \end{aligned} \qquad (A..5)$$

The normalization coefficient $N(J,k,m)$ is determined up to a phase by requiring that $L_1 = L_1^+$; if the phase is fixed to 1, one obtains:

$$N(J,-J,n) = \sqrt{\frac{\Gamma(n-2J)}{\Gamma(n+1)}} \qquad (A..6)$$

The following transposition rule for the matrices

$$D^J(\Lambda)^T = D^J(\Gamma\Lambda^{-1}\Gamma) \tag{A..7}$$

where

$$\Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A..8}$$

satisfies $\Gamma^2 = 1$.

Appendix B

Riemann Surfaces in the Schottky representation

To a Riemann surface $\mathcal{M}$ of genus g it is possible to associate a *Schottky group* $\mathcal{G}_{\mathcal{M}}$, which is a discrete group freely generated by a set of g projective transformations S_μ with $\mu = 1, \ldots, g$ called *Schottky generators.* Any group element, with the exception of the unity, can be written in the following general form:

$$T_\alpha = S_{\mu 1}^{n_1} S_{\mu 2}^{n_2} \ldots S_{\mu_{n_r}} \qquad r = 0, 1, \ldots \qquad n_i \in Z - 0 \;\; \mu_i \neq \mu_{i+1} \tag{B..1}$$

The number of generators or their inverses in the element (B..1) is called its *order* n_α and it is given by:

$$n_\alpha = \sum_{i=1}^{r} |n_i|. \tag{B..2}$$

A projective transformation can be defined by:

$$T(z) = \frac{az + b}{cz + d} \tag{B..3}$$

with $ad - bc = 1$ or, alternatively, by:

$$\frac{T(z) - \eta}{T(z) - \xi} = k \frac{z - \eta}{z - \xi} \tag{B..4}$$

where k is the so called *multiplier* while η and ξ are the *fixed points*, which are just unchanged by the transformation (B..3). Clearly $T'(\eta) = k$ and $T'(\xi) = k^{-1}$; since one can always take $k \leq 1$, this is the reason why η is called *attractive* fixed point and ξ *repulsive* fixed point. Furthermore one has that $T' = ATA^{-1}$ has the same multiplier of T and fixed points $A(\eta)$ and $A(\xi)$.

Two elements T_α and $T_{\alpha'}$ of the Schottky group belong to the same *coniugacy class* if they go into each other by a cyclic permutation of the generators which define it. All the elements of the same coniugacy class have the same multiplier.

Given a genus g Riemann surface $\mathcal{M}$ it is possible to associate to it a Schottky group generated by g projective transformation S_μ.

In the *Schottky representation* one uses the fact that a genus g Riemann surface is equivalent to the compactified complex plane in which g circles C_μ and g circles C'_μ, called *isometric circles*, have been cut out and identified. There exist g projective transformations S_μ ($\mu = 1, \ldots, g$) which map the circles C_μ into C'_μ. If we denote:

$$S_\mu(z) = \frac{A_\mu(z) + B_\mu}{C_\mu z + D_\mu}$$

$$S_\mu^{-1} = \frac{-D_\mu z + B_\mu}{C_\mu z - A_\mu} \tag{B..5}$$

with $A_\mu B_\mu - B_\mu C_\mu = 1$ ($\mu = 1, \ldots, g$); then the points of the circles C_μ and C'_μ satisfy the relations:

$$\left| \frac{dS_\mu}{dz} \right|^{-1/2} = |C_\mu z + D_\mu| = 1$$

$$\left| \frac{dS_\mu^{-1}}{dz} \right|^{-1/2} = |C_\mu z - A_\mu| = 1 \tag{B..6}$$

respectively; their centers J_μ are given by:

$$J_\mu = -\frac{D_\mu}{C_\mu} = \frac{\eta_\mu - k_\mu \xi_\mu}{1 - k_\mu} \qquad J'_\mu = \frac{A_\mu}{C_\mu} = \frac{\xi_\mu - k_\mu \eta_\mu}{1 - k_\mu} \tag{B..7}$$

while their radii are defined by the following equations:

$$R_\mu = R'_\mu = \sqrt{|k_\mu|}\left|\frac{\xi_\mu - \eta_\mu}{1 - k_\mu}\right| \tag{B..8}$$

Any point outside the circle C_μ will be mapped by S_μ into a point inside the circle C'_μ, while any point outside the circle C'_μ will be mapped by S_μ^{-1} into a point inside the circle C_μ.

This implies that the attractive fixed point η_μ is inside the circle C'_μ, while the repulsive fixed point ξ_μ is inside the circle C_μ.

A genus g Riemann surface $\mathcal{M}_g$ is identified with the region outside the $2g$ isometric circles or in other words it is equal to:

$$\mathcal{M}_g = \frac{\mathcal{C} \cup \{\infty\}}{G_g} \tag{B..9}$$

where G_g is the Schottky group associated to a genus g Riemann surface.

One can show that the representation (B..9) of conformally inequivalent Riemann surfaces is one to one, apart for an overall projective transformation and modular transformations.

Going around a cycle a_μ in the Riemann surface corresponds in going around C_μ or C'_μ (in clockwise and anticlock direction respectively), while a path that brings from a point z of C_μ to the point $S_\mu(z)$ of C'_μ corresponds to going around a b_μ cycle.

By means of the Schottky group one can define the geometrical objects of a Riemann surface. The simplest ones are the g abelian differentials defined by:

$$\omega_\mu(z) = \sum_{T_\alpha}^{(\mu)} \left(\frac{1}{z - T_\alpha(\eta_\mu)} - \frac{1}{z - T_\alpha(\xi_\mu)} \right) dz \tag{B..10}$$

where $\sum_{T_\alpha^\mu}$ means that we sum over all elements of the Schottky group, that do not have S_μ^n, $n \in Z - \{0\}$ as the rightmost element.

The g abelian differentials satisfy the following important properties:

i) $\omega_\mu(z)$ is holomorphic in the entire Riemann surface; it has simple poles in the isometric circles C_ν and C'_ν, $\nu = 1, \ldots, g$ and therefore they are in the complex plane, nut not in the Riemann surface.

ii) The abelian differentials are normalized as follows:

$$\oint_{a_\nu} \omega_\mu(z) = \oint_{C'_\nu} \omega_\mu(z) = 2\pi i \delta_{\mu\nu} \tag{B..11}$$

that follows from the fact that the circle C'_ν contains two simple poles of $\omega_\mu(z)$ with opposite residues for those T_α, that have a factor $S^n_\nu, n > 0$, as the leftmost element. Their contribution will cancel except in the case of the identity, that gives the normalization (B..11):

$$\oint_{C'_\nu} \omega_\mu(z) = \oint_{C'_{nu}} \left(\frac{1}{z-\eta_\mu} - \frac{1}{z-\xi_\mu} \right) dz = 2\pi i \delta_{\mu\nu} \tag{B..12}$$

iii) The period matrix $\tau_{\mu\nu}$ is given by the following relation:

$$(2\pi i)\tau_{\mu\nu} = \oint_{b_\nu} \omega_\mu(z) \tag{B..13}$$

It follows from the following steps:

$$(2\pi i)\tau_{\mu nu} = \int_{b_\nu} \omega_\mu(z) = \int_{z_0}^{S_\nu(z_0)} \omega_\mu(z) =$$

$$\sum_{T_\alpha}{}^{(\mu)} \log \frac{S_\nu(z_0) - T_\alpha(\eta_\mu)}{S_\nu(z_\alpha) - T_\alpha(\xi_\mu)} \frac{z_0 - T_\alpha(\xi_\mu)}{z_0 - T_\alpha(\eta_\mu)} =$$

$$\delta_{\mu\nu} \log k_\mu - {}^{(\nu)}\sum_{T_\alpha}{}'^{(\mu)} \log \frac{\eta_\nu - T_\alpha(\xi_\mu)}{\eta_\nu - T_\alpha(\eta_\mu)} \frac{\xi_\nu - T_\alpha(\eta_\mu)}{\xi_\nu - T_\alpha(\xi_\mu)} \tag{B..14}$$

${}^{(\nu)}\sum'_{T_\alpha}{}^{\mu}$ means a sum over the elements of the Schottky group, that do not have any factor S^n_μ as the rightmost element and S^m_ν as the leftmost element for arbitrary $n, m \neq 0$ and that the identity is not present for $\mu = \nu$. The period matrix is therefore independent from the point z_0.

Another important object defined on a Riemann surface is the prime form $E(z, w)$, that is the generalization to an arbitrary Riemann surface of what a simple monomial $(z - w)$ is on the complex plane. It is holomorphic on the Riemann surface and has a simple zero when $z \to w$:

$$E(z, w) = 0 \longleftrightarrow z = w \tag{B..15}$$

Since the function $(z - w)$ is not holomorphic already on the sphere, one actually defines a holomorphic differential form with conformal weight $-\frac{1}{2} \times -\frac{1}{2}$, which generalizes

$$\frac{z-w}{\sqrt{dz}\sqrt{dw}} \tag{B..16}$$

which is regular also at the infinity point. The expression of the prime form in terms of the Riemann Theta functions is:

$$E(z,w) = \frac{\theta[\alpha](\int_w^z \omega)}{\sqrt{\sum_\mu \partial_\mu \theta[\alpha](0)\omega_\mu(z)}\sqrt{\sum_\mu \partial_\mu \theta[\alpha](0)\omega_\mu(w)}} \tag{B..17}$$

for an arbitrary odd spin structure α. In the Schottky representation it is given by:

$$E(z,w) = \frac{z-w}{\sqrt{dz}\sqrt{dw}} \prod_\alpha{}' \frac{z-T_\alpha(w)}{z-T_\alpha(z)} \frac{w-T_\alpha(z)}{w-T_\alpha(w)} \tag{B..18}$$

where the product is over all elements of the Schottky group apart from the identity and where the transformations T_α and T_α^{-1} are counted only once.

The prime form does not change if one goes around a cycle a_μ, while one can prove that its change around a cycle b_μ is given by:

$$E\left(S_\mu(z),w\right) = -\exp\left[-2\pi i\left(\frac{1}{2}\tau_{\mu\mu} + \int_w^z \omega_\mu\right)\right] E(z,w). \tag{B..19}$$

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