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Elementary Particles Sector

2 + 1 QUANTUM DE SITTER GRAVITY

Thesis submitted for the degree of

"Doctor Philosophiae"

Candidate:

Federico Zertuche Mones

Supervisors:

Prof. Tullio Regge

Academic Year 1988/89

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To Thalia with all my love and gratitude

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I) Introduction.

Recent studies on 2 + 1 dimensional gravity, [1,2] with or without a cosmological constant, have shown it to be equivalent to Yang-Mills theory with a Chern-Simons action [3,4,5]. 2 + 1 Dimensional gravity is locally trivial since Einstein equations imply that there are no gravitational waves in this low dimension, so that, physical observables are not associated with local fields, but rather, with global integrated fields, the so called integrated connection.

In a recent work [6] Nelson and Regge have studied Poincare' 2 + 1 dimensional quantum gravity ($\lambda = 0$) and have developed commutation relations for the integrated connections Ψ, which define a quantum representation $\Psi : \pi_{1_{2}} \longrightarrow ISO(2,1)$ of the homotopy group $\pi_1(\Sigma^2)$ of the initial data surface Σ^2 on the Poincare' group ISO(2,1) in 2 + 1 dimensions. The physical relevant dynamical degrees of freedom turns out to be the gauge-invariant traces of certain functionals of $\Psi(P)$.

The aim of this work is to do, the direct generalization of ref.[6] and to study the De Sitter ($\lambda > 0$) and anti-De Sitter ($\lambda < 0$) cases, with particular attention to the last case. The integrated connection Ψ now turns to be a representation of the homotopy group $\pi_1(\Sigma^2)$ on the G group with G = SO(3,1) or SO(2,2) for $\lambda > 0$ and $\lambda < 0$ respectively and it happens that the gauge invariant degrees of freedom are the traces of the matrices of the spinor group of G. We recall that the spinor group of SO(3,1) is SL(2,C) while that of SO(2,2) is SL(2,R) $\otimes$ SL(2,R) and in both cases there are six real parameters [7].

Classical 2 + 1 dimensional gravity with a cosmological constant λ is well described by the De Sitter spin connections $\omega^{ab} = \omega^{ab}_{\alpha}\, dx^{\alpha}$ as the field variables, with a,b = 0,1,2,3 , α = 0,1,2 (see Appendix for the conventions). The tangent Minkowski metric η_{ab} has the signature (-1,1,1,k) with k = +1 for De Sitter and k = -1 for anti-De Sitter space. The triad $e^a = \alpha\, \omega^{a3}$ where $\lambda = 1/3\, k\, \alpha^{-2}$ is included in the De Sitter connection, so that

$$\omega^{ab} = \begin{pmatrix} \omega^{AB} & 1/\alpha\, e^A \\ -1/\alpha\, e^B & 0 \end{pmatrix} \tag{1.1}$$

where ω^{AB} A,B = 0,1,2 is the usual Lorentz spin connection. The Riemann curvature is

$$R^{ab} = d\omega^{ab} - \omega^{a}{}_{t} \cap \omega^{tb} \tag{1.2}$$

and the action of the system is given by:

$$S = 1/2\, \alpha \epsilon_{abcd} \int_M (d\omega^{ab} - 2/3\, \omega^{a}{}_{t} \cap \omega^{tb}) \cap \omega^{cd} \tag{1.3}$$

with $M = \Sigma^2 \times R$ the space-time manifold. Using (1.1) we have

$$S = -2 \epsilon_{ABC} \int_M (d\omega^{AB} - \omega^{A}{}_{T} \cap \omega^{TB} + \lambda e^{A} \cap e^{B}) \cap e^{C} \tag{1.4}$$

where $\epsilon_{ABC3} \equiv -\epsilon_{ABC}$. In the sequel we set $\sqrt{k}$ = +1 or +i for k = 1 or -1 unambigously.

The variational equations obtained from (1.3) are $R^{ab} = 0$ which imply that space-time is everywhere locally De Sitter, in components

$$R^{AB} = d\omega^{AB} - \omega^{A}{}_{C} \wedge \omega^{CB} + \alpha^{-2} e^{A} \wedge e^{B} = 0 \tag{1.5a}$$

and

$$R^{A} \equiv \alpha R^{A3} = de^{A} - \omega^{A}{}_{B} \wedge e^{B} = 0 \tag{1.5b}$$

where R^{A} is the torsion.

In order to calculate the Poisson brackets we separate the variables whose time derivative is present in the action and those whose time derivative does not appear [1]. The variables whose time derivative appear are the spatial components of the connection i.e. $\omega^{ab}{}_{i}$ for $i = 1,2$, and those whose time derivate are absent, are the time components $\omega^{ab}{}_{0}$. Rewriting the action (1.3) in this way we obtain

$$S = 1/2\alpha \;\; \epsilon_{abcd}\epsilon^{0ij} \int d^{3}x \left[\partial_{0}\omega^{ab}{}_{i}\omega^{cd}{}_{j} + 2\omega^{cd}{}_{0}(\partial_{i}\omega^{ab}{}_{j} - \omega^{at}{}_{i}\omega_{t}{}^{b}{}_{j})\right] \tag{1.6}$$

The Poisson brackets can now be directly calculated from the first term in (1.6), they are

$$\left\{ \omega^{ab}{}_{i}(x), \omega^{cd}{}_{j}(y)\right\} = k/\alpha \, \epsilon_{ij} \epsilon^{abcd} \delta^{2}(x - y) \tag{1.7}$$

or in components

$$\left\{ e^{A}{}_{i}(x),\ \omega^{BC}{}_{j}(y) \right\} = -\epsilon_{ij}\epsilon^{ABC}\delta^{2}(x - y)$$

$$\left\{ e^{A}{}_{i}(x),\ e^{B}{}_{j}(y) \right\} = \left\{ \omega^{AB}{}_{i}(x),\ \omega^{CD}{}_{j}(y) \right\} = 0$$

where $x, y \in \Sigma^{2}$ are generic points on the $x^{0} = y^{0} =$ const. surface Σ^{2}, and $\epsilon_{ij} = -\epsilon_{ji}$ with $\epsilon_{12} = 1$.

The variables $\omega^{cd}{}_{0}$ in (1.6) are Lagrange multipliers, taking $\delta S/\delta\omega^{cd}{}_{0} = 0$ we obtain the constraint equations

$$\epsilon^{0ij}\left(\partial_{i}\omega^{ab}{}_{j} - \omega^{a}{}_{ti}\omega^{tb}{}_{j} \right) = 0 \qquad (1.8)$$

which generate gauge transformations.

As in ref.[6] we assume that Σ^{2} is a compact surface of genus g (see figure 1). The fundamental group $\pi_1(\Sigma^{2})$ may be presented by means of 2g generators [1,6,8]:

$$u_{i}\ ,\ v_{j} \qquad\qquad i,j = 1,\dots,g \qquad (1.9)$$

satisfying the relation :

$$u_1\, v_1\, u_1^{-1}\, v_1^{-1} \dots u_g\, v_g\, u_g^{-1}\, v_g^{-1} = 1 \qquad (1.10)$$

From now on we shall work with representations ψ of $\pi_1(\Sigma^{2})$ on the group SO(2,2) and SO(3,1), and we will identify a generic closed path ρ with it's homotopy equivalence class $[\rho]$, such that $\rho \equiv [\rho] \in \pi_1(\Sigma^{2})$.

In chapter II we define the integrated connection along a path in Σ^{2} and calculate the Poisson brackets for intersecting paths. By introducing the spinor groups of the group G = SO(2,2), SO(3,1), we derived the Poisson brackets of the trace variables and established the traces algebra. Introducing a convenient parametrization we show in chapter III, that the Poisson traces algebra has a canonical struture. In chapter IV, the quantization of the traces algebra for the anti-De Sitter case is done and a theory of it's representation is developed. Chapter V contains a discussion of the De Sitter case and on chapter VI we conclude with a discussion of multicrossing paths.

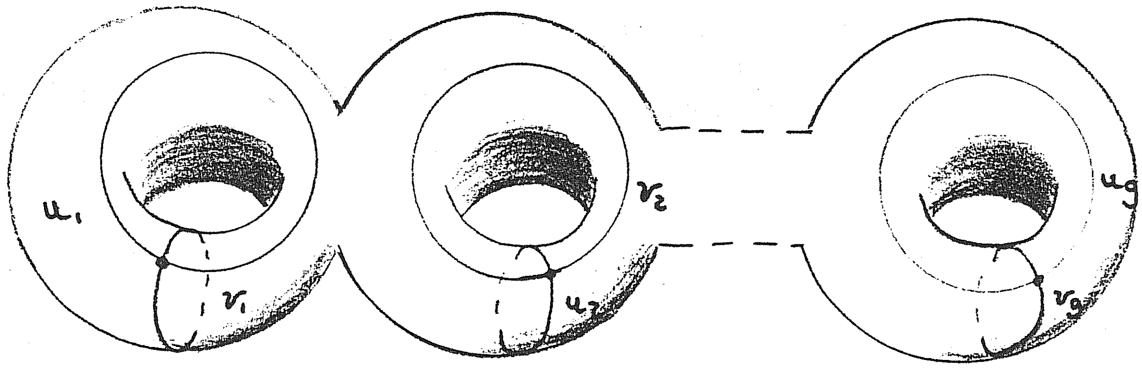

FIGURE 1.

II) The Integrated Connection.

Since space-time is $M = \Sigma^2 \times R$ and R is contractible, $\pi_1(M) \simeq \pi_1(\Sigma^2)$. The field equations (1.5) imply that the connection is given by $\omega = d\psi\psi^{-1}$ with $\psi \in G = SO(2,2)$ or $SO(3,1)$. On Σ^2 consider two generic points $A, B \in \Sigma^2$ and a path γ parametrized as $x(t)$, $0 \leq t \leq 1$ with $x(0) = A$ and $x(1) = B$. Consider the differential equation :

$$d\psi/dt = \omega_t \psi \qquad (2.1)$$

where $\omega_t \equiv \omega_\alpha T^\alpha$ and T is a tangent vector along γ, subject to the boundary condition $\psi(0) = 1$. Since $R^{ab} = 0$, the value of the solution at $t = 1$ will depend on the homotopy class γ only and will be denoted by $\psi(B,A,\gamma)$ or in brief $\psi(\gamma)$. The generic solution to (2.1) is given by

$$\psi(B) = \psi(B,A,\gamma)\ \psi(0) \qquad (2.2)$$

Consider a second path γ' with end points B,C and the solution $\psi(C,B,\gamma')$ of (2.1). The entire solution for the path $\gamma'\gamma$ with end points A,C is

$$\psi(C,A,\gamma'\gamma) = \psi(C,B,\gamma')\,\psi(B,A,\gamma) \qquad (2.3)$$

Setting A = B on (2.2) we obtain a subset of solutions $\psi(A,[\gamma])$ or in short $\psi(\gamma)$, and now γ is an element of $\pi_1(\Sigma^2)$. It is easy to show that

$$\psi : \pi_1(\Sigma^2) \longrightarrow G$$

with the product rule (2.3) defines a group homomorphism.

Now consider in general two oriented open paths $\rho, \sigma \in \Sigma^2$ with end points A, B and C, D respectively and the corresponding integrated connections

$\psi(B,A,\rho)$ $\psi(D,C,\sigma)$. From (1.7) it is possible to calculate its' Poisson brackets. If ρ and σ are homotopic to paths that do not intersect, it is clear that

$$\{ \psi(\rho) , \psi(\sigma) \} = 0 \qquad (2.4)$$

Let ρ and σ be parametrized by t and u respectively, with $0 \leq t \leq 1$, and $0 \leq u \leq 1$. If we assume that they have a single intersection at the point P with $t = t_0$ on ρ and $u = u_0$ on σ, we can divide ρ and σ in three paths each one; so that $\rho = \rho_3\rho_2\rho_1$ and $\sigma = \sigma_3\sigma_2\sigma_1$ with

$$\rho_1 = \left\{ 0 \leq t \leq t_0 - \epsilon \right\} , \qquad \rho_2 = \left\{ t_0 - \epsilon \leq t \leq t_0 + \epsilon \right\} ,$$

$$\rho_3 = \left\{ t_0 + \epsilon \leq t \leq 1 \right\} , \qquad \sigma_1 = \left\{ 0 \leq u \leq u_0 - \epsilon \right\} ,$$

$$\sigma_2 = \left\{ u_0 - \epsilon \leq u \leq u_0 + \epsilon \right\} \text{ and } \sigma_3 = \left\{ u_0 + \epsilon \leq u \leq 1 \right\}$$

for $0 < \epsilon << 1$. We obtain in this way

$$\begin{aligned} \psi(B,A,\rho) &= \psi(\rho_3)\psi(\rho_2)\psi(\rho_1) \\ \psi(D,C,\sigma) &= \psi(\sigma_3)\psi(\sigma_2)\psi(\sigma_1) \end{aligned} \qquad (2.5)$$

With the decomposition the only intersecting paths are ρ_2 , σ_2 and from (2.4)

$$\left\{ \psi(\rho_i) , \psi(\sigma_j) \right\} = 0 \qquad \text{for i or j} = 1,3$$

On the other hand, because $\psi(\rho_2)$ and $\psi(\sigma_2)$ are infinitesimally close to the identity, can be written at the first order in ϵ from (2.1) as:

$$\psi(\rho_2) = 1 + \int_{t_0 - \epsilon}^{t_0 + \epsilon} dt\ \omega_i(x(t))\ dx^i(t)/dt$$

and a similar expression for $\psi\ (\sigma_2)$. From (1.7) we have:

$$\left\{\psi^{ab}(\rho_2),\ \psi^{cd}(\sigma_2)\right\} = k/\alpha \int_{t_0-\epsilon}^{t_0+\epsilon} dt \int_{u_0-\epsilon}^{u_0+\epsilon} du\, J(\rho,\sigma)\ \epsilon^{abcd}\ \delta^2(x(t) - x(u))$$

where $J(\rho,\sigma) = \epsilon_{ij} \dfrac{dx^i}{dt} \dfrac{dx^j}{du}$, defines the orientation of the intersection of ρ and σ, and because

$$J(\rho,\sigma)\ \delta^2(x(t) - x(u)) = -\ s\delta(t - t_0)\ \delta(u - u_0) \tag{2.6}$$

where

$$s \equiv -\ \mathrm{sign}\ J(\rho,\sigma)$$

we obtain

$$\left\{\psi^{ab}(\rho_2)\ ,\ \psi^{cd}(\sigma_2)\right\} = -k/\alpha\ \epsilon^{abcd}s$$

$$\left\{\psi^{ab}(\sigma_2)\ ,\ \psi^{cd}(\rho_2)\right\} = k/\alpha\ \epsilon^{abcd}s \tag{2.7}$$

Using (2.5) we may calculate the Poisson brackets of $\psi(\rho)$ and $\psi(\sigma)$, the result is cumbersome and not of interest, instead we will calculate the Poisson brackets for the spinor groups of G .

In the case of paths ρ and σ with several intersections one can express each path as a product (say $\rho = \rho_1\rho_2\rho_3$...and $\sigma = \sigma_1\sigma_2\sigma_3...$) where each factor has at the most a single intersection, then one can apply (2.4) and (2.7) and do the ensamble.

It is easer to work with the spinor groups of G which are SL(2,C) for the De Sitter case and SL(2,R) ⊗ SL(2,R) for the anti-De Sitter. To do that, we define

$$\Delta(x) = \Delta_\alpha(x)\, dx^\alpha = 1/4\ \omega^{ab}(x)\ \gamma_{ab} \tag{2.8}$$

where $\gamma_{ab} = 1/2(\ \gamma_a,\ \gamma_b)$ and γ_a are the Dirac matrices, and introduce the matrix S(x) such that

$$\psi^{ab}\gamma_b = S^{-1}\gamma^a S \tag{2.9}$$

In the Appendix γ-matrix conventions and useful identities are shown. Using (A.7), it follows that

$$\omega^{ab}\gamma_b = [\ \gamma^a\ ,\ \Delta\ ]$$

and using this relation in (2.1) we see that S obeys the differential equation

$$\frac{dS}{dt} = \Delta S \tag{2.10}$$

From (1.7) the Poisson brackets of Δ are

$$\left\{ \Delta_i(x)\ ,\ \Delta_j(y) \right\} = \frac{1}{16\alpha} k\ \epsilon_{ij}\epsilon^{abcd}\gamma_{ab} \otimes \gamma_{cd}\ \delta^2(x-y) =$$

$$= -\frac{1}{8\alpha} k\ \epsilon_{ij}\ \gamma_5 \gamma_{ab} \otimes \gamma^{ab}\ \delta^2(x-y) \qquad (2.11)$$

The 4×4 representation $\Delta(x)$, $S(x)$ can be decomposed in two 2×2 irreducible parts, using the projectors $p_\pm$ (see in Appendix (A.8))

$$\Delta(x) = \Delta^+(x) \otimes p_+ + \Delta^-(x) \otimes p_-$$
$$S(x) = S^+(x) \otimes p_+ + S^-(x) \otimes p_- \qquad (2.12)$$

so that

$$\frac{dS^\pm(x(t))}{dt} = \Delta^\pm S^\pm(x(t)) \qquad (2.13)$$

From (2.11) and (2.12) the Poisson brackets of the $\Delta^\pm$ are given by

$$\left\{ \Delta^\pm{}_i(x)\ ,\ \Delta^\pm{}_j(y) \right\} = \pm \frac{i}{2\alpha\sqrt{k}} \epsilon_{ij}\ \sigma_m \otimes \sigma_m\ \delta^2(x-y)$$
$$\left\{ \Delta^+{}_i(x)\ ,\ \Delta^-{}_j(y) \right\} = 0 \qquad (2.14)$$

For the integrated connection $S^\pm(\rho)$ it is possible to repeat the same derivation that we have done for (2.7) replacing ψ by S and ω by Δ, and after assembling the paths one obtains the result

$$\left\{ S^{\pm}(\rho)_{\alpha}{}^{\beta} , S^{\pm}(\sigma)_{\gamma}{}^{\tau} \right\} =$$

$$\pm i \frac{s}{2\alpha \sqrt{k}} \left[-S^{\pm}(\rho)_{\alpha}{}^{\beta} S^{\pm}(\sigma)_{\gamma}{}^{\tau} + 2S^{\pm}(\rho_3 \sigma_1)_{\alpha}{}^{\tau} S^{\pm}(\sigma_3 \rho_1)_{\gamma}{}^{\beta} \right]$$

$$\left\{ S^{+}(\rho)_{\alpha}{}^{\beta} , S^{-}(\sigma)_{\gamma}{}^{\tau} \right\} = 0 \qquad (2.15)$$

where $\alpha,\beta,...= 1,2$ are matrix indices.

In the following we will limit to consider closed paths $\rho,\sigma \in \pi_1(\Sigma^2)$ in eq.(2.15). The Poisson brackets (2.15) however do not take into account the constraints of the system, i.e. (1.8). These constraints generate G = SO(2,2) or SO(3,1) gauge transformations on the fields [1,6]. So we must fix the gauge in order that all the constraints become second class and replace Poisson brackets with Dirac brackets [9,10]. It is however easier to work directly with gauge invariant quantities for which Poisson and Dirac brackets coincide, the traces

$$C^{\pm}(\rho) \equiv 1/2 \; S_{\alpha}{}^{\pm\alpha}(\rho) \qquad (2.16)$$

have this property, since

$$C^{\pm}(\rho) = C^{\pm}(\nu\rho\nu^{-1})$$

where ν is any open path. This is equivalent to change the base point A of the elements of $\pi_1(\Sigma^2)$ using the open path ν and then $\pi_1(\Sigma^2)$ changes by an isomorphism (see for instance ref. [8]). In order to calculate the Poisson brackets of $C^{\pm}(\rho)$ we first assume that the base points A and B of ρ and σ are different in order to avoid ambiguous contributions from intersections at the common base point. From figure 2, it is clear that $\nu = \sigma_1^{-1} \rho_1$ is the open path from A to B.

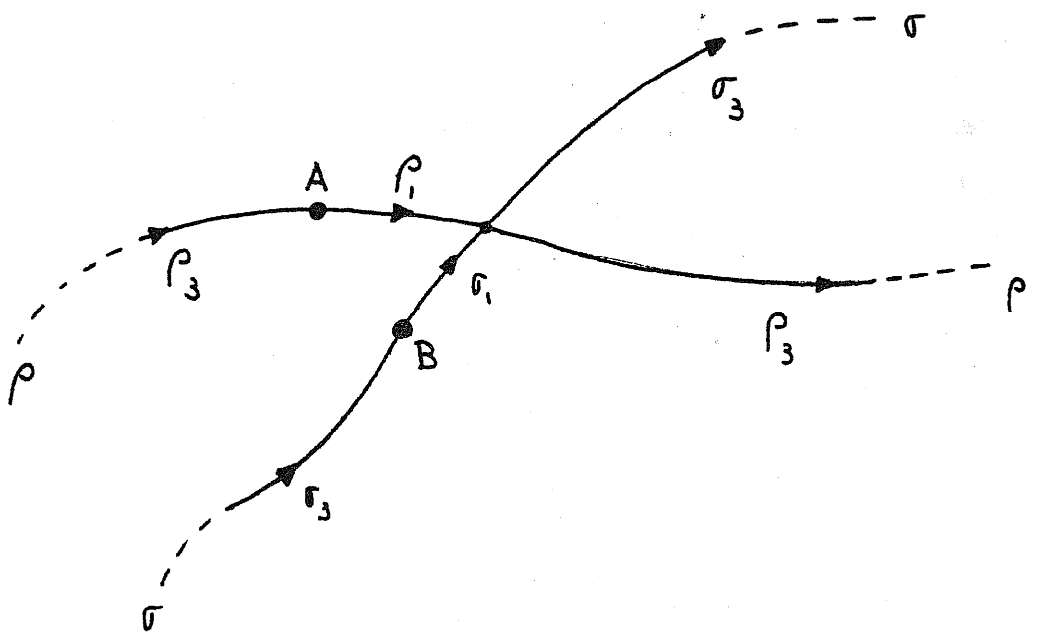

FIGURE 2.

In the right hand side of (2.15) appears the trace

$$C(\rho_3\sigma_1\sigma_3\rho_1) = C(\sigma_3\rho_1\rho_3\sigma_1) = C(\sigma\,\sigma_1^{-1}\rho_1\rho_3\sigma_1) = C(\sigma\rho')$$

where $\sigma = \sigma_3\sigma_1$ and $\rho' = \nu\rho\nu^{-1}$ are now both based in B. We also have $C(\rho') = C(\rho)$ and then we obtain for the Poisson brackets of the traces

$$\left\{ C^{\pm}(\rho)\, ,\, C^{\pm}(\sigma) \right\} = \pm \frac{i\, s}{2\alpha\sqrt{k}} \left[-C^{\pm}(\rho)\, C^{\pm}(\sigma) + C^{\pm}(\rho\sigma) \right]$$

$$\left\{ C^{+}(\rho)\, ,\, C^{-}(\sigma) \right\} = 0$$

The generators u_i, v_j (1.9) have a single intersection for $i = j$, while in al. other cases such as u_i, u_j or v_i, v_j they have not intersection. So we have ir general by setting $u \equiv u_i$, $v \equiv v_i$ $i = 1,2,\ldots,g$ that the trace algebra for the generators of $\pi_1(\Sigma^2)$ is

$$\left\{ C^{\pm}(u)\, ,\, C^{\pm}(v) \right\} = \pm \frac{i\, s}{2\alpha\sqrt{k}} \left[- C^{\pm}(u)\, C^{\pm}(v) + C^{\pm}(uv) \right]$$

$$\left\{ C^{+}(u)\, ,\, C^{-}(v) \right\} = 0 \qquad (2.17)$$

with zero Poisson brackets in all the other cases.

For 2×2 matrices the following identity holds:

$$C^{\pm}(u)\, C^{\pm}(v) = 1/2 \left[C^{\pm}(uv) + C^{\pm}(uv^{-1}) \right] \qquad (2.18)$$

so using it we can compute recursively all the traces $C^{\pm}(w)$ $w \in \pi_1(\Sigma^2)$ jus from the finite subset. For instance all the traces on the subgroup generated b

u,v can be computed from $C^{\pm}(u)$, $C^{\pm}(v)$ and $C^{\pm}(uv)$. It is therefore reasonable to assume that we can derive a representation of the whole traces $C^{\pm}(w)$ once we know an appropriate representation of (2.17).

Using (2.18) in (2.17), we obtain

$$\left\{ C^{\pm}(u) \, , \, C^{\pm}(v) \right\} = \pm \frac{i\, s}{4\alpha\sqrt{k}} \left[C^{\pm}(uv) - C^{\pm}(uv^{-1}) \right] \qquad (2.19)$$

which is an infinite Lie algebra subjet to the non-linear constrains

$$D^{\pm}(u,v) \equiv 1/2 \left[C^{\pm}(uv) + C^{\pm}(uv^{-1}) \right] - C^{\pm}(u)\, C^{\pm}(v) \approx 0 \qquad (2.20)$$

It is easy to check that

$$\left\{ C^{\pm}(w) \, , \, D^{\pm}(u,v) \right\} \approx 0$$

and so the constraints (2.20) form an ideal under the algebra (2.11). The general theory of representations of Lie algebras subject to non-linear constraints of the form (2.20) is yet to be developed. Here we will take a different attitude and work directly with the algebra (2.17) which is not a Lie algebra.

In the case k = 1, G = SO(3,1) with spinor group SL(2,C), and the traces $C^{+}(w)$ and $C^{-}(w)$ are mutually complex conjugate; if k = -1, G = SO(2,2) with spinor group SL(2,R) ⊗ SL(2,R) and then both traces $C^{\pm}(w)$ are real and independent observables. One can see that these statements are consistent with the conjugacy properties of (2.17) with $\sqrt{k} = 1$ for SO(3,1) and $\sqrt{k} = i$

for SO(2,2). From now on we will work directly with the algebra of the $C^+(w)$ and omit the + sign where there is no ambiguity. The algebra of $C^-(w)$ is directly obtained by conjugancy in the De Sitter case, and by a change of sign in the anti De Sitter case. For simplicity we also set $x \equiv C(u)$, $y \equiv C(v)$ and $z \equiv C(uv) = C(vu)$, the expression (2.17) is then

$$\{ x, y \} = \frac{i\, s}{2\alpha\sqrt{k}} (z - xy)$$

$$\{ y, z \} = \frac{i\, s}{2\alpha\sqrt{k}} (x - yz) \tag{2.21}$$

$$\{ z, x \} = \frac{i\, s}{2\alpha\sqrt{k}} (y - zx)$$

It is easy to check that the trace

$$C(uvu^{-1}v^{-1}) = 1 - 2F^2$$

where

$$F^2 = 1 + 2xyz - x^2 - y^2 - z^2 \tag{2.22}$$

lies on the centre of the algebra (2.21). The traces algebra (2.21) is also invariant under the following discrete transformations

$$x \longrightarrow -x \ , \quad y \longrightarrow -y \ , \quad z \longrightarrow z \tag{2.23a}$$
$$x \longrightarrow y \ , \quad y \longrightarrow z \ , \quad z \longrightarrow x \tag{2.23b}$$
$$x \longrightarrow y \ , \quad y \longrightarrow x \ , \quad z \longrightarrow 2xy - z \tag{2.23c}$$

Taking a convenient parametrization of the surface F^2 = const. it is possible to express the trace algebra (2.21) in terms of two canonical conjugate variables ξ and η. In order to parametrize that surface one can take advance of a theorem that asserts that in any cubic surface there exist 27 straight lines. For the surface (2.22) two of them that do not intersect are

$$z = -1, \qquad x + y = - F^2 \tag{2.24a}$$

and

$$z = 1, \qquad x - y = F^2 \tag{2.24b}$$

Introducing the parameters t,w, we have

$$x = t, \qquad y = -F^2 - t \;, \; z = -1$$

for (2.24a), and

$$x = w, \qquad y = -F^2 + w \;, \; z = 1$$

for (2.24b).

The pair (t,w) defines a pair of different points A and B in F^2. The parametric equation of a line passing through A and B is then given by

$$\left.\begin{aligned} x &= \lambda t + (1 - \lambda) w \\ y &= \lambda(- F^2 - t) + (1 - \lambda) (-F^2 + w) \\ z &= 1 - 2\lambda \end{aligned}\right\} \tag{2.25}$$

where λ is a new parameter. Substituting back (2.25) in (2.22) we obtain a cubic algebraic equation for λ, of course $\lambda = 0$ and $\lambda = 1$ are two roots of that equation, the other one is

$$\lambda = \frac{w^2 - (w - t) F^2 - 1}{w^2 - t^2} \qquad (2.26)$$

that defines a third point C of intersection with F^2 of the line passing through A and B. Substituting (2.26) in (2.25) one obtains the following parametrization of F^2 in terms of t and w:

$$\left. \begin{aligned} x &= \frac{1 + wt + F^2 (w - t)}{w + t} \\ y &= \frac{1 - wt}{w - t} \\ z &= \frac{2 + 2F^2 (w - t) - w^2 - t^2}{w^2 - t^2} \end{aligned} \right\} \qquad (2.27)$$

Substitution of this expression in the traces algebra (2.21) gives

$$\{ t, w \} = \pm \frac{i s}{4\alpha \sqrt{k}} (w^2 - t^2) \qquad (2.28)$$

and changing of variables

$$\begin{aligned} t &= e^{\xi} - e^{\eta} \\ w &= e^{\xi} + e^{\eta} \end{aligned} \qquad (2.29)$$

we finally obtain

$$\{ \xi , \eta \} = \pm \frac{i\, s}{2\alpha \sqrt{k}} \tag{2.30}$$

and

$$\begin{aligned} x &= \cosh \xi - 1/2\, e^{2\eta - \xi} + F^2 e^{\eta - \xi} \\ y &= \cosh \eta - 1/2\, e^{2\xi - \eta} \\ z &= 1/2\, e^{-(\xi + \eta)} + F^2 e^{-\xi} - \cosh (\xi - \eta) \end{aligned} \tag{2.31}$$

The canonical structure (2.30) for the traces does not yield a convenient quantization because of ordering problems in (2.31). In fact we have:

$$[x , y] = \pm \frac{i\, s \hbar}{2\alpha \sqrt{k}} \, (z - xy)$$

which is not consistent with the cyclic permutation (2.23b). In chapter IV we will do the quantization procedure directly from the expression (2.21) for the traces algebra and the resulting commutation relations will be invariant under transformations (2.23).

III) The Poisson Traces Algebra in the Anti-De Sitter Theory.

Before we attempt to quantize the traces algebra (2.21) we analyse the structure of traces at the classical level.

Setting $\sqrt{k} = i$ in (2.21) we have for the anti-De Sitter case, the Poisson traces algebra

$$\{ x , y \} = \frac{s}{2 \alpha} (z - xy) \tag{3.1}$$

and cyclical permutation of x, y, z (remember that this are the C^+ traces, the Poisson brackets for the C^- are obtained by a change of sign in the right hand side of (3.1)). As we remarked in the last chapter, x, y, z are traces of SL(2,R) matrices, so they are real. The Casimir :

$$F^2 = 1 + 2\, xyz - x^2 - y^2 - z^2 \tag{3.2}$$

is also real. In principle there is no bound on the value of the trace of an SL(2,R) matrix. However for certain values of F^2 the range of values of the traces will be restricted. Moreover in an SO(4) theory, with spinor group SU(2) ⊗ SU(2) the traces are also real and would have the same Poisson algebra (3.1) with central element (3.2). It is important then to understand how can we distinguish among traces of SL(2,R) matrices and those of SU(2). This can be done by looking at the real locus of the cubic surface

$$F^2 = \Phi = \text{const.} \tag{3.3}$$

If $\Phi > 1$, the surface has four sheets that map into each other by the transformations (2.23a). Writing

$$F^2 = (1 - x^2)(1 - y^2) - (z - xy)^2 \qquad (3.4)$$

we see that if $|x|, |y| \leq 1$ then $F^2 < 1$ and therefore at the most one among the variables x, y, z can have values in the interval $[-1, 1]$. Since F^2 is cyclically symmetric we may suppose $|x| \leq 1$ and $|y|, |z| > 1$. We will find that $F^2 < 0$, so that all three variables x, y, z must have absolute value > 1.

If $0 < \Phi \leq 1$, the surface has now five sheets. Four of this sheets correspond to those already discussed in case (*i*) with $|x|, |y|, |z| > 1$. For the other one let us write

$$1 - z^2 = \Phi + \psi$$

where

$$\psi \equiv x^2 + y^2 - 2xyz.$$

If $|z| \leq 1$ then $\psi \geq 0$ and $1 - z^2 \leq \Phi$ so that $|z| \leq \sqrt{1 - \Phi}$, and by the cyclical symmetry property of F^2 we have $|x|, |y|, |z| \leq \sqrt{1 - \Phi}$. These inequalities define a bounded sheet which contains the limiting case of the angular momentum theory when $x, y, z \ll 1$ and $\Phi \sim 1 - x^2 - y^2 - z^2 \leq 0$, $\{x, y\} \sim z$ with cyclical permutations.

If $\Phi \leq 0$, the central sheet merges with the external ones into a unique unbounded sheet. In short we have:

i) $\Phi > 1$. Four sheets with $|x|, |y|, |z| > 1$.

ii) $0 \leq \Phi \leq 1$. Five sheets, one with $|x|, |y|, |z| \leq 1$, and the other four as on case (*i*).

iii) $\Phi \leq 0$. One unbounded sheet $0 \leq |x|, |y|, |z| < \infty$.

Let x, y, z be traces of SL(2,R) matrices and let us determine the allowed range for Φ The matrices $S(u)$, $S(v) \in SL(2,R)$ may be written as:

$$S(u) = x + \xi_\alpha \tau^\alpha$$

$$S(v) = y + \eta_\beta \tau^\beta \qquad \alpha,\beta = 0,1,2 \tag{3.5}$$

where

$$\xi^2 \equiv \xi_\alpha \xi^\alpha = x^2 - 1$$

$$\eta^2 \equiv \eta_\alpha \eta^\alpha = y^2 - 1 \tag{3.6}$$

and τ^α are the pseudo-Pauli real matrices (see Appendix (A.4)) which satisfies

$$\tau^\alpha \tau^\beta = \eta^{\alpha\beta} + \epsilon^{\alpha\beta\gamma} \tau_\gamma$$

$\eta^{\alpha\beta} = (-1, 1, 1)$. The trace z is then given by

$$z = xy + \xi_\alpha \eta^\alpha$$

and substituting in (3.4) gives

$$F^2 = (x^2 - 1)(y^2 - 1) - (\xi_\alpha \eta^\alpha)^2$$

Let $|y| \geq 1$, from (3.6) we can set $\eta_0 = \eta_1 = 0$, it follows:

$$F^2 = (y^2 - 1)(x^2 - 1 - (\xi_2)^2) \tag{3.7}$$

If $|x| \leq 1$ then $F^2 \leq 0$, while if $|x| > 1$ the sign of F^2 is indefinite. By the cyclical symmetry of x, y, z in F^2 we deduce that if one of the SL(2,R) trace has absolute value less or equal than one then $\Phi \leq 0$.

The case (ii) $0 \leq \Phi \leq 1$ is the only one admitting a central bounded sheet with $|x|$, $|y|$, $|z| \leq 1$ which corresponds to an SO(4) theory.

In fact $S(u)$, $S(v) \in SU(2)$ may be written as

$$S(u) = x + i\xi_\alpha \sigma_\alpha$$

$$S(v) = y + i\eta_\alpha \sigma_\alpha$$

with

$$x^2 + \xi_\alpha \xi_\alpha = 1$$

$$y^2 + \eta_\alpha \eta_\alpha = 1 \ .$$

where σ_α are the Pauli matrices. Now it is clear that $|x|$, $|y|$, $|z| \leq 1$, and doing the same steps as on SL(2,R) case we find that for traces of SU(2) matrices:

$$F^2 = (1 - x^2)(1 - y^2)\sin^2\theta \tag{3.8}$$

where $\cos\theta = \xi_\alpha \eta_\alpha \ / \ |\xi| \ |\eta|$, which is clearly in the interval [0,1]. It is natural to conjecture that on the quantum theory. As we will see in the next chapter, in the quantum theory it is possible to construct an operator F^2 which reduces to the classical one for $\hbar \longrightarrow 0$ and plays the role of the Casimir operator. It is natural to conjecture that the bounded quantum representation of the traces algebra with $0 \leq F^2 \leq 1$ corresponds to an SO(4) theory and must be discarded.

IV) The Quantum Anti-De Sitter Theory.

The Poisson bracket algebra (3.1) of the traces reads:

$$\{x, y\} = \frac{s}{2\alpha}(z - xy)$$

$$\{y, z\} = \frac{s}{2\alpha}(x - yz) \tag{4.1}$$

$$\{z, x\} = \frac{s}{2\alpha}(y - zx)$$

Quantization is achieved by the correspondence principle $\{\ \}\ i\hbar = [\ ]$ and symmetrisation of the xy,yz,zx products. We obtain:

$$[x, y] = i\beta(z - 1/2(xy + yx)) \tag{4.2}$$

where
$$\beta \equiv \frac{s\hbar}{2\alpha}, \tag{4.3}$$

and all the operators are hermitian.

Introducing the variable θ ($-\pi < \theta < \pi$) where

$$\beta = 2\tan\theta/2 \tag{4.4}$$

eq.(4.2) may be written as

$$e^{i\theta/2}\, xy - e^{-i\theta/2}\, yx = 2i \sin \theta/2\, z$$

and by scaling the operators

$$x \longrightarrow \frac{x}{\cos \theta/2} \tag{4.5}$$

we obtain the commutation algebra for the traces operators:

$$\begin{aligned} e^{i\theta/2}\, xy - e^{-i\theta/2}\, yx &= i \sin \theta\, z \\ e^{i\theta/2}\, yz - e^{-i\theta/2}\, zy &= i \sin \theta\, x \\ e^{i\theta/2}\, zx - e^{-i\theta/2}\, xz &= i \sin \theta\, y. \end{aligned} \tag{4.6}$$

The (4.6) are commutation relations for the C^+ traces of SL(2,R) matrices, those associated with C^- traces, are obtained by changing $\theta \longrightarrow -\theta$ in (4.6) . On the following we shall construct hermitian representations of the algebra (4.6). Note that (4.6) is not a Lie algebra, and the classical limit corresponds to $\theta = 0$. The operator :

$$F^2 = \cos \theta + 2e^{i\theta/2}\, xyz - e^{i\theta}\,(x^2 + z^2) - e^{-i\theta} y^2 \tag{4.7}$$

commutes with all the trace operators and plays the role of a Casimir operator for the algebra (4.6), and in the limit $\theta \to 0$ it reduces to the expression (3.2) of the classical Casimir operator. In spite of the appearance F^2 is hermitian and cyclically symmetric in x,y,z, as one can check using (4.6).

The traces algebra (4.6) is invariant under the transformations

$$\left.\begin{array}{l} x \longrightarrow -x \\ y \longrightarrow -y \\ z \longrightarrow z \end{array}\right\} \tag{4.8}$$

$$\left.\begin{array}{l} x \longrightarrow y \\ y \longrightarrow z \\ z \longrightarrow x \end{array}\right\} \tag{4.9}$$

$$\left.\begin{array}{l} x \longrightarrow y \\ y \longrightarrow x \\ z \longrightarrow \dfrac{1}{\cos\theta/2}\,(\,xy + yx\,) - z \end{array}\right\} \tag{4.10}$$

which generalize the classical symmetries (2.23).

a) The raising and lowering operators.

We are now looking for finite dimensional hermitian representations of th algebra (4.6), and with base vectors f_a which are eigenvectors of z :

$$z\, f_a = a\, f_a\ . \tag{4.11}$$

Since $z^{\dagger} = z$, a is real and we can choose the basis f_a to be orthonormal. W write the unitary product of two vectors f_a and f_b as $(\, f_a\, ,\, f_b\,)$.

We construct the raising and lowering operator operator as follows, let K an operator of the form :

$$K = \lambda \, e^{i\theta/2} \, x + \mu y$$

with λ and μ complex constants, and demand that

$$z \, K \, f_a = \xi \, K \, f_a \tag{4.12}$$

with ξ some real number. Using (4.6) we find the homogenous equations

$$\begin{aligned} (a \, e^{-i\theta} - \xi) \, \lambda - i \sin\theta \, \mu &= 0 \\ i \sin\theta \, \lambda + (e^{i\theta} \, a - \xi) \, \mu &= 0 \end{aligned} \tag{4.13}$$

This system has a non trivial solution iff ξ satisfies the equation

$$\xi^2 + a^2 - 2 \cos\theta \, a \, \xi - \sin^2\theta = 0 \tag{4.14}$$

so that

$$\xi = a \cos\theta \pm \sin\theta \sqrt{1 - a^2}$$

ξ is real if $|a| \leq 1$ and we may write $a = \sin\nu$, with $-\pi/2 \leq \nu \leq \pi/2$ and :

$$\xi = \sin (\nu \pm \theta)$$

Substituting this values of a and ξ in the equations (4.13) we find that

$$\frac{\mu}{\lambda} = \pm i \, e^{\pm i\nu}$$

and setting $\lambda = 1$ we obtain

$$K^{\pm}_{\nu} = e^{i\theta/2} x \pm i \, e^{\pm i\nu} y \tag{4.15}$$

for the rising and lowering operators.

The operators (4.15) raise and low only the state f_a with $a = \sin \nu$ Universal raising and lowering operator that do not depend on the state ir which they are acting can be obtained by introducing the hermitian operator N:

$$Nf_a = \nu f_a \quad . \tag{4.16}$$

so that :

$$z = \sin N \tag{4.17}$$

Writing

$$K^{\pm} = e^{i\theta/2} \; x \pm iye^{\pm iN} \tag{4.18}$$

we derive the commutation relations

$$[\, N \, , K^{\pm} \,] = \pm \, \theta K^{\pm}. \tag{4.19}$$

The operators (4.18) are therefore the universal raising and lowerin operators that we were looking for.

The expressions (4.18) lead to :

$$x = 1/2 \, e^{-i\theta/2} \, (\, K^{-} e^{iN} + K^{+} e^{-iN} \,) \, \frac{1}{\cos N} \tag{4.20a}$$

$$y = i/2\ (K^- - K^+)\ \frac{1}{\cos N}. \tag{4.20b}$$

and :

$$e^{iN}\, y\, e^{-iN} = \left(e^{i\theta/2} \sin\theta\, x + y \cos(N+\theta) \right) \frac{1}{\cos N} \tag{4.21a}$$

$$e^{-iN}\, y\, e^{iN} = \left(- e^{i\theta/2} \sin\theta\, x + y \cos(N-\theta) \right) \frac{1}{\cos N} \tag{4.21b}$$

From (4.20) and (4.21) we have:

$$[\,K^+ , K^-\,] = \sin\theta \sin 2N \tag{4.22}$$

and

$$K^+K^- + K^-K^+ = 2\cos\theta \cos^2 N - 2F^2 . \tag{4.23}$$

The operators K^+, K^- and N obey the adjoint relations:

$$(\cos N\, K^+)^\dagger = \cos N\, K^- \tag{4.24a}$$

and

$$N^\dagger = N \tag{4.24b}$$

b) The finite-dimensional representations.

With the help of the operators N, K^+ and K^- it is possible to construct finite dimensional representations of the quantum traces algebra in more or less the same way as one constructs the representations of SO(3) [see for instance ref. 11 chap. 2]. Nevertheless, note that due to (4.22), the algebra of N, K^+ and K^- is not a Lie algebra. Let us define the number operator:

$$H = N/\theta \tag{4.25}$$

In terms of it, equations (4.19) and (4.22) may be written as:

$$[\, H \, , K^{\pm} \,] = \pm\, K^{\pm} \tag{4.26a}$$

and

$$[\, K^{+} \, , K^{-} \,] = \sin\theta \, \sin 2H\theta \tag{4.26b}$$

We rewrite (4.16) in the form:

$$H\, f_p = p\, f_p \qquad\qquad p \in R\,. \tag{4.27a}$$

From (4.26a) we have :

$$K^{-} f_p = \alpha_p f_{p-1} \tag{4.27b}$$

$$K^{+} f_p = \beta_{p+1} f_{p+1} \tag{4.27c}$$

with α_p and β_{p+1} some complex constants. Since we seek a finite dimensiona representation there must exist states f_k and f_l achieving the maximum anc minimum proper values k and l of H respectively, i. e : .

$$l \le p \le k \tag{4.28}$$

and

$$\beta_{k+1} = 0 \tag{4.29a}$$

and

$$\alpha_l = 0 \tag{4.29b}$$

From (4.26b) we have :

$$K^+K^-f_p = (K^-K^+ + \sin\theta \sin 2H\theta) f_p$$

so that:

$$\alpha_p\beta_p - \alpha_{p+1}\beta_{p+1} = \sin\theta \sin 2p\theta$$

this finite difference equation is easily integrated and yields :

$$\alpha_p\beta_p - \alpha_{k+1}\beta_{k+1} = \sin (p + k)\theta \sin (k - p + 1)\theta,$$

and because of (4.29a) we have:

$$\alpha_p\beta_p = \sin (p + k)\theta \sin (k - p + 1)\theta \tag{4.30}$$

From (4.24a) we have

$$(K^+)^\dagger = K^- \frac{\cos (H - 1) \theta}{\cos H \theta} \tag{4.31}$$

The states f_p are orthonormal, so that

$$(f_q , f_p) = \delta_{q,p} \tag{4.32}$$

and then because of :

$$(K^+f_p , f_{p+1}) = (f_p , (K^+)^\dagger f_{p+1})$$

we obtain

$$\beta_p = \alpha_p^* \frac{\cos (p - 1) \theta}{\cos p\theta} \tag{4.33}$$

Substituting (4.33) in (4.30) we have also :

$$|\alpha_p|^2 = \frac{\cos p\theta}{\cos(p-1)\theta} \sin(p+k)\theta \sin(k-p+1)\theta \qquad (4.34)$$

The dimension of the representation is

$$q = k + 1 - l \qquad (4.35)$$

We analise now the conditions to be imposed on l and k which are consistent with a positive r.h.s of (4.34) for $l + 1 \leq p \leq k$.
Since p is discrete the analysis of (4.34) is very difficult, so we shall limit ourselves to consider the case :

$$|p\theta| < \pi/2 \qquad \text{for } l \leq p \leq k. \qquad (4.36a)$$

(4.36a) is satisfied in the semiclassical limit and implies that $\cos p\theta$ and $\cos(p-1)\theta$ are always positive. Since (4.34) is invariant for $\theta \longrightarrow -\theta$, we may choose $\sin(p+k)\theta > 0$ and $\sin(k-p+1)\theta > 0$ for $l + 1 \leq p \leq k$, the first condition holds if (4.36a) holds, and also

$$p + k > 0; \qquad \text{for } l + 1 \leq p \leq k \qquad (4.36b)$$

while the last condition implies $|(k - p + 1)\theta| < \pi/2$ and from (4.35) it follows that

$$q < \frac{\pi}{|\theta|} + 1 \qquad (4.36c)$$

These conditions are very important, because they guarantee the hermiticity of the representation. (4.36a) and (4.29b) can be satisfied in two different ways leading to the inequivalent classes of representations A and B :

$$A) \quad \sin(\,l + k\,)\theta = 0 \tag{4.37a}$$

$$B) \quad \sin(\,k - l + 1)\theta = 0 \tag{4.37b}$$

In both cases we have :

$$|(l - 1)\theta| \neq \pi/2 \tag{4.38}$$

in order that the denominator in (4.34) be different from zero.

i) Case A.

Because of (4.36a), the only way to satisfy (4.37a) is $l = -k$, and the dimension of the representation is

$$q = 2k + 1 \tag{4.39}$$

since $q = 1,2,\ldots,$ it happens that k must be an integral or a half integral positive number as occurs in the angular momentum theory.

ii) Case B.

(4.37b) is incompatible with $l = k + 1$ because $l \leq k$ by hypothesis. The only way to satisfy (4.37b) and (4.36) is to set $(\,k - l + 1\,)\theta = \pm\,\pi$ so that :

$$|\theta| = \pi/q \tag{4.40}$$

θ must be quantized for a representation of type B to exist.
Eq. (4.36) also restrict the values of k to be

$$2k < q \leq 2k + 1 \tag{4.41}$$

and the value of l follows from (4.35). In this case in general k is neither an integral nor a half-integral number, and the choice $q = 2k + 1$ i (4.41) leads to a representation which is also of type A. In table 1 w resume the principal features of the A and B types representations

A	B
$\lvert\theta\rvert < \pi/q-1$	$\lvert\theta\rvert = \pi/q$
$q = 2k + 1$	$2k < q \leq 2k + 1$
$l = -k$	$l = k + 1 - q$
$l \leq p \leq k$	$l \leq p \leq k$
k is an integral or semiintegral positive number	k is a positive real number in general
$F^2 \leq 0$ for $k \geq \dfrac{\pi}{2\lvert\theta\rvert} - 1$	$F^2 < 0$ always
$\lvert z_p\rvert < 1$ for all the proper values of z	$\lvert z_p\rvert < 1$ for all the proper values of z

TABLE 1.

Substituting (4.27) and (4.33) in (4.17) and (4.20) one obtains for the matrix elements of the trace operators :

$$x_{q,p} \equiv (f_q , xf_p) = 1/2\, e^{-i\theta/2} \left(a_p e^{ip\theta} \delta_{q,p-1} + a^{*}_{p+1} e^{-ip\theta} \delta_{q,p+1} \right) \quad (4.42a)$$

$$y_{q,p} \equiv (f_q , yf_p) = i/2 \left(a_p\, \delta_{q,p-1} - a^{*}_{p+1} \delta_{q,p+1} \right) \quad (4.42b)$$

$$z_{q,p} \equiv (f_q, zf_p) = \sin p\theta \delta_{q,p} \quad (4.42c)$$

where

$$a_p \equiv \frac{\alpha_p}{\cos p\, \theta} \quad (4.43)$$

and from (4.34)

$$|a_p|^2 = \frac{\sin (p + k)\theta \sin (k - p + 1)\theta}{\cos p\theta \cos (p - 1)\theta} \quad (4.44)$$

Representations A and B are irreducible by construction because from a given state f_p it is possible to reach any other state using raising and lowering operators. From (4.23), (4.27), (4.33) and (4.43) we have for the Casimir operator that

$$(f_q, F^2 f_p) = \Big\{ \cos\theta \cos^2 p\theta -$$

$$- 1/2 \cos p\theta \, (|a_p|^2 \cos(p-1)\theta + |a_{p+1}|^2 \cos(p+1)\theta) \Big\} \delta_{q,p} \quad (4.45)$$

Substitution of (4.44) gives directly F^2 as a multiple of the identity :

$$(f_q, F^2 f_p) = \cos(k+1)\theta \cos k\theta \, \delta_{q,p} \quad (4.46)$$

For $|\theta| = \pi/q$ all the q-dimensional B-type representations, because o: (4.46), differ in the values of k satisfying $2k < q \leq 2k+1$ and in that o: F^2 as well , they are therefore inequivalent.

The coefficients a_p of the A and B type representations are determined up to a phase and we can set them to be real without loss of generality. In fact, a unitary matrix such that

$$S^\dagger z S = z \quad (4.47)$$

is of the form

$$S_{q,p} = e^{i\tau p} \delta_{r,p} \qquad \tau_p \in R \quad (4.48)$$

and taking

$$S^\dagger x S = x' \qquad S^\dagger y S = y' \quad (4.49)$$

the coefficients a_p are transformed into:

$$a'_p = a_p e^{i(\tau_p - \tau_{p-1})} \quad (4.50)$$

for $l + 1 \leq p \leq k$ (remember that $a_l \equiv 0$). By a proper selection of the τ_p's the a'_p can be set to be real.

We saw on the classical analysis of the traces that, in order that they correspond to a SL(2,R) matrix and not to a SU(2) matrix, the value of F^2 must be negative if the absolute value of one of the traces is less than one. In the A and B type representations because of (4.17), $||z|| \leq 1$ and we must consider the representation for which $F^2 < 0$. The expression (4.46) is indefinite in sign for general values of k and θ however in the semiclassical limit $\theta << 1$

$$F^2 \simeq 1 - (k^2 + k + \tfrac{1}{2}) \theta^2 > 0 \qquad (4.51)$$

so F^2 is positive up to quantum fluctuations and the type A and B representations corressponds to an SO(4) theory. Nevertheless this representations do not have direct physical significance their study is interesting because they are the direct generalizations of the theory of the angular momentum.

c) The type C representations.

Besides the type A and B representations there exists another type of representation, that we will call C, having the property that $|z_p| \leq 1$ for all the proper values of z. Substituting the expressions (4.42) for the matrix elements of x, y and z into the commutation relations (4.6) one obtains the following relations for a_p ;

$$|a_p|^2 \cos(p-1)\theta - |a_{p+1}|^2 \cos(p+1)\theta = 2\sin\theta \sin p\theta. \qquad (4.54)$$

Substitution of (4.44) gives an identity, as it must be, nevertheless it is also possible to satisfy (4.54) setting

$$|a_p|^2 = 1 \qquad (4.55)$$

for all p. Now there is no upper or lower state as in the A and B representations.

The fact that now p ranges over an infinite domain ($p \in Z$) not necessarily implies that the C type representations are of infinite dimension. To see this note that the observables of the system are not the proper values p of the operator H (see (4.25) and (4.27a)), but the proper values of z that are given by $z_p = \sin p\theta$. So if

$$\theta = \frac{2\pi n}{q} \qquad (4.56)$$

with $n \in Z$, $q = 1,2,3,\ldots.,$ it happens that $z_p = z_{p+q}$ and the representation is of dimension q. In order that $-\pi < \theta < \pi$ (see (4.4)) we must impose

$$q > 2\,|n| \qquad (4.57)$$

If (4.56) does not hold the representation is infinite-dimensional. Substituting (4.45) in the expression (4.45) for F^2 one finds that

$$F^2 = 0 \qquad (4.58)$$

for all the type C representations, so that this type corresponds to th particular case in which $S(u)$, $S(v) \in SL(2,R)$ of (3.5) have $\xi^{\alpha} \propto \eta^{\alpha}$ and are restricted to an abelian subgrup of $SL(2,R)$.

In the infinite dimensional case, we can set $a_p = 1$ for all p using a unitary transformation of the form (4.48). In the finite dimensional case $p \in Z_q$ and we must set

$$a_p = a_{p+q}$$

and (4.59)

$$\tau_p = \tau_{p+q}$$

in (4.48). Writing

$$a_p = e^{i\rho_p} \qquad \rho_p \in R \tag{4.60}$$

(also with $\rho_p = \rho_{p+q}$) we find from (4.50) that

$$\rho'_p = \rho_p + \tau_p - \tau_{p-1}$$

and due to (4.59) it happens that

$$\Phi \equiv \sum_{p \in Z_q} \rho'_p = \sum_{p \in Z_q} \rho_p \tag{4.61a}$$

So we see that in the finite-dimensional case, the unitary transformation (4.47) preserves the sum Φ and then, each value of $\Phi \in [\ 0\ ,\ 2\pi\ )$ characterizes inequivalent representations of dimension q. While on the infinite-dimensional case, all the numbers in $[\ -1\ ,\ 1\ ]$ are proper values of z, in the q-dimensional case, one starts with a proper value

$$z_o = \sin \lambda\theta \tag{4.61b}$$

with $0 \le \lambda < 1$ and all the others are given by

$$z_p = \sin(\lambda + p)\theta, \qquad p \in Z_q \qquad (4.61c$$

Different values of λ characterize inequivalent representations. However th
q-dimensional representations are in general reducible when 2n and q i
(4.56) are not relative primes and then λ has not a single value.

It is possible to develop the type-C representations in a very general wa
introducing the hermitian operators ξ and η satisfying the commutator algebra

$$[\xi, \eta] = i \qquad (4.62)$$

and writing for the traces

$$\begin{aligned} x &= 1/2\,(A + A^{\dagger}) \\ y &= 1/2\,(B + B^{\dagger}) \\ z &= 1/2\,(C + C^{\dagger}) \end{aligned} \qquad (4.63)$$

where

$$\begin{aligned} A &= e^{i(\alpha\xi + \phi)} \\ B &= e^{i(\beta\eta + \psi)} \\ C &= e^{-i(\alpha\xi + \beta\eta + \phi + \psi)} \end{aligned} \qquad (4.64)$$

are unitary operators on which $\alpha\beta = -\theta$ and $\phi, \psi \in R$. Using the Bake
Campbell-Hausdorff formula (see (A.1)) it is easy to check that A, B and
satisfy the relations

$$AB = e^{i\theta} BA$$
$$ABC = e^{i\theta/2} \tag{4.65a}$$

and other analogous, by cyclical permutation of A, B and C. For the q-dimensional case when (4.56) holds, using (4.65a) one obtains that A^q, B^q and C^q lie on the centre of the traces algebra (4.6) and their proper values of the form $e^{i\omega}$, with $\omega \in R$ characterizes inequivalent representations in the same way as Φ and λ of (4.61) do. In fact in the particular base in which the matrix elements of the trace operators are given by (4.42), it happens that

$$\begin{aligned} A^q &= e^{-i [\Phi + n\pi (2\lambda - 1)]} \\ B^q &= i^q e^{i\Phi} \\ C^q &= (-i)^q e^{i 2\pi n\lambda} \end{aligned} \tag{4.65b}$$

and of course, due to (4.65a) they are related by $A^q B^q C^q = e^{in\pi}$. The commutator algebra (4.62) has the well known Schroedinger realization.

$$\xi = \xi$$
$$\eta = -i\, \partial/\partial\xi$$

with ξ and η acting on $\Psi(\xi)$ and $\Psi \in L^2(R)$ (the space of square integrable functions).

d) The hybrid representation.

The expressions (4.17) and (4.20) are completely general and using the commutation relations (4.19) and (4.22) one verifies easily that the quantum traces algebra (4.6) is satisfied. The hermitian nature of the trace operators impose certain adjoint relations among the N, K^+ and K^- operators and we took them to be (4.24), however these are not the only adjoint relations that follow from the hermiticity of the trace operators. From equation (4.17) i.e.

$$z = \sin N \tag{4.17}$$

we have that $z^\dagger = z$ if:

$$i) \quad N^\dagger = N \qquad \text{elliptic case} \tag{4.66a}$$

$$ii) \quad N^\dagger = (2n+1)\pi - N \qquad \forall n \in Z \quad \text{hyperbolic case} \tag{4.66b}$$

For the elliptic case it is clear that $|z| \leq 1$, instead, for the hyperbolic case we have that N may be written as $N = (2n+1)\pi/2\ iM$, with M hermitian and then $z = (-1)^n \cosh M$ so that $|z| \geq 1$. Using equations (4.21) one obtains that

$$i) \quad (\cos NK^+)^\dagger = \cos NK^- \qquad \text{elliptic case} \tag{4.67a}$$

$$\left.\begin{aligned} ii) \quad (\cos NK^+)^\dagger &= -\cos NK^+ \\ (\cos NK^-)^\dagger &= -\cos NK^- \end{aligned}\right\} \text{hyperbolic case} \tag{4.67b}$$

The commutator algebra (4.19) and (4.22) can be represented by introducing the operator ζ such that

$$[\,\zeta\,,\,N\,] = i\theta \tag{4.68}$$

and writing the raising and lowering operators as

$$K^{+} = f(N)\;e^{i\zeta} \tag{4.69a}$$

and

$$K^{-} = g(N)\;e^{-i\zeta} \tag{4.69b}$$

where f(N) and g(N) are some complex analytic functions of the general complex operator N. Because of (4.66) and (4.68) it happens that

$$i)\quad \zeta^{\dagger} = \zeta \qquad \text{elliptic case} \tag{4.70a}$$

$$ii)\quad \zeta^{\dagger} = -\,\zeta \qquad \text{hyperbolic case} \tag{4.70b}$$

Equation (4.69) satisfies equations (4.19) identically, while substitution in (4.22) and (4.23) gives for f and g the condition

$$f(N)\,g(\,N - \theta\,) = \cos\,(\,N - \theta\,)\,\cos N - F^{2} \tag{4.71}$$

The conjugation relations (4.67) gives another three relations for f and g:

$$\cos\,(\,N + \theta\,)\;f^{*}(\,N + \theta\,) = \cos N\;g(N) \tag{4.72a}$$

$$\cos\,(\,N - \theta\,)\;f^{*}(\,(\,2n + 1)\,\pi - N + \theta\,) = \cos N\;f\,(N) \tag{4.72b}$$

$$\cos\,(\,N + \theta\,)\;g^{*}(\,(\,2n + 1)\pi - N - \theta\,) = \cos N\;g(N) \tag{4.72c}$$

Note carefully that here and in the following the complex conjugation symbc " $*$ " on the functions means conjugation only of the functions and not of th argument, so that

$$f^{*}(N) \equiv \left[f(N^{\dagger}) \right]^{\dagger}.$$

From equations (4.72a) and (4.72b) follows directly that

$$f(N) = g(\,(2n+1)\pi - N\,) \qquad (4.73)$$

and substitution in (4.71) with $N \longrightarrow N + \theta/2$ gives

$$F^{2} = \cos(N - \theta/2)\cos(N + \theta/2) - $$
$$- f(N + \theta/2)\, f(\,(2n+1)\pi - N + \theta/2) \qquad (4.74)$$

From equations (4.72) and (4.73) one can check that

$$f^{*}(N + \theta/2)\, f^{*}(\,(2n+1)\pi - N + \theta/2) =$$
$$= f(N - \theta/2)\, f(\,(2n+1)\pi - N + \theta/2) \qquad (4.75)$$

holds in general, so that $F^{2\dagger} = F^{2}$ both for $N^{\dagger} = N$ and $N^{\dagger} = (\,2n + 1)\pi -$ N Using (4.72a) in (4.71) and taking $N \longrightarrow N + \theta/2$ we see that F^{2} can t expressed in the convenient form.

$$F^{2} = \cos(N - \theta/2)\,\cos(N + \theta/2) -$$
$$- \frac{\cos(N + \theta/2)}{\cos(N - \theta/2)}\, f^{*}(N + \theta/2)\, f(N + \theta/2) \qquad (4.76)$$

For the elliptic case ($N^\dagger = N$) we have that

$$|f(N + \theta/2)|^2 = \cos^2(N - \theta/2) - \frac{\cos(N - \theta/2)}{\cos(N + \theta/2)} F^2 \qquad (4.77)$$

and the right hand side of the equation must be non negative for all the proper values of N. This gives a restriction in N analogous to that which appears for the A and B type representations. In fact for $0 < F^2 \leq 1$ we can reobtain the A and B type representations (see eq. (4.46)) ; which corresponds to an SO(4) theory. For the hyperbolic case ($N^+ = (2n + 1)\pi N$) there is not an equation as (4.77) and the values of

$$f^*(N + \theta/2)\, f(N + \theta/2)$$

are just given by (4.76). In this case, because $||z|| \geq 1$ the trace operator corresponds always to the anti-De Sitter SO(2,2) theory independently of the sign of F^2.

We can build a Schroedinger representation of the traces algebra consistent with (4.68) by setting:

$$i) \quad N = \eta, \quad \zeta = i\theta \frac{\partial}{\partial \eta} \qquad \text{elliptic case} \qquad (4.78a)$$

and

$$ii) \quad N = \pi/2\,(2n + 1) + i\eta, \quad \zeta = \theta \frac{\partial}{\partial \eta} \qquad \text{hyperbolic case} \qquad (4.78b)$$

and with the space of square integrable functions $\psi(\eta)$ as the Hilbert space.

Using equations (4.17) and (4.20) it is direct to see that in order that an operator $\Lambda(N)$ commutes with all the elements of the algebra, it must happen that

$$\Lambda(N) = \Lambda(N + p\theta)$$

for all $p \in Z$ so the hybrid representation can be reducible in some case where the values of θ are quantized and Λ is a periodic function.

V) The Quantum De Sitter Theory.

In the De Sitter theory (SO(3,1)) the spinor group is SL(2,C) and the traces C^+ and C^- are complex conjugate among them as we saw in chapter II. The Poisson brakets traces algebra is given by (2.21) with $\sqrt{k} = 1$, so that

$$\{ x, y \} = i \frac{s}{2 \alpha} (z - xy) \qquad (5.1)$$

and the cyclical permutations of x, y, z.

As in the last chapter; we do the transition of Poisson brackets into commutators and symmetrize the xy product obtaining

$$[x, y] = -\beta (z - 1/2 (xy + yx))$$

where

$$\beta = \frac{s \hbar}{2 \alpha} \qquad (5.2)$$

Now setting

$$\beta = - 2 \tanh \theta/2 \qquad (5.3)$$

and scaling the operators by

$$x \longrightarrow \frac{x}{\cosh \theta/2} \qquad (5.4)$$

we obtain the commutator algebra

$$e^{\theta/2}xy - e^{-\theta/2}yx = \sinh\theta\, z \tag{5.5}$$

and the cyclical permutations of x, y, z. The commutator algebra for C traces can be obtained simply taking the adjoint in (5.5).

The Casimir operator now turns to be

$$F^2 = \cosh\theta + 2e^{\theta/2}xyz - e^{\theta}(x^2 + z^2) - e^{-\theta}y^2 \tag{5.6}$$

F^2 is cyclically symmetric in x, y, z as in the SO(2,2) theory, but it is no hermitian.

A representation of the algebra (5.5) can be written in terms of th operators $K^{\pm}$ and N by doing the following substitutions

$$\theta \longrightarrow -i\theta$$

$$N \longrightarrow iN$$

$$K^{+} \longrightarrow K^{-}$$

$$K^{-} \longrightarrow K^{+}$$

in the formulae of the past chapter. The resulting formulae are

$$K^{+} = e^{\theta/2} - iye^{N}$$

$$K^{-} = e^{\theta/2} + iye^{-N} \tag{5.7}$$

$$[N, K^{\pm}] = \pm\,\theta K^{\pm}$$

$$[K^{+}, K^{-}] = -\sinh\theta \sinh 2N \tag{5.8}$$

$$\left.\begin{aligned} x &= 1/2\, e^{-\theta/2} \, (K^{+} e^{-N} + K^{-} e^{N}) \frac{1}{\cosh N} \\ y &= i/2 \, (K^{+} - K^{-}) \frac{1}{\cosh N} \\ z &= i \sinh N \end{aligned}\right\} \qquad (5.9)$$

$$K^{+} K^{-} + K^{-} K^{+} = 2\cosh \theta \cosh^{2} N - 2F^{2} \qquad (5.10)$$

Introducing the operator ζ such that

$$[\, \zeta, N \,] = \theta \qquad (5.11)$$

it is possible to construct an analogous of the hybrid representation of the last chapter. We write for $K^{\pm}$:

$$\begin{aligned} K^{+} &= f(N) e^{-\zeta} \\ K^{-} &= g(N) e^{\zeta} \end{aligned} \qquad (5.12)$$

and from equations (5.8) and (5.10) we obtain

$$f(N)\, g(N - \theta) = -\sinh N \sinh (N - \theta) - F^{2} \qquad (5.13)$$

However, now we have no hermiticity relations for anyone of the operators in the De Sitter theory. There remains a lot of work to do on the theory of

representations of this model, but to the moment this is our prese
knowledge.

VI) Conclusion.

We have developed the theory of the commutation relations of the traces obtained from the integrated connection in the 2 + 1 dimensions De Sitter SO(3,1) and anti-De Sitter SO(2,2) theories. This is a direct generalization of the 2 + 1 dimensional Poincare' ISO(2,1) theory that has been developed in ref.[6]. In order to obtain Poincare' gravity from De Sitter gravity we must set $\lambda \longrightarrow 0$, or what is the same $\alpha \longrightarrow \infty$. However one must be careful when taking this limit in several points. The calculations leading to the Poincare' limit in the traces algebra are involved and we will add a few comments only in this respect.

The groups SO(3,1) and SO(2,2) are simple, while ISO(2,1) is not. In fact ISO(2,1) is the semidirect product of SO(2,1) and the abelian group of translations T^3 in 2 + 1 dimensions. In components the De Sitter connection is

$$\omega^a{}_b = \begin{pmatrix} \omega^A{}_B & \dfrac{K}{\alpha}\, e^A \\ -\dfrac{i}{\alpha}\, e_B & 0 \end{pmatrix} \tag{6.1}$$

(see equation (1.1)), while the Poincare' connection is

$$\begin{pmatrix} \omega^A{}_B & e^A \\ 0 & 0 \end{pmatrix} \tag{6.2}$$

so we must take the limit in (6.1) in such a way that $K/\alpha \longrightarrow 1$ as $\alpha \longrightarrow \infty$. In the Poincare' connection (6.2) ω^{AB} has values in the Lie algebra of SO(2,1) while e^{A} in the Lie abelian algebra of T^{3}. The integrated Poincare' connection is then

$$\psi^{ab} = \begin{pmatrix} E^{AB} & J^{A} \\ 0 & 1 \end{pmatrix}$$

with $E \in SO(2,1)$ and $J \in T^{3}$. The gauge invariants of the Poincare' connection turns to be $E^{A}{}_{A}$ and $\epsilon_{ABC}\, J^{A} E^{BC}$. We refer the interested reader to ref. [6 for the details.

As a last comment we speak about the multycrossing paths. In our work we have analysed the traces algebra obtained from the generators $C(u)$, $C(v)$ $C(uv)$ of one genus of all the surface, that has a single intersection (see figure 1). In chapter II we argued that the algebra can be generated from this generators applying recursively the equation (2.18) for the product of two traces of 2×2 matrices, that is

$$C(u)\, C(v) = 1/2\ (\, C(uv) + C(uv^{-1})\,) \tag{6.3}$$

(Here and in what follows we are taking $C \equiv C^{+}$ and we will work in a anti-De Sitter theory).

For the genus generated by u and v, it is possible to calculate just from $x = C(u)$, $y = C(v)$, $z = C(uv)$ any Poisson bracket. In particular one can check that the following formula holds

$$\{ C(uv), C(u^n) \} = n \frac{s}{2\alpha} \left[C(u^{n-1} v^{-1}) - C(uv)\, C(u^n) \right] \tag{6.4}$$

for n = 1,2,... and cyclical permutations of $C(uv)$, $C(u)$, $C(u^{n-1}v^{-1})$; which for n = 1 reduces to the one-intersection formula

$$\{ z, x \} = \frac{s}{2\alpha} (y - zx)$$

Decomposing the traces in terms of x, y, z one can express (6.4) only in terms of them. However when one goes to the quantum level, ordering problems appear and the analysis is not easy. For n = 2 and 3 we have done directly the calculations and obtained the following results.

For n = 2 we have from (6.3) directly that

$$x_2 \equiv C(u^2) = 2x^2 - 1 \tag{6.5}$$

and using the quantum commutation relations (4.6) we obtain

$$e^{i\theta} z x_2 - e^{-i\theta} x_2 z = i \sin 2\theta \, z_2 \tag{6.6a}$$

$$e^{i\theta} x_2 z_2 - e^{-i\theta} z_2 x_2 = i \sin 2\theta \, z \tag{6.6b}$$

$$e^{i\theta} z_2 z - e^{-i\theta} z z_2 = i \sin 2\theta \, x_2 + 2i \sin\theta \, F^2 \tag{6.6c}$$

where

$$z_2 \equiv \frac{1}{\cos \theta/2} (xy + yx) - z \tag{6.7}$$

One can see from the equation (6.6c) the appearance of a central extension just as what happens in the Virasoro algebra [7,12].

For n = 3:

$$x_3 \equiv C(u^3) = 4x^3 - 3x \tag{6.8}$$

and the calculations give

$$e^{i3/2\,\theta} z x_3 - e^{-i3/2\,\theta} x_3 z = i \sin 3\theta \; z_3 \tag{6.9a}$$

$$e^{i3/2\,\theta} x_3 z_3 - e^{-i3/2\,\theta} z_3 x_3 = i \sin 3\theta \; z \tag{6.8b}$$

$$e^{i3/2\,\theta} z_3 z - e^{-i3/2\,\theta} z\, z_3 = i \sin 3\theta \; x_3 +$$

$$+ 8\, i \frac{\sin 3\theta \cos \theta}{2\cos 2\theta + 1} \times F^2 \tag{6.9c}$$

where

$$z_3 \equiv e^{i\theta}(4x^2 y - y - 2xze^{i\theta/2}) \tag{6.10}$$

Now in the equation (6.9c) there appears not just a central element, bu another term.

One can object at this point that there is little interest in the expressions (6.6) and (6.9) once we have found the representation of the operators x, y, z However one could attempt to take the equations (6.6a) and (6.9a) as definin equations for z_2 and z_3 respectively, thus generalizing equation (6.3) for the product of two traces, to the quantum level. In fact one can control that for $\theta = 0$ $z_2 \equiv C(uv^{-1})$ and $z_3 \equiv C(u^2 v^{-1})$ just using (6.3). So in general we coul set

$$C(u^{n-1}v^{-1}) \equiv \frac{1}{i \sin n\theta} \left[e^{in\theta/2} C(uv)\, C(u^n) - e^{-in\theta/2} C(u^n)\, C(uv) \right] \quad (6.11)$$

where $z = C(uv)$ and $C(u^n)$ can be expressed in terms of powers of $x = C(u)$ using (6.3). Note that (6.11) generalizes directly the commutation relations (4.6) for the multycrossing case with $n\theta$ instead of θ.

Appendix.

Baker-Campbell-Hausdorff formulae [7,12]

$$\exp(t\,X)\,\exp(t\,Y) = \exp(t\,Y)\,\exp(t\,X)\,\exp(t^2[X,Y] + O(t^3))$$

$$\exp(t\,X)\,\exp(t\,Y) = \exp(t[x+y] + 1/2t^2[x,y] + O(t^3)) \qquad (A.1)$$

for x and y operators and $t \in C$.

Small latin indices a, b, c,..... take the values 0, 1, 2, 3.

Capital latin indices A, B, C,...... take the values 0, 1, 2.

Greek indices α, β, γ,...... when designate space-time quantities take the values 0, 1, 2.

$$\eta_{ab} = (-1,\ 1,\ 1,\ k\ )$$

$$k = \begin{cases} +1 & \text{for } SO(3,1)\text{, De Sitter theory} \\ -1 & \text{for } SO(2,2)\text{, anti De Sitter theory} \end{cases}$$

$$\sqrt{k} = \begin{cases} 1 & \text{for } SO(3,1) \\ i & \text{for } SO(2,2) \end{cases}$$

$$\eta_{AB} = (-1, 1, 1), \qquad \epsilon_{0123} = -1, \qquad \epsilon^{0123} = k$$

$$\epsilon_{012} = -\epsilon^{012} = 1, \qquad \epsilon_{ABC3} = -\epsilon_{ABC} \tag{A.2}$$

$$\epsilon^{abrs}\,\epsilon_{cdrs} = -2k\left(\delta_c^{\ a}\delta_d^{\ b} - \delta_c^{\ b}\delta_d^{\ a}\right)$$

$$\epsilon^{ABR}\,\epsilon_{CDR} = -\left(\delta_C^{\ A}\delta_D^{\ B} - \delta_C^{\ B}\delta_D^{\ A}\right) \tag{A.3}$$

$$\epsilon^{\alpha\beta\tau}\,\epsilon_{\mu\nu\tau} = -\left(\delta_\mu^{\ \alpha}\delta_\nu^{\ \beta} - \delta_\mu^{\ \beta}\delta_\nu^{\ \alpha}\right)$$

{ , } designate Poisson brakets
[,] designate quantum commutator
(,) designate matrix commutator

The pseudo-Pauli real matrices are

$$\tau^0 = i\sigma^2, \qquad \tau^1 = \sigma^3, \qquad \tau^2 = \sigma^1$$

where σ^i are the Pauli matrices, and satisfy:

$$\tau^\alpha\,\tau^\beta = \eta^{\alpha\beta} + \epsilon^{\alpha\beta\gamma}\,\tau_\gamma$$

$$\eta^{\alpha\beta} = (-1, 1, 1). \tag{A.4}$$

Dirac Matrices conventions.

$$\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$$
$$\gamma_{ab} = 1/2\ (\ \gamma_a, \gamma_b\ ) \tag{A.5}$$

$$\gamma_5 = k\,\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3\ , \qquad \gamma_5^2 = -k$$
$$-1/2\,\epsilon^{abcd}\gamma_a\gamma_b = \gamma_5\,\gamma^{cd} \tag{A.6}$$

$$1/2(\ \gamma_{ab}, \gamma_c\ ) = \gamma_a\eta_{bc} - \gamma_b\eta_{ac} \tag{A.7}$$

In terms of the Pauli matrices the representation is

$$\gamma_0 = i\sigma_2 \otimes \sigma_3, \qquad \gamma_1 = \sigma_3 \otimes \sigma_3, \qquad \gamma_2 = -1 \otimes \sigma_1,$$

$$\gamma_3 = -\sqrt{k}\,\sigma_1 \otimes \sigma_3, \qquad \gamma_5 = i\sqrt{k} \otimes \sigma_2,$$

The proyectors

$$P_\pm = \pm \frac{1}{2i\sqrt{k}}(\ \gamma_5\ \pm i\sqrt{k}\ 1)$$

satisfy $P^2_\pm = P_\pm$, $\quad P_+P_- = P_-P_+ = 0.$ In this

representation

$$P_\pm = 1 \otimes p_\pm, \qquad p_\pm = 1/2(\ 1 \pm \sigma_2\ ) \tag{A.8}$$

It follows that

$$\gamma^{ab} \otimes \gamma_{ab} = -4(\sigma_i \otimes p_+) \otimes (\sigma_i \otimes p_+) - 4(\sigma_i \otimes p_-) \otimes (\sigma_i \otimes p_-) \qquad (A.9)$$

where

$$\sigma_{i\alpha}{}^{\beta}\sigma_{i\mu}{}^{\nu} = -\delta_{\alpha}{}^{\beta}\delta_{\mu}{}^{\nu} + 2\delta_{\alpha}{}^{\nu}\delta_{\mu}{}^{\beta} \qquad (A.10)$$

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