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Approximate Hermitian-Yang-Mills structures and semistability for Higgs bundles

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To my mother and my father.

Deep as the relationship is between mathematics and physics, it would be wrong, however, to think that the two disciplines overlap so much. They do not. And they have their separate aims and tastes. They have distinctly different value judgements, and they have different traditions. At the fundamental conceptual level they amazingly share some concepts, but even there, the life force of each discipline runs along it own veins.

C. N. Yang.

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Contents

CHAPTER 1

Introduction

1.1 Some historical background

The notion of holomorphic vector bundle is common to some branches of mathematics and theoretical physics. In particular, such a notion seems to play a fundamental role in complex differential geometry, algebraic geometry and Yang-Mills theory. In this thesis we study a kind of holomorphic vector bundles over complex manifolds, known as Higgs bundles, and some of their main properties. We restrict such objects to the case when the complex manifold is compact Kähler. On one hand, complex manifolds provide a rich class of geometric objects, which behave rather differently than real smooth manifolds. For instance, one of the main characteristic of a compact complex manifold is that its group of biholomorphisms is always finite dimensional. On the other hand, since the manifolds in which we are interested are compact Kähler, we have that certain invariants associated with the holomorphic bundle can be defined using cohomology classes.

In complex geometry, the Hitchin-Kobayashi correspondence asserts that the notion of (Mumford-Takemoto) stability, originally introduced in algebraic geometry, has a differential-geometric equivalent in terms of special metrics. In its classical version, this correspondence is established for holomorphic vector bundles over compact Kähler manifolds and says that such bundles are polystable if and only if they admit an Hermitian-Einstein[1] structure. This correspondence is also true for Higgs bundles.

The history of this correspondence probably starts in 1965, when Narasimhan and Seshadri [12] proved that a holomorphic bundle over a Riemann surface is stable if and only if it corresponds to a projective irreducible representation of the fundamental group of the surface. Then, in the 80's Kobayashi [3] introduced for the first time the notion of Hermitian-Einstein structure in a holomorphic vector bundle, as a generalization of a Kähler-Einstein metric in the tangent bundle. Shortly after, Kobayashi [4] and Lübke [41] proved that a bundle with an Hermitian-Einstein structure must be necessarily polystable. Donaldson [50] showed that the result of Narasimhan and Seshadri [12] can be formulated in term of metrics and proved that the concepts of polystability and Hermitian-Einstein metrics are equivalent for holomorphic vector bundles over Riemann surfaces. Then, Kobayashi and Hitchin conjectured that the equivalence should be true in general for holomorphic vec-

[1]In the literature Hermitian-Einstein, Einstein-Hermite and Hermitian-Yang-Mills are all synonymous. Sometimes also the terminology Hermitian-Yang-Mills-Higgs is used.

tor bundles over Kähler manifolds. However, the route starting from polystability and showing the existence of special structures in higher dimensions took some time.

The existence of Hermitian-Einstein structures in a stable holomorphic vector bundle was proved by Donaldson for projective algebraic surfaces in [51] and for projective algebraic manifolds in [52]. Finally, Uhlenbeck and Yau showed this for general compact Kähler manifolds in [9] using some techniques from analysis and Yang-Mills theory. Hitchin [16], while studying the self-duality equations over a compact Riemann surface, introduced the notion of Higgs field and showed that the result of Donaldson for Riemann surfaces could be modified to include the presence of a Higgs field. Following the results of Hitchin, Simpson [17] defined a Higgs bundle to be a holomorphic vector bundle together with a Higgs field and proved the Hitchin-Kobayashi correspondence for such an object. Actually, using some sophisticated techniques in partial differential equations and Yang-Mills theory, he proved the correspondence even for non-compact Kähler manifolds, if they satisfy some analytic conditions. As an application of this, Simpson [18] later studied in detail a one-to-one correspondence between stable Higgs bundles over a compact Kähler manifolds with vanishing Chern classes and irreducible representations of the fundamental group of the Kähler manifold.

The Hitchin-Kobayashi correspondence has been extended in several directions. Simpson [17] studied the Higgs case for non-compact Kähler manifolds satisfying some additional requirements and Lübke and Teleman [40] studied the correspondence for compact complex manifolds. Bando and Siu [14] extended the correspondence to torsion-free sheaves over compact Kähler manifolds and introduced the notion of admissible Hermitian metric for such objects. Following the ideas of Bando and Siu, Biswas and Schumacher [24] introduced the notion of admissible Hermitian-Yang-Mills metric in the Higgs case and generalized this extension to torsion-free Higgs sheaves.

In [51] and [52] Donaldson introduced a functional, which is known as the Donaldson functional, and later Simpson [17] defined this functional in his study of the Hitchin-Kobayashi correspondence for Higgs bundles. Kobayashi in [5] constructed the same functional in a different form and showed that it plays a fundamental role in a possible extension of the Hitchin-Kobayashi correspondence. In fact, he proved in [5] that for holomorphic vector bundles over projective algebraic manifolds, the counterpart of semistability is the notion of approximate Hermitian-Einstein structure.

The correspondence between semistability and the existence of approximate Hermitian-Yang-Mills structures in the ordinary case has been originally proposed by Kobayashi. In [5] he proved that for a holomorphic bundle over a compact Kähler manifold a certain boundedness property of the Donaldson functional implies the existence of an approximate Hermitian-Einstein structure and that this implies the semistability of the bundle. Then, using some properties of the Donaldson functional and the Mehta-Ramanathan theorem, he established a boundedness property of the Donaldson functional for semistable holomorphic bundles over compact algebraic manifolds. As a consequence of this, he obtained the correspondence between semistability and the existence of approximate Hermitian-Einstein structures when the base manifold was compact algebraic. Then he conjectured that all three conditions (the boundedness property, the existence of an approximate Hermitian-Einstein structure and the semistability) should be equivalent in general, that is, independently of whether the manifold is algebraic or not.

In the Higgs case and when the manifold is one-dimensional (a compact Riemann surface), the boundedness property of the Donaldson functional follows from the semistability in a similar way to the classical case, since we need to consider only Higgs subbundles and their quotients and we have a decomposition of the Donaldson functional in terms of these objects. The existence of approximate Hermitian-Einstein metrics for semistable holomorphic vector bundles has been recently studied in [46] using some techniques developed by Buchdahl [48], [49] for the desingularization of sheaves in the case of compact complex surfaces. One of the main dificulties in the study of this correspondence in higher dimensions arises from the notion of stability, since for compact Kähler manifolds with dimensions greater or equal than one, it is necessary to consider subsheaves and not only subbundles. On the other hand, properties of the Donaldson functional commonly involves holomorphic bundles. All of these difficulties appear also in the Higgs case, hence, in order to study this correspondence in higher dimensions, it seems natural to introduce first the notion of admissible Hermitian metrics on Higgs sheaves. Then, to define the Donaldson functional for such objects using these metrics and finally, to study how this functional defined for a semistable Higgs bundle, can be decomposed in terms of Higgs subsheaves and their quotients.

1.2 About this thesis

This thesis is organized as follows. In Chapter 2 we study the basic results of Higgs sheaves. These results are important mainly because the notion of stability in higher dimensions (greater than one) makes reference to Higgs subsheaves and not only to Higgs subbundles. The basic properties of Higgs sheaves are studied in the second section; some of them are simple extensions to the Higgs case of classical results on holomorphic vector bundles, however they play an important role in the theory. In the last part of Chapter 2, we establish some results on semistable Higgs sheaves. These properties will be important latter on, when we study the correspondence between semistability and the existence of approximate Hermitian-Yang-Mills structures.

In the first part of Chapter 3, we summarized some properties of metrics and connections on Higgs bundles and introduce the space of Hermitian structures, which is the space where the Donaldson functional is defined. Then we study some vanishing theorems in the context of Higgs bundles. As in the classical case, they have important consequences. In fact, one of these vanishing theorems is crucial to prove the semistability of a Higgs bundle admitting an approximate Hermitian-

Yang-Mills structure. In the second part of Chapter 3, we introduce the notion of (weak) Hermitian-Yang-Mills structure and we study the main properties of Higgs bundles with such metrics. In particular, we prove that any weak Hermitian-Yang-Mills structure can be transformed into an Hermitian-Yang-Mills structure under a conformal change of the Hermitian metric of the Higgs bundle. In the final part of Chapter 3 we introduce the notion of approximate Hermitian-Yang-Mills structure for Higgs bundles. This part is a generalization of the notion of approximate Hermtian-Einstein structure for holomorphic vector bundles studied by Kobayashi [5]. In the final part of this chapter we study some consequences of these notions. In particular we prove a Bogomolov-Lübke inequality for Higgs bundles admiting an approximate Hermitian-Yang-Mills structure.

In Chapter 4, we construct the Donaldson functional for Higgs bundles over compact Kähler manifolds following a construction similar to that of Kobayashi and we present some basic properties of it. In particular, we prove that the critical points of this functional are precisely the Hermitian-Yang-Mills structures, and we show also that its gradient flow can be written in terms of the mean curvature of the Hitchin-Simpson connection. We evaluate this functional using local coordinates and we show that it is in essence the same funtional introduced by Simpson in [17]. We also establish some properties of the solutions of the evolution equation associated with that functional. Next, we study the problem of the existence of approximate Hermitian-Yang-Mills structures and its relation with the algebro-geometric notion of Mumford-Takemoto semistability. We prove that if the Donaldson functional of a Higgs bundle over a compact Kähler manifold is bounded from below, then there exist an approximate Hermitian-Yang-Mills structure on it. This fact, together with a result of Bruzzo-Graña Otero [21], implies the semistability of the Higgs bundle. Then we show that a semistable but not stable Higgs bundle can be included into a short exact sequence with a stable Higgs subsheaf and a semistable Higgs quotient. We use this result in the final part of the chapter when we show that for a Higgs bundle over a compact Riemann surface, the notion of approximate Hermitian-Yang-Mills structure is in fact the differential-geometric counterpart of the notion of semistability. These results cover the classical case if we take the Higgs field equal to zero.

In Chapter 5 we study the notion of admissible Hermitian metric on a torsion-free Higgs sheaf and we briefly review the Hironaka's flattening procedure for resolution of singularities of coherent sheaves. This can be immediately adapted to the Higgs context, as a consequence we can associate with any torsion-free Higgs sheaf a Higgs bundle, which is called a regularization of the Higgs sheaf. Then, we prove that any Hermitian metric on a regularization of a Higgs sheaf induces an admissible metric on the Higgs sheaf. In the second section of this chapter, we study some properties of semistable Higgs sheaves closely related to the notion of adimissible metrics. In particular, we show that the restriction of polystable Higgs sheaves to certains open sets is also polystable, and using this we prove that the tensor product of semistable Higgs sheaves is again semistable. In the third section, following the ideas of Biswas and Schumacher [24] and Simpson [17] we construct the Donaldson functional for torsion-free Higgs sheaves. From this and a result of Simpson for Higgs bundles over non-compact Kähler manifolds we show that the Donaldson functional for a stable Higgs sheaf satisfies a boundedness property. Finally, following ideas of Buchdahl [49] we start the study of the two-dimensional case. We show that a Higgs bundle which is semistable but not stable can be included into an exact sequence with a torsion-free Higgs quotient which in general non locally free. Finally, we show that this sequence can be regularized in a similar form to the ordinary case, and hence we end up with a sequence of Higgs bundles over a certain Kähler manifold. The constructions that has been briefly introduced in the two-dimensional case, can also be made in higher dimensions. These, together with the notion of admissible metrics are in essence the first part of the strategy to prove the existence of approximate Hermitian-Yang-Mills structures for semistable Higgs bundles. In this final part, we do not present a complete proof of the existence of approximate Hermitian-Yang-Mills structures for semistable Higgs bundles in higher dimensions, but only a brief description of the main steps that should to be done to prove this part of the correspondence in a future.

In Appendix A we write some basic properties of coherent sheaves. Then, we review the definition of singularity set of a coherent sheaf and briefly comment some of their main properties concerning the codimension of the singularity set. Then we show the construction of the determinant bundle of a coherent sheaf and write some facts on determinant bundles that are used through this work. In Appendix B we present some general remarks on Higgs bundles and the origin of the Yang-Mills equations. Finally, we summarize some facts about blow-ups and the resolution of singularities.

CHAPTER 2

Higgs sheaves

2.1 Preliminaries

We start with some basic definitions. Let X be an n-dimensional compact Kähler manifold with Kähler form ω, and let Ω^1_X be the cotangent sheaf to X, i.e., it is the sheaf of holomorphic one-forms on X. A Higgs sheaf $\mathfrak{E}$ over X is a coherent sheaf E over X, together with a morphism $\phi : E \to E \otimes \Omega^1_X$ of $\mathcal{O}_X$-modules (that is usually called the Higgs field), such that the morphism $\phi \wedge \phi : E \to E \otimes \Omega^2_X$ vanishes.

Using local coordinates on X we can write $\phi = \phi_\alpha dz^\alpha$, where the index take values $\alpha = 1, ..., n$ and each ϕ_α is an endomorphism of E. Then the condition $\phi \wedge \phi = 0$ is equivalent to $[\phi_\alpha, \phi_\beta] = 0$ for all α, β. This condition, also called the integrability condition, implies that the sequence

$$E \longrightarrow E \otimes \Omega^1_X \longrightarrow E \otimes \Omega^2_X \longrightarrow \cdots$$

naturally induced by the Higgs field is a complex of coherent sheaves. A Higgs subsheaf $\mathfrak{F}$ of $\mathfrak{E}$ is a subsheaf F of E such that $\phi(F) \subset F \otimes \Omega^1_X$, so that the pair $\mathfrak{F} = (F, \phi|_F)$ becomes itself a Higgs sheaf. A Higgs sheaf $\mathfrak{E} = (E, \phi)$ is said to be torsion-free (resp. locally free, reflexive, normal, torsion) if the corresponding coherent sheaf E is torsion-free (resp. locally free, reflexive, normal, torsion). A Higgs bundle $\mathfrak{E}$ is by definition a Higgs sheaf which is locally-free.

Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be two Higgs sheaves over a compact Kähler manifold X. A morphism between $\mathfrak{E}_1$ and $\mathfrak{E}_2$ is a map $f : E_1 \longrightarrow E_2$ such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi_1} & E_1 \otimes \Omega^1_X \\ \downarrow f & & \downarrow f\otimes 1 \\ E_2 & \xrightarrow{\phi_2} & E_2 \otimes \Omega^1_X \end{array}$$

commutes. In the following we will write any morphism of Higgs sheaves simply as

$$f : \mathfrak{E}_1 \longrightarrow \mathfrak{E}_2 \,.$$

Let $\mathfrak{F} = (F, \phi_F)$ be a Higgs subsheaf of $\mathfrak{E} = (E, \phi)$ and let $G = E/F$. Then, in particular

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

is an exact sequence of coherent sheaves. Tensoring this by Ω^1_X we get the following exact sequence

$$F \otimes \Omega^1_X \xrightarrow{f} E \otimes \Omega^1_X \longrightarrow G \otimes \Omega^1_X \longrightarrow 0\,.$$

Since Ω^1_X is locally free, the morphism f is injective (see [36], Ch.V, for details). Therefore, from this one has the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_F} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & F \otimes \Omega^1_X & \longrightarrow & E \otimes \Omega^1_X & \longrightarrow & G \otimes \Omega^1_X & \longrightarrow & 0 \end{array}$$

in which the rows are exact. The morphism ψ in the above diagram is defined by demanding that all diagram becomes commutative (it is in fact well-defined because the rows are exact). It follows from this that ψ is a Higgs field for the quotient sheaf G and we say that the Higgs sheaf $\mathfrak{G} = (G, \psi)$ is a Higgs quotient of $\mathfrak{E}$.

The kernel and the image of morphisms of Higgs sheaves are Higgs sheaves. In fact, if $f : \mathfrak{E}_1 \longrightarrow \mathfrak{E}_2$ is a morphism of Higgs sheaves, $K = \ker f$ and $\iota : K \longrightarrow E_1$ denotes the obvious inclusion, we have the following commutative diagram (with exact rows)

$$\begin{array}{ccccc} K & \xrightarrow{\iota} & E_1 & \xrightarrow{f} & E_2 \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} \\ K \otimes \Omega^1_X & \xrightarrow{\iota'} & E_1 \otimes \Omega^1_X & \xrightarrow{f'} & E_2 \otimes \Omega^1_X \end{array}$$

where $\iota' = \iota \otimes 1$, $f' = f \otimes 1$ and ϕ is the restriction of ϕ_1 to K. In that way the pair $\mathfrak{K} = (K, \phi)$ becomes a Higgs subsheaf of $\mathfrak{E}_1$.

Similarly, if $F = \operatorname{im} f$, we denote by $j : F \longrightarrow E_2$ the inclusion morphism and write $f = j \circ p$, we obtain the following commutative diagram

$$\begin{array}{ccccc} E_1 & \xrightarrow{p} & F & \xrightarrow{j} & E_2 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi_2} \\ E_1 \otimes \Omega^1_X & \xrightarrow{p'} & F \otimes \Omega^1_X & \xrightarrow{j'} & E_2 \otimes \Omega^1_X \end{array}$$

where $p' = p \otimes 1$, $j' = j \otimes 1$ and ψ is the restriction of ϕ_2 to F. From this we get that $\mathfrak{F} = (F, \psi)$ is a Higgs sheaf. Furthermore, from the above diagram it follows that $\mathfrak{F}$ is a Higgs subsheaf of $\mathfrak{E}_2$ and at the same time a Higgs quotient $\mathfrak{E}_1$.

A sequence of Higgs sheaves is a sequence of the corresponding coherent sheaves where each map is a morphism of Higgs sheaves. A short exact sequence of Higgs sheaves (also called an extension of Higgs sheaves or a Higgs extension [18], [26]) is defined in the obvious way.

Let

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{G} \longrightarrow 0 \tag{2.1}$$

be an exact sequence of Higgs sheaves. Since in the ordinary case we identify F with a subsheaf of E, we can see the Higgs field of $\mathfrak{F}$ as a restriction of the Higgs field of $\mathfrak{E}$, in that way we identify $\mathfrak{F}$ with a Higgs subsheaf of $\mathfrak{E}$.

The Higgs field ϕ can be considered as a section of $\operatorname{End} E \otimes \Omega^1_X$ and hence we have a natural dual morphism $\phi^\vee : E^\vee \to E^\vee \otimes \Omega^1_X$ and the pair $\mathfrak{E}^\vee = (E^\vee, \phi^\vee)$ is a Higgs sheaf. On the other hand, if Y is another compact Kähler manifold and $f : Y \to X$ is a holomorphic map, the pair defined by $f^*\mathfrak{E} = (f^*E, f^*\phi)$ is also a Higgs sheaf.

We define the tensor product of two Higgs sheaves $\mathfrak{E}_1$ and $\mathfrak{E}_2$ in the following way:

$$\mathfrak{E}_1 \otimes \mathfrak{E}_2 = (E_1 \otimes E_2\,, \phi) \tag{2.2}$$

where $\phi = \phi_1 \otimes I_2 + I_1 \otimes \phi_2$, and I_1 and I_2 are the identity endomorphisms on E_1 and E_2 respectively. Since

$$\begin{aligned} \phi \wedge \phi &= (\phi_1 \otimes I_2) \wedge (\phi_1 \otimes I_2) + (I_1 \otimes \phi_2) \wedge (I_1 \otimes \phi_2) \\ &\quad + (\phi_1 \otimes I_2) \wedge (I_1 \otimes \phi_2) + (I_1 \otimes \phi_2) \wedge (\phi_1 \otimes I_2) \\ &= 0 \end{aligned}$$

the tensor product defined in (2.2) is automatically a Higgs sheaf.

On the other hand, we define the direct sum of two Higgs sheaves $\mathfrak{E}_1$ and $\mathfrak{E}_2$ as follows

$$\mathfrak{E}_1 \oplus \mathfrak{E}_2 = (E_1 \oplus E_2\,, \operatorname{pr}_1^*\phi_1 + \operatorname{pr}_2^*\phi_2) \tag{2.3}$$

where $\operatorname{pr}_i : E_1 \oplus E_2 \to E_i$ (with $i = 1, 2$) denote the natural projections. If v_1 and v_2 are sections of E_1 and E_2 respectively, then

$$\operatorname{pr}_1^*\phi_1(v_1, v_2) = (\phi_1 v_1, v_2)\,, \qquad \operatorname{pr}_2^*\phi_2(v_1, v_2) = (v_1, \phi_2 v_2)\,. \tag{2.4}$$

From the above it follows that

$$(\operatorname{pr}_1^*\phi_1 + \operatorname{pr}_2^*\phi_2) \wedge (\operatorname{pr}_1^*\phi_1 + \operatorname{pr}_2^*\phi_2) = 0\,,$$

which shows that the direct sum given by (2.3) is a Higgs sheaf.

2.2 Mumford-Takemoto stability

Let X be a compact Kähler manifold and $\mathfrak{E} = (E, \phi)$ a Higgs sheaf over it. We define the degree and the rank of $\mathfrak{E}$, denoted by $\deg \mathfrak{E}$ and $\operatorname{rk} \mathfrak{E}$ respectively, simply as the degree and rank of the coherent sheaf E. Hence, if $\det E$ denotes the determinant bundle of the coherent sheaf E we define

$$\deg \mathfrak{E} = \int_X c_1(\det E) \wedge \omega^{n-1}\,. \tag{2.5}$$

If the rank is positive, we introduce the quotient $\mu(\mathfrak{E}) = \deg\mathfrak{E}/\mathrm{rk}\,\mathfrak{E}$ which is called the slope of the Higgs sheaf $\mathfrak{E}$. In a similar way as in the ordinary case (see for instance [5], [9], [13], [14]) there is a notion of stability for Higgs sheaves [17], [18], [21], [24]. Namely, a Higgs sheaf $\mathfrak{E}$ is said to be ω-stable (resp. ω-semistable) if it is torsion-free and for any Higgs subsheaf $\mathfrak{F}$ with $0 < \mathrm{rk}\,\mathfrak{F} < \mathrm{rk}\,\mathfrak{E}$ we have the inequality $\mu(\mathfrak{F}) < \mu(\mathfrak{E})$ (resp. $\leq$). We say that a Higgs sheaf is ω-polystable if it decomposes into a direct sum of two or more ω-stable Higgs sheaves all these with the same slope. Consequently, a polystable Higgs sheaf is semistable but not stable.

In dimensions greater or equal than two, the notion of stability (resp. semistability) depends on the Kähler form, since the degree depends on it. Now, in dimension one, the degree does not depend on the Kähler form and hence the notion of stability (resp. semistability) does not depend on it and we can establish all our results without any explicit reference to ω. We will see more about this when we study the Hitchin-Kobayashi correspondence for Higgs bundles over compact Riemann surfaces.

On the other hand, the notion of stability (resp. semistability) makes reference only to Higgs subsheaves. Then, in principle we have sheaves which are stable (resp. semistable) in the Higgs case, but not in the ordinary case (see Appendix B for some concrete examples).

Since the degree and the rank of any Higgs sheaf is the same degree and rank of the corresponding coherent sheaf, we have the following (see [5], Ch. V, Lemma 7.3).

Lemma 2.1. *Given an exact sequence of Higgs sheaves (2.1), then*

$$\mathrm{rk}\,\mathfrak{F}\,(\mu(\mathfrak{E}) - \mu(\mathfrak{F})) + \mathrm{rk}\,\mathfrak{G}\,(\mu(\mathfrak{E}) - \mu(\mathfrak{G})) = 0\,. \tag{2.6}$$

From Lemma 2.1 it follows that the condition of stability (resp. semistability) can be written in terms of quotient Higgs sheaves instead of Higgs subsheaves. To be precise we conclude

Corollary 2.2. *Let $\mathfrak{E}$ be a torsion-free Higgs sheaf over a compact Kähler manifold X with Kähler form ω. Then $\mathfrak{E}$ is ω-stable (resp. semistable) if for every quotient Higgs sheaf $\mathfrak{G}$ with $0 < \mathrm{rk}\,\mathfrak{G} < \mathrm{rk}\,\mathfrak{E}$ it follows $\mu(\mathfrak{E}) < \mu(\mathfrak{G})$ (resp. $\leq$).*

From the definition of degree, it follows that any torsion Higgs sheaf $\mathfrak{T}$ has $\deg\mathfrak{T} \geq 0$. Therefore, in a similar way to the classical case, this implies that in the definition of stability (resp. semistability) we do not have to consider all quotient Higgs sheaves. To be precise we have

Proposition 2.3. *Let $\mathfrak{E}$ be a torsion-free Higgs sheaf over a compact Kähler manifold X with Kähler form ω. Then*
(i) *$\mathfrak{E}$ is ω-stable (resp. semistable) if and only if $\mu(\mathfrak{F}) < \mu(\mathfrak{E})$ (resp. $\leq$) for any Higgs subsheaf $\mathfrak{F}$ with $0 < \mathrm{rk}\,\mathfrak{F} < \mathrm{rk}\,\mathfrak{E}$ and such that the quotient $\mathfrak{E}/\mathfrak{F}$ is torsion-free.*
(ii) *$\mathfrak{E}$ is ω-stable (resp. semistable) if and only if $\mu(\mathfrak{E}) < \mu(\mathfrak{G})$ (resp. $\leq$) for any torsion-free quotient Higgs sheaf $\mathfrak{G}$ with $0 < \mathrm{rk}\,\mathfrak{G} < \mathrm{rk}\,\mathfrak{E}$.*

Proof: (i) and (ii) are clear in one direction. For the converse, suppose the inequality between slopes in (i) (resp. in (ii)) holds for proper Higgs subsheaves with torsion-free quotient (resp. for torsion-free quotients Higgs sheaves) and let consider an exact sequence of Higgs sheaves as in (2.1).

Let $\mathfrak{E} = (E, \phi)$ and denote by ψ the Higgs field of $\mathfrak{G}$. That is, $\mathfrak{G} = (G, \psi)$. Now, let T be the torsion subsheaf of G. Since the Higgs field satisfies $\psi(T) \subset T \otimes \Omega^1_X$, the pair $\mathfrak{T} = (T, \psi|_T)$ is a Higgs subsheaf of $\mathfrak{G}$ with Higgs quotient, say $\mathfrak{G}_1$. Then if we define $\mathfrak{F}_1$ by the kernel of the Higgs morphism $\mathfrak{E} \longrightarrow \mathfrak{G}_1$, we have the following commutative diagram of Higgs sheaves

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathfrak{T} & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{F} & \longrightarrow & \mathfrak{E} & \longrightarrow & \mathfrak{G} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{F}_1 & \longrightarrow & \mathfrak{E} & \longrightarrow & \mathfrak{G}_1 & \longrightarrow & 0\\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathfrak{F}_1/\mathfrak{F} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

in which all rows and columns are exact. From this diagram we have that $\mathfrak{F}$ is a Higgs subsheaf of $\mathfrak{F}_1$ with $\mathfrak{T} \cong \mathfrak{F}_1/\mathfrak{F}$. Since $\mathfrak{T}$ is a torsion Higgs sheaf, $\deg \mathfrak{T} \geq 0$ and we also obtain

$$\deg \mathfrak{G} = \deg \mathfrak{T} + \deg \mathfrak{G}_1 \geq \deg \mathfrak{G}_1 \,,$$

$$\deg \mathfrak{F}_1 = \deg \mathfrak{F} + \deg \mathfrak{T} \geq \deg \mathfrak{F} \,.$$

Now, because $\mathfrak{T}$ is torsion we have $\mathrm{rk}\, \mathfrak{G} = \mathrm{rk}\, \mathfrak{G}_1$ and $\mathrm{rk}\, \mathfrak{F}_1 = \mathrm{rk}\, \mathfrak{F}$ and hence finally we obtain

$$\mu(\mathfrak{F}) \leq \mu(\mathfrak{F}_1) \,, \qquad \mu(\mathfrak{G}_1) \leq \mu(\mathfrak{G}) \,.$$

At this point, the converse directions in (i) and (ii) follows from the hypothesis and the last two inequalities. Q.E.D.

Lemma 2.4. *Let $\mathfrak{E}$ be a torsion-free Higgs sheaf over a compact Kähler manifold X, then $\mu(\mathfrak{E}) = -\mu(\mathfrak{E}^\vee)$.*

Proof: Assume that $\mathfrak{E}$ is a torsion-free Higgs sheaf, from a classical result (see [5], Ch.V, Proposition 6.12 for more details) we know that the determinant bundle of

E satisfies $(\det E)^\vee = \det E^\vee$. From this and the definition of the degree of Higgs sheaves we get the following

$$\deg \mathfrak{E}^\vee = \deg (\det E^\vee) = \deg ((\det E)^\vee) = -\deg (\det E) = -\deg \mathfrak{E}$$

and the result follows from the definition of the slope. Q.E.D.

The following is a natural extension to Higgs sheaves of a classical result in Kobayashi [5].

Proposition 2.5. *Let $\mathfrak{E}$ be a torsion-free Higgs sheaf over a compact Kähler manifold X with Kähler form ω. Then*
(i) *If $\operatorname{rk} \mathfrak{E} = 1$, then $\mathfrak{E}$ is ω-stable.*
(ii) *Let $\mathfrak{L}$ be a Higgs line bundle over X. Then $\mathfrak{L} \otimes \mathfrak{E}$ is ω-stable (resp. semistable) if and only if $\mathfrak{E}$ is ω-stable (resp. semistable).*
(iii) *$\mathfrak{E}$ is ω-stable (resp. semistable) if and only if $\mathfrak{E}^\vee$ is ω-stable (resp. semistable).*

Proof: (i) is a direct consequence of the definition of stability and (ii) is equal to the classical case. We will see (iii) in the case of stability (the proof in the case of semistability is similar and is obtained by replacing $<$ by $\leq$ in the inequalities between slopes).

Assume first $\mathfrak{E}^\vee$ is ω-stable and consider an exact sequence of Higgs sheaves

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{G} \longrightarrow 0$$

with $\mathfrak{G}$ torsion-free. Dualizing it, we obtain the following exact sequence of Higgs sheaves

$$0 \longrightarrow \mathfrak{G}^\vee \longrightarrow \mathfrak{E}^\vee \longrightarrow \mathfrak{F}^\vee \,.$$

Since $\mathfrak{E}$ and $\mathfrak{G}$ are both torsion-free, we get from Lemma 2.4 and the above sequence that

$$\mu(\mathfrak{E}) = -\mu(\mathfrak{E}^\vee) < -\mu(\mathfrak{G}^\vee) = \mu(\mathfrak{G})\,,$$

which means that $\mathfrak{E}$ must be ω-stable.

Assume now that $\mathfrak{E}$ is ω-stable and consider an exact sequence of Higgs sheaves

$$0 \longrightarrow \mathfrak{F}' \longrightarrow \mathfrak{E}^\vee \longrightarrow \mathfrak{G}' \longrightarrow 0$$

with $\mathfrak{G}'$ torsion-free. Dualizing it, we obtain again an exact sequence of Higgs sheaves

$$0 \longrightarrow \mathfrak{G}'^\vee \longrightarrow \mathfrak{E}^{\vee\vee} \longrightarrow \mathfrak{F}'^\vee \,.$$

On the other hand, the natural injection $\sigma : \mathfrak{E} \longrightarrow \mathfrak{E}^{\vee\vee}$ defines $\mathfrak{E}$ as a Higgs subsheaf $\mathfrak{E}^{\vee\vee}$. From this and defining the Higgs sheaves $\mathfrak{H}' = \mathfrak{E} \cap \mathfrak{G}'^\vee$ and $\mathfrak{H}'' = \mathfrak{E}/\mathfrak{H}'$

we have the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathfrak{H}' & \longrightarrow & \mathfrak{E} & \longrightarrow & \mathfrak{H}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \sigma} & & & & \\
0 & \longrightarrow & \mathfrak{G}'^{\vee} & \longrightarrow & \mathfrak{E}^{\vee\vee} & \longrightarrow & \mathfrak{F}'^{\vee} & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathfrak{G}'^{\vee}/\mathfrak{H}' & \longrightarrow & \mathfrak{E}^{\vee\vee}/\mathfrak{E} & \longrightarrow & \mathfrak{T}'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

where $\mathfrak{T}''$ is defined such that the sequence on the bottom becomes an exact sequence[1]. In the above diagram all columns and arrows are exact and since $\mathfrak{E}$ is torsion-free, the quotient $\mathfrak{E}^{\vee\vee}/\mathfrak{E}$ is a torsion sheaf supported on a set of codimension at least two, and hence, the same holds also for $\mathfrak{G}'^{\vee}/\mathfrak{H}'$ and $\mathfrak{T}''$. Therefore $\deg \mathfrak{G}'^{\vee} = \deg \mathfrak{H}'$ and $\operatorname{rk} \mathfrak{G}'^{\vee} = \operatorname{rk} \mathfrak{H}'$. Consequently, $\mathfrak{G}'^{\vee}$ and $\mathfrak{H}'$ have the same slope and it follows

$$\mu(\mathfrak{G}') = -\mu(\mathfrak{G}'^{\vee}) = -\mu(\mathfrak{H}') > -\mu(\mathfrak{E}) = \mu(\mathfrak{E}^{\vee})\,,$$

which means that $\mathfrak{E}^{\vee}$ is ω-stable. Q.E.D.

Corollary 2.6. *Let $\mathfrak{E}$ be a torsion-free Higgs sheaf over a compact Kähler manifold X with Kähler form ω. Then $\mathfrak{E}$ is ω-stable (resp. semistable) if and only if the sheaf $\mathfrak{E}^{\vee\vee}$ is ω-stable (resp. semistable).*

The above Corollary is an immediate consequence of the part (iii) of Proposition 2.5. It has been proved independently by Biswas and Schumacher (see [24], Lemma 2.4 for details). Finally, we have a simple result concerning the direct sum of semistable Higgs sheaves.

Theorem 2.7. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be two torsion-free Higgs sheaves over a compact Kähler manifold X with Kähler form ω. Then $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ is ω-semistable if and only if $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are both ω-semistable with $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$.*

Proof: Assume first that $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are both ω-semistable with $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2) = \mu$ and let $\mathfrak{F}$ be a Higgs subsheaf of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$. Then we have the following commutative diagram where the horizontal sequences are exact and the vertical arrows are injective

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{F}_1 & \longrightarrow & \mathfrak{F} & \longrightarrow & \mathfrak{F}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{E}_1 & \longrightarrow & \mathfrak{E}_1 \oplus \mathfrak{E}_2 & \longrightarrow & \mathfrak{E}_2 & \longrightarrow & 0
\end{array}
$$

[1] Notice that the map $\mathfrak{G}'^{\vee}/\mathfrak{H}' \longrightarrow \mathfrak{E}^{\vee\vee}/\mathfrak{E}$ is injective and $\mathfrak{T}''$ is just the corresponding quotient.

where $\mathfrak{F}_1 = \mathfrak{F} \cap (\mathfrak{E}_1 \oplus 0)$ and $\mathfrak{F}_2$ is the image of $\mathfrak{F}$ under $\mathfrak{E}_1 \oplus \mathfrak{E}_2 \longrightarrow \mathfrak{E}_2$. From the above diagram we have

$$\deg(\mathfrak{E}_1 \oplus \mathfrak{E}_2) = \deg \mathfrak{E}_1 + \deg \mathfrak{E}_2 \,. \tag{2.7}$$

Now, since by hypothesis $\mathfrak{E}_1$ and $\mathfrak{E}_2$ have the same slope μ, we have $\mu(\mathfrak{E}_1 \oplus \mathfrak{E}_2) = \mu$ and

$$\deg \mathfrak{F}_1 \leq \mu \cdot \operatorname{rk} \mathfrak{F}_1 \,, \qquad \deg \mathfrak{F}_2 \leq \mu \cdot \operatorname{rk} \mathfrak{F}_2 \,.$$

From these inequalities we obtain

$$\mu(\mathfrak{F}) = \frac{\deg \mathfrak{F}}{\operatorname{rk} \mathfrak{F}} = \frac{\deg \mathfrak{F}_1 + \deg \mathfrak{F}_2}{\operatorname{rk} \mathfrak{F}_1 + \operatorname{rk} \mathfrak{F}_2} \leq \mu$$

and the semistability of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ is follows.

Conversely, suppose $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ is ω-semistable. Since $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are at the same time Higgs subsheaves and quotient Higgs sheaves of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ we necessarily obtain

$$\mu(\mathfrak{E}_1 \oplus \mathfrak{E}_2) = \mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2) \,.$$

A Higgs subsheaf $\mathfrak{G}_1$ of $\mathfrak{E}_1$ is clearly a Higgs subsheaf of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ and hence $\mu(\mathfrak{G}_1) \leq \mu(\mathfrak{E}_1)$, which shows the semistability of $\mathfrak{E}_1$. A similar argument shows the semistability of $\mathfrak{E}_2$. Q.E.D.

In the proof of the above result we showed also that the slope of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ is the same slope of $\mathfrak{E}_1$ and $\mathfrak{E}_2$, which says that the direct sum of semistable Higgs sheaves can never be stable. In fact, even if they are stable, the direct sum is just polystable. From Theorem 2.7 we know that the direct sum of semistable Higgs sheaves with equal slope is again semistable. There is an analog result concerning the tensor product, in that case we do not need the condition on the slopes. Kobayashi [5] obtained this result for holomorphic bundles in the projective case as a direct consequence of the equivalence between semistability and the existence of approximate Hermitian-Yang-Mills structures when X is a projectve algebraic manifold. Simpson [17] proved this for Higgs bundles, again when X is projective. The general result for Higgs sheaves has been proved recently by Biswas and Schumacher [24] using an extension of the Hitchin-Kobayashi correspondence for torsion-free Higgs sheaves. We will see more about this later, when we study the notion of admissible Hermitian-Yang-Mills structure on a Higgs sheaf.

2.3 Semistable Higgs sheaves

The definition of semistability for Higgs sheaves that we have introduced in the preceding section uses only proper Higgs subsheaves, this definition can be reformulated in terms of Higgs sheaves of arbitrary rank (non necessarily proper). Indeed, it was the way in which Kobayashi [5] introduced the notion of semistability for holomorphic vector bundles. Therefore, alternatively we can say that a torsion-free

Higgs sheaf $\mathfrak{E}$ over a compact Kähler manifold X is ω-semistable if and only if $\mu(\mathfrak{F}) \leq \mu(\mathfrak{E})$ for every Higgs subsheaf $\mathfrak{F}$ with $0 < \operatorname{rk}\mathfrak{F} \leq \operatorname{rk}\mathfrak{E}$ or equivalently[2] if and only if $\mu(\mathfrak{E}) \leq \mu(\mathfrak{Q})$ for every quotient Higgs subsheaf $\mathfrak{Q}$ with $0 < \operatorname{rk}\mathfrak{Q} \leq \operatorname{rk}\mathfrak{E}$. Using the above definition it is easy to prove the following, which is a natural extension to Higgs sheaves of a classical result in [5].

Proposition 2.8. *Let $f : \mathfrak{E}_1 \longrightarrow \mathfrak{E}_2$ be a morphism of ω-semistable (torsion-free) Higgs sheaves over a compact Kähler manifold X with Kähler form ω. Then we have the following:*
(i) *If $\mu(\mathfrak{E}_1) > \mu(\mathfrak{E}_2)$, then $f = 0$ (i.e, it is the zero morphism).*
(ii) *If $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$ and $\mathfrak{E}_1$ is ω-stable, then $\operatorname{rk}\mathfrak{E}_1 = \operatorname{rk} f(\mathfrak{E}_1)$ and f is injective unless $f = 0$.*
(iii) *If $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$ and $\mathfrak{E}_2$ is ω-stable, then $\operatorname{rk}\mathfrak{E}_2 = \operatorname{rk} f(\mathfrak{E}_1)$ and f is generically surjective unless $f = 0$.*

Proof: Assume that $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are both ω-semistable with slopes μ_1 and μ_2 and ranks r_1 and r_2 respectively, and let $\mathfrak{F} = f(\mathfrak{E}_1)$; then $\mathfrak{F}$ is a torsion-free quotient Higgs sheaf of $\mathfrak{E}_1$ and a Higgs subsheaf of $\mathfrak{E}_2$.

(i) Suppose that $\mu_1 > \mu_2$ and $f \neq 0$, then

$$\mu(\mathfrak{F}) \leq \mu_2 < \mu_1 \leq \mu(\mathfrak{F}),$$

which is impossible. Therefore f must be the zero morphism.

(ii) Assume $f \neq 0$ and suppose that $\mu_1 = \mu_2$ and $\mathfrak{E}_1$ is ω-stable. If $r_1 > \operatorname{rk}\mathfrak{F}$, then

$$\mu(\mathfrak{F}) \leq \mu_2 = \mu_1 < \mu(\mathfrak{F}).$$

Hence, necessarily $r_1 = \operatorname{rk}\mathfrak{F}$ and f is injective.

(iii) Assume $f \neq 0$ and suppose that $\mu_1 = \mu_2$ and $\mathfrak{E}_2$ is ω-stable. If $r_2 > \operatorname{rk}\mathfrak{F}$, then

$$\mu(\mathfrak{F}) < \mu_2 = \mu_1 \leq \mu(\mathfrak{F}),$$

and consequently $r_2 = \operatorname{rk}\mathfrak{F}$ and the result follows. Q.E.D.

From the above Proposition we have that any extension of semistable Higgs sheaves with the same slope must be semistable. Namely we have

[2] This equivalence is clear from Lemma 2.1. On the other hand, if we assume that a sheaf $\mathfrak{E}$ is semistable according to our original definition and $\mathfrak{F}$ is a Higgs subsheaf with $\operatorname{rk}\mathfrak{F} = \operatorname{rk}\mathfrak{E}$, then we have a sequence

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{Q} \longrightarrow 0$$

with $\mathfrak{Q}$ a torsion sheaf (it is a zero rank sheaf). From the above exact sequence it follows that $\deg\mathfrak{E} = \deg\mathfrak{F} + \deg\mathfrak{Q}$ and since $\deg\mathfrak{Q} \geq 0$, necessarily $\mu(\mathfrak{F}) \leq \mu(\mathfrak{E})$. This means that $\mathfrak{E}$ is semistable with respect to the new definition. The converse direction is immediate, and hence the definition of semistability can be put in terms of Higgs subsheaves (or equivalently quotient Higgs sheaves) of arbitrary rank.

Corollary 2.9. *Let*

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{G} \longrightarrow 0$$

be an exact sequence of torsion-free Higgs sheaves over a compact Kähler manifold X *with Kähler form* ω*. If* $\mathfrak{F}$ *and* $\mathfrak{G}$ *are both* ω*-semistable and* $\mu(\mathfrak{F}) = \mu(\mathfrak{G}) = \mu$*, then* $\mathfrak{E}$ *is also* ω*-semistable and* $\mu(\mathfrak{E}) = \mu$ *.*

Proof: The fact that $\mu(\mathfrak{E}) = \mu$ follows from Lemma 2.1. Suppose now $\mathfrak{E}$ is not semistable and hence there exists a subsheaf $\mathfrak{H}$ destabilizing it, i.e., there exists a proper (non-trivial) Higgs subsheaf $\mathfrak{H}$ such that $\mu(\mathfrak{H}) > \mu$. Without loss of generality we can assume that $\mathfrak{H}$ is semistable[3]. Then we have a morphism $f : \mathfrak{H} \longrightarrow \mathfrak{G}$ with $\mu(\mathfrak{H}) > \mu(\mathfrak{G})$, and from Proposition 2.8 we have $f = 0$. Therefore, there exists a morphism $g : \mathfrak{H} \longrightarrow \mathfrak{F}$ where $\mu(\mathfrak{H}) > \mu(\mathfrak{F})$, and we have again $g = 0$, which means that $\mathfrak{H}$ must be trivial and from this we have a contradiction. Q.E.D.

[3]If it is not, we can destabilize $\mathfrak{H}$ with a Higgs subsheaf $\mathfrak{H}'$. If it is semistable we stop, if it is not, then we repeat this procedure. Clearly this finishes after a finite number of steps, since in the extreme case we get a Higgs sheaf of rank one.

CHAPTER 3

Higgs bundles

3.1 Metrics and connections on Higgs bundles

Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle of rank r over a compact Kähler manifold X with Kähler form ω and let h be an Hermitian metric on it. Let $D_h = D'_h + D''$ be the Chern connection[1] on E, where D'_h and $D'' = d''$ are the (1,0) and (0,1) parts. Using this decomposition of D_h and the Higgs field ϕ, Simpson [17] introduced a connection on $\mathfrak{E}$ in the following way:

$$\mathcal{D}'' = D'' + \phi\,, \qquad \mathcal{D}'_h = D'_h + \bar{\phi}_h\,, \tag{3.1}$$

where $\bar{\phi}_h$ is the usual adjoint of the Higgs field with respect to the Hermitian structure h, that is, it is defined by the formula

$$h(\bar{\phi}_h s, s') = h(s, \phi s') \tag{3.2}$$

where s, s' are sections of the Higgs bundle. Notice however that $\mathcal{D}'_h$ and $\mathcal{D}''$ are not of type $(1,0)$ and $(0,1)$. The resulting connection $\mathcal{D}_h = \mathcal{D}'_h + \mathcal{D}''$ is called the *Hitchin-Simpson connection.* Clearly

$$\mathcal{D}_h = D_h + \phi + \bar{\phi}_h \tag{3.3}$$

depends on the Higgs field ϕ. Even more, it has an extra dependence on h via the adjoint of the Higgs field. The curvature of the Hitchin-Simpson connection is defined by $\mathcal{R}_h = \mathcal{D}_h^2$ and we say that the pair $(\mathfrak{E}, h)$ is Hermitian flat, if this curvature vanishes. Explicitly

$$\begin{aligned}\mathcal{R}_h &= (D_h + \phi + \bar{\phi}_h) \wedge (D_h + \phi + \bar{\phi}_h)\\ &= D_h \wedge D_h + D_h \wedge \phi + \phi \wedge D_h\\ &\quad + D_h \wedge \bar{\phi}_h + \bar{\phi}_h \wedge D_h + \phi \wedge \bar{\phi}_h + \bar{\phi}_h \wedge \phi\,,\end{aligned}$$

then using again the decomposition $D_h = D'_h + D''$ and defining

$$[\phi, \bar{\phi}_h] = \phi \wedge \bar{\phi}_h + \bar{\phi}_h \wedge \phi \tag{3.4}$$

we obtain the following formula of the Hitchin-Simpson curvature in terms of the Chern curvature

$$\mathcal{R}_h = R_h + D'_h(\phi) + D''(\bar{\phi}_h) + [\phi, \bar{\phi}_h]\,. \tag{3.5}$$

[1] Also called the Hermitian connection, this is the unique connection compatible with the metric h and the holomorphic structure of the bundle E, see Chapter I in [5] for more details.

In the above formula, $D'_h(\phi)$ and $D''(\bar{\phi}_h)$ are of type $(2,0)$ and $(0,2)$ respectively, and the $(1,1)$-part is given by

$$\mathcal{R}_h^{1,1} = R_h + [\phi, \bar{\phi}_h] \,. \tag{3.6}$$

Let us consider again a Higgs bundle $\mathfrak{E} = (E, \phi)$ of rank r over an n-dimensional compact Kähler manifold X with Kähler form ω. Consider the usual star operator $* : A^{p,q} \to A^{n-q,n-p}$ and the operator $L : A^{p,q} \to A^{p+1,q+1}$ defined by $L\varphi = \omega \wedge \varphi$, where φ is a form on X of type (p,q). Then we define, as usual,

$$\Lambda = *^{-1} \circ L \circ * : A^{p,q} \to A^{p-1,q-1} \tag{3.7}$$

which is the adjoint of the multiplication by the Kähler form ω. The connection defined by (3.1) satisfies the following identities, which are indeed similar to the Kähler indentities.

Proposition 3.1. *Let h be an Hermitian metric on a Higgs bundle $\mathfrak{E}$ over a compact Kähler manifold X with Kähler form ω, then*

$$i[\Lambda, \mathcal{D}'_h] = -(\mathcal{D}'')^* \,, \qquad i[\Lambda, \mathcal{D}''] = (\mathcal{D}'_h)^* \,, \tag{3.8}$$

where Λ is the adjoint of the multiplication by ω.

Proof: Using local coordinates (see [17] for details) it can be shown that

$$i[\Lambda, \phi] = \bar{\phi}_h^* \,, \qquad i[\Lambda, \bar{\phi}_h] = -\phi^* \,. \tag{3.9}$$

On the other hand, the standard Kähler identities (see [36]) are

$$i[\Lambda, D'_h] = -(D'')^* \,, \qquad i[\Lambda, D''] = (D'_h)^* \,. \tag{3.10}$$

At this point the result follows from (3.9), (3.10) and the decomposition of the Hitchin-Simpson connection given by (3.1). Q.E.D.

Let h be an Hermitian metric on $\mathfrak{E}$, associated to h we have a Hitchin-Simpson curvature $\mathcal{R}_h$. We define the mean curvature of the Hitchin-Simpson connection, just by contraction of this curvature with the operator $i\Lambda$. In other words, $\mathcal{K}_h = i\Lambda\mathcal{R}_h$. The mean curvature is an endomorphism in $\mathrm{End}(E)$. If we consider a local frame field $\{e_i\}_{i=1}^r$ for $\mathfrak{E}$ and a local coordinate system $\{z_\alpha\}_{\alpha=1}^n$ of X, the components of the mean curvature are given by $\mathcal{K}^i_j = \omega^{\alpha\bar{\beta}}\mathcal{R}^i_{j\alpha\bar{\beta}}$, where $\mathcal{R}^i_{j\alpha\bar{\beta}}$ are the components of the $(1,1)$-part of the Hitchin-Simpson curvature.

The mean curvature can be considered also as an Hermitian form by defining

$$\mathcal{K}_h(s, s') = h(s, \mathcal{K}_h s') \,, \tag{3.11}$$

where s, s' are sections of $\mathfrak{E}$.

Let k be an Hermitian metric on $\mathfrak{E}$ and let a be a selfadjoint endomorphism of E (selfadjoint with respect to k). Then, one defines another metric h on $\mathfrak{E}$ by

$$h(s, s') = k(s, as')\,, \tag{3.12}$$

where s, s' are sections of $\mathfrak{E}$. Since a is selfadjoint with respect to k we get

$$h(s, as') = k(s, aas') = k(as, as') = h(as, s')\,,$$

thus a is also selfadjoint with respect to h. The metric defined by (3.12) is usually denoted by $h = ka$. The origin of this notation, which appears quite often in literature, is clear from a local point of view. In fact, using a local frame field $\{e_i\}_{i=1}^r$ on $\mathfrak{E}$ we have

$$h_{ij} = k(e_i, ae_j) = k(e_i, e_l a^l_j) = k_{il} a^l_j$$

and hence the matrix representing h is the product of the matrices representing k and a. Now, with respect to h we have

$$\begin{aligned} h(\bar\phi_h s, s') &= h(s, \phi s') = k(s, a\phi s') = k(as, \phi s') \\ &= k(\bar\phi_k as, s') = h(\bar\phi_k as, a^{-1}s') \\ &= h(a^{-1}\bar\phi_k as, s') \end{aligned}$$

where we have used the fact that a^{-1} is selfadjoint[2] with respect to h. Thus, we obtain the formula

$$\bar\phi_h = a^{-1}\bar\phi_k a\,, \tag{3.13}$$

which relates the adjoint of the Higgs fields of the metrics k and $h = ka$.

3.2 The space of Hermitian structures

In this thesis we work with torsion-free Higgs sheaves over compact Kähler manifolds. Since every torsion-free sheaf is locally free outside a closed analytic subset, it will be necessary to work with certain Higgs bundles over non-compact manifolds. In this section we study some aspects of the space of Hermitian structures which cover this last case. For more deatils see [5], [17].

Let $\mathfrak{E}$ be a Higgs bundle of rank r over a Kähler manifold Y. Let $\mathrm{Herm}(\mathfrak{E})$ denote the space of all C^∞-Hermitian forms on $\mathfrak{E}$. This is an infinite dimensional linear space. If $v \in \mathrm{Herm}(\mathfrak{E})$ and $\{e_i\}_{i=1}^r$ is a local frame field, the matrix representing v and defined by $v_{ij} = v(e_i, e_j)$ is Hermitian.

[2]Notice that since a is selfadjoint with respect to k, we have

$$h(a^{-1}s, s') = k(a^{-1}s, as') = k(s, s') = h(s, a^{-1}s')\,,$$

which says that a^{-1} is selfadjoint with respect to the metric h.

Let $\mathrm{Herm}^+(\mathfrak{E})$ be the space of all Hermitian structures, that is, it is the subset of $\mathrm{Herm}(\mathfrak{E})$ consisting of all positive definite Hermitian forms. If h, k are elements in $\mathrm{Herm}^+(\mathfrak{E})$, the straight line

$$h_t = th + (1-t)k\,, \qquad 0 \le t \le 1 \tag{3.14}$$

remains inside $\mathrm{Herm}^+(\mathfrak{E})$, and hence the latter is a convex space. $\mathrm{Herm}(\mathfrak{E})$ can be considered as the tangent space of $\mathrm{Herm}^+(\mathfrak{E})$ at each point h. That is

$$T_h\mathrm{Herm}^+(\mathfrak{E}) = \mathrm{Herm}(\mathfrak{E}) \tag{3.15}$$

at any Hermitian structure h.

In the case of a compact Kähler manifold X, we introduce a Riemannian metric on the space $\mathrm{Herm}^+(\mathfrak{E})$ in the following way. Let v be an element in $\mathrm{Herm}(\mathfrak{E})$, then one defines $h^{-1}v$ as the endomorphism of $\mathfrak{E}$ satisfying

$$v(s, s') = h(s, h^{-1}vs')\,, \tag{3.16}$$

where s, s' are arbitrary sections of $\mathfrak{E}$. Then one defines a Riemannian structure in $\mathrm{Herm}^+(\mathfrak{E})$ by defining an inner product at each tangent space. Namely, if v, v' are elements in $T_h\mathrm{Herm}^+(\mathfrak{E})$, these can be seen as elements in $\mathrm{Herm}(\mathfrak{E})$ via the identification (3.15) and hence one defines its inner product by

$$(v, v')_h = \int_X \mathrm{tr}(h^{-1}v \cdot h^{-1}v')\, \omega^n/n!\,. \tag{3.17}$$

The norm of any v in $T_h\mathrm{Herm}^+(\mathfrak{E})$ is defined as usual, i.e.,

$$\|v\|_h^2 = (v, v)_h = \int_X |v|_h^2\, \frac{\omega^n}{n!}\,, \tag{3.18}$$

where

$$|v|_h^2 = \mathrm{tr}(h^{-1}v \cdot h^{-1}v)\,. \tag{3.19}$$

Let γ be an smooth curve $h_t\,, a \le t \le b$ in $\mathrm{Herm}^+(\mathfrak{E})$, then each $\partial_t h_t$ is an element in $T_{h_t}\mathrm{Herm}^+(\mathfrak{E})$ and we can consider the functional (usually called the energy integral)

$$E(\gamma) = \int_a^b \|\partial_t h_t\|_{h_t}^2\, dt\,. \tag{3.20}$$

Consider the curve γ_ϵ in $\mathrm{Herm}^+(\mathfrak{E})$ defined by $\epsilon v_t\,, a \le t \le b$ and $\epsilon \ge 0$. Then (see [5] for more details)

$$\frac{d}{d\epsilon}E(\gamma + \gamma_\epsilon)|_{\epsilon=0} = -2\int_a^b (h_t^{-1}\partial_t v_t \cdot [h_t^{-1}\partial_t^2 h_t - h_t^{-1}\partial_t h_t \cdot h_t^{-1}\partial_t h_t])\, dt \tag{3.21}$$

which means that the curve γ is a critical point of E if and only if

$$h_t^{-1}\partial_t^2 h_t - h_t^{-1}\partial_t h_t \cdot h_t^{-1}\partial_t h_t = 0\,, \tag{3.22}$$

or equivalently if and only if

$$\frac{d}{dt}(h_t^{-1}\partial_t h_t) = 0\,. \tag{3.23}$$

From the preceding analysis we conclude that an smooth curve h_t in the space $\mathrm{Herm}^+(\mathfrak{E})$ is a geodesic if and only if the endomorphism $h_t^{-1}\partial_t h_t$, which is associated to the Hermitian form $\partial_t h_t$, is a fixed element in $\mathrm{End}\, E$.

Suppose k is a fixed element in $\mathrm{Herm}^+(\mathfrak{E})$ and that a is a self-adjoint endomorphism of E, then the curve $h_t = ke^{ta}\,, 0 \le t \le 1$ satisfies

$$h_t^{-1}\partial_t h_t = e^{-ta}k^{-1}ke^{ta}a = a \tag{3.24}$$

and hence, it is in a natural way a geodesic in $\mathrm{Herm}^+(\mathfrak{E})$.

Let $\mathrm{Herm}^+_{\mathrm{int}}(\mathfrak{E})$ denote the set of all Hermitian metrics h satisfying

$$\|\mathcal{K}_h\|_{L_1} = \int_Y |\mathcal{K}_h|_h \frac{\omega^n}{n!} < \infty \tag{3.25}$$

where $\mathcal{K}_h$ is the mean curvature of the Hitchin-Simpson connection of h. If Y is compact, this space coincides with $\mathrm{Herm}^+(\mathfrak{E})$. However, in general it is a proper subset of the space of Hermitian structures. The space $\mathrm{Herm}^+_{\mathrm{int}}(\mathfrak{E})$ have been studied by Simpson in [17]. It can be seen as an analytic manifold, which in general is not connected, and has the the following properties. If $k \in \mathrm{Herm}^+_{\mathrm{int}}(\mathfrak{E})$ is a fixed element, then any other metric in the same component is given by $h = ke^a$ with a an smooth endomorphism of E which is selfadjoint with respect to k. Even more, Simpson showed in [17] that for any Higgs bundle $\mathfrak{E}$ over a Kähler manifold Y (satisfying some additional requirements), the solution of the heat equation remains in the same connected component of the initial metric. To be precise, if k is a fixed metric in $\mathrm{Herm}^+_{\mathrm{int}}(\mathfrak{E})$, then the unique solution h_t with $h_0 = k$ is contained in the same connected component as k.

3.3 Vanishing theorems for Higgs bundles

In the study of holomorphic bundles there exists some results on vanishing holomorphic sections if some specific conditions on the curvature apply. These results are called *vanishing theorems* and play an important role in the study of (approximate) Hermitian-Einstein structures (see [5], Ch. III). Some of these results also holds in the context of Higgs bundles, in that case, we must replace the ordinary mean curvature for the Hitchin-Simpson curvature.

Let s be a section of the Higgs bundle $\mathfrak{E} = (E, \phi)$, it is said to be *ϕ-invariant* if there is a holomorphic 1-form λ on X such that $\phi(s) = s \otimes \lambda$. Using ϕ-invariant sections we have a first vanishing theorem for Higgs bundles.

Theorem 3.2. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle over a compact Kähler manifold X and let h be an Hermitian metric on it.*

(i) *If the Hitchin-Simpson mean curvature $\mathcal{K}_h$ is seminegative definite everywhere, then every ϕ-invariant section s of $\mathfrak{E}$ satisfies*

$$\mathcal{K}_h(s,s) = 0 \tag{3.26}$$

and is parallel in the classical sense, that is, $D_h s = 0$ with D_h the Chern connection of h.
(ii) *If the Hitchin-Simpson mean curvature $\mathcal{K}_h$ is seminegative definite everywhere and negative definite at some point of X, then $\mathfrak{E}$ has no nonzero ϕ-invariant sections.*

Proof: Let s be a ϕ-invariant section and assume $\mathcal{K}_h$ is seminegative definite everywhere. From the decomposition of the Hitchin-Simpson curvature (3.5) we have

$$\mathcal{R}_h s = R_h s + D'_h(\phi)s + D''(\bar{\phi}_h)s + [\phi, \bar{\phi}_h]s\,. \tag{3.27}$$

In the above epxression, the terms involving D'_h and D'' are of type $(2,0)$ and $(0,2)$ respectively. On the other hand, since $\phi(s) = s \otimes \lambda$ we have $[\phi, \bar{\phi}_h]s = 0$. Therefore, by applying the operator $i\Lambda$ to the identity (3.27) we obtain

$$\mathcal{K}_h s = i\Lambda\mathcal{R}_h s = i\Lambda R_h s = K_h s\,, \tag{3.28}$$

which in terms of Hermitian forms can be written as

$$\mathcal{K}_h(s,s) = K_h(s,s)\,. \tag{3.29}$$

Now, using the Weitzenböck formula (see [5] or [21] for more details) we have

$$i\Lambda d'd''h(s,s) = h(D'_h s, D'_h s) - h(s, K_h s)\,. \tag{3.30}$$

Since by hypothesis $\mathcal{K}_h$ is a seminegative form, then from (3.29) and (3.30) we conclude that

$$i\Lambda d'd''h(s,s) = |D'_h s|^2_h - \mathcal{K}_h(s,s) \geq 0\,, \tag{3.31}$$

where we denote

$$|D'_h s|^2_h = h(D'_h s, D'_h s)\,.$$

By Hopf's maximum principle it follows

$$i\Lambda d'd''h(s,s) = 0$$

and hence, necessarily $\mathcal{K}_h(s,s) = 0$ and $D'_h s = 0$. Finally, since s is holomorphic and D_h is the Chern connection (which is compatible with the holomorphic structure) we have $D_h s = 0$ and this shows (i).

On the other hand, suppose now s is a nonzero ϕ-invariant section of $\mathfrak{E}$ and assume this time that $\mathcal{K}_h$ is seminegative definite everywhere and negative at some point. Then, from (i) we have $D_h s = 0$ and hence the ϕ-invariant section s never vanishes. By the same part (i) we know $\mathcal{K}_h(s,s) = 0$, and then we have a contradiction since $\mathcal{K}_h$ must be negative at some point of X. Q.E.D.

Theorem 3.3. *Let $\mathfrak{E}_1 = (E_1, \phi_1)$ and $\mathfrak{E}_2 = (E_2, \phi_2)$ be two Higgs bundles over a compact Kähler manifold X and let h_1 and h_2 two Hermitian metrics on $\mathfrak{E}_1$ and $\mathfrak{E}_2$ respectively. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the corresponding main curvatures and $\mathcal{K}_{1\otimes 2}$ the mean curvature of $\mathfrak{E}_1 \otimes \mathfrak{E}_2$. Then*
(i) *If both $\mathcal{K}_1$ and $\mathcal{K}_2$ are seminegative definite everywhere, then every ϕ-invariant section s of $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ satisfies*

$$\mathcal{K}_{1\otimes 2}(s,s) = 0\,, \qquad D_{1\otimes 2}s = 0\,. \tag{3.32}$$

(ii) *If both $\mathcal{K}_1$ and $\mathcal{K}_2$ are seminegative definite everywhere and ether one is negative definite somewhere in X, then $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ admits no nonzero ϕ-invariant sections.*

Proof: In an analogue form to the classical case, we can decompose the Hitchin-Simpson curvature of $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ as

$$\mathcal{R}_{1\otimes 2} = \mathcal{R}_1 \otimes I_2 + I_1 \otimes \mathcal{R}_2\,. \tag{3.33}$$

Taking the trace with respect to ω (that is, multiplying by the operator $i\Lambda$) we have the following expression involving the mean curvatures

$$\mathcal{K}_{1\otimes 2} = \mathcal{K}_1 \otimes I_2 + I_1 \otimes \mathcal{K}_2 \tag{3.34}$$

and at this point the result follows from Theorem 3.2. Q.E.D.

From the above results we have the following

Corollary 3.4. *Let $\mathfrak{E}$ be a Higgs bundle over a compact Kähler manifold X and let h be an Hermitian metric on it. Let $\mathfrak{E}^{\otimes p}$ denote the tensor power of $\mathfrak{E}$ p-times with Higgs field ψ (constructed from the Higgs field of $\mathfrak{E}$). Then*
(i) *If $\mathcal{K}$ is seminegative definite everywhere, then every ψ-invariant section s of $\mathfrak{E}^{\otimes p}$ is parallel in the clasical sense and satisfies*

$$\mathcal{K}_{\otimes p}(s,s) = 0\,, \tag{3.35}$$

where $\mathcal{K}_{\otimes p}$ represents the mean curvature of $\mathfrak{E}^{\otimes p}$.
(ii) *If $\mathcal{K}$ is seminegative definite everywhere and negative definite at some point of X, then $\mathfrak{E}^{\otimes p}$ admits no nonzero ψ-invariant sections.*

We can consider the tangent bundle TX as a trivial Higgs bundle (with Higgs field equal to zero). Since we have an Hermitian metric on TX, we have a natural mean curvature of TX, which we denote by K_{TX}. In this way, $\bigwedge^p T^*X$ is also a trivial Higgs bundle and we denote the mean curvature of it by $K_{\bigwedge^p T^*}$. If $\mathfrak{E} = (E, \phi)$ is a Higgs bundle over X and h is an Hermitian metric on $\mathfrak{E}$ with mean curvature $\mathcal{K}_{\mathfrak{E}}$, one has a mean curvature

$$\mathcal{K}_{\mathfrak{E}\otimes\bigwedge^p T^*} = \mathcal{K}_{\mathfrak{E}} \otimes I_{\bigwedge^p T^*} + I_{\mathfrak{E}} \otimes K_{\bigwedge^p T^*} \tag{3.36}$$

for the Higgs bundle $\mathfrak{E} \otimes \bigwedge^p T^*X$. At this point from Theorem 3.3 and Corollary 3.4 one has the following

Corollary 3.5. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle over a compact Kähler manifold X and let h be an Hermitian metric on it with mean curvature $\mathcal{K}_{\mathfrak{E}}$. Then*
(i) *If $\mathcal{K}_{\mathfrak{E}}$ is seminegative definite and K_{TX} is semipositive definite everywhere, then every ϕ-invariant section s of $\mathfrak{E} \otimes \bigwedge^p T^*X$ is parallel and satisfies*

$$\mathcal{K}_{\mathfrak{E}\otimes\bigwedge^p T^*}(s, s) = 0\,. \tag{3.37}$$

(ii) *If, moreover, either $\mathcal{K}_{\mathfrak{E}}$ is negative definite or K_{TX} is positive definite at some point of X, then $\mathfrak{E} \otimes \bigwedge^p T^*X$ has no nonzero ϕ-invariant sections.*

Let h be an Hermitian metric on $\mathfrak{E}$ and consider a real positive function $a = a(x)$ on X, then, as we said earlier, $h' = ha = ah$ defines another Hermitian metric on $\mathfrak{E}$. Since h' is a conformal change of h, we have in particular $\bar{\phi}_{h'} = \bar{\phi}_h$. Then, we obtain

$$\mathcal{K}'\,\omega^n = in(R' + [\phi, \bar{\phi}_{h'}]) \wedge \omega^{n-1} = \left(K' + i\Lambda[\phi, \bar{\phi}_h]\right)\omega^n. \tag{3.38}$$

Now, in the ordinary case with $\Box_0 = i\Lambda d''d'$ (see [5], Ch. III for details) we obtain the formula $K' = K + \Box_0(\log a)I$ and, by replacing this in (3.38) we get

$$\mathcal{K}' = \mathcal{K} + \Box_0(\log a)I\,, \tag{3.39}$$

and hence, the same relation between mean curvatures also holds in the Higgs case. Using the above expression we obtain the following result, which is an straightforward generalization to the Higgs case of a classical result of holomorphic bundles.

Lemma 3.6. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle over a compact Kähler manifold X and let h be an Hermitian metric on it. Let $\mathcal{K}$ be the Hitchin-Simpson mean curvature and $\lambda_1 < \cdots < \lambda_r$ the corresponding eigenvalues. If*

$$\int_X \lambda_r\,\omega^n < 0\,, \tag{3.40}$$

there exists a real positive function a on X such that the mean curvature $\mathcal{K}'$ of the metric $h' = ah$ is negative definite.

Proof: Let f be a C^∞-function on X such that $\lambda_r < f$ and

$$\int_X f\,\omega^n = 0\,. \tag{3.41}$$

Then, from [5] (see Lemma 1.31, Ch. III) we know the equation $\Box_0 u = -f$ has a solution u. Using this and by defining $a = e^u$, we get

$$\Box_0(\log a) = -f\,. \tag{3.42}$$

Then, from (3.39) and since $\lambda_r < f$ it follows that $\mathcal{K}'$ must be negative definite. Q.E.D.

From Lemma 3.6 and Theorem 3.2 we conclude the following vanishing theorem

Theorem 3.7. *Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle over a compact Kähler manifold X and let h be an Hermitian metric on it. Let $\mathcal{K}$ be the Hitchin-Simpson mean curvature and $\lambda_1 < \cdots < \lambda_r$ the corresponding eigenvalues. If*

$$\int_X \lambda_r\,\omega^n < 0\,, \tag{3.43}$$

then $\mathfrak{E}$ admits no nonzero ϕ-invariant sections.

3.4 Hermitian-Yang-Mills structures

The concept of Hermitian-Einstein structure was introduced by Kobayashi to understand the Bogomolov semistability in a differential geometrical way. In a similar form as in the ordinary case [5], [13], [9], we have a notion of Hermitian-Einstein structure for Higgs bundles [17].

Let h be an Hermitian structure on a Higgs bundle $\mathfrak{E}$, we say that h is a *weak Hermitian-Yang-Mills structure* with factor γ for $\mathfrak{E}$ if

$$\mathcal{K}_h = \gamma \cdot I \,, \tag{3.44}$$

where γ is a real function defined on X and I is the identity endomorphism on E. This means that in components the mean curvature is given by $\mathcal{K}^i_{\ j} = \gamma \cdot \delta^i_{\ j}$. From this definition it follows that any metric h on a Higgs line bundle is necessarily a weak Hermitian-Yang-Mills structure. Also, if h is a weak Hermitian-Yang-Mills structure with factor γ for $\mathfrak{E}$, then the dual metric $h^\vee$ is a weak Hermitian-Yang-Mills structure with factor $-\gamma$ for the dual bundle $\mathfrak{E}^\vee$.

As in the ordinary case, for Higgs bundles we have some simple properties related to the notion of weak Hermitian-Yang-Mills structure. In particular, from the usual formulas for the curvature of tensor products and direct sums we have the following

Proposition 3.8. *Let h_1 and h_2 be two weak Hermitian-Yang-Mills structures with factors γ_1 and γ_2 for Higgs bundles $\mathfrak{E}_1$ and $\mathfrak{E}_2$, respectively. Then*
(i) *$h_1 \otimes h_2$ is a weak Hermitian-Yang-Mills structure with factor $\gamma_1 + \gamma_2$ for the tensor product bundle $\mathfrak{E}_1 \otimes \mathfrak{E}_2$.*
(ii) *The metric $h_1 \oplus h_2$ is a weak Hermitian-Yang-Mills structure for the Whitney sum $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ with factor γ if and only if both metrics h_1 and h_2 are weak Hermitian-Yang-Mills structures with $\gamma_1 = \gamma_2 = \gamma$.*

Proof: (i) Assume that h_1 and h_2 are two weak Hermitian-Yang-Mills structures with factors γ_1 and γ_2 for $\mathfrak{E}_1$ and $\mathfrak{E}_2$ respectively. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the corresponding mean curvatures. Then, the mean curvature of the tensor product bundle $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ is given by

$$\mathcal{K} = \mathcal{K}_1 \otimes I_2 + I_1 \otimes \mathcal{K}_2 \tag{3.45}$$

where I_1 and I_2 denote the identity endomorphisms of E_1 and E_2 respectively. Then from (3.45) it follows that $h_1 \otimes h_2$ is a weak Hermitian-Yang-Mills structure with factor $\gamma_1 + \gamma_2$.

(ii) Since the mean curvature of $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ is given by

$$\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \tag{3.46}$$

the equivalence is clear. Q.E.D.

If we have a weak Hermitian-Yang-Mills structure in which the factor $\gamma = c$ is constant, we say that h is an *Hermitian-Yang-Mills structure* with factor c for $\mathfrak{E}$. From Proposition 3.8 and this definition we get

Corollary 3.9. *If $h \in \mathrm{Herm}^+(\mathfrak{E})$ is a (weak) Hermitian-Yang-Mills structure with factor γ for the Higgs bundle $\mathfrak{E}$. Then*
(i) *The naturally induced Hermitian metric on the tensor product $\mathfrak{E}^{\otimes p} \otimes \mathfrak{E}^{\vee \otimes q}$ is a (weak) Hermitian-Yang-Mills structure with factor $(p-q)\gamma$.*
(ii) *The induced Hermitian metric on $\bigwedge^p \mathfrak{E}$ is a (weak) Hermitian-Yang-Mills structure with factor $p\gamma$ for every $p \leq r = \mathrm{rk}\,\mathfrak{E}$.*

In general, if h is a weak Hermitian-Yang-Mills structure with factor γ, the slope of $\mathfrak{E}$ can be written in terms of γ. To be precise, we obtain

Proposition 3.10. *If $h \in \mathrm{Herm}^+(\mathfrak{E})$ is a weak Hermitian-Yang-Mills structure with factor γ, then*

$$\mu(\mathfrak{E}) = \frac{1}{2n\pi} \int_X \gamma\, \omega^n. \tag{3.47}$$

Proof: Let $\mathcal{R}$ be the Hitchin-Simpson curvature of $\mathfrak{E}$. Then we have the identity

$$in\mathcal{R}_h \wedge \omega^{n-1} = \mathcal{K}_h\, \omega^n\,. \tag{3.48}$$

Now, by hypothesis h is a weak Hermitian-Yang-Mills structure with factor γ, and hence, taking the trace of (3.48) and then integrating over X we get

$$in \int_X \mathrm{tr}\, \mathcal{R}_h \wedge \omega^{n-1} = \int_X (\mathrm{tr}\, \mathcal{K}_h)\, \omega^n = r \int_X \gamma\, \omega^n\,. \tag{3.49}$$

In the above equation, only the $(1,1)$-part of the Hitchin-Simpson curvature contributes with the integral over X. As we have seen before, the $(1,1)$-part of $\mathcal{R}_h$ is given by $\mathcal{R}_h^{1,1} = R_h + [\phi, \bar{\phi}_h]$ where R_h is the Chern curvature of h. Now, using the cyclic property of the trace it follows

$$\begin{aligned} \mathrm{tr}\, [\phi, \bar{\phi}_h] &= \mathrm{tr}\, (\phi \wedge \bar{\phi}_h + \bar{\phi}_h \wedge \phi) \\ &= \mathrm{tr}\, (\phi \wedge \bar{\phi}_h - \phi \wedge \bar{\phi}_h) \\ &= 0\,. \end{aligned}$$

From this we have that only the Chern curvature R_h contributes with the integral on the left-hand side of (3.49). Thus, we have

$$2n\pi \deg \mathfrak{E} = 2n\pi \int_X \frac{i}{2\pi} \mathrm{tr}\, R_h \wedge \omega^{n-1} = r \int_X \gamma\, \omega^n \tag{3.50}$$

and the result follows. Q.E.D.

As we have shown in the last section, if we have a real positive function $a = a(x)$ on X, then $h' = ah$ defines another Hermitian metric on $\mathfrak{E}$ and we have a relation between the Hitchin-Simpson mean curvatures given by (3.39). From this we conclude the following

Lemma 3.11. *Let h be a weak Hermitian-Yang-Mills structure with factor γ for $\mathfrak{E}$ and let a be a real positive definite function on X. Then $h' = ah$ is a weak Hermitian-Yang-Mills structure with factor*

$$\gamma' = \gamma + \Box_0(\log a)\,. \tag{3.51}$$

Making use of the above Lemma we can define a constant c which plays an important role in the definition of the Donaldson functional. Such a constant c is an average of the factor γ of a weak Hermitian-Yang-Mills structure. Namely

Proposition 3.12. *If $h \in \mathrm{Herm}^+(\mathfrak{E})$ is a weak Hermitian-Yang-Mills structure with factor γ, then there exists a conformal change $h' = ah$ such that h' is an Hermitian-Yang-Mills structure with constant factor c, given by*

$$c \int_X \omega^n = \int_X \gamma\, \omega^n\,. \tag{3.52}$$

Such a conformal change is unique up to homothety.

Proof: Let c as in (3.52), then

$$\int_X (c-\gamma)\, \omega^n = 0\,. \tag{3.53}$$

It is sufficient to prove that there is a function u satisfying the equation

$$\Box_0 u = c - \gamma\,, \tag{3.54}$$

where, as we said before $\Box_0 = i\Lambda d''d'$. Because if this holds, then by applying Lemma 3.11 with the function $a = e^u$ the result follows.

Now, from Hodge theory we know that the equation (3.54) has a solution if and only if the function $c-\gamma$ is orthogonal to all $\Box_0$-harmonic functions. Since X is compact, a function is $\Box_0$-harmonic if and only if it is constant. But (3.53) says that $c-\gamma$ is orthogonal to the constant functions and hence the equation (3.54) has a solution u. Finally, the uniqueness follows from the fact that $\Box_0$-harmonic functions are constant. In fact, if u' is another solution to the equation (3.54), then $u'-u$ is $\Box_0$-harmonic and hence it is equal to a constant, say b. Therefore $a' = e^{u'} = e^b a$. Q.E.D.

Since every weak Hermitian-Yang-Mills structure can be transformed into an Hermitian-Yang-Mills structure using a conformal change of the metric, without loss of generality we avoid using weak structures and work directly with Hermitian-Yang-Mills structures.

In [17], [18], Simpson studied an extension of the Hitchin-Kobayashi correspondence for Higgs bundles, his remarkable result is even true for manifolds which are not necessarily compact (satisfying some additional analytic requirements). His result in the case of compact Kähler manifolds is as follows:

Theorem 3.13. *Let $\mathfrak{E}$ be a Higgs bundle over a Kähler manifold X with Kähler form ω. It is ω-polystable if and only if it admits an Hermitian-Yang-Mills structure on it. That is, if and only if there exists an Hermitian metric h such that $\mathcal{K}_h = c \cdot I$ for some constant c.*

Finally, there is another important result, which is an extension of a famous result in complex geometry, known as the Bogomolov-Lübke inequality and which holds for Higgs bundles with Hermitian-Yang-Mills metrics, such a result can be written as:

Theorem 3.14. *Let $\mathfrak{E}$ be a Higgs bundle over a compact Kähler manifold X and suppose that there is an Hermitian-Yang-Mills structure on it, then*

$$\left[2r\, c_2(\mathfrak{E}) - (r-1)c_1(\mathfrak{E})^2\right] \cup [\omega]^{n-2} \geq 0\,. \tag{3.55}$$

Proof: Let h be an Hermitian-Yang-Mills structure for $\mathfrak{E}$. Associated with this metric we have closed $2k$-forms $c_k(\mathfrak{E}, h)$ representing the k-th Chern classes. In particular, for the first two representatives one has

$$c_1(\mathfrak{E}, h) = \frac{i}{2\pi}\,\mathrm{tr}\, R_h\,, \qquad c_2(\mathfrak{E}, h) = \frac{1}{8\pi^2}\left[\mathrm{tr}(R_h \wedge R_h) - (\mathrm{tr}\, R_h)^2\right]\,, \tag{3.56}$$

where R_h is the Chern curvature corresponding to the metric h. Therefore

$$2r\, c_2(\mathfrak{E}, h) - (r-1)\, c_1(\mathfrak{E}, h)^2 = \frac{1}{4\pi^2}\left[r\,\mathrm{tr}(R_h \wedge R_h) - (\mathrm{tr}\, R_h)^2\right]\,. \tag{3.57}$$

Now, multiplying by the Kähler form ω and using local unitary frame fields on X and $\mathfrak{E}$, we have expressions for both traces on the right-hand side of the last formula. Namely we get (see [5], Ch.IV, for details)

$$n(n-1)(\mathrm{tr}\, R_h)^2 \wedge \omega^{n-2} = \left[|\rho_h|^2 - \sigma_h^2\right]\omega^n\,, \tag{3.58}$$

$$n(n-1)\,\mathrm{tr}(R_h \wedge R_h) \wedge \omega^{n-2} = \left[|R_h|^2 - |K_h|^2\right]\omega^n\,, \tag{3.59}$$

where $|K_h|$ represents the *pointwise norm* of the (classical) mean curvature K_h. In other words, this is the norm defined by $|K_h|^2 = \mathrm{tr}\, K_h^2$. The other terms are given by $\sigma_h = \mathrm{tr}\, K_h$ and

$$|R_h|^2 = \sum_{i,j,\alpha,\beta} |R^i_{j\alpha\bar{\beta}}|^2\,, \qquad |\rho_h|^2 = \sum_{i,\alpha,\beta} |R^i_{i\alpha\bar{\beta}}|^2\,, \tag{3.60}$$

where $R^i_{j\alpha\bar{\beta}}$ are the components of the Chern curvature R_h. Hence, integrating over X we can rewrite the right hand side of (3.57) in the following form

$$\int_X (2r\, c_2(\mathfrak{E}, h) - (r-1)\, c_1(\mathfrak{E}, h)^2) \wedge \frac{\omega^{n-2}}{(n-2)!} = \int_X \left[r(|R_h|^2 - |K_h|^2) + \sigma_h^2 - |\rho_h|^2\right]\frac{\omega^n}{n!}\,.$$

On the other hand, since in general we have the inequality $r|R_h|^2 \geq |\rho_h|^2$, we obtain

$$\int_X (2r\, c_2(\mathfrak{E}, h) - (r-1)\, c_1(\mathfrak{E}, h)^2) \wedge \frac{\omega^{n-2}}{(n-2)!} \geq \int_X \left[\sigma_h^2 - r\,|K_h|^2\right]\frac{\omega^n}{n!}\,. \tag{3.61}$$

Since h is an Hermitian-Yang-Mills structure, $\mathcal{K}_h = cI$ for some constant c and one has

$$\sigma_h = \mathrm{tr}\, K_h = \mathrm{tr}(\mathcal{K}_h - i\Lambda[\phi, \bar{\phi}_h]) = \mathrm{tr}\,\mathcal{K}_h = cr \tag{3.62}$$

and

$$
\begin{aligned}
|K_h|^2 &= \operatorname{tr}\left[(\mathcal{K}_h - i\Lambda[\phi,\bar\phi_h])\cdot(\mathcal{K}_h - i\Lambda[\phi,\bar\phi_h])\right] \\
&= \operatorname{tr}\left[\mathcal{K}_h^2\right] - 2\operatorname{tr}\left[\mathcal{K}_h\cdot i\Lambda[\phi,\bar\phi_h]\right] + \operatorname{tr}\left[i\Lambda[\phi,\bar\phi_h]\cdot i\Lambda[\phi,\bar\phi_h]\right] \\
&= c^2 r - 2c\, i\Lambda\operatorname{tr}[\phi,\bar\phi_h] + (i\Lambda)^2\operatorname{tr}\left[[\phi,\bar\phi_h]^2\right] .
\end{aligned}
$$

We know that $\operatorname{tr}[\phi,\bar\phi_h]=0$, and hence we have only a contribution involving a trace of the Higgs fields. Now, since $\phi\wedge\phi = 0 = \bar\phi_h\wedge\bar\phi_h$ we get

$$
\begin{aligned}
\operatorname{tr}\left[[\phi,\bar\phi_h]^2\right] &= \operatorname{tr}\left[(\phi\wedge\bar\phi_h + \bar\phi_h\wedge\phi)\wedge(\phi\wedge\bar\phi_h + \bar\phi_h\wedge\phi)\right] \\
&= \operatorname{tr}\left[\phi\wedge\bar\phi_h\wedge\phi\wedge\bar\phi_h + \bar\phi_h\wedge\phi\wedge\bar\phi_h\wedge\phi\right] \\
&= 0\,,
\end{aligned}
$$

where we have used again the cyclic property of the trace. Therefore $|K_h|^2 = c^2 r$. This equation together with (3.62) implies that

$$
\sigma_h^2 - r\,|K_h|^2 = 0 \tag{3.63}
$$

if h is an Hermitian-Yang-Mills structure. Using this the inequality (3.61) becomes

$$
\int_X (2r\,c_2(\mathfrak{E},h) - (r-1)\,c_1(\mathfrak{E},h)^2)\wedge\frac{\omega^{n-2}}{(n-2)!}\geq 0\,. \tag{3.64}
$$

Since X is compact Kähler, the integral on the left-hand side is independent of the metric h and coincides with the left-hand side term in the inequality (3.55). Hence the result follows. Q.E.D.

In the first part of the proof of the Bogomolov-Lübke inequality we use some standard formulas. We can use some of them to establish results for Higgs bundles with Hermitian metrics. In fact, from (3.56) we get

$$
2\,c_2(\mathfrak{E},h) - \,c_1(\mathfrak{E},h)^2 = \frac{1}{4\pi^2}\operatorname{tr}(R_h\wedge R_h)\,. \tag{3.65}
$$

On the other hand, if we denote by R_h^0 the trace-free part of the Chern curvature, i.e., R_h^0 is defined by the formula

$$
R_h^0 = R_h - \frac{\operatorname{tr} R_h}{r}\,I\,, \tag{3.66}
$$

where I is the identity endomorphism of E, we can rewrite the identity (3.57) as follows:

$$
2r\,c_2(\mathfrak{E},h) - (r-1)\,c_1(\mathfrak{E},h)^2 = \frac{r}{4\pi^2}\operatorname{tr}(R_h^0\wedge R_h^0)\,. \tag{3.67}
$$

At this point, applying (3.59) for R_h and its trace-free part R_h^0, on (3.65) and (3.67) respectively and then integrating over X, we conclude the following

Proposition 3.15. *Let $\mathfrak{E}$ be a Higgs bundle over a compact Kähler manifold X and let h be an Hermitian structure on it, then*

$$
\left[2\,c_2(\mathfrak{E}) - c_1(\mathfrak{E})^2\right]\cup[\omega]^{n-2} = \frac{1}{4\pi^2 n(n-1)}\int_X\left[|R_h|^2 - |K_h|^2\right]\omega^n\,, \tag{3.68}
$$

$$
\left[2r\,c_2(\mathfrak{E}) - (r-1)c_1(\mathfrak{E})^2\right]\cup[\omega]^{n-2} = \frac{r}{4\pi^2 n(n-1)}\int_X\left[|R_h^0|^2 - |K_h^0|^2\right]\omega^n\,. \tag{3.69}
$$

Notice that in the above proposition h is any Hermitian metric on $\mathfrak{E}$, i.e., it is not necessarily Hermitian-Yang-Mills. Therefore, h is the same thing as an Hermitian metric on the corresponding holomorphic bundle E, and hence the above result is in fact a property of holomorphic bundles. We write this proposition in terms of Higgs bundles, because it will be important latter on, when we study the notion of admissible Hermitian metric for Higgs sheaves (see [14], [24] for details).

3.5 Approximate Hermitian-Yang-Mills structures

As we have seen in the preceding section, if we have an Hermitian-Yang-Mills structure with factor c, this constant can be evaluated directly from (3.47) and we have

$$c = \frac{2\pi\,\mu(\mathfrak{E})}{(n-1)!\,\mathrm{vol}\,X}\,. \tag{3.70}$$

On the other hand, regardless if we have an Hermitian-Yang-Mills structure or not on $\mathfrak{E}$, we can always define a constant c just by (3.70). Introduced in such a way, c depends on $c_1(\mathfrak{E})$ and the cohomology class of ω, but not on h.

Let consider now the endomorphism $\mathcal{K} - cI$, as we said before the (pointwise) norm of this endomorphim is given by the formula

$$|\mathcal{K} - cI|^2 = \mathrm{tr}\left[(\mathcal{K} - cI)^2\right]\,. \tag{3.71}$$

We say that a Higgs bundle $\mathfrak{E}$ over a compact Kähler manifold X admits an *approximate Hermitian-Yang-Mills structure* or an *ϵ-Hermitian-Yang-Mills metric*, if for any $\epsilon > 0$ there exists a metric h_ϵ such that

$$\max_X |\mathcal{K}_\epsilon - cI| < \epsilon\,, \tag{3.72}$$

where $\mathcal{K}_\epsilon$ is an abbreviation of the mean curvature $\mathcal{K}_{h_\epsilon}$. From the above definition it follows that $\mathfrak{E}^\vee$ admits an approximate Hermitian-Yang-Mills structure if $\mathfrak{E}$ does. This notion also satisfies some simple properties with respect to tensor products and direct sums.

Proposition 3.16. *If $\mathfrak{E}_1$ and $\mathfrak{E}_2$ admit approximate Hermitian-Yang-Mills structures, so does their tensor product $\mathfrak{E}_1 \otimes \mathfrak{E}_2$. Furthermore, if $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$, so does their Whitney sum $\mathfrak{E}_1 \oplus \mathfrak{E}_2$*

Proof: Assume that $\mathfrak{E}_1$ and $\mathfrak{E}_2$ admit approximate Hermitian-Yang-Mills structures with factors c_1 and c_2 respectively and let $\epsilon > 0$. Then, there exist h_1 and h_2 such that

$$\max_X |\mathcal{K}_1 - c_1 I_1| < \frac{\epsilon}{2\sqrt{r_2}}\,, \qquad \max_X |\mathcal{K}_2 - c_2 I_2| < \frac{\epsilon}{2\sqrt{r_1}}\,,$$

where r_1, r_2 and I_1, I_2 are the ranks and the identity endomorphisms of $\mathfrak{E}_1$ and $\mathfrak{E}_2$ respectively. Now, let $\mathcal{K}$ be the Hitchin-Simpson mean curvature of $\mathfrak{E}_1 \otimes \mathfrak{E}_2$

associated with the metric $h = h_1 \otimes h_2$. Then, by defining $c = c_1 + c_2$ and $I = I_1 \otimes I_2$ it follows

$$\begin{aligned} |\mathcal{K} - cI| &= |\mathcal{K}_1 \otimes I_2 + I_1 \otimes \mathcal{K}_2 - (c_1 + c_2) I_I \otimes I_2| \\ &\leq |(\mathcal{K}_1 - c_1 I_1) \otimes I_2| + |I_1 \otimes (\mathcal{K}_2 - c_2 I_2)| \\ &\leq \sqrt{r_2}\, |\mathcal{K}_1 - c_1 I_1| + \sqrt{r_1}\, |\mathcal{K}_2 - c_2 I_2| \\ &< \epsilon \end{aligned}$$

and hence the tensor product $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ admits an approximate Hermitian-Yang-Mills structure.

On the other hand, if $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$, necesarily the constants c_1 and c_2 coincide. Then, taking this time $c = c_1 = c_2$, $I = I_1 \oplus I_2$ and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, we have

$$\begin{aligned} |\mathcal{K} - cI| &= |\mathcal{K}_1 \oplus \mathcal{K}_2 - c\, I_1 \oplus I_2| \\ &= \sqrt{\mathrm{tr}\,(\mathcal{K}_1 - c_1 I_1)^2 + \mathrm{tr}\,(\mathcal{K}_2 - c_2 I_2)^2} \\ &\leq |\mathcal{K}_1 - c_1 I_1| + |\mathcal{K}_2 - c_2 I_2|\,. \end{aligned}$$

From this inequality it follows that $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ admits an approximate Hermitian-Yang-Mills structure. Q.E.D.

Corollary 3.17. *If $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills structure, so do the tensor product bundle $\mathfrak{E}^{\otimes p} \otimes \mathfrak{E}^{\vee \otimes q}$ and the exterior product bundle $\bigwedge^p \mathfrak{E}$ whenever $p \leq r$.*

Using some vanishing theorems we have a first result about approximate Hermitian-Yang-Mills structures. Namely

Corollary 3.18. *Let $\mathfrak{E}$ be a Higgs bundle over a compact Kähler manifold X which admits an approximate Hermitian-Yang-Mills structure. If $\deg \mathfrak{E} < 0$, then $\mathfrak{E}$ has no nonzero ϕ-invariant sections.*

Proof: Here we reproduce with some more detail the proof written in [21]. Suppose $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills structure, then for each ϵ there exists a metric h_ϵ on $\mathfrak{E}$ such that

$$\max_X |\mathcal{K}_\epsilon - cI| < \epsilon\,,$$

where $\mathcal{K}_\epsilon$ represents here the Hitchin-Simpson mean curvature associated to h_ϵ. This implies that the mean curvature, seen as an Hermitian form, must satisfy the following inequality

$$-\epsilon\, h_\epsilon < \mathcal{K}_\epsilon - c\, h_\epsilon < \epsilon\, h_\epsilon\,.$$

If we assume that $\deg \mathfrak{E} < 0$, then $c < 0$ in the above inequality and hence for some sufficient small ϵ the mean curvature $\mathcal{K}_\epsilon$ is negative definite. At this point the result follows from the vanishing Theorem 3.2. Q.E.D.

From the above we have the following

Proposition 3.19. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be two Higgs bundles over a compact Kähler manifold X admiting approximate Hermitian-Yang-Mills structures. Then there are no nonzero Higgs morphisms from $\mathfrak{E}_1$ to $\mathfrak{E}_2$ if $\mu(\mathfrak{E}_1) > \mu(\mathfrak{E}_2)$.*

Proof: Suppose $\mathfrak{E}_1$ and $\mathfrak{E}_2$ admit both approximate Hermitian-Yang-Mills structures, then from Proposition 3.16 we know that there exists an approximate Hermitian-Yang-Mills structure naturally induced on $\mathfrak{E}_1^\vee \otimes \mathfrak{E}_2$. If additionally $\mu(\mathfrak{E}_1) > \mu(\mathfrak{E}_2)$, then we have

$$\deg(\mathfrak{E}_1^\vee \otimes \mathfrak{E}_2) < 0\,.$$

At this point, the result is a straightforward consequence of Corollary 3.18. Q.E.D.

Proposition 3.19 should be compared with the first part of Proposition 2.8, which is in fact the corresponding result on semistability for Higgs sheaves.

Finally, in a similar way as in the classical case, we have a version of the Bogomolov-Lübke inequality (3.55) also for Higgs bundles admiting an approximate Hermitian-Yang-Mills structure (see [5] and [45] for more details). To be precise, we have the following result

Theorem 3.20. *Let $\mathfrak{E}$ be a Higgs bundle over a compact Kähler manifold X and suppose that $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills structure, then*

$$\left[2r\, c_2(\mathfrak{E}) - (r-1)c_1(\mathfrak{E})^2\right] \cup [\omega]^{n-2} \geq 0\,. \tag{3.73}$$

Proof: Assume that $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills structure. Let $\epsilon > 0$ and suppose h_ϵ is a metric on $\mathfrak{E}$ satisfying (3.72). Then, we have closed $2k$-forms $c_k(\mathfrak{E}, h_\epsilon)$ representing the k-th Chern classes and, in a similar way that in the proof of Theorem 3.14, we obtain

$$(2r\, c_2(\mathfrak{E}, h_\epsilon) - (r-1)\, c_1(\mathfrak{E}, h_\epsilon)^2) \wedge \frac{\omega^{n-2}}{(n-2)!} = \left[r(|R_\epsilon|^2 - |K_\epsilon|^2) + \sigma_\epsilon^2 - |\rho_\epsilon|^2\right] \frac{\omega^n}{n!}$$

where this time the quantities on the right-hand side are associated to the metric h_ϵ. That is, $|K_\epsilon|^2 = \operatorname{tr} K_\epsilon^2$ and $\sigma_\epsilon = \operatorname{tr} K_\epsilon$, and using the formulas defined in (3.60) we obtain the corresponding expressions for $|R_\epsilon|^2$ and $|\rho_\epsilon|^2$. Now, again one has $r|R_\epsilon|^2 \geq |\rho_\epsilon|^2$, and hence integrating over X we obtain

$$\int_X (2r\, c_2(\mathfrak{E}, h_\epsilon) - (r-1)\, c_1(\mathfrak{E}, h_\epsilon)^2) \wedge \frac{\omega^{n-2}}{(n-2)!} \geq \int_X \left[\sigma_\epsilon^2 - r\,|K_\epsilon|^2\right] \frac{\omega^n}{n!}\,. \tag{3.74}$$

Since h_ϵ is an approximate Hermitian-Yang-Mills structure, we have

$$\epsilon^2 > |\mathcal{K}_\epsilon - cI|^2 = |\mathcal{K}_\epsilon|^2 - 2c\,\sigma_\epsilon + c^2 r\,. \tag{3.75}$$

On the other hand,

$$\begin{aligned} |\mathcal{K}_\epsilon|^2 &= \operatorname{tr}\left[(K_\epsilon + i\Lambda[\phi, \bar{\phi}_\epsilon]) \cdot (K_\epsilon + i\Lambda[\phi, \bar{\phi}_\epsilon])\right] \\ &= |K_\epsilon|^2 + 2\,i\Lambda \operatorname{tr}\left[K_\epsilon \cdot [\phi, \bar{\phi}_\epsilon]\right] + (i\Lambda)^2 \operatorname{tr}\left[[\phi, \bar{\phi}_\epsilon]^2\right] \\ &= |K_\epsilon|^2 + 2\,i\Lambda \operatorname{tr}\left[\mathcal{K}_\epsilon \cdot [\phi, \bar{\phi}_\epsilon]\right]\,. \end{aligned}$$

Now, since h_ϵ is an approximate Hermitian-Yang-Mills structure, $\mathcal{K}_\epsilon = cI + \epsilon\, A$ with A a selfadjoint endomorphism of E and hence we can estimate the term involving the trace in the last expression as

$$\operatorname{tr}\left[\mathcal{K}_\epsilon \cdot [\phi, \bar{\phi}_\epsilon]\right] = c \operatorname{tr}[\phi, \bar{\phi}_\epsilon] + \epsilon \operatorname{tr}\left[A \cdot [\phi, \bar{\phi}_\epsilon]\right] = \epsilon\, \eta \tag{3.76}$$

where the $(1,1)$-form $\eta = \operatorname{tr}\left[A \cdot [\phi, \bar{\phi}_\epsilon]\right]$. Consequently

$$|\mathcal{K}_\epsilon|^2 = |K_\epsilon|^2 + 2\epsilon\,(i\Lambda\eta)\,. \tag{3.77}$$

Finally, from (3.75) and (3.77) it follows

$$\sigma_\epsilon^2 - r\,|K_\epsilon|^2 > (\sigma_\epsilon - cr)^2 + f(\epsilon)$$

where $f(\epsilon) = r\epsilon(2\,(i\Lambda\eta) - \epsilon)$. Then, by replacing this last expression in (3.74) we conclude

$$\int_X (2r\, c_2(\mathfrak{E}, h_\epsilon) - (r-1)\, c_1(\mathfrak{E}, h_\epsilon)^2) \wedge \frac{\omega^{n-2}}{(n-2)!} > \int_X f(\epsilon)\, \frac{\omega^n}{n!}\,. \tag{3.78}$$

Now, the integral on the left-hand side is independent of the metric h_ϵ. On the other hand, the above inequality holds for all $\epsilon > 0$ and clearly $f(\epsilon) \to 0$ as $\epsilon \to 0$. Therefore, one has the inequality (3.73) if $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills metric. Q.E.D.

According to Bogomolov [29] and Gieseker [30], the inequality (3.55) holds for any holomorphic semistable bundle over an algebraic surface. In fact, because of that Kobayashi proposed in [5] the notion of approximate Hermitian-Yang-Mills structure as the differential-geometric counterpart of the notion of semistability. This seems to be the case also for Higgs bundles, we will discuss more about this equivalence in the context of Higgs bundles in future sections.

Finally, we have the following natural extension to Higgs bundles of a classical result concerning coverings of compact Kähler manifolds. Its proof is identical to the ordinary case (see [5] for more details).

Proposition 3.21. *Let $\pi : \tilde{X} \longrightarrow X$ be a finite unramified covering of a compact Kähler manifold X with Kähler form ω and let $\pi^*\omega$ be the Kähler form of $\tilde{X}$. Then*
(i) *If a Higgs bundle $\mathfrak{E}$ over X admits and approximate Hermitian-Yang-Mills structure, then so does its pullback $\pi^*\mathfrak{E}$ over $\tilde{X}$.*
(ii) *If a Higgs bundle $\tilde{\mathfrak{E}}$ over $\tilde{X}$ admits and approximate Hermitian-Yang-Mills structure, then so does its pushforward $\pi_*\tilde{\mathfrak{E}}$ to X.*

CHAPTER 4

The Donaldson functional

4.1 Donaldson's functional and secondary characteristic classes

We want to construct a functional $\mathcal{L}$ on $\mathrm{Herm}^+(\mathfrak{E})$ whose gradient will be related with the mean curvature of the Hitchin-Simpson connection. Such a functional has been introduced by Donaldson [50],[52] in his study of stable holomorphic bundles over projective algebraic manifolds. In [5], Kobayashi constructed this functional in detail for holomorphic bundles over compact Kähler manifolds and called it the Donaldson functional. Following the ideas of Donaldson, Simpson [17] introduced an analog functional for Higgs bundles and used it to prove the Hitchin-Kobayashi correspondence in the Higgs case.

Here, we introduce this functional in the Higgs case following the construction of Kobayashi, we will see that such a functional coincides (up to a constant) with the functional introduced by Simpson. The construction will be similar to the ordinary case. However, there will be some differences, which in essence are due to the extra terms involving the Higgs field ϕ in the Hitchin-Simpson curvature.

Given two Hermitian structures h, k in $\mathrm{Herm}^+(\mathfrak{E})$, we connect them by a curve h_t, $0 \leq t \leq 1$, in $\mathrm{Herm}^+(\mathfrak{E})$ so that $k = h_0$ and $h = h_1$. We set

$$Q_1(h,k) = \log(\det(k^{-1}h))\,, \qquad Q_2(h,k) = i\int_0^1 \operatorname{tr}(v_t \cdot \mathcal{R}_t)\,dt\,, \tag{4.1}$$

where $v_t = h_t^{-1}\partial_t h_t$ is the endomorphism associated with $\partial_t h_t \in \mathrm{Herm}(\mathfrak{E})$ at the point h_t and $\mathcal{R}_t$ denotes the curvature of the Hitchin-Simpson connection associated with h_t. The functionals $Q_1(h,k)$ and $Q_2(h,k)$ are usually called secondary characteristic classes (see [13] for more details).

Notice that $Q_1(h,k)$ does not involve the curve (in fact, it is the same functional of the ordinary case). On the other hand, the definition of $Q_2(h,k)$ involves the curve explicitly and differs from the ordinary case because of the extra terms in (3.5). We define the Donaldson functional by

$$\mathcal{L}(h,k) = \int_X \left[Q_2(h,k) - \frac{c}{n}Q_1(h,k)\,\omega\right] \wedge \omega^{n-1}/(n-1)!\,, \tag{4.2}$$

where c is the constant given by

$$c = \frac{2\pi\,\mu(\mathfrak{E})}{(n-1)!\operatorname{vol} X}\,. \tag{4.3}$$

Notice that the components of $(2,0)$ and $(0,2)$ type of $\mathcal{R}_t$ do not contribute to the functional $\mathcal{L}(h,k)$. This means that, in practice, it is enough to consider in the definition of $Q_2(h,k)$ just the components of $(1,1)$-type. To be more precise, in computations involving integration over X, we can always replace the curvature by

$$\mathcal{R}_t^{1,1} = R_t + [\phi, \bar{\phi}_{h_t}]\,. \tag{4.4}$$

The following Lemma and the subsequent Proposition are straightforward generalizations of a result of Kobayashi (see [5], Ch.VI, Lemma 3.6) to the Higgs case. Part of the proof of this Lemma is similar to the proof presented in [5], however some differences arise because of the term involving the commutator in the Hitchin-Simpson curvature.

Lemma 4.1. *Let $h_t, a \leq t \leq b$, be any differentiable curve in* $\mathrm{Herm}^+(\mathfrak{E})$ *and k any fixed Hermitian structure of $\mathfrak{E}$. Then, the (1,1)-component of*

$$i\int_a^b \mathrm{tr}(v_t \cdot \mathcal{R}_t)\,dt + Q_2(h_a,k) - Q_2(h_b,k) \tag{4.5}$$

is an element in $d'A^{0,1} + d''A^{1,0}$.

Proof: In a similar form as is shown by Kobayashi in [5], we consider the domain Δ in $\mathbb{R}^2$ defined by

$$\Delta = \{(t,s)/a \leq t \leq b\,, 0 \leq s \leq 1\} \tag{4.6}$$

and let $h : \Delta \to \mathrm{Herm}^+(\mathfrak{E})$ be a smooth mapping such that

$$h(t,0) = k\,, \qquad h(t,1) = h_t\,, \quad \text{for} \quad a \leq t \leq b\,. \tag{4.7}$$

Let $h(a,s)$ and $h(b,s)$ line segments curves[1] from k to h_a and respectively from k to h_b. We define the endomorphisms $u = h^{-1}\partial_s h$, and $v = h^{-1}\partial_t h$ and we put

$$\mathcal{R} = d''(h^{-1}d'h) + [\phi, \bar{\phi}_h]\,, \qquad \Psi = i\,\mathrm{tr}[h^{-1}\tilde{d}h\mathcal{R}]\,, \tag{4.8}$$

where $\tilde{d}h = \partial_s h\,ds + \partial_t h\,dt$ is considered as the exterior differential of h in the domain Δ. It is convenient to rewrite Ψ in the form

$$\Psi = i\,\mathrm{tr}[(u\,ds + v\,dt)\mathcal{R}]\,. \tag{4.9}$$

Applying the Stokes formula to Ψ (which is considered here as a 1-form in the domain Δ) we get

$$\int_\Delta \tilde{d}\Psi = \int_{\partial\Delta} \Psi\,. \tag{4.10}$$

The right hand side of the above expression can be computed straightforward from definition. In fact, after a short computation we obtain

$$\int_{\partial\Delta} \Psi = i\int_a^b \mathrm{tr}(v_t \cdot \mathcal{R}_t^{1,1})\,dt + Q_2^{1,1}(h_a,k) - Q_2^{1,1}(h_b,k)\,. \tag{4.11}$$

[1]Notice that in general a line segment curve from k to h_t can be written as $h(t,s) = sh_t + (1-s)k$.

Therefore, we need to show that the left hand side of (4.10) is an element in $d'A^{0,1} + d''A^{1,0}$, and hence, it suffices to show that $\tilde{d}\Psi \in d'A^{0,1} + d''A^{1,0}$.

Now, from the definition of Ψ we have

$$\begin{aligned}\tilde{d}\Psi &= i\,\mathrm{tr}[\tilde{d}(u\,ds + v\,dt)\,\mathcal{R} - (u\,ds + v\,dt)\tilde{d}\mathcal{R}] \\ &= i\,\mathrm{tr}\,[(\partial_s v - \partial_t u)\mathcal{R} - u\,\partial_t\mathcal{R} + v\,\partial_s\mathcal{R}]\,ds \wedge dt\,.\end{aligned}$$

On the other hand, a simple computation shows that

$$\partial_t u = -vu + h^{-1}\partial_t\partial_s h\,, \qquad \partial_s v = -uv + h^{-1}\partial_s\partial_t h\,, \tag{4.12}$$

$$\partial_t\mathcal{R} = d''D'v + [\phi, \partial_t\bar{\phi}_h]\,, \qquad \partial_s\mathcal{R} = d''D'u + [\phi, \partial_s\bar{\phi}_h]\,. \tag{4.13}$$

Replacing these expressions in the formula for $\tilde{d}\Psi$ and writing $\mathcal{R} = R + [\phi, \bar{\phi}_h]$ we obtain

$$\begin{aligned}\tilde{d}\Psi &= i\,\mathrm{tr}\left[(vu - uv)R - u\,d''D'v + v\,d''D'u\right] ds \wedge dt \\ &\quad + i\,\mathrm{tr}\left[v\,[\phi, \partial_s\bar{\phi}_h] - u\,[\phi, \partial_t\bar{\phi}_h] + (vu - uv)[\phi, \bar{\phi}_h]\right] ds \wedge dt\,.\end{aligned}$$

The first trace in the expression above does not depend on Higgs field ϕ (in fact, it is the same expression that is found in [5] for the ordinary case). The second trace is identically zero. In order to prove this, we need first to find explicit expressions for $\partial_t\bar{\phi}_h$ and $\partial_s\bar{\phi}_h$.

On the other hand, omitting the parameter t for simplicity, from the identity (3.13) it follows that

$$\bar{\phi}_{h_{s+\delta s}} = u_0^{-1}\bar{\phi}_{h_s}u_0 = \bar{\phi}_{h_s} + u_0^{-1}[\bar{\phi}_{h_s}, u_0] \tag{4.14}$$

where u_0 is the selfadjoint endomorphism such that $h_{s+\delta s} = h_s u_0$. In other words, it is the endomorphism defined by (3.16) with $h_{s+\delta s} \in \mathrm{Herm}(\mathfrak{E})$ and $h_s \in \mathrm{Herm}^+(\mathfrak{E})$. Now, in general we can write

$$h_{s+\delta s} = h_s + \partial_s h_s \cdot \delta s + \mathcal{O}(\delta s^2) \tag{4.15}$$

and hence, at first order in δs, we obtain $u_0 = 1 + u \cdot \delta s$ and consequently $\partial_s\bar{\phi}_h = [\bar{\phi}_h, u]$. In a similar way we obtain the formula $\partial_t\bar{\phi}_h = [\bar{\phi}_h, v]$. Therefore, using these relations, the Jacobi identity and the cyclic property of the trace, we see that

$$\mathrm{tr}\left[v\,[\phi, \partial_s\bar{\phi}_h] - u\,[\phi, \partial_t\bar{\phi}_h]\right] = \mathrm{tr}\left[(uv - vu)[\phi, \bar{\phi}_h]\right]$$

and hence the second trace in the last expression for $\tilde{d}\Psi$ is identically zero. On the other hand, the term involving the curvature R can be rewritten in terms of u, v and their covariant derivatives. So, finally we get

$$\tilde{d}\Psi = -i\,\mathrm{tr}[v\,D'd''u + u\,d''D'v]\,ds \wedge dt\,. \tag{4.16}$$

As is shown in [5], defining the (0,1)-form $\alpha = i\,\mathrm{tr}[v\,d''u]$ we obtain

$$\tilde{d}\Psi = -[d'\alpha + d''\bar{\alpha} + i\,d''d'\mathrm{tr}(vu)]\,ds \wedge dt \tag{4.17}$$

and hence $\tilde{d}\Psi$ is an element in $d'A^{0,1} + d''A^{1,0}$. Q.E.D.

As a consequence of Lemma 4.1 we get an important result for piecewise differentiable closed curves. Namely, we have

Proposition 4.2. *Let h_t, $\alpha \le t \le \beta$, be a piecewise differentiable closed curve in* $\mathrm{Herm}^+(\mathfrak{E})$*. Then*

$$i \int_\alpha^\beta \mathrm{tr}\left(v_t \cdot \mathcal{R}_t^{1,1}\right) dt = 0 \mod d'A^{0,1} + d''A^{1,0}. \tag{4.18}$$

Proof: Let $\alpha = a_0 < a_1 \cdots < a_p = \beta$ be the values of t where h_t is not differentiable. Now take a fixed point k in $\mathrm{Herm}^+(\mathfrak{E})$. Then, Lemma 4.1 applies for each triple $k, h_{a_j}, h_{a_{j+1}}$ with $j = 0, 1, ..., p-1$ and the result follows. Q.E.D.

4.2 Main properties of the Donaldson functional

From Proposition 4.2 we have some properties of the Donaldson functional. In particular we have the following results

Corollary 4.3. *The Donaldson functional $\mathcal{L}(h,h')$ does not depend on the curve joining h and h'.*

Proof: Clearly, from the definition of Q_1

$$Q_1(h,h') + Q_1(h',h) = 0. \tag{4.19}$$

On the other hand, if γ_1 and γ_2 are two differentiable curves from h to h', then applying Proposition 4.2 to $\gamma_1 - \gamma_2$ we obtain

$$Q_2^{1,1}(h,h') + Q_2^{1,1}(h',h) = 0 \mod d'A^{0,1} + d''A^{1,0}, \tag{4.20}$$

and the result follows integrating over X the identities (4.19) and (4.20). Q.E.D.

Proposition 4.4. *For any metric h in* $\mathrm{Herm}^+(\mathfrak{E})$ *and any constant $a > 0$, the Donaldson functional satisfies $\mathcal{L}(h, ah) = 0$.*

Proof: Clearly

$$Q_1(h, ah) = \log\det[(ah)^{-1}h] = -r \log a.$$

Now, let $b = \log a$ and consider the curve $h_t = e^{b(1-t)}h$ from ah to h. For this curve $v_t = -bI$ and we have

$$\mathcal{R}_t^{1,1} = d''(h_t^{-1}d'h_t) + [\phi, \bar{\phi}_t] = d''(h^{-1}d'h) + [\phi, \bar{\phi}_t],$$

where $\bar{\phi}_t$ is an abbreviation for $\bar{\phi}_{h_t}$. Therefore, the (1,1)-component of $Q_2(h, ah)$ becomes

$$Q_2^{1,1}(h, ah) = i\int_0^1 \mathrm{tr}(v_t \cdot \mathcal{R}_t^{1,1})\, dt = i\int_0^1 \mathrm{tr}\left[-b(R + [\phi, \bar{\phi}_t])\right] dt = -ib\,\mathrm{tr}\, R$$

and hence, from the above we obtain

$$\frac{c}{n}\int_X Q_1(h,ah)\,\omega\wedge\omega^{n-1}/(n-1)! = -crb\,\mathrm{vol}\,X\,,$$

$$\begin{aligned}\int_X Q_2(h,ah)\wedge\omega^{n-1}/(n-1)! &= \frac{-ib}{(n-1)!}\int_X \mathrm{tr}\,R\wedge\omega^{n-1}\\ &= \frac{-2\pi b}{(n-1)!}\deg\mathfrak{E}\end{aligned}$$

and the result follows from the definition of the constant c. Q.E.D.

Lemma 4.5. *For any differentiable curve h_t and any fixed point k in* $\mathrm{Herm}^+(\mathfrak{E})$ *we have*

$$\partial_t Q_1(h_t,k) = \mathrm{tr}(v_t)\,, \tag{4.21}$$

$$\partial_t Q_2^{1,1}(h_t,k) = i\,\mathrm{tr}(v_t\cdot\mathcal{R}_t^{1,1}) \mod d'A^{0,1}+d''A^{1,0}\,. \tag{4.22}$$

Proof: Since k does not depend on t, we get

$$\begin{aligned}\partial_t Q_1(h_t,k) &= \partial_t\log(\det k^{-1})+\partial_t\log(\det h_t)\\ &= \partial_t\log(\det h_t)\\ &= \mathrm{tr}(v_t)\,.\end{aligned}$$

Considering b in (4.5) as a variable, and differentiating that expression with respect to b, we obtain the formula for $Q_2^{1,1}(h_t,k)$. Q.E.D.

Notice that in the above Lemma h_t is any differentiable curve and k is an arbitrary metric. In other words, for this curve not necessarily $h_0 = k$. Using Lemma 4.5 we have a formula for the derivative with respect to t of Donaldson's functional. Namely

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(h_t,k) &= \int_X\left[i\,\mathrm{tr}(v_t\cdot\mathcal{R}_t^{1,1})-\frac{c}{n}\mathrm{tr}(v_t)\,\omega\right]\wedge\frac{\omega^{n-1}}{(n-1)!}\\ &= \int_X\left[\mathrm{tr}(v_t\cdot\mathcal{K}_t)-c\,\mathrm{tr}(v_t)\right]\frac{\omega^n}{n!}\\ &= \int_X\mathrm{tr}\left[(\mathcal{K}_t-cI)v_t\right]\frac{\omega^n}{n!}\,.\end{aligned}$$

Since $v_t = h_t^{-1}\partial_t h_t$ and we can consider the endomorphism $\mathcal{K}_t$ as an Hermitian form by defining $\mathcal{K}_t(s,s') = h_t(s,\mathcal{K}_t s')$, for any fixed Hermitian metric k and any differentiable curve h_t in $Herm^+(\mathfrak{E})$ we obtain[2]

$$\frac{d}{dt}\mathcal{L}(h_t,k) = (\mathcal{K}_t - c\,h_t, \partial_t h_t)\,, \tag{4.23}$$

[2]Notice that from the definition (3.16), the endomorphism $\mathcal{K}_t$ can be written formally as $\mathcal{K}_t = h_t^{-1}\mathcal{K}_t(\cdot,\cdot)$ where $\mathcal{K}_t(\cdot,\cdot)$ denotes this time the mean curvature as a form. Therefore, we can express the derivative of the functional as an inner product of the forms $\mathcal{K}_t - c\,h_t$ and $\partial_t h_t$ as in (3.17).

where $\mathcal{K}_t$ is consider here as a form. For each t, we can consider $\partial_t h_t \in \mathrm{Herm}(\mathfrak{E})$ as a tangent vector of $\mathrm{Herm}^+(\mathfrak{E})$ at h_t, and hence the differential $d\mathcal{L}$ of the functional evaluated at $\partial_t h_t$ is given by

$$d\mathcal{L}(\partial_t h_t) = \frac{d}{dt}\mathcal{L}(h_t, k)\,. \tag{4.24}$$

Therefore, the gradient of $\mathcal{L}$ (i.e., the vector field on $\mathrm{Herm}^+(\mathfrak{E})$ dual to the form $d\mathcal{L}$ with respect to the invariant Riemannian metric introduced before) is given by $\nabla\mathcal{L} = \mathcal{K} - ch$. From the above analysis we conclude the following

Theorem 4.6. *Let k be a fixed element in* $\mathrm{Herm}^+(\mathfrak{E})$*. Then, h is a critical point of $\mathcal{L}$ if and only if $\mathcal{K} - c\,h = 0$. i.e., if and only if h is an Hermitian-Yang-Mills structure for $\mathfrak{E}$.*

As we said in Chapter 3, in order to derive some properties of $\mathcal{L}$ is convenient to divide the Hichin-Simpson connection in two terms (see [17], [18]) in the form $\mathcal{D}'_h = D'_h + \bar{\phi}_h$ and $\mathcal{D}'' = D'' + \phi$. In fact, using the above decomposition, it is not difficult to show that all critical points of $\mathcal{L}$ correspond to an absolute minimum.

Theorem 4.7. *Let k be a fixed Hermitian structure of $\mathfrak{E}$ and $\tilde{h}$ a critical point of $\mathcal{L}(h,k)$, then the Donaldson functional attains an absolute minimum at $\tilde{h}$.*

Proof: Let $h_t\,, 0 \le t \le 1\,,$ be a differentiable curve such that $h_0 = \tilde{h}$, then we can compute straightforward the second derivative of $\mathcal{L}$

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{L}(h_t,k) &= \frac{d}{dt}\int_X \mathrm{tr}\left[(\mathcal{K}_t - cI)v_t\right]\frac{\omega^n}{n!}\\ &= \int_X \mathrm{tr}\left[\partial_t\mathcal{K}_t\cdot v_t + (\mathcal{K}_t - cI)\partial_t v_t\right]\frac{\omega^n}{n!}\,.\end{aligned}$$

Since h_0 is a critical point of the functional, then $\mathcal{K}_t - cI = 0$ at $t = 0$, and hence

$$\frac{d^2}{dt^2}\mathcal{L}(h_t,k)\big|_{t=0} = \int_X \mathrm{tr}(\partial_t\mathcal{K}_t\cdot v_t)\,\frac{\omega^n}{n!}\big|_{t=0}\,. \tag{4.25}$$

On the other hand, $\partial_t\mathcal{K}_t$ can be written in terms of the endomorphism v_t in the following way

$$\begin{aligned}\mathcal{D}''\mathcal{D}'v_t &= \mathcal{D}''(D'v_t + [\bar{\phi}_t, v_t])\\ &= D''D'v_t + [\phi, D'v_t] + D''[\bar{\phi}_t, v_t] + [\phi,[\bar{\phi}_t, v_t]]\,,\end{aligned}$$

and since $\partial_t\phi_t = [\bar{\phi}_t, v_t]$ we get

$$\partial_t\mathcal{R}^{1,1}_t = \partial_t R_t + [\phi, \partial_t\bar{\phi}_t] = D''D'v_t + [\phi,[\bar{\phi}_t, v_t]]\,. \tag{4.26}$$

Therefore, taking the trace with respect to ω (i.e., applying the $i\Lambda$ operator) we obtain

$$i\Lambda\mathcal{D}''\mathcal{D}'v_t = i\Lambda\partial_t\mathcal{R}^{1,1}_t = \partial_t\mathcal{K}_t\,. \tag{4.27}$$

Hence, replacing this in the expression for the second derivative of $\mathcal{L}$ we find

$$\frac{d^2}{dt^2}\mathcal{L}(h_t,k)\big|_{t=0} = \int_X \mathrm{tr}(i\Lambda\mathcal{D}''\mathcal{D}'v_t\cdot v_t)\frac{\omega^n}{n!}\big|_{t=0} = \|\mathcal{D}'v_t\|^2_{t=0}\,, \tag{4.28}$$

(that is, h_0 must be at least a local minimum of $\mathcal{L}$). Now suppose in addition that h_1 is an arbitrary element in $\mathrm{Herm}^+(\mathfrak{E})$ and joint them by a geodesic h_t, and hence $\partial_t v_t = 0$. Therefore, for a such a geodesic we have

$$\frac{d^2}{dt^2}\mathcal{L}(h_t,k) = \int_X \mathrm{tr}(\partial_t\mathcal{K}_t\cdot v_t)\,\frac{\omega^n}{n!}\,. \tag{4.29}$$

Following the same procedure we have done before, but this time at t arbitrary, we get for $0 \leq t \leq 1$

$$\frac{d^2}{dt^2}\mathcal{L}(h_t,k) = \|\mathcal{D}'v_t\|^2_{h_t} \geq 0 \tag{4.30}$$

(since there is an implicit dependence on t on the right hand side via $\mathcal{D}'$, we write a subscript h_t in the norm) and it follows that $\mathcal{L}(h_0,k) \leq \mathcal{L}(h_1,k)$. Now if we assume that h_1 is also a critical point of $\mathcal{L}$, we necessarily obtain the equality. Therefore, it follows that the minimum defined for any critical point of $\mathcal{L}$ is an absolute minimum. Q.E.D.

Let h_0 be a fixed Hermitian structure, any Hermitian metric h will be of the form h_0e^v for some section v of $\mathrm{End}(E)$ over X. We can join h_0 to h by the geodesic $h_t = h_0e^{tv}$ where $0 \leq t \leq 1$ (note that here $v_t = h_t^{-1}\partial_t h_t = v$ is constant, i.e., it does not depends on t). The associated endomorphism $e^{tv} = h_0^{-1}h_t$ is Hermitian with respect to both h_t and h_0. Now, in the proof of Theorem 4.7, we really got an expression for the second derivative of $\mathcal{L}(h_t,k)$. Namely

$$\frac{d^2}{dt^2}\mathcal{L}(h_t,k) = \int_X \mathrm{tr}\left[\partial_t\mathcal{K}_t\cdot v_t + (\mathcal{K}_t - cI)\partial_t v_t\right]\frac{\omega^n}{n!}\,. \tag{4.31}$$

Since in our case the chosen curve is a geodesic, $\partial_t v_t = 0$, an hence

$$\frac{d^2}{dt^2}\mathcal{L}(h_t,k) = \int_X \mathrm{tr}(\partial_t\mathcal{K}_t\cdot v)\frac{\omega^n}{n!} = \|\mathcal{D}'v\|^2_{h_t}\,. \tag{4.32}$$

Therefore, following [13], the idea is to find a simple expression for $\|\mathcal{D}'v\|^2_{h_t}$ or equivalently for $\|\mathcal{D}''v\|^2_{h_t}$ and to integrate it twice with respect to t. We can do this using local coordinates, in fact, at any point in X we can choose a local frame field so that $h_0 = I$ and $v = \mathrm{diag}(\lambda_1,...,\lambda_r)$. In particular, using such a local frame field we have $h_t^{ij} = e^{-\lambda_j t}\delta_{ij}$, and hence (after a short computation) we obtain

$$\|\mathcal{D}''v\|^2_{h_t} = \int_X \sum_{i,j=1}^{r} |\mathcal{D}''v^i_j|^2 e^{(\lambda_i-\lambda_j)t}\,\frac{\omega^{n-1}}{(n-1)!}\,. \tag{4.33}$$

Now, take in particular $k = h_0$, then at $t = 0$ the functional $\mathcal{L}(h_t,k)$ vanishes. On the other hand, since h_0 is not necessarily an Hermitian-Yang-Mills structure we have

$$\frac{d}{dt}\mathcal{L}(h_t,k)\big|_{t=0} = \int_X \mathrm{tr}\left[(\mathcal{K}_0 - cI)v\right]\frac{\omega^n}{n!}\,. \tag{4.34}$$

Then, integrating twice the expression (4.32) and using the formulas (4.33) and (4.34) we obtain

$$\mathcal{L}(h_t, k) = t \int_X \operatorname{tr}\left[(\mathcal{K}_0 - cI)v\right] \frac{\omega^n}{n!} + \int_X \sum_{i,j=1}^{r} \psi_t(\lambda_i, \lambda_j) |\mathcal{D}'' v^i_j|^2 \frac{\omega^{n-1}}{(n-1)!} \tag{4.35}$$

where ψ_t is a function given by

$$\psi_t(\lambda_i, \lambda_j) = \frac{e^{(\lambda_i - \lambda_j)t} - (\lambda_i - \lambda_j)t - 1}{(\lambda_i - \lambda_j)^2} \,. \tag{4.36}$$

In particular, at $t = 1$ the expression (4.35) corresponds (up to a constant term) to the definition of Donaldson's functional given by Simpson in [17]. Notice also that if the initial metric h_0 is Hermitian-Yang-Mills, the first term of the right hand side of (4.35) vanishes and the functional coincides with the Donaldson functional used by Siu in [13].

4.3 The evolution equation

For the construction of Hermitian-Yang-Mills structures, the standard procedure is to start with a fixed Hermitian metric h_0 and try to find from it an Hermitian metric satisfying $\mathcal{K} = ch$ using a curve h_t, $0 \leq t < \infty$. In other words, we try to find that metric by deforming h_0 through 1-parameter family of Hermitian metrics and we expect that at $t = \infty$, the metric will be Hermitian-Yang-Mills.

At this point we introduce the Laplacian operator

$$\Delta_h = i\Lambda \mathcal{D}'' \mathcal{D}'_h \tag{4.37}$$

which depends on the Kähler metric ω via the adjoint multiplication Λ and also on the metric h. Using this operator, we can rewrite (4.27) as

$$\partial_t \mathcal{K}_t = \Delta_t v_t \,, \tag{4.38}$$

where the subscript t here remember us the dependence of the Laplacian on the metric h_t.

As we said before, to get an Hermitian-Yang-Mills metric we want to make $\mathcal{K} - cI$ vanish. Therefore, a natural choice is to go along the global gradient direction of the functional given by the global L^2-norm of $\mathcal{K}_t - cI$. Now, taking the derivative of this functional we obtain

$$\begin{aligned}
\frac{d}{dt} \|\mathcal{K}_t - cI\|^2 &= \int_X 2 \operatorname{tr}\left(\partial_t \mathcal{K}_t \cdot (\mathcal{K}_t - cI)\right) \frac{\omega^n}{n!} \\
&= 2 \int_X \operatorname{tr}\left(\Delta_t v_t \cdot (\mathcal{K}_t - cI)\right) \frac{\omega^n}{n!} \\
&= 2 \int_X \operatorname{tr}\left(v_t \cdot \Delta_t \mathcal{K}_t\right) \frac{\omega^n}{n!} \,,
\end{aligned}$$

and the equation that naturally emerges (i.e., the associated steepest descent curve) is $v_t = -\Delta_t \mathcal{K}_t$, or equivalently

$$h_t^{-1} \partial_t h_t = -\Delta_t \mathcal{K}_t \,. \tag{4.39}$$

Since $\mathcal{K}_t$ is of degree two, the right hand side of the above equation becomes a term of degree four and we get at the end a non-linear equation of degree four. To do the analysis, it is easier to deal with an equation of lower degree. In fact, this is the main reason for introducing the Donaldson functional. Following the same argument we did before, but this time using the functional $\mathcal{L}(h_t, k)$ with k fixed, in place of the functional $\|\mathcal{K}_t - cI\|^2$, we end up with a non-linear equation of degree two (the heat equation), to be more precise, we obtain directly from (4.23) the equation

$$\partial_t h_t = -(\mathcal{K}_t - c h_t) \,, \tag{4.40}$$

where this time $\mathcal{K}_t$ represents the associated two form, and not an endomorphism[3]. It is well-known that in the ordinary case the evolution equation (4.40) has a solution (see [5], Ch.VI). Now, Simpson [17] has shown that also for the Higgs case, we have always solutions to the above non-linear evolution equation. Indeed he proved this for non-compact Kähler manifolds satisfying some additional conditions. That proof covers the compact Kähler case without any change. Then, from the result of Simpson we know there is a solution to the evolution equation (4.40). To be precise we have the following

Theorem 4.8. *Given an Hermitian structure h_0 of $\mathfrak{E}$, the non-linear evolution equation*

$$\partial_t h_t = -(\mathcal{K}_t - c h_t)$$

has a unique smooth solution defined for all time $0 \le t < \infty$.

In the rest of this section, we study some properties of the solutions of the evolution equation. In particular, we are interested in the study of the mean curvature when the paramater t goes to infinite.

Proposition 4.9. *Let $h_t \,, 0 \le t < \infty$, be a 1-parameter family of* $\mathrm{Herm}^+(\mathfrak{E})$ *satisfying the evolution equation. Then*
(i) *For any fixed Hermitian structure k of $\mathfrak{E}$, the functional $\mathcal{L}(h_t, k)$ is a monotone decreasing function of t; more precisely*

$$\frac{d}{dt} \mathcal{L}(h_t, k) = -\|\mathcal{K}_t - cI\|^2 \le 0 \,; \tag{4.41}$$

(ii) $\max |\mathcal{K}_t - cI|^2$ *is a monotone decreasing function of t;*
(iii) *If $\mathcal{L}(h_t, k)$ is bounded below, i.e., if there exists a real constant A such that $\mathcal{L}(h_t, k) \ge A > -\infty$ for $0 \le t < \infty$, then*

$$\max_X |\mathcal{K}_t - cI|^2 \to 0 \quad as \quad t \to \infty \,. \tag{4.42}$$

[3]Notice that the equivalent equation involving endomorphisms will be $v_t = -(\mathcal{K}_t - cI)$.

Proof: (i) From the proof of Lemma 4.5 we really know that

$$\frac{d}{dt}\mathcal{L}(h_t, k) = (\mathcal{K}_t - c\,h_t, \partial_t h_t)\ . \tag{4.43}$$

Since h_t is a solution of the evolution equation (4.40), by Theorem 4.8 we know that for $0 \leq t < \infty$

$$\frac{d}{dt}\mathcal{L}(h_t, k) = -\,(\mathcal{K}_t - c\,h_t, \mathcal{K}_t - c\,h_t) = -\|\mathcal{K}_t - c\,h_t\|^2 \tag{4.44}$$

and the result follows from the definition of the metric on $\mathrm{Herm}^+(\mathfrak{E})$ (consider this time as a metric for endomorphisms).

The proofs of (ii) and (iii) are similar to the proof in the classical case [5], but we need to work this time with the operator $\Delta_h = i\Lambda\mathcal{D}''\mathcal{D}'_h$ in place of the operator $\Box_h = i\Lambda D'' D'_h$.

(ii) As we have shown before, with this definition $\Delta_t v_t = \partial_t\mathcal{K}_t$. On the other hand, from the evolution equation we have $v_t = -(\mathcal{K}_t - cI)$ and hence we get $\Delta_t v_t = -\Delta_t\mathcal{K}_t$, and we obtain

$$\Delta_t\mathcal{K}_t = -\partial_t\mathcal{K}_t\,. \tag{4.45}$$

Now,

$$\begin{aligned}\mathcal{D}''\mathcal{D}'|\mathcal{K}_t - cI|^2 &= \mathcal{D}''\mathcal{D}'\mathrm{tr}(\mathcal{K}_t - cI)^2 \\ &= 2\,\mathrm{tr}((\mathcal{K}_t - cI)\mathcal{D}''\mathcal{D}'\mathcal{K}_t) + \mathrm{tr}(\mathcal{D}''\mathcal{K}_t \cdot \mathcal{D}'\mathcal{K}_t) + \mathrm{tr}(\mathcal{D}'\mathcal{K}_t \cdot \mathcal{D}''\mathcal{K}_t)\end{aligned}$$

and taking the trace of this with respect to ω (i.e., multiplying by the operator $i\Lambda$) we get

$$\begin{aligned}\Delta_t|\mathcal{K}_t - cI|^2 &= 2\,\mathrm{tr}((\mathcal{K}_t - cI)\Delta_t\mathcal{K}_t) + i\Lambda\,\mathrm{tr}(\mathcal{D}''\mathcal{K}_t \cdot \mathcal{D}'\mathcal{K}_t) + i\Lambda\,\mathrm{tr}(\mathcal{D}'\mathcal{K}_t \cdot \mathcal{D}''\mathcal{K}_t) \\ &= -2\,\mathrm{tr}((\mathcal{K}_t - cI)\partial_t\mathcal{K}_t) - |\mathcal{D}''\mathcal{K}_t|^2 - |\mathcal{D}'\mathcal{K}_t|^2 \\ &= -\partial_t|\mathcal{K}_t - cI|^2 - |\mathcal{D}''\mathcal{K}_t|^2 - |\mathcal{D}'\mathcal{K}_t|^2\,.\end{aligned}$$

So, finally we have

$$(\partial_t + \Delta_t)|\mathcal{K}_t - cI|^2 = -|\mathcal{D}''\mathcal{K}_t|^2 - |\mathcal{D}'\mathcal{K}_t|^2 \leq 0 \tag{4.46}$$

and the result follows from the maximum principle[4] applied to $|\mathcal{K}_t - cI|^2$.

[4] The maximum principle for parabolic equations says that on a Riemannian manifold X, if a function $f : X \times [0, \infty) \to \mathbb{R}$ is of class C^1 with continuous Laplacian and satisfies the inequality $(\partial_t + b\Delta)f \leq 0$, with b a positive constant. Then the function $f(t)$, defined by

$$f(t) = \max_{x\in X} f(x, t)$$

is a monotone decreasing function in t.

(iii) This follows from (ii) and (i) (this proof is an adaptation to the Higgs case of the proof of Kobayashi in [5], see that reference for more details). By integrating the inequality in (i) from 0 to s we obtain

$$\mathcal{L}(h_s, k) - \mathcal{L}(h_0, k) = -\int_0^s \|\mathcal{K}_t - cI\|^2 dt\,.$$

Since $\mathcal{L}(h_s, k)$ is bounded below by a constant independent of s, we have

$$\int_0^s \|\mathcal{K}_t - cI\|^2 dt < \infty\,.$$

In particular, from this we can conclude that

$$\|\mathcal{K}_t - cI\| \to 0 \quad \text{as} \quad t \to \infty\,. \tag{4.47}$$

Let $H(x, y, t)$ be the heat kernel of $\partial_t + \Delta_t$ and for $(x, t) \in X \times [0, \infty)$ define the function

$$f(x, t) = |\mathcal{K}_t - cI|^2. \tag{4.48}$$

Fix $t_0 \in [0, \infty)$ and set

$$u(x, t) = \int_X H(x, y, t - t_0) f(y, t_0)\, dy\,,$$

where $dy = \omega^n/n!$ .Then, $u(x, t)$ is of class C^∞ on $X \times (t_0, \infty)$ and extends to a continuous function on $X \times [t_0, \infty)$. Hence for every $(x, t) \in X \times (t_0, \infty)$ we have

$$(\partial_t + \Delta_t)u(x, t) = 0\,, \quad u(x, t_0) = f(x, t_0)\,.$$

Considering this together with (4.46) we obtain

$$(\partial_t + \Delta_t)(f(x, t) - u(x, t)) \leq 0 \quad \text{for } (x, t) \in X \times (t_0, \infty)\,.$$

Then, by the maximum principle, for $t \geq t_0$ we have

$$\max_{x\in X}(f(x, t) - u(x, t)) \leq \max_{x\in X}(f(x, t_0) - u(x, t_0)) = 0\,,$$

and hence, with $t = t_0 + a$ we get

$$\begin{aligned}\max_{x\in X} f(x, t) &\leq \max_{x\in X} u(x, t_0 + a) = \max_{x\in X} \int_X H(x, y, a) f(y, t_0)\, dy \\ &\leq C_a \int_X f(y, t_0)\, dy = C_a \|\mathcal{K}_{t_0} - cI\|^2\,,\end{aligned}$$

where the constant C_a is given by

$$C_a = \max_{x,y\in X} H(x, y, a)\,.$$

Now, fix for instance $a = 1$ and let $t_0 \to \infty$. Then the result follows from (4.47). Q.E.D.

At this point we have some estimatives involving the mean curvature of the Hitchin-Simpson connection. In fact, in a similar form as we did in the part (ii) of the proof of the preceding proposition, we find

$$\partial_t |\mathcal{K}_t|^2 = 2\,\mathrm{tr}(\partial_t \mathcal{K}_t \cdot \mathcal{K}_t)\,, \tag{4.49}$$

$$\Delta_t |\mathcal{K}_t|^2 = 2\,\mathrm{tr}(\Delta_t \mathcal{K}_t \cdot \mathcal{K}_t) - |\mathcal{D}'' \mathcal{K}_t|^2 - |\mathcal{D}' \mathcal{K}_t|^2 \tag{4.50}$$

and hence, using (4.45) we obtain

$$2\,|\mathcal{K}_t| \cdot \partial_t |\mathcal{K}_t| = \partial_t |\mathcal{K}_t|^2 = -\Delta_t |\mathcal{K}_t|^2 - |\mathcal{D}'' \mathcal{K}_t|^2 - |\mathcal{D}' \mathcal{K}_t|^2\,. \tag{4.51}$$

The last expression can be rewritten[5] as

$$\partial_t |\mathcal{K}_t| = -\Delta_t |\mathcal{K}_t| - \frac{(|\mathcal{D}'' \mathcal{K}_t|^2 + |\mathcal{D}' \mathcal{K}_t|^2)}{2\,|\mathcal{K}_t|}\,. \tag{4.52}$$

Summarizing, from the identity (4.45), the expressions (4.51) and (4.52), we conclude the following

Proposition 4.10. *Let* $h_t\,, 0 \leq t < \infty$, *be a 1-parameter family of* $\mathrm{Herm}^+(\mathfrak{E})$ *satisfying the evolution equation. Then the mean curvature* $\mathcal{K}_t$ *of* h_t *satisfies the following:*

$$(\partial_t + \Delta_t)\mathcal{K}_t = 0\,; \quad (\partial_t + \Delta_t)|\mathcal{K}_t| \leq 0\,; \quad \textit{and} \quad (\partial_t + \Delta_t)|\mathcal{K}_t|^2 \leq 0\,. \tag{4.53}$$

From this proposition we obtain some simple results. In particular, as a consequence of the maximum principle, we know that in Proposition 4.10 the maximum on X of $|\mathcal{K}_t|$ and $|\mathcal{K}_t|^2$ are monotone decreasing functions of t.

From standard properties of the heat equation (see [24], [14] for details) we know the following

$$\int_X |\mathcal{K}_t|(y)\,dy \leq \int_X |\mathcal{K}_0|(y)\,dy\,, \tag{4.54}$$

$$|\mathcal{K}_t|(x) \leq \int_X H(x,y,t)|\mathcal{K}_0|(y)\,dy\,, \tag{4.55}$$

where again $dy = \omega^n/n!$ and $H(x,y,t)$ is the heat kernel for differentiable functions on X. The inequality (4.54) means that $|\mathcal{K}_t|$ is L^1-bounded for $t \geq 0$. The Proposition 4.10 and the formulas (4.54) and (4.55) play an important role in the study of the heat equation for Higgs sheaves.

At this point we introduce the main result of the section. This establishes, in part, a relation between the boundedness property of Donaldson's functional and the existence of approximate Hermitian Yang-Mills structures.

[5]The result is clear if $|\mathcal{K}_t|$ is strictly positive everywhere. However, $|\mathcal{K}_t|$ may eventually vanish at some points. To get rid of this difficulty we can define $|\mathcal{K}|_\epsilon = \sqrt{\mathrm{tr}(\mathcal{K}^2) + \epsilon}$ with $\epsilon > 0$ and since this is nowhere vanishing, the formula holds for this new norm.

Theorem 4.11. *Let $\mathfrak{E}$ be a Higgs bundle over a compact Kähler manifold X with Kähler form ω . Then we have the implications* (i) $\rightarrow$ (ii) $\rightarrow$ (iii) *for the following statements:*
(i) *for any fixed Hermitian structure k in $\mathfrak{E}$, there exist a constant B such that $\mathcal{L}(h,k) \geq B$ for all Hermitian structure h in $\mathfrak{E}$;*
(ii) $\mathfrak{E}$ *admits an approximate Hermitian-Yang-Mills structure, i.e., given $\epsilon > 0$ there exists an Hermitian structure h in $\mathfrak{E}$ such that*

$$\max|\mathcal{K} - cI| < \epsilon\,; \tag{4.56}$$

(iii) $\mathfrak{E}$ *is ω-semistable.*

Proof: Assume (i). Then, in particular there exists a constant A such that

$$\mathcal{L}(h_t, k) \geq A > -\infty$$

for h_t, $0 \leq t < \infty$, a solution of the evolution equation. Thus, from Proposition 4.9 part (iii), it follows that given any $\epsilon > 0$, there exists a t_0 such that[6]

$$\max|\mathcal{K}_t - cI| < \epsilon \quad \text{for} \quad t > t_0\,, \tag{4.57}$$

which shows that (i) implies (ii).

On the other hand, that (ii) implies (iii) has been proved by Bruzzo and Graña-Otero in [21], here we reproduce their proof.

Assume (ii) and let $\mathfrak{F}$ be a proper nontrivial Higgs subsheaf of $\mathfrak{E}$. Then $\operatorname{rk}\mathfrak{E} = p$ for some $0 < p < r$ and the inclusion $\mathfrak{F} \longrightarrow \mathfrak{E}$ induces a morphism $\det\mathfrak{F} \longrightarrow \bigwedge^p \mathfrak{E}$. Tensoring by $(\det\mathfrak{F})^{-1}$ we have a section s of the Higgs bundle

$$\mathfrak{G} = \bigwedge^p \mathfrak{E} \otimes (\det\mathfrak{F})^{-1}\,. \tag{4.58}$$

If ψ represents the Higgs field naturally defined on $\mathfrak{G}$ by the Higgs fields of $\mathfrak{E}$ and $\mathfrak{F}$, then s is ψ-invariant. Now, since by hypothesis $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills structure, then from Proposition 3.16 we know that so does $\mathfrak{G}$ and, in particular, the constant $c_{\mathfrak{G}}$ associated with the Higgs bundle $\mathfrak{G}$ becomes

$$c_{\mathfrak{G}} = \frac{2p\pi(\mu(\mathfrak{E}) - \mu(\mathfrak{F}))}{(n-1)!\operatorname{vol}X}\,. \tag{4.59}$$

From Corollary 3.18 we necessarily obtain $\deg\mathfrak{G} \geq 0$. Therefore $c_{\mathfrak{G}}$ is non-negative and hence $\mathfrak{E}$ must be ω-semistable, which shows (iii). Q.E.D.

Notice that by the Hitchin-Kobayashi correspondence if $\mathfrak{E}$ is ω-stable, it has an Hermitian-Yang-Mills structure and as we have seen before this metric must be a minimum of the Donaldson functional. Hence the for stable Higgs bundles the Donaldson functional is bounded from below and (iii) implies (i).

[6] Notice that indeed given an $\epsilon > 0$, any Hermitian metric $h = h_{t_1}$ with $t_1 > t_0$ satisfies the inequality (4.57).

4.4 Semistable Higgs bundles

This section is in essence a natural extension to Higgs bundles of some classical results on holomorphic vector bundles. Some of them are written in detail in [13],[5] and [40].

As we have shown before, the direct sum of semistable Higgs sheaves with the same slope is semistable. Therefore, this automatically holds for Higgs bundles and hence, as a simple consequence of Proposition 2.7 we have

Proposition 4.12. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be two Higgs bundles over a compact Kähler manifold X with Kähler form ω. If they are both ω-semistable with $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2) = \mu$, then $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ is also ω-semistable and $\mu(\mathfrak{E}_1 \oplus \mathfrak{E}_2) = \mu$.*

Let

$$0 \longrightarrow \mathfrak{E}' \xrightarrow{\iota} \mathfrak{E} \xrightarrow{p} \mathfrak{E}'' \longrightarrow 0 \tag{4.60}$$

be an exact sequence of Higgs bundles. As in the ordinary case, an Hermitian structure h in $\mathfrak{E}$ induces Hermitian structures h' and h'' in $\mathfrak{E}'$ and $\mathfrak{E}''$ respectively. We have also a second fundamental form $A_h \in A^{1,0}(\mathrm{Hom}(E', E''))$ and its adjoint $B_h \in A^{0,1}(\mathrm{Hom}(E'', E'))$ where, as usual, $B_h^* = -A_h$. In a similar way, some properties which holds in the ordinary case, also holds in the Higgs case.

Proposition 4.13. *Given an exact sequence (4.60) and a pair of Hermitian structures h, k in $\mathfrak{E}$. The function $Q_1(h, k)$ and the form $Q_2(h, k)$ satisfies the following properties:*
(i) $Q_1(h, k) = Q_1(h', k') + Q_1(h'', k'')$,
(ii) $Q_2(h, k) = Q_2(h', k') + Q_2(h'', k'') - i\,\mathrm{tr}[B_h \wedge B_h^* - B_k \wedge B_k^*]$
$\mod d'A^{0,1} + d''A^{1,0}$.

Proof: (i) is straightforward from the definition of Q_1. On the other hand, (ii) follows from an analysis similar to the ordinary case.

Since the sequence (4.60) is in particular an exact sequence of holomorphic vector bundles, for any h we have a splitting of the exact sequence by C^∞-homomorphisms (see [5], Ch.I for details) $\mu_h : E \to E'$ and $\lambda_h : E'' \to E$ with

$$B_h = \mu_h \circ d'' \circ \lambda_h \,. \tag{4.61}$$

We consider now a curve of Hermitian structures $h = h_t$, $0 \le t \le 1$ such that $h_0 = k$ and $h_1 = h$. Corresponding to h_t we have a family of homomorphisms μ_t and λ_t. We define the homomorphism $S_t : E'' \to E'$ given by

$$\lambda_t - \lambda_0 = \iota \circ S_t \,. \tag{4.62}$$

Then $\partial_t B_t = d''(\partial_t S_t)$ and choosing convenient orthonormal local frames for $\mathfrak{E}'$ and $\mathfrak{E}''$, the endomorphism v_t can be represented by the matrix

$$v_t = \begin{pmatrix} v_t' & -\partial_t S_t \\ -(\partial_t S_t)^* & v_t'' \end{pmatrix} .$$

Here v'_t, v''_t are the natural endomorphisms associated to h'_t, h''_t respectively. Now, from the ordinary case we have

$$R_t = \begin{pmatrix} R'_t - B_t \wedge B^*_t & D' B_t \\ -D'' B^*_t & R''_t - B^*_t \wedge B_t \end{pmatrix}$$

where R'_t and R''_t are the Chern curvatures of $\mathfrak{E}'$ and $\mathfrak{E}''$ associated to the metrics h'_t and h''_t respectively.

On the other hand, the (1,1)-part of the curvature is given by $\mathcal{R}^{1,1}_t = R_t + [\phi, \phi_t]$ and since $\mathfrak{E}'$ and $\mathfrak{E}''$ are Higgs subbundles of $\mathfrak{E}$, we obtain a simple expression for the (1,1)-component of the Hitchin-Simpson curvature

$$\mathcal{R}^{1,1}_t = \begin{pmatrix} \mathcal{R}'^{1,1}_t - B_t \wedge B^*_t & D' B_t \\ -D'' B^*_t & \mathcal{R}''^{1,1}_t - B^*_t \wedge B_t \end{pmatrix}$$

where

$$\mathcal{R}'^{1,1}_t = R'_t + [\phi, \phi_t]_{E'}\,, \qquad \mathcal{R}''^{1,1}_t = R''_t + [\phi, \phi_t]_{E''}\,.$$

Hence, at this point we can compute the trace

$$\begin{aligned} \operatorname{tr}(v_t \cdot \mathcal{R}^{1,1}_t) &= \operatorname{tr}(v'_t \cdot \mathcal{R}'^{1,1}_t) + \operatorname{tr}(v''_t \cdot \mathcal{R}''^{1,1}_t) \\ &\quad + \operatorname{tr}(\partial_t S_t \cdot D'' B^*_t) - \operatorname{tr}((\partial_t S_t)^* \cdot D' B_t) \\ &\quad + \operatorname{tr}(v'_t \cdot B_t \wedge B^*_t) - \operatorname{tr}(v''_t \cdot B^*_t \wedge B_t)\,. \end{aligned}$$

The last four terms are exactly the same than in the ordinary case. Finally get that, modulo an element in $d'A^{0,1} + d''A^{1,0}$

$$\operatorname{tr}(v_t \cdot \mathcal{R}^{1,1}_t) = \operatorname{tr}(v'_t \cdot \mathcal{R}'^{1,1}_t) + \operatorname{tr}(v''_t \cdot \mathcal{R}''^{1,1}_t) - \partial_t \operatorname{tr}(B_t \wedge B^*_t)\,. \tag{4.63}$$

Then, multiplying by i and integrating from $t = 0$ to $t = 1$ the last expresion we obtain (ii). Q.E.D.

As a consequence of Proposition 4.13 we get an important result for Higgs bundles over compact Kähler manifolds when $\mathfrak{E}$ and $\mathfrak{E}'$ have the same slope. Indeed, in that case using Lemma 2.1 we know that also $\mathfrak{E}''$ has the same slope of $\mathfrak{E}$ and hence the constants c' and c'' of $\mathfrak{E}'$ and $\mathfrak{E}''$ are equal to the constant c of $\mathfrak{E}$. Then, integrating $Q_1(h,k)$ and $Q_2(h,k)$ over X and since

$$-i \operatorname{tr}(B \wedge B^*) \wedge \omega^{n-1} = |B|^2 \omega^n / n! \tag{4.64}$$

we obtain an identity involving the Donadson functionals $\mathcal{L}(h,k)\,, \mathcal{L}(h',k')$ and $\mathcal{L}(h'',k'')$. To be precise, we obtain the following

Corollary 4.14. *Given an exact sequence (4.60) over a compact Kähler manifold X with $\mu(\mathfrak{E}) = \mu(\mathfrak{E}')$ and a pair of Hermitian structures h and k in $\mathfrak{E}$. The functional $\mathcal{L}(h,k)$ satisfies the following relation*

$$\mathcal{L}(h,k) = \mathcal{L}(h',k') + \mathcal{L}(h'',k'') + \|B_h\|^2 - \|B_k\|^2\,. \tag{4.65}$$

Let $\mathfrak{E}$ be a torsion-free Higgs sheaf which is ω-semistable but not ω-stable and let $\mathfrak{E}'$ be a proper Higgs subsheaf with equal slope as $\mathfrak{E}$ and torsion-free quotient, say $\mathfrak{E}''$. Then we have a short exact sequence of sheaves

$$0 \longrightarrow \mathfrak{E}' \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{E}'' \longrightarrow 0\,. \tag{4.66}$$

Suppose now that $\mathfrak{E}$ is locally-free, clearly the corresponding sequence (4.66) involving destabilizing subsheaves is not necessarily a sequence of Higgs bundles, so even when the slopes are equal, we cannot apply apriori the decomposition of Corollary (4.14). Now, in general for semistable Higgs sheaves which are not stable we have the following result

Lemma 4.15. *Let $\mathfrak{E}$ be a torsion-free Higgs sheaf over a compact Kähler manifold X with Kähler form ω and suppose $\mathfrak{E}$ is ω-semistable but not ω-stable. Let consider an exact sequence (4.66) with $\mathfrak{E}'$ of minimal rank among all proper Higgs subsheaves with the same slope as $\mathfrak{E}$. Then $\mathfrak{E}'$ is ω-stable and $\mathfrak{E}''$ is ω-semistable.*

Proof: Assume $\mathfrak{E}$ is ω-semistable and not ω-stable, and let $\mathfrak{E}'$ be a Higgs subsheaf with minimal rank among all proper non-trivial Higgs subsheaves with the same slope as $\mathfrak{E}$ with torsion-free quotient.

Suppose $\mathfrak{E}'$ is not ω-stable, then there exists a proper Higgs subsheaf $\mathfrak{F}'$ of $\mathfrak{E}'$ with $\mu(\mathfrak{F}') \geq \mu(\mathfrak{E}')$ and since $\mathfrak{F}'$ is clearly a subsheaf of $\mathfrak{E}$ which is ω-semistable, we necessarily obtain $\mu(\mathfrak{F}') = \mu(\mathfrak{E})$ and we get a contradiction, because $\mathfrak{E}'$ was chosen with minimal rank. This shows that $\mathfrak{E}'$ is ω-stable.

Now let $\mathfrak{E}'' = \mathfrak{E}/\mathfrak{E}'$, then from Lemma 2.1 it follows that $\mu(\mathfrak{E}'') = \mu(\mathfrak{E})$ and ω-semistable. In fact, if $\mathfrak{E}''$ is not ω-semistable, then there exists a proper Higgs subsheaf $\mathfrak{H}$ of $\mathfrak{E}''$ with $\mu(\mathfrak{H}) > \mu(\mathfrak{E}'')$. Then, using again Lemma 2.1 we have $\mu(\mathfrak{E}'') > \mu(\mathfrak{E}''/\mathfrak{H})$ and defining $\mathfrak{K}$ as the kernel of the morphism $\mathfrak{E} \longrightarrow \mathfrak{E}''/\mathfrak{H}$, we get the exact sequence

$$0 \longrightarrow \mathfrak{K} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{E}''/\mathfrak{H} \longrightarrow 0$$

and since $\mu(\mathfrak{E}) = \mu(\mathfrak{E}'')$, using again the same Lemma we conclude that $\mu(\mathfrak{K}) > \mu(\mathfrak{E})$, which contradicts the ω-semistability of $\mathfrak{E}$. Q.E.D.

4.5 Higgs bundles over Riemann surfaces

As we said before in the one-dimensional case, when X is a compact Riemann surface, the notion of stability (resp. semistability) does not depend on the Kähler form ω, therefore we can establish our results without make reference to any ω. In this section we establish a boundedness property for the Donaldson functional for semistable Higgs bundles over Riemann surfaces, which is in fact the main result of the section. To be precise we have

Theorem 4.16. *Let $\mathfrak{E}$ be a Higgs bundle over a compact Riemann surface X. If it is semistable, then for any fixed Hermitian structure k in $\mathrm{Herm}^+(\mathfrak{E})$ the set $\{\mathcal{L}(h,k), h \in \mathrm{Herm}^+(\mathfrak{E})\}$ is bounded below.*

Proof: In a similar way to the ordinary case (see [5], Ch.VI for a proof of this for holomorphic bundles), the proof runs by induction on the rank of $\mathfrak{E}$. Fix first an Hermitian metric k and assume that $\mathfrak{E}$ is semistable.

If it is also stable, then by Theorem 3.13 it follows that there exists an Hermitian-Yang-Mills structure $\tilde{h}$ on it. Now, by Theorem 4.7 we know that the Donaldson functional must attain an absolute minimum at $\tilde{h}$. In other words, for any other metric h

$$\mathcal{L}(h,k) \geq \mathcal{L}(\tilde{h},k)$$

and hence the set is clearly bounded below.

Suppose now that $\mathfrak{E}$ is not stable, from Lemma 4.15 we know there exists a short exact sequence

$$0 \longrightarrow \mathfrak{E}' \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{E}'' \longrightarrow 0$$

with $\mathfrak{E}'$ stable and $\mathfrak{E}''$ semistable. Since $\mathfrak{E}'$ and $\mathfrak{E}''$ are torsion-free and $\dim X = 1$, they are also locally free and hence the sequence is in fact an exact sequence of Higgs bundles. Assume now that h is an arbitrary Hermitian metric on $\mathfrak{E}$, then by applying the Corollary 4.14 to the metrics h and k we obtain

$$\mathcal{L}(h,k) = \mathcal{L}(h',k') + \mathcal{L}(h'',k'') + \|B_h\|^2 - \|B_k\|^2 \,, \tag{4.67}$$

where h', k' and h'', k'' are the Hermitian structures induced by h, k in $\mathfrak{E}'$ and $\mathfrak{E}''$ respectively. If the rank of $\mathfrak{E}$ is one, it is stable and hence $\mathcal{L}(h,k)$ is bounded below by a constant which depends on k. If the rank of $\mathfrak{E}$ is greater than one, then by the inductive hypothesis, $\mathcal{L}(h',k')$ and $\mathcal{L}(h'',k'')$ must be bounded below by constants depending only on k' and k'' respectively. Then $\mathcal{L}(h,k)$ is bounded below by a constant depending only on k. Q.E.D.

From Theorem 4.16, we get that all three conditions in Theorem 4.11, are equivalent in the one-dimensional case. In particular from this we have

Corollary 4.17. *Let $\mathfrak{E}$ be a Higgs bundle over a compact Riemann surface X. Then $\mathfrak{E}$ is semistable if and only if $\mathfrak{E}$ admits an approximate Hermitian-Yang-Mills structure.*

This equivalence between the notions of approximate Hermitian-Yang-Mills structures and semistability is one version of the so called *Hitchin-Kobayashi correspondence* for Higgs bundles. As a consequence of the Corollary 4.17 we see that in the one-dimensional case, all results about Higgs bundles written in terms of approximate Hermitian-Yang-Mills structures can be traslated in terms of semistability. In particular we have

Corollary 4.18. *Let $\pi : \tilde{X} \longrightarrow X$ be a finite unramified covering of a compact Riemann surface X. Then*
(i) *If a Higgs bundle $\mathfrak{E}$ over X is semistable, then so does its pullback $\pi^*\mathfrak{E}$ over $\tilde{X}$.*
(ii) *If a Higgs bundle $\tilde{\mathfrak{E}}$ over $\tilde{X}$ is semistable, then so does its pushforward $\pi_*\tilde{\mathfrak{E}}$ to the compact Riemann surface X.*

Corollary 4.19. *If X is a compact Riemann surface and $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are semistable Higgs bundles over X, so is the tensor product $\mathfrak{E}_1 \otimes \mathfrak{E}_2$. Furthermore, if $\mu(\mathfrak{E}_1) = \mu(\mathfrak{E}_2)$, so is the Whitney sum $\mathfrak{E}_1 \oplus \mathfrak{E}_2$.*

Notice that the second part of the above Corollary is in essence the Proposition 4.12 in the one-dimensional case. Therefore, we do not need the correspondence between semistability and the existence of approximate Hermitian-Yang-Mills structures to prove this. However, that correspondence becomes useful in the proof of first part. As a consequence of the first part of Corollary 4.19 we get the following property

Corollary 4.20. *If $\mathfrak{E}$ is semistable, so is the tensor product bundle $\mathfrak{E}^{\otimes p} \otimes \mathfrak{E}^{*\otimes q}$ and the exterior product bundle $\bigwedge^p \mathfrak{E}$ whenever $p \leq r$.*

The equivalence between the existence of approximate Hermitian-Yang-Mills structures and the semistability must be true also in higher dimensions. However, since torsion-free sheaves over compact Kähler manifolds with $\dim X \geq 2$ may not be locally free (they are locally free only outside its singularity set) we need to consider exact sequences of Higgs sheaves. To be precise, in order to use the inductive hypothesis to prove Theorem 4.16 in higher dimensions, it is necessary to find Higgs subbundles; but, in general one can expect to find at most Higgs subsheaves which are Higgs subbundles only outside their singularity set. To get subbundles, these singular sets have to be blown-up and appropriate metrics must be constructed.

Some of these aspects in higher dimensions, has been recently studied for holomorphic vector bundles by Jacob in [46], he uses some techniques on geometric analyisis and an extension of a method of regularization of sheaves that was introduced by Buchdahl [49] in the case of compact complex surfaces.

CHAPTER 5

Admissible metrics

5.1 Admissible metrics

Let us consider a torsion-free Higgs sheaf $\mathfrak{E}$ over a compact Kähler manifold X and let $S = S(\mathfrak{E}) \subset X$ be the locus where $\mathfrak{E}$ is not locally free, i.e., S is the singularity set of the Higgs sheaf $\mathfrak{E}$. As we know, S is a complex analytic subset with $\mathrm{codim} S \geq 2$. Following [24] (see also [14]), an *admissible metric* on $\mathfrak{E}$ is an Hermitian metric h on the bundle $\mathfrak{E}|_{X\backslash S}$ with the following two properties:

(i) The Chern curvature R_h is square-integrable, and
(ii) The mean curvature $K_h = i\Lambda R_h$ is L^1-bounded.

Let us consider now the natural embedding of $\mathfrak{E}$ into its double dual $\mathfrak{E}^{\vee\vee}$; since

$$S(\mathfrak{E}^{\vee\vee}) \subset S(\mathfrak{E})$$

an admissible metric on $\mathfrak{E}^{\vee\vee}$ restricts to an admissible metric on $\mathfrak{E}$. An admissible metric h on a Higgs sheaf $\mathfrak{E}$ is called an *Hermitian-Yang-Mills structure* on $\mathfrak{E}$, if on $X\backslash S$ the mean curvature of its Hitchin-Simpson connection is proportional to the identity. In other words, if

$$\mathcal{K}_h = K_h + i\Lambda[\phi, \bar{\phi}_h] = c \cdot I \tag{5.1}$$

is satisfied on $X\backslash S$ for some constant c. It is important to note here that the admissibility of a metric on a Higgs sheaf depend only on conditions imposed on the Chern curvature. However, the notion of Hermitian-Yang-Mills structure does depend on the Higgs field conditions imposed on the Hitchin-Simpson curvature.

The notion of admissibility can be relaxed to make only reference to certain open sets in the following way: an admissible metric on a Higgs sheaf $\mathfrak{E}$ is an Hermitian metric h defined on an open set U, such that $X\backslash U$ is a complex analytic subset of codimension at least two, which contains the singularity set of $\mathfrak{E}$. Using this modified definition, any admissible metric on $\mathfrak{E}$ induces an admissible metric on $\mathfrak{E}^{\vee\vee}$. From now on (if necessary) we will understand admissible metrics in this modified version.

Proposition 5.1. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be two torsion-free Higgs sheaves over a compact Kähler manifold X and let h_1 and h_2 be two admissible metrics on these Higgs sheaves. Then, $h_1 \otimes h_2$ is an admissible metric on $\mathfrak{E}_1 \otimes \mathfrak{E}_2$.*

Proof: Suppose h_1 and h_2 are admissible metrics and let S_1 and S_2 the singularity sets of $\mathfrak{E}_1$ and $\mathfrak{E}_2$, respectively. Then, h_1 and h_2 are Hermitian metrics on $\mathfrak{E}_1|_{U_1}$

and $\mathfrak{E}_2|_{U_2}$ for some open sets U_1 and U_2 in X, where $X\backslash U_1$ and $X\backslash U_2$ are complex analytic subsets of codimension greater or equal than two, containing the sets S_1 and S_2 respectively.

Since $X\backslash U_1 \cap U_2$ is the union of $X\backslash U_1$ and $X\backslash U_2$, it is a closed analytic subset of codimension at least two containing $S_1 \cup S_2$. From this we obtain that $h_1 \otimes h_2$ is an Hermitian metric on $\mathfrak{E}_1 \otimes \mathfrak{E}_2|_{U_1\cap U_2}$.

On the other hand, if $K_{1\otimes 2}$ denotes the Chern mean curvature of $h_1 \otimes h_2$, we have from classical identities (see [5], Ch.I) that

$$\begin{aligned} |K_{1\otimes 2}| &\leq |K_1 \otimes I_2| + |I_1 \otimes K_2| \\ &\leq \sqrt{r_2}|K_1| + \sqrt{r_1}|K_2| \,. \end{aligned}$$

Since K_1 and K_2 are L^1-bounded, by integrating this inequality over $U_1 \cap U_2$ it follows that $K_{1\otimes 2}$ is also L^1-bounded. Similarly, for the Chern curvature $R_{1\otimes 2}$ we obtain

$$\begin{aligned} |R_{1\otimes 2}|^2 &\leq |R_1 \otimes I_2|^2 + |I_1 \otimes R_2|^2 + 2|R_1 \otimes I_2||I_1 \otimes R_2| \\ &\leq r_2|R_1|^2 + r_1|R_2|^2 + 2\sqrt{r_1 r_2}|R_1||R_2| \,. \end{aligned}$$

Now, since R_1 and R_2 are square-integrable and the product $|R_1||R_2|$ is L^1-bounded, the square-integrability of $R_{1\otimes 2}$ follows integrating the above inequility over $U_1 \cap U_2$. Q.E.D.

Let $\mathfrak{E} = (E, \phi)$ be a torsion-free Higgs sheaf over X. By Biswas and Schumacher [24] (see also [39]), there exists a finite sequence of blowups with smooth centers

$$\pi_j : X_j \longrightarrow X_{j-1} \,,$$

with $j = 1, ..., k$ and $X_0 = X$, such that the pullback of the sheaf $E^\vee$ to X_k modulo torsion is locally free and $\pi_1 \cdots \pi_k$ outside S is a biholomorphism. In other words, setting $\tilde{X} = X_k$ and

$$\pi = \pi_1 \cdots \pi_k : \tilde{X} \longrightarrow X \,, \tag{5.2}$$

and denoting by T the torsion part of $\pi^* E^\vee$, then $\pi^* E^\vee / T$ is a holomorphic bundle over $\tilde{X}$ and π restricted to $\tilde{X} \backslash \pi^{-1}(S)$ is a biholomorphism.

Let $\tilde{E}$ be the dual of the bundle $\pi^* E^\vee / T$. Clearly, the morphism ϕ defines a Higgs field $\psi = \pi^* \phi^\vee$ on $\pi^* E^\vee$ and since $\psi(T) \subset T \otimes \Omega^1_{\tilde{X}}$, the morphism ψ is well defined on the quotient $\pi^* E^\vee / T$ and we have a morphism

$$\psi^\vee : \tilde{E} \longrightarrow \tilde{E} \otimes \Omega^1_{\tilde{X}} \,.$$

From the above analysis we conclude that $\tilde{\mathfrak{E}} = (\tilde{E}, \psi^\vee)$ is a Higgs bundle over $\tilde{X}$. We say that $\tilde{\mathfrak{E}}$ is a *regularization* of the Higgs sheaf $\mathfrak{E}$ and that the map π, defined by (5.2), is a morphism regularizing $\mathfrak{E}$.

If ω is a Kähler metric on X, its pullback $\pi^*\omega$ is degenerate along the exceptional divisor $\pi^{-1}(S)$ and hence it is not a Kähler metric on $\tilde{X}$. Following [24] we can define a Kähler metric closely related to the form $\pi^*\omega$ as follows. Let η be an arbitrary Kähler metric on $\tilde{X}$ and $0 \leq \epsilon \leq 1$ a parameter, then we define

$$\omega_\epsilon = \pi^*\omega + \epsilon\eta \,. \tag{5.3}$$

This is a Kähler metric for each $\epsilon > 0$. Such a metric can be used to prove some simple properties involving admissible metrics. In particular, we have the following result

Proposition 5.2. *Let $\mathfrak{E}$ be a torsion-free Higgs sheaf over a compact Kähler manifold X and $\tilde{\mathfrak{E}}$ a regularization of it. Then, any Hermitian metric on $\tilde{\mathfrak{E}}$ induces an admissible metric on $\mathfrak{E}$.*

Proof: Let $\tilde{h}$ be an Hermitian metric on $\tilde{\mathfrak{E}}$ and denote by S the singularity set of $\mathfrak{E}$. Let π be the morphism regularizing $\mathfrak{E}$. The Hermitian metric $\tilde{h}$ induces an Hermitian metric h on $\mathfrak{E}|_{X\setminus S}$.

Let $K = i\Lambda R$ be the (classical) mean curvature of $\mathfrak{E}|_{X\setminus S}$ associated with the metric h. Since $\tilde{h}$ is defined on all $\tilde{X}$, the pullback of R, denoted by $\tilde{R}$, extends to all $\tilde{X}$ as the curvature of the Hermitian metric $\tilde{h}$ and hence, on each point of $\tilde{X}$, we have

$$in\tilde{R} \wedge \omega_\epsilon^{n-1} = i\Lambda_\epsilon \tilde{R}\,\omega_\epsilon^n = \tilde{K}_\epsilon\,\omega_\epsilon^n \tag{5.4}$$

where $i\Lambda_\epsilon$ denotes this time the adjoint of the multiplication by ω_ϵ and $\tilde{K}_\epsilon$ represents the corresponding mean curvature of $\tilde{\mathfrak{E}}$. Now, since $\tilde{X}$ is compact, for some positive constant C we have

$$i\tilde{R} \leq iC\omega_\epsilon I \tag{5.5}$$

where I here is the identity endomorphism of $\tilde{E}$. Hence, by applying $i\Lambda_\epsilon$ to (5.5) we have that $Ci\Lambda_\epsilon\,\omega_\epsilon I - \tilde{K}_\epsilon$ must be a semi-positive definite endomorphism of $\tilde{E}$. Therefore we get

$$\begin{aligned}
|\tilde{K}_\epsilon|\,\omega_\epsilon^n &\leq |\tilde{K}_\epsilon - Ci\Lambda_\epsilon\omega_\epsilon I|\,\omega_\epsilon^n + |Ci\Lambda_\epsilon\omega_\epsilon I|\,\omega_\epsilon^n \\
&\leq \mathrm{tr}\,[i\Lambda_\epsilon(C\omega_\epsilon I - \tilde{R})]\,\omega_\epsilon^n + \mathrm{tr}\,(Ci\Lambda_\epsilon\omega_\epsilon I)\,\omega_\epsilon^n \\
&\leq in\,\mathrm{tr}\,(2C\omega_\epsilon I - \tilde{R})\,\omega_\epsilon^{n-1} \\
&\leq in\,\mathrm{tr}\,(2C\omega_1 I - \tilde{R})\,\omega_1^{n-1} \,.
\end{aligned}$$

From this we conclude that $\tilde{K}_\epsilon$ is uniformly integrable with respect to $0 < \epsilon \leq 1$ and hence, taking the limit $\epsilon \to 0$, it follows that K is L^1-bounded.

On the other hand, from the theory of holomorphic bundles (see [14], Lemma 6) we have

$$\left[2\,c_2(\tilde{E}) - c_1(\tilde{E})^2\right] \cup [\omega_\epsilon]^{n-2} = \frac{1}{4\pi^2 n(n-1)} \int_{\tilde{X}} \left[|\tilde{R}|^2 - |\tilde{K}_\epsilon|^2\right] \omega_\epsilon^n \,, \tag{5.6}$$

and hence, taking the limit $\epsilon \to 0$, it follows that R is also square-integrable. Q.E.D.

5.2 More about Higgs sheaves

Let $\mathfrak{E}$ be a torsion-free Higgs sheaf over a compact Kähler manifold X. We know that the codimension of its singularity set S is greater or equal than two. The (non-compact) manifold $X\backslash S$ satisfies all assumptions Simpson imposes in [17] (see in particular Propositions 2.1 and 2.2 in that reference). Therefore, we can see $\mathfrak{E}$ as a Higgs bundle over the non-compact manifold $X\backslash S$. Thus, in studying torsion-free Higgs sheaves we are considering implicitly Higgs bundles over non-compact Kähler manifolds.

By Simpson [17], Proposition 3.3, we know that a torsion-free Higgs sheaf over a compact Kähler manifold with an Hermitian-Yang-Mills structure must be at least semistable. However, as Biswas and Schumacher showed in [24], this is just one part of a stronger result. In fact, using the evolution equation, Biswas and Schumacher obtained the Hitchin-Kobayashi correspondence for polystable Higgs sheaves. Namely, they proved the following

Theorem 5.3. *A torsion-free Higgs sheaf $\mathfrak{E}$ over a compact Kähler manifold X with Kähler form ω is ω-polystable if and only if there exists an Hermitian-Yang-Mills structure on it.*

From Theorem 5.3 it follows that any restriction of a stable Higgs sheaf $\mathfrak{E}$ to $X\backslash S$ is ω-polystable. In fact, using the modified definition of admissibility this also holds for restrictions to certain open subsets of X. To be precise we have the following result

Proposition 5.4. *Let $\mathfrak{E}$ be a torsion-free Higgs sheaf over a compact Kähler manifold X with Kähler form ω and denote by S its singularity set. Let $U \subset X$ be an open set such that $X\backslash U$ is a closed analytic subset of codimension at least two containing S. Then $\mathfrak{E}|_U$ is ω-polystable if $\mathfrak{E}$ is ω-polystable.*

Proof: Let $\mathfrak{E}$ be a torsion-free sheaf over X and assume first that it is ω-stable. Then, from Theorem 5.3, there exists an Hermitian-Yang-Mills structure h on it. Let U be an open subset of X such that $X\backslash U$ is a closed analytic subset with codimension greater or equal than two and suppose that $S \subset X\backslash U$. Thus, h is in particular an Hermitian-Yang-Mills metric on the Higgs bundle $\mathfrak{E}|_U$ and hence, from Proposition 3.3 in [17], it must be ω-polystable.

Assume now that $\mathfrak{E}$ is ω-polystable. Then, it can be decomposed as a direct sum of ω-stable Higgs sheaves with the same slope as $\mathfrak{E}$. From the first part of the proof, we know that each restriction of these stable Higgs sheaves to U must be ω-polystable and hence the result follows. Q.E.D.

Lemma 5.5. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be two torsion-free Higgs sheaves over a compact Kähler manifold X with Kähler form ω. If both are ω-polystable, then $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ modulo torsion is also ω-polystable.*

Proof: Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be ω-polystable. Then, from Theorem 5.3 we know there exist Hermitian-Yang-Mills structures h_1 and h_2. Now, by Proposition 5.1, it follows that $h = h_1 \otimes h_2$ is an admissible metric on $\mathfrak{E}_1 \otimes \mathfrak{E}_2$. Clearly, it is an Hermitian-Yang-Mills structure and hence, using again Theorem 5.3, such a tensor product (modulo torsion) must be ω-polystable. Q.E.D.

As a consequence of the above Lemma, Biswas and Schumacher [24] proved that the tensor product of two semistable sheaves is again semistable. Here we present a different proof. Notice first that from Lemma 5.5 we have the following

Lemma 5.6. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be two torsion-free Higgs sheaves over a compact Kähler manifold X with Kähler form ω. If $\mathfrak{E}_1$ is ω-semistable and $\mathfrak{E}_2$ is ω-polystable, then $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ modulo torsion is ω-semistable.*

Proof: Assume that $\mathfrak{E}_2$ is ω-polystable an $\mathfrak{E}_1$ is ω-semistable. Following Simpson [19] (see also [24]), there exists a filtration of $\mathfrak{E}_1$ by Higgs subsheaves

$$0 = \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \cdots \subset \mathfrak{F}_k = \mathfrak{E}_1\,, \tag{5.7}$$

in which the quotients $\mathfrak{F}_j/\mathfrak{F}_{j-1}$ for $j = 1, ..., k$ are ω-polystable and they have all the same slope as $\mathfrak{E}_1$. Now, let $U \subset X$ be the open set in which all terms of the filtration (5.7), all quotients $\mathfrak{F}_j/\mathfrak{F}_{j-1}$, and also $\mathfrak{E}_2$ are locally free. Then $X \backslash U$ is a closed analytic subset of codimension greater or equal than two, and on U we have the sequence

$$0 \longrightarrow \mathfrak{F}_1 \longrightarrow \mathfrak{F}_2 \longrightarrow \mathfrak{F}_2/\mathfrak{F}_1 \longrightarrow 0 \tag{5.8}$$

as a sequence of locally free Higgs sheaves. Then, tensoring the above sequence by $\mathfrak{E}_2$ we obtain the sequence

$$0 \longrightarrow \mathfrak{F}_1 \otimes \mathfrak{E}_2 \longrightarrow \mathfrak{F}_2 \otimes \mathfrak{E}_2 \longrightarrow (\mathfrak{F}_2/\mathfrak{F}_1) \otimes \mathfrak{E}_2 \longrightarrow 0 \tag{5.9}$$

which is again an exact sequence of locally free Higgs sheaves over U. Since $\mathfrak{F}_1$ and $\mathfrak{F}_2/\mathfrak{F}_1$ are both ω-polystable, and also $\mathfrak{E}_2$ is ω-polystable by hypothesis, we have by Proposition 5.4 that they are all ω-polystable over U. Therefore, it follows from Lemma 5.5 that $\mathfrak{F}_1 \otimes \mathfrak{E}_2$ and $(\mathfrak{F}_2/\mathfrak{F}_1) \otimes \mathfrak{E}_2$ are both ω-polystable with equal slopes (in particular they are ω-semistable). Therefore, from this and Corollary 2.9, we obtain the semistability of the Higgs sheaf $\mathfrak{F}_2 \otimes \mathfrak{E}_2$ over the open set U.

Now, we consider the exact sequence

$$0 \longrightarrow \mathfrak{F}_2 \longrightarrow \mathfrak{F}_3 \longrightarrow \mathfrak{F}_3/\mathfrak{F}_2 \longrightarrow 0\,. \tag{5.10}$$

Since over U this is an exact sequence of locally free Higgs sheaves, tensoring again by $\mathfrak{E}_2$ we obtain over U the following exact sequence of locally free Higgs sheaves:

$$0 \longrightarrow \mathfrak{F}_2 \otimes \mathfrak{E}_2 \longrightarrow \mathfrak{F}_3 \otimes \mathfrak{E}_2 \longrightarrow (\mathfrak{F}_3/\mathfrak{F}_2) \otimes \mathfrak{E}_2 \longrightarrow 0\,. \tag{5.11}$$

Using again Lemma 5.5 we have that $(\mathfrak{F}_3/\mathfrak{F}_2) \otimes \mathfrak{E}_2$ is ω-polystable, in particular it is ω-semistable and since $\mathfrak{F}_2 \otimes \mathfrak{E}_2$ is also ω-semistable, we obtain (again by Corollary 2.9) that $\mathfrak{F}_3 \otimes \mathfrak{E}_2$ is ω-semistable. Continuing this process we get at the end that $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ is ω-semistable. Since all of this holds over U, whose complement has codimension greater or equal than two, it can be extended on all X and $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ is ω-semistable on X as well. Q.E.D.

Theorem 5.7. *Let $\mathfrak{E}_1$ and $\mathfrak{E}_2$ be two torsion-free Higgs sheaves over a compact Kähler manifold X with Kähler form ω. If both are ω-semistable, then $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ modulo torsion is also ω-semistable.*

Proof: The Higgs sheaf $\mathfrak{E}_1$ has a filtration by Higgs subsheaves as in (5.7), with ω-polystable quotients with the same slope as $\mathfrak{E}_1$. Now, let $U \subset X$ be an open subset such that all terms of the filtration, all quotients and also $\mathfrak{E}_2$ are locally free. Then, we have the exact sequences (5.8) and (5.9) and since $\mathfrak{E}_2$ is ω-semistable, the result follows by applying Lemma 5.6. Q.E.D.

5.3 Donaldson's functional for Higgs sheaves

Let h be an admissible metric on a torsion-free Higgs sheaf $\mathfrak{E}$ over a compact Kähler manifold X. From [24] we know the Higgs field ϕ is bounded on $X\backslash S$, and hence the Hitchin-Simpson curvature

$$\mathcal{K}_h = K_h + i\Lambda[\phi, \bar{\phi}_h] \tag{5.12}$$

is L^1-bounded. This means that any admissible metric h satisfies

$$\int_{X\backslash S} |\mathcal{K}_h|\, \omega^n < \infty\,. \tag{5.13}$$

According to Simpson [17], we can define the Donaldson functional on $\mathfrak{E}|_{X\backslash S}$ for metrics satisfying (5.13).

Let $\mathrm{Herm}^+(\mathfrak{E}_{X\backslash S})$ be the space of all smooth metrics on $\mathfrak{E}_{X\backslash S}$ satisfying the condition (5.13) and suppose that h and k are two metrics in the same connected component of the space $\mathrm{Herm}^+(\mathfrak{E}_{X\backslash S})$. Then $h = ke^v$ for some endomorphism v of $E|_{X\backslash S}$ and following Simpson [17], we can write the Donaldson functional as

$$\mathcal{L}(ke^v, k) = \int_{X\backslash S} \mathrm{tr}\,[v(\mathcal{K}_k - cI)]\,\frac{\omega^n}{n!} + \int_{X\backslash S} \sum_{i,j=1}^{r} \psi_1(\lambda_i, \lambda_j)|\mathcal{D}'' v^i_j|^2 \frac{\omega^{n-1}}{(n-1)!} \tag{5.14}$$

where the function ψ_1 is given by (4.36). Equivalently, we can write this functional using Kobayashi's approach [5], which we used earlier for Higgs bundles in the compact case. Therefore, if $h_t\,, 0 \leq t \leq 1$ is a curve in a connected component of $\mathrm{Herm}^+(\mathfrak{E}_{X\backslash S})$ such that $h_0 = k$ and $h_1 = h$, we write

$$\mathcal{L}(h, k) = \int_{X\backslash S} \left[Q_2(h,k) - \frac{c}{n} Q_1(h,k)\,\omega\right] \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{5.15}$$

where

$$Q_2(h,k) = i\int_0^1 \mathrm{tr}(v_t \cdot \mathcal{R}_t)\,dt\,, \qquad Q_1(h,k) = \log(\det(k^{-1}h))\,. \tag{5.16}$$

We define the *Donaldson functional on the Higgs sheaf* $\mathfrak{E}$ just as the corresponding functional (5.14), or equivalently (5.15), defined on the Higgs bundle $\mathfrak{E}|_{X\backslash S}$.

In [17] Simpson found an inequality between the supremum of the endomorphisms v relating the metrics h and k and the Donaldson functional for Higgs bundles over (non necessarily) compact Kähler manifolds. To be precise he showed

Proposition 5.8. *Let k be an Hermitian metric on a Higgs bundle $\mathfrak{E}$ over a Kähler manifold Y with Kähler form ω and suppose $\sup_Y|\mathcal{K}_k| \le B$ for certain fixed constant B. If $\mathfrak{E}$ is ω-stable, then there exist constants C_1 and C_2 such that*

$$\sup_Y|v| \le C_1 + C_2\,\mathcal{L}(ke^v, k) \tag{5.17}$$

for any selfadjoint endomorphism v with $\mathrm{tr}v = 0$ and $\sup_Y|v| < \infty$ and such that $\sup_Y|\mathcal{K}_{ke^v}| \le B$.

Since the Donaldson functional for a Higgs sheaf is just the corresponding functional for the associated Higgs bundle and an admissible metric k satisfies (5.13), this result can be immediately adapted to Higgs sheaves; hence we have

Corollary 5.9. *Let k be an admissible metric on a torsion-free Higgs sheaf $\mathfrak{E}$ over a compact Kähler manifold X with Kähler form ω and suppose $\sup_{X\backslash S}|\mathcal{K}_k| \le B$ for certain fixed constant B. If $\mathfrak{E}$ is ω-stable, then there exist constants C_1 and C_2 such that*

$$\sup_{X\backslash S}|v| \le C_1 + C_2\,\mathcal{L}(ke^v, k) \tag{5.18}$$

for any selfadjoint endomorphism v with $\mathrm{tr}v = 0$ and $\sup_{X\backslash S}|v| < \infty$ and such that $\sup_{X\backslash S}|\mathcal{K}_{ke^v}| \le B$.

5.4 Higgs bundles over Kähler surfaces

Assume that $\dim X = 2$ (that is, X is a compact Kähler surface) and denote by ω the Kähler form of X. Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle over X. Then, its degree is given by

$$\deg \mathfrak{E} = \int_X c_1(\det E) \wedge \omega\,, \tag{5.19}$$

and hence, it depends on ω. Suppose that $\mathfrak{E}$ is ω-semistable but not ω-stable and let $\mu(\mathfrak{E}) = \mu$. Then, from Lemma 4.15, we know there are Higgs sheaves $\mathfrak{F}$ and $\mathfrak{G}$ such that

$$0 \longrightarrow \mathfrak{F} \longrightarrow \mathfrak{E} \longrightarrow \mathfrak{G} \longrightarrow 0 \tag{5.20}$$

is an exact sequence over X, where $\mathfrak{F}$ is ω-stable, $\mathfrak{G}$ is ω-semistable and

$$\mu(\mathfrak{F}) = \mu(\mathfrak{G}) = \mu\,.$$

We know that $\mathfrak{G}$ is torsion-free and $\mathfrak{E}$ is reflexive (it is in fact locally free), then $\mathfrak{F}$ is normal and since it is also torsion-free, it is reflexive. Consequently, it has a singularity set $S(\mathfrak{F})$ of codimension at least three and hence $\mathfrak{F}$ must be locally free.

From the above analysis we have that in the exact sequence (5.20) $\mathfrak{F}$ and $\mathfrak{E}$ are Higgs bundles, but $\mathfrak{G}$ is only a torsion-free Higgs sheaf. This means for instance, that we cannot a priori apply the decomposition of the Donaldson's functional that we used in the one-dimensional case, since it holds only for Higgs bundles with equal slopes. Therefore, if we want to use a decomposition of the Donaldson functional, we need to find first an exact sequence of Higgs bundles closely related to the sequence (5.20). In principle, this procedure could be done as follows.

If F and G are the corresponding coherent sheaves of $\mathfrak{F}$ and $\mathfrak{G}$, from (5.20) it follows that

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0 \tag{5.21}$$

is an exact sequence of coherent sheaves over the Kähler surface X. Now, for compact complex surfaces, Buchdahl (see [48], Section 3, for details) showed that it was possible to make a modification of X, such that we end up with an exact sequence of holomorphic bundles over a modified manifold. To be precise, Buchdahl proved in [48] that for any complex surface X and any given sequence (5.21), there exists a modification $\pi : \tilde{X} \longrightarrow X$ of X consisting of finitely many blow-ups and locally free coherent sheaves $\tilde{F}$ and $\tilde{G}$ over $\tilde{X}$ such that the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*F & \longrightarrow & \pi^*E & \longrightarrow & \pi^*G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{F} & \longrightarrow & \pi^*E & \longrightarrow & \tilde{G} & \longrightarrow & 0 \end{array}$$

commutes, has exact rows, and outside the exceptional divisor $\pi^{-1}(S)$ the vertical arrows are isomorphisms. Buchdahl in [48] called such a diagram a *desingularization* of the torsion-free sheaf G and used this diagram to study the behaviour of stability under pullbacks of blow-ups, and to prove the correspondence between polystability and the existence of Einstein-Hermitian metrics for holomorphic bundles over compact complex surfaces.

On the other hand, since $\mathfrak{F}$ is a Higgs subsheaf of $\mathfrak{E}$, it follows that $\pi^*\mathfrak{F}$ is a Higgs subsheaf of $\pi^*\mathfrak{E}$; hence π^*G is the coherent sheaf of a Higgs sheaf $\pi^*\mathfrak{G}$ and the upper row in the above diagram becomes a short exact sequence of Higgs sheaves. Furthermore, from this diagram we know that the rank of $\tilde{G}$ is equal to the rank of π^*G. Consequently, if we denote by K the kernel of the morphism $\pi^*G \to \tilde{G}$, we know that K is the torsion subsheaf of $\pi^*\mathfrak{G}$ and hence the pair $\mathfrak{K} = (K, \eta)$, with η the restriction of $\pi^*\phi_G$ to K, is a (torsion) Higgs sheaf[1]. Therefore, it follows that $\tilde{\mathfrak{G}} = \pi^*\mathfrak{G}/\mathfrak{K}$ is a Higgs bundle. Finally, since the diagram is also commutative, the morphism $\pi^*E \to \tilde{G}$ is clearly a composition of Higgs morphisms and the lower row

[1]Notice that in the proof of Proposition 2.3, we show that the torsion subsheaf of any Higgs sheaf is a Higgs sheaf.

in the above diagram becomes a short exact sequence of Higgs bundles.

Summarizing, from all this we see that if the exact sequence (5.21) comes from a sequence of Higgs sheaves, the above diagram is actually a diagram of Higgs sheaves. In other words, for the exact sequence (5.20) there exists a modification $\pi : \tilde{X} \to X$ of X (consisting of finitely many blow-ups) and Higgs bundles $\tilde{\mathfrak{F}}$ and $\tilde{\mathfrak{G}}$ over $\tilde{X}$, such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\mathfrak{F} & \longrightarrow & \pi^*\mathfrak{E} & \longrightarrow & \pi^*\mathfrak{G} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{\mathfrak{F}} & \longrightarrow & \pi^*\mathfrak{E} & \longrightarrow & \tilde{\mathfrak{G}} & \longrightarrow & 0 \end{array}$$

is commutative, with exact rows and such that outside $\pi^{-1}(S)$ the vertical arrows are isomorphisms.

An Hermitian metric h on $\mathfrak{E}$ induces an Hermitian metric π^*h on π^*E, and since in the diagram of Higgs sheaves the lower row is an exact sequence of Higgs bundles, we have induced Hermitian metrics $\tilde{h}'$ and $\tilde{h}''$ in $\tilde{\mathfrak{F}}$ and $\tilde{\mathfrak{G}}$ respectively. In particular, $\tilde{h}''$ induces an Hermitian metric h'' on $\mathfrak{G}_{X\setminus S}$ and a similar argument to the one used in the proof of Proposition 5.2 shows that the induced metric is an admissible one for the torsion-free Higgs sheaf $\mathfrak{G}$.

At this point the strategy to prove the existence of approximate Hermitian-Yang-Mills structures for semistable Higgs bundles would be as follows: first we must identify the Donaldson functional of $\mathfrak{E}$ with the Donaldson functional of $\pi^*\mathfrak{E}$ (constructed using the form $\pi^*\omega$) and hence, using the Corollary 4.14, we can decompose it in terms of the functionals of the Higgs bundles $\tilde{\mathfrak{F}}$ and $\tilde{\mathfrak{G}}$. The Donaldson functional of $\tilde{\mathfrak{F}}$ should be the same functional of the stable Higgs bundle $\mathfrak{F}$ and therefore, it must be bounded from below. On the other hand, since $\mathfrak{G}$ is only a semistable (torsion-free) Higgs sheaf, we cannot apply the same argument for the functional of $\tilde{\mathfrak{G}}$. However, since we have now an exact sequence of Higgs bundles, we can apply an induction procedure similar to the one used in the one-dimensional case; clearly, doing this, after a finite number of steps we get a stable Higgs quotient. Finally we will need to check that this last functional is bounded from below.

CHAPTER A

Coherent sheaves

A.1 Coherent sheaves

Coherent sheaves appear in sheaf theory and play an important role in Complex Analisis. In this section we summarize some general notions about coherent sheaves, some of them are already used throughout this work (see [5] and [2] for more details).

Let X be a complex manifold of dimension n and $\mathcal{O} = \mathcal{O}_X$ its structure sheaf; we denote by $\mathcal{O}^p$ the direct sum of p copies of the structure sheaf of X. An *analytic sheaf* over X is a sheaf of $\mathcal{O}$-modules over X. We say that an analytic sheaf E over X is *coherent* if for any point $x \in X$, there exist a neighborhood U of x and integers p and q such that

$$\mathcal{O}^q|_U \longrightarrow \mathcal{O}^p|_U \longrightarrow E|_U \longrightarrow 0 \tag{A.1}$$

is an exact sequence. An analytic sheaf E is called a *locally free sheaf*, if for any $x \in X$ there is a neighborhood U of x such that $E|_U \cong \mathcal{O}^r|_U$ for some r. Just by definition, locally free sheaves are coherent.

We have also the following important result (known as the *Three Lemma*),

Lemma A.1. *Let*

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

be an exact sequence of analytic sheaves over X. Then all sheaves are coherent if two of them are coherent.

As a consequence of the Three Lemma we have that the Whitney sum of coherent sheaves is coherent and we obtain also the following

Proposition A.2. *If $f : F \longrightarrow E$ is a morphism of coherent sheaves, then* $\operatorname{Ker} f, \operatorname{Im} f$ *and* $\operatorname{Coker} f$ *are all coherent sheaves.*

If F and E are coherent sheaves over X, the sheaf $\mathcal{H}om(E, F)$ is also coherent. From this and taking $F = \mathcal{O}$, we get that the dual sheaf $E^\vee = \mathcal{H}om(E, \mathcal{O})$ is coherent and hence $E^{\vee\vee}$ is also coherent. From this and Proposition A.2 it follows that the kernel of the natural map $\sigma : E \longrightarrow E^{\vee\vee}$ is coherent. The sheaf $\operatorname{Ker} \sigma$ plays an important role in the theory of sheaves, it is in fact the sheaf of torsion elements of the coherent sheaf E.

One of the most important properties of coherent sheaves is that locally they admit free resolutions. To be precise, for any point x there exist a neighborhood U such that

$$0 \longrightarrow \mathcal{O}^{p_d}|_U \longrightarrow \cdots \longrightarrow \mathcal{O}^{p_0}|_U \longrightarrow E|_U \longrightarrow 0 \tag{A.2}$$

with $d \leq n$ is an exact sequence. Such free resolutions are used to define the determinant bundle of a coherent sheaf.

A.2 Singularity sets

For any integer $0 \leq m \leq n$, we define the *m-th singularity set* of a coherent sheaf E by

$$S_m(E) = \{x \in X \,/\, \mathrm{dh}(E_x) \geq n - m\}\,, \tag{A.3}$$

where dh means the homological dimension (the length of a minimal free resolution of the corresponding module). All of these singularity sets are closed analytic and we have the following chain of inclusions

$$S_0(E) \subset S_1(E) \subset \cdots \subset S_{n-1}(E) \subset S_n(E) = X\,. \tag{A.4}$$

We call $S_{n-1}(E)$ the *singularity set* of E, hence we have

$$S_{n-1}(E) = \{x \in X \,/\, \mathrm{dh}(E_x) \geq 1\} = \{x \in X \,/\, E_x \,\text{is not free}\}\,. \tag{A.5}$$

Therefore, any coherent sheaf E is locally free outside $S_{n-1}(E)$. We define the rank of a coherent sheaf E, denoted by $\mathrm{rk}\, E$, as $\mathrm{rk}\, E_x$ with $x \in X \backslash S_{n-1}(E)$. A coherent sheaf E is *torsion-free*, if every stalk E_x is torsion-free and we say that it is *reflexive*, if it is isomorphic to $E^{\vee\vee}$.

There exist some standard results concerning the above singularity sets. In particular, we have some results for torsion-free and reflexive sheaves.

Theorem A.3. *If E is a torsion-free coherent sheaf over a compact complex manifold X of dimension n, then for all $0 \leq m \leq n$*

$$\dim S_m(E) \leq m - 1\,. \tag{A.6}$$

The above result implies that the dimension of the singularity set of a torsion-free coherent sheaf is less or equal than $n-2$. In other words, for any torsion-free coherent sheaf E we have

$$\mathrm{codim}\, S_{n-1}(E) \geq 2\,. \tag{A.7}$$

From this we have in particular that every torsion-free coherent sheaf over a compact Riemann surface is locally free.

Theorem A.4. *If E is a reflexive coherent sheaf over a compact complex manifold X of dimension n, then for all $0 \leq m \leq n$*

$$\dim S_m(E) \leq m - 2\,. \tag{A.8}$$

This means that the dimension of the singularity set of a reflexive coherent sheaf is less or equal than $n-3$, or equivalently, that for any reflexive coherent sheaf E we have

$$\operatorname{codim} S_{n-1}(E) \geq 3\,. \tag{A.9}$$

In particular, from (A.9) we know that every reflexive coherent sheaf over a compact analytic surface is locally free.

A coherent sheaf E over X is said to be *normal* if for every open set $U \subset X$ and every analytic subset $A \subset U$ of codimension at least two, the restriction map

$$\Gamma(U, E) \longrightarrow \Gamma(U \backslash A, E) \tag{A.10}$$

is an isomorphism, where $\Gamma(U, E)$ and $\Gamma(U \backslash A, E)$ are the set of sections of E over U and $U \backslash A$ respectively. By Hartog's extension theorem, the structure sheaf $\mathcal{O}_X$ of any compact complex manifold X is normal.

We have some results relating the above notions

Proposition A.5. *A coherent sheaf E over a complex manifold X is reflexive if and only if it is torsion-free and normal.*

Proposition A.6. *Let*

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

be an exact sequence of coherent sheaves. If E is reflexive and G is torsion-free, then F is normal.

Since in the above Proposition, F is also torsion-free (because it is a subsheaf of a reflexive sheaf, which is in particular torsion-free), using Proposition A.5 we conclude that F is also reflexive.

A.3 Determinant bundles

Let V be a holomorphic vector bundle of rank r over a compact Kähler manifold X; its *determinant bundle* bundle, $\det V$, is by definition the line bundle $\bigwedge^r V$. As it is well known, there is a correspondence between holomorphic vector bundles and locally free coherent sheaves. In our case, one associates with V the sheaf $\mathcal{O}(V)$ of holomorphic sections of V. Moreover, every exact sequence (see [5], Ch.V for details)

$$0 \longrightarrow V_m \longrightarrow \cdots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0 \tag{A.11}$$

of holomorphic vector bundles over X, induces an exact sequence of locally free coherent sheaves

$$0 \longrightarrow E_m \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow 0 \tag{A.12}$$

over X, where $E_j = \mathcal{O}(V_j)$ for $j = 0, 1, ..., m$, and we have the following

Lemma A.7. *Given the exact sequence of holomorphic vector bundles (A.11), the line bundle* $\bigotimes_{j=0}^{m}(\det V_j)^{(-1)^j}$ *is isomorphic to the trivial line bundle.*

Let E be a coherent sheaf over a compact Kähler manifold X, then there exists an open subset $U \subset X$ and an exact sequence

$$0 \longrightarrow E_n \longrightarrow \cdots \longrightarrow E_1 \longrightarrow E_0 \longrightarrow E|_U \longrightarrow 0 \tag{A.13}$$

over U, where each E_j with $j = 0, 1..., n$ is a locally free coherent sheaf (it is a resolution of $E|_U$ by locally free coherent sheaves). For each j, let V_j be the corresponding holomorphic vector bundle of E_j. Then, the *determinat bundle* of E is given by the formula

$$\det E = \bigotimes_{j=0}^{m} (\det V_j)^{(-1)^j} . \tag{A.14}$$

The first Chern class of E is by definition the first Chern class of its determinant bundle, that is, $c_1(E) = c_1(\det E)$. The degree of E is given by

$$\deg E = \int_X c_1(\det E) \wedge \omega^{n-1} . \tag{A.15}$$

Associated with determinant bundles of coherent sheaves we have the following important result

Proposition A.8. *If*

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

is an exact sequence of coherent sheaves, then there is a canonical isomorphism

$$\det E \simeq \det F \otimes \det G . \tag{A.16}$$

Assume now that E is a torsion-free sheaf of rank r, then there exists a canonical isomorphism between $\det E$ and $(\bigwedge^r E)^{\vee\vee}$ and from this it follows

Proposition A.9. *If E is a torsion-free coherent sheaf, then there exists a canonical isomorphism*

$$(\det E)^{\vee} \simeq \det E^{\vee} . \tag{A.17}$$

If $E \longrightarrow E'$ is a monomorphism between torsion-free coherent sheaves of the same rank, then it induces a sheaf monomorphism $\det E \longrightarrow \det E'$ between its determinant bundles.

A coherent sheaf E over X is said to be a torsion sheaf, if for every $x \in X$ the corresponding stalk E_x is a torsion module. For torsion sheaves we have the following

Proposition A.10. *If E is a torsion sheaf over X, then* $\det E$ *admits a non-trivial holomorphic section. Moreover, if*

$$\text{supp}(E) = \{x \in X,\, E_x \neq 0\} \tag{A.18}$$

has codimension at least two, then $\det E$ *is a trivial line bundle.*

For torsion-free sheaves over Kähler manifolds, there exists a result which plays an important role in the theory and that is commonly cited as *the Harder-Narasimhan filtration theorem*. This result can be written as

Theorem A.11. *Let E be a coherent sheaf over a compact Kähler manifold X with Kähler form ω. Then there exists a unique filtration of E by subsheaves*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{k-1} \subset E_k = E \tag{A.19}$$

such that, for every $1 \leq j \leq k$, the quotients E_j/E_{j-1} are ω-semistable and the slopes $\mu(E_j/E_{j-1})$ are strictly decreasing.

Finally, for semistable sheaves we have the following result, which is called *the Jordan-Hölder theorem.*

Theorem A.12. *Let E be as in Theorem A.11 and assume that it is ω-semistable. Then there exists a filtration of E by subsheaves*

$$0 = E_s \subset E_{s-1} \subset \cdots \subset E_1 \subset E_0 = E \tag{A.20}$$

such that, for every $0 \leq i \leq s-1$, the quotients E_i/E_{i+1} are ω-stable and we have $\mu(E_i/E_{i+1}) = \mu(E)$. Moreover,

$$\mathrm{Gr}(E) = (E_0/E_1) \oplus (E_1/E_2) \oplus \cdots \oplus (E_s/E_{s+1}) \tag{A.21}$$

is uniquely determined by E up to an isomorphism.

CHAPTER B

Some remarks on Higgs bundles

B.1 The Yang-Mills equations and the origin of Higgs bundles

The Yang-Mills equations arose in theoretical physics at the end of the 70's and were defined initially on $\mathbb{R}^4$, the physically relevant solutions (the so called 'instantons') played an important role in quantum field theory. By imposing invariance under translation in one direction, one gets equations on $\mathbb{R}^3$ whose solutions may be interpreted as monopoles. In the 80's, Atiyah [11] and Hitchin [16] studied in detail the Yang-Mills equations formulated on $\mathbb{R}^2$, in that case the equations were conformal invariant and, beacuse of that, they were formulated in terms of Riemann surfaces.

The Yang-Mills equations can be introduced as follows (see [11], [16]). Let P be a principal G bundle over $\mathbb{R}^4$ and let A be a connection on it and R_A its curvature. Let $\mathfrak{g}$ be the Lie algebra of G and write $\operatorname{ad} P = P \times_G \mathfrak{g}$ for the vector bundle associated to the adjoint representation. Denoting by $\Omega^2_{\mathbb{R}^4}(\operatorname{ad} P)$ the space of two forms with coefficients in $\operatorname{ad} P$, we have that R_A is an element in $\Omega^2_{\mathbb{R}^4}(\operatorname{ad} P)$ and the Hodge star operator becomes

$$* : \Omega^2_{\mathbb{R}^4}(\operatorname{ad} P) \longrightarrow \Omega^2_{\mathbb{R}^4}(\operatorname{ad} P)\,. \tag{B.1}$$

A connection A is said to satisfy the *self-dual Yang-Mills equations* if its curvature is invariant under the Hodge star operator, that is if

$$*R_A = R_A\,. \tag{B.2}$$

Using coordinates $(x_\alpha)_{\alpha=1,2,3,4}$ over $\mathbb{R}^4$ and a local trivialization of P, the curvature R_A can be written as

$$R_A = \sum_{\alpha<\beta} R_{\alpha\beta}\, dx^\alpha \wedge dx^\beta \tag{B.3}$$

and the self-dual Yang-Mills equations (B.2) become

$$R_{12} = R_{34}\,, \qquad R_{13} = R_{42}\,, \qquad R_{14} = R_{23}\,. \tag{B.4}$$

Using this trivialization the connection is given by a one form $A = A_\alpha dx^\alpha$ with coefficients in $\operatorname{ad} P$ and we have

$$R_A = dA + A \wedge A\,. \tag{B.5}$$

Introducing the notation of covariant derivatives, $D_\alpha = \partial_\alpha + A_\alpha$, the components of the curvature can be written as

$$R_{\alpha\beta} = [D_\alpha, D_\beta]\,. \tag{B.6}$$

Now, we assume that A_α are independent of x_3 and x_4 and hence they are all functions $A_\alpha(x_1, x_2)$ on $\mathbb{R}^2$. In theroretical physics, such a process is usually called dimensional regularization. From a physical point of view, this process is equivalent ot consider solutions to the self-dual Yang-Mills equations on $\mathbb{R}^4$ invariant under translations in $(x_3, x_4) \in \mathbb{R}^2$. Applying this dimensional regularization procedure, A_1 and A_2 define a connection

$$A = A_1\, dx^1 + A_2\, dx^2 \tag{B.7}$$

over $\mathbb{R}^2$, and the components A_3 and A_4, that we denote by ϕ_1 and ϕ_2 respectively, becomes auxiliary fields[1] over $\mathbb{R}^2$ which are Lie algebra valued. Hitchin called ϕ_1 and ϕ_2 Higgs fields because they played a similar role to the Higgs field in theoretical physics. Then, from (B.6) and using (B.4) we see that the self-dual Yang-Mills equations can be rewritten as:

$$[D_1, D_2] = [\phi_1, \phi_2]\,, \qquad [D_1, \phi_1] = [\phi_2, D_2]\,, \qquad [D_1, \phi_2] = [D_2, \phi_1]\,. \tag{B.8}$$

Introducing the field $\phi = \phi_1 - i\phi_2$, that is usually called the *complex Higgs field*, we can rewrite the equations in the form

$$2R = i[\phi, \phi^*]\,, \qquad [D_1 + iD_2, \phi] = 0\,. \tag{B.9}$$

From the point of view of the induced connection on the principal bundle P over $\mathbb{R}^2$, the curvature R is a two-form with coefficients in $\operatorname{ad} P$ and ϕ is a section of $(\operatorname{ad} P) \otimes \mathbb{C}$. If we write $z = x_1 + ix_2$ and define

$$\Phi = \frac{1}{2}\, \phi\, dz \in \Omega^{1,0}(\mathbb{R}^2, (\operatorname{ad} P) \otimes \mathbb{C})\,, \tag{B.10}$$

the self-dual Yang-Mills equations (B.9) become respectively

$$R + [\Phi, \Phi^*] = 0\,, \qquad D''\Phi = 0\,. \tag{B.11}$$

The second equation simply says that Φ is holomorphic. These equations are conformally invariant, and thus may be studied on a compact Riemann surface (see [16]). Hitchin studied in fact principal G bundles over a compact Riemann surface when $G = SU(2)$, and he found that a solution of the Yang-Mills equations defined a pair (V, ϕ) consisting of a rank two vector bundle V and a morphism $\phi : V \longrightarrow V \otimes \Omega^1_X$. Soon after, Simpson [17],[18] generalized these ideas to holomorphic bundles.

Simpson studied pairs (E, ϕ), where E is a holomorphic bundle of arbitrary rank over n-dimensional Kähler manifolds X (not necessarily compact, but satisfying some additional conditions) and $\phi : E \longrightarrow E \otimes \Omega^1_X$ is a morphism of $\mathcal{O}_X$-modules satisfying the condition[2] $\phi \wedge \phi = 0$. Simpson called such pairs *Higgs bundles*.

[1]This terminology has been used by Hitchin in [16]. This means that A_3 and A_4 are extra fields, which do not form part of the connection. However, notice that such fields give contributions to the Lagrangian of Yang-Mills involving derivatives, and hence, they are not auxiliary fields in the sense of quantum field theory.

[2]Notice that such a condition is automatically satisfied in the one-dimensional case (over a compact Riemann surface).

Using local coordinates on X, we can write

$$\phi(z) = \sum_{\alpha=1}^{n} \phi_\alpha(z)\, dz^\alpha \,, \tag{B.12}$$

where each $\phi_\alpha(z)$ is an endomorphism of the fiber E_z. Hence

$$\begin{aligned}(\phi \wedge \phi)(z) &= \sum_\alpha \phi_\alpha(z)\, dz^\alpha \wedge \sum_\beta \phi_\beta(z)\, dz^\beta \\ &= \sum_{\alpha<\beta} [\phi_\alpha(z), \phi_\beta(z)]\, dz^\alpha \wedge dz^\beta \,.\end{aligned}$$

Thus, the condition $\phi \wedge \phi = 0$ is equivalent to

$$[\phi_\alpha(z), \phi_\beta(z)] = 0 \tag{B.13}$$

for all α, β and all z. In other words, $\phi \wedge \phi = 0$ is equivalent to the commutativity of the endomorphisms ϕ_α of E.

Let X be a compact Riemann surface and K the canonical line bundle on it. Then, we can construct a Higgs bundle as follows: let $K^{1/2}$ be a holomorphic line bundle such that $K = K^{1/2} \otimes K^{1/2}$ and let $E = K^{1/2} \oplus K^{-1/2}$. Then E is a holomorphic vector bundle of rank two over X. Now, let $\phi : E \longrightarrow E \otimes \Omega^1_X$ be the morphism given by

$$\phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where 1 represents the canonical section of $\mathrm{Hom}(K^{1/2}, K^{-1/2}) \otimes K$ (notice that $\mathrm{Hom}(K^{1/2}, K^{-1/2}) = K^{-1}$ and hence it is well defined). Since $\dim X = 1$, automatically $\phi \wedge \phi = 0$; Furthermore,

$$c_1(E) = c_1(K^{1/2}) + c_1(K^{-1/2}) = 0$$

and hence, the pair $\mathfrak{E} = (E, \phi)$ becomes a Higgs bundle of rank two over X with zero degree. Suppose now that X has a genus $g > 1$, then $\deg K^{1/2} > 0$ and E is not stable in the ordinary sense. Now, since

$$\phi(K^{-1/2}) \subset K^{-1/2} \otimes \Omega^1_X$$

and it is the only ϕ-invariant subbundle of E, such a bundle is stable as a Higgs bundle.

This example shows some general fact about Higgs sheaves. Namely, a stable Higgs sheaf is not necessarily stable in the ordinary case. Now, clearly from the definition the converse is always true, i.e., if a Higgs sheaf is stable in the ordinary sense, it is stable as a Higgs sheaf.

B.2 Blow-ups

The notion of blow-up is an important tool in the resolution of singularities, in this Appendix we briefly review some basic facts about the projective space and the notion of blow-up (for more details see the references [36], [1], [39]).

On $\mathbb{P}^n$ there is a natural Kähler metric, called the *Fubini-Study metric*, which can be constructed as follows. Let $z_0, ..., z_n$ be coordinates on $\mathbb{C}^{n+1}$ and denote by $\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$ the natural projection map. Let $U \subset \mathbb{P}^n$ be an open set and $Z : U \longrightarrow \mathbb{C}^{n+1} - \{0\}$ a lifting of U, i.e., a holomorphic map with $\pi \circ Z = \mathrm{id}_U$, and consider the (1,1)-form

$$\eta = \frac{i}{2\pi} d'd'' \log \|Z\|^2 \,. \tag{B.14}$$

This form is closed. Indeed, it is exact because it can be rewritten as

$$\begin{aligned} \eta &= \frac{i}{4\pi}(d' + d'')(d'' - d')\log\|Z\|^2 \\ &= \frac{i}{4\pi} d(d' - d'')\log\|Z\|^2 \,. \end{aligned}$$

The form η defined by (B.14) is also independent of the lifting. In fact, if $Z' : U \longrightarrow \mathbb{C}^{n+1} - \{0\}$ is another lifting, then necessarily $Z' = fZ$ with f a nonzero holomorhic function and

$$\log\|Z'\|^2 = \log\|Z\|^2 + \log f + \log \bar{f} \,. \tag{B.15}$$

Since f is holomorphic, $d''f = d'\bar{f} = 0$ and hence Z' defines the same form η. Now, this form is also positive definite. In order to prove this, it is sufficient to prove the positivity at one point (the unitary group $U(n+1)$ leaves η invariant and acts transitively on $\mathbb{P}^n$). Let $w_\alpha = z_\alpha / z_0$ be the coordinates on the open set $U_0 = \{z_0 \neq 0\}$ in $\mathbb{P}^n$ and consider the lifting on U_0 given by $Z = (1, z_1, ..., z_n)$, we obtain (ommiting the summation symbol for simplicity)

$$\begin{aligned} \eta &= \frac{i}{2\pi} d'd'' \log\left[1 + w_\alpha \bar{w}_\alpha\right] \\ &= \frac{i}{2\pi} d' \left[\frac{w_\alpha d\bar{w}_\alpha}{1 + w_\alpha \bar{w}_\alpha}\right] \\ &= \frac{i}{2\pi}\left[\frac{dw_\alpha \wedge d\bar{w}_\alpha}{1 + w_\alpha \bar{w}_\alpha} - \frac{\bar{w}_\alpha dw_\alpha \wedge w_\beta d\bar{w}_\beta}{(1 + w_\alpha \bar{w}_\alpha)^2}\right] . \end{aligned}$$

At the point $[1, 0, ..., 0]$ in $\mathbb{P}^n$

$$\eta = \frac{i}{2\pi} \sum dw_\alpha \wedge d\bar{w}_\alpha \,, \tag{B.16}$$

which is clearly positive definite[3] and hence is a Kähler metric on the projective space.

[3]Remember that in general, using holomorphic coordinates $w_1, ..., w_n$, a (1,1)-form ω is said to be positive if the corresponding matrix $g_{ij}(z)$ is a positive definite matrix for each w.

Now we review the notion of blow-up at a point. Let consider the space $\mathbb{C}^n$ with $n \geq 2$ and let $z = (z_1, ..., z_n)$ be the standard coordinates of $\mathbb{C}^n$ and $w = [w_1, ..., w_n]$ be the normal coordinates of $\mathbb{P}^{n-1}$. If B is a small ball center at $0 \in \mathbb{C}^n$, we define

$$\tilde{B} = \{(z, w) \in B \times \mathbb{P}^{n-1} / z_i w_j = z_j w_i\} \,. \tag{B.17}$$

This is a complex submanifold of $B \times \mathbb{P}^{n-1}$ of dimension n. We have natural mappings $\pi : \tilde{B} \hookrightarrow B \times \mathbb{P}^{n-1} \to B$ and $\pi' : \tilde{B} \hookrightarrow B \times \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$. The surjective mapping π is called the *blow-up of B at* 0. This map has the following important property:

$$\pi^{-1}(z) = (z, [z]) \quad \text{if} \quad 0 \neq z \in B \qquad \text{and} \quad \pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1} \,. \tag{B.18}$$

The map π restricted to $\tilde{B} \backslash \pi^{-1}(0)$ is a biholomorphism, $\pi^{-1}(0)$ is a codimension one submanifold of $\tilde{B}$ and is usually called the *exceptional divisor* of the blow-up.

Now, let X be a complex manifold of dimension $n \geq 2$ and $x \in X$ a fixed point and consider a neighborhood U of x biholomorphic to B in $\mathbb{C}^n$. Let $\tilde{X}$ be the manifold obtained from X by replacing $U \cong \tilde{B}$, then we have a surjective holomorphic map from $\tilde{X}$ onto X, which we still denote by π, with

$$L \equiv \pi^{-1}(x) \cong \mathbb{P}^{n-1} \tag{B.19}$$

and such that restricted to $\tilde{X} \backslash L$ it is a biholomorphism. The map π is called the *blow-up of X at x*.

It is possible also to consider the blow-up along a submanifold (see [36] or [49]). If S is a complex submanifold of X with codimension greater or equal than two, we can blow-up X along S. As a consequence of this, we obtain a complex manifold $\tilde{X}$ and a surjective holomorphic map $\pi : \tilde{X} \longrightarrow X$ which is called the *blow-up of X along S*; again π restricted to $\tilde{X} \backslash \pi^{-1}(S)$ is a biholomorphism, the hypersurface $\pi^{-1}(S)$ is called the *exceptional divisor* and S is called the *center of the blow-up.*

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