ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

ATTESTATO DI RICERCA

DI

"DOCTOR PHILOSOPHIAE"

DIFFERENTIAL EQUATIONS WITH JUMPING NON LINEARITIES

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Anno Accademico 1981/82

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1. INTRODUCTION

In recent years, several people studied the nonlinear elliptic boundary value problems of the form

$$\text{(BVP)} \qquad \begin{aligned} -\Delta u &= f(x,u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

where Ω is a regular open bounded domain in $\mathbb{R}^{m}$ and f is a smooth function on $\Omega \times \mathbb{R}$ and valued in $\mathbb{R}$. We are interested in finding conditions which ensure the existence of solutions of (BVP) or yield multiplicity phenomena. To this aim it is of particular importance to notice the relationship between the spectrum σ of the differential operator $-\Delta$ and the range of the derivative of the nonlinearity with respect to the last variable. In particular, a classical result for (BVP) ([32],[10]) concerns the case in which the range of $\partial_{m+1}f$ is disjoint from σ . In this case one can study (BVP) via a global inversion method proved by a local analysis: it is apparent that our hypothesis ensures in particular that the linearization of (BVP) has nowhere nontrivial solutions (see e.g. [36]).

A more interesting case is obtained modifying f for u in a bounded interval. In this case one easily proves, using Leray-Shauder degree that (BVP) always has a solution and can study the multiplicity of the solutions under various assumptions. In all these cases, any boundedness hypotheses on $\partial_{m+1}f$ can be considerably weakened but here we are not interested on this problem.

The existence results are generally lost if the interaction between σ and $\partial_{m+1}f$ is assumed,roughly speaking, in an asymptotical sense as we do therein.

Before stating the assumptions, a last remark, although obvious, must be done. We will assume later the bounds we will need in a stronger form, since we are more interested in the application of the methods of nonlinear functional analysis than in the technical problems of finding sharp a priori estimates; in the same way, $-\Delta$ can be replaced by a more general strongly elliptic operator.

Throughout the paper, we shall assume that

$$f_{\pm} = \lim_{s \to \pm\infty} \partial_{m+1} f(x,s)$$

exist uniformly with respect to $x \in \Omega$ and are real numbers. Our aim will be to give, under various subcases, some recent results about the solvability of (BVP), in all these subcases we will have $]f_-, f_+[\cap \sigma \neq \emptyset$.

We briefly state the main notation and outline the paper. We denote by $\lambda_1, \lambda_2, \dots, \lambda_k, \dots$ the eigenvalues of $-\Delta$ on Ω with the homogeneous Dirichlet boundary conditions. By ϕ_j we denote a normalized eigenfunction related to the j-th eigenvalue λ_j. In the sequence of the λ_j's each eigenvalue is repeated so many times as its multiplicity, therefore we can choose the sequence of eigenfunctions ϕ_j such to be a complete orthogonal system in L^2. It is well-known that, by the Krain-Rutmann theorem, [27], the first eigenvalue λ_1, is simple and the eigenfunction ϕ_1 is of constant sign on Ω, we will take $\phi_1 > 0$; by this choice, ϕ_1 will be strongly positive (i.e. will belong to the interior of the positive cone, see e.g. [2]) in $C_0^{1,\alpha}(\Omega)$.

We will sometimes denote the set of the λ_j's by σ.

The paper is divided into two parts. The first part is devoted to the case $f_- < \lambda_1 < f_+$. We first prove essentially the first result on this class of problem, following [9], [18], to the aim either of giving an immediate model for the results which will be obtained later on in a more general form, and of introducing the most used tools in this paper. Subsequently, we treat the main existence and multiplicity results in this case.

The second part is concerned with the case $\lambda_1 < f_{\pm}$. We first briefly study this problem in general, afterwords we confine us to the case: $\lambda_{k-1} < f_- < \lambda_k < f_+ < \lambda_{k+1}$, for some $k > 1$, following essentially [20] . This type of results will be then applied also to some problems with suitable symmetries. Finally, we shall discuss some recent multiplicity results.

2. THE CASE $f_- < \lambda_1 < f_+$.

In this part we are concerned with the most studied case for our problem and that in which the strongest results about the nonexistence and the multiplicity of solutions are known. The first paper on this subject was [6] in which $f(x,u) = h(x) + g(u)$ is taken and

$$g \in C^2(\mathbb{R}), \quad g'' > 0 \tag{2.1}$$

$$0 < f_- < \lambda_1 < f_+ < \lambda_2 \tag{2.2}$$

Fixed g, under (2.1) – (2.2), in [6] it is proven the existence of C^1 closed connected manifold Γ_1 of codimension 1 in $C^{\alpha}(\Omega)$ which splits the space into two open sets Γ_0 and Γ_2 and such that (BVP) has exactly i(classical) solutions for $h \in \Gamma_i$, $i = 0, 1, 2$.

This result has been restated and improved in several subsequent papers, we recall e.g. [9], [20], [35] and the survey papers [15], [5]. The phenomena of existence of zero two solutions (eventually in a weak sense) also hold under weaker regularity assumption on g and the first inequality in (2.2) can be dropped; however in this more general setting, the result cannot be stated in the same geometrically meaningful form. The method used in [6] is closely related to the global inversion method, taking into account that the linearized problem near a point u_0 has a nontrivial solution if u_0 solves (BVP) for $h \in \Gamma_1$. We will never follow this method (and will not prove completely the result) and we invite the reader to consult [6][7]. The convexity hypothesis $g'' > 0$ it is essential to have the sharp estimate of the number of solutions.

In [26] assumption (2.2) is considerably relaxed being only asked

$$f_- < \lambda_1 < f_+ \tag{2.3}$$

We write $h = h_1 + t\phi_1$ with $h_1 \perp \phi_1$; in [26] it is proved the existence of $t(h_1) \in \mathbb{R}$ such that (BVP) has a solution if $t < t(h_1)$ and no solution for $t(h_1) < t$. This result has been improved independently in [13] – [3]; it is proved there the solvability of (BVP) for $t = t(h_1)$ and the existence of two distinct solutions for $t < t(h_1)$. Higher multiplicity results were proved later in [30], [50] under the further assumption $\lambda_2 < f_+ < \lambda_3$ and in [25][48] for $\lambda_2 < f_+$, $f_+ \neq \lambda_j$.

In this part we give an unitary exposition of the above results.

§ 1. APPLICATION OF THE LYAPUNOV-SCHMIDT METHOD.

We shall work in this section with the Hilbert space $E = L^2(\Omega, \mathbb{R}^n)$ ordered by assuming $u \leq v$ iff the range of $v - u$ is contained in the positive cone of $\mathbb{R}^n$, and let a linear selfadjoint operator $A: \mathcal{D}(A) \to E$ be given. The generality of our setting is required by further applications of the results in this section which will be given later on. We use the notation

$$u^+ = \sup(u, 0) \qquad\qquad u^- = (-u)^+$$

Let $\alpha, \beta \in \mathbb{R} - \sigma(A)$, denoting by $\sigma(A)$ the spectrum of A, and $(P_\lambda)_{\lambda \in \mathbb{R}}$ be the spectral resolution of A [16], [51]. We write:

$$P = \int_\alpha^\beta dP_\lambda \qquad\qquad Q = I - P = \int_{-\infty}^\alpha dP_\lambda + \int_\beta^\infty dP_\lambda$$

$$V = \operatorname{Im} P \qquad\qquad W = \operatorname{Im} Q$$

Therefore we have $E = V \oplus W$.

Let $G : E \to E$ be such that

$$\alpha(u-v)^+ - \beta(u-v)^- \leq Gu - Gv \leq \beta(u-v)^+ - \alpha(u-v)^- \tag{2.4}$$

<u>Lemma 2.5</u> <u>The equation</u>

$$Au = Gu + h \tag{2.6}$$

<u>with h fixed in E, is equivalent, for $u = v+w(v)$, to</u>

$$Av = PG(v+w(v)) + Ph \tag{2.7}$$

<u>where w(v) is the unique solution of the equation</u>

$$Aw = QG(v+w) + Qh \tag{2.8}$$

Moreover $w: v \to w(v)$ is a Lipschitz map from V to W and there exist $c > 0$ such that

$$\|w(v)\| \leq c(\|v\| + \|Qh\|) \tag{2.9}$$

for any given $v \in V$, $h \in E$.

Proof It is apparent that (2.6) is equivalent to the system (2.7)-(2.8). About the solvability of (2.8) we remark that it is equivalent to:

$$(A - \frac{\alpha + \beta}{2} I)w = Q(G - \frac{\alpha + \beta}{2} I)(v+w) + Qh$$

and $Q(G - \frac{\alpha+\beta}{2} I)(v + \cdot)$ is a Lipschitz continuous mapping on W, with Lipschitz constant $\frac{\beta - \alpha}{2}$, while $A - \frac{\alpha+\beta}{2} I$ is invertible on W, with inverse K_W, and $\| K_W \| < \left(\frac{\beta - \alpha}{2}\right)^{-1}$. Therefore the statement is a straightforward consequence of Banach's theorem on contractive mappings.■

Using the previous lemma, it appears very easy to prove the existence results when the range of $\partial_{m+1} f$ is contained in a compact set disjoint from σ, as we stated before. We now use it to prove a theorem of existence of zero-two solutions of the type of [6], a similar approach was used in [8]. We state an easy corollary of the lemma.

First we assume $f: \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function i.e.

(F_1) $f(x,t)$ is continuous with respect to the t variable for a.e. $x \in \Omega$ and it is measurable in $x \; \forall \; t \in \mathbb{R}$

and let

(F_2) f is bounded on bounded subset of $\Omega \times \mathbb{R}$

$$f_{\pm} = \lim_{t \to \pm\infty} \frac{f(x,t)}{t} \tag{2.11}$$

there exist uniformly with respect to $x \in \Omega$ and $f_- < \lambda_1 < f_+$, moreover let

$$-\infty < \inf_{t \neq s} \frac{f(x,t) - f(x,s)}{t - s} = \alpha \leq \beta = \sup_{t \neq s} \frac{f(x,t) - f(x,s)}{t - s} < \lambda_2 \tag{2.12}$$

Of course (F_2) holds when (2.12) is true and f is bounded on x. Let V and W be as in the lemma (therefore V = spanned (ϕ_1)) , let w: $\mathbb{R} \to W$ defined by:

$$- \Delta w(s) = Qf(s\,\phi_1 + w(s)) \quad \text{in } E \qquad (2.13)$$
$$w(s) \in H_0^1(\Omega)$$

and

$$\xi(s) = (f(s\,\phi_1 + w(s)), \phi_1) - \lambda_1 s \qquad (2.14)$$

We parametrize (BVP) fixing $\Psi \in E$ and considering

$$(P_t) \qquad \begin{aligned} -\Delta u &= f(x,u) + t\Psi && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

<u>Corollary 2.15</u> <u>u is a solution of (P_t) for $\Psi = \phi_1$ iff $u = s\phi_1 + w(s)$ with $\xi(s) = -t$.</u>

The corollary is a re-statement of the lemma in a simpler setting. We remark that our approach is not more restrictive than assuming $f(x,u) = g(u) + h(x)$ and considering h split into the form $h = h_1 + t\phi_1$ with $h_1 \perp \phi_1$ since h_1 can be inserted in the nonlinearity, without changing (2.12). If nothing is said, we shall always assume $\psi = \phi_1$ in (P_t).

<u>Lemma 2.16</u> <u>Under the above assumptions ξ is a continuous real function and</u>

$$\lim_{|s| \to +\infty} \xi(s) = +\infty \qquad (2.17)$$

<u>Proof</u> We choose $f'_\pm, c \in \mathbb{R}$, $f_- < f'_- < \lambda_1$, $\lambda_1 < f'_+ < f_+$ and such that

$$f(x,t) \geq f'_+ t^+ - f'_- t^- + c \qquad (2.18)$$

Therefore we have:

$$\xi(s) \geq (f'_+ u^+ - f'_- u^- + c, \phi_1) - \lambda_1 s \tag{2.19}$$

taking into account the positiveness of ϕ_1 and denoting $s\phi_1 + w(s)$ by u.

From (2.19), by easy computations:

$$\begin{aligned} \xi(s) &\geq (\delta|u| + c, \phi_1) \geq \\ &\geq \delta|s| + (c, \phi_1) \end{aligned} \tag{2.20}$$

for $\delta = \min(\lambda_1 - f'_-, f'_+ - \lambda_1) > 0$. From (2.20) we get (2.17); the first part of the statement is a straightforward consequence of lemma 2.5.■

Theorem 2.21 Let f be given and (F_1), (2.11), (2.12), $f_- < \lambda_1 < f_+$ hold. Then $\exists\, t_o \in \mathbb{R}$ such that (P_t) has:

no solution for $t > t_o$

at least a solution for $t = t_o$

at least two solutions for $t < t_o$

Proof It follows easily from lemma 2.15 - 2.17 taking $t_o = - \inf \xi$.■

In § 3 we shall treat the problem of finding the exact number of solutions of (P_t) for convex f.

§ 2. THE GENERAL CASE

Our goal in this section will be to eliminate assumption (2.12), asking only $-\infty < \alpha$, assumptions $(F_1 - F_2)$ still hold. We assume, as before, $h_1 = 0$, this condition is now equivalent to $h_1 \in L^\infty(\Omega)$ since we need (F_2) hold completely. Condition $-\infty < \alpha$ and the weak maximum principle ([22] cap. 8 § 1) imply the existence of a solution of (P_t) when one can find a subsolution $\underline{u}$ and a supersolution $\bar{u}$, $\underline{u} \leq \bar{u}$, see [2].

Lemma 2.22 $\forall t \in \mathbb{R}$ (P_t) has a subsolution $\underline{u}$ such that, if $\bar{u}$ is a supersolution of (P_t), then $\underline{u} \leq \bar{u}$.

Proof. Fix f'_- as in lemma 2.16. We have:

$$f(x,t) \geq f'_- + c \qquad \forall (x,t) \in \Omega \times \mathbb{R} \tag{2.23}$$

with a suitable constant c. Let $\underline{u}$ be a solution of

$$\begin{aligned} -\Delta \underline{u} - f' \underline{u} &= c && \text{in } \Omega \\ \underline{u} &= 0 && \text{on } \partial\Omega \end{aligned} \tag{2.24}$$

Of course $\underline{u}$ is a subsolution of (P_t); to prove the second part of the statement let $\bar{u}$ be a supersolution, using (2.23), (2.24):

$$\begin{aligned} -\Delta(\bar{u} - \underline{u}) - f' (\bar{u} - \underline{u}) &\geq 0 && \text{in } \Omega \\ \bar{u} - \underline{u} &\geq 0 && \text{on } \partial\Omega \end{aligned} \tag{2.25}$$

From (2.25) and the weak maximum principle we get $\underline{u} \leq \bar{u}$.■

We are following closely [15] . A further lemma will allow us to give a first extension of Theorem 2.21.

<u>Lemma 2.26</u> <u>$\exists\, t \in \mathbb{R}$ such that (P_t) has at least a solution.</u>

<u>Proof</u>. By the previous lemma, it will be enough to find a supersolution of (P_t) for some t. To this aim, let $m = \sup |f|$ on $\Omega \times [-1,1]$. Let Ω' be the trace on Ω of a neighbourhood of $\partial\Omega$ and $X_{\Omega'}$ be its characteristic function. Let $\bar{u}$ be the solution of

$$\begin{aligned} -\Delta\bar{u} &= m X_{\Omega'} && \text{in } \Omega \\ \bar{u} &= 0 && \text{on } \partial\Omega \end{aligned} \tag{2.27}$$

Of course, using L^p estimates for linear elliptic equations [22] and Sobolev's embedding theorems [1], one can get $\|\bar{u}\|_{L^\infty} \leq 1$ by taking Ω' small enough in measure. In this case, by the definition of m, we have:

$$-\Delta\bar{u} \geq f(x,\bar{u}) \qquad \text{on } \Omega' \tag{2.28}$$

Since, on $\Omega - \Omega'$, $f(x,\bar{u})$ is bounded and ϕ_1 is bounded from below by a positive constant, we can take t such that

$$f(x,\bar{u}) + t\,\phi_1 \leq 0 \quad \text{on } \Omega - \Omega' \tag{2.29}$$

by (2.28) - (2.29):

$$-\Delta \bar{u} \geq f(x,\bar{u}) + t\phi_1 \quad \text{in } \Omega \tag{2.30}$$

and therefore $\bar{u}$ is the desired supersolution for (P_t).■

Theorem 2.31 Assume $(F_1 - F_2)$, $-\infty < \alpha$, $f_- < \lambda_1 < f_+$ hold. Then $\exists\, t_o \in \mathbb{R}$ such that (P_t) has

no solution for $t > t_o$

at least a solution for $t < t_o$

Proof We remark that if u is a solution of (P_t), then it is a supersolution of $(P_{t'})$ for $t' < t$. Therefore, by lemma 2.22, we see that if (P_t) is solvable for some t, then it is solvable for all t', $t' < t$. We must only show that

$$t_o = \sup \{ t \in \mathbb{R} \mid (P_t) \text{ is solvable} \} \in \mathbb{R}$$

and the statement will follow from the above considerations.

The previous lemma implies that $-\infty < t_o$; to see that $t_o < +\infty$ fix f'_-, f'_+, c as in lemma 2.16 (of course the condition $f'_+ < \lambda_2$ must be dropped) and suppose that u is a solution of (P_t). We have:

$$\begin{aligned} t &= (-\Delta u - f(x,u), \phi_1) \leq \\ &\leq (-\Delta u - f'_+ u^+ + f'_- u^- - c, \phi_1) = \\ &= (\lambda_1 - f'_+)(u^+, \phi_1) + (f'_- - \lambda_1)(u^-, \phi_1) - c\int_\Omega \phi_1 \leq \\ &\leq -c\int_\Omega \phi_1 \end{aligned} \tag{2.32}$$

and, taking the supremum, $t_o \leq -c\int \phi_1$.■

We remark that to get this kind of results it is not necessary to take $\Psi = \phi_1$ see e.g.[23]

Theorem 2.21 can be completely extended to the case $\lambda_2 \leq f_+$. To this aim, it will be convenient to work with the Schauder's space $F = C_o^{1,\alpha}(\Omega)$; it will require some more regularity assumption on f. Precisely, we ask:

f is a C^1 function on $\mathbb{R}$ such that

(F_3) $\quad f_\pm = \lim_{s \to \pm\infty} \frac{f(s)}{s}$ there exist in $\mathbb{R}$ and $f_- < \lambda_1 < f_+$

(F_3) ensures that the mapping T_t defined by :

$$-\Delta(T_t u) + (\alpha + 1)T_t u = f(u) + (\alpha + 1)u + h + t\phi_1 \tag{2.33}$$

on $C_0^{1,\alpha}$, for given $t \in \mathbb{R}$, $h \in C^{0,\alpha}$, is completely continuous and strongly increasing. Here, coherently with (2.12), we denote by α the inf f' ; T_t strongly increasing means that whenever $u < v$ (i.e. $u \leq v$ in the pointwise ordering and $u \neq v$), then $T_t u << T_t v$ (i.e. $T_t(v) - T_t(u)$ is an interior point of the positive cone $P = \{u \in C_0^{1,\alpha} | u \geq 0 \}$).The properties of T_t are easily deduced by the continuity of the Nemytskii operator induced by f, the Schauder's estimates for linear elliptic equations , the Schauder's embedding theorems and the strong maximum principle, see e.g. [22] .

The complete continuity of T_t allows us to use the topological degree of $I - T_t$ on bounded subset of F; [31], [46], [38], [33]. Englobing h in the nonlinearity f, the zeroes of $I - T_t$ are precisely the solutions of (P_t). We will extend the theorem using the following lemma.

<u>Lemma 2.34</u> <u>For any given constant $\tau \in \mathbb{R}$, the set of the zeroes of $I - T_t$, for $\tau < t$, is bounded in F.</u>

<u>Proof</u> Arguing as in the theorem 2.31 we find f'_+ , f'_-, $c \in \mathbb{R}$, $f'_- < \lambda_1 < f'_+$ such that (2.18) holds.From (2.32), letting $\delta = \min (f'_+ - \lambda_1, \lambda_1 - f'_-) > 0$ we see:

$$\int_\Omega |u| \phi_1 \leq m \tag{2.35}$$

where m is the constant $- \delta^{-1} (\tau + c \int \phi_1)$

We now use the Hardy-Sobolev inequality

$$\left\| \frac{u}{\phi_1^\eta} \right\|_{L^P} \leq c \, \| \nabla u \|_{L^2} \tag{2.36}$$

for $u \in H_0^1$, $0 \leq \eta \leq 1$, $\frac{1}{p} = \frac{1}{2} - \frac{1-\eta}{n}$

Using (2.3) with $\eta = 1$, and Holder inequality:

$$\int_{\Omega_P} | u |^2 \leq \left(\int_\Omega |u| \phi_1 \right)^{1/2} \left(\int_\Omega |u|^2 \frac{|u|}{\phi_1}\right)^{1/2} \leq (mc)^{1/2} \| u \|_{L^4} \, \| \nabla u \|^{1/2}_{L^2} \tag{2.37}$$

By the L^P estimates for linear elliptic equations one has, for a suitable constant c'

$$\| u \|^2_{L^2} \leq c' \, \| u \|_{L^2}^{3/2} \tag{2.38}$$

Therefore we have a bound in $L^2(\Omega)$. Using standards bootstraps arguments one extend it to the $C^{1,\alpha}$ norm.■

Lemma 2.39 Let t_o be as in theorem 2.31 and $t < t_o$. Then there exist a non empty open bounded set $B_t \subset F$ such that

$$\deg(I - T_t, B_t, 0) = 1 \tag{2.40}$$

Proof We remark that if $t < t_o$, using the arguments of this section, we can find a strict subsolution $\underline{u}$ and a strict supersolution $\bar{u}$ of (P_t), in F. Since $[\underline{u}, \bar{u}]$ is bounded in the L^∞-norm, using a bootstrap argument, we see that the set of the solutions $u \in [\underline{u}, \bar{u}]$ of the equation $u = sT_t u$, for $s \in [0,1]$ is bounded by a constant $r_t - 1$, in the F-norm. We claim that, by the strong maximum principle, the solution u of (P_t) in $[\underline{u}, \bar{u}]$ belongs to $B_t = B(o, r_t) \cap \cap [\underline{u}, \bar{u}]$. To prove (2.40) we use the homotopy $H : [0,1] \times \bar{B} \rightarrow F$ defined

$$H(s,u) = u - sT_t u \tag{2.41}$$

To prove the admissibility of H for the computation of $\deg(I - T_t, B_t, 0)$ we first note that, by the strong maximum principle, $H(s,.)$ sends $[\underline{u}, \bar{u}]$ into its interior; therefore if $u \in \partial B_t$ is a zero of $H(s,.)$ then u must belong to $\partial B(0, r_t) \cap [\underline{u}, \bar{u}]$. This is impossible because, by the choice of r_t, we have $\| u \| \leq r_t - 1$. We have showed that H is admissible, since $H(0,.) = I$, (2.40) follows.■

We are now in a position to prove the main result in this section.

Theorem 2.42 Let $(F_1 - F_3)$ hold. Then there exist $t_o \in \mathbb{R}$ such that (P_t) has

no solution for $t > t_o$

at least a solution for $t = t_o$

at least two solutions for $t < t_o$

Proof Of course, t_o is the same that in theorem (2.31). We have only to show that there exist at least a solution of (P_{t_o}) and at least two solutions of (P_t) for $t < t_o$.

For the first part let u_n be a solution of $(P_{t_o - \frac{1}{n}})$, whose existence is ensured by theorem (2.31). Lemma (2.34) ensures that $(u_n)_n$ is a bounded sequence in F; by Schauder's estimates and Schauder's embedding theorems, we can prove that $(u_n)_n$ has a compact subsequence $(u_{n_k})_k$. It is apparent that $u = \lim_k u_{n_k}$ is a solution of (P_{t_o}). Let now $t < t_o$. By lemma (2.39), there exists B_t such that (2.40) holds; by lemma (2.34) we can choose R_t such that

$$B_t \subset B(0, R_t)$$

and if u is such that $u = T_s u$ for $t \leqslant s$, then: $\| u \| < R_t$. We consider the homotopy $H: [t, t_o + 1] \times B(0, R_t) \to F$ defined by:

$$H(s,u) = u - T_s u$$

Our choice of R_t proves the admissibility of H. Therefore:

$$\deg(I - T_t, B(0,R_t), 0) = \deg(I - T_{t_o+1}, B(0,R_t), 0) = 0 \qquad (2.44)$$

the last equality holds since (P_{t_o+1}) has no solution. From (2.40), (2.43) it follows that $I - T_t$ has a solution on B_t and a solution on $B(0,R_t) - B_t$.■

§ 3. FURTHER MULTIPLICITY RESULTS.

The previous theorem shows the existence of at least two solutions of (P_t), for $t < t_o$. Actually, if we take t small enough, we can prove in many cases the existence of more solutions. The first result in this sense was in [30] , we present it here under weaker assumptions following [48] . We ask (2.11) in the stronger form

$$f(x,t) - f_+ t^+ + f_- t^- \qquad \text{is bounded} \qquad (2.45)$$

Condition (2.45) which, of course, implies (F_2), can be considerably relaxed; we assume it here to simplify the computations of the "a priori" estimates which would be otherwise much more tedious.

We denote $f(x,t) - f_+ t^+ + f_- t^-$ by $g(x,t)$. For given $t \in \mathbb{R}, \Psi \in L^2(\Omega)$ we consider

$$\begin{aligned} -\Delta v &= \lambda v + g(x, v + t\Psi) && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega \end{aligned} \qquad (2.46_t)$$

We state an easy lemma whose proof is immediate by the L^P estimates for the linear elliptic equations.

Lemma 2.47 Let v be a solution of (2.46_t) for some $t \in \mathbb{R}$ and $\lambda \neq \lambda_j$. Then there exists a constant C, depending only on g and such that

$$\| u \|_{1,\infty} \leq C. \tag{2.48}$$

From lemma 2.47 it follows:

Lemma 2.49 Let $\lambda \neq \lambda_j$ and consider

$$\begin{aligned} -\Delta u &= \lambda u + g(x,u) + t\phi_1 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{2.50_t}$$

If $\lambda < \lambda_1$, there exist $\tau_{1,2} \in \mathbb{R}$ such that every solution of (2.50_t) is negative if $t < \tau_1$ and is positive if $\tau_2 < t$. If $\lambda_1 < \lambda$, there exist $\tau_{1,2} \in \mathbb{R}$ such that every solution of (2.50_t) is positive if $t < \tau_1$ and is negative if $\tau_2 < t$.

Proof Let $\lambda < \lambda_1$ and let u be a solution of (2.50_t); then $v = u - (\lambda - \lambda_1)^{-1}\phi_1$ is a solution of (2.46_t) for $\Psi = (\lambda_1 - \lambda)^{-1}\phi_1$. Using the bound (2.48) and the strong positiveness of ϕ_1, we see that u is negative for t large negative and positive fot t large positive.■

Proposition 2.51 Let (F_1), (2.45) hold with $f_- < \lambda_1 < f_+ \neq \lambda_j$. Then $\exists \tau \in \mathbb{R}$ such that (P_t) has a positive and a negative solution if $t < \tau$.

Proof We solve (2.50_t) taking $\lambda = f_-$. For $\tau \leq \tau_2$, every solution of (2.50_t) is negative and therefore as one can easily see, is a solution of (P_t). Taking $\lambda = f_+$, we find a positive solution.■

We have seen that we can solve (P_t) reducing it to an asymptotically linear problem of the type (2.50_t). Our next step will be to recognize that these solutions preserve their variational characterisations as solutions of (2.50_t).

We will see later on that also the local degree is preserved.

We shall work with the Hilber space $H = H^1_0(\Omega)$. We define on H the functionals:

$$I_t(u) = \frac{1}{2}(\|u\|^2 - f_+|u^+|^2 - f_-|u^-|^2) - \int_\Omega G(x,u)dx - t\int_\Omega \phi_1 u$$

From now on, we denote by $\|\cdot\|$ the norm of H (i.e.: $\|u\|^2 = \int_\Omega |\nabla u|^2$) and by $|\cdot|$ the L^2 norm. We set $G(x,s) = \int_0^s g(x,r)dr$. It is well-known that, given t, I_t is a C^1 functional on E and that the critical points of I_t are precisely the solutions of (P_t) for general references about variational methods see[43], [44],[37] . Let:

$$B_{t,r} = B\left(\frac{t}{\lambda_1 - f_-}\phi_1 , r\right).$$

We will prove that, if r is suitably chosen, than $B_{t,r}$ contains a local minimum of I_t. To this aim, it will be convenient to consider on H the functionals:

$$I_t^{\pm}(u) = \frac{1}{2}(\|u\|^2 - f_\pm|u|^2) - \int_\Omega G(x,u) - t\int_\Omega u\phi_1$$

$$J_t^{\pm}(u) = \frac{1}{2}(\|u\|^2 - f_\pm|u|^2) - \int_\Omega G_t^{\pm}(x,u)$$

where: $G_t^{\pm}(x,s) = \int_0^s g(x,r + \frac{t}{\lambda_1 - f_\pm}\phi_1(x))dr.$

<u>Lemma 2.52</u> <u>There exist $\bar{r} \in \mathbb{R}_+$ such that $\forall t \in \mathbb{R}$, $\forall u \in E$, $\|u\| > \bar{r}$:</u>

$$J^-(u) \geq \frac{\lambda_1 - f_-}{4\lambda_1}\|u\|^2$$

<u>Proof</u> It is easy to see, by (2.45), that $\int_\Omega G_t(x,u) \leq c|u|$, where the the constant c does not depend on t. Therefore:

$$J_t^-(u) \geq \frac{\lambda_1 - f_-}{2\lambda_1}\|u\|^2 - c|u| \geq \frac{\lambda_1 - f_-}{4\lambda_1}\|u\|^2$$

provided: $||u|| > \bar{r} = \dfrac{\lambda_1 - f_-}{4\lambda_1 c}$.■

Let us denote by $V_n = Sp\{\phi_1, \phi_2, \ldots, \phi_n\}$ the space spanned by the first n eigenfunctions of $-\Delta$. We remark that, by the strong positivity of ϕ_1 in $C^1_o(\Omega)$, we can prove

<u>Lemma 2.53</u> <u>$\forall r \in \mathbb{R}$ $\forall n \in \mathbb{N}$, $\exists a(r,n) \in \mathbb{R}$ such that $\forall t \in \mathbb{R}$, $t < a(r,n)$:</u>

$$B_{t,r} \cap V_n \subset -P$$

P, as before, denotes the set of the positive valued functions on Ω.

We are now in a position to prove the existence of a local minimum in $B_{t,r}$.

It is an immediate consequence of the next proposition.

<u>Proposition 2.54</u> <u>If $t < a(r,n)$, $r \geq \bar{r}$: $\forall u \in \partial B_{t,r}$</u>

$$I_t(u) - I_t\left(\frac{t}{\lambda_1 - f_-}\phi_1\right) \geq \left(\frac{\lambda_1 - f_-}{4\lambda_1} - \frac{f_+ - f_-}{2\lambda^2_{n+1}}\right) r^2 \tag{2.55}$$

<u>Proof</u> We note that the functionals

$$I^{\pm}_t(u) - J^{\pm}_t\left(u - \frac{t}{\lambda_1 - f_\pm}\phi_1\right)$$

are constants on u, i.e., since $J^{\pm}_t(0) = 0$

$$I^{\pm}_t(u) - I^{\pm}_t\left(\frac{t}{\lambda_1 - f_\pm}\phi_1\right) = J^{\pm}_t\left(u - \frac{t}{\lambda_1 - f_\pm}\phi_1\right) \tag{2.56}$$

For $t \leq 0$, since $\phi_1 \geq 0$ and $f_- < \lambda_1 < f_+$, we have $I_t\left(\frac{t}{\lambda_1 - f_\pm}\phi_1\right) = I^{\pm}_t\left(\frac{t}{\lambda_1 - f_\pm}\phi_1\right)$.
We get, using (2.56) and lemma 2.52, for $u \in \partial B_{t,r}$

$$I^-_t(u) - I^-_t\left(\frac{t}{\lambda_1 - f_-}\phi_1\right) \geq \frac{\lambda_1 - f_-}{4\lambda_1} r^2 \tag{2.57}$$

and, as a consequence

$$I_t(u) - I_t\left(\frac{t}{\lambda_1 - f_-}\phi_1\right) \geq \frac{\lambda_1 - f_-}{4\lambda_1} r^2 + I_t(u) - I^-_t(u). \tag{2.58}$$

Therefore, we only need an estimate for $I_t(u) - I^-_t(u)$. This for, write $u = v_n + w_n$, with $v_n \in V_n$, $w_n \in V_n^\perp$. Noting that $t < a(r,n)$ implies $v_n \leq 0$, we have:

$$I_t^-(u) - I_t(u) = \frac{1}{2}(f_+ - f_-)\,|u^+|^2 \leq \frac{1}{2}(f_+ - f_-)\,|w_n^+|^2 \leq$$

$$\leq \frac{1}{2}(f_+ - f_-)\,|w_n|^2 \leq \frac{f_+ - f_-}{2\,\lambda_{n+1}^2}\, r^2$$

From the last inequality and (2.58), (2.55) follows.■

<u>Corollary 2.59</u> <u>If n is chosen large enough, for $t < a(r,n)$, I_t has a local minimum on $B_{t,r}$.</u>

<u>Remark 2.60</u> It can be shown that this local minimum can be chosen as the negative solution found in proposition 2.51. In fact, the negative solution is a local minimum of I_t^- and is strongly negative in $C_o^1(\Omega)$. By standard regularity arguments, it follows that it is also a local minimum for I_t. This provides an alternative proof of corollary 2.59. Since we want to prove the existence of a third solution of (P_t), we can suppose that the solutions which we find by variational methods are the same that in proposition 2.51. This will allow us to use the sign characterisation when it will be convenient.■

We denote by $v_{t,r}$ the absolute minimum of I_t in $B_{t,r}$.
We now have to study the behaviour of I_t in a neighbourhood of the positive solution. From now on we are assuming that $\lambda_2 < f_+ \neq \lambda_j$ and, therefore, proposition 2.51 holds. Let $k \geq 2$ be the positive integer such that $\lambda_k < f_+ < \lambda_{k+1}$.
We set:

$$\rho = \frac{\lambda_{k+1}(f_+ - f_-)}{f_+(\lambda_{k+1} - f_-)} + 1$$

and

$$C_r = (B(0,r)\cap V) \times (B(0,\rho r)\cap W)$$
$$\partial_1 C_r = (\partial B(0,r)\cap V) \times (B(0,\rho r)\cap W)$$
$$\partial_2 C_r = (B(0,r)\cap V) \times (B(0,\rho r)\cap W)$$

where $V = V_k = Sp\{\phi_1, \phi_2, \ldots, \phi_k\}$ and $W = V^{\perp}$.

Finally we use the notation:

$$C_{t,r} = \frac{t}{\lambda_1 - f_+}\phi_1 + C_r \qquad \partial_i C_{t,r} = \frac{t}{\lambda_1 - f_+}\phi_1 + \partial_i C_r \qquad i = 1,2$$

For every $t,r \in \mathbb{R}$ we can choose, since $\lim_{\alpha\to+\infty} I_t(\alpha\phi_1) = -\infty$, $u_{t,r} \in H \setminus (B_{t,r} \cup C_{t,r})$ such that: $I_t(u_{t,r}) \leq I_t(v_{t,r})$. Let $\Gamma_{t,r}$ be the set of all the paths in E joining $v_{t,r}$ and $u_{t,r}$. Let:

$$c_{t,r} = \inf_{\gamma\in\Gamma_{t,r}} \sup_{\gamma} I_t$$

$$\Gamma'_{t,r} = \{\gamma\in\Gamma_{t,r} \mid \gamma \cap \mathring{C}_{t,r} = \emptyset\}$$

$$c'_{t,r} = \inf_{\gamma\in\Gamma'_{t,r}} \sup_{\gamma} I_t$$

We wish to prove:

Proposition 2.61 There exist $\tau, r \in \mathbb{R}$ such that $\forall t \leq \tau$: $C_{t,r}$ contains in its interior a critical point $\bar{u}_{t,r}$ which, if moreover $c_{t,r} \neq c'_{t,r}$, can be chosen such that $I_t(\bar{u}_{t,r}) \neq c_{t,r}$.

We will prove this result by a series of lemma.

Lemma 2.62 There exists $\bar{r} \in \mathbb{R}$ such that $\forall t \in \mathbb{R}$, $\forall r \geq \bar{r}$, I_t^+ has a critical point $\bar{u}_{t,r}$ such that:

$$\inf_{\partial_2 C_{t,r}} I_t^+ > I_t^+(\bar{u}_{t,r}) \tag{2.63}$$

Proof We use (2.56) and therefore we have to prove the existence of $\bar{r} \in \mathbb{R}$ such that $\forall t\in\mathbb{R}$, $\forall r \geq \bar{r}$, there exist a critical point $\bar{u}$ such that:

$$\inf_{\partial_2 C_r} J_t^+ > J_t^+(\bar{u}) \tag{2.64}$$

By (2.56), $\bar{u}_{t,r} = \bar{u} + \dfrac{t}{\lambda_1 - f_+}\phi_1$ satisfies (2.63). To find $\bar{u}$, we consider all the regular deformations η on E which leave fixed $\partial B_r \cap V$. It is easy proved that $\exists\, \bar{r}$ such that for $r \geq \bar{r}$: $\quad s_{t,r} = \inf_{\eta} \sup_{\eta(B_r\cap V)} J_t^+ > \sup_{\partial B_r \cap V} J_t^+$

It is well-known that this implies the existence of a critical point $\bar{u}$ at level $s_{t,r}$. If $\bar{r}$ is chosen large enough, one can also prove that $\bar{u} \in C_r$. All this estimates are independent on t. To prove (2.64) it is enough to take η = identity and to observe that

$$\inf_{\partial_2 C_r} J_t^+ > \sup_{B_r \cap V} J_t^+ \tag{2.65}$$

For the proof of (2.65) let $u \in \partial_2 C_r$ and $v' \in B_r \cap V$. We write: $u = v + w$, $v \in V$ $w \in W$.

$$J_t^+(u) - J_t^+(v) \geq \frac{\lambda_{k+1} - f_+}{\lambda_{k+1}} ||w||^2 - \frac{f_+ - \lambda_1}{\lambda_1} ||v||^2 - \int_\Omega G_t u + \int_\Omega G_t v' \geq$$

$$\geq \rho\left(\frac{\lambda_{k+1} - f_+}{\lambda_{k+1}} - \frac{f_+ - \lambda_1}{\lambda_1}\right) r^2 - cr$$

Our choice of ρ is such that: $c' = \rho\left(\frac{\lambda_{k+1} - f_+}{\lambda_{k+1}} - \frac{f_+ - \lambda_1}{\lambda_1}\right) > 0$.

For $\bar{r} = cc'^{-1}$ one gets (2.65) when $r \geq \bar{r}$.■

We set: $\ell = \frac{\lambda_k(\lambda_1 - f_+)}{\lambda_1(\lambda_k - f_+)} + 1$ and consider:

$S_r = (1 + \ell)\, \partial B_r \cap V$. Given $u \in \partial_1 C_{t,r}$, if one sets $u - \frac{t}{\lambda_1 - f_+}\phi_1 = v + w$, $v \in V$ $w \in W$, the projection of u on $S_{t,r} = S_r + \frac{t}{\lambda_1 - f_+}\phi_1$ is $v'_{t,r} = \frac{t}{\lambda_1 - f_+}\phi_1 + (1 + \ell)v$. Therefore, the straight line path joining u with $v'_{t,r}$ has equation:

$$\sigma_{t,r}(\alpha) = \frac{t}{\lambda_1 - f_+}\phi_1 + (1 - \alpha)(v + w) + \alpha(\ell + 1)v =$$

$$= \frac{t}{\lambda_1 - f_+}\phi_1 + (1 + \alpha\ell)v + (1 - \alpha)w$$

for $\alpha \in [0,1]$.

Lemma 2.66 There exist $\bar{r} \in \mathbb{R}$ such that $\forall t \in \mathbb{R}$, $\forall r \geq \bar{r}$:

i) $\forall u \in \partial_1 C_{t,r}$ $\forall \alpha \in [0,1]$: $(\nabla I_t^+(\sigma_{t,r}(\alpha)), v'_{t,r} - u) \leq -cr^2$

ii) $\sup_{S_{t,r}} I_t^+ < \inf_{C_{t,r}} I_t^+$

Proof Using (2.56), one sees that (i) and (ii) are respectively equivalent to:

$$\left(\nabla J_t^+\left(\sigma_{t,r}(\alpha) - \frac{t}{\lambda_1 - f_+}\phi_1\right), \ell v - w\right) \leq -cr^2 \qquad (2.67)$$

$$\sup_{S_r} J_t < \inf_{C_r} J_t \qquad (2.68)$$

in (2.67) we have used: $v'_{t,r} - u = \ell v - w$.

(2.67) follows from:

$$\left(\nabla J_t^+\left(\sigma_{t,r}(\alpha) - \frac{t}{\lambda_1 - f_+}\phi_1\right), \ell v - w\right) = \left(\nabla J_t^+((1+\alpha\ell)v + (1-\alpha)w), \ell v - w\right) \leq$$

$$\leq \frac{\lambda_k - f_+}{\lambda_k}\|v\|^2 + \int_\Omega g((1+\alpha\ell)v + (1-\alpha)w)\,(\ell v - w) \leq \frac{\lambda_k - f_+}{\lambda_k} r^2 + c'r \leq -cr$$

if $r > \bar{r} = \dfrac{c'\lambda_k}{f_+ - \lambda_k}$

To prove (2.68) fix $v' \in S_r$, $u = v + w \in C_r$:

$$J_t^+(u) - J_t^+(v') \geq \frac{\lambda_1 - f_+}{\lambda_1}\|v\|^2 - \frac{\lambda_k - f_+}{\lambda_k}\|v'\|^2 - \int_\Omega G_t(u) + \int_\Omega G_t(v') \geq$$

$$\geq \left(\frac{f_+ - \lambda_k}{\lambda_k}\ell + \frac{\lambda_1 - f_+}{\lambda_1}\right) r^2 - cr.$$

By definition of ℓ: $c' = \dfrac{f_+ - \lambda_k}{\lambda_k}\ell + \dfrac{\lambda_1 - f_+}{\lambda_1} > 0$

Therefore we get (2.68) for $r > \bar{r} = cc'^{-1}$. ■

We have to transfer the estimates in the previous proposition from the functional I_t^+ to the functional I_t. In order to do so, we argue as in lemma 2.53. We denote by $T_{t,r}$ the convex hull of $C_{t,r} \cup S_{t,r}$ and remark that, given $n \in \mathbb{N}$, we can choose $b(r,n) \in \mathbb{R}$ such that $\forall t < b(r,n)$: $T_{t,r} \cap V_n \subset P$. The following lemma is analogous to proposition (2.54).

Lemma 2.69 Let $r > \bar{r}$ be given. There exists $\tau \in \mathbb{R}$ such that $\forall t < \tau$ there exists a critical point $\bar{u}_{t,r}$ of I_t in $C_{t,r}$ such that:

i) $\inf_{\partial_2 C_{t,r}} I_t > I_t(\bar{u}_{t,r})$

ii) $\forall u \in C_{t,r}$, $\forall \alpha \in [0,1]: (\nabla I_t(\sigma_{t,r}(\alpha), v'_{t,r} - u) \leq cr^2$

iii) $\sup_{S_{t,r}} I_t < \inf_{C_{t,r}} I_t$

Proof We apply lemma 2.49 to prove that the critical point $\bar{u}_{t,r}$ found in 2.62 is positive. Therefore $\bar{u}_{t,r}$ is a critical point of I_t and (i) follows trivially from (2.63) since $I_t^+ - I_t$ and $I_t(\bar{u}_{t,r})$. To prove (ii) and (iii) we apply lemma 2.66 arguing as in proposition 2.54 – corollary 2.59.■

We can now prove proposition 2.61.

Proof of proposition 2.61 We take $\bar{u}_{t,r}$ as in the previous lemma. Suppose, by contradiction, that $I_t(\bar{u}_{t,r}) = c_{t,r}$. Using (i), we choose $\gamma \in \Gamma_{t,r}$ such that $\sup_\gamma I_t < \inf_{\partial_2 C_{t,r}} I_t$. Since, of course, $c_{t,r} \leq c'_{t,r}$, we can, by contradiction, also suppose $\sup_\gamma I_t < c'_{t,r}$. It follows: $\gamma \cap \dot{C}_{t,r} \neq \emptyset$ and $\gamma \cap \partial_2 C_{t,r} = \emptyset$.
Let $s_1 = \inf\{ s \in [0,1]: \gamma(s) \in C_{t,r}\}$ and $s_2 = \sup\{s \in [0,1] : \gamma(s) \in C_{t,r}\}$.
Since $\gamma \cap \partial_2 C_{t,r} = \emptyset$ and $v_{t,r}, u_{t,r} \notin C_{t,r}$, we deduce that $\gamma(s_i) \in \partial_1 C_{t,r}$, $i = 1,2$. Therefore, we can consider the projections v_i of $\gamma(s_i)$ on $S_{t,r}$. We define:

- γ_1 the restriction of γ to $[0,s_1]$
- γ_2 the straight-line path from $\gamma(s_1)$ to v_1
- γ_3 a path joining v_1 and v_2 in $S_{t,r}$
- γ_4 the straight-line path from v_2 to $\gamma(s_2)$
- γ_5 the restriction of γ to $[s_2, 1]$

and, finally, γ' the path obtained joining γ_i, $i = 1\ldots5$. We claim that $\sup_{\gamma'} I_t \leq \sup_\gamma I_t$. It is equivalent to show $\sup_{\gamma_i} I_t \leq \sup_\gamma I_t$ for $i = 1\ldots5$. This is trivial for $i = 1$ and $i = 5$, while (ii) shows that I_t is decreasing along γ_2 and increasing along γ_4 and therefore proves the cases $i = 2,4$, (iii) proves the case $i = 3$ since $\gamma_3 \subset S_{t,r}$ and $\gamma \cap C_{t,r} \neq \emptyset$. We have so found $\gamma' \varepsilon \Gamma'_{t,r}$ such that $\sup_{\gamma'} I_t < c'_{t,r} = \inf_{\gamma \varepsilon \Gamma'_{t,r}} \sup_{\gamma} I_t$, a contradiction.■

We are now ready to prove that (P_t) has at least three solutions if $f_- < \lambda_1 < \lambda_2 < f_+ \neq \lambda_j$ and t is large negative. The proof will be carried out by using a variational method, precisely a variant of the mountainpass theorem of Ambrosetti-Rabinowitz ([8] theorem 2.1). Let us recall a well-known definition.

<u>Definition 2.70. Let I be a C^1 functional on H. We say that I satisfies the [P.S.] condition iff every given sequence $(u_n)_n$ of points of H such that:</u>

(1) $I(u_n)$ is bounded

(2) $\lim_n \nabla I(u_n) = 0$

<u>has a convergent subsequence.</u>

<u>Proposition 2.71</u> <u>$\forall t \varepsilon \mathbb{R}$; I_t verifies [P.S] .</u>

<u>Proof</u> Actually, to prove that $(u_n)_n$ has a convergent subsequence, we use only (2). We claim that, under (2), $|u_n|$ is bounded. Suppose by contradiction that this is not true, i.e. passing to a subsequence $|u_n| \to +\infty$ and set: $v_n = |u_n|^{-1} u_n$. Since:

$$\nabla I_t(u_n) = - \Delta u_n - f_+ u_n^+ + f_- u_n^- - gu_n + t\phi_1 \tag{2.72}$$

dividing by $|u_n|$:

$$- \Delta v_n - f_+ v_n^+ + f_- v_n^- = |u_n|^{-1}(\nabla I_t(u_n) + gu_n - t\phi_1) \tag{2.73}$$

From our assumptions, it follows that the right-hand side of (2.73) converges to 0 in $H^{-1}(\Omega)$ as $n \to +\infty$. Therefore, $\| \Delta v_n \|_{-1}$ is bounded and $(v_n)_{n \varepsilon \mathbb{N}}$ has a convergent subsequence in $L^2(\Omega)$ to a limit point $v \varepsilon H$, $|v| = 1$. Taking the limit of both sides of (2.73), one finds:

$$- \Delta v = f_+ v^+ - f_- v^- \tag{2.74}$$

this is a contradiction, since (2.74) cannot have a nontrivial solution v if $f_- < \lambda_1 < f_+$. Therefore, $(u_n)_n$ is bounded and, passing to a subsequence, $(u_n)_n$ is convergent in $L^2(\Omega)$. This, by (2) - (2.72), implies that Δu_n converges in $H^{-1}(\Omega)$ and finally that $(u_n)_n$ is convergent in E.■

In [8] is proved the following deformation lemma. Given I, we set $\forall c \in \mathbb{R}$:

$$K_c = \{ u \in H \mid \nabla I(u) = 0,\ I(u) = c \}$$

$$A_c = \{ u \in H \mid I(u) \leq c \}$$

Proposition 2.75 Let I satisfy [P.S.] and N be a given neighbourhood of K_c for $c \in \mathbb{R}$. there exist $\varepsilon > 0$ as small as we want, and a deformation $\eta : [0,1] \times E \to E$ such that:

(i) $\eta(0,x) = x \qquad \forall\, x \in H$

(ii) $\eta(t,x) = x \qquad \forall\, x \in A_{c-2\varepsilon} \cup (H \setminus A_{c+2\varepsilon}) \quad \forall t \in [0,1]$

(iii) $\eta(1,x) \in A_{c-\varepsilon} \qquad \forall\, x \in A_{c+\varepsilon} \setminus N$

The following is a first improvement of Theorem 2.42.

Theorem 2.76 Let (F_1), (2.45) hold and $f_- < \lambda_1 < \lambda_2 < f_+ \neq \lambda_j$. Then $\exists\, \tau \in \mathbb{R}$, such that, for $t < \tau$, (P_t) has at least three distinct solutions.

Proof By corollary 2.59, for t small enough, I_t has a local minimum $v_{t,r}$. By [8] theorem 2.1, $c_{t,r}$ is a critical level and, of course, $I_t(v_{t,r}) < c_{t,r}$. For t possibly smaller, we have also the solution $\bar{u}_{t,r}$, therefore, if $I_t(\bar{u}_{t,r}) \neq c_{t,r}$, we have three solutions.

Suppose $I_t(\bar{u}_{t,r}) = c_{t,r}$. By proposition (2.61): $c_{t,r} = c'_{t,r}$, in this case we find three solutions if we show that $K_{c_{t,r}} \neq \{ \bar{u}_{t,r} \}$. Suppose by contradiction that this is not the case, then proposition 2.75 holds with $N = C_{t,r}$. By definition of $c'_{t,r}$ we can take $\gamma' \in \Gamma'_{t,r}$, $\gamma' \subset A_{c+\varepsilon} \setminus N$. For $\varepsilon < \frac{1}{2} \min (c_{t,r} - I(v_{t,r}),\ c_{t,r} - I(u_{t,r}))$, $\gamma = \eta(1,\,.) \circ \gamma' \in \Gamma_{t,r}$. By proposition 2.75 (iii) $\sup_\gamma I_t \leq c_{t,r} - \varepsilon$, a contradiction.■

In [25] it is proved that the number of solutions is actually four if I_t is C^2. We do not prove this result but will give an idea of the proof and will show a simpler but weaker related result in [50] . First we have to compute the Leray-Schauder degree of some mappings. Since we will need these results later on in a more general setting, we are assuming on $f_\pm$ only that $f_\pm \neq \lambda_j$, and denoting by $k_\pm$ the positive integers such that: $\lambda_{k_\pm} < f_\pm < \lambda_{k_\pm + 1}$ ($k_\pm = 0$ if $f_\pm < \lambda_1$). Let K be the resolvent operator $(-\Delta)^{-1}$ on Ω , with the homogeneous Dirichlet boundary conditions, and

$$F_t(u) = u - K(f(x,u) + t\,\psi) \tag{2.77}$$

$$F_t^{\pm}(s,u) = u - K(f_{\pm}\, u + sg(x,u + \frac{t}{\lambda_1 - f_{\pm}}\,\phi_1)) \tag{2.78}$$

From now on we are assuming the conditions (F_1) and (2.45) and are taking $\psi = \phi_1$. The functions F_t, $F_t^{\pm}$ are defined on $E = L^2(\Omega)$. Our aim is, roughly speaking, to compute the local degree of F_t in $v_{t,r}$ and $\bar{u}_{t,r}$. We begin, following [49], with a technical lemma. From now on, in this section, by $\|\cdot\|$ is denoted the norm of E.

Lemma 2.79 $\exists\, \varepsilon, r \in \mathbb{R}_+$ such that $\forall u \in E$ which verifies:

$$\| F_t^{\pm}(s,u) \| < \varepsilon \|u\| \tag{2.80}$$

for some $t \in \mathbb{R}$, $s \in [0,1]$, it holds: $\|u\| > r$.

Proof Since $f_{\pm} \neq \lambda_j$ $\exists\, \varepsilon \in \mathbb{R}_+$ such that: $\| u - f_{\pm} K u \| > 2\varepsilon \|u\|$. If (2.80) holds with such a ε, we get: $\| g(u + \frac{t}{\lambda_1 - f_{\pm}}\phi_1) \| \geq \varepsilon \|u\|$ and, therefore, the statement with $r = \varepsilon^{-1} |\Omega|^{1/2} \max|g|$. ■

We fix r as in the previous lemma and set

$$i_t^{\pm} = \deg(F_t^{\pm}(1,.), B(0,r), 0)$$

$i_t^{\pm}$ are well defined by lemma (2.79) and the complete continuity of K. It is easy to compute $i_t^{\pm}$:

Lemma 2.81 $\forall t \in \mathbb{R} : \; i_t^{\pm} = (-1)^{k_{\pm}}$

Proof $F_t^{\pm}(0,.)$ is a linear operator and:

$$\deg(F_t(0,.), B(0,r), 0) = (-1)^{k_{\pm}}.$$

Lemma 2.79 ensures the admissibility of the homotopy $F_t^{\pm}$. This proves the statement. ■

Now we set, as before, $V_n = Sp\{\phi_1, \ldots, \phi_n\}$ and claim that, by the strong positivity of ϕ_1, $\forall n \in \mathbb{N}$ $\exists\, a_n^{\pm} \in \mathbb{R}$ such that:

$$B_t^{\pm} = B(\frac{t}{\lambda_1 - f_{\pm}}\,\phi_1, r) \cap V_n \subset \pm P$$

for $t < a_n^+$, in the + case, and $a_n^- < t$ in the $-$case if $\lambda_1 < f_{\pm}$ and for $t < a_n^{\pm}$ if $f_- < \lambda_1 < f_+$. We use this to prove:

Lemma 2.81 $\exists n \in \mathbb{N}$ such that:

$$\deg(F_t, B_t^{\pm}, 0) = (-1)^{k_\pm} \tag{2.82}$$

for $t < a_n^+$ or $a_n^- < t$, if $\lambda_1 < f_\pm$, and for $t < a_n^\pm$ if $f_- < \lambda_1 < f_+$.

Proof We prove the +case. Consider the homotopy

$$\eta_t(s,u) = F_t(u) + s(f_+ - f_-)\, Ku^- \tag{2.83}$$

Note that $\eta_t(1,u) = F_t^+(1,u - \frac{t}{\lambda_1 - f_+}\phi_1)$, therefore, by lemma 2.81, we have by translation:

$$\deg(\eta_t(1,.), B_t^+, 0) = (-1)^{k_\pm} \tag{2.84}$$

(2.82) follows from (2.84) if we prove that η_t is an admissible homotopy if $t < a_n^+$ for a suitable n. To this aim we note that, by lemma 2.79, $\forall t \in \mathbb{R}$, $\forall u \in \partial B_t^+$:

$$\| \eta_t(1,u) \| \geq \varepsilon \| u - \frac{t}{\lambda_1 - f_+}\phi_1 \| \tag{2.85}$$

By contradiction, suppose $\eta(s,u) = 0$ for some $t < a_n^+$, $s \in [0,1]$, $u \in \partial B_t^+$. Write: $u = v_n + w_n$, with $v_n \in V_n$, $w_n \in W_n$. From (2.85) we get:

$$\|u^-\| \geq \frac{\varepsilon}{(f_+ - f_-)\|K\|} \quad \|u - \frac{t}{\lambda_1 - f_+}\phi_1\| = \frac{\varepsilon r}{\|K\|(f_+ - f_-)} \tag{2.86}$$

since $t < a_n^+$: $v_n \in B_t^+ \cap V_n \subset P$; therefore

$$\|u^-\| = \|(v_n + w_n)^-\| \leq \|w_n^-\| \leq \|w_n\| \tag{2.87}$$

by (2.86) - (2.87):

$$\|w_n\| \geq \frac{\varepsilon r}{\|K\|(f_+ - f_-)}$$

and, therefore:

$$\| -\Delta u - f_+ u - t\phi_1 \| \geq \| \Delta w_n - f_+ w_n \| \geq \frac{\varepsilon r(\lambda_{n+1} - f_+)}{\|K\|(f_+ - f_-)}$$

while, using again $\eta_t(s,u) = 0$, we get:

$$\|-\Delta u - f_+ u - t\phi_1\| \leq (1-s)(f_+ - f_-)\,\|u^-\| + \|gu\| \leq (f_+ - f_-)r + c \qquad (2.89)$$

where (2.87) is used to show: $\|u^-\| \leq r$ and c is a constant whose existence is stated in (2.45). From (2.88) - (2.89) we get:

$$\frac{\varepsilon r(\lambda_{n+1} - f_+)}{\|K\|\,(f_+ - f_-)} - (f_+ - f_-)r - c \leq 0 \qquad (2.90)$$

which is a contradiction if n is chosen large enough. The -case is treated in a similar way in both cases $\lambda_1 < f_-$ or $f_- < \lambda_1$.■

Since B_t^+ is a neighbourhood of $\bar{u}_{t,r}$ and $\bar{B}_t$ is a neighbourhood of $v_{t,r}$ we have showed that these two solutions have odd local degree, for t large negative. It follows, in view of lemma 2.34 and 2.40, that whenever also the third solution found in theorem 2.76 has odd local degree we must have also a fourth solution. In [25] is showed, by using at the same time the degree theory, the calculus of variations and the order methods, that the solution found by the mountainpass theorem has local degree -1. We refer to [25] for the interested reader and treat here a particular case (i.e. asking $\lambda_2 < f_+ < \lambda_3$ inside of $\lambda_2 < f_+ \neq \lambda_j$) in which the existence of four solutions can be proved using a simpler device which will be useful also later on, following [50]. First we use some "a priori" bounds for a class of parametrized problems, assuming : $f_- < \lambda_1 < \lambda_2 < f_+ < \lambda_3$.

<u>Lemma 2.91</u> <u>Let ε be a fixed real positive number, which we suppose chosen very small. Consider the problem:</u>

$$(2.92) \quad -\Delta u = f_+ u^+ - f_- u^- + s(\phi_1 + \varepsilon\phi_2) + \phi_2 - \varepsilon\phi_1 \qquad \text{in } \Omega$$

$$u = 0 \qquad \text{on } \partial\Omega$$

Let S be the closed hyperplane: $\mathbb{R}\cdot(\phi_1 + \varepsilon \dfrac{\lambda_1 - f_+}{\lambda_2 - f_+} \phi_2) + \{\phi_1, \phi_2\}^{\perp}$. Then $\forall\, s \in \mathbb{R}$: (2.92) has no solution $u \in S$.

Proof. Let Q be the orthogonal projector on E such that: $\ker Q = \mathbb{R}\cdot(\phi_2 - \varepsilon\phi_1)$. Suppose by contraddiction that $u \in S$ solves (2.92) for some $s \in \mathbb{R}$. Projecting (2.92) by Q we find that u is also a solution of:

$$-Q(\Delta v) = Q(f_+ v^+ - f_- v^-) + s(\phi_1 + \varepsilon\phi_2) \qquad \text{in } \Omega$$

(2.93)

$$v = 0 \qquad \text{on } \partial\Omega$$

We remark that (2.93) is a problem of the type considered in section 1. Precisely, we use lemma 2.5 to prove that v is a solution of (2.93) if $v = r(\phi_1 + \varepsilon \dfrac{\lambda_1 - f_+}{\lambda_2 - f_+} \phi_2) + w(r)$ where $\xi(r) = -s$ and ξ and w are two functions given by a Lyapunov-Schmidt reduction as in section 1. If we note that the projected equation on $\{\phi_1, \phi_2\}^{\perp}$ is positively homogeneous, we find easily that ξ is of the form:

(2.94) $\xi(r) = a_+ r^+ - a_- r^-$

with $\pm a_{\pm} > 0$. By the above arguments, we therefore see that (2.93) has exactly two solutions for $s < 0$, exactly one for $s = 0$ and no solution for $s > 0$. Let us try to find directly the solution of (2.93), for $s \leq 0$, of the form $v = r(\phi_1 + \varepsilon \frac{\lambda_1 - f_+}{\lambda_2 - f_+} \phi_2)$. With simple computations we find that v is a solution of (2.93) if :

(2.95) $b_+ r^+ - b_- r^- = s$

where:

$$\pm b_{\pm} = (1+\varepsilon^2)^{-1}\left(\lambda_1 + \varepsilon^2 \lambda_2 \frac{\lambda_1 - f_+}{\lambda_2 - f_+} - f_{\pm}\left(1 + \varepsilon^2 \frac{\lambda_1 - f_+}{\lambda_2 - f_+}\right)\right)$$

This implies, by the condition: $f_- < \lambda_{1,2} < f_+$, that $b_{\pm} < 0$, and, as a consequence, that (2.95) has the two solutions $r_{\pm} = \pm b_{\pm}^{-1} s$ of opposite sign, when $s \le 0$, and, therefore, that the two solutions of (2.93) for $s \le 0$ are:

$\pm b_{\pm}^{-1} s(\phi_1 + \varepsilon \frac{\lambda_1 - f_+}{\lambda_2 - f_+} \phi_2)$. Let us come back to our hypothesis that for some s, (2.92) has a solution $u \in S$. We have shown that u must be equal to $\pm b_{\pm}^{-1} s\, (\phi_1 + \varepsilon \frac{\lambda_1 - f_+}{\lambda_2 - f_+} \phi_2)$ for some $s \le 0$. Substituting in (2.92) and taking the scalar product of both sides with $\phi_2 - \varepsilon\phi_1$, we find:

$$(2.96)\quad \pm b_{\pm}^{-1}\, \varepsilon\, \left((\lambda_2 - f_{\pm}) \frac{\lambda_1 - f_+}{\lambda_2 - f_+} - \lambda_1 + f_{\pm}\right) s = 1 + \varepsilon^2$$

which gives: $s > 0$, a contraddiction.■

<u>Lemma 2.97. Let $c > 0$ be a fixed constant.</u>
<u>There exists $\bar{m} > 0$ such that the equation</u>

$$(2.98)\quad -\Delta u = f(x,u) + s\,(\phi_1 + \varepsilon\phi_2) + m\,(\phi_2 - \varepsilon\phi_1)$$

<u>has no solution $u \in S$, $u = 0$ on $\partial\Omega$, for some m,s such that $|s| \le c\, m$, $m \ge \bar{m}$</u>

Proof. Suppose by contraddiction that $\forall n \in \mathbb{N}$, (2.98) has a solution (u_n, s_n, m_n) with $m_n > n$, $|s_n| < cm_n$. We divide (2.98) by m_n obtaining, setting $v_n = m_n^{-1} u_n$:

$$(2.99) \quad -\Delta v_n = f_+ v_n^+ - f_- v_n^- + m_n^{-1} g(u_n) + m_n^{-1} s_n (\phi_1 + \varepsilon\phi_2) + \phi_2 - \varepsilon\phi_1$$

By our assumptions we can suppose, passing to a subsequence, that $m_n^{-1} s_n$ is convergent to some $s \in \mathbb{R}$. Suppose in addition that $(v_n)_n$ is a bounded sequence. Using (2.99) and passing to a subsequence, we can suppose $v_n \to v \in S$ in E. In this case v is a solution of (2.92) which is false. Therefore $(v_n)_n$ is unbounded. We divide (2.99) by $||v_n||$ and, taking $w_n = ||v_n||^{-1} v_n$, we can suppose, as before $w_n \to w$ in E, and therefore $||w|| = 1$. Taking the limit in (2.99) we see that:

$$(2.100) \quad -\Delta w = f_+ w^+ - f_- w^-$$

But it is easily seen that (2.100) cannot have non trivial solutions for $f_- < \lambda_1 < f_+$, since, taking the scalar product of (2.100) with ϕ_1, we get:

$$(2.101) \quad (f_+ - \lambda_1)\ (w^+, \phi_1) + (\lambda_1 - f_-)\ (w^-, \phi_1) = 0$$

which implies $\int_\Omega |w|\lambda_1 = 0$ and therefore $w = 0$. ■

Let us use the notation:

$$B^{\pm}(0,r) = \{u \in B(0,r) \mid \pm (u, \varepsilon \frac{\lambda_1 - f_+}{\lambda_2 - f_+} \phi_1 - \phi_2) > 0 \}$$

Lemma 2.102 $\exists \tau \in \mathbb{R}$ such that $\forall t < \tau \ \exists r_t$ such that:

$$(2.103) \quad \deg (F_t, B^+(0, r), 0) = 0.$$

Proof. Let us note that: $\phi_1 = \frac{1}{1 + \varepsilon^2} (\phi_1 + \varepsilon\phi_2) - \frac{\varepsilon}{1 + \varepsilon^2} (\phi_2 - \varepsilon\phi_1)$. We use the previous lemma with $c = \varepsilon^{-1}$ and take $\tau = -\bar{m}\, \varepsilon^{-1}(1 + \varepsilon^2)$. With this notation, we write $t\phi_1$, $t < \tau$ as $-cm\, (\phi_1 + \varepsilon\phi_2) + m\, (\phi_2 - \varepsilon\phi_1)$, with $m > \bar{m}$. We consider the homotopy:

$$(2.104) \quad H(s,u) = u - K(f(x,u) + s\, (\phi_1 + \varepsilon\phi_2) + m\, (\phi_2 - \varepsilon\phi_1))$$

defined on $[-cm, cm] \times E$. The previous lemma shows that H has no zeros on

$[-cm, cm]\times S$, while the results of section 2 show that H has no zeros with $||u|| \geq r_t$ for a suitable r_t. Therefore:

(2.105) $\deg(H(s, \cdot), B^{\pm}(0, r), 0)$ is constant for $s \in [-cm, cm]$

$\forall r > r_t$. Taking τ smaller, we can also suppose that $F(cm, \cdot)$ has no zeros on E. This can be seen taking the scalar product of both sides of (2.104) with ϕ_1 and using (2.45). It follows (2.103), since $F_t = H(-cm, \cdot)$.■

Lemma 2.81 - 2.102 allows us to prove the existence of four solutions of (P_t) when t is large negative.

Theorem 2.106 $\exists \tau \in \mathbb{R}$ such that (P_t) has at least four distinct solutions, for $t<\tau$.

Proof. We take τ such that $\tau \leq a_{\pm}(n)$, with n as in lemma 2.81, and such that $B_t^{\pm} \cap S \neq \emptyset$. This implies $B_t^{\pm} \subset B^{\pm}(0,r)$ for r large enough. Taking $\bar{r} \geq r_t$, where r_t is the constant found in the previous lemma and using (2.82), (2.103) and the additivity of topological degree we get

(2.107) $\deg(F_t, B, 0) \neq 0$

when $B = B_t^{\pm}$ or $B = B^{\pm}(o, r) \setminus B_t^{\pm}$.

This shows the existence of four distinct solutions.■

We close this section showing that, if $f_+ < \lambda_2$, it can happen that $\forall t \in \mathbb{R}$: (P_t) has no more than two solutions, this is precisely the case when f is convex on the t-variable. In fact, under this additional assumption theorem 2.21 holds in the following sharper form.

Theorem 2.107 Let f be given and (F_1), (2.11), $f_- < \lambda_1 < f_+$ hold, moreover assume that f is strictly convex on its last variable. Then $\exists t_o \in \mathbb{R}$ such that (P_t) has:

no solution for $t > t_o$

exactly a solution for $t = t_o$

exactly two solutions for $t < t_o$.

Proof. Since (P_t) is equivalent to $\xi = -t$, to prove the statement it is sufficient to show that $\forall t \in \mathbb{R}$: (P_t) has at most two solutions and then to apply theorem 2.21.

Let us consider the eigenvalue problem with weight:

$$(2.108) \qquad \begin{aligned} -\Delta u &= \mu g u \quad &&\text{in } \Omega \\ u &= 0 &&\text{on } \partial\Omega \end{aligned}$$

It is well known, [28], [35], that the positive values of μ for which (2.108) has non-trivial solutions are a diverging sequence:

$\mu_1(g) < \mu_2(g) < \ldots < \mu_k(g) < \ldots$ that μ_1 is simple and the corresponding eigenvectors have constant sign, and that if $g_1 < g_2$ then $\mu_k(g_2) < \mu_k(g_1)$ $\forall k$. Suppose, by contraddiction that there exists $t \in \mathbb{R}$, such that (P_t) has three distinct solutions u_i, $i = 1, 2, 3$. Since, $\forall$ i, j : $u_i - u_j$ is a solution of (2.108) for $\mu = 1$ and $g(x) = \dfrac{f(x, u_i(x)) - f(x, u_j(x))}{u_i(x) - u_j(x)} = g_{ij}(x)$ where $u_i(x) \neq$ $\neq u_j(x)$, and $g_{ij} \leq f_+ < \lambda_2$ by the convexity of f and (2.12) we have $\mu_k(g_{ij}) > \mu_k(\lambda_2) > 1$ for $k \geq 1$. Therefore $1 = \mu_1(g_{ij})$ and $u_i - u_j$ has constant sign. By this, we can suppose $u_1 < u_2 < u_3$. Therefore, by the strict convexity of f, we get: $g_{12} < g_{23}$ and, as a consequence, $\mu_k(g_{23}) < \mu_k(g_{12})$ $\forall k$. This is a contraddiction since we have shown that $\mu_1(g_{ij}) = 1$ $\forall i,j$. ■

3. THE CASE $\lambda_1 < f_\pm$

The general assumption made in this part makes the study of (BVP) somewhat different than in the previous case. For instance, it is not true that if $[f_-, f_+] \cap \sigma \neq \emptyset$ then phenomena of existence of zero-two solutions must occur; we shall prove later several results of existence of at least a solution for every known term. We have seen at the end of the last section that the equation (2.100) cannot have non-trivial solutions for $f_- < \lambda_1 < f_+$ (however we have used this implicitly many times, as to show that the functionals I_t satisfy the [P.S.] condition or to prove the "a priori" bounds in 2§2) while, assuming $\lambda_1 \leq f_\pm$, we shall see that it is in general a hard problem to understand whether or not (2.100) can have non-trivial solutions.

This larger variety of phenomena makes it more difficult to give a systematic exposition of the results under this assumption, therefore we shall restrict ourselves to consider some subcases of particular interest. In the first section we briefly treat the general problem of homogeneous equations; for every further comment we shall refer to [12]. In the second section we shall make the additional assumption that $[f_-, f_+]$ contains exactly an eigenvalue of $-\Delta$, which is simple. This is a case first studied in [41] and recently, more in general, in [20], [45]. After we show, following [47], how this approach can be used with eigenvalues of higher multiplicity, when the problem presents some particular symmetries. This result will be applied to some ordinary differential vector valued equations and to the wave equation. We shall obtain some existence results which hold in a much more general form for the scalar ordinary differential equations.

In the last section we shall give some multiplicity results obtained with similar methods to those used in the first part.

§1 STUDY OF THE HOMOGENEOUS EQUATION

In this section we are concerned with equations of the type (2.100) without any restrictive assumption of $f_\pm$. As we have said before, we are able to give only a few general answers to this question, that is, in general, one of the main open problems on this argument. Let us consider:

$$(3.1)\qquad \begin{aligned} -\Delta u &= \lambda_+ u^+ - \lambda_- u^- && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

and let: $\Sigma = \{(\lambda_+, \lambda_-) \in \mathbb{R}^2 \mid (3.1) \text{ has non-trivial solutions}\}$. We see trivially that Σ is not empty and $\Sigma \cap \{(\lambda_+, \lambda_-) \in \mathbb{R}^2 \mid \lambda_+ = \lambda_-\} = \sigma^2$. We shall see in the following sections that many existence problems are essentially a question on a local behaviour of Σ; this gives an idea of the importance of any general information about the structure of Σ.

<u>Theorem 3.2</u> <u>$\mathbb{R}^2 - \Sigma$ is an open disconnected set.</u>

<u>Proof</u>. The disconnectness of $\mathbb{R}^2 - \Sigma$ will follow from the continuity of the

Leray-Schauder index, which will be defined later on, and by prop. 3.7. We prove that $\mathbb{R}^2 - \Sigma$ is open. This will follow from the following lemma:

Lemma 3.3 Let $K = (-\Delta)^{-1}$, $(\lambda_+, \lambda_-) \notin \Sigma$. Then $\exists c > 0$ such that:

$$(3.4) \quad ||u - K(\lambda_+ u^+ - \lambda_- u^-)|| \geq c\, ||u|| \qquad \forall\, u \in E = L^2(\Omega)$$

Proof. (3.4) is trivially equivalent to:

$$(3.5) \quad ||u - K(\lambda_+ u^+ - \lambda_- u^-)|| \geq c > 0 \qquad \forall\, u \in E,\ ||u|| = 1.$$

This is obvious, because a map whose difference from the identity is completely continuous, is proper on bounded closed sets; from this it follows that it maps closed bounded sets into closed sets (see e.g. [7]). Therefore the set:

$$C = \{u - K(\lambda_+ u^+ - \lambda_- u^-) \mid u \in E,\ ||u|| = 1\}$$

is closed in E. The condition $(\lambda_+, \lambda_-) \notin \Sigma$ it is equivalent to $0 \notin C$. Therefore $\exists c > 0$, such that $C \cap B(0, c) \neq \emptyset$. It follows (3.5).■

We come back to the proof of theorem 3.2.
Take $(\lambda_+', \lambda_-') \in \mathbb{R}^2$, dist $((\lambda_+', \lambda_-'), (\lambda_+, \lambda_-)) < \dfrac{c}{2||k||}$.
Using (3.4) we get, $\forall\, u \in E$:

$$||u - K(\lambda_+' u^+ - \lambda_-' u^-)|| \geq$$
$$\geq ||u - K(\lambda_+ u^+ - \lambda_- u^-)|| - ||K((\lambda_+ - \lambda_+')u^+ - (\lambda_- - \lambda_-')u^-)||$$
$$\geq c||u|| - ||K||\,(|\lambda_+ - \lambda_+'| + |\lambda_- - \lambda_-'|)\,||u|| > 0.$$

The last inequality proves the statement.■

We set: $F(\lambda_+, \lambda_-, u) = u - K(\lambda_+ u^+ - \lambda_- u^-)$.
If $(\lambda_+, \lambda_-) \notin \Sigma$, it is well defined:

$$(3.6) \quad i(\lambda_+, \lambda_-) = \deg\,(F(\lambda_+, \lambda_-, \cdot), B(0, r), 0),$$

since the right-hand side of (3.6) has meaning and is independent on r. By the homotopy invariance of Leray-Schauder degree it follows trivially:

Proposition 3.6 The function i is constant on the connected components of $\mathbb{R}^2 - \Sigma$.

It is easy to compute i on some particular subsets of $\mathbb{R}^2$, as we see in the following proposition.

Proposition 3.7 $\forall$ k positive integer let: $A_k =]\lambda_k, \lambda_{k+1}[^2$, $(A_o =]-\infty, \lambda_1[^2)$.

Let: $B \equiv \{(\lambda_+, \lambda_-) \mid \lambda_- < \lambda_1 < \lambda_+ \text{ or } \lambda_+ < \lambda_1 < \lambda_-\}$. Then

A_k, $B \subset \mathbb{R}^2 - \Sigma$ $\forall k \geq 0$. Moreover $i = (-1)^k$ on A_k, $i = 0$ on B.

Proof. $A_k \subset \mathbb{R}^2 - \Sigma$ it is a consequence of lemma 2.5 in which it is stated that 2.8 has a unique solution. In this case: $G u = \lambda_+ u^+ - \lambda_- u^-$, $v = 0$ Q = identity. It follows that 0 is the unique solution for h = 0. If $(\lambda_+, \lambda_-) \in B$ we argue as in lemma 2.97 to prove that (2.100) has only the trivial solution, and we get: $B \subset \mathbb{R}^2 - \Sigma$. To compute i, we note that if $A_k \neq \emptyset$ we can take $\lambda \in]\lambda_k, \lambda_{k+1}[$ and it holds: $(\lambda, \lambda) \in A_k$. It is well known that

(3.8) $\deg(I - \lambda K, B(0, r), 0) = (-1)^k$

Since $I - \lambda K = F(\lambda, \lambda, \cdot)$, it follows $i(\lambda, \lambda) = (-1)^k$. Since A_k is connected, $i = (-1)^k$ everywhere in A_k. If $(\lambda_+, \lambda_-) \in B$, suppose, by contraddiction, that u solves

$$
(3.9) \quad \begin{cases} -\Delta u = \lambda_+ u^+ - \lambda_- u^- + (\lambda_+ - \lambda_-)\phi_1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}
$$

Taking the scalar product of both sides of (3.9) with ϕ_1 we find:

$$(\lambda_+ - \lambda_1)\cdot(u^+, \phi_1) + (\lambda_1 - \lambda_-)\cdot(u^-, \phi_1) + \lambda_+ - \lambda_- = 0$$

which is a contraddiction, since $\lambda_+ - \lambda_1$, $\lambda_1 - \lambda_-$, $\lambda_+ - \lambda_-$ have the same sign if $(\lambda_+, \lambda_-) \in B$. Therefore (3.9) has no solution. We shall see in the next proposition that this is impossible if $i(\lambda_+, \lambda_-) \neq 0$.■

We now consider a nonlinear operator $G : E \to E$. We shall say that G is sublinear if:

$$(3.10)\qquad \lim_{r \to +\infty} r^{-1} \sup_{||u|| \leq r} ||Gu|| = 0$$

<u>Proposition 3.11</u> <u>Let $(\lambda_+, \lambda_-) \in \mathbb{R}^2 - \Sigma$, and G be a sublinear operator on E, and let $F: E \to E$ be defined by: $Fu = u - K(\lambda_+ u^+ - \lambda_- u^-) - Gu$. Then F is a proper mapping. If moreover $i(\lambda_+, \lambda_-) \neq 0$, F is surjective.</u>

<u>Proof</u>. Using (3.4) and the sublinearity of G, we see that the counterimage by F of a bounded set is bounded. By this and by the properness of F on bounded sets it follows that F is proper on the whole space.

To prove the second part of the statement, consider the homotopy:

$$(3.12)\qquad H(s, u) = F(\lambda_+, \lambda_-, u) - sGu$$

We claim that the set $\{u \in E \mid \exists\, s \in [0, 1]$ such that $H(s, u) = 0\}$ is bounded. In fact if $H(s, u) = 0$, using (3.4) one gets:

$$(3.13)\qquad ||G(u)|| \geq c\, ||u||$$

By (3.10) $\exists\, r_0$ such that if $||u|| \geq r_0$:

$||u||^{-1}\, ||G(u)|| < c$. By (3.13) : $||u|| < r_0$.

Using the homotopy invariance of the topological degree, we get:

$$(3.14)\qquad \deg(F, B(0, r_0), 0) = \deg(H(1,\cdot), B(0, r_0), 0) =$$

$$= \deg(H(0, \cdot), B(0, r_0), 0) = \deg(F(\lambda_+, \lambda_-,\cdot), B(0, r_0), 0) =$$

$$= i(\lambda_+, \lambda_-)$$

Therefore, if $i(\lambda_+, \lambda_-) \neq 0$, F takes the value 0 in B(0, r). The same argument shows that F takes any fixed value $\bar{u} \in E$, since $G + \bar{u}$ is also sublinear.■

<u>Corollary 3.15</u> <u>Let f verify (F_1) and (2.45) with $(f_+, f_-) \notin \Sigma$. If $i(f_+, f_-) \neq 0$ than (P_t) is solvable $\forall\, t \in \mathbb{R}, \psi \in E$.</u>

Proof. We need only to note that $u \to K(g(x, u) + t\psi)$ is a sublinear operator.■

It is clear in the above proof that (2.45) can be considerably weakened.

We shall close this section pointing out that the study of the connected components of $\mathbb{R}^2 - \Sigma$ and of the behaviour of i, in view of corollary 3.15, is of fundamental importance for the solvability of (P_t). We shall see later on that, if we take $\psi = \phi_1$, it is always true that when $i(f_+, f_-) = 0$ the problem (P_t) has at least two solutions for t large negative; however, if $f_\pm > \lambda_1$ it has also two solutions for t large positive. We shall show in the next section that, with a different choice of ψ, we shall have similar results to those in the previous part, in some particular cases.

We could say more on the general structure of Σ. For instance, using the continuity of i, it is easily seen that if λ_k is an eigenvalue of $-\Delta$ of odd multiplicity, then (λ_k, λ_k) must belong to an unbounded connected component of Σ. Actually the oddness requirement is not necessary if one uses a more general index [14], see also [11], [4]. Further general information regarding Σ can be found in [12].

It is much easier to study the one-dimensional case. One can completely study the behaviour of Σ by direct computations, either for the two point than for the periodic solutions problem [17]. In this last case the only connected components of $\mathbb{R}^2$ which do not contain pairs (λ_+, λ_-) with $\lambda_+ = \lambda_-$ are the two component of B. Therefore, out of B, i is never 0 and one can have results of existence of solution as corollary 3.15, [17], [19]. We shall obtain local similar results for vector valued equations and for the wave equation, for which a direct computation of the pairs in Σ seems to be not so easy.

§2 THE JUMP ON A SIMPLE EIGENVALUE

In this section we shall study (BVP) under the additional assumption that $[f_-, f_+] \cap \sigma$ is reduced to one eigenvalue of multiplicity one. For further application we work as in 2§1 with an abstract symmetric operator A on the space $E = L^2(\Omega, \mathbb{R}^n)$. We shall refer sometimes to A the notations introduced for $-\Delta$ up to now, with obvious meaning. We assume that λ is a given eigenvalue of A, such that $\underline{\lambda} = \sup \sigma \cap]-\infty, \lambda[< \lambda$ and $\bar{\lambda} = \inf \sigma \cap]\lambda, +\infty[> \lambda$. The main problem we have to solve here is to study Σ on the open square $]\underline{\lambda}, \bar{\lambda}[^2$.

We use the notation V and W as in 2§1, taking $(\alpha, \beta) \in]\underline{\lambda}, \lambda[\times]\lambda, \bar{\lambda}[$, moreover we set:

$$Q_1 = \int_{-\infty}^{\alpha} dP_\lambda, \quad Q_2 = \int_{\beta}^{+\infty} dP_\lambda, \quad W_i = \operatorname{Im}(Q_i), \; i = 1, 2.$$

Given $u \in E$, when no confusion can occur, we often write v for Pu, w for Qu, w_i for $Q_i u$ $i = 1, 2$. Using the notation in lemma 2.5, with $Gu = \lambda_+ u^+ - \lambda_- u^-$, we set $\phi(\lambda_+, \lambda_-, v) = \lambda||v||^2 - (\lambda_+ \cdot (v + w(v))^+ - \lambda_- \cdot (v + w(v))^-, v)$. It is easy to see that the function ϕ so defined is continuous on its three variables. An important property of ϕ is proved in the following lemma (see also [20]).

<u>Lemma 3.16</u> <u>ϕ is decreasing on its first two variables; if v changes sign ϕ is strictly decreasing.</u>

<u>Proof</u>. Suppose $\lambda_+^1 < \lambda_+^2$ and let $w^i(v)$ be the solution of (2.8) for $Gu = \lambda_+^i u^+ - \lambda_- u^-$, $u^i = v + w^i(v)$, for $i = 1,2$. Let $Q_i w^j = w_i^j$ $i, j = 1, 2$.

By easy computations:

$$\phi(\lambda_+^i, \lambda_-, v) = (-\Delta v - \lambda_+^i (u^i)^+ + \lambda_- (u^i)^-, v) =$$
$$= (-\Delta u - \lambda_+^i (u^i)^+ + \lambda_- (u^i)^-, u^i) = J^i(u^i)$$

where the last step is justified by (2.8), with $Gu = \lambda_+^i u^+ - \lambda_- u^-$, since $u - v \in W$. We easily see that $J^2 \leq J^1$. Let us consider J^i as two functions of the three variables (w_1, v, w_2); by (2.4) it is apparent that they are strictly concave in their first variable and strictly convex in their last variable. By (2.8) u^1 and u^2 are stationary points of J^1 and J^2 on $u_i + W$ therefore they maximize the respective functional with respect to w_1 and minimize them with respect to w_2, strictly.

Therefore:

$$\phi(\lambda_+^2, \lambda_-, v) = J^2(w_1^2 + v + w_2^2) \leq J^2(w_1^2 + v + w_2^1) \leq$$

$$\leq J^1(w_1^2 + v + w_2^1) \leq J^1(w_1^1 + v + w_2^1) = \phi(\lambda_+^1, \lambda_-, v)$$

If the equality holds we must have $w_i^1 = w_i^2$, $i = 1, 2$, i.e. $u^1 = u^2$. Substituting in (2.7)-(2.8) for $i = 1, 2$ and taking the difference we get: $(u^1)^+ = 0$. Therefore, by (2.8):

(3.17) $Q(Au^1) = Q(\lambda_- u^1)$

which is equivalent to:

(3.18) $Aw^1 = \lambda_- w^1$

Since λ_- is not an eigenvalue of A on W, from (3.18) it follows that $w^1 = 0$ and therefore that $v = u^1$ has constant sign.■

We remark that in the case in which v has constant sign the study of ϕ is trivial, since $w(v) = 0$ for all pairs (λ_+, λ_-).

Since ϕ is positively homogeneous with respect to its first variable and we are interested in the sign of ϕ we can confine ourselves to the normalized vectors v.

Using the previous lemma we see that $\forall\, v \in V$, the set $\Gamma(v) = \{(\lambda_+, \lambda_-) \mid \phi(\lambda_+, \lambda_-, v) = 0\}$ is a continuous curve which crosses (λ, λ) and splits $]\underline{\lambda}, \bar{\lambda}[^2$ into two connected components $\Gamma\pm(v) = \{(\lambda_+ . \lambda_-) \in]\underline{\lambda}, \bar{\lambda}[^2 \mid \pm \phi(\lambda_+, \lambda_-, v) > 0\}$.

We also see trivially that $\Sigma \cap]\underline{\lambda}, \bar{\lambda}[^2 \subset \bigcup_{v \in V} \Gamma(v)$ and that the sets $\Gamma^\pm = \bigcap_{v \in V} \Gamma^\pm(v)$ are also connected.

Let us come now to the case that $\lambda = \lambda_k$ is a simple eigenvalue of $-\Delta$. In this case the possible choices of a normalized vector $v \in V$ are two i.e. $v = \phi_k$ or $v = -\phi_k$. In this case the two curves $\Gamma(\pm\phi_k)$ split $]\lambda_{k-1}, \lambda_{k+1}[$ in at most four connected components. Since $]\lambda_{k-1}, \lambda_k[^2 \subset \Gamma^+$ and $]\lambda_k, \lambda_{k+1}[^2 \subset$

$\subset \Gamma^-$, as can be easily seen using lemma 3.16 and the relation $\phi(\lambda_k, \lambda_k, v) = 0 \ \forall v \in V$, and $\Sigma \cap]\lambda_{k-1}, \lambda_{k+1}[= \Gamma(\phi_k) \cup \Gamma(-\phi_k)$, i is defined on $\Gamma^+ \cup \Gamma^-$ and $i = (-1)^{k-1}$ on Γ^+ and $i = (-1)^k$ on Γ^-. Therefore a straightforward application of corollary 3.15 gives the following.

Theorem 3.19 Let f verify (F_1) - (2.45) and assume $(f_+, f_-) \in \Gamma^+ \cup \Gamma^-$. Then (BVP) has at least a solution.

We now consider briefly the case $(f_+, f_-) \notin \Gamma^+ \cup \Gamma^-$. Suppose $(f_+, f_-) \in \Gamma^+(-\phi_k) \cap \Gamma^-(\phi_k)$.

Proposition 3.20 Let (F_1), (2.45) hold an $(f_+, f_-) \in \Gamma^+(-\phi_k) \cap \Gamma^-(\phi_k)$. Then, for $\psi = \phi_k$, $\exists \tau_1 \in \mathbb{R}$ such that P_t is not solvable for $t > \tau_1$.

Proof. By definition of ϕ we see that the problem

$$(3.21) \quad -\Delta u = f_+ u^+ - f_- u^- + t\phi_k \qquad u_{|\partial\Omega} = 0$$

has no solution for $t > 0$. Arguing as in lemma 3.3, since $(f_+, f_-) \in \Sigma$, from (3.21) we get:

$$(3.22) \quad ||u - K(f_+ u^+ - f_- u^- + t\,\phi_k)|| \geq c\,t$$

with a positive constant c, $\forall t \geq 0$.

For $\tau_1 = c^{-1} ||K|| \cdot |\Omega|^{1/2} \sup |g|$ and from (3.22) we get the statement.■

Let $B_k^{\pm}(o, r) = \{ u \in B(o,r) | \pm(u, \phi_k) \geq 0\}$.

Lemma 3.23 Suppose $(\lambda_+, \lambda_-) \in \Gamma^{\pm}(\pm\phi_k)$. Then $\exists \bar{r}_o > 0$ such that $\forall r \geq \bar{r}_o$:

$$(3.24) \quad \deg(F(\lambda_+, \lambda_-, \cdot), B_k^{\pm}(0, r), \phi_k) = \pm(-1)^{k-1}$$

Proof. We prove the + case.

Suppose $\lambda_+ \leq \lambda_-$. We claim that the homotopy H, defined on $[\lambda_+, \lambda_-] \times E$

in E by:

(3.25) $H(s, u) = F(\lambda_+, s, u)$

is admissible for the computation of the degree in (3.24). This is true since, for all $s \in [\lambda_+, \lambda_-]$, the problem (3.1) has no nontrivial solution u such that $(u, \phi_k) \geq 0$, if λ_- is replaced by s. Moreover, $H(s,\cdot)$ has no zeros u on $(\phi_k)^\perp$, since, by lemma 2.5, we would have u = 0. Therefore, by the homotopy invariance of the Leray-Schauder degree:

(3.26) $$\deg (F(\lambda_+, \lambda_-,\cdot), B_k^+(0, r), \phi_k) =$$
$$= \deg (F(\lambda_+, \lambda_+,\cdot), B_k^+(0, r), \phi_k).$$

Since $F(\lambda_+, \lambda_+,\cdot) = I - \lambda_+ K$, $\lambda_+ < \lambda_k$, (3.24) is easily proved. ∎

From (3.24) it follows:

<u>Lemma 3.27</u> <u>Let (F_1) and (2.45) hold, $(f_+, f_-) \in \Gamma^\pm (\pm\phi_k)$. Then $\exists \tau \in \mathbb{R}$ such that, $\forall\, t \geq \tau \; \exists r_t > 0$ such that if $r \geq r_t$:</u>

(3.28) $$\deg (F_t, B_k^\pm (0, r), 0) = \pm(-1)^{k-1}$$

<u>for $\psi = \phi_k$.</u>

<u>Proof.</u> Note that for $t \geq 0$, setting $v = t^{-1}u$:

(3.29) $$F_t(u) = u - K (f_+ u^+ - f_- u^- + gu + t\phi_k) =$$
$$= t (v - K(f_+ v^+ - f_- v^- + t^{-1}g (tu) + \phi_k)) =$$
$$= t (F(f_+, f_-, v) - t^{-1}Kg(tu) - \lambda_k^{-1} \phi_k)$$

From (3.24) we have that for $r \geq \lambda_k^{-1} \bar{r}_o = t^{-1} r_t$

(3.30) $$\deg (F(f_+, f_-,\cdot), B_k^\pm(0, r), \lambda_k^{-1}\phi_k) = \pm(-1)^{k-1}$$

Since g is bounded, for t large enough, the perturbation $t^{-1}Kg(t\cdot)$ does not change the degree in the left-hand side of (3.30). Therefore:

(3.31) $\deg (F(f_+, f_-,\cdot), B_k^{\pm}(0, r), \lambda_k^{-1}\phi_k) =$

$$= \deg (F(f_+, f_-,\cdot) - t^{-1}K (g(t\cdot)) - \lambda_k^{-1}\phi_k, B_k^{\pm}(0, r), 0) =$$

$$= \deg (F_t, B_k^{\pm}(0,tr), 0)$$

The statement follows from (3.30) - (3.31).■

<u>Corollary 3.32 Let (F_1), (2.45) hold, $(f_+, f_-) \in \Gamma^{\pm}(\pm\phi_k)$. Then $\exists\, \tau_2 \in \mathbb{R}$ such that $\forall\, t > \tau_2$, (P_t) has at least a solution u such that $\pm(u, \phi_k) > 0$, for $\psi = \phi_k$.</u>

The previous corollary provides a result of existence of zero-two solutions for the case: $(f_+, f_-) \in (\Gamma^+(-\phi_k) \cap \Gamma^-(\phi_k)) \cup (\Gamma^+(\phi_k) \cap \Gamma^-(-\phi_k))$ and for the semi-resonance case: $(f_+, f_-) \in (\Gamma(\phi_k) \setminus \Gamma(-\phi_k)) \cup (\Gamma(-\phi_k) \setminus \Gamma(\phi_k))$.

<u>Proposition 3.33</u> <u>Let (F_1), (2.45) hold and $(f_+, f_-) \in \Gamma^+(-\phi_k) \cap \Gamma^-(\phi_k)$. Then $\exists\, \tau_{1,2} \in \mathbb{R}$ such that, for $\psi = \phi_k$, (P_t) has:</u>

<u>no solution for $t > \tau_1$</u>

<u>at least two solutions for $t < \tau_2$</u>

The proof follows at once by proposition 3.20 and corollary 2.32 (changing ϕ_k with $-\phi_k$). Corollary 3.33 contains quite explicitly the informations for the semiresonance case. We have so studied all the possible cases for $(f_+, f_-) \in$ $\in]\lambda_{k-1}, \lambda_{k+1}[$, but the resonance case $(f_+, f_-) \in \Gamma(\phi_k) \cap \Gamma(-\phi_k)$, which includes as a particular subcase $f_+ = f_- = \lambda_k$. In this last case conditions of the Landesman-Lazer type [29] can be done (see [21]).

§3 THE JUMP ON AN EIGENVALUE OF HIGHER MULTIPLICITY: THE SYMMETRIC CASE

We now come back to the case of a general operator A, without asking that λ is simple. The problem is, in general, hard because we have an infinity of curves $\Gamma(v)$, given by normalized $v \in V$. However, in many applications it happens that $\Gamma(v)$ does not change changing v in V. In this case we get easily some existence result in [47].

We assume that H is a set of linear isometries.

Definition 3.34 Let $b: E \to E$. We shall say that b is H-equivariant if $\forall h \in H$: $: bh = hb$.

We shall ask that H satisfies the condition:

(H_1) A, P and $(\cdot)^+$ are H equivariant

(H_2) $\forall v_{1,2} \in V$, $||v_1|| = ||v_2||$ $\exists h \in H$ such that $h(v_1) = v_2$

Proposition 3.35 Let ($H_1 - H_2$) hold. $\forall v_1, v_2 \in V-\{0\}$: $\Gamma(v_1) = \Gamma(v_2)$.

Proof. Using (H_1) one sees that $\forall (\lambda_+, \lambda_-) \in]\underline{\lambda}, \bar{\lambda}[^2$, $\forall v \in V$, $\forall h \in H$: $: w(hv) = h(w(v))$. Using (H_2) and taking into account that h preserves the scalar product:

$$\begin{aligned}\phi(\lambda_+, \lambda_-, v_2) &= \lambda||v_2||^2 - (\lambda_+(v_2 + w(v_2))^+ - \lambda_-(v_2 + w(v_2))^-, v_2) = \\ &= \lambda||v_1||^2 - (\lambda_+(h(v_1) + w(h(v_1)))^+ - \lambda_-(h(v_1) + w(h(v_1)))^-, h(v_1)) = \\ &= \lambda||v_1||^2 - (h(\lambda_+(v_1 + w(v_1))^+ - \lambda_-(v_1 + w(v_1))^-), h(v_1)) = \\ &= \phi(\lambda_+, \lambda_-, v_1) \blacksquare\end{aligned}$$

Therefore, under the assumptions of the previous proposition, $]\underline{\lambda}, \bar{\lambda}[^2 \setminus \Sigma$ has at most two connected components. They are exactly two every time there is an index like i which changes from $]\underline{\lambda}, \lambda[^2$ to $]\lambda, \bar{\lambda}[^2$. However, we can prove it directly when H verifies a further condition, used in [42] to give simpler proofs of various classical bifurcation theorems for conservative systems.

Proposition 3.36 Let (H_1) and

(H_3) $\forall v_1, v_2 \in V$, linearly independent $\exists h \in H$ such that:

$$h(v_1) = v_1 \ , \quad h(v_2) \neq v_2$$

Then $\forall\, v \in V \setminus \{0\}$: $\Gamma(v) \subset \Sigma$

Proof. We prove that $v + w(v)$ is a solution of (3.1) if $(\lambda_+, \lambda_-) \in \Gamma(v)$. To see this we must only prove that $v_2 = v - K(\lambda_+(v + w(v))^+ - \lambda_- (v + w(v))^-)$ and $v_1 = v$ are linearly dependent, since $\phi(\lambda_+, \lambda_-, v) = 0$ implies: $v_2 = = ||v||^{-2} (v, v_2)v = ||v||^{-2} \phi(\lambda_+, \lambda_-, v)\, v = 0$. Suppose, by contraddiction, that v_1 and v_2 are linearly independent. In this case (H_3) holds. Using (H_0) and the first part of (H_3) we get:

$$h(v_2) = h(v_1 - K(\lambda_+(v_1 + w(v_1))^+ - \lambda_- (v_1 + w(v_1))^-) =$$

$$= h(v_1) - K(\lambda_+(h(v_1) + w(h(v_1)))^+ - \lambda_- (h(v_1) + w(h(v_1)))^-)$$

$$= v_1 - K(\lambda_+(v_1 + w(v_1))^+ - \lambda_- (v_1 + w(v_1))^-) = v_2$$

which is a contraddiction to (H_3). ∎

The symmetry conditions $(H_1 - H_2 - H_3)$ are often verified by problems of existence of periodic solutions. We state explicitly some results for the systems of ordinary differential equations. Let B be a n x n symmetric matrix with eigenvalues $-\omega_1^2 > -\omega_2^2 > \ldots > -\omega_m^2$, where $\omega_h \in \mathbb{R} \setminus \{0\}$, $h = 1, \ldots, m$. Suppose that, for a given i, $-\omega_i^2$ is a simple eigenvalue and that $\omega_i \notin \omega_j \cdot \mathbb{Z}$, $\forall\, j \neq i$, and denote by ξ_i a fixed eigenvector of B related to $-\omega_i^2$. Let $\Omega = = [o, 2\pi]$, $E = L^2(\Omega, \mathbb{R}^n)$ and L be the linear operator on E defined by $Lu = = B \dfrac{d^2 u}{dt^2}$, on the subspace D(L) constituted by the functions in $H^2(\Omega, \mathbb{R}^n)$ which verify the periodicity conditions. It is easy to see that L has compact resolvent. Moreover:

Proposition 3.37 ω_i^2 is a double eigenvalue of L which corresponding eigenspace V generated by ξ_i sin and ξ_i cos.

Proof. To calculate the eigenvalues of L we use the Fourier decomposition: $u = \sum_{k=0}^{\infty} (u_k \cos kt + u_{-k} \sin kt)$, by which the equation $Lu = \lambda u$ becomes:

$$\sum_{k=0}^{\infty} - k^2 \; (Bu_k \cos kt + Bu_{-k} \sin kt) =$$

$$= \sum_{k=0}^{\infty} \lambda \; (u_k \cos kt + u_{-k} \sin kt)$$

and therefore, it is equivalent to $-K^2 Bu_k = \lambda u_k$, $\forall\, k \in \mathbb{Z}$ This implies that, when $u \neq 0$, it must exist $h \in \{1, 2, \ldots, m\}$ and $k \in \mathbb{Z}$ such that $\lambda = k^2 \omega_h^{\,2}$ and $u_k \neq 0$. In this case $u_k \cos kt$ and $u_k \sin kt$ are eigenvectors of L, with eigenvalue λ. The previous analysis shows that $\omega_i^{\,2}$ is an eigenvalue of L and that $\xi_i \sin$ and $\xi_i \cos$ are related eigenvectors. Moreover, it shows also that, if we had a third independent eigenvector, we must have $\omega_i^{\,2} = k^2 \omega_j^2$ for some $j \neq i$, a contraddiction.■

We show that the set H of the transformation on E of the type:

$$[h(u)](t) = u(a - t)$$

for any given $a \in \mathbb{R}$ verify $(H_1 - H_2 - H_3)$. (H_1) is obvious, to see (H_3) let $v_1, v_2 \in V = \ker(\omega_i^{\,2} I - L)$. We have shown in proposition 3.37 that v_j must be of the form $v_j = \alpha_j \; \xi_i \cos(a_j + t)$, $j = 1, 2$. The hypothesis that v_1 and v_2 are linearly independent is equivalent to $a_1 - a_2 \notin \pi \cdot \mathbb{Z}$. We take in the definition of h : $a = -2 a_1$. We get:

$$[h(v_1)]\,(t) = v_1\,(-2a_1 - t) = \alpha_1 \xi_1 \cos(a_1 - 2a_1 - t) =$$
$$= \alpha_1 \; \xi_1 \cos(-a_1 - t) = v_1 \; (t)$$

which means : $h(v_1) = v_1$. By similar computations : $[h(v_2)](t) = v_2(2a_1 - a_2 + t)$, by which it follows that $h(v_2) = v_2$ is equivalent to the condition : $2a_1 - 2a_2 \in \; \in 2\pi \cdot \mathbb{Z}$, which contradicts the linear independence of v_1 and v_2. To prove (H_2) we make similar computations for $a = -(a_1 + a_2)$.

Before stating the application of the abstract results we remark that, using the notation of §2, we have:

$$\underline{\lambda} = \sup \{ k^2 \omega^2_j \mid k \in \mathbb{Z} \;\; j = 1, 2, \ldots, m;\; k^2 \omega^2_j < \omega^2_i \}$$

$$\overline{\lambda} = \inf \{ k^2 \omega^2_j \mid k \in \mathbb{Z},\; j = 1, 2, \ldots, m;\; k^2 \omega^2_j > \omega^2_i \}$$

Theorem 3.38 There exists a relatively closed subset $\Gamma \subset]\underline{\lambda}, \overline{\lambda}[$ which crosses (ω_i^2, ω_i^2), has at most a point on every straight line of positive direction and which splits $]\underline{\lambda}, \overline{\lambda}[^2$ into two connected components, such that: if f verifies (F_1), (2.45) with $(f_+, f_-) \in]\underline{\lambda}, \overline{\lambda}[^2 \setminus \Gamma$, the equation:

$$(3.39) \qquad B \frac{d^2 u}{dt^2} = f(t, u)$$

has at least a 2π - periodic solution.

Proof. It is an easy variant of Proposition 3.20, since, in our case, the Leray-Schauder index can be used and the connected components of $]\underline{\lambda}, \overline{\lambda}[^2 \setminus \Gamma$ contain respectively $]\underline{\lambda}, w_i^2[^2$ and $]w_i^2, \overline{\lambda}[^2$, on which the index is not 0. Note that in this theorem we have used t to indicate the variable of $\Omega = [0, 2\pi]$. ■

We remark that the set H verify $(H_1 - H_2 - H_3)$ with every autonomous differential operator of even order. For the odd order differential operators the set H of the traslations verify $(H_1 - H_2)$ and therefore the main results of the abstract theory are also applicable. We wish now to give an application to the one dimensional wave equation. The main difference from the previous equation is that we use an operator which has not compact resolvent and, therefore, the Leray- Schander topological degree is no longer applicable. However, with some further restrictions on f, we can use the generalized degree for the condensing perturbations of the identity. We introduce the necessary terminology. Given a subset C of a metric space we call non-compactness measure of C the positive number $\alpha(C) = \inf\{\varepsilon > 0 \mid C \text{ has an } \varepsilon\text{ - net}\}$. It is clear that a set is bounded iff it has finite non-compactness measure while is precompact iff its non compactness measure is 0. A mapping f between two metric spaces is called an α-contradiction of modulus β if $\forall C : \alpha(f(C)) \leq \beta(\alpha(C))$, is called condensing if $\forall C : \alpha(f(C)) < \alpha(C)$, when $\alpha(C) \notin \{0, +\infty\}$.

One sees at once that an α-contraction of modulus $\beta < 1$ is condensing and that f is completely continuous iff it is an α-contraction of modulus 0.

It is well known that the topological degree can be defined for the condensing perturbation of identity (for further details see e.g. [35]). Let $\Omega = [0, 2\pi] \times [0, \pi]$, we shall denote by t the first variable of the pairs in Ω and by x the second one. It is well known that the operator $\Box = \frac{d^2}{dt^2} - \frac{d^2}{dx^2}$, defined on the elements of $H^2(\Omega)$ which verify the periodicity conditions on t and the Diriclet homogeneous conditions on x, has as eigenvalues the numbers of the form $k^2 - h^2$ with k and h integers. Moreover, if c is not an eigenvalue of $\Box$, then $K_c = (cI - \Box)^{-1}$ is an α-contraction of modulus $|c|^{-1}$ (see e.g. [34]). Therefore, a generalized topological index can be defined for all the pairs $(\lambda_+, \lambda_-) \in \mathbb{R}^2 - \Sigma$ such that $\lambda_+\lambda_- > 0$. In fact, suppose $\lambda_+ > 0$, $\lambda_- > 0$ and choose $c > 2^{-1}\max(\lambda_+, \lambda_-)$. In this case the map G_c which sends u in $K_c(\lambda_+ u^+ - \lambda_- u^- - c\,u) = K_c((\lambda_+ - c)\,u^+ - (\lambda_- - c)\,u^-)$ is an α-contraction of modulus $|c|^{-1}\max((\lambda_+ - c), (\lambda_- - c)) < 1$. Therefore we can use as $i(\lambda_+, \lambda_-)$ the number: $\deg(I - G_c, B(0, r), 0)$, where $r > 0$ is arbitrarily chosen and c is large enough. We do not treat more this problem, noting also that, using the generalized degree as shown above, one can easily repeat the analysis made via Leray-Schauder index.

In this way one proves the following result when λ is a number which can be written in a unique way in the form: $k^2 - h^2$ with h and k integers and

$$\underline{\lambda} = \sup\{k^2 - h^2 \mid k, h \in \mathbb{Z}, k^2 - h^2 < \lambda\}$$

$$\overline{\lambda} = \inf\{k^2 - h^2 \mid k, h \in \mathbb{Z}, k^2 - h^2 > \lambda\}$$

<u>Theorem 3.40</u> <u>There exists a relatively closed subset $\Gamma \subset]\underline{\lambda}, \overline{\lambda}[^2$ which crosses (λ, λ), has at most a point on every straight line of positive direction and which splits $]\underline{\lambda}, \overline{\lambda}[^2$ into two connected components, such that: if f verifies (F_1), (2.12) and (2.45), with $\alpha\beta > 0$ and $(f_+, f_-) \in]\underline{\lambda}, \overline{\lambda}[^2 \setminus \Gamma$, the equation</u>

$$\Box u = f(t, x, u) \tag{3.41}$$

<u>has at least a solution on Ω, 2π - periodic on t and with null trace on $[0, 2\pi] \times \{0, \pi\}$.</u>

The proof of theorem 3.40 is the same that of theorem 3.38, the only difference being the use of the generalized degree. We remark only that the condition (2.12) with $\alpha\beta > 0$ allows the use of the degree with the function f since, supposing e.g. $\alpha > 0$, when $c > 2^{-1}\max(\alpha,\beta)$ then $K_c(f - cI)$ is an α-contraction of modulus $|c|^{-1}\max(|\alpha - c|, |\beta - c|) < 1$.

§4 SOME MULTIPLICITY RESULTS FOR THE CASE $\lambda_1 \leq f_\pm$

What we want to do in this section is to study the parametrized problem (P_t) with $\psi = \phi_1$, as already done in the case $f_- < \lambda_1 < f_+$. Also here we get results of existence of multiple solutions for suitable values of the parameter.

Using the results of 2§3 we get easily two multiplicity results under general assumptions.

<u>Proposition 3.42</u> <u>Let f verify (F_1), (2.45) with $(f_+, f_-) \notin \Sigma$, $\lambda_1 < f_\pm \neq \lambda_j$. Then if (P_t) is not solvable $\forall t \in \mathbb{R}$, $\exists\ \tau > 0$ such that for $|t| > \tau$ (P_t) has at least two distinct solutions.</u>

<u>Proof</u>. Suppose that (P_t) is not solvable $\forall t \in \mathbb{R}$. Then by corollary 3.15 we get $i(f_+, f_-) = 0$, and (3.14), taking $F = F_t$, shows the existence of $r_t \in \mathbb{R}$ such that $\forall\ r > r_t$:

(3.43) $\deg(F_t, B(0,r), 0) = 0$

Fix τ in such a way that (2.82) holds with the + or - sign according to the case $\pm t < \pm(-\tau)$.

Let $t < -\tau$; we take $r > r_t$ large enough to have $B_t^+ \subset B(0, r)$. By (2.82), (3.43) and the additivity of the topological degree, F_t has non zero degree on the sets B_t^+ and $B(0, r) \setminus B_t^+$. Therefore, the equation $F_t = 0$ has at least two distinct solutions. The case $\tau < t$ is obviously similar. ■

Proposition 3.44 Let f verify (F_1), (2.45) with $(f_+, f_-) \notin \Sigma$, $\lambda_1 < f_\pm \neq \lambda_j$ and let the total number of eigenvalues of $-\Delta$ in $[f_-, f_+]$ be odd. Then $\exists \tau \in \mathbb{R}$ such that (P_t) has at least two solutions either for $t < -\tau$ or for $\tau < t$.

Proof. Suppose by contraddiction that the statement is not true; then we can choose two sequences of real numbers $(t_n^\pm)_n$ such that $t_n^\pm \to \pm\infty$ and (P_t) has at most a solution for $t = t_n^\pm$. As in the previous proposition, we can suppose that (2.82) holds, with the + or − sign for $t = t_n^\pm$. By our assumption and by the excision property of the topological degree, from (2.82) it follows that $\deg(F_t, B(0, r), 0) = (-1)^{k\pm}$ for $t = t_n^\pm$ and if $r \geq r_n$, where r_n is chosen in such a way that $B_{t_n^\pm}^\pm \subset B(0, r_n)$. But, by (3.14), when r is large enough, since $(f_+, f_-) \notin \Sigma$, we must have: $\deg(F_t, B(0, r), 0) = i(f_+, f_-)$. Therefore, by the above equalities, we get: $(-1)^{k_+} = (-1)^{k_-}$ which means that $k_+ - k_-$ is an even number. This is a contraddiction, since $k_+ - k_-$ is the number of eigenvalues in $[f_-, f_+]$, which we have assumed to be odd. ∎

We can have better results taking the additional assumption that $[f_-, f_+]$ contains exactly one, simple, eigenvalue of $-\Delta$. Note that, in this case, the assumption $(f_+, f_-) \notin \Sigma$ is explained by the results in section 2. The result is similar to theorem 2.106 and we first need a lemma of the type of lemma 2.91. From now on, we essentially follow[50].

Lemma 3.45 Let ε be a fixed real positive number, which we suppose chosen very small. Let $k \neq 1$ be fixed and : $\lambda_{k-1} < f_- < \lambda_k < f_+ < \lambda_{k+1}$; consider the problem:

$$(3.46) \quad \begin{cases} -\Delta u = f_+ u^+ - f_- u^- + s(\phi_1 + \varepsilon \phi_k) + \phi_k - \varepsilon \phi_1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Let S be the closed hyperplane: $\mathbb{R}\left(\phi_1 + \varepsilon \frac{\lambda_1 - f_+}{\lambda_k - f_+} \phi_k\right) + \{\phi_1, \phi_k\}^\perp$. Then $\forall s \in \mathbb{R}$: (3.46) has no solution $u \in S$.

Proof. The proof is, obviously, somehow similar to that of lemma 2.91. Let Q be the orthogonal projector on E such that $\operatorname{Ker} Q = \mathbb{R}\,(\phi_k - \varepsilon\,\phi_1)$. If u solves (3.46) then it solves also:

$$(3.47)\qquad \begin{cases} -Q\,(\Delta v) = Q(f_+ v^+ - f_- v^-) + s\,(\phi_1 + \varepsilon\phi_k) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

It is an easy variant of lemma (2.5) to show, by using the Banach's theorem for contractive mappings, that $\forall\, s \in \mathbb{R}$: (3.47) has exactly one solution $v \in S$. We try to find directly a solution of (3.47) of the form $v = r(\phi_1 + \varepsilon \dfrac{\lambda_1 - f_+}{\lambda_k - f_+}\phi_k)$. It is not difficult to see that, for ε small enough, v is a solution of (3.47) if:

$$(3.48)\qquad b_+ r^+ - b_- r^- = s$$

where the coefficients $b_\pm$ are given by:

$$(3.49)\qquad \pm\, b_\pm = (1 + \varepsilon^2)^{-1}\left(\lambda_1 - \varepsilon^2 \lambda_k \frac{\lambda_1 - f_+}{\lambda_k - f_+} - f_\pm \left(1 + \varepsilon^2 \frac{\lambda_1 - f_+}{\lambda_k - f_+}\right)\right)$$

and when ε is small : $b_\pm < 0$. Therefore (3.48) is equivalent to

$$(3.50)\qquad r = b_-^{-1} s^+ - b_+^{-1} s^-$$

We have so shown that if u is a solution of (3.46) u must be equal to: $(b_-^{-1} s^+ - b_+^{-1} s^-)\,(\phi_1 + \varepsilon \dfrac{\lambda_1 - f_+}{\lambda_k - f_+}\phi_k)$ for some $s \in \mathbb{R}$. We substitute this value in (3.46) and, taking the scalar product of both members with $\phi_k - \varepsilon\,\phi_1$, we find:

$$(3.51)\qquad (b_-^{-1} s^+ - b_+^{-1} s^-)\left((\lambda_k - f_\pm)\frac{\lambda_1 - f_+}{\lambda_k - f_+} - (\lambda_1 - f_\pm)\right) = 1 + \varepsilon^2$$

where the + or − sign of f has to be taken according to the sign of r (i.e. of $b_\pm$) and not of s. Therefore, from (3.51) it follows: $b_-^{-1} s^+ > 0$; a contradiction, since we have remarked that $b_\pm < 0$. ∎

From the previous lemma, we get:

Lemma 3.52. Let $c > 0$ be a fixed constant. There exists $\bar{m} > 0$ such that the equation

$$(2.98) \qquad -\Delta u = f(x, u) + s(\phi_1 + \varepsilon\phi_2) + m(\phi_2 - \varepsilon\phi_1)$$

has no solution $u \in S$, $u = 0$ on $\partial\Omega$ for some m, s such that $|s| < cm$, $m > \bar{m}$, if f verifies (F_1)- (2.45) with $\lambda_{k-1} \leq f_- < \lambda_k < f_+ < \lambda_{k+1}$.

The proof of lemma 3.52 is completely similar to that of lemma 2.97. We set:

$$B^{\pm}_{\varepsilon,k}(0, r) = \{u \in B(0, r) \mid \pm(u, \varepsilon \frac{\lambda_1 - f_+}{\lambda_k - f_+}\phi_1 - \phi_k)\} > 0.$$

Lemma 3.53 Let all be assumptions of the previous theorem hold. $\exists \tau \in \mathbb{R}$ such that $\forall t > \tau$ $\exists r_t$ such that if $(f_+, f_-) \in \Gamma^{\pm}(\pm\phi_k)$, for $\psi = \phi_1$, $r \geq r_t$:

$$(3.54) \quad \deg(F_t, B^{\pm}_{\varepsilon,k}(0, r), 0) = \pm(-1)^{k-1}$$

Proof. As in lemma 2.102, we consider the homotopy:

$$(3.55) \quad H(s, u) = u - K(f(x, u) + s(\phi_1 + \varepsilon\phi_k) + m(\phi_k - \varepsilon\phi_1))$$

with $m = -t\varepsilon(1 + \varepsilon^2)^{-1}$. Using lemma 3.52 and arguing as in lemma 2.102, one finds:

$$(3.56) \quad \deg(H(-\varepsilon^{-1}m, \cdot), B^{\pm}_{\varepsilon,k}(0, r), 0) = \deg((H(0, \cdot), B^{\pm}_{\varepsilon,k}(0, r), 0)$$

As in lemma 2.102: $H(-\varepsilon^{-1}m, \cdot) = F_t$, taking $\psi = \phi_1$ in the parametrization, while $H(0, \cdot) = F_m$, taking $\psi = \phi_k - \varepsilon\phi_1$. From the above considerations, and in particular (3.56), (3.54) follows from (3.28), for ε small, in view of the continuity of the topological degree, taking eventually large $-\tau$ (and subsequently m). ■

We are now ready to prove our main multiplicity result in this section:

Theorem 3.57 Let (F_1), (2.45) hold and suppose that $\exists k > 1$ such that:

$\lambda_{k-1} < f_- < \lambda_k < f_+ < \lambda_{k+1}$. Moreover assume that $(f_+, f_-) \notin \Sigma$ and consider the problem (P_t) parametrized taking $\psi = \phi_1$. Then one of the following three cases is true:

(i) $\exists \tau \in \mathbb{R}$ such that (P_t) has at least three solutions for $t < \tau$.

(ii) $\exists \tau \in \mathbb{R}$ such that (P_t) has at least two solutions for $|t| > \tau$.

(iii) $\exists \tau \in \mathbb{R}$ such that (P_t) has at least three solutions for $\tau < t$.

Proof. By the results of section 2, the hypothesis $(f_+, f_-) \in \Sigma$ is equivalent to $(f_+, f_-) \in \Gamma(\phi_k) \cup \Gamma(-\phi_k)$. If we suppose $(f_+, f_-) \in (\Gamma^+(\phi_k) \cap \Gamma^-(-\phi_k)) \cup (\Gamma^+(-\phi_k) \cap \Gamma^-(\phi_k))$, then, by proposition 3.20, (P_t) is not solvable $\forall\, t \in \mathbb{R}$, therefore, by proposition 3.42, the case (ii) must be true.

Assume, on the other side, that $(f_+, f_-) \in \Gamma^-(\phi_k) \cap \Gamma^-(-\phi_k)$. For t large negative (3.54) holds with the - sign and $B^+_t \subset B^+_{\varepsilon, k}(0, r)$ for r large. Moreover, by what stated in section 2, $i(f_+, f_-) = (-1)^k$, therefore, with an eventually larger r, by the additivity of the topological degree, $\deg(F_t, B^+_{\varepsilon,k}(0, r), 0) = 0$. Using this, (3.54-) and (2.82) one gets the existence of at least a solution in each of the three disjoint sets: $B^-_{\varepsilon,k}(0, r)$, B^+_t, $B^+_{\varepsilon, k}(0, r) \setminus B^+_t$. The last case we have to consider that $(f_+, f_-) \in \Gamma^+(\phi_k) \cap \Gamma^+(-\phi_k)$. In this case, arguing as before, one shows that (iii) must be true. However, to this aim, one must change also the lemma in such a way to have an analogous of (5.54) for t large positive. ∎

We remark that it is implicit, by proposition 3.42, that, in the cases (i), (iii), (P_t) has also at least a solution $\forall\, t \in \mathbb{R}$.

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