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Variational and Nonvariational Relaxed Dirichlet Problems

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Introduction

The aim of this thesis is to develope some subjects related to the study of the theory of relaxed Dirichlet problems. The notion of "relaxed Dirichlet problem" was introduced in [33] to describe the asymptotic behaviour of the solutions of classical Dirichlet problems in strongly perturbed domains.
Given a bounded open subset Ω of $\mathbf{R}^N$, $N \geq 2$, and an elliptic operator L in divergence form with bounded measurable coefficients on Ω, a relaxed Dirichlet problem can be written in the form

$$\begin{cases} Lu + \mu u = G & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $G \in H^{-1}(\Omega)$ and μ belongs to the space $\mathcal{M}_0(\Omega)$ of all positive Borel measures on Ω which do not charge any set of capacity zero. The problem (1) has to be interpreted in a weak sense. Namely a function u is the solution of (1) if u belongs to $H_0^1(\Omega) \cap L^2(\Omega, \mu)$ and satisfies

$$\langle Lu, v \rangle + \int_\Omega uv \, d\mu = \langle G, v \rangle \,, \tag{2}$$

for every $v \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$. Here and in the following $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. The class of relaxed Dirichlet problems contains all the classical Dirichlet problems in subdomains of Ω. More precisely, for every open subset Ω' of Ω a function u is the solution to the problem

$$\begin{cases} Lu = G & \text{in } \Omega', \\ u = 0 & \text{on } \partial\Omega', \end{cases} \tag{3}$$

if and only if its prolongation to zero in $\Omega \setminus \Omega'$ is the unique solution to the relaxed Dirichlet problem

$$\begin{cases} Lu + \infty_{\Omega\setminus\Omega'} u = G & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\infty_{\Omega\setminus\Omega'}(E) = \begin{cases} 0, & \text{if } \operatorname{cap}(E\setminus\Omega',\Omega)=0, \\ +\infty & \text{otherwise,} \end{cases} \tag{4}$$

for every subset E of Ω (here "cap" denotes the harmonic capacity). Thus the framework of the relaxed Dirichlet problems allows us to treat problems (3) and (1) in a unified way.

On the class $\mathcal{M}_0(\Omega)$ we can introduce a notion of convergence, called γ^L-convergence (see [34], [23], [30]): a sequence of measures $\{\mu_h\}$ of $\mathcal{M}_0(\Omega)$ γ^L-converges to a measure $\mu \in \mathcal{M}_0(\Omega)$ if for every $G \in H^{-1}(\Omega)$ the sequence $\{u_{\mu_h}\}$ of the solutions to the problems

$$\begin{cases} Lu_{\mu_h} + \mu_h u_{\mu_h} = G & \text{in } \Omega, \\ u_{\mu_h} = 0 & \text{on } \partial\Omega, \end{cases}$$

converge strongly in $L^2(\Omega)$ to the solution u of (1).

The main result concerning relaxed Dirichlet problems is the following compactness theorem (see [34], Theorem 4.14, and [30], Theorem 4.5): every sequence $\{\mu_h\}$ admits a γ^L-converging subsequence. In the particular case $\mu_h = \infty_{\Omega\setminus\Omega_h}$, this implies that for every sequence $\{\Omega_h\}$ of open subsets of Ω there exist a subsequence, still denoted by $\{\Omega_h\}$, and a measure $\mu \in \mathcal{M}_0(\Omega)$, such that for every $G \in H^{-1}(\Omega)$ the solutions u_h of the Dirichlet problems

$$\begin{cases} Lu_h = G & \text{in } \Omega_h, \\ u_h = 0 & \text{on } \partial\Omega_h, \end{cases} \tag{5}$$

extended to 0 on $\Omega\setminus\Omega_h$, converge in $L^2(\Omega)$ to the unique solution u of (1). Moreover, the following density theorem holds (see [34], Theorem 4.16 for the symmetric case): for every $\mu \in \mathcal{M}_0(\Omega)$ there exists a sequence $\{\Omega_h\}$ of open subsets of Ω such that for every $G \in H^{-1}(\Omega)$ the solution u of (1) is the limit in $L^2(\Omega)$ of the sequence $\{u_h\}$ of the solutions of (5). The proof of this density theorem provides an explicit approximation only when μ is the Lebesgue measure, while it is rather indirect in the other cases, and does not suggest any efficient method for the construction of the sets Ω_h.

The subjects developed in this thesis can be splitted into two parts. The first one is contained in Chapters 2 and 3 and deals with the problems (1)

that we briefely call "variational relaxed Dirichlet problems", due to their variational form (2).
In Chapter 2 we present an explicit approximation scheme for the relaxed Dirichlet problem (1) by means of sequences of classical Dirichlet problems of the form (5).
In Chapter 3 we apply this approximation result to some shape optimization problems with cost functionals of the form $J(A) = F(A, u_A)$ where A, the unknown of the problem, is an open subset of Ω and u_A is the solution of (3) with $\Omega' = A$, and we characterize the behaviour of the minimizing sequences for J (for the explicit form of J see (9) below).

The second part of this thesis deals with relaxed Dirichlet problems with a measure in the right–hand side, that we call "nonvariational relaxed Dirichlet problems".

In Chapter 4 we prove some regularity results for the solution of (1), depending on the regularity of G, and we introduce, by a duality method, a notion of solution for the problem (1) if G is a measure with bounded variation, while in Chapter 5 we study the asymptotic behaviour of these solutions in perforated domains.

Let us give more details about the contents of each chapter and some references.

In Chapter 2 we assume that $\mu \in \mathcal{M}_0(\Omega)$ is a Radon measure, and we construct explicitly a sequence $\{\Omega_h\}$ of open subsets of Ω such that for every $G \in H^{-1}(\Omega)$ the solutions to the problems (5) prolonged to zero in $\Omega \setminus \Omega_h$ converge in $L^2(\Omega)$ to the solution u of (1). In Section 2.2 the sets Ω_h will be obtained by removing an array of small balls from the set Ω. The geometric construction is quite simple. For every $h \in \mathbf{N}$ we fix a partition $\{Q_h^i\}_i$ of $\mathbf{R}^N$ composed of cubes with side $1/h$, and we consider the set $N(h)$ of all indices i such that $Q_h^i \subset\subset \Omega$. For every $i \in N(h)$ let B_h^i be the ball with the same center as Q_h^i and radius $1/2h$, and let E_h^i be another ball with the same center such that

$$\mathrm{cap}^L(E_h^i, B_h^i) = \mu(Q_h^i)\,.$$

Finally, let $E_h = \bigcup_{i \in N(h)} E_h^i$ and $\Omega_h = \Omega \setminus E_h$. Note that the size of the hole E_h^i contained in the cube Q_h^i depends only on the operator L and on

the value of the measure μ on Q_h^i.
By using a very general version of the Poincaré inequality proved by P. Zamboni [67], we shall show that, if μ belongs to the Kato space $K_N^+(\Omega)$, i.e., the potential generated by μ is continuous, then the method introduced by D. Cioranescu and F. Murat [25] can be applied, so that for every $G \in H^{-1}(\Omega)$ the solutions u_h of the Dirichlet problems (5) converge in $L^2(\Omega)$ to the solution u of the relaxed Dirichlet problem (1). To prove that the same result holds also when μ is an arbitrary Radon measure of the class $\mathcal{M}_0(\Omega)$ we use the method of μ-capacities introduced in [33] and [23]. Finally, if μ is a Radon measure and $\mu \notin \mathcal{M}_0(\Omega)$, then we prove that our construction leads to the approximation of the solutions of the relaxed Dirichlet problem

$$\begin{cases} Lu + \mu_0 u = G & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where μ_0 is the greatest measure of the class $\mathcal{M}_0(\Omega)$ which is less than or equal to μ.

In Section 2.3 we show that for every Radon measure $\mu \in \mathcal{M}_0(\Omega)$ and for every nonatomic, nonnegative Radon measure λ it is possible to construct a sequence $\{E_h\}$ of subsets of Ω such that E_h is the union of $(N-1)$-dimensional balls centered in a cubical lattice with $\lambda(E_h) = 0$, and the sequence $\{\infty_{E_h}\}$ γ^L-converges to μ. While the proof of the approximation result in Section 2.2 is based on the method of oscillating test functions as in [25], in this case we follow a completely different line and we use directly the capacitary method introduced in [23] for symmetric operators and generalized to the nonsymmetric case in [31].

The study of the asymptotic behaviour of solutions of partial differential equations in perforated domains is strictly related to the study of some shape optimization problems with cost functionals of the form

$$J(A) = F(u_A), \tag{6}$$

where A varies in a suitable class $\mathcal{A}$ of open subsets of a given bounded domain $\Omega \subseteq \mathbf{R}^N$, and u_A is the solution of a partial differential equation in A (see, for instance, [57], [58], [69], [60], [66], [21], [14], [15], [16]). As it was shown in the papers [20], [18], [24], [19], [37], the framework of relaxed Dirichlet problems turns out to be especially useful in the case where $\mathcal{A}$ is the family of all open sets A contained in Ω, and u_A is the solution of the problem

$$(7) \qquad Lu_A = G \quad \text{in } A, \qquad u_A = 0 \quad \text{in } \overline{\Omega}\backslash A,$$

and $F(u)$ in (6) is an integral functional, continuous on $L^2(\Omega)$, of the form

$$F(u) = \int_\Omega j(x,u)\,dx.$$

In these papers it was pointed out that in general the corresponding minimization problem

$$(8) \quad \min_A \left\{ \int_\Omega j(x,u_A)\,dx : A \text{ open} \subseteq \Omega, Lu_A = G \text{ in } A,\ u_A = 0 \text{ in } \overline{\Omega}\backslash A \right\}$$

does not admit any solution. An explanation of this fact is that, if $\{A_h\}$ is a sequence of open subsets of Ω, in particular a minimizing sequence of (8), then the corresponding sequence $\{u_{A_h}\}$ of solutions of (7) has a subsequence converging to a function u in $L^2(\Omega)$, but, in general, there is no open set A such that $u = u_A$. Nevertheless, by the compactness theorem mentioned above, there exists a measure $\mu \in \mathcal{M}_0(\Omega)$ such that u is the solution of the problem (1). The relaxed form of (8) is studied in [20] and it is given by

$$\min_\mu \left\{ \int_\Omega j(x,u_\mu)\,dx \ : \ \mu \in \mathcal{M}_0(\Omega),\ Lu_\mu + \mu u_\mu = G \text{ in } \Omega,\ u_\mu = 0 \text{ on } \partial\Omega \right\},$$

Moreover, if we identify each open set A with the measure $\mu_A = \infty_{\Omega\backslash A}$ defined in (4), the functional $\tilde{J}(\mu) = \int_\Omega j(x,u_\mu)\,dx$ which appears in the relaxed problem turns out to be the lower semicontinuous envelope of the functional $J(A) = \int_\Omega j(x,u_A)\,dx$ with respect to the γ^L-convergence. It is possible to prove that the topological space $(\mathcal{M}_0(\Omega), \gamma^L)$ is actually a compact metrizable space (see [33], [30]), and hence the relaxation of problem (8) studied in [20] can be considered in the general framework of [5].

In Chapter 3 we treat a case where the cost functionals J depend on the unknown domain A not only through the solution u_A as in (6). More precisely, we consider functionals J of the form

$$J(A) = \begin{cases} \int_A j(x, u_A)\, d\lambda, & \text{if } \lambda(A) \in T, \\ +\infty, & \text{otherwise,} \end{cases} \tag{9}$$

where λ is a bounded measure in $\mathcal{M}_0(\Omega)$, and $T = [m, M]$ is a subinterval of $[0, \lambda(\Omega)]$ possibly degenerating to a point. Notice that the functional J depends on A through the domain of integration, through the solution u_A of the differential equation, and through the constraint $\lambda(A) \in T$. We shall prove that also in this case the minimum problem

$$\min\{J(A)\colon\ A \text{ open},\ A \subseteq \Omega\}$$

has, in general, no solution, and that the relaxed problem can be written as

$$\min\left\{\overline{J}(\mu)\colon\ \mu \in \mathcal{M}_0(\Omega)\right\},$$

where $\overline{J}$ is the lower semicontinuous envelope of J in $\mathcal{M}_0(\Omega)$ with respect to the γ^L-convergence.
The main result of Chapter 3 is an explicit integral representation of $\overline{J}$ in terms of the integrand j and of the constraint T (Theorem 3.3.1). The relevant new difficulty with respect to [20] lies in the fact that A appears in J also in the domain of integration. This requires a substantial change in the proof, which is based on some new measure theoretical arguments.

In Chapter 4, following the lines of a paper by G. Stampacchia (see [65]), we find that the solution of (1) is more regular if the datum is more regular: as for the regularity of u in Lebesgue spaces $L^p(\Omega)$ we obtain the exact analogue of the results of [65]; furthermore we prove the regularity of u in $L^p(\Omega, \mu)$. In particular, we find that, if G belongs to $H^{-1,p}(\Omega)$ with $p > N$, then u is in $L^\infty(\Omega) \cap L^\infty(\Omega, \mu)$.
The latter result allows us to study the relaxed Dirichlet problem (1) if $G = \nu$ is a measure with bounded variation. In particular we introduce a notion of solution (that gives, if $\mu = 0$, the solution given by G. Stampacchia in [64]),

proving an existence and uniqueness result. We also show that in the Lebesgue equivalence class of this solution there is a representative, defined up to a set of harmonic capacity zero depending on the datum ν, that coincides with the limit of its convolutions with any sequence of positive, spherically symmetric, δ-approximating convolution kernels with compact support. Moreover, we study the regularity properties of the solution, both in Sobolev spaces $H_0^{1,p}(\Omega)$ and in Lebesgue spaces $L^p(\Omega,\mu)$, if ν belongs to $L^1(\Omega)$ or to $L^m(\Omega)$, with m "small".
The nonvariational existence result allows us to define the Green function G_μ for relaxed Dirichlet problems. We show that it is possible to define $G_\mu(x,y)$ pointwise in $\Omega\times\Omega$ outside the diagonal, and that this representative is upper semicontinuous in each variable, has the usual symmetry property, and the classical representation formula for solutions of (1) with measure data holds. The main difficulty in proving these properties consists in overcoming the lack of continuity of G_μ.

Finally, in Chapter 5 we study the asymptotic behaviour of the solutions to the problems

$$\begin{cases} Lv_h = \nu & \text{in } \Omega_h, \\ v_h = 0 & \text{on } \partial\Omega_h, \end{cases} \tag{10}$$

where ν is a measure with bounded variation. It can be easily seen that for every $1 \le p < \frac{N}{N-1}$ the sequence $\{v_h\}$ is bounded in the Sobolev space $H_0^{1,p}(\Omega)$, so that there exists a subsequence, still denoted by $\{v_h\}$, which converges to a function v in the weak topology of $H_0^{1,p}(\Omega)$. On the other hand the compactness result mentioned before guarantees that, possibly passing to a futher subsequence still denoted by $\{\Omega_h\}$, there exists $\mu \in \mathcal{M}_0(\Omega)$ such that for every $G \in H^{-1}(\Omega)$ the solutions u_h to the problems

$$\begin{cases} Lu_h = G & \text{in } \Omega_h, \\ u_h = 0 & \text{on } \partial\Omega_h, \end{cases}$$

converge in $L^2(\Omega)$ to the solution u of the relaxed Dirichlet problem (1). Our goal will be to establish whether v is the solution of

$$\begin{cases} Lv + \mu v = \nu & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where μ is the same measure which appears in (1). An easy example due to Murat shows that this does not occur in general. It is enough to consider $\Omega_h = \Omega \setminus \overline{B}(x_0, 1/h)$, where $\overline{B}(x_0, 1/h)$ is the closed ball centered in x_0 and with radius $1/h$, and $\nu = \delta_{x_0}$. In this case it is easily seen that $v = 0$, while $\mu = 0$, so that the solution of (11) is the Green function of the operator L in Ω with pole x_0.

In this chapter we prove that if the sequence $\{\infty_{\Omega\setminus\Omega_h}\}$ γ^L-converges to μ, then for every ν which does not charge polar sets the sequence $\{v_h\}$ of solutions to the problems (10) admits a limit v in the weak topology of $H_0^{1,p}(\Omega)$, $1 \leq p < \frac{N}{N-1}$, and v coincides with the solution v of (11). Thus, for example, the asymptotic behaviour of solutions of Dirichlet problems with datum in the Lebesgue space $L^1(\Omega)$ and defined in perforated domains is characterized by the asymptotic behaviour of the solutions of the corresponding problems with datum in $H^{-1}(\Omega)$.

On the other hand we prove that, if the operator L has regular coefficients and and the limit measure μ has a density f with respect to the Lebesgue measure, with $f \in L^p(\Omega)$, $p > N/2$, then for every measure ν with bounded variation in Ω, there exists a subsequence $\{v_{h_k}\}$ of the sequence $\{v_h\}$ of the solutions solutions to the problems (10) which converges to the solution to the problem

$$\begin{cases} Lv + \mu v = \lambda & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ is a measure with bounded variation in Ω depending on ν and on the subsequence $\{\Omega_{h_k}\}$, while μ is the measure obtained in (1).

Since the solutions of problems with right–hand side measure are characterized by a duality identity, the method of oscillating test functions used in [30] to prove the compactness result cannot be applied. Our approach deals with a corrector result which could be interesting by itself. For every $g \in L^\infty(\Omega)$ we consider the solutions u_h, u_μ to the problems

$$\begin{cases} Lu_h = g & \text{in } \Omega_h, \\ u_h = 0 & \text{on } \partial\Omega_h, \end{cases} \tag{12}$$

$$\begin{cases} Lu_\mu + \mu u_\mu = g & \text{in } \Omega, \\ u_\mu = 0 & \text{on } \partial\Omega, \end{cases} \tag{13}$$

and we denote by w_h, w_μ the solutions of (12), (13) corresponding to $g = 1$. Since L and μ are regular, the function w_μ is continuos in Ω and $w_\mu(x) > 0$ for every $x \in \Omega$. Thus we can construct the functions

$$z_h = u_h - \frac{u_\mu}{w} w_h.$$

In [30] it was proved that, if the sequence $\{u_h\}$ converges weakly in $H_0^1(\Omega)$ to u_μ for every $g \in L^\infty(\Omega)$, then

$$\lim_{h\to\infty} \|z_h\|_{H^1(\Omega')} = 0 ,$$

for every open set Ω' compactly contained in Ω. Thus

$$u_h = u_\mu + \left(u_\mu(\frac{w_h}{w_\mu} - 1) \right) + z_h ,$$

and the remainder z_h tends to zero strongly in $H^1_{loc}(\Omega)$. This allows us to regard $\{u_\mu(\frac{w_h}{w_\mu} - 1)\}$ as a sequence of correctors in $H^1_{loc}(\Omega)$ for the problems considered above. Under the previous regularity assumptions on L and μ we obtain

$$\lim_{h\to\infty} \|z_h\|_{L^\infty(\Omega')} = 0 ,$$

for every open set Ω' compactly contained in Ω, and this will be the basic tool in order to pass to the limit in the equations satisfied by v_h. We underline that the regularity assumptions on L and μ are suggested by the tecnique of the proof, and we hope that they could be removed by using a different approach. However our result generalizes the ones in [56] and [63]. In these papers is was proved that, under the same regularity assumption on L and μ, and for sequences $\{\Omega_h\}$ with a special geometry, there exists a sequence of correctors (different from $\{z_h\}$) which converges uniformly to zero. For results of this type we refer also to [38] and [46].
The corrector result is still valid if we suppose that for every $g \in L^\infty(\Omega)$ the sequence $\{u_h\}$ of the solutions of (12) converges weakly in $H_0^1(\Omega)$ to the solution u_μ to the problem

$$\begin{cases} Lu_\mu + \mu u_\mu = g & \text{in } A, \\ u_\mu = 0 & \text{in } \overline{\Omega} \setminus A, \end{cases}$$

where A is an open subset of Ω, and μ has the same regularity assumed before. Then we can also prove that in this case for every measure ν with bounded variations in Ω such that $|\nu|(\Omega \cap \partial A) = 0$, there exists a subsequence $\{v_{h_k}\}$ of the sequence $\{v_h\}$ of the solutions to the problems (10) which converges to the solution v_μ to the problem

$$\begin{cases} Lv_\mu + \mu v_\mu = \lambda & \text{in } A, \\ v_\mu = 0 & \text{in } \overline{\Omega} \setminus A. \end{cases}$$

Moreover, if ∂A is smooth, the result is proved for every measure ν with bounded variation.

The approximation result stated in Section 2.2 was obtained in collaboration with Prof. G. Dal Maso and it is published in [32], while Section 2.3 is a generalized version of the result obtained in collaboration with Prof. A. Braides of the University of Brescia and published in [12].
The application to shape optimization problems contained in Chapter 3 was investigate in collaboration with Prof. G. Dal Maso, Prof. G. Buttazzo of the University of Pisa, and with Dott. A. Garroni of SISSA, and it is published in [22].
The results of Chapter 4 were obtained in collaboration with Dott. L. Orsina of the University of Roma I, and they are published in [52], while Chapter 5 contains results published in [53].

Chapter 1
Definitions and preliminary results

In this chapter we collect the main preliminary results and notation needed in the sequel. The preliminaries related to only one chapter will be contained in the first section of the chapter itself.

1.1. Capacity and Sobolev spaces

Throughout this thesis Ω will be a bounded open subset of $\mathbf{R}^N$, $N \geq 2$, and $B_r(x)$ (or $B(x,r)$) will be the open ball of center $x \in \mathbf{R}^N$ and radius r. For every set E, $\mathbf{1}_E$ will be the characteristic function of E, that is the function defined as

$$\mathbf{1}_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

We shall denote by $L^q(\Omega, \mu)$, $1 \leq q \leq +\infty$, the usual Lebesgue spaces with respect to a Borel measure μ. If $\mu = \mathcal{L}$ is the Lebesgue measure on $\mathbf{R}^N$, we set $L^q(\Omega, \mathcal{L}) = L^q(\Omega)$. We shall denote by $H^{1,q}(\Omega)$, $H_0^{1,q}(\Omega)$, $1 \leq q < +\infty$, the usual Sobolev spaces, and with $H^1(\Omega)$ (resp. $H_0^1(\Omega)$) the space $H^{1,2}(\Omega)$ (resp. $H_0^{1,2}(\Omega)$). The dual space of $H_0^{1,q}(\Omega)$ will be denoted by $H^{-1,q'}(\Omega)$ where q' is the conjugate exponent of q. The dual space of $H_0^1(\Omega)$ will be denoted by $H^{-1}(\Omega)$. A function u belongs to $L^q_{loc}(\Omega)$ (resp. $H^{1,q}_{loc}(\Omega)$) if $u \in L^q(\Omega')$ (resp. $H^{1,q}(\Omega')$) for every open set Ω' compactly contained in Ω ($\Omega' \subset\subset \Omega$). The duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ will be denoted by $\langle \cdot, \cdot \rangle$.

If Ω' is an open set compactly contained in Ω and $u \in H_0^{1,q}(\Omega')$, then we can extend u to Ω by setting $u = 0$ in $\Omega \setminus \Omega'$. We shall always identify u with this extension, which is an element of $H_0^{1,q}(\Omega)$.

Definition 1.1.1 *Let q be a real number, $1 \leq q < +\infty$. Let A be an open subset of Ω. The q-capacity of A with respect to Ω is defined as:*

$$(1.1.1) \quad \operatorname{cap}_q(A, \Omega) = \inf \left\{ \int_\Omega |Du|^q \, dx : u \in H_0^{1,q}(\Omega),\ u \geq \mathbf{1}_A \text{ a.e. in } \Omega \right\},$$

We use the convention that $\inf \emptyset = +\infty$.

This definition can be extended to any subset E of Ω in the following way:

$$\text{(1.1.2)} \qquad \operatorname{cap}_q(E,\Omega) = \inf \left\{ \operatorname{cap}_q(A,\Omega),\ A \text{ open},\ E \subseteq A \right\} .$$

Using Hölder inequality, it is easily seen by (1.1.1) that, if $p < q$ and if A is an open subset of Ω, then

$$\text{(1.1.3)} \qquad \left(\operatorname{cap}_p(A,\Omega)\right)^{\frac{1}{p}} \leq |\Omega|^{\frac{1}{p}-\frac{1}{q}} \left(\operatorname{cap}_q(A,\Omega)\right)^{\frac{1}{q}} ;$$

by (1.1.2), this inequality clearly holds for every subset E of Ω.

We say that a property $\mathcal{P}(x)$ holds cap_q-quasi everywhere (cap_q-q.e.) in Ω if there exists a subset E of Ω with q-capacity zero such that $\mathcal{P}(x)$ holds for every x in $\Omega \setminus E$. The expression "almost everywhere" (a.e.) refers, as usual, to the analogous property for the Lebesgue measure.

We remark that cap_2 coincides with the harmonic capacity studied in potential theory (see e.g. [7], [44] and [49]). In the following we always write cap instead of cap_2 and q.e. instead of cap_2-q.e. .

A function $u : \Omega \to \mathbf{R}$ is said to be cap_q–quasi continuous (or quasi continuous for $q = 2$) if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$ with $\operatorname{cap}_q(E,\Omega) \leq \varepsilon$ and such that the restriction of u to $\Omega \setminus E$ is continuous. We recall that for every u in $H_0^{1,q}(\Omega)$, $1 \leq q < +\infty$, there exists a cap_q-quasi continuous function $\tilde{u}$, unique up to subsets of q-capacity zero, such that $u = \tilde{u}$ almost everywhere (see, e.g., [68], Theorem 3.1.4). In the following we will always choose in the equivalence class of a function u belonging to $H_0^{1,q}(\Omega)$ its cap_q-quasi continuous representative $\tilde{u}$.

Now we give some properties of the cap_q-quasi continuous representative we shall use in the following.

Theorem 1.1.2 *Let q be a real number, $1 \leq q < +\infty$. Let $\{u_n\}$ be a sequence of $H_0^{1,q}(\Omega)$ functions that converges strongly to u in the same space. Then there exists a subsequence $\{u_{n_k}\}$ that converges to u cap_q-q.e.. Moreover, if u and v are two functions in $H^{1,q}(\Omega)$ such that $u \leq v$ almost everywhere, then $u \leq v$ cap_q-q.e..*

Proof. See [39], Theorem 2.1. □

1.2. Measures

By a Borel measure on Ω we mean a positive, countably additive set function with values in $\overline{\mathbf{R}}$ defined on the σ–field $\mathcal{B}(\Omega)$ of all Borel subsets of Ω. By a Radon measure on Ω we mean a Borel measure which is finite on every compact subset of Ω. If μ is a Borel measure and $E \in \mathcal{B}(\Omega)$, the Borel measure $\mu \llcorner E$ is defined by $(\mu \llcorner E)(B) = \mu(E \cap B)$ for every set $B \in \mathcal{B}(\Omega)$. If μ is a Borel measure, and h is a Borel measurable function, we shall denote by $h\mu$ the Borel measure defined by $(h\mu)(B) = \int_B h\,d\mu$ for every set $B \in \mathcal{B}(\Omega)$. For every signed measure μ, μ^+ and μ^- will be respectively the positive and negative part of μ, and $|\mu| = \mu^+ + \mu^-$ will be its total variation.

Using the notion of capacity, we can define a class of Borel measures.

Definition 1.2.1 *We denote by $\mathcal{M}_0(\Omega)$ the set of all nonnegative Borel measures μ on Ω such that $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ with* $\operatorname{cap}(B,\Omega) = 0$.

For every subset E of Ω we shall denote by ∞_E the measure in $\mathcal{M}_0(\Omega)$ defined by

$$\infty_E(B) = \begin{cases} 0, & \text{if } \operatorname{cap}(B \cap E, \Omega) = 0, \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.2.1}$$

for every Borel set $B \subseteq \Omega$.

Definition 1.2.2 *We say that a nonnegative Radon measure ν on Ω belongs to $H^{-1}(\Omega)$ if there exists $f \in H^{-1}(\Omega)$ such that*

$$\langle f, \varphi \rangle = \int_\Omega \varphi\, d\nu \qquad \forall \varphi \in C_0^\infty(\Omega).$$

We shall always identify f and ν.

It is well known that every nonnegative Radon measure which belongs to $H^{-1}(\Omega)$ belongs also to $\mathcal{M}_0(\Omega)$ (see [68], Section 4.7).

Another class of measures we are interested in is the Kato space.

Definition 1.2.3 *The Kato space* $K_N^+(\Omega)$ *is the cone of all positive Radon measures* μ *on* Ω *such that*

$$\lim_{r\to 0^+} \sup_{x\in\Omega} \int_{\Omega\cap B_r(x)} G_N(y-x)\, d\mu(y) = 0\,,$$

where G_N *is the fundamental solution of the Laplace operator* $-\Delta$.

For every $\mu \in K_N^+(\Omega)$ and for every Borel set $E \subseteq \Omega$ we define

$$\|\mu\|_{K_N^+(E)} = \sup_{x\in E} \int_E |y-x|^{2-N}\, d\mu(y)\,, \qquad \text{if } N \geq 3\,,$$

$$\|\mu\|_{K_N^+(E)} = \sup_{x\in E} \int_E \log\left(\frac{\operatorname{diam}(E)}{|y-x|}\right) d\mu(y) + \mu(E)\,, \qquad \text{if } N = 2.$$

For every $\mu \in K_N^+(\Omega)$ it is easy to see that $\|\mu\|_{K_N^+(\Omega)} < +\infty$ and $\|\mu\|_{K_N^+(E)}$ tends to zero as $\operatorname{diam}(E)$ tends to zero. Finally, we recall that every measure in $K_N^+(\Omega)$ is bounded and belongs to $H^{-1}(\Omega)$. For more details about this subject we refer to [1], [33], [47], [61].

Theorem 1.2.4 *For every* $\mu \in \mathcal{M}_0(\Omega)$ *there exists a positive, Borel measurable function* h, *and a positive Kato measure* γ *such that* $\int_\Omega f\, d\mu = \int_\Omega f\, h\, d\gamma$ *for every quasi continuous function* $f \in L^1(\Omega,\mu)$.

Proof. See [4], Proposition 2.5, and [28], Theorem 2.6. □

1.3. Varational relaxed Dirichlet problems

Let $A = A(x) = (a_{ij})_{i,j=1,\dots,N}$, with $a_{ij}\colon \mathbf{R}^N \to \mathbf{R}$, be a matrix with measurable coefficients such that

$$A(x)\,\xi\cdot\xi \geq \theta\,|\xi|^2 \qquad \text{for a.e. } x \in \mathbf{R}^N,\ \forall \xi \in \mathbf{R}^N\,, \tag{1.3.1}$$

and

$$\|a_{ij}\|_{L^\infty(\Omega)} \leq \Theta \tag{1.3.2}$$

for some constants $0 < \theta \leq \Theta$. Let $L\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ be the operator defined by $L\,u = -\operatorname{div}(A\,Du)$; thanks to the hypotheses on A, L is a uniformly elliptic and bounded operator. We shall denote by L^* the adjoint operator associated to L, i.e., $L^* u = -\mathrm{div}\,(A^* Du)$, where $A^* = (a_{ji})_{i,j=1,\dots,N}$ is the transposed matrix of A.

We give the definition of relaxed Dirichlet problems for the operator L as it was given in [33].

Definition 1.3.1 *We say that a function u is a local solution of the equation $Lu + \mu u = f$ in Ω, with $f \in H^{-1}(\Omega)$ and $\mu \in \mathcal{M}_0(\Omega)$, if $u \in H^1_{\mathrm{loc}}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega, \mu)$, and*

$$\int_\Omega A\,DuDv\,dx + \int_\Omega u\,v\,d\mu = \langle f, v\rangle\,, \tag{1.3.3}$$

for every $v \in H^1(\Omega) \cap L^2(\Omega, \mu)$ with compact support in Ω. We say that u is a solution of the relaxed Dirichlet problem

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3.4}$$

with $f \in H^{-1}(\Omega)$ and $\mu \in \mathcal{M}_0(\Omega)$, if u is a local solution and u belongs to $H^1_0(\Omega)$.

Theorem 1.3.2 *Suppose that $f \in H^{-1}(\Omega)$ and $\mu \in \mathcal{M}_0(\Omega)$. Then there exists a unique solution u of the problem*

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, u belongs to $H^1_0(\Omega) \cap L^2(\Omega, \mu)$, satisfies

$$\int_\Omega A\,DuDv\,dx + \int_\Omega u\,v\,d\mu = \langle f, v\rangle\,, \tag{1.3.5}$$

for every $v \in H^1_0(\Omega) \cap L^2(\Omega, \mu)$, and

$$\|u\|_{H^1_0(\Omega)} + \|u\|_{L^2(\Omega,\mu)} \le c\,\|f\|_{H^{-1}(\Omega)}\,,$$

for some positive constant c depending only on N, θ, and Θ.

Proof. See [33], Theorem 2.4. □

Remark 1.3.3 The solution of (1.3.5) given by the preceding theorem may not be a solution in the sense of distributions. Actually, as has been shown in [34], Remark 3.11, there exist a measure μ such that $H_0^1(\Omega) \cap L^2(\Omega, \mu) \neq \{0\}$, while $C_0^\infty(\Omega) \cap L^2(\Omega, \mu) = \{0\}$.

Remark 1.3.4 Let $\mu \in \mathcal{M}_0(\Omega)$, and let h and γ be respectively a Borel function and a Kato measure as in Theorem 1.2.4. Since the functions in $H_0^1(\Omega)$ are quasi continuous, then, by Theorem 1.2.4, $\int_\Omega u\, v\, d\mu = \int_\Omega u\, v\, h\, d\gamma$ for every u and v in $H_0^1(\Omega) \cap L^2(\Omega, \mu)$. Hence, the solution u of (1.3.5) is also the solution of the relaxed Dirichlet problem with μ is replaced by $h\gamma$.

Remark 1.3.5 If E is a closed subset of Ω, then a function u is the solution of (1.3.5) corresponding to $\mu = \infty_E$ if and only if $u = 0$ q. e. in E and u is the solution in $\Omega \setminus E$ of the classical Dirichlet problem

$$\begin{cases} L\, u = f & \text{in } \Omega \setminus E, \\ u = 0 & \text{on } \partial(\Omega \setminus E), \end{cases}$$

(see, for example [BuDM1], Section 2).

The solutions of relaxed Dirichlet problems satisfy a maximum principle.

Theorem 1.3.6 *Let μ_1 and μ_2 be in $\mathcal{M}_0(\Omega)$, with $\mu_1 \le \mu_2$. Let f_1 and f_2 be in $H^{-1}(\Omega)$, with $0 \le f_2 \le f_1$. Let u_1 and u_2 be the solutions of*

$$\begin{cases} L\, u_i + \mu_i\, u_i = f_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1, 2.$$

Then $0 \le u_2 \le u_1$ almost everywhere in Ω. Moreover, every solution u of (1.3.5) with positive datum f satisfies $L\, u \le f$ in the sense of distributions on Ω.

Proof. See [33], Proposition 2.6 and Theorem 2.10. □

For every $\mu \in \mathcal{M}_0(\Omega)$ we define w_μ to be the unique solution in the sense of (1.3.5) of the problem

$$\text{(1.3.6)} \qquad \begin{cases} Lw_\mu + \mu w_\mu = 1 & \text{in } \Omega, \\ w_\mu = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 1.3.7 If we apply Theorem 1.3.6 with $f_1 = f_2 = 1$, $\mu_1 = 0$, $\mu_2 = \mu$, and the regularity results for classical Dirichlet problems (see [64], Théorème 4.2), we obtain that there exists a constant c, depending only on θ, Θ, N and Ω such that $w_\mu(x) \leq c$ for almost every $x \in \Omega$. Moreover, again by Theorem 1.3.6 $w_\mu \geq 0$ q.e. in Ω, and for every solution of problem (1.3.4), with $f \in L^\infty(\Omega)$, we have $|u(x)| \leq \|f\|_\infty w_\mu(x)$ for every $x \in \Omega$.

As a consequence of Theorem 1.3.6 we obtain the following result.

Lemma 1.3.8 *If $\mu_1, \mu_2 \in \mathcal{M}_0(\Omega)$ and $\mu_1 \geq \mu_2$, then $w_{\mu_1} \leq w_{\mu_2}$ q.e. in Ω.*

The main tool for the study of the asymptotic behaviour of Dirichlet problems in perforated domains is the following notion of convergence in $\mathcal{M}_0(\Omega)$.

Definition 1.3.9 *Let $\{\mu_h\}$ be a sequence of measures of $\mathcal{M}_0(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. We say that $\{\mu_h\}$ γ^L-converges to μ (in Ω) if the sequence $\{u_h\}$ of the solutions to the problems*

$$\begin{cases} Lu_h + \mu_h u_h = f & \text{in } \Omega, \\ u_h = 0 & \text{on } \partial\Omega, \end{cases}$$

converges weakly in $H_0^1(\Omega)$ to the solution u of the problem

$$\begin{cases} Lu + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for every $f \in H^{-1}(\Omega)$.

Remark 1.3.10 This convergence of measures is the natural extension of the notion of γ^L-convergence introduced in [34], when L is the Laplace operator, and in [23] when L is symmetric.

Then main properties of γ^L-convergence are stated in the following propositions.

Proposition 1.3.11 *(Compactness). Every sequence of measures of $\mathcal{M}_0(\Omega)$ contains a γ^L-convergent subsequence.*

Proof. See [30], Theorem 4.5. □

Proposition 1.3.12 *The sequence* $\{\mu_h\}$ γ^L*-converges to* μ *if and only if* $\{\mu_h\}$ γ^{L^*}*–converges to* μ.

Proof. See[30], Theorem 4.3. □

Chapter 2
Approximation of relaxed Dirichlet problems by boundary value problems in perforated domains

The aim of this chapter is to present an explicit approximation scheme for relaxed Dirichlet problems by means of sequences of classical Dirichlet problems in perforated domains. We assume that $\mu \in \mathcal{M}_0(\Omega)$ is a Radon measure, and in Sections 2.1 and 2.2 we consider only the case when L is a symmetric operator. Nevertheless, using the results in [31] about μ–capacities associated to a nonsymmetric operator and with other minor changes, Theorem 2.2.9 below can be extended to the case of nonsymmetric operators.

2.1. Preliminaries

Let $L\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ be a linear elliptic operator in divergence form

$$Lu = -\mathrm{div}\,(A\,Du)\,,$$

where $A = A(x) = (a_{ij}(x))$ is a symmetric $N \times N$ matrix satisfying (1.3.1) and (1.3.2) with $\theta = \alpha$ and $\Theta = \alpha^{-1}$ for a suitable constant $\alpha > 0$, and let $\mu \in \mathcal{M}_0(\Omega)$.

A set function cap_μ^L can be associated with every measure μ in the class $\mathcal{M}_0(\Omega)$.

Definition 2.1.1 *Let $\mu \in \mathcal{M}_0(\Omega)$. For every open set $A \subseteq \Omega$ and for every Borel set $E \subseteq A$ we define the μ-capacity of E in A corresponding to the operator L as*

$$\mathrm{cap}_\mu^L(E, A) = \min\left\{ \langle Lu, u\rangle + \int_E u^2\, d\mu : u - 1 \in H_0^1(A) \right\},$$

The μ-capacity corresponding to $L = -\Delta$ will be denoted by cap_μ, while the μ-capacity with respect to $\mu = \infty_\Omega$ will be denoted by cap^L. The latter coincides with the classical capacity relative to the operator L according to the definition of [64] and [50]. If $L = -\Delta$ and $\mu = \infty_\Omega$, then cap_μ^L coincides with the harmonic capacity introduced at the beginning of this thesis. If $\mu = \infty_F$ for some $F \subseteq \Omega$, and L is any elliptic operator, then $\text{cap}_\mu^L(E, A) = \text{cap}^L(E \cap F, A)$ for every $E \subseteq A$.

Some of the properties of cap_μ^L are stated in the following proposition.

Proposition 2.1.2 *Let $\mu \in \mathcal{M}_0(\Omega)$, A, B open subsets of Ω and E, F subsets of A. Then*

(i) $\text{cap}_\mu^L(\emptyset, A) = 0$;

(ii) $E \subseteq F \Longrightarrow \text{cap}_\mu^L(E, A) \leq \text{cap}_\mu^L(F, A)$;

(iii) $\text{cap}_\mu^L(E \cup F, A) \leq \text{cap}_\mu^L(E, A) + \text{cap}_\mu^L(F, A)$;

(iv) $A \subseteq B \Longrightarrow \text{cap}_\mu^L(E, A) \geq \text{cap}_\mu^L(E, B)$;

(v) $\alpha\, \text{cap}_\mu(E, A) \leq \text{cap}_\mu^L(E, A) \leq \alpha^{-1}\, \text{cap}_\mu(E, A) \leq \alpha^{-1} \text{cap}(E, A)$;

(vi) if $\{E_h\}$ is an increasing sequence of subsets of A and $E = \cup_h E_h$, then $\text{cap}_\mu^L(E, A) = \sup_h \text{cap}_\mu^L(E_h, A)$.

Proof. See [33], Proposition 3.11, Theorem 3.10 and [28], Theorem 2.9. □

Since we are dealing with symmetric operators, the γ^L-convergence of a sequence of measures $\{\mu_h\}$ is related to the Γ-convergence of the quadratic functionals

$$F_{\mu_h}(u) = \langle Lu, u \rangle + \int_\Omega u^2\, d\mu_h\,,$$

as it is stated in the following theorem.

Theorem 2.1.3 *A sequence $\{\mu_h\}$ in $\mathcal{M}_0(\Omega)$ γ^L-converges to the measure $\mu \in \mathcal{M}_0(\Omega)$, if and only if the following conditions are satisfied for every $u \in H_0^1(\Omega)$:*

(a) for every sequence $\{u_h\}$ in $H_0^1(\Omega)$ converging to u in $L^2(\Omega)$

$$F_\mu(u) \leq \liminf_{h \to \infty} F_{\mu_h}(u_h)\,;$$

(b) there exists a sequence $\{u_h\}$ in $H_0^1(\Omega)$ converging to u in $L^2(\Omega)$ such that

$$F_\mu(u) = \lim_{h \to \infty} F_{\mu_h}(u_h).$$

Proof. See [4], Proposition 2.9. □

Our definition of γ^L-convergence coincides with the definition considered in [28]. As shown in [4], Proposition 2.8, if properties (a) and (b) hold on Ω, then they also hold for every open set $\Omega' \subseteq \Omega$. Conversely, if (a) and (b) hold for every open set $\Omega' \subset\subset \Omega$, then they hold on Ω. So our definition of γ^L-convergence differs from the definition given in [23] only in the fact that now the ambient space is Ω instead of $\mathbf{R}^N$. When $L = -\Delta$, our definition coincides with the definition given in [33].

Remark 2.1.4 Let $\{\lambda_h\}$ and $\{\mu_h\}$ be two sequences in $\mathcal{M}_0(\Omega)$ which γ^L-converge to λ and μ, respectively. If $\lambda_h \leq \mu_h$ for every h, by Theorem 2.1.3 we have $\int_\Omega u^2\, d\lambda \leq \int_\Omega u^2\, d\mu$ for every $u \in H_0^1(\Omega)$. In particular, if μ is a Radon measure, then $\lambda \leq \mu$.

Theorem 2.1.5 *Let $\{\mu_h\}$ be a sequence in $\mathcal{M}_0(\Omega)$ which γ^L-converges to a measure μ in $\mathcal{M}_0(\Omega)$. Then*

$$\mathrm{cap}_\mu^L(A,B) \leq \liminf_{h\to\infty} \mathrm{cap}_{\mu_h}^L(A,B),$$

for every pair of open sets A, B, with $A \subseteq B \subseteq \Omega$.

Proof. See [28], Proposition 5.7. □

We consider now a sufficient condition for the γ^L-convergence of a sequence of measures of the form $\{\infty_{E_h}\}$, where $\{E_h\}$ is a sequence of compact subsets of Ω. In this case, if $\Omega_h = \Omega\backslash E_h$, the solution u_h coincides with the solution of the classical problem

$$\begin{cases} L u_h = f & \text{in } \Omega_h, \\ u_h = 0 & \text{on } \partial\Omega_h\,, \end{cases}$$

prolonged to zero outside Ω_h.

Assume that $\{E_h\}$ satisfies the following hypotheses, studied by D. Cioranescu and F. Murat: there exist a measure $\mu \in H^{-1,\infty}(\Omega)$, a sequence $\{w_h\}$ in $H^1(U)$, and two sequences of positive measures of $H^{-1}(\Omega)$, $\{\nu_h\}$

and $\{\lambda_h\}$, such that

$$w_h \rightharpoonup 1 \qquad \text{weakly in } H^1(U)\,,$$
$$w_h = 0 \qquad \text{q.e. in } E_h\,,$$
$$L w_h = \nu_h - \lambda_h\,,$$
$$\nu_h \to \mu \qquad \text{strongly in } H^{-1}(\Omega)\,,$$
$$\lambda_h \rightharpoonup \mu \qquad \text{weakly in } H^{-1}(\Omega)\,,$$

and $\langle \lambda_h, v\rangle = 0$ for every $h \in \mathbf{N}$ and for every $v \in H_0^1(\Omega)$, with $v = 0$ q.e. in E_h.

Under these hypotheses the sequence $\{u_h\}$ converges weakly in $H_0^1(\Omega)$ to the weak solution u of the problem

$$\begin{cases} L u + \mu u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(see [25], Théorème 1.2). Later, H. Kacimi and F. Murat pointed out that the hypothesis $\mu \in H^{-1,\infty}(\Omega)$ can be replaced by $\mu \in H^{-1}(\Omega)$ (see [46], Rémarque 2.4).
In conclusion, using the language introduced in Definition 1.3.9, the following theorem holds.

Theorem 2.1.6 *If $\{E_h\}$ satisfies the hypotheses considered above, with $\mu \in H^{-1}(\Omega)$, then the sequence of measures $\{\infty_{E_h}\}$ γ^L-converges to the measure μ.*

We shall use in the following a Poincaré inequality involving Kato measures.

Lemma 2.1.7 *Let A be a Borel subset of a ball $B_R = B_R(x_0)$ such that $\operatorname{diam}(A) \geq q R$ for some $q \in (0,1)$, and let $\mu \in K_N^+(A)$. Then there exists a positive constant c, depending only on q and on the dimension N of the space, such that*

$$\int_A u^2\, d\mu \leq c\, \|\mu\|_{K_N^+(A)} \int_{B_R} |\nabla u|^2\, dx$$

for every $u \in H_0^1(B_R)$.

Proof. An inequality of this kind was proved by P. Zamboni in the case $N \geq 3$, $A = B_R$, and μ absolutely continuous with respect to the Lebesgue measure. The same arguments can be adapted, up to minor modifications, also to the general case. The main change in the case $N = 2$ is the use of the inequality

$$\int_{B_R} \frac{1}{|x-y|\,|z-y|}\,dy \leq c_q \left(\log\left(\frac{\operatorname{diam}(A)}{|x-z|}\right)+1\right) \qquad \forall x, z \in A,$$

which can be proved by direct computation. □

Finally we need a sort of dominated convergence theorem for measures in $H^{-1}(\Omega)$.

Lemma 2.1.8 *Let $\{\mu_h\}$ be a sequence of positive measures belonging to $H^{-1}(\Omega)$ that converges to 0 in the weak* topology of measures and suppose that there exists $\mu \in H^{-1}(\Omega)$ such that $\mu_h \leq \mu$. Then the sequence $\{\mu_h\}$ converges to 0 strongly in $H^{-1}(\Omega)$.*

Proof. This result could be obtained easily by using the strong compactness of the order intervals in $H^{-1}(\Omega)$. However, we give here a self-contained elementary proof. Let us define $\nu_h = \mu - \mu_h$. Clearly $\|\nu_h\|_{H^{-1}(\Omega)} \leq \|\mu\|_{H^{-1}(\Omega)}$ and so, up to a subsequence, $\{\nu_h\}$ converges to μ weakly in $H^{-1}(\Omega)$. The previous inequality, together with the lower semicontinuity of the norm, implies that $\|\nu_h\|_{H^{-1}(\Omega)}$ converges to $\|\mu\|_{H^{-1}(\Omega)}$. This shows that $\{\nu_h\}$ converges to μ strongly in $H^{-1}(\Omega)$ and concludes the proof of the lemma. □

2.2. The approximation result

In this section we prove that for every Radon measure $\mu \in \mathcal{M}_0(\Omega)$ the general approximation rule outlined in the introduction provides a sequence of measures of the form $\{\infty_{E_h}\}$ which γ^L-converges to μ according to Definition 1.3.9.

To deal with the case $\mu \in K_N^+(\Omega)$, we need the following lemmas.

Lemma 2.2.1 *Let U and V be open subsets of Ω, with $V \subset\subset U \subset\subset \Omega$, and let w be the L-capacitary potential of V with respect to U, i.e., the* *unique solution of*

$$\begin{cases} w \in H_0^1(U), & w \geq 1 \text{ q.e. on } V, \\ \langle Lw, v - w\rangle \geq 0, & \forall v \in H_0^1(U),\ v \geq 1 \text{ q.e. on } V. \end{cases}$$

Let us extend w to Ω by setting $w = 0$ on $\Omega\backslash U$. Then $w \in H_0^1(\Omega)$ and $w = 1$ q.e. on V. Moreover there exist two positive Radon measures γ and ν belonging to $H^{-1}(\Omega)$ such that $\operatorname{supp}\gamma \subseteq \partial V$, $\operatorname{supp}\nu \subseteq \partial U$, $Lw = \gamma - \nu$ *in Ω, and $\nu(\Omega) = \gamma(\Omega) = \operatorname{cap}^L(V, U)$.*

We call γ (resp. ν) the inner (resp. outer) L-capacitary distribution of V with respect to U.

Proof of Lemma 2.2.1. It is well known (see [64], Section 3) that there exists a positive Radon measure $\gamma \in H^{-1}(U)$, with $\operatorname{supp}\gamma \subseteq \partial V$, such that $Lw = \gamma$ in Ω and $\gamma(\Omega) = \operatorname{cap}^L(V, U)$. Let us consider now the following obstacle problem

$$\begin{cases} z \in H_0^1(\Omega), & z \geq 0 \text{ q.e. in } \Omega\backslash U, \\ \langle Lz + \gamma, v - z\rangle \geq 0 & \forall v \in H_0^1(\Omega),\ v \geq 0 \text{ q.e. in } \Omega\backslash U. \end{cases}$$

It is well known that there exists a unique solution z of this problem, that z is a supersolution of $L+\gamma$, i.e., $Lz+\gamma = \nu \geq 0$ for some positive measure $\nu \in H^{-1}(\Omega)$, and that $z \leq \zeta$ for every supersolution $\zeta \in H^1(\Omega)$ of $L+\gamma$ with $\zeta \geq 0$ q.e. in $\Omega\backslash U$ (see [48], Section II.6). Since γ is a positive measure, 0 is a supersolution of $L + \gamma$. Consequently $z \leq 0$ q.e. in Ω. As $z \geq 0$ q.e. in $\Omega\backslash U$, we conclude that $z = 0$ q.e. in $\Omega\backslash U$, hence $z \in H_0^1(U)$. On the other hand $Lz + \gamma = 0$ in U. As $Lw = \gamma$ on U, by uniqueness we can conclude that $z = -w$ in U, hence in Ω. This implies $Lw = \gamma - \nu$ in Ω. As $Lw - \gamma = 0$ in U and in $\Omega\backslash\overline{U}$ we conclude that $\operatorname{supp}\nu \subseteq \partial U$. Since $Lw = \gamma - \nu$ in Ω, we have

$$\int_\Omega A\, Dw D\varphi\, dx = \int_\Omega \varphi\, d\gamma - \int_\Omega \varphi\, d\nu \qquad \forall \varphi \in H_0^1(\Omega).$$

Let ψ be a cut-off function of class $C_0^\infty(\Omega)$ such that $\psi(x) = 1$ in $\overline{U}$. Choosing $\varphi = \psi(w-1)$ as test function we obtain

$$\int_\Omega A\, DwDw\, \psi\, dx + \int_\Omega A\, DwD\psi\, (w-1)\, dx = \int_\Omega \psi\, (w-1)\, d\gamma + \int_\Omega \psi\, (1-w)\, d\nu$$

and, using the fact that $w = 1$ γ-a.e. in Ω and $\psi(1-w) = 1$ q.e. on $\operatorname{supp}\nu$, we obtain $\int_\Omega ADwDw\,dx = \nu(\Omega)$. As $\gamma(\Omega) = \mathrm{cap}^L(V,U) = \int_\Omega ADwDw\,dx$, we conclude that $\nu(\Omega) = \gamma(\Omega) = \mathrm{cap}^L(V,U)$. □

Let us fix $x^0 \in \Omega$. For every $\rho > 0$ let $B_\rho = B_\rho(x^0)$ and let Q_ρ be the open cube $\{x \in \mathbf{R}^N\colon -\rho < x_k - x_k^0 < \rho \text{ for } k = 1,\dots,N\}$. If $0 < \rho < r$ and $B_r \subset\subset \Omega$, let w_r^ρ be the L-capacitary potential of B_ρ with respect to B_r, and let ν_r^ρ be the corresponding outer L-capacitary distribution.

Lemma 2.2.2 *For every $q \in (0,1)$ there exists a constant $c = c(q,\alpha,N)$, independent of the operator L, such that, if $B_r \subset\subset \Omega$ and $0 < \rho \le qr$, then*

$$\frac{1}{\nu_r^\rho(\partial B_r)} \int_{\partial B_r} \varphi\, d\nu_r^\rho \le c\, \frac{1}{\nu_r^{qr}(\partial B_r)} \int_{\partial B_r} \varphi\, d\nu_r^{qr}$$

for every $\varphi \in H^1(Q_r)$ with $\varphi \ge 0$ q.e. in Q_r.

Proof. Let us fix q, ρ, r, φ as required, and let $u \in H_0^1(\Omega)$ be a function whose restriction to B_r is a solution of the Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } B_r\,, \\ u - \varphi \in H_0^1(B_r)\,. \end{cases}$$

We may assume that $u = \varphi$ q.e. on the annulus $B_R \setminus \overline{B}_r$ for some $R > r$, so that $u = \varphi$ q.e. on $B_R \setminus B_r$. By De Giorgi's theorem, we have $u \in C^0(B_r)$. For every $s \in (0,r)$ we want to prove that

$$\frac{1}{\gamma_r^s(\partial B_s)} \int_{\partial B_s} u\, d\gamma_r^s = \frac{1}{\nu_r^s(\partial B_r)} \int_{\partial B_r} \varphi\, d\nu_r^s, \tag{2.2.1}$$

where γ_r^s is the inner L-capacitary distribution associated with w_r^s. Using the symmetry of the operator L, we get

$$\begin{aligned} 0 &= \int_{B_r} ADuDw_r^s\, dx = \int_\Omega ADw_r^s Du\, dx = \\ &= \int_\Omega u\, d\gamma_r^s - \int_\Omega u\, d\nu_r^s = \int_{\partial B_s} u\, d\gamma_r^s - \int_{\partial B_r} \varphi\, d\nu_r^s. \end{aligned}$$

Since $\nu_r^s(\partial B_r) = \mathrm{cap}^L(B_s, B_r) = \gamma_r^s(\partial B_s)$, we obtain (2.2.1).

Now we remark that, by the maximum principle, $u \geq 0$ on B_r. On the other hand, by Harnack's inequality,

$$\sup_{B_{qr}} u \leq c \inf_{B_{qr}} u,$$

where the constant c depends only on N, q, α, (see [64], Theorem 8.1). If we apply (2.2.1) with $s = \rho$ and $s = qr$, we obtain

$$\frac{1}{\nu_r^\rho(\partial B_r)} \int_{\partial B_r} \varphi \, d\nu_r^\rho = \frac{1}{\gamma_r^\rho(\partial B_\rho)} \int_{\partial B_\rho} u \, d\gamma_r^\rho \leq \sup_{B_{qr}} u \leq$$

$$\leq c \inf_{B_{qr}} u \leq c \frac{1}{\gamma_r^{qr}(\partial B_{qr})} \int_{\partial B_{qr}} u \, d\gamma_r^{qr} = c \frac{1}{\nu_r^{qr}(\partial B_r)} \int_{\partial B_r} \varphi \, d\nu_r^{qr},$$

and the lemma is proved. □

For every $0 < \rho < r$, with $B_r \subset\subset \Omega$, let $M_r^\rho : H^1(Q_r) \to \mathbf{R}$ be the linear function defined by

$$M_r^\rho u = \frac{1}{\nu_r^\rho(\partial B_r)} \int_{\partial B_r} u \, d\nu_r^\rho, \tag{2.2.2}$$

where ν_r^ρ is the outer L-capacitary distribution of B_ρ with respect to B_r.

Lemma 2.2.3 *For every $q \in (0,1)$ there exists a constant $c = c(q, \alpha, N)$ such that, if $Q_r \subset\subset \Omega$ and $0 < \rho \leq qr$, then*

$$\|u - M_r^\rho u\|_{L^2(Q_r)} \leq c\, r \, \|Du\|_{L^2(Q_r)},$$

for every $u \in H^1(Q_r)$.

Proof. Let us fix q, ρ, r as required. It is not restrictive to assume $x^0 = 0$. Let $Q = Q_1$ and $B = B_1$. Let us consider the operator L_r defined by $L_r u = -\mathrm{div}\,(A_r Du)$, where $A_r(y) = A(ry)$. It is easy to check that, if $w_r^\rho(x)$ is the L-capacitary potential of B_ρ with respect to B_r, then $v_r^\rho(y) = w_r^\rho(ry)$ is the L_r-capacitary potential of $B_{\rho/r}$ with respect to B. By Lemma 2.2.1 we can write $L_r v_r^\rho = \lambda_r^\rho - \mu_r^\rho$, with $\mathrm{supp}\,\lambda_r^\rho \subseteq \partial B_{\rho/r}$ and $\mathrm{supp}\,\mu_r^\rho \subseteq \partial B$. We want to prove that for every $u \in H^1(Q_r)$ we have

$$\int_{\partial B_r} u \, d\nu_r^\rho = r^{N-2} \int_{\partial B} u_r \, d\mu_r^\rho, \tag{2.2.3}$$

where $u_r(y) = u(ry)$. Let us fix $u \in H^1(Q_r)$ and let $\psi \in C_0^\infty(\Omega)$ be a cut-off function such that $\psi = 1$ on ∂B_r and $\psi = 0$ on $\overline{B}_\rho$. If $\psi_r(y) = \psi(ry)$, then

$$\int_{\partial B_r} u\, d\nu_r^\rho = \int_{\partial B_r} u\,\psi\, d\nu_r^\rho = -\int_{B_r} A\, Dw_r^\rho D(u\,\psi)\, dx =$$
$$= -r^{N-2} \int_B A_r\, Dv_r^\rho D(u_r\,\psi_r)\, dy = r^{N-2} \int_{\partial B} u_r\,\psi_r\, d\mu_r^\rho = r^{N-2} \int_{\partial B} u_r\, d\mu_r^\rho,$$

which proves (2.2.3). Taking $u = 1$ we get $\nu_r^\rho(\partial B_r) = r^{N-2}\,\mu_r^\rho(\partial B)$, so that the previous equality gives

$$\frac{1}{\nu_r^\rho(\partial B_r)} \int_{\partial B_r} u\, d\nu_r^\rho = \frac{1}{\mu_r^\rho(\partial B)} \int_{\partial B} u_r\, d\mu_r^\rho \tag{2.2.4}$$

for every $u \in H^1(Q_r)$. Finally, we recall that, if P is a projection from $H^1(Q)$ into $\mathbf{R}$, then the following Poincarè inequality holds for every u in $H^1(Q)$:

$$\|u - P(u)\|_{L^2(Q)} \le \beta\,\|P\|_{(H^1(Q))'}\,\|Du\|_{L^2(Q)},$$

where $(H^1(Q))'$ is the dual space of $H^1(Q)$ and the constant β depends only on the dimension N of the space (see [68], Theorem 4.2.1). Applying this result to

$$P_r^\rho(u) = \frac{1}{\mu_r^\rho(\partial B)} \int_{\partial B} u\, d\mu_r^\rho,$$

and using (2.2.4), we obtain

$$\begin{aligned} \|u - M_r^\rho u\|^2_{L^2(Q_r)} &= r^N \int_Q (u_r - P_r^\rho(u_r))^2\, dy \le \\ &\le \beta^2\, r^N \left(\frac{1}{\mu_r^\rho(\partial B)}\,\|\mu_r^\rho\|_{(H^1(Q))'}\right)^2 \int_Q |Du_r|^2\, dy = \\ &= \beta^2\, r^2 \left(\frac{1}{\mu_r^\rho(\partial B)}\,\|\mu_r^\rho\|_{(H^1(Q))'}\right)^2 \int_{Q_r} |Du|^2\, dx. \end{aligned} \tag{2.2.5}$$

It remains to estimate $\frac{1}{\mu_r^\rho(\partial B)}\|\mu_r^\rho\|_{(H^1(Q))'}$. By Lemma 2.2.2, applied to L_r, we obtain

$$\left|\frac{1}{\mu_r^\rho(\partial B)} \int_{\partial B} \varphi\, d\mu_r^\rho\right| \le c\,\frac{1}{\mu_r^{qr}(\partial B)} \int_{\partial B} |\varphi|\, d\mu_r^{qr}$$

for every $\varphi \in H^1(Q)$, so that

$$\frac{1}{\mu_r^\rho(\partial B)}\|\mu_r^\rho\|_{(H^1(Q))'} \leq c\,\frac{1}{\mu_r^{qr}(\partial B)}\|\mu_r^{qr}\|_{(H^1(Q))'}. \tag{2.2.6}$$

By Proposition 2.1.2(v) and by Lemma 2.2.1 we have

$$\mu_r^{qr}(\partial B) = \mathrm{cap}^{L_r}(B_q, B) \geq \alpha\,\mathrm{cap}(B_q, B). \tag{2.2.7}$$

Let $\zeta \in C_0^\infty(\mathbf{R}^N)$ be a cut-off function such that $\zeta = 1$ on ∂B, $\zeta = 0$ on $\overline{B}_q$, $0 \leq \zeta \leq 1$ on B, and $|D\zeta| \leq c_q = 2/(1-q)$ on B. Then, using again Proposition 2.1.2(v), for every $\varphi \in H^1(Q)$ we obtain

$$\begin{aligned} \int_Q \varphi\, d\mu_r^{qr} = \int_{\partial B} \varphi\zeta\, d\mu_r^{qr} = -\int_B A_r\, Dv_r^{qr} D(\varphi\zeta)\, dy \leq \\ \leq c_q\,\alpha^{-1/2}\left(\mathrm{cap}^{L_r}(B_q, B)\right)^{1/2} \|\varphi\|_{H^1(Q)} \leq \\ \leq c_q\,\alpha^{-1}\left(\mathrm{cap}(B_q, B)\right)^{1/2} \|\varphi\|_{H^1(Q)}. \end{aligned} \tag{2.2.8}$$

From (2.2.6), (2.2.7), (2.2.8) we obtain

$$\frac{1}{\mu_r^\rho(\partial B)}\|\mu_r^\rho\|_{(H^1(Q))'} \leq k(q, \alpha, N),$$

which, together with (2.2.5), concludes the proof of the lemma. □

For every $r > 0$ let $\hat{Q}_r$ be the cube $\{x \in \mathbf{R}^N\colon\ -r \leq x_k - x_k^0 < r \text{ for } k = 1, \ldots, N\}$, so that Q_r is the interior of $\hat{Q}_r$.

Lemma 2.2.4 *Let μ be a measure of $K_N^+(\Omega)$. For every $r > 0$, with $Q_r \subset\subset \Omega$, let $\rho = \rho(r) \in (0, r)$ be the radius such that $\mathrm{cap}^L(B_\rho, B_r) = \mu(\hat{Q}_r)$, and let $M_r = M_r^{\rho(r)}$, where $M_r^{\rho(r)}$ is the average defined in (2.2.2). Then there exists a function $\omega_\mu\colon \mathbf{R}_+ \to \mathbf{R}_+$, with $\lim\limits_{r\to 0+} \omega_\mu(r) = 0$, such that*

$$\|u - M_r u\|_{L^2_\mu(Q_r)} \leq \omega_\mu(r)\,\|Du\|_{L^2(Q_r)} \tag{2.2.9}$$

for every $u \in H^1(\Omega)$.

Proof. First of all we prove that for every $q \in (0,1)$ there exists $r_q > 0$ such that $\rho(r) \leq qr$ for $r \leq r_q$. We consider only the case $N \geq 3$; the case $N = 2$ is analogous. Since μ is a Kato measure, for every $r > 0$ we have

$$\mu(\Omega \cap B_r)\, r^{2-N} \leq \int_{\Omega \cap B_r} |y - x^0|^{2-N} \, d\mu(y) \leq \psi(r),$$

where ψ is an increasing function with $\lim_{r \to 0+} \psi(r) = 0$. If $\rho = \rho(r) > qr$, then, recalling that $\operatorname{cap}(B_{qr}, B_r) = c_q\, r^{N-2}$, and using Proposition 2.1.2(v), we obtain

$$\alpha c_q\, r^{N-2} \leq \alpha \operatorname{cap}(B_\rho, B_r) \leq \operatorname{cap}^L(B_\rho, B_r) = \mu(\hat{Q}_r).$$

So we can write $\alpha c_q\, r^{N-2} \leq \mu(\hat{Q}_r) \leq \mu(\Omega \cap B_{N\,r}) \leq \beta_N\, \psi(N\, r)\, r^{N-2}$. Choosing r_q such that $\psi(N\, r_q) < \alpha c_q / \beta_N$, we obtain a contradiction for $r \leq r_q$. Therefore, there exists $r_q > 0$, with $Q_{r_q} \subset\subset \Omega$, such that $\rho(r) \leq qr$ for every $r \leq r_q$. Since $c_q \to +\infty$ as $q \to 1$, we can choose r_q so that for every $r > 0$, with $Q_r \subset\subset \Omega$, there exists $q \in (0,1)$, with $r \leq r_q$.

Let us fix $q \in (0,1)$. It is clearly enough to prove (2.2.9) for every $r \leq r_q$. As $\mu \in K_N^+(\Omega)$, by Lemma 2.1.7, there exists a constant $c_N > 0$ such that, if $Q_r \subset\subset \Omega$, then

$$\int_{Q_r} u^2 \, d\mu \leq c_N \, \|\mu\|_{K_N^+(Q_r)} \int_{B_{N\,r}} |Du|^2 \, dx \tag{2.2.10}$$

for every $u \in H_0^1(B_{Nr})$.

Let us fix a bounded extension operator $\Pi \colon H^1(Q_1) \to H_0^1(B_N)$, and for every $r > 0$ let us define the extension operator $\Pi_r \colon H^1(Q_r) \to H_0^1(B_{Nr})$ by $(\Pi_r u)(x) = (\Pi u_r)(x/r)$, where $u_r(y) = u(ry)$. It is easily seen that the boundedness of Π implies the existence of a constant $k_N > 0$ such that

$$\int_{B_{N\,r}} |D(\Pi_r\, v)|^2 \, dx \leq k_N \left(\int_{Q_r} |Dv|^2 \, dx + \frac{1}{r^2} \int_{Q_r} v^2 \, dx \right) \tag{2.2.11}$$

for every $v \in H^1(Q_r)$. Note that, if $v \in H^1(\Omega)$ and $Q_r \subset\subset \Omega$, then $v = \Pi_r\, v$ q.e. on $\hat{Q}_r$, since both functions are quasi continuous and coincide on Q_r.

Using (2.2.10) and (2.2.11), for every $u \in H^1(\Omega)$ we obtain

$$\int_{Q_r} (u - M_r u)^2 \, d\mu \le c_N \|\mu\|_{K_N^+(Q_r)} \int_{B_{Nr}} \left(D\left(\Pi_r(u - M_r u)\right)\right)^2 dx$$
$$\le c_N \, k_N \|\mu\|_{K_N^+(Q_r)} \left(\int_{Q_r} |Du|^2 \, dx + \frac{1}{r^2} \int_{Q_r} (u - M_r u)^2 \, dx \right).$$

As $r \le r_q$, we have $\rho = \rho(r) \le qr$, so that Lemma 2.2.3 implies that

$$\frac{1}{r^2} \int_{Q_r} (u - M_r u)^2 \, dx \le c^2 \int_{Q_r} |Du|^2 \, dx,$$

hence

$$\int_{Q_r} (u - M_r u)^2 \, d\mu \le c_N \, k_N \, (1 + c^2) \, \|\mu\|_{K_N^+(Q_r)} \int_{Q_r} |Du|^2 \, dx,$$

for every $r \le r_q$ and for every $u \in H^1(\Omega)$. Since $\|\mu\|_{K_N^+(Q_r)}$ tends to zero as r tends to zero, the statement is proved. □

We are now in a position to prove our result for Kato measures. Let $\{Q_h^i\}_{i \in \mathbf{Z}^N}$ be the partition of $\mathbf{R}^N$ composed of the cubes

$$Q_h^i = \{x \in \mathbf{R}^N \colon \, i_k/h \le x_k < (i_k + 1)/h \text{ for } k = 1, \dots, N\}.$$

Theorem 2.2.5 *Let $\mu \in K_N^+(\Omega)$. Let $N(h)$ be the set of all indices i such that $Q_h^i \subset\subset \Omega$. For every $i \in N(h)$ let B_h^i be the ball with the same center as Q_h^i and radius $1/2h$, and let E_h^i be another ball with the same center such that*

$$\mathrm{cap}^L(E_h^i, B_h^i) = \mu(Q_h^i).$$

Define $E_h = \bigcup_{i \in N(h)} E_h^i$. Then the measures ∞_{E_h} γ^L-converge to μ as $h \to \infty$.

Proof. Let v_h^i be the L-capacitary potential of E_h^i with respect to B_h^i, extended to 0 on Ω, and let $w_h^i = 1 - v_h^i$. By Lemma 2.2.1, we obtain $L w_h^i = \nu_h^i - \lambda_h^i$ in Ω, with $\nu_h^i,\ \lambda_h^i \in H^{-1}(\Omega)$, $\nu_h^i \geq 0$, $\lambda_h^i \geq 0$, $\operatorname{supp} \nu_h^i \subseteq \partial B_h^i$, $\operatorname{supp} \lambda_h^i \subseteq \partial E_h^i$, and

$$\nu_h^i(Q_h^i) = \lambda_h^i(Q_h^i) = \operatorname{cap}^L(E_h^i, B_h^i) = \mu(Q_h^i). \tag{2.2.12}$$

Let us define $w_h \in H^1(\Omega)$ as

$$w_h = \begin{cases} w_h^i & \text{in } B_h^i \backslash E_h^i, \\ 0 & \text{in } E_h^i, \\ 1 & \text{elsewhere} \end{cases} \tag{2.2.13}$$

and the measures ν_h and λ_h as

$$\nu_h = \sum_{i \in N(h)} \nu_h^i, \qquad \lambda_h = \sum_{i \in N(h)} \lambda_h^i. \tag{2.2.14}$$

We want to prove that all hypotheses of Theorem 2.1.6 hold for w_h and ν_h.

First of all, we prove that w_h converges weakly to 1 in $H^1(\Omega)$. Since, by the maximum principle, $0 \leq w_h \leq 1$ in Ω, we have that $\{w_h\}$ is bounded in $L^2(\Omega)$. On the other hand,

$$\alpha \int_\Omega |Dw_h|^2\, dx \leq \sum_{i \in N(h)} \operatorname{cap}^L(E_h^i, B_h^i) = \sum_{i \in N(h)} \mu(Q_h^i) \leq \mu(\Omega).$$

Thus $\{w_h\}$ is bounded in $H^1(\Omega)$ so that there exist a subsequence (still denoted $\{w_h\}$) and a function $w \in H^1(\Omega)$, such that $\{w_h\}$ converges to w weakly in $H^1(\Omega)$, and hence strongly in $L^2(\Omega)$. We are going to show that $w = 1$ a.e. in Ω, using the arguments of D. Cioranescu and F. Murat (see [25], Théorème 2.2). Let us consider the family $\{C_h^i\}_{i \in \mathbf{Z}^N}$ of all open balls with radius $(\sqrt{N} - 1)/2h$ and centers in the vertices i/h of the cubes Q_h^i. In these balls we have $w_h = 1$. Therefore, if we define C_h as the union of the balls C_h^i contained in Ω, we have $w_h \mathbf{1}_{C_h} = \mathbf{1}_{C_h}$, where $\mathbf{1}_{C_h}$ is the characteristic function of C_h. Since $\{\mathbf{1}_{C_h}\}$ converges to a positive constant in the weak* topology of $L^\infty(\Omega)$, passing to the limit in the equality $w_h \mathbf{1}_{C_h} = \mathbf{1}_{C_h}$ we obtain $w = 1$ a.e. in Ω.

It remains to prove that the measures ν_h defined in (2.2.14) converge to μ in the strong topology of $H^{-1}(\Omega)$. Indeed, since w_h converges to 1 weakly in $H^1(\Omega)$, this implies also that λ_h converges weakly to μ in $H^{-1}(\Omega)$.

For every $h \in \mathbf{N}$ we introduce the polyrectangle $P_h = \bigcup_{i\in N(h)} Q_h^i$ and we define $S_h = \Omega \setminus P_h$. Moreover, for every $\varphi \in H_0^1(\Omega)$, we consider the function

$$\varphi_h = \sum_{i\in N(h)} (M_h^i \varphi)\, \mathbf{1}_{Q_h^i},$$

where, according to (2.2.2),

$$M_h^i \varphi = \frac{1}{\nu_h^i(\partial B_h^i)} \int_{\partial B_h^i} \varphi \, d\nu_h^i ,$$

and we define $\varepsilon_h = \|\mu \llcorner S_h\|_{H^{-1}(\Omega)}$. Note that $\{\varepsilon_h\}$ tends to zero by Lemma 2.1.8. Recalling that $\mu(Q_h^i) = \nu_h^i(\partial B_h^i)$ and using the Poincarè inequality (2.2.9), we have that,

$$\begin{aligned}
|\langle \nu_h, \varphi\rangle - \langle \mu, \varphi\rangle| &= \left| \sum_{i\in N(h)} \frac{\mu(Q_h^i)}{\nu_h^i(\partial B_h^i)} \int_{\partial B_h^i} \varphi\, d\nu_h^i - \sum_{i\in N(h)} \int_{Q_h^i} \varphi\, d\mu - \int_{S_h} \varphi\, d\mu \right| \le \\
&\le \int_{P_h} |\varphi - \varphi_h|\, d\mu + \int_{S_h} |\varphi|\, d\mu \le \\
&\le \left(\mu(\Omega) \int_{P_h} (\varphi - \varphi_h)^2 \, d\mu\right)^{1/2} + \|\mu \llcorner S_h\|_{H^{-1}(\Omega)} \|\varphi\|_{H_0^1(\Omega)} = \\
&= \left(\mu(\Omega) \sum_{i\in N(h)} \|\varphi - M_h^i \varphi\|^2_{L^2_\mu(Q_h^i)}\right)^{1/2} + \varepsilon_h \|\varphi\|_{H_0^1(\Omega)} \le \\
&\le \left(\mu(\Omega) \sum_{i\in N(h)} \omega(1/h)^2 \|D\varphi\|^2_{L^2(Q_h^i)}\right)^{1/2} + \\
&+ \varepsilon_h \|\varphi\|_{H_0^1(\Omega)} \le \left(\omega(1/h)\,\mu(\Omega)^{1/2} + \varepsilon_h\right) \|\varphi\|_{H_0^1(\Omega)}.
\end{aligned}$$

Thus we obtain

$$\|\nu_h - \mu\|_{H^{-1}(\Omega)} \le \mu(\Omega)^{1/2} \omega(1/h) + \varepsilon_h,$$

hence $\{\nu_h\}$ converges to μ strongly in $H^{-1}(\Omega)$. Therefore $\{\infty_{E_h}\}$ γ^L-converges to μ by Theorem 2.1.6. □

In order to generalize this result to every Radon measure we need the following results.

Proposition 2.2.6 *For every Radon measure $\mu \in \mathcal{M}_0(\Omega)$ there exist a measure $\nu \in K_N^+(\Omega)$ and a positive Borel function $g\colon \Omega \to [0,+\infty]$ such that $\mu = g\nu$.*

Proof. The result follows from Theorem 1.2.4 and from the fact that μ is a Radon measure (see, e.g., [28], Section 3). □

Proposition 2.2.7 *Let $\lambda \in \mathcal{M}_0(\Omega)$, let μ be a Radon measure in $\mathcal{M}_0(\Omega)$; for every $x \in \Omega$ let*

$$f(x) = \liminf_{r\to 0} \frac{\mathrm{cap}_\lambda^L(B_r(x), B_{2r}(x))}{\mu(B_r(x))}.$$

Assume that f is bounded. Then λ is a Radon measure and we have $\lambda = f\mu$.

Proof. See [23], Theorem 2.3. □

Proposition 2.2.8 *Let μ be a positive Radon measure on Ω. Then there exists a unique pair (μ_0, μ_1) of Radon measures on Ω such that:*

(i) $\mu = \mu_0 + \mu_1$;

(ii) $\mu_0 \in \mathcal{M}_0(\Omega)$;

(iii) $\mu_1 = \mu \llcorner N$, *for some Borel set N with* $\mathrm{cap}(N,\Omega) = 0$.

Proof. See [40], Lemma 2.1. □

We are now in a position to prove our main result in its most general form.

Theorem 2.2.9 *Let μ be a positive Radon measure on Ω. Let $\{Q_h^i\}$ and $\{E_h\}$ be defined as in Theorem 2.5. Then the following results hold:*

(i) if μ belongs to $\mathcal{M}_0(\Omega)$, then $\{\infty_{E_h}\}$ γ^L-converges to μ;

(ii) if $\mu = \mu_0 + \mu_1$, with μ_0 and μ_1 as in Proposition 2.2.8, then $\{\infty_{E_h}\}$ γ^L-converges to μ_0.

Proof. If μ is a Radon measure in $\mathcal{M}_0(\Omega)$, then, by Proposition 2.2.6, $\mu = g\nu$, where $\nu \in K_N^+(\Omega)$ and g is a positive Borel function. By Theorem 2.1.3, there exists a subsequence, still denoted by $\{E_h\}$, and a measure $\lambda \in \mathcal{M}_0(\Omega)$, such that $\{\infty_{E_h}\}$ γ^L-converges to λ. Let $x \in \Omega$ and let $r > 0$ such that $B_{2r}(x) \subseteq \Omega$. We want to prove that for every Borel set $E \subseteq B_{2r}$

$$\mathrm{cap}_\lambda^L(E, B_{2r}(x)) \leq \mu(E). \tag{2.2.15}$$

If A and A' are two open sets such that $A' \subset\subset A \subseteq B_{2r}(x)$ and h is small enough we have

$$\bigcup_{E_h^i \cap A' \neq \emptyset} Q_h^i \subseteq A,$$

hence, by Proposition 2.1.2,

$$\mathrm{cap}^L(E_h \cap A', B_{2r}(x)) \leq \sum_{E_h^i \cap A' \neq \emptyset} \mathrm{cap}^L(E_h^i, B_{2r}(x)) \leq$$
$$\leq \sum_{E_h^i \cap A' \neq \emptyset} \mathrm{cap}^L(E_h^i, B_h^i) = \sum_{E_h^i \cap A' \neq \emptyset} \mu(Q_h^i) \leq \mu(A).$$

Using Theorem 2.1.5 we obtain,

$$\mathrm{cap}_\lambda^L(A', B_{2r}(x)) \leq \liminf_{h \to \infty} \mathrm{cap}^L(E_h \cap A', B_{2r}(x)) \leq \mu(A)$$

and, as $A' \nearrow A$, we obtain $\mathrm{cap}_\lambda^L(A, B_{2r}(x)) \leq \mu(A)$ for every open set $A \subseteq B_{2r}(x)$ (see Theorem 2.1.2(vi)). Since μ is a Radon measure, this inequality can be easily extended to all Borel subsets of $B_{2r}(x)$. So (2.2.15) is proved. Choosing $E = B_r(x)$ in (2.2.15) and applying Proposition 2.2.7, we obtain that λ is a Radon measure and that $\lambda \leq \mu$.

Define, for $k \in \mathbf{N}$, the measures $\mu^k = g^k \nu$, where $g^k(x) = \min(g(x), k)$. As $\mu^k \in K_N^+(\Omega)$, by Theorem 2.5 for every k there exists a sequence $\{E_{k,h}\}_h$ such that $\{\infty_{E_{k,h}}\}_h$ γ^L-converges to μ^k. Since $\mu^k \leq \mu$, the construction of Theorem 2.5 implies that $E_{k,h} \subseteq E_h$ for every h and k. By Remark 2.1.4 this implies $\lambda \geq \mu^k$ for every k, hence $\lambda \geq \mu$. As the opposite inequality has already been proved, we obtain $\lambda = \mu$. Since the γ^L-limit does not depend on the subsequence, the whole sequence $\{\infty_{E_h}\}$ γ^L-converges to μ.

Let now μ be any Radon measure on Ω. By Proposition 2.2.8, we can write $\mu = \mu_0 + \mu_1$, with $\mu_0 \in \mathcal{M}_0(\Omega)$ and $\mu_1 = \mu \llcorner N$, where N is a Borel set with $\operatorname{cap}(N,\Omega)=0$. Arguing as before, let λ be the γ^L-limit of a subsequence of $\{\infty_{E_h}\}$. If $x\in\Omega$ and $r>0$ is such that $B_{2r}(x)\subseteq\Omega$, we have

$$\operatorname{cap}^L_\lambda(B_r(x),B_{2r}(x))=\operatorname{cap}^L_\lambda(B_r(x)\setminus N,B_{2r}(x)),$$

since $\operatorname{cap}(N,B_{2r}(x))=0$ (see Proposition 2.1.2). Therefore (2.2.15), applied with $E=B_r(x)\setminus N$, gives

$$\operatorname{cap}^L_\lambda(B_r(x),B_{2r}(x))\le\mu(B_r(x)\setminus N)=\mu_0(B_r(x)).$$

By applying again Proposition 2.2.7 we obtain $\lambda\le\mu_0$.

Since μ_0 is a Radon measure of $\mathcal{M}_0(\Omega)$, by the first part of this theorem we can construct the holes $E_{0,h}$ such that $\{\infty_{E_{0,h}}\}$ γ^L-converges to μ_0. Since $\mu(Q^i_h)\ge\mu_0(Q^i_h)$, we have $E_{0,h}\subseteq E_h$, hence, by Remark 2.1.4, $\lambda\ge\mu_0$. As the opposite inequality has already been proved, we obtain $\lambda=\mu_0$. As before, this implies that the whole sequence $\{\infty_{E_h}\}$ γ^L-converges to μ_0. □

We want to underline that if we take $L=-\Delta$, μ as the Lebesgue measure, and $N\ge 3$, then we have $\rho^i_h=\rho_h$ independent of i, and

$$\operatorname{cap}(E^i_h,B^i_h)=(N-2)\sigma_N\frac{\rho_h^{N-2}\left(\frac{1}{2h}\right)^{N-2}}{\left(\frac{1}{2h}\right)^{N-2}-\rho_h^{N-2}},$$

where σ_N is the surface area of the unit sphere of $\mathbf{R}^N$, and we obtain

$$\rho_h=h^{-\frac{N}{N-2}}\Big(\frac{2^{N-2}h^{2N-2}}{1+\sigma_N(N-2)2^{N-2}h^{2N-2}}\Big)^{\frac{1}{N-2}},$$

which yields the same approximating sequence obtained by Cioranescu and Murat in [25].

Finally the following example shows that if the measure μ is not a Radon measure, our construction cannot be applied.

Remark 2.2.10 Let (x_h) be a renumbering of $\mathbf{Q}^N\cap\Omega$, and let us consider the union $U=\bigcup_h B(x_h,r_h)$ of the balls with center on x_h and with radii r_h

such that $\sum_h \text{cap}^L(B(x_h, r_h), B_R) < \text{cap}^L(\Omega, B_R)$, where B_R is a ball such that $\Omega \subset\subset B_R$. If we consider the measure $\mu \in \mathcal{M}_0(\Omega)$ defined by

$$\mu(B) = \begin{cases} 0 & \text{if } \text{cap}^L(B \cap U, B_R) = 0, \\ +\infty & \text{otherwhise,} \end{cases}$$

for every Borel set $B \subseteq \Omega$, then our construction gives a sequence $\{\infty_{E_h}\}$ which γ^L-converges to the measure $\beta \in \mathcal{M}_0(\Omega)$ defined as $\beta(B) = +\infty$ for every Borel set B of positive capacity. Hence, in this case μ does not coincide with the γ^L-limit of $\{\infty_{E_h}\}$.

2.3. Approximation by problems in domains with holes of measure zero

In Section 2.2 we proved that every Radon measure $\mu \in \mathcal{M}_0(\Omega)$ can be approximated, in the sense of γ^L-convergence, by measures of the type ∞_{E_h} where $E_h \subseteq \Omega$ is the union of N-dimensional balls centered in a cubical lattice and with radius depending on the mass of the measure μ in each cube of the lattice.

In this section we shall show that for every Radon measure $\mu \in \mathcal{M}_0(\Omega)$ and for every nonatomic, nonnegative Radon measure λ it is possible to construct a sequence $\{E_h\}$ of subsets of Ω such that E_h is the union of $(N-1)$-dimensional balls centered in the cubical lattice, with $\lambda(E_h) = 0$, and such that the sequence $\{\infty_{E_h}\}$ γ^L-converges to μ.

Let us remark that while the proof of the previous approximation result is based on the method of oscillating test functions as in [25], allowed by the use of a general version of the Poincaré inequality on Kato spaces, in this section we shall follow a completely different line and we use directly the capacitary method introduced in [23] for symmetric operators and generalized to the nonsymmetric case in [31]. An application of this result to shape optimization problems will be seen in Chapter 3

From now on L will be the operator introduced in Section 1.3, without the symmetry assumption. Let $\{Q_h^i\}_{i\in Z^n}$, $\{B_h^i\}$ and $N(h)$ as in Section 2.2. For every $i \in N(h)$ we fix a constant $c_h^i \geq 0$ and we consider the function

$$\varphi_h(x) = \sum_{i\in N(h)} c_h^i \, \mathbf{1}_{Q_h^i}(x). \tag{2.3.1}$$

The main theorem of this section is the following.

Theorem 2.3.1 *Let λ be a nonatomic, nonnegative Radon measure, and let μ be a Radon measure belonging to $\mathcal{M}_0(\Omega)$. Suppose that the functions φ_h defined in (2.3.1) converge to a function φ in the weak* topology of $L^\infty_\mu(\Omega)$. Then for every $h \in \mathbf{N}$ there exists a compact set E_h obtained as a union of closed $(N-1)$-dimensional spheres $E^i_h = \overline{S}(x^i_h, \rho^i_h)$ with the same center of Q^i_h, radius ρ^i_h, and lying in a suitable $(N-1)$-dimensional hyperplane, with the property*

$$\text{(2.3.2)} \qquad \operatorname{cap}^L(E^i_h, B^i_h) = c^i_h \mu(Q^i_h) = (\varphi_h \mu)(Q^i_h),$$

$\lambda(E^i_h) = 0$ for every $i \in N(h)$, and such that the sequence $\{\infty_{E_h}\}$ γ^L-converges to the measure $\varphi\mu$.

In order to construct the sequence $\{E_h\}$ we need the following lemma.

Lemma 2.3.2 *Let λ be a nonnegative nonatomic bounded measure defined in a bounded open set Ω of $\mathbf{R}^N$, $N \geq 2$, and let us fix $x \in \Omega$. Then there exists a $(N-1)$-dimensional hyperplane P_{N-1} containing x and such that $\lambda(\Omega \cap P_{N-1}) = 0$.*

Proof. We prove the result by induction on the dimension N of the space. If $N = 2$, the set $\Omega \setminus \{x\}$ can be written as the union $\bigcup_\omega (P^\omega_1 \setminus \{x\})$ of all straight lines passing through x and deprived of the point x. Since λ is finite and nonatomic, there exist at most countably many indeces ω such that $\lambda(\Omega \cap P^\omega_{N-1}) > 0$. Thus the result is proved for $N = 2$

If we suppose that there exists a $(N-2)$-dimensional hyperplane P_{N-2} containing x and such that $\lambda(\Omega \cap P_{N-2}) = 0$, then we can consider the family $\{P^\omega_{N-1} \setminus P_{N-2}\}_\omega$ of all the $(N-1)$-dimensional hyperplanes containing P_{N-2} and deprived of P_{N-2}. As before, there exists at most countably many indeces ω such that $\lambda(\Omega \cap P^\omega_{N-1}) > 0$, and this concludes the proof. □

By Lemma 2.3.2, for every $h \in \mathbf{N}$ and $i \in N(h)$ we can find a $(N-1)$-dimensional hyperplane P^i_h, passing through the center x^i_h of Q^i_h such that $\lambda(P^i_h \cap Q^i_h) = 0$. Notice that the choice of P^i_h does not depend on the measure μ considered. We define E^i_h to be the $(N-1)$-dimensional sphere lying on

P_h^i, centered in x_h^i and with radius ρ_h^i such that $E_h^i = \overline{S}(x_h^i, \rho_h^i)$ verifies (2.3.2). The existence of ρ_h^i is due to the continuity of the function $\rho \mapsto \mathrm{cap}^L(\overline{S}(x_h^i, \rho), B_h^i)$. Setting $E_h = \bigcup_{i\in N(h)} E_h^i$, it remains to prove that the sequence $\{\infty_{E_h}\}$ γ^L-converges to the measure $\varphi\mu$.

In the following we need some properties of the capacity associated to the operator L.

Proposition 2.3.3 *Let U, V be open subsets of $\mathbf{R}^N$ and let E, F be subsets of U. We have*
(i) $\mathrm{cap}^L(\emptyset, U) = 0$;
(ii) $E \subseteq F \Longrightarrow \mathrm{cap}^L(E,U) \leq \mathrm{cap}^L(F,U)$;
(iii) $\mathrm{cap}^L(E\cup F,U) \leq \mathrm{cap}^L(E,U) + \mathrm{cap}^L(F,U)$;
(iv) $U \subseteq V \Longrightarrow \mathrm{cap}^L(E,U) \geq \mathrm{cap}^L(E,V)$;
(v) $\theta\,\mathrm{cap}(E,U) \leq \mathrm{cap}^L(E,U) \leq \Theta\,\mathrm{cap}(E,U)$;

Proof. If L is symmetric these properties follow from Proposition 2.1.2 with $\mu = 0$. The general case is studied in [51]. □

The proof of Theorem 2.3.1 is based on the following characterization of the γ^L-limit of measures.

Theorem 2.3.4 *For every sequence $\{\mathcal{E}_h\}$ of compact subsets of Ω there exist a subsequence, still denoted by $\{\mathcal{E}_h\}$ and a measure β belonging to $\mathcal{M}_0(\Omega)$ such that the sequence of measures $\{\infty_{\mathcal{E}_h}\}$ γ^L-converges to β. Moreover β is the least superadditive set function such that*

$$\beta(A) \geq \inf_{\substack{U \text{ open} \\ A\subseteq U}} \; \sup_{\substack{K \text{ compact} \\ K\subseteq U}} \; \limsup_{h\to\infty} \mathrm{cap}^L(\mathcal{E}_h \cap K, \Omega), \tag{2.3.3}$$

for every Borel subset $A \subseteq \Omega$.

Proof. See [28], Theorems 6.1 and 6.3, and [34], Theorem 4.14 for the symmetric case, and [31], Theorem 5.10 for the general case. □

In order to prove Theorem 2.3.1 it will suffice to show that for every fixed subsequence $\{E_{k_h}\}$, taking $\mathcal{E}_h = E_{k_h}$ in Theorem 2.3.4, we obtain $\beta = \varphi\mu$. First of all we prove the easier inequality.

Proposition 2.3.5 *If μ, φ and $\{E_h\}$ are as in Theorem 2.3.1, then $\beta \le \varphi\mu$.*

Proof. Let A and A' be two open subsets of Ω such that $A' \subset\subset A$ and let h be large enough to have $\bigcup_{E_h^i \cap A' \neq \emptyset} Q_h^i \subseteq A$. Then, by Proposition 2.3.3(iii), (iv), we obtain

$$\mathrm{cap}^L(E_h \cap A', \Omega) \le \sum_{E_h^i \cap A' \neq \emptyset} \mathrm{cap}^L(E_h^i, \Omega) \le$$

$$\le \sum_{E_h^i \cap A' \neq \emptyset} \mathrm{cap}^L(E_h^i, B_h^i) = \sum_{E_h^i \cap A' \neq \emptyset} (\varphi_h \mu)(Q_h^i) \le (\varphi_h \mu)(A) = \int_A \varphi_h \, d\mu .$$

Passing to the limit as h goes to $+\infty$ we get

$$\limsup_{h \to \infty} \mathrm{cap}^L(E_h \cap A', \Omega) \le \int_A \varphi \, d\mu$$

so that

$$\beta(A) = \sup_{\substack{K \text{ compact} \\ K \subseteq A}} \limsup_{h \to \infty} \mathrm{cap}^L(E_h \cap K, \Omega) =$$

$$= \sup_{\substack{A' \text{ open} \\ A' \subset\subset A}} \limsup_{h \to \infty} \mathrm{cap}^L(E_h \cap A', \Omega) \le (\varphi\mu)(A) ,$$

for every open set A. By the very definition of β and since μ is a Radon measure, we obtain $\beta \le \varphi\mu$. □

From now on we shall consider only the case $N \ge 3$. The changes in dimension $N = 2$ will be explained in Remark 2.3.9.

In order to prove the inequality $\varphi\mu \le \beta$ we need a sort of "local almost-superadditivity" of the capacity of the sets E_h that holds if μ belongs to $H^{-1}(\Omega)$, as it was done in other papers on this subject (see [3], [12] and [13]). The main tools in studying this property will be the following three lemmas.

Lemma 2.3.6 *Let μ, ν be two positive measures belonging to $H^{-1}(\Omega)$, let $\{\mathcal{E}_h\}$ be a sequence of compact subsets of Ω, and let β be defined as in Theorem 2.3.4. Assume that there exist two positive constants k_1, k_2 such that for every open set $A \subset\subset \Omega$*

$$\beta(\overline{A}) \ge k_1 \, \nu(A) - k_2 \iint_{\overline{A} \times \overline{A}} G(x,y) \, d\mu(x) d\mu(y)$$

where G is the fundamental solution for the Laplace operator in $\mathbf{R}^N$ and $\overline{A}$ is the closure of A; then $\beta \geq k_1 \nu$.

Proof. Since μ and ν are Radon measures, and by the definition of β, we have

$$\beta(A) = \sup_{\substack{U \text{ open} \\ U \subset\subset A}} \beta(\overline{U}) \geq k_1 \nu(A) - k_2 \iint_{\overline{A}\times\overline{A}} G(x,y)\, d\mu(x) d\mu(y) ,$$

for every open set $A \subseteq \Omega$.

Now let us fix $h \in \mathbf{N}$, and let $\{A_i^h\}$ be a finite family of open sets such that $\bigcup_i A_i^h = A$, $\operatorname{diam}(A_i^h) \leq 1/h$, and with the property that there exists a positive constant c such that for every fixed j the number of the indexes i such that $A_i^h \cap A_j^h \neq \emptyset$ does not exceed c.

Thanks to the superadditivity of β we obtain

$$\beta(A) \geq \sum_i \beta(A_i^h) \geq k_1 \nu(A) - ck_2 \iint_{K_h} G(x,y)\, d\mu(x) d\mu(y) , \tag{2.3.4}$$

where $K_h = \{(x,y) \in A \times A : \operatorname{dist}((x,y),\Delta) \leq 1/h\}$ and Δ is the diagonal in $\mathbf{R}^N \times \mathbf{R}^N$. Since μ belongs to $H^{-1}(\Omega)$, the measure $G\mu \otimes \mu$ is finite in $A \times A$. Moreover, since μ is nonatomic and by Fubini theorem, we obtain $(G\mu \otimes \mu)(\Delta) = 0$. Thus the term $\iint_{K_h} G(x,y)\, d\mu(x) d\mu(y)$ tends to zero as h goes to $+\infty$. Eventually, a passage to the limit in (2.3.4) gives $\beta(A) \geq k_1 \nu(A)$ for any open set $A \subset\subset \Omega$, and hence the thesis. □

Lemma 2.3.7 *Let μ be a positive measure belonging to $H^{-1}(\Omega)$. Given a positive real number δ and an open subset A of Ω, we define the set of indices*

$$J_h^{A,\delta} = \Big\{ j \in \mathbf{Z}^N : \sum_{\substack{i \in \mathbf{Z}^N \ i \neq j \\ Q_h^i \cap A \neq \emptyset}} \frac{\mu(Q_h^i)}{|x_h^i - x_h^j|^{N-2}} + \mu(Q_h^j) h^{N-2} > \delta \,,\ Q_h^j \cap A \neq \emptyset \Big\} .$$

Then, if we set $A_h = \bigcup\{Q_h^i \ :\ Q_h^i \cap A \neq \emptyset\}$, we have

$$\mu\Big(\bigcup_{j \in J_h^{A,\delta}} (Q_h^j \cap A) \Big) \leq \frac{c(N)}{\delta} \iint_{A_h \times A_h} G(x,y)\, d\mu(x) d\mu(y) .$$

Proof. Let us fix $j \in \mathbf{Z}^N$; note that for every x, ξ in Q_h^j we have $|x-\xi| \leq |x-x_h^j|+|\xi-x_h^j| \leq 2\sqrt{N}/h$, and so

$$\mu(Q_h^j)\, h^{N-2} \leq (2\sqrt{N})^{N-2} \int_{Q_h^j} \frac{1}{|x-\xi|^{N-2}}\, d\mu(\xi) \qquad \forall x \in Q_h^j\,.$$

Moreover, if $i \in \mathbf{Z}^N$ is such that $\sup_{k=1,\ldots,N} |j_k - i_k| \leq 1$, then for every $x \in Q_h^j$ and for every $\xi \in Q_h^i$ we have

$$\begin{aligned} &|x-\xi| \leq |x-x_h^j| + |x_h^j - x_h^i| + |x_h^i - \xi| \leq \\ &\leq (2\sqrt{N})/h + |x_h^j - x_h^i| \leq (2\sqrt{N}+1)\, |x_h^j - x_h^i|\,, \end{aligned}$$

so that

$$\frac{\mu(Q_h^i)}{|x_h^i - x_h^j|^{N-2}} \leq (2\sqrt{N}+1)^{N-2} \int_{Q_h^i} \frac{1}{|x-\xi|^{N-2}}\, d\mu(\xi)$$

for every $x \in Q_h^j$, i such that $\sup_{k=1,\ldots,N} |j_k - i_k| \leq 1$.

Finally, if $i \in \mathbf{Z}^N$ is such that $\sup_{k=1,\ldots,N} |j_k - i_k| > 1$, one obtains, in a similar way, the estimate

$$\frac{\mu(Q_h^i)}{|x_h^i - x_h^j|^{N-2}} \leq (3\sqrt{N})^{N-2} \int_{Q_h^i} \frac{1}{|x-\xi|^{N-2}}\, d\mu(\xi)$$

So we can conclude that for every $j \in J_h^{A,\delta}$ there exists $c = c(N) > 0$ such that

$$\delta \leq c \sum_{\substack{i \in \mathbf{Z}^N \\ Q_h^i \cap A \neq \emptyset}} \int_{Q_h^i} \frac{1}{|x-\xi|^{N-2}}\, d\mu(\xi) = c \int_{A_h} G(x,\xi)\, d\mu(\xi) \qquad \forall x \in Q_h^j\,.$$

Integrating the last inequality on the set $\bigcup_{j \in J_h^{A,\delta}} (Q_h^j \cap A)$ we achieve the result.

□

Lemma 2.3.8 *If μ, φ and E_h are as in Theorem 2.3.1, and in addition μ belongs to $H^{-1}(\Omega)$, then $\beta \geq \varphi\mu$.*

Proof. Let A be an open set compactly contained in Ω. Fixed $\delta > 0$ we shall prove the estimate

$$\beta(\overline{A}) \geq (1-c\delta)^2\, (\varphi\mu)(A) - \frac{c}{\delta} \iint_{\overline{A}\times\overline{A}} G(x,y)\, d\mu(x) d\mu(y)\,. \tag{2.3.5}$$

In fact (2.3.5) and Lemma 2.3.6 yield $\beta \geq (1-c\delta)^2\varphi\mu$, and then the thesis follows by taking the limit as $\delta \to 0_+$.

Since φ_h converge in the weak* topology of $L^\infty(\Omega,\mu)$, then there exists $M>0$ such that $\|\varphi_h\|_{L^\infty(\Omega,\mu)} \leq M$ for every $h \in \mathbf{N}$. Thus, since μ belongs to $H^{-1}(\Omega)$, for every $h\in\mathbf{N}$ the measure $\varphi_h\mu$ also belongs to $H^{-1}(\Omega)$. By Lemma 2.3.7 and by the definition of E_h^i, setting $\mu_h = \varphi_h\mu$ and

$$I_h^{A,\delta} = \{i \in \mathbf{Z}^N \setminus J_h^{A,\delta} \ : Q_h^i \cap A \neq \emptyset\},$$

for every open set A compactly contained in Ω and for h large enough to have $A_h = \bigcup_{Q_h^i\cap A\neq\emptyset} Q_h^i \subset\subset \Omega$, we get

$$\mu_h(A) \leq \mu_h\Big(\bigcup_{i\in J_h^{A,\delta}} Q_h^i \cap A\Big) + \sum_{i\in I_h^{A,\delta}} \mu_h(Q_h^i) \leq$$

$$\leq \frac{c}{\delta}\iint_{A_h\times A_h} G(x,y)\,d\mu_h(x)d\mu_h(y) + \sum_{i\in I_h^{A,\delta}} \operatorname{cap}^L(E_h^i, B_h^i).$$

Since $\|\varphi_h\|_{L^\infty(\Omega,\mu)} \leq M$, we obtain

$$\mu_h(A) \leq \frac{c_1}{\delta}\iint_{A_h\times A_h} G(x,y)\,d\mu(x)d\mu(y) + \sum_{i\in I_h^{A,\delta}} \operatorname{cap}^L(E_h^i, B_h^i). \tag{2.3.6}$$

It remains to estimate the quantity $\sum_{i\in I_h^{A,\delta}} \operatorname{cap}^L(E_h^i, B_h^i)$. By Lemma 6.3 of [31] if the L-capacitary potential u of $\bigcup_{i\in I_h^{A,\delta}} E_h^i$ with respect to Ω is not greater that $c\delta$ on each ∂B_h^i, $i \in I_h^{A,\delta}$, then

$$\sum_{i\in I_h^{A,\delta}} \operatorname{cap}^L(E_h^i, B_h^i) \leq (1-c\delta)^{-2}\operatorname{cap}^L(E_h\cap A_h, \Omega). \tag{2.3.7}$$

The inequality (2.3.7) together with (2.3.6) gives the estimate

$$\mu_h(A) \leq \frac{c_1}{\delta}\iint_{A_h\times A_h} G(x,y)\,d\mu(x)d\mu(y) + \frac{1}{(1-c\delta)^2}\operatorname{cap}^L(E_h\cap A_h, \Omega),$$

which yields (2.3.5). It remains to prove that $u \leq c\delta$ on ∂B_h^i, $i \in I_h^{A,\delta}$, for a suitable constant $c = c(N,\lambda)$. Let us fix $\Omega' \supset\supset \Omega$ and extend L by setting

$a_{ij} = \delta_{ij}$ in $\Omega' \setminus \Omega$; if we denote by $G' = G^L_{\Omega'}$ the Green function of the operator L on Ω', then there exists a constant M such that

$$\frac{1}{M}\frac{1}{|x-y|^{N-2}} \le G'(x,y) \le M\frac{1}{|x-y|^{N-2}}, \qquad \text{for all } x,y \in \Omega, \tag{2.3.8}$$

(see [50], Theorem 7.1). If we define for every $i \in N(h)$ such that $Q^i_h \cap A \ne \emptyset$ the function

$$w_i(x) = \frac{\gamma}{\rho^i_h}\int_{E^i_h} G'(x,y)\,d\mathcal{H}^{N-1}(y),$$

then we can choose $\gamma = \gamma(N) > 0$ such that w_i is larger than or equal to 1 on E^i_h and satisfies $w_i(x) \le c'(\rho^i_h)^{N-2}|x^i_h - x|^{2-N}$ for every $x \in \partial B^i_h$. Hence, the function $w = \sum_i w_i$ is a supersolution for the operator L and $w \ge 1$ on $E_h \cap A$. By the properties of the L-capacitary potential, then $u \le w$ (see [48], Section II.6). Moreover by Proposition 2.3.3(v)(iv) we have

$$\mathrm{cap}^L(E^i_h, B^i_h) \ge \theta\,\mathrm{cap}(E^i_h, B^i_h) \ge \theta\,\mathrm{cap}(E^i_h, \mathbf{R}^N)$$

and $\mathrm{cap}(E^i_h, \mathbf{R}^N) = (\rho^i_h)^{N-2}\mathrm{cap}(E, \mathbf{R}^N)$, where E is the (N-1)-dimensional closed sphere centered in 0 and with unit radius (see [35], Section 4.7, Theorem 2). Thus $(\rho^i_h)^{N-2} \le c(\lambda, N)\mathrm{cap}^L(E^i_h, B^i_h) \le c(\lambda, N)\mu(Q^i_h)$, and we have

$$w(x) \le c'\Big(\sum_{\substack{i \in \mathbf{Z}^N \ i \ne j \\ Q^i_h \cap A \ne \emptyset}} \frac{\mu(Q^i_h)}{|x^i_h - x^j_h|^{N-2}} + \mu(Q^j_h)h^{N-2}\Big),$$

and so $u \le w \le c'\delta$ on ∂B^j_h for all $j \notin J^{A,\delta}_h$. □

Proof of Theorem 2.3.1. By Proposition 2.3.5 and Lemma 2.3.8 the result is proved for measures belonging to $H^{-1}(\Omega)$. If μ is a Radon measure of $\mathcal{M}_0(\Omega)$, then by Proposition 2.3.5 we have $\beta \le \varphi\mu$. In order to prove the opposite inequality we need the decomposition result in Theorem 2.2.6: there exist a Borel function g and a measure $\nu \in H^{-1}(\Omega)$ such that $\varphi\mu = g\nu$. Now we can consider for every $k \in \mathrm{N}$ the measure $\mu_k = \min(g,k)\nu$, which belongs to $H^{-1}(\Omega)$. By the first part of the proof we have $\mu_k = \beta_k$, where β_k is defined as in Theorem 2.3.4, with $\mathcal{E}_h = E_{h,k}$, $\{E_{h,k}\}$ being the sequence given by Theorem 2.3.1 for the measure μ_k. Since $\mu_k \leq \varphi\mu$ for every $k \in \mathbf{N}$, then $E_{h,k} \subseteq E_h$ for every $k \in \mathbf{N}$ and so $\mu_k = \beta_k \leq \beta$. Passing to the limit as k tends to infinity and using the fact that $\{\mu_k\}$ increases to $\varphi\mu$, we obtain $\varphi\mu \leq \beta$. □

Remark 2.3.9 The procedure described above is still valid in dimension $N = 2$, and gives the same results: as for the proof, it is enough to consider logarithmic potentials instead of the newtonian ones. For example, the set $J_h^{A,\delta}$ of the "bad" cubes will take the form

$$\Big\{ j \in \mathbf{Z}^N : \sum_{\substack{i \in \mathbf{Z}^N \ i \neq j \\ Q_h^i \cap A \neq \emptyset}} \mu(Q_h^i) \log\Big(\frac{2\,d}{|x_h^i - x_h^j|}\Big) + \mu(Q_h^j)\log(2h\,d) > \delta,\ Q_h^j \cap A \neq \emptyset \Big\},$$

where $d = \mathrm{diam}(\Omega)$. More details can be found in [13].

Chapter 3
An application of the theory of relaxed Dirichlet problems to some shape optimization problems

In this chapter we apply the approximation result proved in Section 2.3 to the study of the minimum problem

$$\min\{J(A), A \text{ open } A \subseteq \Omega\}$$

where

$$J(A) = \begin{cases} \int_A j(x, u_A)\, d\lambda, & \text{if } \lambda(A) \in T, \\ +\infty, & \text{otherwise,} \end{cases}$$

λ is a bounded measure not charging polar sets, and $T = [m, M]$ is a subinterval of $[0, \lambda(\Omega)]$, and u_A is the solution of a Dirichlet problem in A. Notice that the functional J depends on A through the domain of integration, through the solution u_A of the differential equation, and through the constraint $\lambda(A) \in T$. We shall prove that the minimum problem has, in general, no solution, and that the relaxed problem can be written as

$$\min\left\{\overline{J}(\mu) \colon \ \mu \in \mathcal{M}_0(\Omega)\right\} ,$$

where $\overline{J}$ is the lower semicontinuous envelope of J in $\mathcal{M}_0(\Omega)$ with respect to the γ^L-convergence.

The main result of the chapter is an explicit integral representation of $\overline{J}$ in terms of the integrand j and of the constraint T.

3.1. Preliminaries

We say that a nonnegative Borel measure μ is nonatomic if $\mu(\{x\}) = 0$ for every $x \in \Omega$. It is well known that, if μ is nonnegative and nonatomic, then for every $B \in \mathcal{B}(\Omega)$ there exists a Borel subset B_1 of B such that

$0 < \mu(B_1) \leq \frac{1}{2}\mu(B)$, and, by induction, for every $k \in \mathbf{N}$ there exists $B_k \subseteq B$ such that

$$0 < \mu(B_k) \leq \frac{1}{2^k}\mu(B).$$

It is also well known that a nonnegative, nonatomic bounded measure has the following continuity property: for every choice of $A, B \in \mathcal{B}(\Omega)$, with $A \subseteq B$, and for every α in the interval $[\mu(A), \mu(B)]$ there exists a set $C \in \mathcal{B}(\Omega)$ such that $A \subseteq C \subseteq B$ and $\mu(C) = \alpha$ (see, e.g., (43)). In the sequel we shall need the following continuity result, involving only open sets.

Lemma 3.1.1 *Let μ be a nonnegative nonatomic bounded measure in Ω and let A, B be two open subsets of Ω such that $A \subseteq B$. Then for every α such that $\mu(A) \leq \alpha \leq \mu(B)$ there exists an open set C such that $A \subseteq C \subseteq B$ and $\mu(C) = \alpha$.*

Proof. Let $\{C_k\}$ be an increasing sequence of open subsets of B such that $C_0 = A$ and $\alpha \geq \mu(C_{k+1}) \geq s_k - 1/k$, where

$$s_k = \sup\{\mu(U) : \ U \ \text{ open}, \ C_k \subseteq U \subseteq B, \ \mu(U) \leq \alpha\}.$$

If we define $C = \bigcup_k C_k$ and

$$s = \sup\{\mu(U) : \ U \ \text{ open}, \ C \subseteq U \subseteq B, \ \mu(U) \leq \alpha\},$$

then C is open, $A \subseteq C \subseteq B$, and $0 \leq s \leq s_k$ for every k. As $\{C_k\}$ is an increasing sequence, we have

$$\mu(C) = \lim_{k\to\infty} \mu(C_k) \geq \lim_{k\to\infty} s_k \geq s,$$

hence $\mu(C) = s$ and $\mu(U) = s$ for every open set U such that $C \subseteq U \subseteq B$ and $\mu(U) \leq \alpha$.

It remains to prove that $s = \alpha$. By contradiction, if $s < \alpha$, then $\mu(B \setminus C) > 0$. Let us fix $0 < \beta < \alpha - s$. Since μ is nonatomic, there exists a set $E \in \mathcal{B}(\Omega)$ such that $E \subseteq B \setminus C$ and $0 < \mu(E) < \beta$. Moreover, since μ is a Radon measure, there exists an open set V such that $E \subseteq V \subseteq B$ and $\mu(E) \leq \mu(V) \leq \beta$. If we denote $U = C \cup V$, then $C \subseteq U \subseteq B$, $\mu(U) \leq s$, and

$$\mu(U) \geq \mu(C \cup E) = \mu(C) + \mu(E) > \mu(C) = s,$$

which gives a contradiction. Thus $\mu(C) = \alpha$, and this concludes the proof. □

In the sequel we shall deal with measures belonging to $H^{-1}(\Omega)$. We underline that, having identified each function u in $H_0^1(\Omega)$ with its quasi continuous representative, for every nonnegative Radon measure $\nu \in H^{-1}(\Omega)$ we have $H_0^1(\Omega) \subseteq L^1_\nu(\Omega)$, and $\langle f, u\rangle = \int_\Omega u\,d\nu$ for every $u \in H_0^1(\Omega)$. Moreover the injection of $H_0^1(\Omega)$ into $L^1_\nu(\Omega)$ is compact. Indeed, if $\{v_h\}$ is a sequence of functions which converges to a function v weakly in $H_0^1(\Omega)$, then $|v_h - v|$ converges to 0 weakly in $H_0^1(\Omega)$, so that $\int_\Omega |v_h - v|\,d\nu = \langle \nu, |v_h - v|\rangle$ tends to 0.

A subset A of Ω is said to be quasi open if for every $\varepsilon > 0$ there exists an open subset U_ε of Ω, with $\mathrm{cap}(U_\varepsilon, \Omega) < \varepsilon$, such that $A \cup U_\varepsilon$ is open.

We denote by $\tilde{\mathcal{M}}_0(\Omega)$ the set of all nonnegative Borel measures μ on Ω such that

(i) $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ with $\mathrm{cap}(B, \Omega) = 0$,

(ii) $\mu(B) = \inf\{\mu(A) : A \text{ quasi open}\,,\ B \subseteq A\}$ for every Borel set $B \subseteq \Omega$.

Since all quasi open sets differs from a Borel set by a set of capacity zero, all quasi open sets are μ–measurable for every nonnegative Borel measure μ which satisfies (i). Therefore $\mu(A)$ is well defined when A is quasi open, and condition (ii) makes sense.

Remark 3.1.2 If we say that two measures μ_1 and μ_2 of $\mathcal{M}_0(\Omega)$ are equivalent if $\int_\Omega u^2\,d\mu_1 = \int_\Omega u^2\,d\mu_2$ for every $u \in H_0^1(\Omega)$, then for every $\mu \in \mathcal{M}_0(\Omega)$ there exists a unique measure $\tilde{\mu} \in \tilde{\mathcal{M}}_0(\Omega)$ equivalent to μ. Moreover if two measures μ_1, μ_2 of $\mathcal{M}_0(\Omega)$ are equivalent, then the a function u is the solution of a relaxed Dirichlet problem corresponding to μ_1 if and only if it is the solution of the relaxed Dirichlet problem corresponding to μ_2. Finally $\tilde{\mathcal{M}}_0(\Omega)$ coincides with the class $\mathcal{M}_0^*(\Omega)$ introduced in [28].

It is easy to see that if μ belongs to $\tilde{\mathcal{M}}_0(\Omega)$ and E is a closed subset of Ω, then the measures $\mu \llcorner E$ and ∞_E belong to $\tilde{\mathcal{M}}_0(\Omega)$. This is not true, in general, when E is not closed.

For every quasi open set $A \subseteq \Omega$ we denote by μ_A the measure defined by

$$\mu_A(B) = \begin{cases} 0, & \text{if } \operatorname{cap}(B \setminus A, \Omega) = 0, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.1.1}$$

Notice that μ_A belongs to $\tilde{\mathcal{M}}_0(\Omega)$. Indeed condition (i) is clearly satisfied and (ii) is trivial whenever $\mu_A(B) = +\infty$. Moreover, if $\mu_A(B) = 0$, then $\operatorname{cap}(B \setminus A, \Omega) = 0$, so that $B \cup A$ is a quasi open set containing B with $\mu_A(A \cup B) = \mu_A(B)$. This implies (ii).

In the sequel we shall use the following result.

Proposition 3.1.3 *For every measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ there exist a nonnegative Borel measurable function h and a nonnegative measure $\nu \in H^{-1}(\Omega)$ such that $\mu(A) = (h\nu)(A)$ for every quasi open subset A of Ω.*

Proof. It is enough to apply Theorem 1.2.4 with $f = \mathbf{1}_A$. □

Let w_μ be the function defined in (1.3.6). By A_μ we shall denote the set $\{x \in \Omega : w_\mu(x) > 0\}$. Notice that A_μ is defined only up to a set of capacity zero, hence all the equalities or inclusions involving A_μ are intended up to sets of capacity zero. Since w_μ is quasi continuous, A_μ is quasi open.

Lemma 3.1.4 *Let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ and let w_μ be the solution of problem (1.3.6). Then $\mu(B) = +\infty$ for every Borel subset B of Ω with $\operatorname{cap}(B \setminus A_\mu, \Omega) > 0$.*

Proof. See (30), Lemma 3.2. □

Remark 3.1.5 It is easy to see that, if μ is a Radon measure of $\tilde{\mathcal{M}}_0(\Omega)$, then $A_\mu = \Omega$. If $\mu = \mu_A$ and A is open, then $A_\mu = A$ by Remark 1.3.5 and by the strong maximum principle.

Theorem 3.1.6 *Let $\{\mu_h\}$ be a sequence of measures of $\tilde{\mathcal{M}}_0(\Omega)$ and let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Then $\{\mu_h\}$ γ^L-converges to μ (in Ω) if and only if the sequence $\{w_{\mu_h}\}$ converges to w_μ weakly in $H_0^1(\Omega)$.*

Proof. See [30], Theorem 4.5. □

As a consequence of Theorem 2.3.1 we obtain the following density result.

Proposition 3.1.7 *Let us fix a nonnegative nonatomic Radon measure* λ. *Then for every* $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ *there exists a sequence* $\{E_h\}$ *of compact subsets of* Ω *such that* $\{\mu_{\Omega \setminus E_h}\}$ γ^L*-converges to* μ, *and* $\lambda(E_h) = 0$ *for every* $h \in \mathbb{N}$.

Proof. Since by Proposition 3.7 of [30] every measure of $\tilde{\mathcal{M}}_0(\Omega)$ can be approximated in γ^L-convergence by a sequence of Radon measures of $\tilde{\mathcal{M}}_0(\Omega)$, here it is not restrictive to suppose that μ is a Radon measure. Thus we can apply Theorem 2.3.1 of Chapter 2, obtaining the result. □

Finally, let us consider a real valued functional J defined on the class of all open subsets of Ω. With every open subset A of Ω we can associate the measure μ_A. Thus the functional J can be considered as a functional defined on the subclass $\{\mu_A : A \text{ open}, \ A \subseteq \Omega\}$ of $\tilde{\mathcal{M}}_0(\Omega)$.

Definition 3.1.8 *We shall call relaxation of* J *in* $\tilde{\mathcal{M}}_0(\Omega)$ *with respect to the* γ^L*-convergence, and we shall denote it by* $\overline{J}$, *the greatest* γ^L*-lower semi-continuous functional defined on* $\tilde{\mathcal{M}}_0(\Omega)$ *such that* $\overline{J}(\mu_A) \leq J(A)$ *for every open set* $A \subseteq \Omega$.

Remark 3.1.9 One can check that

$$\overline{J}(\mu) = \inf\{\liminf_{h\to\infty} J(A_h) \ : A_h \text{ open}, \{\mu_{A_h}\} \ \gamma^L\text{-converging to } \mu\}, \tag{3.1.2}$$

for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. The previous formula characterizes the relaxation $\overline{J}$ as the unique functional which satisfies the following properties for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$:

(i) for every sequence $\{A_h\}$ of open sets with $\{\mu_{A_h}\}$ γ^L-converging to μ in $\tilde{\mathcal{M}}_0(\Omega)$

$$\overline{J}(\mu) \leq \liminf_{h\to\infty} J(A_h);$$

(ii) there exists a sequence $\{A_h\}$ of open sets such that $\{\mu_{A_h}\}$ γ^L-converges to μ in $\tilde{\mathcal{M}}_0(\Omega)$ and

$$\overline{J}(\mu) \geq \limsup_{h\to\infty} J(A_h).$$

The relaxation $\overline{J}$ describes the behaviour of the minimizing sequences of J. More precisely $\overline{J}$ is γ^L-lower semicontinuous and so, by the direct method of calculus of variations, $\overline{J}$ has a minimum point on the γ^L-compact set $\tilde{\mathcal{M}}_0(\Omega)$. Moreover

$$\min_{\tilde{\mathcal{M}}_0(\Omega)} \overline{J}(\mu) = \inf_{\substack{A \text{ open} \\ A \subseteq \Omega}} J(A),$$

every cluster point of a minimizing sequence for J is a minimum point for $\overline{J}$ in $\tilde{\mathcal{M}}_0(\Omega)$, and every minimum point for $\overline{J}$ in $\tilde{\mathcal{M}}_0(\Omega)$ is the limit of a minimizing sequence for J (in the last statements we identify every open set $A \subseteq \Omega$ with the corresponding measure μ_A).

For a more general treatment of this subject see, e.g., [17] and [29].

3.2. Lower Semicontinuity

In this section we study the lower semicontinuity, with respect to the γ^L-convergence, of some functionals defined on $\tilde{\mathcal{M}}_0(\Omega)$. More precisely, fixed a bounded measure λ in $\tilde{\mathcal{M}}_0(\Omega)$, a function g in $L^1(\Omega, \lambda)$, and a closed (possibly degenerating to a point) subinterval $T = [m, M]$ of $[0, \lambda(\Omega)]$, we consider the functional

$$G_T(\mu) = \inf\left\{ \int_B g\, d\lambda \ : B \in \mathcal{B}(\Omega),\, A_\mu \subseteq B,\, \lambda(B) \in T \right\}, \tag{3.2.1}$$

where A_μ is the quasi open set introduced in Section 3.1. We shall always use the convention $\inf \emptyset = +\infty$.

Theorem 3.2.1 *The functional G_T defined in (3.2.1) is lower semicontinuous in $\tilde{\mathcal{M}}_0(\Omega)$ with respect to the γ^L-convergence.*

Remark 3.2.2 Let us fix a constant c such that $0 \leq c \leq \lambda(\Omega)$. If the set T takes the form $T = \{c\}$, $T = [0, c]$, or $T = [c, \lambda(\Omega)]$, then the functional G_T becomes respectively

$$G_c(\mu) = \inf\left\{ \int_B g\, d\lambda \ : \ B \in \mathcal{B}(\Omega)\,,\ A_\mu \subseteq B\,,\ \lambda(B) = c \right\},$$

$$G_{[0,c]}(\mu) = \inf\left\{\int_B g\,d\lambda \,:\, B\in\mathcal{B}(\Omega)\,,\ A_\mu\subseteq B\,,\ \lambda(B)\le c\right\},$$

$$G_{[c,\lambda(\Omega)]}(\mu) = \inf\left\{\int_B g\,d\lambda \,:\, B\in\mathcal{B}(\Omega)\,,\ A_\mu\subseteq B\,,\ \lambda(B)\ge c\right\}.$$

Notice that, in general, G_c, $G_{[0,c]}$, and $G_{[c,\lambda(\Omega)]}$ are different as it can be easily seen by choosing $g\equiv 1$ in each functional. Indeed in this case we have

$$G_c(\mu) = \begin{cases} c, & \text{if } \lambda(A_\mu)\le c,\\ +\infty, & \text{otherwise,}\end{cases}$$

$$G_{[0,c]}(\mu) = \begin{cases} \lambda(A_\mu), & \text{if } \lambda(A_\mu)\le c,\\ +\infty, & \text{otherwise,}\end{cases}$$

$$G_{[c,\lambda(\Omega)]}(\mu) = \sup\{c,\lambda(A_\mu)\}.$$

Moreover, for every $g\in L^1(\Omega,\lambda)$ the functional $G_{[0,c]}$ can be rewritten as

$$G_{[0,c]}(\mu) = \int_{A_\mu} g^+\,d\lambda - \sup\int_B g^-\,d\lambda\,,$$

where g^+ and g^- are the positive and the negative part of g respectively, and the supremum is taken over all the Borel subset B of Ω such that $A_\mu\subseteq B$, and $\lambda(B)=c$. When $c=\lambda(\Omega)$, it reduces to the functional

$$G_{[0,\lambda(\Omega)]}(\mu) = \int_{A_\mu} g^+\,d\lambda - \int_\Omega g^-\,d\lambda = \int_{A_\mu} g\,d\lambda - \int_{\Omega\setminus A_\mu} g^-\,d\lambda\,,$$

which then turns out to be γ^L-lower semicontinuous on $\tilde{\mathcal{M}}_0(\Omega)$.

In order to prove Theorem 3.2.1 we need a result of measure theory.

Lemma 3.2.3 *Let λ be a nonnegative, nonatomic bounded Borel measure on Ω, and let $g\in L^1(\Omega,\lambda)$. Fixed a closed subinterval $T=[m,M]$ of $[0,\lambda(\Omega)]$, let us consider the functional*

$$\mathcal{G}(A) = \inf\left\{\int_B g\,d\lambda \,: B\in\mathcal{B}(\Omega),\, A\subseteq B,\, \lambda(B)\in T\right\} \tag{3.2.2}$$

defined on the class of all Borel subsets A *of* Ω. *Then, for every* A_1, A_2 *such that* $\lambda(A_1) \leq M$ *and* $\lambda(A_2) \leq M$, *we have*

$$\mathcal{G}(A_1) \leq \mathcal{G}(A_2) + 2\omega(\lambda(A_1 \setminus A_2)), \tag{3.2.3}$$

where $\omega(\delta) = \sup\{\int_B |g|\, d\lambda \,:\, \lambda(B) \leq \delta\}$, *and hence* $\omega(\delta) \to 0$ *as* $\delta \to 0$.

Proof. Let B_2 be any Borel set such that $A_2 \subseteq B_2$ and $\lambda(B_2) \in T$. Since λ is nonatomic and $\lambda(B_2 \setminus A_1) = \lambda(B_2) - \lambda(A_1) + \lambda(A_1 \setminus B_2)$, if $\lambda(B_2) - \lambda(A_1) \geq 0$ one can find a Borel set $E \subseteq B_2 \setminus A_1$ such that $\lambda(E) = \lambda(A_1 \setminus B_2)$. Thus, setting $F = B_2 \setminus (A_1 \cup E) = (B_2 \setminus A_1) \setminus E$ and $B_1 = A_1 \cup F = (A_1 \cup B_2) \setminus E$, we have $A_1 \subseteq B_1$, and

$$\lambda(B_1) = \lambda(A_1) + \lambda(F) = \lambda(A_1) + \lambda(B_2 \setminus A_1) - \lambda(A_1 \setminus B_2) = \lambda(B_2),$$

and hence $\lambda(B_1)$ belongs to T. Moreover, as $A_1 \cup B_2 = B_1 \cup E$ and $B_1 \cap E = \emptyset$, we have

$$\int_{B_1} g\, d\lambda = \int_{B_2} g\, d\lambda + \int_{A_1 \setminus B_2} g\, d\lambda - \int_E g\, d\lambda.$$

Since $\lambda(A_1 \setminus B_2) = \lambda(E) \leq \lambda(A_1 \setminus A_2)$ we get

$$\mathcal{G}(A_1) \leq \int_{B_1} g\, d\lambda \leq \int_{B_2} g\, d\lambda + 2\omega(\lambda(A_1 \setminus A_2)).$$

If $\lambda(B_2) - \lambda(A_1) \leq 0$, then $\lambda(B_2) \leq \lambda(A_1) \leq M$, so that $\lambda(A_1) \in T$ and

$$\mathcal{G}(A_1) \leq \int_{A_1} g\, d\lambda.$$

Since in this case $\lambda(B_2 \setminus A_1) \leq \lambda(A_1 \setminus B_2)$, we have

$$\mathcal{G}(A_1) \leq \int_{B_2} g\, d\lambda + \int_{A_1 \setminus B_2} g\, d\lambda - \int_{B_2 \setminus A_1} g\, d\lambda \leq \int_{B_2} g\, d\lambda + 2\omega(\lambda(A_1 \setminus A_2)).$$

Therefore (3.2.3) follows by taking the infimum over all admissible B_2. The fact that $\omega(\delta) \to 0$ as $\delta \to 0$ is a consequence of the absolute continuity of the integral. □

We consider now a first lower semicontinuity result for functionals defined on $\tilde{\mathcal{M}}_0(\Omega)$.

Lemma 3.2.4 *Let $\lambda \in \tilde{\mathcal{M}}_0(\Omega)$ and let $\{\mu_h\}$ be a sequence in $\tilde{\mathcal{M}}_0(\Omega)$ γ^L-converging to $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Then*

$$\lambda(A_\mu) \leq \liminf_{h\to\infty} \lambda(A_{\mu_h}).$$

If, in addition, $\lambda(A_\mu) < +\infty$, then $\lim_{h\to\infty} \lambda(A_\mu \setminus A_{\mu_h}) = 0$.

Proof. By Proposition 3.1.3 there exist a nonnegative measure $\nu \in H^{-1}(\Omega)$ and a nonnegative Borel function h such that $\lambda = h\nu$ on the class of all quasi open subsets of Ω. In particular $\lambda(A_\mu) = \int_{A_\mu} h\,d\nu$ for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Let $f : \Omega \times \mathbf{R} \to [0,+\infty]$ be the Borel function defined by

$$f(x,s) = \begin{cases} h(x), & \text{if } s > 0, \\ 0, & \text{if } s \leq 0. \end{cases}$$

Then $f(x,\cdot)$ is lower semicontinuous, and by definition of A_μ we have

$$\lambda(A_\mu) = \int_\Omega f(x, w_\mu)\,d\nu \qquad \forall \mu \in \tilde{\mathcal{M}}_0(\Omega),$$

where w_μ is the solution of problem (1.3.6). Let now $\{\mu_h\}$ be a sequence in $\tilde{\mathcal{M}}_0(\Omega)$ which γ^L-converges to a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$, i.e., the sequence $\{w_{\mu_h}\}$ converges to w_μ weakly in $H_0^1(\Omega)$ (see Theorem 3.1.6). Since $\nu \in H^{-1}(\Omega)$, $\{w_{\mu_h}\}$ converges to w_μ in the strong topology of $L^1_\nu(\Omega)$. Thus, possibly passing to a subsequence, $\{w_{\mu_h}\}$ converge to w_μ ν-a.e. in Ω. Therefore, by Fatou's lemma and the lower semicontinuity of $f(x,\cdot)$, we obtain

$$\lambda(A_\mu) = \int_\Omega f(x,w_\mu)\,d\nu \leq \liminf_{h\to\infty} \int_\Omega f(x, w_{\mu_h})\,d\nu = \liminf_{h\to\infty} \lambda(A_{\mu_h}),$$

and the proof of the first statement is complete. Let us suppose now that $\lambda(A_\mu) < +\infty$. As we have shown before, the γ^L-convergence of $\{\mu_h\}$ to μ implies the strong convergence in $L^1(\Omega,\nu)$ of $\{w_{\mu_h}\}$ to w_μ. Hence, using Fatou's lemma again, we get

$$\limsup_{h\to\infty} \lambda(A_\mu \setminus A_{\mu_h}) = \limsup_{h\to\infty} [\lambda(A_\mu) - \lambda(A_{\mu_h} \cap A_\mu)] =$$

$$= \int_{A_\mu} f(x,w_\mu)\,d\nu - \liminf_{h\to\infty} \int_{A_\mu} f(x,w_{\mu_h})\,d\nu \leq 0,$$

which concludes the proof. □

Remark 3.2.5 If the measure λ does not belong to $\tilde{\mathcal{M}}_0(\Omega)$, the conclusion of Lemma 3.2.4 may be false. In fact, take $N \geq 2$, $x_0 \in \Omega$ and let λ be the Dirac measure δ_{x_0}. It is easy to see that the measures $\mu_h = \mu_{\Omega \setminus B(x_0,1/h)}$ γ^L-converge to the measure μ which is identically zero on Ω. On the other hand we have $A_{\mu_h} = \Omega \setminus B(x_0, 1/h)$ and $A_\mu = \Omega$, so that $\lambda(A_\mu) = 1 > 0 = \liminf_{h\to\infty} \lambda(A_{\mu_h})$.

Proof of Theorem 3.2.1. Let μ be a fixed measure in $\tilde{\mathcal{M}}_0(\Omega)$, and let $\{\mu_h\}$ be a sequence in $\tilde{\mathcal{M}}_0(\Omega)$ which γ^L-converges to μ. It is not restrictive to suppose that $G_T(\mu_h) < +\infty$ for every $h \in \mathbf{N}$. This implies that $\lambda(A_{\mu_h}) \leq M$ for every $h \in \mathbf{N}$, and then, by Lemma 3.2.4, we have also $\lambda(A_\mu) \leq M$. Since for every quasi open set A there exists a Borel set B containing A, with $\operatorname{cap}(B \setminus A, \Omega) = 0$, we can apply Lemma 3.2.3 with $A_1 = A_\mu$, and $A_2 = A_{\mu_h}$, and we get

$$G_T(\mu) \leq G_T(\mu_h) + 2\omega(\lambda(A_\mu \setminus A_{\mu_h}))\,.$$

The γ^L-lower semicontinuity of G_T follows now from the second part of Lemma 3.2.4. □

3.3. Relaxation

In this section we apply the previous lower semicontinuity results in order to obtain an explicit representation of the relaxation of some cost functionals in optimal shape design.

Let us fix a functional $f \in H^{-1}(\Omega)$, a bounded measure λ in $\tilde{\mathcal{M}}_0(\Omega)$, a closed interval $T = [m, M]$ of $[0, \lambda(\Omega)]$ with $M > 0$, and a Borel function $j\colon \Omega \times \mathbf{R} \to \mathbf{R}$ satisfying the following conditions:

(i) for every $u \in H_0^1(\Omega)$ the function $j(x, u(x))$ belongs to $L^1(\Omega, \lambda)$;

(ii) the map $u \mapsto \int_\Omega j(x,u)\,d\lambda$ is sequentially continuous in the weak topology of $H_0^1(\Omega)$.

We shall consider the functional J defined on the class of all open subsets of Ω as

$$J(A) = \begin{cases} \int_A j(x, u_A)\,d\lambda, & \text{if } \lambda(A) \in T, \\ +\infty, & \text{otherwise,} \end{cases} \tag{3.3.1}$$

where u_A is the unique solution of the Dirichlet problem

$$(3.3.2)\qquad \begin{cases} Lu_A = f & \text{on } A, \\ u_A = 0 & \text{on } \partial A. \end{cases}$$

Different hypotheses on λ and j can be made in order to fulfill (i) and (ii). For instance, if we assume that $j(x,s)$ is measurable in x and continuous in s, we can require one of the following properties:

(1) λ is the Lebesgue measure and $|j(x,s)| \le c(1+|s|^p)$ for λ-a.e. x in Ω and for every $p < 2N/(N-2)$. Namely, in this case, by Sobolev imbedding (i) and (ii) are obviously fulfilled.

(2) λ is the $(N-1)$-dimensional Hausdorff measure restricted on a smooth $(N-1)$-dimensional hypersurface $S \subseteq \Omega$ and $|j(x,s)| \le c(1+|s|^2)$ for λ-a.e. x in S. In this case (i) and (ii) follow from the compactness of the trace operator between $H_0^1(\Omega)$ and $L^2(S)$.

(3) λ belongs to $H^{-1}(\Omega)$ and j has linear growth in s. In this case, (i) and (ii) follow from the compactness of the injection of $H_0^1(\Omega)$ into $L_\lambda^1(\Omega)$.

It is well known that, under these very weak assumptions, the optimal design problem

$$\min_{\lambda(A)\in T} J(A)$$

in general has no solution (see Examples 3.3.11 and 3.3.12). Thus, in order to investigate the asymptotic behaviour of the minimizing sequences of J, we are interested in its relaxation. To this aim, for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ we denote by u_μ the unique solution in the sense of (1.3.5) of the problem

$$(3.3.3)\qquad Lu_\mu + \mu u_\mu = f \quad \text{in } \Omega, \qquad u_\mu \in H_0^1(\Omega) \cap L_\mu^2(\Omega).$$

The main result of this section is the following.

Theorem 3.3.1 *Let $f \in H^{-1}(\Omega)$, let λ be a bounded measure in $\tilde{\mathcal{M}}_0(\Omega)$, and let $j:\Omega\times\mathbf{R} \to \mathbf{R}$ be a Borel function satisfying (i) and (ii). Fixed a closed subinterval $T=[m,M]$ of $[0,\lambda(\Omega)]$ with $M>0$, we consider the functional*

J defined by (3.3.1) for every open set $A \subseteq \Omega$, where u_A is the solution of problem (3.3.2). Then its relaxation $\overline{J}$ in $\tilde{\mathcal{M}}_0(\Omega)$ is given by

$$\begin{aligned}\overline{J}(\mu) = \int_{A_\mu} j(x,u_\mu)\,d\lambda + \\ + \inf\left\{\int_{B\setminus A_\mu} j(x,0)\,d\lambda : B\in\mathcal{B}(\Omega),\ A_\mu \subseteq B,\ \lambda(B)\in T\right\},\end{aligned} \tag{3.3.4}$$

with the convention $\inf\emptyset = +\infty$.

Example 3.3.2 If there exists a constant k such that $j(x,0) = k$ for every λ-a.e. $x \in \Omega$, then (3.3.4) can be simplified. Namely, if k is positive, then (3.3.4) becomes

$$\overline{J}(\mu) = \begin{cases} \int_{A_\mu} j(x,u_\mu)\,d\lambda + k(m-\lambda(A_\mu))^+, & \text{if } \lambda(A_\mu) \le M, \\ +\infty, & \text{otherwise,} \end{cases}$$

while, if k is negative, we get

$$\overline{J}(\mu) = \begin{cases} \int_{A_\mu} j(x,u_\mu)\,d\lambda + k(M-\lambda(A_\mu)), & \text{if } \lambda(A_\mu) \le M, \\ +\infty, & \text{otherwise.} \end{cases}$$

In particular, if $T = \{c\}$, then (3.3.4) takes the form

$$\overline{J}(\mu) = \begin{cases} \int_\Omega j(x,u_\mu)\,d\lambda + k(c-\lambda(\Omega)), & \text{if } \lambda(A_\mu) \le c, \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark 3.3.3 The functional J can be written as

$$J(A) = \int_\Omega j(x,u_A)\,d\lambda - \int_\Omega j(x,0)\,d\lambda + \int_A j(x,0)\,d\lambda\,,$$

where u_A is extended to 0 on $\Omega\setminus A$. For every fixed $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ and for every sequence $\{A_h\}$ of open subsets of Ω such that $\{\mu_{A_h}\}$ γ^L-converges to μ, we have that the sequence $\{u_{A_h}\}$ of the solutions of the Dirichlet problems on

A_h converges to the solution u_μ of the relaxed Dirichlet problem (3.3.3) in the weak topology of $H_0^1(\Omega)$ (see Remarks 1.3.5 and 1.3.10), so that, by (ii),

$$\lim_{h\to\infty}\int_\Omega j(x,u_{A_h})\,d\lambda-\int_\Omega j(x,0)\,d\lambda=\int_\Omega j(x,u_\mu)\,d\lambda-\int_\Omega j(x,0)\,d\lambda\,.$$

Hence, by (3.1.2), in order to exhibit the relaxation $\overline{J}$ of J, it is enough to relax the functional

$$J_0(A)=\begin{cases}\int_A j(x,0)\,d\lambda, & \text{if } \lambda(A)\in T,\\ +\infty, & \text{otherwise.}\end{cases} \tag{3.3.5}$$

Since the solution u_μ of (3.3.3) is zero q.e. in $\Omega\backslash A_\mu$ (Lemma 3.1.4), to conclude the proof it is enough to show that the relaxation $\overline{J}_0$ of J_0 coincides with the functional

$$\tilde{J}_0(\mu)=\inf\left\{\int_B j(x,0)\,d\lambda : B\in\mathcal{B}(\Omega),\ A_\mu\subseteq B\,,\ \lambda(B)\in T\right\}. \tag{3.3.6}$$

By Theorem 3.2.1, $\tilde{J}_0$ is γ^L-lower semicontinuous. Since $\tilde{J}_0(\mu_A)\le J_0(A)$ for every open set $A\subseteq\Omega$, by definition of $\overline{J}_0$, we have $\tilde{J}_0\le\overline{J}_0$.

Suppose that, for every $\mu\in\tilde{\mathcal{M}}_0(\Omega)$ with $\tilde{J}_0(\mu)<+\infty$ and for every Borel set B containing A_μ with $\lambda(B)\in T$, we are able to find a sequence $\{A_h\}$ of open subsets of Ω such that $\lambda(A_h)\in T$, $\{\mu_{A_h}\}$ γ^L-converges to μ, and

$$\lim_{h\to\infty}\int_{A_h} j(x,0)\,d\lambda=\int_B j(x,0)\,d\lambda\,.$$

Then by (3.1.2) we get

$$\overline{J}_0(\mu)\le\lim_{h\to\infty}J_0(A_h)=\int_B j(x,0)\,d\lambda\,.$$

Taking the infimum over all admissible B we obtain $\overline{J}_0(\mu)\le\tilde{J}_0(\mu)$, and hence $\overline{J}_0(\mu)=\tilde{J}_0(\mu)$.

In order to construct the sequence $\{A_h\}$ we shall require that the set B is open. As shown in the following lemma, this condition is not restrictive for a large class of measures in $\tilde{\mathcal{M}}_0(\Omega)$.

Lemma 3.3.4 *Let g, λ, and G be as in Theorem 3.2.1 and let $T = [m, M] \subseteq [0, \lambda(\Omega)]$. Then*

$$G(\mu) = \inf\left\{\int_B g\, d\lambda \ : B \text{ open }, A_\mu \subseteq B, \lambda(B) \in T\right\} \tag{3.3.7}$$

for every measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ such that $\lambda(A_\mu) < M$.

Proof. Let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ with $\lambda(A_\mu) < M$. Let us denote by $H(\mu)$ the right–hand side of (3.3.7). It is enough to prove that $G(\mu) \geq H(\mu)$, since the opposite inequality is trivial. Since $\lambda(A_\mu) < M$, λ is nonatomic, and T is an interval, $G(\mu)$ coincides with the infimum taken over all Borel sets B such that $\lambda(B) \in T$, $A_\mu \subseteq B$, and $\lambda(B) > \lambda(A_\mu)$.

Given one of these sets B, we shall exhibit a sequence $\{A_h\}$ of open subsets of Ω, with $\lambda(A_h) = \lambda(B)$ and $A_\mu \subseteq A_h$, such that

$$\lim_{h\to\infty} \int_{A_h} g\, d\lambda = \int_B g\, d\lambda. \tag{3.3.8}$$

Since $H(\mu) \leq \int_{A_h} g\, d\lambda$, taking the limit as $h \to \infty$ we obtain

$$H(\mu) \leq \int_B g\, d\lambda, \tag{3.3.9}$$

and taking the infimum with respect to B we get $H(\mu) \leq G(\mu)$. It remains to construct the sequence A_h. Since λ is bounded, we can find a sequence $\{U_h\}$ of open subsets of Ω such that $A_\mu \subseteq U_h$ for every $h \in \mathbf{N}$, and $\lambda(U_h \setminus A_\mu) \leq 1/h$. Moreover for every $h \in \mathbf{N}$ there exists an open set $B_h \supseteq B$ such that $\lambda(B_h \setminus B) \leq 1/h$. It is not restrictive to suppose $U_h \subseteq B_h$, since one can always replace B_h with $B_h \cup U_h$, and

$$\lambda((B_h \cup U_h)\setminus B) \leq \lambda(B_h\setminus B) + \lambda(U_h\setminus B) \leq \lambda(B_h\setminus B) + \lambda(U_h\setminus A_\mu) \leq 2/h.$$

Thus, for h large enough, $\lambda(U_h) \leq \lambda(A_\mu) + 1/h < \lambda(B) \leq \lambda(B_h)$, so that, by Lemma 3.1.1 we can find an open set A_h such that $U_h \subseteq A_h \subseteq B_h$ and $\lambda(A_h) = \lambda(B)$. Moreover we have that $A_\mu \subseteq U_h \subseteq A_h$ and

$$\int_{A_h} g\, d\lambda = \int_B g\, d\lambda - \int_{B\setminus A_h} g\, d\lambda + \int_{A_h\setminus B} g\, d\lambda.$$

Since $\lambda(A_h) = \lambda(B)$, we have $\lambda(B \setminus A_h) = \lambda(A_h \setminus B) \leq \lambda(B_h \setminus B) \leq 1/h$. Then, as $g \in L^1_\lambda(\Omega)$, (3.3.8) follows from the absolute continuity of the integral. □

Lemma 3.3.5 *Let λ and T be as in Theorem 3.3.1, and let $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. Then for every open set B containing A_μ, with $\lambda(B) \in T$, there exists a sequence $\{A_h\}$ of open subsets of B, with $\lambda(B \setminus A_h) = 0$, such that the sequence $\{\mu_{A_h}\}$ γ^L-converges to μ.*

Proof. Applying Proposition 3.1.7 with Ω replaced by B, it is possible to find a sequence $\{A_h\}$ of open subsets of B with $\lambda(B \setminus A_h) = 0$, such that the sequence $\{\mu_{A_h}\}$ γ^L-converges to μ in B, i.e., replacing Ω by B in Theorem 3.1.6, the sequence $\{w_h\}$ of the solutions to the problems

$$(3.3.10) \qquad Lw_h = 1 \quad \text{in } A_h\,, \qquad w_h = 0 \quad \text{on } \partial A_h\,,$$

extended to zero outside A_h weakly converges in $H_0^1(B)$ to the solution w in the sense of (1.3.5) of the problem

$$(3.3.11) \qquad Lw + \mu w = 1 \quad \text{in } B\,, \qquad w \in H_0^1(B) \cap L_\mu^2(B)$$

extended to zero outside B. Since $A_\mu \subseteq B$, the solution w_μ of (1.3.6) in Ω equals zero q.e. on $\Omega \setminus B$, so that $w_\mu \in H_0^1(B) \cap L_\mu^2(B)$ and satisfies (3.3.11). As $A_h \subseteq B$, this shows that $\{\mu_{A_h}\}$ γ^L-converges to μ in Ω. □

We are now in a position to prove (3.3.4) for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ such that $\lambda(A_\mu) < M$.

Proposition 3.3.6 *Let j, λ, and T be as in Theorem 3.3.1 and let $\tilde{J}_0$ be the functional defined in (3.3.6). Then $\overline{J}_0(\mu) = \tilde{J}_0(\mu)$ for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ with $\lambda(A_\mu) < M$.*

Proof. Let us fix $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ such that $\lambda(A_\mu) < M$. As pointed out in Remark 3.3.3, it is enough to show that $\overline{J}_0(\mu) \le \tilde{J}_0(\mu)$. Let us consider an open set B, containing A_μ, with $\lambda(B) \in T$. By Lemma 3.3.5 there exists a sequence $\{A_h\}$ of open subsets of B, with $\lambda(A_h) = \lambda(B)$, such that $\{\mu_{A_h}\}$ γ^L-converges to μ. Thus, as in Remark 3.3.3, we obtain $\overline{J}_0(\mu) \le \int_B j(x,0)\,d\lambda$. Taking the infimum over all admissible open sets B, by Lemma 3.3.4 applied to $g(x) = j(x,0)$, we obtain $\overline{J}_0(\mu) \le \tilde{J}_0(\mu)$. □

In order to extend the equality $\overline{J}_0(\mu) = \tilde{J}_0(\mu)$ to every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$, we need the following lemma.

Lemma 3.3.7 *Let $\{A_h\}$ and $\{\tilde{A}_h\}$ be two sequences of quasi open subsets of Ω such that $A_h \subseteq \tilde{A}_h$ for every h and $\mathrm{cap}(\tilde{A}_h \setminus A_h, \Omega) \to 0$. Let w_h and $\tilde{w}_h$ be the solutions of problem (1.3.6) with $\mu = \mu_{A_h}$ and $\mu = \mu_{\tilde{A}_h}$. If $\{w_h\}$ and $\{\tilde{w}_h\}$ converge weakly to w and $\tilde{w}$, then $w = \tilde{w}$ q.e. in Ω.*

Proof. Since $A_h \subseteq \tilde{A}_h$, by Lemma 1.3.8 we have $w_h \leq \tilde{w}_h$ q.e. in Ω for every $h \in \mathbf{N}$, and then $w \leq \tilde{w}$ q.e. in Ω. Let us prove the opposite inequality. To this aim, following the ideas of Stampacchia (see [64]), for every subset E of Ω we denote with K_E the set of all functions $v \in H_0^1(\Omega)$ such that $v \geq 1$ q.e. in E and, if K_E is nonempty, we consider the unique solution z of the variational inequality

$$\begin{cases} z \in K_E\,, \\ \langle Lz, v - z\rangle \geq 0 \qquad \forall v \in K_E. \end{cases}$$

The function z is called the L-capacitary potential of E in Ω and the L-capacity of E in Ω is defined by $\mathrm{cap}^L(E,\Omega) = \langle Lz, z\rangle$. We set $\mathrm{cap}^L(E,\Omega) = +\infty$ if $K_E = \emptyset$. It is easy to see that

$$\langle Lz, \psi\rangle = 0 \qquad \forall \psi \in H_0^1(\Omega)\,, \text{ with } \psi = 0 \text{ q.e. in } E\,.$$

Moreover, by the maximum principle, one can check that $z = 1$ q.e. in E. Finally $\mathrm{cap}^L(E,\Omega) \leq c\,\mathrm{cap}(E,\Omega)$ so that, by hypothesis, $\mathrm{cap}^L(\tilde{A}_h \setminus A_h, \Omega) \to 0$. Then by the ellipticity assumption on L the sequence $\{z_h\}$ of the L-capacitary potentials of the sets $\tilde{A}_h \setminus A_h$ in Ω converges to zero strongly in $H_0^1(\Omega)$. Let c be a positive constant such that $\tilde{w}_h \leq c$ q.e. in Ω for every $h \in \mathbf{N}$. We claim that for every $h \in \mathbf{N}$

$$\tilde{w}_h \leq w_h + cz_h \qquad \text{q.e. in } \Omega\,. \tag{3.3.12}$$

As $z_h = 1$ q.e. in $\tilde{A}_h \setminus A_h$, $w_h \geq 0$, and $\tilde{w}_h = 0$ q.e. in $\Omega \setminus \tilde{A}_h$, (3.3.12) is trivially satisfied in $\Omega \setminus A_h$. Since $\langle Lz_h, \psi\rangle = 0$ for every $\psi \in H_0^1(\Omega)$ with $\psi = 0$ q.e. in $\tilde{A}_h \setminus A_h$, in particular, we have

$$\langle L(\tilde{w}_h - w_h - cz_h), \psi\rangle = 0 \tag{3.3.13}$$

for every $\psi \in H^1_0(\Omega)$ with $\psi = 0$ q.e. in $\Omega \setminus A_h$. Taking in (3.3.13) $\psi = (\tilde{w}_h - w_h - c\,z_h)^+$, by the ellipticity assumption on L, we obtain that $(\tilde{w}_h - w_h - c\,z_h)^+ = 0$ q.e. in Ω, which proves (3.3.12). □

Proof of Theorem 3.3.1. By Remark 3.3.3 it is enough to characterize the relaxation of the functional J_0 defined by (3.3.5). By Proposition 3.3.6 $\tilde{J}_0(\mu) = \overline{J}_0(\mu)$ for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ with $\lambda(A_\mu) < M$. Let us consider now a measure $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ with $\lambda(A_\mu) = M$. Let $\lambda_{|A_\mu}$ be the measure on Ω defined by $(\lambda_{|A_\mu})(B) = \lambda(A_\mu \cap B)$. Since $M > 0$, there exists a point $x \in \operatorname{supp} \lambda_{|A_\mu}$, that is $\lambda(B(x,r) \cap A_\mu) > 0$ for every $r > 0$. Setting $B_h = B(x, 1/h)$, $A_h = A_\mu \setminus B_h$, and $\mu_h = \mu + \mu_{\Omega \setminus B_h}$ we have that $A_{\mu_h} \subseteq A_h$ and $\lambda(A_{\mu_h}) < M$. By Lemma 3.3.7 applied to $\tilde{A}_h = A_\mu$, the sequence $\{\mu_h\}$ γ^L-converges to μ. Thus by the γ^L-lower semicontinuity of $\overline{J}_0$ we have

$$\overline{J}_0(\mu) \le \liminf_{h\to\infty} \overline{J}_0(\mu_h).$$

As $\lambda(A_{\mu_h}) < M$, by Proposition 3.3.6 $\overline{J}_0(\mu_h) = \tilde{J}_0(\mu_h)$ for every $h \in \mathbf{N}$. Moreover, since $A_{\mu_h} \subseteq A_\mu$ and $\lambda(A_\mu) = M$, we have

$$\tilde{J}_0(\mu_h) \le \int_{A_\mu} j(x,0)\, d\lambda = \tilde{J}_0(\mu)$$

for every h. Therefore

$$\overline{J}_0(\mu) \le \liminf_{h\to\infty} \tilde{J}_0(\mu_h) \le \tilde{J}_0(\mu),$$

and hence $\overline{J}_0(\mu) = \tilde{J}_0(\mu)$ for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. □

Remark 3.3.8 If $T = \{0\}$ and λ is the Lebesgue measure, then the functional J_0 defined in (3.3.5) takes the form

(3.3.14)
$$J_0(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ +\infty, & \text{otherwise.} \end{cases}$$

Thus its relaxation $\overline{J}_0$ is the functional defined by

(3.3.15)
$$\overline{J}_0(\mu) = \begin{cases} 0, & \text{if } \mu = \mu_\emptyset, \\ +\infty, & \text{otherwise.} \end{cases}$$

It turns out that $\overline{J}_0$ coincides with the functional $\tilde{J}_0$ defined in (3.3.6). Indeed $\tilde{J}_0$ is finite and, more precisely, takes the value zero, only for $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ with $\lambda(A_\mu) = 0$. It is well known that every quasi open set with Lebesgue measure zero has capacity zero, so that $\lambda(A_\mu) = 0$ if and only if $\mu = \mu_\emptyset$.

The following counterexample shows that, for a general λ, the functional

$$\tilde{J}_0(\mu) = \begin{cases} 0, & \text{if } \lambda(A_\mu) = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

does not coincide with the relaxation of the functional J_0 when $T = \{0\}$.

Example 3.3.9 Let $\{q_h\} = \Omega \cap \mathbf{Q}^N$ and let $\{r_h\}$ a sequence of positive number such that $\mathrm{cap}(B(q_h, r_h), \Omega) < 1/2^h$. Let $V = \bigcup_h B(q_h, r_h)$ and let u be the function in $H_0^1(\Omega)$ such that $u = 1$ q.e. in V and $\int_\Omega |Du|^2 dx = \mathrm{cap}(V, \Omega)$. Let us consider the measure λ defined as

$$\lambda(B) = \mathcal{L}^N(V \cap B)$$

for every $B \in \mathcal{B}(\Omega)$, where $\mathcal{L}^N$ denotes the N-dimensional Lebesgue measure. Since for every open subset A of Ω with $A \neq \emptyset$ we have $\lambda(A) > 0$, the functional J_0 defined in (3.3.5) corresponding to this choice of λ takes the form (3.3.14) and its relaxation $\overline{J}_0$ is given by (3.3.15). On the other hand, if we consider the quasi open set $A = \{x \in \Omega : u(x) < 1/2\}$, since $u = 1$ q.e. in V we have $\mathrm{cap}(A \cap V, \Omega) = 0$ and hence $\lambda(A) = 0$. Finally A has positive Lebesgue measure, and then positive capacity, so that $\mu_A(A) = 0 \neq +\infty = \mu_\emptyset(A)$. Thus $\tilde{J}_0(\mu_A) = 0$, but $\overline{J}_0(\mu_A) = +\infty$.

Remark 3.3.10 If $T = [0, \lambda(\Omega)]$, then by Remarks 3.2.2 and 3.3.3 the relaxed functional $\overline{J}$ can be written as

$$\overline{J}(\mu) = \int_{A_\mu} j(x, u_\mu)\, d\lambda - \int_{\Omega \setminus A_\mu} j^-(x, 0)\, d\lambda.$$

The following example shows that for every fixed $\nu \in \tilde{\mathcal{M}}_0(\Omega)$ there exists a functional J_ν as in Theorem 3.3.1, with $f = 1$ and $T = [0, \lambda(\Omega)]$, such that

ν is the unique minimum point of $\overline{J}_\nu$. If $\nu \neq \mu_A$ for every open set $A \subseteq \Omega$, then the minimum problem

$$\min_{\substack{A \text{ open} \\ A \subseteq \Omega}} J_\nu(A)$$

has no solution. Indeed $J_\nu(A) \geq \overline{J}_\nu(\mu_A) > \overline{J}_\nu(\nu)$ for every open set $A \subseteq \Omega$, and

$$\inf_{\substack{A \text{ open} \\ A \subseteq \Omega}} J_\nu(A) = \min_{\mu \in \tilde{\mathcal{M}}_0(\Omega)} \overline{J}_\nu(\mu) = \overline{J}_\nu(\nu)$$

by Remark 3.1.9.

Example 3.3.11 Let w_0 be the solution of the Dirichlet problem

$$w_0 \in H_0^1(\Omega)\,, \qquad Lw_0 = 1 \quad \text{in } \Omega\,.$$

Then $w_0 \in L^\infty(\Omega)$ and $w_\mu \leq w_0$ q.e. in Ω by the comparison principle (Lemma 1.3.8). Let us fix $\nu \in \tilde{\mathcal{M}}_0(\Omega)$ and let $g_\nu \in L^\infty(\Omega)$ be the function defined by

$$g_\nu = \begin{cases} w_\nu & \text{in } A_\nu\,, \\ 3k & \text{in } \Omega \backslash A_\nu\,, \end{cases}$$

where $k \in \mathbf{R}$ and $k \geq \|w_0\|_{L^\infty(\Omega)}$. Let $j_\nu : \Omega \times \mathbf{R} \to \mathbf{R}$ be the function defined by

$$j_\nu(x,s) = |s - g_\nu(x)|^2 - k^2\,.$$

Finally let $f = 1$, let λ be the Lebesgue measure, and let $T = [0, \lambda(\Omega)]$. Then the functional defined by (3.3.1) is given by

$$J_\nu(A) = \int_A (|w_A - g_\nu|^2 - k^2)\,dx\,,$$

for every open set $A \subseteq \Omega$. By Remark 3.3.10 the relaxation $\overline{J}_\nu$ of J_ν takes the form

$$\overline{J}_\nu(\mu) = \int_{A_\mu} (|w_\mu - g_\nu|^2 - k^2)\,dx - \int_{\Omega \backslash A_\mu} (g_\nu^2 - k^2)^-\,dx \tag{3.3.16}$$

for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. We want to prove that $\overline{J}_\nu(\mu) > \overline{J}_\nu(\nu) = -k^2\lambda(A_\nu)$ for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ with $\mu \neq \nu$. By (3.3.16) we can write

$$\overline{J}_\nu(\mu) = \int_{A_\mu \cap A_\nu} (|w_\mu - w_\nu|^2 - k^2)\, dx + \int_{A_\mu \setminus A_\nu} (w_\mu^2 - 6kw_\mu + 8k^2)\, dx$$
$$- \int_{\Omega \setminus (A_\mu \cup A_\nu)} (9k^2 - k^2)^-\, dx - \int_{A_\nu \setminus A_\mu} (w_\nu^2 - k^2)^-\, dx\,,$$

Since $w_\nu = 0$ q.e. in $A_\mu \setminus A_\nu$, $w_\mu = 0$ q.e. in $A_\nu \setminus A_\mu$, and $0 \le w_\mu \le k$ q.e. in Ω, we have $w_\mu^2 = |w_\mu - w_\nu|^2$ q.e. in $A_\mu \setminus A_\nu$, $-6kw_\mu + 8k^2 \ge 0$ q.e. in Ω, and $-(w_\nu^2 - k^2)^- = w_\nu^2 - k^2 = |w_\mu - w_\nu|^2 - k^2$ q.e. in $A_\nu \setminus A_\mu$. Hence

$$\overline{J}_\nu(\mu) \ge \int_{A_\mu \cap A_\nu} |w_\mu - w_\nu|^2\, dx - k^2\lambda(A_\mu \cap A_\nu) + \int_{A_\mu \setminus A_\nu} |w_\mu - w_\nu|^2\, dx$$
$$+ \int_{A_\nu \setminus A_\mu} |w_\mu - w_\nu|^2\, dx - k^2\lambda(A_\nu \setminus A_\mu) = \int_\Omega |w_\mu - w_\nu|^2 - k^2\lambda(A_\nu)\,.$$

This shows that $\overline{J}_\nu(\mu) \ge \overline{J}_\nu(\nu) = -k^2\lambda(A_\nu)$ for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$. If $\overline{J}_\nu(\mu) = \overline{J}_\nu(\nu)$, then $\int_\Omega |w_\mu - w_\nu|^2 = 0$, hence $w_\mu = w_\nu$ a.e. in Ω, and this implies $\mu = \nu$ by Lemma 3.3.3 of [30].

In the following example, which is a particular case of the previous one, the function j is continuous, and even $C^\infty(\Omega)$ if the coefficients of the operator L are $C^\infty(\Omega)$.

Example 3.3.12 Let w_0, k, λ, f, T be as in Example 3.3.11, let $j(x,s) = |s - \frac{1}{2}w_0|^2 - k^2$, and let

$$J(A) = \int_A (|w_A - \frac{1}{2}w_0|^2 - k^2)\, dx$$

be the corresponding functional defined for every open set $A \subseteq \Omega$. Then

$$\inf_{\substack{A \text{ open} \\ A \subseteq \Omega}} J(A) = -k^2\lambda(\Omega)\,, \tag{3.3.17}$$

where λ is the Lebesgue measure, but the minimum in (3.3.17) is not achieved. To prove this fact it is enough to notice that $w_0 > 0$ in Ω by the strong maximum principle, and that

$$L(\frac{1}{2}w_0) + \frac{1}{2}\frac{w_0}{w_0} = 1 \quad \text{in } \Omega\,,$$

hence $\frac{1}{2}w_0$ is the solution of (1.3.6) corresponding to the measure $\nu = \lambda/w_0$. Therefore $\frac{1}{2}w_0 = w_\nu$. As $A_\nu = \Omega$, using the notation of the Example 3.3.11 we have $g_\nu = \frac{1}{2}w_0$, $j_\nu = j$, $J_\nu = J$. Therefore $\overline{J}(\mu) > \overline{J}(\nu) = -k^2\lambda(\Omega)$ for every $\mu \in \tilde{\mathcal{M}}_0(\Omega)$ with $\mu \neq \nu$. The conclusion follows from (3.3.16) and (3.3.17).

Chapter 4
Relaxed Dirichlet problems with measure data

The aim of this chapter is to prove regularity results for the solution of variational relaxed Dirichlet problems and to introduce, by a duality method, a notion of solution for relaxed Dirichlet problems with a measure with bounded variation as datum. In particular we introduce a notion of solution (that gives, if $\mu = 0$, the solution given by G. Stampacchia in [64]), proving an existence and uniqueness result. This nonvariational existence result allows us to define the Green function G_μ for relaxed Dirichlet problems. We show that it is possible to define pointwise $G_\mu(x,y)$ in $\Omega \times \Omega$ outside the diagonal, and that this representative is upper semicontinuous in each variable, has the usual symmetry property, and the representation formula for solutions of relaxed Dirichlet problems with measure data holds. The main difficulty in proving these properties consists in overcoming the lack of continuity of G_μ.

4.1. Preliminaires

We shall define $\mathcal{M}_b(\Omega)$ as the space of all signed measures on Ω with bounded total variation; we shall denote by δ_x the Dirac mass concentrated in $x \in \Omega$. By $\mathcal{M}_0^b(\Omega)$ we shall denote the class of measure with bounded variation and not charging polar sets.

Definition 4.1.1 *Let $\{\nu_n\}$ be a sequence of measures in $\mathcal{M}_b(\Omega)$. We say that ν_n converges to a measure $\nu \in \mathcal{M}_b(\Omega)$ in the weak* topology of measures, if $\int_\Omega f(x)\, d\nu_n(x)$ converges to $\int_\Omega f(x)\, d\nu(x)$ for every continuous function f with compact support in Ω.*

Remark 4.1.2 If ν_n converges to ν in the sense of Definition 4.1.1, and $\operatorname{supp} \nu_n \subseteq K$ for every $n \in \mathbf{N}$, where K is a compact subset of Ω, then $\int_\Omega f\, d\nu_n$ converges to $\int_\Omega f\, d\nu$ for every continuous function f on Ω.

Theorem 4.1.3 *Let $\{\nu_n\}$ be a sequence of measures of $\mathcal{M}_b(\Omega)$, and suppose that there exists a positive constant c such that $|\nu_n|(\Omega) \le c$ for every n in $\mathbf{N}$. Then there exists a subsequence $\{\nu_{n_k}\}$, and a measure $\nu \in \mathcal{M}_b(\Omega)$, such that ν_{n_k} converges to ν in the weak* topology of measures. Moreover, if $\{\nu_n\}$ is a sequence of positive measures that converges to $\nu \in \mathcal{M}_b(\Omega)$ in the weak* topology of measures, then*

$$\liminf_{n\to\infty} \int_\Omega f(x)\, d\nu_n(x) \ge \int_\Omega f(x)\, d\nu(x)$$

for every continuous function f on Ω, bounded from below.

Proof. See [2], Theorem 4.5.1. □

We recall that for every positive measure μ the weak-$L^p(\Omega,\mu)$ space $L^p_w(\Omega,\mu)$ is the space of all functions f such that

$$\|f\|_{L^p_w(\Omega,\mu)} = \sup_{\sigma>0} [\sigma^p \mu(\{x \in \Omega : |f| > \sigma\})]^{\frac{1}{p}} < +\infty .$$

A useful result about weak-$L^p(\Omega,\mu)$ spaces is the Marcinkiewicz interpolation theorem.

Theorem 4.1.4 *Let μ and ν be two positive measures and let T be a linear continuous mapping between $L^{p_0}(\Omega,\mu)$ and $L^{q_0}_w(\Omega,\nu)$, and between $L^{p_1}(\Omega,\mu)$ and $L^{q_1}_w(\Omega,\nu)$ with $p_0 \ne p_1$. Define, for a real number θ in (0,1),*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

and assume that $p \le q$. Then T is continuous between $L^p(\Omega,\mu)$ and $L^q(\Omega,\nu)$.

Proof. See [6], Theorem 1.3.1. □

4.2. Regularity results for the variational case

In order to define relaxed Dirichlet problems with measure data, we need some regularity results for the solutions of variational relaxed Dirichlet problems with respect to the regularity of the datum. We begin with a useful result, whose proof is straightforward by a theorem due to G. Stampacchia (see [64], Lemme 1.1).

Lemma 4.2.1 *Let $s: \mathbf{R} \to \mathbf{R}$ be a lipschitz function such that $s(0) = 0$; then the operator S, defined by $S(u)(x) = s(u(x))$ for every measurable function u, maps $H_0^1(\Omega) \cap L^2(\Omega, \mu)$ into itself.*

Another basic tool in the proof of the regularity theorem will be the following.

Lemma 4.2.2 *Let $\phi: [0, +\infty) \to [0, +\infty)$ be a decreasing function such that*

$$\phi(h) \leq \frac{c}{(h-k)^\alpha} [\phi(k)]^\beta \qquad \forall h > k \geq 0,$$

where c, α and β are positive constants. Then the following hold:
(i) if $\beta > 1$, then $\phi(d) = 0$ where $d^\alpha = c[\phi(0)]^{\beta-1} 2^{\frac{\alpha\beta}{\beta-1}}$;
(ii) if $\beta < 1$, then $\phi(h) \leq 2^{\frac{\tau}{1-\beta}} c^{\frac{1}{1-\beta}} h^{-\tau}$ for every $h > 0$, where $\tau = \frac{\alpha}{1-\beta}$.

Proof. See [64], Lemme 4.1. □

Now we prove the regularity result.

Theorem 4.2.3 *Let μ be a measure in $\mathcal{M}_0(\Omega)$. Let u be the solution of the relaxed Dirichlet problem*

$$\text{(4.2.1)} \qquad \begin{cases} Lu + \mu u = -\text{div}\,(g) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the following hold:
(i) if the function $|g|$ belongs to $L^m(\Omega)$, with $m > N$, then u belongs to $L^\infty(\Omega) \cap L^\infty(\Omega, \mu)$ and the continuity estimate holds:

$$\|u\|_{L^\infty(\Omega)} \leq c_1 \|g\|_{L^m(\Omega)}, \qquad \|u\|_{L^\infty(\Omega,\mu)} \leq c_2 \|g\|_{L^m(\Omega)},$$

where c_1 and c_2 are positive constants that do not depend on u and g;
(ii) if the function $|g|$ belongs to $L^m(\Omega)$, with $2 \leq m < N$, u belongs to $L^{m^*}(\Omega) \cap L^p(\Omega, \mu)$, with $m^* = \frac{Nm}{N-m}$, and $p = \frac{N-2}{N} m^*$, and the continuity estimate holds:

$$\|u\|_{L^{m^*}(\Omega)} \leq c_1 \|g\|_{L^m(\Omega)}, \qquad \|u\|_{L^p(\Omega,\mu)} \leq c_2 \|g\|_{L^m(\Omega)},$$

where c_1 and c_2 are positive constants that do not depend on u and g.

Proof. Let us define the set $A(k) = \{x \in \Omega : |u(x)| > k\}$, where k is a positive real number, and choose as test function in (4.2.1) $v_k(x) = \max(|u(x)| - k, 0)\,\mathrm{sgn}(u(x))$. We obtain, by (1.3.1),

$$\theta \int_{A(k)} |Dv_k|^2\, dx + \int_{A(k)} |u| v_k\, d\mu \le \int_{A(k)} g D v_k\, dx\,.$$

Using Young inequality in the right hand side and simplifying equal terms we have

$$\frac{\theta}{2} \int_{A(k)} |Dv_k|^2\, dx + \int_{A(k)} |u| v_k\, d\mu \le \frac{1}{2\theta} \int_{A(k)} |g|^2\, dx\,.$$

If $h > k$ then $A(h) \subset A(k)$, and $v_k \ge h - k$ in $A(h)$; thus, by Sobolev embedding,

$$\int_{A(k)} |Dv_k|^2\, dx \ge c \left(\int_{A(k)} |v_k|^{2^*}\, dx \right)^{\frac{2}{2^*}} \ge c\,(h-k)^2 \mathcal{L}\,(A(h))^{\frac{2}{2^*}}\,,$$

where $2^* = \frac{2N}{N-2}$ if $N \ge 3$, or any real number greater than $\frac{2m}{m-2}$ if $N = 2$, and

$$\int_{A(k)} |u| v_k\, d\mu \ge \int_{A(k)} |v_k|^2\, d\mu \ge (h-k)^2\, \mu\,(A(h))\,.$$

On the other hand, using Hölder inequality,

$$\int_{A(k)} |g|^2\, dx \le \|g\|^2_{L^m(\Omega)}\, (\mathcal{L}\,(A(k)))^{1-\frac{2}{m}}\,.$$

Finally, putting all the estimates together, we obtain the relation

$$c(h-k)^2\,(\mathcal{L}\,(A(h)))^{\frac{2}{2^*}} + (h-k)^2 \mu\,(A(h)) \le \|g\|^2_{L^m(\Omega)}\, (\mathcal{L}\,(A(k)))^{1-\frac{2}{m}}\,,$$

and so, for every $h > k \ge 0$,

(4.2.2) $$\mathcal{L}\,(A(h)) \le c^{-\frac{2^*}{2}} \frac{1}{(h-k)^{2^*}} \|g\|^{2^*}_{L^m(\Omega)}\, (\mathcal{L}\,(A(k)))^{\frac{2^*}{2}(1-\frac{2}{m})}\,,$$

(4.2.3) $$(h-k)^2 \mu\,(A(h)) \le \|g\|^2_{L^m(\Omega)}\, (\mathcal{L}\,(A(k)))^{1-\frac{2}{m}}\,.$$

Applying Lemma 4.2.2 with $\phi(h) = \mathcal{L}(A(h))$, $\alpha = 2^*$, and $\beta = (1 - \frac{2}{m})\frac{2^*}{2}$ we conclude that if $m > N$ then, by (4.2.2), $\mathcal{L}(A(d)) = 0$, where $d = c^{-\frac{1}{2}} \|g\|_{L^m(\Omega)}$, and, by (4.2.3), $\mu(A(h)) = 0$ for every $h > d$. Thus, the first part of the theorem is proved.

If $2 \leq m < N$ then, by (4.2.2), $\mathcal{L}(A(k)) \leq c_1 \|g\|^{m^*}_{L^m(\Omega)} k^{-m^*}$ and, choosing $h = 2k$ in (4.2.3), $\mu(A(2k)) \leq c_2 \|g\|^{(1-2/m)m^*}_{L^m(\Omega)} (2k)^{-(N-2)m^*/N}$. Hence, the mapping $g \mapsto R(g)$, where $R(g)$ is the solution of the relaxed Dirichlet problem with datum $-(g)_{x_i}$, is linear and continuous between $L^m(\Omega)$ and $L^{m^*}_{w}(\Omega, \mathcal{L})$, and between $L^m(\Omega)$ and $L_w^{\frac{N-2}{N}m^*}(\Omega, \mu)$, for every m in $[2, N)$. By Marcinkiewicz interpolation theorem this linear map is continuous between $L^m(\Omega)$ and $L^{m^*}(\Omega)$, and between $L^m(\Omega)$ and $L^{\frac{N-2}{N}m^*}(\Omega, \mu)$ for every m in $(2, N)$. Moreover, if $m = 2$ the continuity of the operator R between $L^2(\Omega)$ and $L^{2^*}(\Omega)$, and between $L^2(\Omega)$ and $L^2(\Omega, \mu)$ is well known (see Theorem 1.3.2), and this concludes the proof of the theorem. □

Remark 4.2.4 It is possible to obtain the result of Theorem 4.2.3(ii) without using the Marcinkiewicz interpolation theorem; for example, the techniques used in [10], Teorema 1.5, can be adapted in a straightforward way to this kind of problems.

The following theorem can be proved in the same way.

Theorem 4.2.5 *Let μ be a measure in $\mathcal{M}_0(\Omega)$. Let u be the solution of the relaxed Dirichlet problem (1.3.5). Then the following hold:*

(i) if the datum f belongs to $L^m(\Omega)$, with $m > \frac{N}{2}$, then u belongs to $L^\infty(\Omega) \cap L^\infty(\Omega, \mu)$ and

$$\|u\|_{L^\infty(\Omega)} \leq c_1 \|f\|_{L^m(\Omega)}, \qquad \|u\|_{L^\infty(\Omega,\mu)} \leq c_2 \|f\|_{L^m(\Omega)},$$

where c_1 and c_2 are two positive constants independent of u and f;

(ii) if the datum f belongs to $L^m(\Omega)$, $\frac{2N}{N+2} \leq m < \frac{N}{2}$, u belongs to $L^{m^{**}}(\Omega) \cap L^p(\Omega, \mu)$, with $p = \frac{N-2}{N} m^{**} = \frac{(N-2)m}{N-2m}$ and

$$\|u\|_{L^{m^{**}}(\Omega)} \leq c_1 \|f\|_{L^m(\Omega)}, \qquad \|u\|_{L^p(\Omega,\mu)} \leq c_2 \|f\|_{L^m(\Omega)},$$

where c_1 and c_2 are two positive constants independent of u and f.

4.3. Some results about Dirichlet problems with measure data

In the following, we give a definition of solution of an elliptic equation with measure data, and prove some of its properties.

Definition 4.3.1 *The resolvent operator of L is the operator $R: L^\infty(\Omega) \to L^\infty(\Omega)$ that associates to every $L^\infty(\Omega)$ function f the unique solution $u = R(f)$ of the classical Dirichlet problem*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The operator R is well defined thanks to Théorème 4.2 of [64]. We denote by R^* the resolvent operator of L^*.

Remark 4.3.2 We note explicitly that, by Théorème 7.1 of [64], $R(f)$ belongs to $C^0(\Omega)$, and not only to $L^\infty(\Omega)$. Moreover, if one assumes that $\partial\Omega$ is regular (see Définition 6.2 of [64]), then $R(f)$ belongs to $C^0(\overline{\Omega})$.

Definition 4.3.3 *Let γ be a measure in $\mathcal{M}_b(\Omega)$ with compact support in Ω. A function u in $L^1(\Omega)$ is a solution of*

$$\begin{cases} Lu = \gamma & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if

$$\int_\Omega u(x)\, g(x)\, dx = \int_\Omega R^*(g)(x)\, d\gamma(x) \qquad \forall g \in L^\infty(\Omega). \tag{4.3.1}$$

Remark 4.3.4 The right hand side of (4.3.1) is well defined thanks to Remark 4.3.2.

Remark 4.3.5 Our definition of solution is slightly different from the usual one given in [64], Section 9, where (4.3.1) is required to hold for every $g \in C^0(\overline{\Omega})$ and $\partial\Omega$ must be regular (in a sense specified in Définition 6.2 of [64]). In this paper we overcome these problems with the hypothesis of compact support for the measure γ.

Anyway, the proof of existence and uniqueness of solutions given in [64] remains valid also in our case. Thus, there exists one and only one function

u satisfying (4.3.1), and this function obviously coincides with the solution found in [64], so that it enjoys all of its properties. In particular, it belongs to $H_0^{1,q}(\Omega)$ for every $1 \leq q < \frac{N}{N-1}$, satisfies the continuity estimate

$$\|u\|_{H_0^{1,q}(\Omega)} \leq c_q\, |\gamma|(\Omega)\,, \tag{4.3.2}$$

for some positive constant c_q, and is the solution in the sense of distributions if $\gamma \in H^{-1}(\Omega)$. Moreover, if $G(x,y)$ is the Green function of the operator L in Ω, that is, the solution in the sense of Definition 4.3.3 of $L^* u = \delta_x$, then the solution of $L\,u = \gamma$ with homogeneous Dirichlet boundary conditions can be written as

$$u(x) = \int_\Omega G(x,y)\,d\gamma(y) \qquad \text{for a.e. } x \in \Omega \tag{4.3.3}$$

(please note that in [64], Définition 9.2, there is a typing error: δ_y should be replaced by δ_x). Finally we recall that G is positive, continuous (with extended real values) in $\Omega \times \Omega$, and that $G(x,y) = G^*(y,x)$ for every x and y in Ω, where G^* is the Green function of the operator L^*.

Remark 4.3.6 A question arises whether or not the solution u of (4.3.1) is also a solution in the sense of distributions of $L\,u = \gamma$, i.e. if u is such that

$$\int_\Omega A\,Du D\varphi\,dx = \int_\Omega \varphi\,d\gamma \qquad \forall \varphi \in C_0^\infty(\Omega)\,.$$

In [8] it is proved that the limit of an approximating sequence of solutions of Dirichlet problems with regular data converging to γ in the weak* topology of measures is a solution in the sense of distributions. Anyway, the solution in the sense of distributions is not unique, as a counterexample by Serrin (see [62]) shows, and so it may be different from the (unique) solution of (4.3.1). The following result shows that the solution of (4.3.1) can be obtained by approximation, and so is a solution in the sense of distributions.

Lemma 4.3.7 *Let ν belong to $\mathcal{M}_b(\Omega)$, and suppose that ν has compact support in Ω. Let $\{\varphi_n\}$ be a sequence of positive, spherically symmetric, δ-approximating convolution kernels with compact support, and define $f_n = \varphi_n * \nu$. Let $\{u_n\}$ be the sequence of solutions of*

$$\begin{cases} L\,u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, for every real number q such that $1 \le q < \frac{N}{N-1}$, u_n converges strongly in $H_0^{1,q}(\Omega)$ to the solution u of (4.3.1). Moreover

$$\int_\Omega A\,DuD\varphi\,dx = \int_\Omega \varphi\,d\gamma \qquad \forall\varphi \in H_0^{1,q'}(\Omega)\,. \tag{4.3.4}$$

Proof. By Lemma 1 of [8], u_n converges strongly in $H_0^{1,q}(\Omega)$ (hence in $L^1(\Omega)$) to a function u that satisfies (4.3.4). Moreover, since $f_n \in L^2(\Omega)$, we have

$$\int_\Omega u_n\,g\,dx = \int_\Omega R^*(g)\,f_n\,dx \qquad \forall g \in L^\infty(\Omega),\ \forall n \in \mathbf{N}\,.$$

Since $R^*(g) \in C^0(\Omega)$, we can use Remark 2.2 and pass to the limit in both sides of the preceding relations, obtaining that u is the solution of (4.3.1). □

Definition 4.3.8 *Given a positive measure $\gamma \in \mathcal{M}_b(\Omega)$ we define the set N_γ as*

$$N_\gamma = \{x \in \Omega : \int_\Omega G(x,y)\,d\gamma(y) = +\infty\}\,.$$

It is proved in [7], Chapter 2, Sections 2-3, and Chapter 6, Section 1, that N_γ has harmonic capacity zero; moreover, since $G(x,x) = +\infty$, then $\gamma(\{x\}) = 0$ for every $x \notin N_\gamma$, and $N_\gamma \subseteq \operatorname{supp}\gamma$, since $G(x,y)$ is bounded outside a neighbourhood of the diagonal.

If γ is a Kato measure, $N_\gamma = \emptyset$, and the solution of (4.3.1), defined pointwise by (4.3.3), is continuous in Ω (see [33], Theorem 4.11).

Remark 4.3.9 From now on we will always choose in the almost everywhere equivalence class of the solution of $Lu = \gamma$, the representative given by $\int_\Omega G(x,y)\,d\gamma(y)$. We remark that it is well defined in Ω if γ is a positive measure, while, in general, it is not defined in $N_{\gamma^+} \cap N_{\gamma^-}$. Furthermore, if γ is a positive measure, then this representative is lower semicontinuous, as can be easily seen by Fatou lemma. This is a more precise choice of the pointwise value of u, which, *a priori*, is defined only cap_q-quasi everywhere, while $N_{\gamma^+} \cap N_{\gamma^-}$ is a set with 2-capacity zero. Moreover, this representative is independent from the choice of u in its Lebesgue class, as we prove in the following lemma.

Lemma 4.3.10 *Let $\gamma \in \mathcal{M}_b(\Omega)$ be a positive measure with compact support in Ω, and let u be the solution of (4.3.1). Then, for every x in Ω,*

$$\lim_{n\to\infty} \int_\Omega u(y)\,\varphi_n(x-y)\,dy = \int_\Omega G(x,y)\,d\gamma(y)\,, \tag{4.3.5}$$

where $\{\varphi_n\}$ is any sequence of positive, spherically symmetric, δ-approximating convolution kernels with compact support.

Proof. By (4.3.3) and Fubini theorem, we have

$$\int_\Omega u(y)\,\varphi_n(x-y)\,dy = \int_\Omega \left(\int_\Omega G(y,z)\,\varphi_n(x-y)\,dy\right)\,d\gamma(z)\,.$$

By Fatou lemma, since γ is a positive measure and $G(\cdot,z)$ is continuous and positive, we have

$$\int_\Omega G(x,z)\,d\gamma(z) \le \liminf_{n\to\infty} \int_\Omega u(y)\,\varphi_n(x-y)\,dy\,,$$

so that (4.3.5) is proved if its right hand side is infinite, i.e., if $x \in N_\gamma$.

Suppose that $G(x,\cdot)$ belongs to $L^1(\Omega,\gamma)$. We recall that, if Γ is the Green function for the Laplace operator on Ω, then $\Gamma(x,\cdot)$ is a superharmonic function, and so we obtain $\int_\Omega \Gamma(y,z)\,\varphi_n(x-y)\,dy \le \Gamma(x,z)$ for every $y \in \Omega$ since φ_n is radially symmetric; moreover, there exists a positive constant c, depending on x and $\operatorname{supp}\gamma$, such that $G(x,y) \le c\,\Gamma(x,y) \le c^2\,G(x,y)$ for every $y \in \operatorname{supp}\gamma$ (see [64], formula (9.23)). Thus, for every $y \in \operatorname{supp}\gamma$, we have $\int_\Omega G(y,z)\,\varphi_n(x-y)\,dy \le c^2\,G(x,z)$. Since $G(\cdot,z)$ is continuous, formula (4.3.5) follows from Lebesgue theorem. □

Remark 4.3.11 The same result holds if we use convolution kernels of the form $\varphi_n(x) = n^N\,\varphi(nx)$, $\varphi \in L^\infty(B_1(0))$. Actually, $\int_\Omega \Gamma(y,z)\,\varphi_n(x-y)\,dy \le c\,\Gamma(x,z)$ for every $y \in \Omega$, where c depends only on φ, and this is the only point of the proof that has to be changed.

Lemma 4.3.12 *Let $\{\gamma_n\}$ be a sequence of measures of $\mathcal{M}_b(\Omega)$. Suppose that there exists a compact set $K \subset\subset \Omega$ such that $\operatorname{supp}\gamma_n \subseteq K$ for every n in $\mathbf{N}$. Let w_n be the solution of (4.3.1) with datum γ_n, and let q be any real number such that $1 \le q < \frac{N}{N-1}$. Then the following holds:*

(i) if γ_n converges to a measure γ in the weak topology of measures, then w_n converges to a function w weakly in $H_0^{1,q}(\Omega)$, and w is the solution of (4.3.1) with datum γ;*

(ii) if w_n converges to a function w weakly in $H_0^{1,q}(\Omega)$, then γ_n converges to a measure γ in the weak topology of measures, and w is the solution of (4.3.1) with datum γ;*

(iii) if γ_n converges to a measure γ in the weak topology of measures, γ_n is a positive measure for every n in $\mathbf{N}$, and w_n is an increasing sequence of functions, then the pointwise limit w is such that*

$$w(x) = \int_\Omega G(x,y)\,d\gamma(y) \qquad \forall x \in \Omega\,.$$

Proof. (i) Since, by (4.3.2), $\|w_n\|_{H_0^{1,q}(\Omega)} \le |\gamma_n|(\Omega) \le c$, then, up to subsequences, w_n converges weakly in $H_0^{1,q}(\Omega)$ to a function w, and so, by Rellich theorem, w_n converges to w strongly in $L^1(\Omega)$. Using Remark 2.2, we can pass to the limit in the identities

$$\int_\Omega w_n\, g\, dx = \int_\Omega R^*(g)\, d\gamma_n \qquad \forall g \in L^\infty(\Omega)\,,$$

and obtain that w is the solution of (4.3.1) with γ as datum. Since the limit does not depend on the subsequence, then the whole sequence $\{w_n\}$ converges to w.

(ii) Since, by Lemma 4.3.7, w_n is the solution in the sense of distributions of $L\,w_n = \gamma_n$ with Dirichlet boundary conditions, then

$$\int_\Omega A\,Dw_n D\varphi\, dx = \int_\Omega \varphi\, d\gamma_n \qquad \forall \varphi \in C_0^\infty(\Omega)\,.$$

Choosing φ such that $\varphi \equiv 1$ on K, and using the boundedness of $\{w_n\}$ in $H_0^{1,q}(\Omega)$, we obtain that $|\gamma_n|(\Omega) \le c$. Hence, up to subsequences, γ_n converges in the weak* topology of measures to a measure γ by Theorem 4.1.3. By (i), this implies the result. Once again, by uniqueness, the limit γ does not depend on the subsequence, and so the whole sequence $\{\gamma_n\}$ converges.

(iii) By (i), $w(x) = \int_\Omega G(x,y)\,d\gamma(y)$ for almost every $x \in \Omega$. Since

$$w_n(x) = \int_\Omega G(x,y)\,d\gamma_n(y) \qquad \forall x \in \Omega,\ \forall n \in \mathbf{N}\,,$$

we have, passing to the limit, and using Theorem 4.1.3,

$$w(x) \geq \int_\Omega G(x,y)\,d\gamma(y) \qquad \forall x \in \Omega\,.$$

Let x be a fixed point in Ω; for every $r > 0$ there exists $x_r \in B_r(x)$ such that $w(x_r)\,\mathcal{L}\,(B_r(x)) \leq \int_{B_r(x)} w(y)\,dy$. Thus, since w is lower semicontinuous as limit of an increasing sequence of lower semicontinuous functions, we have, by Lemma 4.3.10,

$$\begin{aligned} w(x) &\leq \liminf_{r\to 0^+} w(x_r) \leq \liminf_{r\to 0^+} \frac{1}{\mathcal{L}\,(B_r(x))} \int_{B_r(x)} w(y)\,dy \\ &= \lim_{r\to 0^+} \frac{1}{\mathcal{L}\,(B_r(x))} \int_{B_r(x)} \left(\int_\Omega G(y,z)\,d\gamma(z)\right) dy = \int_\Omega G(x,z)\,d\gamma(z)\,, \end{aligned}$$

which concludes the proof. □

4.4. Nonvariational relaxed Dirichlet problems

The aim of this section is to introduce the notion of relaxed Dirichlet problems in a nonvariational setting, giving existence and regularity results. We will use the results of Theorems 4.2.3 and 4.2.5, and the ideas of Section 4.3, to give the definition of a solution for this kind of problems if the datum ν belongs to $\mathcal{M}_b(\Omega)$.

Definition 4.4.1 *Let μ be a measure in $\mathcal{M}_0(\Omega)$. We define the resolvent $R_\mu\colon L^\infty(\Omega) \to L^\infty(\Omega)$ as the operator that associates to every $L^\infty(\Omega)$ function f the unique solution $u = R_\mu(f)$ of the relaxed Dirichlet problem*

$$\begin{cases} L\,u + \mu\,u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense of Definition 1.3.1. The operator R_μ is well defined thanks to Theorem 4.2.5. We define R^*_μ the resolvent operator of L^*.

Theorem 4.4.2 *Let f be a positive function in $L^\infty(\Omega)$. Then there exists a positive measure $\gamma \in H^{-1}(\Omega)$ such that $L\,(R(f) - R_\mu(f)) = \gamma$ in the sense of Definition 4.3.3 in Ω. Thus, $R_\mu(f)$ can be defined pointwise as*

$$R_\mu(f)(x) = R(f)(x) - \int_\Omega G(x,y)\,d\gamma(y) \qquad \forall x \in \Omega\,. \tag{4.4.1}$$

This pointwise value coincides with the limit of its convolutions with any sequence of positive, spherically symmetric, δ-approximating convolution kernels with compact support, and

$$|R_\mu(f)(x)| \leq \|R_\mu(f)\|_{L^\infty(\Omega)} \qquad \forall x \in \Omega\,. \tag{4.4.2}$$

Proof. By Theorem 1.3.6, $R_\mu(f)$ is non-negative, and $L\,R_\mu(f) \leq f$ in the sense of distributions, so that $L\,(R(f) - R_\mu(f)) \geq 0$. Hence, there exists a positive measure γ such that $L\,(R(f) - R_\mu(f)) = \gamma$ in the sense of distributions, and (since $\gamma \in H^{-1}(\Omega)$) in the sense of Definition 4.3.3. Thus, we define pointwise $R(f) - R_\mu(f)$ as $\int_\Omega G(x,y)\,d\gamma(y)$; since $R(f)$ is continuous, (4.4.1) gives a pointwise definition for $R_\mu(f)$. By Lemma 4.3.10, this value coincides with the limit of its convolutions, and so formula (4.4.2) follows. □

Remark 4.4.3 Since the equation is linear, we can give a pointwise definition of $R_\mu(f)$ for every $f \in L^\infty(\Omega)$.

Definition 4.4.4 *Let ν belong to $\mathcal{M}_b(\Omega)$, and suppose that ν has compact support in Ω. We say that a function u in $L^1(\Omega)$ is a solution of the nonvariational relaxed Dirichlet problem*

$$\begin{cases} L\,u + \mu\,u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if

$$\int_\Omega u(x)\,g(x)\,dx = \int_\Omega R^*_\mu(g)(x)\,d\nu(x) \qquad \forall g \in L^\infty(\Omega)\,. \tag{4.4.3}$$

We point out that the right hand side of (4.4.3) is well defined thanks to Theorem 4.4.2. If $\mu = 0$ the above definition coincides with the definition given in Section 4.3.

Remark 4.4.5 If $\nu \in H^{-1}(\Omega)$ then the solution u of the variational relaxed Dirichlet problem (1.3.5) is the solution of (4.4.3), since $R^*_\mu(g)$ can be chosen as test function in the variational formulation of the equation satisfied by u and vice versa.

Now we give an existence and uniqueness result for solutions of (4.4.3).

Theorem 4.4.6 *There exists one and only one solution u of (4.4.3), with u belonging to $H_0^{1,q}(\Omega)$ for every real number q such that $1 \le q < \frac{N}{N-1}$, and*

$$\|u\|_{H_0^{1,q}(\Omega)} \le c_q \, |\nu|(\Omega) , \tag{4.4.4}$$

for some positive constant c_q independent of u.

Proof. We follow the ideas developed in [64]. Let us consider the linear operator $F: L^\infty(\Omega) \to \mathbf{R}$ defined as

$$F(g) = \int_\Omega R_\mu^*(g)(x)\, d\nu(x) \qquad g \in L^\infty(\Omega) .$$

Since $L^\infty(\Omega)$ is embedded in $H^{-1,p}(\Omega)$, $p > N$, we have, by Theorem 4.2.3,

$$|F(g)| \le |\nu|(\Omega)\, \|R_\mu^*(g)\|_{L^\infty(\Omega)} \le c\, |\nu|(\Omega)\, \|g\|_{H^{-1,p}(\Omega)} . \tag{4.4.5}$$

Since the embedding is dense, F is a continuous linear operator on $H^{-1,p}(\Omega)$. So, there exists a unique function u belonging to the space $H_0^{1,q}(\Omega)$, $1 \le q < \frac{N}{N-1}$, that represents F, and, in particular,

$$\int_\Omega u(x)\, g(x)\, dx = \int_\Omega R_\mu^*(g)(x)\, d\nu(x) \qquad \forall g \in L^\infty(\Omega) .$$

Formula (4.4.4) easily follows from (4.4.5). □

We show that the solution of (4.4.3) can be obtained as the limit of solutions of variational relaxed Dirichlet problems.

Theorem 4.4.7 *Let ν belong to $\mathcal{M}_b(\Omega)$, and suppose that ν has compact support in Ω. Let $\{\varphi_n\}$ be a sequence of positive, spherically symmetric, δ-approximating convolution kernels with compact support, and define $f_n = \varphi_n * \nu$. Let $\{u_n\}$ be the sequence of solutions of the problems (1.3.5) with f_n as data. Then, for every real number q such that $1 \le q < \frac{N}{N-1}$, u_n converges strongly in $H_0^{1,q}(\Omega)$ to the solution u of problem (4.4.3).*

Proof. Since f_n converges to ν in the weak* topology of measures, we have that $\{f_n\}$ is bounded in $L^1(\Omega)$, and so, by (4.4.4),

$$\|u_n\|_{H_0^{1,q}(\Omega)} \le c\, \|f_n\|_{L^1(\Omega)} . \tag{4.4.6}$$

Using the technique of [8], Lemma 1, with minor changes due to the presence of the measure μ, we can prove that $\{Du_n\}$ is a Cauchy sequence in measure; this and (4.4.6) imply, by Vitali theorem, that u_n strongly converges to u in $H_0^{1,q}(\Omega)$, for every real number q such that $1 \leq q < \frac{N}{N-1}$.

It remains to prove that u is the solution of the problem (4.4.3). Actually, for every g in $L^\infty(\Omega)$ and for every n in $\mathbf{N}$, we have $\int_\Omega u_n \, g \, dx = \int_\Omega R_\mu^*(g) \, f_n \, dx$. Clearly, the left hand side converges to $\int_\Omega u \, g \, dx$. On the other hand, since $\operatorname{supp} \nu$ is compact in Ω, we obtain, for every n such that the distance between the support of ν and the boundary of Ω is greater than $1/n$,

$$\int_\Omega R_\mu^*(g)(x) \, f_n(x) \, dx = \int_\Omega R_\mu^*(g)(x) \int_\Omega \varphi_n(x-y) \, d\nu(y) \, dx =$$

$$\int_\Omega \left(\int_\Omega R_\mu^*(g)(x) \, \varphi_n(x-y) \, dx \right) d\nu(y) = \int_\Omega (R_\mu^*(g) * \varphi_n)(y) \, d\nu(y) \, .$$

Using Theorem 4.4.2 and the boundedness of $R_\mu^*(g)$, we can pass to the limit by Lebesgue theorem, thus achieving the result. □

Remark 4.4.8 If $\mu = 0$, we obtain again the result of Lemma 4.3.7: the solution of (4.3.1) can be obtained by approximation.

Our next result is about the regularity of the solution of a nonvariational relaxed Dirichlet problem outside the support of the datum ν.

Theorem 4.4.9 *Let $\nu \in \mathcal{M}_b(\Omega)$, suppose that ν has compact support in Ω, and let $\Omega' \subseteq \Omega$ be an open set such that $|\nu| \llcorner \Omega' \in H^{-1}(\Omega)$. If u is the solution of the nonvariational relaxed Dirichlet problem with ν as datum in the sense of Definition 4.4.4, then $u \in H^1_{\mathrm{loc}}(\Omega') \cap L^2_{\mathrm{loc}}(\Omega', \mu)$, and it is a local solution of $L u + \mu u = \nu$ in Ω' in the sense of Definition 1.3.1.*

Proof. Since the equation is linear, it is not restrictive to suppose $\nu \geq 0$. We begin by proving that, if E is the support of ν, then u is in $H^1_{\mathrm{loc}}(\Omega \backslash E) \cap L^\infty_{\mathrm{loc}}(\Omega \backslash E) \cap L^2_{\mathrm{loc}}(\Omega \backslash E, \mu)$, and is a local solution of $L u + \mu u = 0$ in $\Omega \backslash E$. Consider the approximating sequence $\{u_n\}$ of solutions of the problems

$$\text{(4.4.7)} \qquad \begin{cases} L u_n + \mu u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_n = \varphi_n * \nu$ and $\{\varphi_n\}$ is as in the statement of Lemma 4.3.10. Let $E_n = \operatorname{supp} f_n$, so that, for n large enough, $\{E_n\}$ is a decreasing sequence of compact subsets of Ω. Let K be any compact subset of $\Omega \backslash E$, and let $n_0 = n_0(K)$ be an integer such that $E_n \cap K = \emptyset$ for every $n \geq n_0$. Let $\{v_n\}$ be the sequence of positive solutions of

$$\begin{cases} L v_n = f_n & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $E_n \subset E_{n_0}$ for every $n \geq n_0$ then $L v_n = 0$ in $\Omega \backslash E_{n_0}$ for every $n \geq n_0$. By Harnack inequality (see [64], Section 8), there exists a positive constant $c = c(\theta, \Theta, K, \Omega \backslash E_{n_0})$ such that $\sup_K v_n \leq c \inf_K v_n$; using (4.4.4), and Sobolev embedding,

$$\sup_K v_n \leq \frac{c}{\mathcal{L}(K)} \int_K v_n \, dx \leq c_1 \|v_n\|_{L^1(\Omega)} \leq c_2 \|f_n\|_{L^1(\Omega)} .$$

Since $u_n \leq v_n$ almost everywhere in Ω by Theorem 1.3.6, we have

$$\|u_n\|_{L^\infty(K)} \leq \sup_K v_n \leq c(K) , \tag{4.4.8}$$

and so u belongs to $L^\infty(K)$ for every compact subset K of $\Omega \backslash E$.

Let K and $n_0 = n_0(K)$ be as before, and consider a compact set K' with $K \subset\subset K' \subset\subset \Omega \backslash E_{n_0}$. Let $\alpha \in C_0^\infty(K')$ be such that $0 \leq \alpha \leq 1$ everywhere, $\alpha \equiv 1$ on K, and set $\alpha = 0$ in $\Omega \backslash K'$. Choosing $\alpha^2 u_n$ as test function in (4.4.7), we obtain, by (1.3.1), by Young inequality, and by (4.4.8),

$$\int_{K'} |Du_n|^2 \alpha^2 \, dx + \int_{K'} u_n^2 \alpha^2 \, d\mu \leq c(K') \int_{K'} |D\alpha|^2 u_n^2 \, dx \leq c(K') .$$

Since α is positive, and $\alpha \equiv 1$ on K we have

$$\int_K |Du_n|^2 \, dx + \int_K u_n^2 \, d\mu \leq c(K') ,$$

which implies that u belongs to $H^1(K) \cap L^2(K, \mu)$, and it is the weak limit of $\{u_n\}$ in the same space. Choosing as test function in the variational formulation of (4.4.7) any function φ belonging to $H^1(\Omega \backslash E) \cap L^2(\Omega \backslash E, \mu)$, with $\operatorname{supp} \varphi \subset \Omega \backslash E$, and passing to the limit as n goes to infinity, we obtain that u is a local solution of the equation $L u + \mu u = 0$ in $\Omega \backslash E$.

To conclude the proof, it suffices to observe that, by linearity, $u = v + w$, where v and w are the solutions of

$$\begin{cases} Lv + \mu v = \nu \llcorner \Omega' & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} Lw + \mu w = \nu - \nu \llcorner \Omega' & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

and that $\operatorname{supp}(\nu - \nu \llcorner \Omega') = \operatorname{supp}\nu \backslash \Omega'$. □

It was proved in [33], [34] and [41] that, even if the datum f is smooth, the presence of the measure μ in (4.4.3) may produce discontinuity points for the solutions in Ω. Hence, we do not have, *a priori*, a standard way to provide a pointwise value for a solution u of (4.4.3). On the other hand, since u belongs to $H_0^{1,q}(\Omega)$, then it has a pointwise value, but defined up to sets of q-capacity zero. The following result states that u can be also defined pointwise outside N_ν, a set of 2-capacity zero, and that this value has some useful properties.

Theorem 4.4.10 *Let μ be a measure of $\mathcal{M}_0(\Omega)$, and let Ω' be a bounded open set such that $\Omega \subset\subset \Omega'$. Let ν be a positive measure in $\mathcal{M}_b(\Omega)$ with compact support in Ω, let u and v be the solutions of*

$$\begin{cases} Lu + \mu u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} Lv = \nu \llcorner \Omega & \text{in } \Omega', \\ v = 0 & \text{on } \partial\Omega', \end{cases}$$

and define $\tilde{u}$ the prolongation of u to zero in $\Omega' \backslash \Omega$. Then there exists a positive measure $\gamma = \gamma(\Omega', \nu)$, with $\operatorname{supp}\gamma \subseteq \overline{\Omega}$, such that $L(v - \tilde{u}) = \gamma$ in Ω' in the sense of Definition 4.3.3. Thus, u will be defined pointwise in $\Omega \backslash N_\nu$ as

$$u(x) = \int_{\Omega'} G'(x,y)\, d(\nu \llcorner \Omega)(y) - \int_{\Omega'} G'(x,y)\, d\gamma(y),$$

where G' is the Green function of the operator L in Ω'. Moreover, this pointwise value coincides with the limit of its convolutions with every sequence of positive, spherically symmetric, δ-approximating convolution kernels with compact support, and does not depend on the choice of Ω'.

Proof. Let us consider the solutions u_n and v_n of

$$\begin{cases} Lu_n + \mu u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} Lv_n = f_n & \text{in } \Omega', \\ v_n = 0 & \text{on } \partial\Omega', \end{cases}$$

where $\{f_n\}$ is, as in Lemma 4.3.10, the sequence of positive $C_0^\infty(\Omega)$ functions that approximates ν in the weak* topology of measures. By Theorem 4.4.7 and Lemma 4.3.7, the sequences $\{u_n\}$ and $\{v_n\}$ converge respectively to u and v. By Theorem 1.3.6, $L u_n \leq f_n$ in the sense of distributions in Ω, and so, by the result of [9], Appendix A, $L \tilde{u}_n \leq f_n$ in the sense of distributions in Ω', where $\tilde{u}_n$ is the prolongation to zero in Ω' of u_n. Thus, there exists a positive measure γ_n, with $\operatorname{supp}\gamma_n \subseteq \overline{\Omega}$, such that $L(v_n - \tilde{u}_n) = \gamma_n$ in Ω', in the sense of Definition 4.3.3. By Lemma 4.3.12(ii), there exists a measure γ such that $L(v - \tilde{u}) = \gamma$ in the sense of Definition 4.3.3. Moreover, one can easily check that $\operatorname{supp}\gamma \subseteq \overline{\Omega}$.

As last step, in order to prove that u is well defined outside N_ν, we show that $N_\nu = N'_\nu$, where N'_ν is the set of points $x \in \Omega$ such that $\int_{\Omega'} G'(x,y)\, d(\nu \llcorner \Omega)(y) = +\infty$. Actually, if w is such that $Lw = \nu$ in the sense of Definition 4.3.3 in Ω, then, by the result of [24] mentioned above, $L\tilde{w} \leq \nu$ in the sense of distributions on Ω', where $\tilde{w}$ is the prolongation of w to zero in $\Omega' \setminus \Omega$. Hence, the solution of $L w_1 = \nu \llcorner \Omega$ in the sense of Definition 4.3.3 in Ω' can be written as $\tilde{w}(x) + \int_{\Omega'} G'(x,y)\, d\gamma(y)$, where γ is the positive measure of $\mathcal{M}_b(\Omega')$, with support contained in $\partial\Omega$, such that $L(w_1 - \tilde{w}) = \gamma$ in the sense of Definition 4.3.3 in Ω'. If $x \in \Omega$, then $\int_{\Omega'} G'(x,y)\, d\gamma(y)$ is finite, since γ has support in $\partial\Omega$. Hence, $N_\nu = N'_\nu$.

If Ω'' is another open subset of $\mathbf{R}^N$, with $\Omega \subset\subset \Omega''$, and G'' is the Green function of L in Ω'', then $\int_{\Omega'} G'(x,y)\, d(\nu \llcorner \Omega)(y) - \int_{\Omega'} G'(x,y)\, d\gamma'(y)$ and $\int_{\Omega''} G''(x,y)\, d(\nu \llcorner \Omega)(y) - \int_{\Omega''} G''(x,y)\, d\gamma''(y)$ coincide almost everywhere in Ω, since they both solve the equation satisfied by u. On the other hand, they are both equal in $\Omega \setminus N_\nu$ to the limit of their convolutions, and so they are equal on this set. □

Remark 4.4.11 If $\operatorname{supp}\mu$ is compact in Ω, we can choose $\Omega' = \Omega$, since $\operatorname{supp}\gamma_n \subseteq \operatorname{supp}\mu$. The proof is the same, but we do not need to apply the result of [24].

Our next result proves that a solution of (4.4.3) is continuous outside the support of the datum ν if μ is a Kato measure.

Theorem 4.4.12 *Let μ be a Kato measure, ν a measure of $\mathcal{M}_b(\Omega)$ with* *compact support in* Ω, $f \in L^\infty(\Omega)$, *and let* u *be the solution of*

$$\begin{cases} Lu + \mu u = \nu + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense of Definition 4.4.4. If γ *is the measure given by Theorem 4.4.10, and if* $E = \operatorname{supp}\nu$, *then* $\gamma \llcorner (\Omega\backslash E) = u\,\mu \llcorner (\Omega\backslash E)$, *and* u *is a continuous function on* $\Omega\backslash E$.

Proof. By Theorem 4.4.9 u is a local solution of $Lu + \mu u = f$ in $\Omega\backslash E$. Thus, since $C_0^1(\Omega\backslash E) \subset H_0^1(\Omega\backslash E) \cap L^2(\Omega\backslash E, \mu)$,

$$\int_\Omega A\, Du D\psi\, dx = \int_\Omega f\,\psi\, dx - \int_\Omega u\,\psi\, d\mu \qquad \forall \psi \in C_0^1(\Omega\backslash E).$$

By Lemma 4.3.7, if v is the solution of $Lv = \nu + f$ in the sense of Definition 4.3.3, then

$$\int_\Omega A\, D(v-u) D\psi\, dx = \int_\Omega \psi\, d\gamma, \qquad \int_\Omega A\, Dv D\psi\, dx = \int_\Omega f\,\psi\, dx,$$

for every $\psi \in C_0^1(\Omega\backslash E)$. Thus, $\int_\Omega \psi\, d\gamma = \int_\Omega \psi\, u\, d\mu$ for every $\psi \in C_0^1(\Omega\backslash E)$, and so $\gamma \llcorner (\Omega\backslash E) = u\,\mu \llcorner (\Omega\backslash E)$. Moreover, since u belongs to $L^\infty(\Omega\backslash E)$, $f - u\,\mu \llcorner (\Omega\backslash E)$ is a Kato measure, and so

$$w(x) = \int_\Omega G(x,y)\, f(y)\, dy - \int_\Omega G(x,y)\, u(y)\, d\mu(y)$$

is continuous on $\Omega\backslash E$. On the other hand, by Theorem 4.4.10, and by the fact that $N_\nu \cap (\Omega\backslash E) = \emptyset$, $u(x) = \int_\Omega G(x,y)\, f(y)\, dy - \int_\Omega G(x,y)\, d\gamma(y)$, and so $u \equiv w$ on $\Omega\backslash E$.

□

Remark 4.4.13 Applying the preceding result with $\nu = 0$, we have that $R_\mu(f)$ is continuous on Ω for every f in $L^\infty(\Omega)$ if μ is a Kato measure.

In the following, we prove the analogous result of Theorem 1.3.6 for non-variational relaxed Dirichlet problems.

Theorem 4.4.14 *Let μ_1 and μ_2 in $\mathcal{M}_0(\Omega)$, with $\mu_1 \le \mu_2$. Let ν_1 and ν_2 in $\mathcal{M}_b(\Omega)$, with $0 \le \nu_2 \le \nu_1$. Let u_1 and u_2 be the solutions, in the sense of Definition 4.4.4, of*

$$\begin{cases} L u_i + \mu_i u_i = \nu_i & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1,2.$$

Then $0 \le u_2(x) \le u_1(x)$ for every $x \in \Omega \backslash N_{\nu_1}$.

Proof. Let $\{f_{1,n}\}$ and $\{f_{2,n}\}$ be the sequences of $C_0^\infty(\Omega)$ functions that approximate ν_1 and ν_2 in the weak* topology of measures, and let $\{u_{1,n}\}$ and $\{u_{2,n}\}$ be the sequences of solutions of

$$\begin{cases} L u_{i,n} + \mu_i u_{i,n} = f_{i,n} & \text{in } \Omega, \\ u_{i,n} = 0 & \text{on } \partial\Omega, \end{cases} \qquad i = 1,2.$$

Then, by Theorem 1.3.6, $0 \le u_{2,n}(x) \le u_{1,n}(x)$ for almost every $x \in \Omega$; hence, by Lemma 4.3.10, for every $x \in \Omega$. Since $u_{i,n}$ converges to u_i almost everywhere by Theorem 4.4.7, then $0 \le u_1(x) \le u_2(x)$ for almost every $x \in \Omega$, and so the same inequality holds for every $x \in \Omega \backslash N_{\nu_1}$ by Theorem 4.4.10 (we remark that $N_{\nu_2} \subseteq N_{\nu_1}$ since $\nu_2 \le \nu_1$). □

4.5. Regularity results for nonvariational relaxed Dirichlet problems

Up to now we have not said anything about the regularity of the solution u of (4.4.3) with respect to the measure μ. First of all, we need an estimate of the norm in $L^1(\Omega,\mu)$ of a solution u of a variational relaxed Dirichlet problem (1.3.5).

Lemma 4.5.1 *Let μ be a measure of $\mathcal{M}_0(\Omega)$, and let u be the solution of problem (1.3.5) with $f \in L^2(\Omega)$. Then the following estimate holds:*

$$\|u\|_{L^1(\Omega,\mu)} \le \|f\|_{L^1(\Omega)}. \tag{4.5.1}$$

Proof. Let $n \in \mathbb{N}$, consider the function S_n defined by

$$S_n(s) = \begin{cases} -1 & \text{if } s \le -\frac{1}{n}, \\ ns & \text{if } -\frac{1}{n} < s < \frac{1}{n}, \\ 1 & \text{if } s \ge \frac{1}{n}, \end{cases}$$

and choose $S_n(u)$ as test function in the equation satisfied by u. This can be done by Lemma 4.2.1. We obtain, dropping the positive term that contains derivatives,

$$\int_\Omega u\, S_n(u)\, d\mu \le \int_\Omega f\, S_n(u)\, dx \le \|f\|_{L^1(\Omega)} .$$

Since $S_n(u)$ converges to $\mathrm{sgn}\,(u)$ μ-almost everywhere, we obtain the result by Fatou lemma. □

Lemma 4.5.2 *Let $\mu \in \mathcal{M}_0(\Omega)$ and let $\nu \in \mathcal{M}_b(\Omega)$ with compact support in Ω. Let u be the solution, in the sense of Definition 4.4.4, of $Lu + \mu u = \nu$. Then $u \in L^1(\Omega,\mu)$.*

Proof. Let h and γ be respectively the Borel function and the Kato measure associated to μ as in Theorem 1.2.4. By Remark 1.3.4, $R_\mu(f) = R_{h\,\gamma}(f)$ for every $f \in L^2(\Omega)$. Hence, we shall work with $h\,\gamma$. Let $\{\varphi_n\}$ be as in Theorem 4.4.7, and let $f_n = \varphi_n * \nu$. Let $u_n = R_{h\,\gamma}(f_n)$. Then, by Theorem 4.4.7, u_n strongly converges to u in $H_0^{1,q}(\Omega)$, with $1 \le q < \frac{N}{N-1}$. Let $k \in \mathbf{R}_+$, and let $v_n^k = T_k(u_n^+) = \min(u_n^+, k)$. Since v_n^k belongs to $H_0^1(\Omega) \cap L^2(\Omega, h\,\gamma)$ by Lemma 4.2.1, we can test with it the equation solved by u_n; we obtain, using (1.3.1), and dropping the positive term that contains the measure $h\,\gamma$,

$$\lambda \int_\Omega |Dv_n^k|^2\, dx \le \int_\Omega f_n\, v_n^k\, dx \le k\, \|f_n\|_{L^1(\Omega)} ,$$

so that $\{v_n^k\}$ is bounded in $H_0^1(\Omega)$ for every k. Hence it weakly converges, up to subsequences, to a function $v \in H_0^1(\Omega)$. Since u_n converges to u strongly in $L^1(\Omega)$, then $v = T_k(u^+)$. We show that v_n^k converges to v strongly in $L^1(\Omega,\gamma)$. Actually,

$$\int_\Omega |v_n^k - v|\, d\gamma = \langle \gamma, |v_n^k - v| \rangle ,$$

and the right hand side tends to zero as n tends to infinity since $\gamma \in H^{-1}(\Omega)$ and $|v_n^k - v|$ converges to 0 weakly in $H_0^1(\Omega)$. Hence, up to subsequences, v_n^k converges to v γ-almost everywhere, and so $h\,\gamma$-almost everywhere. Recalling (4.5.1), using the positivity of $h\,\gamma$ and v_n^k, and applying Fatou lemma, we obtain

$$\int_\Omega T_k(u^+)\, h\, d\gamma \le \liminf_{n\to\infty} \int_\Omega T_k(u_n^+)\, h\, d\gamma \le$$

$$\leq \liminf_{n\to\infty} \int_\Omega u_n^+ \, h \, d\gamma \leq \liminf_{n\to\infty} \|f_n\|_{L^1(\Omega)} \leq c\,,$$

so that $T_k(u^+)$ belongs to $L^1(\Omega, h\,\gamma)$ for every $k \geq 0$. Since $T_k(u^+) \in H_0^1(\Omega)$, then $\int_\Omega T_k(u^+)\,d\mu = \int_\Omega T_k(u^+)\,h\,d\gamma$, and so $T_k(u^+)$ belongs to $L^1(\Omega,\mu)$. Letting k tend to infinity implies, again by Fatou lemma, that u^+ belongs to $L^1(\Omega,\mu)$. The same computations with $v_n^k = T_k(u_n^-)$ yield that $u^- \in L^1(\Omega,\mu)$, and this concludes the proof. □

Remark 4.5.3 We note explicitly that $\int_\Omega |u|\,d\mu$ has a perfect meaning since, by Theorem 4.4.10, u is defined up to subsets of 2-capacity zero, and μ does not charge these sets.

Remark 4.5.4 In the proof of Lemma 4.5.2 we have shown that if $u_n = R_\mu(f_n)$ is a sequence of solutions of relaxed Dirichlet problems that converges to a function u weakly in $H_0^1(\Omega)$, and if h and γ are associated to μ as in Theorem 1.2.4, then, up to subsequences, u_n converges to u $h\,\gamma$-almost everywhere.

We obtain further properties on u making stronger assumptions on ν.

Theorem 4.5.5 *Let q be a real number with $1 \leq q < \frac{N}{N-1}$, let μ be a measure of $\mathcal{M}_0(\Omega)$, and suppose that ν has a density f with respect to the Lebesgue measure with f in $L^1(\Omega)$. Then the solution u of problem (4.4.3) satisfies*

$$\int_\Omega A\,DuDv\,dx + \int_\Omega u\,v\,d\mu = \int_\Omega f\,v\,dx\,,$$

for every v in $H_0^{1,q'}(\Omega) \cap L^2(\Omega,\mu)$. Moreover, u can be uniquely obtained as the limit of a sequence of solutions of problems (1.3.5) with data f_n in $L^2(\Omega)$ converging to f in $L^1(\Omega)$.

Proof. Since the equation is linear, it is not restrictive to suppose that f is positive. Let $\{f_n\}$ be any sequence of $L^2(\Omega)$ functions that converges to f in $L^1(\Omega)$, and let u_n be the solution of problem (1.3.5) with datum f_n. Since $\{f_n\}$ is a Cauchy sequence in $L^1(\Omega)$, then, by linearity, by (4.3.2) and by (4.5.1), we have that $\{u_n\}$ is a Cauchy sequence in $H_0^{1,q}(\Omega) \cap L^1(\Omega,\mu)$. Hence there exist two functions, u and v, such that u_n converges to u strongly in

$H_0^{1,q}(\Omega)$, and u_n converges to v strongly in $L^1(\Omega,\mu)$. We claim that $u = v$ μ-almost everywhere. To prove it, fix n and m in $\mathbf{N}$, and subtract the equations satisfied by u_n and u_m; let $k \geq 0$, and choose as test function $T_k(u_n - u_m)$, where $T_k(s) = \min(k, \max(s, -k))$ for every $s \in \mathbf{R}$; using (1.3.1) we obtain, dropping the positive term that contains the measure μ,

$$\theta \int_\Omega |DT_k(u_n - u_m)|^2 \, dx \leq k \, \|f_n - f_m\|_{L^1(\Omega)} .$$

If we define, for n fixed, $v_{m,n} = T_k(u_n - u_m)$, then $\{v_{m,n}\}$ is bounded in $H_0^1(\Omega)$, and so it weakly converges (up to subsequences) to a function v_n; since u_m converges strongly in $L^1(\Omega)$ to u, then $T_k(u_n - u_m)$ converges strongly in $L^1(\Omega)$ to $T_k(u_n - u)$; hence, $v_n = T_k(u_n - u)$. By lower semicontinuity, this implies

$$\theta \int_\Omega |DT_k(u_n - u)|^2 \, dx \leq \liminf_{m \to \infty} \theta \int_\Omega |DT_k(u_n - u_m)|^2 \, dx \leq k \, \|f_n - f\|_{L^1(\Omega)} ,$$

and so, for every k, $T_k(u_n - u)$ converges strongly to zero in $H_0^1(\Omega)$; hence, it converges to zero quasi everywhere, and so μ-almost everywhere. This fact implies that $u_n - u$ converges to zero μ-almost everywhere (this can be seen by an easy contradiction argument, using the continuity of T_k); since u_n converges to v μ-almost everywhere, we have that $u = v$ μ-almost everywhere.

A passage to the limit in the approximate equations yields the result, since $\int_\Omega u_n \, v \, d\mu$ converges to $\int_\Omega u \, v \, d\mu$ because $H_0^{1,q'}(\Omega) \cap L^2(\Omega,\mu)$ is a subset of $L^\infty(\Omega,\mu)$.

Obviously, the function u does not depend on the choice of the approximating sequence. □

Remark 4.5.6 The latter result shows that the solution of a nonvariational relaxed Dirichlet problem with $L^1(\Omega)$ datum satisfies an identity similar to (4.3.4). Again, it may not be the unique function that satisfies it (the counterexample of Serrin is still valid), but it is anyway the unique that can be obtained by means of approximating techniques. For other result of this kind about classical Dirichlet problems, see e.g. [26] and [59].

If the density of ν with respect to the Lebesgue measure is more summable than $L^1(\Omega)$, then the solution u of (4.4.3) is more regular.

Theorem 4.5.7 *Let $N \geq 3$. Let m be such that $1 < m < \frac{2N}{N+2}$. Assume that ν has a density f with respect to the Lebesgue measure $\mathcal{L}$, with f in $L^m(\Omega)$, and that μ is a measure of $\mathcal{M}_0(\Omega)$. Then the solution u of (4.4.3) belongs to $H_0^{1,m^*}(\Omega) \cap L^p(\Omega, \mu)$, with $p = \frac{N-2}{N} m^{**}$.*

Proof. Let $\{f_n\}$ be a sequence of $L^2(\Omega)$ functions converging to f in $L^m(\Omega)$ and let $\{u_n\}$ be the sequence of solutions of problem (1.3.5) with f_n as data. Let r be a real number such that $0 < r < 1$ and define $v_n = ((1 + |u_n|)^{1-r} - 1)\operatorname{sgn}(u_n)$. Choosing v_n as test function in the approximate problems and using (1.3.1) we obtain the following estimates:

$$\theta\,(1-r) \int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^r}\, dx \leq \int_\Omega f_n\, v_n\, dx\,, \tag{4.5.2}$$

$$\int_\Omega |u_n|\,((1+|u_n|)^{1-r} - 1)\, d\mu \leq \int_\Omega f_n\, v_n\, dx\,. \tag{4.5.3}$$

From (4.5.2) we obtain, working as in [59], Lemma 2.1, that $\{u_n\}$ is is a bounded sequence in $H_0^{1,m^*}(\Omega)$, and that $\int_\Omega f_n\, v_n\, dx \leq c$, for some positive constant c. This is achieved by choosing r in such a way that $2 - r = p$. With this choice of r we have, by (4.5.3) and by Lemma 4.5.1,

$$\begin{aligned}\int_\Omega |u_n|^p\, d\mu &\leq \int_\Omega |u_n|(1+|u_n|)^{1-r}\, d\mu = \\ &= \int_\Omega |u_n|((1+|u_n|)^{1-r} - 1)\, d\mu + \int_\Omega |u_n|\, d\mu \leq c\,,\end{aligned}$$

and this yields our result. □

Remark 4.5.8 We point out that the regularity of the solution with respect to μ is the same obtained in the variational case (see Theorem 4.2.5).

4.6. The Green function for relaxed Dirichlet problems

Definition 4.6.1 *Let μ be a measure in $\mathcal{M}_0(\Omega)$. We call μ-Green function for the operator L the solution $G_\mu(x,\cdot)$, $x \in \Omega$, given by Theorem 4.4.6, of the nonvariational relaxed Dirichlet problem*

$$\begin{cases} L^* G_\mu(x,\cdot) + \mu\, G_\mu(x,\cdot) = \delta_x & \text{in } \Omega, \\ G_\mu(x,\cdot) = 0 & \text{on } \partial\Omega. \end{cases}$$

We define $G^*_\mu(x,\cdot)$ the μ-Green function for the operator L^*.

The following result is technical, and will be used in the proof of the symmetry formula for the μ-Green function.

Lemma 4.6.2 *Let $\mu \in \mathcal{M}_0(\Omega)$, and let $\{g_n\}$ be an increasing sequence of positive Borel measurable functions on Ω that converges to 1 everywhere in Ω, possibly except the set E of the points $x \in \Omega$ such that $R_\mu(g)(x) = 0$ for every g in $L^\infty(\Omega)$. Then $R_{g_n\mu}(f)$ converges to $R_\mu(f)$ strongly in $L^2(\Omega)$ for every f in $L^\infty(\Omega)$.*

Proof. Since the equation is linear we can limit ourselves to the case of a positive f. If h and γ are respectively a Borel function and a Kato measure as in Theorem 1.2.4, then, by Remark 1.3.4, $R_\mu(f) = R_{h\,\gamma}(f)$ and $R_{g_n\mu}(f) = R_{g_n h\,\gamma}$ for every $f \in L^2(\Omega)$. Hence, we will define $\mu_n = g_n h\,\gamma$, and prove the result for μ_n and $h\,\gamma$. Let us define $u_n = R_{\mu_n}(f)$. By Theorem 1.3.2, $\{u_n\}$ is bounded in $H^1_0(\Omega)$, and so it converges weakly in the same space, and strongly in $L^2(\Omega)$, to a function u. Since $L^2(\Omega, h\,\gamma) \subseteq L^2(\Omega, \mu_n)$, from the variational formulation of the problem solved by u_n we have that

$$\int_\Omega A\,Du_n Dv\,dx + \int_\Omega u_n\,v\,d\mu_n = \int_\Omega f\,v\,dx \qquad \forall v \in H^1_0(\Omega) \cap L^2(\Omega, h\,\gamma).$$

The first term converges to $\int_\Omega A\,DuDv\,dx$. We are going to show that $\int_\Omega u_n\,v\,d\mu_n$ converges to $\int_\Omega u\,v\,h\,d\gamma$, so that, by uniqueness, $u = R_{h\,\gamma}(f) = R_\mu(f)$. Recalling the definition of μ_n, and using the fact that u belongs to $L^2(\Omega, \mu_n)$ for every n since $0 \le u \le u_n$, and that it is zero on E, we have

$$\int_\Omega u_n\,v\,d\mu_n - \int_\Omega u\,v\,h\,d\gamma = \int_\Omega (u_n - u)\,g_n\,v\,h\,d\gamma + \int_\Omega u\,v\,(g_n - 1)\,h\,d\gamma\,,$$

and the last term tends to zero by Lebesgue theorem. Since u_n converges to u weakly in $H_0^1(\Omega)$, then, by Remark 4.5.4, and up to subsequences, u_n converges to u $h\gamma$-almost everywhere, so that $(u_n - u)g_n$ tends to 0 $h\gamma$-almost everywhere. Moreover, since $g_n \leq 1$ and $\|u_n\|_{L^2(\Omega,\mu_n)}$ is bounded,

$$\int_\Omega (u_n - u)^2 g_n^2 \, h \, d\gamma \leq \int_\Omega (u_n - u)^2 g_n \, h \, d\gamma \leq c\,.$$

Standard measure theory arguments imply that $(u_n - u)g_n$ tends weakly to 0 in the space $L^2(\Omega, h\gamma)$, and this concludes the proof. □

Remark 4.6.3 If the operator L is symmetric, the result of Lemma 4.6.2 follows from the theory of γ-convergence of measures in $\mathcal{M}_0(\Omega)$ (see [33] and [34]).

Theorem 4.6.4 *The function G_μ is upper semicontinuous outside the diagonal in each variable, and*

$$G_\mu(x,y) = G_\mu^*(y,x) \qquad \forall x, y \in \Omega, \; x \neq y\,.$$

Proof. First of all we prove that, for every $y \in \Omega$, there exists a set $E(y) \subset \Omega$ such that $\mathcal{L}(E(y)) = 0$, and $G_\mu(x,y) = G_\mu^*(y,x)$ for every $x \in \Omega \setminus E(y)$. Actually, observing that $\frac{\mathbf{1}_{B_r(0)}}{\mathcal{L}(B_r(0))}$ is a positive, spherically symmetric, δ-approximating convolution kernel with compact support, we have, by Theorem 4.4.10 applied with $\nu = \delta_x$, by the definition of G_μ, and by Theorem 4.4.7,

$$\begin{aligned} G_\mu(x,y) &= \lim_{r \to 0^+} \frac{1}{\mathcal{L}(B_r(y))} \int_{B_r(y)} G_\mu(x,z)\,dz = \\ &= \lim_{r \to 0^+} R_\mu \left(\frac{\mathbf{1}_{B_r(y)}}{\mathcal{L}(B_r(y))} \right)(x) = G_\mu^*(y,x)\,, \end{aligned}$$

and the last equality holds for almost every $x \in \Omega$. Exchanging the rôles of G_μ and G_μ^*, we have that for every $x \in \Omega$ there exists a set $E(x)$, with $\mathcal{L}(E(x)) = 0$, such that $G_\mu(x,y) = G_\mu^*(y,x)$ for every $y \in \Omega \setminus E(x)$.

Suppose that μ is a Kato measure on Ω, with $\operatorname{supp}\mu \subset\subset \Omega$. Then, by Theorem 4.4.12 applied with $\nu = \delta_x$ and $f = 0$, for every fixed $x \in \Omega$, $G_\mu(x,\cdot)$ and $G^*_\mu(x,\cdot)$ are continuous in $\Omega\backslash\{x\}$.

Moreover, for every fixed $y \in \Omega$, $G_\mu(\cdot,y)$ and $G^*_\mu(\cdot,y)$ are upper semicontinuous in $\Omega\backslash\{y\}$. Actually, if x is a fixed point in Ω, and $\{x_n\}$ is a sequence in Ω that converges to x as n tends to infinity, then, by (4.4.4), $\{G_\mu(x_n,\cdot)\}$ is bounded in $H^{1,q}_0(\Omega)$ for every $1 \le q < \frac{N}{N-1}$, and so there exists a function f such that $G_\mu(x_n,\cdot)$ converges to f in the strong topology of $L^1(\Omega)$. Passing to the limit in the identities $\int_\Omega G_\mu(x_n,y)\, g(y)\, dy = R_\mu(g)(x_n)$, that hold for every g in $L^\infty(\Omega)$, and since $R_\mu(g)$ is continuous by Remark 4.4.13, we obtain

$$\int_\Omega f(y)\, g(y)\, dy = R_\mu(g)(x) \qquad \forall g \in L^\infty(\Omega)\,,$$

so that $G_\mu(x,y) = f(y)$ for almost every $y \in \Omega$. From Theorem 4.4.10, and Remark 4.4.11, it follows that there exists a sequence $\gamma_n = \gamma(x_n)$ of positive measures with $\operatorname{supp}\gamma_n \subseteq \operatorname{supp}\mu$, and a positive measure $\gamma = \gamma(x)$, such that

$$L^*(G(x_n,\cdot) - G_\mu(x_n,\cdot)) = \gamma_n\,,$$

$$L^*(G(x,\cdot) - G_\mu(x,\cdot)) = L^*(G(x,\cdot) - f(\cdot)) = \gamma\,,$$

in the sense of Definition 4.3.3. By Lemma 4.3.12(ii), γ_n converges to γ in the weak* topology of measures. Thus, since $G(y,\cdot)$ is positive and continuous, by Theorem 4.4.10 and Theorem 4.1.3 we have,

$$\begin{aligned}\liminf_{n\to\infty}\,(G(x_n,y) - G_\mu(x_n,y)) &= \liminf_{n\to\infty}\int_\Omega G(y,z)\, d\gamma_n(z)\\ &\ge \int_\Omega G(y,z)\, d\gamma(z) = G(x,y) - G_\mu(x,y)\,.\end{aligned}$$

Since $G(\cdot,y)$ is continuous and finite in $\Omega\backslash\{y\}$, then $G_\mu(\cdot,y)$ is upper semicontinuous in $\Omega\backslash\{y\}$.

Let $(x_0,y_0) \in \Omega\times\Omega$, $x_0 \ne y_0$. Let $\{x_n\}$ be a sequence of points in Ω whose limit is x_0, and such that $G_\mu(x_n,y_0) = G^*_\mu(y_0,x_n)$ for every $n \in \mathbf{N}$. Passing to the limit, and using the properties of G_μ and G^*_μ, we obtain $G_\mu(x_0,y_0) \ge G^*_\mu(y_0,x_0)$. The opposite inequality is obtained identically, choosing a sequence $\{y_n\}$ whose limit is y_0.

Let $\mu \in \mathcal{M}_0(\Omega)$, with $\operatorname{supp}\mu \subset\subset \Omega$. By Remark 1.3.4, $R_\mu(f) = R_{h\nu}(f)$ for every $f \in L^2(\Omega)$, where h and ν are respectively a Borel measurable function and a Kato measure as in Theorem 1.2.4. Thus, we will work with $h\,\nu$. Let us consider the sequence $\{\mu_n\}$ of Kato measures defined by $\mu_n = h_n\,\nu$, where $h_n(x) = T_n(h(x)) = \min(h(x), n)$. Obviously, $\{\mu_n\}$ is an increasing sequence of measures whose limit is $h\,\nu$, and $\operatorname{supp}\mu_n \subseteq \operatorname{supp}(h\,\nu)$. By Theorem 4.4.14, for every fixed $x \in \Omega$, $\{G_{\mu_n}(x,\cdot)\}$ is a decreasing sequence of functions, whose pointwise limit we will denote with f. Since, by (4.4.4), $\{G_{\mu_n}(x,\cdot)\}$ is bounded in $H_0^{1,q}(\Omega)$ for every $1 \le q < \frac{N}{N-1}$, then $G_{\mu_n}(x,\cdot)$ converges to f weakly in the same space, and strongly in $L^1(\Omega)$. Moreover, G_{μ_n} is such that

$$\int_\Omega G_{\mu_n}(x,y)\,g(y)\,dy = R_{\mu_n}(g)(x) \qquad \forall g \in L^\infty(\Omega),\ \forall x \in \Omega\,. \tag{4.6.1}$$

If we define $g_n = h_n/h$, we have that g_n converges to 1 on $\Omega\setminus E$, where $E = \{x \in \Omega : h(x) = +\infty\}$, and that $R_\mu(g) = 0$ in E, for every g in $L^\infty(\Omega)$. Thus, applying Lemma 4.6.2, we conclude that $R_{\mu_n}(g)$ converges to $R_\mu(g)$ strongly in $L^2(\Omega)$, and almost everywhere; moreover, if g is positive (which is not restrictive since the equation is linear), Theorem 1.3.6 and the continuity of R_{μ_n} imply that $R_{\mu_n}(g)$ decreases everywhere to a upper semicontinuous function η. Furthermore, by Theorem 4.4.2, there exists a sequence $\{\gamma_n\}$ of positive measures, and a positive measure γ, such that

$$L\,(R(g) - R_{\mu_n}(g)) = \gamma_n, \qquad L(R(g) - R_\mu(g)) = L(R(g) - \eta) = \gamma\,,$$

in the sense of Definition 4.3.3. By Lemma 4.3.12(ii), γ_n converges to γ in the weak* topology of measures. Thus, $w_n = R(g) - R_{\mu_n}(g)$ and $w = R(g) - \eta$ satisfy the hypotheses of Lemma 4.3.12(iii), and so $\eta(x) = R_\mu(g)(x)$ for every x in Ω. Passing to the limit in (4.6.1) yields

$$\int_\Omega f(y)\,g(y)\,dy = R_\mu(g)(x) \qquad \forall g \in L^\infty(\Omega),\ \forall x \in \Omega\,,$$

which implies that $f(y) = G_\mu(x,y)$ for almost every $y \in \Omega$. It is easily seen that $w_n = G(x,\cdot) - G_{\mu_n}(x,\cdot)$ and $w = G(x,\cdot) - f(\cdot)$ satisfy the hypotheses of Lemma 4.3.12(iii), and so $f(y) = G_\mu(x,y)$ for every y in Ω, $y \neq x$.

Thus, $G_{\mu_n}(x,y)$ tends to $G_\mu(x,y)$ as n tends to infinity, and the same holds for $G^*_{\mu_n}(x,y)$; since $G_{\mu_n}(x,y) = G^*_{\mu_n}(y,x)$ for every x and y, $x \neq y$, we obtain the result for a measure μ in $\mathcal{M}_0(\Omega)$, with compact support in Ω. Moreover, both G_μ and G^*_μ are upper semicontinuous functions of their arguments.

Let μ be a measure in $\mathcal{M}_0(\Omega)$, let $\{K_n\}$ be an increasing sequence of compact subsets of Ω such that $\bigcup K_n = \Omega$, and define $\mu_n = \mu \llcorner K_n$. Since μ_n increases to μ, then, for every fixed x in Ω, there exists a function f such that $G_{\mu_n}(x,\cdot)$ decreases to $f(\cdot)$. By means of Theorem 4.4.10, Lemma 4.3.12(iii) applied in some Ω', with $\Omega \subset\subset \Omega'$, and Lemma 4.6.2 applied with $g_n = \mathbf{1}_{K_n}$, we show as before that $f(y) = G_\mu(x,y)$ for every y in Ω, $y \neq x$, and so the theorem follows. □

Remark 4.6.5 If $\{\varphi_n\}$ is as in the statement of Lemma 4.3.10, we have, as a consequence of this theorem,

$$G_\mu(x,y) = \lim_{n\to\infty} \int_\Omega G_\mu(z,y)\,\varphi_n(x-z)\,dz \qquad \forall x,\ y \in \Omega,\ x \neq y,$$

as can be checked applying Theorem 4.4.10.

Theorem 4.6.6 *Let $\mu \in \mathcal{M}_0(\Omega)$ and $\nu \in \mathcal{M}_b(\Omega)$ be a positive measure with compact support in Ω. Then the pointwise value in $\Omega\backslash N_\nu$, given by Theorem 4.4.10, of the solution u of the nonvariational relaxed Dirichlet problem (4.4.3) with ν as datum is such that*

$$u(x) = \int_\Omega G_\mu(x,y)\,d\nu(y) \qquad \forall x \in \Omega\backslash N_\nu, \tag{4.6.2}$$

so that $\int_\Omega G_\mu(x,y)\,d\nu(y)$ gives a representative of the solution u in its almost everywhere equivalence class.

Proof. Let g be a function in $L^\infty(\Omega)$. Then, by definition of G^*_μ,

$$R^*_\mu(g)(x) = \int_\Omega G^*_\mu(x,y)\,g(y)\,dy \qquad \forall x \in \Omega,$$

and so, by Fubini theorem, and by Theorem 4.6.4,

$$\int_\Omega u(x)\, g(x)\, dx = \int_\Omega R^*_\mu(g)(x)\, d\nu(x) =$$

$$= \int_\Omega \left(\int_\Omega G^*_\mu(x,y)\, g(y)\, dy \right) d\nu(x) = \int_\Omega \left(\int_\Omega G_\mu(y,x)\, d\nu(x) \right) dy\,,$$

so that (4.6.2) holds for almost every $x \in \Omega$. Let $v(x) = \int_\Omega G_\mu(x,y)\, d\nu(y)$ for every $x \in \Omega$, and let $x \in \Omega \setminus N_\nu$. We have

$$\int_\Omega v(y)\, \varphi_n(x-y)\, dy = \int_\Omega \left(\int_\Omega G_\mu(y,z)\, \varphi_n(x-y)\, dy \right) d\nu(z)\,,$$

and, by Remark 4.6.5, $\int_\Omega G_\mu(y,z)\, \varphi_n(x-y)\, dy$ converges to $G_\mu(x,z)$ for every $z \neq x$. Since $\nu(\{x\}) = 0$, then $\int_\Omega G_\mu(y,z)\, \varphi_n(x-y)\, dy$ converges to $G_\mu(x,z)$ ν-almost everywhere. Moreover, by Theorem 4.4.14, and by the result of [64] mentioned in the proof of Lemma 4.3.10, there exists a positive constant c such that

$$\int_\Omega G_\mu(y,z)\, \varphi_n(x-y)\, dy \le \int_\Omega G(y,z)\, \varphi_n(x-y)\, dy \le c^2\, G(x,z) \qquad \forall z \in \Omega\,,$$

and $G(x,\cdot)$ belongs to $L^1(\Omega,\nu)$ if $x \in \Omega \setminus N_\nu$. Hence, by Lebesgue theorem,

$$\lim_{n\to\infty} \int_\Omega v(y)\, \varphi_n(x-y)\, dy = \int_\Omega G(x,z)\, d\nu(z) = v(x)$$

for every $x \in \Omega \setminus N_\nu$, and so $u = v$ on the same set. □

Remark 4.6.7 We have always supposed that the datum ν has compact support in Ω. This assumption can be removed. Indeed, fixed a bounded open set $\Omega' \supset\supset \Omega$, for every $\mu \in \mathcal{M}_0(\Omega)$ we can consider the measure $\mu_1 = \mu + \infty_{\Omega' \setminus \Omega}$. It can be easily seen that for every measure $\nu \in \mathcal{M}_b(\Omega)$, if we set $\nu_1(B) = \nu(B \cap \Omega)$ for every Borel set B, a function u is the solution to the problem

$$\begin{cases} Lu + \mu_1 u = \nu_1 & \text{in } \Omega', \\ u = 0 & \text{on } \partial\Omega', \end{cases}$$

if and only if $u = 0$ q.e. in $\Omega' \setminus \Omega$ and it is the solution to the problem

$$\begin{cases} Lu + \mu u = \nu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since ν has compact support in Ω' all the results remain valid.

Chapter 5
Asymptotic behaviour of Dirichlet problems with measure data in perforated domains

In this chapter we study the asymptotic behaviour of the solutions of elliptic equations with measure data and with Dirichlet boundary conditions in perforated domains. We prove that if a sequence $\{\mu_h\}$ of measures of $\mathcal{M}_0(\Omega)$ γ^L-converges to a measure μ, then for every measure ν with bounded variation in Ω and which does not charge polar sets, the sequence $\{v_h\}$ of solutions to the problems

$$(5.0.1) \qquad \begin{cases} Lv_h + \mu_h v_h = \nu & \text{in } \Omega, \\ v_h = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a limit v in the weak topology of $H_0^{1,p}(\Omega)$, $1 \leq p < \frac{N}{N-1}$, and v coincides with the solution v_μ of

$$\begin{cases} Lv_\mu + \mu v_\mu = \nu & \text{in } \Omega, \\ v_\mu = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand we prove that, if the operator L has regular coefficients and and the limit measure μ has a density f with respect to the Lebesgue measure, with $f \in L^p(\Omega)$, $p > N/2$, then for every measure ν with bounded variation in Ω, there exists a subsequence $\{v_{h_k}\}$ of the sequence $\{v_h\}$ of the solutions solutions to the problems (5.0.1) which converges to the solution to the problem

$$\begin{cases} Lv + \mu v = \lambda & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega_h, \end{cases}$$

where λ is a measure with bounded variation in Ω depending on ν and on the subsequence $\{\mu_{h_k}\}$.

Since the solutions of problems with right–hand side measure are characterized by a duality identity, the method of oscillating test functions used in [30] to prove the compactness result cannot be applied. Our approach deals with a corrector result (see Section 5.2).

5.1. Data not charging polar sets

Let $\{\mu_h\}$ be a sequence of measures of $\mathcal{M}_0(\Omega)$ which γ^L-converges to a measure $\mu \in \mathcal{M}_0(\Omega)$, and let $\nu \in \mathcal{M}^b(\Omega)$. Let $\{v_{\mu_h}\}$ be the sequence of the solutions in the sense of Definition 4.4.4 to the problems

$$\begin{cases} Lv_{\mu_h} + \mu_h v_{\mu_h} = \nu & \text{in } \Omega, \\ v_{\mu_h} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1.1}$$

By the very definition of γ^L-convergence, for every $G \in H^{-1}(\Omega)$ the sequence $\{u_{\mu_h}\}$ of solutions of the variational relaxed Dirichlet problems with G as datum converges weakly in $H_0^1(\Omega)$ to u_μ, solution of (1.3.5). We want to investigate the asymptotic behaviour of the solutions of the non-variational relaxed Dirichlet problems corresponding to the measure μ_h and with the measure ν as datum. By the continuity estimate (4.4.4) the sequence $\{v_{\mu_h}\}$ is bounded in $H_0^{1,p}(\Omega)$ so that it admits a subsequence weakly converging to a function $v \in H_0^{1,p}(\Omega)$. We shall show in Section 5.3 that in general we cannot expect that v solves the relaxed Dirichlet problem corresponding to μ and with ν as datum. Nevertheless this occurs whenever ν belongs to $\mathcal{M}_0(\Omega)$.

Theorem 5.1.1 *Let $\{\mu_h\}$ be a sequence of measures of $\mathcal{M}_0(\Omega)$ which γ^L-converges to a measure $\mu \in \mathcal{M}_0(\Omega)$, and let $\nu \in \mathcal{M}_0^b(\Omega)$. Then the sequence $\{v_{\mu_h}\}$ of solutions of (5.1.1) converges weakly in $H_0^{1,p}(\Omega)$, $1 \le p < \frac{N}{N-1}$, to the solution v_μ to the problem*

$$\begin{cases} Lv_\mu + \mu v_\mu = \nu & \text{in } \Omega \\ v_\mu = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1.2}$$

Proof. Thanks to the linearity of the problem it is not restrictive to suppose that ν is positive. Since $\|v_{\mu_h}\|_{H_0^{1,p}(\Omega)} \le c_p \nu(\Omega)$ for every $1 \le p < \frac{N}{N-1}$, there exists a subsequence, still denoted by $\{v_{\mu_h}\}$, weakly converging in the same space to a function v. We want to prove that v coincides with the solution v_μ of (5.1.2). Let us consider the Borel function f and the measure $\lambda \in H^{-1}(\Omega)$ as in Theorem 1.2.4, and for every $k \in \mathbf{N}$ let us define the measure $\nu_k = (f \wedge k)\lambda \in H^{-1}(\Omega)$, where $f(x) \wedge k = \min(f(x), k)$ for every

$x \in \Omega$. By the dominate convergence theorem, for every continuous function φ with compact support in Ω and such that $|\varphi| \leq 1$ we have

$$\lim_{k\to\infty} \left| \int_\Omega \varphi \, d\nu_k - \int_\Omega \varphi \, d\nu \right| \leq \lim_{k\to\infty} \int_\Omega |f \wedge k - f| \, d\lambda = 0$$

so that $\lim_{k\to\infty} |\nu_k - \nu|(\Omega) = 0$. Thus the solutions v^k of the variational relaxed Dirichlet problems

$$\begin{cases} Lv^k + \mu v^k = \nu_k & \text{in } \Omega \\ v^k = 0 & \text{on } \partial\Omega. \end{cases}$$

converge weakly in $H_0^{1,p}(\Omega)$, $1 \leq p < \frac{N}{N-1}$, to the solution v_μ of (5.1.2). Indeed for every $k \in \mathbf{N}$ and for every $g \in L^\infty(\Omega)$ we have

$$\int_\Omega v^k g \, dx = \int_\Omega R_\mu^*(g) \, d\nu_k \,, \tag{5.1.3}$$

and by the continuity estimate (4.4.4) $\|v^k\|_{H_0^{1,p}(\Omega)} \leq c|\nu_k|(\Omega) \leq c_1$, so that there exists a subsequence converging to a function z in the weak topology of $H_0^{1,p}(\Omega)$. Moreover

$$\left| \int_\Omega R_\mu^*(g) \, d\nu_k - \int_\Omega R_\mu^*(g) \, d\nu \right| \leq$$
$$\int_\Omega |R_\mu^*(g)| \, d|\nu_k - \nu| \leq \|R_\mu^*(g)\|_{L^\infty(\Omega)} |\nu_k - \nu|(\Omega) \,,$$

and the last term tends to zero as k goes to $+\infty$. Thus we can pass to the limit in (5.1.3), getting $v_\mu = z$.

Let us consider the solutions $v_h^k \in H_0^1(\Omega)$ to the problems

$$\begin{cases} Lv_h^k + \mu_h v_h^k = \nu_k & \text{in } \Omega \\ v_h^k = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1.4}$$

Since $\{\mu_h\}$ γ^L-converges to μ, for every fixed k the sequence $\{v_h^k\}$ converges weakly to v^k in $H_0^1(\Omega)$. Finally, by the continuity estimate (4.4.4), $\|v_h^k - v_{\mu_h}\|_{H_0^{1,p}(\Omega)} \leq c|\nu_k - \nu|(\Omega)$, where c does not depend on h or k.

Let now v be the weak limit of v_{μ_h} in $H_0^{1,p}(\Omega)$; we have

$$\|v_\mu - v\|_{L^1(\Omega)} \leq \|v_\mu - v^k\|_{L^1(\Omega)} + \|v^k - v_h^k\|_{L^1(\Omega)} + \|v_h^k - v_{\mu_h}\|_{L^1(\Omega)} + \|v_{\mu_h} - v\|_{L^1(\Omega)}$$

so that, passing to the limit in each term of the right–hand side (first as $h \to \infty$ and then as $k \to \infty$) we can conclude that $v_\mu = v$ a.e. in Ω. □

5.2. $L^\infty(\Omega)$ estimates for the correctors

Let us consider a sequence $\{\mu_h\}$ of measures belonging to $\mathcal{M}_0(\Omega)$, and a measure $\mu \in \mathcal{M}_0(\Omega)$. Fixed $g \in L^\infty(\Omega)$, let u_{μ_h} and u_μ be the solutions of the relaxed Dirichlet problems corresponding to μ_h, μ and with g as datum, and let w_{μ_h}, w_μ be the functions introduced in (1.3.6) corresponding to μ_h and μ. From now on we shall assume that the following condition is satisfied:

(H1) $\mu = f\mathcal{L}$, with $f \in L^p(\Omega)$, $p > N/2$.

Let us define the functions

$$z_h = u_{\mu_h} - w_{\mu_h}\frac{u_\mu}{w_\mu}\,. \tag{5.2.1}$$

By De Giorgi's regularity theorem (see [42], Theorem 8.22), we have that for every $g \in L^\infty(\Omega)$ the solution u_μ to the problem

$$\begin{cases} Lu_\mu + fu_\mu = g & \text{in } \Omega, \\ u_\mu = 0 & \text{on } \partial\Omega, \end{cases}$$

is continuous, and, since by Remark 1.3.7 $w_\mu \geq 0$ in Ω, by Harnack inequality (see [42], Theorem 8.20), $w_\mu > 0$ in Ω, so that z_h is well defined.

In order to investigate the asymptotic behaviour of the solutions of problems (5.1.1), we need the local uniform convergence of the functions z_h.

Lemma 5.2.1 *Let $\{\mu_h\}$ be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^L-converging to μ. Then for every open set $\Omega' \subset\subset \Omega$ and for every $h \in \mathbf{N}$ the functions z_h defined by (5.2.1) belong to $H^1(\Omega') \cap L^\infty(\Omega)$ and satisfy*

$$\begin{aligned} \langle Lz_h, v\rangle + \int_\Omega z_h v\, d\mu_h &= -\langle \operatorname{div}(A^* D(\frac{u_\mu}{w_\mu})), w_{\mu_h} v\rangle + \\ &+\langle g - \frac{u_\mu}{w_\mu} + \operatorname{div}(w_{\mu_h}(A + A^*)D(\frac{u_\mu}{w_\mu})), v\rangle\,, \end{aligned} \tag{5.2.2}$$

for every $v \in H_0^1(\Omega') \cap L^2(\Omega', \mu_h) \cap L^\infty(\Omega')$.

Proof. Since both u_μ and w_μ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and there exists $\varepsilon > 0$ such that $w_\mu(x) \geq \varepsilon$ for every $x \in \Omega'$, the function z_h belongs to $H^1(\Omega')$. Moreover, by Remark 1.3.7, the function $\frac{u_\mu}{w_\mu}$ is bounded in Ω, so that z_h also belongs to $L^2(\Omega, \mu_h) \cap L^\infty(\Omega)$.

We have to compute $L(w_{\mu_h}\frac{u_\mu}{w_\mu})$. For every $v\in H_0^1(\Omega')\cap L^2(\Omega',\mu_h)\cap L^\infty(\Omega')$ we have

$$
\begin{aligned}
\int_\Omega AD(w_{\mu_h}\frac{u_\mu}{w_\mu})Dv\,dx &= \int_\Omega w_{\mu_h}AD(\frac{u_\mu}{w_\mu})Dv\,dx+\\
+\int_\Omega \frac{u_\mu}{w_\mu}ADw_{\mu_h}Dv\,dx &= \int_\Omega w_{\mu_h}AD(\frac{u_\mu}{w_\mu})Dv\,dx+\\
+\int_\Omega ADw_{\mu_h}D(v\frac{u_\mu}{w_\mu})\,dx &- \int_\Omega vADw_{\mu_h}D(\frac{u_\mu}{w_\mu})\,dx\,.
\end{aligned}
\tag{5.2.3}
$$

Since the function $v\frac{u_\mu}{w_\mu}$ is an admissible test function in the equation satisfied by w_{μ_h}, we get

$$
\int_\Omega ADw_{\mu_h}D(v\frac{u_\mu}{w_\mu})\,dx = -\int_\Omega w_{\mu_h}\frac{u_\mu}{w_\mu}v\,d\mu_h + \int_\Omega \frac{u_\mu}{w_\mu}v\,dx\,.
\tag{5.2.4}
$$

Moreover we have

$$
\begin{aligned}
&\int_\Omega vADw_{\mu_h}D(\frac{u_\mu}{w_\mu})\,dx =\\
&= \int_\Omega A^*D(\frac{u_\mu}{w_\mu})D(w_{\mu_h}v)\,dx - \int_\Omega w_{\mu_h}A^*D(\frac{u_\mu}{w_\mu})D(v)\,dx\,.
\end{aligned}
\tag{5.2.5}
$$

By (5.2.3), (5.2.4) and (5.2.5) we get

$$
\begin{aligned}
\int_\Omega AD(w_{\mu_h}\frac{u_\mu}{w_\mu})Dv\,dx &= \int_\Omega w_{\mu_h}(A+A^*)D(\frac{u_\mu}{w_\mu})Dv\,dx+\\
-\int_\Omega w_{\mu_h}\frac{u_\mu}{w_\mu}v\,d\mu_h &+ \int_\Omega \frac{u_\mu}{w_\mu}v\,dx - \int_\Omega A^*D(\frac{u_\mu}{w_\mu})D(w_{\mu_h}v)\,dx\,.
\end{aligned}
$$

Adding this integral equation to the equation satisfied by u_{μ_h} we obtain (5.2.2). □

Another basic tool needed in the following is a regularity result for local solutions of relaxed Dirichlet problems which generalizes Theorem 4.2.3. As in Chapter 4, we obtain for this kind of problems the same results obtained in [64] for local solutions of Dirichlet problems.

Theorem 5.2.2 *Let $\mu \in \mathcal{M}_0(\Omega)$, let $u \in H^1_{loc}(\Omega) \cap L^2_{loc}(\Omega,\mu)$ be a local solution to the problem $Lu + \mu u = 0$ in Ω, and let $B_R = B(x_0, R) \subset\subset \Omega$. Then we have*

$$\|u\|_{L^\infty(B_{R/2})} \le c\left\{R^{-N}\int_{B_R} u^2\,dx\right\}^{1/2}, \tag{5.2.6}$$

where the constant c depends only on θ, Θ and N.

Proof. For every $0 < \rho < R$ let $\varphi \in C_0^\infty(\Omega)$ be such that $\varphi(x) = 1$ for every $x \in B_\rho$, $\varphi(x) = 0$ for every $x \in \Omega \setminus B_R$, $0 \le \varphi \le 1$ and $|D\varphi| \le 2/(R-\rho)$. Let us consider the function $v = \varphi^2 \max(|u| - k, 0)\, u/|u|$. Clearly v belongs to $H^1_0(\Omega)$ and it has compact support in Ω. Moreover

$$\int_\Omega v^2\,d\mu \le \int_{\{|u|>k\}} (|u|-k)^2\,d\mu \le \int_\Omega u^2\,d\mu$$

so that v belongs to $L^2(\Omega,\mu)$, and we can choose it as test function in the equation satisfied by u, obtaining

$$\int_{\{|u|>k\}} \varphi^2 ADu\,Du\,dx + 2\int_{\{|u|>k\}} (|u|-k)u/|u|\varphi ADu\,D\varphi\,dx + \\ + \int_{\{|u|>k\}} \varphi^2 |u|(|u|-k)\,d\mu = 0\,.$$

Dropping the positive term involving μ and using (1.3.2), we get

$$\int_{\{|u|>k\}} \varphi^2 ADu\,Du\,dx \le 2\Theta \int_{\{|u|>k\}} \varphi(|u|-k)|Du|\,|D\varphi|\,dx\,.$$

Now we can follow the lines of the proof of the analogous theorem for local solutions of Dirichlet problems (see [64], Théorème 5.1), getting the estimate (5.2.6). □

Corollary 5.2.3 *Let μ be a measure in $\mathcal{M}_0(\Omega)$ and let $u \in H^1(\Omega) \cap L^2(\Omega,\mu)$ be a local solution to the problem $Lu + \mu u = G$ in Ω. Then the following hold:*

(i) if G belongs to $H^{-1,q}(\Omega)$, with $q > N$, then u belongs to $L^\infty_{loc}(\Omega)$ and for every $\Omega' \subset\subset \Omega$

$$\|u\|_{L^\infty(\Omega')} \le c\left\{\|u\|_{L^2(\Omega)} + \|G\|_{H^{-1,q}(\Omega)}\right\},$$

where c is a positive constant that depends only on θ, Θ, N, q, and Ω';

(ii) if G belongs to $H^{-1,q}(\Omega)$, with $2 \le q < N$, then u belongs to $L^{q^*}_{loc}(\Omega)$, with $q^* = \frac{Nq}{N-q}$, and for every $\Omega' \subset\subset \Omega$

$$\|u\|_{L^{q^*}(\Omega')} \le c\left\{\|u\|_{L^2(\Omega)} + \|G\|_{H^{-1,q}(\Omega)}\right\},$$

where c is a positive constant that depends only on θ, Θ, N, q, and Ω'.

Proof. If $\Omega' = B(x_0, R) = B_R$, using Theorems 5.2.2 and 4.2.3, and following the lines of the proof of Théorème 5.4 in [64], we get

$$\|u\|_{L^{q^*}(B_{R/2})} \le c\left\{\left(R^{N(2/q^*-1)}\|u\|_{L^2(B_R)}\right)^{1/2} + \|G\|_{H^{1,q}(B_R)}\right\},$$

if $2 \le q < N$, and

$$\|u\|_{L^\infty(B_{R/2})} \le c\left\{\left(R^N\|u\|_{L^2(B_R)}\right)^{1/2} + R^{1-N/q}\|G\|_{H^{1,q}(B_R)}\right\},$$

if $q > N$. The result for $\Omega' \subset\subset \Omega$ can be obtained by standard compactness arguments. □

Remark 5.2.4 We want to underline that the conclusions of Corollary 5.2.3 are still valid if we consider a function $u \in H^1(\Omega) \cap L^2(\Omega, \mu)$ which satisfies the equality

$$\langle Lu, v\rangle + \int_\Omega uv\, d\mu = \langle G, v\rangle,$$

only for every $v \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega)$ with compact support in Ω and which also belongs to $L^\infty(\Omega)$. Indeed both the proof of Theorem 5.2.2 and the proof of Theorem 4.2.3 are based on the choice of a suitable test function which belongs to $L^\infty(\Omega)$.

We are now in a position to state and proof the main result of this section. Notice that the technique we shall use to obtain the uniform convergence to zero of the functions z_h exploits the classical regularity results for solutions of elliptic equations in divergence form. In order to apply these results we need the hypothesis (H1) and the following regularity assumption on the coefficients of the matrix A:

(H2) the coefficients of the matrix A belong to $C^1(\overline{\Omega})$;

The assumptions (H1) and (H2) are satisfied in most of the explicit examples treated in literature, as the classical periodic case studied in [25] and its generalization developed in [63].

Theorem 5.2.5 *Let $\{\mu_h\}$ be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^L-converging to μ. Then, under the assumptions (H1) and (H2),*

$$\lim_{h\to\infty} \|z_h\|_{L^\infty(\Omega')} = 0\,,$$

for every open subset $\Omega' \subset\subset \Omega$.

Proof. Let us fix an open set Ω'' with smooth boundary and such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. Since w_μ is positive and continuous in Ω, there exists $\varepsilon > 0$ such that $w_\mu(x) \geq \varepsilon$ for every $x \in \Omega''$. Moreover, since u_μ and w_μ are bounded, by (H1) the products fu_μ and fw_μ belong to $L^p(\Omega)$. The classical L^p-estimates for second derivatives (see e.g. [42], Corollary 9.18) guarantee that u_μ and w_μ belong to $H^{2,p}(\Omega'')$. For the rest of the proof we assume that $N/2 < p < N$, the case $p \geq N$ being easier. By the Sobolev embedding we have also that u_μ and w_μ belong to $H^{1,p^*}(\Omega'')$. As $p > N/2$ we have $p^* > N$. Let $q = \min\{p, p^*/2\}$. Then $q > N/2$ and $\frac{u_\mu}{w_\mu}$ belongs to $H^{2,q}(\Omega'')$. Therefore the functions

$$w_{\mu_h} \mathrm{div}(A^* D(\frac{u_\mu}{w_\mu}))$$

belong to $L^q(\Omega'')$ and, consequently, to $H^{-1,q^*}(\Omega'')$, with $q^* > N$. Since $q^* \leq p^*$ and, by Remark 1.3.7, the sequence $\{w_{\mu_h}\}$ is bounded in $L^\infty(\Omega)$ and converges to w_μ in $L^r(\Omega)$ for every $1 \leq r < \infty$, we obtain that the functionals

$$G_h = g - \frac{u_\mu}{w_\mu} + \mathrm{div}(w_{\mu_h}(A + A^*)D(\frac{u_\mu}{w_\mu})) - w_{\mu_h} \mathrm{div}(A^* D(\frac{u_\mu}{w_\mu})) \tag{5.2.7}$$

belong to $H^{-1,q^*}(\Omega'')$ and converge to

$$G = g - \frac{u_\mu}{w_\mu} + \mathrm{div}(w_\mu(A + A^*)D(\frac{u_\mu}{w_\mu})) - w_\mu \mathrm{div}(A^* D(\frac{u_\mu}{w_\mu}))\,,$$

strongly in $H^{-1,q^*}(\Omega'')$. By using the equations satisfied by u_μ and w_μ we have

$$\langle G,v\rangle = \int_\Omega gv\,dx - \int_\Omega \frac{u_\mu}{w_\mu} v\,dx - \int_\Omega ADu_\mu\,Dv\,dx + \int_\Omega \frac{u_\mu}{w_\mu} ADw_\mu\,Dv\,dx +$$

$$+ \int_\Omega vADw_\mu\,D(\frac{u_\mu}{w_\mu})\,dx = \int_\Omega uv\,d\mu - \int_\Omega \frac{u_\mu}{w_\mu} v\,dx + \int_\Omega ADw_\mu\,D(\frac{u_\mu}{w_\mu} v)\,dx = 0\,,$$

for every $v \in C_0^\infty(\Omega'')$. Therefore $\{G_h\}$ converges to 0 strongly in $H^{-1,q^*}(\Omega'')$.

By Lemma 5.2.1, the function $z_h \in H^1(\Omega'')\cap L^2_{\mu_h}(\Omega'')$ is a local solution (in the sense of test functions belonging to $H^1_0(\Omega'')\cap L^2_{\mu_h}(\Omega)\cap L^\infty(\Omega)$ with compact support in Ω'') of $Lz_h + \mu_h z_h = G_h$ in Ω''. Thus by Corollary 5.2.3, and Remark 5.2.4 we have that

$$\|z_h\|_{L^\infty(\Omega')} \le c\left\{\|z_h\|_{L^2(\Omega'')} + \|G_h\|_{H^{-1,q^*}(\Omega'')}\right\}\,. \tag{5.2.8}$$

As $\{u_{\mu_h}\}$ converges to u_μ and $\{w_{\mu_h}\}$ converges to w_μ strongly in $L^2(\Omega)$, then $\{z_h\}$ converges to 0 strongly in $L^2(\Omega'')$. Therefore, by (5.2.8) the sequence $\{z_h\}$ converges to 0 in $L^\infty(\Omega')$. □

5.3. Asymptotic behaviour of nonvariational relaxed Dirichlet problems

Now we are in a position to investigate the asymptotic behaviour of solutions of relaxed Dirichlet problems with a measure as datum. As a first step we consider the case of "fully homogenized" limit problems, that is we assume that the sequence $\{\mu_h\}$ γ^L-converges to a measure μ satisfying (H1) and (H2).

In the following $w^*_{\mu_h}$ (resp. w^*_μ) will denote the solution of the relaxed Dirichlet problems corresponding to the operator L^*, the measure μ_h (resp. μ), and with datum $g = 1$.

Theorem 5.3.1 *Let $\{\mu_h\}$ be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^L-converging to a measure μ which satisfies (H1) and (H2). Let v_{μ_h} be the solution to the problem*

$$\begin{cases} Lv_{\mu_h} + \mu_h v_{\mu_h} = \nu & \text{in } \Omega, \\ v_{\mu_h} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists a subsequence $\{v_{\mu_{h_k}}\}$ *which converges weakly in* $H_0^{1,p}(\Omega)$, $1 \le p < \frac{N}{N-1}$, *to the solution* v *to the problem*

$$\text{(5.3.1)} \qquad \begin{cases} Lv + \mu v = \lambda & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda = (1/w_\mu^*)\alpha$, *and* α *is the weak* limit in* $\mathcal{M}^b(\Omega)$ *of the measures* $w_{\mu_{h_k}}^* \nu$.

Proof. Notice that by Proposition 1.3.12 we can use all the result stated in the previous sections replacing L with L^* in the equations. By the continuity estimate (4.4.4), for every $1 \le p < \frac{N}{N-1}$, $\|v_{\mu_h}\|_{H_0^{1,p}(\Omega)} \le c|\nu|(\Omega)$, so that there exists a subsequence $\{v_{\mu_{h_k}}\}$ weakly converging to a function v in $H_0^{1,p}(\Omega)$. Moreover, since $|w_{\mu_{h_k}}^* \nu| \le c|\nu|$, we can apply Theorem 4.1.3 and we obtain that there exists a further subsequence, still denoted by $\{w_{\mu_{h_k}}^* \nu\}$, and a measure $\alpha \in \mathcal{M}^b(\Omega)$ such that $\{w_{\mu_{h_k}}^* \nu\}$ converges weakly* to α. It remains to prove that v is the solution of (5.3.1). We want to pass to the limit in the equations

$$\int_\Omega v_{\mu_h} g \, dx = \int_\Omega R_{\mu_h}^*(g) \, d\nu \qquad \forall g \in L^\infty(\Omega).$$

Thanks to the linearity with respect to g of the problems it is not restrictive to consider only the case $g \ge 0$. Clearly we can pass to the limit in the left hand side, since $\{v_{\mu_h}\}$ converges to v in $L^1(\Omega)$.

For every $\eta > 0$ let us consider an open set $\Omega' \subset\subset \Omega$ such that $|\nu|(\Omega \setminus \Omega') \le \eta$. Let us fix another open set Ω'' such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. By Theorem 5.2.5 we have $\lim_{h\to\infty} \|z_h\|_{L^\infty(\Omega'')} = 0$. Fixed $\varphi \in C_0^\infty(\Omega'')$ such that $\varphi = 1$ in Ω' and $0 \le \varphi \le 1$ in Ω'', we have

$$\text{(5.3.2)} \qquad \begin{aligned} \int_\Omega R_{\mu_{h_k}}^*(g) \, d\nu &= \int_\Omega (1-\varphi) \, R_{\mu_{h_k}}^*(g) \, d\nu + \int_\Omega \varphi \, R_{\mu_{h_k}}^*(g) \, d\nu = \\ &= \int_\Omega (1-\varphi) \, R_{\mu_{h_k}}^*(g) \, d\nu + \int_\Omega \varphi \, z_{h_k}^* \, d\nu + \int_\Omega \varphi \, R_\mu^*(g) \frac{w_{\mu_{h_k}}^*}{w_\mu^*} \, d\nu \,, \end{aligned}$$

where

$$z_{h_k}^* = R_{\mu_{h_k}}^*(g) - w_{\mu_{h_k}}^* \frac{R_\mu^*(g)}{w_\mu^*} \,.$$

Since this function is integrated with respect to the (possibly singular) measure ν, we have to define its pointwise value at every point of Ω. The functions $R^*_{\mu_{h_k}}(g)$ and $w_{\mu_{h_k}}$ are positive and pointwise defined as the limit of their averages, and $\frac{R^*_\mu(g)}{w^*_\mu}$ is continuous and bounded in Ω'', so that for every $h \in \mathbf{N}$ the function $z^*_{h_k}$ has a pointwise value given by the limit of its averages. Thanks to this choice we also have

$$\int_\Omega \varphi\, z^*_{h_k}\, d\nu \le |\nu|(\Omega)\|z^*_{h_k}\|_{L^\infty(\Omega'')}\,.$$

Moreover, since $R^*_{\mu_h}(g)(x) \le \|R^*_{\mu_h}(g)\|_{L^\infty(\Omega)} \le c$ for every $x \in \Omega$, we have

$$\int_\Omega (1-\varphi)\, R^*_{\mu_h}(g)\, d\nu \le c\,|\nu|(\Omega \setminus \Omega')\,.$$

Finally, by the very definition of α, we get

$$\lim_{k\to\infty} \int_\Omega \varphi\, R^*_\mu(g) \frac{w^*_{\mu_{h_k}}}{w^*_\mu}\, d\nu = \int_\Omega \varphi\, R^*_\mu(g) \frac{1}{w^*_\mu}\, d\alpha\,,$$

and

$$\int_\Omega \varphi\, R^*_\mu(g) \frac{1}{w^*_\mu}\, d\alpha = \int_\Omega R^*_\mu(g) \frac{1}{w^*_\mu}\, d\alpha + \int_\Omega (\varphi - 1)\, R^*_\mu(g) \frac{1}{w^*_\mu}\, d\alpha\,.$$

Since $w_{\mu_h}(x) \le \|w_{\mu_h}\|_{L^\infty(\Omega)} \le c$ for every $x \in \Omega$ and $|R^*_\mu(g)| \le \|g\|_{L^\infty(\Omega)} w^*_\mu$, we have $\alpha \le c\nu$ and

$$\int_\Omega (\varphi - 1) R^*_\mu(g) \frac{1}{w^*_\mu}\, d\alpha \le c\|g\|_{L^\infty(\Omega)}\nu(\Omega \setminus \Omega') \le c\|g\|_{L^\infty(\Omega)}\eta\,.$$

Thus, if we pass to the limit in (5.3.2) in h and then we let η tend to zero, we obtain the result. □

Example 5.3.3 shows that for an arbitrary $\nu \in \mathcal{M}^b(\Omega)$ the limit problem depends on the choice of the subsequence.

Lemma 5.3.2 *If $\{E_h\}$ is a sequence of compact subsets of Ω such that*

$$\lim_{h\to\infty} \operatorname{cap}(E_h, \Omega) = 0\,,$$

then the sequence $\{\infty_{E_h}\}$ γ^L*-converges to* $\mu = 0$*. Moreover for every* $G \in H^{-1}(\Omega)$ *the sequence of the solutions* u_h *to the problems*

$$\begin{cases} Lu_h = G & \text{in } \Omega \setminus E_h, \\ u_h = 0 & \text{on } \partial(\Omega \setminus E_h), \end{cases}$$

converge strongly in $H_0^1(\Omega)$ *to the solution* u *to the problem*

$$\begin{cases} Lu = G & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. See [55], Corollary 2. □

Example 5.3.3 Fixed x_1, x_2 in Ω, $x_1 \neq x_2$, let us consider $\mu_h = \infty_{E_h}$, where E_h is the closed ball $\overline{B}(x_1, 1/h)$ if h is odd, and $E_h = \overline{B}(x_2, 1/h)$ if h is even. Then, by Lemma 5.3.2 the measures μ_h γ^L-converge to 0. If we choose $\nu = \delta_{x_1}$, then $v_{\mu_h} = 0$ for every odd h, while, for every even h, v_{μ_h} is the solution of the problem

$$\begin{cases} Lv_{\mu_h} = \delta_{x_1} & \text{in } \Omega \setminus \overline{B}(x_2, 1/h), \\ v_{\mu_h} = 0 & \text{in } \partial(\Omega \setminus \overline{B}(x_2, 1/h)). \end{cases}$$

Thus, if we consider the subsequence $\{\Omega_{2k+1}\}$, then the limit problem is clearly

$$\begin{cases} Lv = 0 & \text{in } \Omega, \\ v = 0 & \text{in } \partial\Omega, \end{cases}$$

while, if we consider the subsequence $\{\Omega_{2k}\}$ we obtain the limit problem

$$\begin{cases} Lv = \delta_{x_1} & \text{in } \Omega, \\ v = 0 & \text{in } \partial\Omega. \end{cases}$$

It remains to establish if, given a γ^L-converging sequence $\{\mu_h\}$, one can extract a subsequence $\{w_{\mu_{h_k}}\}$ such that for every $\nu \in \mathcal{M}^b(\Omega)$ the sequence $\{w_{\mu_{h_k}}\nu\}$ converges weakly* in $\mathcal{M}^b(\Omega)$. The Example 5.3.5 shows that this is not possible in general. More precisely we shall exhibit a sequence $\{\Omega_h\}$ of open subsets of Ω such that for every subsequence $\{\Omega_{h_k}\}$ there exists a point $x \in \Omega$ with the property

$$\limsup_{k\to\infty} w_{\mu_{h_k}}(x) \geq M > 0, \qquad \liminf_{k\to\infty} w_{\mu_{h_k}}(x) = 0.$$

For every set E we shall denote by $\mathbf{1}_E$ the characteristic function of E, that is

$$\mathbf{1}_E(x) = \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$$

Lemma 5.3.4 *Let E be a Borel subset of $[0,1]$ such that $0 < \mathcal{L}(E) < 1$, and let $\{\varepsilon_h\}$ a decreasing sequence of positive numbers converging to zero and such that $(1+\varepsilon_h)E \subseteq [0,1]$. For every $h \in \mathbf{N}$ we define the sets*

$$E_h = \bigcup_{k=0}^{h-1} h^{-1}(E+k), \qquad F_h = \bigcup_{k=0}^{h-1} h^{-1}((1+\varepsilon_h)E+k).$$

Then for every subsequence $\{h_k\}$ there exists a set B with Lebesgue measure zero such that

$$\limsup_{k\to\infty} \mathbf{1}_{E_{h_k}}(x) = 1, \qquad \liminf_{k\to\infty} \mathbf{1}_{F_{h_k}}(x) = 0,$$

for every $x \in [0,1] \setminus B$.

Proof. It is well known that the sequence $\{\mathbf{1}_{E_h}\}$ converges weakly in $L^1(\Omega)$ to the constant $\mathcal{L}(E)$, so that for every subsequence $\{h_k\}$ there exists a set B' with Lebesgue measure zero such that

$$\limsup_{k\to\infty} \mathbf{1}_{E_{h_k}}(x) = 1, \qquad \liminf_{k\to\infty} \mathbf{1}_{E_{h_k}}(x) = 0,$$

for every $x \in [0,1] \setminus B'$. Moreover $\{\mathbf{1}_{F_{h_k}} - \mathbf{1}_{E_{h_k}}\}$ is a sequence converging to zero in $L^1(0,1)$. Thus there exists a subsequence, still denoted by $\{h_k\}$, and a set N with Lebesgue measure zero such that

$$\lim_{k\to\infty} \left(\mathbf{1}_{F_{h_k}}(x) - \mathbf{1}_{E_{h_k}}(x)\right) = 0,$$

for every $x \notin N$. Then for every $x \in [0,1] \setminus (B' \cup N)$ we have

$$\liminf_{k\to\infty} \mathbf{1}_{F_{h_k}}(x) = \liminf_{k\to\infty} \mathbf{1}_{E_{h_k}}(x) = 0,$$

and the result is proved choosing $B = B' \cup N$. □

Example 5.3.5 Let Ω be an open set in $\mathbf{R}^N$, $N \geq 3$, such that the cylinder $\{(x_1,\hat{x}) \in \mathbf{R}^N \colon 0 \leq x_1 \leq 1,\ |\hat{x}| < \rho_0\}$ is compactly contained in Ω for a suitable $\rho_0 > 0$, where $\hat{x} = (x_2,\ldots,x_N)$. Let E_h, F_h be constructed as in Lemma 3.3.1 starting from the set $E = (1/4, 3/4)$ and with $\varepsilon_h = (2h)^{-2}$, and for every $0 < \rho \leq \rho_0$, and h large enough, let C_h, $D_h^\rho \in \Omega$ be the sets

$$C_h = \{(x_1,\hat{x}) \in \mathbf{R}^N \colon x_1 \in F_h\,, |\hat{x}| < h^{-2}\}\,,$$

$$D_h^\rho = \{(x_1,\hat{x}) \in \mathbf{R}^N \colon x_1 \in E_h\,, |\hat{x}| < \rho\}\,.$$

It is easily seen that for every fixed h, $\lim_{\rho\to 0} \operatorname{cap}(D_h^\rho,\Omega) = 0$, so that by Lemma 5.3.2 the sequence of solutions $\{w_h^\rho\}$ to the problems

$$Lw_h^\rho = 1 \quad \text{in } \Omega \setminus D_h^\rho\,, \qquad w_h^\rho = 0 \quad \text{on } \partial(\Omega \setminus D_h^\rho)\,,$$

converges strongly in $H_0^1(\Omega)$, as ρ tends to zero, to the solution w to the problem

$$Lw = 1 \quad \text{in } \Omega\,, \qquad w = 0 \quad \text{on } \partial\Omega\,.$$

Moreover, for every $h \in \mathbf{N}$ we consider a set C_h' such that $D_h^{\rho_h} \subset\subset C_h' \subset\subset C_h$ for some $\rho_h > 0$. Then for every $\rho \leq \rho_h$, the function w_h^ρ is a local solution in $\Omega \setminus C_h'$ to the problem $Lw_h^\rho = 1$. By the De Giorgi's regularity result (see, e.g., [42], Theorem 8.22), we have that for every open set Ω' such that $C_h \subset\subset \Omega' \subset\subset \Omega$, the family $\{w_h^\rho\}_{\rho\leq\rho_h}$ is equicontinuous in $\Omega' \setminus C_h$ with respect to ρ, so that it converges to w uniformly in $\Omega' \setminus C_h$ as $\rho \to 0$.

By the strong maximum principle, there exists $m > 0$ such that $w(x) \geq m$ for every $x \in S = \{x \in \Omega', \hat{x} = 0\}$. Thus for every $h \in \mathbf{N}$ there exists $r_h \leq \rho_h$ such that $w_h^{r_h}(x) > m/2$ for every $x \in (\Omega' \setminus C_h) \cap S$. Finally by Lemma 3.3.1 for every subsequence $\{h_k\}$ there exists $x \in S$ such that $x \in D_{h_k}^{r_{h_k}}$ for infinitely many k, but $x \notin C_{h_k}$ for infinitely many k. Thus

$$\limsup_k w_{h_k}^{r_{h_k}}(x) \geq m/2 > 0\,, \qquad \liminf_k w_{h_k}^{r_{h_k}}(x) = 0\,. \tag{5.3.3}$$

Since $\lim_{h\to\infty} \operatorname{cap}(D_h^{r_h},\Omega) = 0$, by Lemma 5.3.2 the measures $\mu_h = \infty_{D_h^{r_h}}$ γ^L-converge to $\mu = 0$, but (5.3.3) guarantees that there exist no subsequence of $\{w_{h_k}^{r_{h_k}}\}$ pointwise converging in Ω.

For every $x \in \Omega$ let δ_x be the Dirac mass at x. Since $\{w_{h_k}^{r_{h_k}} \delta_x\}$ converges weakly* in $\mathcal{M}^b(\Omega)$ if and only if $\{w_{h_k}^{r_{h_k}}(x)\}$ converges in $\mathbf{R}$, we conclude that for every subsequence $\{w_{h_k}^{r_{h_k}}\}$ there exists $x \in \Omega$ such that $\{w_{h_k}^{r_{h_k}} \delta_x\}$ does not converge weakly* in $\mathcal{M}^b(\Omega)$.

In order to generalized Theorem 5.3.1 to the case in which some "holes" appear in the limit problem we need the following lemmas.

Lemma 5.3.6 *Let $\{\mu_h\}$, μ be measures of $\mathcal{M}_0(\Omega)$ such that $\{\mu_h\}$ γ^L-converges to μ. Let $g \in L^\infty(\Omega)$ be a positive function and u_{μ_h}, u_μ be the solution of (1.3.5) corresponding to μ_h, μ and with g as datum. Then*

$$\limsup_{h\to\infty} u_{\mu_h}(x) \le u_\mu(x)$$

for every $x \in \Omega$.

Proof. Let u_0 be the solution of

$$Lu_0 = g \quad \text{in } \Omega\,, \qquad u_0 = 0 \quad \text{on } \partial\Omega \tag{5.3.4}$$

and let γ_h and γ be respectively the positive measures of $H^{-1}(\Omega)$ such that

$$L(u_0 - u_{\mu_h}) = \gamma_h \quad \text{in } H^{-1}(\Omega)\,, \qquad L(u_0 - u) = \gamma \quad \text{in } H^{-1}(\Omega)\,. \tag{5.3.5}$$

Then we have the representations

$$u_{\mu_h}(x) = u_0(x) - \int_\Omega G(x,y)\,d\gamma_h(y)\,, \qquad u(x) = u_0(x) - \int_\Omega G(x,y)\,d\gamma(y)\,,$$

where G is the Green's function of L in Ω with Dirichlet boundary conditions. From (5.3.5) it follows that $\{\gamma_h\}$ converges to γ weakly in $H^{-1}(\Omega)$, and the boundedness in $H^{-1}(\Omega)$ implies that the hypotheses of Theorem 4.1.3 are satisfied. Then we conclude easily that $\{\gamma_h\}$ converges weakly* in $\mathcal{M}^b(\Omega)$ to γ. Since G is a positive continuous function, by Theorem 4.1.3 we get

$$\int_\Omega G(x,y)\,d\gamma(y) \le \liminf_{h\to\infty} \int_\Omega G(x,y)\,d\gamma_h(y)\,. \tag{5.3.6}$$

By the continuity of the function u_0, the result follows from (5.3.6). □

Corollary 5.3.7 *Let $\{u_{\mu_h}\}$ and u_μ be as in Lemma 5.3.6, and let ν be a positive measure of $\mathcal{M}^b(\Omega)$. Then the sequence of measures $\{u_{\mu_h}\nu\}$ admits a subsequence which converges weakly* to a measure λ in $\mathcal{M}^b(\Omega)$. Moreover $\lambda \leq u_\mu \nu$.*

Proof. By Theorem 1.3.6 we have $u_{\mu_h}(x) \leq u_0(x)$ for every $x \in \Omega$, where u_0 is the continuous function defined in (5.3.4). Then $\{u_{\mu_h}\}$ is equiintegrable with respect to ν, and we can apply Theorem 4.1.3, obtaining that there exists $\lambda \in \mathcal{M}^b(\Omega)$ such that, up to a subsequence, $\{u_{\mu_h}\nu\}$ converges weakly* to λ.

By Fatou Lemma and Lemma 5.3.6, for every positive function $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \varphi \, d\lambda = \lim_{h\to\infty} \int_\Omega \varphi u_{\mu_h} \, d\nu \leq \int_\Omega \varphi \limsup_{h\to\infty} u_{\mu_h} \, d\nu \leq \int_\Omega \varphi u_\mu \, d\nu \,,$$

which gives the result. □

We are now in a position to study the asymptotic behaviour of nonvariational relaxed Dirichlet problems for which the corresponding variational problems converge to a measure that can be infinite in some subdomain of Ω. More precisely we have the following assumption:

(H3) there exists an open set Ω_μ of Ω such that $\mu = f\mathcal{L} \llcorner \Omega_\mu + \infty_{\Omega\setminus\Omega_\mu}$, with $f \in L^p(\Omega_\mu)$, $p > N/2$.

Remark 5.3.8 The hypothesis (H3) implies that for every $G \in H^{-1}(\Omega)$ the solution u_μ of (1.3.4) coincides with the solution to the problem

$$\begin{cases} Lu + fu = G & \text{in } H^{-1}(\Omega_\mu), \\ u = 0 & \text{on } \partial\Omega_\mu, \end{cases}$$

(see [20], Section 2).

Theorem 5.3.9 *Let $\{\mu_h\}$ be a sequence of measures of $\mathcal{M}_0(\Omega)$ γ^L-converging to a measure μ, and let $\nu \in \mathcal{M}^b(\Omega)$ be such that $|\nu|(\Omega \cap \partial\Omega_\mu) = 0$. Suppose that the hypotheses (H2) and (H3) hold. For every $h \in \mathbf{N}$ let v_{μ_h} be the solution to the problem*

$$\begin{cases} Lv_{\mu_h} + \mu_h v_{\mu_h} = \nu & \text{in } \Omega, \\ v_{\mu_h} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists a subsequence $\{v_{\mu_{h_k}}\}$ *of* $\{v_{\mu_h}\}$ *which converges weakly in* $H_0^{1,p}(\Omega)$, $1 \le p < \frac{N}{N-1}$, *to the solution* v_μ *of the problem*

$$\begin{cases} Lv_\mu + \mu v_\mu = \lambda & \text{in } \Omega, \\ v_\mu = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3.7}$$

where the measure λ *is given by*

$$\lambda = \begin{cases} (1/w_\mu^*)\alpha, & \text{in } \Omega_\mu, \\ 0 & \text{in } \Omega \setminus \Omega_\mu, \end{cases}$$

and α *is the weak* limit in* $\mathcal{M}^b(\Omega)$ *of the measures* $w_{\mu_{h_k}}^* \nu$.

Proof. It is not restrictive to suppose that ν is a positive measure. In this case by Corollary 5.3.7 we obtain that there exists a subsequence $\{w_{\mu_{h_k}}^* \nu\}$ which converges weakly* to some measure α with $0 \le \alpha \le w_\mu \nu$.

The only change we have to do with respect to the proof of Theorem 5.3.1 is to split

$$\int_\Omega R_{\mu_{h_k}}^*(g)\, d\nu = \int_{\Omega_\mu} R_{\mu_{h_k}}^*(g)\, d\nu + \int_{\Omega\setminus\Omega_\mu} R_{\mu_{h_k}}^*(g)\, d\nu .$$

By Remark 5.3.8 we can apply the same arguments of the proof of Theorem 5.3.1, obtaining

$$\lim_{k\to\infty} \int_{\Omega_\mu} R_{\mu_{h_k}}^*(g)\, d\nu = \int_{\Omega_\mu} \frac{R_\mu^*(g)}{w_\mu^*}\, d\alpha .$$

By Corollary 5.3.7, $\alpha = 0$ in $\Omega \setminus \overline{\Omega}_\mu$, and

$$\int_{\Omega_\mu} \frac{R_\mu^*(g)}{w_\mu^*}\, d\alpha = \int_\Omega R_\mu^*(g)\, d\lambda .$$

On the other hand , by Lemma 5.3.6 and since $R_\mu^*(g)(x) = 0$ for $x \in \Omega \setminus \overline{\Omega}_\mu$, for every positive $g \in L^\infty(\Omega)$ and for every $x \in \Omega \setminus \overline{\Omega}_\mu$ we have

$$\lim_{k\to\infty} R_{\mu_{h_k}}^*(g)(x) = 0 .$$

As $\nu(\Omega \cap \partial\Omega_\mu) = 0$, by the dominated convergence theorem we get

$$\lim_{k\to\infty} \int_{\Omega\setminus\Omega_\mu} R_{\mu_{h_k}}^*(g)\, d\nu = 0 \qquad \forall g \ge 0 . \tag{5.3.8}$$

By the linearity of $R_{\mu_{h_k}}^*(g)$ with respect to g (5.3.8) holds for every $g \in L^\infty(\Omega)$, which concludes the proof. □

Remark 5.3.10 In Theorem 5.3.9 the fact that the measure ν does not charge the set $\Omega \cap \partial\Omega_\mu$ was used only in order to get (5.3.8). The same result can be obtained assuming that $|\nu|(\Omega \cap (\partial\Omega_\mu \setminus E)) = 0$, where E is the set of all the Wiener points of Ω_μ, that is all the points $x \in \partial\Omega_\mu$ such that

$$\int_0^1 \frac{\operatorname{cap}(B(x,\rho) \setminus \Omega_\mu, B(x,2\rho))}{\rho^{N-1}} d\rho = +\infty .$$

Indeed it is well known that if $x \in E$, then $R_\mu^*(g)(x) = 0$ for every $g \in L^\infty(\Omega)$ (see e.g. [33], Section 5), and this is enough to conclude the proof of Theorem 5.3.9.

In particular if $\partial\Omega_\mu$ is smooth, then the conclusion of Theorem 5.3.9 remains valid with no assumption on ν.

Remark 5.3.11 As an application of Theorem 5.3.9 we can consider a sequence $\{\Omega_h\}$ of open subsets of Ω such that the sequence $\{\infty_{\Omega\setminus\Omega_h}\}$ γ^L-converges to the measure $\mu = \infty_{\Omega\setminus\Omega_\infty}$, where Ω_∞ is an open subset of Ω. Our result guarantees that for every $x \notin \Omega \cap \partial\Omega_\infty$ (and also for every $x \in \Omega \cap \partial\Omega_\infty$ which is a Wiener point for Ω_∞) there exists a subsequence $\{\Omega_{h_k}\}$ and a constant $c(x) \geq 0$, both depending on x, such that the sequence $\{G_{h_k}(x,\cdot)\}$ of the Green's functions corresponding to the operator L with Dirichlet boundary conditions on $\partial\Omega_{h_k}$ converges to the function $c(x)G_\infty(x,\cdot)$, where $G_\infty(x,\cdot)$ is the Green function corresponding to the operator L and with Dirichlet boundary conditions on $\partial\Omega_\infty$. Moreover $c(x)$ is given by

$$c(x) = \lim_{k\to\infty} \int_\Omega G_{h_k}(x,y)\, dy = \lim_{k\to\infty} w_{h_k}(x) , \tag{5.3.9}$$

where w_h is the solution of the problem

$$Lw_h = 1 \quad \text{in } \Omega_h , \qquad w_h = 0 \quad \text{on } \partial\Omega_h .$$

Since for every $x \in \Omega$ the function $G_h(x,\cdot)$ satisfies

$$\int_\Omega G_h(x,y)g(y)\, dy = R_{\mu_h}(g)(x)$$

and there exists a subsequence $\{\mu_{h_k}\}$ such that

$$\lim_{k\to\infty} R^*_{\mu_{h_k}}(g)(x) = R^*_\mu(g)(x)$$

for almost every $x \in \Omega$, we can find a subsequence $\{G_{h'_k}(x,\cdot)\}$ such that $G_{h_k}(x,\cdot)$ converge to $G_\infty(x,\cdot)$ for almost every $x \in \Omega$.

On the other hand, since $c(x)$ is given by (5.3.9), Example 5.3.5 shows that in general it is not possible to find a subsequence $\{\Omega_{h_k}\}$ such that the sequence $\{G_{h_k}(x,\cdot)\}$ converges for every $x \in \Omega$.

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