ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

RADIATION DAMPING IN CLOSED UNIVERSES

Thesis submitted for the degree of
Doctor Philosophiae

Candidate: Armando Bernui

Academic Year 1988 / 89

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I STUDI AVANZATI
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" Nelle questioni naturali .... la cognizione
degli effetti e' quella che ci conduce
all'investigazione e ritrovamento
delle cause "

G. Galilei

A mis padres y hermanas;
alle famiglie Soldati e Castellari;
a Maria Letizia, Fabio, Franco e Lorenzo;
a mis Profesores;
a Franco e Silvia;
a todos mis amigos cuyos nombres me gustaria
escribir aqui, pero toda la pagina no bastaria.

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Notation

μ,ν,η (Greg indices) = 0,1,2,3; m,n (Latin indices) = 1,2,3.

x^μ, x'^μ : four (or spacetime) - coordinates; $\underline{x}$, $\underline{x}'$: three (or space) - coordinates.

$\mathbb{R}$: the real line; $\mathbb{R}^+$: the real positive line without zero.

M : a four - dimensional manifold [11]; $\Omega \subset M$: an open set of M.

N : a three - dimensional manifold

S^3 : a three sphere.

$g = \det [g_{\mu\nu}]$; $h = \det [h_{mn}]$.

$d^4x \sqrt{-g}$ is the four-dimensional volume element.

$d^3x \sqrt{h}$ is the three-dimensional volume element.

C^n : space of functions which have the n-derivative continuous.

$I \equiv [t_0 , \infty) \subset \mathbb{R}$.

$F(t) \equiv dF(t) / dt$, $F \in C^1$; dot always denote derivative with respect to the (time) coordinate t.

$F'(s) \equiv dF / ds = \dot{F}\, dt / ds$, $F \in C^1$; prime denotes derivative with respect to the argument, except when used for set of coordinates or for constants.

$\partial_\mu \phi(x^\nu) \equiv \partial \phi(x^\nu) / \partial x^\mu$.

$\theta(x) = 0$, for $x < 0$; $\theta(x) = 1$, for $x \geq 0$.

rhs : right hand side; lhs : left hand side.

⌘ : indicates either the end of a Definition or the end of the proof of a Proposition.

INTRODUCTION

The controversy about the origin of irreversible processes in physics probably started with Einstein and Ritz in 1909 [1]. When the expansion of the universe was discovered, the idea arose that irreversibilities can originate from or at least be related with this cosmological event. Since then a lot of work has been done on this problem [2]. Part of such effort has been devoted to the search for the primary cause of irreversible processes in cosmology, studying the role that cosmological events play in the evolution of physical systems.

The present work results from an attempt to discover the (possible) relationship between the following irreversible features of a physical phenomenon: the radiation (i.e. emission of energy) of a harmonic oscillator and the expansion of the universe. More specifically we were motivated to investigate the validity of the conjecture that the expansion of the universe (with closed topology) could be sufficient to imply radiation damping.

With this purpose in mind we have studied the dynamics of a linear coupled particle-field model [3]. As embedding spacetimes, we have considered a class of Robertson-Walker (RW) spacetimes (we will be specially interested in those with closed topology, i.e. k=+1, and in monotone expansion), and we have assumed that the gravitational field is external (that is we neglect the back-reaction of the system on the metric of the spacetime). This coupled model has been previously studied for embedding spacetimes with open space slices [4-7] and for the Einstein universe [8], while an analysis of the RW (k=+1) spacetime case is unknown.

This physical system describes the interchange of energy between a non-relativistic harmonic oscillator Q (i.e. the particle) with uncoupled constant frequency ω_0, and a classical (i.e. unquantized) massless scalar (radiation) field ϕ; the system is embedded in a spacetime (M,g) of RW type, where $M = \mathbb{R} \times N$, (N,h) being the 3-space. The spacetime location of the oscillator is given by a form-factor ρ. The coupling parameter is the real positive constant λ. The evolution equations of this physical system are two coupled linear (each one in Q and ϕ) differential equations.

The strategy to study this model consists in eliminating from these coupled equations the field variables. Thus we arrive at an integro-differential equation in which only the oscillator variables appear: the so-called radiation reaction equation. The solutions of this equation (for arbitrary initial-data) provide the details of the radiation process of the particle and throw light on the role played by the topological-metrical properties of the spacetime. This will be the focus of our attention. Perhaps the most interesting application, due to its similarity with classical electrodynamics [9], arises when the interaction between field and particle is confined to a point: the point-like case . In such a case: $\rho \to \delta$ and the oscillator becomes a point-like particle. Thus, we shall restrict our attention to the study of the radiation reaction equation in the point-like case .

However, to correctly obtain the radiation reaction equation in the point-like limit $\rho \to \delta$, we have to proceed with care: in fact, due to the 1/r dependence (r represents the dimensions of the particle) of the field, this equation becomes singular in the point-like limit. To avoid this divergence one has to renormalize (to an infinite value) the oscillator's frequency: this procedure is called the renormalization problem [10]. Here we are interested in non-static spacetimes (specifically, expanding universes). If we consider a non-static spacetime, e.g. the RW k=0 (i.e. flat space sections) spacetime, instead of the (static) Minkowski spacetime, we notice that such a renormalization procedure does not exist. This fact is crucial since in such a case one fails to assign mathematical (and also physical) sense to the radiation reaction equation in the point-like limit.

However, we have to stress here that this inconvenience is model-dependent. By this we mean that one could modify the original model - e.g. (i) allowing the harmonic oscillator to have a time-dependent frequency (i.e. $\omega_0 = \omega_0(t)$) or (ii) allowing the coupling "constant" of the model to be time-dependent (i.e. $\lambda = \lambda(t)$) - in order to construct a correct renormalization procedure. However, these modifications of the coupled model do not seem to have physical interpretations. Therefore we have preferred not to consider them here. Moreover one could object that, because the renormalization problem arises in the limit of a point-like source ($\rho \rightarrow \delta$), why not consider an extended source? Unfortunately, the radiation reaction equation for extended sources in non-static spacetimes appears to be very complicated.

Therefore, since there is no renormalization procedure to obtain the radiation-reaction equation in the point-like limit ($\rho \rightarrow \delta$) for expanding (hence non-static) spacetimes, we have to conclude that our original plan fails.

However, performing this research we have found a class of linear second order discontinuous differential equations with functional arguments (also known in the literature as delay differential equations [18]) that are interesting *di per se*. As far as we know, the only time that these equations have appeared in the literature was in [8], where it is conjectured that these equations represent the radiation reaction equation in the point-like limit for the coupled model in non-static spacetimes with closed topology. However, due to the fact that there is no renormalization procedure to correctly obtain the radiation reaction equation in the point-like limit, we have been unable to prove this conjecture. Nevertheless, what is true about this class of equations is that a sub-class of it turns out to be the radiation-reaction equation for the Einstein universe case in the point-like limit (which is correctly obtained because this is a static spacetime).

Since equations of this type have not been previously discussed in the literature, we have constructed here an (original) approach to analyze the late time behavior (for arbitrary initial-data) of the solutions of these equations. This approach is based on the construction of a suitable family of Lyapunov functions which has been used to study the stability and asymptoticity properties of the zero solution. We have found that (whenever certain conditions are satisfied): (i) the zero solution is uniformly stable (u.s.), and (ii) the zero solution is globally uniformly asymptotically stable (g.u.a.s.).

The thesis has been arranged as follows.

Chapter I is devoted to the investigation of the renormalization problem in both static and non-static spacetimes. First we introduce the physical system: the coupled field-particle model in RW (k=0) spacetimes which is described by a pair of coupled equations. We do not consider here the initial-value problem of these equations, as it has been widely discussed in [4-7]. Eliminating the field variables from these equations we arrive at the radiation reaction equation for general ρ. Then we want to study the point-like limit $\rho \to \delta$ of this equation in two cases: for a static spacetime (Minkowski spacetime) and for a non-static spacetime (RW, k=0 spacetime). Our approach to perform the point-like limit ($\rho \to \delta$) in the Minkowski spacetime case consists in considering a δ-sequence of functions $\rho = \rho(\varepsilon)$, such that $\rho \to \delta$ when $\varepsilon \to 0$, and a suitable ε-family of coupled models, where the oscillator's frequency $\omega_0 = \omega_0(\varepsilon)$ is such that $\omega_0 \to \infty$ when $\varepsilon \to 0$. In this form we obtain an ε-family of radiation reaction equations and we formally show that this family converges to a limit ("renormalized") equation in which the oscillator's frequency has a finite renormalized value. Then we have attempted to carry out a similar procedure for the RW k=0 (a non-static) spacetime case, but in this case we have found that such a procedure does not exist.

In Chapter II we study the dynamics of the coupled model in RW k=+1 spacetimes ($M = \mathbb{R} \times S^3$), that is spacetimes with closed topology. To solve the field equation in terms of the oscillator variable we first calculate the retarded Green function of the Laplace-Beltrami operator in RW k=+1 spacetimes. We use this function to calculate the field solution and then to replace it in the oscillator equation and thus obtain the radiation reaction equation for a general source ρ. Then we restrict our attention to the Einstein universe (a static spacetime) and perform the point-like limit of the radiation-reaction equation (as we did in Chapter I). The result is a second order discontinuous differential equation with retarded arguments. In a suitable limit this equation reduces to the radiation reaction equation in the point-like limit for the Minkowski spacetime case.

We devote Chapter III to the study of the stability properties of the solutions of a class of linear second order differential equations which are parametrized by a real function (of real variable) $R = R(t)$. Perhaps the most important feature of these equations is that, putting the function R to be a constant, namely the constant radius of the Einstein universe, then this class of equations reduces to the radiation reaction equation obtained in Chapter II (for the coupled system embedded in the Einstein spacetime and for the point-like case: $\rho = \delta$). Regarding this class of equations, we first concentrate on the study of the initial-value problem. We then analyze the stability and asymptoticity of its zero solution. At the end of the Chapter we perform some computer simulations for different functions R, including the Einstein universe case: $R =$ constant.

I THE RENORMALIZATION PROBLEM IN STATIC AND NON-STATIC SPACETIMES

The aim of this Chapter is to study the point-like limit ($\rho \to \delta$) of the radiation reaction equation, with static and non-static embedding spacetimes. This is known as the renormalization problem.

In the first section we introduce the physical model. For the embedding spacetime we consider a class of RW, k=0 spacetimes, which are parametrized by a real function R. We obtain the evolution equations which are two coupled linear (in Q and ϕ) differential equations. Eliminating the field variables from these equations, we are left with an integro-differential equation in which only the particle variables appear: this is the so-called radiation-reaction equation.

In the second section we shall study the point-like limit $\rho \to \delta$ of this equation for the case in which the embedding spacetime is the Minkowski spacetime (which corresponds to the case R = 1 of the class of RW k=0 spacetimes). Our approach in performing the point-like limit consists in considering a δ-sequence of functions $\rho = \rho(\varepsilon)$, such that $\rho \to \delta$ when $\varepsilon \to 0$, and a suitable ε-family of coupled models, where the oscillator's frequency $\omega_0 = \omega_0(\varepsilon)$ is such that $\omega_0 \to \infty$ when $\varepsilon \to 0$. In this form we obtain an ε-family of radiation reaction equations. Then we formally show that this family converges to a limit "renormalized" equation in which the oscillator's frequency has a finite renormalized value. In this form we assign mathematical (and physical) meaning to the limit "renormalized" equation: in fact it represents the radiation reaction equation of the coupled system in Minkowski spacetime for the point-like case $\rho = \delta$. Finally we show a similar ε-convergency for the field variable.

In the third section we have attempted to perform a similar renormalization procedure for the RW k=0 (non-static) spacetime case. In this case we show that such a procedure does not exist.

I.1 THE PHYSICAL MODEL AND ITS EVOLUTION EQUATIONS

In this section we introduce the coupled physical system: harmonic oscillator - scalar field. We give the action from which we obtain the evolution equations. We discuss them as an initial-value problem. We eliminate the field variables from these equations and thus arrive at an equation in which only the oscillator's variables appear.

First of all we present the class of RW k=0 spacetimes. The RW line element with Lorentz signature (+,-,-,-) is [11]:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = dt^2 - R^2(t)\, d\underline{x}^2 \ , \qquad \text{(I.1)}$$

where $g_{\mu\nu} = g_{\mu\nu}(x^\eta)$, are the components of the metric; $x^\eta \in$ M and $\underline{x} \in$ N. R is the real function of real variable that parametrizes the class of RW spacetimes, R: $I \equiv [t_0, \infty) \rightarrow \mathbb{R}^+$, $R \in C^2(I)$; and $d\underline{x}^2$ is the line element of N:

$$d\underline{x}^2 = h_{mn}\, dx^m dx^n \ , \qquad \text{(I.2)}$$

where $h_{mn} = h_{mn}(x^j)$. Then

$$\sqrt{-g(t,\underline{x})} = R^3(t) \sqrt{h(\underline{x})} \ . \qquad \text{(I.3)}$$

The coupled model is described by the action [5] ($g = g(t,\underline{x})$):

$$\begin{aligned} S = {} & 1/2 \int d^4x \sqrt{-g}\,(\partial_\mu\phi\, \partial_\nu \phi\, g^{\mu\nu} - {}^1\!/_6\, \hat{R}\, \phi^2) \\ & + 1/2 \int dt\, (\dot{Q}^2 - \omega_0{}^2\, Q^2) + \lambda \int d^4x \sqrt{-g}\, \rho\, Q\, \phi \ , \end{aligned} \qquad \text{(I.4)}$$

where $Q = Q(t)$: $I \rightarrow \mathbb{R}$; $Q(t)$ represents the oscillating amplitude of the harmonic oscillator at the time t, ω_0 being the constant (i.e. time-independent) uncoupled frequency. ϕ is the scalar field, ϕ: $\Omega \subset M \rightarrow \mathbb{R}$. $\hat{R}$: $\Omega \rightarrow \mathbb{R}$ is the 4-dimensional scalar curvature of the spacetime. $\lambda \in \mathbb{R}^+$ is the coupling constant of the model. $\rho = \rho(t,\underline{x})$: $\Omega \to \mathbb{R}^+$ is the form-factor, which is chosen such that the time evolution of the universe does not afffect its size, that is

$$\int d^3x \sqrt{-g}(t,\underline{x})\, \rho(t,\underline{x}) = \int d^3x \sqrt{h(\underline{x})}\, \rho(\underline{x}) \ .$$

For simplicity we assume that ρ is normalized so that the value of this integral is 1, that is:

$$\rho(t,\underline{x}) = \rho(\underline{x}) \,/\, R^3(t) \ , \qquad \text{(I.5)}$$

where $\rho(\underline{x})$: $N \to \mathbb{R}^+$. Moreover we choose $\rho(\underline{x})$ to be a function both peaked at and spherically symmetric around the location of the oscillator ($r = |\underline{x}|$):

$$\rho(\underline{x}) = \rho(r) \,/\, 4\pi \ . \qquad \text{(I.6)}$$

Next we define the initial-data of the field and particle.

<u>Definition I.1.1</u> The pair,

$$(\psi_0(\underline{x}),\upsilon_0(\underline{x})) = (\phi(t,\underline{x}),\partial_t\phi(t,\underline{x}))\,|_{t=to} \ , \qquad \text{(I.7)}$$

$\forall\, \underline{x} \in N$, is called the initial-data of the field. We assume $\psi_0(\underline{x}),\upsilon_0(\underline{x}) \in C^\infty(N)$. The pair:

$$(Q_0,\, \dot{Q}_0) = (Q(t),\, \dot{Q}(t))\,|_{t=to} \ , \qquad \text{(I.8)}$$

is called the initial-data of the oscillator. ⌘

From the action (4) we obtain the coupled evolution equations of the system (with initial-data (7)-(8)):

$$(\Box_g + {}^1\!/_6\, \hat{R})\, \phi(t,\underline{x}) \;=\; \lambda\, \rho(t,\underline{x})\, Q(t) \ , \qquad \text{(I.9)}$$

$$\ddot{Q}(t) + \omega_0^2 Q(t) = \lambda \int d^3x \sqrt{h(\underline{x})}\, \rho(\underline{x})\, \phi(t,\underline{x}) \quad , \tag{I.10}$$

where $\Box_g \equiv (\sqrt{-g})^{-1} \partial_\mu (\sqrt{-g}\, g^{\mu\nu} \partial_\nu)$ is the covariant D'Alembertian operator and $\Box_g + {}^1\!/_6 \hat{R}$ is the Laplace-Beltrami (LB) operator.

Our strategy to solve this system of equations is the following: from equation (9), for given initial-data $(\psi_0(\underline{x}),\upsilon_0(\underline{x}))$, using Green's identities (see [9,12]), we solve ϕ in terms of Q: $\phi = \phi(Q)$; after that we replace $\phi(Q)$ into equation (10) to obtain an equation in which only Q-variables appear.

Consider the field equation (9) as an initial-value problem. Its solution for given initial-data (7) is given by:

$$\phi(t,\underline{x}) = \phi_I(t,\underline{x};t_0) + \phi_H(t,\underline{x};t_0) \quad . \tag{I.11}$$

ϕ_I is the inhomogeneous solution of (11) and is given by [12] $(q(t',\underline{x}') = \lambda Q(t')\, \rho_\varepsilon(t',\underline{x}'))$, where $t' \in [t_0,t]$, $t \in I$:

$$\phi_I(t,\underline{x};t_0) = \int d^4x' \sqrt{-g(t',\underline{x}')}\, \mathbf{G}(x^\mu,x'^\mu)\, q(t',\underline{x}') \quad . \tag{I.12}$$

ϕ_H is the homogeneous solution which is uniquely determined by the initial-data of the field; it is given by [12] $(\mathbf{G} = \mathbf{G}(x^\mu,x'^\mu))$, for $t \geq t_0$:

$$\phi_H(t,\underline{x};t_0) = \int d^3x' \sqrt{-g(t',\underline{x}')}\, \{\mathbf{G}\, \upsilon_0(\underline{x}') - \partial_{t'}\mathbf{G}\, \psi_0(\underline{x}')\}\, |_{t'=t_0} \quad , \tag{I.13}$$

where $\phi_H(t,\underline{x};t_0) \in C^\infty(M)$ (because $\psi_0(\underline{x}),\upsilon_0(\underline{x}) \in C^\infty(N)$). $\mathbf{G}$ is the retarded Green function which satisfies

$$(\Box_g + {}^1\!/_6 \hat{R})\, \mathbf{G}(x^\mu,x'^\mu) = \delta(t - t')\, \delta(\underline{x} - \underline{x}') / \sqrt{-g(t,\underline{x})} \quad . \tag{I.14}$$

Then using (11), (12) and (13) in (10) we obtain:

$$\ddot{Q}(t) + \omega_0^2 Q(t) = \lambda^2 \int dt'\, \Delta(t,t')\, Q(t') + \lambda\, f(t;t_0) \quad , \tag{I.15}$$

where

$$\Delta(t,t') \equiv \int d^3x \sqrt{h(\underline{x})}\, \rho(\underline{x})\, d^3x' \sqrt{h(\underline{x}')}\, \rho(\underline{x}')\, \mathbf{G}(x^\mu,x'^\mu) \ , \qquad (I.16)$$

$$f(t;t_0) \equiv \int d^3x \sqrt{h(\underline{x})}\, \rho(\underline{x})\, \phi_H(t,\underline{x};t_0) \ . \qquad (I.17)$$

I.2 THE RENORMALIZATION PROBLEM: MAKING SENSE OF THE POINT-LIKE LIMIT IN MINKOWSKI SPACETIME

In this section we shall study the point-like limit $\rho \to \delta$ of equation (15) for the case in which the embedding spacetime is the Minkowski spacetime (which corresponds to the case R = 1 of the class of RW k=0 spacetimes). Our approach to perform the point-like limit consists in considering a δ-sequence of functions $\rho = \rho(\varepsilon)$, such that $\rho \to \delta$ when $\varepsilon \to 0$, and a suitable ε-family of coupled models, where the oscillator's frequency $\omega_0 = \omega_0(\varepsilon)$ is such that $\omega_0 \to \infty$ when $\varepsilon \to 0$. In this form we obtain an ε-family of radiation reaction equations. Then we formally show that this family converges to a limit "renormalized" equation in which the oscillator's frequency has a finite renormalized value. In this form we assign mathematical (and physical) meaning to the limit "renormalized" equation: in fact, it represents the radiation reaction equation of the coupled system in Minkowski spacetime for the point-like case $\rho = \delta$. We also show a similar ε-convergency for the field variable. Then we conclude that the coupled system (Q,ϕ) has a correct ε-convergency when $\varepsilon \to 0$.

First note that the Minkowski spacetime is obtained by putting R = 1 in equations (1) and (3). To perform the ε-convergency it is useful to assume that the functions $\rho(r)$ are chosen to be positive definite δ-sequences: $\forall\, \varepsilon > 0$, $\rho(r) = \rho_\varepsilon(r) > 0$, $\rho_\varepsilon(r) \in C^\infty(\mathbb{R}^+)$ (e.g. $\rho_\varepsilon(r) = \exp(-r/\varepsilon)/\varepsilon^2 r$), such that ρ_ε converges to δ in the δ-sequence sense, i.e.

$$\lim_{\varepsilon\to 0} \int r^2\, dr\, \rho_\varepsilon(r)\, F(r) = \int dr\, \delta(r)\, F(r) = F(0) \ , \qquad (I.18)$$

for any function F that is continuous at r = 0.

<u>Definition I.2.1</u> The convergence:

$$\lim_{\varepsilon\to 0} \rho_\varepsilon(r) = \delta(r)/r^2 \ , \qquad (I.19)$$

in the δ-sequence sense (see (18)) is called <u>the point-like limit</u> . This convergency will be simply denoted by: $\rho_\varepsilon \to \delta$. By this limit we mean physically that the extended source ρ_ε converges to the point-like source δ when $\varepsilon \to 0$ (in the δ-sequence sense) [13]. ⌘

The retarded Green function **G** in Minkowski spacetime for $t > t_0$, is [12]:

$$\mathbf{G}(t,t',|\underline{x} - \underline{x}'|) = [1/4\pi|\underline{x} - \underline{x}'|]\ \theta(t - t')\ \delta(t - t' - |\underline{x} - \underline{x}'|) \ , \qquad (I.20)$$

(we observe that it is singular at $\underline{x} = \underline{x}'$). We now calculate $\int dt'\Delta(t,t')Q(t')$ by using (20) in (16) ($\Delta = \Delta_\varepsilon$, $f = f_\varepsilon$ from now on):

$$\int dt'\Delta_\varepsilon(t,t')Q(t') = \int d^3x\sqrt{h(\underline{x})}\rho_\varepsilon(\underline{x})\ d^3x'\sqrt{h(\underline{x}')}\rho_\varepsilon(\underline{x}')\ [1/4\pi|\underline{x}- \underline{x}'|]\ Q(t - |\underline{x} - \underline{x}'|).$$

Performing the point-like limit $\varepsilon \to 0$ (in the δ-sequence sense) we obtain :

$$\lim_{\varepsilon\to 0} \int dt'\ \Delta_\varepsilon(t,t')\ Q(t') = \int dr\ \delta(r)\ [1/4\pi r]\ Q(t - r)$$

$$= - [1/4\pi]\ \dot{Q}(t) + [1/4\pi]\int dr\ \delta(r)\ [1/r]\ Q(t) \ ,$$

where we observe that the last term diverges (it is formally infinite), and

$$\lim_{\varepsilon\to 0} f_\varepsilon(t;t_0) = \int d^3x \sqrt{h(\underline{x})}\rho_\varepsilon(\underline{x})\ \phi_H(t,\underline{x};t_0) = \phi_H(t,\underline{0};t_0) \equiv f_0(t;t_0) \ , \qquad (I.21)$$

uniformly in t, for $t > t_0$. Therefore equation (15) in the limit $\varepsilon \to 0$ turns out to be ($2\Gamma \equiv \lambda^2/4\pi$), $t > t_0$:

$$\ddot{Q}(t) + \omega_0^2\ Q(t) = - 2\Gamma\ \dot{Q}(t) + 2\Gamma Q(t) \int dr\ \delta(r)\ [1/r] + \lambda\ f_0(t;t_0) \ , \qquad (I.15_\infty)$$

where we observe that the term $2\Gamma\int dr\ \delta(r)\ [1/r] \equiv \omega_\infty{}^2$ is formally infinite. The way to get around from this divergence problem is to think of $\omega_0{}^2$ as an epsilon-dependent frequency: $\omega_0{}^2 = \omega_0{}^2(\varepsilon)$ such that $\omega_0{}^2(\varepsilon) \rightarrow \infty$ when $\varepsilon \rightarrow 0$, and to think of $\omega_{ren}{}^2 \equiv \omega_0{}^2 - \omega_\infty{}^2$ as the (finite positive) "renormalized frequency". Thus, equation (15_∞) converts into

$$\ddot{Q}(t) + 2\Gamma\ \dot{Q}(t) + \omega_{ren}{}^2\ Q(t) = \lambda\ f_0(t;t_0) \quad , \qquad (I.15_{ren})$$

and one is motivated to regard a solution of (15_{ren}) as a renormalized solution to equation (15_∞) with renormalized frequency $\omega_{ren}{}^2$. However this notion is meaningless unless one gives a precise connection between solutions of (15) (valid $\forall\ \varepsilon > 0$, with $0 < \omega_0{}^2 < \infty$) and solutions of ($15_{ren}$) (valid for $\omega_{ren}{}^2 > 0$, i.e. $\omega_0{}^2 = \infty$). Before considering this connection we shall need the following definitions.

Definition I.2.2 Consider an epsilon-family of equations (15) with $\omega_0{}^2 = \omega_0{}^2(\varepsilon) > 0$, $\forall\varepsilon > 0$, for $t > t_0$ (we denote $Q = Q_\varepsilon$ due to the dependence on ε):

$$\ddot{Q}_\varepsilon(t) + \omega_0{}^2(\varepsilon)\ Q_\varepsilon(t) = \lambda^2\int dt'\ \Delta_\varepsilon(t,t')\ Q_\varepsilon(t') + \lambda\ f_\varepsilon(t;t_0) \ . \qquad (I.15_\varepsilon)$$

Fix the initial-data $(Q_0, \dot{Q}_0)$ and $f_\varepsilon(t;t_0)$. The function $Q_\varepsilon = Q_\varepsilon(t): I \rightarrow \mathbb{R}$, which satisfies identically (15_ε) and at $t = t_0$ satisfies: $(Q_\varepsilon(t_0), \dot{Q}_\varepsilon(t_0)) = (Q_0, \dot{Q}_0), \forall\varepsilon > 0$, is called a solution of (15_ε), for $t > t_0$.

Consider similarly equation (15_{ren}) with $\omega_{ren}{}^2 > 0$:

$$\ddot{Q}_{ren}(t) + 2\Gamma\ \dot{Q}_{ren}(t) + \omega_{ren}{}^2\ Q_{ren}(t) = \lambda\ f_0(t;t_0) \ , \qquad (I.15_{ren})$$

fix the initial-data $(Q_0, \dot{Q}_0)$ and $f_0(t;t_0) = \phi_H(t,\underline{0};\ t_0)$. The function $Q_{ren} = Q_{ren}(t)$: $I \rightarrow \mathbb{R}$, which satisfies identically equation (15_{ren}) and for $t = t_0$ satisfies: $(Q_{ren}(t_0), \dot{Q}_{ren}(t_0)) = (Q_0, \dot{Q}_0)$ is called a solution of (15_{ren}) or a renormalized solution, for $t > t_0$. ⌘

The procedure to bypass the ∞ - divergence can now be formulated. We have to consider an ε - family of equations (15) instead of just ONE equation

(15): this family of equations is (15_ε). Then we have to show that the correct limit of these equations, for $\varepsilon \to 0$, is (15_{ren}) and not (15_∞). In this form we give mathematical (and physical) meaning to the above notion of a renormalized solution. We shall do this in the following proposition.

Proposition I.2.3 Fix $\omega_{ren}^2 > 0$, $(Q_0, \dot{Q}_0)$ and $(\psi_0(\underline{x}),\upsilon_0(\underline{x}))$ (and hence $\phi_H(t,\underline{x};t_0)$ is determined, see (13)). Define

$$\omega_0^2(\varepsilon) \equiv \omega_{ren}^2 + 2\Gamma \int d^3x \sqrt{h(\underline{x})}\, \rho_\varepsilon(\underline{x})\, d^3x' \sqrt{h(\underline{x}')}\, \rho_\varepsilon(\underline{x}')\, [1/|\,\underline{x} - \underline{x}'|\,] , \quad (I.22)$$

(note that $\omega_0^2(\varepsilon) \to \infty$ when $\varepsilon \to 0$). Then the solution $Q_\varepsilon(t)$ converges uniformly to the solution $Q_{ren}(t)$ when $\varepsilon \to 0$, i.e.

$$\lim_{\varepsilon \to 0} Q_\varepsilon(t) = Q_{ren}(t) \ , \quad (I.23)$$

uniformly in t, for $t > t_0$.

Proof. First note that

$$\lambda^2 \int dt' \Delta_\varepsilon(t,t') Q_\varepsilon(t') = 2\Gamma \int d^3x \sqrt{h(\underline{x})} \rho_\varepsilon(\underline{x}) d^3x' \sqrt{h(\underline{x}')} \rho_\varepsilon(\underline{x}') [Q_\varepsilon(t - |\underline{x} - \underline{x}'|)/|\underline{x} - \underline{x}'|]. \quad (I.24)$$

Define: $P(t) \equiv Q_\varepsilon(t) - Q_{ren}(t)$; then substracting ($15_\varepsilon$) - ($15_{ren}$), and using (22) and (24) we obtain :

$$\ddot{P}(t) + \omega_{ren}^2 P(t) - 2\Gamma \dot{Q}_{ren}(t)$$

$$= 2\Gamma \int d^3x \sqrt{h(\underline{x})}\, \rho_\varepsilon(\underline{x}) d^3x' \sqrt{h(\underline{x}')}\, \rho_\varepsilon(\underline{x}') [(Q_\varepsilon(t - |\underline{x} - \underline{x}'|) - Q_\varepsilon(t)) / |\underline{x} - \underline{x}'|] + \lambda\, [f_\varepsilon(t;t_0) - f_0(t;t_0)] \ . \quad (I.25)$$

Adding to both sides of (25) the term: $2\Gamma \int d^3x \sqrt{h(\underline{x})}\, \rho_\varepsilon(\underline{x}) d^3x' \sqrt{h(\underline{x}')} \rho_\varepsilon(\underline{x}')\, \dot{Q}_\varepsilon(t)$, (25) converts into:

$$\ddot{P}(t) + 2\Gamma \dot{P}(t) + \omega_{ren}^2 P(t)$$

$$= 2\Gamma \int d^3x \sqrt{h(\underline{x})}\, \rho_\varepsilon(\underline{x}) d^3x' \sqrt{h(\underline{x}')} \rho_\varepsilon(\underline{x}')\{ \dot{Q}_\varepsilon(t) - ([Q_\varepsilon(t) - Q_\varepsilon(t - |\underline{x} - \underline{x}'|)]/|\underline{x} - \underline{x}'|)\}$$
$$+ \lambda\, [f_\varepsilon(t;t_0) - f_0(t;t_0)] \ . \qquad (I.26)$$

Observe that equation (26) is an inhomogeneous ordinary differential equation. Its solution is given by $P = P_H + P_I$, where the homogeneous solution $P_H = P_H(t)$ satisfies:

$$\ddot{P}_H(t) + 2\Gamma\, \dot{P}_H(t) + \omega_{ren}^2\, P_H(t) = 0 \ .$$

Because equation (15_ε) and (15_{ren}) has the same initial-data, equation (26) has zero initial-data, that is: $(P(t_0), \dot{P}(t_0)) = (0,0)$. This homogeneous equation is well-known [14-16]; the unique solution with zero initial-data, is $P_H(t) = 0$, for all $t \geq t_0$.

Now consider the inhomogeneous solution; it is given by ($s \in [t_0,t]$, $t < \infty$) [15] :

$$P_I(t) = \int ds\, G(t - s)\, F_\varepsilon(s) \ + \lambda \int ds\, G(t - s)\, [f_\varepsilon(s;t_0) - f_0(s;t_0)] \ , \qquad (I.27)$$

where

$$F_\varepsilon(t) \equiv 2\Gamma \int d^3x \sqrt{h(\underline{x})} \rho_\varepsilon(\underline{x}) d^3x' \sqrt{h(\underline{x}')} \rho_\varepsilon(\underline{x}')\{ \dot{Q}_\varepsilon(t) - ([Q_\varepsilon(t) - Q_\varepsilon(t - |\underline{x} - \underline{x}'|)]/|\underline{x} - \underline{x}'|)\}, \qquad (I.28)$$

and G is the Green function of the differential operator $D^2 + 2\Gamma D + \omega_{ren}^2 I$ and is given by [13-16]: $G(t - s) = \omega_1^{-1}\{\sin[\omega_1(t - s)] \exp[-\Gamma(t - s)]\}$, where $\omega_1^2 \equiv \omega_{ren}^2 - \Gamma^2$. We consider first the integral $\int ds\, G(t - s)\, [f_\varepsilon(s;t_0) - f_0(s;t_0)]$:

$$\left|\int ds\, G(t - s)\, [f_\varepsilon(s;t_0) - f_0(s;t_0)]\right| \leq \int ds\, |G(t - s)|\, |f_\varepsilon(s;t_0) - f_0(s;t_0)|$$

$$\leq \sup_{z \in [t_0,t]} |f_\varepsilon(z;\, t_0) - f_0(z;\, t_0)| \int ds\, |G(t - s)|$$

$$= |f_\varepsilon(t^*;\, t_0) - f_0(t^*;\, t_0)| \int ds\, |G(t - s)| \leq (\Gamma\omega_1)^{-1} |f_\varepsilon(t^*;\, t_0) - f_0(t^*;\, t_0)| \ , \qquad (I.29)$$

where $\sup_{z \in [t_0,t]} |f_\varepsilon(z; t_0) - f_0(z; t_0)| = |f_\varepsilon(t^*; t_0) - f_0(t^*; t_0)|$, for $t^* \in [t_0,t]$, and $\forall\, t \in I$. Since $\lim_{\varepsilon \to 0} |f_\varepsilon(t; t_0) - f_0(t; t_0)| = 0$ uniformly in t, we use this result in (29) to obtain, in the limit when $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} |\int ds G(t-s)[f_\varepsilon(s;t_0) - f_0(s;t_0)]| \leq (\Gamma\omega_1)^{-1} \lim_{\varepsilon \to 0} |f_\varepsilon(t^*;t_0) - f_0(t^*;t_0)| = 0, \tag{I.30}$$

for $t^* \in [t_0,t]$, $\forall\; t \in I$, that is uniformly in t. Similarly, we consider the term $\int ds\, G(t - s)\, F_\varepsilon(s)$:

$$| \int ds\, G(t - s)\, F_\varepsilon(s) | \leq \int ds\, |G(t - s)|\, |F_\varepsilon(s)|$$

$$= \sup_{z \in [t_0,t]} | F_\varepsilon(z) | \int ds\, |G(t - s)| \leq (\Gamma\omega_1)^{-1} | F_\varepsilon(s^\wedge) | \;, \tag{I.31}$$

for some $s^\wedge \in [t_0,t]$, where $\sup_{z \in [t_0,t]} | F_\varepsilon(z) | = | F_\varepsilon(s^\wedge) |$. To analyze $| F_\varepsilon(s^\wedge) |$ in the limit $\varepsilon \to 0$, consider the following δ - sequence: $\rho_\varepsilon(r) = 0$, for $r > \varepsilon$; $\rho_\varepsilon(r) = 1/(\varepsilon r^2)$, for $0 \leq r < \varepsilon$, which is C^∞ everywhere except at $r = \varepsilon$ [13]. In such a case we observe that: $|\underline{x} - \underline{x}'| < |\underline{x}| + |\underline{x}'| < 2\varepsilon$. Moreover:

$$|\int d^3x\sqrt{h(\underline{x})}\, \rho_\varepsilon(\underline{x}) d^3x'\sqrt{h(\underline{x}')}\, \rho_\varepsilon(\underline{x}')\{([Q_\varepsilon(t) - Q_\varepsilon(t - |\underline{x} - \underline{x}'|)] / |\underline{x} - \underline{x}'|)\} |$$

$$\leq \sup_{\sigma \in [0,2\varepsilon]} | ([Q_\varepsilon(t) - Q_\varepsilon(t - \sigma)] / \sigma) | \int d^3x\sqrt{h(\underline{x})}\, \rho_\varepsilon(\underline{x}) d^3x'\sqrt{h(\underline{x}')}\, \rho_\varepsilon(\underline{x}')$$

$$= \sup_{\sigma \in [0,2\varepsilon]} | ([Q_\varepsilon(t) - Q_\varepsilon(t - \sigma)] / \sigma) | \;.$$

Therefore $\forall\, \varepsilon > 0, \exists\, \delta = 2\varepsilon > 0$, such that

$$| \dot{Q}_\varepsilon(t) - \int d^3x\sqrt{h(\underline{x})}\rho_\varepsilon(\underline{x}) d^3x'\sqrt{h(\underline{x}')}\rho_\varepsilon(\underline{x}')\{([Q_\varepsilon(t) - Q_\varepsilon(t - |\underline{x} - \underline{x}'|)]/|\underline{x} - \underline{x}'|)\} | < \varepsilon,$$

whenever $|\underline{x} - \underline{x}'| < |\underline{x}| + |\underline{x}'| < 2\varepsilon = \delta$. In the limit, and using the definition of F_ε (see (28)):

$\lim_{\varepsilon\to 0}$ [$|F_\varepsilon(t)| / 2\Gamma$]

$$= \lim_{\varepsilon\to 0} | \dot{Q}_\varepsilon(t) - \int d^3x\sqrt{h(\underline{x})}\, \rho_\varepsilon(\underline{x}) d^3x'\sqrt{h(\underline{x}')}\, \rho_\varepsilon(\underline{x}')\{([Q_\varepsilon(t) - Q_\varepsilon(t - |\underline{x} - \underline{x}'|)] / |\underline{x} - \underline{x}'|)\} | = 0. \tag{I.32}$$

$\forall\, t \in$ I. Therefore from (31) and (32):

$$\lim_{\varepsilon\to 0} | \int ds\, G(t - s)\, F_\varepsilon(s) | = 0 \ , \tag{I.33}$$

$\forall\, t \in$ I, that is uniformly in t. Thus, using (30) and (33) in (27) we conclude that: $\lim_{\varepsilon\to 0} |P_I(t)| = 0$, uniformly in t. Since $P_H(t) = 0$, $\forall\, t \in$ I, then $\lim_{\varepsilon\to 0} |P(t)| = 0$ uniformly in t. Therefore:

$$\lim_{\varepsilon\to 0} Q_\varepsilon(t) = Q_{ren}(t) \ ,$$

uniformly in t, as we claim. ⌘

Now, we want to complete the programm of analysis of the point-like limit (renormalization problem) of the coupled system, studying the convergence of the field.

Proposition I.2.4 Given the initial-data (7) for the equation (9), with $\rho = \rho_\varepsilon$:

$$\Box_g\, \phi_\varepsilon(t,\underline{x}) = \lambda\, \rho_\varepsilon(t,\underline{x})\, Q_\varepsilon(t) \ , \tag{I.9}$$

then: $\lim_{\varepsilon\to 0} \phi_\varepsilon(t,\underline{x}) = \phi_0(t,\underline{x})$, $\forall\, t \in$ I, where

$$\phi_0(t,\underline{x}) = (\lambda\, /4\pi r)\, \theta(t - t_0 - r)\, Q_{ren}(t - r) + \phi_H(t,\underline{x};t_0) \ .$$

and $Q_{ren} = Q_{ren}(t)$ is the solution of (15_{ren}), with initial-data (8).

Proof. Using (20) in (12) we obtain (remember that: $\sqrt{-g}(t',\underline{x}') = R^3(t')\sqrt{h(\underline{x}')} = R^3(t')\, r'^2 \sin\theta'$, $t' \in [t_0,t]$, $q(t',\underline{x}') = \lambda\, Q(t')\rho_\varepsilon(t',\underline{x}')$, $\rho_\varepsilon(t,\underline{x}) = \rho_\varepsilon(\underline{x})/R^3(t) = \rho_\varepsilon(r)/4\pi\, R^3(t)$):

$$\phi_{I\varepsilon}(t,\underline{x};t_0) = \int dt'd^3x'R^3(t')r'^2\sin\theta'\ [1/4\pi|\underline{x}-\underline{x}'|]\ \theta(t-t')\ \delta(t-t'-|\underline{x}-\underline{x}'|)\ \lambda Q_\varepsilon(t')\rho_\varepsilon(t',\underline{x}')$$

$$= \int dt'\ d^3x'\ R^3(t')\ r'^2\sin\theta'\ [1/4\pi|\underline{x}-\underline{x}'|]\ \theta(t-t')\theta(t'-t_0)\ \delta(t-t'-|\underline{x}-\underline{x}'|)\ .$$
$$.\ \lambda\ Q_\varepsilon(t')\ [\rho_\varepsilon(r')/4\pi\ R^3(t')]$$

$$= \lambda \int d^3x'\ r'^2\sin\theta'\ [1/4\pi|\underline{x}-\underline{x}'|]\ \theta(t - t_0 - |\underline{x}-\underline{x}'|)\ Q_\varepsilon(t - |\underline{x}-\underline{x}'|)\ [\rho_\varepsilon(r')/4\pi]\ ,$$

that is,

$$\phi_{I\varepsilon}(t,\underline{x};t_0) = (\lambda/4\pi) \int dr'\ r'^2\ [1/|\underline{x}-\underline{x}'|]\theta(t - t_0 - |\underline{x}-\underline{x}'|)\ Q_\varepsilon(t-|\underline{x}-\underline{x}'|)\rho_\varepsilon(r')\ . \quad (I.34)$$

In the limit $\varepsilon \to 0$, using (18) in (34), we obtain:

$$\lim_{\varepsilon\to 0} \phi_{I\varepsilon}(t,\underline{x}) = (\lambda/4\pi)\ \lim_{\varepsilon\to 0} \int dr'r'^2[1/|\underline{x}-\underline{x}'|]\ \theta(t-t_0-|\underline{x}-\underline{x}'|)Q_\varepsilon(t - |\underline{x}-\underline{x}'|)\rho_\varepsilon(r')$$

$$= (\lambda/4\pi)\ \lim_{\varepsilon\to 0} \int dr'\ [1/|\underline{x}-\underline{x}'|]\ \theta(t-t_0-|\underline{x}-\underline{x}'|)Q_\varepsilon(t - |\underline{x}-\underline{x}'|)\ \delta(r')$$

$$= (\lambda/4\pi)\ [1/r]\ \theta(t - t_0 - r)\ Q_\varepsilon(t - r)\ ,$$

since $[\ \theta(t - t_0 - |\underline{x}-\underline{x}'|)Q_\varepsilon(t - |\underline{x}-\underline{x}'|) / |\underline{x}-\underline{x}'|\ ]$ is continuous at $r' = 0$, whenever $(t - t_0) > r$ (see (18)). For the case $(t - t_0) = r$, continuity conditions are necessary (for a detailed discussion of this point see [5]). Therefore, since $\phi_\varepsilon(t,\underline{x}) = \phi_{I\varepsilon}(t,\underline{x};t_0) + \phi_H(t,\underline{x};t_0)$, ($\phi_H(t,\underline{x};t_0)$ depends on the initial-data and does not depends on ε, see (13)) we finally obtain: $\lim_{\varepsilon\to 0} \phi_\varepsilon(t,\underline{x}) = \phi_0(t,\underline{x})$, $\forall\ t \in I$. ⌘

Proposition I.2.5 Given the initial-data (7)-(8) for the coupled system (9)-(10), with $\rho = \rho_\varepsilon$:

$$\Box_g\ \phi_\varepsilon(t,\underline{x}) = \lambda\ \rho_\varepsilon(t,\underline{x})\ Q_\varepsilon(t)\ , \quad (I.9_\varepsilon)$$

$$\ddot{Q}_\varepsilon(t) + \omega_0^2(\varepsilon)Q_\varepsilon(t) = \lambda \int d^3x\sqrt{h(\underline{x})}\ \rho_\varepsilon(\underline{x})\ \phi_\varepsilon(t,\underline{x})\ , \quad (I.10_\varepsilon)$$

then: $\lim_{\varepsilon\to 0} (Q_\varepsilon(t),\phi_\varepsilon(t,\underline{x})) = (Q_{ren}(t),\phi_0(t,\underline{x})),\ \forall\ t \in I$, where

$$\phi_0(t,\underline{x}) = (\lambda\ /4\pi r)\ \theta(t - t_0 - r)\ Q_{ren}(t - r) + \phi_H(t,\underline{x};t_0)\ ,$$

and $Q_{ren} = Q_{ren}(t)$ satisfy equation (15_{ren}).

Proof. This follows from Propositions I.2.3 and I.2.4. ⌘

I.3 THE POINT-LIKE LIMIT IN RW k=0 SPACETIME

Now let us analyze the point-like limit in the RW (k=0) spacetime: $N = \mathbb{R}^3$, $R = R(t): I \to \mathbb{R}^+$. In this case the retarded Green function is [5,12,13]:

$$\mathbf{G}(t,t',|\ \underline{x} - \underline{x}'|) = [1/4\pi|\underline{x} - \underline{x}'|R(t)\ ]\ \theta(t - t')\ \delta(t' - f^{-1}(f(t) - |\underline{x} - \underline{x}'|))\ , \qquad (I.35)$$

where $\tau : [0,\infty) \to \mathbb{R}^+$, is defined by :

$$\tau \equiv f(t) \equiv \int_{t_0}^{t} ds\ /\ R(s)\quad .$$

Then, similar to the above discussion, we are attempted to define

$$\omega_0{}^2(\varepsilon) \equiv \omega_{ren}{}^2 + 2\Gamma \int d^3x \sqrt{h(\underline{x})}\ \rho_\varepsilon(\underline{x})\ d^3x'\sqrt{h(\underline{x}')}\ \rho_\varepsilon(\underline{x}')\ [1/|\ \underline{x} - \underline{x}'|R(t)\ ]\ ,$$

but now $\omega_0{}^2(\varepsilon)$ should be time dependent $\omega_0{}^2(\varepsilon) = \omega_0{}^2(\varepsilon,t)$, which is not allowed for the models here considered. Thus, there is no analogue of Proposition I.2.3 for non-static spacetimes.

II THE COUPLED MODEL IN RW SPACETIMES WITH CLOSED TOPOLOGY

In this Chapter we study the dynamics of the coupled model introduced in Chapter I, but now considering as background a class of RW k=+1 spacetimes (M = $\mathbb{R} \times S^3$, that is spacetimes with closed topology). The new feature now is that the retarded Green function of the Laplace-Beltrami operator in this class of spacetimes is unknown. One of the main results of this Chapter is the calculation of this function. The other principal result is the calculation of the radiation reaction equation in the point-like limit in the case in which these RW k=+1 spacetimes go over to (because this is a particular case) the Einstein universe.

In the first section we present the physical system and the evolution equations. We discuss the general solution as an initial-value problem. Eliminating from these equations the field variables we obtain the radiation reaction equation, for general ρ in which only the oscillator's variables appear (as in the RW k=0 spacetimes).

We devote the second section to the calculation of the retarded Green function of the Laplace-Beltrami operator in RW k=+1 spacetimes. To perform this more easily we define a new time variable: $\tau \equiv \int ds / R(s)$, where R is the real function that parametrizes the class of RW spacetimes.

In the third section we restrict our attention to the Einstein universe (a static spacetime), which is obtained from the class of RW k=+1 spacetimes by taking R = constant (= the radius of the Einstein universe). Then we perform the point-like limit of the radiation reaction equation. Since the Einstein universe is a static spacetime, a similar procedure can be followed to perform this limit correctly. The result is a very interesting second order linear discontinuous differential equation with retarded arguments [18-21]. An interesting feature of this equation is that, in the suitable limit $R \to \infty$ we recover the radiation reaction equation in the point-like limit for the Minkowski spacetime case.

II.1 THE PHYSICAL MODEL AND ITS EVOLUTION EQUATIONS

Here we consider the evolution equations of the couple model in RW, k=+1 spacetimes (i.e. $N = S^3$ then $M = \mathbb{R} \times S^3$). The line element of this class of RW spacetimes is given by:

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = dt^2 - R^2(t)\,\{d\alpha^2 + \sin^2\alpha\,[d\theta^2 + \sin^2\theta d\varphi^2]\}\ , \qquad (II.1)$$

where $g_{\mu\nu} = g_{\mu\nu}(x^\eta)$; $t \in I \equiv [t_0, \infty)$, $\alpha \in [0,\pi]$, $\theta \in [0,\pi]$, $\varphi \in [0,2\pi]$, $R: I \to \mathbb{R}^+$, $R \in C^2(I)$. We denote $x^\eta = (t,\alpha,\theta,\varphi) \in M$, $\underline{x} = (\alpha,\theta,\varphi) \in S^3$. Then

$$\sqrt{-g(t,\underline{x})} = R^3(t)\,\sqrt{h(\underline{x})} = R^3(t)\,\sin^2\alpha\,\sin\theta\quad . \qquad (II.2)$$

If we let $R(t) = R_0$ = constant then (1) describes the line element of the Einstein universe.

The system is described by the action ($g = g(t,\underline{x})$):

$$S = 1/2 \int d^4x\,\sqrt{-g}\,(\,\partial_\mu\phi\,\partial_\nu\phi\,g^{\mu\nu} - 1/_6\,\hat{R}\,\phi^2\,) + 1/2 \int dt\,(\,\dot{Q}^2 - \omega_0{}^2\,Q^2\,) + \lambda \int d^4x\,\sqrt{-g}\,\rho\,Q\,\phi\quad , \qquad (II.3)$$

As in Chapter I, we assume that the form-factor $\rho = \rho(t,\underline{x})$: $\Omega \subset M \to \mathbb{R}^+$, is chosen such that the time evolution of the universe does not afffect its size, that is

$$\int d^3x\,\sqrt{-g(t,\underline{x})}\;\rho(t,\underline{x}) = \int d^3x\,\sqrt{h(\underline{x})}\;\rho(\underline{x})\quad .$$

For simplicity we assume that ρ is normalized so that the value of this integral is 1, therefore we choose: $\rho(t,\underline{x}) = \rho(\underline{x}) / R^3(t)$, where $\rho(\underline{x})$: $S^3 \to \mathbb{R}^+$. Moreover we choose $\rho(\underline{x})$ as a function peaked at and spherically symmetric around the location of the oscillator ($r = |\underline{x}|$): $\rho(\underline{x}) = \rho(r) / 4\pi$.

From (3), the coupled equations of motion of the system are:

$$(\Box_g + {}^1/_6\,\hat{R})\,\phi(t,\underline{x}) = \lambda\,\rho(t,\underline{x})\,Q(t) \quad , \tag{II.4}$$

$$\ddot{Q}(t) + \omega_0{}^2\,Q(t) = \lambda \int d^3x \sqrt{h(\underline{x})}\,\rho(\underline{x})\,\phi(t,\underline{x}) \quad , \tag{II.5}$$

where $\Box_g \equiv (\sqrt{-g})^{-1}\,\partial_\mu\,(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu)$ is the covariant D'Alembertian operator, $\Box_g + {}^1/_6\,\hat{R}$ is the Laplace-Beltrami (LB) operator, $\hat{R}$ is the scalar curvature of M. The initial-data for the field is:

$$(\phi(t,\underline{x}),\partial_t\phi(t,\underline{x}))\,|_{t=to} = (\psi_0(\underline{x}),\upsilon_0(\underline{x})) \quad , \tag{II.6}$$

$\forall\, \underline{x} \in S^3$, where $\psi_0(\underline{x})$, $\upsilon_0(\underline{x}) \in C^\infty(S^3)$; and the initial-data for the oscillator is:

$$(Q(t),\,\dot{Q}(t))\,|_{t=to} = (Q_0,\,\dot{Q}_0) \quad . \tag{II.7}$$

We write the solution ϕ of (7) in the form

$$\phi(t,\underline{x}) = \phi_I(t,\underline{x};t_0) + \phi_H(t,\underline{x};t_0) \quad , \tag{II.8}$$

where ϕ_I is the inhomogeneous solution and ϕ_H is the homogeneous solution, that is it satisfies

$$(\,\Box_g + {}^1/_6\,\hat{R}\,)\,\phi_H(t,\,\underline{x};\,t_0) = 0 \quad . \tag{II.9}$$

Using Green's identities [9,12], the ϕ_H for $t \geq t_0$, is given by (we denote : $\mathbf{G} = \mathbf{G}(x^\mu,x'^\mu)$):

$$\phi_H(t,\underline{x};\,t_0) = \int d^3x'\,\sqrt{-g(t',\underline{x}')}\,\{\,\mathbf{G}\,\upsilon_0(\underline{x}') - \partial_{t'}\mathbf{G}\,\psi_0(\underline{x}')\,\}\,|_{t'=to} \quad . \tag{II.10}$$

To calculate the first integral we need the retarded Green function of the LB operator: $\mathbf{G} = \mathbf{G}(x^\mu,x'^\mu)$, and initial-data of ϕ, at $t = t_0$. Similarly, the inhomogeneous solution ϕ_I for $t \geq t_0$, is given by:

$$\phi_I(t,\underline{x};t_0) = \int d^4x' \sqrt{-g(t',\underline{x}')}\, \mathbf{G}(x^\mu,x'^\mu)\, q(t',\underline{x}') \quad , \qquad \text{(II.11)}$$

where $\underline{x}' \in S^3$, $t' \in [t_0,t]$; $q(t',\underline{x}') = \lambda\, Q(t')\, \rho(t',\underline{x}')$, q represents the source or inhomogeneity of the differential equation (4).

Now, writing $\phi = \phi_H + \phi_I$, with ϕ_H given in (10) and ϕ_I given in (11), and substituting in the oscillator's equation (5) we obtain:

$$\ddot{Q} + \omega_0^2 Q = \lambda \int d^3x \sqrt{-g(t,\underline{x})}\rho(t,\underline{x}) \int d^4x' \sqrt{-g(t',\underline{x}')}\, \lambda\, \mathbf{G}(t,t',|\alpha - \alpha'|)\, Q(t')\, \rho(t',\underline{x}')$$
$$+ \lambda \int d^3x \sqrt{-g(t,\underline{x})}\;\; \rho(t,\underline{x})\, \phi_H(t,\underline{x};t_0) \quad . \qquad \text{(II.12)}$$

This equation is called the radiation-reaction equation of the coupled system.To analyze the solutions of this equation we need to calculate the retarded Green function **G**.

II.2 THE RETARDED GREEN FUNCTION IN RW k=+1

Now let us calculate the Green's function of the LB operator $\mathbf{G} = \mathbf{G}(x^\mu,x'^\mu)$, x^μ, $x'^\mu \in \Omega$ (that is, for $t > t_0$). For the RW spacetimes this Green function is more easily calculated if we define a new time-coordinate, the so-called proper time τ :

$$\tau \;\equiv\; f_0(t) \;\equiv\; \int_{t_0}^{t} ds / R(s) \quad , \qquad \text{(II.13)}$$

$\tau : [0,\infty) \to \mathbb{R}^+$ (in the Einstein universe case: R = constant, then $\tau = (t - t_0)/R$). Then, the line element (1) in $(\tau,\alpha,\theta,\varphi)$-coordinates reads:

$$ds^2 = \overline{R}^2(\tau)\, (d^2\tau \; - d^2\alpha - \sin^2\alpha\, (d^2\theta + \sin^2\theta \;\; d^2\varphi)) \quad , \qquad \text{(II.14)}$$

where $\overline{R}(\tau) \equiv R(f^{-1}(\tau))$. Then the metric in these coordinates takes the form: $\overline{g}_{\mu\nu} = \text{diag}(\overline{R}^2(\tau), -\overline{R}^2(\tau), -\overline{R}^2(\tau)\sin^2\alpha, -\overline{R}^2(\tau)\sin^2\alpha\, \sin^2\theta)$. Let us define $\overline{\mathbf{G}}(\tau,\underline{x},\tau',\underline{x}') \equiv \mathbf{G}(t,\underline{x},t',\underline{x}')$, where the bar means that the function has to be considered a function of τ instead of t . In RW spacetimes $\hat{R}$ is given by :

$1/_6\,\hat{R} = (1/\overline{R}^3(\tau))\,[\overline{R}(\tau)'' + k\overline{R}(\tau)]$, where $\overline{R}'(\tau) \equiv d\,\overline{R}(\tau)\,/\,d\tau$. Then for RW k = +1, we have

$$1/_6\,\hat{R} = (1/\overline{R}^3(\tau))[\overline{R}(\tau)'' + \overline{R}(\tau)] \quad , \qquad \text{(II.15)}$$

(in the Einstein universe case: $k = +1$, $R(\tau)' = R(\tau)'' = 0$, we have $1/_6\,\hat{R} = (1/R^2)$).

Proposition II.2.1 Let $\overline{G} = \overline{G}(\tau,\tau',|\underline{x} - \underline{x}'|)$ be the Green function in $(\tau,\alpha,\theta,\varphi)$ -coordinates that satisfies :

$$(\Box_g + 1/_6\,\hat{R})_\tau\, \overline{G}(\tau,\tau',|\underline{x} - \underline{x}'|) = [1/2\pi|\underline{x} - \underline{x}'|^2\, \overline{R}^2(\tau)\overline{R}^2(\tau')]\ \delta(\tau-\tau')\delta(|\underline{x}-\underline{x}'|), \quad \text{(II.16)}$$

then :

$$\overline{G}(\tau,\tau',|\alpha - \alpha'|) = [1\,/\,4\pi\,\overline{R}(\tau)\overline{R}(\tau')\,\sin|\alpha - \alpha'|]\ \sum_{n=0}^{\infty}\ \delta(\tau - \tau' - |\alpha - \alpha'| - 2\pi n)$$
$$- \ \delta(\tau - \tau' + |\alpha - \alpha'| - 2\pi n)\ ] \quad . \qquad \text{(II.17)}$$

Proof. Take for simplicity $x'^\mu = (0,0,0,0)$, then $|\underline{x} - \underline{x}'| = |\underline{x}| = \sin\alpha$. From the definition

$$(\Box_g + 1/_6\,\hat{R})_\tau \equiv (1/\overline{R}^4(\tau))[\,\partial_\tau\,\overline{R}^2(\tau)\partial_\tau\,] - (1/\overline{R}^2(\tau)\sin^2\alpha)\ [\,\partial_\alpha \sin^2\alpha\ \partial_\alpha\,]$$
$$+ (1/\overline{R}^3(\tau))[\overline{R}(\tau)'' + \overline{R}(\tau)] \quad . \qquad \text{(II.18)}$$

Then combining equations (18) and (16) we obtain:

$$\{(1/\overline{R}^4(\tau))(\partial_\tau\,\overline{R}^2(\tau)\partial_\tau) - (1/\,\overline{R}^2(\tau)\sin^2\alpha)\ (\partial_\alpha \sin^2\alpha\ \partial_\alpha)$$
$$+ (\,\overline{R}''(\tau) + \overline{R}(\tau))/\,\overline{R}^3(\tau)\}\ \overline{G}(\tau,\alpha) = \ [1/2\pi\overline{R}(0)\overline{R}^3(\tau)\ \sin^2\alpha]\ \delta(\tau)\ \delta(\alpha) \quad . \quad \text{(II.19)}$$

We note that :

$$(\partial_\tau\,\overline{R}^2(\tau)\partial_\tau)\ \overline{G} = 2\ \overline{R}(\tau)\ \overline{R}'(\tau)\ \partial_\tau\overline{G} + \overline{R}^2(\tau)\ \partial_\tau^2\ \overline{G} \quad , \qquad \text{(II.20)}$$

and

$$(\partial_\alpha \sin^2\alpha\, \partial_\alpha)\, \overline{\mathbf{G}} = 2\sin\alpha\, \cos\alpha\, \partial_\alpha \overline{\mathbf{G}} + \sin^2\alpha\, \partial_\alpha{}^2\, \overline{\mathbf{G}}\ . \qquad \text{(II.21)}$$

If we define: $\tilde{\mathbf{G}}(\tau,\alpha) \equiv 4\pi R_0 \overline{R}(\tau)\sin\alpha\, \overline{\mathbf{G}}(\tau,\alpha)$, where $R_0 \equiv \overline{R}(0) \equiv R(t_0)$ we can transform (20) and (21) into, respectively:

$$\begin{aligned}(\partial_\tau\, \overline{R}^2(\tau)\partial_\tau)\, \overline{\mathbf{G}} &= [\ 1/4\pi R_0 \sin\alpha\ ]\ \{\ 2\, \overline{R}'\, \partial_\tau \tilde{\mathbf{G}} - 2\, \overline{R}'^2\, (\tilde{\mathbf{G}} / \overline{R}\,) + \\ &\quad \overline{R}\, \partial_\tau{}^2\, \tilde{\mathbf{G}} - 2\, \overline{R}'\, \partial_\tau\, \tilde{\mathbf{G}} - \overline{R}''\, \tilde{\mathbf{G}} + 2\, \overline{R}'^2\, (\tilde{\mathbf{G}} / \overline{R}\,)\ \}\ , \\ &= [\ 1/4\pi\ \overline{R_0}\, \sin\alpha\ ]\ \{\ \overline{R}\, \partial_\tau{}^2\, \tilde{\mathbf{G}}\ -\ \overline{R}''\, \tilde{\mathbf{G}}\ \}\ ,\end{aligned} \qquad \text{(II.22)}$$

$$\begin{aligned}(\partial_\alpha \sin^2\alpha\, \partial_\alpha)\, \overline{\mathbf{G}} &= [1/4\pi\ \overline{R_0 R}\,]\ \{\ 2\cos\alpha\, \partial_\alpha\, \tilde{\mathbf{G}} - 2\cos^2\alpha\, (\tilde{\mathbf{G}} / \sin\alpha) + \sin\alpha\, \partial_\alpha{}^2\, \tilde{\mathbf{G}} \\ &\quad - 2\cos\alpha\, \partial_\alpha\, \tilde{\mathbf{G}} + \sin\alpha\, \tilde{\mathbf{G}} + 2\cos^2\alpha\, (\tilde{\mathbf{G}} / \sin\alpha)\ \} \\ &= [\ 1/4\pi\ \overline{R_0}\ \overline{R}\,]\ \{\ \sin\alpha\, \partial_\alpha{}^2\, \tilde{\mathbf{G}} + \sin\alpha\, \tilde{\mathbf{G}}\ \}\ .\end{aligned} \qquad \text{(II.23)}$$

Therefore, using (22) and (23) the lhs of (19) converts into :

$$\begin{aligned}&\{(1/\overline{R}^4(\tau))(\partial_\tau\, \overline{R}^2(\tau)\partial_\tau) - (1/\, \overline{R}^2(\tau)\sin^2\alpha)\, (\partial_\alpha \sin^2\alpha\, \partial_\alpha) \\ &\quad + (\overline{R}''(\tau) + \overline{R}(\tau))/\overline{R}^3(\tau)\}\ (\ \tilde{\mathbf{G}}(\tau,\alpha) / 4\pi\ \overline{R_0 R}\, \sin\alpha)\end{aligned}$$

$$\begin{aligned}&= (1/\overline{R}^4(\tau))\ [1/4\pi\ \overline{R_0}\, \sin\alpha]\ (\ \overline{R}\, \partial_\tau{}^2\, \tilde{\mathbf{G}} - \overline{R}''\, \tilde{\mathbf{G}}\ ) - (1/\overline{R}^2(\tau)\, \sin^2\alpha)\ [1/4\pi\ \overline{R_0}\ \overline{R}\,] \\ &\quad (\sin\alpha\, \partial_\alpha{}^2\, \tilde{\mathbf{G}} + \sin\alpha\, \tilde{\mathbf{G}}) + [\ \tilde{\mathbf{G}}\ /4\pi\ \overline{R_0 R}\, \sin\alpha\ ]\ \{(\overline{R}''(\tau) + \overline{R}(\tau))/\overline{R}^3(\tau)\}\end{aligned}$$

$$=\ [1/4\pi\ \overline{R_0}\ \overline{R}^3(\tau)\, \sin\alpha]\ (\ \partial_\tau{}^2 - \partial_\alpha{}^2\ )\ \tilde{\mathbf{G}}(\tau,\alpha)\ . \qquad \text{(II.24)}$$

On the other hand, the rhs of (19), using $\delta(\alpha) / \sin\alpha = -\, \delta'(\alpha)$ converts into:

$$[\ -\ 1 / 2\pi\ \overline{R_0}\, \overline{R}^3(\tau)\ \sin\alpha\ ]\ \delta(\tau)\ \delta'(\alpha)\ . \qquad \text{(II.25)}$$

Therefore from (24) and (25) we transform (19) into:

$$(\ \partial_\tau{}^2 - \partial_\alpha{}^2\ )\ \tilde{\mathbf{G}}(\tau,\alpha)\ =\ -\ 2\ \delta(\tau)\ \delta'(\alpha)\ . \qquad \text{(II.26)}$$

To solve this equation it is convenient to choose a better set of coordinates; for this we define: $\zeta \equiv \alpha - \tau$, $\sigma \equiv \alpha + \tau$, then the Green function in the new coordinates is: $\breve{G}(\gamma,\sigma) \equiv \tilde{G}(\tau,\alpha)$. In these coordinates we have:

$$\partial_\tau{}^2 - \partial_\alpha{}^2 = - 4\, \partial_\zeta\, \partial_\sigma \;;$$

and

$$\delta(\tau)\, \delta'(\alpha) = \; 2\, [\, \delta'(\zeta)\, \delta(\sigma) + \delta(\zeta)\, \delta'(\sigma)\,] \; .$$

Thus, equation (26) transforms into :

$$\partial_\zeta\, \partial_\sigma\, \breve{G}(\zeta,\sigma) = \; [\, \delta'(\zeta)\, \delta(\sigma) + \delta(\zeta)\, \delta'(\sigma)\,] \; . \qquad \text{(II.27)}$$

Now we can integrate this equation, where the limits of integration are chosen in such a way that the initial conditions for $\overline{G}$ are satisfied [12]; for this we note that the initial conditions to be satisfied by $\overline{G}$ are that $\overline{G}(\tau,\alpha)$ and $\partial_\tau\, \overline{G}(\tau,\alpha)$ has to be zero for $\tau < 0$. Note, from (24), that the contributions to $\overline{G}$ come only when $\zeta \equiv \alpha - \tau = 0$ <u>or</u> $\sigma \equiv \alpha + \tau = 0$ (we said <u>or</u> and not <u>and</u> because they can not be simultaneously zero, unless $\alpha = \tau = 0$ which is a point). Since $\alpha \in [0,\pi]$, $\zeta \equiv \alpha - \tau = 0$, for $\tau \in [0,\pi)$, $\sigma = \alpha + \tau - 2\pi = 0$, for $\tau \in [\pi,2\pi)$, and also $\zeta = \alpha - \tau + 2\pi = 0$, for $\tau \in [2\pi,3\pi)$, etc; in general $\zeta = \alpha - \tau + 2\pi n = 0$, for $\tau \in [2\pi n, 2\pi n + \pi)$, and $\sigma = \alpha + \tau - 2\pi n = 0$, for $\tau \in [2\pi n+\pi,\, 2\pi(n+1))$, $n = 0,1,2,\ldots$. Then integrating (27) between the appropriate limits, we obtain :

$$\breve{G}(\zeta,\sigma) \;=\; \Big\{ \int d\zeta' \int d\sigma'\, \delta'(\zeta')\, \delta(\sigma') \;+\; \int d\zeta' \int d\sigma'\, \delta(\zeta')\, \delta'(\sigma') \Big\}$$

$$+ \Big\{ \int d\zeta' \int d\sigma'\, \delta'(\zeta')\, \delta(\sigma') \;+\; \int d\zeta' \int d\sigma'\, \delta(\zeta')\, \delta'(\sigma') \Big\} + \ldots;$$

that is:

$$\breve{G}(\zeta,\sigma) \; = \theta(\sigma)\, \delta(\zeta) + \theta(\zeta + 2\pi)\, \delta(\sigma - 2\pi) + \theta(\sigma - 2\pi)\, \delta(\zeta + 2\pi) + \ldots;$$

or equivalently,

$$\breve{\mathbf{G}}(\zeta,\sigma) = \sum_{n=0}^{\infty} \{ \theta(\sigma - 2\pi n)\, \delta(\zeta + 2\pi n) - \theta(\zeta + 2\pi n)\, \delta(\sigma - 2\pi n) \}.$$

Returning to the (τ,α)-coordinates, the Green function takes the form:

$$\overline{\mathbf{G}}(\tau,\alpha) = [1/4\pi\, \overline{R_0}\, \overline{R(\tau)} \sin\alpha] \sum_{n=0}^{\infty} \{\theta(\tau + \alpha - 2\pi n)\, \delta(\tau - \alpha - 2\pi n) - \theta(\alpha - \tau + 2\pi n)\, \delta(\tau + \alpha - 2\pi n)\}$$

$$= [1/4\pi\, \overline{R_0}\, \overline{R(\tau)} \sin\alpha] \sum_{n=0}^{\infty} \{ \delta(\tau - \alpha - 2\pi n) - \delta(\tau + \alpha - 2\pi n)\} \quad . \qquad \text{(II.28)}$$

This Green function $\mathbf{G}$ is interpreted as a pulse (or wave) of unitary amplitude originated at a point-like source located at $x'^{\mu} = (0,0,0,0)$ which propagates along geodesics of the spacetime; in a spacetime with open space slices this pulse expands forever, whereas in a spacetime with closed spatial sections (like S^3) waves go through the space and turn around (as it does in S^1 or S^2, which provides better visualization of this fact). As a consequence the Green function of the LB operator contains the contributions of many delta-"functions" each one being valid for precise intervals of τ, i.e. for $\tau \in [\pi n, \pi(n+1)]$, $n = 0,1,2,....$ Note that the infinite sum is actually a finite sum for $\tau < \infty$, due to the properties of the δ-"functions".

From (27), requiring $\overline{\mathbf{G}}$ to satisfy the so-called reciprocity conditions (see [12]) we extend its validity to arbitrary x'^{μ}, and taking into account that $\overline{\mathbf{G}}(\tau,\tau',\alpha,\alpha') = \overline{\mathbf{G}}(\tau,\tau',|\alpha - \alpha'|)$, we finally obtain:

$$\overline{\mathbf{G}}(\tau,\tau',|\alpha - \alpha'|) = [1 / 4\pi\, \overline{R(\tau)}\, \overline{R(\tau')} \sin|\alpha - \alpha'|\,] \sum_{n=0}^{\infty} \delta(\tau - \tau' - |\alpha - \alpha'| - 2\pi n) - \delta(\tau - \tau' + |\alpha - \alpha'| - 2\pi n)\,] \,, \qquad \text{(II.29)}$$

as we claim . ⌘

From (29) we can obtain the Green function in (t,α) - coordinates $(\mathbf{G}(t,t',\alpha,\alpha') = \mathbf{G}(t,t',|\alpha - \alpha'|))$:

$$\mathbf{G}(t,t',|\alpha - \alpha'|) = [1/4\pi R(t)R(t') \sin|\alpha - \alpha'|] \sum_{n=0}^{\infty} [\delta(f(t) - f(t') - |\alpha - \alpha'| - 2\pi n) - \delta(f(t) - f(t') + |\alpha - \alpha'| - 2\pi n)] \ . \qquad \text{(II.30)}$$

To end this section we shall proceed to calculate the homogeneous solution ϕ_H. For this we replace the Green function **G** given in (30) into equation (10), $t \geq t_0$:

$$\phi_H(t,\underline{x};\, t_0) = \int d^3x' \sqrt{-g}(t',\underline{x}') \{ \mathbf{G}\ \upsilon_0(\underline{x}') - \partial_{t'}\mathbf{G}\ \psi_0(\underline{x}') \} |_{t'=t_0} \ . \qquad \text{(II.10)}$$

From this we observe that: $\phi_H(t,\underline{x};t_0) \in C^\infty(M)$. It is convenient to define:

$$\gamma_0(\sin\alpha) \equiv (R_0 \sin\alpha/4\pi) \int d\Omega\ \psi_0(\underline{x}) \ , \qquad \text{(II.31)}$$

$$\beta_0(\sin\alpha) \equiv d/dt\, [(R(t) \sin\alpha/4\pi) \int d\Omega\ \phi(t,\underline{x})] |_{t\,=\,t_0} \ . \qquad \text{(II.32)}$$

Note that, because $\psi_0(\underline{x}), \upsilon_0(\underline{x}) \in C^\infty(S^3)$ then $\gamma_0(\sin\alpha), \beta_0(\sin\alpha) \in C^\infty(S^3)$. Using these definitions we can eliminate the angular degrees of freedom in the integrals in (10), obtaining after some calculations, for $t \geq t_0$:

$$\phi_H(t,\underline{x};t_0) = (1/R(t)) \int d\alpha' \, [\sin\alpha'/ \sin|\alpha - \alpha'|] \{R_0\beta_0(\sin\alpha')\delta(|f(t) - 2\pi N| - |\alpha - \alpha'|) + \gamma_0(\sin\alpha')\delta'(|f(t) - 2\pi N| - |\alpha - \alpha'|)\}, \qquad \text{(II.33)}$$

where $N \equiv [(f(t) + \pi)/2\pi] = 0,1,2,...$(where [s] means the maximum integer of s).

In the point-like limit case $\alpha = 0$, for $t \geq t_0$, we obtain:

$$\phi_H(t,\underline{0};t_0) = (1/R(t))\ [R_0\beta_0(\sin|f(t) - 2\pi N|) + \gamma'_0(\sin|f(t) - 2\pi N|)] \ . \qquad \text{(II.34)}$$

From (34) we observe that:

$$\phi_H(t,\underline{0};t_0) = [1/R(t)]\ P(f(t)) \ , \qquad \text{(II.35)}$$

for $t \geq t_0$, where $P: [0,\infty) \rightarrow \mathbb{R}$, is a periodic function of $f(t)$, with period 2π. As we will see in the next section, $\lambda\phi_H$ represents the inhomogeneous term of the radiation reaction equation (12).

II.3 THE POINT-LIKE LIMIT FOR THE EINSTEIN UNIVERSE

In this section we apply the results of the previous section to the point-like limit case: $\rho \to \delta$, for the Einstein universe case (i.e. $R = R_0$ = constant). The renormalization problem appears here because the Green function has the 1/r dependence. However, since the Einstein universe is static, we can similarly assign mathematical meaning and physical significance to the equations of motion in the same way as we did for the point-like case in Minkowski spacetime (see section I.2). Therefore, in this section: $R = R_0$ = constant.

Let us start calculating the Green function (30) for the case $R = R_0$. For this, notice that $f(t) = (t - t_0) / R_0$. Then

$$\mathbf{G}(t,t',|\alpha - \alpha'|) = [1/4\pi R_0 \sin|\alpha - \alpha'|] \sum_{n=0}^{\infty} [\delta(t - t' - |\alpha - \alpha'|R_0 - 2\pi n R_0) - \delta(t - t' + |\alpha - \alpha'|R_0 - 2\pi n R_0)] \quad . \qquad \text{(II.36)}$$

Now, it is convenient to assume that the functions $\rho(r)$ are chosen to be positive-definite δ-sequences: $\forall\, \varepsilon > 0$, $\rho(r) = \rho_\varepsilon(r) > 0$, $\rho_\varepsilon(r) \in C^\infty(\mathbb{R}^+)$ (a.e. $\rho_\varepsilon(r) = \exp(-r/\varepsilon) / \varepsilon^2 r$), such that ρ_ε converges to δ in the δ-sequence sense, i.e.

$$\lim_{\varepsilon\to 0} \int r^2\, dr\, \rho_\varepsilon(r)\, F(r) = \int dr\, \delta(r)\, F(r) = F(0) \quad , \qquad \text{(II.37)}$$

for any function F that is continuous in $r = 0$.

Our aim is to evaluate (12), with $\rho = \rho_\varepsilon$:

$$\ddot{Q} + \omega_0^2 Q = \lambda \int d^3x \sqrt{-g(t,\underline{x})}\rho(t,\underline{x}) \int d^4x' \sqrt{-g(t',\underline{x}')}\, \lambda\, \mathbf{G}(t,t',|\alpha - \alpha'|)\, Q(t')\rho(t',\underline{x}') + \lambda \int d^3x \sqrt{-g(t,\underline{x})}\, \rho(t,\underline{x})\, \phi_H(t,\underline{x};t_0) \quad , \qquad \text{(II.12)}$$

in the limit where ρ converges to δ in the δ-sequence sense. To calculate the rhs of (12), we first calculate the integral:

$$U = U_\varepsilon(t,\alpha) \equiv \int d^4x' \sqrt{-g(t',\underline{x}')}\ \mathbf{G}(t,t',|\alpha - \alpha'|)\ \rho_\varepsilon(t',\underline{x}')\ Q(t')\ . \qquad (II.38)$$

Introducing the Green function given in (36), and the information about the form-factor: $\rho(t,\underline{x}) = \rho(\underline{x})/R_0{}^3 = \rho(\alpha)/4\pi R_0{}^3$, we have

$$U = [1/4\pi R_0] \sum_{n=0}^{\infty} \int d^4x' \sqrt{-g(t',\underline{x}')}Q(t')\ \rho_\varepsilon(t',\underline{x}')\ [1/\sin|\alpha - \alpha'|\ ]\ [\ \delta(t - t' - |\alpha - \alpha'|R_0 - 2\pi n R_0) - \delta(t - t' + |\alpha - \alpha'|R_0 - 2\pi n R_0)]$$

$$= [1/16\pi^2 R_0] \sum_{n=0}^{\infty} \int d^4x' \sqrt{h(\underline{x}')}\ \ Q(t')\ \rho_\varepsilon(\alpha')\ [1/\sin|\alpha - \alpha'|]\ [\ \delta(t - t' - |\alpha - \alpha'|R_0 - 2\pi n R_0) - \delta(t - t' + |\alpha - \alpha'|R_0 - 2\pi n R_0)]$$

$$= [1/4\pi R_0] \int d\alpha' \sin^2\alpha'\ \rho_\varepsilon(\alpha')\ [1/\sin|\alpha - \alpha'|]\ [\ Q(t - |\alpha - \alpha'|R_0)$$
$$+ [1/4\pi R_0] \sum_{n=1}^{\infty} \int d\alpha' \sin^2\alpha'\ \rho_\varepsilon(\alpha')\ [1/\sin|\alpha - \alpha'|]\ [\ Q(t - |\alpha - \alpha'|R_0 - 2\pi n R_0) - Q(t + |\alpha - \alpha'|R_0 - 2\pi n\ R_0)\ ]\ .$$

Then in the point-like limit: $\lim_{\varepsilon\to 0} \rho_\varepsilon(r) = \delta(r)/r^2$, we obtain:

$$U = \lim_{\varepsilon\to 0} [1/4\pi R_0] \int d\alpha' \sin^2\alpha' \rho_\varepsilon(\alpha')\ [1/\sin|\alpha - \alpha'|]\ [\ Q(t - |\alpha - \alpha'|R_0)$$
$$+ (1/4\pi R_0) \sum_{n=1}^{\infty} \int d\alpha' \sin^2\alpha'\ \rho_\varepsilon(\alpha')\ [1/\sin|\alpha - \alpha'|]\ [\ Q(t - |\alpha - \alpha'|R_0 - 2\pi n R_0) - Q(t + |\alpha - \alpha'|R_0 - 2\pi n R_0)\ ]$$

$$= (1/4\pi R_0)\ [1/\sin\alpha]\ [\ Q(t - \alpha R_0)$$
$$+ (1/4\pi R_0) \sum_{n=1}^{\infty}\ [1/\sin\alpha]\ [\ Q(t - \alpha R_0 - 2\pi n R_0) - Q(t + \alpha R_0 - 2\pi n R_0)\ ]\ .$$

Therefore the first term of the rhs of (12) in the limit when $\varepsilon \to 0$ is ($2\Gamma \equiv \lambda^2/4\pi$):

$$\lim_{\varepsilon\to 0}\ \lambda^2 \int d^3x \sqrt{-g(t,\underline{x})}\ \rho_\varepsilon(t,\underline{x})\ U_\varepsilon(t,\alpha)$$
$$= (2\Gamma/R_0) \lim_{\varepsilon\to 0} \int d^3x \sqrt{-g(t,\underline{x})}\ \rho_\varepsilon(t,\underline{x})\ [1/\sin\alpha]\ [\ Q(t - \alpha R_0)$$

$$+ (2\Gamma/R_0) \sum_{n=1}^{\infty} \lim_{\varepsilon\to 0} \int d^3x \sqrt{-g(t,\underline{x})}\, \rho_\varepsilon(t,\underline{x})\, [1/\sin\alpha]\, [\, Q(t - \alpha R_0 - 2\pi n R_0) - Q(t + \alpha R_0 - 2\pi n R_0)\,]$$

$$= (2\Gamma/R_0) \left\{ \int d\alpha\, \delta(\alpha)[1/\sin\alpha]Q(t) - R_0\, \dot{Q}(t) - 2R_0 \sum_{n=1}^{\infty} \dot{Q}(t - t_n) \right\} . \qquad \text{(II.39)}$$

Similarly, the second term in the rhs of (12) in the limit gives:

$$\lambda \lim_{\varepsilon\to 0} \int d^3x \sqrt{-g(t,\underline{x})}\, \rho_\varepsilon(t,\underline{x})\, \phi_H(t,\underline{x};t_0) = \lambda\, \phi_H(t,\underline{0};t_0) \;, \qquad \text{(II.40)}$$

where $\phi_H(t,\underline{0};t_0)$ is given by relation (34). We observe that this term represents the inhomogeneous term of (12). Then replacing (39) and (40) into the rhs of (12) we obtain ($t_n \equiv 2\pi n R_0$):

$$\ddot{Q}(t) + 2\Gamma\, \dot{Q}(t) + 4\Gamma \sum_{n=1}^{\infty} \theta(t - t_n)\, \dot{Q}(t - t_n) + \omega_0^2 Q(t) - (2\Gamma/R_0) \int d\alpha\, \delta(\alpha)[1/\sin\alpha]\, Q(t) = \lambda\phi_H(t,\underline{0};t_0) \,. \qquad \text{(II.41)}$$

We observe that the coefficient of $Q(t)$: $2\Gamma R_0^{-1} \int d\alpha\, \delta(\alpha)\, (1/\sin\alpha) \equiv \omega_\infty^2$ is (again; see ($I.15_\infty$)) formally infinite. As before we need to think of ω_0^2 as an epsilon-dependent constant: $\omega_0^2 = \omega_0^2(\varepsilon)$ such that $\omega_0^2(\varepsilon) \to \infty$ when $\varepsilon \to 0$, and think of $\omega_{ren}^2 \equiv \omega_0^2(\varepsilon) - \omega_\infty^2$ as the (finite, positive) "renormalized frequency". We proceed as follows.

Fix the real numbers $\omega_{ren}^2 > 0$, $(Q_0, \dot{Q}_0)$, and the function $\phi_H(t,\underline{0};t_0)$, given in (34), is well-defined by the initial-data (6); define

$$\omega_0^2(\varepsilon) \equiv \omega_{ren}^2 + 2\Gamma R_0^{-1} \int d^3x \sqrt{h(\underline{x})}\, \rho_\varepsilon(\underline{x}) d^3x' \sqrt{h(\underline{x}')}\, \rho_\varepsilon(\underline{x}')\, [1/\sin|\alpha - \alpha'|] \,. \qquad \text{(II.42)}$$

With this definition, (41) converts into:

$$\ddot{Q}(t) + 2\Gamma\, \dot{Q}(t) + 4\Gamma \sum_{n=1}^{\infty} \theta(t - t_n)\, \dot{Q}(t - t_n) + \omega_{ren}^2 Q(t) = \lambda\phi_H(t,\underline{0};t_0) \,. \qquad (\text{II}.41_{ren})$$

Again we are motivated to regard any solution of (41_{ren}) (with renormalized constant ω_{ren}^2) as a renormalized solution of equation (41) . However this notion is meaningless unless one gives a precise connection between solutions of (41) (valid $\forall\ \varepsilon > 0$, and $0 < \omega_0^2 < \infty$) and solutions of (41_{ren}) (valid for $\omega_{ren}^2 > 0$, i.e. $\omega_0^2 = \infty$). This connection can be constructed in a similar way as was done in Proposition I.2.3 for the Minkowski spacetime case. To repeat this procedure here will not add insight to our discussion, so that we assume that this precise connection is given and from now on we will consider equation (41_{ren}) as the point-limit case of the radiation reaction equation (12) in the Einstein universe case. The important issue of the well-posed initial-value problem of equation (41_{ren}) will be considered in the next Chapter, when a class of differential equations will be studied; in fact this class of equations includes as a particular case equation (41_{ren}).

In the limit $R_0 \to \infty$, we have $t_n \to \infty$, $n \geq 1$. The fact that $t_n \to \infty$, $n \geq 1$, implies that $t - t_n < 0$, $\forall\ t \in I$, therefore $\theta(t - t_n) = 0$, $\forall\ n \geq 1$. Then equation (41_{ren}) converts into:

$$\ddot{Q}(t) + 2\Gamma\,\dot{Q}(t) + \omega_{ren}^2 Q(t) = \lambda\phi_H(t,\underline{0};t_0)\ ,$$

which coincides with the radiation reaction equation ($I.15_{ren}$) obtained in the point-like limit for the Minkowski spacetime. Physically the limit $R_0 \to \infty$ means that the closed space turn out to be open, then one is motivated to think that in limit the radiation reaction equation should be that one corresponding to the Minkowski spacetime.

III A CLASS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENTS

This final Chapter is devoted to the study of the stability properties of the zero solution of a certain class of differential equations. These equations are linear second order discontinuous differential equations with functional retarded arguments (DEFRA from now on) [18-21]. As far as we know, the only time that the class of equations here considered have appeared in the literature was in [8]. In [8] it is conjectured that these equations represent the radiation reaction equation in the point-like limit for the coupled model (introduced in Chapter I) in non-static spacetimes with closed topology. However, we have been unable to prove this conjecture due to the fact that there is no renormalization procedure (see section I.3) to correctly perform the point-like limit of the radiation reaction equation (II.12).

The parameter of this class of equations is a real function (of real variable) R = R(t). Here we shall restrict our attention to the sub-class of equations for which $\ddot{R} \geq 0$. Perhaps the most important feature of these equations is that, if we let the function R be a constant, namely the constant radius of the Einstein universe, then this class of equations reduces to the radiation reaction equation (II.41_{ren}) (obtained in Chapter II) for the coupled system embedded in the Einstein spacetime and for the point-like case: $\rho = \delta$.

In the first section we analyze the important issue of the initial-value problem of this class of equations. We show that these equations are a special type of DEFRA equations because they are discontinuous. In fact, to have a well-posed initial-value problem continuous DEFRA equations need initial-data in an interval. Here we show that, as for ordinary differential equations (ODE from now on), initial Cauchy-data determines a well-posed initial-value problem for the discontinuous DEFRA equations considered here.

Since equations of this type has not been previously discussed in the literature, we have constructed here an (original) approach to analyze the late time behavior (for arbitrary initial-data) of the solutions of these equations. This approach is based on the construction of a suitable family of Lyapunov functions which has been used to study the stability and asymptoticity properties of the zero solution. Thus, the second section is devoted to the use of these Lyapunov functions to show that the zero solution is uniformly stable (u.s.) [14-18].

In the third section we demonstrate that, if (in addition to $\ddot{R} \geq 0$) $\dot{R} > 0$, then the zero solution of this class of equations is globally uniformly asymptotically stable (g.u.a.s.) [14-18].

At the end of the Chapter we perform some computer simulations for different functions R, including the Einstein universe case: R = constant.

III.1 THE INITIAL-VALUE PROBLEM

We begin this section by presenting the class of DEFRA equations that will be deeply analyzed in this Chapter:

$$\ddot{Q}(t) + 2\Gamma\,\dot{Q}(t) + 4\Gamma\sum_{n=1}^{\infty}\theta(t - t_n)\,\dot{Q}(g_n(t)) + \omega^2 Q(t) = F(t) \ , \qquad \text{(III.1)}$$

where $Q: I \rightarrow \mathbb{R}$, $I \equiv [t_0, \infty)$; $\Gamma, \omega^2 \in \mathbb{R}^+$. The functions $g_n: I \rightarrow I$, $t_n \equiv f^{-1}(2\pi n)$, n = 1,2,.., are defined by:

$$g_n(t) \equiv f^{-1}(f(t) - 2\pi n) \ , \qquad \text{(III.2)}$$

where

$$\tau \equiv f(t) \equiv \int_{t_o}^{t} ds\,/\,R(s) \ , \qquad \text{(III.3)}$$

and $R = R(t): I \rightarrow \mathbb{R}^+$, $R \in C^2$. The inhomogeneous term $F: I \rightarrow \mathbb{R}$, $F \in C^\infty$, is of the form:

$$F(t) = [1/R(t)]\ P(f(t)) \ , \qquad \text{(III.4)}$$

where P is a periodic function of f(t). The homogeneous equation (1) is:

$$\ddot{Q}(t) + 2\Gamma\,\dot{Q}(t) + 4\Gamma \sum_{n=1}^{\infty} \theta(t - t_n)\,\dot{Q}(g_n(t)) + \omega^2 Q(t) = 0 \ . \qquad \text{(III.1h)}$$

Notice that due to the θ-functions, the infinite sum is actually a finite sum. In fact, for $t \in [t_0,t_{N+1})$, with $N < \infty$, the sum has exactly N terms, where $N \equiv [f(t)/2\pi]$, i.e. N is the maximum integer of the real number $f(t)/2\pi$. Notice also that, equation (1) is not a continuous ordinary differential equation due to the terms $\theta(t - t_n)\,\dot{Q}(g_n(t))$.

Previous to the study of the initial-value problem, let us rapidly verify that the class of equations (1) reduces to equation $(II.41_{ren})$ for the case $R = R_0$ = the constant radius of the Einstein universe. In this case: $\tau \equiv f(t) \equiv \int ds / R_0 = (t - t_0)/R_0$. Then the functions g_n take the form: $g_n(t) = t - t_n$, where $t_n = 2\pi n R_0$. We chose $P(f(t)) = \lambda R_0 \phi_H(t,\underline{x};t_0)$ (hence $F(t) = \lambda\phi_H(t,\underline{0};t_0)$) and $\omega^2 = \omega^2{}_{ren}$. With this information equations (1) becomes ($t > t_0$):

$$\ddot{Q}(t) + 2\Gamma\,\dot{Q}(t) + 4\Gamma \sum_{n=1}^{\infty} \theta(t - t_n)\,\dot{Q}(t - t_n) + \omega^2 Q(t) = \lambda\phi_H(t,\underline{0};t_0) \ ,$$

which is the radiation reaction equation $(II.41_{ren})$ obtained in section II.3.

Now we will concentrate our efforts on the analysis of the initial-value problem of equations (1). For this, let us consider some properties of the functions g_n.

Proposition III.1.1 Let $g_n \in C^2$, $\forall\ n \geq 1$, be the functions defined in (2). Then g_n: $[t_{m-1}, t_m] \to [t_{m-n-1},t_{m-n}]$, for $m \geq 2$ and $n \geq 1$ (with $m - n - 1 \geq 0$). Moreover $g_n(t_N) = t_{N-n}$, for $N-n \geq 0$.

Proof. According to the definition: $\tau \equiv f(t) \equiv \int ds/R(s)$, $s \in [t_0,t]$, then for $t = t_N \Rightarrow f(t_N) = 2\pi N$, and for $t = t_{N+1} \Rightarrow f(t_{N+1}) = 2\pi(N+1)$, hence for $t \in [t_{m-1},t_m]$, $\tau \equiv f(t) \in [2\pi(m-1),2\pi m]$. Thus, for $t \in [t_{m-1},t_m] \Rightarrow f(t) - 2\pi n \in [2\pi(m-n-1),2\pi(m-n)]$, with $m - n - 1 \geq 0$. Therefore, for $s \in [2\pi(m-n-1), 2\pi(m-n)] \Rightarrow f^{-1}(s) \in [t_{m-n-1},t_{m-n}]$ as we wanted to show. Hence, for given m and for $t \in [t_{m-1},t_m]$, $g_n(t) \in [t_{m-n-1},t_{m-n}]$, $m \geq 2$ and $n \geq 1$, with $m - n - 1 \geq 0$.

Moreover, from its definition ($g_n(t) \equiv f^{-1}(f(t) - 2\pi n)$):

$$g_n(t_m) = f^{-1}(f(t_m) - 2\pi n) = f^{-1}(2\pi m - 2\pi n) = f^{-1}(2\pi(m-n)) = t_{m-n} ,$$

for $m, n \geq 1$, with $m - n \geq 0$. ⌘

Define: $I_m \equiv [t_m, t_{m+1})$, $m \geq 0$. Then, for $t \in I_N$, $\dot{Q}(g_n(t))$, $n \geq 1$, is the derivative (with respect to t) of the function Q evaluated at the "retarded" point $g_n(t) \in I_{N-n}$, where for each $n \geq 1$, I_{N-n} represents n-intervals precedent to I_N. For this reason we call $g_n = g_n(t)$ the retarded or delay functions [18-25] and $\dot{Q}(g_n(t))$ (for each n) a function with retarded argument .

Thus, for $t \in I_N$, the sum in (1):

$$\sum_{n=1}^{\infty} \theta(t - t_n)\, \dot{Q}(g_n(t)) = \dot{Q}(g_1(t)) + \dot{Q}(g_2(t)) + ... + \dot{Q}(g_N(t)) \quad ,$$

depends on the values that $\dot{Q}$ takes at the retarded points $g_1(t), g_2(t), ..., g_N(t)$. For this reason, equation (1) for $t \in I_N$, is called a differential equation with N functional retarded arguments: DEFRA.

In the literature (e.g. [18-19]), this name is reserved for continuous m-order differential equations. From (1) we observe that it is a second order differential equation, but due to the θ-functions it is a discontinuous differential equation: $Q \notin C^2(I)$. This makes a great difference regarding the initial-value problem. In fact, DEFRA equations need inital-data in an interval (the so-called initial-interval) to determine a well-posed initial-value problem, while ODEs need only initial-data at one point (say $t = t_0$, the so-called initial-time). We will show in the next Proposition that, due to the θ-functions, the initial-value problem of equation (1) is well-posed by just giving initial-data at $t = t_0$.

Proposition III.1.2 The initial-value problem [14-16] for equation (1) is well-posed for (Cauchy) initial-data:

$$(Q(t_0), \dot{Q}(t_0)) = (Q_H(t_0), \dot{Q}_H(t_0)) = (Q_0, \dot{Q}_0) \quad . \qquad \text{(III.5)}$$

Proof. Notice from (1) that $Q \notin C^2([a,b])$ for $t_0 \leq a < b < \infty$, if there exist an integer m such that: $t_m \in [a, b]$. Therefore we have to look for solutions of (1): $Q \in C^1([a, b])$ for arbitrary a, b: $t_0 \leq a < b < \infty$, (observe that $Q \in C^2([t_{m-1}, t_m))$, for $m \geq 1$).

Let us perform our analysis step by step, i.e. for $t \in I_m \equiv [t_m, t_{m+1})$, for m $= 0,1,\ldots,N$. Fix a real $\varepsilon > 0$.

Consider the following equation for $t \in [t_0, t_1+\varepsilon]$:

$$\ddot{Q}(t) + 2\Gamma\,\dot{Q}(t) + \omega^2 Q(t) = F(t) \quad , \qquad \text{(III.6)}$$

with initial-data (5). Note that for $t \in I_0 \subset [t_0, t_1+\varepsilon]$, this equation coincides with (1), hence so does its solution. This is a well-known second order differential equation whose initial-value problem is well-posed for initial-data (5) [14-16]; its C^2-solution is then given by $Q = Q_H + Q_I \in C^2([t_0, t_1+\varepsilon])$, where:

$$Q_H(t) = e^{-\Gamma(t - t_0)} \{Q_0[(\Gamma/\tilde{\omega})\sin\tilde{\omega}(t - t_0) + \cos\tilde{\omega}(t - t_0)] + (1/\tilde{\omega})\,\dot{Q}_0 \sin\tilde{\omega}(t - t_0)\} \,, \qquad \text{(III.7)}$$

where $\tilde{\omega}^2 \equiv \omega^2 - \Gamma^2$, (we assume $\tilde{\omega}^2 > 0$) and

$$Q_I(t) = (1/\tilde{\omega}) \int dt' \sin\tilde{\omega}(t - t')\, e^{-\Gamma(t - t')} F(t') \quad , \qquad \text{(III.8)}$$

for $t' \in [t_0, t]$. At $t = t_1$ we have the data:

$$(Q(t_1), \dot{Q}(t_1)) = (Q_H(t_1), \dot{Q}_H(t_1)) = (Q_1, \dot{Q}_1) \,. \qquad \text{(III.9)}$$

The next step now is to consider the following equation, for $t \in [t_1, t_2+\varepsilon]$:

$$\ddot{Q}(t) + 2\Gamma\,\dot{Q}(t) + \omega^2 Q(t) = F(t) - 4\Gamma\,\dot{Q}(g_1(t)) \equiv F_1(t) \ , \qquad \text{(III.10)}$$

with initial-data (9), and $\dot{Q}(g_1(t))$ given by

$$\dot{Q}(g_1(t)) = d/dt\ [Q_H(g_1(t)) + Q_I(g_1(t))] \ ,$$

with $g_1(t) \in [t_0, g_1(t_2+\varepsilon)]$, where $Q_H(g_1(t))$ ($Q_I(g_1(t))$) is the homogeneous (inhomogeneous) solution of equation (6) calculated in the preceding step and given by (7) ((8)) :

$$Q_H(g_1(t)) = e^{-\Gamma(g_1(t) - t_0)}\ \{Q_0[(\Gamma/\tilde{\omega})\sin\tilde{\omega}(g_1(t) - t_0) + \cos\tilde{\omega}(g_1(t) - t_0)] + (1/\tilde{\omega})\ \dot{Q}_0 \sin\tilde{\omega}(g_1(t) - t_0)\} \ ,$$

$$(\ Q_I(g_1(t)) = (1/\tilde{\omega}) \int dt'\ \sin\tilde{\omega}(g_1(t) - t')\ e^{-\Gamma(g_1(t) - t')}\ F(t')\ ) \ .$$

Note that equation (10) coincides with (1) for $t \in I_1 \subset [t_1, t_2+\varepsilon]$, therefore for identical initial-data its solution is the same. We find the solution of equation (10) by noting that the form of (10) is the same as that of equation (6) except that the inhomogeneous term in (10) is F_1 instead of F in (6). Therefore its C^2-solution is then given by $Q = Q_H + Q_I \in C^2([t_1, t_2+\varepsilon])$, where $Q_H = Q_H(t)$ is given by (7) for $t \in [t_1, t_2+\varepsilon]$ with initial-data (9) and $Q_I = Q_I(t)$ for $t \in [t_1, t_2+\varepsilon]$, is given by (8) with F replaced by F_1 and thus again in this second step the initial-value problem is well-posed for initial-data (9). At $t = t_2$ we obtain the data:

$$(Q(t_2), \dot{Q}(t_2)) = (Q_H(t_2), \dot{Q}_H(t_2)) = (Q_2, \dot{Q}_2) \ , \qquad \text{(III.11)}$$

which will be the initial-data for the next interval.

Continuing this procedure, assume that we have performed (N-1) steps and have solved the equation

$$\ddot{Q}(t) + 2\Gamma\,\dot{Q}(t) + \omega^2 Q(t) = F(t) - 4\Gamma\Sigma\ \dot{Q}(g_n(t)) \equiv F_{N-1}(t) \ , \qquad \text{(III.12)}$$

for $t \in [t_{N-1}, t_N+\varepsilon]$, and therefore can calculate at $t = t_N$ the data:

$$(Q(t_N), \dot{Q}(t_N)) = (Q_H(t_N), \dot{Q}_H(t_N)) = (Q_N, \dot{Q}_N) \ , \qquad \text{(III.13)}$$

which will be the initial-data for the next interval: $t \in [t_{N-1}, t_N+\varepsilon]$.

Finally, consider the differential equation (1) for $t \in [t_N, t_{N+1}+\varepsilon]$:

$$\ddot{Q}(t) + 2\Gamma\, \dot{Q}(t) + \omega^2 Q(t) = F(t) - 4\Gamma \sum_1^N \dot{Q}(g_n(t)) \equiv F_N(t) \ , \qquad \text{(III.14)}$$

with initial-data (13):

$$(Q(t_N), \dot{Q}(t_N)) = (Q_H(t_N), \dot{Q}_H(t_N)) = (Q_N, \dot{Q}_N) \ . \qquad \text{(III.13)}$$

In the inhomogeneous term $F_N(t)$, the sum $\sum_1^N \dot{Q}(g_n(t))$ is calculated as follows:

$$\sum_1^N \dot{Q}(g_n(t)) = \sum_1^N d/dt\, [Q_H(g_n(t)) + Q_I(g_n(t))] \quad ,$$

where $g_n(t) \in [g_n(t_N), g_n(t_{N+1}+\varepsilon)]$, n=1,2,...N, and each term of this sum is calculated by differentiating the calculated solutions in the (N-1) preceding intervals . Note that equation (12) coincides with (1) for $t \in I_N \subset [t_N, t_{N+1}+\varepsilon]$, therefore for identical initial-data its solution is the same. We find the solution of equation (12) by noting that (12) has the same form as equation (6) except that the inhomogeneous term in (12) is F_N instead of F in (6). Therefore its C^2-solution is then given by $Q = Q_H + Q_I \in C^2([t_N, t_{N+1}+\varepsilon])$, where $Q_H = Q_H(t)$ is given by (7) for $t \in [t_N, t_{N+1}+\varepsilon]$ with initial-data (13) and $Q_I = Q_I(t)$ for $t \in [t_N, t_{N+1}+\varepsilon]$, is given by (8) with F replaced by F_N. Thus again in this N-step the initial-value problem is well-posed for initial-data (13) . At $t = t_{N+1}$ we can obtain the data:

$$(Q(t_{N+1}), \dot{Q}(t_{N+1})) = (Q_H(t_{N+1}), \dot{Q}_H(t_{N+1})) = (Q_{N+1}, \dot{Q}_{N+1}) \ ,$$

which will be the initial-data for the next interval. Thus, we can continue to solve equation (1) step by step, finding in each case that the initial-value problem is well-posed for the interval in question. The solution can be extended in unique fashion past the discontinuities, and both Q and $\dot{Q}$ remain continuous. That is, we have constructed $C^1([a,b])$ - solutions of (1), for given intial Cauchy-data (5), for arbitrary a, b such that: $t_0 \leq a < b < \infty$. ⌘

III.2 THE STABILITY ANALYSIS OF THE ZERO SOLUTION

In this section we shall study the stability properties of the zero solution of the class of equations (1). Initially we shall prove that the zero solution of the homogeneous equation (1h) is uniformly stable. At the end we will see that, since the inhomogeneous term of (1) is uniformly bounded, then also the zero solution of (1) is uniformly stable.

We start this section investigating some interesting (and further useful) properties of the functions $\dot{g}_n(t)$, for $n \geq 1$.

Proposition III.2.1 (i) Let R be a C^2-function of time $R = R(t)$: $[t_0,\infty) \to \mathbb{R}^+$. Then the time derivative of the functions g_n, defined in (2), satisfies:

$$\dot{g}_n(t) = R(g_n(t)) \,/\, R(t) \,,$$

$\forall\, t \in I$, $\forall\, n \geq 1$.

(ii) If $\ddot{R} \geq 0$, $\forall\, t \in I$, then $\ddot{g}_n(t) \leq 0$,$\forall\, t \in I$, $\forall\, n \geq 1$.

Proof. (i) From (3) we deduce that $df(x)/dx = 1/R(x)$, $\forall\, t \in I$; hence

$$1 = df^{-1}(f(x))/dx = [df^{-1}(f(x))/df(x)]\,[df(x)/dx] = [df^{-1}(f(x))/df(x)][1/R(x)] \,,$$

$$\Rightarrow \qquad [df^{-1}(y)/dy] = R(f^{-1}(y)) \,,$$

for $y = f(x)$. Now, taking the time derivative to $g_n(t) \equiv f^{-1}(f(t) - 2\pi n)$, $n \geq 1$, we have:

$$dg_n(t)/dt = [df^{-1}(f(t) - 2\pi n)/d(f(t) - 2\pi n)][d(f(t) - 2\pi n)/dt] \quad ,$$

and using the relation: $[df^{-1}(y)/dy] = R(f^{-1}(y))$ with $y = f(t) - 2\pi n$, we obtain:

$$\Rightarrow \quad dg_n(t)/dt = [df^{-1}(y)/dy]\,[d(f(t) - 2\pi n)/dt] = R(f^{-1}(y)) \,/\, R(t)$$

$$= R(f^{-1}(f(t) - 2\pi n)) \,/\, R(t) = R(g_n(t)) \,/\, R(t) \,,$$

for $n \geq 1$, as we claim.

(ii) From part (i): $\dot{g}_n(t) = R(g_n(t))/R(t)$. Differentiating this expresion with respect to t we obtain:

$$\ddot{g}_n(t) = R^{-2}(t)\,[R(t)\,\dot{R}(g_n(t)) - R(g_n(t))\,\dot{R}(t)]$$

$$= R^{-2}(t)\,[R(t)\,\dot{R}(g_n(t)) - R(g_n(t))\,\dot{R}(t)] = R^{-2}(t)\,[R(t)\,\dot{g}_n(t)R'(g_n(t)) - R(g_n(t))\,\dot{R}(t)]$$

$$= R^{-2}(t)\,[R(g_n(t))\,R'(g_n(t)) - R(g_n(t))\,\dot{R}(t)] = R^{-2}(t)R(g_n(t))[R'(g_n(t)) - \dot{R}(t)] \quad ,$$

however from condition $\ddot{R} \geq 0$, we conclude that $\dot{R}(t) \geq R'(g_n(t))$, therefore using this relation above :

$$\ddot{g}_n(t) = R^{-2}(t)\,R(g_n(t))\,\,[R'(g_n(t)) - \dot{R}(t)] \leq 0 \,,$$

as we claim . ⌘

<u>Proposition III.2.2</u> Let $R = R(t)$: $[t_0,\infty) \rightarrow \mathbb{R}^+$ be a function such that $\ddot{R} \geq 0$. Then $\dot{g}_n(t) \leq r^n$, for $t \in [t_{N},{}_{N+1}]$, $N \geq n \geq 1$, where $R(t_0)/R(t_1) \equiv r \leq 1$.

<u>Proof.</u> From Proposition III.2.1 we know that: $\dot{g}_n(t) = R(g_n(t))/R(t)$ and that $\ddot{g}_n(t) \leq 0$, if $\ddot{R} \geq 0$. Integrating for $s \in [t_N,t]$, $t \in [t_N,t_{N+1}]$ and $N \geq n \geq 1$; we obtain :

$$\Rightarrow \quad \dot{g}_n(t) - \dot{g}_n(t_N) = \int \ddot{g}_n(s)\,ds \;\leq 0 \quad ,$$

$$\Rightarrow \quad \dot{g}_n(t) \le \dot{g}_n(t_N) = R(g_n(t_N))/R(t_N)$$
$$= R(t_{N-n})/R(t_N) \ , \qquad \text{(III.15)}$$

Then for $N = 1 = n$, $t \in [t_1,t_2]$, relation (15) reads: $\dot{g}_1(t) \le R(t_0)/R(t_1) \equiv r$. But from Proposition III.2.1 : $\dot{g}_1(t) = R(g_1(t))/R(t) \Rightarrow$ for $t = t_2$:

$$\dot{g}_1(t_2) = R(t_1)/R(t_2) \le r \ .$$

For $N = 2$ and $n = 2$, $t \in [t_2,t_3]$, relation (15) reads: $\dot{g}_2(t) \le R(t_0)/R(t_2)$,

$$\Rightarrow \quad \dot{g}_2(t) \le R(t_0)/R(t_2) = [R(t_0)/R(t_1)]\ [R(t_1)/R(t_2)] = r\ [R(t_1)/R(t_2)] \le r^2 \ .$$

Now suppose that $\dot{g}_n(t) \le r^n$, $N \ge n \ge 1$, for $t \in [t_N,t_{N+1}]$. But $\dot{g}_N(t) = R(g_N(t))/R(t)$, then for $t = t_{N+1}$:

$$\dot{g}_N(t_{N+1}) = R(t_1)/R(t_{N+1}) \le r^N \ .$$

Therefore for $n = N+1 \ge 2$, $t \in [t_{N+1},t_{N+2}]$, from $\ddot{g}_{N+1} \le 0$:

$$\dot{g}_{N+1}(t) = R(g_{N+1}(t))/R(t) \le \dot{g}_{N+1}(t_{N+1})$$

$$= R(t_0)/R(t_{N+1}) = [R(t_0)/R(t_1)]\ [R(t_1)/R(t_{N+1})]$$

$$= r\ [R(t_1)/R(t_{N+1})] \le r^{N+1} \ ,$$

therefore $\dot{g}_n(t) \le r^n$, for $t \in [t_N,t_{N+1}]$, $N \ge n \ge 1$, as we claim . ⌘

We shall need the following definitions for our future analysis of the zero solution of (1h).

Definitions III.2.3 We denote by

$$\underline{Q}(t) \equiv (Q(t),\ \dot{Q}(t)) \ , \qquad \text{(III.16)}$$

the pair of functions $(Q(t), \dot{Q}(t))$ where $Q = Q(t)$ is the C^1-solution of (1), for $t \in I$. Moreover, if $t \in I_{N-1} \equiv [t_{N-1}, t_N)$, $N \geq 1$, we write

$$\underline{Q}_N(t) \equiv (Q_N(t), \dot{Q}_N(t)) \ , \qquad \text{(III.17)}$$

to indicate the solution of (1) (and its time derivative) only in the interval I_{N-1}, e.g. $\underline{Q}_3 = \underline{Q}_3(t)$ is the solution in the interval I_2, $t \in [t_2, t_3)$. Notice that $(Q_N(t), \dot{Q}_N(t))$ represents a pair of functions, and should not be confused with the initial-data $(Q(t_N), \dot{Q}(t_N)) = (Q_N, \dot{Q}_N)$, which is a pair of real numbers.
Define the (square of the) norm of $\underline{Q}_N$ by:

$$\| \underline{Q}_N(t) \|^2 \equiv \tfrac{1}{2} \left[\dot{Q}_N^2(t) + \omega^2 Q_N^2(t) \right] \ , \qquad \text{(III.18)}$$

for $t \in I_{N-1}$. A brief (and useful) notation for this will be

$$\xi_N(t) \equiv \| \underline{Q}_N(t) \|^2 \ , \qquad \text{(III.19)}$$

with values at the points $t = t_{N-1}$, $N \geq 1$:

$$E_{N-1} \equiv \xi_N(t_{N-1}) \ . \qquad \text{(III.20)}$$

Define a positive-definite function $V_{N-1}: [t_{N-1}, t_N) \times \mathbb{R}^2 \rightarrow \mathbb{R}^+$,

$$V_{N-1}(t, Q, \dot{Q}) \equiv \sum_1^N r^{n-1} \xi_{N-n+1}(g_{n-1}(t)) \ , \qquad \text{(III.21)}$$

where $g_0(t) \equiv t$, for $N \geq 1$. Note that $\forall\, N \geq 1$, $V_{N-1} = 0 \Leftrightarrow \xi_{N-n+1} = 0 \Leftrightarrow Q = \dot{Q} = 0$. For simplicity of notation we shall write: $V_{N-1}(t, Q, \dot{Q}) \equiv V_{N-1}(t)$. ⌘

Now let us concentrate on the stability analysis of the equation (1h). To show the uniform stability of the zero solution we need to show that for each $N \geq 1$, the functions $\xi_N(t) \equiv \| \underline{Q}_N(t) \|^2$ are bounded by a constant independent of N and of t_0. Our treatment relies on the construction of positive-definite functions

V_{N-1}, defined for $t \in [t_{N-1},t_N)$ for each $N \geq 1$, which have the property to posses a non-positive definite (that is, less or equal to zero) time derivative along the solutions of equations (1h), as we will show in Proposition III.2.4. This however is not enough. This result only asserts that: $V_{N-1}(t_N) \leq V_{N-1}(t) \leq V_{N-1}(t_{N-1})$, for $t \in [t_{N-1},t_N)$, $\forall\ N \geq 1$. Our task in Proposition III.2.5 will be to analyze these inequalities for each N: in fact combining them in a suitable way we will show, using mathematical induction, that $\xi_N \leq E_0 \equiv \xi_1(t_0)$, for each $N \geq 1$, that is all the functions ξ_N are bounded by a constant independent of N and of t_0. After that we easily show that the zero solution of (1h) is uniformly stable.

Proposition III.2.4 Let $V_{N-1} = V_{N-1}(t)$ be the function defined in (21). If $\ddot{R} \geq 0$, for $t \in$ I, then the time derivative of V_{N-1}, for $t \in I_{N-1} = [t_{N-1},t_N)$, $\forall\ N \geq 1$ along the solutions of (1h) is a non-positive-definite quantity [17], i.e. $\dot{V}_{N-1} \leq 0$, for $N \geq 1$.

Proof. We will prove this Proposition by mathematical induction.

For $N = 1$, $t \in [t_0,t_1)$, all the θ-functions in equation (1h) are zero, then it reads:

$$\ddot{Q}_1(t) + 2\Gamma\, \dot{Q}_1(t) + \omega^2\, Q_1(t) = 0\ . \qquad \text{(III.1h}_1\text{)}$$

In this case from (21) we have:

$$V_0(t) = \xi_1(t) = \tfrac{1}{2}\ (\, \dot{Q}_1{}^2(t) + \omega^2 Q_1{}^2(t)\,)\ .$$

It is not difficult to see that: $\dot{V}_0(t) = -\, 2\Gamma\, \dot{Q}_1{}^2(t) \leq 0$.

For $N = 2$, $t \in [t_1,t_2)$, equation (1h) reads:

$$\ddot{Q}_2(t) + 2\Gamma\, \dot{Q}_2(t) + \omega^2\, Q_2(t) = -\, 4\Gamma\, \dot{Q}_1(g_1(t))\ , \qquad \text{(III.1h}_2\text{)}$$

so that in this interval we shall consider the positive definite function defined in (21) for $N = 2$:

$$V_1(t) = \xi_2(t)\ +\ r\, \xi_1(g_1(t))$$

$$= \tfrac{1}{2}\,(\,\dot{Q}_2{}^2(t) + \omega^2 Q_2{}^2(t)\,) + \tfrac{1}{2}\,r\,(Q'_1{}^2(g_1(t)) + \omega^2 Q_1{}^2(g_1(t))\,)\;.$$

Let us take the time derivative of V_1:

$$\dot{V}_1(t) = \dot{Q}_2(t)\ddot{Q}_2(t) + \omega^2 Q_2(t)\,\dot{Q}_2(t) + r\,\{\,\dot{g}_1(t)Q'_1(g_1(t))Q''_1(g_1(t)) + \omega^2\,\dot{g}_1(t)Q_1(g_1(t))Q'_1(g_1(t))\}\;,$$

along the solutions of ($1h_2$). This means that here we have to use ($1h_2$) to eliminate $\ddot{Q}_2$ and ($1h_1$) in order to eliminate Q''_1, obtaining:

$$\dot{V}_1(t) = -\;2\Gamma\,\{\,\dot{Q}_2{}^2(t) + 2\,\dot{g}_1(t)\,\dot{Q}_2(t)Q'_1(g_1(t)) + r\,\dot{g}_1(t)Q'_1{}^2(g_1(t))\}$$

$$= -\;2\Gamma\,\{\,\dot{Q}_2{}^2(t) + 2\,\dot{g}_1(t)\,\dot{Q}_2(t)Q'_1(g_1(t)) + [r/\,\dot{g}_1(t)]\,\dot{g}_1{}^2(t)Q'_1{}^2(g_1(t))\}\;. \tag{III.22}$$

Then we use Proposition III.2.2 to see that $\dot{g}_1(t) \le r \Rightarrow -\,[r\,/\,\dot{g}_1(t)] < -1$; therefore:

$$\begin{aligned} &-2\Gamma\,\{\,\dot{Q}_2{}^2(t) + 2\,\dot{g}_1(t)\,\dot{Q}_2(t)Q'_1(g_1(t)) + [r/\,\dot{g}_1(t)]\,\dot{g}_1{}^2(t)Q'_1{}^2(g_1(t))\} \\ &\le -\;2\Gamma\,\{\,\dot{Q}_2{}^2(t) + 2\,\dot{g}_1(t)\,\dot{Q}_2(t)\,Q'_1(g_1(t)) + \dot{g}_1{}^2(t)\,Q'_1{}^2(g_1(t))\} \\ &= -\;2\Gamma\,\{\,\dot{Q}_2(t) + \dot{g}_1(t)Q'_1(g_1(t))\}^{\,2}\;, \end{aligned}$$

that is

$$\begin{aligned} &-\;2\Gamma\,\{\,\dot{Q}_2{}^2(t) + 2\,\dot{g}_1(t)\,\dot{Q}_2(t)Q'_1(g_1(t)) + r\,\dot{g}_1(t)Q'_1{}^2(g_1(t))\} \\ &\le -\;2\Gamma\,\{\,\dot{Q}_2(t) + \dot{g}_1(t)Q'_1(g_1(t))\}^{\,2}\;. \end{aligned} \tag{III.23}$$

Therefore using (23) in (22), we arrive at:

$$\dot{V}_1(t) \le -\;2\Gamma\,\{\,\dot{Q}_2(t) + \dot{g}_1(t)Q'_1(g_1(t))\}^{\,2} \le 0\,,$$

that is: $\dot{V}_1(t) \le 0$.

For $N = 3$, $t \in [t_2,t_3)$, equation (1h) is :

$$\ddot{Q}_3(t) + 2\Gamma\,\dot{Q}_3(t) + \omega^2 Q_3(t) = -4\Gamma(\dot{Q}_2(g_1(t)) + \dot{Q}_1(g_2(t)))\ . \qquad (III.1h_3)$$

In this case we consider the positive-definite function V_2:

$$V_2(t) = \xi_3(t) + r\,\xi_2(g_1(t)) + r^2\xi_1(g_2(t))$$

$$= \tfrac{1}{2}(\dot{Q}_3^2(t) + \omega^2 Q_3^2(t)) + \tfrac{1}{2} r(Q'^2_2(g_1(t)) + \omega^2 Q_2^2(g_1(t)))$$
$$+ \tfrac{1}{2} r^2(Q'^2_1(g_2(t)) + \omega^2 Q_1^2(g_2(t)))\ .$$

Taking the time derivative of V_2 and using equations ($1h_3$), ($1h_2$) and ($1h_1$) we obtain:

$$\dot{V}_2(t) = \dot{Q}_3(t)\ddot{Q}_3(t) + \omega^2 Q_3(t)\,\dot{Q}_3(t)$$
$$+ r\{\dot{g}_1(t)Q'_2(g_1(t))\,Q''_2(g_1(t)) + \omega^2\,\dot{g}_1(t)Q_2(g_1(t))Q'_2(g_1(t))\}$$
$$+ r^2\{\dot{g}_2(t)Q'_1(g_2(t))\,Q''_1(g_2(t)) + \omega^2\,\dot{g}_2(t)Q_1(g_2(t))Q'_1(g_2(t))\}$$

$$= -2\Gamma\{\dot{Q}_3(t)^2 + 2\,\dot{Q}_3(t)\{\dot{g}_1(t)Q'_2(g_1(t)) + \dot{g}_2(t)Q'_1(g_2(t))\} + r\,\dot{g}_1(t)Q'_2(g_1(t))^2$$
$$+ 2r\,\dot{g}_1(t)Q'_2(g_1(t))[\dot{g}_2(t)/\dot{g}_1(t)]Q'_1(g_2(t)) + r\,\dot{g}_1(t)r[\dot{g}_2(t)/\dot{g}_1(t)]\,Q'_1(g_2(t))^2\}$$

$$= -2\Gamma\{\dot{Q}_3(t)^2 + 2\,\dot{Q}_3(t)\{\dot{g}_1(t)Q'_2(g_1(t)) + \dot{g}_2(t)Q'_1(g_2(t))\}$$
$$+ [r\dot{g}_1(t)]\,\{Q'^2_2(g_1(t))$$
$$+ 2\,[\dot{g}_2(t)/\dot{g}_1(t)]\,Q'_2(g_1(t))Q'_1(g_2(t)) + r\,[\dot{g}_2(t)/\dot{g}_1(t)]\,Q'^2_1(g_2(t))\}\}.$$

Now perform the change of variable: $s = g_1(t)$. Then we have $g_k(t) = g_{k-1}(s)$ (recall that by the definition of g: $g_m(t) \equiv f^{-1}(f(t) - 2\pi m)$, hence $g_m(g_k(t)) = f^{-1}(f(g_k(t)) - 2\pi m) = f^{-1}(f(t) - 2\pi k - 2\pi m) = f^{-1}(f(t) - 2\pi(m+k)) = g_{m+k}(t)$), $Q'_m(g_k(t)) = dQ_m(g_k(t)) / dg_k(t) = dQ_m(g_{k-1}(s)) / dg_{k-1}(s)$, and $\dot{g}_k(t)/\dot{g}_1(t) = dg_k(t)/dg_1(t) = dg_k(t)/ds = dg_{k-1}(s) / ds = g'_{k-1}(s)$. Hence

$$\dot{V}_2(t) = -2\Gamma\{\dot{Q}_3(t)^2 + 2\,\dot{Q}_3(t)\{\dot{g}_1(t)Q'_2(g_1(t)) + \dot{g}_2(t)Q'_1(g_2(t))\}$$
$$+ [r\dot{g}_1(t)]\,\{Q'^2_2(s)$$

$$+ 2\, g'_1(s)\; Q'_2(s)Q'_1(g_1(s)) + rg'_1(s)\;\; Q'_1{}^2(g_1(s))\} \;\} \;. \qquad \text{(III.24)}$$

However from (23) we observe that:

$$\begin{aligned} - 2\Gamma\; \{Q'_2{}^2(s) + 2\, g'_1(s)\; Q'_2(s)Q'_1(g_1(s)) + rg'_1(s)\;\; Q'_1{}^2(g_1(s))\} \\ \le - 2\Gamma\; \{Q'_2(s) + g'_1(s)Q'_1(g_1(s))\}^2 \;. \end{aligned} \qquad \text{(III.25)}$$

Now, returning to the old variable $g_1(t) = s$, and inserting this result in (24):

$$\begin{aligned} \dot{V}_2(t) \le - 2\Gamma \;\{\; \dot{Q}_3(t)^2 + 2\, \dot{Q}_3(t)\{\; \dot{g}_1(t)Q'_2(g_1(t)) + \dot{g}_2(t)Q'_1(g_2(t))\} \\ +\;\; [r/\, \dot{g}_1(t)] \;\{\; \dot{g}_1(t)Q'_2(g_1(t)) + \dot{g}_2(t)Q'_1(g_2(t))\;\}^2 \;\} \end{aligned}$$

$$\le - 2\Gamma \;\{\; \dot{Q}_3(t) + \dot{g}_1(t)Q'_2(g_1(t)) + \dot{g}_2(t)Q'_1(g_2(t))\}^2 \;, \qquad \text{(III.26)}$$

where we have used Proposition III.2.2: - $[r\, /\, \dot{g}_1(t)] < -1$. Therefore we conclude that $\dot{V}_2(t) \le 0$.

Now consider equation (1h) for $t \in I_{m-1} \equiv [t_{m-1}\, ,t_m)$, that is for $N = m$ we have:

$$\ddot{Q}(t) + 2\Gamma\; \dot{Q}(t) + \omega^2\; Q(t) = - 4\Gamma \sum_1^N \theta(t - t_n)\; \dot{Q}(g_n(t)) \;. \qquad \text{(III.1h}_m\text{)}$$

Consider here the functional :

$$V_{m-1}(t) = \Sigma\;\; r^{n-1}\xi_{m-n+1}(g_{n-1}(t)) \;.$$

where $1 \le n \le m$. Now suppose that for $N = m$ is true that:

$$\dot{V}_{m-1}(t) \le - 2\Gamma\{\; \dot{Q}_m(t) + \dot{g}_1(t)Q'_{m-1}(g_1(t)) + \ldots + \; \dot{g}_{m-1}(t)Q'_1(g_{m-1}(t))\}^2 \;. \qquad \text{(III.27)}$$

Let us take the time derivative of V_{m-1} along the solutions of ($1h_m$):

$$\dot{V}_{m-1}(t) = \dot{Q}_m(t)\ddot{Q}_m(t) \;\; + \omega^2 Q_m(t)\; \dot{Q}_m(t)$$

$$+ r(\dot{g}_1(t)Q'_{m-1}(g_1(t))Q''_{m-1}(g_1(t)) + \omega^2 \dot{g}_1(t)Q_{m-1}(g_1(t))Q'_{m-1}(g_1(t))) + \ldots$$
$$+ r^{m-1}(\dot{g}_{m-1}(t)Q'_1(g_{m-1}(t))Q''_1(g_{m-1}(t)) + \omega^2 \dot{g}_{m-1}(t)Q_1(g_{m-1}(t))Q'_1(g_{m-1}(t))) ,$$

and using equation (1h) for N = 1,2,...m (remember that Q_N is the solution of (1h) for $t \in [t_{N-1},t_N)$, $N \geq 1$), we obtain:

$$\dot{V}_{m-1}(t) = - 2\Gamma\{ \dot{Q}_m^2(t) + 2 \dot{Q}_m(t)\{ \dot{g}_1(t)Q'_{m-1}(g_1(t))+ \dot{g}_2(t)Q'_{m-2}(g_2(t)) +\ldots + \dot{g}_{m-1}(t)Q'_1(g_{m-1}(t))\} + r \dot{g}_1(t)Q'^2_{m-1}(g_1(t))$$
$$+ 2 [r / \dot{g}_1(t)]\dot{g}_1(t)Q'_{m-1}(g_1(t))\{ \dot{g}_2(t)Q'_{m-2}(g_2(t)) +\ldots+ \dot{g}_{m-1}(t)Q'_1(g_{m-1}(t))\}$$
$$+ r^2 \dot{g}_2(t)Q'^2_{m-2}(g_2(t)) + 2[r^2 / \dot{g}_2(t)] \dot{g}_2(t)Q'_{m-2}(g_2(t))\{ \dot{g}_3(t)Q'_{m-3}(g_3(t)) +\ldots$$
$$+ \dot{g}_{m-1}(t)Q'_1(g_{m-1}(t))\} +\ldots+ r^{m-1} \dot{g}_{m-1}(t)Q'^2_1(g_{m-1}(t)) \} .$$

Then using (27) we obtain that:

$$- 2\Gamma\{ \dot{Q}_m^2(t) + 2 \dot{Q}_m(t)\{ \dot{g}_1(t)Q'_{m-1}(g_1(t))+ \dot{g}_2(t)Q'_{m-2}(g_2(t)) +\ldots + \dot{g}_{m-1}(t)Q'_1(g_{m-1}(t))\} + r \dot{g}_1(t)Q'^2_{m-1}(g_1(t))$$
$$+ 2 [r / \dot{g}_1(t)]g_1(t)Q'_{m-1}(g_1(t))\{ \dot{g}_2(t)Q'_{m-2}(g_2(t)) +\ldots+ \dot{g}_{m-1}(t)Q'_1(g_{m-1}(t))\}$$
$$+ r^2 \dot{g}_2(t)Q'^2_{m-2}(g_2(t)) + 2[r^2 / \dot{g}_2(t)] \dot{g}_2(t)Q'_{m-2}(g_2(t))\{ \dot{g}_3(t)Q'_{m-3}(g_3(t)) +\ldots$$
$$+ \dot{g}_{m-1}(t)Q'_1(g_{m-1}(t))\} +\ldots+ r^{m-1} \dot{g}_{m-1}(t)Q'^2_1(g_{m-1}(t)) \}$$

$$\leq - 2\Gamma\{ \dot{Q}_m(t) + \dot{g}_1(t)Q'_{m-1}(g_1(t))+\ldots+ \dot{g}_{m-1}(t)Q'_1(g_{m-1}(t))\}^2 . \qquad \text{(III.28)}$$

Now consider equation (1h) for $t \in I_{m-1} \equiv [t_{m-1} ,t_m)$, that is for N = m + 1:

$$\ddot{Q}(t) + 2\Gamma \dot{Q}(t) + \omega^2 Q(t) = - 4\Gamma \sum_1^{m+1}\theta(t - t_n) \dot{Q}(g_n(t)) . \qquad \text{(III.1h}_{m+1})$$

Consider here the functional :

$$V_m(t) = \Sigma\ r^{n-1}\xi_{m-n+2}(g_{n-1}(t)) ,$$

where $1 \leq n \leq m+1$. Now perform the time derivative of $V_m(t)$ along the solutions of $(1h_{m+1})$:

$$\dot{V}_m(t) = -2\Gamma \{ \dot{Q}_{m+1}^2(t) + 2\,\dot{Q}_{m+1}(t)[\,\dot{g}_1(t)Q'_m(g_1(t)) + \dot{g}_2(t)Q'_{m-1}(g_2(t)) + \ldots + \dot{g}_m(t)Q'_1(g_m(t))] + d/dt\,[(-2\Gamma)^{-1} \sum_n r^n \xi_{m-n+1}(g_n(t))] \}, \tag{III.29}$$

where $1 \leq n \leq m$, and

$$\begin{aligned} &d/dt\,[(-2\Gamma)^{-1} \sum_n r^n \xi_{m-n+1}(g_n(t))] \\ &= r\,\dot{g}_1(t)Q'^2_m(g_1(t)) \\ &+ 2\,r\,\dot{g}_1(t)Q'_m(g_1(t))[[dg_2(t)/dg_1(t)]Q'_{m-1}(g_2(t)) + [dg_3(t)/dg_1(t)]Q'_{m-2}(g_3(t)) + \ldots \\ &\quad + [dg_m(t)/dg_1(t)]Q'_1(g_m(t))\,] + r^2\,\dot{g}_2(t)Q'^2_{m-1}(g_2(t)) \\ &+ 2\,r^2\,\dot{g}_2(t)Q'_{m-1}(g_2(t))[[dg_3(t)/dg_2(t)]Q'_{m-2}(g_3(t)) + \ldots \\ &\quad + [dg_m(t)/dg_2(t)]Q'_1(g_m(t))] + r^3\,\dot{g}_3(t)\,Q'^2_{m-2}(g_3(t)) \\ &+ 2\,r^3\,\dot{g}_3(t)Q'_{m-2}(g_3(t))\,[[dg_4(t)/dg_3(t)]Q'_{m-3}(g_4(t)) + \ldots \\ &\quad + [dg_m(t)/dg_3(t)]\,Q'_1(g_m(t)] + \ldots + r^m\,\dot{g}_m(t)Q'^2_1(g_m(t)) \}\,. \end{aligned}$$

Notice that $[dg_k(t)/dg_j(t)] = [\,\dot{g}_k(t)/\,\dot{g}_j(t)]$. Then factorizing $[r\,\dot{g}_1(t)]$, we obtain:

$$\begin{aligned} &d/dt\,[(-2\Gamma)^{-1} \sum_n r^n \xi_{m-n+1}(g_n(t))] \\ &= r\,\dot{g}_1(t)\,\{Q'^2_m(g_1(t)) \\ &+ 2\,Q'_m(g_1(t))[[\,\dot{g}_2(t)/\,\dot{g}_1(t)]Q'_{m-1}(g_2(t)) + [\,\dot{g}_3(t)/\,\dot{g}_1(t)]Q'_{m-2}(g_3(t)) + \ldots \\ &\quad + [\,\dot{g}_m(t)/\,\dot{g}_1(t)]Q'_1(g_m(t))] + r\,[\,\dot{g}_2(t)/\,\dot{g}_1(t)]Q'^2_{m-1}(g_2(t)) \\ &+ 2\,r\,Q'_{m-1}(g_2(t))[[\,\dot{g}_3(t)/\,\dot{g}_1(t)]Q'_{m-2}(g_3(t)) + \ldots \\ &\quad + [\,\dot{g}_m(t)/\,\dot{g}_1(t)]Q'_1(g_m(t))] + r^2\,[\,\dot{g}_3(t)/\,\dot{g}_1(t)]Q'^2_{m-2}(g_3(t)) \\ &+ 2\,r^2 Q'_{m-2}(g_3(t))\,[[\,\dot{g}_4(t)/\,\dot{g}_1(t)]\,Q'_{m-3}(g_4(t)) + \ldots + [\,\dot{g}_m(t)/\,\dot{g}_1(t)]Q'_1(g_m(t)] + \ldots \\ &\quad + r^{m-1}\,[\,\dot{g}_m(t)/\,\dot{g}_1(t)]Q'^2_1(g_m(t))\,\}\,. \end{aligned}$$

Performing once again the change of variable $s = g_1(t)$, we obtain $g_k(t) = g_{k-1}(s)$,

$$Q'_m(g_k(t)) = dQ_m(g_k(t))\,/dg_k(t) = dQ_m(g_{k-1}(\dot{s}))\,/dg_{k-1}(s)\,,$$

and

$$\dot{g}_k(t)/\dot{g}_1(t) = dg_k(t)/dg_1(t) = dg_k(t)/ds = dg_{k-1}(s) / ds = g'_{k-1}(s) \;;$$

therefore the last expression converts into :

$$\begin{aligned} &d/dt\, [(-2\Gamma)^{-1} \sum_n r^n \xi_{m-n+1}(g_n(t))] \\ &\quad = r\,\dot{s}\, \{\, Q'_m{}^2(s) + 2\, Q'_m(s)[g'_1(s)Q'_{m-1}(g_1(s)) + g'_2(s)Q'_{m-2}(g_2(s)) + \ldots \\ &\qquad + g'_{m-1}(s)\, Q'_1(g_{m-1}(s))] + rg'_1(s)Q'_{m-1}{}^2(g_1(s)) \\ &\quad + 2\, [r/g'_1(s)]g'_1(s)Q'_{m-1}(g_1(s))[g'_2(t)Q'_{m-2}(g_2(s)) + \ldots \\ &\qquad + g'_{m-1}(s)Q'_1(g_{m-1}(s))] + r^2 g'_2(s)\, Q'_{m-2}{}^2(g_2(s)) \\ &\quad + 2\, [r^2/g'_2(s)]g'_2(s)Q'_{m-2}(g_2(s))\, [g'_3(s)Q'_{m-3}(g_3(s)) + \ldots + g'_{m-1}(s)Q'_1(g_{m-1}(s))] + \ldots \\ &\qquad + r^{n-1} g'_{m-1}(s)Q'_1{}^2(g_{m-1}(s))\, \}\, . \end{aligned}$$

Therefore using (28):

$$\begin{aligned} &d/dt\, [(-2\Gamma)^{-1} \sum_n r^n \xi_{m-n+1}(g_n(t))] \\ &\quad \geq r\,\dot{s}\, \{Q'_m(s) + g'_1(s)Q'_{m-1}(g_1(s)) + \ldots + g'_{m-1}(s)Q'_1(g_{m-1}(s))\}^2\, . \end{aligned}$$

Therefore, using this result in the expresion (29), with $s = g_1(t)$, we obtain:

$$\begin{aligned} \dot{V}_m(t) \leq -\, 2\Gamma\, \{\, &\dot{Q}_{m+1}{}^2(t) + 2\, \dot{Q}_{m+1}(t)\, [\dot{g}_1(t)Q'_m(g_1(t)) + \dot{g}_2(t)Q'_{m-1}(g_2(t)) + \ldots \\ &+ \dot{g}_m(t)Q'_1(g_m(t))] + r\, \dot{g}_1(t)[Q'_m(g_1(t)) + g'_1(s)Q'_{m-1}(g_2(t)) + \ldots \\ &+ g'_{m-1}(s)Q'_1(g_m(t))]^2\, \} \end{aligned}$$

$$\begin{aligned} = -\, 2\Gamma\, \{\, &\dot{Q}_{m+1}{}^2(t) + 2\, \dot{Q}_{n+1}(t)\, [\, \dot{g}_1(t)Q'_m(g_1(t)) + \dot{g}_2(t)Q'_{m-1}(g_2(t)) + \ldots \\ &+ \dot{g}_m(t)Q'_1(g_m(t))] + [r/\, \dot{g}_1(t)]\, [\, \dot{g}_1(t)Q'_m(g_1(t)) + \dot{g}_1(t)g'_1(s)\, Q'_{m-1}(g_2(t)) + \ldots \\ &+ \dot{g}_1(t)g'_{m-1}(s)Q'_1(g_m(t))\,]^2\, \}\, , \end{aligned}$$

but since $\dot{g}_1(t) \leq r \Rightarrow -\, [r / \dot{g}_1(t)] < -1$ (Proposition III.2.2), we have

$$\dot{g}_1(t)\, g'_m(s) = (dg_1(t)/dt)(dg_m(s)/ds) = dg_m(s)/dt = dg_{m+1}(t)/dt\, .$$

Then

$$\dot{V}_m(t) \le - 2\Gamma \{ \dot{Q}_{m+1}{}^2(t) + 2 \dot{Q}_{m+1}(t)[\dot{g}_1(t)Q'_m(g_1(t)) + \dot{g}_2(t)Q'_{m-1}(g_2(t)) +... + \dot{g}_m(t)Q'_1(g_m(t))] + [r/ \dot{g}_1(t)] [\dot{g}_1(t)Q'_m(g_1(t)) + \dot{g}_2(t)Q'_{m-1}(g_2(t)) +... + \dot{g}_m(t)Q'_1(g_m(t))]^2 \}$$
$$\le - 2\Gamma \{ \dot{Q}_{m+1}(t) + \dot{g}_1(t)Q'_m(g_1(t)) + \dot{g}_2(t)Q'_{m-1}(g_2(t)) +... + \dot{g}_m(t)Q'_1(g_m(t))\}^2 = - 2\Gamma\{ \sum_{k=0}^{m} \dot{g}_k(t)\, Q'_{m+1-k}(g_k(t)) \}^2 , \quad \text{(III.30)}$$

that is $\dot{V}_m(t) \le 0$. Since $m \ge 0$ is arbitrary, we conclude that the $\dot{V}_m(t)$ are, for all $m \ge 0$, non-positive-definite quantities, as asserted. ⌘

Now we can conclude that $V_{N-1}(t_N) \le V_{N-1}(t) \le V_{N-1}(t_{N-1})$ (observe that due to the continuity conditions of Q, i.e. $Q \in C^1(I)$, we can put $V_{N-1}(t_N) = V_N(t_N)$; this will be useful in the next section). In the next Proposition we combine these inequalities to show by mathematical induction that $\xi_N(t) \le E_0$, $\forall N \ge 1$; that is the functions ξ_N are uniformly bounded, $\forall N \ge 1$.

Proposition III.2.5 [26] If $\ddot{R}(t) \ge 0$, then for $t \in I_{N-1} \equiv [t_{N-1}, t_N)$, $\forall N \ge 1$: $\xi_N(t) \equiv \| \underline{Q}_N(t) \|^2 \le E_0$, where $Q = Q(t)$ satisfies the homogeneous equation (1h).

Proof. If $\ddot{R}(t) \ge 0 \Rightarrow \dot{V}_{N-1}(t) \le 0$ (Proposition III.2.4); this implies that:

$$\sum_1^N r^{n-1}E_{N-n+1} \le V_{N-1}(t) = \sum_1^N r^{n-1}\xi_{N-n+1}(g_{n-1}(t)) \le \sum_1^N r^{n-1}E_{N-n} . \quad \text{(III.31)}$$

Define , for m = 1,2,... :

$$\sigma_m \equiv - 2^{m-2}r^{m-1} + E_{m,0} + rE_{m-1,0} + r(2r-1)E_{m-2,0} + ...+ 2^{m-3}r^{m-2}(2r-1)E_{1,0} , \quad \text{(III.32)}$$

(m+1-terms) and

$$\rho_{m-1} \equiv 2^{m-2}r^{m-1} +(1 - r)E_{m-1,0} + r(1-2r)E_{m-2,0} +....+ 2^{m-3}r^{m-2}(1-2r)E_{1,0} , \quad \text{(III.33)}$$

(m-terms) where $\rho_0 \equiv 1$, $E_m / E_0 \equiv E_{m,0}$, for m = 1,2,... .

Now we want to prove by induction that:

$$\sigma_m \leq \xi_{m,0}(t) \equiv \xi_m(t) / E_0 \leq \rho_{m-1} \ , \qquad \text{(III.34)}$$

for m = 1,2,... .

For m = 1, this is trivial ($E_{1,0} \leq \xi_{1,0}(t) \leq 1$). For m = N = 2 we know that (see (23)) $\dot{V}_1(t) \leq 0$, i.e.: $V_1(t_2) \leq V_1(t) \leq V_1(t_1)$ then:

$$E_{2,0} + rE_{1,0} \leq \xi_{2,0}(t) + r\xi_{1,0}(t) \leq E_{1,0} + r \ . \qquad \text{(III.35)}$$

But using $E_{1,0} \leq \xi_{1,0}(t) \leq 1$ we arrive at

$$\sigma_2 = E_{2,0} + rE_{1,0} - r \leq \xi_{2,0}(t) \leq (1 - r)E_{1,0} + r = \rho_1 \ , \qquad \text{(III.36)}$$

which is relation (34) for m = 2.

Suppose now that for m = k > 1 we have:

$$\sigma_k \leq \xi_{k,0}(t) \leq \rho_{k-1} \ . \qquad \text{(III.37)}$$

From (63) we have (n = 1,2,...k+1):

$$\xi_{k+1,0}(t) \leq \sum_n r^{n-1} E_{k+1-n,0} - \sum_n r^n \xi_{k+1-n,0}(g_{n-1}(t))$$
$$\leq E_{k,0} + rE_{k-1,0} + r^2E_{k-2,0} + \dots + r^{k-1}E_{1,0} + r^k - r\xi_{k,0}(t) - r^2\xi_{k-1,0}(t) - \dots$$
$$- r^k \xi_{1,0}(t) \ .$$

Using (37) and (32) for m = 1,2,...k, we obtain:

$$\xi_{k+1,0}(t)$$
$$\leq E_{k,0} + rE_{k-1,0} + r^2 E_{k-2,0} + \dots + r^{k-1} E_{1,0} + r^k - r\sigma_k - r^2 \sigma_{k-1} - \dots - r^k \sigma_1 \leq$$
$$(1-r)E_{k,0} + r(1-2r)E_{k-1,0} + 2r^2(1-2r) E_{k-2,0} + \dots + 2^{k-2}r^{k-1}(1-2r)E_{1,0} + 2^{k-1}r^k =$$
$$\rho_k \ ,$$

the equality following from (33). Operating similarly for the lower bound , $\sigma_{k+1} \leq \xi_{k+1,0}(t)$, we finally obtain:

$$\sigma_{k+1} \leq \xi_{k+1,0}(t) \leq \rho_k , \tag{III.38}$$

which is (34) for m = k + 1, as we wanted to show.

Finally let us show by induction that $\rho_m \leq 1$, m = 1,2,... . For m = 1, we know that $E_{1,0} < 1$, then from the definition of ρ_1, $\rho_1 \equiv E_{1,0}(1-r) + r \leq 1$. Suppose that $E_{m,0} < 1$, m = 2,3,...; then

$$\rho_{k-1} = (1-r)E_{k-1,0} + r(1-2r)E_{k-2,0} +....+ 2^{k-3}r^{k-2}(1-2r)E_{1,0} + 2^{k-2}r^{k-1}$$

$$\leq (1-r) + r(1-2r) +.... + 2^{k-4}r^{k-3}(1-2r) + 2^{k-3}r^{k-2}(1-2r) + 2^{k-2}r^{k-1} = 1.$$

From (32) and (34),

$$\sigma_k \equiv E_{k,0} + rE_{k-1,0} + r(2r-1)E_{k-2,0} +...+ 2^{k-3}r^{k-2}(2r-1)E_{1,0} - 2^{k-2}r^{k-1}$$
$$< \rho_{k-1} \leq 1 ,$$

then ($E_{k,0} < \rho_{k-1} - rE_{k-1,0} - r(2r-1)E_{k-2,0} + ...$) :

$$\begin{aligned} E_{k,0} &< (1-r)E_{k-1,0} + r(1-2r)E_{k-2,0} +....+ 2^{k-3}r^{k-2}(1-2r)E_{1,0} \\ &+ 2^{k-2}r^{k-1} - rE_{k-1,0} + r(1-2r)E_{k-2,0} + ...+ 2^{k-3}r^{k-2}(1-2r)E_{1,0} + 2^{k-2}r^{k-1} \\ &= (1-2r)E_{k-1,0} + 2r(2r-1)E_{k-2,0} +....+ 2^{k-2}r^{k-2}(2r-1)E_{1,0} - 2^{k-1}r^{k-1} , \end{aligned}$$

since $E_{m,0} < 1$, m = 1,2,..., we obtain:

$$E_{k,0} < (1-2r) + 2r(2r-1) +....+ 2^{k-2}r^{k-2}(2r-1) - 2^{k-1}r^{k-1} \leq 1 ;$$

that is, $E_{m,0} < 1$ for m = k. Then using $E_m < E_0$, m = 1,2,...,k, in (30):

$$\rho_k = (1 - r)E_{k,0} + r(1 - 2r)E_{k-1,0} + \dots + 2^{k-2}r^{k-1}(1-2r)E_{1,0} + 2^{k-1}r^k$$

$$\leq (1 - r) + r(1 - 2r) + \dots + 2^{k-3} r^{k-2} (1 - 2r) + 2^{k-2} r^{k-1}(1 - 2r) + 2^{k-1}r^k = 1;$$

therefore $\rho_m \leq 1$, $m = 1,2,\dots$, as we claim.

Therefore we conclude, using this result in (34), that:

$$\xi_n(t) / E_0 \leq \rho_n \leq 1 ,$$

$n = 1,2,\dots$, and therefore:

$$\| \underline{Q}_n(t) \|^2 = \xi_n(t) \leq E_0 , \qquad \text{(III.39)}$$

for $n = 1,2,\dots$. ⌘

With this information, we are ready to prove now that the zero solution of (1h) is uniformly stable.

<u>Corollary III.2.6</u> If $\ddot{R} \geq 0$, then the zero solution of equation (1h) is u.s.

<u>Proof</u>. $\forall\, \varepsilon > 0$, there exist a $\delta = \delta(\varepsilon) > 0$, such that $\forall\, t_0 \in \mathbb{R}$, whenever

$$\| \underline{Q}_1(t_0) \| \equiv (E_0)^{1/2} < \delta(\varepsilon) , \qquad \text{(III.40)}$$

the solution exists and (from (39) and (40)) satisfies :

$$\| \underline{Q}_n(t) \| \leq (E_0)^{1/2} < \delta(\varepsilon) \leq \varepsilon , \qquad \text{(III.41)}$$

$\forall\, n \geq 1$ and thus the zero solution of (1h) is stable. Moreover, since δ does not depend on t_0 then the zero solution of (1h) is u.s. as we claim. ⌘

Finally we will show that, because the inhomogeneous term of (1) is uniformly bounded, then also the zero solution of (1) is uniformly stable.

<u>Corollary III.2.7</u> Consider now the inhomogeneous equation (1). Then its zero solution is uniformly stable.

<u>Proof.</u> Since the inhomogeneous term $F = F(t) = P(f(t)) / R(t)$, where P is

periodic (with finite period, see (4)) and $F \in C^\infty(I)$, then F is bounded, that is there is a $b \in \mathbb{R}^+$ such that $|F(t)| \leq b$, $\forall\, t \in I$.

The inhomogeneous solution of (1) for $t \in I_{N-1} \equiv [t_{N-1}, t_N)$, $N \geq 1$, corresponding to the inhomogeneous term F, is given by:

$$Q_{IN}(t) = (1/\tilde{\omega}) \int dt' \sin \tilde{\omega}(t - t')\, e^{-\Gamma(t - t')} F(t') \quad , \qquad \text{(III.8)}$$

for $t' \in [t_0, t]$, $t \in I_{N-1}$, $N \geq 1$. Then

$$| Q_{IN}(t) | \leq (1/\tilde{\omega}) \int dt'\, e^{-\Gamma(t - t')} | F(t') |$$

$$\leq (b/\tilde{\omega}) \int dt'\, e^{-\Gamma(t - t')} \leq (b/\tilde{\omega}\Gamma) \quad ,$$

$\forall\, t \in I_{N-1}$, $N \geq 1$. We obtain a similar result for $| \dot{Q}_I(t) |$. Hence there exist a $B \in \mathbb{R}^+$ such that: $\| \underline{Q}_{IN}(t) \| \leq B$, for all $t \in I$. Since $Q_N = Q_{HN} + Q_{IN}$, then

$$\| \underline{Q}_N(t) \| \leq \| \underline{Q}_{HN}(t) \| + \| \underline{Q}_{IN}(t) \| \quad ,$$

thus from (39) and the boundedness of $\| \underline{Q}_I(t) \|$ we have:

$$\| \underline{Q}_N(t) \| \leq (E_0)^{1/2} + B \equiv C \quad .$$

Therefore a similar proof to that one of Corollary III.2.6 follows, with the upper bound C instead of $(E_0)^{1/2}$. ⌘

Finally, notice that the radiation reaction equation (41_{ren}) obtained for the Einstein spacetime case (and which is included in the class of equations (1)), satisfy all the Propositions of this section. This is because for R = constant, then $\dot{R} = 0$ and $\ddot{R} = 0$, hence the hypothesis: $\ddot{R} \geq 0$, is satisfied. This permit us to conclude that the zero solution of the equation (41_{ren}) is uniformly stable.

III.3 THE ASYMPTOTIC ANALYSIS OF THE ZERO SOLUTION

In this section we will show that, if (in addition of $\ddot{R} \geq 0$): $\dot{R} > 0$, then the zero solution of the class of equations (1) is globally uniformly asymptotically stable (g.u.a.s.) [14-18]. Remember that uniformly means $\forall$ intial-time t_0 and globally means $\forall$ intial-data $E_0 > 0$.

Initially we shall work with the homogeneous equation (1h). First we show that the length of the intervals $[t_N,t_{N-1}]$, i.e. $(t_N - t_{N-1})$, $N \geq 1$ is directly related with the function $\dot{R}$: if $\dot{R} > 0 \; \forall \; t \in I$ then the length of the intervals $[t_N,t_{N-1}]$ increases with N. The next step will be to show that this fact implies that $V_N(t)$ and therefore $\| \underline{Q}_N(t) \|$ (which are solutions of (1h)), converges suitably to zero with N, that is with t. Here we also analyze this convergence under slightly different hypothesis, namely with a suitable condition on $\ddot{g}_n(t)$ and for the particular case $\dot{R}$ = constant > 0. At the end we consider the global uniform convergence of the zero solution of the inhomogeneous equation (1). Observe that the radiation reaction equation (41_{ren}) corresponding to the Einstein universe is not included in the analysis of this section, because in this case $\ddot{R} \geq 0$ but $\dot{R} = 0$ hence it does not satisfy $\dot{R} > 0, \forall \; t \in I$. Then the zero solution of (41_{ren}) is not globally uniformly asymptotically stable.

Proposition III.3.1 If $\ddot{R} \geq 0$ and $\dot{R} > 0 \; \forall \; t \in I$, then the length of any interval (i.e. $(t_m - t_{m-1})$, $m \geq 2$) is greater than the length of the precedent interval (i.e. $(t_{m-1} - t_{m-2})$, $m \geq 2$). Moreover the rate of increase is greater than or equal to $1/r$, where r was already defined (see Proposition III.2.2): $r \equiv R(t_0)/R(t_1)$.

Proof. If $\dot{R} > 0$, then $r < 1$. Hence from Proposition II.2.2, if $\ddot{R} \geq 0$ and $\dot{R} > 0 \Rightarrow \dot{g}_n(t) \leq r^n < 1$. Suppose now that $t \in [t_{m-1},t_m]$, then integrating $\dot{g}_1(t) \leq r$:

$$\dot{g}_1(t) \leq r < 1 \Rightarrow \; g_1(t_m) - g_1(t_{m-1}) \leq r \, (t_m - t_{m-1}) \, ,$$

but $g_k(t_m) = t_{m-k}$, then $(t_{m-1} - t_{m-2}) \leq r \, (t_m - t_{m-1}) < (t_m - t_{m-1})$. Moreover,

$$(t_m - t_{m-1}) / (t_{m-1} - t_{m-2}) \geq (1/r) \,. \qquad ⌘$$

Definition III.3.2 Define a useful norm:

$$\| \underline{Q}_N(t) \|_0^2 \equiv \tfrac{1}{2} \left[(Q_N(t) + \Gamma \dot{Q}_N(t))^2 + (\Gamma^2 + \omega^2) Q_N^2(t) \right] . \qquad \text{(III.43)}$$

Remember the definition (III.21) of the positive definite function V_{N-1}: $[t_0,t_N] \times \mathbb{R}^2 \to \mathbb{R}^+$,

$$V_{N-1}(t,Q,\dot{Q}) \equiv V_{N-1}(t) \equiv \sum_1^N r^{n-1} \xi_{N-n+1}(g_{n-1}(t)) \;. \qquad \text{(III.21)}$$

for $N \geq 1$, with $g_0(t) \equiv t$. Note that $\forall\ N \geq 1$, $V_{N-1} = 0 \Leftrightarrow \xi_{N-n+1} = 0 \Leftrightarrow Q = \dot{Q} = 0$. Define the positive definite function W_{N-1}: $[t_0,t_N] \times \mathbb{R}^2 \to \mathbb{R}^+$,

$$W_{N-1}(t,Q,\dot{Q}) \equiv W_{N-1}(t) \equiv \| \underline{Q}_N(t) \|_0^2 + \beta \sum_n r^{n-1} \xi_{N-n}(g_n(t)), \qquad \text{(III.44)}$$

where $N\text{-}1 \geq n \geq 1$ (then the sum appear only for $N \geq 2$), $\beta \equiv 4(1+\omega^2)$, which vanishes $\Leftrightarrow \xi_{N-n+1} = 0$ and $Q_N = \dot{Q}_N = 0 \Leftrightarrow Q_m = \dot{Q}_m = 0$, $m \geq 1$. For simplicity in the notation, from now on we will write: $W_{N-1}(t) \equiv W_{N-1}(t,Q,\dot{Q})$. Moreover we denote $W_{N-1}(t_{N-1}) \equiv W_{N-1,N-1}$. ⌘

Now using some algebra calculations we find a relation between the norms $\| \underline{Q}_N(t) \|_0$ and $\| \underline{Q}_N(t) \|$.

Proposition III.3.3 From the above definitions it holds that

$$r_m \| \underline{Q}_N(t) \|^2 \leq \| \underline{Q}_N(t) \|_0^2 \leq r_M \| \underline{Q}_N(t) \|^2 , \qquad \text{(III.45)}$$

where $0 < r_m < 1 < r_M < 2(1-h)^{-1}$, $h \equiv \omega^{-2} \Gamma^2 < 1$.

Proof. This relation between the norms is a direct aplication of the algebraic property that says: if $b^2 \leq ac \Rightarrow ax^2 + 2bxy + cy^2 \geq 0$, where $a, c \in \mathbb{R}^+$, $b \in \mathbb{R}$.

⌘

Then this relation permit us to find a useful relation between the positive

-definite functions $W_{N-1}(t)$ and $V_{N-1}(t)$.

Corollary III.3.4 If $\ddot{R} \geq 0$ and $\dot{R} > 0$, $\forall\, t \in I$, then for $t \in [t_{N-1},t_N]$, $N \geq 1$, we have:

(i) $$\beta_m\, V_{N-1}(t) \leq W_{N-1}(t) \leq \beta_M\, V_{N-1}(t)\,,$$

where $\beta_m \equiv \min\{r_m,\beta\}$ and $\beta_M \equiv \max\{r_M,\beta\}$; and

(ii) $$W_{N-1}(t) \leq E_0\, \beta_M \sum_1^N r^n < E_0\, \beta_M \sum_{n=1}^{\infty} r^n \equiv D\,,$$

where $D \in \mathbb{R}^+$.

Proof. (i) From (44) and (45):

$$r_m \| \underline{Q}_N(t) \|^2 + \beta \sum_n r^{n-1} \xi_{N-n}(g_n(t)) \leq W_{N-1}(t)$$
$$\leq r_M \| \underline{Q}_N(t) \|^2 + \beta \sum_n r^{n-1} \xi_{N-n}(g_n(t))\;;$$

and using (III.21) :

$$\beta_m\, V_{N-1}(t) \leq W_{N-1}(t) \leq \beta_M\, V_{N-1}(t)\,,$$

where $\beta_m \equiv \min\{r_m,\beta\}$ and $\beta_M \equiv \max\{r_M,\beta\}$.

(ii) From Proposition III.2.4:

$$V_{N-1}(t_N) \leq V_{N-1}(t) \leq V_{N-1}(t_{N-1})$$

for $t \in [t_{N-1},t_N]$, $N \geq n \geq 1$. From (21): $V_{N-1}(t_{N-1}) = \sum_1^N r^n \xi_{N-n}(g_n(t_{N-1}))$.

$$V_{N-1}(t_N) \leq V_{N-1}(t) \leq \sum_1^N r^n \xi_{N-n}(g_n(t_{N-1}))\;.$$

Using this result in (i) we obtain: $W_{N-1}(t) \leq \beta_M \sum_1^N r^n \xi_{N-n}(g_n(t_{N-1}))$, but from Proposition III.2.5: $\xi_{N-n}(g_n(t_{N-1})) \leq E_0$, therefore

$$W_{N-1}(t) \leq E_0\, \beta_M \sum_1^N r^n \leq E_0\, \beta_M \sum_{n=1}^{\infty} r^n \equiv D\,,$$

for $t \in [t_{N-1}, t_N]$, $N \geq 1$, where $D \in \mathbb{R}^+$. In particular, by continuity:

$$W_{N-1}(t_{N-1}) \equiv W_{N-1,N-1} \leq D \ . \qquad ⌘$$

The next step will be, profiting all these relations and some calculations from Proposition III.2.4, to show that $\dot{W}_{N-1}(t) \leq - \Gamma \| Q_N(t) \|^2$.

Proposition III.3.5 If $\ddot{R} \geq 0$ and $\dot{R} > 0$, $\forall\ t \in I$, then

$$\dot{W}_{N-1}(t) \leq - \Gamma \| Q_N(t) \|^2 \quad ,$$

for $t \in [t_{N-1}, t_N)$, $N \geq n \geq 1$, along the solutions of equation (1h).

Proof. If $\ddot{R} \geq 0$, from Proposition II.2.4, we know that:

$$d/dt\ [(-2\Gamma)^{-1} \sum_n r^n \xi_{N-n}(g_n(t))] \geq (\sum_n \dot{g}_n(t) Q'_{N-n}(g_n(t)))^2 \geq 0 \ , \qquad \text{(III.46)}$$

for $t \in [t_{N-1}, t_N)$, $N-1 \geq n \geq 1$, which is a non-negative definite quantity.

Now let us calculate the time derivative of t along the solutions of (1h):

$$\begin{aligned} \dot{W}_{N-1}(t) = &- \Gamma \| Q_N(t) \|^2 - ½\Gamma\, \dot{Q}_N(t)^2 - ½\Gamma\omega^2 Q_N(t)^2 \\ &- 4\Gamma(\dot{Q}_N(t) + \omega^2 Q_N(t))\ [\sum_n \dot{g}_n(t) Q'_{N-n}(g_n(t))] \\ &+ \beta\ d/dt\ [\sum_n r^n \xi_{N-n}(g_n(t))] \ , \end{aligned}$$

and using (46):

$$\begin{aligned} \dot{W}_{N-1}(t) = &- \Gamma \| Q_N(t) \|^2 - ½\Gamma\, \dot{Q}_N(t)^2 - ½\Gamma\omega^2 Q_N(t)^2 \\ &- 4\Gamma\, \dot{Q}_N(t)[\sum_n \dot{g}_n(t) Q'_{N-n}(g_n(t))] - 4\Gamma\omega^2 Q_N(t)[\sum_n \dot{g}_n(t) Q'_{N-n}(g_n(t))] \\ &- 2\Gamma\beta\ [\sum_n \dot{g}_n(t) Q'_{N-n}(g_n(t))]^2 \ , \end{aligned}$$

remembering that $\beta \equiv 4(1+\omega^2)$, we have :

$$\dot{W}_{N-1}(t) = - \Gamma \| Q_N(t) \|^2$$

$$- \tfrac{1}{2}\Gamma\,\dot{Q}_N(t)^2 - \tfrac{1}{2}\Gamma\omega^2 Q_N(t)^2$$
$$- 4\Gamma\,\dot{Q}_N(t)[\sum_n \dot{g}_n(t)Q'_{N-n}(g_n(t))] - 4\Gamma\omega^2 Q_N(t)[\sum_n \dot{g}_n(t)Q'_{N-n}(g_n(t))\ ]$$
$$- 8\Gamma\,[\sum_n \dot{g}_n(t)Q'_{N-n}(g_n(t))]^2 - 8\Gamma\omega^2\,[\sum_n \dot{g}_n(t)Q'_{N-n}(g_n(t))]^2$$

$$= - \Gamma \,\|\, \underline{Q}_N(t) \,\|^2 - \tfrac{1}{2}\Gamma(\, \dot{Q}_N(t) + 4\sum_n \dot{g}_n(t)Q'_{N-n}(g_n(t))\,)^2$$
$$- \tfrac{1}{2}\Gamma\omega^2(\, Q_N(t) + 4\sum_n \dot{g}_n(t)Q'_{N-n}(g_n(t))\,)^2 \ ,$$

therefore :

$$\dot{W}_{N-1}(t) \leq - \Gamma \,\|\, \underline{Q}_N(t) \,\|^2 \quad , \tag{III.47}$$

as we claim. ⌘

Now with this information we are ready to show that $\lim_{t\to\infty} V_{N-1}(t) = 0$, and from this result will not be difficult to show the convergence of $\|\, \underline{Q}_N(t) \,\|$. Notice that there exist two classes of functions $R = R(t)$ satisfying $\ddot{R} \geq 0$, and $\dot{R} > 0$; those for which (remember that $N = [f(t)/2\pi]$ indicates the number of intervals until time t): (a) $\lim_{t\to\infty} [f(t)/2\pi] = \infty$ and (b) $\lim_{t\to\infty} [f(t)/2\pi] < \infty$. Here we will consider case (a).

<u>Proposition III.3.6</u> If $\ddot{R} \geq 0$ and $\dot{R} > 0$, $\forall\, t \in I$, such that $\lim_{t\to\infty} [f(t)/2\pi] = \infty$, then

$$\lim_{N\to\infty} V_{N-1}(t) = 0 \ ,$$

uniformly in t and globally (i.e. $\forall\, E_0$).

<u>Remark.</u> In the N-finite case this Proposition reads: if $\ddot{R} \geq 0$ and $\dot{R} > 0$, $\forall\, t \in I$, such that $\lim_{t\to\infty} [f(t)/2\pi] \equiv N < \infty$, then

$$\lim_{t\to\infty} V_{N-1}(t) = 0 \ ,$$

uniformly in t and globally. The proof is similar to the following one, with the obvious change that the analysis, instead of be done in the (sufficiently large but finite) interval $[t_{N-1},t_N)$, is performed in the infinite interval $[t_{N-1},\infty)$, because now $t_N = \infty$. (For another proof of this N-finite case see [26].)

Proof. To prove this we will use the absurtio method. The proof follows $\forall\ \varepsilon > 0$. Fix an $\varepsilon > 0$, choose an $\varepsilon' > 0$ such that : $\varepsilon' \leq \varepsilon / 2\sigma$ where $\sum_{n=1}^{\infty} r^{n-1} \equiv \sigma < \infty$ ($\sum_{n=1}^{\infty} r^{n-1}$ is a convergent series because $r < 1$). Suppose that:

$$\| Q_N(t) \|^2 \geq \varepsilon' , \qquad \text{(III.48)}$$

for all $N \geq 1$, with $t \in [t_{N-1},t_N)$, then using this relation in (47) we have:

$$\dot{W}_{N-1}(t) \leq - \Gamma \| Q_N(t) \|^2 \leq - \Gamma\varepsilon' , \qquad \text{(III.49)}$$

and integrating between $[t_{N-1},t]$:

$$W_{N-1}(t) \leq - \Gamma\varepsilon' (t - t_{N-1}) + W_{N-1}(t_{N-1}) , \qquad \text{(III.50)}$$

and from Corollary (III.3.4) (ii) ($W_{N-1,N-1} \leq [r_M + \beta \sum_n r^n]E_0 \equiv D < \infty$):

$$W_{N-1}(t) \leq - \Gamma\varepsilon' (t - t_{N-1}) + D . \qquad \text{(III.51)}$$

But since the length of the interval increases with N (Proposition III.3.1), we can always find an N_ε such that for $N \geq N_\varepsilon$: $D < \Gamma\varepsilon\ (t_N - t_{N-1})$, or

$$(t_N - t_{N-1}) > (\Gamma\varepsilon')^{-1}D .$$

Then there exist a $T \in [t_{N-1},t_N)$ defined by $T \equiv t_{N-1} + (\Gamma\varepsilon')^{-1}D$ such that, for all $t > T$, it follows

$$- \Gamma\varepsilon\ (t - t_{N-1}) + D < 0 ,$$

and therefore, from (12) for $t > T \in [t_{N-1},t_N\}$:

$$W_{N-1}(t) < 0 ,$$

which contradicts the definition of $W_{N-1} \geq 0$ (see (44)). This lead us to conclude that is false that for all N≥1, $\| \underline{Q}_N(t) \|^2 \geq \varepsilon'$. Then for $N \geq N_\varepsilon$, $t \in [t_{N-1},t_N)$:

$$\| \underline{Q}_N(t) \|^2 < \varepsilon' \ . \qquad \text{(III.52)}$$

This implies that for all $N \geq N_\varepsilon$ (remember $E_N \equiv \| \underline{Q}_N(t_N) \|^2$) :

$$E_N \equiv \| \underline{Q}_N(t_N) \|^2 < \varepsilon' \ . \qquad \text{(III.53)}$$

Then from (III.21) for all $N > N_\varepsilon$:

$$V_{N-1}(t) = \sum_1^N r^{n-1}\xi_{N-n+1}(g_{n-1}(t_N))$$

$$\leq \sum_1^N r^{n-1}\xi_{N-n+1}(g_{n-1}(t_N)) + r^N \sum_{N+1}^{\infty} r^{n-1}\xi_{N-n+1}(g_{n-1}(t_N)) \ .$$

From (53): $\xi_{N-n+1}(g_{n-1}(t_N)) = E_N < \varepsilon'$, for n=1,2,...N. Furthermore from Proposition III.2.5: $\xi_{N-n+1}(g_{n-1}(t_N)) \leq E_0$, for all $n \geq 1$, then

$$V_{N-1}(t) \leq \varepsilon' \sum_1^N r^{n-1} + r^N E_0 \sum_{N+1}^{\infty} r^{n-1} \leq \varepsilon' \sigma + r^N \sigma E_0 \ ,$$

since $\varepsilon' > 0$ was choosed such that: $\varepsilon' \leq \varepsilon / 2\sigma$, $\forall \varepsilon > 0$, then: $\varepsilon' \sigma \leq 1/2\ \varepsilon$, moreover for all $N \geq \ln[\varepsilon / 2\sigma E_0] / \ln r$ follows that: $r^N \sigma E_0 \leq 1/2\ \varepsilon$; therefore we conclude that for all $N \geq \max\{ N_\varepsilon , \ln[\varepsilon / 2\sigma E_0] / \ln r \}$

$$V_{N-1}(t) \leq \varepsilon' \sigma + r^N \sigma E_0 \leq 1/2\ \varepsilon + 1/2\ \varepsilon = \varepsilon \ . \qquad \text{(III.54)}$$

Since this result follows $\forall\ \varepsilon > 0$, then we conclude that

$$\lim_{N \to \infty} V_{N-1}(t) = 0 \ , \qquad \text{(III.55)}$$

uniformly in t (since the prove follows for all intial-time t_0) and globally (since the proof follows $\forall E_0 > 0$). ⌘

Now consider the case in which the condition $\dot{R} > 0$, $\forall t \in I$, is replaced by a condition on $g_n(t)$. The result is an exponential convergence, with N, of $\xi_N(t)$.

Proposition III.3.6' If $\ddot{R} \geq 0\ \forall t \in I$, and $g_n(t)$ is such that, there exist an $\varepsilon > 0$ that satisfies $\ddot{g}_n(t) \leq -\varepsilon$, $\forall t \in [t_{N-1}, t_N]$, $N \geq n \geq 1$, then

$$\lim_{N\to\infty} V^*_{N-1}(t) = 0 \ ,$$

uniformly in t and globally, where

$$V^*_{N-1}(t, Q, \dot{Q}) \equiv V^*_{N-1}(t) \equiv \sum_n \dot{g}_{n-1}(t) \xi_{N-n+1}(g_{n-1}(t)) \ .$$

Proof. It is not difficult to see that using V^*_{N-1} instead of V_{N-1} in Proposition III.2.4 we arrive at

$$\dot{V}^*_{N-1}(t) \leq \sum_n \ddot{g}_{n-1}(t) \xi_{N-n+1}(g_{n-1}(t)) \ . \qquad \text{(III.56)}$$

Then using $\ddot{g}_n(t) \leq -\varepsilon$ in (56) we obtain :

$$\dot{V}^*_{N-1}(t) \leq -\varepsilon \sum_n \xi_{N-n+1}(g_{n-1}(t)) \ .$$

From Proposition III.2.2: $\dot{g}_{n-1}(t) \leq r$, hence: $V^*_{N-1}(t) \equiv \sum_n \dot{g}_{n-1}(t)\ \xi_{N-n+1}(g_{n-1}(t)) \leq r \sum_n \xi_{N-n+1}(g_{n-1}(t))$. Then we use this to find a relation between V^* and $\dot{V}^*$:

$$\dot{V}^*_{N-1}(t) \leq -\varepsilon \sum_n \xi_{N-n+1}(g_{n-1}(t)) \leq -(\varepsilon / r)\ V^*_{N-1}(t) \ . \qquad \text{(III.57)}$$

Integrating (57) for $t \in [t_{N-1}, t_N]$ we obtain :

$$V^*_{N-1}(t_N) \leq V^*_{N-1}(t_{N-1})\ e^{-\Gamma'(t_N - t_{N-1})} \quad ,$$

where $\Gamma' \equiv (\varepsilon / r)$. Since $\lim_{N\to\infty} (t_N - t_{N-1}) = \infty$ (see Proposition III.3.1), then in the limit

$$\lim_{N\to\infty} V^*_{N-1}(t) = 0 \quad ,$$

uniformly in t and globally. ⌘

Now consider the particular case in which R is a linear function of t, that $R(t) = mt + b$, m (> 0) and b real constants, then $\dot{R} = m > 0$.

Proposition III.3.6" If $\dot{R}$ = constant > 0, $\forall\, t \in I$, then

$$\lim_{N\to\infty} V_{N-1}(t) = 0 \quad ,$$

uniformly in t and globally.

Proof. If $\dot{R} = m$ = constant > 0, then $R = m\,t + b$, then calculating $g_n(t)$ from its definition: $\dot{g}_n(t) = r^n$, $N \geq n \geq 1$. Now define a positive-definite function W^, (similar to W):

$$W^\wedge_{N-1}(t,Q,\dot{Q}) \equiv W^\wedge_{N-1}(t) \equiv \|\, Q_N(t)\, \|_0{}^2 + \beta \sum_n \dot{g}_n(t)\xi_{N-n}(g_n(t)) + (R_{N-1}/R(t))\sum_n \dot{g}_n(t)\xi_{N-n}(g_n(t)) \quad . \qquad \text{(III.44^)}$$

Taking the time derivative of W^ along the solutions of (1h) (using the calculations performed in Proposition III.3.5) we arrive to:

$$\dot{W}^\wedge_{N-1}(t) = -\Gamma \,\|\, Q_N(t)\, \|^2 - \tfrac{1}{2}\Gamma(\dot{Q}_N(t) + 4\sum_n \dot{g}_n(t)Q'_{N-n}(g_n(t)))^2 - \tfrac{1}{2}\Gamma\omega^2(Q_N(t) + 4\sum_n \dot{g}_n(t)Q'_{N-n}(g_n(t)))^2 + (R_{N-1}/R(t)) \sum_n \ddot{g}_n(t)\xi_{N-n}(g_n(t)) - (mR_{N-1}/R^2(t))\sum_n \dot{g}_n(t)\xi_{N-n}(g_n(t))$$

but $\dot{g}_n(t) = r^n$ and $\ddot{g}_n(t) = 0$, $N \geq n \geq 1$ then

$$\dot{W}^\wedge_{N-1}(t) \leq -\Gamma \,\|\, Q_N(t)\, \|^2 - (mR_{N-1}/R^2(t))\sum_n r^n\xi_{N-n}(g_n(t))$$
$$\leq -\Gamma \,\|\, Q_N(t)\, \|^2 - [R_{N-1}/R_N](m/R(t))\sum_n r^n\xi_{N-n}(g_n(t))$$

$$= - \Gamma \parallel \underline{Q}_N(t) \parallel^2 - (rm/R(t)) \sum_n r^n \xi_{N-n}(g_n(t))$$
$$\leq - R^- [\parallel \underline{Q}_N(t) \parallel^2 + \sum_n r^n \xi_{N-n}(g_n(t))] , \qquad \text{(III.58)}$$

where $R^- = \min\{\Gamma,(rm/R_N)\}$, $R_{N-1,N} / R_N \geq r$. Note that $R^- = R^-(N)$. Using (45) in (44^), we obtain:

$$W\wedge_{N-1}(t) = \parallel \underline{Q}_N(t) \parallel_0^2 + \beta \sum_n \dot{g}_n(t)\xi_{N-n}(g_n(t)) + (R_{N-1}/R(t)) \sum_n \dot{g}_n(t)\xi_{N-n}(g_n(t))$$
$$\leq r_M \parallel \underline{Q}_N(t) \parallel^2 + \beta \sum_n r^n \xi_{N-n}(g_n(t)) + (R_{N-1}/R(t)) \sum_n r^n \xi_{N-n}(g_n(t))$$
$$\leq r_M \parallel \underline{Q}_N(t) \parallel^2 + (\beta +1) \sum_n r^n \xi_{N-n}(g_n(t))$$
$$\leq \beta_M [\parallel \underline{Q}_N(t) \parallel^2 + \sum_n r^n \xi_{N-n}(g_n(t))] , \qquad \text{(III.59)}$$

where $\beta^+ = \max\{r_M,(\beta +1)\}$. Then using (59) in (58) we obtain :

$$\dot{W}\wedge_{N-1}(t) \leq - \Gamma'' W\wedge_{N-1}(t) ,$$

where $\Gamma'' = (R^-/\beta^+)$. Integrating easily this expresion we obtain :

$$W\wedge_{N-1}(t) \leq W\wedge_{N-1}(t_{N-1})\, e^{-\Gamma''[t-t_{N-1}]} . \qquad \text{(III.60)}$$

Here we have to proceed carefully because it can happens that $\Gamma''= \Gamma''(N)$, and for $N \to \infty$ we would have $\Gamma''(N) \to 0$, and therefore might not be convergence of W^. However, from an easy calculation $R_n = R_0\, r^{-n}$ and $t_n = (R_0 r^{-n} - b)/m$,

$$\Rightarrow \qquad (t_N-t_{N-1})/R_N = [(1 - r)/m]\, R_0\, r^{-N} /R_N = [(1 - r)/m] .$$

Therefore for the intervals where N is such that $\Gamma \leq (rm/R_N) = (rm/R_0)r^N$, we have $R^- = \Gamma$, then $\Gamma''= \Gamma/\beta^+ \in \mathbb{R}^+$, hence $\Gamma'' \neq \Gamma''(N)$, and thus relation (21) is useful; in such a case the decayment of W^ is of exponential type. Instead for $\Gamma >$ (rm/R_N), we have $\Gamma''= (rm/R_N\beta^+)$, then

$$- \Gamma''(t_N-t_{N-1}) = - (rm/\beta^+)\, (t_N-t_{N-1})/R_N = - (r/\beta^+)(1 - r) = \ln \rho ,$$

where ρ is a real number less than one because - $(r/\beta^+)(1 - r)$ is negative-definite: i.e. $\rho < 1$. Then for $t = t_N$, in (60) we obtain

$$W^\wedge_{N-1}(t_N) \leq W^\wedge_{N-1}(t_{N-1})\, e^{-\Gamma''[t_N - t_{N-1}]} = W^\wedge_{N-1}(t_{N-1})\, \rho \,.$$

Therefore for the N+M-th interval, that is for $t \in [t_{N+M-1}, t_{N+M}]$, $N+M \geq n \geq 1$, we will obtain for the end-point of the interval $t = t_{N+M}$:

$$W^\wedge_{N+M-1}(t_{N+M}) \leq W^\wedge_{N-1}(t_{N-1})\, \rho^{M+1} \,,$$

and we observe that for $M \to \infty$, since $\rho < 1$, then $W^\wedge_{N+M-1}(t_{N+M}) \to 0$. Therefore, from the definition of $W^\wedge$ one arrives to a relation similar to Corollary III.3.4 (i), from which one concludes that:

$$\lim_{N\to\infty} V_{N-1}(t, Q, \dot{Q}) = 0 \;,$$

uniformly in t and globally. ⌘

Now we will show the global uniform convergence of $\| \underline{Q}_N(t) \|$.

Corollary III.3.7 If $\ddot{R} \geq 0$ and $\dot{R} > 0$, $\forall\, t \in I$, then

$$\lim_{N\to\infty} \| \underline{Q}_N(t) \| = 0 \;,$$

uniformly in t (i.e. $\forall\, t_0$) and globally (i.e. $\forall\, E_0$). That is, the zero solution of (1h) is globally uniformly asymptotic (g.u.a.).

Proof. It follows by using Proposition III.3.6 in the definition of $V_{N-1}(t)$. ⌘

This result means that the zero solution of (1h) is g.u.a.. Since for $\ddot{R} \geq 0$ $\forall\, t \in I$, the zero solution is u.s. (see section III.2), we conclude that

Corollary III.3.8 If $\ddot{R} \geq 0$ and $\dot{R} > 0$, $\forall\, t \in I$, then the zero solution of (1h) is g.u.a.s..

Finally we also show that the zero solution of the inhomogeneous solution (1) is g.u.a.s..

Corollary III.3.9 If $\ddot{R} \geq 0$ and $\dot{R} > 0$, $\forall$ t $\in$ I, then the zero solution of (1) is g.u.a.s..

Proof. We have showed in Corollary III.2.7 that for $\ddot{R} \geq 0$ the zero solution of (1) is u.s. Now we only have to see that: $\lim_{N\to\infty} \| \underline{Q}_{IN}(t) \| = 0$, uniformly in t and globally, where $Q_{IN} = Q_{IN}(t)$ for $t \in I_{N-1} \equiv [t_{N-1}, t_N)$ $N \geq 1$, is the inhomogeneous solution corresponding to the inhomogeneous term F = F(t).

We know that: F(t) = [1/R(t)] P(f(t)) where P is periodic (with finite period) and $F \in C^\infty(I)$, then there exist a real $b \in \mathbb{R}^+$ such that:

$$| F(t) | = [1/R(t)] \; P(f(t)) \leq b / R(t) \, ,$$

$\forall$ t $\in$ I. Since $\ddot{R} \geq 0$ and $\dot{R} > 0$ $\forall$ t $\in$ I, then $\lim_{t\to\infty} R(t) = \infty$, that is $\forall \, \varepsilon > 0$, we can always find an $N_\varepsilon = (N-1)$, such that: $| F(t) | < \varepsilon \equiv (b / R(t_{N-1}))$, for all $t > t_{N-1}$.

On the other hand the inhomogeneous solution of (1) for $t \in I_{N-1} \equiv [t_{N-1}, t_N)$, $N \geq 1$, corresponding to the inhomogeneous term F, is given by:

$$Q_{IN}(t) = (1/\tilde{\omega}) \int dt' \, \sin \tilde{\omega}(t - t') \, e^{-\Gamma(t - t')} \, F(t') \quad , \qquad \text{(III.8)}$$

for $t' \in [t_0, t]$ and $\forall$ t $\in$ I. Then

$$| Q_{IN}(t) | \leq (1/\tilde{\omega}) \int dt' \, e^{-\Gamma(t - t')} \, | F(t') |$$

$$\leq (b/\tilde{\omega}) \int dt' \, e^{-\Gamma(t - t')} \leq (b/\tilde{\omega}\Gamma R(t_{N-1}))$$

$$= (1/\tilde{\omega}\Gamma) \, \varepsilon \; ,$$

and we can obtain a similar result for $| \dot{Q}_{IN}(t) |$, where $\Gamma, \tilde{\omega} \in \mathbb{R}^+$ are fixed constants. Then there exist a $B \in \mathbb{R}^+$ (which depends on Γ and $\tilde{\omega}$ but not on N, t_0 or E_0) such that: $\| \underline{Q}_{IN}(t) \| \leq B \, \varepsilon$, whenever $t > t_{N-1}$. Since this result is valid $\forall \, \varepsilon > 0$, then we conclude that

$$\lim_{N \to \infty} \| \underline{Q}_{IN}(t) \| = 0 \ ,$$

uniformly in t (i.e. $\forall\, t_0$) and globally (i.e. $\forall\, E_0$). Then the proof is complete. ⌘

III.4 THE COMPUTER SIMULATIONS

We have performed many computer simulations (the corresponding grafics Q vs. t and $\dot{Q}$ vs. t are shown in the next pages) of the radiation-reaction equations (III.39) and (III.1), for arbitrary initial-data. It is pleasent to see (as we can observe from the graphics) that the simulations are in perfect agreement with the results predicted in the Propositions proved in this work about the convergency of the solutions of the radiation-reaction equations .

FIGURE CAPTIONS

In table 1 we show the retarded functions and its derivatives coresponding to three cases: R = constant (Einstein universe case), R = R(t) such that $N = \infty$ and R = R(t) such that $N < \infty$, where N is the number of terms with retarded arguments. In figure 1 we simulate the homogeneous equation ($II.41_{ren}$) (Einstein universe case).

In figures 2 and 3 we simulate equation (III.1h), for different values of r [27].

$R(t)$	$f(t)$	$g_n(t)$
$R_0 \in \mathbb{R}^+$	$(t - t_0)/R_0$	$t - 2\pi R_0 n$
$mt,\ m \in \mathbb{R}^+$ $(N = \infty)$	$(1/m) \ln t$	$e^{-2\pi mn}\, t$
$e^{at}, 0 < a < (2\pi N)^{-1}$ $(N < \infty)$	$(e^{-at_0} - e^{-at})/a$	$- \ln(e^{-at} + 2\pi an)/a$

Table 1

Fig. 1. R = const., $\dot{g}_n(t) = 1$

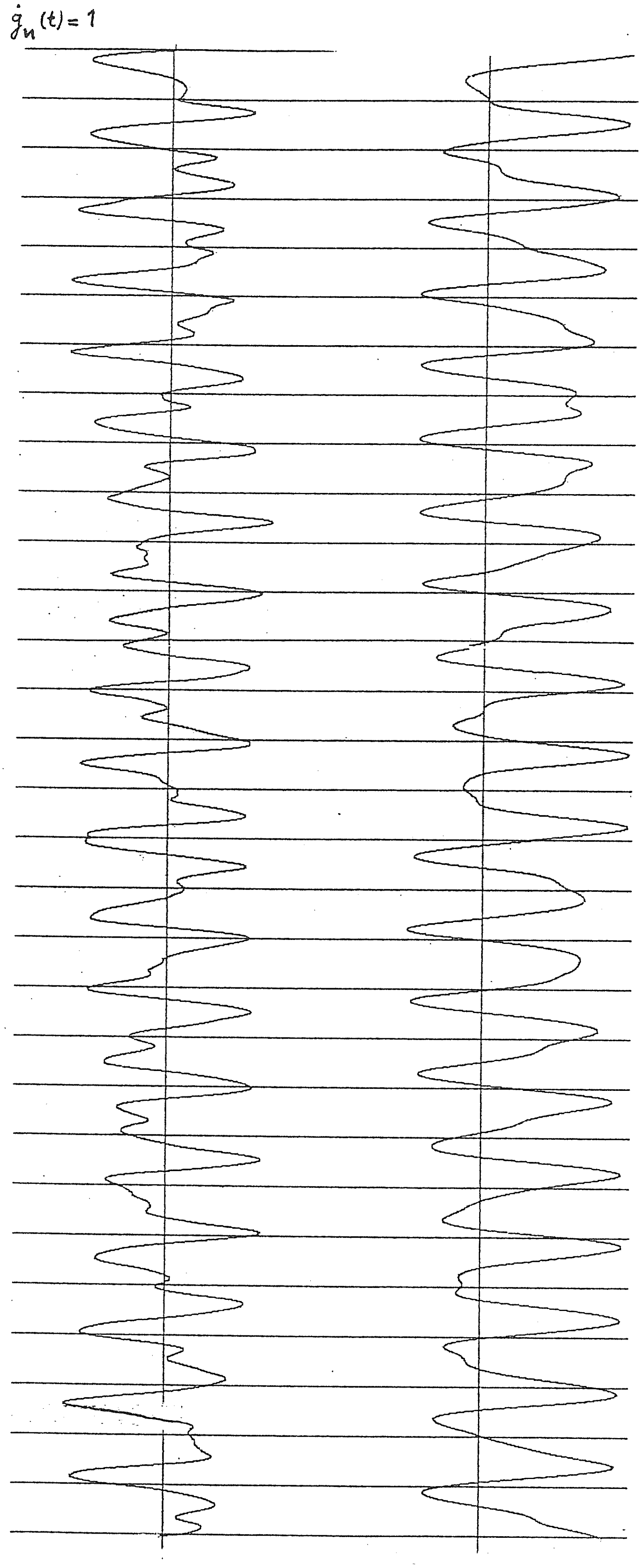


Fig. 2. $r \equiv R(t_0)/R(t_1) = 0.9, 0.5$

$\Gamma = 0.01$

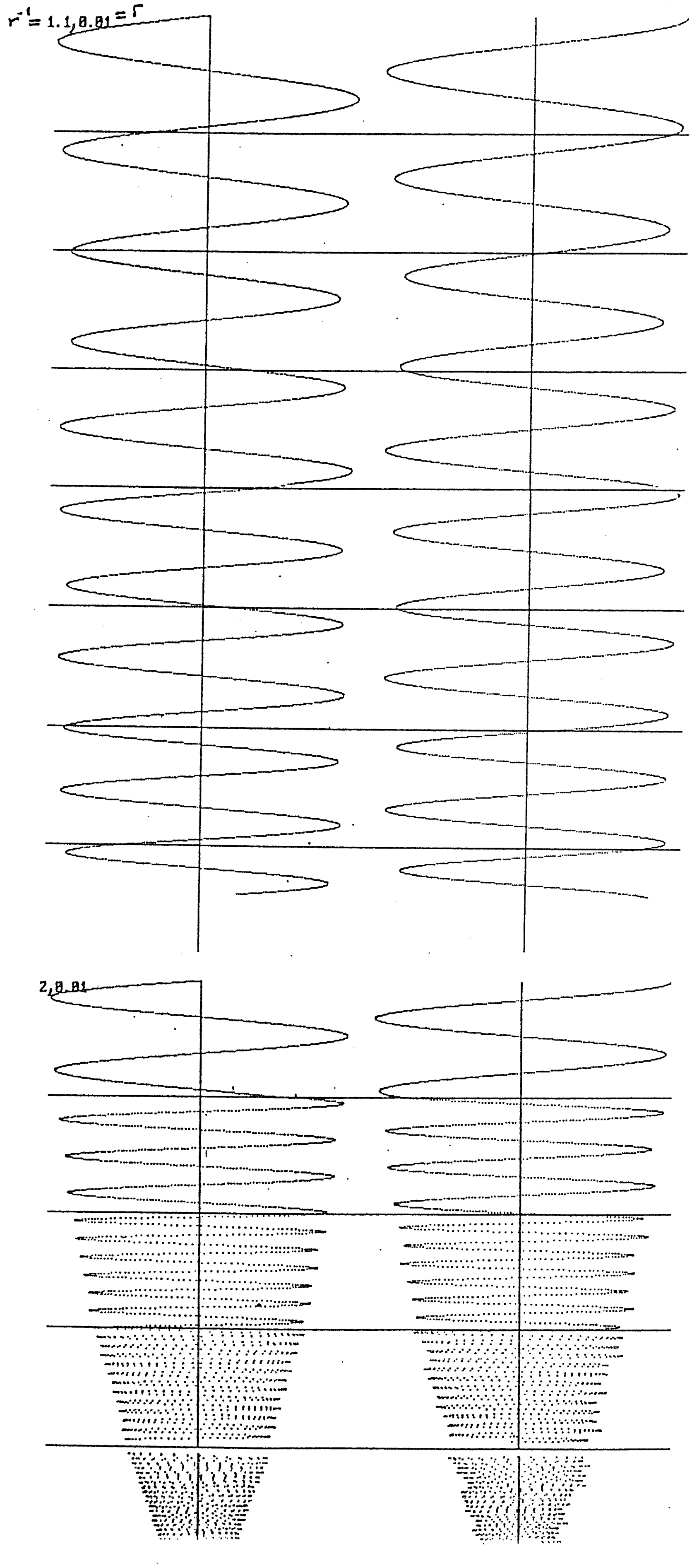
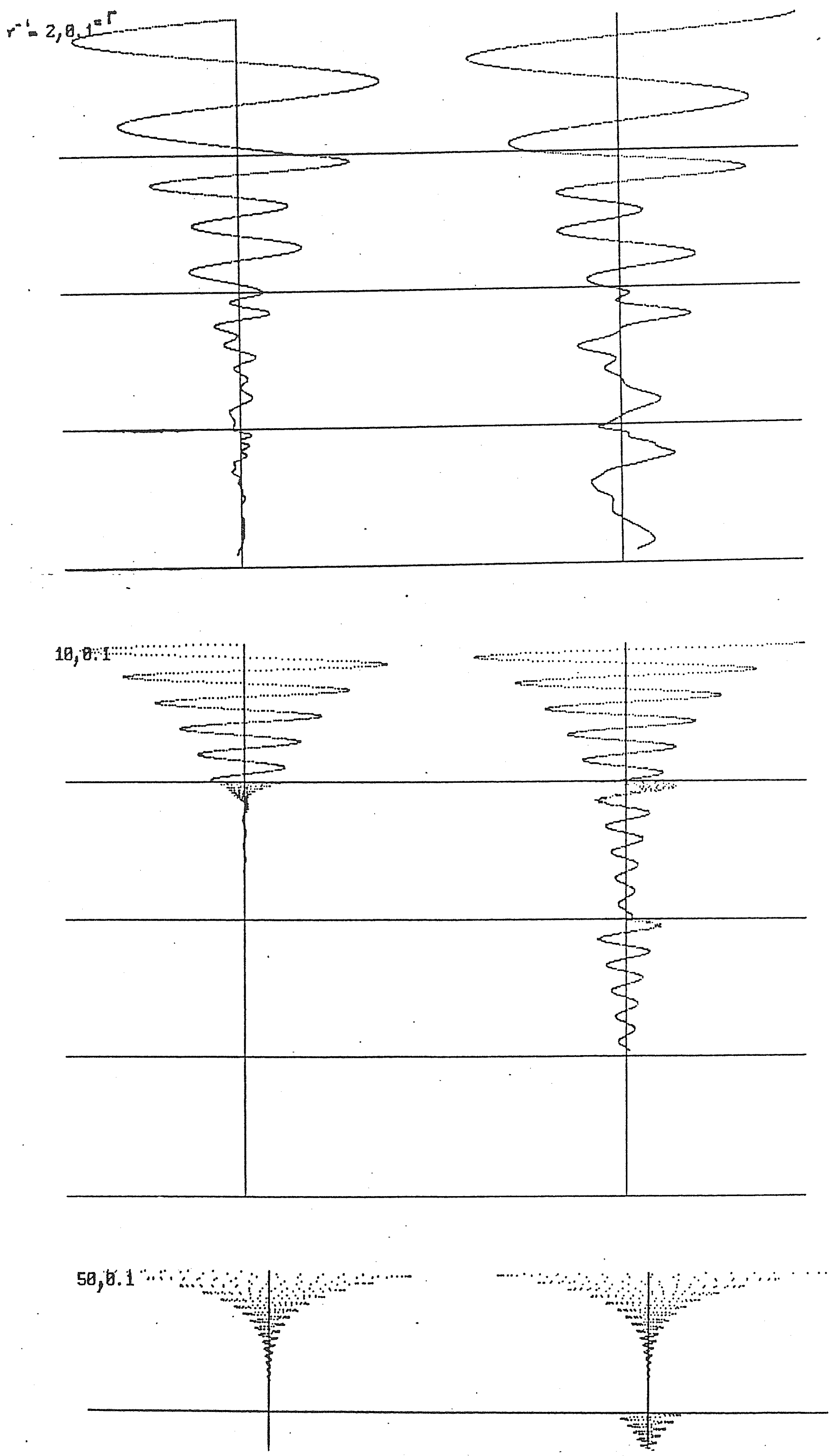


Fig. 3. r = 0.5, 0.1, 0.02

Γ = 0.1



III.5 CONCLUSIONS

(1) Despite the effort spent in the study of the radiation damping in closed expanding universes, we have to conclude that the problem is still open. We have seen that the problem remains in the logical gap between the radiation reaction equation for a general source ρ and for a point-like source $\rho = \delta$. We think that the problem could be bypassed in two ways: either (i) assuming some simple form of ρ such that the radiation reaction equation can be solved easily, thus not having to perform the point-like limit; or (ii) modifying the coupled model (e.g. allowing the harmonic-oscillator to have a time-dependent frequency) in such a way that the renormalization process can be done correctly.

(2) We have obtained (see Chapter II) the radiation reaction in the point-like limit for the Einstein spacetime case (a static one). This is a second order linear (in the oscillator variable) discontinuous differential equation with retarded arguments, a type of differential equations which is virtually unknown. It is very interesting to observe that the terms with retarded arguments represent precisely the effect of the radiation that comes back to the source (recharging it) due to the fact that the space is closed (remember that $M = \mathbb{R} \times S^3$). Therefore physically one is motivated to think that the energy of the oscillator would never escape from the spacetime. Then it is reassuring that, from a mathematical point of view, we have found that the zero solution of this equation is uniformly stable (section III.2) but not asymptotically stable (section III.3).

(3) Probably our major contribution resides in the stability analysis of a certain class of differential equations considered in Chapter III. We have to admit that for a certain time we did not adhere to the rule: "new problems need new solutions". In fact, we originally attempted to use some of the (known) machinery of the stability theory of differential equations to treat this completely unknown (hence "new") class of DEFRA equations. Then we realized the necessity for new tools, and in this manner we have constructed the original approach that is contained in Chapter III. We were (strongly) motivated to perform this stability analysis by the results obtained from computer simulations of these equations.

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