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ATTESTATO DI RICERCA

"DOCTOR PHILOSOPHIAE"

PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS IN THE RESONANCE CASE

CANDIDATA:

Dott.ssa Addolorata Salvatore

RELATORE:

Prof. Donato FORTUNATO

Anno Accademico 1983/84

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INTRODUCTION

During the past few years several people studied the Hamiltonian system of the form

$$\text{(FHS)} \qquad \begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q}(t,p,q) \\ \\ \dot{q} = \dfrac{\partial H}{\partial p}(t,p,q) \end{cases}$$

where p, q $\in \mathbb{R}^n$, $H : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$ is differentiable and T-periodic and $\cdot$ denotes $\dfrac{d}{dt}$.

This system can be represented more concisely as

$$\text{(FHS)} \qquad \dot{z} = JH_z(t,z)$$

where $z=(p,q)$, $H_z = \dfrac{\partial H}{\partial z}$ and $J = \begin{pmatrix} O & -I \\ I & O \end{pmatrix}$, I being the identity matrix in $\mathbb{R}^n$.

If H depends explicity on t, we shall speak of "forced Hamiltonian systems"; if H(z) do not depends on t, the Hamiltonian system

$$\text{(HS)} \qquad \dot{z} = JH_z(z)$$

is called autonomous.

There are many types of questions both local and global in the study of periodic solutions of (FHS) and (HS) (cf. e.g. the review articles [2],[13],[36] and their references). A first kind of problems is the existence of solutions of (HS) having a prescribed period and the solutions of (FHS) having the given period of forcing (i.e. the period of H). Another question is the existence of periodic solutions of (HS) on a given energy level (let us observe that H is an integral of the motion for (HS), i.e. if $z(t)$ is a solution of (HS), $H(z(t))$ is independent on t).

The aim of this thesis is to deal with the first problem: we shall study periodic solutions of (HS) and (FHS) in the large by using a variational approach; namely the solutions found are the critical points of a suitable functional. In the section 1, we recall some abstract critical points theorems: in these theorems a weaker version of the Palais-Smale condition is utilized. Section 2 and 3 are devoted to the study of asymptotically quadratic Hamiltonian systems, i.e. we assume that

$$H_z(t,z) = b_\infty(t)z + g(t,z)$$

and

$$\frac{g(t,z)}{|z|} \longrightarrow 0 \text{ as } |z| \longrightarrow +\infty \quad \text{uniformly in } t \in \mathbb{R}.$$

If we denote the "linearized operator at infinity" by L_∞, i.e. $L_\infty z = -J\dot{z}-b_\infty(t)z$ (for a more precise definition see section 2), we shall say that "resonance" occurs if $0 \notin \sigma(L_\infty)$, $\sigma(L_\infty)$ being the spectrum of L_∞.

In section 2 we shall find at least one T-periodic solution of (FHS) under a "noresonance" assumption; in section 3 we shall study the case in which the "strong resonance" occurs, i.e. there is the resonance and the non-

linear part goes very rapidly to zero at infinity.
The resonance assumption prevents in general to get suitable a priori bounds which assure that the set of critical level is bounded, and therefore the (P-S) condition could not be satisfied.
We shall prove that the strong resonance assumption implies a weaker version of the (P-S) condition and this permit to use the abstract framework of § 1.
In section 4 we shall deal with the Hamiltonian function H(t,p,q) of the form

$$(0.1) \qquad H(t,p,q) = \sum_{i,j=1}^{n} a_{ij}(t,q)p_i p_j + V(t,q)$$

where $\{a_{ij}(t,q)\}$ is a positive definite matrix and V(t,q) is bounded.
Such hamiltonian function often occurs in the study of mechanical systems.

First we shall assume $a_{ij}(q)$ constant: then the Hamiltonian system (FHS) can be written as

$$(0.2) \qquad -\ddot{x} = \nabla V(t,x) \ .$$

This problem has been studied by many authors when V(t,x) has a superquadratic or subquadratic growth at infinity.
If V is asymptotically quadratic, it is known that there exist T-periodic solutions under a nonresonance assumption or, if the resonance occurs, under a "Landesmann Lazer type condition" for the nonlinearity (cf. [6] and its bibliographie).
Now we are dealing with a nonlinearity which rapidly goes

to 0 (strong resonance case): arguing as in section 3, we shall find existence and multiplicity results for (0.2).

In section 5 we consider the Hamiltonian function of type (0.1) with a_{ij} depending on q: T-periodic solutions of (HS) and (FHS) are found under the assumption that V(q) goes to infinity as $|q| \to +\infty$.

On the other hand if V(q) is bounded, we do not know a direct proof of the generalized Palais-Smale condition. Then in section 6 we have studied this problem by restricting the action functional to a suitable subspace which has trivial intersection with the linearized operator at infinity. So we find some solutions of (HS) in the case in which V(q) and $\nabla V(q)$ are bounded. Amoung the various physical problems, to which the results of § 6 can be applied, we shall recall the equations of the "double pendulum".

§ 1. Some abstract critical points theorems

In this section we recall some critical point theorems we shall need in the following for a real functional f on a real Hilbert space.

Let us give the following notations and definitions. We denote by E a real Hilbert space, by $(\cdot,\cdot)$ the inner product in E and by $\|\cdot\|$ the corresponding norm. By $C^1(E,\mathbb{R})$ we denote the space of continuously Fréchet differentiable maps from E to $\mathbb{R}$ and by $f'(u)$ the derivative of f at $u \in E$. We shall identify E with its dual E'. For $u \in E$ and $R > 0$, we set $B_R(u)=\{v \in E \mid \|v-u\|< R\}$ $B_R = B_R(0)$, $S_R = \partial B_R = \{u \in E \mid \|u\| = R\}$.

Classical critical points theorems have been proved under the assumption that $f \in C^1(E,\mathbb{R})$ satisfies the well-known Palais-Smale condition, which can be expressed as follows:

Definition 1.1.

f satisfies the Palais-Smale condition in $]c_1 c_2[$ $(-\infty \le c_1 < c_2 \le +\infty)$ if

(PS) every sequence $\{u_k\} \subset f^{-1}(]c_1,c_2[)$ for which $f(u_k)$ is bounded and $f'(u_k) \to 0$, possesses a convergent subsequence.

Obviously (PS) can be expressed in a equivalent way as follows:

(i) every bounded sequence $\{u_k\} \subset f^{-1}(]c_1,c_2[)$ for which $\{f(u_k)\}$ is bounded and $f'(u_k) \longrightarrow 0$, possesses a convergent subsequence;

(ii) $(\{u_k\} \subset f^{-1}(]c_1,c_2[), \{f(u_k)\}$ bounded and $\|u_k\| \to +\infty$ for $k \to +\infty) \Longrightarrow (\|f'(u_k)\| \geq \alpha > 0$ for k sufficiently large).

Condition (i) is a "compactness" condition which is satisfied by a large class of functionals (cf. remark 1.7). Condition (ii) is easy to verify in problems "non resonant" at infinity, i.e. with a linear part at infinity invertible; on the other hand the following "weakening" of (ii) is needed to study problems with "strong resonance" at infinity (cf. section 2).

Definition 1.2 We shall say that $f \in C^1(X, \mathbb{R})$ satisfies the condition (C) in $]c_1,c_2[$ if

(i) holds, and

(ii)' $\forall\, c \in\,]c_1,c_2[\ \exists\, \sigma, R, \alpha > 0$ s.t. $[c-\sigma, c+\sigma] \subset\,]c_1,c_2[$ and $\forall\, u \in f^{-1}([c-\sigma,c+\sigma]), \|u\| \geq R : \|f'(u)\|\,\|u\| \geq \alpha$

In a more compact form the condition (C) becomes "Every sequence $\{u_k\} \subset f^{-1}(]c_1,c_2[)$ for wich $\{f(u_k)\}$ is bounded and $\|f'(u_k)\|\ \|u_k\| \to 0$ possesses a convergent subsequence".

A condition similar to (C) has been introduced by Cerami in [23] and applied to the search for critical points of a functional on an unbounded Riemannian manifold.

Let us assume now that the functional is invariant for the action of a compact group, more precisely let us consider a functional f even. The following theorem holds (cf. [4]).

Theorem 1.3 Suppose that $f \in C^1(E, \mathbb{R})$ satisfies the following properties:

(f_1) f satisfies condition (C) in $]0,+\infty[$ and $f(0) \geq 0$;

(f_2) there exist two closed subspaces V and W of E, with codim $V < +\infty$, and two constants $c_\infty > c_0 > f(0)$ such that

a) $f(u) \geq c_0 \qquad \forall u \in S_\rho \cap V$

b) $f(u) < c_\infty \qquad \forall u \in W$

(f_3) f is even.

Then, if dim W $\geq$ codim V, f possesses at least m = dim W-codim V distinct pairs of critical points whose corresponding critical values belong to $[c_0, c_\infty]$

Theorem 1.3 is a generalization of theorem 2.13 of [3]. Ambrosetti and Rabinowitz have used Palais-Smale condition instead of the weaker assumption (C); moreover they have replaced the assumption (f_2)(b) with the stronger requirement that for any finite dimensional space $E_k \subset E$ the set $\{u \in E_k | f(u) \geq 0\}$ is bounded.

In the case in which the functional do not exhibit symmetries, linking arguments need. Let S be a closed set in E and Q an Hilbert manifold with boundary ∂Q. We shall say that S and ∂Q link if:

(L_1) $S \cap \partial Q = \emptyset$;

(L_2) if Φ is a continuous map of E into itself such that $\Phi(u) = u \quad \forall u \in \partial Q$, then $\Phi(Q) \cap S \neq \emptyset$.

Examples of linking sets are given in [4].
The following theorem holds:

Theorem 1.4 *Suppose that* $f \in C^1(E,\mathbb{R})$ *satisfies the following properties*

(f_1) f *satisfies condition* (C) in $]0,+\infty[$;

(f_4) *there exists a closed subset* S *and a Hilbert manifold* Q *with boundary* ∂Q *such that*

a) S *and* ∂Q *link*;

b) *there exist two constants* $\beta > \alpha \geq 0$ s.t. $f(u) \leq \alpha \; \forall u \in \partial Q$ *and* $f(u) \geq \beta \; \forall u \in S$;

c) $\sup_{u \in Q} f(u) < +\infty$

Then f *possesses a critical value* $c \geq \beta$.

Linking arguments have been used by many authors (cf. [3],[5],[12],[32],[33] and [34]) under the Palais-Smale condition.

We shall apply theorems (1.3)-(1.4) in order to study the periodic solutions of the second order Hamiltonian systems.
In these cases the functional of the action is semidefinite, i.e. is bounded from above (or from below) modulo weakly continuous perturbations. Infact, if we denote E^+(respectively E^-) the subspace of E where the quadratic part of f is positive (resp. negative) definite, it results that $\dim E^- < +\infty$ or $\dim E^+ < +\infty$, and therefore we can write f as a quadratic positive (or negative) part plus a functional with compact derivative.
We shall consider now the case in which f can be strongly

"indefinite", i.e. E^+ and E^- are both infinite dimensional, as it occurs in the study of periodic solutions of Hamiltonian systems. In this case, we have to assume that f has a particular form. First we recall the following theorems for a functional with symmetry.

Theorem 1.5 Let E be a real Hilbert space, on which an unitary representation T_g of the group S^1 acts. Let $f \in C^1(E, \mathbb{R})$ be a functional on E satisfying the following assumptions:

(I_1) $f(u)=\frac{1}{2}(Lu,u) - \psi(u)$ where

(i) L is a continuous self-adjoint operator on E,

(ii) $\psi \in C^1(E,\mathbb{R})$, $\psi(0)=0$ and ψ' is a compact operator,

(iii) L and ψ' are S^1-equivariant.

(I_2) 0 does not belong to the essential spectrum of L;

(I_3) every sequence $\{u_n\} \subset E$, for which $\{f(u_n)\} \to c \in]0,+\infty[$ and $\|f'(u_n)\| \, \|u_n\| \to 0$, possesses a bounded subsequence;

(I_4) there exist two S^1-invariant closed subspaces V and W of E s.t.

(i) $\dim (V \cap W) < +\infty$, $\operatorname{codim} (V+W) < +\infty$

(ii) $\operatorname{Fix}(S^1) \subset V$ or $\operatorname{Fix}(S^1) \subset W$

(iii) there exist two positive constants c_o and ρ s.t.

$$f(u) \geq c_o \quad \text{for every } u \in V \cap S\rho$$

(iv) there exists $c_\infty \in \mathbb{R}$ s.t. $f(u) \leq c_\infty$ for every $u \in W$,

(v) $f(u) < c_o$ for $u \in \operatorname{Fix}(S^1)$ s.t. $f'(u) = 0$.

Then there exist at least

$$\frac{1}{2}\left[\dim(V \cap W) - \operatorname{codim}(V + W)\right]$$

orbits of critical point, with critical values in $[c_o, c_\infty]$.

Theorem 1.6 Let $f \in C^1(E,R)$ be a functional satisfying the preceding assumptions with (I_1)(iii) and (I_4) replaced by

$(I_1)(\overline{iii})$ ψ' is odd;

$(\bar{I}_4)$ there exist two closed linear subspaces $V, W \subset E$ which satisfy (I_4)(i),(ii),iv).

Then there exist at least

$$\dim(V \cap W) - \operatorname{codim}(V + W)$$

pairs of nonzero critical points with critical values greater or equal than c_o.

Remark 1.7 We shall prove that $(I_1)(I_2)$ and (I_3) implie condition (C) on $]0,+\infty[$.

Namely, let $\{u_n\} \subset f^{-1}(]0,+\infty[)$ for which $\{f(u_n)\}$ is bounded and $\|f'(u_n)\| \, \|u_n\| \to 0$. By (I_3), there exists a bounded subsequence which we shall denote always by $\{u_n\}$. If $u_n \to 0$, the proof is achieved, otherwise $\|f'(u_n)\| \to 0$. Then

$$\tfrac{1}{2}(L\, u_n, u_n) - \psi(u_n) \text{ is bounded}$$
$$L\, u_n - \psi'(u_n) \to 0$$

Obviously, we can select a subsequence $\{u'_n\}$ weakly converging to $u_o \in E$.

By (I_2), $0 \leq \dim \ker L < +\infty$.

If $\dim \ker L = 0$, there exists $L^{-1}: E \to E$ continuous s.t.

$$u'_n - L^{-1}(\psi'(u'_n)) \to 0$$

and $L^{-1}(\psi'(u_n))$ converging to $L^{-1}(\psi'(u_o))$ by compactness of ψ', the convergence of u'_n to u_o follows.
If $0 < \dim\ker L < +\infty$, it can be proved as above that $\tilde{u}'_n$ is convergent, $\tilde{u}'_n = u_n - u^\circ_n$, $u^\circ_n \in \ker L$; moreover the boundeness of u°_n implies its convergence in E and therefore the conclusion follows. ■

Remark 1.8 We recall that in our applications the functional of the action will be always of the type

$$f' = L + \psi' \qquad \psi' \text{ compact} \qquad 0 \notin \sigma_e(L) .$$

So we shall replace condition (C) by the weaker assumption (I_3) also in the semidefinite case.

Remark 1.9 Theorem 1.5 generalizes theorem 4.1 of [7] in two points: the condition (I_2)-(I_3) are more general than (P-S) and (f_4)-(iii) is replaced by the stronger assumption Fix $S^1 \subset W$.
In the case in which the functional f is indefinite and does not exibit any symmetry, we shall need the following theorem (cf. [14]).

Theorem 1.10 Given $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, let $f \in C^1(E,\mathbb{R})$ a functional satisfying the assumptions (I_1) (i)-(ii), (I_2) and (I_3) for any $c \in]\alpha,+\infty[$.
Moreover suppose that

(I_5) there exist a constant $R > 0$ and two closed L-invariant subspaces E_1 and E_2 such that $E = E_1 \oplus E_2$ and if we set $Q = B_R \cap E_1$, $S = q + E_2$ (with $q \in Q$, $\|q\| < R$) assume that (i) $f(u) \geq \beta$ on S

(ii) $f(u) \leq \alpha$ on ∂Q

(iii) $\sup_Q f(u) = c_\infty$ where $c_\infty < +\infty$

Then f possesses at least a critical value $c \in [\beta, c_\infty]$.

Remark 1.11 This theorem generalizes theorem (0.1) of Benci-Rabinowitz (cf. [12]) because $(I_1)-(I_3)$ are weaker assumptions than the respective assumptions in [12].

§ 2. Asymptotically linear and non resonant Hamiltonian systems

In this section we are looking for T-periodic solutions of (FHS) in the case in which $H(t,z)$ is asymptotically quadratic, i.e. there exists a symmetric matrix $2n \times 2n$ $b_\infty(t)$ for any $t \in [0,T]$ such that

$$(H_1) \quad \begin{cases} H_z(t,z) = b_\infty(t)z + g(t,z) \\ \text{and} \\ g(t,z)/|z| \to 0 \quad \text{as} \ |z| \to +\infty \quad \text{uniformly in } t \in \mathbb{R} \end{cases}$$

where $|\cdot|$ denotes the norm in R^{2n} and $(\cdot,\cdot)$ the corresponding inner product.
If we denote by L_∞ the linearized operator at infinity, i.e. $L_\infty z = -J\dot{z} - b_\infty(t)z$ (for a more precise definition let us see the following) we assume that (FHS) is not resonant, i.e.

$$(H_2) \quad 0 \notin \sigma(L_\infty)$$

The following results are contained in [38].
First, we shall state the following theorem:

Theorem 2.1 *If $(H_1),(H_2)$ hold, then (FHS) has at least one T-periodic solution.*

The solution found can be constant. If we suppose that 0 is an equilibrium point of the Hamiltonian vector field, it is interesting to find a T-periodic and nontrivial solution. Precisely, we shall require that

(H_3) $H(t,0)=H_z(t,0)=0$ for any $t\in R$ H is C^2 at $z=0$.

In this case, we can write

(2.2) $H_z(t,z)=b_0(t)z + O(|z|)$ as $|z|\to 0$

where

(2.3) $b_0(t) = b_\infty(t) + g_z(t,0)$.

We set

(2.4) $G(t,z)=H(t,z)-\frac{1}{2}(b_\infty(t)z,z)=\int_0^1 (g(t,sz)|z)ds.$

Let be denote by $\tilde{\lambda}_1^\infty$ (resp. $\tilde{\lambda}_{-1}^\infty$) the smallest positive (resp. the greatest negative) eigenvalue of L_∞ in $W^{\frac{1}{2}}([0,T],\mathbb{R}^{2n})$ (cf. the following for its definition).

The following theorem holds:

<u>Theorem</u> 2.5 <u>Under the assumptions</u> $(H_1),(H_2),(H_3)$ <u>and</u>

(H_4) $G(t,z)\leq 0$ $t\in\mathbb{R}, z\in\mathbb{R}^{2n}$ (<u>resp</u>. (H_4') $G(t,z)\geq 0$)

(H_5) $\bar{\lambda}=\max\limits_{0\leq t\leq T}[\max \sigma(g_z(t,0))]<\tilde{\lambda}_{-1}^\infty$,

(<u>resp</u>. (H_5') $\underline{\lambda}=\min\limits_{0\leq t\leq T}[\min \sigma(g_z(t,0))]>\tilde{\lambda}_1^\infty$)

<u>there exists at least one T-periodic nontrivial solution of</u> (FHS).

Analogous results have been obtained by Amann and Zendher in [1] under the assumptions that b_0 and b_∞ do not depend on t, the Hamiltonian function $H(t,z)$ is C^2 and the Hessian $H_{zz}(t,z)$ is uniformly bounded. On the other hand in theorem 2.5 we need an additional condition on the sign of G; (H_5) establishes the connection between $b_o(t)$ and $b_\infty(t)$ which guarantees that the solution we find is nontrivial. (H_5) corresponds to the assumption of theorem 2 in [1]

$$(*) \qquad i(b_o, b_\infty, \frac{2\pi}{T}) > 0 ;$$

infact in the special case of two harmonic oscillators with frequencies α° and α^∞, we can easily verify that (H_5) and (*) are equivalent. More recently Conley and Zendher (cf. [24]) have studied the general case in which the linearizations at zero and infinity are time-dependent: they used a generalized Morse theory and assume, as in [1], that H(t,z) is C^2 and the Hessian is uniformly bounded in order to reduce the problem to a finite dimensional problem.

Proof of the theorems

We initially introduce some functional spaces we shall need in the following. If $m \in R$ and $t>1$ we set

$$L^t = L^t(S^1, \mathbb{R}^m).$$

If $s \in R$ we set

$$W^s = \{u \in L^2(S^1, R^{2n}) \mid \sum_{\substack{i \in Z \\ k=1,\dots,2n}} (1+|j|^2)^s |u_{jk}|^2 < +\infty\}$$

where u_{jk} $(j \in Z, k=1,\dots,2n)$ are the Fourier components of u with respect to the basis (in $L^2(S^1, \mathbb{R}^{2n})$)

$$(2.6) \qquad \psi_{jk} = e^{jtJ}\Phi_k = \cos t(jt)\Phi_k + J\mathrm{sen}(jt)\Phi_k$$

where $\{\Phi_k\}$ $(k=1,\dots,2n)$ is the standard basis in $\mathbb{R}^{2n}$. W^s equipped with the inner product

(2.7) $$(u|v)_{W^s} = \sum_{j,k} (1+|j|^2)^s u_{jk} v_{jk}$$

is an Hilbert space. We recall that the embedding $W^s \to L^t$ is compact if $\frac{1}{t} > \frac{1}{2} - s$. So in particular $W^{\frac{1}{2}}$ is compactly embedded in L^t for any $t \geq 1$.

Let us denote by $(\cdot,\cdot)_{L^t}$ and $((\cdot,\cdot))$ the inner products in L^t and $W^{\frac{1}{2}}$ and by $|\cdot|_{L^t}$ and $\|\cdot\|$ the corresponding norms.

Now consider the Hamiltonian system where $H(t,z)$ is T-periodic in t. Making the change of variable $t \to \frac{2\pi t}{T}$, (FHS) becomes

(FHS)-1 $$-J\dot{z} = \omega H_z(\omega t, z) \text{ where } \omega = T/2\pi$$

Obviously the 2π-periodic solutions of (FHS)-1 correspond to the T-periodic solutions of (FHS).

In order to construct the action functional whose critical points are the 2π-periodic solutions of (FHS)-1 we introduce the following bilinear form

$$a'u,v) = \sum_{j \in Z} \sum_{k=1}^{2n} j\, u_{jk} v_{jk} \qquad u,v \in W^{\frac{1}{2}}$$

where u_{jk}, v_{jk} are the Fourier-components of u,v with respect to the basis (2.6). The bilinear form $a(\cdot,\cdot)$ is symmetric and continuous in $W^{\frac{1}{2}}$. Let $L: W^{\frac{1}{2}} \to W^{\frac{1}{2}}$ be the self-adjoint, continuous operator defined by

$$((Lu,v)) = a(u;v) \qquad u,v \in W^{\frac{1}{2}} .$$

Observe that if $u,v \in C^1(S^1, \mathbb{R}^{2n})$

$$((Lu,v))=\int_0^{2\pi}(-J\dot{u},v)dt.$$

Let us consider now the operator L_∞: $W^{\frac{1}{2}} \to W^{\frac{1}{2}}$ s.t.

$$(2.8) \qquad ((L_\infty u,v))=((Lu,v))-\omega\int_0^{2\pi}(b_\infty(\omega t)u,v)dt \qquad u,v \in W^{\frac{1}{2}}$$

By (H_1), standard arguments show that the functional

$$(2.9) \qquad f(z)= \tfrac{1}{2}((L_\infty z,z))-\omega\int_0^{2\pi}G(\omega t,z)dt \qquad z \in W^{\frac{1}{2}}$$

is Frèchet differentiable and that its critical points correspond to the 2π-periodic solutions of (FHS)-1.
For simplicity, in the sequel we shall take $\omega=1$, i.e. $T=2\pi$.
So (2.9) becomes

$$(2.10) \qquad f(z)=\tfrac{1}{2}((L_\infty z,z))-\psi(z)$$

where $\psi(z) = \int_0^{2\pi}G(t,z)dt$.

Since $W^{\frac{1}{2}}$ is compactly embedded in L^t for any $t\geq 1$, by (H_1) we have that the map $z \to G_z(t,z)$ is compact from $W^{\frac{1}{2}}$ on $W^{-\frac{1}{2}}$, then ψ' is compact.
Now it is easy to verify (cf.[11] sect.3) that the spectrum of L consists of the limit points $+1,-1$ and of the eigenvalues

$$\lambda_j = \frac{j}{1+|j|} \qquad j \in \mathbb{Z}$$

and that each eigenvalues has multiplicity 2n.
Since L_∞ is a compact perturbation of L, $\sigma_e(L_\infty)=\sigma_e(L)=\{+1,-1\}$.

So the functional f is strongly indefinite and does not presents any symmetry: therefore we shall apply the abstract theorem 1.10 in order to find its critical points. Assumptions (I_1)(i)-(ii) and (I_2) are obviously satisfied. For completeness we shall prove assumption (I_3), which is always verified under the nonresonance condition (H_2). Since $0 \notin \sigma_e(L_\infty)$, we can denote by $\tilde{\lambda}^\infty_{-1}$ (resp. $\tilde{\lambda}^\infty_1$) the first negative (resp. positive) eigenvalue of L_∞ in $W^{\frac{1}{2}}$ and by λ^∞_{-1} and λ^∞_1 the analogous in L^2. Let be H^-_∞ (resp. H^+_∞) the subspace of $W^{\frac{1}{2}}$ where L_∞ is negative (resp. positive) definite; every $z \in W^{\frac{1}{2}}$ can be decomposed as follows

$$(2.11) \qquad z=z^+ + z^- , \qquad z^\pm \in H^\pm_\infty$$

and it results

$$(2.12) \qquad ((L_\infty z^+,z^+)) \geq \tilde{\lambda}^\infty_1 \|z^+\|^2 , ((L_\infty z^-,z^-)) \leq \tilde{\lambda}^\infty_{-1}\|z^-\|^2$$

The following lemma holds:

<u>Lemma</u> 2.13 <u>Let us assume that</u> (H_1) <u>hold. For any</u> $\varepsilon>0$ <u>there exists two positive constants</u> c_1 <u>and</u> M <u>such that</u>

$$|G(t,z)| \leq c_1|z| + \varepsilon/2|z|^2 \qquad z \in \mathbb{R}^{2n}, \ |z| \geq M.$$

Proof By (H_1) we have that

$$\int_0^1 [H_z(t,sz)-b_\infty(t)(sz,z)]ds = \int_0^1 (g(t,sz)|z)ds$$

and

$$G(t,z)=H(t,z)-\frac{1}{2}(b_\infty(t)z,z)=\int_0^1 (g(t,sz),z)ds .$$

Then

(2.14) $$|G(t,z)| \leq |z|\int_0^1 |g(t,sz)|ds \qquad z \in \mathbb{R}^{2n}.$$

By (H_1), for every $\varepsilon > 0$ there exists $M>0$ such that

(2.15) $$|g(z)| \leq \varepsilon|z| \qquad \text{for } |z|\geq M.$$

Let be $|z|\geq M$ and

$$A_1(z) = \{s \in [0,1] \mid |sz| < M\}$$

$$A_2(z) = \{s \in [0,1] \mid |sz| \geq M\}.$$

Then by (2.15) we have

(2.16) $$\int_0^1 |g(t,sz)|ds=\int_{A_1(z)} |g(t,sz)|ds+\int_{A_2(z)} |g(t,sz)|ds \leq$$

$$\leq c_1+\varepsilon/2|z|$$

where $c_1=\sup\{|g(t,z)|, \ t \in [0,2\pi], \ |z|<M\}$.

By (2.14) and (2.16) the conclusion of lemma follows. ■

We are proving the following lemma

Lemma 2.17 Let us assume that $(H_1)-(H_2)$ hold. Then every sequence $\{u_n\}$, for which $f(u_n) \to c$, $c>0$, and $\|f'(u_n)\| \, \|u_n\| \to 0$, possesses a bounded subsequence.

Proof.- Let be $\{u_n\} \in W^{\frac{1}{2}}$ such that $f(u_n) \to c$ and $\|f'(u_n)\|\|u_n\| \to 0$; it follows that

$$\langle f'(u_n), u_n^+ \rangle \longrightarrow 0$$

where $<\cdot,\cdot>$ denotes the pairing in $W^{\frac{1}{2}}$. Then there exist two positive constants c_1, c_2 such that

$$(2.18) \qquad c_1 \leq \tfrac{1}{2}\,((L_\infty u_n, u_n^+)) - \int_0^{2\pi} g(t,u_n)\,u_n^+\,dt \leq c_2 \;.$$

By (2.16) for $\varepsilon>0$ small there exists $c_3 \in \mathbb{R}_+$ such that

$$(2.19) \qquad \int_0^{2\pi} g(t,u_n)\,u_n^+\,dt \leq c_3 |u_n^+|_{L^1} + \varepsilon |u_n|_{L^2} |u_n^+|_{L^2} \;.$$

By (2.12), (2.18) and (2.19) it follows that

$$(2.20) \qquad \tilde{\lambda}_1^\infty \|u_n^+\|^2 \leq ((L_\infty u_n, u_n^+)) \leq c_3 \|u_n^+\| + \varepsilon \|u_n\| \, \|u_n^+\| + c_2 .$$

Arguing similarly

$$(2.21) \qquad -\tilde{\lambda}_{-1}^\infty \|u_n^-\|^2 \leq -((L_\infty u_n, u_n^-)) \leq c_3 \|u_n^-\| + \varepsilon \|u_n\| \, \|u_n^-\| + c_2 \;.$$

If we take $\tilde{\lambda} = \min\,(\tilde{\lambda}_1^\infty, -\tilde{\lambda}_{-1}^\infty)$, $\tilde{\lambda}>0$, we have adding (2.20) and (2.21)

$$\tilde{\lambda} \|u_n\|^2 \leq c_4 \|u_n\| + \varepsilon \|u_n\|^2 + c_2$$

or
$$(\tilde{\lambda}-\varepsilon) \|u_n\|^2 - c_4 \|u_n\| - c_2 \leq 0$$

This proves that $\|u_n\|$ is bounded. ■

Proof of theorem 2.1

Now we have to show that the geometrical condition (I_5) holds. Let be

$$Q = H_\infty^- \cap B_R \qquad S = H_\infty^+ \;.$$

By lemma 2.13 it follows easily that

$$f(z)=\frac{1}{2}((L_\infty z,z)) - \int_0^{2\pi} G(t,z)dt \geq \beta \quad \text{for any } z \in S$$

Moreover there exist two real constants α and c_∞ such that for $z \in Q$

$$f(z) \leq \frac{1}{2}\, \tilde{\lambda}^\infty_{-1} \|z\|^2 + \varepsilon/2 |z|^2_{L^2} + c_1 |z|_{L^1} + c_2 \leq$$
$$\leq \frac{1}{2}(\tilde{\lambda}^\infty_{-1}+\varepsilon)\|z\|^2 + c_3\|z\| + c_2 < c_\infty$$

and

$$f(z) < \alpha \qquad z \in \partial Q.$$

We can choose R large enough such that $\alpha < \beta$.
Theorem 1.10 assures that f has at least one critical value $c \geq \beta$. Clearly, we can not exclude the trivial solution. ■

<u>Proof of theorem</u> 2.5 Let L_o be the self-adjoint realization of $-J\dot{z}-b_o(t)z$ in $W^{\frac{1}{2}}$.
Let H_o^+ (resp. H_o^-) be te subspace of $W^{\frac{1}{2}}$ where L_o is positive (resp. negative) definite. The following lemma holds:

<u>Lemma</u> 2.22 <u>Under the assumptions of theorem</u> (2.5) <u>it results</u>

$$\text{(i)} \quad H_\infty^- \cap H_o^+ \neq \{0\} \qquad\qquad \text{(ii)} \quad H_\infty^+ \subset H_o^+ .$$

Proof. Let q be an eigenvector corresponding to the eigenvalue λ^∞_{-1} in L^2. Thenit is known that $q \in W^{\frac{1}{2}}$ and q is an eigenvector corresponding to $\tilde{\lambda}^\infty_{-1}$ in $W^{\frac{1}{2}}$. Moreover

$$((L_o q,q)) = ((L_\infty q,q)) - (g_z(t,0)q,q)_{L^2} \geq \tilde{\lambda}^\infty_{-1}\|q\|^2 -$$
$$-(\max \sigma(g_z(t,0))q,q))_{L^2} \geq \tilde{\lambda}^\infty_{-1}\|q\|^2 - \bar{\lambda}|q|^2 =$$
$$= (\tilde{\lambda}^\infty_{-1} - \bar{\lambda})\|q\|^2 > 0.$$

Obviously $q \in H_0^+$ and i) follows. Proving (ii), we observe that if $z \in H_\infty^+$,

$$((L_0 q,q)) = ((L_\infty q,q)) - (g_z(t,0)q,q)_{L^2} \geq \lambda_1^\infty \|q\|^2$$

and by (i) the inclusion (ii) is strict. ■

Let us prove theorem 2.5. It is obvious that the functional (2.10) verifies the assumptions $I_1)-I_3)$ of the abstract theorem 1.10; the geometrical assumptions hold as in the proof of theorem (2.1) setting

$$Q = H_\infty^- \cap B_R \qquad S = q + H_\infty^+$$

where q is an eigenvector of L_∞ corresponding to λ_{-1}^∞ (and to $\tilde{\lambda}_{-1}^\infty$) with $\|q\| < R$. We shall show that, in this case, f is bounded from below on S by a strictly positive constant β.

In fact, taken $z \in S$, $z=q+z_+$, $z_+ \in H_\infty^+$, we have

$$f(z) = \tfrac{1}{2}((L_\infty z,z)) - \int_0^T G(t,z)dt \geq \tfrac{1}{2}(\tilde{\lambda}_1^\infty \|z_+\|^2 + \tilde{\lambda}_{-1}^\infty \|q\|^2 -$$

$$-\int_0^T G(t,z)dt \geq \tfrac{1}{2}(\tilde{\lambda}_1^\infty \|z_+\|^2 + \tilde{\lambda}_{-1}^\infty \|q\|^2) \; .$$

Fixed ε small, we distinguish two cases

a) $\|z_+\|^2 < -\lambda_{-1}^\infty/\lambda_1^\infty \|q\|^2 + \varepsilon$

b) $\|z_+\|^2 \geq -\lambda_{-1}^\infty/\lambda_1^\infty \|q\|^2 + \varepsilon$.

In the first case, if we choose $\|q\|$ small enough, $\|z\|$ is small and it turns out that

$$f(z) = \tfrac{1}{2}(L_0 z,z) + O(\|z\|^2) \geq \tilde{\beta} > 0$$

because $z \in H_0^+$ by (ii) of lemma 2.22 and $\|z\| \geq \|q\| > 0$.
In the second case, it results

$$f(z) \geq \tfrac{1}{2}(\tilde{\lambda}_1^\infty \|z_+\|^2 + \tilde{\lambda}_{-1}^\infty \|q\|^2) \geq \varepsilon \cdot \tilde{\lambda}_1^\infty > 0.$$

Setting $\beta = \min(\tilde{\beta}, \tilde{\lambda}_1^\infty \varepsilon)$ we conclude that

$$f(z) \geq \beta > 0 \qquad \forall z \in S.$$

Thus, there exists a critical value $c \geq \beta > 0$ and therefore, being $f(0)=0$, there exists at least one critical nontrivial point. ■

Remark. If (H_1), (H_2), (H_3) (H_4') and (H_5') hold, we wan prove that $H_\infty^+ \cap H_0^- \neq \{0\}$ and the functional $-f$ satisfies the assumptions of the theorem 1.10 setting $Q = H_\infty^+ \cap B_R$, $S = q + H_\infty^-$, $q \in H_\infty^+ \cap H_0^-$, $\|q\|$ small.

§ 3. Hamiltonian systems with strong resonance at infinity

In this section we shall study now the case in which $H(t,z)$ is asymptotically quadratic and verifies the resonance assumption

$$(R_1) \qquad 0 \in \sigma(L_\infty) .$$

In this case condition (I_3) is not generally true, because we cannot controle the component of z in the Kernel of L_∞. Depending on the growth of nonlinearity at ∞, we have different "degrees" of resonance (cf. [4]). We shall consider a "strong resonance" condition, i.e.

$$(R_2) \quad \begin{cases} G(t,z) \longrightarrow 0 & \text{as } |z| \to +\infty \quad \text{uniformly in } t \in \mathbb{R} \\ (g(t,z),z)_{\mathbb{R}^{2n}} \longrightarrow 0 & \text{as } |z| \to +\infty, \text{ uniformly in } t \in \mathbb{R}. \end{cases}$$

The assumption of g and G has been introduced in [4] for a semidefinite problem. In order to prove (I_3) in this situation, we need the following lemma, which generalizes lemma 3.2 of [4].

<u>Lemma</u> 3.1 <u>Let</u> $\Omega \subset \mathbb{R}^n$ <u>be a bounded open set and</u> Z <u>a finite dimensional subspace of</u> $C(\bar{\Omega}, \mathbb{R}^k)$ <u>such that every</u> $u \in Z \setminus \{0\}$ is different from zero a.e. in Ω. <u>Let</u> $h \in L^\infty(\mathbb{R}^k)$ <u>such that</u>

$$h(x) \to 0 \qquad \text{as } |x| \to +\infty .$$

<u>Let</u> K <u>be a compact subset of</u> $L^p(\Omega,\mathbb{R}^k)$ with $p \geq 1$. <u>Then</u>

$$\lim_{|\lambda| \to +\infty} \int_\Omega |h(\lambda u(x)+v(x)|dx = 0$$

uniformly as $v \in K$ *and* $u \in S$, *where*

$$S = \{u \in Z | \sup_{x \in \Omega} |u(x)| = 1\} .$$

Proof. The proof is analogous to the proof of lemma 3.2. in [4].

We prove the following lemma.

Lemma 3.2. *If* $R_1)$, $R_2)$ hold, then for every $c \in \mathbb{R}_+$ *there exist positive constants* σ, R, α *such that*

$\|f'(u)\| \; \|u\| \geq \alpha$ *for any* $u \in \varphi^{-1}([c-\sigma, c+\sigma])$ $\|u\| \geq R$.

Proof Let be H_∞^+, H_∞^- and $\ker L_\infty$ the subspaces of $W^{\frac{1}{2}}$ where L_∞ is positive, negative and nulle definite; every $z \in W^{\frac{1}{2}}$ can be decomposed as follows

$$z = z^+ + z^- + z^\circ .$$

The operator L_∞ being continuous and bounded in $W^{\frac{1}{2}}$, there exist $\tilde{\lambda}_m^\infty$ and $\tilde{\lambda}_M^\infty$ the smallest and the largest eigenvalue of L_∞. It results, if $\tilde{\lambda}_1^\infty$ and $\tilde{\lambda}_{-1}^\infty$ are the eigenvalues of L_∞ defined in section 2,

$$(3.3) \quad \begin{cases} \tilde{\lambda}_m^\infty \|z^-\|^2 \leq ((L_\infty z^-, z^-)) \leq \tilde{\lambda}_{-1}^\infty \|z^-\|^2 \\ \tilde{\lambda}_1^\infty \|z^+\|^2 \leq ((L_\infty z^+, z^+)) \leq \tilde{\lambda}_M^\infty \|z^+\|^2 \end{cases}$$

and the analogous with L_∞ replaced by L_o.

Now given $c > 0$, let $\delta > 0$ such that

$$(3.4) \quad \int_0^T (g(t,u),v)\,dt \leq \delta \|v\| \qquad \text{for any } u,v \in W^{\frac{1}{2}} .$$

We set (cf [4])

$$(3.5) \qquad \sigma = \frac{c}{2}, \quad \alpha = \min\{3/4(c-\sigma), -\lambda^{\infty}_{-1}\}$$

and we choose $\rho > 0$ such that

$$(3.6) \qquad \frac{\rho^2 - 2(c+\sigma+q)}{\tilde{\lambda}^{\infty}_1 - \tilde{\lambda}^{\infty}_m} \geq \left(\frac{\delta}{-\lambda^{\infty}_{-1}} + 1\right)^2$$

where $q=2\pi \sup|G(t,u)|$. By the compactness of $B_\rho=\{u \in E \,|\, \|u\| \leq \rho\}$ in L^2, lemma 3.1 implies that if $z^+ + z^- \in B_\rho$ then

$$(3.7 \qquad \lim_{\|z_0\| \to +\infty} \int_0^T \left|G\left(t, \|z_0\| \cdot \frac{z_0}{\|z_0\|} + z^- + z^+\right)\right| dt = 0$$

and the analogous with $G(t,z)$ replaced by $(g(t,z),z)$.
By (3.7) these exists $\gamma > 0$ such that

$$(3.8) \quad \begin{cases} \int_0^T |G(t,z)|dt \leq \frac{c-\sigma}{2} & z \in W^{\frac{1}{2}} \ \|z\| > \gamma \\ \int_0^T |(g(t,z),z)|dt \leq \frac{c-\sigma}{4} & \text{and} \quad z^+ + z^- \in B_\rho \,. \end{cases}$$

Now we set $R=\max\{1,\gamma\}$ and in order to prove the lemma we consider $z \in f^{-1}(]c-\sigma,c+\sigma[), \|z\| > R$ and distinguish two cases i) and ii):

i) $z^+ + z^- \in B_\rho$;

in this case we have that

$$(3.9) \qquad c-\sigma \leq \tfrac{1}{2}((L_\infty z,z)) - \int_0^T G(t,z)dt \leq c + \sigma$$

$$(3.10) \qquad \int_0^T |G(t,z)|dt \leq \frac{c-\sigma}{2}$$

hence

(3.11) $\quad ((L_\infty z,z)) \geq c-\sigma.$

By (3.8) and (3.11) we have that

$$\|f'(z)\| \, \|z\| \geq ((L_\infty z,z))-\int_0^T (g(t,z)dt \geq \alpha$$

and the conclusion follows.

ii) $z^+ + z^- \notin B_\rho$;

in this case (3.9) still holds, then we have that

(3.12) $\quad \tilde{\lambda}_1^\infty \|z^+\|^2 + \tilde{\lambda}_m^\infty \|z^-\|^2 \leq ((L_\infty z,z)) \leq 2(c+\sigma+q).$

By (3.12) we deduce that

$$\|z^-\|^2 \geq \frac{\tilde{\lambda}_1^\infty(\|z^+\|^2+\|z^-\|^2) - 2(c+\sigma+q)}{\tilde{\lambda}_1^\infty - \tilde{\lambda}_m^\infty} \geq$$

(3.13)
$$\geq \frac{\tilde{\lambda}_1^\infty \rho^2 - 2(c+\sigma+q)}{\tilde{\lambda}_1^\infty - \tilde{\lambda}_m^\infty} \geq \left(\frac{\delta}{-\tilde{\lambda}_{-1}^\infty} + 1\right)^2$$

Finally by (3.13)

$$\|f'(z)\| \, \|z\| \geq -(f'(z),z^-)\frac{\|z\|}{\|z^-\|} = (-((L_\infty z, z^-)) +$$

$$+ \int_0^T (g(t,z),z^-)dt) \cdot \frac{\|z\|}{\|z^-\|} \geq (-\tilde{\lambda}_{-1}^\infty \|z^-\|^2 - \delta\|z^-\|)\frac{\|z\|}{\|z^-\|} =$$

$$= (-\tilde{\lambda}_{-1}^\infty \|z^-\| - \delta)\|z\| \geq -\tilde{\lambda}_{-1}^\infty \|z\| \geq \alpha\|z\| > \alpha R \geq \alpha,$$

and the lemma is proved. ■

We are looking for T-periodic solutions of (FHS) in the case where strong resonance assumption occurs.

First we consider the problem without symmetry (cf [38]). The following theorem holds:

Theorem 3.14 If $H(t,z)$ satisfies $(H_1),(R_1),(R_2)$, then (FHS) has at least one T-periodic solution. Moreover, if $(H_3),(H_4)$, (H_5) (resp. $(H_4'),(H_5')$) hold too, then there exists at least one T-periodic nontrivial solution.

Proof. We say that the functional (2.9) satisfies $(I_1)-(I_3)$. We set now

$$Q=(\mathrm{Ker}\, L_\infty \oplus H_\infty^-) \cap B_R \qquad S = q + H_\infty^+ ,$$

q being an eigenvector corresponding to $\tilde{\lambda}_{-1}^\infty$ as in section 2, $\|q\|$ small enough and R the constant which will be determined in the following. As usual, ve have

$$f(z) \geq \beta > 0 \quad \text{for every } z \in S.$$

Let $M = 2\pi \sup\{|G(t,z)|, z \in \mathbb{R}^{2n}, 0 \leq t \leq T\}$ and ρ a positive constant such that

$$\tfrac{1}{2}\, \tilde{\lambda}_{-1}^\infty \rho^2 + M < 0 . \tag{3.15}$$

By lemma 3.1 there exists $R>0$ large enough such that for $z \in \mathrm{Ker}\, L_\infty \oplus H_\infty^-$, $\|z\| = R$, $z=z^\circ+z^-$, $z^- \in B_\rho$, we have

$$\int_0^T |G(t,z)|\,dt < \beta/2 \tag{3.16}$$

Taking $z \in \partial Q$, there are two possibilities:

i) $z^- \in B_\rho$ ii) $z^- \notin B_\rho$.

In the first case, by (3.16) we have

$$f(z) = \tfrac{1}{2}((L_\infty z,z)) - \int_0^T G(t,z)\,dt < \beta/2 < \beta .$$

In the second case, by (3.15) it follows that

$$f(z)\leq \frac{1}{2}\tilde{\lambda}^{\infty}_{-1}\|z_-\|^2+M\leq \frac{1}{2}\tilde{\lambda}^{\infty}_{-1}\rho^2+M<0<\beta/2<\beta \ .$$

Then (I_5)(ii) holds with $\alpha=\beta/2$; on the other hand it is obvious that f is bounded from above on Q. So by theorem 1.10, the conclusion of theorem 3.14 follows. ■

Let us consider now an autonomous Hamiltonian system with a strong resonance at infinity. It is known that T-periodic solutions of (HS) correspond to 2π-periodic solutions of

(HS)-1 $\qquad -J\dot{z} = \omega H_z(z)$.

Let L_0 and L_∞ be the operators linearized at zero and at the infinity, i.e.

$$L_0 z = -J\dot{z} - b_0 z \qquad z\in W^{\frac{1}{2}}$$
$$L_\infty z = -J\dot{z} - b_\infty z \qquad z\in W^{\frac{1}{2}} \ .$$

The following theorems hold (cf. [21]).

<u>Theorem</u> 3.17 <u>If</u> H(z) <u>satisfies</u> $(H_1),(H_3),(R_1),(R_2)$ <u>and</u>
(H_6) b_∞ <u>is a positive definite matrix</u>
(H_7) $H(z)\geq 0$ <u>for any</u> $z\in\mathbb{R}^{2n}$ s.t. $H_z(z)=0$,
<u>then</u> (HS) <u>has at least</u> $\frac{1}{2}\dim(H_0^+\cap H_\infty^-)$ <u>non constant</u> T-<u>periodic solutions whenever</u> $H_0^+\cap H_\infty^-\neq\{0\}$.

<u>Theorem</u> 3.18 <u>If</u> H(z) <u>satisfies</u> $(H_1)(H_3)(R_1)(R_2)$ <u>and</u>
(H_6') b_0 <u>is positive definite</u>
(H_7') $H(z)\leq 0$ <u>for any</u> $z\in\mathbb{R}^{2n}$ s.t. $H_z(z)=0$
<u>then</u> (HS) <u>has at least</u> $\frac{1}{2}\dim(H_0^-\cap H_\infty^+)$ <u>non constant</u> T-periodic <u>solutions whenever</u> $H_0^-\cap H_\infty^+\neq\{0\}$.

Proof. The functional of the action being S^1-invariant; in order to find its critical points we shall apply theorem 1.5.

Obviously the functional f verifies $I_1)$ $I_2)$ $I_3)$. Now we shall prove that also (I_4) is satisfied. By (H_6) we have that

$$\{\text{constant functions}\} = \text{Fix } S^1 \subset H_\infty^-$$

then (I_4)-(ii) holds. Since $L_\infty - L_0$ is compact (cf. [8]) also (I_4)-(i) is satisfied. Let $z \in H_0^+$, then

$$f(z) = f(0) + \langle f'(0), z\rangle + \tfrac{1}{2}\langle f''(0)z, z\rangle + O(\|z\|^2) =$$

$$= \tfrac{1}{2}(L_0 z, z) + O(\|z\|^2) \geq \frac{\lambda_1^0}{2}\|z\|^2 + O(\|z\|^2)$$

and (I_4)-(iii) follows. Now let $z \in H_\infty^-$, then by lemma 2.5

$$f(z) \leq \lambda_{-1}^\infty \|z\|^2 - \omega \int G(z)\,dt \leq \lambda_{-1}^\infty \|z\|^2 + \omega(|z|_{L^1} + \varepsilon/2\,|z|_{L^2}^2) + c_2$$

where c_2 is a positive constant depending on ε.

Hence if we choose ε sufficiently small, by the above formula f is bounded from above on H_∞^-, i.e. (I_4)-(iv) holds. Finally by (H_7) also (I_4)-(v) is satisfied and the theorem 3.17 is proved. ■

Remark The proof of theorem 3.18 is analogous to the proof of theorem 2.4.

§ 4. Hamiltonian systems of the second order with strong resonance at infinity

In the previous sections we have studied Hamiltonian systems with an asymptotically quadratic Hamiltonian function; moreover many authors consider the case in which $H(t,z)$ is superquadratic in z, i.e.

$$\frac{H(t,z)}{|z|^2} \longrightarrow +\infty \qquad \text{as } |z| \longrightarrow +\infty$$

(cf. [7], [11], [12], [26], [35])
or subquadratic in z, i.e.

$$\frac{H(t,z)}{|z|^2} \longrightarrow 0 \qquad \text{as } |z| \longrightarrow +\infty$$

(cf. [6], [7], [12] and [14]).
Unfortunately the above results do not cover the classical mechanical problems: namely the Hamiltonian of a mechanical system with holonomous constraints in a conservative field of forces has the form

$$(4.1) \qquad H(t,p,q) = \sum_{i,j=1}^{n} a_{ij}(t,q)p_i p_j + \sum_{i=1}^{n} b_i(t,q)p_i + V(t,q)$$

where $\{a_{ij}(t,q)\}$ is a positive definite matrix for every t and q.
In this case H is quadratic in p, but is not globally quadratic, or superquadratic or subquadratic in z. These Hamiltonian has been studied by Benci-Capozzi-Fortunato in [10] in the case $V(q)$ is superquadratic. The problem has been examined also in [27] in the case $b_i(q)=0$. In

the following we shall distinguish three different situations.

(i) $\{a_{ij}(t,q)\}$ is constant

(ii) $V(q)$ is subquadratic but $V(q) \longrightarrow +\infty$ as $q \longrightarrow +\infty$

(iii) $V(q)$ is bounded.

In this section we deal with an Hamiltonian function of the form (4.1) in the case

$b_i=0$ and $a_{ij}(q)$ constant, i.e.

$$(4.2)\qquad H(t,p,q) = \sum_{i,j=1}^{n} a_{ij}p_ip_j + V(t,q).$$

By a change of variables, (4.2) becomes

$$(4.3)\qquad H(t,p,q) = \tfrac{1}{2}\lambda p^2 + V(t,q)$$

and (HS) is

$$\begin{cases} \dot{p} = \nabla V(t,q) \\ \dot{q} = -\lambda p \end{cases}$$

This system is obviously equivalent to the system of n differential equations of the second order

$$(4.4)\qquad -\ddot{x} = \nabla V(t,x)$$

where $V(t,x) \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, $V(t,x)=V(t+T,x)$ for any $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\nabla V(t,x)$ denotes the gradient of V respect to the space variable.

It is known that T-periodic solutions of (4.4) correspond to 2π-periodic solutions of

$$(4.5)\qquad -\ddot{x} = \omega^2 \nabla V(\omega t,x)$$

This problem has been studied by many authors under different assumptions on the growth on U (cf.[6] and [39] for a rather complete bibliography). Let us assume that the problem (4.4) is asymptotically linear, i.e. there exists for any $t \in [0,T]$ a symmetric matrix $n \times n$ $M(t)$ s.t.

$$(U_1) \quad \begin{cases} \nabla V((t,x) = M(t)x + \nabla U(t,x) \\ \dfrac{\nabla U(t,x)}{|x|} \longrightarrow 0 \quad \text{as} |x| \longrightarrow +\infty \quad \text{uniformly in } t \in \mathbb{R}. \end{cases}$$

Then (4.5) can be written as

$$-\ddot{x} - \omega^2 M(t\omega)x = \omega^2 \nabla U(t\omega,x) \ .$$

Let be $\mathscr{L}$ the selfadjoint realization in L^2 of the operator $x \to -\ddot{x} - \omega^2 M(t\omega)x$ with periodic conditions.
As in section 3, let us assume that the problem has a" strong resonance at infinity",i.e.

$$(U_2) \quad \begin{cases} 0 \in \sigma(\mathscr{L}) & \\ U(t,x) \to 0 \quad \text{as} |x| \to +\infty & \text{uniformly in } t \in \mathbb{R} \\ (\nabla U(t,x),x) \to 0 \quad \text{as} |x| \to +\infty & \text{uniformly in } t \in \mathbb{R} \ . \end{cases}$$

The nonlinear term going rapidly to 0, we shall prove that the generalized Palais-Smale condition (I_3) holds by arguing as in section 3.
Let be L_∞ and L_0 the linearized operators of $x \to -\ddot{x} - \omega^2 \nabla V(\omega t,x)$ at infinity and at origins, i.e.

$$L_\infty x = -\ddot{x} - \omega^2 M(\omega t)x$$
$$L_0 x = -\ddot{x} - \omega^2 M(\omega t)x - \omega^2 V_{xx}(\omega t,0)x.$$

We denote by m_∞ (resp. m_0) the dimension of subspaces where L_∞ (resp. L_0) is negative semidefinite.
The following theorem holds (cf.[19]):

Theorem 4.6 If the problem (4.5) verifies $(U_1),(U_2)$

(U_3) $U(t,x)$ is C^2 at $x=0$, $U(t,0)= \nabla U(t,0) = 0 \quad \forall\, t \in \mathbb{R}$

(U_4) $\mu < 0$ where $\mu = \sup\limits_{[0,T]} (\sup \sigma(U_{xx}(t,0)))$

(U_5) there exists $\lambda_h \in \sigma(\mathcal{L})$ $\lambda_h \leq 0$ s.t. $\lambda_h - \omega^2 \mu > 0$

(U_6) $U(t,x) = U(t-x) \quad \forall\, x \in \mathbb{R}^n, \quad \forall\, t \in \mathbb{R}$

problem (4.5) possesses at least m distinct pairs of nontrivial 2π-periodic solutions with

$$m = m_\infty - m_0 \ .$$

The same results hold in the autonomous case, namely (cf. [20]):

Theorem 4.7 Assume that $\frac{\partial U}{\partial t} = 0$ and $(U_1)-(U_5)$ hold. Moreover

(U_7) M is positive semidefinite or $\nu \leq -\frac{\lambda_h}{\omega^2}$, where $\nu = \max \sigma(M)$

(U_8) $-U(x) \leq \frac{1}{2}(Mx,x) \quad \forall\, x \in \mathbb{R}^n$ s.t. $\nabla U(x) = Mx$.

Then problem (4.5) has at least m distinct orbits of nonconstant solutions with $m = \frac{m_\infty - m_0}{2}$.

Theorem 4.8 We can replace assumptions $(U_4),(U_5),(U_7)$, (U_8) by

(U_4') $\mu > 0$ where $\mu = \min \sigma(U_{xx}(0))$ i.e. $U_{xx}(0)$ is positive definite,

(U_5') there exists $\lambda_s \in \sigma(\mathcal{L})$ $\lambda_s \geq 0$ s.t. $\lambda_s - \omega^2 \mu < 0$

(U_7') M is negative semidefinite or $\nu \geq -\frac{\lambda_s}{\omega^2}$ where $\nu = \min \sigma(M)$

(U_8') $\frac{1}{2}(Mx,x) \leq -U(x) \quad \forall x \in \mathbb{R}^n$ s.t. $\nabla U(x) = Mx$.

Then (4.5) has at least m distinct orbits of nonconstants solutions with

$$m = \frac{m_{\bar{0}} - m_{\infty}}{2}$$

Proof of theorems

Let be $H^1 = H^1([0,2\pi]), \mathbb{R}^n)$ and $H = \{u \in H^1 | u(o) = u(2\pi)\}$ equipped with the scalar product of H^1, i.e.

$$(u,v)_H = (u,v)_{H^1} \quad .$$

It is easy to show that classical solutions of (4.5) are the critical points of the functional defined on H

$$(4.9) \quad f(u) = \tfrac{1}{2}\int_0^{2\pi} |\dot{u}(t)|^2 dt - \frac{\omega^2}{2}\int_0^{2\pi} (M(\omega t)u(t), u(t))dt - \\ - \omega 2\int_0^{2\pi} U(t,u(t))dt.$$

This functional being semidefinite, we shall apply theorems (1.3) and (1.5). We denote by $\beta(t)$ the largest eigenvalue of $M(t)$ and by I_n the identity matrix in $\mathbb{R}^n$ and we set $\beta = \sup_{[0,T]} \beta(t)$. We consider the bilinear form on H

$$a(u,v) = \int_0^{2\pi} (\dot{u},\dot{v})dt + \int_0^{2\pi} (u,v)dt - \omega^2\int_0^{2\pi} (M(\omega t)u,v)dt + \beta\int_0^{2\pi} (u,v)dt.$$

By easy computations it can be proved that a is continuous and coercive on H.

Then by standard theorems (cf. [28]) there exists a unique bounded linear operator $S: H \to H$ with a bounded linear inverse S^{-1} such that $(Su,u)_H = a(u,v) \quad \forall u,v \in H$. We set

$$D(\mathcal{S}) = \{u \in H \mid Su \in L^2\} \quad \text{and} \quad \mathcal{S} = S_{|D(\mathcal{S})}.$$

$\mathcal{S}$ is a linear self-adjoint operator with compact resolvent. Then $\sigma(\mathcal{S})$ consists of a positively divergent sequence of isolated eigenvalues with finite multiplicities (cf. [28]). If we denote by $s_0 < s_1 < \ldots < s_j < \ldots$ the eigenvalues of and by M_j the corresponding eigenspaces, then $L^2 = \bigoplus_j M_j$. If we denote by $\mathcal{L}$ the self-adjoint realization of $-\ddot{x}-\omega^2 M(\omega t)x$ in L^2, we have that $\mathcal{L}=\mathcal{S}-(\beta+1)I$, $I:L^2 \to L^2$ being the identity map and $\lambda_j = s_j-(\beta+1)$. We observe that the eigenspace corresponding to λ_j is M_j for any j. If $m \geq 0$ is an integer, we set

$$H^-(m) = \bigoplus_{j\leq m} M_j \qquad \text{and} \qquad H^+(m) = \overline{\bigoplus_{j\geq m} M_j}$$

where the closure is taken in H.
By (U_2) there exists $\lambda_h \in \sigma(\mathcal{L})$ s.t. $\lambda_h = 0$. Obviously for any $u \in H$

$$u = u^+ + u_0 + u^- ,$$

where $u_0 \in M_k$, $u^+ \in H^+(k+1)$, $u^- \in H^-(k-1)$.
The following lemma holds:

<u>LEMMA</u> 4.10 <u>There exist</u> $\eta, \tau, \nu > 0$ such that

(i) $\quad (\mathcal{L}u, u^+)_{L^2} \geq \eta |u^+|^2_{L^2} \qquad \forall\, u \in H$

(ii) $\quad -\tau |u^-|^2_{L^2} \leq (\mathcal{L}u, u^-)_{L^2} \leq -\nu |u^-|^2_{L^2} \qquad \forall\, u \in H$.

Proof It suffices to take η=the first positive eigenvalue of $\mathcal{L}$, $-\tau$ and $-\nu$ the smallest and the largest negative eigenvalue of $\mathcal{L}$.

Remark 4.11 By lemma (4.10) and by the same arguments used in section 3, we could prove that the strong resonance assumption implies condition (I_3) of theorem 1.5. Moreover the following lemmas hold:

LEMMA 4.12 Under the assumptions $(U_1),(U_3)$ and (U_4), there exist $\rho,\ c_0>0$ s.t.

$$f(u)\geq c_0 \qquad \forall\ u\in H^+(h)\cap S\rho\ .$$

Proof Let be $u\in H^+(h)\cap S\rho$, ρ small enough, then

$$(4.13)\qquad f(u)=f(0)+\langle f'(0),u\rangle+\tfrac{1}{2}(f''(0)u,u)+o(\|u\|^2_H)=$$

$$=\tfrac{1}{2}(\mathscr{L}u,u)_{L^2}-\tfrac{1}{2}\omega^2\int_0^{2\pi}(U_{xx}(\omega t,0)u,u)dt+o(\|u\|^2_H)\geq$$

$$\geq\tfrac{1}{2}(\mathscr{L}u,u^+)_{L^2}-\tfrac{1}{2}\omega^2\mu|u|^2_{L^2}+o(\|u\|^2_H)=$$

$$=\tfrac{1}{2}\sum_{j=h}^{\infty}(\lambda_j-\omega^2\mu)|u_j|^2_{L^2}+o(\|u\|^2_H)\ .$$

There exist $t>h$ and $\delta>0$ s.t.

$$\lambda_j-\omega^2\mu>\delta\lambda_j>0 \qquad \forall\ j>t,$$

then

$$(4.14)\qquad \sum_{j=h}^{\infty}(\lambda_j-\omega^2\mu)|u_j|^2_{L^2}=\sum_{j=h}^{t}\frac{\lambda_j-\omega^2\mu}{\lambda_j}\lambda_j|u_j|^2_{L^2}+$$

$$+\sum_{j=t+1}^{\infty}(\lambda_j-\omega^2\mu)|u_j|^2_{L^2}\geq \text{const}\sum_{j=h}^{t}\lambda_j|u_j|^2_{L^2}+$$

$$+\sum_{j=t+1}^{\infty}\delta\lambda_j|u_j|^2_{L^2}\geq \text{const}\sum_{j=h}^{\infty}\lambda_j|u_j|^2_{L^2}\ .$$

By (4.13) and (4.14) we have

$$f(u) \geq \text{const } \|u\|_H^2 + o(\|u\|_H^2) \geq c_o \ . \qquad \blacksquare$$

<u>LEMMA</u> 4.15 <u>There exists</u> $c_\infty \in \mathbb{R}$ s.t.

$$f(u) < c_\infty \qquad \forall\, u \in H^-(k).$$

Proof. Let $\lambda = \sup\limits_{|0,T|} |U(t,u)|$, then for any $u \in H^-(K)$:

$$f(u) = (\mathscr{L}u^-, u^-)_{L^2} - \omega^2 \int_0^{2\pi} U(\omega t, u)\,dt = 2\pi\omega^2\lambda \ . \blacksquare$$

<u>Proof of theorem</u> 4.6

The functional f being even, we shall apply theorem (1.3). By the remarks (1.7) and (4.10) condition (f_1) holds; lemmas (4.12) and (4.15) implie (f_2) with $V=H^+(h)$ and $W=H^-(k)$. The the problem (4.5) possesses at least

$$m = \dim (M_h \oplus \ldots \oplus M_k)$$

distinct pairs of nontrivial solutions. Obviously $m = m_\infty - m_o$. $\blacksquare$

<u>Remark</u> 4.16 Let us assume, as in [18], that

$$M(t) = \lambda_k I \qquad \lambda_k = k^2 \qquad k=0,1,2,\ldots$$

i.e. $\lambda_k \in \sigma(-\ddot{x})$ in L^2 with periodic conditions. Problem (4.5) becomes

$$(4.17) \qquad -\ddot{x} - \omega^2\lambda_k x = \omega^2 \nabla U(\omega t, x).$$

Assumption (U_3) can be replaced by

"there exists $\lambda_h \in \sigma(-\ddot{x})$, $\lambda_h \leq \lambda_k$ s.t. $\lambda_h - \lambda_k - \omega^2 \mu > 0$ ".

Problem (4.17) possesses at least

$$m = \dim H^-(k) - \operatorname{codim} H^+(h)$$

distinct pairs of nontrivial solutions.
If $\lambda_k = 0$, we obtain the case studied by Thews (cf. [39]).

Proof of theorem (4.7)

We shall use theorem 1.5 for S^1-invariant functionals. Namely the functional of the action is

$$f(u) = \tfrac{1}{2}\int_0^{2\pi} |\dot{u}|^2 dt - \tfrac{1}{2}\,\omega^2 \int_0^{2\pi} (Mu,u)\,dt - \omega^2 \int_0^{2\pi} U(u)\,dt$$

which satisfies obviously (I_1), (I_2) and (I_3). As above, there exist c_0, $c_\infty \in \mathbb{R}_+$ s.t.

$$(I_4)-(iii) \qquad f(u) \geq c_0 \qquad \forall\, u \in V \cap S_\rho \qquad V = H^+(h),$$

$$(I_4)-(iv) \qquad f(u) \leq c_\infty \qquad \forall\, u \in W = H^-(k).$$

(I_4)-(i) is obvious; in order to prove (I_4)-(ii) it is sufficient to prove that all the eigenvectors of M belong to V or W. Let c be an eigenvector of M and λ be the corresponding eigenvalue. It results

$$Lc = -\,\omega^2 Mc = -\,\omega^2 \lambda c,$$

then $-\omega^2\lambda \in \sigma(L)$, where $(Lu,u)_H = (\mathscr{L}u,u)_{L^2}$. If M is positive semidefinite, we have $-\omega^2\lambda \leq 0 = \lambda_k$, therefore $c \in W$; if $\nu \leq -\dfrac{\lambda_h}{\omega^2}$ we have $-\omega^2\lambda \geq -\omega^2\nu \geq \lambda_h$ and $c \in V$.

Finally, if u is a constant and $f'(u)=0$, by (U_8)

$$f(u) = -\frac{\omega^2}{2}\int_0^{2\pi}(Mu,u)\,dt-\omega^2\int_0^{2\pi}U(u)\,dt \leq 0 < c_\infty \quad .$$

Then (f_4)-(iii) is satisfied.

Remark 4.18 Under the assumption of the theorem 4.8, the conclusion follows by applying the abstract theorem (1.5) to the functional $-f$.

§ 5. The case $H(p,q)=a(q)p^2+V(q)$, $V(q)$ unbounded

Let us consider now an Hamiltonian function of the form (4.1) and with subquadratic growth in q (cf.case (ii) of the section 4).

We make the following assumptions:

Assumptions (H_o)

(V_1) There exist constants $R>0$, $\alpha<2$ s.t.

$$(V'(t,q),q)_{\mathbb{R}^n}-\alpha V(t,q)\leq 0 \text{ for } |q|\geq R, \text{ for every } t\in\mathbb{R}.$$

(V_2) There exist constants $R_1, c_1>0$ s.t.

$$|V'(t,q)|\leq c_1 V(t,q) \quad \text{for} \quad |q|\geq R_1 \ , \text{ for every } t\in\mathbb{R}.$$

(V_3) $V(t,q)\to+\infty$ for $|q|\to+\infty$ uniformly in $t\in\mathbb{R}$.

(A_1) There exist a real, continuous function $\nu(q)>0$ and a constant M s.t.

$$\nu(q)|p|^2\leq \sum_{i,j=1}^{n} a_{ij}(t,q)p_ip_j\leq M|p|^2 \quad \text{for every } p,q\in\mathbb{R}^n.$$

(A_2) There exist a constant $\beta\in\,]0,2-\alpha[$ such that

$$\left\{\sum_{k=1}^{n}\frac{\partial}{\partial q_k}a_{ij}(t,q)q_k-\beta a_{ij}(t,q)\right\} \text{ is negative semidefinite.}$$

(A_3) There exists a constant c_2 s.t.

$$\left|\sum_{i,j=1}^{n}\frac{\partial}{\partial q_k}a_{ij}(t,q)p_ip_j\right|\leq c_2\sum_{i,j=1}^{n}a_{ij}(t,q)p_ip_j$$

for every $k=1,\dots,n$ $\quad q\in\mathbb{R}^n$, $t\in\mathbb{R}$.

$(B_1)\ \lim_{|q|\to+\infty} \dfrac{b_i(t,q)^2}{\nu(q)V(t,q)} = 0$ for every $i = 1,\ldots, n$.

$(B_2)\ \lim_{|q|\to\infty} \dfrac{\left|\frac{\partial b_i}{\partial q_k}(t,q)q_k\right|^2}{\nu(q)V(t,q)} = 0$ for every $k,i=1,\ldots,n$.

(H_1) $H(t,z)$ is C^2 in 0 and $(H_{zz}(t,0)z|z)\geq\lambda_1|z|^2$

$\lambda_1>M$, for every $t \in \mathbb{R}$.

Remark. - The assumption (V_1) implies that V is subquadratic in q but, obviously, $H(t,z)$ is not subquadratic in z.

(A_1) is a phisical assumption wich depends on the fact that the "Kinetic energy" is positive. The other assumptions are technical assumptions wich control the growth ad infinity of the coefficients of (4.1).

We shall prove the following theorems (cf. [37]):

Theorem 5.1 - Suppose that H satisfies the assumptions (H_0), $\frac{\partial H}{\partial t} = 0$ and

$H_2)$ $H(0)=H'(0)=0$, $H'(z)\neq 0$ for every $z \neq 0$.

Then there exists $\bar{T} \in \mathbb{R}$ such that (HS) has at least n non-constant T-periodic solutions for every $T>\bar{T}$.

Theorem 5.2 - Suppose that $H(t,z)$ satisfies the assumptions (H_0) and

$H_2')$ $H(t,z)$ is even in z, T-periodic in t and $H(t,0)=H_z(t,0)=0$ for every $t \in \mathbb{R}$.

Then there exists $\bar{T} \in \mathbb{R}$ such that (FHS) has at least $2n$ non trivial T-periodic solutions, if $T > \bar{T}$.

Proof of theorems

As usually, the T-periodic solutions of the Hamiltonian systems (FHS) correspond to the critical points of the functional defined on $W^{\frac{1}{2}}$

$$f(z)=\frac{1}{2}((Lz,z))-\omega\int_0^{2\pi}H(\omega t,z)\,dt\ .$$

By the assumption (H_0), there exist c_3, c_4, s positive constants s.t.

$$|H_z(t,z)|\le c_3+c_4|z|^s \qquad \text{for any } t \text{ and } z.$$

Then standard arguments show that f is Fréchet differentiable and satisfies (I_1) of theorem 1.5. We recall that $\operatorname{Ker} L=\mathbb{R}^{2n}$ and therefore $0\notin\sigma_e(L)$. We shall prove now (I_3) arguing as in [10].

In the sequel we shall use the following shortened notations:

$$a(q)p^2=\sum_{i,j=1}^{n} a_{ij}(t,q)p_ip_j \qquad a'(q)qp^2=\sum_{i,j,k=1}^{n}\frac{\partial}{\partial q_k}a_{ij}(t,q)q_kp_ip_j$$

$$b(q)p=\sum_{i=1}^{n} b_i(t,q)p_i \qquad b'(q)qp=\sum_{i,k=1}^{n}\frac{\partial}{\partial q_k}b_i(t,q)q_kp_i\ .$$

We recall the following lemma:

<u>Lemma</u> 5.3 - <u>Let</u> $\{z_n\}\subset W^{\frac{1}{2}}$, $z_n=(p_n,q_n)$, <u>be a sequence satisfying</u>

(5.4) $$f(z_n)\to c$$

(5.5) $$\|f'(z_n)\|\cdot\|z_n\|\to 0\ .$$

Then the following sequences

(5.6) $$\int_0^{2\pi}(H(t,z_n) - (H_p(t,z_n),p_n))dt$$

(5.7) $$\int_0^{2\pi}(H(t,z_n) - (H_q(t,z_n),q_n))dt$$

are bounded.

Remark. - The Hamiltonian being of the forme (4.1), the sequences (5.6) and (5.7) become

(5.8) $$\int_0^{2\pi}V(t,q_n) - a(q_n)p_n^2\,dt$$

(5.9) $$\int_0^{2\pi}[-a'(q_n)q_np_n^2-b'(q_n)q_np_n-(V'(t,q_n),q_n) + +a(q_n)p_n^2+b(q_n)p_n+V(t,q_n)]dt.$$

In the sequel we omit the index n and we denote by M_i a positive constant.

Lemma (5.10 - Suppose that V_1) A_1) A_2) B_1) B_2) hold and that {z} is a sequence in $W^{\frac{1}{2}}$ satisfying (5.4) and (5.5); then the sequences

$$\int_0^{2\pi}V(t,q)dt \quad \text{and} \quad \int_0^{2\pi}a(q)p^2dt$$

are bounded.

Dim. - Let $\delta>0$ be a constant such that $\alpha+\beta+2\delta=2$.

Multiplying (5.8) by $1-\beta-\delta$ and adding the product to (5.9) we obtain that the sequence

(5.11) $\int_0^{2\pi}[(\beta+\delta)a(q)p^2-a'(q)qp^2+b(q)p-b'(q)qp +$

$+(2-\beta-\delta)V(t,q)-V'(t,q),q)]dt$

is bounded.

By (V_1) and $A_2)$ there exists $M_1>0$ s.t.

$$M_1 \geq \int_0^{2\pi}[\delta a(q)p^2+(2-\alpha-\beta-\delta)V(t,q)+b(q)p-b'(q)qp]dt$$

that is

(5.12) $\int_0^{2\pi}[\delta a(q)p^2+\delta V(t,q)+b(q)p-b'(q)qp]dt\leq M_1$.

By $B_1)$ and $B_2)$ there exists $M_2>0$ s.t.

(5.13) $\dfrac{|b(q)|^2+|b'(q)q|^2}{\delta\nu(q)} < \dfrac{\delta}{2}V(t,q)+M_2$ for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$.

Then, using (5.13) we get

(5.14) $\int_0^{2\pi}[b'(q)qp-b(q)p]dt\leq\int_0^{2\pi}[|b'(q)q|\,|p|+|b(q)\|p|]dt\leq$

$$\leq\int_0^{2\pi}[\frac{|b'(q)q|^2}{\delta\nu(q)}+|p|^2\frac{\delta}{4}\nu(q)+\frac{|b(q)|^2}{\delta\nu(q)}+\frac{\delta}{4}\nu(q)|p|^2]dt\leq$$

$$\leq\int_0^{2\pi}[\frac{\delta}{2}V(t,q)+\frac{\delta}{2}\nu(q)|p|^2]dt+M_3$$

and by (5.12)

$$M_1\geq\int_0^{2\pi}[\delta a(q)p^2+\delta V(t,q)-\frac{\delta}{2}V(t,q)-\frac{\delta}{2}\nu(q)p^2]dt-M_3$$

$$\geq\int_0^{2\pi}[\frac{\delta}{2}a(q)p^2+\frac{\delta}{2}V(t,q)]dt-M_3.$$

The lemma (5.10) is proved. ■

Lemma (5.15) - Under the assumptions $A_1), A_2), A_3)$, $V_1), V_2)$, $V_3)$, $B_1), B_2)$, $f_3)$ holds

Proof. - By lemma (5.10), $V_2)$ and $A_3)$, it follows easily that $\{H_z(t,z)\}$ is bounded in L^1. L^1 is continuously embedded into $W^{-\frac{1}{2}-\frac{\eta}{2}}$, for any $\eta>0$. Then $H_z(t,z)$ is bounded in $W^{-\frac{1}{2}-\frac{\eta}{2}}$.

Assume now that $\{z\}$ is unbounded. Then, by (5.5)

$$Lz-\omega H_z(t,z) \to 0, \quad \text{in} \quad W^{-\frac{1}{2}} .$$

It follows that

$$\{Lz\} \quad \text{is bounded in } W^{-\frac{1}{2}-\frac{\eta}{2}} .$$

If z_0 denotes the component of z belonging to $M_0=\mathrm{Ker}L$, and $\tilde{z}=z-z_0$, it is known that

$$\forall z \in W^{\frac{1}{2}} : \|\tilde{z}\|_{W^{\frac{1}{2}-\frac{\eta}{2}}} \leq \mathrm{cost}\|Lz\|_{W^{-\frac{1}{2}-\frac{\eta}{2}}}$$

and therefore

$$\{\tilde{z}\} \quad \text{is bounded in} \quad W^{\frac{1}{2}-\frac{\eta}{2}} .$$

We prove now that $\{z_0\}$ is bounded in L^1.

If $z_0=(p_0,q_0)$, by lemma (2.2) and by $V_3)$ it follows that $\{q^0\}$ is bounded in L^1. Infact by $V_3)$ there exists a function $\chi:\mathbb{R}_+\to\mathbb{R}$ (cf. [8]) such that

$$(5.16) \quad \begin{aligned} &a) \quad \chi(0)=0 \quad \lim_{s\to\infty}\chi(s)=0 \quad \chi'(s)>0 \\ &b) \quad V(t,q)\geq \chi(|q|)-c \end{aligned}$$

Since KerL is a finite dimensional space, if $\|q^0\|_{L^1}\to+\infty$, then $|q^o| \to +\infty$ and

$$\int_o^{2\pi} \chi(|q^o|)dt \to +\infty .$$

On the other hand, by (5.16) b) we have

$$\int_o^{2\pi} \chi(|q^o|)dt \leq \int_o^{2\pi} V(t,q^o)dt + 2\pi c \leq M_4 .$$

Therefore also $\{q^o\}$ is bounded in L^1. The conclusion follows as in lemma (6.5) of [10]. ∎

We shall prove now the geometrical condition (I_4). Let be

$$W_j^+ = \overline{\bigoplus_{k\geq j} M_{\lambda k}} \qquad W_j^- = \overline{\bigoplus_{k\leq j} M_{\lambda_k}}$$

where M_{λ_k} is the eigenspace corresponding to the eigenvalue $\lambda_k = \frac{k}{1+|k|}$ of $-J\dot{z}$ in $W^{\frac{1}{2}}$ and the closures are taken in $W^{\frac{1}{2}}$.

It is known (cf. section 2) that a basis of $W^{\frac{1}{2}}$ is

$$\psi_{jk} = e^{jtJ}\Phi_k \qquad j \in \mathbb{Z} \qquad k=1,\ldots,2n$$

where Φ_k is the standard basis in $\mathbb{R}^{2n}$. Therefore, if $\psi \in W^{\frac{1}{2}}$, we have

$$\psi = \begin{pmatrix} \sum_{j\in\mathbb{Z},k=1,\ldots,n} a_{ik} \cos jt\Phi_k^* - \sum_{j\in\mathbb{Z},k=1,\ldots,n} b_{jk} \sin it\Phi_k^* \\ \sum_{j\in\mathbb{Z},k=1,\ldots,n} a_{ik} \sin jt\Phi_k^* + \sum_{j\in\mathbb{Z},k=1,\ldots n} b_{jk} \cos jt\Phi_k^* \end{pmatrix}$$

Φ_k^* being the standard basis in $\mathbb{R}^n$.

Let ψ^+, ψ^- and ψ_o be the components of z belonging to W_1^+, W_{-1}^- and M_o.

It results

$$\psi^+ = \begin{pmatrix} \sum\limits_{j\geq 1, k=1,\dots,n} [a_{jk}\cos jt - b_{jk}\sin jt]\Phi_k^* \\ \\ j\geq 1, k=1,\dots,n [a_{jk}\sin jt + b_{jk}\cos jt]\Phi_k^* \end{pmatrix}$$

and

$$\psi^- = \begin{pmatrix} \sum\limits_{j\leq -1, k=1,\dots,n} [a_{jk}\cos jt - b_{jk}\sin jt]\Phi_k^* \\ \\ \sum\limits_{j\leq -1, k=1,\dots,n} [a_{jk}\sin jt + b_{jk}\cos jt]\Phi_k^* \end{pmatrix}$$

Let p^+, q^+, p^-, q^- be the components of ψ^+ and ψ^-; obviously

$$(5.17) \qquad |p^+|^2 = |q^+|^2 \qquad |p^-|^2 = |q^-|^2 \, .$$

In order to prove (I_4), the following lemma need:

<u>Lemma</u> (5.18) - <u>There exists</u> $j \in N$ <u>such that</u> f is <u>bounded from below on</u> W_j^+ .

Proof. - We have

$$(5.19) \qquad \begin{aligned} f(z) &= \tfrac{1}{2}((Lz,z)) - \omega\int_0^{2\pi} a(q)p^2 dt - \omega\int_0^{2\pi} b(q)p dt - \omega\int_0^{2\pi} V(t,q)dt \\ &\geq \tfrac{1}{2} j |z|_{L^2}^2 - \omega M |p|_{L^2}^2 - \omega\int_0^{2\pi} b(q)p dt - \omega\int_0^{2\pi} V(t,q)dt \, . \end{aligned}$$

We observe (cf. (5.17)) that

$$(5.20) \qquad |p|_{L^2}^2 = |q|_{L^2}^2 \qquad \text{for any } z = (p,q) \in W_j^+ \quad \forall j > 0$$

and

$$\int_0^{2\pi} b(q)p\,dt \le \int_0^{2\pi} |b(q)p|\,dt \le \int_0^{2\pi} \frac{|b(q)|}{\sqrt{\varepsilon\nu(q)}} \cdot |\sqrt{\varepsilon\nu(q)}\,p|\,dt \le$$

(5.21) $$\left(\int_0^{2\pi} \frac{b^2(q)}{\varepsilon\nu(q)}\,dt\right)^{\frac{1}{2}} \left(\int_0^{2\pi} \varepsilon\nu(q)p^2dt\right)^{\frac{1}{2}} \le \tfrac{1}{2}\left[\int_0^{2\pi} \frac{b^2(q)}{\varepsilon\nu(q)}dt + \varepsilon\int_0^{2\pi} \nu(q)p^2dt\right] \le$$

$$\le \tfrac{1}{2}\left[\int_0^{2\pi} V(t,q)\,dt + M_5 + M\varepsilon|p|^2_{L^2}\right] .$$

By (5.19),(5.20),(5.21) it follows

$$f(z) \ge j|p|^2_{L^2} - \omega M|p|^2_{L^2} - 2\omega\int_0^{2\pi} V(t,q)\,dt - \omega\frac{\varepsilon}{2}M|p|^2_{L^2} - \tfrac{1}{2}M_5 \ge$$

$$\ge \left(j - \omega M\left(1+\frac{\varepsilon}{2}\right)\right)|q|^2_{L^2} - 2\omega|q|^{\alpha}_{L^{\alpha}} - \tfrac{1}{2}M_5 > M_6 .$$

The last inequality holds for $j - \omega M(1+\frac{\varepsilon}{2}) > 0$ i.e.

(5.22) $$j > \omega M .$$ ■

The lemma holds for j sufficiently large.

<u>Lemma</u> (5.23) - <u>There exist positive constants</u> $\rho,\delta>0$ <u>and</u> $m \in N$ <u>such that</u>

$$f(z) < -\delta \quad \underline{\text{on}} \quad \delta_\rho \cap V \qquad V = W^-_{j+m}$$

Proof. - We have

$$f(z) = f(o) + f'(o)[z] + \tfrac{1}{2}((f''(o)z,z)) + O(\|z\|^2) =$$

$$= \tfrac{1}{2}((Lz,z)) - \frac{\omega}{2}\int_0^{2\pi} (H_{zz}(o,t)z,z)\,dt + O(\|z\|^2) \le$$

$$\le \tfrac{1}{2}\frac{j+m}{1+j+m}\|z^+\|^2 - \frac{1}{4}\|z^-\|^2 - \lambda_1\frac{\omega}{2}|z|^2_{L^2} + O(\|z\|^2) \le$$

$$\le \tfrac{1}{2}\frac{j+m}{1+j+m}\|z^+\|^2 - \frac{1}{4}\|z\|^2 + \frac{1}{4}\|z^+\|^2 + \frac{1}{4}\|z_o\|^2 - \lambda_1\frac{\omega}{2}|z|^2_{L^2} + O(\|z\|^2) =$$

$$= \frac{3(j+m)+1}{4(1+j+m)}\|z^+\|^2-(\tfrac{1}{4}-\varepsilon)\|z\|^2-\varepsilon\|z\|^2-\lambda_1\frac{\omega}{2}|z|^2_{L^2}+\tfrac{1}{4}|z_o|^2_{L^2}+o(\|z\|^2)=$$

$$=\left[\frac{3(j+m)+1}{4(1+j+m)}-(\tfrac{1}{4}-\varepsilon)\right]\|z^+\|^2-(\tfrac{1}{4}-\varepsilon)\|z^-\|^2-(\tfrac{1}{4}-\varepsilon-\tfrac{1}{4})|z_o|^2_{L^2}-\lambda_1\frac{\omega}{2}|z^+|^2_{L^2}-$$

$$\lambda_1\frac{\omega}{2}|z^-|^2-\lambda_1\frac{\omega}{2}|z_o|^2_{L^2}-\varepsilon\|z\|^2+o(\|z\|^2)\ \le$$

$$\left[\frac{3(j+m)+1}{4(1+j+m)}-(\tfrac{1}{4}-\varepsilon)-\frac{\lambda_1\omega}{2(1+j+m)}\right]\|z^+\|^2-(\frac{\lambda_1\omega}{2}-\varepsilon)|z_o|^2-\varepsilon\|z\|^2+o(\|z\|^2)$$

If

(5.24) $$\frac{3(j+m)+1}{4(1+j+m)}-(\tfrac{1}{4}-\varepsilon)-\frac{\lambda_1\omega}{2(1+j+m)}\le 0$$

it follows

$$f(z)\le-\delta \quad \text{su} \quad S_\rho\cap W^-_{j+m}\ .$$

By (5.24) we obtain

$$2(j+m)-2\lambda_1\omega+4\varepsilon(1+j+m)\ \le\ 0,$$

therefore

$$2(j+m)-2\lambda_1\omega\ <\ 0,$$

i.e.

(5.25) $$j+m\ <\ \lambda_1\omega$$ ■

<u>Remark</u> - The term b(q) does not change the proof of the lemma (5.23) being $H_{zz}(o)$ unvaried.

We shall prove that, for ω large, (5.22) and (5.25) hold at the same time and, consequently, lemmas (5.18),(5.23) are verified.

Lemma (5.26) - There exists $\bar{\omega} \in \mathbb{R}$ such that for $\omega > \bar{\omega}$ lemmas (5.22) and (5.25) hold.

Proof. - Lemmas (5.22)(5.25) hold if there exist $j \geq 1$ and $m \geq 0$ such that

$$\frac{j+m}{\lambda_1} < \omega < \frac{j}{M} \quad . \tag{5.27}$$

First, from (5.27) it follows that we must suppose $\lambda_1 > M$. If $m=0$, (5.27) becomes

$$\frac{j}{\lambda_1} < \omega < \frac{j}{M} \quad , \tag{5.28}$$

j being a convenient positive integer.

Since the sequence $\{\frac{j}{j+1}\}$ is increasing and it converges to 1, fixing $\frac{M}{\lambda_1} < 1$, we can say that there exists $j_o \in N$ s.t. for every $j \geq j_o$: $\frac{M}{\lambda_1} < \frac{j}{j+1}$ i.e. there exists $j_o \in N$ s.t. for every $j \geq j_o$: $\frac{j}{M} > \frac{j+1}{\lambda_1}$.

Then it results

$$]\frac{j_o}{\lambda_1}, +\infty [= \bigcup_{j \geq j_o}] \frac{j}{\lambda_1} , \frac{j}{M}[\quad .$$

Finally

There exists j_o large enough s.t. for $\omega > \frac{j_o}{\lambda_1}$ it follows

$$\frac{j}{\lambda_1} < \omega < \frac{j}{M} \quad \text{for a suitable } j \geq j_o$$

The conclusion follows with $\bar{\omega} = \frac{j_o}{\lambda_1}$. ■

Lemma 5.26 implies that the functional $-f$ verifies assumption (I_4) for ω large enough and $V=W_j^-$, $W=W_j^+$. Theorem 1.5 assure the existence of at least

$$\frac{1}{2}\left[\dim (V\cap W) - \operatorname{codim}(V+W)\right] = n$$

nonconstant and geometrically distinct critical points of f.

Remark 5.29 We can improve the thesis of the theorem 5.1 as it follows.
We set

$$M_\omega = \{m \in N \mid \exists j \in N \text{ s.t. } \frac{j+m}{\lambda_1} < \omega < \frac{j}{M}\} \quad .$$

If $\omega > \bar{\omega}$, M_ω is non empty; if we set $m_\omega = \max M_\omega$, we can observe that for $\omega > \bar{\omega}$ there exist at least $n(m_\omega+1)$ critical points of f.

Remark 5.30 The proof of theorem 5.2 directly follows by theorem (1.6).

§ 6. The case $H=a(q)p^2+V(q)$, $a(q)$=const.and $V(q)$ bounded

The Hamiltonian system studied in the section 5,does not correspond to a physical situation, because the potential $V(q)$ is generally bounded in the physical problems. On the other hand if $V(q)$ is bounded, we cannot use the methods of the section 5 in order to prove the P.S condition, because we can not controle the growth of the component of z in the kernel.

The idea exploiteed in this case is to reduce the resonant problem to a one in which no resonance occurs. More precisely, using a trick introduced by Marlin, Mawhin, Coron (cf. [30],[31],[25]) we restrict the functional of the action f to a subspace E disjoint of the kernel such that the critical points of $f_{|E}$ are critical points of f. Therefore it will be sufficient to prove that $f_{|E}$ satisfies (P-S) condition.

First, let us consider the second order Hamiltonian system (case $a(q)$=const.); in the next section we shall deal with the general case $H(p,q)$ of the form (4.1) and $V(q)$ bounded.

Let us given the system

$$-\ddot{x}-k^2x=\omega^2\nabla U(\omega t,x) \tag{6.1}$$

where k is a nonnegative integer number.

If

$$U(t,x)\longrightarrow+\infty \quad \text{for } x\longrightarrow\infty \text{ uniformly in } t\in\mathbb{R}.$$

it is possible to obtain suitable a priori bounds which permit to verify the (P-S) condition (cf.[6]). On the other hand if

$$U(t,x) \text{ is bounded,}$$

the (P-S) condition is not in general satisfied.
In section 4 it has been studied (6.1) under the strong resonance assumption. Here we assume that

(U_9) $U(t,x)$ and $\nabla U(t,x)$ bounded.

The following theorem holds (cf. [16]):

<u>Theorem</u> 6.2 <u>Assume</u> $k \neq 0$, <u>k is even</u> (resp. odd), $U(t,x)$ <u>is</u> $T/2$-<u>periodic in</u> t <u>and satisfies</u> (U_9) <u>and the assumptions</u> (U_3), (U_4), (U_6) <u>of theorem</u> 4.6.
<u>Then there exist at least</u> 2mn <u>pairs of</u> T-<u>periodic nontrivial solutions of</u> (6.1), <u>where</u>

$$m = \# \{ j \in N \mid j \text{ odd (resp. even) s.t. } k^2 + \omega^2\mu < j^2 < k^2 \}$$

<u>Remark</u> 6.3 Obviously m is strictly positive if $T > \bar{T}$, $\bar{T} = 2\pi\bar{\omega}$, $\bar{\omega}^2 = -k^2/\mu$.

<u>Remark</u> 6.4 If we replace (U_4) by

$(\bar{U}_4)$ $\mu > 0$ where $\mu = \inf\limits_{[0,T]} (\inf \sigma(U_{xx}(t,0))$

then the same conclusion of theorem (6.2) follows (in the case k=0 too) with

$$m = \# \{ j \in N \mid j \text{ odd (resp. even) } k^2 < j^2 < k^2 + \omega^2\mu \}.$$

Obviously m is strictly positive if $T > \bar{T} = 2\pi\bar{\omega}$, $\bar{\omega}^2 = \frac{(k+1)^2 - k^2}{\mu}$.

Now we shall consider the more general case in which there exists a symmetric matrix n x n such that

$$\nabla V(t,x) = Mx + \nabla U(t,x),$$

i.e. we shall look for the solutions of

$$(6.5) \qquad -\ddot{x} - Mx = \nabla U(t,x).$$

Let us assume the resonance condition

(R_1) $\left\{\begin{array}{l}\text{there exist } k \in N \text{ and } \mu_i \text{ eigenvalue of } M \text{ such that} \\ k^2-\omega^2\mu_i = 0\end{array}\right.$

Assume k even (resp. odd). Moreover suppose that

(R_2) $\left\{\begin{array}{l}\text{There are not two integers } k_1,k_2,\ k_1 \text{ even and } k_2 \text{ odd} \\ \text{and two eigenvalues } \mu_1 \text{ and } \mu_2 \text{ of } M \text{ s.t.} \\ k_1^2-\omega^2\mu_1=0=k_2^2-\omega^2\mu_2 .\end{array}\right.$

The following theorem holds:

Theorem 6.6 Suppose that $U(t,x)$ verifies all the assumptions of the theorem 6.2 and moreover the assumptions (R_1) and (R_2). Then there exists at least 2mn pairs of T-periodic nontrivial solutions of (6.5) where

$$m = \#\{(j,i)\in N \times \{1,\dots,n\} \mid j \text{ odd(resp.even)} \ 0<j^2-\omega^2\mu_i<\omega^2\mu\}.$$

Remark 6.7 Assumption (R_2) is satisfied for exemple if

1) M is negative semidefinite
2) for any $\mu_i,\mu_j \in \sigma(M)$: $\mu_i/\mu_j \notin Q$
3) for any $\mu_i,\mu_j \in \sigma(M)$: $|\mu_i-\mu_j| \geq 1/\omega^2$.

Proof of theorem (6.2)

As usual, we shall find the critical points of the functional defined on H

$$f(u)=\frac{1}{2}\{\int_0^{2\pi}|\dot{u}(t)|^2dt-k^2\int_0^{2\pi}|u(t)|^2dt\}-\omega^2\int_0^{2\pi}V(t\omega,u)dt .$$

Let us denote by $\mathscr{L}$ the self-adjoint realization in $L^2([0,2\pi],\mathbb{R}^n)$ with periodic conditions of the operator $u\mapsto-\ddot{u}-k^2u$. The spectrum of $\mathscr{L}$ consists of the eigenvalues $\{\lambda_j=j^2-k^2\}$, $j \in N$, and the corresponding eigenvectors are $\{\cos jt\ \Phi_i^*,\ \sin jt\ \Phi_i^*\}$, $\{\Phi_i^*\}$ being the standard basis in $\mathbb{R}^n$. Clearly $\ker L = \mathrm{Span}\{\cos kt\ \Phi_i^*,\ \sin kt\ \Phi_i^*\}$.
The difficultie arising from the resonance assumption and the boundeness of V can be avoided by a trick used in [30], [31] and [25]. More precisely if k is even, we shall restrict the functional f to the subspace

$$E = \{u \in H\ u(t+\pi)=-u(t)\}.$$

U(t,x) being T/2-periodic in t and even in x, easy computations show that E is a closed subspace of H such that

$$(6.8)\quad \begin{cases} \text{(i)} & L(E)\subset E \\ \text{(ii)} & \ker L\cap E = \{0\} \\ \text{(iii)} & E \text{ is invariant under } \nabla U \text{ (i.e. } u \in E \Longrightarrow \nabla U(t,u)\in E) \\ \text{(iv)} & u \in E \iff u=\sum\limits_{\substack{j\in N\\ j\,odd}} (a_j\cos jt\ \Phi_i^*+b_j\sin jt\ \Phi_i^*) \end{cases}$$

(6.8)(i)-(iii) assure that the critical points of $f_{|E}$ are critical points of f on H and therefore solutions of (6.1). By (6.8)(ii) $0 \notin \sigma(L_{|E})$, then standard arguments show that $f_{|E}$ satisfies (I_3).
Let be E^+ (resp E^-) the subspace of E where L is positive (resp. negative) definite, i.e. if we denote as usual by M_j

the eigenspace corresponding to λ_j, then

$$E^+ = \overline{\bigoplus_{\substack{j>k \\ j \text{ odd}}} M_j} \qquad (\text{resp. } E^- = \bigoplus_{\substack{j<k \\ j \text{ odd}}} M_j)$$

where the closure is taken in H. Clearly $E=E^+ \oplus E^-$.
Now we can apply theorem 1.5 to the functional $f_{|E}$. It is known that $f_{|E}$ satisfies the assumptions $(I_1)-(I_3)$; (I_4) follows easily choosing $W = E^-$ and $V = \overline{\bigoplus_{j^2-k^2>\omega^2\mu} M_j}$. By this definition it results that

$$\dim(V\cap W)-\operatorname{codim}(V+W) = \dim(V\cap W) = 2mn ,$$

where m is defined in theorem (6.2).

In the case k is odd, we can repeat the above arguments choosing

$$E = \{u \in H | u(t+\pi) = u(t)\} .$$ ■

<u>Remark</u> 6.9 If we suppose $\mu > 0$, we can apply theorem (1.5) to the functional $-f_{|E}$ taking $V = \overline{\bigoplus_{\substack{j^2-k^2<\omega^2\mu \\ j \text{ odd}}} M_j}$ and $W = E^+$.

<u>Proof of theorem</u> 6.6

In order to prove theorem (6.6) we recall that $\{j^2-\omega^2\mu_i | j \in N, i=1,\dots,n\}$ are the eigenvalues of the operator $L:x \to -\ddot{x}-\omega^2 Mx$ in L^2 (cf.[6]) corresponding to eigenvectors $\{f_i \cos jt, f_i \text{ sen } jt\}$, f_i being the eigenvectors of M. Assumption (R_2) assures that the eigenfunctions of the Kernel of L are $2\pi/k$-periodic, k even (resp. odd) and therefore we can repeat the above arguments and we find at least 2mn pairs of solutions. We shall prove now that m is strictly positive for ω large enough.

By resonance assumption (R_1), we have

$$\mu_M = \max \sigma(M) \geq 0.$$

Then it is sufficient to prove that if ω is large enough,

$$\#\{j \in N \mid j \text{ odd (resp.even)}, 0<j^2-\omega^2\mu_M<\omega^2\mu\} > 0.$$

In the case $\mu_M=0$, the proof is obvious. Assume $\mu_M>0$. Clearly

$$0<j^2-\omega^2\mu_M<\omega^2\mu \Longleftrightarrow \frac{j^2}{\mu_M+\mu} <\omega^2< \frac{j^2}{\mu_M} ;$$

if we fix $\mu_M/\mu_M+\mu <1$, there exists $j_o \in N$, j_o odd, s.t.

$$\frac{(j+1)^2}{\mu_M+\mu} < \frac{j^2}{\mu_M} \quad \text{for every } j \geq j_o,\ j_o \text{ odd.}$$

Then we have

$$\left]\frac{j_o^2}{\mu_M+\mu}, +\infty\right[= \bigcup_{\substack{j\geq j_o \\ j \text{ odd}}} \left] \frac{j^2}{\mu_M+\mu}, \frac{j^2}{\mu_M} \right[$$

and for $\omega^2>\bar{\omega}^2 = j_o^2/\mu_M+\mu$ the conclusion follows. ■

§ 7. The case $H(p,q)=a(q)p^2+V(q)$, $V(q)$ bounded

In this last section, we shall find periodic solutions of Hamiltonian systems (HS) in the case in wich H has the form

$$H(p,q)=\sum_{i,j=1}^{n} a_{ij}(q)p_ip_j+V(q)$$

and $V(q)$ is bounded (cf. case (iii) of section 4). From now on, we essentially follow [17].

Theorem 7.1 Assume that assumption (A_3) of theorem 5.1 hold, moreover

(A_1') there exists a positive constant M such that

$$0<\sum_{i,j=1}^{n} a_{ij}(q)p_ip_j\leq M(p)^2 \quad \text{for any } p,q\in\mathbb{R}^n$$

(V_4) $V(q)$ and $\nabla V(q)$ are bounded

(V_5) $V(0)=\nabla V(0)=0$

(V_6) $V(q)$ is twice differentiable at $q=0$ and

$$(V''(0)q,q)\geq\bar{k}|q|^2 \quad \text{with } \bar{k}^2>M^2/2a(0)$$

where $V''(0)$ denotes the Hessian matrix of V at 0 and $a(0)$ is the smallest eigenvalue of the matrix $\{a_{ij}(0)\}$.

(H) $H(z)=H(-z)$ for any $z\in\mathbb{R}^{2n}$.

Then there exists $\bar{T}>0$ such that (HS) has at least n nonconstant and geometrically distinct T-periodic solutions for any $T>\bar{T}$.

Remarks 7.2 Theorem 7.1 still holds in the not autonomous case with suitable modifications.

Remark 7.3 Let us observe that this theorem is very interesting from a physical point of view because it can be applied to different problems, for exemple to the case of "the double pendulum".

Proof of theorem 7.1

In order to find the critical points of the functional

$$f(z)=\frac{1}{2}((Lz,z)) - \omega\int_0^{2\pi} H(z)\,dt \qquad z \in W^{\frac{1}{2}}$$

we shall apply theorem 1.5.

Obviously the functional $g(z)=-f(z)$ verifies the assumptions (I_1) and (I_2).

Now we have to prove that (I_3) holds. As in section 6, the boundness of the nonlinearity does not permit to verify directly this assumption. Then we restrict g to the closed subspace of $W^{\frac{1}{2}}$

$$F = \{u \in W^{\frac{1}{2}} \mid u(t+\pi)=-u(t)\} .$$

It is easy to see that

$$(7.4)\qquad \begin{array}{ll} \text{(i)} & L(F) \subset F \\ \text{(ii)} & \ker L \cap F = \{0\} \\ \text{(iii)} & F \text{ is invariant under } \nabla V \\ \text{(iv)} & F = \bigoplus_{j\in Z,\, j \text{ odd}} M_{\lambda_j} \end{array}$$

Observe that the eveness of H guarantees (7.4)-(iii) and this is the only point where we need of the assumption (H).

Conditions (7.4)(i)-(iii) assure that the critical points of $g_{|F}$ are critical points of g on $W^{\frac{1}{2}}$, hence they are solutions of (HS).

In the sequel we still denote by g the restriction $g_{|F}$.

By the same arguments used in section 5 it can be proved the following lemma:

Lemma 7.5 *Suppose that* H *verifies* (A_3) *and* (V_4). *Let* $z_n=(p_n,q_n)$ *be a sequence in* F satisfying

(7.6) $\qquad g(z_n) \to c \qquad$ *with* $\quad c > 0$

(7.7) $\qquad \|g'(z_n)\| \, \|z_n\| \to 0$

then there exists a subsequence of $\{z_n\}$ *bounded in* F.

Proof Arguing as in section 5, it can easily seen that the sequences

$$\int_0^{2\pi} a(q_n)p_n^2 dt \qquad \text{and} \qquad \int_0^{2\pi} V(q_n)dt$$

are bounded and therefore $\{H_z(z_n)\}$ is bounded in L^1. The sequence $\{z_n\}$ being in F, the conclusion follows (cf. lemma 5.15) by

$$\|z_n\|_{W^{\frac{1}{2}}-\frac{\eta}{2}} \le \text{cost}\ \|Lz\|_{W^{\frac{1}{2}}-\frac{\eta}{2}}$$ ∎

In order to prove the geometrical condition (I_4), we shall need the following lemmas:

Lemma (7.8) If (A'_1) and (V_4) hold, then for any $j \in N$, j odd, such that

(7.9) $$j > \omega M$$

we have that

(7.10) $$\sup_{F_j^+} g(z) = C_\infty < +\infty \ .$$

Lemma 7.11 - If (A_1), (V_5), (V_6) hold, then there exist two positive constants ρ, δ such that for any $j, m \in N$, j odd and m even, for which

(7.12) $$\omega > \frac{j+m}{(2\bar{k}\ a(0))^{\frac{1}{2}}}$$

(7.13) $$g(z) \geq \delta \quad \text{for any} \quad z \in F_{j+m}^-, \quad \|z\| = \rho \quad .$$

Lemma 7.14 If (A_1), (V_4), (V_5), (V_6) hold, then there exists $\bar{\omega}>0$ and $j,m \in \mathbb{N}$, j odd and m even, such that for any $\omega>\bar{\omega}$ (7.10) and (7.13) hold.

Before proving these lemmas, we conclude the proof of theorem 7.1.

If we set $W=F_j^+$ and $V=F_{j+m}^-$, where j and m are obtained by lemma 7.14, then by theorem 1.5 we have that there exists $\bar{T} = 2\pi\bar{\omega}$ such that for any $T>\bar{T}$ the system (HS) has at least

$$\tfrac{1}{2}\left[\dim(V_n W) - \operatorname{codim}\ (V+W)\right] \geq n$$

nonconstant and geometrically distinct T-periodic solutions.

More precisely, if we set

$$m_\omega = \# \{m \in \mathbb{N} \mid m \text{ even}, \exists j \geq 1 \text{ s.t. } \frac{j+m}{(2\bar{k}\ a(0))^{\frac{1}{2}}} < \omega < \frac{j}{M}\}$$

for $T > \bar{T} = 2\pi\bar{\omega}$ the system (HS) has at least $n(m_\omega + 1)$ nonconstant and geometrically distinct T-periodic solutions.

<u>Proof of lemma</u> 7.8 - Let $V^* = \max V(q)$. Then for any $j \in \mathbb{N}$, j odd, it results

$$(7.15) \qquad g(z) = \omega\int_0^{2\pi} a(q)p^2 dt + \omega\int_0^{2\pi} V(q)dt - \tfrac{1}{2}((Lz,z)) \leq \omega M|p|^2 + 2\pi\omega V^* - \tfrac{1}{2}j|z|^2 \ .$$

Therefore by (7.15) and by (5.20)

$$g(z) \leq (\omega M - j)|p|^2 + 2\pi\omega V^* \ .$$

The conclusion follows with $j \geq \omega M$. ∎

<u>Proof of lemma</u> 7.11 - Let $z \in W^-_{j+m}$, then

$$z = z_- + \tilde{z} \qquad z_- \in E^-_{-j-m-2} \ , \ \tilde{z} \in \bigoplus_{-j-m}^{j+m} M_{\lambda_k}$$

By the definition of $\{\psi_{jh}\}$, $j \in Z$, $h = 1,\dots,2n$, we have

$$z_- = \sum_{\substack{i \leq -j-m-2 \\ j \text{ odd}}} \sum_{k=1}^{n} \begin{pmatrix} [a_{ik}\cos it - b_{ik}\sin it]\Phi_k^* \\ \\ [a_{ik}\sin it + b_{ik}\cos it]\Phi_k^* \end{pmatrix}$$

$$\tilde{z} = \sum_{\substack{i=-j-m \\ i \text{ odd}}}^{j+m} \sum_{k=1}^{n} \begin{pmatrix} [a_{ik}\cos it - b_{ik}\sin it]\Phi_k^* \\ \\ [a_{ik}\sin it + b_{ik}\cos it]\Phi_k^* \end{pmatrix} =$$

$$= \sum_{\substack{i=1 \\ i \text{ odd}}}^{j+m} \sum_{k=1}^{n} \begin{pmatrix} [(a_{ik}+a_{-ik})\cos it - (b_{ik}-b_{-ik})\sin it]\Phi_k^* \\ \\ [(a_{ik}-a_{-ik})\sin it + (b_{ik}+b_{-ik})\cos it]\Phi_k^* \end{pmatrix}.$$

If we set $\tilde{z}=(\tilde{p},\tilde{q})$ it results

$$|\tilde{p}|^2 = \pi \sum_{\substack{i=1 \\ i \text{ odd}}}^{j+m} \sum_{k=1}^{n} [(a_{ik}+a_{-ik})^2 + (b_{ik}-b_{-ik})^2]$$

(7.16) $$|\tilde{q}|^2 = \pi \sum_{\substack{i=1 \\ i \text{ odd}}}^{j+m} \sum_{k=1}^{n} [(a_{ik}-a_{ik})^2 + (b_{ik}+b_{-ik})^2]$$

$$|\tilde{z}|^2 = 2\pi \sum_{\substack{i=1 \\ i \text{ odd}}}^{j+m} \sum_{k=1}^{n} (a_{ik}^2 + a_{-ik}^2 + b_{ik}^2 + b_{-ik}^2)$$

Moreover we have that

$$\tfrac{1}{2}((Lz_-, z_-)) = \tfrac{1}{2} \sum_{\substack{i \leq -j-m-2 \\ i \text{ odd}}} \sum_{k=1}^{n} i|z_i|^2_{L^2} =$$

(7.17) $$= \frac{1}{2} \sum_{\substack{i \le -j-m-2 \\ i \text{ odd}}} \sum_{k=1}^{n} i \int_0^{2\pi} [(a_{ik} \cos it - b_{ik} \sin it)^2 +$$

$$+ (a_{ik} \sin it + b_{ik} \cos it)^2] dt = \pi \sum_{\substack{i \le -j-m-2 \\ i \text{ odd}}} \sum_{k=1}^{n} i (a_{ik}^2 + b_{ik}^2) .$$

$$\frac{1}{2}((L\tilde{z},\tilde{z}) = \frac{1}{2} \sum_{\substack{i=-j-m \\ i \text{ odd}}}^{j+m} \sum_{k=1}^{n} i \int_0^{2\pi} [(a_{ik} \cos it - b_{ik} \sin it)^2 +$$

(7.18) $$+ (a_{ik} \sin it + b_{ik} \cos it)^2] dt = \pi \sum_{\substack{i=-j-m \\ i \text{ odd}}}^{j+m} \sum_{k=1}^{n} i (a_{ik}^2 + b_{ik}^2) =$$

$$= \pi \sum_{\substack{i=1 \\ i \text{ odd}}}^{j+m} \sum_{k=1}^{n} i (a_{ik}^2 + b_{ik}^2 - a_{-ik}^2 - b_{-ik}^2) .$$

Now let be $z \in S_\rho \cap E^-_{j+m}$; if ρ is small enough, it results

$$g(z) = g(0) + g'(0)[z] + \frac{1}{2}((g''(0)z,z)) + O(\|z\|^2) =$$

$$= \frac{\omega}{2} \int_0^{2\pi} (H_{zz}(0)z,z) dt - \frac{1}{2}((Lz,z)) + O(\|z\|^2) \geq$$

$$\geq \omega a(0) |p|^2 + \frac{\omega}{2}(V''(0)q,q) - \frac{1}{2}((Lz_-,z_-)) - \frac{1}{2}((L\tilde{z},\tilde{z})) + O(\|z\|^2) .$$

By (V_6) and (7.18)

$$g(z) \geq \omega a(0) |\tilde{p}|^2 + \frac{\omega}{2}\bar{k} |\tilde{q}|^2 + \frac{1}{2} |z_-|^2 -$$

(7.19) $$- \pi \sum_{\substack{i=1 \\ i \text{ odd}}}^{j+m} \sum_{k=1}^{n} i (a_{ik}^2 + b_{ik}^2 - a_{-ik}^2 - b_{-ik}^2) + O(\|z\|^2) .$$

The conclusion of lemma is achieved if we prove that there exists $\gamma>0$ such that

$$\omega a(0)|\tilde{p}|^2+\frac{\omega}{2}\bar{k}|\tilde{q}|^2+\pi\sum_{\substack{i=1\\ i\ \text{odd}}}^{j+m}\sum_{k=1}^{n} i(-a_{ik}^2-b_{ik}^2+a_{-ik}^2+b_{-ik}^2) >$$

(7.20)

$$> \frac{\gamma\omega}{4}\|\tilde{z}\|^2$$

or equivalently we have to prove

$$\sum_{\substack{i=1\\ i\ \text{odd}}}^{j+m}\sum_{k=1}^{n}\left[\phi_i(a_{ik},a_{-ik})+\phi_i(b_{ik},b_{-ik})\right] > 0 \tag{7.21}$$

where for any $i=1,\dots,j+m$, i odd

$$\phi_i(x,y)=\omega a(0)(x+y)^2+\omega\frac{\bar{k}}{2}(x-y)^2+i(y^2-x^2)-\frac{\gamma\omega}{2}(x^2+y^2)$$

By (7.12) it can be shown that

$$\phi_i(x,y)>0 \qquad i=1,\dots,j+m \qquad i\ \text{odd}$$

then (7.21) holds. ∎

<u>Proof of lemma</u> 7.14 - Lemmas (7.8) and (7.11) hold at the same time if there exist $j,m\in\mathbb{N}$, j odd and m even such that

$$\frac{j+m}{(2\bar{k}\ a(0))^{\frac{1}{2}}} < \omega < \frac{j}{M} \tag{7.22}$$

First of all, (7.22) is possible if it occurs that

$$\frac{j+m}{(2\bar{k}\ a(0))^{\frac{1}{2}}} < \frac{j}{M} \tag{7.23}$$

or equivalently

$$\frac{M}{(2\bar{k}\ a(0))^{\frac{1}{2}}} < 1 \quad . \tag{7.24}$$

Assumption (V_6) implies

$$M^2 < 2\bar{k}\ a(0) \quad , \tag{7.25}$$

so (7.24) is satisfied. We shall prove that there exists $\bar{\omega} \in \mathbb{R}$ such that for $\omega > \bar{\omega}$ (7.22) is verified with m=0.

Namely, since the sequence $\{\frac{j}{j+1} | j \text{ odd}\}$ is incresing and goes to 1 as j goes to infinity, and the number $M/(2\bar{k}\ a(0))^{\frac{1}{2}}$ is less or equal than 1, there exists $j_o \in \mathbb{N}$, $j_o$odd, such that

$$\frac{M}{(2\bar{k}\ a(0))^{\frac{1}{2}}} < \frac{j}{j+1} \quad \text{for every } j \geq j_o,\ j \text{ odd.}$$

Then it results

$$\left]\frac{j_o}{(2\bar{k}\ a(0))^{\frac{1}{2}}}, +\infty\right[= \bigcup_{\substack{j \geq j_o \\ j \text{ odd}}} \left] \frac{j}{(2\bar{k}\ a(0))^{\frac{1}{2}}}, \frac{j}{M} \right[\quad .$$

The conclusion of lemma follows with $\bar{\omega} = \dfrac{j_o}{(2\bar{k}\ a(0))^{\frac{1}{2}}}$. ∎

<u>Remark</u> 7.26 - We recall that if z_1 is a solution of (HS), $z_2 = -z_1$ is still a solution of (HS). Since z_1 and z_2 belong to F, they have the same orbit and therefore correspond to the same solution found in the theorem (7.1).

Now we shall give an application of theorem 7.1 to the equation of the double pendulum.
Consider a system of two unitary masses in a double pendulum constrained to move in a plane. Two angles θ_1 and θ_2 completely specify the position of masses m_1 and m_2 and they can be considered as the "generalized coordinates".
Let us denote by p_i, $i=1,2$, the "generalized momentum" associated wit the generalized coordinates θ_i.

The Hamiltonian of this system (cf.[29]) is

$$H(p,q) = (A(q)p,p) + V(q)$$

where $q = (\theta_1, \theta_2)$

(7.27)

$$A(\theta_1,\theta_2) = \frac{1}{2\left[2-\cos^2(\theta_1-\theta_2)\right]}\begin{pmatrix} 1 & -\cos(\theta_1-\theta_2) \\ -\cos(\theta_1-\theta_2) & 2 \end{pmatrix}$$

$$V(\theta_1,\theta_2) = 3g-2g\cos\theta_1-g\cos\theta_2$$

Let us consider the Hamiltonian system

$$-J\dot{z} = H_z(z) \tag{7.28}$$

where H is as in (7.27). It is easy to verify that

$$\frac{3-\sqrt{5}}{8}\, p^2 \leq (A(q)p,p) \leq \frac{3+\sqrt{5}}{4}\, p^2 \tag{7.29}$$

$$(V''(0)q,q) \geq g|q|^2 \text{ and } g > \left(\frac{3+\sqrt{5}}{4}\right)^2 \frac{1}{2\frac{3-\sqrt{5}}{4}} . \tag{7.30}$$

By (7.27), (7.29) and (7.30) it follows that H verifies the assumptions of the theorem (7.1). Then:

Theorem 7.31 - There exists $\bar{T}>0$ such that for every $T>\bar{T}$ there exist at least two T-periodic solutions of Hamiltonian system (7.28).

Now we want to compare this result with those contained in [8]. We need of the following

Definition 7.32 - We say that $z(t)=(p(t),q(t)$ is a "generalized T-periodic solution" (or "revolutionary solution") of (7.28) if there exist $k_1,k_2 \in \mathbb{N}$ such that

$$q(t+T)-q(t) = 2\pi(k_1,k_2) .$$

Observe that if $k_1=k_2=0$, q is a T-periodic solution of (7.28). Benci in [8] has studied the existence of T-periodic solution of Langrangian systems on manifolds. In particular the configuration space of the double pendulum is $T^2=S^1\times S^1$. Then from the results of Benci, it can be deduced that (7.28) has a generalized T-periodic solution for any $T>0$. Putting togheter the results of Benci and theorem (7.31) we can conclude that there exists To such that for any $T>T_o$ (7.28) has two T-periodic solutions, and for any $T>T_o$ (7.28) has a generalized T-periodic solution.

Obviously it is reasonable to think that the energy E corresponding to the T-periodic solutions, $T>T_o$, is less than max V(q). If E>maxV(q), the double pendulum has a "generalized solution", whose "period" is decreasing as E increases.

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