ASPECTS OF KALUZA-KLEIN THEORIES

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ABSTRACT

The review part of thesis contains detailed discussions of five-dimensional Kaluza-Klein theory (KKT), zero-mode ansatz and six-dimensional model due to Randjbar-Daemi, Salam & Strathdee as well as basic information about different compactification mechanisms, stability problem, treatment of fermions in KKT harmonic expansion on homogeneous spaces, chiral anomalies and model proposed by Candelas & Weinberg. Original contributions are devoted to different aspects of KKT. Compactification of D=10 dimansional SU(3)xU(1) Einstein-Yang-Mills (EYM) theory to M_4xCP(3) is shown to be classically stable. The gauge symmetry seen in four dimensions is SU(4)xU(1), first being the isometry of CP(3) and second - an unbroken part of initial gauge group. The topologically nontrivial background configuration of SU(3)x U(1) gauge fields makes it possible to obtain massless chiral fermions in four dimensions after compactification. Asymmetry in left and right handed zero modes agrees with that predicted by the Atiyah & Singer theorem. A relation between spontaneous compactification mechanism and chiral anomalies is investigated. A simple model (SU(3) EYM theory in D=6 dimensions with M_4xS_2 background geometry) is used to argue that the theory is free from chiral anomalies in higher dimensions if and only if the effective theory of zero modes in four dimensions is also anomaly-free. Correspondence between different types of gauge and gravitational (and mix) anomalies in D=6 and D=4 dimensions is displayed. Two possibilities of getting a natural symmetry breaking mechanism in KKT are investigated. In the framework of KKT with elementary gauge fields in higher dimensions, it is shown that one can obtain a solution of classical field equations with infinitesimally deformed N-sphere as the internal manifold, if a multiplet of scalar fields is added to the theory. In the D=6 dimensional model the symmetry breaking pattern

$O(3) \times U(1) \rightarrow O(2)$ (a subgroup of $O(3)$) results. Masses of initially massless (as deformation vanishes) vector gauge bosons are calculated. They are of order ε/a, where ε is a deformation parameter, and a is a Planck's length. The deformed background configuration can be made classically stable. In the context of the model due to Candelas & Weinberg, a total effective potential for a D=7 dimensional case with massless scalar fields minimally coupled to gravity is calculated. The background configuration is taken to be $M_4 \times S_3$, S_3 being a homogeneously deformed three-sphere with isometry $SU(2) \times U(1)$. The effective potential as a function of two parameters (scale of S_3 and deformation) has a local minimum for a non-zero deformation. The round S_3 corresponds to a local maximum of the potential. Therefore the dynamics itself (quantum fluctuations of scalar fields) can determine the actual shape of the internal manifold.

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I INTRODUCTORY REMARKS

This thesis are devoted to different aspects of Kaluza-Klein theories (KKT). The name is after two physisists who about 60 years ago made first attempt to unify different types of interactions by exploring an assumption that space-time is more than four-dimensional (Kaluza(1921),Klein(1926)). At their time the problem was to unify gravitation with electromagnetism. Five-dimensional gravitation in the space-time with topology $M_4 x S_1$ can do this since, as they showed, it contains usual D=4 dimensional gravity and Maxwell theory. Gauge symmetry arises as a part of general coordinate transformations in five dimensions (D=5). The theory does not give any predictions about the size of the "internal space" (in this case circle).

For the next 50 years not many tried to develop the idea of Kaluza and Klein (Pauli(1933), Einstein & Bergmann (1938), Jordan(1947), De Witt(1965), Rayski(1965), Kerner(1968), Trautman (1975), Cho(1975)). Probably the most important achievment of this period was an observation that with any compact space B_N with an isometry group K in place of S_1 it can be shown that 4+N dimensional gravity contains usual D=4 dimensional gravity and Yang-Mills theory with Gauge group K. Therefore, all the gauge symmetries seen in D=4 may have a purely geometrical origin.

A renewal of an interest in KKT started in late 70-th with a construction of supergravity theories in more than four dimensions. In particular a D=11 supergravity attracted a lot of atension after Witten made an observation that eleven is simultaeously a maximal number of space-time dimensions for a supergravity theory and a minimal one necessary to obtain SU(3)xSU(2)xU(1) gauge symmetry as the isometry of an internal space (Witten (1981a).

Recently string theories are very fashionable. These theories are consistently formulated only in more than four (ten or twenty six) space-time dimensions being in this sense an example of KKT. String theories could unify all the interactions together with gravitation (unlike in Grand Unified Theories) in one finite, anomaly free theory (Green & Schwarz(1984)). They differ from standard KKT by the fact that (at least in a most promising approach - Candelas et al. (1985)) the four dimensional gauge group symmetry does not arise as the isometry of an internal space but rather as an unbroken part of an initial large gauge group (O(32) or $E_8 x E_8$).

KKT can be viewed as theories in which an effect similar to the spontaneous symmetry breaking occurs. A theory formulated originally in 4+N dimensions is supposed to have a ground state with topology $M_4 x B_N$ instead of M_{4+N}. Its symmetry group is therefore $P_4 x G_B$ rather than P_{4+N} (P_4 is a Poincaré group, G_B is the isometry group of B_N). Unfortunately, unlike in the usual Higgs mechanism it is very difficult to compare between different candidates for a ground state of the theory, if these candidates correspond to different topologies. A question whether the $M_4 x B_N$ state is quantum mechanically stable is a very important one - but to much degree open (Witten(1981b)).

The mechanism of symmetry breaking leading to KKT is sometimes called a <u>spontaneous compactification</u>. Usually it is assumed that the ground state is a solution of classical equations of motion. In these theories the size of internal space is undetermined. This size must be small enough to be invisible at present energies (i.e. we do not observe massive excitations on B_N). Typically there will be a relation between this size and the value of the gauge theory coupling constant. If we would like to get the coupling constant not much less than 1, a typical length of B_N would differ at most by few orders of magnitude from the Planck length (10^{-33}cm).

If internal dimensions are so small, quantum effects can become relevant. This is why "self-consistent" models are discussed in the literature. In these models one looks for extrema of an effective action with 1-loop quantum effects included. 1-loop contributions may come from a graviton or some (scalar or fermion) additional fields. Self-consistent models have an important fearture that the size of the internal space is dynamically determined.

One of the biggest problems in KKT was the difficulty in obtaining chiral massless fermions after spontaneous compactification (Palla(1978), Witten(1983)). Several ideas were proposed to solve this problem. The most appealing one was to introduce elementary gauge fields into a 4+N dimensional theory and assume topologically nontrivial vacuum configuration of them. This kind of models will be most often discussed in these thesis.

With chiral fermions in the theory one must deal with a problem of chiral anomalies. The study of anomalies in multidimensional theories has become a very fruitful idea during last few years. The understanding of the structure of all the gauge and gravitational chiral anomalies led to the discovery of anomaly free superstring theories.

Superstring theories are very promising. There is a hope that they can provide a first consistent quantum theory of gravity. A finitness of multiloop amplitudes is however still far from being proved and requires a lot of research. The question about <u>physical</u> relevance of multidimensional theories is also still open. Of course, a range of phenomena which such theories should explain is nowadays much bigger than 60 years ago.

The plan of my thesis is the following:

In Chapter II a general review of the sub ect is given.

In particular D=5 dimensional model (Section A) and a zero-mode ansatz (Section C) are discussed. After a brief explanation of different compactification mechanisms (Section D) the one with elementary gauge fields is presented in detail (Section E). Later, a main idea of a harmonic expansion technique is shown (Section F). Remarks on the stability problem (Section G) are followed by a describtion of a 6-dimensional model due to Randjbar-Daemi, Salam and Strathdee (1983) (Section H). Treatment of fermions in KKT (Section I), anomalies (Section J) and another compactification mechanism - with quantum corrections playing a crutial role (Section K) are remaining topics discussed in Chapter II.

In next Chapters I present my contributions to the subject based on papers either already published or just being prepared for a publication.

In Chapters III and IV a particular D=10 dimensional model with SU(3)xU(1) elementary gauge fields is discussed. The main result is that M_4xCP(3) compactification is classicaly stable and that the gauge group in D=4 is SU(4)xU(1), the first coming as the isometry of CP(3), the other one being an unbroken part of the initial gauge group. It is also shown that chiral fermions are available in this compactification.

In Chapter V I consider a problem of a relation between the spontaneous compactification mechanism and a property of a theory of being free from chiral anomalies. I discuss a D=6 theory with SU(3) gauge symmetry with compactification induced by a magnetic monopole configuration of a Maxwell field on S_2. It turns out that the cancellation of all kinds of anomalies in D=6 is strictly related with the cancellation of anomalies in D=4 for massless fermions produced in the process of spontaneous compactification. The anomalies in D=4 vanish if and only if the anomalies in D=6 also vanish.

In Chapter VI I propose a mechanism for symmetry breaking in D=4 having a geometrical interpretation as the deformation of an internal space. By introducing a scalar multiplet into a theory a solution of classical equations of motion is found with infinitesimally deformed S_N. In a D=6 model this solution leads to the symmetry breaking pattern $O(3)xU(1) \rightarrow O(2)$ (a subgroup of O(3)). In perturbation theory corrections to the masses of vector bosons corresponding to the broken part of the O(3)xU(1) gauge symmetry are calculated. The solution with deformed internal space (S_2) is classically stable.

In Chapters VII and VIII I present results obtained in collaboration with Dr. T.C. Shen. In Chapter VII different analytical continuation schemes useful in self-consistent KKT are discussed. In Chapter VIII the calculation of an effective potential for a massless scalar field minimally coupled to gravity in the M_4xS_3 background is presented. S_3 is here a homogeneously deformed 3-sphere. The main result is the existence of deformed S_3 solutions of the quantum-corrected equations of motion. This may be seen as a kind of "dynamical symmetry breaking". <u>It opens an interesting possibility that in the higher dimensional theories dynamics itself may determine the shape of the internal space.</u>

II REVIEW OF KALUZA-KLEIN THEORIES

Not many review articles on KKT exist. The most interesting are due to Witten (1981b), Van Nieuvenhuizen (1984) (contains almost complete Kaluza-Klein bibliography till 1983),Duff Nilsson & Pope (1986) (contains a detailed discussion of a D=11 supergravity), Strathdee (1986). There is also a book with a collection of articles on KKT : Lee (1985).

A. Maxwell theory from 5-dimensional gravity

A1. Basic calculation

It is an old observation that D=5 gravity contains a usual D=4 gravity and a Maxwell theory. Before giving more precise definition of what is understood by "containing" I would like to remind a standard argument.

Let parametrize a D=5 metric after Chodos(1984)

$$g_{MN}(z) = \phi^{-1/3}(z)\begin{bmatrix} g_{mn}(z) + \phi(z) A_m(z) A_n(z) & \phi(z) A_m(z) \\ \phi(z) A_n(z) & \phi(z) \end{bmatrix} \qquad \text{(II.1)}$$

$M=(m,5)$ $m=0,1,2,3$;we omit an index 5; z is a coordinate of a point in D=5, $z=(x,y)$

For a moment it is just a change of variables and we do not loose any generality.

We assume now $g_{MN}(z) = g_{MN}(x,y) = g_{MN}(x)$ so that the metric does not depend on one (circle-shaped) space-like dimension. This assumption is usually called a zero-mode ansatz (see Subsection IIA4).

One should express now the Einstein-Hilbert action in terms of $g_{mn}(x)$, $A_m(x)$, $\phi(x)$. This calculation is still rather tedious. $\phi(x)$ corresponds to the massless scalar (Brans-Dicke scalar) present in the theory. In this Section we are not interested in scalar fields and put $\phi = 1$ in the computations, even if it is inconsistent with equations of motion. In the convention $R_{MN}{}^{K}{}_{L} = \partial_M \Gamma^K_{NL} - \partial_N \Gamma^K_{ML} + \Gamma^K_{MS}\Gamma^S_{NL} - \Gamma^K_{NS}\Gamma^S_{ML}$ we get

$$R_{5r}{}^{5}{}_{5} = \frac{1}{4} F_t{}^5 F_r{}^t$$

$$R_{5r}{}^{t}{}_{5} = \frac{1}{2} \nabla_r F_5{}^t + \frac{1}{4} F_n{}^t A_5 F_r{}^n$$

$$\begin{aligned} R_{mr}{}^{t}{}_{n} = {} & \frac{1}{2} A_r \nabla_m F_n{}^t + \frac{1}{2} A_n \nabla_m F_r{}^t - \frac{1}{2} A_m \nabla_r F_n{}^t \\ & - \frac{1}{2} A_n \nabla_r F_m{}^t + \frac{1}{4} F_m{}^t F_{nr} - \frac{1}{4} F_r{}^t F_{nm} + \frac{1}{2} F_{mr} F_n{}^t \\ & + \frac{1}{4} A_m A_n F_l{}^t F_r{}^l - \frac{1}{4} A_r A_n F_l{}^t F_m{}^l \\ & + \tilde{R}_{mr}{}^{t}{}_{n} \end{aligned} \tag{II.2}$$

$$R_{m5}{}^{r}{}_{n} = -\frac{1}{2} A_r \nabla_m F_n{}^r + \frac{1}{4} F_n{}^l F_{lm} + \frac{1}{4} A^r A_n F_{lr} F_m{}^l$$

In the above expressions all the indices are raised up and lowered by g^{mn} and g_{mn} : $g^{mn} g_{nk} = \delta^m{}_k$. $\tilde{R}_{mr}{}^{t}{}_{n}$ is a Riemann tensor calculated for $g_{mn}(x)$; $F_{mn} = \partial_m A_n - \partial_n A_m$.

From (II.2) we get

$$R = \tilde{R} + \frac{1}{4} F^2 \tag{II.3}$$

Since

$$\det g_{MN} = \det g_{mn} \tag{II.4}$$

we can integrate over y the E-H action in D=5

$$-\int d^4x\, dy \sqrt{-g}\, \frac{R}{\kappa^2} = -\frac{2\pi a}{\kappa^2} \int d^4x \sqrt{-\tilde{g}} \left(\tilde{R} + \frac{F^2}{4} \right) \tag{II.5}$$

a is a radius of an "internal circle". If we want to introduce a "physical" Maxwell field $A_m(x)$ with dimensionality +1 we must define it as

$$\tilde{A}_m = \sqrt{\frac{2\pi a}{\varkappa^2}}\, A_m \tag{II.6}$$

A2. Gauge symmetry

Equation (II.5) suggests that the D=5 gravity contains electromagnetism. But what about gauge transformation? We know that the D=5 dimensional E-H action is invariant under general coordinate transformations (GCT)

$$\delta g_{MN} = \xi^P \partial_P g_{MN} + \partial_M \xi^P g_{PN} + \partial_N \xi^P g_{MP} \tag{II.7}$$

If we take a parameter (arbitrary) $\xi^P(x,y) = \delta^{P5}\xi^5(x)$ then we obtain

$$\begin{aligned} \delta g_{55} &= 0 \\ \delta g_{m5} &= \phi\, \partial_m \xi^5 \\ \delta g_{mn} &= \phi\left(\partial_m \xi^5 \cdot A_n + \partial_n \xi^5 \cdot A_m\right) \end{aligned} \tag{II.8}$$

so that (look at (II.1))

$$\delta A_m(x) = \partial_m \xi^5(x) \tag{II.9}$$

A3. Scalar field coupled to D=5 dimensional gravity

Consider a complex scalar field φ minimally coupled to gravity in D=5 dimensions. Does this mean that φ is coupled to the $\tilde{A}_m$ field (see Eq.(II.6)) in D=4 dimensions?

$$S = -\int d^4x \int dy \sqrt{-g}\left(g^{MN}\partial_M \varphi^* \partial_N \varphi + M^2 \varphi^* \varphi\right) \tag{II.10}$$

(my metric convention is -++++...+)

$$g^{MN}\partial_M\varphi^*\partial_N\varphi = g^{mn}\partial_m\varphi^*\partial_n\varphi + (1+A^2)\partial_5\varphi^*\partial_5\varphi - A^m\left(\partial_m\varphi^*\partial_5\varphi + \partial_5\varphi^*\partial_m\varphi\right) \tag{II.11}$$

We assume that the fifth space-time dimension (space-like) is a circle of a radius a. Therefore

$$\varphi(x,y) = \sum_{n=-\infty}^{+\infty} \varphi^{(n)}(x) \exp\left(\frac{iny}{a}\right) \tag{II.12}$$

If we use (II.12) in (II.10) and integrate over y the action (II.10) factorizes into parts with different (n) lebels. Terms with $\varphi^{(n)}$ are

$$S = -(2\pi a)\int d^4x\sqrt{-g}\Big[g^{mk}\left(\partial_m\varphi^{*(n)} + i\frac{n}{a}A_m\varphi^{*(n)}\right)\left(\partial_k\varphi - i\frac{n}{a}A_k\varphi^{(n)}\right) + \left(M^2+\frac{n^2}{a^2}\right)\varphi^{*(n)}\varphi^{(n)}\Big] \tag{II.13}$$

We see that $\varphi^{(n)}$ is mimimally coupled to $\tilde{A}_m = A_m\sqrt{\frac{2\pi a}{\varkappa^2}}$ Maxwell field and carries a U(1) charge $\frac{\varkappa n}{\sqrt{2\pi a^3}}$. The mass term for $\varphi^{(n)}$ is modified by a term $\frac{n^2}{a^2}$.

Important point to stress is that in the discussed theory neither a nor the value of the U(1) coupling constant e are fixed. They are related to each other: $e \sim \frac{\varkappa}{a^{3/2}}$.

From (II.5) it follows that the relation between $\varkappa$ and the Newton`s constant is (after integration out extra dimension)

$$\frac{2\pi a}{\varkappa^2} = \frac{1}{16\pi G_4}$$

so that

$$e^2 \sim \frac{G_4}{a^2} \tag{II.14}$$

This is a <u>typical</u> relation between e^2 , a^2 and the Newton's constant.

A4. Computation of the spectrum

In this Subsection I will calculate a full spectrum of the theory. To this aim I compute the part of the action bilinear in fluctuation around the background (which is $M_4 x S_1$).

$$\begin{aligned} g_{MN} &= \eta_{MN} + \varkappa h_{MN} \\ g^{KL} &= \eta^{KL} - \varkappa h^{KL} + \varkappa h^K{}_N h^{NL} \end{aligned} \tag{II.15}$$

$\varkappa$ is introduced for later convenience.
We first get:

$$\Gamma_{PMN} = \tilde{\Gamma}_{PMN} + \frac{\varkappa}{2}\left(\nabla_M h_{NP} + \nabla_N h_{MP} - \nabla_P h_{MN}\right)$$
$$+ \varkappa \tilde{\Gamma}^{L}_{MN} h_{LP} \qquad \text{(II.16)}$$

$$\Gamma^{S}_{MN} = \tilde{\Gamma}^{S}_{MN} + \frac{\varkappa}{2}\eta^{PS}\left(\nabla_M h_{NP} + \nabla_N h_{MP} - \nabla_P h_{MN}\right) -$$
$$- \frac{\varkappa^2}{2} h^{PS}\left(\nabla_M h_{NP} + \nabla_N h_{MP} - \nabla_P h_{MN}\right) \qquad \text{(II.17)}$$

Riemann`s tensor is:

$$R_{MN}{}^{P}{}_{Q} = \tilde{R}_{MN}{}^{P}{}_{Q} + \frac{\varkappa}{2}\eta^{PR}\left(\nabla_M \nabla_N h_{QR} + \nabla_M \nabla_Q h_{NR} + \nabla_M \nabla_R h_{NQ}\right.$$
$$\left. - \nabla_N \nabla_M h_{QR} - \nabla_N \nabla_Q h_{MR} + \nabla_N \nabla_R h_{MQ}\right) + \frac{\varkappa^2}{4}\left(\nabla_M h_R{}^P + \nabla_R h_M{}^P\right.$$
$$\left. - \nabla^P h_{MR}\right)\left(\nabla_N h_Q{}^R + \nabla_Q h_N{}^R - \nabla^R h_{NQ}\right) - \frac{\varkappa^2}{4}\left(\nabla_N h_R{}^P + \nabla_R h_N{}^P - \right.$$
$$\left. - \nabla^P h_{NR}\right)\left(\nabla_M h_Q{}^R + \nabla_Q h_M{}^R - \nabla^R h_{MQ}\right) \qquad \text{(II.18)}$$

The scalar curvature is then calculated to be:

$$R = \tilde{R} - \varkappa \tilde{R}_{MN} h^{MN} + \varkappa \nabla^2 h_M{}^M - \varkappa \nabla_N \nabla_M h^{MN} + \varkappa^2 \tilde{R}_{MQ} h^M{}_L h^{QL}$$
$$+ \frac{\varkappa^2}{2} h_{RN} \nabla^M \nabla^N h_M{}^R - \frac{\varkappa^2}{2} h_N{}^N \nabla^2 h_M{}^M - \frac{\varkappa^2}{2} h_{MR} \nabla^2 h^{MR} \qquad \text{(II.19)}$$

Since:

$$\sqrt{-g} = \sqrt{-\tilde{g}}\left(\frac{\varkappa^2}{4} h_M{}^M \nabla^2 h_N{}^N - \frac{\varkappa^2}{4} h_{MN} \nabla^2 h^{MN} - \frac{\varkappa^2}{2} h_{MN} \nabla_R \nabla^M h^{NR}\right) \qquad \text{(II.20)}$$

we finally get:

$$\left(\sqrt{-g}R\right)_{bil} = \sqrt{-\tilde{g}}\,\varkappa^2\left[\frac{1}{4} h_M{}^M \nabla^2 h_N{}^N - \frac{1}{4} h_{MN} \nabla^2 h^{MN} + \frac{1}{2} h_{MN} \nabla_R \nabla^M h^{NK}\right.$$
$$- \frac{1}{2} h_M{}^M \nabla_K \nabla_N h^{KN} + \tilde{R}\left(\frac{1}{8} h_M{}^M h_N{}^N - \frac{1}{4} h_{MN} h^{MN}\right)$$
$$\left. + \tilde{R}_{MN}\left(h^{MK} h^N{}_K - \frac{1}{2} h^{MN} h_K{}^K\right)\right] \qquad \text{(II.21)}$$

In the above computation:

1/ All the covariant derivatives are calculated with respect to the background geometry (II.15).

2/ $\tilde{\Gamma}^{\,|}_{||}$ $\tilde{R}_{||}{}^{|}{}_{|}$ refer to the background geometry.

3/ My convention for the Ricci tensor is $R_{ML} = R_{MK}{}^{K}{}_{L}$.

Now we use

$$[\nabla_K, \nabla_M] h^{NK} = \tilde{R}_{MK}{}^{TN} h_T{}^K - \tilde{R}_M{}^T h^N{}_T \qquad \text{(II.22)}$$

In the particular case we are interested in ($M_4 x S_1$) $\tilde{R}_{MN\ S}^{\ \ \ P} = 0$. Therefore we obtain:

$$(\sqrt{-g}R)_{b.l} = \varkappa^2 \Big[\frac{1}{4} h_M{}^M \nabla^2 h_N{}^N - \frac{1}{4} h_{MN} \nabla^2 h^{MN} - \\ - \frac{1}{2} h_M{}^M \nabla_K \nabla_L h^{KL} + \frac{1}{2} h_{MN} \nabla^M \nabla_K h^{NK} \Big] \qquad \text{(II.23)}$$

For the purpose of spectrum computation it is very convenient to choose a light-cone gauge (Randjbar-Daemi, Salam, Strathdee (1984b)) so that only <u>physical</u> degrees of freedom remain. After some algebra we get:

$$S_{b.l} = \int d^4x \int dy \Big[\frac{1}{4} h_{jk}^T (\partial^2 + \nabla^2) h_{jk}^T + \\ + \frac{1}{2} h_{j5} (\partial^2 + \nabla^2) h_{j5} + \frac{3}{8} h_{55} (\partial^2 + \nabla^2) h_{55} \Big] \qquad \text{(II.24)}$$

j,k indices refer to four dimensions. In this simplest possible case $\nabla^2 = \frac{\partial^2}{\partial y^2}$.

Now the assumption that the topology of the D=5 dimensional world is $M_4 x S_1$ is used and all the fields are expanded in Fourier series (the simplest harmonic expansion)

$$h_{jk}^T(x,y) = \sum_{n=-\infty}^{+\infty} h_{jk}^{T(n)}(x) \exp\left(\frac{iny}{a}\right) \qquad \text{(II.25)}$$

$$h_{j5}(x,y) = \sum_{n=-\infty}^{+\infty} h_{j5}^{(n)}(x) \exp\left(\frac{iny}{a}\right) \qquad \text{(II.26)}$$

$$h_{55}(x,y) = \sum_{n=-\infty}^{+\infty} h_{55}^{(n)}(x) \exp\left(\frac{iny}{a}\right) \qquad \text{(II.27)}$$

The spectrum is found to be:

1/ Massless helicity 0,1,2 particles.
2/ Masssive spin 2 particles with masses n^2/a^2, $n \in Z$ (modes with the same mass and spin 0,1,2 combine into a massive spin 2 particle).

Massless states are those with n = 0 i.e. not depending on the internal coordinate what justifies a zero-mode ansatz used before.

The fact that the D=4 dimensional spectrum contains a massless scalar particle is rather atypical. This particle is present since oscillations of ϕ correspond to a change in the radius of the fifth dimension (the line element in the fifth dimension is $\phi\, dy^2$; the radius is $2\pi a\sqrt{\phi}$). The classical Einstein equations do not determine the value of the radius, therefore ϕ is classically massless (Witten(1981b)).

A5. Solutions of D=5 dimensional wave equation for a graviton

Massless spectrum can be also computed using different argument (Peskin(1985)). We write a wave equation for a graviton in the D=5 dimensional cylindrical world ($M_4 x S_1$)

$$(\partial^2 + \nabla^2)\, h_{MN}(x,y) = 0$$

The solution is $h_{MN}(x,y) = \epsilon_{MN}\, e^{-ikx}$. The physical degrees of freedom can be identified after choosing a gauge

$$\epsilon_{0M} = \epsilon_{N0} = 0 \qquad \epsilon_M{}^M = 0 \qquad k^N \epsilon_{NM} = 0$$

If we assume for simplicity that $\vec{k} = (1,0,0)$, then five zero modes are found:

a) $\epsilon_{22} = -\epsilon_{33}$

b) $\epsilon_{23} = \epsilon_{32}$

c) $\epsilon_{25} = \epsilon_{52}$

d) $\epsilon_{35} = \epsilon_{53}$

e) $\epsilon_{22} = \epsilon_{33} = -\frac{1}{2}\epsilon_{55}$

Their D=4 dimensional interpretation is: (a) and (b) correspond to two different polarization states of the graviton; (c) and (d) correspond to the massless vector particle and (d) to the scalar.

B. NonAbelian gauge symmetry from extra dimensions

A straightforward generalization of the model discussed in Section II.A leads to the conclusion that D=4+N dimensional gravity contains the usual D=4 dimensional gravity theory and a Yang-Mills theory with a nonAbelian gauge group (De Witt (1965)). To this aim we must assume the space-time topology to be $M_4 x B_N$: B_N being a compact space admitting Killing vectors of the isometry group G: $K^\alpha_\mu(y)$, $\alpha=1,\dots,K$ (rank G). The form of the zero-mode ansatz is now:

$$g_{MN} = \begin{bmatrix} g_{mn} + g^{\mu\nu} K^\alpha_\mu K^\beta_\nu A^\alpha_m A^\beta_n & K^\alpha_\nu A^\alpha_n \\ K^\alpha_\mu A^\alpha_m & g_{\mu\nu} \end{bmatrix} \tag{II.28}$$

Similar to those presented in Section IIA calculations give (see Luciani (1978))

$$R\sqrt{-g} = \sqrt{-g_4}\sqrt{g_N}\left(R_4 + R_N + \frac{1}{4} h^{\alpha\beta} F^\alpha_{\mu\nu} F^{\beta\mu\nu}\right) \tag{II.29}$$

where R_4 and R_N are scalar curvatures calculated for g_{mn} and $g_{\mu\nu}$ respectively and

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - f^{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu \tag{II.30}$$

$f^{\alpha\beta\gamma}$ are the same structure constants as those in the commutator of Killing vector fields

$$K^\alpha = K^{\alpha\mu}\partial_\mu$$

$$[K^\alpha, K^\beta] = f^{\alpha\beta\gamma} K^\gamma \tag{II.31}$$

$h^{\alpha\beta}$ is a tensor obtained from Killing vectors: $h^{\alpha\beta} = K^\alpha_\mu K^{\beta\mu}$.
After integration out extra dimensions we get:

$$S = -\int d^4x \sqrt{-g_4}\left(\frac{R_4}{16\pi G_4} + \Lambda + \frac{F^2}{4g^2}\right) \tag{II.32}$$

where

$$\frac{1}{16\pi G_4} = \frac{V_N}{\varkappa^2} \tag{II.33}$$

V_N is a volume of B_N;

$$\Lambda = \frac{1}{\varkappa^2}\int d^N y \sqrt{g_N} R_N \tag{II.34}$$

Since

$$\int d^N y \sqrt{g_N}\, h^{\alpha\beta} = \delta^{\alpha\beta} V_N \frac{N}{K}$$

(see Luciani(1978)),

$$\frac{1}{g^2} = \frac{V_N}{\varkappa^2} \frac{N}{K} \qquad \text{(II.35)}$$

The determination of an effective Yang-Mills coupling constant is now more subtle than in was in the Section IIA. In order to have Yang-Mills fields $\tilde{A}_m$ with a correct D=4 dimensional dimensionality (+1), K_μ must carry a dimensionality -1 and $h^{\alpha\beta}$ -2. Therefore for the effective coupling constant $\tilde{g}$ we have:

$$\frac{1}{\tilde{g}^2} = \frac{V_N}{\varkappa^2} \frac{N}{K} a^2 = \frac{N}{K} \frac{a^2}{16\pi G_4} \qquad \text{(II.36)}$$

where a is a length scale of the internal manifold. This result may be compared with (II.14). Detailed analysis was done by Weinberg (1983).

The discussion of a nonAbelian gauge symmetry arising from a pure gravitational action in D=4+N dimensions is formal. This is because the theory should have a ground state (at least in the sense of a solution of classical equations of motion) of the geometry $M_4 x B_N$ (if we had discussed quantum corrections, spectrum analysis would become much more involved). For the action

$$S = - \int d^{4+N} z \sqrt{-g}\, \frac{1}{\varkappa^2} (R + \Lambda) \qquad \text{(II.37)}$$

we write Einstein equations

$$R_{MK} - \frac{R}{2} g_{MK} = \frac{1}{2} g_{MK} \Lambda \qquad \text{(II.38)}$$

We want the solution to have mentioned above geometry so that

$$R_{mn} = 0 \qquad R_{m\nu} = 0$$

$$R = R_{\mu\nu} g^{\mu\nu} \qquad \text{(II.39)}$$

Equation(II.38) splits into two: for 4- and N- dimensional parts

$$R + \Lambda = 0 \tag{II.40}$$

$$R_{\mu\nu} = \frac{1}{2} (\Lambda + R) g_{\mu\nu} = 0 \tag{II.41}$$

The internal space turns out to be Ricci flat. If we insist that B_N is a compact space, which is reasonable becouse of properties of differential (elliptic) operators defined on them (hovewer not necessary as advocated by Wetterich(1984)), Moncrief theorem states that B_N cannot support nonAbelian Killing fields.

We can make our assumption weaker and allow M_4 to be other maximally symmetric space i.e. de Sitter ($C<0$) or anti de Sitter ($C>0$) space

$$R_{mn} = C g_{mn} \tag{II.42}$$

$$R = 4C + R_N \tag{II.43}$$

Equations of motion become then (assuming that the internal space in an Einstein space)

$$C = \frac{1}{2} (\Lambda + R) = \frac{1}{2} (\Lambda + 4C + R_N) \tag{II.44}$$

$$2 R_N = N (\Lambda + R) \tag{II.45}$$

We obtain

$$C = - \frac{\Lambda}{N+2} \tag{II.46}$$

$$R_N = - \frac{N\Lambda}{N+2} \tag{II.47}$$

R_N denotes a scalar curvature of the internal space.
If $\Lambda > 0$, M_4 is a de Sitter space which is doubtful as a candidate for a ground state since positive energy theorem cannot be proved for it.
If $\Lambda < 0$, B_N has a positive scalar curvature and then no isometries (Duff(1984)).
We conclude that we cannot get a physically interesting model

via the Kaluza-Klein mechanism from the action (II.37).

C. Zero-mode ansatz

In the Kaluza-Klein literature one can find different forms of a metric of the D=4+N dimensional world. I think it is useful to make few comments about them.

The most basic one is a metric corresponding to the ground state of the theory:

$$g_{MN} = \begin{bmatrix} \eta_{mn} & 0 \\ 0 & g_{\mu\nu}(y) \end{bmatrix} \tag{II.48}$$

Symmetries of the ground state should be reflected in the massless modes present in the theory (in D=4 dimensions). These symmetries typically contain: Poincaré group and some local (i.e. x-dependent) isometry group of the internal space-time (with the metric $g_{\mu\nu}(y)$). Spectrum will contain graviton and massless vector particles. This can be seen by analysing small fluctuations around the background metric (II.48).

Zero-mode ansatz is a form of the metric in which some specific fluctuations (corresponding to massless states) are singled out. Typically one writes:

$$g_{MN}(z) = \begin{bmatrix} g_{mn}(x) + K^{\alpha}_{\mu}(y) K^{\beta}_{\nu}(y) A^{\alpha}_{m}(x) A^{\beta}_{n}(x) g^{\mu\nu}(y) \; ; & K^{\alpha}_{\nu}(y) A^{\alpha}_{m}(x) \\ K^{\alpha}_{\mu}(y) A^{\alpha}_{n}(x) & ; \; g_{\mu\nu}(y) \end{bmatrix} \tag{II.49}$$

One can ask question: what justifies this form of the zero-mode ansatz? Is the same ansatz true in theories containing other matter fields?

The fact that massless vector states should be associated with Killing vectors on B_N can be seen by using the following argument. If we calculate the part of the Hilbert-Einstein action (with cosmological constant Λ), bilinear in fluctuations (in the light-cone gauge), we get among other terms the following one:

$$\frac{1}{2} h_{\alpha j} \left(\partial^2 + \nabla^2 + R + \Lambda - R_{\alpha\beta} \right) h_{\beta j} \tag{II.50}$$

j is a D=4 dimensional Lorentz index; α, β are internal space indices. We can think of $-\nabla^2 - R - \Lambda + R_{\alpha\beta}$ as of a mass operator. The assumption that the D=4 dimensional part of the space -time is flat gives on the level of equations of motion $R + \Lambda = 0$ and the spectrum of vector states is given by the operator $-\nabla^2 + R_{\alpha\beta}$. There is a mathematical theorem (Yano & Bochner (1953)) saying that zero modes of this operator are given by Killing fields, provided that the manifold is compact and orientable*.

In models containing elementary gauge fields in D=4+N dimensions the situation is more subtle. Zero-modes turns out to be linear combinations of two possible vector fluctuations (see e.g. Randjbar-Daemi, Salam & Strathdee (1983)). It is reasonable to say that the zero-mode ansatz is a model-dependent statement.

The form of the zero-mode ansatz proposed before is most often used in the literature but not the only possible one. We can also write it as (Witten(1981b)):

$$g_{MN}(z) = \begin{bmatrix} g_{mn}(x) & A^{\alpha}_{m} K^{\alpha}_{\nu} \\ A^{\alpha}_{n} K^{\alpha}_{\mu} & g_{\mu\nu}(y) \end{bmatrix} \tag{II.51}$$

---

* Strictly speaking the theorem states that K_μ is a Kiliing field if and only if:

1/ $(\nabla^2 - R_\mu{}^\nu) K_\nu = 0$

2/ $\nabla_\mu K^\mu = 0$

This metric also leads to $R_4 + R + \frac{1}{4}F^2$ action after integrating over internal coordinates (in this case one must perform several integrations by parts).

The nice feature of the ansatz (II.49) is that it makes easy to see how gauge symmetry arises as a part of GCT group in D=4+N dimensions:

$$(x^m, y^\mu) \longrightarrow (x^m, y^\mu + \varepsilon^{(\alpha)}(x) K^{(\alpha)\mu}(y)) \tag{II.52}$$

(in anology with (II.8) and (II.9)).

D. Review of compactification mechanisms

The analysis of Section IIB showed that in order to get phenomenologically interesting higher-dimensional theory one must add some extra fields to it. Another strong argument for this is that one would like to be able to obtain chiral fermions in D=4 dimensions and in the framework of purely gravitational theory it is very difficult if not impossible (Witten(1983)) (I will return to this point in Section II.I).

An important class of models are provided by theories with elementary gauge fields in higher dimensions (Cremner, Scherk (1976,1977), Horvath et al.(1977)). The action for these models is given by:

$$S = -\int d^{4+N}z \left(\frac{R}{\varkappa^2} + \Lambda + \frac{(F^j)^2}{4} \right) \tag{II.53}$$

By fine tuning the value of Λ it is possible to find a $M_4 x B_N$ solution of equations of motion. Typically, A^j is supposed to have a nonzero expectation value on B_N. Its topological properties may enable to solve the chirality problem after fermions are introduced (Palla(1978)). I will discuss these models in detail in Section II.E.

A compactification mechanism in a very fashionable few years ago D=11 supergravity theory looks very similar. This theory is interesting since it is a unique supergravity theory in

D=11 dimensions, which a maximal one for a supergravity and a minimal one if SU(3)xSU(2)xU(1) gauge symmetry is to be obtained as isometry group of extra dimensions (Witten(1981a)). Review of the subject may be found in Ref Duff et al. (1986).

The theory contains several bosonic and fermionic fields. One of them is A_{MNP} with completely antisymmetric indices. In the solution of the equations of motion found by Fround & Rubin (1980)

$$\langle F_{mnpr} \rangle \sim \epsilon_{mnpr}$$

where $F_{MNPR} = 4\,\partial_{[M}A_{NPR]}$.
The theory obtained (with different internal spaces, S_7 and M_{pqr} being the most interesting choices) is very unrealistic - huge value of the D=4 dimensional cosmological constant (of the same order of magnitude as the curvature of the internal space); no hope for obtaining chiral fermions. It was however a useful laboratory for developing techniques of harmonic expansion, spectrum analysis etc.

Another interesting possibility is provided by "self-consistent models". A background $M_4 x B_N$ is now a solution of corrected eqations of motion. Corrections are due to quantum effects (a là Casimir effect) for graviton or for some other extra fields (Appelquist & Chodos(1983)). I will describe this mechanism in detail in Section II.K. A very interesting feature of self-consistent models is that the effective D=4 dimensional coupling constant and the radius of the internal space are no longer free parameters: their values are determined by dynamics (Candelas & Weinberg(1983)).

Yet another mechanism was proposed by Orzalesi. It is based on the assumption that due to quantum effects a fermion condensate is created. This mechanism requires that one start from the generalized gravity theory in D=4+N dimensions which allows non-zero torsion to be present (Einstein-Cartan theory). A review of this approach can be found in the Ref Destri et al. (1983). Some particular models were investigated, including one with S_3 as an internal manifold (Camporesi et al. (1986)). The results are however somehow bizzare: despite a fact that the background configuration is SO(4) invariant, the massless spectrum contains only three SO(3) vector bosons. I do not know the explanation of this phenomenon, since it is reasonable to expect that each symmetry of the background state leads to the appearence of one vector zero mode, i.e. that the symmetry is unbroken.

Also σ-models admit $M_4 x B_N$ solutions of classical field equations by identification of the space in which scalar fields take values with compact internal space B_N (Omero & Percacci (1980)). In this model (as noticed already in the original paper) the symmetry of the background configuration is only global so one cannot expect to obtain massless vector fields after compactification. This was confirmed in explicit calculation by Aulakh & Sahdev (1985).

E. Models with elementary gauge fields

Theories with elementary gauge fields were proposed by Cremner & Scherk (1976), Horvath et al. (1977) as first models exhibiting the phenomenon of spontaneous compactification. They were then systematically studied by Randjbar Daemi, Salam & Strathdee (1983,1984), Schellekens (1984,1985), Pilch & Schellekens (1985). In my exposition I will mainly follow Refs Salam & Strathdee (1982) and Percacci & Randjbar Daemi (1982).

The starting point is an Einstein-Yang-Mills theory in D=4+N dimensions with a gauge group K:

$$S = - \int d^{4+N}z \; \sqrt{-g} \; \left(\frac{R}{\varkappa^2} + \frac{\bar{F}^2}{4g^2} + \lambda \right) \qquad \text{(II.54)}$$

where $F^j_{MN} = \partial_M A^j_N - \partial_N A^j_N + C^{jkl} A^k_M A^l_N$; $\bar{F}^2 = F^j_{MN} F^{jMN}$;

$F_{MN} = F^j_{MN} t^j$; $A_N = A^j_N t^j$; $\varkappa^2$, g^2, λ are constants; j is a K gauge group index. C_{kjl} are K-Lie algebra structure constants. We assume K to be semisimple so that C`s can be made antisymmetric.

The equations of motion are:

$$R_{MN} - \frac{R}{2} g_{MN} = \frac{\varkappa^2}{2}\left(g_{MN}\left(\lambda + \frac{F^2}{4g^2}\right) - \frac{1}{g^2} F^j_{ML} F^{jL}_{N}\right) \qquad \text{(II.55)}$$

$$g^{LM}\nabla_L F_{MN} = g^{LM}\left(\partial_L F_{MN} - \Gamma^P_{LM} F_{PN} - \Gamma^P_{LN} F_{MP} + \right.$$
$$\left. + [A_L, F_{MN}]\right) = 0 \qquad \text{(II.56)}$$

We would like to find a solution of the equations of motion such that the space-time factorizes into $M_4 x G/H$. For G/H we assume that it is a symmetric homogeneous space. It means that for G-Lie algebra structure constants $C_{\hat\alpha\hat\beta}{}^{\hat\gamma}$ $\hat\alpha = (\alpha, \tilde\alpha)$ $\tilde\alpha$ indices correspond to a $\mathcal{H}$ subalgebra of $\mathcal{G}$ and α to the rest, we have:

$$C_{\tilde\alpha\tilde\beta}{}^{\gamma} = C_{\alpha\tilde\beta}{}^{\gamma} = C_{\alpha\beta}{}^{\gamma} = 0 \qquad \text{(II.57)}$$

This is true for example for S_N = SO(N+1)/SO(N) and CP(N) = SU(N+1)/SU(N)xU(1) manifolds. A nonsymmetric space solutions are analysed by Forgacs et al. (1985). We will assume also G to be semisimple and compact.

The statement which we will prove is that <u>for K$\supset$H a standard G-invariant solution of the equations of motion (II.55,56) exists.</u>

First we introduce coset representatives on G/H, which we think of the space of left cosets gH, $g \in G$. Let $U \subset G/H$. $L: U \longrightarrow G$ such that $\pi(L(y)) = y$ where π is a canonical projection $\pi : G \longrightarrow G/H$. (II.58)

Next we construct

$$e(y) = L(y)^{-1}\, dL(y) = Q_{\hat\alpha}\, e^{\hat\alpha}(y) \qquad \text{(II.59)}$$

$Q_{\hat\alpha}$ are generators of $\mathcal{G}$ Lie algebra. They must be antihermitian if L(y) is a unitary matrix:

$$(L^{-1}dL)^{+} = dL^{+}(L^{-1})^{+} = dL^{-1}\,L =$$

$$= d(L^{-1}L) - L^{-1}dL = -L^{-1}dL \qquad \text{(II.60)}$$

$e^{\hat{\alpha}}(y)$ are 1-forms so that $e^{\hat{\alpha}}(y) = e^{\hat{\alpha}}_{\mu}(y)\,dy^{\mu}$.

We will show that

$$g_{\mu\nu}(y) = e^{\alpha}_{\mu}(y)\,e^{\alpha}_{\nu}(y) \qquad \text{(II.61)}$$

and

$$A^{\bar{\alpha}}_{\mu}(y) = \frac{1}{a}\,e^{\bar{\alpha}}_{\mu}(y) \qquad \text{(II.62)}$$

are solutions of the equations (II.55,56)

From (II.59) we get:

$$de(y) = -\frac{1}{a}\,e(y)\wedge e(y) \qquad \text{(II.63)}$$

$$de^{\hat{\alpha}} = -\frac{1}{2a}\,C_{\hat{\beta}\hat{\gamma}}{}^{\hat{\alpha}}\,e^{\hat{\beta}}\wedge e^{\hat{\gamma}} \qquad \text{(II.64)}$$

a of dimension -1 was introduced in order to make dimensionalities balanced in (II.63).

In particular:

$$de^{\alpha} = -\frac{1}{2a}\,C_{\beta\bar{\gamma}}{}^{\alpha}\,e^{\beta}\wedge e^{\bar{\gamma}} - \frac{1}{2a}\,C_{\bar{\gamma}\beta}{}^{\alpha}\,e^{\bar{\gamma}}\wedge e^{\beta} \qquad \text{(II.65)}$$

The torsion free spin connection 1-form is:

$$B^{\alpha}{}_{\beta} = \frac{1}{a}\,C_{\bar{\gamma}\beta}{}^{\alpha}\,e^{\bar{\gamma}} \qquad \text{(II.66)}$$

(It satisfies $de^{\alpha} + B^{\alpha}{}_{\beta}\wedge e^{\beta} = 0$)

The curvature 2-form is

$$R^{\alpha}{}_{\beta} = dB^{\alpha}{}_{\beta} + B^{\alpha}{}_{\gamma}\wedge B^{\gamma}{}_{\beta} = \frac{1}{a}\,C_{\bar{\gamma}\beta}{}^{\alpha}\,de^{\bar{\gamma}} +$$

$$+ \frac{1}{a^{2}}\,C_{\bar{\delta}\gamma}{}^{\alpha}\,C_{\bar{\varphi}\beta}{}^{\gamma}\,e^{\bar{\delta}}\wedge e^{\bar{\varphi}} = -\frac{1}{2a^{2}}\,C_{\bar{\gamma}\beta}{}^{\alpha}\,C_{\delta\varphi}{}^{\bar{\gamma}}\,e^{\delta}\wedge e^{\varphi} +$$

$$+ \frac{1}{2a^{2}}\left(C^{\alpha}_{\bar{\delta}\gamma}\,C_{\bar{\varphi}\beta}{}^{\gamma} - C_{\bar{\delta}\gamma}{}^{\beta}\,C_{\bar{\varphi}\alpha}{}^{\gamma} - C_{\bar{\gamma}\beta}{}^{\alpha}\,C_{\bar{\delta}\bar{\varphi}}{}^{\bar{\gamma}}\right) =$$

$$= -\frac{1}{2a^{2}}\,C^{\alpha}_{\bar{\gamma}\beta}\,C_{\delta\varphi}{}^{\bar{\gamma}}\,e^{\delta}\wedge e^{\varphi} = \frac{1}{2}\,R^{\alpha}{}_{\beta\delta\varphi}\,e^{\delta}\wedge e^{\varphi} \qquad \text{(II.67)}$$

In the computation the Jacobi identity was used.

$$R^{\alpha}{}_{\beta\bar{\gamma}\varphi} = \frac{1}{a^2} C_{\beta\bar{\gamma}}{}^{\alpha} C_{\bar{\gamma}\varphi}{}^{\bar{\gamma}} \tag{II.68}$$

$$R_{\alpha\beta} = \frac{1}{a^2} C_{\beta\bar{\gamma}}{}^{\varphi} C_{\alpha\varphi}{}^{\bar{\gamma}} = -\frac{1}{2a^2} \delta_{\alpha\beta} \tag{II.69}$$

(normalization is $C_{\hat{\alpha}\hat{\gamma}}{}^{\hat{\delta}} C_{\hat{\beta}\hat{\delta}}{}^{\hat{\gamma}} = -\frac{1}{2} \delta_{\hat{\alpha}\hat{\beta}}$)

$$R = -\frac{N}{2a^2} \tag{II.70}$$

For $A = t^{\bar{\alpha}} e^{\bar{\alpha}}$ ($t^{\bar{\alpha}}$ is a $\mathcal{H}$ generator)

$$F = \frac{1}{a}\left(dA + \frac{1}{2a}[A,A]\right) = t^{\bar{\alpha}}\left(de^{\bar{\alpha}} + \frac{1}{2a} C_{\bar{\beta}\bar{\gamma}}^{\bar{\alpha}} e^{\bar{\beta}} \wedge e^{\bar{\gamma}}\right) \tag{II.71}$$

$$F = -\frac{1}{2a^2} t^{\bar{\alpha}} C_{\beta\gamma}{}^{\bar{\alpha}} e^{\beta} \wedge e^{\gamma} \tag{II.72}$$

so that

$$F^{\bar{\alpha}}_{\beta\gamma} = -\frac{1}{a^2} C_{\beta\gamma}{}^{\bar{\alpha}} \tag{II.73}$$

$$F^2 = \frac{N}{2a^4} \tag{II.74}$$

$$F^{\bar{\alpha}}_{\beta\gamma} F^{\bar{\alpha}}_{\alpha\gamma} = \frac{1}{2a^4} \delta_{\alpha\beta} \tag{II.75}$$

The Einstein equations give

$$-\frac{N}{a^2} + \varkappa^2\left(\lambda + \frac{N}{8g^2a^4}\right) = 0 \tag{II.76}$$

$$\frac{1}{2a^2} - \frac{\varkappa^2}{2g^2}\frac{1}{2a^4} = 0 \tag{II.77}$$

$$a^2 = \frac{\varkappa^2}{2g^2} \qquad \lambda = \frac{3}{2} N \frac{g^2}{\varkappa^4} \tag{II.78}$$

Einstein equations turn out to be equivalent to algebraical relations between $a^2, \varkappa^2, g^2$ and λ.

We still must show that the Yang-Mills equation (II.56) is satisfied

$$\nabla_\xi F^{\bar\alpha}_{\beta\gamma} = \partial_\xi F^{\bar\alpha}_{\beta\gamma} - B_{\xi,\beta\sigma} F^{\bar\alpha}_{\sigma\gamma} - B_{\xi,\gamma\sigma} F^{\bar\alpha}_{\beta\sigma} + C^{\bar\alpha}_{\bar\tau\bar\omega} A^{\bar\tau}_{\xi} F^{\bar\omega}_{\beta\gamma} =$$

$$= \frac{1}{a^3}\left\{ e^{\bar\tau}_{\xi} C_{\beta\bar\tau\sigma} C_{\bar\alpha\sigma\gamma} + e^{\bar\tau}_{\xi} C_{\gamma\bar\tau\sigma} C_{\bar\alpha\beta\sigma} - e^{\bar\tau}_{\xi} C_{\bar\alpha\bar\tau\bar\omega} C_{\bar\omega\beta\gamma} \right\} \tag{II.80}$$

$$\delta_{\xi\beta} \nabla_\xi F^{\alpha}_{\beta\gamma} = \frac{1}{a^3} e^{\bar\tau}_{\beta} \left(C_{\beta\bar\tau\sigma} C_{\bar\alpha\beta\gamma} + C_{\gamma\bar\tau\sigma} C_{\bar\alpha\beta\sigma} - C_{\bar\alpha\bar\tau\bar\omega} C_{\bar\omega\beta\gamma}\right) =$$

$$= \frac{1}{a^3} e^{\bar\tau}_{\beta} \left(C_{\beta\bar\tau\hat\delta} C_{\bar\alpha\hat\delta\gamma} + C_{\gamma\bar\tau\hat\delta} C_{\bar\alpha\beta\hat\delta} - C_{\bar\alpha\bar\tau\hat\delta} C_{\hat\delta\beta\gamma}\right) = 0 \tag{II.81}$$

In the last line the fact that G/H is a symmetric space and Jacobi identities for $\mathcal{G}$ Le algebra were used.

It is important to investigate symmetry properties of the solution (II.61,62).

The action of the group G on the coset space is defined by

$$g L(y) = L(y') h \tag{II.82}$$

The transformation laws for e^{α} and $e^{\bar\alpha}$ may be read from this

$$L^{-1}(y)\, dL(y) \xrightarrow{g} L^{-1}(y')\, dL(y') = (gLh^{-1})^{-1} d(gLh^{-1}) =$$

$$= h L^{-1}(g^{-1}dg) L h^{-1} + h(L^{-1}dL)h^{-1} + h\, dh^{-1} \tag{II.83}$$

We introduced above a matrix of the adjoint representation of G

$$g^{-1} Q_{\hat\alpha}\, g = D_{\hat\alpha}{}^{\hat\beta}(g)\, Q_{\hat\beta} \tag{II.84}$$

We get

$$e^{\hat\alpha} Q_{\hat\alpha} \longrightarrow (g^{-1}dg)^{\hat\alpha} D_{\hat\alpha}{}^{\hat\beta}(Lh^{-1}) Q_{\hat\beta} + e^{\hat\alpha} D_{\hat\alpha}{}^{\hat\beta}(h^{-1}) Q_{\hat\beta} +$$
$$+ (h^{-1}dh)^{\bar\alpha} Q_{\bar\alpha} \tag{II.85}$$

$$e^{\alpha}(y') = e^{\beta}(y) D_{\beta}{}^{\alpha}(h^{-1}) + (g^{-1}dg)^{\hat\beta} D_{\hat\beta}{}^{\alpha}(Lh^{-1}) \tag{II.86}$$

$$e^{\bar\alpha}(y') = e^{\bar\beta}(y) D_{\bar\beta}{}^{\bar\alpha}(h^{-1}) + (h\,dh^{-1})^{\bar\alpha} + (g^{-1}dg)^{\hat\beta} D_{\hat\beta}{}^{\bar\alpha}(Lh^{-1}) \tag{II.87}$$

(since $D_{\alpha}{}^{\bar\beta}(h) = D_{\bar\alpha}{}^{\beta}(h) = 0$)

In general g can be x-dependent (we discuss local G transformations). Then

$$e^{\alpha}(y') = e^{\alpha}_{\nu}(y')\, dy'^{\nu} = e^{\alpha}_{\nu}(y)\left(\frac{\partial y'^{\nu}}{\partial y^{\mu}} dy^{\mu} + \frac{\partial y'^{\nu}}{\partial x^{m}} dx^{m}\right) \qquad \text{(II.88)}$$

$$e^{\bar{\alpha}}(y') = e^{\bar{\alpha}}_{\nu}(y')\, dy'^{\nu} = e^{\bar{\alpha}}_{\nu}(y')\left(\frac{\partial y'^{\nu}}{\partial y^{\mu}} dy^{\mu} + \frac{\partial y'^{\nu}}{\partial x^{m}} dx^{m}\right) \qquad \text{(II.89)}$$

Finally we get for e^{α}_{μ} and $e^{\bar{\alpha}}_{\mu}$

$$e^{\alpha}_{\mu}(y') = \frac{\partial y^{\nu}}{\partial y'^{\mu}}\, e^{\beta}_{\nu}(y)\, D_{\beta}{}^{\alpha}(h^{-1}) \qquad \text{(II.90)}$$

$$e^{\bar{\alpha}}_{\mu}(y') = \frac{\partial y^{\nu}}{\partial y'^{\mu}}\left(e^{\bar{\beta}}_{\nu}(y)\, D_{\bar{\beta}}{}^{\bar{\alpha}}(h^{-1}) + (h\,\partial_{\nu} h^{-1})^{\bar{\alpha}}\right) \qquad \text{(II.91)}$$

We see that:

1/ If $D_{\beta}{}^{\alpha}(h^{-1})$ is an orthogonal matrix, the metric is G-invariant.

2/ $e^{\bar{\alpha}}_{\mu}$ transforms as a connection 1-form on G/H.

F. Harmonic expansion

In all KKT we are interested in a theory which is effectively seen in D=4 dimensions. In particular we are interested in the effective theory of massless modes since all the excitations modes have a very big mass due to the small typical length of the internal space.

Massive excitations although not relevant at present energies play a very important role in the ultraviolet limit of the theory. For example, it is believed that some superstring theories are finite even if effective theories of massless particles corresponding to them are not so. Some authors, however, discuss consistent truncations in KKT i.e. selfconsistent methods of elimination of all the massive modes from the theory (Duff et al. (1984)).

Zero modes may be analyzed using purely topological methods (Candelas et al. (1985)). In order to compute a full spectrum of a theory one must know spectra of several differential operators acting on the internal space. One of the motivations to study homogeneous spaces G/H as internal spaces is that (with the G-invariant metric) these spectra are calculable.

The prototype of the analysis I am going to discuss was a Fourier expansion in the D=5 dimensional KKT with S_1 as the internal space (II.27). In this Section I will mainly follow Refs Salam & Strathdee(1982) and Strathdee(1983).

In the case in which an internal space is a group manifold, a Peter & Weil theorem (see Barut & Rączka (1977)) tells us that a complete set of orthonormal functions on it is given by <u>all the matrix elements of all the unitary irreducible representations of G</u>:

$$\varphi(x,y) = \sum_{l} \sum_{pq} \varphi^{(l)}_{pq}(x)\, D^{(l)}_{pq}(y) \qquad \text{(II.92)}$$

(l) stands for representation index and p,q are matrix indices.

On G/H we will deal with functions transforming in a definite way under the action of H.

Using coset representatives one may represent y as an element of G : y $\longrightarrow L_y^{-1} \in$ G. The expansion (II.92) applied to the function on G/H should be restricted in order to satisfy

$$\mathbb{D}_{IJ}(h)\,\varphi_J(x,y) = \sum_{l} \sum_{pq} \varphi^{(l)}_{Ipq}(x)\, D^{(l)}_{pr}(h)\, D^{(l)}_{rq}(L_y^{-1}) \qquad \text{(II.93)}$$

$\mathbb{D}_{IJ}$ is a matrix of the representation to which φ_J belongs.

Let us consider a simple example (we follow a Ref (Randjbar-Daemi et al. (1983))).

Let G/H = SU(2)/U(1)

$$\begin{aligned} L_{\theta\varphi} &= e^{-\varphi Q_3}\, e^{-\theta Q_2}\, e^{\varphi Q_3} \\ h &= e^{\alpha Q_3} \end{aligned} \qquad \text{(II.94)}$$

($Q_{\hat{\alpha}}$'s are antihermitian)

Suppose, φ_I carries a U(1) charge c

$$\mathbb{D}_{IJ}(e^{\alpha Q_3})\varphi_J(x,y) = e^{i\alpha c}\varphi_I(x,y) \tag{II.95}$$

$D^{(l)}_{pr}$ is a matrix of SU(2) representation given by (Edmonds (1957));

$$D^{(l)}_{pr}(e^{\gamma Q_3}e^{\beta Q_2}e^{\alpha Q_1}) = e^{ip\gamma}\, d^{(l)}_{pr}(\beta)\, e^{ir\alpha} \tag{II.96}$$

Finally

$$\varphi_c(x,y) = \sum_l \sum_{q} \varphi^{(l)}_{c,cq}(x)\, D^{(l)}_{cq}(L_y^{-1}) \tag{II.97}$$

Instead of all the matrix elements of $D^{(l)}$, <u>only one row of D_{pr} matrix labells spherical harmonics in this example</u>. This result can be generalized (Salam & Strathdee (1982)) and

$$\varphi_i(x,y) = \sum_l \sum_{q} D^{(l)}_{iq}(L_y^{-1})\,\varphi^{(l)}_{i,q}(x) \tag{II.98}$$

In the expression above φ_i form a certain H group multiplet. $D^{(l)}_{iq}$ refers to only those rows in a unitary representation of G (index l), which if we restrict G to H will give a representation of H.

Spherical harmonics introduced in (II.98) are eigenfunctions of Laplace operator on G/H. This can be seen as follows

$$\begin{aligned}\nabla_\alpha \varphi_i(x,y) &= e_\alpha^{\ \mu}\partial_\mu\varphi_i - \tfrac{1}{2} B_{\alpha,\beta\gamma}\,\mathbb{D}_{ij}(\Sigma_{\beta\gamma})\varphi_j = \\ &= e_\alpha^{\ \mu}\partial_\mu\varphi_i - \tfrac{1}{2} e_\alpha^{\ \mu} e_\mu^{\ \bar\alpha}\, C_{\bar\alpha\beta\gamma}\,\mathbb{D}_{ij}(\Sigma_{\beta\gamma})\varphi_j \end{aligned} \tag{II.99}$$

Notice that

$$Q_{\bar\gamma} = -\tfrac{1}{2} C_{\bar\gamma\alpha\beta}\Sigma_{\alpha\beta} \tag{II.100}$$

(it is essentially a consequence of (II.90)). We have

$$\nabla_\alpha\varphi_i(x,y) = e_\alpha^{\ \mu}\partial_\mu\varphi_i + e_\alpha^{\ \mu} e_\mu^{\ \bar\beta}\,\mathbb{D}_{ij}(Q_{\bar\beta})\varphi_j \tag{II.101}$$

Since

$$\partial_\mu \varphi_i(x,y) = \partial_\mu \sum_{l} \sum_{q} \mathbb{D}^{(l)}_{iq}(L_y^{-1})\varphi^{(l)}_{iq}(x) =$$

$$= \sum_{l} \sum_{q} \mathbb{D}^{(l)}_{iq}(\partial_\mu L_y^{-1})\varphi^{(l)}_{iq}(x) = \sum_{l} \sum_{q} \mathbb{D}^{(l)}_{iq}(-L_y^{-1}\partial_\mu L_y L_y^{-1})\varphi^{(l)}_{iq}(x) =$$

$$= -e_\mu^{\hat\beta} \sum_{lq} \sum_{r} \mathbb{D}^{(l)}_{ir}(Q_{\hat\beta})\, \mathbb{D}^{(l)}_{rq}(L_y^{-1})\varphi^{(l)}_{iq}(x) \qquad \text{(II.102)}$$

finally we get:

$$\nabla_\alpha \varphi_i(x,y) = -e_\alpha^{\mu} e_\mu^{\beta}\, \mathbb{D}_{ij}(Q_\beta)\varphi_j(x,y) = -\,\mathbb{D}_{ij}(Q_\alpha)\,\varphi_j(x,y) \qquad \text{(II.103)}$$

Therefore

$$\nabla^2 \mathbb{D}^{(l)}_{ij}(L_y^{-1}) = -\left(C_2(G) - C_2(H)\right) \mathbb{D}^{(l)}_{ij}(L_y^{-1}) \qquad \text{(II.104)}$$

where $C_2(G)$, $C_2(H)$ are second order Casimir operator eigenvalues for groups G and H , in the representation labelled by l.

G. Stability

In all conventional field theories one assumes that the ground state is that of the lowest energy. In theories involving gravity it is not clear what does such statement mean, since the definition of energy depends on asymptotic behavior of the space-time. Therefore one cannot compare energy of a $M_4 x B_N$ topology candidate for the ground state with that of M_{4+N} topology (Witten (1981,1982)).

An understanding of stability of all the higher-dimensional theories (including superstrings) is unsatisfactory. The analysis is usually restricted to the classical stability (stability against small perturbations). Such calculations were done in a lot of different models including higher-dimensional supergravity theories* and models with elementary gauge fields.

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* In the D=11 dimensional supergravity there are subtleties about what does stability mean for anti DeSitter background.

First two such models to be analyzed were D=6 and D=8 dimensional EYM theories with magnetic monopole and instanton compactifications to $M_4 x S_2$ and $M_4 x S_4$. Both are classically stable (Randjbar-Daemi, Salam & Strathdee (1983,1984)). Important contributions to the subject are due to Schellekens. He discussed compactifications to any S_N and CP(N) internal space (Schellekens (1984,1986)). All of them except S_3 are stable. A background configuration used in all these papers is that described in Section II.E. A CP(N) results confirmed previous result obtained for a particular CP(3) model (Sobczyk(1985)). Another development was a systematic study of possible instabilities due to enlargement of a gauge group from the "minimal" one H (if the internal space is G/H) to K : $K \supset H$ (Schellekens (1984)). Recently a compactification of a theory with additional gravitational Gauss-Bonnet term to $M_4 x S_2$ was shown to be stable (Mignemi (1986)).

The actual calculation is greately simplified with a specific gauge choice for gauge and gravitational fields, namely with a light-cone gauge (Randjbar-Daemi, Salam, Strathdee(1984), Lee (1986). In the light-cone gauge only physical degrees of freedom propagate. After expanding all the fields around their background values one calculates a part of the action bilinear in fluctuations and then looks for signs of kinetic terms and mass terms - for possible ghosts and tachyons. The simplest example of such computation was already discussed in Section II.A. In Section II.H I will present a more interesting model, namely a D=6 dimensional Einstein-Maxwell theory with magnetic monopole compactification to $M_4 x S_2$.

Supergravity theories can have some advantage if ground state admits a supersymmetry transformation ie if compactification does not break completely supersymmetry. By a general argument a Hamiltonian of supersymmetric theory is always positive definite, what is equivalent to a statement that the ground state is perturbatively stable.

A demonstration of the classical stability does not conclude the analysis. One should discuss also a stability against quantum tunneling effect (Coleman (1977), Coleman & Callan (1977) Frampton (1976)). As I have already mentioned it is a difficult and almost open problem. From existing results the most interesting is one due to Witten (1982). He demonstrated that D=5 dimensional KKT is semiclassically unstable. Witten constructed explicitely a solution of Einstein equations ($R_{MN} = 0$) in the Euclidean space-time, which asymptotically aproaches a $R_4 x S_1$ solution. A certain fluctuation of this configuration has a negative energy which is a sign of instability (Perry (1980)). Detailed analysis shows that $M_4 x S_1$ solution decays into nothing. Introduction of fermions to the theory may hovewer make it stable.

It is difficult to generalize this result to a theory with a $F_{MNPR}F^{MNPR}$ term and a Freund & Rubin type background ansatz for F_{MNPR} field (Young (1983)). Other interesting references to the problem of semiclassical stability are due to Frieman & Kolb (1985) and Maeda (1986).

H. Six-dimensional model

In this Section I will discuss a D=6 dimensional Einstein Maxwell theory with magnetic monopole compactification to $M_4 x S_2$ (Randjbar-Daemi, Salam & Strathdee (1983)). This theory illustrates well computational techniques used in KKT. The spectrum calculation will be performed in the light-cone gauge, not in the harmonic gauge like in the original paper. In the original (and may be more physical) approach one couples all the fields of the theory to external sources and investigates the effective theory of sources by looking for poles and residua due to exchange of physical (after elimination all the gauge degrees of freedom) fields.

First steps in the discussion are exactly those done in Section II.E. The model is

$$S = -\int d^4x \int d^2y \sqrt{-g}\left(\frac{R}{\kappa^2} + \frac{F^2}{4g^2} + \lambda\right) \qquad \text{(II.105)}$$

$F_{MN} = \partial_M A_N - \partial_N A_N$; g^2, λ, κ^2 are constants. A solution of the equations of motion (II.55,56) is

$$\langle g_{\mu\nu} dy^\mu dy^\nu \rangle = d\theta^2 + \sin^2\theta \, d\varphi^2 \qquad \text{(II.106)}$$

$$\langle A_\mu(y) dy^\mu \rangle = \frac{n}{2a}(\cos\theta \pm 1) d\varphi \qquad \text{(II.107)}$$

$$\langle g_{mn} \rangle = \eta_{mn} \qquad \text{(II.108)}$$

The A_μ configuration is of the magnetic monopole type. Plus and minus signs refer to two coordinate patches necessary in order to cover S_2, n must be an integer (see Eguchi, Gilkey, Hanson (1980); a is a length scale of S_2.

The Eistein equations are satisfied if

$$\frac{1}{\kappa^2} = \frac{1}{2g^2a^2} \qquad \text{(II.109)}$$

$$\lambda = \frac{1}{2g^2a^4} \qquad \text{(II.110)}$$

Fluctuations are defined as

$$\begin{aligned} g_{MN} &= \langle g_{MN} \rangle + \kappa h_{MN} \\ A_M &= \langle A_M \rangle + V_M \end{aligned} \qquad \text{(II.111)}$$

The part of the action bilinear in the fluctuations is given by

$$S_{bil} = \int d^6z \sqrt{-\langle g \rangle}\Big[\frac{1}{4} h_{AB}\nabla^2 h_{AB} - \frac{1}{4} h_{AA}\nabla^2 h_{BB} - \frac{1}{2} h_{AB}\nabla_A\nabla_C h_{BC}$$

$$+ \frac{1}{2} h_{AA}\nabla_B\nabla_C h_{BC} + \langle R_{AB}\rangle\left(\frac{1}{2} h_{AB}h_{CC} - \frac{1}{2} h_{AC}h_{BC}\right) +$$

$$+ \frac{1}{2}\langle R_{AB}\rangle V_A V_B - \frac{1}{2}\left(\langle R_{ADCB}\rangle - \frac{\kappa^2}{2}\langle F_{AC}\rangle\langle F_{BD}\rangle\right) h_{AB}h_{CD}$$

$$+\langle F_{AB}\rangle\langle F_{AC}\rangle\left(\frac{\varkappa^2}{4}h_{BC}h_{DD}-\frac{\varkappa^2}{2}h_{BD}h_{CD}\right)+\frac{1}{2}V_A\nabla^2V_A$$

$$+\frac{1}{2}\nabla_AV_A\nabla_BV_B+\left(\nabla_AV_B-\nabla_BV_A\right)\frac{\varkappa}{2}\left(\langle F_{CB}\rangle h_{CA}-\langle F_{CA}\rangle h_{CB}\right)$$

$$\left.-\frac{\varkappa}{4}h_{CC}\langle F_{AB}\rangle\left(\nabla_AV_B-\nabla_BV_A\right)\right] \qquad \text{(II.112)}$$

where $\langle R_{ABCD}\rangle$, $\langle F_{AB}\rangle$ stand for the background configuration given in (II.106-108). We introduce orthonormal basis

$$\langle e_{\varphi+}\rangle=\frac{e^{-i\varphi}}{\sqrt{2}}(-\sin\theta) \qquad \langle e_{\varphi-}\rangle=\langle e_{\varphi+}\rangle^*$$
$$\langle e_{\theta+}\rangle=\frac{e^{-i\varphi}}{\sqrt{2}}(-i) \qquad \langle e_{\theta-}\rangle=\langle e_{\theta+}\rangle^* \qquad \text{(II.113)}$$

for internal space components of $A=(a,\alpha)$; $a=0,1,2,3$; $\alpha=+,-$
In this basis

$$\langle R_{+-}\rangle=-\frac{1}{a^2} \qquad \text{(II.114)}$$

$$\varkappa\langle F_{+-}\rangle=\frac{i\sqrt{2}}{a} \qquad \text{(II.115)}$$

In the light-cone gauge we put

$$V_-=h_{A-}=0 \qquad \text{(II.116)}$$

(light-cone +- components should not be confused with those defined in (II.113)). In turns out that V_+ and h_{A+} components do not propagate (their equations of motion are merely algebraical). Physical degrees of freedom are

$$h^T_{jk},\ h_{\alpha j},\ V_j,\ h_{\alpha\beta},\ h_{\alpha\alpha}=-h_{jj} \qquad \text{(II.117)}$$

(where $h^T_{jk}=h_{jk}-\frac{1}{2}\delta_{jk}h_{ll}$) or in +- components for α

$$h^T_{jk}\ \ h_{\mp j},\ V_j\ ,\ h_{++},\ h_{--},\ V_+,\ V_- \qquad \text{(II.117a)}$$

h^T_{jk}, V_j, h_{jj} are real and

$$
\begin{aligned}
(h_{+j})^* &= h_{-j} \\
(h_{++})^* &= h_{--} \qquad\qquad (II.118)\\
(V_+)^* &= V_-
\end{aligned}
$$

A final expression for the bilinear action is

$$
\begin{aligned}
S_{b.l} = \int d^6 z \sqrt{-\langle g \rangle}\ \Big[& \tfrac{1}{4} h_{jk}^T (\partial^2 + \nabla^2) h_{jk}^T + \\
& + h_{j-} (\partial^2 + \nabla^2 + R_{+-}) h_{j+} + \tfrac{1}{2} V_j (\partial^2 + \nabla^2) V_j + \\
& + \varkappa F_{+-} V_j (\nabla_+ h_{-j} - \nabla_- h_{+j}) + V_- (\partial^2 + \nabla^2 + R_{+-} + \varkappa^2 (F_{+-})^2) V_+ \\
& + \tfrac{1}{2} h_{--} (\partial^2 + \nabla^2 + 2 R_{+-} - \varkappa^2 (F_{+-})^2 - 2 R_{+--+}) h_{++} \\
& + \tfrac{1}{4} h_{jj} (\partial^2 + \nabla^2 + R_{+-} + R_{+--+} + \tfrac{\varkappa^2}{2} (F_{+-})^2) h_{jj} \\
& - 2\varkappa h_{++} V_- \nabla_- F_{-+} - 2\varkappa h_{--} V_+ \nabla_+ F_{+-} \\
& + \varkappa F_{+-} (h_{++} \nabla_- V_- - h_{--} \nabla_+ V_+) \\
& + \tfrac{\varkappa}{2} h_{jj} F_{+-} (\nabla_+ V_- - \nabla_- V_+) \Big] \qquad (II.119)
\end{aligned}
$$

where

$$\nabla^2 = \nabla_+ \nabla_- + \nabla_- \nabla_+ \qquad (II.120)$$

By applying a technique of harmonic expansion (Section II.F) we can calculate a full spectrum of the theory.

$$\Phi_\lambda (x, \varphi, \theta) = \sum_{l \geq |\lambda|} \sum_m \mathbb{D}^{(l)}_{\lambda m} (\varphi, \theta)\, \phi^{(l)}_{\lambda m} (x) \qquad (II.121)$$

Φ_λ stands for any field present in (II.117). $\mathbb{D}^{(l)}_{\lambda m}$ is a matrix element of 2l+1 dimensional unitary irreducible representation of SU(2). λ refers to the effective U(1) charge (S_2 is taken to be SU(2)/U(1)). +- basis was chosen is such a way that for h_{++}, h_{--} the value of U(1) charge is ± 2

$$\begin{array}{lr} V_+ ,\ h_{+j} & +1 \\ V_- ,\ h_{-j} & -1 \\ h^T_{jk} ,\ h_{jj} ,\ V_j & 0 \end{array} \qquad \text{(II.122)}$$

Using orthonormal properties of the spherical harmonics introduced in (II.121)

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \ \sin\theta \ \mathbb{D}^{*(l)}_{m_1 m_2}(\varphi,\theta)\, \mathbb{D}^{(l')}_{n_1 n_2}(\varphi,\theta) =$$

$$= \delta_{ll'}\, \delta_{n_2+m_2,0}\, \delta_{n_1+m_1,0} \qquad \text{(II.123)}$$

(I use different normalization of $\mathbb{D}^{(l)}$\`s than authors of the original paper). $\mathbb{D}^{(l)}_{\lambda m}$ are all eigenfunctions of the Laplace operator ∇^2 with eigenvalues given by (II.104).

From the form of (II.119) it follows that states with different spin/helicity decouple.

In the spin/helicity 2 sector there is one field only: h^T_{jk}. After being expanded in a harmonic series it gives rise to one massless ($\mathbb{D}^{(0)}_{00}$) and to a "tower" of massive ($\mathbb{D}^{(l)}_{0m}$) modes.

In the spin/helicity 1 sector there are three fields: V_j, $h_{\pm j}$ which mix among themselves. l=0 harmonic hovewer is present in the expansion of V_j only. It is a first vector zero mode. For l=1 the mass matrices for three sets of fields $(V^{(1)}_{j0} , h^{(1)}_{\mp j0} , h^{(1)*}_{\pm j0})$, $(V^{(1)}_{j1} , h^{(1)}_{+j1} , h^{(1)*}_{+j-1})$, $(V^{1}_{j1} , h^{(1)*}_{+j1} , h^{(1)}_{+j-1})$ are

$$\mathcal{M} = \begin{bmatrix} \partial^2 - \frac{2}{a^2} & \frac{i\sqrt{2}}{a^2} & \mp \frac{i\sqrt{2}}{a^2} \\ -\frac{i\sqrt{2}}{a^2} & \partial^2 - \frac{2}{a^2} & 0 \\ \pm \frac{i\sqrt{2}}{a^2} & 0 & \partial^2 - \frac{2}{a^2} \end{bmatrix} \qquad \text{(II.124)}$$

The eigenvalues of $\partial^2 = M^2$ (the mass operator) are

$$M_1^2 = 0,\ M_2^2 = \frac{2}{a},\ M_3^2 = \frac{4}{a}.$$

There are three zero modes in the theory. Graviton is a SU(2) scalar (l=0). Vector states are: SU(2) scalar (l=0) and SU(2) triplet (l=1 i.e. adjoint representation of SU(2)). In D=4 dimensions there is SU(2)xU(1) gauge group. All the other states have positive M^2 of the order of a^{-2}. There are no massless scalars in the theory since now, unlike in the D=5 dimensional theory the scale of S_2 is constrained by (II.109,110). We conclude that the background solution (II.106-108) is <u>classically stable.</u>

I.Fermions

In addition to SU(3)xSU(2)xU(1) gauge fields and gravitation we must also have fermionic matter fields in the effective D=4 dimensional theory. One may hope to explain a complexity of fermionic representations of the gauge group and existence of three families by properties of B_N and Yang-Mills fields background configurations on it.

I begin the discussion from a D=5 dimensional model (Peskin (1985)). The γ-matrices are chosen to be

$$\gamma^\mu = i\begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^\mu = (1, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma}) \tag{II.125}$$

D=5 dimensional Dirac eqution

$$i\gamma^\mu \partial_\mu \Psi = 0 \tag{II.126}$$

in the Kaluza-Klein world $M_4 x S_1$ for $\Psi = \xi e^{-ikx}$ gives

$$(k_\mu \gamma^\mu - i \frac{n}{a} \gamma^5) \xi = 0 \qquad \text{(II.126a)}$$

(n is an integer; a is the radius of S_1). This is unitary equivalent to the ordinary Dirac equation in D=4 dimensions with the mass $\frac{n}{a}$ since

$$\mathcal{U} = \exp\left(-i \frac{\pi}{8} \gamma^5\right)$$

satisfies.

$$\mathcal{U} \gamma^\mu \mathcal{U}^+ = -i \gamma^5 \gamma^\mu \qquad \text{(II.127)}$$

For n=0 there are two solutions of the equation (II.126a)

$$k_\mu \gamma^\mu \psi = 0 \quad ; \quad k^2 = 0 \qquad \text{(II.128)}$$

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \qquad \text{(II.129)}$$

$$\begin{aligned} (k_0 - \bar{k}\bar{\sigma}) \psi_R &= 0 \\ (k_0 + \bar{k}\bar{\sigma}) \psi_L &= 0 \end{aligned} \qquad \text{(II.130)}$$

This very simple model illustrates two important features of Dirac fields in KKT:

1/ D-dimensional Dirac equation splits

$$i \gamma^A D_A \psi = (i \gamma^a D_a + i \gamma^\alpha D_\alpha) \psi$$

and the "internal" Dirac operator plays role of the mass operator

2/ It is very difficult to obtain zero modes <u>other than in left-right pairs</u> i.e. it is difficult to obtain chiral fermions in D=4 dimensions.

There is also another more general difficulty expressed by the Lichnerowicz theorem (Lichnerowicz (1963)).

Consider a Dirac equation in a D-dimensional space:

$$\Gamma^m D_m \eta = 0 \qquad \text{(II.131)}$$

where

$D_m = \nabla_m - \frac{i}{2} \omega_{mab} \Sigma_{ab}$, Σ_{ab} is a matrix acting in a space in which η is defined.

We calculate

$$\Gamma^n D_n \Gamma^m D_m = \Gamma^n (\nabla_n - \frac{i}{2}\omega_{nab}\Sigma_{ab}) e^m_c \Gamma^c (\partial_m - \frac{i}{2}\omega_{mde}\Sigma_{de}) =$$

$$= \Gamma^n \Gamma^m D_n D_m + \Gamma^n \{\nabla_n e^m_c \Gamma^c - \frac{i}{2}\omega_{nab} e^m_c [\Sigma_{ab}, \Gamma^c]\} D_m =$$

$$= (\eta^{nm} + \frac{1}{2}[\Gamma^n, \Gamma^m]) D_n D_m = D^n D_n + \frac{1}{4}[\Gamma^n, \Gamma^m][D_n, D_m] =$$

$$= D^2 + \frac{i}{8} R_{nmcd} [\Gamma^n, \Gamma^m] \Sigma_{cd} =$$

$$= D^2 + \frac{1}{32} R_{abcd} [\Gamma_a, \Gamma_b][\Gamma_c \Gamma_d] =$$

$$= D^2 + \frac{R}{4} \qquad \text{(II.132)}$$

We have used the following formulas:

$$[D_n, D_m] = \frac{i}{2} R_{mncd} \Sigma_{cd} \qquad \text{(II.133a)}$$

$$\Sigma_{cd} = \frac{i}{4} [\Gamma_c, \Gamma_d] \qquad \text{(II.133b)}$$

$$R_{abcd} [\Gamma_a \Gamma_b][\Gamma_c \Gamma_d] = 8R \qquad \text{(II.133c)}$$

The last identity follows from:

$$R_{abcd} \Gamma_b \Gamma_c \Gamma_d = 2 R_{ab} \Gamma_b \qquad \text{(II.133d)}$$

It holds since

$$R_{abcd} \Gamma_b \Gamma_c \Gamma_d = - R_{abcd} \Gamma_b \Gamma_d \Gamma_c = 2 R_{ac} \Gamma_c +$$

$$+ R_{abcd} \Gamma_d \Gamma_b \Gamma_c = 4 R_{ac} \Gamma_c - R_{abcd} \Gamma_d \Gamma_c \Gamma_b \qquad \text{(II.134)}$$

We compute:

$$R_{abcd}(\Gamma_b\Gamma_c\Gamma_d + \Gamma_d\Gamma_c\Gamma_b) = 4R_{ac}\Gamma_c$$

$$(R_{abcd} + R_{adcb})\Gamma_b\Gamma_c\Gamma_d = 4R_{ac}\Gamma_c$$

$$(2R_{abcd} + R_{acdb})\Gamma_b\Gamma_c\Gamma_d = 4R_{ac}\Gamma_c$$

and

$$R_{acdb}\Gamma_b\Gamma_c\Gamma_d = R_{acdb}(2\eta_{bc}\Gamma_d - \Gamma_c\Gamma_b\Gamma_d) =$$

$$= -2R_{ac}\Gamma_c - R_{acdb}\Gamma_c\Gamma_b\Gamma_d = -2R_{ac}\Gamma_c + R_{acdb}\Gamma_c\Gamma_d\Gamma_b \tag{II.135}$$

Hence (II.133d) follows immediately.

We now come back to (II.132). Since D^2 is negative definite (we assume positive Euclidean signature for B_N), if $R < 0$ everywhere, the Dirac operator cannot have zero eigenvalues at all.

In the important paper due to Witten (1983) it is shown that the existence of a metric with $R<0$ is related to the problem whether it is posssible to obtain chiral fermions in D=4 dimensions. If B_N has an isometry group G and if one of G invariant metrics has $R<0$, then for any G-invariant metric zero modes of the Dirac operator form L-R pairs (for example G/H with the canonical G-invariant metric has (in my conventions) $R<0$).

Morover, a Lawson & Yau theorem states that on any compact manifold B_N with a nonAbelian isometry group G there is a G-invariant metric with negative scalar curvature.

One may also have a Rarita-Schwinger field in D=4+N dimensions. Then it was proved (always by Witten), that if B_N is a homogeneous space, no massless chiral fermions in D=4 dimensions are available.

One way out of these no-go theorems is to introduce elementary gauge fields in D=4+N dimensions (possible as a part of a supergravity theory). Other possibilities are: to discuss non-Riemannian geometry on $M_4 x B_N$ as it was proposed by Weinberg (1984), or to investigate noncompact, finite volume B_N as it was suggested by Wetterich (1983).

With elementary gauge fields for topological reasons one can get chiral fermions in D=4 dimensions (if gauge fields are assumed to have topologically nontrivial background configuration). A nice example in this context is provided by a two-dimensional superconductor (Peskin (1985)). Far from the flux tub $\bar{B} = 0$ but possible $\bar{A}$ may be different from zero. In fact one finds:

$$\oint d\bar{l}\, \bar{A} = 2\pi \frac{n}{e} \qquad n \text{ - integer} \tag{II.136}$$

This equation is solved by:

$$\bar{A} = \frac{f(r)}{er} \left(\frac{x_2}{r}, -\frac{x_1}{r} \right) \tag{II.137}$$

$$f(r) \underset{r\to 0}{\longrightarrow} 0 \qquad\qquad f(r) \underset{r\to\infty}{\longrightarrow} 1$$

We can now consider a Dirac field coupled to $\bar{A}$. We look for zero modes of the operator:

$$i\gamma^\mu (\partial_\mu - ieA_\mu) \tag{II.138}$$

acting on $\Psi = \begin{pmatrix} \Psi_R \\ \Psi_L \end{pmatrix}$

where γ matrices are chosen to be

$$\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad\qquad \gamma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(II.138) ie equivalent to two equations

$$\left[-\partial_1 + i\partial_2 + \frac{f(r)}{r} \left(\frac{ix_2}{r} - \frac{x_1}{r} \right) \right] \Psi_R = 0 \tag{II.139a}$$

$$\left[-\partial_1 - i\partial_2 + \frac{f(r)}{r} \left(\frac{ix_2}{r} + \frac{x_1}{r} \right) \right] \Psi_L = 0 \tag{II.139b}$$

Try to find a solution of the form

$$\Psi_R = g_1(r) \qquad \Psi_L = g_2(r) \tag{II.140}$$

Then

$$g_1(r) \sim \exp\left(-\int_0^r dr' \frac{f(r')}{r'}\right) \tag{II.141a}$$

$$g_2(r) \sim \exp\left(\int_0^r dr' \frac{f(r')}{r'}\right) \tag{II.141b}$$

We conclude that $g_2(r)$ does not belong to the Hilbert space. Hence we have obtained <u>only one zero mode of a definite chirality</u> (Ψ_R)

Similar is the mechanism thanks to which chiral zero modes appear in models investigated by Ranjbar-Daemi, Salam & Strathdee. I will present a typical argument folowing a paper due to Randjbar-Daemi (1983).

We assume to have a Dirac spinor Ψ in a D=4+N dimensional space-time transforming also according to a certain representation of gauge group K.

$$S_{fermion} = -\int d^{4+N}z \quad \left(i\bar{\Psi}\Gamma^A D_A \Psi + h.c.\right) \tag{II.142}$$

$$\Gamma^A D_A = \Gamma^A\left(e_A{}^M \partial_M + \omega_A + A_A\right) \tag{II.143}$$

ω_A, A_A denote Riemannian and Yang-Mills connections on $M_4 x B_N$. We assume also N to be even and choose Dirac matrices as

$$\begin{aligned} \Gamma_a &= \gamma_a \otimes \mathbb{1} \qquad & a &= 0, \dots, 3 \\ \Gamma_\alpha &= \gamma_5 \otimes \gamma_\alpha \qquad & \alpha &= 5, \dots, 4+N \end{aligned} \tag{II.144}$$

γ_a and γ_α are 4 and N dimensional Dirac matrices

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}$$

$$\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta}$$

$$\bar{\Gamma} = \Gamma_0 \cdot \Gamma_1 \cdots \Gamma_{3+N} = \gamma_5 \otimes \bar{\gamma} \qquad \text{(II.145)}$$

is used to distinguish two inequivalent representations of SO(1,3+N).

$$\bar{\gamma} = \gamma_5 \cdots \gamma_{4+N} \qquad \text{(II.146)}$$

Irreducible SO(1,3+N) spinor ($\bar{\Gamma}$ = +1) branches into two irreducible pieces with respect to SO(1,3) and SO(N):

$$\begin{array}{lll} (\Psi_L) & \gamma_5 = +1 & \bar{\gamma} = +1 \\ (\Psi_R) & \gamma_5 = -1 & \bar{\gamma} = -1 \end{array} \qquad \text{(II.147)}$$

Ψ_L, Ψ_R belong to different SO(N) representations. We expand both Ψ_L and Ψ_R in harmonic series on G/H:

$$\Psi_{R,L}(x,y) = \sum_{\ell m} \mathbb{D}^{(\ell)}_{\lambda m}(y)\, \Psi^{(\ell)}_{R,L,m}(x) \qquad \text{(II.148)}$$

Ψ_R and Ψ_L have different transformation properties with respect to H, and some harmonics D^j may be missing in the expansion for Ψ_R or Ψ_L . Then, since the mass operator has a form

$$i \bar{\Psi}_L \Gamma^\alpha \nabla_\alpha \Psi_R + h.c. \qquad \text{(II.149)}$$

a corresponding (to the mentioned D^j) field will be a zero mode for Ψ_L or Ψ_R .

On the other hand Atiyah & Singer theorem gives us immediately the assymetry in left and right handed zero modes of the Dirac operator in terms of the background Yang-Mills field configuration (see e.g. Eguchi, Gilkey & Hanson (1980)).

$$n_R - n_L = \int_{B_N} \hat{A}(B_N) \wedge ch(V) \qquad \text{(II.150)}$$

$\hat{A}$ is a A-roof genus on B_N

ch is a Chern character of V, associated fibre bundle of which Ψ is a cut.

J. Anomalies

With fermions in KKT one must face a problem of chiral anomalies. The simplest manifestation of chiral anomaly is given by a triangle graph representing an interaction of a Dirac field with two vector and one axial-vector currents (such graphs are not present in QED, but they are in the Weinberg-Salam model).

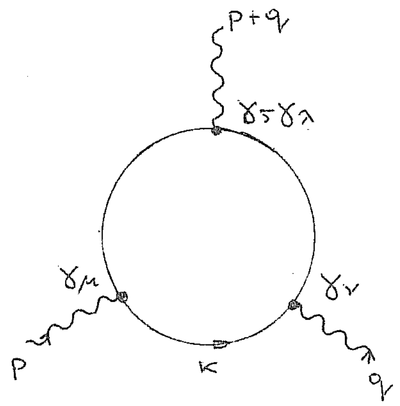

Figure 1

The amplitude for such a process is given by

$$S_{\lambda\nu\mu}(p,q) = -\int \frac{d^4k}{(2\pi)^4}\,(k^2-m^2+i\varepsilon)^{-1}\,((k+q)^2-m^2+i\varepsilon)^{-1}\,((k-p)^2-m^2+i\varepsilon)^{-1}$$

$$\times\Big[\mathrm{Tr}\,\gamma_5\gamma_\lambda(m+\hat{k}+\hat{q})\gamma_\nu(m+\hat{k})\gamma_\mu(m+\hat{k}-\hat{p}) +$$

$$+\mathrm{Tr}\,\gamma_5\gamma_\lambda(m-\hat{k}+\hat{p})\gamma_\mu(m-\hat{k})\gamma_\nu(m-\hat{k}-\hat{q})\Big] \qquad \text{(II.151)}$$

where $S_{\lambda\nu\mu}(q,p)$ is symmetric under exchange $(p,\mu)\leftrightarrow(q,\nu)$. After some algebra one gets:(Rosenberg (1963))

$$S_{\lambda\nu\mu}(q,p) = -\frac{i}{\pi^2}\Big\{A_1\,\epsilon_{\lambda\nu\mu\xi}\,q^\xi + A_2\,\epsilon_{\lambda\nu\mu\xi}\,p^\xi +$$

$$+p_\mu\,\epsilon_{\lambda\nu\sigma\xi}\,p^\sigma q^\xi\,(I_{20}-I_{10}) + q_\nu\,\epsilon_{\lambda\mu\sigma\xi}\,q^\sigma p^\xi\,(I_{20}-I_{10})$$

$$-q_\mu\,\epsilon_{\lambda\nu\sigma\xi}\,p^\sigma q^\xi\,I_{11} - p_\nu\,\epsilon_{\lambda\mu\sigma\xi}\,q^\sigma p^\xi\,I_{11}\Big\} \qquad \text{(II.152)}$$

where A_1 and A_2 are <u>formally divergent integrals</u> and

$$I_{sp}(p,q) = \int d\alpha \int d\beta \;\; \Theta(\alpha)\Theta(\beta)\,\Theta(1-\alpha-\beta)\,\alpha^s \beta^r$$
$$\times \left[q^2\beta(1-\beta)+p^2\alpha(1-\alpha)+2pq\alpha\beta - m^2 + i\varepsilon\right]^{-1} \quad \text{(II.153)}$$

We must use some regularization procedure in order to remove divergences from $S_{\lambda\nu\mu}$ (contained formally in A_1 and A_2). For example we can demand that after regularization both vector currents to which the Dirac fields is coupled are conserved:

$$\begin{aligned} p^\mu S_{\lambda\nu\mu} &= 0 \\ q^\nu S_{\lambda\nu\mu} &= 0 \end{aligned} \quad \text{(II.154)}$$

These conditions fix A_1 and A_2 to be:

$$\begin{aligned} A_1 &= pq\, I_{11} + p^2\left(I_{10} - I_{20}\right) \\ A_2 &= -pq\, I_{11} + q^2\left(I_{20} - I_{10}\right) \end{aligned} \quad \text{(II.155)}$$

Then for the divergence of the axial-vector current we get:

$$(p+q)^\lambda S_{\lambda\nu\mu} = -\frac{i}{\pi^2}\,\epsilon_{\nu\mu\varsigma\varsigma}\, p^\lambda q^\varsigma \left(2pq\, I_{11} - (p^2+q^2)(I_{20}-I_{10})\right)$$
$$\xrightarrow[m^2\to 0]{} \; -\frac{i}{\pi^2}\,\epsilon_{\nu\mu\lambda\varsigma}\, p^\lambda q^\varsigma \quad \text{(II.156)}$$

One <u>cannot</u> regularize $S_{\lambda\nu\mu}(q,p)$ in such a way that all three currents are conserved - this is an anomaly (Adler (1969)).

In general one speaks about anomaly <u>if quantum effects spoil a symmetry present in the classical theory</u>.

One can think about the effective action:

$$\Gamma = S_{cl} + \hbar\,\Gamma_1 + \dots \quad \text{(II.157)}$$

where Γ_1 <u>is not</u> invariant under the symmetry transformation of S_{cl}.

We will concentrate ourselves on chiral anomalies. They arises since there is a "disagreement" between chiral symmetry and all the regularization procedures: no chirality preserving regularization exists.

In the theory with chiral fermions it means that gauge symmetry is broken. One can write

$$\delta\Gamma = \int d^4x \; \delta A^{\alpha}_{\mu} \; J^{\alpha}_{\mu} = \int d^4x \; v^{\alpha} \nabla_{\mu} J_{\mu}^{\alpha} = \int d^4x \; v^{\alpha} a^{\alpha} \tag{II.158}$$

$$a^{\alpha} = \nabla_{\mu} J_{\mu}^{\alpha}$$

The correspondence between the current nonconservation and $\delta\Gamma \neq 0$ is clear. One must remember hovewer that it is always possible to redefine Γ by adding a finite counter-term to it. We conclude that the anomaly is present if and only if for every counterterm $\Delta\Gamma$:

$$\delta(\Gamma + \Delta\Gamma) \neq 0$$

Which amplitudes are anomalous? First they must contain a $\epsilon_{\mu_1 \cdots \mu_D}$ tensor since they are parity-violating. Suppose there are N external vector lines. They provide N polarization vectors and N-1 independent momenta. We get a condition

$$N + (N-1) \geqslant D \qquad \text{i.e.} \qquad N \geqslant \frac{D+1}{2} \tag{II.159}$$

For example if D=4, $N \geqslant 3$.

There exists a theorem stating that all the anomalies with nonminimal N can be derived from the lowest N one.

We observe also that if D is odd then there is no chiral anomaly since Pauli & Villars regularization is always possible.

Anomalies are of <u>true</u> physical importance. In its early years they made possible to calculate a $\pi^0 \longrightarrow 2\gamma$ amplitude. Later they played essential role in solving U(1) problem in QCD.

The point is that they are "good" and "bad" anomalies. Good are called U(1) or Abelian anomalies and are associated with transformations.

$$\psi(x) \to \exp\left(-i\alpha(x)\gamma^5\right)\psi(x) \tag{II.160}$$

Bad are called nonAbelian anomalies and are related to transformations

$$\psi(x) \to \exp\left(-i\alpha^j(x)\, t^j \gamma^5\right)\psi(x) \tag{II.161}$$

where t^j is a generator of the Lie algebra in the representation according to which ψ transforms.

NonAbelian anomaly in the theory with chiral fermions spoils gauge covariance which is a basic ingredient in the proof of renormalizability and unitarity of the theory. Therefore we do not want nonAbelian anomalies to be present. In the theory with several chiral fermions the only possibility to achieve this is that contributions from different species of fermionic matter <u>cancell each other algebraically</u>. This happens for example in the Weinberg - Salam model within each family with three colors (Gross & Jackiw (1972)).

Chiral anomalies in multidimensional theories were first studied by Frampton & Kephart (1983). As D grows one must calculate diagrams with more and more external lines (see II.159).

In theories containing gravity (as KKT) there is also a possibility of gravitational (Lorentz) and mixed gauge-gravitational anomalies (Alvarez Gaumé & Witten (1983)). Vanishing of all kinds of chiral anomalies is a consistency condition for a theory formulated in a given number of dimensions. It is not clear how relevant is this condition if $D > 4$. Strictly speaking we should only require that effective theory obtained after spontaneous compactification in D=4 dimensions is free from all inconsistencies.

The structure of all kinds of chiral anomalies was understood during last few years (Zumino (1984), Alvarez Gaumé, Ginsparg (1985)). It turned out that nonAbelian anomaly in D dimensions is related to the index theorem in D+2 dimensions (it may be surprising that purely quantum phenomenon is understood is the language of topology).

All these discoveries culminated in the observation that some superstring theories and field theories obtained from them are free from chiral anomalies (Green & Schwarz (1984)). Since then a great interest towards the structure of string theories started.

K. Quantum Kaluza-Klein theories

The smallness of extra dimensions in a $M_4 x B_N$ Kaluza-Klein world makes it difficult to believe that quantum effects are negligible. One can thus try to construct a consistent Kaluza-Klein model in the sense that the background space-time will appear to be a solution of quantum corrected equations of motion. In general

$$S = S_{cl} + \Gamma^{(1)} \qquad \text{(II.162)}$$

$$S_{cl} = -\frac{1}{16\pi G_0}\int d^{4+N}z\,\sqrt{-g}\,(R+\Lambda) + S_{matter} \qquad \text{(II.163)}$$

$\Gamma^{(1)}$ is a 1-loop contribution.

We want the space-time to have a structure $M_4 x S_N$, from which it follows that

$$R = R_4 - \frac{N(N-1)}{a^2} \qquad \text{(II.164)}$$

R_4 being a D=4 dimensional curvature scalar.

S_{matt} may contain free scalar or Dirac fields coupled (minimally, conformally) to gravity. For dimensional reasons we expect that

$$\Gamma^{(1)} = - \int d^4x \sqrt{-g} \; A \, a^{-4} \qquad \text{(II.165)}$$

A is a calculable, model dependent constant*.

By combining together (II.164) and (II.165) we get in D=4 dimensions

$$S = - \int d^4x \sqrt{-g} \left(\frac{V_N}{16\pi G_0} \left(R + \lambda - \frac{N(N-1)}{a^2} \right) + \frac{A}{a^4} \right) \qquad \text{(II.166)}$$

where

$$V_N = \frac{2\pi^{N+1/2}}{\Gamma(N+1/2)} \, a^N$$

is a volume of S_N.

We look for solutions of the effective field equations

$$\left. \frac{\delta \Gamma}{\delta g^{mn}} \right|_{g_{mn} = \eta_{mn}} = 0$$

$$\left. \frac{\delta \Gamma}{\delta g^{\mu\nu}} \right|_{g_{mn} = \eta_{mn}} = 0 \qquad \text{(II.167)}$$

The last one is equivalent to

$$\frac{\partial \Gamma}{\partial a} = 0$$

The first leads to:

$$- \frac{N(N-1)}{a^2} + \lambda + \frac{16\pi G_0}{V_N} \frac{A}{a^4} = 0 \qquad \text{(II.168)}$$

which means a vanishing of D=4 dimensional cosmological constant. (II.167a) gives:

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* Usually one computes the one-loop effective potential only in an odd number of dimensions. The curvature invariant from which renormalization counterterms can be constructed are all of even dimensionality. We expect no divergences to be present in the odd number of dimensions (by a more precise argument due to Duff & Toms (1983,1984) only odd-loop amplitudes in dimensional regularization are finite)

$$-4\,\frac{A}{a^4} + \left(\lambda - \frac{(N-1)(N-2)}{a^2}\right)\frac{N V_N}{16\pi G_0} = 0 \qquad \text{(II.169)}$$

The calculation of the ceefficient A depends on the model. We will compute a contribution to A coming from a single massless mimimally coupled to gravity scalar field. This calculation was first done by Candelas & Weinberg (1983).

A massless scalar field on $M_4 x S_3$ background (we will specialize to S_3 from now on) is seen in D=4 dimensions as a "tower" of massive scalar particles with massess given by a spectrum of the Laplace operator on S_3. Its eigenvalues are given by:

$$M_n^2 = \frac{1}{a^2}(n^2-1) \qquad n = 1,2,\ldots \qquad \text{(II.170)}$$

with multiplicity n^2.

The effective potential is known to be

$$V(a) = \sum_{n=1}^{\infty} n^2 \left(\frac{-i}{2(2\pi)^4}\right)\int d^D k \ln\left(k^2 + M_n^2 - i\varepsilon\right) =$$

$$= -\sum_{n=1}^{\infty} n^2 \frac{\Gamma(-D/2)\, M_n^D}{2\,(4\pi)^{D/2}} \qquad \text{(II.171)}$$

(dimensional regularization is understood; $D \longrightarrow 4$)

We must make sense of:

$$\Gamma(-D/2)\sum_{n=1}^{\infty} n^2 (n^2-1)^{D/2} \qquad \text{(II.172)}$$

for $D \longrightarrow 4$, while as it stands (II.172) is well defined only for $D < -3$.

We do analytical continuation in D by using the following formula (Gradshteyn & Ryzhik (1980)):

$$n\left(n^2-a^2\right)^{D/2} = \frac{a\sqrt{\pi}}{\Gamma(-D/2)}\int_0^{\infty} dt\, e^{-nt}\left(\frac{t}{2a}\right)^{-D/2-1/2} I_{-D/2-3/2}(at) \qquad \text{(II.173)}$$

which is valid for $n > a$ and $D < -1$

After doing summation over n we get:

For D even integer the integrand is an even function of t. It behaves near t=0 as $t^{-D/2-5/2}$, so for $D<-3$

$$\Gamma(-D/2)\sum_{n=1}^{\infty} n^2 (n^2-a^2)^{D/2} =$$

$$= \sqrt{\pi}\, 2^{(D-1)/2} \int_C dt \frac{t^{-D-2}}{(2\,sh\, t/2)^2} (at)^{(D+3)/2}\, I_{-(D+3)/2}(at) \qquad \text{(II.174)}$$

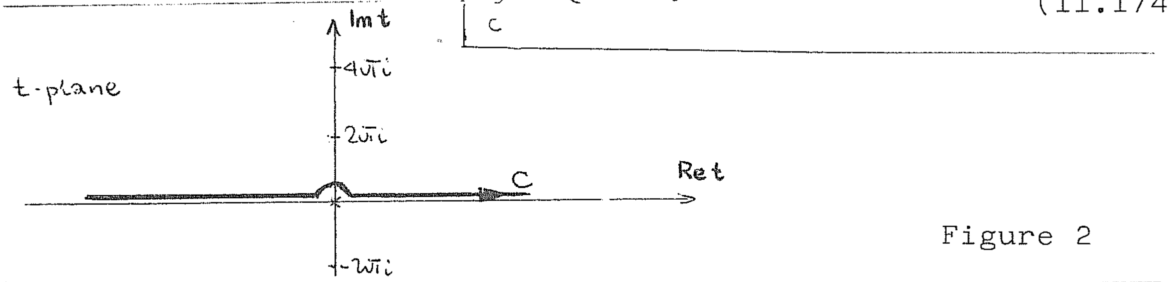

Figure 2

This expression makes sense also for D=4, $a^2<1$.

If we use the explicit form of $I_{-7/2}(z)$

$$I_{-7/2}(z)\, z^{7/2} = \sqrt{\frac{2}{\pi}}\left[sh(z)(15z+z^3) - ch(z)(15+6z^2)\right] \qquad \text{(II.175)}$$

then for D=4 the expression defined in (II.172) is equal to

$$4\int_C \frac{dt}{(2\,sh\,t/2)^2}\left[sh(at)\left(\frac{15a}{t^5}+\frac{a^3}{t^3}\right) - ch(at)\left(\frac{15}{t^6}+\frac{6a^2}{t^4}\right)\right] =$$

$$= 8\pi \sum_{p=1}^{\infty}\left[\sin(2\pi p a)\left(-\frac{90a}{(2\pi p)^6}+\frac{9a^3}{(2\pi p)^4}\right) + \cos(2\pi p a)\times \right.$$

$$\left. \times\left(\frac{39a^2}{(2\pi p)^5} - \frac{a^4}{(2\pi p)^3} - \frac{90}{(2\pi p)^7}\right)\right] \qquad \text{(II.176)}$$

The final representation was obtained by closing the contour C in the upper (or lower) half plane and summing over contributions from residua from poles at $2\pi p i$, p - integer. In particular for $a^2=1$ we get

$$8\pi\left[\frac{39}{(2\pi)^5}\zeta(5) - \frac{1}{(2\pi)^3}\zeta(3) - \frac{90}{(2\pi)^7}\zeta(7)\right] \qquad \text{(II.177)}$$

In the effective potential

$$V_0(a) = \frac{A}{a^4}$$

we can calculate a numerical value of A

$$A = 0.7568 \cdot 10^{-4} \qquad \text{(II.178)}$$

Looking back to (II.166), G_0/V_3 can be identified as Newton's constant (this is actually true only up to quantum corrections (Toms (1983))).Then equations (II.168) and (II.169) fix the value of a in terms of A and the Newton's constant. This is a very interesting result: values of the coupling constant and the radius of S_3 are <u>fixed by dynamics.</u>

It is in general very difficult to investigate the stability of the background configuration in this kind of compactification. Only stability against some very particular modes was investigated. Candelas & Weinberg (1983) discussed fluctutions corresponding to D=4 dimensional graviton and O(N+1) gauge vectors and obtained the result that it is necessary to introduce some fermions to the the theory in order to stabilize it. The stability against infinitesimal deformations of S_N was studied by a lot of authors: Page (1983), Lim (1985), Shen (1985), Okada (1986), Shiraishi (1986).

Some new results obtained recently by Shen & Sobczyk (1986) will be discussed in Chapter VIII.

An interesting review of quantum KKT is provided by Ref Toms (1986).

III.D=10 DIMENSIONAL MODEL WITH CP(3) AS THE INTERNAL SPACE

In this Chapter I will present calculations of a D=10 dimensional model of EYM theory with SU(3)xU(1) gauge fiels. These results were published by the author (1985). The discussion of CP(3) compactification can also be found in the paper due to Watamura (1983) without hovewer computation of the classical stability of the ground state which is the main topic presented here. After these results were published, Schellekens, using different methods, managed to prove that all CP(N) compactifications of analogous models are perturbatively stable (1985). In this Chapter I discuss only bosonic fields. Fermions will be considered independently in Chapter IV.

I start my exposition from few remarks about CP(3) space. It is a compact space which can be realized in different ways as a homogeneous space (action of different groups can be defined on this).

I will look at CP(3) as a SU(4)/U(3) homogeneous space. The action of the group SU(4) can be defined as a matrix multiplication from the right on four-dimensional complex vectors (z_1, z_2, z_3, z_4), satisfying condition:

$$\sum_{j=1}^{4} |z_j|^2 = \text{const}$$

with identification: $(z_1, z_2, z_3, z_4) \sim e^{i\phi}(z_1, z_2, z_3, z_4)$

An isotropy group of a given point - e.g. (0,0,0,1) consists from:

1/ SU(3) subgroup of SU(4) chosen as

$$\begin{pmatrix} SU(3) & 0 \\ 0 & 1 \end{pmatrix}$$

2/ U(1) subgroup of SU(4)

$$\begin{pmatrix} 1_3 e^{i\varphi} & 0 \\ 0 & e^{-3i\varphi} \end{pmatrix}$$

We observe that subgroups defined in 1/ and 2/ have a common factor Z_3, so that the isotropy group is*

$$\frac{SU(3) \times U(1)}{Z_3} = U(3)$$

(CP(3) can also be realized as other homogeneous spaces e.g. SO(5)/SO(3)xU(1))

A theory is defined by a following action principle

$$S = -\Omega \int d^{10}z \sqrt{-g} \left(\frac{R}{\varkappa^2} + \lambda + \right.$$

$$\left. + \frac{1}{4e^2} B_{MN} B^{MN} + \frac{1}{4g^2} F^{o}_{MN} F^{o\,MN} \right) \qquad \text{(III.1)}$$

I have introduced a dimensional constant Ω so that all the fields and coupling constants have standard D=4 dimensional dimensionalities;

$$B_{MN} = \partial_M B_N - \partial_N B_M \qquad \text{(III.2)}$$

$$F^{o}_{MN} = \partial_M A^{o}_N - \partial_N A^{o}_M + f^{okl} A^k_M A^l_N$$

e,g are U(1) and SU(3) coupling constants; f^{jkl} are SU(3) structure constants; M,N run from 0 to 3 and from 5 to 10, they are world indices; $z^M = (x^m, y^\mu)$; m=0,...,3 μ = 5,...,10. Later on I will mainly use orthonormal frame indices A = (a,α); a=0,...,3 , α =5,...,10 ; my metric convention is (-++...+).

SU(4) Lie algebra generators are given by $Q_{\hat\alpha}$

$$[Q_{\hat\alpha}, Q_{\hat\beta}] = f_{\hat\alpha\hat\beta\hat\gamma} Q_{\hat\gamma} \quad ; \quad \hat\alpha = 1, \dots, 15 \qquad \text{(III.3)}$$

$f_{\hat\alpha\hat\beta\hat\gamma}$ are totally antisymmetric

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* I thank Dr. Ludwik Dąbrowski for clarifying this point to me

Normalization I choose is:

$$f_{\hat{\alpha}\hat{\beta}\hat{\gamma}} f_{\hat{\sigma}\hat{\gamma}\hat{\beta}} = - \delta_{\hat{\alpha}\hat{\sigma}} \qquad \text{(III.4)}$$

SU(3)xU(1) subalgebra of SU(4) is generated by Q_j, $j = 1,\dots,8$ and Q_{15} ; they are altogether denoted as $Q_{\bar{\alpha}}$, and others as Q_α .

We have:

$$[Q_{\bar{\alpha}}, Q_{\bar{\beta}}] \sim Q_{\bar{\gamma}}$$

$$[Q_{\bar{\alpha}}, Q_\beta] \sim Q_\gamma \quad \text{(CP(3) is reductive)} \qquad \text{(III.5)}$$

$$[Q_\alpha, Q_\beta] \sim Q_{\bar{\gamma}} \quad \text{(CP(3) is symmetric)}$$

For example, the fundamental representation of SU(4) is given by ($Q_{\hat{\alpha}}$ are antihermitian):

$$Q_1 = \frac{1}{4}\begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \dots \quad Q_8 = \frac{1}{4\sqrt{3}}\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$Q_9 = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \quad \dots \quad Q_{15} = \frac{1}{4\sqrt{6}}\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -3i \end{bmatrix} \qquad \text{(III.6)}$$

The action (III.1) leads to following equations of motion:

$$R_{MN} - \frac{R}{2} g_{MN} = - \frac{\varkappa^2}{2} \Big(\frac{1}{g^2} \big(F^j_{MK} F^{jK}_N - \frac{1}{4} g_{MN} F^j_{KL} F^{jKL} \big) + \frac{1}{e^2} \big(B_{MK} B_N{}^K - \frac{1}{4} g_{MN} B_{KL} B^{KL} \big) - \lambda g_{MN} \Big) \qquad \text{(III.7)}$$

$$g^{LM} \nabla_L B_{MN} = g^{LM} (\partial_L B_{MN} - \Gamma^P_{LM} B_{PN} - \Gamma^P_{LN} B_{MP}) = 0 \qquad \text{(III.8)}$$

$$g^{LM} \nabla_L F^j_{MN} = g^{LM} (\partial_L F^j_{MN} - \Gamma^P_{LM} F^j_{PN} - \Gamma^P_{LN} F^j_{MP} + f^{jkl} A^k_L F^l_{MN}) = 0 \qquad \text{(III.9)}$$

According to the general discussion presented in Section II.E the set of equations (III.7-9) has a standard SU(4) symmetric solution which can be expressed in terms of SU(4) structure constants (in orthonormal basis)

$$R_{\alpha\beta\gamma\delta} = \frac{1}{a^2}\left(f_{j\alpha\beta} f_{j\gamma\delta} + f_{15\alpha\beta} f_{15\gamma\delta} \right) \tag{III.10}$$

$$F^{j}_{\alpha\beta} = \frac{1}{a^2} f_{j\beta\alpha} \tag{III.11}$$

$$B_{\alpha\beta} = \frac{\xi}{a^2} f_{15\beta\alpha} \tag{III.12}$$

a is a length scale of CP(3) $(R=-3/a^2)$; ξ is a "magnetic monopole" number for U(1) field configuration on CP(3) (locally a solution of Maxwell equation can be multiplied by arbitrary constant; the requirement that it is a U(1) connection induces constraints on ξ ; it turns out that ξ is quantized*; in my normalization SU(4) branching into SU(3)xU(1) is given by

$\underline{4} = \underline{3}_{1/4} \oplus \underline{1}_{-3/4}$

and $\xi = \frac{3}{4}n$, $n \in Z$)

In order that (III.10-12) solve equations (III.7-9), the parameters of the theory must satisfy:

$$\lambda = \frac{3}{2}\,\frac{1}{a^2\varkappa^2} \tag{III.13}$$

$$\frac{\varkappa^2}{a^2} = \frac{6}{\frac{2}{g^2} + \frac{\xi}{e^2}} \tag{III.14}$$

For later convenience we introduce dimensionfree parameter Y

$$Y = \frac{\varkappa^2}{2a^2g^2} = \frac{3}{2+\left(\frac{\xi g}{e}\right)^2} \quad ; \quad 0 \leq Y \leq \frac{3}{2} \tag{III.15}$$

$Y \longrightarrow 0$ corresponds to $\xi g \gg e$

$Y \longrightarrow \frac{3}{2}$ corresponds to $\xi g \ll e$

We assume the background configuration to have topology $M_4 x CP(3)$ so that $R_{abcd}=0$

As it was discussed in Section II.E, the requirement

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* I thank S.Randjbar-Daemi for help at this point

that the solution (III.7-9) is SU(4) invariant leads to the definition of the imbedding SU(3)xU(1) $\subset$ SO(6) (Salam & Strathdee (1982)):

$$Q_{\bar{\gamma}} = -\frac{1}{2} \rho_{\bar{\gamma}\alpha\beta} \Sigma_{\alpha\beta} \tag{III.16}$$

where $\Sigma_{\alpha\beta}$ is a SO(6) generator.

In the orthonormal basis all the quantities are expressed as SO(1,3) and SO(6) tensors. We would like to "translate" SO(6) tensor indices into SU(3)xU(1) ones using relation (III.16). Explicit calculations give (Sobczyk (1984)):

$\frac{1}{\sqrt{2}} \begin{pmatrix} \phi_9 - i\phi_{10} \\ \phi_{11} - i\phi_{12} \\ \phi_{13} - i\phi_{14} \end{pmatrix}$ transforms like $\underline{3}^*$ with U(1) charge -1

$\frac{1}{\sqrt{2}} \begin{pmatrix} \phi_9 + i\phi_{10} \\ \phi_{11} + i\phi_{12} \\ \phi_{13} + i\phi_{14} \end{pmatrix}$ transforms like $\underline{3}$ with U(1) charge +1

We want to compute a part of the action (III.1) bilinear in fluctuations around the background solution (III.10-12; II.61-62):

$$\begin{aligned} g_{MN} &= \langle g_{MN} \rangle + \varkappa h_{MN} \\ A_M^{j} &= \langle A_M^{j} \rangle + W_M^{j} \\ B_M &= \langle B_M \rangle + V_M \end{aligned} \tag{III.17}$$

Calculations are performed in the light-cone gauge. We get:

$$S = \int d^{10}z \Big[\frac{1}{4} h^T_{jk} (\partial^2 + \nabla^2) h^T_{jk} + \frac{1}{2} h_{rs} \Big(\partial^2 + \nabla^2 - \frac{1}{a^2} +$$

$$+ \frac{Y-1}{2a^2}\Big) h_{\bar{r}\bar{s}} + \frac{1}{2} h_{\bar{r}s} \Big(\partial^2 + \nabla^2 - \frac{1}{a^2} + \frac{3}{4a^2}(Y-1)\Big) h_{rs}$$

$$+ h_{jr} \Big(\partial^2 + \nabla^2 - \frac{1}{2a^2}\Big) h_{j\bar{r}} + \frac{1}{6} h_{jj} \Big(\partial^2 + \nabla^2 - \frac{1}{4a^2}\Big) h_{kk}$$

$$
\begin{aligned}
&+\frac{1}{2e^2} V_j (\partial^2+\nabla^2) V_j + \frac{1}{2g^2} W_{jrs} (\partial^2+\nabla^2) W_{j\dot{r}\dot{s}} + \frac{1}{e^2} V_r \Big(\partial^2+\nabla^2-\frac{1}{2a^2} + \\
&+\frac{1}{3a^2}(2Y-3)\Big) V_{\dot{r}} + \frac{1}{g^2} W_{rs\dot{t}} \Big(\partial^2+\nabla^2-\frac{3}{4a^2}\Big) W_{t\dot{r}\dot{s}} + \\
&+\frac{2}{g^2} W_{pr} \Big(\partial^2+\nabla^2-\frac{1}{4a^2}\Big) W_{\dot{p}\dot{r}} + \frac{3}{8g^2} W_r \Big(\partial^2+\nabla^2+\frac{1}{4a^2}+\frac{2}{3}\frac{Y}{a^2}\Big) W_{\dot{r}} \\
&+\frac{iY}{\varkappa\sqrt{2}} \Big[h_{jr} \nabla_{\dot{s}} W_{js\dot{r}} - h_{j\dot{r}} \nabla_s W_{jr\dot{s}} + h_{rs} \nabla_{\dot{t}} W_{t\dot{r}\dot{s}} - h_{\dot{r}\dot{s}} \nabla_t W_{rs\dot{t}} \\
&\qquad + \frac{1}{8} h_{rs} (\nabla_{\dot{r}} W_{\dot{s}} + \nabla_{\dot{s}} W_{\dot{r}}) - \frac{1}{8} h_{\dot{r}\dot{s}} (\nabla_r W_s + \nabla_s W_r) + \\
&\qquad + h_{\dot{r}s} \nabla_{\dot{t}} W_{rt\dot{s}} - h_{r\dot{t}} \nabla_s W_{t\dot{r}s} + \epsilon_{rtp} h_{\dot{r}s} \nabla_{\dot{t}} W_{\dot{p}s} \\
&\qquad - \epsilon_{rsp} h_{r\dot{t}} \nabla_s W_{pt} + \frac{1}{8} h_{r\dot{s}} (\nabla_s W_{\dot{r}} - \nabla_{\dot{r}} W_s) + \\
&\qquad + \frac{1}{6} h_{jj} (\nabla_s W_{\dot{s}} - \nabla_{\dot{s}} W_s) \Big] - \frac{1}{\sqrt{3}} \frac{Y(3-2Y)}{\varkappa^2 \xi} (W_{\dot{r}} V_r + W_r V_{\dot{r}}) \\
&+ i\sqrt{\frac{2}{3}} \frac{3-2Y}{\varkappa\xi} \Big[h_{jr} \nabla_{\dot{r}} V_j - h_{j\dot{r}} \nabla_r V_j + \frac{1}{2} h_{rs} (\nabla_{\dot{s}} V_{\dot{r}} + \nabla_{\dot{r}} V_{\dot{s}}) \\
&\qquad\qquad - \frac{1}{2} h_{\dot{r}\dot{s}} (\nabla_r V_s + \nabla_s V_r) + h_{\dot{r}s} (\nabla_{\dot{s}} W_r - \nabla_r W_{\dot{s}}) \\
&\qquad\qquad + \frac{1}{6} h_{jj} (\nabla_t V_{\dot{t}} - \nabla_{\dot{t}} V_t) \Big] \Big]
\end{aligned}
\qquad \text{(III.18)}
$$

Indices j,k run 1,2 (they refer to Minkowski space); $r, \dot{r}$ are SU(3) indices.

In the action (III.18) all the fields belong to a certain irreducible SU(3)xU(1) representation. Below in Table 1 we give a list of these representations

TABLE 1

SU(3)xU(1) representation *	Fields
$\underline{1}_0$	h_{jj}, h^T_{jk}, V_j
$\underline{3}_1$	h_{jr}, V_r, W_r
$\underline{3}^*_{-1}$	$h_{j\dot r}$, $V_{\dot r}$, $W_{\dot r}$
$\underline{8}_0$	$h^T_{r\dot s}$, $W^T_{jr\dot s}$
$\underline{6}_2$	h_{rs}
$\underline{6}^*_{-2}$	$h_{\dot r\dot s}$
$\underline{6}_{-1}$	W_{rs}
$\underline{6}^*_1$	$W_{\dot r\dot s}$
$\underline{15}_1$	$W_{rs\dot t}$
$\underline{15}^*_{-1}$	$W_{\dot r st}$

* subscript denotes a value of U(1) charge

Results for W-fields follow from the tensor products $\underline{3} \otimes \underline{8}$ and $\underline{3}^* \otimes \underline{8}$, since W transforms as a SU(3) octet being a fluctuation of the gauge group and as a triplet because it can be also a SO(6) vector.

We are now almost ready to apply harmonic expansion in order to compute a spectrum of the theory. To this aim hovewer one must first solve a group theoretical problem: given a representation SU(3)xU(1), find all the representations of SU(4) containing the particular one.

This problem is most easily solved in the language of Gelfand & Zetlin patterns (see Barut & Rączka (1977)). Any irreducible SU(4) Lie algebra representation is given by four integers (m_1,m_2,m_3,m_4) ; $m_1 \geq m_2 \geq m_3 \geq m_4$. (m_1,m_2,m_3,m_4) and $(m_1+k,m_2+k,m_3+k,m_4+k)$ describe the same representation. In the case of SU(4) we need <u>three independent integers</u> to describe a representation.

Similarly, a SU(3) Lie algebra representation is given by three integers (n_1,n_2,n_3) , two of them being independent.

A representation space is given by all the possible Gelfand & Zetlin patterns obtained according to the rule

$$\begin{bmatrix} m_1 & & m_2 & & m_3 & & m_4 \\ & n_1 & & n_2 & & n_3 & \\ & & k_1 & & k_2 & & \\ & & & l_1 & & & \end{bmatrix} \qquad \begin{array}{l} m_1 \geq n_1 \geq m_2 \geq n_2 \geq m_3 \geq n_3 \geq m_4 \\ n_1 \geq k_1 \geq n_2 \geq k_2 \geq n_3 \\ k_1 \geq l_1 \geq k_2 \end{array} \tag{III.19}$$

(n_1,n_2,n_3) give all the SU(3) representations contained in SU(4) under branching SU(4) $\longrightarrow$ SU(3)xU(1). The dimensionality of (m_1,m_2,m_3,m_4) representation is

$$\frac{1}{12}(m_1-m_2+1)(m_1-m_3+2)(m_1-m_4+3)(m_2-m_3+1)\times (m_2-m_4+2)(m_3-m_4+1) \tag{III.20}$$

In the case of SU(3) representation (n_1,n_2,n_3) the formula for dimensionality is

$$\frac{1}{2}(n_1-n_2+1)(n_1-n_3+2)(n_2-n_3+1) \tag{III.21}$$

The simplest example is supplied by a (1,0,0) representation. Gelfand & Zetlin patterns are

$$\begin{bmatrix} 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix} \qquad \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \qquad \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \tag{III.22}$$

(1,0,0) describes a triplet representation.

The remaining problem is to define a U(1) charge in the language of Gelfand & Zetlin patterns. We normalized a U(1) charge in such a way that the following decomposition takes place

$$\underline{4} = \underline{3}_{1/4} \oplus \underline{1}_{-3/4} \tag{III.23}$$

In the fundamental representation of SU(4)

$$Q_{15} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} = \frac{1}{4}(A_{11}+A_{22}+A_{33}-3A_{44}) \tag{III.24}$$

In Ref Barüt & Rączka (1977) we can find algebraic expressions defining values of A_{jj} generators of GL(4,R) in action on Gelfand & Zetlin pattern (III.19)

$$\begin{aligned} A_{11} &= l_1 \\ A_{22} &= k_1 + k_2 - l_1 \\ A_{33} &= n_1 + n_2 + n_3 - k_1 - k_2 \\ A_{44} &= m_1 + m_2 + m_3 + m_4 - n_1 - n_2 - n_3 \end{aligned} \qquad \text{(III.25)}$$

(all A_{jj} are diagonal).

Thus in the action on the pattern (III.19)

$$Q_{15} = (n_1 + n_2 + n_3) - \frac{3}{4}(m_1 + m_2 + m_3 + m_4) \qquad \text{(III.26)}$$

We are now ready to identify all the series of SU(4) representations which are present in the harmonic expansions (II.98).

Example 1

Consider $\underline{1}_0$ representation. A SU(3) representation (0,0,0) is contained in the following SU(4) representations:

(m,0,0,n) $\quad m \geqslant 0 \quad n \leqslant 0$

A U(1) charge is given by $-\frac{3}{4}(m+n) = 0$. Thus $m=-n$.

$$(0,0,0)_0 \subset (n,0,0,-n) \qquad \text{(III.27)}$$

Example 2

Consider $\underline{3}_1$ representation. A SU(3) representation (1,0,0) is contained in the following SU(4) representations:

1/ (m,1,0,n) $\quad m \geqslant 1 \quad n \leqslant 0$

2/ (m,0,0,n) $\quad m \geqslant 0 \quad n \leqslant 0$

In the case 1/ a U(1) charge is $1 - \frac{3}{4}(m+n+1)$ so that

$m+n+1 = 0$ and $m=-n-1$

In the case 2/ a U(1) charge is $1 - \frac{3}{4}(m+n)$ so that $m+n = 0$ and $m=-n$

Finally

$$(1,0,0)_1 \subset (n,0,0,-n) + (n,1,0,-n-1). \tag{III.28}$$

In the Table 2 I give a list of all the SU(4) representations which lebel harmonics present in the expansions of all the fields of the theory

TABLE 2

SU(3)xU(1) representation	SU(4) harmonics present in the expansion
$\underline{0}_0, \underline{3}_1, \underline{3}^*_{-1}, \underline{8}_0, \underline{15}_1, \underline{15}^*_{-1}, \underline{6}_{-1}, \underline{6}^*_1$	$(n,0,0,-n)$
$\underline{8}_0\ \underline{3}_1\ \underline{6}_2\ \underline{6}^*_1\ \underline{15}_1\ \underline{15}^*_{-1}$	$(n,1,0,-n-1)$
$\underline{8}_0\ \underline{3}^*_{-1}\ \underline{6}_{-1}\ \underline{6}^*_{-2}\ \underline{15}_1\ \underline{15}^*_{-1}$	$(n+1,0,-1,-n)$
$\underline{6}_2\ \underline{15}_1$	$(n+1,2,0,-3-n)$
$\underline{6}^*_{-2}\ \underline{15}^*_{-1}$	$(n+3,0,-2,-n)$
$\underline{8}_0\ \underline{6}_{-1}\ \underline{6}^*_1\ \underline{15}_1\ \underline{15}^*_{-1}$	$(n,1,-1,-n)$
$\underline{6}_{-1}$	$(n+2,-1,-1,-n)$
$\underline{6}^*_1$	$(n,1,1,-2-n)$
$\underline{15}_1$	$(n+1,2,-1,-2-n)$
$\underline{15}^*_{-1}$	$(n+2,1,-2,-1-n)$

In Section II.F it was shown that the action of covariant derivative on SU(4) harmonics is equivalent to a certain algebraical operation:

$$\nabla_\alpha \mathbb{D}^{(\lambda)}_{pr}(L_y^{-1}) = -\frac{i}{a}\, \mathbb{D}^{(\lambda)}_{ps}(Q_\alpha)\, \mathbb{D}^{(\lambda)}_{sr}(L_y^{-1}) \tag{III.29}$$

From a general theory (Barut & Rączka (1977)) we can learn how do GL(n,R) generators (from which SU(n) generators are built) act on Gelfand & Zetlin patterns. Let

$$m = \begin{bmatrix} m_{1n} \quad m_{2n} & \cdots & \cdots & m_{n-1n} \quad m_{nn} \\ m_{1n-1} & & \cdots & m_{n-1n-1} \\ & & & \\ & & m_{12} \quad m_{22} & \\ & & m_{11} & \end{bmatrix} \qquad \text{(III.30)}$$

and A_{kl} be a n x n matrix defined as:

$$(A_{kl})_{ij} = \delta_{ki}\,\delta_{lj}\,. \qquad \text{(III.31)}$$

Then

$$\begin{aligned} A_{k,k-1}(m) &= \sum_{j}^{k-1} a^{j}_{k-1}[m]\; m^{j}_{k-1} \\ A_{k-1,k}(m) &= \sum_{j}^{k-1} b^{j}_{k-1}[m]\; \hat{m}^{j}_{k-1} \end{aligned} \qquad \text{(III.32)}$$

m^{j}_{k-1} is a pattern in which element $m_{j,k-1}$ is replaced by $m_{j,k-1}-1$; $\hat{m}^{j}_{k-1}$ is a pattern in which element $m_{j,k-1}$ is replaced by $m_{j,k-1}+1$.

$$\begin{aligned} a^{j}_{k-1}[m] &= \left[- \frac{\prod_{i=1}^{k}(l_{ik}-l_{jk-1}+1)\,\prod_{i=1}^{k-2}(l_{ik-2}-l_{jk-1})}{\prod_{i\neq j}(l_{ik}-l_{jk-1}+1)(l_{ik-1}-l_{jk-1})} \right]^{1/2} \\ b^{j}_{k-1}[m] &= \left[- \frac{\prod_{i=1}^{k}(l_{ik}-l_{jk-1})\,\prod_{i=1}^{k-2}(l_{ik-2}-l_{jk-1}-1)}{\prod_{i\neq j}(l_{ik-1}-l_{jk-1})(l_{ik-1}-l_{jk-1}-1)} \right]^{1/2} \end{aligned} \qquad \text{(III.33)}$$

where $l_{jk} = m_{jk} - j$

Next we find a correspondence between G&Z patterns and irreducible SU(3) tensors present in (III.18). By direct inspection we obtain:

$$\underline{3} \qquad \Phi_1 = \begin{bmatrix} 1 & 0 & 0 \\ & 1 \quad 0 & \\ & 1 & \end{bmatrix} \qquad \Phi_2 = \begin{bmatrix} 1 & 0 & 0 \\ & 1 \quad 0 & \\ & 0 & \end{bmatrix} \qquad \Phi_3 = \begin{bmatrix} 1 & 0 & 0 \\ & 0 \quad 0 & \\ & 0 & \end{bmatrix}$$

$$\underline{3}^{*} \qquad \Phi_1 = \begin{bmatrix} 0 & 0 & -1 \\ & 0 \quad -1 & \\ & -1 & \end{bmatrix} \qquad \Phi_2 = \begin{bmatrix} 0 & 0 & -1 \\ & 0 \quad -1 & \\ & 0 & \end{bmatrix} \qquad \Phi_3 = \begin{bmatrix} 0 & 0 & -1 \\ & 0 \quad 0 & \\ & 0 & \end{bmatrix}$$

$\underline{8}$ $\qquad \Phi_{1\dot{3}} = \begin{bmatrix} 1 & 0 & -1 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \qquad \Phi_{2\dot{3}} = \begin{bmatrix} 1 & 0 & -1 \\ & 1 & 0 \\ & & 0 \end{bmatrix}$ etc

$$\Phi_{1\dot{1}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ & 1 & -1 \\ & & 0 \end{bmatrix} - \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 0 & -1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$

$$\Phi_{2\dot{2}} = -\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ & 1 & -1 \\ & & 0 \end{bmatrix} - \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 0 & -1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$

$$\Phi_{3\dot{3}} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & 0 & -1 \\ & 0 & 0 \\ & & 0 \end{bmatrix}$$

$\underline{6}$ $\qquad \Phi_{11} = \sqrt{2} \begin{bmatrix} 2 & 0 & 0 \\ & 2 & 0 \\ & & 2 \end{bmatrix} \qquad \Phi_{12} = \begin{bmatrix} 2 & 0 & 0 \\ & 2 & 0 \\ & & 1 \end{bmatrix}$ etc

(III.34)

We find also that (see Section II.F)

$$\nabla_r \sim \frac{1}{a} \frac{i}{2\sqrt{2}} A_{r4}$$

$$\nabla_{\dot{r}} \sim \frac{1}{a} \frac{i}{2\sqrt{2}} A_{4r} \qquad \text{(III.35)}$$

Eigenvalues of the Laplace operator are given by the difference between values of two quadratic Casimir operators in appropriate representations of SU(4) and SU(3)xU(1). These values are given in Table 3

TABLE 3

SU(3)xU(1) representation	Casimir operator eigenvalue
$\underline{3}_1$	$-\frac{1}{2a^2}$
$\underline{8}_0$	$-\frac{3}{4a^2}$
$\underline{6}_2$	$-\frac{3}{2a^2}$
$\underline{6}_{-1}$	$-\frac{1}{a^2}$
$\underline{15}_1$	$-\frac{3}{2a^2}$

SU(4) representation	Casimir operator eigenvalue
$(n,0,0,-n)$	$-\frac{1}{4a^2}(n^2+3n)$
$(n,1,0,-1-n)$	$-\frac{1}{4a^2}(n^2+4n+3)$
$(n+1,2,-1,-2-n)$	$-\frac{1}{4a^2}(n^2+6n+11)$
$(n,1,1,-n-2)$	$-\frac{1}{4a^2}(n^2+5n+7)$
$(n+1,2,0,-n-3)$	$-\frac{1}{4a^2}(n^2+7n+14)$
$(n,1,-1,-n)$	$-\frac{1}{4a^2}(n^2+3n+2)$

It is now possible to analyze - sector by sector - the spectrum of the theory. We will see that modes corresponding to different values of spin|helicity decouple.

I resign here from giving all the details of calculations. They can be found elsewhere (Sobczyk (1984)). I am going to explain only how do zero modes arise in D=4 dimensions. All of them appear in the sector (n,0,0,-n) for n = 0,1.

For n=0 there are two massless modes: $h^{T\circ}_{jk}(x)$ (graviton) and $V^{\circ}_{j}(x)$ (U(1) vector). Both are SU(4) singlets (representation (0,0,0,0)).

For n=1 in the spin/helicity 1 sector there are four fields: $h_{\underline{3}j}$, $h_{\underline{3}^*j}$, V_j, $W_{\underline{8}j}$. They give rise to the following mass matrix*:

$$\mathcal{M} = \begin{bmatrix} \partial^2-\frac{1}{4a^2}(n^2+3n) & 0 & B & -B \\ 0 & \partial^2-\frac{1}{4a^2}(n^2+3n-3) & C & -C \\ B & C & \partial^2-\frac{1}{4a^2}(n^2+3n) & 0 \\ -B & -C & 0 & \partial^2-\frac{1}{4a^2}(3n+3n) \end{bmatrix}$$

where (III.36)

* There is a misprint in my original paper.

$$B = \frac{1}{2\sqrt{3}} \frac{e(3-2Y)}{a\varkappa\xi} \sqrt{n^2+3n}$$
$$C = \frac{1}{4} \frac{gY}{a\varkappa} \sqrt{\tfrac{2}{3}(n^2+3n)} \qquad \text{(III.37)}$$

We calculate

$$\begin{aligned}\mathrm{Det}\, \mathcal{M} = {} & \left(\partial^2 - \frac{1}{4a^2}(n^2+3n)\right)\left(\left(\partial^2 - \frac{1}{4a^2}(n^2+3n)\right)^2\left(\partial^2 - \frac{1}{4a^2}(n^2+3n-3)\right)\right.\\ & - \left(\partial^2 - \frac{1}{4a^2}(n^2+3n-3)\right)\frac{1}{12a^4}(3-2Y)(n^2+3n)\\ & - \left(\partial^2 - \frac{1}{4a^2}(n^2+3n)\right)\frac{Y}{24a^2}(n^2+3n)\end{aligned} \qquad \text{(III.38)}$$

In particular for n=1

$$\begin{aligned}\mathrm{Det}\, \mathcal{M} = {} & \left(\partial^2 - \frac{1}{a^2}\right)\left[\left(\partial^2 - \frac{1}{a^2}\right)^2\left(\partial^2 - \frac{1}{4a^2}\right) - \right.\\ & \left. - \left(\partial^2 - \frac{1}{4a^2}\right)\frac{3-2Y}{3a^4} - \left(\cdot\ ^2 - \frac{1}{a^2}\right)\frac{Y}{6a^4}\right]\end{aligned} \qquad \text{(III.39)}$$

Solutions of the equation:

$$(\partial^2)^3 - \frac{9}{4}\frac{(\partial^2)^2}{a^2} + \frac{\partial^2}{a^4}\frac{1+Y}{2} = 0 \qquad \text{(III.40)}$$

are given by:

$$\begin{aligned}\partial^2 &= 0\\ \partial^2_{\pm} &= \frac{1}{a^2}\left(\frac{9}{8} \pm \frac{\sqrt{81-32(1+Y)}}{8}\right)\end{aligned} \qquad \text{(III.41)}$$

The zero mode is found in the sector (1,0,0,-1) i.e. it transform as the adjoint representation of SU(4). These are transformation properties of SU(4) gauge bosons. Our conjecture that SU(4), as the symmetry group of the background solution (III.10-12) should be seen in D=4 dimensions as the gauge group is therefore confirmed.

To conclude I will collect main results of this Section:

1/ The spectrum turns out to be perturbatively stable (i.e. there are no ghosts and tachyons in the spectrum)

2/ SU(4)xU(1) gauge symmetry is seen in D=4 dimensions

3/ An effective SU(4) coupling constant is computed to be (Sobczyk (1984)) $f^2 = \frac{20\, G_4}{a^2(1+Y)}$ which can be compared with (II.14).

IV. CHIRAL FERMIONS IN D=10 DIMENSIONAL EINSTEIN-YANG-MILLS SU(3)xU(1) THEORY COMPACTIFIED TO CP(3)*

In Chapter III a complete study of the bosonic sector of D=10 dimensional E-Y-M theory with SU(3)xU(1) gauge fields was presented. The topological properties of the background configuration of Yang-Mills fields on CP(3) make it possible to obtain massless chiral fermions after spontaneous compactification.

A general introduction to the subject was given in Section II.I.

We introduce a Dirac field into the D=10 dimensional action (later we will assume definite D=10 dimensional chirality of this field):

$$S = -\Omega \int d^{10}z \sqrt{-g} \left(\frac{R}{\varkappa^2} + \Lambda + \frac{1}{4e^2} B_{MN} B^{MN} + \right. \tag{IV.1}$$
$$\left. + \frac{1}{4g^2} F^{\delta}_{MN} F^{\delta MN} + i \bar{\Psi} \Gamma^M \nabla_M \Psi + h.c. \right)$$

Notation is the same as in Chapter III.

We choose some realization of D=10 dimensional Dirac matrices (32 x 32 matrices)

$$\begin{aligned} \Gamma^a &= \gamma^a \otimes \mathbb{1} \\ \Gamma^\alpha &= \hat{\gamma}^5 \otimes \gamma^\alpha \end{aligned} \tag{IV.2}$$

γ^a are 4 x 4 Dirac matrices satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$

γ^α are 8 x 8 Dirac matrices satisfying $\{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta}$

Moreover we define:

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* Based on the paper: Sobczyk(1985)

$$\hat{\gamma}^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \qquad (\hat{\gamma}^5)^2 = \mathbb{1}$$

$$\hat{\Gamma} = -i\gamma^5 \cdots \gamma^{10} \qquad (\hat{\Gamma})^2 = \mathbb{1} \tag{IV.3}$$

$$\tilde{\Gamma} = \Gamma^0 \cdots \Gamma^{10} \qquad (\tilde{\Gamma})^2 = \mathbb{1}$$

The background configuration of the Dirac field is $\langle\Psi\rangle = 0$ (we require Poincarè invariance of the background).

Ψ has both SU(3)xU(1) (U(3)) and SO(1,9) indices. We assume that SU(3) transformation properties are (a,b) in Young tableau notation, and (a+b,b,0) in Gelfand & Zetlin notation. The value of U(1) charge is chosen to be 1.

The background covariant derivative of Ψ has a form

$$\nabla\Psi = d\Psi + S\Psi - iB\Psi - iA^j t^j(a,b)\Psi \tag{IV.4}$$

where $t^j(a,b)$ is a SU(3) matrix in the (a,b) representation;

$$S = \frac{1}{8} S_{[AB]} [\gamma^A, \gamma^B] = \frac{i}{2} S_{[AB]} \Sigma^{AB} \tag{IV.5}$$

is a spin connection.

We are interested in terms in the action bilinear in small fluctuations around the background configuration. Since $\langle\Psi\rangle = 0$, the fermionic part (which is bilinear in Ψ) will contain background values of all the fields to which Ψ is coupled.

I remind the background configuration of the Yang-Mills fields and of the spin connection (II.62,66)

$$\langle A^j \rangle = e^j(y)$$

$$\langle B \rangle = \xi e^{15}(y) \tag{IV.6}$$

$$\langle S_{[AB]} \rangle = e^{\bar{\gamma}} C_{\bar{\gamma}\alpha\beta}$$

$C_{\bar{\gamma}\alpha\beta}$ are structure constants of SU(4).

The embedding (see Section II.E) of U(3) in SO(6) gives:

$$\underset{\sim}{Q}^{\bar{\gamma}} = -\frac{i}{2} C_{\bar{\gamma}\alpha\beta} \Sigma^{\alpha\beta} \tag{IV.7}$$

and the background covariant derivative takes a form:

$$\begin{aligned} \bar{\nabla}\psi = d\psi & - i e^{\dot{\delta}}(y) \left(\underset{\sim}{Q}^{\dot{\delta}} + t^{\dot{\delta}} \right) \psi - \\ & - i e^{15}(y) \left(\underset{\sim}{Q}^{15} + \frac{3}{4} n \right) \psi \end{aligned} \tag{IV.8}$$

This expresses the fact that ψ has some <u>effective</u> transformation properties with respect to SU(3)xU(1). These must be found in order to use them in harmonic expansion.

We start from a Weyl spinor in D=10 dimensions. To be definite

$$\tilde{\Gamma}\psi = +\psi \tag{IV.9}$$

Since $\tilde{\Gamma} = \hat{\gamma}^5 \otimes \hat{\Gamma}$, after branching SO(1,9) ---> SO(1,3)$\otimes$SO(6) the positive chirality (in D=4 dimensions) spinor has positive D=6 dimensional chirality and negative D=4 dimensional spinor has a negative D=6 dimensional chirality. This is why ψ_R and ψ_L have <u>different</u> effective transformation properties with respect to SU(3)xU(1).

Two fundamental SO(6) spinors have (due to embedding) the following transformation properties with respect to SU(3)x U(1)

$$\begin{aligned} \psi_+ &\sim \underline{3}_{-1/2} \oplus \underline{1}_{3/2} \\ \psi_- &\sim \underline{3}^*_{1/2} \oplus \underline{1}_{-3/2} \end{aligned} \tag{IV.10}$$

The effective transformation properties of $\psi_{R,L}$ with respect to SU(3)xU(1) are

$$\begin{aligned} \psi_R \sim (a,b)_{\frac{3n}{4}+\frac{3}{2}} & \oplus (a+1,b)_{\frac{3n}{4}-\frac{1}{2}} \oplus \\ & \oplus (a-1,b+1)_{\frac{3n}{4}-\frac{1}{2}} \oplus (a,b-1)_{\frac{3n}{4}-\frac{1}{2}} \end{aligned} \tag{IV.11}$$

$$\Psi_L \sim (a,b)_{\frac{3n}{4}-\frac{3}{2}} \oplus (a,b+1)_{\frac{3n}{4}+\frac{1}{2}} \oplus$$

$$\oplus\ (a-1,b)_{\frac{3n}{4}-\frac{1}{2}} \oplus (a+1,b-1)_{\frac{3n}{4}+\frac{1}{2}} \qquad \text{(IV.12)}$$

According to the analysis of Section II.I we must find harmonics present only in the expansion for Ψ_R or Ψ_L but not in both. These harmonics (in the Gelfand & Zetlin notation) are given in Table 4.

TABLE 4

Harmonics present in the expansion of Ψ_L only	Harmonic present in the expansion of Ψ_R only
$3n < -6-2a-b$ $(-n-2+k,\ a+b,\ b,\ 0)$	$3n > 2b+a+6$ $(a+b-1,\ b-1,\ -1,\ k-n+1)$
$a+3-b < 3n < 2b+a$ $(a+b-1,\ b-1,\ k-n,\ 0)$	$-2a-b < 3n < a-b-3$ $(a+b-1,\ k-n-1,\ b,\ 0)$

An interesting observation is that harmonic expansion exists only if $a+2b=3k$ where k is an integer (for other values of U(1) charge, this constraint would have a different form).

Dimensionalities of all the SU(4) representations given in Table 4 are equal to

$$d = \frac{1}{12}(a+1)(b+1)(a+b+2)(k-n+1)(k-n-b)(k-n-a-b-1) \qquad \text{(IV.13)}$$

Consider two examples.

1/ $a = b = k = 0$

$n \leqslant -2$ $(-n-2,0,0,0)$ harmonic is present in the expansion of Ψ_L only

$n \geqslant 2$ $(-1,-1,-1,-n+1)$ harmonic is present in the expansion of Ψ_R only

Dimensionalities of both SU(4) representations are

$$D= \frac{1}{6}|n|(n^2-1). \tag{IV.14}$$

2/ $a = b = k = 1$

$n = 1$ $(1,0,0,0)$
$n \leqslant -3$ $(-n-1,2,1,0)$ } harmonics are present in the expansion of Ψ_L only

$n = -1$ $(0,0,0,-1)$
$n > 3$ $(1,0,-1,2-n)$ } harmonics are present in the expansion of Ψ_R only

Dimensionalities of all SU(4) representations present in the Example 2 are

$$D = \frac{4}{3}|n|\,|n^2-4| \tag{IV.15}$$

It is interesting to try to derive above results directly from the Atiyah & Singer theorem (see II.150).

$$n_R - n_L = \int_{\mathbb{C}P(3)} \hat{A}\,(\mathbb{C}P(3)) \wedge ch_1(V) \tag{IV.16}$$

We consider now a topologically nontrivial principal fibre bundle $U(3) = \frac{SU(3) \times U(1)}{Z_3}$ over CP(3). We use a normalized two-form z on CP(3)

$$\int_{\mathbb{C}P(3)} z^3 = 1 \tag{IV.17}$$

From Eguchi, Gilkey & Hanson (1980)

$$\hat{A} = 1 - \frac{z^2}{6} \tag{IV.18}$$

For a=b=0 $ch(a=0,b=0)=1+nz+\frac{n^2}{2}z^2 + \frac{n^3}{6}z^3$

and

$$(n_R - n_L) = \frac{n}{6}(n^2-1) \tag{IV.19}$$

For $a = b = 1$

$$ch(a=1,b=1) = 8 + 8nz + (4n^2-4)z^2 + (\frac{4}{3}n^3-4n)z^3$$

and

$$(n_R - n_L) = \frac{4}{3}n(n^2-4) \tag{IV.20}$$

We see that both results are in agreement (also signs are OK). We did not compute the general expression for $n_R - n_L$ as the function of (a,b). It is hovewer not difficult to reproduce its values for every (a,b) using the fact that ch(v) has simple properties under tensor multiplication of irreducible SU(3) representations

$$V \otimes W = \oplus \sum_{\delta} T_{\delta} \tag{IV.21}$$

where T_j are irreducible representations,

$$ch(V) \wedge ch(W) = \sum_{\delta} ch(T_{\delta}) \tag{IV.22}$$

as forms on CP(3).

Similar study with CP(N) (with arbitrary N) internal space was afterwards done by Bailin & Love (1985). For N=3 both results agree.

Before, a particular case (a=b=0) was investigated by Chapline & Grossman (1984).

V. ANOMALIES AND SPONTANEOUS COMPACTIFICATION*

Studies of anomalies (introduction to anomalies is presented in Section II.J) in higher-dimensional theories have become a very fruitful idea in the last few years. The requirement of algebraic cancellation of all the gauge and gravitational (and mix) anomalies appears to be a very strong constraint restricting the class of reasonable physical models (Alvarez-Gaumè & Witten (1983), Green & Schwarz (1984). In this context a natural question arises of whether a spontaneous compactification mechanism (which is necessary in going to the four physical dimensions) is such that no new anomalies are introduced.

There are some results related to this problem in the literature. Witten (1984) showed that for O(32) string theory the topological condition connecting curvature and Yang-Mills two-forms

$$\int \mathrm{tr}\, R \wedge R = \frac{1}{30} \int \mathrm{Tr}\, F \wedge F \qquad \text{(V.1)}$$

(tr and Tr refer to fundamental and adjoint representations) leads to the absence of anomalies in D=4 dimensions. Green, Schwarz & West (1985) discussed compactification of the superstring theory to six space-time dimensions with K3 taken to be an internal manifold. They showed (using a similar to Witten`s argument) that again, the absence of anomalies in D=4 dimensions is guaranteed.

I would like to address this problem in a different way by considering a particular example which gives some insight to the problem.

I am going to discuss the case of D=6 dimensional E-Y-M

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* Based on the paper: Sobczyk (1986)

theory with SU(3) gauge group (generalization to SU(N) gauge group seems to be straightforward). I assume that spontaneous compactification to $M_4 x S_2$ is induced by a magnetic monopole background configuration on S_2 of one of SU(3) gauge fields (for details see Section II.H).

I assume some chiral fermions exist in D=6 dimensions. They transform in a definite way under SU(3) transformation. After compactification in D=4 dimensions we will see SU(2)\`x SU(2)xU(1) gauge group, SU(2)\` being the isometry of a two-sphere (viewed as a homogeneous space SU(2)/U(1)) and SU(2)x U(1) being a remnant of the initial gauge group SU(3) (unbroken part of this).

Before analysing the structure of all possible anomalies in the theory, let us recall that we are not worried by the fact that enlarging the "minimal" gauge group from U(1) to SU(3) may, and in fact will, make initially stable theory unstable (Schellekens (1984)).We are only interested in anomaly cancellations and up to our knowledge problems of anomalies and instabilities are completely separated.

In D=6 dimensions there are three types of chiral anomalies appearing in diagrams with four external lines:

(a) purely gauge anomaly with four external SU(3) vectors

(b) mixed gauge and gravitational anomaly with two gravitons and two SU(3) vector particlas

(c) purely gravitational anomaly with four external gravitons.

We need not to bother about a purely gravitational anomaly. This is proportional to the difference between the total number of left and right handed Weyl spinors in D=6 dimensions. This can be always made zero by adding an appropriate number of left or right handed SU(3) singlets which after compactification will not produce any massless chiral fermions in D=4 dimensions (Frampton & Yamamoto (1984)).

In D=4 dimensions we look for anomalies due to massless chiral fermions produced while spontaneous compactification is performed.

In D=4 dimensions there are four types of anomalous diagrams with three external lines:

(A) with two gravitons and one U(1) vector

(B) with three U(1) vectors

(C) with two SU(2) vectors and one U(1) vector

(D) with two SU(2)` vectors and one U(1) vector

My argument is the following:

Every representation of SU(3) is described by two numbers (m,n). Therefore every multiplet of fermion fields of definite (say positive) chirality transforming according to (m,n) representation of SU(3) gives some contribution to anomalies (a) and (b) which can be algebraically expressed as polynomials in m and n - say $f_a(m,n)$ and $f_b(m,n)$ (we do not bother about overall normalization here). On the other hand, by a definition of the embedding of the SU(2)xU(1) subgroup into SU(3) as $\underline{3} = \underline{2}_1 + \underline{1}_{-2}$ the representation (m,n) branches into a series of SU(2)xU(1) representations corresponding to some massless chiral fermions in D=4 dimensions and therefore giving some contributions to (A) --- (D) defined above. We will show that these contributions are <u>linear combinations</u> of f_a and f_b. This is exactly what we need because if we start in D=6 dimensions from the anomaly free representation (theory) i.e.

$$\sum_{j} f_a(m_j, n_j) = 0 \qquad\qquad \sum_{j} f_b(m_j, n_j) = 0 \tag{V.2}$$

then it <u>automatically</u> follows that there are no anomalies in D=4 dimensions. This argument can be also inverted.

I will give now some technical details. The problem of SU(3) branching into SU(2)xU(1) can be systematically studied using Gelfand & Zetlin patterns (see Barut & Rączka (1977)). This technique is explained in Chapter III. A SU(3) representation (m,n,0) after branching SU(3) $\longrightarrow$ SU(2)xU(1) will give rise to SU(2) representations described by (k,l) where $m \geq k \geq n \geq l \geq 0$ with a U(1) charge equal to (see III.26)

$$Q = 3(k+l) - 2(m+n) \tag{V.3}$$

In D=6 dimensions the contributions to (a) and (b) are proportional to $Str(t_\alpha t_\beta t_\gamma t_\delta)$ and $tr(t_\alpha t_\beta)$ where t_α's are generators of SU(3) in the (m,n,0) representations and Str denotes a symmetrized trace.

$$tr(t_\alpha t_\beta) = \delta_{\alpha\beta} f_b \tag{V.4}$$

$$Str(t_\alpha t_\beta t_\gamma t_\delta) = \frac{1}{3}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) f_a \tag{V.5}$$

where

$$f_b = \frac{1}{24}(m^2 + n^2 - mn + 3m)(m^2n - mn^2 + 2mn + 3m - 2n^2 + 2) \tag{V.6}$$

$$f_a = \frac{1}{480}(m^2n - mn^2 + m^2 + 2mn + 3m - 2n^2 + 2) \cdot$$
$$\cdot (m^2 + n^2 - mn + 3m)(m^2 + n^2 - mn - \frac{3}{2} + 3m) \tag{V.7}$$

This was calculated using quadratic Casimir operator eigenvalue in the (m,n,0) representation:

$$\mathcal{C}_2 = \frac{1}{3}(m^2 + n^2 - mn + 3m) \tag{V.8}$$

In order to analyze D=4 dimensional anomalies we must first discuss rules according to which massless fermions appear in D=4 dimensions (Randjbar-Daemi, Salam & Starthdee (1983)). Assume that magnetic monopole number of U(1) field backgroud configuration is $\xi > 0$ (integer). If there is a positive chirality Weyl spinor in D=6 dimensions having a U(1) charge Q (Q must be an integer in our normalization), then in D=4 dimensions there will appear a massless chiral multiplet with a U(1) charge Q, chirality sgn(Q) transforming according to |Q| dimensional representation of SU(2). Starting from a (m,n,0) representation of SU(3) several massless chiral multiplets will appear in D=4 dimensions with the following transformation properties with respect to the gauge group SU(2)`xSU(2)x U(1)·

$$(\xi|Q|\,,\ k-l+1\,,\ Q\,) \tag{V.9}$$

where Q is given by (V.3). They are labelled by (k,l) where k and l take values

$$m \geqslant k \geqslant n \geqslant l \tag{V.10}$$

These multiplets (with all the possible values of k and l) give the following contributions to four types of anomalies in D=4 dimensions:

$$\text{(A)}\quad Q\,(k-l+1)\,\xi\,|Q|\,\mathrm{sgn}\,Q = \xi\,Q^2\,(k-l+1) \tag{V.11}$$

$$\text{(B)}\quad Q^3\,(k-l+1)\,\xi\,|Q|\,\mathrm{sgn}\,Q = \xi\,Q^4\,(k-l+1) \tag{V.12}$$

$$\text{(C)}\quad Q\,\mathrm{tr}^{(k-l+1)}(S_\alpha S_\beta)\,\xi\,|Q|\,\mathrm{sgn}\,Q =$$

$$= \delta_{\alpha\beta}\,\xi\,Q^2\,\frac{1}{12}\,(k-l)(k-l+1)(k-l+2) \tag{V.13}$$

$$\text{(D)}\quad Q\,(k-l+1)\,\mathrm{tr}^{(\xi|Q|)}(S_\alpha S_\beta)\,\mathrm{sgn}\,Q =$$

$$= \delta_{\alpha\beta}\,\xi\,Q^2\,(k-l+1)\,\frac{1}{12}\,(\xi^2 Q^2 - 1) \tag{V.14}$$

where

$$\mathrm{tr}^{(m)}(S_\alpha S_\beta) = \frac{1}{12}\,\delta_{\alpha\beta}\,m\,(m^2-1) \tag{V.15}$$

s_α, s_β are generators of SU(2) Lie algebra in a m-dimensional representation.

We calculate now:

$$\rho_A = \sum_{k=n}^{m} \sum_{l=0}^{n} \xi (k-l+1) \left[3(k+l) - 2(m+n) \right]^2 = 12 \xi f_b \tag{V.16}$$

$$\rho_B = \sum_{k=n}^{m} \sum_{l=0}^{n} \xi (k-l+1) \left[3(k+l) - 2(m+n) \right]^4 = 144 \xi f_a \tag{V.17}$$

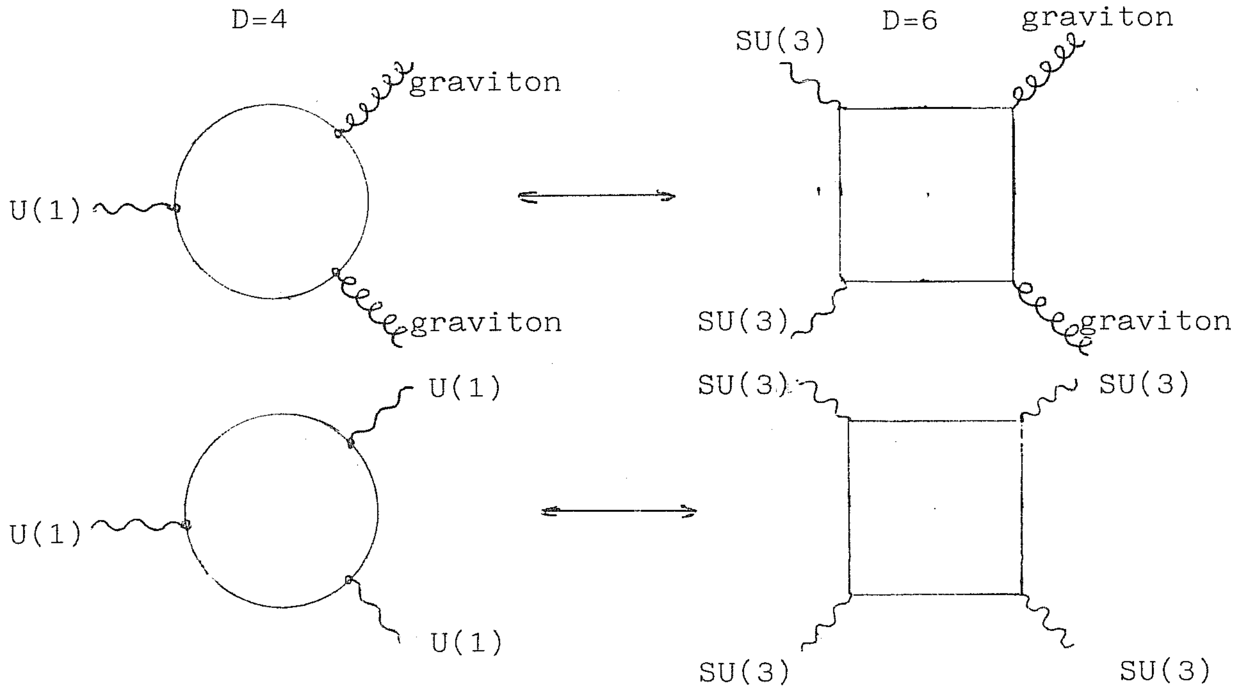
$$\rho_C = \sum_{k=n}^{m} \sum_{l=0}^{n} \frac{\xi}{12} \left[3(k+l) - 2(m+n) \right]^2 (k-l)(k-l+1)(k-l+2) =$$
$$= 4 \xi f_a \tag{V.18}$$

$$\rho_D = \sum_{k=n}^{m} \sum_{l=0}^{n} \left\{ \frac{\xi^3}{12} \left[3(k+l) - 2(m+n) \right]^4 (k-l+1) - \right.$$
$$\left. - \frac{\xi}{12} (k-l+1) \left[3(k+l) - 2(m+n) \right]^2 \right\} = 12 \xi^3 f_a - \xi f_b \tag{V.19}$$

where f_a and f_b are defined in (V.6-7).

If we start from a negative chirality Weyl spinor in D=6 dimensions the contributions just <u>change sign.</u> We see that if we have an anomaly free theory in D=6 dimensions (the sipmlest choice for fermions is: $\underline{8}_L \oplus \underline{3}_R \oplus \underline{6}_R \oplus \underline{1}_L$) we obtain also an anomaly free theory in D=4 dimensions.

Let us look also which anomalies in D=6 dimensions and D=4 dimensions are related to each other (see V.16-19).



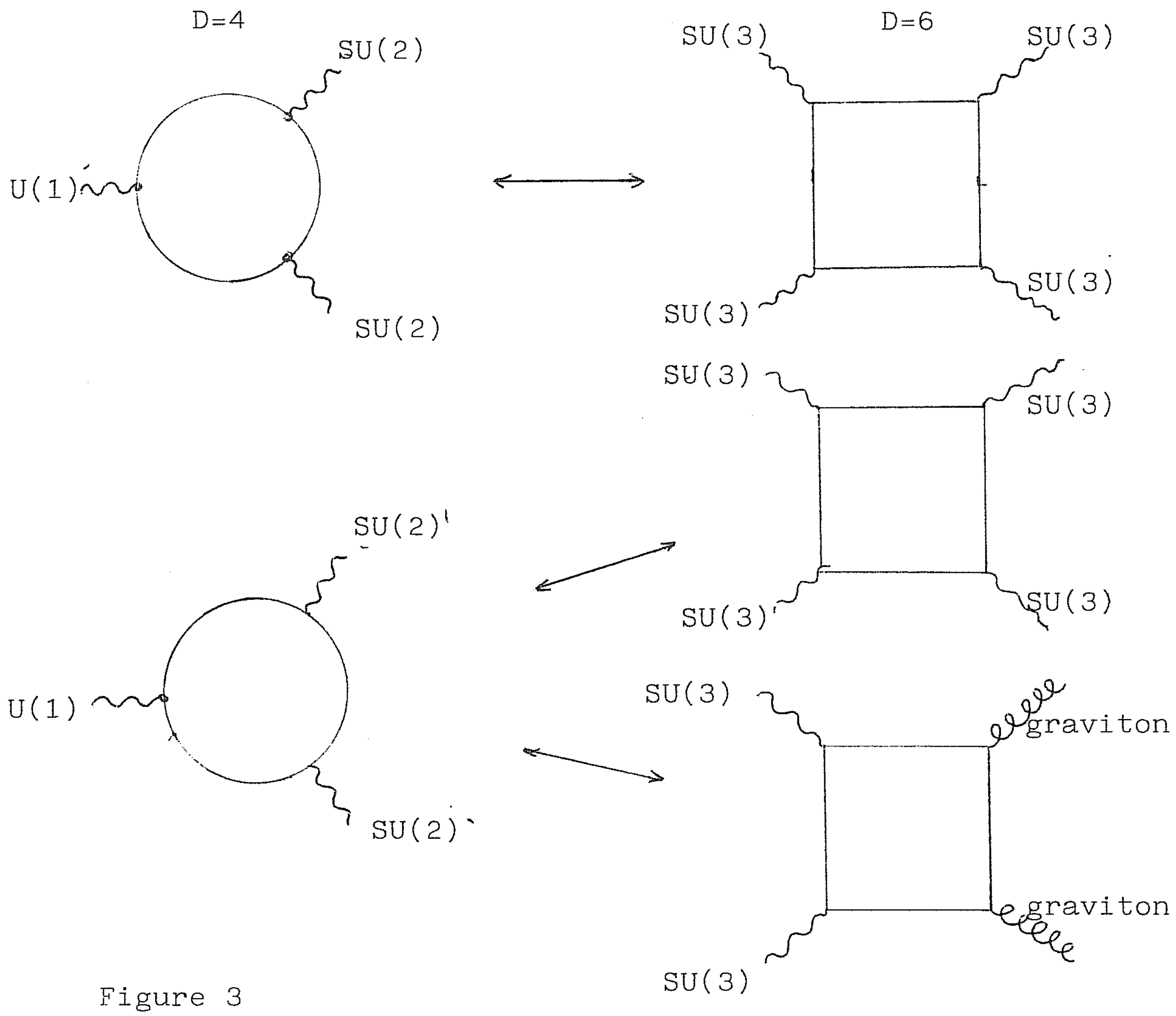


Figure 3

Only last relation seems to be nontrivial. This can be hovewer explained by a fact that physical SU(2)`gauge vectors in this model are linear combinations of nondiagonal excitations in the metric field and vector excitation of the Maxwell field (see Section II.H).

It is certainly a nonobvious result that the theory remains to be anomaly free after neglecting all the massive modes produced while performing dimensional reduction.

VI. SYMMETRY BREAKING IN KALUZA-KLEIN THEORIES*

KKT provide us means of understanding the origin of D=4 dimensional gauge symmetries as isometries of an internal manifold of a very small and therefore invisible size (see Chapter II). But what we really need to obtain in D=4 dimensions is not only a gauge group but also a mechanism by which it can be broken. In the context of Kaluza-Klein theories there is an interesting possibility to be investigated, namely that the gauge group is broken due to small deformation of the internal manifold.

I will describe such a mechanism to break the gauge symmetry and will explain why it is necessary to introduce some new fields (scalars) in the theory. This mechanism might be seen as an alternative to the usual Higgs mechanism. I will then, in the context of a specific model, give a general solution of field equtions with a deformed N-sphere as the internal space with the isometry O(N) rather than the round N-sphere with O(N+1) isometry. The solution will be expressed in terms of a small parameter ε such that in the limit $\varepsilon \to 0$ we recover the undeformed N-sphere solution with O(N+1) isometry. I will then specialize to the case of six dimensions (N=2) and discuss the resulting theory in detail. Examining the spectrum I will find (in perturbative theory in ε) that two of the three O(3) vector bosons acquire a real non-zero mass. This corresponds to breaking the symmetry of the solution of field equations from O(3) to O(2). I will also discuss the stability of the scalar sector and show how all the tachyonic modes can be eliminated. In the particular six-dimensional model that I work with there is an extra U(1) gauge symmetry present in D=4 dimensions in the limit $\varepsilon \to 0$ that does not arise from the isometry of the internal manifold. I will show that this U(1)

*Based on the paper: Sobczyk (1985)

symmetry is broken for the infinitesimally deformed N-sphere. This breaking is due to the nonzero vacuum expectation value for the complex scalar field carrying some U(1) charge which is a necessary part of the model. Thus the overall symmetry breaking pattern is $\underline{O(3)\times U(1) \longrightarrow U(1)}$, the unbroken U(1) being a subroup of O(3).

Because I have achieved a mechanism for symmetry breaking by introduction of a complex scalar field with a tachyonic mass (it is required by field equations), as in the usual Higgs mechanism, there is also another possible ground state in the theory. It corresponds to the undeformed two-sphere as the internal space with the scalar field ground state expectation value minimizing $-M^2\phi^*\phi + \xi(\phi^*\phi)^2$ terms in the action (i.e. $\langle\phi\rangle = \frac{M}{\sqrt{2\xi}}$). It turns out that this background breaks completely the local O(3)xU(1) symmetry.

.A Specification of the model and background solution

Phenomenologically interesting Kaluza-Klein models generally contain gravitation and elementary gauge fields (see Section II.E). In our case we will work with a KKT consisting of gravity, elementary gauge fields and some scalar fields coupled to gauge fields and minimally coupled to gravity. Without scalars in the theory, the internal space would have necessarily a constant scalar curvature. This result follows from the argument due to Randjbar-Daemi & Wetterich (1984).

Consider a D=4+N dimensional theory of gravity and Yang-Mills fields with nonzero D=4+N dimensional cosmological constant:

$$S = -\int d^{4+N}z\,\sqrt{-g}\,\left(\frac{R}{\kappa^2} + \frac{F^2}{4g^2} + \lambda\right) \qquad (\text{VI}.1)$$

The topology of the background configuration is assumed to be $M_4 \times B_N$.

Einstein equations

$$R_{MK} - \frac{R}{2} g_{MK} = \frac{\varkappa^2}{2}\left(-\frac{1}{g^2} F_{ML} F_{K}{}^{L} + g_{MK}\left(\lambda + \frac{F^2}{4g^2}\right)\right) \tag{VI.2}$$

can be rewritten as two equations *

$$\frac{R}{\varkappa^2} + \frac{F^2}{4g^2} + \lambda = 0 \tag{VI.3}$$

$$\frac{R_{mn}}{\varkappa^2} + \frac{1}{2g^2} F_{mk} F_{n}{}^{k} = 0 \tag{VI.4}$$

From these it follows that

$$R = -2\lambda\varkappa^2 \tag{VI.5}$$

Let N=2. Every two-dimensional manifold admits a metric with constant scalar curvature, but if $R = \text{const} < 0$ (as in our case), S_2 and RP_2 (real projective space; a nonorientable manifold) are only possibilities. Since we always have in mind introduction chiral fermions to the theory, we want internal space to be orientable. Thus for N=2 there is only one possibility: B_2 must be a round two-sphere with O(3) isometry.

For $N \geqslant 3$ the situation is more complcated. It is well known that for example S_3 can be deformed in a homogeneous way so that it remains a space of a constant scalar curvature (see e.g. Coquearaux (1984)). It appears hovewer impossible to satisfy additionally Yang-Mills equations as well as equetions (VI.3-4).

Hence we decide to introduce scalars to the theory. Later on we will see that equations of motion will not be satisfied unless the scalar fields have a tachyonic mass.

Our model is described by the action

$$S = -\int d^{4+N}z \sqrt{-g}\left(\frac{R}{\varkappa^2} + \lambda + \frac{F^2}{4g^2} + (\nabla\phi)^+\nabla\phi - M^2\phi^+\phi + \xi(\phi^+\phi)^2\right) \tag{VI.6}$$

where F^{j}_{MN} is a gauge field strength; ϕ belongs to a certain

* In Sections VI.B and VI.E latin indices refer to internal space

representation of the gauge group generated by t^j (antihermitian).

$$\nabla_M \phi = \partial_M \phi + A^j_M t^j \phi \tag{VI.7}$$

We would like to find a solution of field equations with background topology $M_4 x S_N$ with O(N) invariance in terms of an infinitesimal parameter ε . In the limit $\varepsilon \longrightarrow 0$ we want to recover the O(N+1) invariant solution. The field equations for the action (VI.6) are

$$\Delta \phi + M^2 \phi - 2\xi (\phi^+ \phi)\phi = 0 \tag{VI.8a}$$

$$\frac{1}{g^2} \nabla_m F^{j\,m}_{n} + (\nabla_n \phi)^+ t^j \phi - \phi^+ t^j \nabla_n \phi = 0 \tag{VI.8b}$$

$$\frac{R_{mn}}{\varkappa^2} + \frac{1}{2g^2} F^j_{mk} F^{jk}_{n} + (\nabla_m \phi)^+ \nabla_n \phi = 0 \tag{VI.8c}$$

$$\frac{F^2}{4g^2} = \lambda - M^2 \phi^+ \phi + \xi (\phi^+ \phi)^2 \tag{VI.8d}$$

In the absence of scalars a O(N+1) symmetric solution is given by

$$\langle g_{kl} \rangle = \frac{\delta_{kl}}{\gamma^2} - \frac{y_k y_l}{\gamma^4} \tag{VI.9}$$

$$\langle A^{(ab)}_n \rangle = \frac{1}{a} \frac{\delta_{bn} y_a - \delta_{an} y_b}{\gamma(\gamma+1)} \tag{VI.10}$$

$$\frac{1}{\varkappa^2} = \frac{1}{2a^2 g^2} \quad ; \quad \frac{(N-1)^2 + (N-1)}{4g^2 a^4} = \lambda \tag{VI.11}$$

where we have used projective coordinates on S_N : y_k, k=1,..,N; $\gamma = (1+y^2)$; t^{ab} is a O(N) generator defined as N x N matrix with +1 in the a^{th} row and b^{th} column, and -1 in the b^{th} row and a^{th} column; a is a length scale of S_N.

Equation $\frac{1}{\varkappa^2} = \frac{1}{2g^2 a^2}$ tells us essentially that if $g \sim 1$, then a is of order of Planck`s length (see Section II.B)

In the O(N+1) symmetric case the number of massless vector fields in D=4 dimensions is $\frac{N(N+1)}{2}$, while in the O(N) symmetric case - only $\frac{N(N-1)}{2}$. If the symmetry is broken from O(N+1) to O(N), N massless vector fields become massive. We expect (by counting degrees of freedom) that in order to achieve the symmetry breaking in this way a multiplet of N scalars must be introduced. In fact, the following solution (up to O(ε^3)) is O(N) symmetric. Technical details will be given in the Section VI.E.

$$\langle g_{mk} \rangle = \frac{\delta_{mk}}{\gamma^2} + y_k y_l \left(-\frac{1}{\gamma^4} + \frac{\varepsilon^2}{\gamma^6} \right) \tag{VI.12}$$

$$\langle \phi_k \rangle = \varepsilon y_k \frac{(N-1)^{1/2}}{\varkappa \gamma} \tag{VI.13}$$

$$\langle A_m^{ab} \rangle = \frac{1}{a} \frac{\delta_{bm} y_a - \delta_{am} y_b}{\gamma(\gamma+1)} \tag{VI.14}$$

$$M^2 = \frac{1}{a^2} \quad ; \quad \frac{1}{\varkappa^2} = \frac{1}{2g^2a^2} \quad ; \quad \frac{(N-1)^2 + (N-1)}{4g^2a^4} = \lambda \tag{VI.15}$$

(ϕ is assumed to transform as a vector representation of O(N)).

We note that another solution of field equations (VI.8) can be found:

$$\langle \phi \rangle = \frac{M}{\sqrt{2\xi}} \tag{VI.16}$$

$$\langle g_{kl} \rangle = \frac{\delta_{kl}}{\gamma^2} = \frac{y_k y_l}{\gamma^4} \tag{VI.17}$$

$$\langle A_m^{ab} \rangle = \frac{1}{a} \frac{\delta_{bm} y_a - \delta_{am} y_b}{\gamma(\gamma+1)} \tag{VI.18}$$

$$\frac{1}{\varkappa^2} = \frac{1}{2g^2a^2} \quad ; \quad \lambda - \frac{M^2}{4\xi} = \frac{(N-1)^2 + (N-1)}{4g^2a^4} \tag{VI.19}$$

This solution and the earlier one correspond to different "vacua". We can see that in the limit $\varepsilon \longrightarrow 0$ they are not the same. For the first solution M^2 is not a free parameter. Relations between $\varkappa^2, g^2, a^2$ and λ given by (VI.15) and (VI.19) for two solutions are also different. The solution described by Eqs (VI.16-19) (even if corresponds to the round internal N-sphere) breaks completely the local O(N+1) symmetry. This is because ϕ is an O(N) gauge group multiplet. Hence due to embedding of the gauge group into the group of tangent space rotations (also O(N)), the background is <u>not invariant</u> under adjoint action of an O(N) transformation followed by a tangent space rotation.

We will see that by arranging the value of the parameter ξ (at least in the six dimensional model) the background configuration described by Eqs (VI.12-15) can be made perturbatively stable. Even if it corresponds only to a local minimum of energy, so that it can decay through quantum tunelling effect to the stable configuration (VI.16-19), it may still play important role when used in cosmological models.

B. Six-dimensional case.

It is reasonable to expect that for the deformed solution (VI.12-15) some of the massless vector particles would later become massive with small mass of order ε^2. We will check this hypothesis in the six-dimensional case (see Section II.H).

The action is

$$S = -\int d^6 z \sqrt{-g} \left(\frac{R}{\varkappa^2} + \frac{F^2}{4} + \lambda + (\nabla\phi)^* \nabla\phi - \right.$$
$$\left. - M^2 \phi^* \phi + \xi (\phi^* \phi)^2 \right) \qquad \text{(VI.20)}$$

ϕ is a complex scalar field (or O(2) real doublet) and

$$\nabla_M = \partial_M \phi + ie A_M \phi \qquad \text{(VI.21)}$$

The solution of field equations with only O(2) symmetry is (in spherical coordinates)

$$<g_{\theta\theta}> = 1 + \varepsilon^2 \sin^{|n|}\theta \qquad \text{(VI.22a)}$$

$$<g_{\varphi\varphi}> = \sin^2\theta \qquad \text{(VI.22b)}$$

$$<A> = \frac{n}{2ae}(1-\cos\theta)d\varphi \qquad \text{(VI.22c)}$$

$$<\phi> = \frac{\varepsilon}{a}\left(\frac{|n|}{4e^2}\right)^{1/2} \exp\left(-\frac{i}{2}n\varphi\right) \sin^{|n|/2}\theta \qquad \text{(VI.22d)}$$

$$M^2 = \frac{|n|}{2}\frac{1}{a^2} \qquad \text{(VI.22e)}$$

In the limit $\varepsilon \longrightarrow 0$ and in the absence of scalar field we recognize the solution given in (II.106-108). n is a magnetic monopole number . In order that $<\phi>$ be a single-valued function of φ n must be an <u>even integer</u> and not any integer as before. We introduce local frames

$$<e_{\varphi+}> = \frac{e^{-i\varphi}}{\sqrt{2}}(-\sin\theta)$$

$$<e_{\theta+}> = \frac{e^{-i\varphi}}{\sqrt{2}}\left(-i - \frac{i}{2}\varepsilon^2\sin^{|n|}\theta\right) \qquad \text{(VI.23)}$$

$$<e_{\varphi-}> = <e_{\varphi+}>^*$$

$$<e_{\theta-}> = <e_{\theta+}>^* \qquad \text{(VI.24)}$$

As usual we domand the background configuration to be symmetric under the action of some group up to a local rotation in the tangent space. The symmetry group of the background is O(2) generated by a Killing vector with components $K^\theta = 0$ and $K^\varphi = 1$. It is a symmetry only if $e_{\mu+}$ $e_{\mu-}$ (where $\mu = \varphi, \theta$) and ϕ have definite transformation properties with respect to the tangent space rotation group O(2), viz. $e_{\mu+} \sim +1$, $e_{\mu-} \sim -1$, $\phi \sim n/2$. These numbers become important in the harmonoc expansion on S_2 viewed as the homogeneous space O(3)/O(2).

In order to analyze the spectrum of the theory we expand all the fields around their background values

$$
\begin{aligned}
g_{MN} &= \langle g_{MN} \rangle + \varkappa h_{MN} \\
A_M &= \langle A_M \rangle + V_M \qquad\qquad (VI.25)\\
\phi &= \langle \phi \rangle + \varphi
\end{aligned}
$$

It is convenient to work in orthonormal frame coordinates and to choose a light-cone gauge for h_{AB} and V_A fields (see Section II.H). More technical details will be given in Section VI.F. Here we only write a part of action bilinear in fluctuations with fields of spin/helicity 1 and 2 as well as terms bilinear in the scalar field fluctuations

$$
\begin{aligned}
S_{hel\,1,2} = \int d^4x \int d^2y \sqrt{-\langle g \rangle}\, \Big[& \tfrac{1}{4} h^T_{jk} (\partial^2 + \nabla^2) h^T_{jk} \\
& + \tfrac{1}{2} h_{j\alpha} (\partial^2 + \nabla^2) h_{j\alpha} + \tfrac{1}{2} V_j (\partial^2 + \nabla^2) V_j \\
& + \tfrac{1}{2} h_{j\alpha} h_{j\beta} \langle R_{\alpha\beta} \rangle - e^2 \langle \phi^* \phi \rangle V_j V_j + \\
& + \varkappa V_j \langle F_{\alpha\beta} \rangle \nabla_\beta h_{j\alpha} \Big] \qquad (VI.26)
\end{aligned}
$$

$$
\begin{aligned}
S_{\varphi\varphi} = \int d^4x \int d^2y \sqrt{-\langle g \rangle}\, \Big[& \varphi^* (\partial^2 + \nabla^2 - M^2) \varphi \\
& + \tfrac{e^2}{2} (\langle \phi^* \rangle \varphi - \varphi^* \langle \phi \rangle)^2 - \tfrac{\varkappa^2}{2} ((\nabla_\alpha \langle \phi \rangle)^* \varphi + \varphi^* \nabla_\alpha \langle \phi \rangle)^2 \\
& - \xi \,(4 \langle \phi \rangle^* \langle \phi \rangle \varphi^* \varphi + \langle \phi \rangle^{*2} \varphi^2 + \varphi^{*2} \langle \phi \rangle^2 \Big] \qquad (VI.27)
\end{aligned}
$$

Before expanding h, V, φ in series of harmonics on S_2 they should be written in terms of irreducible O(2) representations i.e. in +- basis defined in (VI.23-24).

The background solution gives

$$
\langle R_{+-} \rangle = \frac{1}{a^2} \left(-1 + \varepsilon^2 \sin^{|n|}\theta - \frac{|n|}{2} \varepsilon^2 \sin^{|n|}\theta \, \frac{\cos^2\theta}{\sin^2\theta} \right)
$$

$$\varkappa\langle F_{+-}\rangle = \frac{i\sqrt{2}}{a}\left(1 - \frac{\varepsilon^2}{2}\sin^{|n|}\Theta\right)$$

$$\langle\phi^*\phi\rangle = \frac{1}{a^2}\,\frac{\varepsilon^2|n|}{4e^2}\sin^{|n|}\Theta$$

$$(\nabla_+\langle\phi\rangle)^*\nabla_-\langle\phi\rangle = 0$$

$$(\nabla_+\langle\phi\rangle)^*\nabla_+\langle\phi\rangle = \frac{1}{8}(n-|n|)^2\,\frac{\cos^2\Theta}{\sin^2\Theta}\,\frac{\langle\phi^*\phi\rangle}{a^2} \tag{VI.28}$$

$$(\nabla_-\langle\phi\rangle)^*\nabla_-\langle\phi\rangle = \frac{1}{8}(n+|n|)^2\,\frac{\cos^2\Theta}{\sin^2\Theta}\,\frac{\langle\phi^*\phi\rangle}{a^2}$$

$$\langle\phi^*\phi^*\rangle = \frac{\varepsilon^2|n|}{4a^2e^2}\sin^{|n|}\Theta\, e^{in\varphi}$$

$$\langle\phi\,\phi\rangle = \frac{\varepsilon^2|n|}{4a^2e^2}\sin^{|n|}\Theta\, e^{-in\varphi}$$

Harmonic expansions are (see Section II.H)

$$h^T_{jk}(x,y) = \sum_{\substack{l\geq 0\\ m}} \mathbb{D}^{(l)}_{0m}(y)\, h^{(l)}_{jk,m}(x) \tag{VI.29a}$$

$$h_{+j}(x,y) = \sum_{\substack{l\geq 1\\ m}} \mathbb{D}^{(l)}_{1m}(y)\, h^{(l)}_{+jm}(x) \tag{VI.29b}$$

$$V_j(x,y) = \sum_{\substack{l\geq 0\\ m}} \mathbb{D}^{(l)}_{0m}(y)\, V^{(l)}_{jm}(x) \tag{VI.29c}$$

$$\varphi(x,y) = \sum_{\substack{l\geq |n|/2\\ m}} \mathbb{D}^{(l)}_{n/2\,m}(y)\, V \tag{VI.29d}$$

where y stands for (φ,Θ) and $|m|\leq 1$. $\mathbb{D}^{(l)}_{m_1m_2}$ are spherical harmonics on S_2 normalized as follows

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\Theta \sin\Theta\; \mathbb{D}^{*(l)}_{m_1m_2}(\varphi,\Theta)\,\mathbb{D}^{(l')}_{n_1n_2}(\varphi,\Theta) = \tag{VI.30}$$

$$= \delta_{ll'}\,\delta_{m_2+n_2,0}\,\delta_{m_1+n_1,0} \tag{VI.30}$$

C Calculation of the spectrum

A full analysis of the theory can be done systematically as explained in Section II.H. In the limit $\varepsilon \to 0$ we get the following massless states:

(a) for $l=0$ a helicity 2 state - graviton (h^{o}_{jk})
(b) for $l=0$ a helicity 1 state - a U(1) vector (V^{o}_{j})
(c) for $l=1$ helicity 1 states - O(3) triplet
(d) for $l=|n|/2$ helicity 0 states - multiplet of $|n|+1$ massless complex scalars

For the background with only O(2) symmetry ($\varepsilon \neq 0$) all the mass matrices become complicated and computation of their eigenvalues more involved. We will display some details of computation in Section VI.G. Here we wish to make few remarks on the most important points of calculation.

1/ We are interested in corrections to masses only up to $O(\varepsilon^3)$

2/ States with different l`s no longer decouple. Hovewer terms which mix among themselves are of order ε^2, so that in the determinant of the mass matrix they give rise to corrections of the order $O(\varepsilon^4)$ - which can be ignored.

3/ It is a priori not very clear how to define harmonic expansion on the internal manifold which we can think of a deformed two-sphere or as a two-sphere with infitesimally deformed metric. Should the harmonics $\mathcal{D}^{(\omega)}_{m_1 m_2}$ be the same as in a nondeformed case? Or should they be redefined as eigenfunctions of the deformed Laplace operator or by a requirement that they must satisfy orthonormality condition (the integration measure is changed since $\det \langle g \rangle = \sin\Theta \left(1 + \frac{\varepsilon^2}{2} \sin^{|n|}\Theta\right)$). Fortunately, up to $O(\varepsilon^3)$ all three definitions seems to lead to the same answer. In the actual computation we have used the first possibility.

4/ In the scalar sector we are interested in the mass matrix for states with $l=|n|/2$. Full analysis is not simple since there are seven fields involved (h_{++}, h_{--}, V_+, V_-, h_{jj}, φ , φ^*) so the mass matrix is a 7 x 7 matrix.

We are interested hovewer only in the lowest order corrections to massless (in the limit $\varepsilon \longrightarrow 0$) states which we identified as $\varphi_m^{|n|/2}$, $\varphi_m^{|n|/2 *}$. The lowest order corrections are given by the matrix elements

$$< \varphi_m^{|n|/2 *} \, | M^2 | \, \varphi_m^{|n|/2} >$$

so that it is sufficient to calculate corrections to M^2 only in terms bilinear in φ^*, φ.

Results of computation are following:

1/ In the helicity/spin 2 sector graviton remains massless.

2/ In the helicity/spin 1 sector only one state remains massless. This is a linear combination

$$\frac{1}{\sqrt{2}} \left(V_{j0}^{(1)} + \frac{i}{\sqrt{2}} \left(h_{+j0}^{(1)*} - h_{+j0}^{(1)} \right) \right) \tag{VI.31}$$

Two other initially (as $\varepsilon \longrightarrow 0$) massless states, forming with the one mentioned before a O(3) multiplet, acquire masses

$$M^2 = \frac{1}{a^2} \frac{\varepsilon^2}{8} \left(3|n| + 9 \right) Z_{|n|+2} \tag{VI.32}$$

$$Z_k = \frac{2^{k/2+1} \, (k/2)!}{(k+1)!!}$$

If we want massess to be of order $\sim 10^2$ GeV, then (since a^{-1} is of order $\sim 10^{-19}$ GeV), $\varepsilon \sim 10^{-17}$. <u>Thus à posteriori perturbation theory is justified.</u>

3/ U(1) vector boson acquires a mass

$$M^2 = \frac{|n|}{4} Z_{|n|} \frac{\varepsilon^2}{a^2} \tag{VI.33}$$

4/ Masses of $\varphi_m^{|n|/2}$ scalars are proportional to the integral

$$M^2 \sim \varepsilon^2 \int_0^{\pi} d\theta \, \sin\theta \, (1-\cos\theta)^{2|m|} \sin^{|n|-2|m|}\theta \cdot \sin^{|n|}\theta \, \times$$

$$\times \left(\frac{m^2}{\sin^2\theta} + |n||m| \frac{\cos\theta}{\sin^2\theta} + \frac{|n|}{2} \left(\frac{|n|}{2} + 2 \right) \frac{\cos^2\theta}{\sin^2\theta} + |n| \left(\eta - \frac{1}{4} \right) \right) \tag{VI.34}$$

where $\eta = \xi / e^2$ is a dimensionless parameter. We observe that

$$m^2 + |n||m|\cos\theta + \frac{|n|}{2}\left(\frac{|n|}{2}+2\right)\cos^2\theta \geq 0 \qquad \text{(VI.35)}$$

for $|m| \leqslant |n|/2$.

The only possible source of imaginary mass for $\varphi_m^{|n|/2}$ modes is the last term $|n|(\eta - 1/4)$. We found that in fact for $\eta = 0$ there are tachyonic modes unless $|n| = 2$. On the other hand if $\eta \geqslant 1/4$ we can be sure that the background configuration (VI.22) is perturbatively stable.

D Concluding remarks

The mechanism desribed in this Chapter can be most probably applied to other internal manifolds, in particular with some minor modifications to CP(n) manifolds. Furthermore, by introducing different scalar multiplets one may obtain other symmetry breaking patterns, the breakdown O(N+1) to O(N) being the simplest example of this mechanism.

The classical stability of our solution has one more interesting aspect. We notice that by putting $\varepsilon = 0$ one obtains an unstable theory in D=4 dimensions because the background value for the scalar field is then $\langle\phi\rangle = \frac{M}{\sqrt{2\xi}}$ rather than $\langle\phi\rangle = 0$. Only when $\varepsilon \neq 0$ the requirement that $\langle\phi\rangle$ be O(2) invariant forces us to associate with ϕ some <u>definite</u> transformation properties withrespect to the tangent space rotation group (viz. a charge of $|n|/2$) from which it follows that in the harmonic series we must start from $l = |n|/2$. Since

$$\left(\nabla^2 + \frac{|n|}{2}\right) D^{|n|/2}_{n/2\, m}(y) = 0 \qquad \text{(VI.36)}$$

for ∇^2 being a "round" Laplace operator on two-sphere. As $\varepsilon \longrightarrow 0$ but $\varepsilon \neq 0$ there is a multiplet of scalar massless states for $l = |n|/2$ and massive ones for $l > |n|/2$.

Finally I would like to mention that the symmetry breaking mechanism obtained in the case $\varepsilon = 0$ (with $\langle\phi\rangle = \frac{M}{\sqrt{2\xi}}$) seems to be similar to results obtained by Shin (1986). In his model gravity is coupled to Maxwell field and a nonlinear σ-field (not carrying U(1) charge).

E Appendix A

A N-sphere in N+1 dimensional Euclidean space is described by equation

$$\sum_{j=1}^{N+1} x_j^2 = 1 \tag{VI.37}$$

We introduce projective coordinates on S_N

$$y_j = \frac{x_j}{x_{N+1}} \tag{VI.38}$$

in the domain in which $x_{N+1} \neq 0$ (we need N coordinate patches to cover a N-sphere in this way).

The induced metric in projective coordinates has a form

$$g_{kl} = \frac{\delta_{kl}}{1+y^2} - \frac{y_k y_l}{(1+y^2)^2} \tag{VI.39}$$

From the isometry group of this metric (which is O(N+1)) we pick an O(N) subgroup. This subgroup is defined in such a way that M_{mn} element of o(N) (Lie algebra) generates transformations

$$\begin{aligned} \delta y_m &= \tau y_n \\ \delta y_n &= -\tau y_m \end{aligned} \tag{VI.40}$$

τ - is an infinitesimal parameter.

The $\frac{N(N-1)}{2}$ generators of O(N), generate transformations of S_N leaving the solution (VI.12-15) invariant. Killing vectors are given by $K^m = y_n$; $K^n = y_m$ $(m < n)$. One verifies that Lie derivatives of (VI.12-14) followed by tangent space rotations (A^{ab} is in the adjoint representation of O(N) and

ϕ in the vector representation) are zero.

We now turn to verification that the solution given in (VI.12-14) is in fact the sulution of field equations (VI.8). This is fairly straithforward but tedious.

We first obtain (up to $O(\varepsilon^3)$)

$$R_{mp} = \frac{g_{mp}}{a^2}\left[(N-1)\left(-\frac{1}{\gamma^2}-\frac{\varepsilon^2}{\gamma^4}\right)+\varepsilon^2\frac{\gamma^2-1}{\gamma^4}\right] + \frac{y_m y_p}{a^2}\left[(N-1)\left(\frac{1}{\gamma^4}+\frac{\varepsilon^2}{\gamma^6}\right)-\frac{\varepsilon^2}{\gamma^4}\right] \tag{VI.41}$$

$$F^{o}_{mn}F^{on}_{p} = \frac{g_{mp}}{a^4}\left[(N-1)\frac{1}{\gamma^2}-\varepsilon^2\frac{\gamma^2-1}{\gamma^4}\right] + \frac{y_m y_p}{a^4}\left[-(N-1)\frac{1}{\gamma^4}+\frac{\varepsilon^2}{\gamma^4}\right] \tag{VI.42}$$

$$(\nabla_m\phi)^+\nabla_p\phi = g_{mp}\frac{\varepsilon^2(N-1)}{\gamma^4\varkappa^2a^2} + y_m y_p\left(-\frac{\varepsilon^2(N-1)}{a^2\varkappa^2\gamma^6}\right) \tag{VI.43}$$

From these relations the following equations can be derived

$$\frac{1}{a^2\varkappa^2} = \frac{1}{2g^2a^4} \tag{VI.44}$$

$$F^2 = \frac{1}{a^4}\left[(N-1)^2+(N-1)\left(1+\frac{2\varepsilon^2}{\gamma^2}-2\varepsilon^2\right)\right] \tag{VI.45}$$

$$\phi^+\phi = \varepsilon^2\,\frac{N-1}{\varkappa^2}\,\frac{\gamma^2-1}{\gamma} \tag{VI.46}$$

$$\lambda = \frac{(N-1)^2+N-1}{4a^4g^2} \tag{VI.47}$$

As far as Eq (VI.8b) is concerned we get

$$\nabla_m F_k^{(a,b)m} = -\frac{1}{a^3}\left(y_b\delta_{ak}-y_a\delta_{bk}\right)(N-1)\,\frac{\varepsilon^2}{\gamma^3} \tag{VI.48}$$

$$(\nabla_k\phi)^+t^{ab}\phi-\phi^+t^{ab}\nabla_k\phi = \varepsilon^2\left(y_b\delta_{ak}-y_a\delta_{bk}\right)\frac{2(N-1)}{a\varkappa^2\gamma^3} \tag{VI.49}$$

Finally one verifies that

$$\nabla^2\phi_k = -\frac{1}{a^2}\phi_k \tag{VI.50}$$

F Appendix B

A part of the action bilinear in fluctuations defined in Eq (VI.25) is

$$
\begin{aligned}
S_{bil} = \int d^6 z \sqrt{-\langle g \rangle}\, \Big[& \tfrac{1}{4} h_{AB} \nabla^2 h_{AB} - \tfrac{1}{4} h_{AA} \nabla^2 h_{BB} - \tfrac{1}{2} h_{AB} \nabla_A \nabla_C h_{BC} \\
& + \tfrac{1}{2} h_{AA} \nabla_B \nabla_C h_{BC} + \langle R_{AB} \rangle \left(\tfrac{1}{2} h_{AB} h_{CC} - \tfrac{1}{2} h_{AC} h_{BC} \right) \\
& + \tfrac{1}{2} \langle R_{AB} \rangle V_A V_B - \tfrac{1}{2} \left(\langle R_{ADCB} \rangle + \tfrac{\varkappa^2}{2} \langle F_{AC} \rangle \langle F_{BD} \rangle \right) h_{AB} h_{CD} \\
& + \langle F_{AB} \rangle \langle F_{AC} \rangle \left(\tfrac{\varkappa^2}{4} h_{BC} h_{DD} - \tfrac{\varkappa^2}{2} h_{BD} h_{CD} \right) + \tfrac{1}{2} V_A \nabla^2 V_A \\
& + \tfrac{1}{2} \nabla_A V_A \nabla_B V_B + (\nabla_A V_B - \nabla_B V_A) \tfrac{\varkappa}{2} \left(\langle F_{CB} \rangle h_{CA} - \langle F_{CA} \rangle h_{CB} \right) \\
& - \tfrac{\varkappa}{4} h_{CC} \langle F_{AB} \rangle (\nabla_A V_B - \nabla_B V_A) + \tfrac{\varkappa^2}{2} h_{AA} h_{BC} \langle \nabla_B \Phi \rangle^* \langle \nabla_C \Phi \rangle \\
& - \tfrac{\varkappa}{2} h_{AA} \left((\nabla \varphi)^* \langle \nabla \Phi \rangle + (\nabla \Phi)^* \nabla \varphi \right\} + ie V_B \left(\langle \nabla_B \Phi \rangle^* \langle \Phi \rangle - \Phi^* \langle \nabla_B \Phi \rangle \right. \\
& - \varkappa^2 h_{AB} h_{AC} \langle \nabla_B \Phi \rangle^* \langle \nabla_C \Phi \rangle + \varphi^* \nabla^2 \varphi + e^2 V_A V_A \langle \Phi^* \Phi \rangle \\
& + ie V_A \left(\langle \Phi \rangle^* \nabla_A \varphi - (\nabla_A \varphi)^* \langle \Phi \rangle \right) \\
& + ie V_A \left(\varphi^* \langle \nabla_A \Phi \rangle - \langle \nabla_A \Phi \rangle^* \varphi \right) \\
& + \varkappa h_{AB} \Big(\langle \nabla_A \Phi \rangle^* \nabla_B \varphi + (\nabla_A \varphi)^* \langle \nabla_B \Phi \rangle + \\
& \qquad + ie V_A \left(\langle \nabla_B \Phi \rangle^* \langle \Phi \rangle - \langle \Phi \rangle^* \langle \nabla_B \Phi \rangle \right) \Big) \\
& + \tfrac{\varkappa}{2} h_{AA} \left(\left(\langle \Phi \rangle^* \varphi + \varphi^* \langle \Phi \rangle \right) \left(M^2 - 2 \xi \langle \Phi^* \Phi \rangle \right) + \varphi^* \varphi M^2 \right. \\
& - 4 \xi \langle \Phi^* \Phi \rangle \varphi^* \varphi - \xi \langle \Phi^* \Phi^* \rangle \varphi \varphi - \xi \langle \Phi \Phi \rangle \varphi^* \varphi^* \Big]
\end{aligned}
\tag{VI.51}
$$

We have used orthonormal frame indices; $A = (a,\alpha)$, a runs 0,1,2,3 α runs 5,6.

In order to go to light-cone gauge we use coordinates $a=(+-j)$ $j=1,2$ and put $V_-=h_{A-}=0$ (see SectionII.H). It turns out that field equations for + components of V_A and h_{AB} are algebraic equations (they do not describe propagation of physical degrees of freedom). We obtain

$$
\begin{aligned}
& h_{\alpha\alpha} + h_{jj} = 0 \\
& \partial_- V_+ = \nabla_\alpha V_\alpha + \partial_k V_k - ie\,(\phi^* \varphi - \varphi^* \phi) \\
& \partial_- h_{+j} = \nabla_\alpha h_{\alpha j} + \partial_k h_{kj} \\
& \partial_- h_{+\alpha} = \partial_j h_{\alpha j} + \nabla_\beta h_{\alpha\beta} - \varkappa F_{\alpha\beta} V_\beta - \\
& \qquad - \varkappa \left((\nabla_\alpha \phi)^* \varphi + \varphi^* \nabla_\alpha \phi \right)
\end{aligned}
\tag{VI.52}
$$

We decompose now all the internal O(2) tensors in terms of irreducible representations of tangent space O(2) group and get

$$
\begin{aligned}
S_{b.l} = \int d^6 z \sqrt{-\langle g \rangle} \Big[& \frac{1}{4} h_{jk}^T (\partial^2 + \nabla^2) h_{jk}^T \\
& + h_{j-} (\partial^2 + \nabla^2 + \langle R_{+-} \rangle) h_{j+} + \frac{1}{2} V_j (\partial^2 + \nabla^2 - 2e^2 \langle \phi^* \phi \rangle) V_j \\
& + \varkappa \langle F_{+-} \rangle V_j (\nabla_+ h_{-j} - \nabla_- h_{+j}) + \\
& + V_- (\partial^2 + \nabla^2 + \langle R_{+-} \rangle + \varkappa^2 \langle F_{+-} \rangle^2 - 2e^2 \langle \phi^* \phi \rangle) V_+ \\
& + \frac{1}{2} h_{--} (\partial^2 + \nabla^2 + 2\langle R_{+-} \rangle - \varkappa^2 \langle F_{+-} \rangle^2 - 2 \langle R_{+--+} \rangle) h_{++}
\end{aligned}
$$

$$+\frac{1}{4}h_{jj}\left(\partial^2+\nabla^2+\langle R_{+-}\rangle+\langle R_{+--+}\rangle+\frac{\varkappa^2}{2}\langle F_{+-}\rangle^2\right)h_{kk}$$

$$-2\varkappa h_{++}V_-\langle\nabla_- F_{-+}\rangle-2\varkappa h_{--}V_+\langle\nabla_+ F_{+-}\rangle$$

$$+\varphi^*(\partial^2+\nabla^2)\varphi\quad+\varkappa\langle F_{+-}\rangle(h_{++}\nabla_- V_- - h_{--}\nabla_+ V_+)$$

$$+\frac{\varkappa}{2}h_{jj}\langle F_{+-}\rangle(\nabla_+ V_- - \nabla_- V_+)$$

$$+\frac{e^2}{2}\left(\langle\phi\rangle^*\varphi-\varphi^*\langle\phi\rangle\right)^2$$

$$-\varkappa h_{++}\left(\langle\nabla_+\nabla_+\phi\rangle^*\varphi+\varphi^*\langle\nabla_-\nabla_-\phi\rangle\right)$$

$$-\varkappa h_{--}\left(\langle\nabla_-\nabla_-\phi\rangle^*\varphi+\varphi^*\langle\nabla_+\nabla_+\phi\rangle\right)$$

$$-\frac{\varkappa M^2}{2}h_{jj}\left(\langle\phi\rangle^*\varphi+\varphi^*\langle\phi\rangle\right)-\xi\left(\langle\phi\rangle^*\right)^2\varphi^2$$

$$-\xi\left(\langle\phi\rangle\right)^2(\varphi^*)^2\quad-4\xi\langle\phi^*\phi\rangle\varphi^*\varphi$$

$$-\varkappa^2\Big(\varphi^*\varphi\left(\langle\nabla_+\phi\rangle^*\langle\nabla_+\phi\rangle+\langle\nabla_-\phi\rangle^*\langle\nabla_-\phi\rangle\right)$$

$$+\varphi^2\,\xi\,\langle\nabla_+\phi\rangle^*\langle\nabla_-\phi\rangle^*$$

$$+(\varphi^*)^2\langle\nabla_+\phi\rangle\langle\nabla_-\phi\rangle\Big)\Big]\qquad\text{(VI.53)}$$

where

$$\nabla^2=\nabla_+\nabla_-+\nabla_-\nabla_+$$

and

$$h^T_{jk}=h_{jk}-\frac{\delta_{jk}}{2}h_{mm}.\qquad\text{(VI.54)}$$

G Appendix C

We need explicit form of some spherical harmonics on S_2

$$D^0_{00} = \sqrt{\frac{1}{4\pi}}$$

$$D^1_{00} = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad D^1_{01} = -\sqrt{\frac{3}{8\pi}}\, e^{i\varphi} \sin\theta$$

$$D^1_{01} = -D^{1\,*}_{0-1} \qquad D^1_{10} = i\sqrt{\frac{3}{8\pi}}\, e^{-i\varphi} \sin\theta$$

$$D^1_{1-1} = i\sqrt{\frac{3}{16\pi}}\,(1+\cos\theta) \qquad D^1_{11} = i\sqrt{\frac{3}{16\pi}}\, e^{-2i\varphi}\,(1-\cos\theta)$$

$$D^1_{-10} = D^{1\,*}_{10} \qquad D^1_{-1-1} = -D^{1\,*}_{11} \tag{VI.55}$$

$$D^1_{-11} = -D^{1\,*}_{1-1}$$

$$D^{|n|/2}_{n/2\,m} = C^{|n|/2}_m\, e^{-i\varphi(n/2+m)}\,(1-\cos\theta)^m \sin^{|n|/2-m}\theta$$

($C^{|n|/2}_m$ is a irrelevant normalization constant)

They are all eigenfunctions of "round" Laplace operator. We see that (using spin connection $\omega_{\varphi+-} = i(\cos\theta - 1)$)

$$\nabla^2 D^1_{0m} = -\frac{2}{a^2} D^1_{0m}$$

$$\nabla^2 D^1_{1m} = -\frac{1}{a^2} D^1_{1m} \tag{VI.56}$$

$$\nabla^2 D^{|n|/2}_{n/2\,m} = -\frac{|n|}{2}\frac{1}{a^2} D^{|n|/2}_{n/2\,m}$$

For the deformed S_2 the metric is

$$\langle g_{\theta\theta}\rangle = 1 + \varepsilon^2 \sin^{|n|}\theta$$

$$\langle g_{\varphi\varphi}\rangle = \sin^2\theta \tag{VI.57}$$

and the spin connection takes a form

$$\omega_{\varphi+-} = i\left(\cos\Theta - 1 - \frac{\varepsilon^2}{2}\cos\Theta\,\sin^{|n|}\Theta\right) \qquad \text{(VI.58)}$$

which of course modifies a ∇^2 operator. We can show that

$$\nabla^2 D^1_{00} = \frac{1}{a^2}\left(-2 + \varepsilon^2\left(2 + \frac{|n|}{2}\right)\sin^{|n|}\Theta\right) D^1_{00}$$

$$\nabla^2 D^1_{01} = \frac{1}{a^2}\left(-2 + \varepsilon^2 \sin^{|n|}\Theta - \varepsilon^2\left(1 + \frac{|n|}{2}\right)\frac{\cos^2\Theta}{\sin^2\Theta}\sin^{|n|}\Theta\right) D^1_{01}$$

$$(\nabla^2 + R_{+-}) D^1_{10} = \frac{1}{a^2}\left(-2 + \varepsilon^2 \sin^{|n|}\Theta\left(2 - |n|\frac{\cos^2\Theta}{\sin^2\Theta}\right)\right) D^1_{10}$$

$$(\nabla^2 + R_{+-}) D^1_{11} = \frac{1}{a^2}\left(-2 + \varepsilon^2 \frac{\sin^{|n|}\Theta}{\sin^2\Theta}(1 - \cos\Theta - 2\cos^2\Theta) - \right.$$

$$\left. - \varepsilon^2 \frac{|n|}{2}\frac{\sin^{|n|}\Theta}{\sin^2\Theta}(\cos\Theta - 2\cos^2\Theta)\right) D^1_{11}$$

$$\nabla^2 D^{|n|/2}_{n/2\, m} = \frac{1}{a^2}\left(-\frac{|n|}{2} + \varepsilon^2 \sin^{|n|}\Theta\left(\frac{|n|}{2} - \frac{n^2}{4} - \right.\right.$$

$$\left.\left. - |n| m \frac{\cos\Theta}{\sin^2\Theta} - \frac{m^2}{\sin^2\Theta}\right)\right) D^{|n|/2}_{n/2\, m} \qquad \text{(VI.59)}$$

As a typical example of the type of calculations involved we will demonstrate in detail how one from three O(3) massless vector bosons <u>remains massless.</u>

In the bilinear part of the action we have

$$\int d^6 z \left(\frac{1}{2} V_j (\partial^2 + \nabla^2 - 2e^2 \langle \phi^* \phi \rangle) V_j + \right.$$

$$+ h_{j-}(\partial^2 + \nabla^2 + R_{+-}) h_{j+} +$$

$$\left. + \varkappa F_{+-} V_j (\nabla_+ h_{-j} - \nabla_- h_{+j})\right) \qquad \text{(VI.60)}$$

From harmonic expansions for V_j and $h_{j\pm}$ we pick out the V^1_{jo}, $h^1_{+j\,o}$, $h^{1*}_{+j\,o}$ elements which mix among themselves only (up to $O(\varepsilon^3)$). After integration over $d\varphi$ we obtain

$$2\pi \int d^4 x \int_0^{\pi} d\Theta \sin\Theta \left(1 + \frac{\varepsilon^2}{2}\sin^{|n|}\Theta\right) \times$$

$$\times \left[\frac{1}{2} V^1_{jo} D^1_{oo}\left(\partial^2 + \frac{1}{a^2}(-2 + 2\varepsilon^2 \sin^{|n|}\Theta)\right) V^1_{jo} D^1_{oo} + \right.$$

$$+ h^{1*}_{j+0} D^{0}_{-10} \left(\partial^2 + \frac{1}{a^2}\left(-2 + \varepsilon^2 \sin^{|n|}\theta \left(2 - |n| \frac{\cos^2\theta}{\sin^2\theta}\right)\right)\right) h^{1}_{+j0} D^{1}_{10} +$$

$$+ \frac{i\sqrt{2}}{a^2}\left(1 - \frac{\varepsilon^2}{2}\sin^{|n|}\theta\right) V^{1}_{j0} D^{1}_{00}\left(1 - \frac{\varepsilon^2}{2}\sin^{|n|}\theta\right) \times$$

$$\times \left(h^{1}_{+j0} - h^{1*}_{+j0}\right) D^{1}_{00} \Big] \qquad \text{(VI.61)}$$

Using the relation

$$\nabla_{-} D^{1}_{10} = -\frac{1}{a}\left(1 - \frac{\varepsilon^2}{2}\sin^{|n|}\theta\right) D^{1}_{00} \qquad \text{(VI.62)}$$

we get

$$S = \int d^4x \Big[\frac{i\sqrt{2}}{a^2} V^{1}_{j0}\left(h^{1}_{+j0} - h^{1*}_{+j0}\right)\left(1 + \frac{3}{4}\varepsilon^2\left(Z_{|n|+2} - Z_{|n|}\right)\right)$$

$$+ \frac{1}{2} V^{1}_{j0}\left(\left(1 + \frac{3}{4}\varepsilon^2\left(Z_{|n|} - Z_{|n|+2}\right)\right)\partial^2 + \frac{1}{a^2}\left(-2 + \frac{3}{2}\varepsilon^2\left(Z_{|n|} - Z_{|n|+2}\right)\right)V^{1}_{j0}$$

$$+ h^{1*}_{+j0}\left(\left(1 + \frac{3}{8}\varepsilon^2 Z_{|n|+2}\right)\partial^2 + \frac{1}{a^2}\left(-2 + \frac{3}{4}\left(1 + |n|\right)\varepsilon^2 Z_{|n|+2} - \right.\right.$$

$$\left.\left. - \frac{3}{4}|n|\,\varepsilon^2 Z_{|n|}\right) h^{1}_{+j0} \right] \qquad \text{(VI.63)}$$

where

$$Z_{|n|} = \int_0^{\pi} d\theta \, \sin\theta \, \sin^{|n|}\theta = \frac{2^{|n|/2+1}\left(\frac{|n|}{2}\right)!}{(|n|+1)!!} \qquad \text{(VI.64)}$$

Z_n satisfies relation

$$|n|\left(Z_{|n|+2} - Z_{|n|}\right) = 2Z_{|n|} - 3Z_{|n|+2} \qquad \text{(VI.65)}$$

We introduce redefined fields $\tilde{h}_{+j}$ and $\tilde{V}_j$

$$\tilde{V}_j = V^{1}_{j0}\left(1 + \frac{3}{8}\varepsilon^2 Z_{|n|} - \frac{3}{8}\varepsilon^2 Z_{|n|+2}\right)$$

$$\tilde{h}_{+j} = h^{1}_{+j0}\left(1 + \frac{3}{16}\varepsilon^2 Z_{|n|+2}\right) \qquad \text{(VI.66)}$$

In terms of V_j and h_{+j} the action has a form

$$S = \int d^4x \, \Big[\tfrac{1}{2} \tilde{V}_j \Big(\partial^2 + \frac{1}{a^2} \big(-2 + 3\varepsilon^2 z_{|n|} -$$

$$- 3\varepsilon^2 z_{|n|+2}\big)\Big) \tilde{V}_j + \tilde{h}^*_{+j} \Big(\partial^2 + \frac{1}{a^2} \big(-2 + \tfrac{3}{2} \varepsilon^2 z_{|n|} -$$

$$- \tfrac{3}{4} \varepsilon^2 z_{|n|+2}\big)\Big) \tilde{h}_{+j} + \frac{i\sqrt{2}}{a^2} \tilde{V}_j \big(\tilde{h}_{+j} - \tilde{h}^*_{+j} \big) \Big(1 - \tfrac{9}{8} \varepsilon^2 z_{|n|} +$$

$$+ \tfrac{15}{16} \varepsilon^2 z_{|n|+2} \Big) \Big] \qquad \text{(VI.67)}$$

The mass matrix for $\tilde{V}_j$, $\tilde{h}_{+j}$, $\tilde{h}^*_{+j}$ is

$$\mathcal{M} = \begin{bmatrix} \partial^2 - M_1 & i M_3 & -i M_3 \\ -i M_3 & \partial^2 - M_2 & 0 \\ i M_3 & 0 & \partial^2 - M_2 \end{bmatrix} \qquad \text{(VI.68)}$$

where

$$M_1 = \frac{1}{a^2} \big(2 + 3\varepsilon^2 (z_{|n|+2} - z_{|n|} \big)$$

$$M_2 = \frac{1}{a^2} \big(2 + \tfrac{3}{4} \varepsilon^2 z_{|n|+2} - \tfrac{3}{2} \varepsilon^2 z_{|n|} \big) \qquad \text{(VI.69)}$$

$$M_3 = \frac{\sqrt{2}}{a^2} \big(1 - \tfrac{9}{8} \varepsilon^2 z_{|n|} + \tfrac{15}{16} \varepsilon^2 z_{|n|+2} \big)$$

A determinant of the matrix (VI.68) is

$$\det \mathcal{M} = \big[(\partial^2 - M_1)(\partial^2 - M_2) - 2M_3^2 \big] (\partial^2 - M_2) \qquad \text{(VI.70)}$$

and its eigenvalues are

$$0$$

$$\frac{1}{a^2} \big(2 + \tfrac{3}{4} \varepsilon^2 z_{|n|+2} - \tfrac{3}{2} \varepsilon^2 z_{|n|} \big) = M_2 \qquad \text{(VI.71)}$$

$$\frac{1}{a^2} \big(4 - \tfrac{9}{2} \varepsilon^2 z_{|n|} + \tfrac{15}{4} \varepsilon^2 z_{|n|+2} \big)$$

As we already saw in Section II.K, in quantum KKT we need to make sense of certain expessions through the procedure of analytical continuation in the variable D corresponding to the number of space-time dimensions (i.e. we consider a limit $D \longrightarrow 4$). In this Chapter I am going to explain different ways of performing analytical continuation.

The basic expression I have in mind is

$$F(D, A^2) = \Gamma(-D/2) \sum_{n=1}^{\infty} n^2 (n^2 - A^2)^{D/2} \tag{VII.1}$$

where $A^2 < 1$. $F(D,A^2)$ represents (up to normalization constant) an effective potential in D=4 dimensions of a free, minimally coupled to gravity scalar field with a mass m^2 in the background $M_4 x S_3$;

$$A^2 = 1 - m^2 a^2 \tag{VII.2}$$

a is a length scale of S_3 (round three-sphere).

$F(D,A^2)$ is well defined only for $D < -3$. It is possible hovewer to express F in the integral form and obtain representations which make sense also for other values of D.

The first method based on the Laplace transformation was already discussed in Section II.K. After applying this transformation the dependence on n is in the factor e^{-nt}. Therefore the sum over n can be performed easily. Let me remind here the formula (II.176)

$$F(D, A^2) \underset{D \to 4}{\longrightarrow} 8\pi \sum_{p=1}^{\infty} \Big[\sin(2\pi p A) \Big(-\frac{90A}{(2\pi p)^6} + \frac{9A^3}{(2\pi p)^4} \Big) + \cos(2\pi p A) \Big(\frac{39A^2}{(2\pi p)^5} - \frac{A^4}{(2\pi p)^3} - \frac{90}{(2\pi p)^7} \Big) \Big] \tag{VII.3}$$

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* Based on the paper: Shen & Sobczyk (1986), Appendix B

and (II.177)

$$F(D,A^2) \xrightarrow[A^2\to 1]{D\to 4} 8\pi \left[\frac{39}{(2\pi)^5}\zeta(5) - \frac{1}{(2\pi)^3}\zeta(3) - \frac{90}{(2\pi)^7}\zeta(7) \right] \qquad \text{(VII.4)}$$

The method we describe now was proposed in Ref. Candelas Weinberg (1984). It cannot be applied when the sum in (VII.1) starts from n=2 or when D is odd. In the later case it is not clear how to choose an appropriate integration contour.

The second method is also based on the Laplace transformation. It was proposed by Critchley & Dowker (1981) and later by Sarmadi (1986). To employ this method we first calculate $F(4,-A^2)$ and then we will make a "Wick rotation" and obtain the expression for $F(4,A^2)$.

We begin with the relation

$$F(D,-A^2) = \Gamma(-D/2) \sum_{n=1}^{\infty} \left[(n^2+A^2)^{D/2+1} - A^2 (n^2+A^2)^{D/2} \right] \qquad \text{(VII.5)}$$

We use the formula

$$(n^2+A^2)^{\nu} = \frac{1}{\Gamma(-\nu)} \int_0^{\infty} dt\, e^{-t(n^2+A^2)}\, t^{-\nu-1} \qquad \text{(VII.6)}$$

valid for $\nu < 0$. On using the property of theta function

$$\sum_{n=-\infty}^{+\infty} e^{-sn^2} = \sqrt{\frac{\pi}{s}} \sum_{n=-\infty}^{\infty} e^{-\pi^2 n^2/s} \qquad \text{(VII.7)}$$

one can derive the following representation of F:

$$F(D,-A^2) = -\sum_{n=1}^{\infty} \sqrt{\pi} \int_0^{\infty} ds\, e^{-sA^2} e^{-\pi^2 n^2/s} \left[A^2 s^{\frac{-D-3}{2}} + \left(1+\frac{D}{2}\right) s^{\frac{-D-5}{2}} \right]$$

$$- \frac{\sqrt{\pi}}{2} \int_0^{\infty} ds\, e^{-sA^2} \left[A^2 s^{\frac{-D-3}{2}} + \left(1+\frac{D}{2}\right) s^{\frac{-D-5}{2}} \right] \qquad \text{(VII.8)}$$

It is here that the minus sign at $-A^2$ is necessary in order to make the integral converge at infinity. One can now rewrite (VII.8) by an integral representation of the Bessel function K_ν (Gradshteyn & Ryzhik (1980))

$$K_\nu(xz) = \frac{z^\nu}{2} \int_0^{\infty} e^{-\frac{x}{2}\left(t+\frac{z^2}{t}\right)}\, t^{-\nu-1}\, dt \qquad \text{(VII.9)}$$

and explicit forms of $K_{7/2}$ and $K_{5/2}$. The result is

$$F(4,-A^2) = \frac{\sqrt{\pi}}{4} A^7 \Gamma(-7/2) - 8\sqrt{\pi} \sum_{p=1}^{\infty} \exp(-2\pi p A) \left[\frac{A^4}{(2\pi p)^3} + \right.$$

$$\left. + \frac{9A^3}{(2\pi p)^4} + \frac{39A^2}{(2\pi p)^5} + \frac{90A}{(2\pi p)^6} + \frac{90}{(2\pi p)^7} \right] \quad \text{(VII.10)}$$

We now can do the "Wick rotation" ($A^2 \to -A^2$) on (VII.10) to return to our original expression (VII.3). Because of the $A \to -A$ symmetry in the original expression we need to symmetrize the final expression too (we call it a "symmetric prescription").

One can also start from the expression (II.176) and arrive at (VII.10) by a symmetrized $A^2 \to -A^2$ transformation and performing the integral in (II.176). The contour C should be closed in the upper-half plane for terms with e^{itA} factor and in the lower-half plane for terms with e^{-itA}. The residue at zero yields the first term on the RHS of (VII.10).

This method can be used when the sum (VII.1) is from $n \geqslant 2$. Moreover even if D is odd in which case method described before completely fails, this method can single out the divergent part which will be of the form $\Gamma(\frac{-D-1}{2})$, and the regular part can be evaluated numerically.

The third method of performing analytic continuation in D is based on the Sommerfeld-Watson integral (Kikkawa et al. (1985)). The basic equality is

$$F(D, A^2) = \frac{i}{2} \Gamma(-D/2) \int_C dz\, z^2 (z^2 - A^2)^{D/2} \cot(\pi z) \quad \text{(VII.11)}$$

where path C and later C` are shown in figure below.

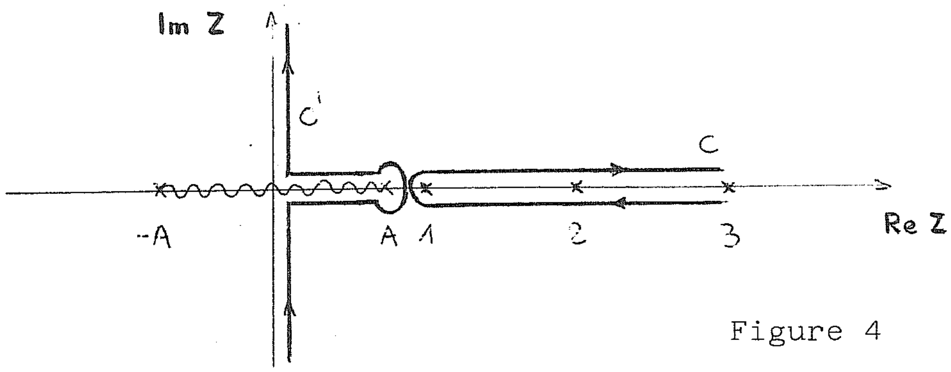

Figure 4

The equality follows since the integrand on the RHS of (VII.11) has simple poles at z = 1, 2, 3, ... For $D<-3$ one can change the integration contour of (VII.11) from C to C` (see Fig.) and obtain

$$F(D,A^2) = \Gamma(-D/2)\ \sin\frac{\pi D}{2}\left[\int_0^\infty dy\ y^2(y^2+A^2)^{D/2}\coth(\pi y) + \right.$$

$$+ \int_0^A dx\ x^2(A^2-x^2)^{D/2}\cot(\pi x) + \begin{Bmatrix}\text{contribution from the} \\ \text{"bubble" integral at A}\end{Bmatrix} \tag{VII.12}$$

The first integral on the RHS of (VII.12) is formally divergent as $D \to 4$. We can use the relation

$$\coth(\pi y) = 1 + \frac{2}{e^{2\pi y}-1} \tag{VII.13}$$

to separate out the singular part and proceed to regularize it. The singular part can be written as

$$\int_0^\infty dy\ y^2(y^2+A^2)^{D/2} = \frac{A^{3+D/2}}{2}\ \frac{\Gamma(3/2)\,\Gamma(\frac{-D-3}{2})}{\Gamma(-\frac{D}{2})} \tag{VII.14}$$

This representation gives regular result in the limit of $D \to 4$ when it is put into (VII.12). The contribution from the bubble integral at point A is negligible as $D \to 4$. After dropping terms of higher order in (D-4) we have

$$F(4,A^2) = -\frac{\pi}{2}\left[2\int_0^\infty dy\ \frac{y^6+2y^4A^2+y^2A^4}{e^{2\pi y}-1} + \right.$$

$$\left. + A^7\int_0^1 dx\ x^2(1-x^2)^2\cot(\pi x A)\right] = -\pi\left[720\frac{\zeta(7)}{(2\pi)^7} + 48A^2\frac{\zeta(5)}{(2\pi)^5} + \right.$$

$$\left. + 2A^4\frac{\zeta(3)}{(2\pi)^3} + \frac{A^7}{2}\int_0^1 dx\ x^2(1-x^2)^2\cot(\pi x A)\right] \tag{VII.15}$$

which is to be compared with (VII.3). In particular for A=1 we have

$$F(D, A^2) \underset{\substack{D\to 4\\ A^2\to 1}}{\longrightarrow} -\pi \left[720 \frac{\zeta(7)}{(2\pi)^7} + 48 \frac{\zeta(5)}{(2\pi)^5} + 2 \frac{\zeta(3)}{(2\pi)^3} + \right.$$

$$\left. + \frac{1}{2} \int_0^1 dx\, x^2 (1-x^2)^2 \cot(\pi x) \right] \qquad \text{(VII.16)}$$

Making use of the fact that for positive integer n

$$\int_0^1 x^n (1-x)^n \cot(\pi x) = 0 \qquad \text{(VII.17)}$$

and properties of Bernoulli polynomials $B_n(x)$ (Abramovitz & Stegun (1965))

$$\int_0^1 B_{2n+1}(x) \cot(\pi x)\, dx = (-)^{n+1} \frac{2(2n+1)!\,\zeta(2n+1)}{(2\pi)^{2n+1}} \qquad \text{(VII.18)}$$

one recovers the expression given in (VII.4).

This method still holds if we generalize the lower limit of the sum in (VII.1) into n=m and $m-1 \leqslant A \leqslant m$. If $A < m-1$ the residue contribution from poles like $m-1$ in the C^k contour integral will be finite and the overall $\Gamma(\frac{-D}{2})$ factor leads to infinity when D is even. When D is odd, this method can also single out the singular part which is just (VII.14).

The last method of analytic regularization I will discuss, is making use of the infinite Plana sum formula (Lindelöf (1947))

$$\sum_{n=m}^{\infty} f(n) = \frac{f(m)}{2} + \int_m^{\infty} f(\tau)\,d\tau + i \int_0^{\infty} d\tau\, \frac{f(m+it) - f(m-it)}{e^{2\pi t} - 1} \qquad \text{(VII.19)}$$

which is valid if

1/ f(z) is regular for $\mathcal{R}e(z) > 0$

2/ $\lim_{t\to\infty} e^{-2\pi|t|} f(\tau + it) = 0$ uniformly for $0 \leqslant \tau < \infty$

3/ $\lim_{\tau\to\infty} \int_{-\infty}^{+\infty} e^{-2\pi|t|} |f(\tau + it)| = 0$

Using (VII.19) for our function (VII.1) we have

$$F(D,A^2) = \Gamma(-D/2)\left[\frac{1}{2}(1-A^2)^{D/2} + \int_1^\infty t^2(t^2-A^2)^{D/2}\,dt + \right.$$
$$\left. + i\int_0^\infty dt\,\frac{(1+it)^2\left[(1+it)^2-A^2\right]^{D/2} - (1-it)^2\left[(1-it)^2-A^2\right]^{D/2}}{e^{2\pi p}-1}\right] \tag{VII.20}$$

One must expand the RHS of (VII.20) in powers of (D-4). The leading term which is potentially problematic because $\lim_{D\to 4}\Gamma(-D/2) = -\frac{1}{D-4}$, turns out to be zero. The linear term in the bracket combined with gamma factor gives a <u>finite result.</u> The only formally divergent term on the RHS of (VII.20) (as $D \to 4$) is the $(1,\infty)$ integral. It can be regularized by using the following representation

$$\int_1^\infty dt\, t^2(t^2-A^2)^{D/2} = -\int_1^{A^{-1}} t^2(t^2-A^2)^{D/2}dt + \frac{A^{3+D}}{2}\,\frac{\Gamma(\frac{-D-3}{2})\,\Gamma(\frac{D+2}{2})}{\Gamma(-1/2)} \tag{VII.21}$$

which is clearly finite as $D \to 4$.

The advantage of this method is that it singles out the divergence (essentially the $\Gamma(\frac{-D-3}{2})$ factor) quickly in the odd D case. It is useful when one is interested in the renormalization aspect of certain theories. The evaluation of the finite part seems to be more involved than in other methods. This method can be also used when the summation is from $n \geq 2$.

For some values of D and A, $F(D,A^2)$ can give a <u>divergent expression.</u> For example if $D \to 0$ and $A \to 1$, then F diverges as ln(1-A) in all four methods. We will encounter this case in the perturbative representation of the effective potential calculated in Chapter VIII. Fortunately that expression will be multiplied by some polynomial in (1-A) and perturbative expression turns out to be finite (at least up to $O(\alpha^2)$). In the minimally coupled case $1-A^2 = m^2a^2$, so that $A^2 \to 1$ limit is a consequence of the massless limit that we will consider. The logarythmic divergence can hence be viewed as an <u>infrared divergence.</u>

VIII. SYMMETRY BREAKING IN QUANTUM KALUZA-KLEIN THEORIES*

Few years ago an interesting model of quantum KKT was proposed by Candelas & Weinberg (1984). The idea of this paper is presented in Section II.K together with detailed calculation of a particular D=7 dimensional model. Originally, a solution of quantum corrected field equations was found with the background geometry $M_4 x S_N$, S_N being a round N-sphere (one assumes N to be odd since then in dimensional regularization one-loop effective potential is finite (Duff & Toms (1983,1984))). The stability of this compactification was discussed by a lot of authors (some references are given at the end of Section II.K). The stability against deformations of S_N is of particular importance for reasons explained in Chapter VI: in KKT it is natural to think of symmetry breaking mechanism as of having a geometrical interpretation in deformations of the internal space.

In this Chapter by considering a very simple model I will try to show that this kind of mechanism can be obtained in the framework proposed by Candelas & Weinberg. I will present the calculation of the effective potential for a scalar field minimally coupled to gravity in the M_4x(homogeneously deformed S_3) background. I will show that the effective potential as a function of a deformation parameter α has a local minimum for $\alpha \neq 0$. The $\alpha = 0$ solution (round S_3) corresponds to a local maximum of the potential (see Figure12 at the end of this Chapter - p.122).

This result opens an interesting possibility that dynamics itself (quantum vacuum energy of matter fields) is responsible for the actual shape of the internal space. This effect can be

* Based on the paper: Shen & Sobczyk (1986)

relevant in every higher-dimensional theory including superstrings.

A. Model

We consider a scalar field in a seven-dimensional space-time. The classical action of the system has a form

$$S = \int dV_7 \left[\frac{1}{\varkappa^2} (\bar{R} + \bar{\Lambda}) + \frac{1}{2} \phi (\bar{\Box} - m^2) \phi \right] \qquad \text{(VIII.1)}$$

where dV_7 is a volume element and barred quantities refer to D=7 dimensional space-time. We assume that this system admits a $M_4 x S_3$ geometry as a solution of the quantum corrected field equations. S_3 denotes here a three-sphere with possible homogenous deformation. The line element on S_3 is given by

$$ds^2 = \sum_{a=1}^{3} (l_a \sigma^a)^2 \qquad \text{(VIII.2)}$$

where σ^a form a basis one form on S_3 satisfying the structure equation $d\sigma^a = \frac{1}{2} \epsilon_{abc} \sigma^b \wedge \sigma^c$. l_a's are principal curvature radii of the homogeneous internal space. The case in which all l_a's are equal corresponds to the usual round S_3 with isometry SU(2)xSU(2). The case when two l_a's are equal corresponds to the Taub space with isometry SU(2)xU(1). This is the case we shall consider here. Let $l_1 = l_2 \neq l_3$. Then the background geometry depends on two parameters only: the scale a and deformation α defined by

$$a = l_1 \qquad \text{(VIII.3)}$$

$$\alpha = \left(\frac{l_1}{l_3}\right)^2 - 1 \qquad \text{(VIII.4)}$$

The range of α is $-1 < \alpha < \infty$ and $\alpha = 0$ corresponds to the round S_3.

The Laplace operator in the background geometry $M_4 x S_3$ can be written as a sum of two operators (corresponding to M_4 and

S_3 respectively)

$$\bar{\Box} = \Box + \tilde{\Box} \qquad \text{(VIII.5)}$$

The scalar field $\Phi(x,y)$ where y denotes internal coordinates can then be expanded by a complete orthonormal set of eigenfunctions $Y_M(y)$ of the operator $-\tilde{\Box} + m^2$ with eigenvalues $\{\lambda_M\}$ as

$$\Phi(x,y) = \sum_M \Phi_M(x) Y_M(y) \qquad \text{(VIII.6)}$$

The eigenvalues $\{\lambda_M\}$ are obtained as $(K = -J, \dots, J)$

$$\lambda_M = \frac{J(J+1)}{l_1^2} + \left(\frac{1}{l_3^2} - \frac{1}{l_1^2}\right) K^2 + m^2 \qquad \text{(VIII.7)}$$

On making use of the orthonormality of Y_M, the D=7 dimensional scalar field action can be reduced to a D=4 dimensional one with infinite number of massive scalar fields

$$S_{scalar} = -\frac{1}{2} \int d^4x \sum_M \Phi_M(x) (-\Box + \lambda_M) \Phi_M(x) \qquad \text{(VIII.8)}$$

The effective potential for the action (VIII.8) can be found to be

$$\Gamma = -i \ln \int D\Phi \, e^{iS[\Phi]} =$$

$$= \frac{\Omega_D M^{4-D}}{2(2\pi)^{D/2}} \Gamma(-D/2) \sum_M (\lambda_M)^{D/2} \qquad \text{(VIII.9)}$$

where D is a complex variable corresponding to the number of space-time dimensions. We assume effective potential to be an analytical function of D in order to perform analytic continuation from the domain in which it is well defined to D=4. One can then define the matter effective potential in the Minkowski space by $V = -\Gamma/\Omega_D$

It proves convenient to use the following definitions

$$n = 2J+1 \qquad q = J-K \qquad \sigma = m^2 a^2 - 1 \qquad \text{(VIII.10)}$$

The eigenvalues can thus be written as

$$\lambda_M = \left[n^2 + \sigma + \alpha (n-1-2q)^2 \right] / a^2 \qquad \text{(VIII.11)}$$

We shall consider only massless scalar field, so that σ reduces to -1 and the effective potential follows from (VIII.9) and (VIII.10)

$$V(a,\alpha,D) = - \frac{\mu^{4-D}\,\Gamma(-D/2)}{2a^D (4\pi)^{D/2}} \sum_{n=1}^{\infty} n \sum_{q=0}^{n-1} \left[n^2 - 1 + \alpha (n-1-2q)^2 \right]^{D/2} \qquad \text{(VIII.12)}$$

The effective potential as it stands is well defined only for Re(D) $<$ -3. We shall continue V as function of D analytically to D=4. Since this is a one-loop computation in odd dimensional space-time we expect that dimensional regularization will lead to <u>finite result.</u> Details of calculations will be given in Section VIII.C. In Section VIII.B we shall first present a perturbation evoluation of V(a,α) following Ref. Shen (1985).

B. <u>Perturbative calculation</u>

If the deformation is small, one can calculate the effective potential (VIII.12) in power series of the deformation parameter α . The calculation of mode sums in this way is much simpler than the exact computation and may hence serve as a useful test to verify it.

One gets $\quad (\sigma = -1 + m^2 a^2 \; ; \; m^2 \to 0)$

$$\begin{aligned} V(a,\alpha,D) = - \frac{\mu^{4-D}\,\Gamma(-D/2)}{2a^D (4\pi)^{D/2}} \sum_{n=1}^{\infty} \Big[& n^2 (n^2+\sigma)^{D/2} + \\ & + \alpha \frac{D}{6} \left(n^2 (n^2+\sigma)^{D/2} - (\sigma+1)\, n^2 (n^2+\sigma)^{D/2-1} \right) + \\ & + \alpha^2 \Big(\frac{D(D-2)}{40} n^2 (n^2+\sigma)^{D/2} - \frac{D(D-2)}{60} (5+3\sigma)\, n^2 (n^2+\sigma)^{D/2-1} \\ & + \frac{D(D-2)}{120} (7+10\sigma+3\sigma^2)\, n^2 (n^2+\sigma)^{D/2-2} \Big) + O(\alpha^3) \Big] \end{aligned} \qquad \text{(VIII.13)}$$

To evaluate the effective potential (VIII.13) one can use the analytical continuation scheme due to Candelas & Weinberg (1984) (see also Section II.K). It appears quite useful to keep a non-zero mass throughout the calculation and only at the very end of them to consider a massless limit $m^2 \to 0$. Alternatively, one can sum from n=2 at the expense that the analytic continuation method must be changed (see Chapter VII).

Introduce the same expression as in Chapter VII

$$F(D, A^2) = \Gamma(-D/2) \sum_{n=1}^{\infty} n^2 (n^2 - A^2)^{D/2} \tag{VIII.14}$$

By applying a method described in Section II.K one gets

$$F(D, A^2) \xrightarrow[A^2 \to 1]{D \to 4} 4 \left[\frac{39}{(2\pi)^4} \zeta(5) - \frac{90}{(2\pi)^6} \zeta(7) - \frac{1}{(2\pi)^2} \zeta(3) \right] \tag{VIII.15}$$

$$F(D, A^2) \xrightarrow[A^2 \to 1]{D \to 2} -2 \left[\frac{12}{(2\pi)^4} \zeta(5) - \frac{5}{(2\pi)^2} \zeta(3) \right] \tag{VIII.16}$$

As we have mentioned at the end of Chapter VII, for the limit

$$\lim_{\substack{D \to 0 \\ A^2 \to 1}} \Gamma(-D/2) \sum_{n=1}^{\infty} n^2 (n^2 - A^2)^{D/2 - 2} \tag{VIII.17}$$

one obtains a logarithmic divergence in (1-A). When we look at (VIII.13) we observe hovewer that it is finite in the $A \to 1$ (or $m^2 \to 0$) limit. The final result is

$$V(a, \alpha) = \frac{1}{a^4} \left[(1 + \tfrac{2}{3}\alpha) \cdot 0.7568 \cdot 10^{-4} - \alpha^2 \cdot 1.0657 \cdot 10^{-4} + O(\alpha^3) \right] \tag{VIII.18}$$

C Exact calculation

In order to calculate $V(a, \alpha)$, we first do the summation over q using the Plana sum formula*. For this purpose one must investigate analytic properties of the function one applies it to. In our case the function has a form

---

* Plana (1820)

$$\Phi(q) = \left[n^2 - 1 + \alpha\,(n - 1 - 2q)^2 \right]^{D/2} \tag{VIII.19}$$

For $\alpha > 0$, $\Phi(q)$ has branch points at

$$q = \frac{n-1}{2} \pm \frac{i}{2}\sqrt{\frac{n^2-1}{\alpha}} \tag{VIII.20}$$

For $\alpha < 0$, $\Phi(q)$ has branch points at

$$q = \frac{n-1}{2} \pm \frac{1}{2}\sqrt{\frac{n^2-1}{|\alpha|}} \tag{VIII.21}$$

We can choose branch cuts as shown in Figures below

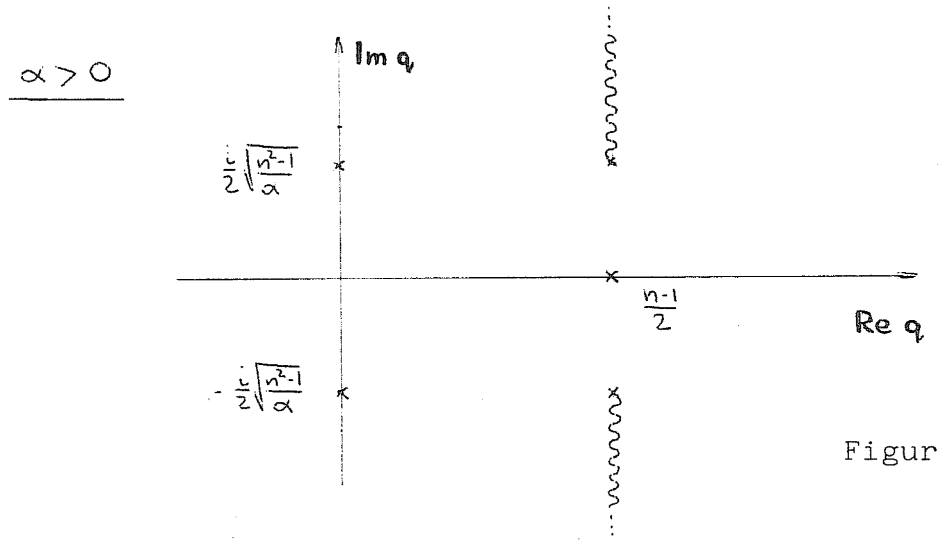

Figure 5

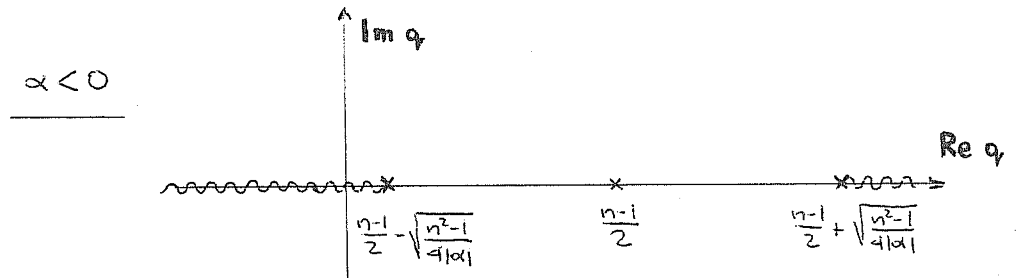

Figure 6

The Plana formula states that

$$\sum_{k=n}^{m} \phi(k) = i\int_0^\infty \frac{dy}{e^{2\pi y}+1}\left[\phi(m+\tfrac{1}{2}+iy) - \phi(n-\tfrac{1}{2}+iy) - (y \to -y)\right] + \int_{n-1/2}^{m+1/2} \phi(x)\,dx \tag{VIII.22}$$

In the proof one calculates contributions from contours: C_1' for the function $\Phi(z)/(\exp(2\pi iz)-1)$, and C_2' for the function $\Phi(z)/(\exp(2\pi iz)-1)$. Contours C_1' and C_2' are shown below

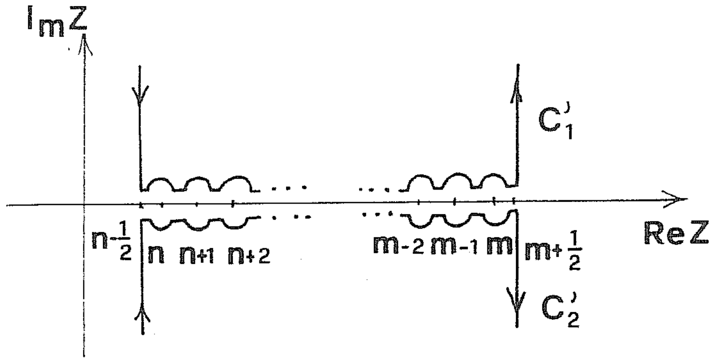

Figure 7

In the proof one requires that $\phi(z)$ is analytical "inside" contours C_1' and C_2'. For this reason the case $\alpha > 0$ is more difficult to consider - additional contribution in (VIII.22) appears

$$-4\sin\left(\frac{\pi D}{2}\right)(4\alpha)^{D/2}\int_p^\infty \frac{(y^2-p^2)^{D/2}}{e^{i\pi(k-1)+2\pi y}-1} \quad ; \quad p=\frac{1}{2}\sqrt{\frac{n^2-1}{\alpha}} \qquad \text{(VIII.23)}$$

Let us concentrate first on the case $\alpha < 0$. We must still be sure that the cut is outside regions surrounded by C_1' and C_2'. This turns out to be true for

$$-\frac{3}{4} < \alpha < 0 \qquad \text{(VIII.24)}$$

For α satisfying this condition we get

$$V = -\frac{\mu^{4-D}\,\Gamma(-D/2)}{2\,a^D(4\pi)^{D/2}}\left[\int_0^1 dy\, F(y)\,(1+\alpha y^2)^{D/2} + 2i\,(1+\alpha)^{D/2}\int_0^\infty \frac{G(y)\,dy}{e^{2\pi y}+1}\right] \qquad \text{(VIII.25)}$$

where

$$F(y) = \sum_{n=2}^\infty (n^2-A^2)^{D/2}\, n^2 \qquad \text{(VIII.26)}$$

$$G(y) = \sum_{n=2}^\infty n\left\{\left[(n+iB)^2-E^2\right]^{D/2} - \left[(n-iB)^2-E^2\right]^{D/2}\right\} \qquad \text{(VIII.27)}$$

$$A = (1+\alpha y^2)^{-1/2} \; ; \; B = \frac{2y\alpha}{1+\alpha} \; , \; E^2 = \frac{4\alpha y^2}{(1+\alpha)^2} + \frac{1}{1+\alpha} \qquad \text{(VIII.28)}$$

Since n=1 is a zero mode, it is neglected in Eqs (VIII.26-27)

Next we find that the infinite sum in F(y) can be converted into an integral in the complex plane by a Sommerfeld-Watson transformation (see Chapter VII)

$$\sum_{n=2}^{\infty} n^2(n^2-A^2)^{D/2} = \frac{i}{2}\int_C dz\, z^2(z^2-A^2)^{D/2} \cot(\pi z) \qquad \text{(VIII.29)}$$

where C is the contour on Figure below

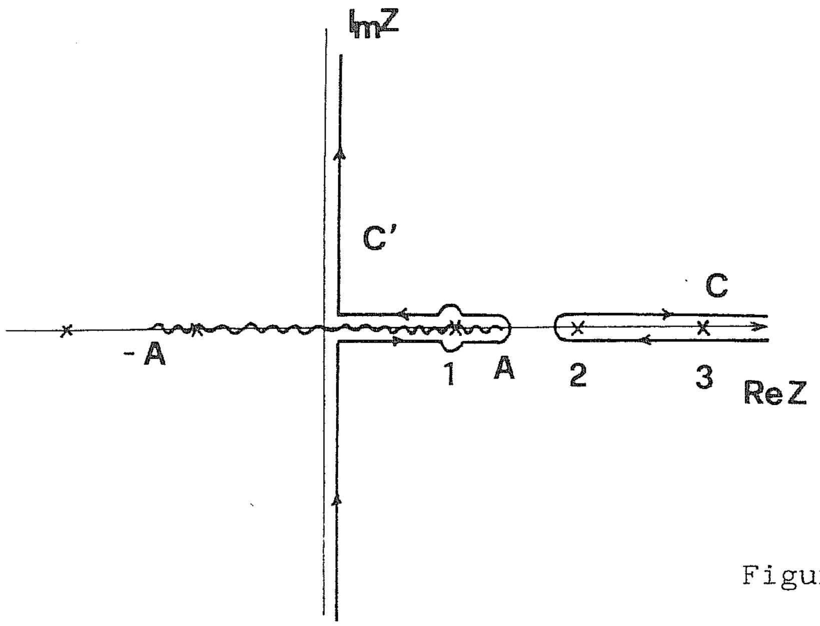

Figure 8

Because α is negative and $y \in (0,1)$, we find that $1 \leqslant A \leqslant 2$. The regularization consists of two steps. First, one changes the integration path from C to C' (see Figure above). Eq. (VIII.29) can now be expressed as

$$F(y) = \sin\frac{\pi D}{2}\left[P\int_0^A x^2(A^2-x^2)^{D/2} \cot\pi x\, dx \; + \right.$$

$$\left. + \int_0^{\infty} x^2(A^2+x^2)^{D/2} \coth(\pi x) dx\right] - (A^2-1)^{D/2} \cos\frac{\pi D}{2} \qquad \text{(VIII.30)}$$

Because z=1 is a simple pole on the integration path, the (0,A) integral has to be defined by a principal value. The last term of (VIII.30) comes from the residue of the z=1 pole. The second integral in (VIII.30) is divergent when the value D=4 is taken. Hence further regularization is called for. Rewriting $\coth(\pi x)$ as $1 + 2/(\exp(2\pi x)-1)$, one has

$$\int_0^\infty dx\, x^2 (x^2+A^2)^{D/2} \coth(\pi x) =$$

$$= 2\int_0^\infty dx \frac{x^2(x^2+A^2)^{D/2}}{e^{2\pi x}-1} + \frac{A^{D+3}\,\Gamma(3/2)\,\Gamma(-\frac{D+3}{2})}{2\,\Gamma(-D/2)} \qquad \text{(VIII.31)}$$

Substituing (VIII.31) into (VIII.30) and taking $D \to 4$ limit, we find

$$F(y) = \lim_{\varepsilon\to 0} \pi\varepsilon \left[\frac{1}{2} P\int_0^A x^2 (A^2-x^2)^2 \cot(\pi x)\, dx + \frac{\Gamma(7)\zeta(7)}{(2\pi)^7} + \right.$$

$$\left. + 2A^2 \frac{\Gamma(5)\zeta(5)}{(2\pi)^5} + A^4 \frac{\Gamma(3)\zeta(3)}{(2\pi)^3} \right] - \lim_{D\to 4} \frac{\pi}{\cos\frac{\pi D}{2}} (A^2-1)^{D/2} \qquad \text{(VIII.32)}$$

where $\varepsilon = D-4$. Note that there is an overall $\Gamma(-\frac{D}{2})$ factor in (VIII.25) which gives $-\frac{1}{\varepsilon}$ in the limit of $D \to 4$. We conclude that (VIII.32) gives finite contribution to the effective potential except for the last term which will be treated separately.

It remains now to evaluate the function G in (VIII.27). Because the branch cuts are different, we have to separate the $(0,\infty)$ integral into $(0,u)$ and (u,∞) with $u = \sqrt{(1+\alpha)/(-4\alpha)}$. From (VIII.28) it is easy to see that when $0 \leq y \leq u$, $E^2 \geq 0$. Branch cuts of functions $[(z \pm iz)^2 - E^2]^{D/2}$ are shown below.

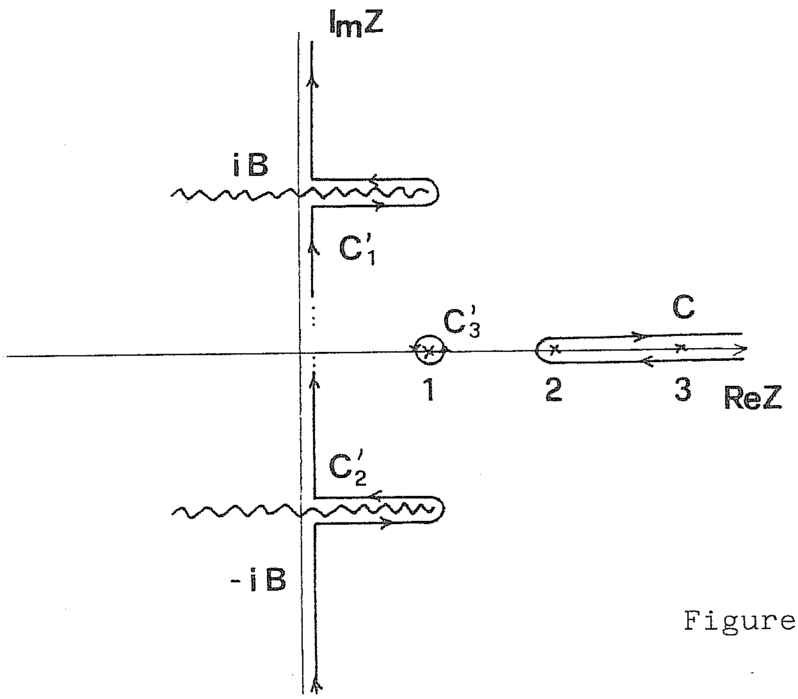

Figure 9

One can do similar Sommerfeld-Watson transformation and G(y) takes the form

$$2iG(y) = \int_C dz\, z \cot(\pi z) \left\{ \left[(z-iB)^2 - E^2\right]^{D/2} - \left[(z+iB)^2 - E^2\right]^{D/2} \right\} \tag{VIII.33}$$

where C is a contour in Figure above. One can observe that C can be replaced by contours $(C_1' + C_3')$ and $(C_2' + C_3')$ respectively for two terms in (VIII.33). This is obviously allowed if $D<-3$. After manipulations similar to those done before, we obtain for $0 \leq y \leq u$

$$2iG(y) = \lim_{\varepsilon \to 0} 2\pi\varepsilon \left\{ \int_0^E dx\, (E^2-x^2)^2 \frac{B\sin(2\pi x) - x\sinh(2\pi B)}{\cosh(2\pi B) - \cos(2\pi x)} \right.$$

$$+ 8B \left[\frac{\Gamma(5)\zeta(5)}{(2\pi)^5} + (B^2+E^2) \frac{\Gamma(3)\zeta(3)}{(2\pi)^3} \right]$$

$$\left. + \int_0^B dx\, \coth(\pi x) \left[(x-B)^2 + E^2\right]^2 x \right\}$$

$$- \lim_{D\to 4} \left\{ \left[(1+iB)^2 - E^2\right]^{D/2} - \left[(1-iB)^2 - E^2\right]^{D/2} \right\} \tag{VIII.34}$$

In the case $y \geq u$, (VIII.28) yields $E^2 \leq 0$. The branch cuts associated with (VIII.27) will be along the imaginary axis (see Figure below)

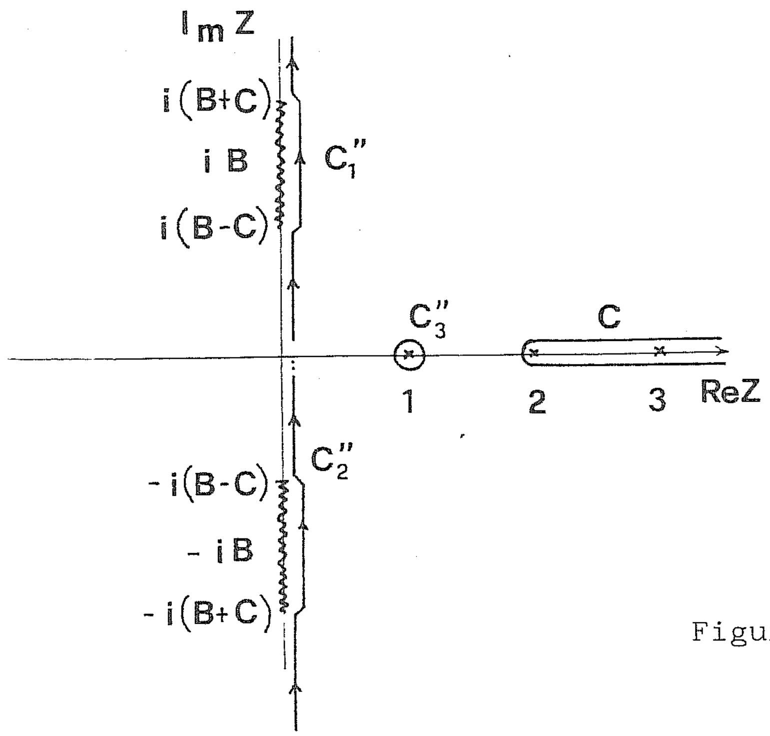

Figure 10

We then replace the contour C of the Figure above by $C_1'' + C_3''$ and $C_2'' + C_3''$ for two integrals in (VIII.33) respectively. By carefully keeping track of the phase changes when the integration path is along the cut, we find that (VIII.33) leads to

$$2iG(y) = \lim_{\varepsilon\to 0} 2\pi\varepsilon \left\{ \frac{8}{15} BF^2 + 8B\left[\frac{\Gamma(5)\zeta(5)}{(2\pi)^5} + (B^2+E^2)\frac{\Gamma(3)\zeta(3)}{(2\pi)^3} \right] \right.$$
$$\left. - \int_0^{F-B} dx \, \frac{x\left[(x+B)^2+E^2\right]^2}{e^{2\pi x}-1} + \int_0^{F-B} dx \, \frac{x\left[(x-B)^2+E^2\right]^2}{e^{2\pi x}-1} \right\}$$
$$- \lim_{D\to 4} \left\{ \left[(1+iB)^2 - E^2\right]^{D/2} - \left[(1-iB)^2 - E^2\right]^{D/2} \right\} \qquad \text{(VIII.35)}$$

where $F = (-E^2)^{1/2}$ and $y \geqslant u$.

Substituting the last terms of (VIII.32), (VIII.34) and (VIII.35) and performing the y-integration, we find that altogether they give a finite contribution to the effective potential after $D \to 4$ limit is taken. More explicitly, we have

$$\lim_{D\to 4} - \Gamma(-D/2) \left\{ \int_0^1 dy \, (1-A^{-2})^{D/2} \cos\frac{\pi D}{2} + 2i \int_0^\infty \frac{dy \, (1+\alpha)^2}{e^{2\pi y}+1} \, \times \right.$$
$$\left. \times \left[\left[(1+iB)^2 - E^2\right]^{D/2} - \left[(1-iB)^2 - E^2\right]^{D/2} \right] \right\} =$$

$$= -\frac{1}{25} + 16 \int_0^\infty \frac{dy}{e^{2\pi y}+1} \, y(4y^2-1)\ln(4y^2+1) \, +$$
$$+ 2 \int_0^\infty \frac{dy}{e^{2\pi y}+1} \, (16y^4 - 24y^2 + 1) \cdot \Theta \qquad \text{(VIII.36)}$$

where $\tan(\Theta/2) = 1/(2y)$. On using the results of (VIII.32) to (VIII. 35) in (VIII. 25) we obtain the final expression for the regularized effective potential

$$V = \frac{1}{32\pi a^4} \left\{ \frac{1}{2} \int_0^1 dy \, (1+\alpha y^2)^2 \; P\int_0^A x^2 (A^2 - x^2)^2 \cot(\pi x) \, dx \, + \right.$$
$$+ 2(1+\alpha)^2 \int_0^u \frac{W_2(y)}{e^{2\pi y}+1} \, dy \, +$$

$$+ 2(1+\alpha)^2 \int_u^\infty \frac{dy}{e^{2\pi y}+1} \left[\frac{8}{15} BF^2 - \int_0^{F-B} dx \frac{x[(x+B)^2+E^2]^2}{e^{2\pi x}+1} \right.$$

$$\left. + \int_0^{F+B} dx \frac{x[(x-B)^2+E^2]^2}{e^{2\pi x}-1} \right] + W_1(\theta) \Bigg\} \tag{VIII.37}$$

where

$$W_1(\theta) = \frac{\alpha^2}{\pi} \left[\frac{1}{25} - 16 \int_0^\infty \frac{dy}{e^{2\pi y}+1} y(4y^2-1) \ln(4y^2+1) \right.$$

$$\left. - 2\int_0^\infty \frac{dy}{e^{2\pi y}+1} (16y^4-24y^2+1)\theta \right] + \frac{45\zeta(7)}{8\pi^6} \left(1 + \frac{2\alpha}{3} + \frac{\alpha^2}{5}\right)$$

$$+ \frac{\zeta(5)}{2\pi^4}(3+2\alpha+\alpha^2) + \frac{\zeta(3)}{\pi^2}\left(\frac{1}{4} + \frac{\alpha}{6} + \frac{7\alpha^2}{60}\right) \tag{VIII.38}$$

and

$$W_2(\theta) = \int_0^E dx \,(E^2-x^2)^2 \frac{B\sin(2\pi x) - x\sinh(2\pi B)}{\cosh(2\pi B) - \cos(2\pi x)}$$

$$+ \int_0^B dx \coth(\pi x) \left[(x-B)^2+E^2\right]^2 x \tag{VIII.39}$$

For $\alpha > 0$ similar computation with an additional term given by a Plana formula (VIII.23) leads to

$$V = \frac{1}{32\pi a^4} \Bigg\{ \frac{1}{2} \int_0^1 dy\,(1+\alpha y^2)^2 \int_0^A x^2(A^2-x^2)^2 \cot(\pi x)\,dx + W_1(\theta')$$

$$+ 2(1+\alpha)^2 \int_0^\infty \frac{W_2(y)}{e^{2\pi y}+1} - 32\alpha^2 \sum_{n=2}^\infty n \int_p^\infty dy \frac{(y^2-p^2)^2}{e^{i\pi(n-1)+2\pi y}-1} \Bigg\} \tag{VIII.40}$$

where $\tan(\theta'/2) = -2y$

All the integrals in (VIII.37) and (VIII.40) can be evaluated numerically. It turns out to be convenient for later analysis if we replace the scale a by the internal space volume via the relation

$$\Omega = \frac{2\pi^2 a^3}{\sqrt{1+\alpha}} \tag{VIII.41}$$

We can then write the effective potential as

$$V = \frac{Y(\alpha)}{\Omega^{4/3}} \qquad \text{(VIII.42)}$$

The graph of $Y(\alpha)$ is plotted in Figure below

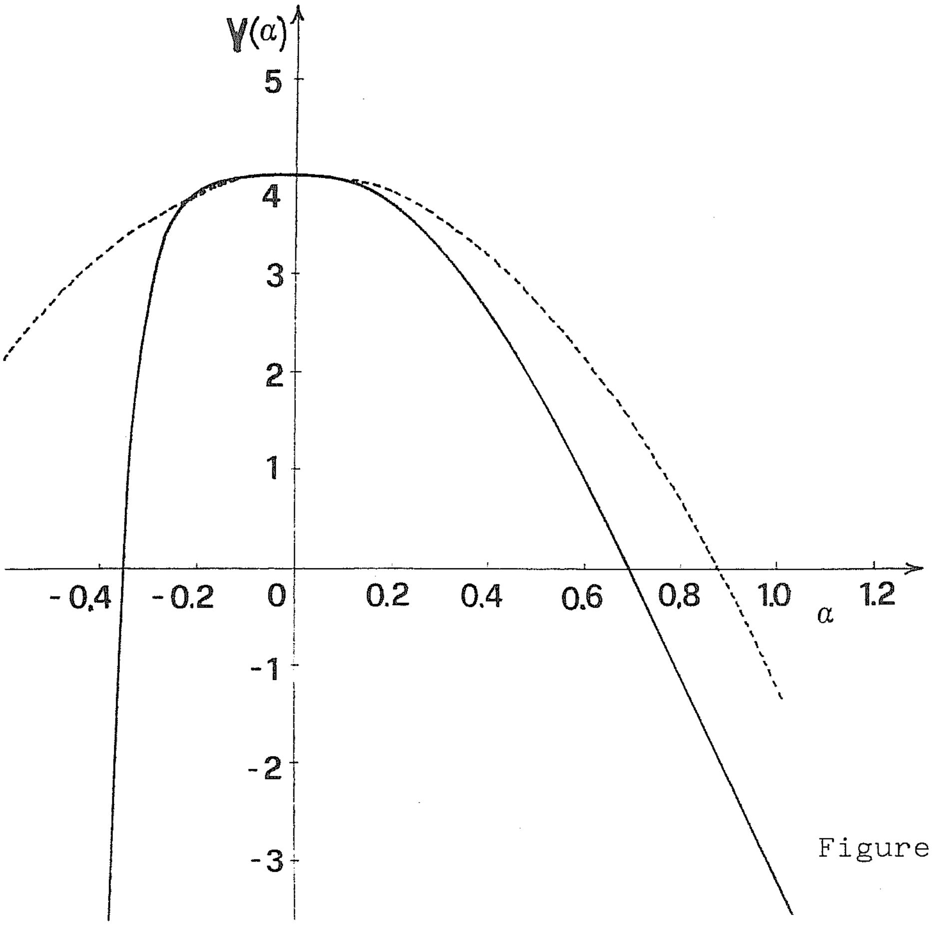

Figure 11

The dotted line denotes the small α perturbative result (VIII.18).

It is interesting to note that even if the topology of the manifold is still S_3, deformations in both prolate ($\alpha < 0$) and oblate ($\alpha > 0$) directions induce large negative Casimir energy similar to the well-known case of two parallel plates, which corresponds to the topology of S_1. The asymptotic behavior of $Y(\alpha)$ seems to be independent of the coupling constant of scalar field to gravity.

We have compared the present result with that of a small α perturbation. The perturbative calculation agrees with the exact one within 1% up to $|\alpha| = 0.14$.

D. Solutions of field equations

We have calculated the effective potential of a massless minimally coupled scalar field in a M_4 x Taub geometry without asking how to obtain this background in the first place. In this Section we shall show that certain configurations of the background geometry can be sustained by this potential.

The action in general can be written as

$$S = \int dV_x \int dV_y \left[\frac{\bar{R} + \bar{\Lambda}}{\varkappa^2} - \bar{V} \right] \tag{VIII.43}$$

where $\bar{V}$ is a D=7 dimensional matter effective potential. In particular, we are seeking a ground state solution with D=4 dimensional Poincarè invariance and a static homogeneous internal space. The action can hence be reduced to

$$S = \int dV_x \left[\frac{\Omega}{\varkappa^2} (\tilde{R} + \bar{\Lambda}) - V \right] \tag{VIII.44}$$

There is no kinetic term in the Lagrangian, so the total potential can be read off immediately

$$V_{tot} = \frac{1}{\varkappa^2} \left[-\Omega \tilde{R} - \Omega \bar{\Lambda} + \varkappa^2 V \right] \tag{VIII.45}$$

We can take (VIII.45) as a potential for a classical system with two dynamical variables: internal space volume Ω and deformation parameter α. $\bar{\Lambda}$ is a D=7 dimensional cosmological constant which will be determined by the value of Ω and α through (VIII.47). Static solutions must satisfy the following field equations

$$\left.\frac{\partial V_{tot}}{\partial \alpha}\right|_{\alpha_0 \Omega_0} = \left.\frac{\partial V_{tot}}{\partial \Omega}\right|_{\alpha_0 \Omega_0} = 0 \tag{VIII.46}$$

$$\left. V_{tot} \right|_{\alpha_0 \Omega_0} = 0 \tag{VIII.47}$$

Eqs. (VIII.46) and (VIII.47) are the (ij) and (00) components of Einstein equations, respectively. (VIII.47) signifies that cosmological constant in D=4 dimensions is zero. From the general form of the effective potential (VIII.45), we get the following set of algebraic equations

$$-\left(\frac{2\pi^2}{\Omega_0}\right)^{2/3} \frac{3+4\alpha_0}{3(1+\alpha_0)^{4/3}} - \frac{\bar{\Lambda}}{2} - \frac{2\bar{\varkappa}^2 Y}{3\Omega_0^{7/3}} = 0 \qquad \text{(VIII.48)}$$

$$\frac{(2\pi^2)^{2/3}}{3}\,\Omega_0^{1/3}\,\frac{4\alpha_0}{(1+\alpha_0)^{7/3}} + \frac{\bar{\varkappa}^2 Y'}{2\Omega_0^{4/3}} = 0 \qquad \text{(VIII.49)}$$

$$-(2\pi^2)^{2/3}\,\Omega_0^{1/3}\,\frac{3+4\alpha_0}{(1+\alpha_0)^{4/3}} - \frac{\bar{\Lambda}\Omega_0}{2} + \frac{\bar{\varkappa}^2 Y}{2\Omega_0^{4/3}} = 0 \qquad \text{(VIII.50)}$$

where

$$Y' = \left.\frac{dY}{d\alpha}\right|_{\alpha_0} \qquad \text{(VIII.51)}$$

From these equations we first determine the solution through

$$Y' = -\frac{14\alpha_0}{3(1+\alpha_0)(3+4\alpha_0)}\,Y \qquad \text{(VIII.52)}$$

Then, Ω_0 and $\bar{\Lambda}$ follow accordingly

$$\Omega_0^{5/3} = \frac{7\bar{\varkappa}^2 Y(\alpha_0)(1+\alpha_0)^{4/3}}{4(3+4\alpha_0)(2\pi^2)^{2/3}} \qquad \text{(VIII.53)}$$

$$\bar{\Lambda} = -\frac{5\bar{\varkappa}^2 Y(\alpha_0)}{2\,\Omega_0^{7/3}} \qquad \text{(VIII.54)}$$

From (VIII.45) we see that the D=4 dimensional Newton's constant is given by $\varkappa^2 = \bar{\varkappa}^2/\Omega$ (actually this is true only up to quantum corrections (Toms (1983)). Numerical calculation of (VIII.52) gives three solutions

$$\alpha_0 = 0\ , \quad -0.05941\ , \quad -0.1861 \qquad \text{(VIII.55)}$$

The total potential V_T is shown in Figure at the end of this Chapter, where $\bar{\Lambda}$ and Ω_0 are evaluated through (VIII.53) and (VIII.54) with $\alpha_0 = -0.05941$. For two other values of α, the shape of the potential V_T is similar except that the fine tuning effect of the cosmological constant shifts the value of V_T to zero at $\alpha = 0$ and -0.1861 respectively. It may be interesting to note that at $\alpha = 0$ $Y'/Y = 0$ for any matter fields in this background. This relation follows directly from (VIII.52).

Ω_0 as it stands in (VIII.53) leads to a scale $a_0 \sim 5 \times 10^{-3}$ lp (Planck length) which is beyond the validity regime of semi-classical approximation. One way to justify our result is to consider b scalar fields. The scale a_0 will increase with $b^{1/2}$, hovewer the shape α_0 is preserved.

E. Discussion

It is important to ask whether the ground-state solutions we find in Section VIII.D are stable. Unfortunately, for vacuum energy compactification models it is quite involved to calculate the response of the effective potential to an arbitrary metric perturbation. It seems that only $\alpha_0 = -0.05941$ solution is a reasonable candidate for a stable background configuration in linear perturbative sense. One must be careful, hovewer. If the system is reduced to one with only two degrees of freedom, coefficients at kinetic terms for α and Ω become important, so that quantum (through effective potential) corrections to them must be taken into account. The work in this direction is now in progress.

If any of the solutions of (VIII.52) is stable, it is a candidate for the ground state of the theory. Consider a general metric perturbation on this background. The $h_{\mu m}$ part of the perturbation in the direction of a Killing vector on S_3 can be identified as gauge field A_m. The gauge symmetry will be the same as the isometry of the internal space (see Chapter II), because there are no other fields to break this symmetry. For $\alpha \neq 0$ solution gauge symmetry is SU(2)xU(1) and hence there are four gauge fields only. Using the zero-mode ansatz it can be shown that masses gained by two other gauge fields (if $\alpha = 0$) are proportional to the deformation α

$$\propto \left(a\sqrt{1+\alpha} \right)^{-1} \tag{VIII.56}$$

Of course, to be realistic one should include fermions and graviton in the discussion. Also more dimensions are required if one wishes to consider more interesting models. Moreover, internal space should be (most probably) even-dimensional. This Chapter provides only the simplest possible example how a ground state of higher-dimensional theory with a non-maximally symmetric internal space geometry can arise and to investigate possible physical implications of that.

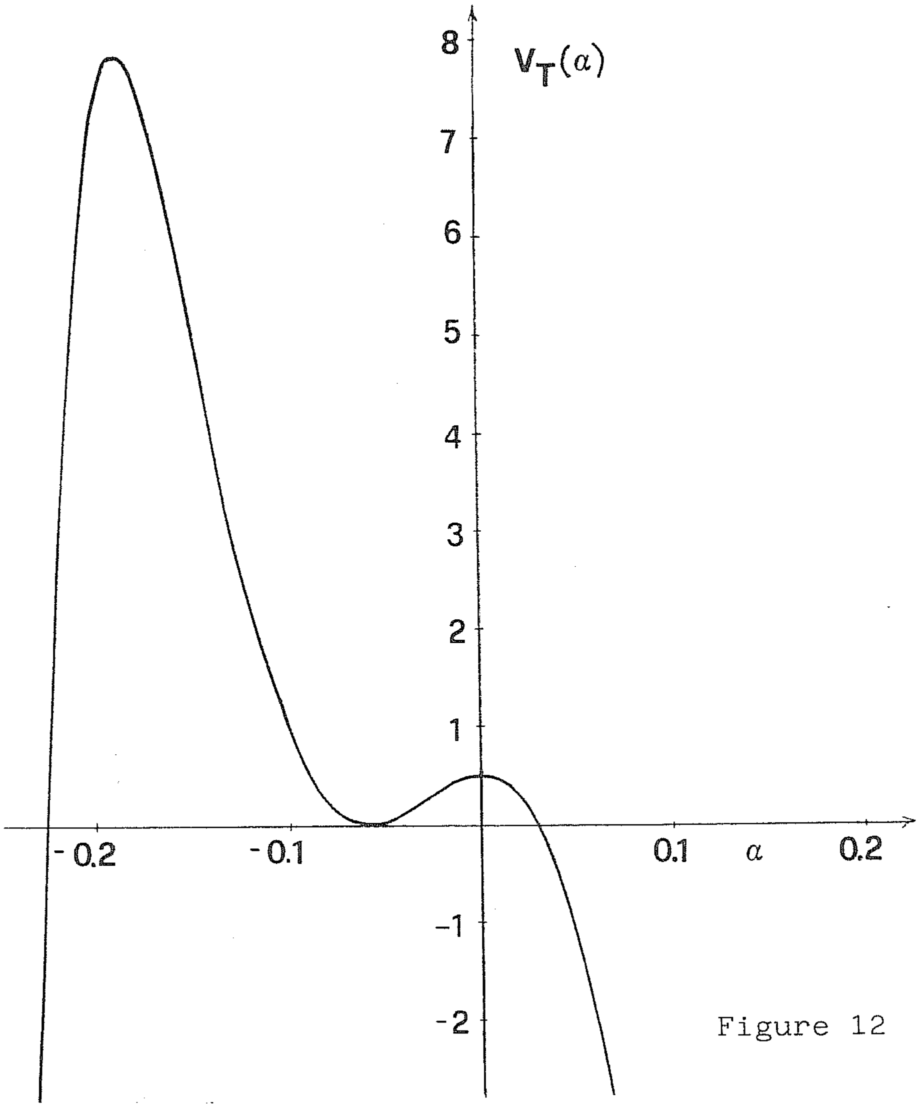

Figure 12

The total effective potential given by Eq (VIII.45) for the solution $\alpha = -0.05941$.

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