SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

Asymptotic safety and the electroweak interactions

Ph.D. THESIS

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Chapter 1

Introduction

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The existence of a nontrivial fixed point for the renormalization group flow can make a quantum field theory consistent up to arbitrarily high energies. The good ultraviolet (UV) limit is ensured by the finiteness of all dimensionless couplings when energy goes to infinity. This is achieved by requiring that the theory lies on a renormalization group trajectory that flows towards the fixed point in the UV. This property was called 'asymptotic safety' by Weinberg in [Weinberg 1976] and is equivalent to a generalized version of renormalizability.

Asymptotic safety was introduced as a way of constructing a consistent quantum field theory for general relativity [Weinberg 1979a], but for this idea to give a physically viable theory it is necessary that also other interactions should behave in this way. Strong interactions are already described by an asymptotically safe theory and there are reasons to believe that this result is not ruined by the coupling to gravity [Folkerts 2012]. On the other hand, electroweak interactions are not UV complete because some of their perturbative beta functions are positive. In this case there is a room for application of asymptotic safety. If the world is described by an asymptotically safe theory, there are two main possibilities. In the first case, each interaction is asymptotically safe by itself and reaches the fixed point at its own characteristic energy scale. The second case is that asymptotic safety is an inherently gravitational phenomenon which would manifest itself at the Planck scale and the coupling to gravity makes all other interactions safe. In the thesis the first scenario will be mainly explored, but in this introduction a general description of possible asymptotic safety applications is presented.

The problem with general relativity is that a fully consistent quantum field theory of gravity does not exist. This does not mean that it is not possible to compute quantum gravitational predictions: at least at low energy, quantum gravity can be described by an effective field theory based on metric degrees of freedom, as was first shown in [Donoghue 1994]. Effective field theory techniques [Weinberg 1979b, Georgi 1984, Gasser 1985, Donoghue 1992, Pich 1998] have become a powerful tool used in particle physics. Effective field theory can be seen as a procedure of organizing calculations which separates out the effects of high energy physics from the known quantum effects at low energy. General relativity is a field theory in which this treatment can be naturally applied. The known manifestation of an effective field theory which is close to gravity is chiral perturbation theory [Weinberg 1979b, Gasser 1984, Pich 1995] where the pion dynamics is described by a nonlinear sigma model. Gravity and the nonlinear sigma model are both nonpolynomially interacting theories and, from the power counting point of view, they have exactly the same structure.

The guiding principle of general relativity is that of local invariance under coordinate transformations. In following this principle, one is forced to introduce a geometry and to define an action for the theory using invariant terms under the general coordinate transformations. Since many quantities are invariant, the action for gravity can be organized in powers of curvatures:

$$\mathcal{S}_{grav}[g] = \int d^dx\,\sqrt{-g}\left\{-\frac{1}{\kappa^2}\Lambda + \frac{1}{2\kappa^2}R + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + \cdots\right\} \tag{1.1}$$

In eq. (1.1) the metric field is denoted by $g_{\mu\nu}$, the quantity $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, $R_{\mu\nu}$ is the Ricci tensor and the dots represent invariant terms containing higher powers of R, $R_{\mu\nu}$, $R_{\mu\nu\delta\rho}$ and their covariant derivatives. The coupling constant is $\kappa^2 = 8\pi G_N$, where G_N is the Newton constant and has mass dimension $2-d$, Λ is the cosmological constant term with mass dimension 2 and c_i are dimensionless coefficients.

Physical principles and experimental indications can enter in order to simplify the action. Experimental measures [Nakamura 2010] show that the cosmological constant is a very small quantity in Planck units, then the action in eq. (1.1) can be simplified by setting $\Lambda = 0$. Furthermore, setting $c_1 = c_2 = 0$ and forbidding all higher curvature terms one obtains Einstein's theory:

$$\mathcal{S}_{EH}[g] = \frac{1}{2\kappa^2}\int d^dx\,\sqrt{-g}\,R\,. \tag{1.2}$$

Experiments say very little about the size of the coefficients c_1, c_2 and the coefficients of the terms with higher powers of curvatures have essentially no constraints. In practice, there is no reason to require c_i to vanish completely. However, nonzero values for c_i do not influence physics at very low energy since the action in eq. (1.1) can be seen as organized in an energy expansion and their contribution is suppressed by a factor $\kappa^2E^2 \sim E^2/M^2_{Planck}$, where E is the typical energy of the process and M_{Pl} is the Planck mass ($G_N \simeq M_{Pl}^{-2}$). In order to set up the energy expansion, it is important to note that the connection $\Gamma^\lambda_{\alpha\beta} = g^{\lambda\sigma}(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta})/2$ is first order in derivatives and the curvature $R_{\mu\nu} = \partial_\nu\Gamma^\lambda_{\mu\lambda} - \partial_\lambda\Gamma^\lambda_{\mu\nu} + \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\lambda\sigma}$ is second order. When on-shell matrix elements are computed, derivatives turns into factors of momentum $\partial_\mu \sim ip_\mu$, so that the leading term of the Einstein-Hilbert action of eq. (1.2) is said to be of order $\mathcal{O}(p^2)$, while the other terms of eq. (1.1) involving two powers of curvature are said to be of order $\mathcal{O}(p^4)$. Using eq. (1.2), it is possible to compute, at tree level, the graviton-graviton scattering amplitude. In the helicity basis all the amplitude for the process $1{+}2 \to 3{+}4$ vanish except those related by crossing to the amplitude $A(++;++)$ which is given by [DeWitt 1967, Berends 1975]:

$$A(++;++) = \frac{i}{4}\frac{\kappa^2 s^3}{tu}\,, \tag{1.3}$$

where $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$ are the usual Mandelstam variables.

No matter which theory actually describes high-energy quantum gravity, in the infrared (IR) limit any physically valid theory must reproduce the results found in the effective field theory framework.

In the last years, however, the possibility that the ultraviolet completion of gravity can still be described in terms of the metric as fundamental degrees of freedom have been taken in consideration by many authors. They investigate the possibility that gravity may be asymptotically safe computing the renormalization group flow of the theory within a functional approach [Niedermaier 2006, Reuter 2006, Percacci 2009, Codello 2009b].

In the study of the renormalization group flow of quantum gravity one encounters many technical complications mainly due to the gauge fixing issue. It is often desirable to test the machinery in a simpler setting which in the case of gravity is represented by the nonlinear sigma model. The lowest order term in the derivative expansion is:

$$\mathcal{S}_{NL}[\varphi] = \frac{1}{2f^2}\int d^d x\, h_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\,, \tag{1.4}$$

where f is the nonlinear sigma model coupling with mass dimension $(2-d)/2$. As already said, gravity and the nonlinear sigma model have striking similarities: both are nonlinear and non-renormalizable theories with dimensionful coupling constant. The structure of the interactions in both theories is nonpolynomial and they admit a derivative expansion. The nonlinear sigma model is therefore a good theoretical laboratory where one can study various technical aspects of the nonperturbative renormalization of gravity without having to consider the complications due to gauge fixing.

Although the existence of the fixed point was already known in the nonlinear sigma model in $d = 2 + \varepsilon$ [Bardeen 1976], its presence has been obscured by the widespread use of dimensional regularization. The use of this regularization method artificially removes power divergences, which give important contributions to the beta function of dimensionful couplings such as f. These contributions are essential in generating the nontrivial fixed point. More recently, functional renormalization group techniques have been applied to study the system and indications about the existence of a nontrivial fixed point for the nonlinear sigma model with two derivatives have been found in [Codello 2009a]. This result persist also including higher derivative operators as shown in [Hasenfratz 1989, Percacci 2010]. The one-loop result for the beta function of the dimensionless nonlinear sigma model coupling $\tilde{f}$ is shown in Fig.1.1.

Understanding the UV behavior of the nonlinear sigma model may shed some light on the

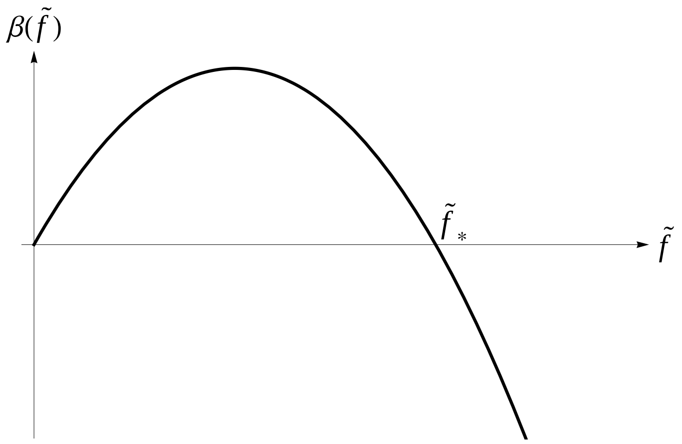


Figure 1.1: One-loop beta function for the dimensionless nonlinear sigma model coupling $\tilde{f}$ in $d > 2$. Beside a Gaussian fixed point, the model admits a nontrivial UV attractive one at $\tilde{f}_*$.

analogous issue for gravity. On the other hand, the problem of its UV completion is also important because of its application in particle phenomenology. The best known application of the nonlinear sigma model in particle physics phenomenology is chiral perturbation theory: it describes the low energy effective theory of pions, regarded as Goldstone bosons of the flavor symmetry $SU(N)_L \times SU(N)_R$, broken to the diagonal subgroup by the quark condensate. In this specific case the UV completion of the chiral model is QCD, which is an asymptotically free theory (and then safe) and there is no reason to look further.

On the other hand, when the nonlinear sigma model is coupled to gauge fields, the physical interpretation for the Goldstone bosons changes completely with respect to the ungauged case. The most important phenomenological application of this idea is electroweak chiral perturbation theory. It is similar to chiral perturbation theory, except that the 'pions' are identified with the angular degrees of freedom of the standard model (SM) Higgs field and are coupled to the electroweak gauge fields. The pion decay constant is identified with the Higgs VEV υ. The target space is $(SU(2)_L \times U(1)_Y)/U(1)_Q \sim SU(2)$, just as in the simplest chiral perturbation theory. The electroweak chiral lagrangian provides the most general low-energy parametrization of the Higgs phenomenon for the spontaneous breaking of the $SU(2)_L \times U(1)_Y$ symmetry in terms of the minimal number of degrees of freedom, namely, the three would-be Goldstone bosons. At tree level and to lowest order in the derivative expansion, the effective low-energy theory for the electroweak sector is the well known lagrangian of the gauged nonlinear sigma model given by [Applequist 1980, Longhitano 1980, Longhitano 1981]:

$$\mathcal{L}_{GNL} = \frac{\upsilon^2}{4} \mathrm{Tr}\,[D_\mu U D_\mu U^\dagger] + \mathcal{L}_G \,, \tag{1.5}$$

where $\mathcal{L}_G$ is the kinetic lagrangian of the gauge fields W^a_μ and B_μ. In eq. (1.5) the Goldstone bosons π^a are encoded in a matrix valued field $U = e^{i\sigma_a \pi^a / v}$. The coupling of the Goldstone bosons to the gauge bosons is obtained through the covariant derivative $D_\mu U = \partial_\mu U - igW_\mu U + ig' U B_\mu$. At tree level eq. (1.5) can be viewed as the SM Higgs sector in the limit when the quartic coupling $\lambda \to \infty$ at fixed VEV v, so that the mass of the Higgs field goes to infinity. Electroweak chiral perturbation theory can be seen as an approximation used in the SM where the energy is sufficiently low that the Higgs degree of freedom cannot be excited. This model is perfectly adequate to give mass to the gauge bosons and it leads to the proper low-energy theorems for the scattering of longitudinal vector bosons [Chanowitz 1985]. The lagrangian in eq. (1.5) is usually regarded as low energy effective theory thought to be valid up to a cutoff scale $\Lambda = 4\pi v$ and the theory becomes less and less useful for increasing energy and eventually the perturbative procedure breaks down for momenta of order Λ.

To incorporate effects coming from new physics at higher scale, new effective operators have to be considered. The complete $SU(2)_L \times U(1)_Y$ invariant chiral lagrangian containing the whole set of invariant operators up to dimension four can be written as:

$$\mathcal{L}_{EW\chi} = \mathcal{L}_{GNL} + \sum_i a_i \mathcal{L}_i \,, \tag{1.6}$$

where $\mathcal{L}_i$ are the $\mathcal{O}(p^4)$ terms. The explicit form of these operators is reported in Appendix C. In eq. (1.6) the arbitrary coefficients a_i have to be fixed by the experiments or by matching the theory with an UV completion.

In particular, some of these are related to the electroweak precision measurements since they contribute to the S, T and U oblique parameters [Peskin 1992] and therefore they are directly contrained by LEP. If the nonlinear sigma model was really equivalent to the $m_H \to \infty$ limit of the Higgs model [Herrero 1994], this would be enough to essentially rule it out, since a very heavy Higgs particle is disfavored by precision electroweak data. The issue of the compatibility of the electroweak chiral lagrangian with electroweak precision measurements has been analyzed in [Bagger 2000].

An interesting theoretical possibility is the UV completion of the model described by the lagrangian in eq. (1.6) given by asymptotic safety. This model is minimal in the sense that no Higgs field or other exotic particle involved in electroweak symmetry breaking is present or integrated out. This model represents a starting point for many possible extensions that can be dictated from what the LHC can give us in terms of results. Requiring that the theory lies on a renormalization group trajectory that hits a nontrivial fixed point in the UV will force to tune the value of some coefficients a_i in terms of the others that are taken as free parameters, these are predictions that can be compared with the experimental bounds.

In a more realistic picture one has to take into account fermions, which are assumed to coupled to the nonlinear sigma model only via (proto)-Yukawa interaction:

$$\mathcal{L}_{Yuk} = -m_i \bar{\psi}_{Li} U \psi_{Ri} + h.c. \tag{1.7}$$

where ψ_i are $SU(2)$ SM fermions. In this case the parameter space enlarges with the introduction of the Yukawa couplings $h_i = m_i/\upsilon$ and the system of coupled equations for the beta function becomes more involved. Considering fermions and study their influence on the fixed point is important if one wants to build a realistic model.

Another important issue is the scattering amplitude. Making use of the equivalence theorem [Cornwall 1974, Chanowitz 1985] it is possible to use eq. (1.5) to compute the scattering of longitudinal gauge bosons in a range of energies $m_W^2 \ll s \ll \Lambda^2$. The elastic scattering for the pion scattering process $\pi^a\pi^b \to \pi^c\pi^d$ is given by a single tree level amplitude [Gasser 1984]:

$$A(s,t,u) = \frac{s}{\upsilon^2} \tag{1.8}$$

where s, t, u are the usual Mandelstam variables. This amplitude has a common feature with the graviton amplitude computed in eq. (1.3), both amplitudes increase quadratically with the energy of the process leading to a violation of perturbative unitarity at a certain energy scale which in the case of electroweak interactions occurs at $\sim \sqrt{8\pi}\upsilon$. In the SM the problem is solved by embedding the nonlinear sigma model into a complex dublet transforming linearly under $SU(2)$. This is achieved introducing an extra degree of freedom (the Higgs boson) which is responsible for unitarizing the theory. In doing this, one makes the theory perturbatively renormalizable although not fully UV complete, due to the positive beta function for the quartic coupling λ. In strongly interacting theories unitarity is restored thanks to the presence of resonance states, which soften the UV behavior of the amplitude.

In the case of the nonlinear sigma model, a qualitative argument in favor of unitarity is obtained by noticing that the existence of a nontrivial fixed point for the Goldstone boson coupling υ implies that its scaling for $k \to \infty$ is given by $\tilde{\upsilon}_* k$, where $\tilde{\upsilon}_*$ is a fixed point value for the dimensionless coupling $\tilde{\upsilon} = \upsilon k^{-1}$. Identifying $k^2 = s$ and substituting into eq. (1.8) one obtains that the amplitude reaches a constant value for $s \to \infty$ depending on the fixed point value:

$$A(s,t,u) = \frac{s}{\tilde{\upsilon}_*^2 s} \sim \frac{1}{\tilde{\upsilon}_*^2}\,. \tag{1.9}$$

The full study of the Goldstone boson scattering amplitude in the context of asymptotic safety is important for the reliability of the minimal electroweak model at all energies but can also give some hints about what is expected to happen in gravity, where the structure of the amplitude is similar.

The discovery of an Higgs-like particle with mass around 125 GeV/c^2 has been recently announced by the LHC experiments CMS [CMS 2012] and ATLAS [ATLAS 2012]. Data collected during the 2011 and 2012 are not sufficient to completely pin down the details of this new particle and to tell us whether it is the true SM Higgs. Thus, one is forced to extend the minimal model in eq. (1.6) and to consider the case in which a light neutral scalar H exists in addition to the known matter and gauge fields. The most general description of such Higgs-like particle is obtained by considering the electroweak chiral lagrangian and adding all possible interactions involving H [Giudice 2007, Contino 2010]. The lowest order lagrangian reads:

$$\mathcal{L}_{NLH} = \frac{1}{2}\partial_\mu H \partial^\mu H + \frac{v^2}{4}\mathrm{Tr}\,[D_\mu U D_\mu U^\dagger]\left(1 + 2a\frac{H}{v} + b\frac{H^2}{v^2} + \cdots\right) + V(H) + \mathcal{L}_G + \mathcal{L}_{Yuk}\,, \quad (1.10)$$

where the dots represents terms including higher powers of H. In eq. (1.10) $V(H)$ is the scalar potential and $\mathcal{L}_{Yuk}$ is the extension of the Yukawa lagrangian in eq. (1.7) obtained by including the H field. This is the most general extension of the minimal Goldstone boson model which includes the SM particles already discovered and the new Higgs-like scalar H. The SM case consist of having $a = b = 1$ and all other higher couplings zero in eq. (1.10). In this case the three Goldstone bosons and the Higgs field can be recasted in a $SU(2)$ dublet which transforms linearly.

The main reason why the SM uses a linearly transforming Higgs field, rather than a nonlinear one, is that the nonlinear sigma model and its extension in eq. (1.10) are perturbatively nonrenormalizable . However, this theory could be made renormalizable in a nonperturbative sense if asymptotically safe. Asymptotic safety can also manifest itself with a linearly realized Higgs field in the presence of Yukawa interactions [Gies 2009, Gies 2010b, Gies 2010a]. In this case asymptotic safety at a nontrivial fixed point can lead to a reduction of physical parameter and hence make the model to have more predictive power.[1]

The aim of this thesis is to study the construction of an asymptotically safe electroweak model, where the Higgs sector is parametrized in a minimal way by a nonlinear sigma model. After having supported the existence of a nontrivial fixed point in the electroweak gauged nonlinear sigma model, the general point of view is to assume the existence of such a fixed point at nonperturbative level and to work out the phenomenological consequences of this assumption. It is important to remark that, given the subsequent discovery of an Higgs-like particle at the LHC, this model turned out to be too minimal to describe the real world. However, it represents a building block of any possible extensions, giving some insights about the general picture of the asymptotic safety construction in the case of electroweak theory. In addition, there remain motivations to study the nonlinear sigma model, at more theoretical level, because it can give some indications of what is expected to happen in general relativity given the similar structure of the theories.

[1] Taking into account also gravity and assuming that there are no intermediate energy scales between the Fermi and Planck scale, the authors of [Shaposhnikov 2010] predict an Higgs boson with mass about 126 GeV/c^2.

1.1 Asymptotic safety

Asymptotic safety was introduced as a set of requirements, based on the existence of a nontrivial fixed point for the renormalization group flow, which would make a quantum field theory consistent up to arbitrarily high energies.

Consider a quantum field theory defined by a scale dependent effective action $\Gamma_k[\varphi]$ which is assumed to describe the physics of the system at scale k. This notation will be made more precise in Chapter 2. The action $\Gamma_k[\varphi]$ can be parametrized in terms of a complete set of operators $\mathcal{O}_{i,k}$ consistent with the underlying symmetries of the theory, as follows:

$$\Gamma_k[\varphi] = \sum_i g_{i,k}\mathcal{O}_{i,k}[\varphi]\,, \tag{1.11}$$

where the coefficients $g_{i,k}$, including masses and wave function renormalizations, are called coupling constants. In general this basis of operators is an infinite dimensional k-dependent set. The operators $\mathcal{O}_{i,k}$ have some canonical mass dimension D_i which implies that the relative couplings are, in general, dimensionful. In particular $g_{i,k}$ have mass dimension $d_i = -D_i$.

The k-derivative of the action $\Gamma_k[\varphi]$ may be called 'beta functional' and gives the scale dependence of the theory:

$$k\frac{\partial}{\partial_k}\Gamma_k[\varphi] = k\frac{\partial}{\partial_k}\sum_i g_{i,k}\mathcal{O}_{i,k}[\varphi]\,. \tag{1.12}$$

Consider, for simplicity, a basis of operators that does not flow:

$$\Gamma_k[\varphi] = \sum_i g_{i,k}\mathcal{O}_i[\varphi]\,. \tag{1.13}$$

In this case one has

$$k\frac{\partial}{\partial_k}\Gamma_k[\varphi] = \sum_i \left(k\frac{\partial}{\partial_k}g_{i,k}\right)\mathcal{O}_i[\varphi]\,, \tag{1.14}$$

where the k-derivative of the coupling is called beta function and is denoted by β_i:

$$k\frac{\partial}{\partial_k}g_{i,k} = \beta_i(g,k)\,. \tag{1.15}$$

The beta function is, in general, a function of the couplings of the theory and the scale k. Given $g_{i,k}$, it is possible to define the dimensionless coupling $\tilde{g}_{i,k}$ as follows:

$$\tilde{g}_{i,k} = k^{-d}g_{i,k}. \tag{1.16}$$

The set of all the variables $\tilde{g}_{i,k}$ form an infinite dimensional space that is called 'theory space' because it parametrizes all the possible actions. Parameterizing the theory space with the dimensionless couplings $\tilde{g}_{i,k}$ just means that one is using the cutoff k as a unit of mass. Introducing

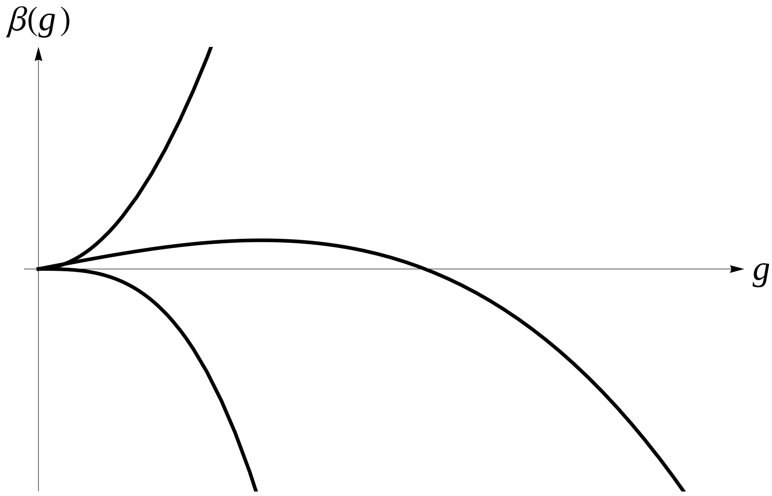


Figure 1.2: Possible forms of the beta function $\beta(g)$. The first upper curve shows a positive beta function which drives the coupling g away from zero making it to diverge at finite scale (Landau pole). The second mid curve represents a beta function with a nontrivial zero that is UV attractive. The last curve is a negative beta function where $g = 0$ is the UV attractive fixed point (asymptotic freedom).

these quantity is well justified from the physical point of view in which measured quantities are always obtained with respect to a reference scale. In the theory of renormalization group, dimensionless couplings are densitized cupling obtained by rescaling after the coarse graining procedure. The corresponding beta functions for the dimensionless coupling are:

$$\tilde{\beta}_i(\tilde{g}) = k\frac{\partial}{\partial_k}\tilde{g}_{i,k} = -d\tilde{g}_{i,k} + k^{-d_i}\beta_i(g,k)\,. \tag{1.17}$$

A simple scaling argument can be used to show that the beta function for a dimensionless coupling can only depend on dimensionless quantities $\tilde{g}$ and not on k. Examples of beta functions are shown in Fig.1.2.

A fixed point of the renormalization group flow, denoted by $\tilde{g}_*$, is defined by the values of the dimensionless couplings for which all the corresponding beta functions vanish identically:

$$\tilde{\beta}_i(\tilde{g}_*) = 0\,. \tag{1.18}$$

A fixed point is called Gaussian when $\tilde{g}_* = 0$, it describes a free theory. It is obvious that if one takes $\tilde{g}_*$ as initial condition for the flow, the theory remains at the fixed point at any scale. A fixed point defines a conformal field theory which by definition is a theory invariant under conformal transformations[2]. In such a theory the result of any experiment is completely determined by the fixed point.

[2]Recent discussions about the possibility of having scale invariance without conformal invariance can be found in [Fortin 2012].

The good UV behavior of the theory is ensured by the existence of a fixed point and by requiring that it lies on a trajectory that flows toward the fixed point in the UV. Such a trajectory could describe physics up to arbitrarily high energy and it is said to be 'asymptotically safe' or 'renormalizable'.

Asymptotic safety is usually formulated in terms of the behavior of the couplings in the action but, more physically, it should be formulated in terms of the behavior of observable quantities. The two formulations are related by noticing that any observable quantity $\mathcal{F} = \mathcal{F}(g_i, p_j, k)$ is a function of the couplings g_i, the external momenta p_j and the mass scale k. By dimensional analysis one can rewrite $\mathcal{F}(g_i, p_j, k) = k^d f(\tilde{g}_i, x_j)$, where d is the mass dimension of $\mathcal{F}$ and f is a dimensionless function of $\tilde{g}_i$ and the kinematic variables x_j. If $\tilde{g}_i$ have a finite UV limit then also the quantity $f(\tilde{g}_i, x_j)$ is expected to be finite and the observable $\mathcal{F}$ behaves for k that goes to infinity like powers of k and does not develop any unphysical singularity.

It is important to notice that the condition of asymptotic safety alone is not sufficient to guarantee the predictivity of the theory. In particular, if all couplings of the theory space were attracted towards the fixed point, then one would have a good UV limit irrespective to the initial conditions. This would leave infinitely many arbitrary couplings to be determined by experiments and the theory would lose predictivity.

In order to characterize the fixed point one needs to study the behavior of a theory in the vicinity of the fixed point by linearizing the flow around the fixed point itself. In doing this it is customary to introduce the stability matrix B_{ij} defined by

$$B_{ij} = \left. \frac{\partial \tilde{\beta}_i}{\partial \tilde{g}_j} \right|_{\tilde{g}_*} . \tag{1.19}$$

Defining $\delta g = \tilde{g} - \tilde{g}_*$, the linearized flow equation takes the following form:

$$k \frac{\partial}{\partial_k} \tilde{g}_{i,k} = B_{ij}(\tilde{g}_*) \delta g_j + \mathcal{O}(\delta g_j^2) . \tag{1.20}$$

The solution of the linearized system can be written as:

$$\tilde{g}_i(k) = \tilde{g}_{i*} + \sum_a c_a v_i^a \left(\frac{k}{k_0} \right)^{b_a} \tag{1.21}$$

where b_a and v_i^a are eigenvalues and eigenvectors of the stability matrix $B_{ij} v_j^a = b_a v_i^a$ and c_a are constants of integration fixed by the relation $\tilde{g}(k_0) = \tilde{g}_{i*} + \sum_a c_a v_i^a$. Eigenvectors v_i^a whose eigenvalues have negative real part are said to be 'UV-attractive'. By inspection of eq. (1.21) it is easy to see that for $Re[b_a] < 0$ then $\tilde{g}_i(k)$ flows towards its fixed point value $\tilde{g}_{i*}$ for $k \to \infty$ independently of the initial condition. On the contrary, eigenvectors v_i^a whose eigenvalues have positive real part are said to be 'UV-repulsive'. In this case the couplings will reach the fixed point $\tilde{g}_{i*}$ for $k \to \infty$ only if $c_a = 0$. The operators associated to UV-attractive (-repulsive) couplings are called relevant (irrelevant), since their importance in driving the theory away from the UV fixed point increases (decreases) as one flows towards the IR.

If the fixed point has a finite number n of attractive directions, then the family of trajectories that are physically acceptable has dimension $n-1$. The set of all points belonging to this trajectories is a n-dimensional surface called 'UV-critical surface'. The most predictive case is $n = 1$, where there is only one physical acceptable trajectory. In this case, after having fixed the value of the attractive coupling at some scale, the requirement of asymptotic safety determines uniquely the values of the other couplings at any scale. More generally, for $n > 1$, one has to perform n experiments to fix the value of the attractive couplings at some scale. Everything else can be, in principle, determined and constitute the genuine prediction of the theory, that can be verified experimentally.[3]

Perturbation theory corresponds to the case where the fixed point is Gaussian. In this case the tangent space to the critical surface, obtained by the linearized flow, is spanned by the couplings that have positive or vanishing mass dimension, i.e. those that are power counting renormalizable. Thus asymptotic safety at the Gaussian fixed point is equivalent to perturbative renormalizability plus asymptotic freedom. It is widely agreed that a theory with these properties makes sense up to arbitrarily high energies and therefore can be regarded as a fundamental theory. Asymptotic safety is a generalization of this behavior to the case when the fixed point is not a free theory.

The best known example of asymptotic safety at the Gaussian fixed point (asymptotic freedom) is QCD. In this theory only a finite number of couplings are attracted towards the fixed point in the UV, namely the gauge coupling and the quark masses. All other couplings must be set to zero in order to have a good UV limit of the model. Another example of asymptotic safety is the Gross-Neveu model in two dimensions with a $p^{-2+\varepsilon}$ propagator, which is perturbatively renormalizable and has been shown to be renormalizable at a nontrivial fixed point [Gawedzki 1985].

1.2 Outline

In this chapter, the concept of asymptotic safety and the motivations to look for it in the case of the electroweak interactions have been presented. The rest of the thesis is organized as follows.

In Chapter 2, the functional renormalization group machinery is introduced. The effective average action is defined, it is a scale dependent version of the usual effective action obtained by implementing a cutoff kernel in the functional integral definition. The nice property of

[3] Actually, only the essential couplings of the theory must have the good UV behavior. If a coupling is inessential, namely it can be eliminated by a field redefinition, then it does not need to have a finite UV limit (a typical example is the wave function renormalization constant).

this functional is that its scale dependence can be studies in terms of a simple exact functional equation. The beta functional is UV and IR finite and can therefore be calculated unambiguously for any theory. Gauge interactions are also accommodated by defining the background effective average action. This action is constructed by implementing a cutoff kernel constructed with covariant Laplacians that respects the symmetries of the background fields and obeys, as well as the usual effective average action, to an exact renormalization group equation. The last section of the chapter is devoted to two important examples of application of the exact functional renormalization group equation. The Yang-Mills beta function and the renormalization group flow of the nonlinear sigma model are computed. Both these models admit an UV attractive fixed point. This results are the starting point for the calculations presented in the next chapters of this thesis.

In Chapter 3, the functional renormalization group study of the gauged nonlinear sigma model is presented and the specific case of a chiral $SU(N)$ model is considered. The chapter deals with the construction of the background effective average action for the model and the solution of the functional equation it obeys. The detailed computations of the renormalization group flow obtained by taking into account different schemes of regularization and paying attention to the gauge dependence of the results is discussed. The fixed points of the model are studied and the results are compared with the ungauged case. Comments on possible relevance for phenomenology are finally reported.

In Chapter 4, some phenomenological applications of the gauged nonlinear sigma model in the case of electroweak interactions is presented. The first part is devoted to the study of the renormalization group flow of the $SU(2)_L \times U(1)_R$ gauged nonlinear sigma model using the functional methods introduced in the previous chapters. The possibility that the model might be asymptotically safe is considered. . The predictivity of the model will be tested enlarging the theory space by including higher order operators in the truncation. Therefore, dimension four operators, related to the electroweak S and T parameters, are taken into account and the renormalization group flow of the theory is studied by the same functional methods. The predictions obtained from the asymptotic safety picture are presented and the compatibility of the model with precision measurements is also discussed.

In a realistic model one needs to accommodate also SM fermions coupling them in a chiral invariant way to the Goldstone fields. In this way it is possible to provide a mass for quarks and leptons. In the third part, the renormalization group flow of the nonlinear sigma model coupled to fermions is studied. In this case, a one-loop computation shows that the inclusion of fermions drastically modifies the asymptotic properties of the nonlinear sigma model. The modifications one has to provide in order to preserve asymptotic safety of the theory is discussed. In particular, the good UV limit of the theory is ensured by adding effective four fermion interactions.

The final part is devoted to computing the Goldstone boson scattering amplitude using the functional formalism. The effective action has been computed solving the one-loop flow equation, rather than performing a functional integral. Divergences appear integrating the flow and some renormalization conditions are necessary to remove the infinities. In this way, the result for the scattering amplitude turns out to be the same as in perturbation theory.

In Chapter 5, final comments about the results obtained from the study of the minimal electroweak model are presented. The main open issues and future research directions are also discussed.

Chapter 2

Exact functional renormalization group

Contents

It is a well appreciated fact that the behavior of physical systems depends upon the length scale at which they are probed. One of the greatest insights of modern theoretical physics was the realization that it is possible to encode this scale dependence into the measurable parameters of the system, i.e. the coupling constants of the theory [Gell-Mann 1954, Bogoliubov 1959]. The development of these ideas culminated in Wilson's formulation of the renormalization group theory [Wilson 1971a, Wilson 1971b, Wilson 1972, Wilson 1974] where the scale dependence of the couplings is built-in in the formalism. In Wilson's method, which is based on the functional approach to quantum field theory presented in Appendix A, one imposes a floating finite ultraviolet cutoff Λ in the integral definition of the partition function and instead of taking $\Lambda \to \infty$, one requires that the bare constants of the theory depend on Λ in such a way that all observable quantities are cutoff-independent. In Euclidean space one has:

$$Z = \int \mathcal{D}\phi_\Lambda \, e^{-\mathcal{S}_\Lambda[\phi, g_\Lambda]} = \prod_{|q|<\Lambda} \int d\phi(q) \, e^{-\mathcal{S}_\Lambda[\phi, g_\Lambda]} \tag{2.1}$$

where the measure $\mathcal{D}\phi_\Lambda$ is defined in such a way that the integration involves only field fluctuations $\phi(q)$ with $|q| \leq \Lambda$ and $\phi(q) = 0$ for $|q| > \Lambda$. In eq. (2.1) g_Λ represents the set of coupling constants and the sources are neglected for simplicity. The cutoff Λ makes the integral expression finite and represents the UV scale at which the theory is fully described by the action

$\mathcal{S}_\Lambda$. Wilson's method consist on momentum shell mode elimination which is carried out by integrating over high-momentum degrees of freedom of ϕ. Introducing a scale $k < \Lambda$, it is possible to define an action $\mathcal{S}_k$ which is supposed to be a good description of the physics at scale k after performing the integration over the modes with $k \leq |q| \leq \Lambda$:

$$Z = \int \mathcal{D}\phi_k \, e^{-\mathcal{S}_k[\phi, g_k]} \tag{2.2}$$

where

$$e^{-\mathcal{S}_k[\phi, g_k]} = \prod_{k \leq |q| < \Lambda} \int d\phi(q) \, e^{-\mathcal{S}_\Lambda[\phi, g_\Lambda]} \,. \tag{2.3}$$

In order to make a careful comparison of the new functional integral in eq. (2.2) with the initial one in eq. (2.1) it is convenient to rescale distances and momenta in eq. (2.2) according to $q' = q\Lambda/k$ and $x' = xk/\Lambda$. The operation of integrating out high-momentum degrees of freedom combined with the rescaling leads to a transformation of the original action in which the contribution of the field fluctuations with $k \leq |q| \leq \Lambda$ can be absorbed by adjusting, or renormalizing, the couplings g_k at scale k. Continuing this procedure, it is possible to integrate over another shell of momentum space and transform the action further. In this way the corrections coming from high-momentum fluctuations are introduced slowly and systematically. If the shells of momentum integration are infinitesimally thin, the transformation becomes a continuous and can be described as a flow in theory space. For historical reasons, these continuously generated transformation are referred to as the renormalization group even if they do not form a group in the formal sense. It is important to notice that even if the initial UV action $\mathcal{S}_\Lambda$ is local, the resulting $\mathcal{S}_k$ is, in general, a complicated non-local action containing all possible invariant terms compatible with the symmetries of the theory. In this way the renormalization group theory naturally introduces the space of all possible actions compatible with the symmetries of the theory (theory space). The renormalization group framework is an extremely powerful tool in theoretical physics and its application has led to important results in studying a variety of classical and quantum systems [Zinn-Justin 2002], from solid state [Fisher 1998] to high energy physics [Polchinski 1984].

The aim of this chapter is to introduce the functional renormalization group methods that will be applied in this thesis. They focus on the mode elimination procedure of Wilson, but in place of integrating over finite momentum shells, one encodes the integration over an infinitesimal momentum shell in a differential equation describing how the effective action changes as the cutoff is varied. The striking and fundamental point is that it is possible to write an exact functional equation describing this process. In particular, it turns out to be convenient to study a scale dependent generalization of the effective action, called effective average action Γ_k. In the language of statistical physics, Γ_k is a type of coarse-grained free energy with a coarse graining length scale ($\sim k^{-1}$). In this way, one can work directly with the mean or average fields, which have a clear and direct physical interpretation. Moreover, the exact flow equation for the effective average action turns out to be extremely compact and powerful. However, the flow equation is a very complicated functional integro-differential equation, which can be treated only at the cost of making some approximation. The effective average action is also suited to be applied to physical systems in presence of background gauge fields and it is possible to derive an exact renormalization group equation for it by introducing a cutoff kernel constructed with covariant Laplacians that respects the symmetries of the background. The last section of the chapter will be devoted to present two examples of application of this formalism in the case of Yang-Mills theory and the nonlinear sigma model. For a general reference about the effective average action see [Wetterich 1993, Berges 2002] while for an introduction to the formalism see [Litim 2001, Gies 2006].

2.1 Effective average action

The effective average action Γ_k is a simple generalization of the usual effective action (see Appendix A). It is obtained by implementing an infrared cutoff, which depends on some scale k, in the defining functional integral, such that only fluctuations with momenta $q^2 > k^2$ are included. The effective average action Γ_k can be seen as a scale-dependent effective action which arises from integrating out all field fluctuations with momenta larger than k. By definition, the average action is equal to the standard effective action for $k = 0$, i.e. $\Gamma_0 = \Gamma$; in the limit $k = 0$ the IR cutoff is absent and all fluctuations are included. On the other hand, in a model with a physical ultraviolet cutoff Λ it is possible to associate Γ_Λ with the microscopic or classical action $\mathcal{S}$. No fluctuations with momenta below Λ are effectively included if the IR cutoff k equals the UV cutoff Λ. Thus the average action Γ_k is a functional that interpolates between the classical action $\mathcal{S}$ and the effective action Γ as k is lowered from the ultraviolet cutoff Λ to zero:

$$\Gamma_\Lambda \approx \mathcal{S} \qquad , \qquad \lim_{k\to 0} \Gamma_k = \Gamma \, . \tag{2.4}$$

The ability to solve the theory is equivalent to the ability to follow the evolution of Γ_k from $k = \Lambda$ to $k \to 0$.

The most important point is that there exists a well-defined (*i.e.* UV and IR finite) exact non-perturbative flow equation, which will be presented in the next section, that describes the dependence of the average action on the scale k.

In the following, the construction of the effective average action for a simple model of one real scalar field ϕ will be derived. Consider a theory defined by an Euclidean classical action $\mathcal{S}[\phi]$ in space-time dimension d, one starts with the path integral representation of the generating functional for correlation functions $Z_k[J]$ in the presence of an IR cutoff $\Delta\mathcal{S}_k$ and a nonhomogeneous

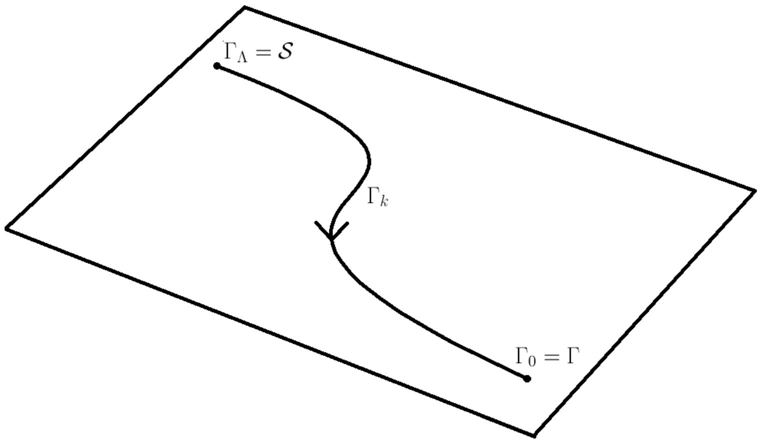


Figure 2.1: The flow of the effective average action. In the limit $k \to 0$ it is possible to recover the full quantum effective action Γ as the result of the integration of the flow.

source J:

$$Z_k[J] = \int \mathcal{D}\phi \exp\left(-\mathcal{S}[\phi] - \Delta\mathcal{S}_k[\phi] + \int d^d x\, J(x)\phi(x)\right) . \tag{2.5}$$

Correlation functions in presence of a source can be computed by taking functional derivatives of Z_k with respect to J:

$$\langle \phi(x_1)\,\phi(x_2)\dots\phi(x_n)\rangle_{kJ} = \frac{1}{Z_k[J]}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_n)} Z_k[J] . \tag{2.6}$$

As usual, one is most interested in the generating functional for the connected correlation functions which is denoted by $W_k[J]$ and is given by the logarithm of the partition function $Z_k[J]$:

$$W_k[J] = \log Z_k[J] = \log \int \mathcal{D}\phi \exp\left(-\mathcal{S}[\phi] - \Delta\mathcal{S}_k[\phi] + \int d^d x\, J(x)\phi(x)\right) . \tag{2.7}$$

The n-point regularized connected correlation function can be computed using W_k as generating functional:

$$G_k(x_1, x_2 \dots, x_n) = \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_n)} W_k[J] . \tag{2.8}$$

The 2-point regularized connected correlation function $G_k(x_1, x_2)$ is called 'regularized propagator', it is given by:

$$G_k(x_1, x_2) = \frac{\delta^2 W_k[J]}{\delta J(x_1)\delta J(x_2)} = \langle \phi(x_1)\phi(x_2)\rangle_{kJ} - \langle\phi(x_1)\rangle_J\langle\phi(x_2)\rangle_{kJ} . \tag{2.9}$$

The only modification in eq. (2.7) compared to the construction of the standard effective action is the presence of an additional IR cutoff term $\Delta\mathcal{S}_k[\phi]$ which is taken to be quadratic in the fields:

$$\Delta\mathcal{S}_k[\phi] = \frac{1}{2}\int d^dx \int d^dy\, \phi(x)\mathcal{R}_k(x,y)\phi(y)\,, \tag{2.10}$$

where $\mathcal{R}_k(x,y) = R_k(\Delta)\delta(x-y)$. In general, the 'cutoff shape function' $R_k(\Delta)$ can be taken as function of a proper covariant Laplace operator $\Delta = -\nabla^2 + U$, where ∇ is a covariant derivative and U is a field dependent endomorphism. In the simple case in which $\Delta = -\partial^2$ it is possible to rewrite $\Delta\mathcal{S}_k[\phi]$ in momentum space as:

$$\Delta\mathcal{S}_k[\phi] = \frac{1}{2}\int d^dq\, \phi(q)R_k(q^2)\phi(q)\,. \tag{2.11}$$

The cutoff shape function $R_k(q^2)$ is an arbitrary function of q^2, apart from the requirement that, at fixed q^2, it monotonically interpolates between $R_k = 0$ for $k \to 0$ and $R_k \sim k^2$ for $k \to \infty$ (or $k \to \Lambda$). An example of such a cutoff shape function, mostly used in this thesis, is provided by the so called 'optimized cutoff' [Litim 2001]:

$$R_k^{opt}(q^2) = (k^2 - q^2)\theta(k^2 - q^2)\,. \tag{2.12}$$

Other examples of cutoff shape functions are the 'exponential' and the 'mass-type':

$$R_k^{exp}(q^2) = \frac{q^2}{e^{\frac{q^2}{k^2}} - 1} \qquad , \qquad R_k^{mass}(q^2) = k^2\,. \tag{2.13}$$

As a result, for fluctuations with small momenta $q^2 < k^2$ this cutoff behaves as $R_k(q^2) \sim k^2$ and since $\Delta\mathcal{S}_k[\phi]$ is quadratic in the fields, all Fourier modes of ϕ with momenta smaller than k acquire an effective mass term $\sim k$ which acts as an effective IR cutoff for the low momentum modes. In contrast, for $q^2 \gg k^2$ the function $R_k(q^2)$ goes rapidly to zero so that the functional integration of the high momentum modes is not affected. The term $\Delta\mathcal{S}_k[\phi]$ added to the classical action is the main ingredient for the construction of an effective action that includes all fluctuations with momenta $q^2 \gtrsim k^2$, whereas fluctuations with $q^2 \lesssim k^2$ are suppressed. As usual, it is possible to introduce the average field φ which is the expectation value of ϕ in the presence of $\Delta\mathcal{S}_k[\phi]$ and a source J:

$$\varphi(x) = \langle\phi(x)\rangle_{kJ} = \frac{\delta W_k[J]}{\delta J(x)}\,. \tag{2.14}$$

Notice that the relation between φ and J is k-dependent, namely $\varphi = \varphi_k[J]$ and $J = J_k[\varphi]$. The effective average action is defined in terms of W_k via a modified Legendre transform:

$$\Gamma_k[\varphi] = -W_k[J] + \int d^dx J(x)\varphi(x) - \Delta\mathcal{S}_k[\varphi]\,. \tag{2.15}$$

In eq. (2.15), the term $\Delta\mathcal{S}_k[\varphi]$ has been subtracted on the right hand side, this is crucial for the definition of a reasonable coarse-grained effective action with the property $\Gamma_\Lambda \approx \mathcal{S}$. It guarantees that the only difference between Γ_k and Γ is the effective infrared cutoff in the fluctuations. Furthermore, it has the consequence that Γ_k does not have to be convex for nonvanishing k, whereas a pure Legendre transform is always convex by definition. The Legendre transform can be inverted using the relation:

$$\frac{\delta\Gamma_k[\varphi]}{\delta\varphi(x)} = J(x) - \frac{\delta\Delta\mathcal{S}_k[\varphi]}{\delta\varphi(x)}\,. \tag{2.16}$$

Taking another derivative of eq. (2.16) with respect to φ, it is possible to compute the Hessian of Γ_k:

$$\frac{\delta^2\Gamma_k[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)} = \frac{\delta J(x_2)}{\delta\varphi(x_1)} - \frac{\delta^2\Delta\mathcal{S}_k}{\delta\varphi(x_1)\delta\varphi(x_2)} = \frac{\delta J(x_2)}{\delta\varphi(x_1)} - \mathcal{R}_k(x_1,x_2)\,. \tag{2.17}$$

Using eq. (2.9) and eq. (2.14) one can rewrite the regularized propagator as:

$$G_k(x_1,x_2) \equiv \frac{\delta^2 W_k}{\delta J(x_1)\delta J(x_2)} = \frac{\delta\varphi(x_1)}{\delta J(x_2)}\,. \tag{2.18}$$

Putting together eq. (2.17) and eq. (2.18), one obtains that the modified Hessian of Γ_k is the inverse of the regularized propagator:

$$\int d^dx\, G_k(x_1,x)\left(\frac{\delta^2\Gamma_k[\varphi]}{\delta\varphi(x)\delta\varphi(x_2)} + \mathcal{R}_k(x,x_2)\right) = \delta(x_1-x_2)\,. \tag{2.19}$$

It is possible to derive an integro-differential equation satisfied by Γ_k inserting eq. (2.15) into eq. (2.7):

$$e^{-\Gamma_k[\varphi]} = \int \mathcal{D}\phi \exp\left(-\mathcal{S}[\phi] - \Delta\mathcal{S}_k[\phi] + \int J(\phi-\varphi) + \Delta\mathcal{S}_k[\varphi]\right)\,. \tag{2.20}$$

Now, shifting the integration variable $\chi = \phi - \varphi$ and using the inverse Legendre transform relation in eq. (2.16), it is possible to write the functional integral representation for the effective average action Γ_k:

$$e^{-\Gamma_k[\varphi]} = \int \mathcal{D}\chi \exp\left(-\mathcal{S}[\chi+\varphi] - \Delta\mathcal{S}_k[\chi] + \int \frac{\delta\Gamma_k[\varphi]}{\delta\varphi}\chi\right)\,, \tag{2.21}$$

where the condition $\langle\chi\rangle = 0$ is understood. It is an integro-differential equation for the effective average action Γ_k, it can be also used in place of eq. (2.15) as a starting point to define the effective average action. This expression resembles closely the expression for the implicit definition of the effective action in eq. (A.16) except for the modification given by the term $\Delta\mathcal{S}_k$.

As already mentioned, the effective average action interpolates smoothly between the classical action $\mathcal{S}$ at UV scale and the full quantum effective action Γ at the IR scale. The limit $k \to 0$

is easy to study since in this limit $\Delta\mathcal{S}_k$ vanishes and then by definition $\Gamma_0 = \Gamma$. To study the limit $k \to \infty$, it is important to notice that the cutoff shape function diverges and the term $\Delta\mathcal{S}_k$ behaves as $Ck^2 \int d^dx\chi^2$, with C a cutoff shape dependent constant. Redefining the fluctuation field as $\chi \to \chi/k$ and using the relation $\Delta\mathcal{S}_k[\chi/k] = \Delta\mathcal{S}_k[\chi]/k^2$, which follows from eq. (2.10), it is possible to show that for $k \to \infty$ eq. (2.21) behaves as:

$$e^{-\Gamma_k[\varphi]} \quad \to \quad e^{-\mathcal{S}[\varphi]} \int \mathcal{D}\chi \exp\left(-\frac{1}{2}C \int d^dx\chi^2\right) , \tag{2.22}$$

where one has assumed that $\delta\Gamma_k[\varphi]/\delta\varphi$ is finite in the limit $k \to \infty$. The functional integral one needs to evaluate is a Gaussian one and it is just a multiplicative constant, then

$$\Gamma[\varphi]_{k\to\infty} = \mathcal{S}[\varphi] + const\,, \tag{2.23}$$

which can be seen as a UV boundary condition for the effective average action. This shows that Γ_k interpolates between the bare action $\mathcal{S}$ for $k \to \infty$ and the full quantum effective action Γ for $k \to 0$.

Applying standard perturbation theory to the effective average action as defined in eq. (2.21) one obtains the one-loop order average action:

$$\Gamma_k[\varphi] = \mathcal{S}[\varphi] + \frac{1}{2}\mathrm{Tr}\,\log\left(\frac{\delta^2\mathcal{S}[\varphi]}{\delta\varphi\delta\varphi} + R_k\right) . \tag{2.24}$$

This resembles the standard loop expansion for the quantum effective action with the modification introduced by the presence of the cutoff term R_k.

Chiral fermions can be incorporated easily in this formalism since chirally invariant cutoffs can be formulated [Wetterich 1990, Bornholdt 1992]. Effective average actions for gauge theories can be formulated as well [Reuter 1993, Reuter 1994a, Reuter 1994b] even though $\Delta\mathcal{S}_k$ may not be gauge invariant. In this case it is possible to derive closed expressions for corrections to the usual Ward identities [Ellwanger 1994]. They appear as counterterms in Γ_Λ and are crucial for preserving gauge invariance of physical quantities.

The effective average action presents many analogies with the Wilsonian effective action $\mathcal{S}_\Lambda$ but there is a conceptual difference. The Wilsonian effective action describes a family of actions parametrized by Λ for the same model, the n-point functions are independent of Λ and have to be computed from $\mathcal{S}_\Lambda$ by further functional integration. On the other hand, for any value of k, the functional Γ_k can be viewed as the generating functional of one-particle-irreducible correlation functions for a model with different action $\mathcal{S}_k = \mathcal{S} + \Delta\mathcal{S}_k$ and the n-point functions depend on k. The Wilsonian effective actions does not generate the one-particle-irreducible Green functions [Sumi 2000].

2.1.1 Exact flow equation for the effective average action

The most important feature of the effective average action of eq. (2.15) is that it is possible to write down an exact functional equation which describes the dependence of Γ_k on the cutoff scale k. In order to derive this functional equation it is useful to consider

$$\Gamma_k[\varphi] = \tilde{\Gamma}_k[\varphi] - \Delta\mathcal{S}_k[\varphi] \tag{2.25}$$

where according to eq. (2.15)

$$\tilde{\Gamma}_k[\varphi] = -W_k[J] + \int d^d x J(x)\varphi(x)\,. \tag{2.26}$$

The scale dependence of $\tilde{\Gamma}_k$ is given by

$$\frac{\partial}{\partial k}\tilde{\Gamma}_k[\varphi] = -\left(\frac{\partial W_k}{\partial k}\right)[J] - \int d^d x\, \frac{\delta W_k}{\delta J(x)}\frac{\partial J(x)}{\partial k} + \int d^d x\, \varphi(x)\frac{\partial J(x)}{\partial k}\,. \tag{2.27}$$

Using eq. (2.14) it is easy to show that the last two terms in eq. (2.27) cancel. The k-derivative of W_k is obtained from its defining functional integral in eq. (2.7) where all the k-dependence is encoded inside $\Delta\mathcal{S}_k$, this yields to:

$$\frac{\partial}{\partial k}\tilde{\Gamma}_k[\varphi] = \langle\frac{\partial\Delta\mathcal{S}_k}{\partial k}[\phi]\rangle = \frac{1}{2}\langle\int d^d x\int d^d y\, \phi(x)\frac{\partial\mathcal{R}_k(x,y)}{\partial k}\phi(y)\rangle\,. \tag{2.28}$$

Using eq. (2.9) and eq. (2.14) , the scale dependence of $\tilde{\Gamma}_k$ can be expressed as

$$\begin{aligned}\frac{\partial}{\partial k}\tilde{\Gamma}_k[\varphi] &= \frac{1}{2}\int d^d x\int d^d y\left[\frac{\partial\mathcal{R}_k(x,y)}{\partial k}G_k(y,x) + \varphi(x)\frac{\partial\mathcal{R}_k(x,y)}{\partial k}\varphi(y)\right]\\ &= \frac{1}{2}\mathrm{Tr}\left[G_k\frac{\partial R_k}{\partial k}\right] + \frac{\partial\Delta\mathcal{S}_k}{\partial k}[\varphi].\end{aligned} \tag{2.29}$$

The flow equation of the effective average action is then obtained subtracting the contribution coming from the k-derivative of $\Delta\mathcal{S}_k$. Using the relation in eq. (2.19) it is possible to rewrite the exact flow equation for Γ_k as

$$\frac{\partial}{\partial k}\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2\Gamma_k[\varphi]}{\delta\varphi\delta\varphi} + R_k\right)^{-1}\frac{\partial R_k}{\partial k}\right]\,. \tag{2.30}$$

The renormalization group flow of the effective average action in eq. (2.30) is described in a closed form by a functional differential equation. This equation is also called Wetterich equation and was first derived in [Wetterich 1993]. Moreover, it is exact since no approximations where made in its derivation.

The dependence of Γ_k on the scale k is given in terms of the inverse average propagator

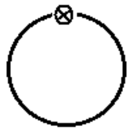
$$\frac{\partial}{\partial k}\Gamma_k[\varphi] = \frac{1}{2}$$ 

Figure 2.2: Graphical representation of the exact renormalization group flow of eq. (2.30). The continuous line represents the regularized propagator while the cross represents the insertion of $\partial_k R_k$.

$\Gamma_k^{(2)} \equiv \delta^2\Gamma_k/\delta\varphi\delta\varphi$ and has a simple graphical expression as a one-loop equation, as shown in Fig.2.2, where the full k-dependent propagator is associated to the close solid line and the dot denotes the insertion of $\partial_k R_k$. In order to obtain a one-loop like flow equation it is crucial that the cutoff action in eq. (2.10) is quadratic in the fields, other forms for $\Delta\mathcal{S}_k$ lead to higher order vertices of the effective average action on the right hand side of eq. (2.30) that spoil the one-loop structure of the equation.

In order to obtain a formulation of eq. (2.30) which resembles the usual beta functions one just replaces the partial k-derivative by a partial derivative with respect the logarithmic variable $t = \log(k/k_0)$, where k_0 is some reference scale. Taking the t-derivative of eq. (2.24), one gets the one-loop flow equation for the effective average action:

$$\frac{\partial}{\partial t}\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2\mathcal{S}[\varphi]}{\delta\varphi\delta\varphi} + R_k\right)^{-1}\frac{\partial R_k}{\partial t}\right]. \tag{2.31}$$

It is remarkable that the full renormalization group improvement $\mathcal{S}^{(2)} \to \Gamma_k^{(2)}$ turns the one-loop expression into an exact identity which incorporates effects of higher loops as well as genuinely non-perturbative ones. Standard one-loop renormalization group improved perturbation theory can be recovered after replacing the propagator and vertices appearing $\Gamma_k^{(2)}$ by the ones derived from the classical action and expanding the result to lowest order in the couplings which are taken as k-dependent.

The presence of the cutoff shape function R_k with the properties mentioned in Section 2.1.1 ensures that the trace of eq. (2.30) is both infrared and ultraviolet finite. In particular, for momenta $q^2 \ll k^2$ the cutoff acts as a mass term $R_k \sim k^2$ in the inverse average propagator curing potential infrared problems. On the other hand, ultraviolet finiteness is ensured by the fast decay of $\partial_k R_k$ for $q^2 \gg k^2$.

The flow equation (2.30) is an differential equation that is, in general, very difficult to solve exactly. This is mainly due to the fact that Γ_k is a functional defined in an infinite dimensional theory space which is parametrized by the coupling constants of all interaction terms consistent with the symmetries of the theory. Since it is impossible to follow the flow of such an infinite number of couplings, then some approximation schemes are required to solve the renormalization group equation. The usual way to proceed is to truncate the effective average action, making an ansatz for Γ_k which only retains a subset of all the possible terms of the theory and substituting this ansatz into the flow equation. Then one projects the result of the flow onto the subspace of the truncation. One common way to truncate the effective action is based on the vertex expansion. In this case, differentiating eq. (2.30) with respect to the field φ, one obtains an hierarchy of vertex flow equations which are taken to some finite order n. Another useful truncation is the derivative expansion, in this scheme the effective average action is expanded in powers of the derivatives to some finite order n.

The flow equation can be also solved by using an iterative method. In this case one chooses as initial ansatz $\Gamma_{k,0}$ for the effective average action and plugs it into eq. (2.30) in order to obtain the flow of the next approximation $\Gamma_{k,1}$. Integrating the flow and imposing the initial condition $\Gamma_{\Lambda,1} = \mathcal{S}$ one computes $\Gamma_{0,1}$. The obtained result is then used as a new seed into eq. (2.30) and the procedure is repeated. In this way one generates a series of approximation $\Gamma_{0,n}$ that may converge to the full effective action. It is important to notice that the initial ansatz can be chosen to have some given scale dependence. If it is chosen to be the bare action, namely $\Gamma_{k,0} = \mathcal{S}$, then one generates the perturbative loop expansion.

2.2 Effective average action for gauge theories

In this section, the construction of the effective average action is generalized to the case of gauge theories, in particular to non-abelian gauge theories. The important point in the construction is obviously that gauge invariance has to be preserved after the introduction of the cutoff. In the previous section was pointed out the importance for the cutoff action to be quadratic in the fields in order to obtain a one-loop like flow equation. This means that if one tries to introduce in gauge theories a cutoff by simply taking as cutoff kernel a function of the covariant Laplacian, this will spoil the simple one loop structure of the flow. Moreover the effective average action will not be gauge invariant because of the non-covariant coupling of the gauge field to the source. The way out is to employ the background field method for the k-dependent gauge effective action. This implementation was was first proposed in [Reuter 1994a]. In the following, the background field method for gauge theories is reviewed and the application of this method for the construction of the effective average action is presented. Finally, the flow equation for the background effective average action is derived.

2.2.1 Background field method for gauge theories

The background field method [Honerkamp 1972, 't Hooft 1976, Abbot 1981] is a technique widely used in dealing with gauge theories that allows to compute quantum effects without losing explicit gauge invariance.

Consider the classical gauge invariant Yang-Mills action for the field A in dimension d:

$$\mathcal{S}_0[A] = \frac{1}{4g^2} \int d^d x \, F_i^{\mu\nu} F^i_{\mu\nu} \,. \tag{2.32}$$

In eq. (2.32) g is the gauge coupling constant with mass dimension $(4-d)/2$ and $F^i_{\mu\nu}$ is the gauge field strength tensor:

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + f^i{}_{jk} A^j_\mu A^k_\nu \,, \tag{2.33}$$

where $f^i{}_{jk}$ are the structure constants of the gauge group. The basic idea of the background field method is to write the gauge field A appearing in eq. (2.32) as a sum of background field $\bar{A}$ and a quantum fluctuation a which will be the new integration variable of the functional integral:

$$A = \bar{A} + a \,. \tag{2.34}$$

The classical gauge invariant action evaluated at the shifted field reads:

$$\mathcal{S}_0[\bar{A}+a] = \frac{1}{4g^2} \int d^d x \, F_i^{\mu\nu}[\bar{A}+a] F^i_{\mu\nu}[\bar{A}+a] \,, \tag{2.35}$$

with

$$F^i_{\mu\nu}[\bar{A}+a] = \bar{F}^i_{\mu\nu} + \bar{D}_\mu a^i_\nu - \bar{D}_\nu a^i_\mu + f^i{}_{jk} a^j_\mu a^k_\nu \,, \tag{2.36}$$

where

$$\bar{D}_\mu a^i_\nu = \partial_\mu a^i_\nu + f^i{}_{jk} \bar{A}^j_\mu a^k_\nu \qquad , \qquad \bar{F}^i_{\mu\nu} = \partial_\mu \bar{A}^i_\nu - \partial_\nu \bar{A}^i_\mu + f^i{}_{jk} \bar{A}^j_\mu \bar{A}^k_\nu \,. \tag{2.37}$$

The classical Yang-Mills action in eq. (2.32) is invariant under the infinitesimal gauge transformation

$$\delta_\omega A^i_\mu = \partial_\mu \omega^i + f^i{}_{jk} A^j_\mu \omega^k \equiv D_\mu \omega^i \,. \tag{2.38}$$

This transformation can be split such that the background field $\bar{A}$ transforms inhomogeneously as a gauge field while the fluctuation a transforms homogeneously as a tensor in the adjoint representation:

$$\delta_\omega \bar{A}^i_\mu = \partial_\mu \omega^i + f^i{}_{jk} \bar{A}^j_\mu \omega^k \equiv \bar{D}_\mu \omega^i \tag{2.39}$$

$$\delta_\omega a^i_\mu = f^i{}_{jk} a^j_\mu \omega^k \,. \tag{2.40}$$

The splitting in eq. (2.34) allows to define a background field-dependent generating functional $W[J,\bar{A}]$ as [Abbot 1981]:

$$W[J,\bar{A}] = \log \int \mathcal{D}a \det\left[\frac{\delta' G^i[\bar{A};a]}{\delta\omega^j}\right] \exp\left(-\mathcal{S}_0[\bar{A}+a] - S_{gf}[\bar{A};a] + \int Ja\right) \,, \tag{2.41}$$

where

$$S_{gf}[\bar{A};a] = \frac{1}{2\alpha g^2}\int d^dx\, G^i[a;\bar{A}]G^i[a;\bar{A}] \tag{2.42}$$

is the gauge fixing term which depends on the gauge fixing condition $G^i[a;\bar{A}]$ and on the gauge fixing parameter α. The choice is to work in the so called background field gauge which retains explicitly the gauge invariance in terms of the background field $\bar{A}$ and it is defined by taking:

$$G^i[a;\bar{A}] = \bar{D}^\mu a^i_\mu\,. \tag{2.43}$$

The Fadeev-Popov determinant is obtained from the gauge variation of $G^i[a;\bar{A}]$ keeping the background field $\bar{A}$ fixed. More precisely the variation δ' corresponds to the gauge transformation in eq. (2.38) acting only on the fluctuation field a:

$$\delta'_\omega \bar{A}^i_\mu = 0 \tag{2.44}$$

$$\delta'_\omega a^i_\mu = \partial_\mu\omega^i + f^i{}_{jk}(a^j_\mu + \bar{A}^j_\mu)\omega^k = D_\mu\omega^i\,, \tag{2.45}$$

then

$$\det\left[\frac{\delta' G^i[\bar{A};a]}{\delta\omega^j}\right] = \det\left[\bar{D}^\mu\bar{D}_\mu\delta^i_l + f^i{}_{kj}\bar{D}^\mu a^k_\mu\right]\,. \tag{2.46}$$

As usual one can rewrite the determinant in eq. (2.46) as a functional integral over anticommuting ghost fields:

$$\det\left[\frac{\delta' G^i[\bar{A};a]}{\delta\omega^j}\right] = \int \mathcal{D}\bar{c}\mathcal{D}c\,\exp\left(-\mathcal{S}_{gh}[\bar{A};a,\bar{c},c]\right)\,. \tag{2.47}$$

The ghost action is given by:

$$S_{gh}[\bar{A};a,\bar{c},c] = \int d^dx\,\bar{c}_i\left(-\bar{D}^\mu\bar{D}_\mu\delta^i_l - f^i{}_{kj}\bar{D}^\mu a^k_\mu\right)c^j\,, \tag{2.48}$$

where the ghost covariant derivative is

$$\bar{D}_\mu c^i = \partial_\mu c^i + f^i{}_{jk}\bar{A}^j_\mu c^k. \tag{2.49}$$

Using an invariant measure $\int\mathcal{D}a$ for the functional integration, one has that the generating functional $W[J,\bar{A}]$ in eq. (2.41) is invariant under the background field transformation in eq. (2.39) provided that the the current J transforms as an adjoint tensor:

$$\delta_\omega J^\mu_i = f_i{}^{kl}J^\mu_k\,\omega^l\,. \tag{2.50}$$

At this point, it is possible to define the background effective action via the Legendre transform:

$$\Gamma[\bar{A};\tilde{a}] = -W[J,\bar{A}] + \int d^dx J(x)\tilde{a}(x)\,, \tag{2.51}$$

where the average field $\tilde{a}$ is the expectation value of a in the presence of J and the background field $\bar{A}$:

$$\tilde{a}(x) = \langle a(x)\rangle_J = \frac{\delta W[J,\bar{A}]}{\delta J(x)} \qquad , \qquad \frac{\delta\Gamma[\bar{A};\tilde{a}]}{\delta\tilde{a}(x)} = J(x)\,. \tag{2.52}$$

In eq. (2.52) the variations are performed at fixed background $\bar{A}$. The background effective action in eq. (2.51) is invariant under the transformation (2.39) for $\bar{A}$ and the simultaneous homogeneous transformation (2.40) for $\tilde{a}$. In particular, $\Gamma[\bar{A};0]$ must be an explicit gauge invariant functional of $\bar{A}$ since (2.39) is just an ordinary gauge transformation of the background field. One can define a functional $\bar{\Gamma}[\bar{A}]$, that is called gauge invariant effective action, by setting $\tilde{a} = 0$ in eq. (2.51):

$$\bar{\Gamma}[\bar{A}] = \Gamma[\bar{A};0]\,, \tag{2.53}$$

this is the gauge invariant quantity one usually computes in the background field method. It is possible to show [Abbot 1981] that it is equal to the usual effective action calculated in an unconventional gauge which depends on $\bar{A}$.

2.2.2 Background effective average action

In this section, the generalization of the background field method to the case of the effective average action is introduced. One usually starts by defining the path integral representation of the k-dependent generating functional $W_k[J,\bar{A}]$ in the presence of the background field $\bar{A}$ and the source J:

$$W_k[J,\bar{A}] = \log\, C_k[\bar{A}] \int \mathcal{D}a \det\left[\frac{\delta' G^i[\bar{A};a]}{\delta\omega^j}\right] \exp\left(-\mathcal{S}_0[\bar{A}+a] - S_{gf}[\bar{A};a] - \Delta\mathcal{S}_k[\bar{A};a] + \int Ja\right)\,, \tag{2.54}$$

where $\mathcal{S}_0[\bar{A}+a]$, $S_{gf}[\bar{A};a]$ and $\det[\delta G/\delta\omega]$ are given in eq. (2.35), eq. (2.42) and eq. (2.46) respectively. The cutoff action $\Delta\mathcal{S}_k[\bar{A};a]$ is taken to be quadratic in the fluctuation field a:

$$\Delta\mathcal{S}_k[\bar{A};a] = \frac{1}{2}\int d^dx\, a^i_\mu(x)\mathcal{R}_k[\bar{A}]\delta^{\mu\nu}_{ij} a^j_\nu(x). \tag{2.55}$$

The quantity $C_k[\bar{A}]$ provides an infrared cutoff for the ghosts of the nonabelian gauge theory, it depends on $\bar{A}$ through the ghost covariant derivative $\bar{D}$ defined in eq. (2.49). Moreover its form is dictated by the chosen gauge fixing term and for the background gauge fixing choice of eq. (2.43) one has that $C_k[\bar{A}]$ is explicitly given by [Reuter 1994a]:

$$C_k[\bar{A}] = \det[1 + (-\bar{D}^2)^{-1}R_k(-\bar{D}^2)]\,. \tag{2.56}$$

Combined with the Fadeev-Popov determint of eq. (2.46) it leads to an effective modified inverse propagator for the ghosts of the nonabelian gauge theory.

The specific form of the cutoff action and the gauge fixing term makes the generating functional $W_k[J,\bar{A}]$ in eq. (2.54) invariant under the simultaneous infinitesimal transformations, where $\bar{A}$ transforms inhomogeneously as in eq. (2.39) and J as in eq. (2.50). It is possible now to introduce the average field $\tilde{a}$ which is the background dependent expectation value of a in the presence of $\Delta\mathcal{S}_k[\bar{A};a]$ and J:

$$\tilde{a}(x) = \langle a(x)\rangle_J = \frac{\delta W_k[J,\bar{A}]}{\delta J(x)} \tag{2.57}$$

and define the background field effective average action via a modified Legendre transform:

$$\Gamma_k[\bar{A};\tilde{a}] = -W_k[J,\bar{A}] + \int d^dx J(x)\tilde{a}(x) - \Delta\mathcal{S}_k[\bar{A};\tilde{a}]\,. \tag{2.58}$$

The Legendre transform can be inverted by considering the relation

$$\frac{\delta\Gamma_k[\bar{A};\tilde{a}]}{\delta\tilde{a}(x)} = J(x) - \frac{\delta\Delta\mathcal{S}_k[\bar{A};\tilde{a}]}{\delta\tilde{a}(x)}\,. \tag{2.59}$$

The background effective average action $\Gamma_k[\bar{A};\tilde{a}]$ is invariant under the following simultaneous transformations:

$$\delta_\omega \bar{A}^i_\mu = \bar{D}_\mu\omega^i \tag{2.60}$$
$$\delta_\omega \tilde{a}^\mu_i = f_i{}^{kl} a^\mu_k \omega^l\,. \tag{2.61}$$

As for the background effective action introduced in eq. (2.51), one has that $\Gamma_k[\bar{A};0]$ must be an explicit gauge invariant functional of $\bar{A}$, since the transformation in eq. (2.60) is just an ordinary gauge transformation of the background field. One can define a functional $\Gamma_k[\bar{A}]$ that is called gauge invariant effective average action by setting $\tilde{a}=0$ in eq. (2.58):

$$\Gamma_k[\bar{A}] = \Gamma_k[\bar{A};0]\,. \tag{2.62}$$

This is the gauge-invariant quantity one wants to compute with the background field method.

2.2.3 Exact flow equation for the background effective average action

Following the derivation of Section 2.1.1, it is possible to derive an exact flow equation that the background effective average action of eq. (2.58) satisfies. Consider

$$\tilde{\Gamma}_k[\tilde{a};\bar{A}] = \Gamma_k[\bar{A};\tilde{a}] + \Delta\mathcal{S}_k[\bar{A};\tilde{a}] = -W_k[J;\bar{A}] + \int d^dx J(x)\tilde{a}(x)\,, \tag{2.63}$$

with scale dependence given by:

$$\frac{\partial}{\partial t}\tilde{\Gamma}_k[\bar{A};\tilde{a}] = -\left(\frac{\partial W_k}{\partial t}\right)[J;\bar{A}]\,. \tag{2.64}$$

The t-derivative of W_k is obtained from its defining functional integral in eq. (2.54). All the k-dependence is encoded inside $\Delta\mathcal{S}_k[\bar{A};\tilde{a}]$ and C_k, this yields to:

$$\frac{\partial}{\partial t}\tilde{\Gamma}_k[\bar{A};\tilde{a}] = \langle\frac{\partial\Delta\mathcal{S}_k}{\partial t}[\bar{A};a]\rangle - \frac{\partial\log C_k[\bar{A}]}{\partial t} \tag{2.65}$$

Repeating the same steps of Section 2.1.1, the scale dependence of $\tilde{\Gamma}_k[\bar{A};\tilde{a}]$ can be expressed as

$$\frac{\partial}{\partial t}\tilde{\Gamma}_k[\bar{A};\tilde{a}] = \frac{1}{2}\mathrm{Tr}\left[G_k[\bar{A};\tilde{a}]\frac{\partial\mathcal{R}_k[\bar{A}]}{\partial t}\right] + \frac{\partial\Delta\mathcal{S}_k}{\partial t}[\bar{A};\tilde{a}] - \mathrm{Tr}\,\frac{\partial C_k[\bar{A}]}{\partial t}\,. \tag{2.66}$$

The exact flow equation for the effective average action then follows from eq. (2.63) by subtracting the contribution coming from the k-derivative of $\Delta\mathcal{S}_k[a;\bar{A}]$:

$$\frac{\partial}{\partial t}\Gamma_k[\bar{A};\tilde{a}] = \frac{1}{2}\mathrm{Tr}_{\,a}\left[\left(\frac{\delta^2\Gamma_k[\bar{A};\tilde{a}]}{\delta\tilde{a}\delta\tilde{a}} + \mathcal{R}_k[\bar{A}]\right)^{-1}\frac{\partial\mathcal{R}_k[\bar{A}]}{\partial t}\right] - \mathrm{Tr}_{\,c}\left[\frac{\partial_t R_k[-\bar{D}^2]}{-\bar{D}^2 + R_k[-\bar{D}^2]}\right], \tag{2.67}$$

where

$$G_k[\bar{A};\tilde{a}] = \left(\frac{\delta^2\Gamma_k[\bar{A};\tilde{a}]}{\delta\tilde{a}\delta\tilde{a}} + \mathcal{R}_k[\bar{A}]\right)^{-1}\,. \tag{2.68}$$

From eq. (2.67), it is possible to write down the flow equation for the gauge invariant effective action defined in eq. (2.62):

$$\frac{\partial}{\partial t}\Gamma_k[\bar{A}] = \frac{1}{2}\mathrm{Tr}_{\,a}\left[\left(\frac{\delta^2\Gamma_k[\bar{A};0]}{\delta\tilde{a}\delta\tilde{a}} + \mathcal{R}_k[\bar{A}]\right)^{-1}\frac{\partial\mathcal{R}_k[\bar{A}]}{\partial t}\right] - \mathrm{Tr}_{\,c}\left[\frac{\partial_t R_k[-\bar{D}^2]}{-\bar{D}^2 + R_k[-\bar{D}^2]}\right]. \tag{2.69}$$

The corresponding one-loop result can be obtained by replacing Γ_k with $\mathcal{S}$ on the right hand side of eq. (2.69).

2.3 Examples

In this section, two applications of the exact renormalization group equations are presented in order to show how the machinery works with specific examples. The first application concerns the computation of the Yang-Mills beta function. Then, the functional methods are applied to study the renormalization group flow of the nonlinear sigma model. These two examples are of particular importance, since they represent the starting point of what is going to be discussed in the next chapters of this thesis.

2.3.1 Yang-Mills

Recall the classical Yang-Mills

$$\mathcal{S}_0[A] = \frac{1}{4g^2}\int d^dx\, F_i^{\mu\nu} F^i_{\mu\nu}\,. \tag{2.70}$$

The computation of the renormalization group flow of this theory is done by solving the beta functional equation for the background effective average action (see Section 2.2.3). One expands the gauge field A around nonconstant background $\bar{A}$ as $A(x) = \bar{A}(x) + a(x)$, then the classical Yang-Mills action can be written as functional Taylor series around $\bar{A}$:

$$\mathcal{S}_0[\bar{A}+a] = \mathcal{S}_0[\bar{A}] + \mathcal{S}_0^{[1]}[\bar{A},a] + \mathcal{S}_0^{[2]}[\bar{A},a] + \cdots \tag{2.71}$$

where $\mathcal{S}^{[n]}$ is of order n in the fluctuations a. The second order piece is:

$$S_0^{[2]}[\bar{A};a] = \frac{1}{2g^2}\int d^dx\, a^i_\mu \Big(-\bar{D}^2\delta_{ij}\delta^{\mu\nu} + \bar{D}^\nu\bar{D}^\mu\delta_{ij} + \bar{F}^{\ell\mu\nu} f_{\ell ij}\Big) a^j_\nu\,, \tag{2.72}$$

where $\bar{D}^\mu a^i_\nu$ and $\bar{F}^{\ell\mu\nu}$ are defined in eq. (2.37). The gauge fixed action $\mathcal{S}[\bar{A};a]$ is obtained by adding to $\mathcal{S}_0[\bar{A}+a]$ the gauge fixing term defined by eq. (2.42) and eq. (2.43):

$$\mathcal{S}[\bar{A};a] = \mathcal{S}_0[\bar{A}+a] + \mathcal{S}_{gf}[\bar{A};a]. \tag{2.73}$$

The quadratic part of the gauged fixed action then reads:

$$\mathcal{S}^{[2]}[\bar{A};a] = \frac{1}{2g^2}\int d^dx\, a^i_\mu \Big(-\bar{D}^2\delta_{ij}\delta^{\mu\nu} + \Big(1-\frac{1}{\alpha}\Big)\delta_{ij}\bar{D}^\mu\bar{D}^\nu - 2\bar{F}^{\ell\mu\nu} f_{i\ell j}\Big) a^j_\nu\,. \tag{2.74}$$

The Faddeev-Popov determinant is given by eq. (2.46):

$$\det\left[\frac{\delta' G^i[\bar{A};a]}{\delta\omega^j}\right] = \det\left[\bar{D}^\mu\bar{D}_\mu\delta^i_l + f^i{}_{kj}\bar{D}^\mu a^k_\mu\right]\,. \tag{2.75}$$

In order to simplify the computation one works in the 't Hooft-Feynman gauge $\alpha = 1$, then

$$\mathcal{S}^{[2]}[\bar{A};a] = \frac{1}{2g^2}\int d^dx\, a^i_\mu \Big(-\bar{D}^2\delta_{ij}\delta^{\mu\nu} - \bar{E}^{\mu\nu}_{ij}\Big) a^j_\nu\,, \tag{2.76}$$

where $\bar{E}^{\mu\nu}_{ij} = 2\bar{F}^{\ell\mu\nu} f_{i\ell j}$.

To compute the renormalization group flow of the theory one starts from the effective average action $\Gamma_k[A;a]$, which is assumed to have the same form of the original action $\mathcal{S}[A;a]$, where the bare coupling is replaced by the renormalized one dependent on k. The flow equation for the gauge invariant effective action $\Gamma_k[A] = \Gamma_k[A;0]$ is then given by:

$$\frac{\partial}{\partial t}\Gamma_k[A] = \frac{1}{2}\mathrm{Tr}_{\,a}\left[\left(\frac{\delta^2\Gamma_k[A;0]}{\delta a\delta a} + \mathcal{R}_k[A]\right)^{-1}\frac{\partial\mathcal{R}_k[A]}{\partial t}\right] - \mathrm{Tr}_{\,c}\left[\frac{\partial_t R_k[-D^2]}{-D^2 + R_k[-D^2]}\right].$$

From here onwards all bars from background quantities are dropped, since no confusion should arise. The cutoff kernel is chosen to be:

$$\mathcal{R}_k[A] = \frac{1}{g^2}R_k(-D^2 - E)\,, \tag{2.77}$$

where R_k is taken to be the optimized cutoff shape function of eq. (2.12). In the terminology of [Codello 2009b] this choice is called 'type II cutoff'. The t-derivative of the cutoff is:

$$\frac{\partial\mathcal{R}_k[A]}{\partial t} = \frac{1}{g^2}\left[\partial_t R_k(z) + \eta_a R_k(z)\right], \tag{2.78}$$

where $z = -D^2 - E$ and $\eta_a = -\partial_t \log g^2$ is the so called 'anomalous dimension'. The one-loop result is obtained by setting $\eta_a = 0$. The cutoff term for the ghost fields is given by eq. (2.56), combined with the Faddeev-Popov determinant it leads to a modified propagator for the ghost fields. Combining the quadratic action with the cutoff, the beta functional equation takes the following form:

$$\frac{\partial}{\partial t}\Gamma_k[A] = \frac{1}{2}\mathrm{Tr}_{\,a}\left[\frac{\partial_t R_k(z) + \eta_a R_k(z)}{P_k(z)}\right] - \mathrm{Tr}_{\,c}\left[\frac{\partial_t R_k[-D^2]}{-D^2 + R_k[-D^2]}\right], \tag{2.79}$$

where $P_k(z) = z + R_k(z)$. The computation of the traces in eq. (2.79) is performed using heat kernel methods presented in Appendix B. The relevant contribution to the first trace of eq. (2.79) comes from the B_4 coefficient of the heat kernel expansion:

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}_{\,a}\left[\frac{\partial_t R_k(z) + \eta_a R_k(z)}{P_k(z)}\right] &\supset \frac{1}{2}\frac{1}{(4\pi)^{d/2}}Q_{\frac{d}{2}-2}\left(\frac{\partial_t R_k(z) + \eta_a R_k(z)}{P_k(z)}\right)B_4(z)\\
&= \frac{1}{2}\frac{1}{(4\pi)^{d/2}}\frac{2k^{d-4}}{\Gamma(\frac{d}{2}-1)}\left(1 + \frac{\eta_a}{d-2}\right)\int d^dx\,\mathrm{tr}\left[\frac{1}{12}\Omega^{(a)}_{\mu\nu}\Omega^{\mu\nu}_{(a)} + \frac{1}{2}E^2\right]\\
&= \frac{C_2(G)}{(4\pi)^{d/2}}\frac{k^{d-4}}{\Gamma(\frac{d}{2}-1)}\left(1 + \frac{\eta_a}{d-2}\right)\left[-\frac{d}{12} + 2\right]\int d^dx\,F_i^{\mu\nu}F^i_{\mu\nu}\,.
\end{aligned} \tag{2.80}$$

In eq. (2.80) the quantity $\Omega^{(a)}_{\mu\nu}$ is the commutator of the background covariant derivative defined in eq. (2.37):

$$[D_\mu, D_\nu]a^i_\sigma = \Omega^{(a)i\rho}_{\mu\nu}{}_{k\sigma}a^k_\rho \quad , \quad \Omega^{(a)i\rho}_{\mu\nu}{}_{k\sigma} = \delta^\rho_\sigma f^i{}_{jk}F^j_{\mu\nu}\,. \tag{2.81}$$

Using the defining relation of the adjoint Casimir constant $C_2(G)$

$$f^i{}_{lm} f_{jlm} = C_2(G)\delta^i_j \,, \tag{2.82}$$

one has that

$$\mathrm{tr}\,\Omega^{(a)}_{\mu\nu}\Omega^{\mu\nu}_{(a)} = -dC_2(G)F^{\mu\nu}_i F^i_{\mu\nu} \qquad , \qquad \mathrm{tr}\;E^2 = E^{ij}_{\mu\nu}E^{\nu\mu}_{ji} = 4C_2(G)F^{\mu\nu}_i F^i_{\mu\nu} \,. \tag{2.83}$$

The Q-functional is computed using eq. (B.16)

$$Q_{\frac{d}{2}-2}\left(\frac{\partial_t R_k(z) + \eta_a R_k(z)}{P_k(z)}\right) = \frac{2}{\Gamma(\frac{d}{2}-1)}\left(1+\frac{\eta_a}{d-2}\right)k^{d-4} \,. \tag{2.84}$$

The relevant contribution to the second trace of eq. (2.79) comes again from the B_4 coefficient of the heat kernel expansion:

$$\begin{aligned}
\mathrm{Tr}_c\left[\frac{\partial_t R_k(-D^2)}{P_k(-D^2)}\right] \;&\supset\; \frac{1}{(4\pi)^{d/2}} Q_{\frac{d}{2}-2}\left(\frac{\partial_t R_k(-D^2)}{P_k(-D^2)}\right) B_4(-D^2) \\
&= \frac{1}{(4\pi)^{d/2}}\frac{2}{\Gamma(\frac{d}{2}-1)}k^{d-4}\int d^dx\,\mathrm{tr}\left[\frac{1}{12}\Omega^{(c)}_{\mu\nu}\Omega^{\mu\nu}_{(c)}\right] \\
&= -\frac{C_2(G)}{12}\frac{1}{(4\pi)^{d/2}}\frac{2}{\Gamma(\frac{d}{2}-1)}k^{d-4}\int d^dx\,F^{\mu\nu}_i F^i_{\mu\nu} \,.
\end{aligned} \tag{2.85}$$

In eq. (2.85) the quantity $\Omega^{(c)}_{\mu\nu}$ is the commutator of the ghost covariant derivative defined in eq. (2.49):

$$[D_\mu, D_\nu]c^i = \Omega^{(c)i}_{\mu\nu}{}_k c^k \qquad , \qquad \Omega^{(c)i}_{\mu\nu}{}_k = f^i{}_{jk}F^j_{\mu\nu} \,. \tag{2.86}$$

Then:

$$\mathrm{tr}\,\Omega^{(c)}_{\mu\nu}\Omega^{\mu\nu}_{(c)} = -C_2(G)F^{\mu\nu}_i F^i_{\mu\nu} \,. \tag{2.87}$$

Summing up the contributions coming from the traces in eq. (2.80) and eq. (2.85), reading off the coefficient of $1/4\int d^dx\,F^{\mu\nu}_i F^i_{\mu\nu}$, it is possible to extract the beta function of g^2:

$$\partial_t g^2 = -\frac{C_2(G)}{(4\pi)^{d/2}}\frac{4}{\Gamma(\frac{d}{2}-1)}k^{d-4}\left[\left(2-\frac{d}{12}\right)\left(1+\frac{\eta_a}{d-2}\right)+\frac{1}{6}\right]g^4 \,. \tag{2.88}$$

Substituting on the right hand side of eq. (2.88) the expression for η_a and solving the algebraic equation for $\partial_t g^2$ it is easy to obtain the expression for the beta function for g^2. Consider the case in which the gauge group is $G = SU(N)$, then $C_2(G) = N$. For $d = 4$ and $\eta_a = 0$, one obtains the well know one loop result:

$$\partial_t g^2 = -\frac{N}{(4\pi)^2}\frac{22}{3}g^4 \,. \tag{2.89}$$

Considering η_a and solving eq. (2.88) one gets a nonperturbative improvement for the beta function [Reuter 1994a]:

$$\partial_t g^2 = -\frac{N}{(4\pi)^2}\frac{22}{3}g^4\left(1-\frac{N}{(4\pi)^2}\frac{10}{3}g^2\right)^{-1} \,. \tag{2.90}$$

Expanding for small g^2, it is possible to compare beta function

$$\partial_t g^2 = -\frac{N}{(4\pi)^2}\frac{22}{3}g^4 - \frac{N^2}{(4\pi)^4}\frac{220}{9}g^6 + \mathcal{O}(g^8)\,, \tag{2.91}$$

with the well know two-loop result [Gross 1973, Politzer 1973, Jones 1974, Caswell 1974]:

$$\partial_t g^2 = -\frac{N}{(4\pi)^2}\frac{22}{3}g^4 - \frac{N^2}{(4\pi)^4}\frac{68}{3}g^6 + \mathcal{O}(g^8)\,. \tag{2.92}$$

There is a disagreement in the two-loop coefficient. Since the regularization scheme adopted is massive, one does not expect to reproduce the two-loop contribution which is universal for mass-independent regularization schemes such as dimensional regularization. Recently, a SUSY inspired beta function of the same form of eq. (2.90) has been proposed in [Ryttov 2008]. The authors of [Pica 2011] prove the existence of an all orders beta function for Yang-Mills theories assuming a linear relation between the beta function and the gauge field anomalous dimension. They have an equation of the same form of eq. (2.88) which yields a beta function that is similar, in shape, to the one in eq. (2.90). Using, instead, the functional renormalization group formalism, introduced in the previous sections, the linear relation between the beta function and the anomalous dimension is obtained as a consequence of the particular structure of the exact equation itself. This shows that this tool is powerful and able to capture some nonperturbative dynamics of the theory.

2.3.2 Nonlinear sigma model

Nonlinear sigma models are very rich class of theories [Ketov 2000]. They are widely used in high energy physics, where the most important application is chiral perturbation theory [Weinberg 1979b, Gasser 1984]. It describes the dynamics of the pions, regarded as Goldstone bosons of the flavor symmetry $SU(N)_L \times SU(N)_R$ broken to the diagonal subgroup $SU(N)_D$. On the other hand, nonlinear sigma models find applications also in condensed matter physics [Fradkin 1991] and string theory [Polyakov 1975].

From the mathematical point of view, the nonlinear sigma model describes the dynamics of a map φ from a d-dimensional base manifold $\mathcal{M}$ to a D-dimensional target manifold $\mathcal{N}$. Given a coordinate system $\{x^\mu\}$ on $\mathcal{M}$ and $\{y^\alpha\}$ on $\mathcal{N}$, the map φ is represented by D scalar fields $\varphi^\alpha(x)$. Physics must be independent of the choice of coordinates on $\mathcal{N}$, forcing the action to be a functional constructed with tensorial structure on $\mathcal{N}$. Only derivative interactions are allowed. The Euclidean action of the nonlinear sigma model can be expanded in derivatives and the lowest-order term is

$$\mathcal{S}[\varphi] = \frac{1}{2f^2}\int d^dx\, h_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta, \tag{2.93}$$

where $h_{\alpha\beta}$ is a dimensionless riemannian metric on $\mathcal{N}$ and f is the nonlinear sigma model coupling with mass dimension $(2-d)/2$. It usually assumed that $h_{\alpha\beta}$ is a positive-definite field-dependent matrix in order to ensure the absence of negative norm states. Moreover, the scalars are assumed to take their values in a compact usually symmetric space $\mathcal{N}$. From the field theory point of view, the nonlinear sigma model metric $h_{\alpha\beta}(\varphi)$ is just a set of given functions of φ. After being expanded in powers of φ,

$$h_{\alpha\beta}(\varphi) = h_{\alpha\beta}(0) + \partial_\gamma h_{\alpha\beta}(0)\varphi^\gamma + \frac{1}{2}\partial_\gamma\partial_\sigma h_{\alpha\beta}(0)\varphi^\gamma\varphi^\sigma + \cdots, \tag{2.94}$$

the action in eq. (2.93) thus represents a field theory with a generically infinite number of interactions and coupling constants. The action in eq. (2.93) is formally invariant under field reparametrizations (diffeomorphism invariance on $\mathcal{N}$)

$$\varphi^\alpha \to \varphi^{\alpha\prime}(\varphi)\,, \tag{2.95}$$

provided that the metric transforms as a second-rank tensor:

$$h'_{\alpha\beta}(\varphi') = \frac{\partial\varphi^\gamma}{\partial\varphi^{\alpha\prime}}\frac{\partial\varphi^\sigma}{\partial\varphi^{\beta\prime}}h_{\gamma\sigma}(\varphi). \tag{2.96}$$

The computation of the beta function for the nonlinear sigma model coupling is carried on using the background field method, where the full quantum field φ^α is expanded around a nonconstant background configuration $\bar\varphi^\alpha$ as $\varphi^\alpha(x) = \bar\varphi^\alpha(x) + \pi^\alpha(x)$. Since the field $\pi^\alpha(x)$ is a difference of coordinates it does not have good transformation properties, so it is convenient to express the background field expansion in terms of normal coordinates $\xi^\alpha(x)$, which are taken as quantum fields centered at $\bar\varphi^\alpha(x)$, *i.e.* $\mathrm{Exp}_{\bar\varphi(x)}(\xi(x)) = \varphi(x)$ [Honerkamp 1972, Alvarez-Gaume 1981]:

$$\varphi^\alpha = \bar\varphi^\alpha + \xi^\alpha - \frac{1}{2}\bar\Gamma_\beta{}^\alpha{}_\gamma\xi^\beta\xi^\gamma + \dots\,, \tag{2.97}$$

where $\bar\Gamma_\beta{}^\alpha{}_\gamma$ are the Christoffel symbols of the metric $\bar h_{\alpha\beta} = h_{\alpha\beta}(\bar\varphi)$. The background field expansions for the geometric objects entering in eq. (2.93) are given by [Honerkamp 1972, Alvarez-Gaume 1981]:

$$
\begin{aligned}
h_{\alpha\beta}(\varphi) &= \bar{h}_{\alpha\beta} - \frac{1}{3}\bar{R}_{\alpha\varepsilon\beta\eta}\xi^{\varepsilon}\xi^{\eta} + \cdots \\
\partial_\mu \varphi^{\alpha} &= \partial_\mu \bar{\varphi}^{\alpha} + \bar{\nabla}_\mu \xi^{\alpha} - \frac{1}{3}\partial_\mu \bar{\varphi}^{\gamma}\bar{R}_{\gamma\varepsilon}{}^{\alpha}{}_{\eta}\xi^{\varepsilon}\xi^{\eta} + \cdots
\end{aligned}
\tag{2.98}
$$

In eq. (2.98) the Riemann tensor $\bar{R}_{\gamma\varepsilon}{}^{\alpha}{}_{\eta}$ is constructed using the Christoffel symbols evaluated on the background $\bar{\varphi}$. The covariant derivative is defined as follows:

$$
\bar{\nabla}_\mu \xi^{\alpha} = \partial_\mu \xi^{\alpha} + \partial_\mu \bar{\varphi}^{\beta}\bar{\Gamma}_{\beta}{}^{\alpha}{}_{\gamma}\xi^{\gamma}. \tag{2.99}
$$

The nonlinear sigma model action in eq. (2.93) can be expanded in functional Taylor series around the background

$$
\mathcal{S}[\varphi] = \mathcal{S}[\bar{\varphi}] + \mathcal{S}^{[1]}[\bar{\varphi};\xi] + \mathcal{S}^{[2]}[\bar{\varphi};\xi] + \cdots \tag{2.100}
$$

where $\mathcal{S}^{[n]}$ is of order n in the fluctuations ξ. It is convenient to define the quantum fields $\xi^{a} = e^{a}_{\alpha}\xi^{\alpha}$, where e^{a}_{α} is a vielbein for the metric. The second order piece reads

$$
\mathcal{S}^{[2]}[\bar{\varphi};\xi] = \frac{1}{2f^2}\int d^d x\, \xi^{a}\left(-\bar{D}^2\delta_{ab} - \bar{M}_{ab}\right)\xi^{b}\,, \tag{2.101}
$$

where

$$
\bar{M}_{ab} = e^{\alpha}_{a} e^{\beta}_{b} \bar{D}_\mu \bar{\varphi}^{\varepsilon}\bar{D}^{\mu}\bar{\varphi}^{\eta}\bar{R}_{\varepsilon\alpha\eta\beta}\,. \tag{2.102}
$$

From here onwards all bars from background quantities will be dropped, since no confusion should arise. The beta function of f is obtained by solving the beta functional equation for the background invariant effective action $\Gamma[\varphi] = \Gamma_k[\varphi;0]$, which is assumed to have the same form of the original action:

$$
\frac{\partial}{\partial t}\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2\Gamma_k[\varphi;0]}{\delta\xi\delta\xi} + \mathcal{R}_k[\varphi]\right)^{-1}\frac{\partial \mathcal{R}_k[\varphi]}{\partial t}\right]. \tag{2.103}
$$

The cutoff kernel is chosen to be:

$$
\mathcal{R}_k[\varphi] = \frac{1}{f^2}R_k(-D^2)\,, \tag{2.104}
$$

where R_k is taken to be the optimized cutoff shape function of eq. (2.12). In the terminology of [Codello 2009b] this choice is called 'type I cutoff'. The t-derivative of the cutoff is:

$$
\frac{\partial \mathcal{R}_k[\varphi]}{\partial t} = \frac{1}{f^2}\left[\partial_t R_k(z) + \eta_\xi R_k(z)\right]\,, \tag{2.105}
$$

where $z = -D^2$ and $\eta_\xi = -\partial_t \log f^2$ is the so called 'anomalous dimension'. Combining the quadratic action with the cutoff, the flow equation for the background effective average action takes the following form:

$$\begin{aligned}\frac{\partial}{\partial t}\Gamma_k[\varphi] &= \frac{1}{2}\mathrm{Tr}\left[\frac{\partial_t R_k(z)+\eta_\xi R_k(z)}{P_k(z)-M}\right] \\ &= \frac{1}{2}\mathrm{Tr}\left[\frac{\partial_t R_k(z)+\eta_\xi R_k(z)}{P_k(z)}+\frac{\partial_t R_k(z)+\eta_\xi R_k(z)}{P_k^2(z)}M+\mathcal{O}(M^2)\right],\end{aligned} \tag{2.106}$$

where $P_k(z) = z + R_k(z)$. In eq. (2.106), the argument of the trace has been expanded in powers of $M/P_k(z)$ and the term containing two derivatives of φ is the second. This trace is evaluated using heat kernel methods presented in Appendix B. The relevant contribution comes from the B_0 coefficient of the heat kernel expansion:

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\left[\frac{\partial_t R_k+\eta_\xi R_k}{P_k^2}M\right] &\supset \frac{1}{2}\frac{1}{(4\pi)^{d/2}}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k+\eta_\xi R_k}{P_k}\right)\int d^dx\,\mathrm{tr}M \\ &= \frac{1}{2}\frac{1}{(4\pi)^{d/2}}\frac{2}{\Gamma(\frac{d}{2}+1)}\left(1+\frac{\eta_\xi}{d+2}\right)k^{d-2}\int d^dx\,R_{\alpha\beta}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\,.\end{aligned} \tag{2.107}$$

By inspection of eq. (2.106) one obtains a kind of Ricci flow [Codello 2009a]:

$$\partial_t\frac{1}{f^2}h_{\alpha\beta} = \frac{1}{(4\pi)^{d/2}}\frac{2}{\Gamma(\frac{d}{2}+1)}\left(1+\frac{\eta_\xi}{d+2}\right)k^{d-2}R_{\alpha\beta}\,. \tag{2.108}$$

If one sets $\eta_\xi = 0$ in eq. (2.108), the one loop result is represented by the 'geometric flow' in which the running of the metric is given by the Ricci tensor. It is possible to show that for $d = 2$ the coefficient of the beta function is scheme independent while for $d > 2$ there is a dependence on the cutoff choice but it does not affect the qualitative properties of the beta function.

The result of eq. (2.108) can be applied to homogeneous spaces of the form $\mathcal{N} = G/H$ admitting a single invariant Einstein metric $h_{\alpha\beta}$, up to scalings. In this case, it is convenient to think $h_{\alpha\beta}$ as being fixed and interpret the flow as affecting only f^2. The Ricci tensor of the metric is $R_{\alpha\beta} = \frac{R}{D}h_{\alpha\beta}$, where R is the Ricci scalar, therefore:

$$\partial_t f^2 = -\frac{1}{(4\pi)^{d/2}}\frac{2}{\Gamma(\frac{d}{2}+1)}k^{d-2}\left(1+\frac{\eta_\xi}{d+2}\right)\frac{R}{D}f^4\,. \tag{2.109}$$

When eq. (2.109) is solved for $\partial_t f^2$ one obtains a rational beta function. In terms of dimensionless coupling $\tilde{f}^2 = k^{d-2} f^2$, the beta function reads:

$$\partial_t \tilde{f}^2 = (d-2)\tilde{f}^2 - \frac{\frac{1}{(4\pi)^{d/2}} \frac{2}{\Gamma(\frac{d}{2}+1)} \frac{R}{D} \tilde{f}^4}{1 - \frac{1}{(4\pi)^{d/2}} \frac{2}{\Gamma(\frac{d}{2}+1)} \frac{R}{D} \frac{\tilde{f}^2}{d+2}} \,. \tag{2.110}$$

Looking first at the one-loop flow, one has that for $d > 2$ and $R > 0$ there is a nontrivial fixed point at $\tilde{f}^2_* = (4\pi)^{d/2}\Gamma(\frac{d}{2}+1)\frac{d-2}{2}\frac{D}{R}$. For large R this fixed point value occurs at small coupling, where perturbation theory is reliable. The derivative of the beta function at the fixed point is:

$$\left.\frac{\partial}{\partial \tilde{f}^2}\beta_{\tilde{f}^2}\right|_{\tilde{f}^2_*} = -(d-2)\,, \tag{2.111}$$

so this fixed point is UV attractive for any $d > 2$ and the mass critical exponent $\nu = 1/(d-2)$ is mean field-like. This shows that the nonlinear sigma model with positive Ricci curvature is an asymptotic safe theory. For $\mathcal{N} = SO(N+1)/SO(N)$ one has that $R = D(D-1)$ and it is possible to reproduce the results of $2+\varepsilon$ expansion [Polyakov 1975]. For $\mathcal{N} = SU(N)$ in $d = 4$ one has that $R = N(N^2-1)/4$ and $D = N^2 - 1$, then $\tilde{f}^2_* = 8(4\pi)^2/N$.

When one considers the full beta function in eq. (2.110) the one-loop fixed point is shifted at $\tilde{f}^2_* = (4\pi)^{d/2}\Gamma(\frac{d}{2}+1)\frac{D(d^2-4)}{2dR}$ and it is still UV attractive for $d > 2$. The mass critical exponent is now smaller then the mean-field value:

$$\nu = \frac{d+2}{2D(d-2)} < \frac{1}{d-2}\,. \tag{2.112}$$

Numerically, in $d = 4$ the results do not differ very much from the one-loop ones, but since the derivation is not based on perturbation theory, its validity does not depend on the coupling being small, this indicates that general nonlinear sigma model may be asymptotically safe even in $d = 4$. The truncation of the effective action considered here is very restrictive, then this result has to be considered just as an indication of the existence of the nontrivial fixed point for the nonlinear sigma model and further studies in which one takes into account different approximation schemes are needed to prove its existence.

Chapter 3

Gauged nonlinear sigma model

Contents

Any theory where a global symmetry G is spontaneously broken to some subgroup H at some characteristic energy scale Λ can be described at energies $E < \Lambda$ by a nonlinear sigma model, a theory describing the dynamics of a set of scalars with values in the coset space G/H [Weinberg 1968, Coleman 1969, Callan 1969, Salam 1969]. These scalars are the Goldstone bosons. Because the coset space is (in general) not a linear space, the physics of the Goldstone bosons is rather different from that of scalars carrying linear representations of G. The most important phenomenological application of this theory is chiral perturbation theory [Weinberg 1979b, Gasser 1984], it describes the dynamics of the pions, regarded as Goldstone bosons of the flavor symmetry $SU(N)_L \times SU(N)_R$ which, in QCD, is broken to the diagonal subgroup $SU(N)_D$ by the quark condensate. The theory is characterized by a mass scale F_π and, for energies $E < F_\pi$, terms with n derivatives give contributions that are suppressed by factors $(E/4\pi F_\pi)^n$, so one can usefully expand the action in powers of derivatives.

When such a theory is coupled to gauge fields of the group G, the physical interpretation for the Goldstone bosons changes completely with respect to the ungauged case. The Goldstone bosons are acted upon transitively by the gauge group, which means that any field configuration can be transformed into any other field configuration by a gauge transformation. So, in a sense, they are now gauge degrees of freedom. It is then possible to fix the gauge in such a way that the

Goldstone bosons disappear completely from the spectrum. In this unitary gauge no residual gauge freedom is left, so the spectrum of the theory consists just of massive gauge fields, the masses originating from the covariant kinetic term of the Goldstone bosons. This is the essence of the Higgs phenomenon, but in this variant where the scalars carry a nonlinear realization of G, there is no physical Higgs field left over. The most important phenomenological application of this idea is electroweak chiral perturbation theory [Applequist 1980, Longhitano 1980]. This model is perfectly adequate to give mass to the gauge bosons and can be seen as an approximation used in the SM where the energy is sufficiently low that the Higgs degree of freedom cannot be excited.[1]

For these reasons, it is important to have a good understanding of the UV behavior of this model. Here, the possibility that the gauged nonlinear sigma model could be asymptotically safe is explored. This would make the theory UV complete and predictive. Indications about the existence of a nontrivial fixed point for the nonlinear sigma model with two derivatives have been found in [Codello 2009a] (see also Section 2.3.2). This fixed point is preserved considering higher derivative terms as shown in [Percacci 2010]. The novelty presented in this chapter is the presence of the gauge fields coupled with the Goldstone boson degrees of freedom. There is no try to derive any phenomenological consequence, but merely consider the theoretical problem of the $SU(N)$ chiral model coupled to $SU(N)_L$ gauge fields [Fabbrichesi 2011b].

3.1 Gauged action

Consider the Euclidean action of the nonlinear sigma model introduced in eq. (2.93)

$$\mathcal{S}_0[\varphi] = \frac{1}{2f^2}\int d^dx\, h_{\alpha\beta}(\varphi)\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\,, \tag{3.1}$$

where φ^α are (dimensionless) coordinates on G/H. The infinitesimal transformation

$$\delta\varphi^\alpha = K^\alpha(\varphi) \tag{3.2}$$

induces a variation of the action given by

$$\delta\mathcal{S}_0 \;=\; \frac{1}{2f^2}\int d^dx\,\delta h_{\alpha\beta}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta + \frac{1}{f^2}\int d^dx\, h_{\alpha\beta}\delta\partial_\mu\varphi^\alpha\partial^\mu\phi^\beta\,. \tag{3.3}$$

Since

$$\delta h_{\alpha\beta} = \partial_\gamma h_{\alpha\beta}K^\gamma \quad , \qquad \delta\partial_\mu\phi^\alpha = \partial_\beta K^\alpha\partial_\mu\phi^\beta\,, \tag{3.4}$$

[1]In the SM also the QCD pions are responsible for giving mass to the gauge bosons, but their contribution is very small compared to tho one provided by the Higgs scalar.

then

$$\delta\mathcal{S}_0 = \frac{1}{f^2}\int d^dx\, \nabla_\alpha K_\beta \partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta \,. \tag{3.5}$$

In the equation above, the explicit definition of a metric-compatible connection has been used $\Gamma^\gamma_{\alpha\rho} = \frac{1}{2}h^{\gamma\sigma}(\partial_\alpha h_{\beta\rho} + \partial_\rho h_{\beta\alpha} - \partial_\beta h_{\alpha\rho})$. The vector K^α is called a 'Killing vector' if the following relation holds:

$$\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0 = -\mathcal{L}_K h_{\alpha\beta}\,. \tag{3.6}$$

If K^α is a Killing vector then the transformation in eq. (3.2) is called an isometry. Using the relation in eq. (3.6) into eq. (3.5) it is easy to show that $\delta\mathcal{S}_0 = 0$. Isometries form a finite dimensional subgroup of the diffeomorphism group on $\mathcal{N}$ (isometry group) and represent the global symmetries of the theory.

Consider the case where $\mathcal{N} = SU(N)$, endowed with a left- and right-invariant metric $h_{\alpha\beta}$. In order to describe this geometry one chooses the matrix generators $\{T_i\}$ in the fundamental representation satisfying $[T_i, T_j] = f_{ij}{}^kT_k$, where $f_{ij}{}^k$ are the structure constants of the symmetry group. The Ad-invariant Cartan-Killing form is $B_{ij} = \mathrm{Tr}\,(Ad(T_i)Ad(T_j)) = f_{i\ell}{}^k f_{jk}{}^\ell = -N\delta_{ij}$, whereas in the fundamental representation $\mathrm{Tr}\,(T_iT_j) = (1/2)\delta_{ij}$. The choice is to work with the inner product in the Lie algebra $-(1/N)B_{ij} = \delta_{ij}$. The Ad-invariance of this inner product implies that $f_{ijk} = f_{ij}{}^\ell\delta_{\ell k}$ is totally antisymmetric. Under the identification of the Lie algebra with the tangent space to the group at the identity, to each abstract generator T_i there corresponds a left-invariant vectorfield L^α_i and a right-invariant vectorfield R^α_i, coinciding with T_i at the identity. They form fields of bases on the group and satisfy the commutation relations:

$$[L_i, L_j] = f_{ij}{}^kL_k\ , \qquad [R_i, R_j] = -f_{ij}{}^kR_k\ . \tag{3.7}$$

The dual bases L^i_α and R^i_α are defined by

$$L^i_\alpha L^\alpha_j = \delta^i_j\ , \qquad R^i_\alpha R^\alpha_j = \delta^i_j\ , \qquad R^i_\alpha L^\alpha_j = Ad(U)^i{}_j\ , \tag{3.8}$$

where U denotes the $n\times n$ matrix corresponding to the group element with coordinate φ. The dual bases are the components of the Maurer-Cartan forms: $L^iT_i = U^{-1}dU$, $R^iT_i = dUU^{-1}$. The metric $h_{\alpha\beta}$ on the group is defined as the unique left- and right-invariant metric that coincides with the inner product in the Lie algebra: $\delta_{ij} = h(\mathbf{1})(R_i, R_j) = h(\mathbf{1})(L_i, L_j)$. Thus, the vectorfields R_i and L_i are Killing vectors, generating $SU(N)_L$ and $SU(N)_R$ respectively, and they are also orthonormal fields of frames on the group:

$$h_{\alpha\beta} = R^i_\alpha R^j_\beta\delta_{ij} = L^i_\alpha L^j_\beta\delta_{ij}\ . \tag{3.9}$$

Here, the choice is to gauge only $SU(N)_L$ and the corresponding gauge fields are denoted by A_μ. Restricting to terms containing two derivatives of the fields, the Euclidean action of the

d-dimensional gauged nonlinear sigma model is:

$$\mathcal{S}_0[\varphi, A] = \frac{1}{2f^2}\int d^dx\, h_{\alpha\beta} D_\mu\varphi^\alpha D^\mu\varphi^\beta + \frac{1}{4g^2}\int d^dx\, F^i_{\mu\nu}F_i^{\mu\nu}, \tag{3.10}$$

where f is the Goldstone coupling with mass dimension $(2-d)/2$ and g is the gauge coupling with mass dimension $(4-d)/2$. The Goldstone covariant derivative and the gauge field strength tensor are given by:

$$D_\mu\varphi^\alpha = \partial_\mu\varphi^\alpha + A^i_\mu R^\alpha_i(\varphi)\,, \tag{3.11}$$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + f^i{}_{jl}A^j_\mu A^l_\nu\,, \tag{3.12}$$

where $f^i{}_{jl}$ are $SU(N)$ structure constants. The action in eq. (3.10) is invariant under local $SU(N)_L$ infinitesimal transformations

$$\delta_{\varepsilon_L}\varphi^\alpha = -\varepsilon^i_L R^\alpha_i(\varphi) \qquad\qquad \delta_{\varepsilon_L}A^i_\mu = \partial_\mu\varepsilon^i_L + f_{j\ell}{}^i A^j_\mu\varepsilon^\ell_L\,. \tag{3.13}$$

3.1.1 Background field expansion and gauge fixing

The computation of the beta functions is performed using the background field method. The full quantum field $\varphi^\alpha(x)$ is expanded, using normal coordinates $\xi^\alpha(x)$, around nonconstant background field configuration $\bar\varphi^\alpha(x)$ as in eq. (2.97). The background field expansions of the geometric objects entering in eq. (3.10) are given by eq. (2.98) together with

$$R^\alpha_i(\varphi) = \bar R^\alpha_i + \xi^\varepsilon\bar\nabla_\varepsilon\bar R^\alpha_i - \frac{1}{3}\bar R^\alpha{}_{\varepsilon\gamma\eta}\bar R^\gamma_i\xi^\varepsilon\xi^\eta + \cdots \tag{3.14}$$

where the background covariant derivative $\bar\nabla_\varepsilon\bar R^\alpha_i$ is defined as:

$$\bar\nabla_\varepsilon\bar R^\alpha_i = \partial_\varepsilon\bar R^\alpha_i + \bar\Gamma_\varepsilon{}^\alpha{}_\gamma\bar R^\gamma_i\,. \tag{3.15}$$

The gauge field is expanded as $A^i_\mu(x) = \bar A^i_\mu(x) + a^i_\mu(x)$. The background field expansion for the gauge field strength tensor is given by:

$$F^i_{\mu\nu} = \bar F^i_{\mu\nu} + \bar D_\mu a^i_\nu - \bar D_\nu a^i_\mu + f^i_{j\ell}a^j_\mu a^\ell_\nu\,, \tag{3.16}$$

where $\bar D_\mu a^i_\nu$ and $\bar F^i_{\mu\nu}$ are defined in eq. (2.37). The action in eq. (3.10) can be expanded in functional Taylor series around the background:

$$\mathcal{S}_0[\varphi, A] = \mathcal{S}_0[\bar\varphi, \bar A] + \mathcal{S}_0^{[1]}[\bar\varphi, \bar A; \xi, a] + \mathcal{S}_0^{[2]}[\bar\varphi, \bar A; \xi, a] + \ldots \tag{3.17}$$

where $S_0^{[n]}$ is of order n in the fluctuations. The second order piece is

$$\begin{aligned} S_0^{[2]}[\bar\varphi,\bar A;\xi,a] &= \frac{1}{2f^2}\int d^dx\,\xi^\alpha\left(-\bar D^2\bar h_{\alpha\beta}-\bar D_\mu\bar\varphi^\varepsilon\bar D^\mu\bar\varphi^\eta\bar R_{\varepsilon\alpha\eta\beta}\right)\xi^\beta \\ &+ \frac{1}{f^2}\int d^dx\,a^i_\mu\left(\bar h_{\alpha\gamma}\bar D^\mu\bar\varphi^\alpha\bar\nabla_\beta\bar R^\gamma_i+\bar h_{\alpha\beta}\bar R^\alpha_i\bar D^\mu\right)\xi^\beta \\ &+ \frac{1}{2g^2}\int d^dx\,a^i_\mu\left(-\bar D^2\delta_{ij}\delta^{\mu\nu}+\bar D^\nu\bar D^\mu\delta_{ij}+\frac{g^2}{f^2}\bar h_{\alpha\beta}\bar R^\alpha_i\bar R^\beta_j\delta^{\mu\nu}+\bar F^{\ell\mu\nu}f_{\ell ij}\right)a^j_\nu\,, \end{aligned} \tag{3.18}$$

where

$$\bar D_\mu\xi^\alpha=\bar\nabla_\mu\xi^\alpha+\bar A^i_\mu\bar\nabla_\beta\bar R^\alpha_i\xi^\beta\,. \tag{3.19}$$

The covariant derivative $\bar\nabla_\mu\xi^\alpha$ is given in eq. (2.99) and $\bar D_\mu\bar\varphi^\alpha=\partial_\mu\bar\varphi^\alpha+\bar A^i_\mu\bar R^\alpha_i$. The second integral in eq. (3.18) is a cross term between the Goldstone boson and the gauge fluctuations, this is the generalization to the background field method of the term one usually obtains in the case of spontaneously broken gauge symmetries. This term contains a nonminimal derivative piece that can be removed by taking a suitable gauge fixing condition. To this end it is convenient to perform an integration by parts, making use of the Killing property $\bar\nabla_\alpha\bar R^i_\beta=-\bar\nabla_\beta\bar R^i_\alpha$, in order to rewrite:

$$\int d^dx\,a^i_\mu\left(\bar h_{\alpha\gamma}\bar D^\mu\bar\varphi^\alpha\bar\nabla_\beta\bar R^\gamma_i+\bar h_{\alpha\beta}\bar R^\alpha_i\bar D^\mu\right)\xi^\beta=\int d^dx\,\left(2a^i_\mu\bar h_{\alpha\gamma}\bar D^\mu\bar\varphi^\alpha\bar\nabla_\beta\bar R^\gamma_i-\bar h_{\alpha\beta}\bar R^\alpha_i\bar D^\mu a^i_\mu\right)\xi^\beta\,. \tag{3.20}$$

The quadratic part of the gauge action in eq. (3.18) contains a nonminimal second order differential operator and an explicit mass term for the gauge fields, that can be read using the orthonormality condition for the Killing vectors in eq. (3.9). Since the gauge symmetry is fully broken all the gauge fields become massive, their square mass is

$$m_A^2=\frac{g^2}{f^2}. \tag{3.21}$$

The gauge fixed action $\mathcal S[\bar\varphi,\bar A;\xi,a]$ is obtained by adding to the classical action of eq. (3.17) the background gauge fixing term $\mathcal S_{gf}[\bar\varphi,\bar A;\xi,a]$:

$$\mathcal S[\bar\varphi,\bar A;\xi,a]=\mathcal S_0[\bar\varphi,\bar A;\xi,a]+\mathcal S_{gf}[\bar\varphi,\bar A;\xi,a]. \tag{3.22}$$

The explicit form of the gauge fixing action is given by

$$\mathcal S_{gf}[\bar\varphi,\bar A;\xi,a]=\frac{1}{2\alpha g^2}\int d^dx\,\delta_{ij}\chi^i\chi^j\qquad\text{with}\qquad\chi^i=\bar D^\mu a^i_\mu+\beta\frac{g^2}{f^2}\bar h_{\alpha\beta}\bar R^\alpha_i\xi^\beta\,, \tag{3.23}$$

where α and β are gauge fixing parameters. The quadratic part of the gauge fixed action then reads:

$$\begin{aligned}\mathcal{S}^{[2]}[\bar\varphi,\bar A;\xi,a] &= \frac{1}{2f^2}\int d^dx\,\xi^\alpha\left(-\bar D^2\bar h_{\alpha\beta}-\bar D_\mu\bar\varphi^\varepsilon\bar D^\mu\bar\varphi^\eta\bar R_{\varepsilon\alpha\eta\beta}+\frac{\beta^2}{\alpha}\frac{g^2}{f^2}\bar h_{\alpha\beta}\right)\xi^\beta\\&+\frac{1}{2g^2}\int d^dx\,a^i_\mu\left(-\bar D^2\delta_{ij}\delta^{\mu\nu}+\left(1-\frac{1}{\alpha}\right)\delta_{ij}\bar D^\mu\bar D^\nu+\frac{g^2}{f^2}\delta_{ij}\delta^{\mu\nu}-2\bar F^{\ell\mu\nu}f_{i\ell j}\right)a^j_\nu\\&+\frac{2}{f^2}\int d^dx\,a^i_\mu\bar h_{\alpha\gamma}\bar D^\mu\varphi^\alpha\bar\nabla_\beta\bar R^\gamma_i\xi^\beta+\frac{1}{f^2}\left(\frac{\beta}{\alpha}-1\right)\int d^dx\,\bar h_{\alpha\beta}\bar R^\alpha_i\bar D^\mu a^i_\mu\xi^\beta\,.\end{aligned}\tag{3.24}$$

The Faddeev-Popov determinant is obtained from the variation of the gauge fixing term χ^i with respect to the infinitesimal gauge transformation, keeping the background fields fixed:

$$\delta_{\varepsilon_L}\chi^i=\bar D_\mu\delta_{\varepsilon_L}a^i_\mu+\beta\frac{g^2}{f^2}\bar h_{\alpha\beta}\bar R^\alpha_i\delta_{\varepsilon_L}\xi^\beta\,.\tag{3.25}$$

The variation of the gauge field can be read directly from the transformation properties of A^i_μ in eq. (3.13):

$$\delta_{\varepsilon_L}a^i_\mu\equiv D_\mu\varepsilon^i_L\,.\tag{3.26}$$

The variation of the normal coordinates ξ^α can be worked out using the transformation properties of φ^α in eq. (3.13) together with the relation in eq. (2.97) and inverting the series:

$$\delta\varphi^\alpha=\delta\xi^\alpha-\Gamma_\beta{}^\alpha{}_\gamma\xi^\beta\delta\xi^\gamma+\dots\tag{3.27}$$

In this case only the first pieces coming from the variations in eq. (3.25) matter:

$$\delta_{\varepsilon_L}\chi^i=\bar D_\mu\bar D^\mu\varepsilon^i_L-\beta\frac{g^2}{f^2}\bar h_{\alpha\beta}\bar R^\alpha_i\bar R^\beta_j\varepsilon^j_L+\dots\tag{3.28}$$

where the dots stand for terms containing the field a and higher powers of ξ. Using again eq. (3.9), the Faddeev-Popov determinant can be written as

$$\det\left[\frac{\delta\chi}{\delta\varepsilon_L}\right]=\det\left[\bar D^2-\beta\frac{g^2}{f^2}\right]\,.\tag{3.29}$$

As usual, one can rewrite the determinant in eq. (3.29) as a functional integral over anticommuting ghost fields c^i:

$$\det\left[\frac{\delta\chi}{\delta\varepsilon_L}\right]=\det\left[\bar D^2-\beta\frac{g^2}{f^2}\right]=\int\mathcal{D}\bar c\mathcal{D}c\,\exp-\mathcal{S}_{gh}[\bar A;\bar c,c]\,.\tag{3.30}$$

The ghost action is given by:

$$\mathcal{S}_{gh}[\bar{A};\bar{c},c] = \int d^4x\, \bar{c}^i \left(-\bar{D}^2\delta_{ij} + \beta\frac{g^2}{f^2}\delta_{ij}\right) c^j\,, \tag{3.31}$$

where

$$\bar{D}_\mu c^i = \partial_\mu c^i + f^i{}_{jl}\bar{A}^j_\mu c^l\,. \tag{3.32}$$

From here onwards all bars from background quantities will be dropped, since no confusion should arise, and the gauge fixing parameters are fixed to be equal, namely $\alpha = \beta$. This particular choice is called 'α-gauge fixing', it is a generalization to the background field method of what is usually known as R_ξ-gauge. The quadratic gauge fixed action in the α-gauge is then given by:

$$\begin{aligned} \mathcal{S}^{[2]}[\varphi,A;\xi,a] &= \frac{1}{2f^2}\int d^dx\,\xi^\alpha\left(-D^2h_{\alpha\beta} - D_\mu\varphi^\varepsilon D^\mu\varphi^\eta R_{\varepsilon\alpha\eta\beta} + \alpha\frac{g^2}{f^2}h_{\alpha\beta}\right)\xi^\beta \\ &+ \frac{1}{2g^2}\int d^dx\,a^i_\mu\left(-D^2\delta_{ij}\delta^{\mu\nu} + \left(1-\frac{1}{\alpha}\right)\delta_{ij}D^\mu D^\nu + \frac{g^2}{f^2}\delta_{ij}\delta^{\mu\nu} - 2F^{\ell\mu\nu}f_{i\ell j}\right)a^j_\nu \\ &+ \frac{2}{f^2}\int d^dx\,a^i_\mu h_{\alpha\gamma}D^\mu\varphi^\alpha\nabla_\beta R^\gamma_i\xi^\beta\,. \end{aligned} \tag{3.33}$$

In this gauge the cross term between the Goldstone bosons simplifies and the nonminimal derivative piece exactly cancels out. It is important to notice that the nonminimal differential piece acting on the gauge fluctuations is absent in the 't Hooft-Feynman gauge where the gauge fixing parameter is $\alpha = 1$. Moreover, the sigma model fields ξ acquire a mass given by:

$$m^2_\xi = \alpha g^2/f^2. \tag{3.34}$$

3.2 Beta functions

The computation of the beta functions of the $SU(N)$ gauged nonlinear sigma model is performed using functional methods presented in Chapter 2. In this case, one starts from the background effective average action $\Gamma_k[\varphi,A;\theta]$ which is assumed to have the same form of the original action $\mathcal{S}[\varphi,A;\theta]$, where the bare couplings f and g are replaced by renormalized couplings that depend on k. The functional $\Gamma_k[\varphi,A;\theta]$ depends on the background fields φ, A and on the classical average fields θ (the variables that are Legendre conjugated to the sources coupled linearly to the quantum field in the path integral definition). The collective field $\theta^T = (\xi^i, a^i_\mu)$ is a $D(1+d)$ component bosonic field and $\xi^i = R^i_\alpha\xi^\alpha$. From $\Gamma_k[\varphi,A;\theta]$ it is possible to construct the functional $\Gamma_k[\varphi,A] = \Gamma_k[\varphi,A;0]$, which is manifestly both diffeomorphism and gauge invariant.

It obeys a functional differential equation, which in the present context reads:

$$\frac{\partial}{\partial t}\Gamma_k[\varphi, A] = \frac{1}{2}\mathrm{Tr}_{\,\theta}\left[\left(\frac{\delta^2\Gamma_k[\varphi, A; 0]}{\delta\theta\delta\theta} + \mathcal{R}_k[\varphi, A]\right)^{-1}\frac{\partial\mathcal{R}_k[\varphi, A]}{\partial t}\right] - \frac{\partial}{\partial t}\log C_k[A]\,, \tag{3.35}$$

where $t = \log(k/k_0)$.

From eq. (3.33) one reads the second variation of the effective average action:

$$\Gamma^{[2]}_k[\varphi, A; \theta] = \frac{1}{2}\int d^dx\, \theta^T\, Q[\varphi, A]\, \theta\,, \tag{3.36}$$

where $Q[\varphi, A]$ is a background dependent covariant quadratic differential operator:

$$Q[\varphi, A] = \begin{pmatrix} \frac{1}{f^2}(-D^2\delta_{ij} + \alpha\frac{g^2}{f^2}\delta_{ij} - M_{ij}) & -\frac{1}{f^2}B^{\mu}_{ij} \\ -\frac{1}{f^2}B^{T\mu}_{ij} & \frac{1}{g^2}\left(-D^2\delta^{\mu\nu}_{ij} + \frac{g^2}{f^2}\delta^{\mu\nu}_{ij} + \left(\frac{\alpha-1}{\alpha}\right)D^{\mu}D^{\nu}\delta_{ij} - 2F^{\mu\nu}_{ij}\right) \end{pmatrix}. \tag{3.37}$$

In eq. (3.37), the explicit form of the endomorphism matrices is:

$$M_{ij} = R^{\alpha}_i R^{\beta}_j D_{\mu}\varphi^{\varepsilon} D^{\mu}\varphi^{\eta} R_{\varepsilon\alpha\eta\beta}\ ; \qquad F^{\mu\nu}_{ij} = F^{\ell\mu\nu} f_{i\ell j}\ ; \qquad B^{\mu}_{ij} = -2h_{\alpha\gamma}D^{\mu}\varphi^{\alpha}\nabla_{\beta}R^{\gamma}_i R^{\beta}_j. \tag{3.38}$$

In eq. (3.35) one needs to specify the form of the cutoff kernel $\mathcal{R}_k[\varphi, A]$ and to fix the quantity $C_k[A]$, which provides an infrared modification for the ghost propagator as described in Section 2.2.2.

There is a lot of freedom in the choice of the cutoff kernels. Generally, one chooses them in such a way as to make the calculations simpler, but it is also interesting to examine the dependence of the results on such choices. This is the called 'scheme dependence', because in the context of perturbation theory it is closely related to the dependence of results on the renormalization scheme. Results that have a direct physical significance should be scheme independent. The beta functions will be calculated in two different cases. The first calculation uses the 't Hooft-Feynman gauge $\alpha = 1$ and is valid in any dimension. The second calculation is in an arbitrary α-gauge but is restricted to four dimensions. It will be convenient to adopt slightly different schemes in the two cases. Then a comparison of the two calculations in four dimensions will be presented.

3.2.1 Arbitrary dimension, 't Hooft-Feynman gauge

In this subsection the gauge fixing parameter is set to be $\alpha = 1$ and the space-time dimension d is kept arbitrary. In this gauge, the operator of eq. (3.37) reduces to:

$$Q[\varphi, A] = \begin{pmatrix} \frac{1}{f^2}(-D_\xi^2 + \frac{g^2}{f^2} - M) & -\frac{1}{f^2}B \\ -\frac{1}{f^2}B^T & \frac{1}{g^2}\left(-D_a^2 + \frac{g^2}{f^2} - 2F\right) \end{pmatrix} . \tag{3.39}$$

In this gauge, the quantity $Q[\varphi, A]$ becomes a minimal second order operator (the highest order part is a Laplacian) and this simplifies the calculation significantly. The bosonic cutoff kernel $\mathcal{R}_k[\varphi, A]$ entering in eq. (3.35) is chosen to be:

$$\mathcal{R}_k[\varphi, A] = \begin{pmatrix} \frac{1}{f^2}R_k(z) & 0 \\ 0 & \frac{1}{g^2}R_k(w) \end{pmatrix} . \tag{3.40}$$

In eq. (3.40) $z = -D_\xi^2$ and $w = -D_a^2$, where D_ξ and D_a are defined in eq. (3.19) and eq. (2.37) respectively. The cutoff for the ghost sector is provided by the quantity $C_k[A]$ which is chosen to be:

$$C_k[A] = \det[1 + (y + g^2/f^2)^{-1}R_k(y)] , \tag{3.41}$$

where $y = -D_c^2$ and D_c is defined in eq. (3.32). The cutoff profile functions R_k are taken to be functions only of the background covariant Laplacians, in the terminology of [Codello 2009b] this is called a type I cutoff. The form of R_k is chosen to be the optimized one [Litim 2001]:

$$R_k(z) = (k^2 - z)\theta(k^2 - z) , \tag{3.42}$$

which ensures that the integrations over momenta are explicitly calculable. Being constructed with the background Laplacians, this cutoff prescription preserves the background invariance. The t-derivative of the bosonic cutoff kernel $\mathcal{R}_k[\varphi, A]$ is given by:

$$\frac{\partial \mathcal{R}_k[\varphi, A]}{\partial t} = \begin{pmatrix} \frac{1}{f^2}\left[\partial_t R_k(z) + \eta_\xi R_k(z)\right] & 0 \\ 0 & \frac{1}{g^2}\left[\partial_t R_k(w) + \eta_a R_k(w)\right] \end{pmatrix} , \tag{3.43}$$

where

$$\eta_\xi = -2\partial_t \log f \qquad \text{and} \qquad \eta_a = -2\partial_t \log g \tag{3.44}$$

are the so called 'anomalous dimensions', they give the nonperturbative contribution to the beta functions. The t-derivative of the ghost cutoff term is given by:

$$\frac{\partial}{\partial t}\log C_k[A] \quad = \quad \mathrm{Tr}_c\left[\left(P_k(y) + g^2/f^2\right)^{-1}\partial_t R_k(y)\right] . \tag{3.45}$$

At this point it is convenient to rewrite modified propagator entering in eq. (3.35) separating the differential part form the endomorphism part by writing:

$$\frac{\delta^2\Gamma_k[\varphi, A; 0]}{\delta\theta\delta\theta} + \mathcal{R}_k[\varphi, A] = Q[\varphi, A] + \mathcal{R}_k[\varphi, A] = \Pi_k[\varphi, A] - E[\varphi, A]\,. \tag{3.46}$$

In eq. (3.46) the quantity $\Pi_k[\varphi, A]$ is the matrix of the modified inverse propagators

$$\Pi_k[\varphi, A] = \begin{pmatrix} \frac{1}{f^2}(P_k(z) + \frac{g^2}{f^2}) & 0 \\ 0 & \frac{1}{g^2}(P_k(w) + \frac{g^2}{f^2}) \end{pmatrix}, \tag{3.47}$$

where $P_k(z) = z + R_k(z)$.

The quantity $E[\varphi, A]$ is the endomorphism block-matrix given by:

$$E[\varphi, A] = \begin{pmatrix} \frac{1}{f^2}M & \frac{1}{f^2}B \\ \frac{1}{f^2}B^T & \frac{2}{g^2}F \end{pmatrix}, \tag{3.48}$$

where the quantities M, B and F are defined in eq. (3.38). With the notation introduced above it is possible to rewrite eq. (3.35) as follows:

$$\frac{\partial}{\partial t}\Gamma_k[\varphi, A] = \frac{1}{2}\mathrm{Tr}_{\,\theta}\left[(\Pi_k - E)^{-1}\,\partial_t\mathcal{R}_k\right] - \mathrm{Tr}_{\,c}\left[\left(P_k + g^2/f^2\right)^{-1}\partial_t R_k\right]. \tag{3.49}$$

To solve the beta functional equation for the model one needs to compute the traces in eq. (3.49). The first trace is evaluated by expanding the argument in powers of $(\Pi_k)^{-1}E$:

$$\frac{1}{2}\mathrm{Tr}_\theta\left[(\Pi_k + E)^{-1}\,\partial_t\mathcal{R}_k\right] = \frac{1}{2}\mathrm{Tr}_\theta\left[\left(\Pi_k^{-1} + \Pi_k^{-1}E\Pi_k^{-1} + \Pi_k^{-1}E\Pi_k^{-1}E\Pi_k^{-1} + \ldots\right)\partial_t\mathcal{R}_k\right] \tag{3.50}$$

Note that in doing so the entire dependence on g^2/f^2 has been kept in the inverse propagator Π_k but the trace is expanded in powers of E, which depends on the background fields. The computation is restricted to the first three terms of the expansion since they are the only ones that give contribution to the beta functions of f and g. These traces can be evaluated using heat kernel methods and the detailed computation is presented in Appendix B.

The first trace gives:

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}_{\,\theta}\left[\Pi_k^{-1}\partial_t\mathcal{R}_k\right] \;\supset\; & -\frac{N}{48}\frac{1}{(4\pi)^{d/2}\Gamma(d/2-1)}\frac{k^{d-2}}{k^2+\frac{g^2}{f^2}}\left(1+\frac{\eta_\xi}{d-2}\right)\int d^dx\, F^I_{\mu\nu}F^{\mu\nu}_I \\ & - \frac{Nd}{12}\frac{1}{(4\pi)^{d/2}\Gamma(d/2-1)}\frac{k^{d-2}}{k^2+\frac{g^2}{f^2}}\left(1+\frac{\eta_a}{d-2}\right)\int d^dx\, F^{\mu\nu}_I F^I_{\mu\nu}\,.\end{aligned} \tag{3.51}$$

The second trace gives:

$$\frac{1}{2}\mathrm{Tr}_{\theta}\left[\Pi_k^{-1}E\Pi_k^{-1}\partial_t\mathcal{R}_k\right] \supset \frac{N}{4}\frac{1}{(4\pi)^{d/2}\Gamma(d/2+1)}\frac{k^{d+2}}{(k^2+\frac{g^2}{f^2})^2}\left(1+\frac{\eta_\xi}{d+2}\right)\int d^dx\, h_{\alpha\beta}D_\mu\varphi^\alpha D^\mu\varphi^\beta\,. \tag{3.52}$$

The third trace gives:

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}_{\theta}\left[\Pi_k^{-1}E\Pi_k^{-1}E\Pi_k^{-1}\partial_t\mathcal{R}_k\right] \supset\;& \frac{N\frac{g^2}{f^2}}{(4\pi)^{d/2}\Gamma(d/2+1)}\frac{k^{d+2}}{(k^2+\frac{g^2}{f^2})^3}\left(1+\frac{\eta_\xi}{d+2}\right)\int d^dx\, h_{\alpha\beta}D_\mu\varphi^\alpha D^\mu\varphi^\beta\\ &+\frac{N\frac{g^2}{f^2}}{(4\pi)^{d/2}\Gamma(d/2+1)}\frac{k^{d+2}}{(k^2+\frac{g^2}{f^2})^3}\left(1+\frac{\eta_a}{d+2}\right)\int d^dx\, h_{\alpha\beta}D_\mu\varphi^\alpha D^\mu\varphi^\beta\\ &+\frac{4N}{(4\pi)^{d/2}\Gamma(d/2+1)}\frac{k^{d+2}}{(k^2+\frac{g^2}{f^2})^3}\left(1+\frac{\eta_a}{d+2}\right)\int d^dx\, F_i^{\mu\nu}F^i_{\mu\nu}\,.\end{aligned} \tag{3.53}$$

The ghost trace gives:

$$\mathrm{Tr}_c\left[\left(P_k+g^2/f^2\right)^{-1}\partial_t R_k\right] \supset -\frac{N}{6}\frac{1}{(4\pi)^{d/2}\Gamma(d/2-1)}\frac{k^{d-2}}{k^2+\frac{g^2}{f^2}}\int d^dx\, F_i^{\mu\nu}F^i_{\mu\nu}\,. \tag{3.54}$$

The system of coupled equations for the beta functions of g^2 and f^2 is obtained by summing up all contributions coming from eq. (3.51), eq. (3.52), eq. (3.53), eq. (3.54) and reading off the coefficients of $(1/2)\int d^dx\, h_{\alpha\beta}D_\mu\varphi^\alpha D^\mu\varphi^\beta$ and $(1/4)\int d^dx\, F^i_{\mu\nu}F^{i\mu\nu}$:

$$\partial_t f^2 = -\frac{N}{2}\frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2+1)}\frac{k^{d-2}}{(1+\frac{\tilde g^2}{\tilde f^2})^2}\left[1+\frac{\eta_\xi}{d+2}+\frac{4\frac{\tilde g^2}{\tilde f^2}}{1+\frac{\tilde g^2}{\tilde f^2}}\left(2+\frac{\eta_\xi+\eta_a}{d+2}\right)\right]f^4 \tag{3.55}$$

$$\begin{aligned}\partial_t g^2 \;=\;& \frac{1}{(4\pi)^{d/2}}\frac{N}{3}\frac{1}{\Gamma(d/2-1)}\frac{k^{d-4}}{1+\frac{\tilde g^2}{\tilde f^2}}\left[-\frac{192}{d(d-2)}\frac{1}{(1+\frac{\tilde g^2}{\tilde f^2})^2}\left(1+\frac{\eta_a}{d+2}\right)+d-2\right.\\ &\left.+\frac{1}{4}+\frac{\eta_\xi/4+d\eta_a}{d-2}\right]g^4\end{aligned} \tag{3.56}$$

where $\tilde f^2 = f^2k^{d-2}$ and $\tilde g^2 = g^2k^{d-4}$ are the dimensionless couplings.

A few comments are in order at this point. As they stand, these are not yet explicit beta functions, because the right hand sides contain the beta functions themselves inside the factors of η_ξ and η_a. Thus, these can be regarded as algebraic equations for the beta functions that can easily be obtained by solving the above equations. Omitting the terms containing η_ξ and η_a in the right hand sides, one obtains the one loop beta functions.

These equations give the beta functions of the dimensionful couplings. The corresponding beta functions of the dimensionless combinations $\tilde{f}^2$ and $\tilde{g}^2$ can be obtained by simple algebra. Note that on the right hand side the dimensions are carried just by the explicit powers of k, all the rest is dimensionless.

As mentioned earlier, the only approximation made in this calculation consists in neglecting higher derivative terms. This is a good approximation at sufficiently low energy and one is implicitly assuming that it remains a reasonably good approximation also at higher energy. Provided this important assumption is true, these beta functions are valid at all energy scales: having used a mass-dependent renormalization, one gets automatically the effect of thresholds, which are represented by the factors $1/(1+g^2/\tilde{f}^2)$ (note that g^2/f^2 has dimensions of mass squared in any dimension). For $k^2 \gg g^2/f^2$ these factors become equal to one, whereas for $k^2 \ll g^2/f^2$ the denominators become large and suppress the running, reflecting the decoupling of the corresponding massive field modes.

Finally it is important to observe that (3.56) has an apparent pole at $d=2$, which is actually cancelled by the pole of the function $\Gamma(d/2-1)$ in the denominator.

3.2.2 Four dimensions, generic α-gauge

In this section, a generic α-gauge is considered, it is the generalization to the background field method of what is usually known as R_ξ gauge, where the parameter ξ is now called α in order not to generate confusion with the Goldstone modes. Due to the increased complication, the space-time dimension is fixed to be $d=4$. In this case the operator $Q[\varphi, A]$ of eq. (3.37) is nonminimal (meaning that the highest order terms are not simply a Laplacian). A standard way of dealing with nonminimal operators is to decompose the field they act on in irreducible components, in the present case the longitudinal and transverse parts of a^i_μ. The resulting operators acting on the irreducible subspaces are typically of Laplace type. One thus defines the operators $\mathcal{D}_L$ and $\mathcal{D}_T$ by

$$\mathcal{D}_{T\mu\nu} = -D^2\delta_{\mu\nu} - 2F_{\mu\nu} \ ; \qquad \mathcal{D}_{L\mu\nu} = -D_\mu D_\nu \ , \tag{3.57}$$

where it is understood that F acts to the fields in the adjoint representation, as in equation (3.38). Assuming that the background gauge field is covariantly constant, one can easily prove that the following operators are projectors:

$$\mathbf{P}_L = \mathcal{D}_T^{-1}\mathcal{D}_L \qquad , \qquad \mathbf{P}_T = \mathbf{1} - \mathbf{P}_L \ . \tag{3.58}$$

Introducing these projectors, it is possible to rewrite the operator $Q[\varphi, A]$ of eq. (3.37) as follows:

$$Q[\varphi, A] = \begin{pmatrix} \frac{1}{f^2}(-D_\xi^2 + \alpha\frac{g^2}{f^2} - M) & -\frac{1}{f^2}B \\ -\frac{1}{f^2}B^T & \frac{1}{g^2}\left[(\mathcal{D}_T + \frac{g^2}{f^2})\mathbf{P}_T + \frac{1}{\alpha}(\mathcal{D}_T + \alpha\frac{g^2}{f^2})\mathbf{P}_L\right] \end{pmatrix}, \tag{3.59}$$

where the matrices M and B are defined in eq. (3.38). The quantity $\mathcal{R}_k[\varphi, A]$ entering in eq. (3.35) is chosen such that the cutoff is introduced separately in the transverse and longitudinal subspaces as follows:

$$\mathcal{R}_k[\varphi, A] = \begin{pmatrix} \frac{1}{f^2}R_k(z) & 0 \\ 0 & \frac{1}{g^2}\left[R_k(w)\mathbf{P}_T + \frac{1}{\alpha}R_k(w)\mathbf{P}_L\right] \end{pmatrix}, \tag{3.60}$$

where $z = -D_\xi^2 - M$ and $w = \mathcal{D}_T$. In eq. (3.60), the optimized form for the cutoff profile function R_k given in eq. (3.42) is used. Note that the cutoff is now a function of the kinetic operator acting in each irreducible subspace, including the background-dependent terms M and F, but not the mass-like term g^2/f^2. Following the terminology of [Codello 2009b], this is called a type II cutoff. For the ghosts one uses the same cutoff defined in eq. (4.32) with the difference that now the ghost mass is $\alpha g^2/f^2$.

The t-derivative of the bosonic cutoff kernel $\mathcal{R}_k[\varphi, A]$ is given by:

$$\frac{\partial\mathcal{R}_k[\varphi, A]}{\partial t} = \begin{pmatrix} \frac{1}{f^2}\left[\partial_t R_k(z) + \eta_\xi R_k(z)\right] & 0 \\ 0 & \frac{1}{g^2}\left[(\partial_t R_k(w) + \eta_a R_k(w))\mathbf{P}_T + \frac{1}{\alpha}(\partial_t R_k(w) + \eta_a R_k(w))\mathbf{P}_L\right] \end{pmatrix}, \tag{3.61}$$

where

$$\eta_\xi = -2\partial_t \log f \qquad \text{and} \qquad \eta_a = -2\partial_t \log g \tag{3.62}$$

are the so called anomalous dimensions. The t-derivative of the ghost cutoff term is given by:

$$\frac{\partial}{\partial t}\log C_k[A] \;=\; \mathrm{Tr}_c\left[\left(P_k(y) + \alpha g^2/f^2\right)^{-1}\partial_t R_k(y)\right]. \tag{3.63}$$

In order to compute the beta function of the Goldstone boson coupling it is convenient to set $A_\mu = 0$. In this case the projectors of eq. (3.58) reduce to

$$\mathbf{P}_{L\mu\nu} = \frac{\partial_\mu\partial_\nu}{\partial^2} \quad , \quad \mathbf{P}_{T\mu\nu} = 1 - \frac{\partial_\mu\partial_\nu}{\partial^2} \tag{3.64}$$

and one can use standard momentum space techniques. The ghost part does not give contribution to the beta function of f, in this case eq. (3.35) reduces to:

$$\frac{\partial}{\partial t}\Gamma_k[\varphi, 0] = \frac{1}{2}\mathrm{Tr}_\theta\left[\left(\frac{\delta^2\Gamma_k[\varphi, 0; 0]}{\delta\theta\delta\theta} + \mathcal{R}_k[\varphi, 0]\right)^{-1}\frac{\partial\mathcal{R}_k[\varphi, 0]}{\partial t}\right]. \tag{3.65}$$

The modified inverse propagator entering in eq. (3.65) is:

$$\frac{\delta^2\Gamma_k[\varphi,0;0]}{\delta\theta\delta\theta}+\mathcal{R}_k[\varphi,0]=\begin{pmatrix}\frac{1}{f^2}(P_k(z)+\alpha\frac{g^2}{f^2}) & -\frac{1}{f^2}\tilde{B}\\ -\frac{1}{f^2}\tilde{B}^T & \frac{1}{g^2}\left[(P_k(w)+\frac{g^2}{f^2})\mathbf{P}_T+\frac{1}{\alpha}(P_k(w)+\alpha\frac{g^2}{f^2})\mathbf{P}_L\right]\end{pmatrix},$$

where $\tilde{B}$ is the matrix B of eq. (3.38) evaluated at $A_\mu=0$, $z=-\nabla_\xi^2-M$ and $w=-\partial^2$. As in the previous section, it is convenient to separate the derivative part form the diffeomorphism part in eq. (3.66) by writing:

$$\frac{\delta^2\Gamma_k[\varphi,0;0]}{\delta\theta\delta\theta}+\mathcal{R}_k[\varphi,0]=\Pi_k[\varphi]-E[\varphi]\,, \tag{3.66}$$

where

$$E=\begin{pmatrix}0 & \frac{1}{f^2}\tilde{B}\\ \frac{1}{f^2}\tilde{B}^T & 0\end{pmatrix}. \tag{3.67}$$

The functional bosonic trace of eq. (3.65) is evaluated by expanding in powers of $(\Pi_k)^{-1}E$:

$$\frac{1}{2}\mathrm{Tr}_\theta\left[(\Pi_k-E)^{-1}\,\partial_t\mathcal{R}_k\right]=\frac{1}{2}\mathrm{Tr}_\theta\left[\left(\Pi_k^{-1}+\Pi_k^{-1}E\Pi_k^{-1}+\Pi_k^{-1}E\Pi_k^{-1}E\Pi_k^{-1}+\ldots\right)\partial_t\mathcal{R}_k\right]. \tag{3.68}$$

These traces can be computed using heat kernel methods presented in Appendix B. Contributions to the beta function of f come from the first and the third trace of eq. (3.68).

The first trace gives:

$$\frac{1}{2}\mathrm{Tr}_\theta\left[\Pi_k^{-1}\partial_t\mathcal{R}_k\right]\;\supset\;\frac{N}{4}\frac{1}{(4\pi)^2}\frac{k^4}{k^2+\alpha\frac{g^2}{f^2}}\left(1+\frac{\eta_\xi}{4}\right)\int d^4x\,h_{\alpha\beta}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\,. \tag{3.69}$$

The third trace gives:

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\left[\Pi_k^{-1}E\Pi_k^{-1}E\Pi_k^{-1}\partial_t\mathcal{R}_k\right]\;\supset\;&\frac{N}{8}\frac{1}{(4\pi)^2}\frac{g^2}{f^2}\left[3\frac{k^6}{(k^2+\alpha\frac{g^2}{f^2})^2(k^2+\frac{g^2}{f^2})}\left(1+\frac{\eta_\xi}{6}\right)\right.\\ &+\alpha\frac{k^6}{(k^2+\alpha\frac{g^2}{f^2})^3}\left(1+\frac{\eta_\xi}{6}\right)+3\frac{k^6}{(k^2+\frac{g^2}{f^2})^2(k^2+\alpha\frac{g^2}{f^2})}\left(1+\frac{\eta_a}{6}\right)\\ &\left.+\alpha\frac{k^6}{(k^2+\alpha\frac{g^2}{f^2})^3}\left(1+\frac{\eta_a}{6}\right)\right]\int d^4x\,h_{\alpha\beta}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta\,.\end{aligned} \tag{3.70}$$

Summing all contributions coming from eq. (3.69), eq. (3.70) and reading off the coefficients of

$(1/2)\int d^4x\, h_{\alpha\beta}\partial_\mu\varphi^\alpha\partial^\mu\varphi^\beta$, one obtains the following beta function for f^2:

$$\partial_t f^2 = -\frac{N}{(4\pi)^2}\frac{k^2}{2}\frac{1+\frac{\eta_\xi}{4}}{1+\alpha\frac{g^2}{\tilde{f}^2}}f^4 - \frac{N}{(4\pi)^2}g^2 f^2\Bigg[\left(1+\frac{\eta_\xi}{6}\right)\left(\frac{3}{4}\frac{1}{(1+\frac{g^2}{f^2})(1+\alpha\frac{g^2}{\tilde{f}^2})^2}+\frac{\alpha}{4}\frac{1}{(1+\alpha\frac{g^2}{\tilde{f}^2})^3}\right)$$
$$+\left(1+\frac{\eta_a}{6}\right)\left(\frac{3}{4}\frac{1}{(1+\frac{g^2}{\tilde{f}^2})^2(1+\alpha\frac{g^2}{f^2})}+\frac{\alpha}{4}\frac{1}{(1+\alpha\frac{g^2}{\tilde{f}^2})^3}\right)\Bigg]. \tag{3.71}$$

At this point, in order to compute the running of the gauge coupling g, it is convenient to set $D_\mu\varphi^\alpha = 0$. In this case $B = 0$, $M = 0$ and eq. (3.35) reduces to:

$$\frac{\partial}{\partial t}\Gamma_k[0,A] = \frac{1}{2}\mathrm{Tr}_\xi\left[\frac{\partial_t R_k+\eta_\xi R_k}{P_k+\alpha\frac{g^2}{f^2}}\right]+\frac{1}{2}\mathrm{Tr}_a\left[\frac{\partial_t R_k+\eta_a R_k}{P_k+\frac{g^2}{f^2}}\mathbf{P}_T\right]+\frac{1}{2}\mathrm{Tr}_a\left[\frac{\partial_t R_k+\eta_a R_k}{P_k+\alpha\frac{g^2}{f^2}}\mathbf{P}_L\right]$$
$$-\ \mathrm{Tr}_c\left[\frac{\partial_t R_k}{P_k+\alpha\frac{g^2}{f^2}}\right]. \tag{3.72}$$

These traces can be computed using heat kernel methods presented in Appendix B.

The first trace gives:

$$\frac{1}{2}\mathrm{Tr}_\xi\left[\frac{\partial_t R_k+\eta_\xi R_k}{P_k+\alpha\frac{g^2}{f^2}}\right]\supset -\frac{N}{(4\pi)^2}\frac{k^2}{k^2+\alpha\frac{g^2}{f^2}}\left(\frac{2+\eta_\xi}{96}\right)\int d^4x\, F^i_{\mu\nu}F_i^{\mu\nu}. \tag{3.73}$$

The second trace gives:

$$\frac{1}{2}\mathrm{Tr}_a\left[\frac{\partial_t R_k+\eta_\xi R_k}{P_k+\frac{g^2}{f^2}}\mathbf{P}_T\right]\supset \frac{N}{(4\pi)^2}\frac{k^2}{k^2+\frac{g^2}{f^2}}\left(\frac{2+\eta_a}{2}\right)\frac{7}{4}\int d^4x\, F^i_{\mu\nu}F_i^{\mu\nu}. \tag{3.74}$$

The third trace gives:

$$\frac{1}{2}\mathrm{Tr}_a\left[\frac{\partial_t R_k+\eta_\xi R_k}{P_k+\alpha\frac{g^2}{f^2}}\mathbf{P}_L\right]\supset -\frac{N}{(4\pi)^2}\frac{k^2}{k^2+\alpha\frac{g^2}{f^2}}\left(\frac{2+\eta_a}{2}\right)\frac{1}{12}\int d^4x\, F^i_{\mu\nu}F_i^{\mu\nu}. \tag{3.75}$$

The ghost trace gives:

$$\mathrm{Tr}_c\left[\frac{\partial_t R_k}{P_k+\alpha\frac{g^2}{f^2}}\right]\supset -\frac{N}{(4\pi)^2}\frac{k^2}{k^2+\alpha\frac{g^2}{f^2}}\frac{1}{6}\int d^4x\, F_i^{\mu\nu}F^i_{\mu\nu}. \tag{3.76}$$

Summing all contributions coming from eq. (3.73), eq. (3.74), eq. (3.75), eq. (3.76) and reading off the coefficients of $(1/4)\int d^4x F^i_{\mu\nu}F^{i\mu\nu}$, one obtains the following beta function for g^2:

$$\partial_t g^2 = -\frac{N}{(4\pi)^2}\left[\frac{7}{2}\frac{2+\eta_a}{1+\frac{g^2}{f^2}} + \frac{1}{3}\left(2 - \frac{2+\eta_a}{2} - \frac{2+\eta_\xi}{8}\right)\frac{1}{1+\alpha\frac{g^2}{f^2}}\right] g^4\,. \tag{3.77}$$

The first piece appearing on the right hand side of eq. (3.77) is the transverse gauge boson contribution, the second piece is the contribution of the ghosts, the third is the longitudinal gauge boson contribution and the fourth is the contribution of the Goldstone bosons. As for the results of the previous section, the final expressions obtained in eq. (4.66) and eq. (3.77) are not the explicit beta functions, because the right hand sides contain the beta functions themselves inside the factors of η_ξ and η_a. These can be regarded as algebraic equations for the beta functions that can easily be obtained by solving those equations.

3.2.3 Comparison

At this point, some observations concerning the gauge- and scheme-dependence of the beta functionsone are in order. Specializing equation (3.55) to the case $d = 4$ one obtains

$$\partial_t f^2 \;=\; -\frac{N}{(4\pi)^2}\frac{k^2}{4}\frac{1}{(1+\frac{g^2}{f^2})^2}\left[1 + \frac{\eta_\xi}{6} + \frac{4\frac{g^2}{f^2}}{1+\frac{g^2}{f^2}}\left(2 + \frac{\eta_\xi+\eta_a}{6}\right)\right] f^4\,. \tag{3.78}$$

Focusing on the one loop result ($\eta_\xi = \eta_a = 0$), it is useful to study the high energy limit of the beta function ($k^2 \gg g^2/f^2$). Expanding for small $\frac{g^2}{f^2}$ and retaining the first leading terms, eq. (3.78) becomes

$$\partial_t f^2 = -\frac{1}{(4\pi)^2}\frac{3}{2}Ng^2 f^2 - \frac{1}{(4\pi)^2}\frac{N}{4}k^2 f^4\,, \tag{3.79}$$

whereas equation (4.66) reduces in the same limit to

$$\partial_t f^2 = -\frac{1}{(4\pi)^2}\frac{3}{2}Ng^2 f^2 - \frac{1}{(4\pi)^2}\frac{N}{2}k^2 f^4\,. \tag{3.80}$$

At the first order in the gauge coupling, the beta function of f^2 is gauge independent and reduces to the one of the unguaged nonlinear sigma model in the limit $g = 0$, In this case, the leading terms of these beta functions are scheme dependent. As already observed in [Percacci 2010], the difference in the coefficient is the effect of passing from the type I cutoff to a type II cutoff. Note also that in this approximation, the difference could be absorbed in a redefinition of k if one wanted.

On the other hand, specializing equation (3.56) to the case $d = 4$ one obtains

$$\partial_t g^2 = -\frac{N}{(4\pi)^2}\left[\frac{8}{(1+\frac{g^2}{\tilde{f}^2})^3}\left(1+\frac{\eta_a}{6}\right)-\frac{1}{3}\left(\frac{9}{4}+2\eta_a+\frac{1}{8}\eta_\xi\right)\frac{1}{1+\frac{g^2}{\tilde{f}^2}}\right]g^4 . \tag{3.81}$$

This does not agree with equations (3.77), specialized to the case $\alpha = 1$. However, if one considers the one loop part of the beta function, *i.e.* neglects all the terms involving η_ξ and η_a, and energies much larger than the threshold ($\frac{g^2}{\tilde{f}^2} \ll 1$), then the beta function of the gauge coupling reduces to

$$\partial_t g^2 = -\frac{N}{(4\pi)^2}\frac{29}{4}g^4 . \tag{3.82}$$

This is the same in both calculations and illustrates the universality of these beta functions.

3.3 Results

3.3.1 Fixed points in $d = 4$

In this subsection the space-time dimension is fixed to be $d = 4$. As mentioned before, due to the presence of the terms involving η_a and η_ξ, equations (3.55), (3.56), (4.66), (3.77) are not the beta functions themselves but linear equations for the beta functions. They do become the one loop beta functions if one drops all the terms involving η_a and η_ξ. Otherwise, before solving for the flow, one has to solve them. The general structure of the beta functions is

$$\partial_t f^2 = -(A_1 k^2 + B_{11}k^2\eta_\xi + B_{12}k^2\eta_a)f^4 , \tag{3.83}$$

$$\partial_t g^2 = -(A_2 + B_{21}\eta_\xi + B_{22}\eta_a)g^4 , \tag{3.84}$$

where A_i and B_{ij} are (dimensionless) functions of $\tilde{f}$ and g that one can easily read off from equations (3.55), (3.56), (4.66), (3.77). The solution of these algebraic equations has the form

$$\partial_t \tilde{f} = \tilde{f} - \frac{1}{2}\frac{(A_1 + (B_{12}A_2 - B_{22}A_1)g^2)\tilde{f}^3}{1 - B_{22}g^2 - B_{11}\tilde{f}^2 + (B_{11}B_{22} - B_{12}B_{21})g^2\tilde{f}^2} , \tag{3.85}$$

$$\partial_t g = -\frac{1}{2}\frac{(A_2 + (B_{21}A_1 - B_{11}A_2)\tilde{f}^2)g^3}{1 - B_{11}\tilde{f}^2 - B_{22}g^2 + (B_{11}B_{22} - B_{12}B_{21})\tilde{f}^2 g^2} . \tag{3.86}$$

Notice that it is the beta functions of the dimensionless couplings that have to vanish in the definition of fixed point. In the one loop approximation one just sets all the B_{ij} coefficients to zero, so that the denominators simplify to one, and in the numerators only the terms A_1 and A_2 survive. Comparison of equations (3.77) and (3.81) shows that even at one loop the beta function of g is scheme- and gauge-dependent. However, this dependence only affects the threshold behavior due to the fact that this model describes massive gauge fields. For $k^2 \gg g^2/f^2$

the massive modes decouple and this is reflected in the large denominators, which effectively switch off the beta functions. If one considers the regime $g^2/\tilde{f}^2 \ll 1$, the denominators reduce to one. In this case the beta function of g is given by equation (3.82):

$$\partial_t g = -\frac{1}{2} A_2 g^3 \; ; \tag{3.87}$$

with a universal coefficient $A_2 = \frac{N}{(4\pi)^2}\frac{29}{4}$. Note that $29/4$ differs from the coefficient $22/3$ of the pure gauge theory by the Goldstone boson contribution $-1/12$. This contribution is quite small and does not spoil the asymptotic freedom of g. On the other hand, in the same limit the beta function of $\tilde{f}$ becomes

$$\partial_t f = \tilde{f} - \frac{1}{2} A_1 \tilde{f}^3 \tag{3.88}$$

with $A_1 = \frac{1}{(4\pi)^2}\frac{N}{4}$ or $A_1 = \frac{1}{(4\pi)^2}\frac{N}{2}$ for cutoffs of type I or II respectively. This beta function has a nontrivial fixed point at $\tilde{f}_* = \sqrt{2/A_1}$.

The solution of the beta functions (3.85) and (3.86), including the improvement, due to the η-terms, requires a bit more work. In addition to the Gaussian fixed point at $g = 0$, $\tilde{f} = 0$, there is always also a non-Gaussian fixed point where $\tilde{f} \neq 0$. The position of this fixed point and the scaling exponents θ_i (defined as minus the eigenvalues of the linearized flow equations) are given in the following table:

cutoff and gauge	$\tilde{f}_*$	g_*	θ_1	θ_2
type I, $\alpha = 1$	$4\pi\sqrt{6/N}$	0	8/3	0
type II, $\alpha = 1$	$8\pi\sqrt{2/3N}$	0	3	0
type II, $\alpha = 0$	$8\pi\sqrt{2/3N}$	0	3	0

This gives an idea of the scheme- dependence of the results. Note that g is always asymptotically free and when one sets $g = 0$ the beta function of f becomes α-independent. Therefore, the position of the fixed point is actually gauge independent. Fig. 3.1 shows the solutions of eq. (3.87) and eq. (3.88) in the case $N = 2$:

$$\tilde{f}(t) = \frac{8\pi e^t}{\sqrt{16\pi^2 + e^{2t}}} \; , \tag{3.89}$$

$$g(t) = \frac{4\pi}{\sqrt{1 + \frac{29}{2} t}} \; , \tag{3.90}$$

where $\tilde{f}(0) = 2$ and $g(0) = 4\pi$.

At high energies the Goldstone boson coupling $\tilde{f}$ reaches a nontrivial fixed point at $\tilde{f}_* = 8\pi$ while g goes to zero.

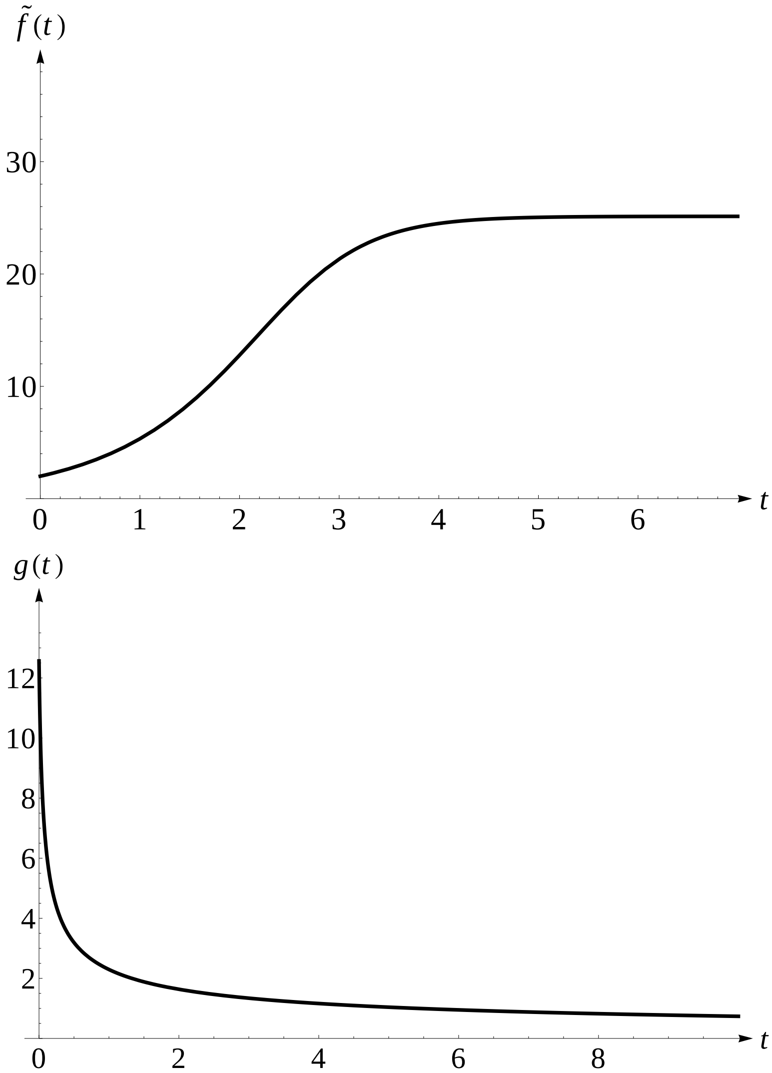


Figure 3.1: Scale dependence of the $SU(2)$ gauged nonlinear sigma model couplings $\tilde{f}$ and g.

3.3.2 Fixed points in other dimensions

The beta functions in eq. (3.55) and eq. (3.56) admit solutions also in arbitrary dimension. The existence of nontrivial fixed points in Yang-Mills theories in $d > 4$ has been discussed earlier in [Kazakov 2003, Gies 2003]. It is due to the nontrivial dimensionality of the gauge coupling. One would expect it to be there also in the presence of the Goldstone bosons. As usual, the simplest way to see this is to consider the one loop beta functions

$$\partial_t \tilde{f} = \frac{d-2}{2}\tilde{f} - \frac{1}{2}A_1 \tilde{f}^3; \tag{3.91}$$

$$\partial_t \tilde{g} = \frac{d-4}{2}\tilde{g} - \frac{1}{2}A_2 g^3 \tag{3.92}$$

where $\tilde{f} = k^{(d-2)/2} f$ and $\tilde{g} = k^{(d-4)/2} g$. From (3.56) and (3.55) one finds

$$A_1 = \frac{1}{(4\pi)^{d/2}} \frac{N}{2} \frac{1}{\Gamma\left(\frac{d}{2}+1\right)} \frac{1}{(1+\frac{\tilde{g}^2}{\tilde{f}^2})^2} \left[1 + \frac{8\tilde{g}^2/\tilde{f}^2}{1+\frac{\tilde{g}^2}{\tilde{f}^2}}\right] , \tag{3.93}$$

$$A_2 = \frac{1}{(4\pi)^{d/2}} \frac{N}{3} \frac{1}{\Gamma\left(\frac{d}{2}-1\right)} \frac{1}{1+\frac{\tilde{g}^2}{\tilde{f}^2}} \left[-\frac{1}{4} - d + 2 + \frac{192}{d(d-2)} \frac{1}{(1+\frac{\tilde{g}^2}{\tilde{f}^2})^2}\right] \tag{3.94}$$

In the limit $k^2 \gg g^2/f^2$, A_1 and A_2 become positive constants implying a fixed point at

$$\tilde{f}_* = \sqrt{\frac{d-2}{A_1}} \; ; \qquad \tilde{g}_* = \sqrt{\frac{d-4}{A_2}} \; . \tag{3.95}$$

It is important to note that the value of $\frac{\tilde{g}^2}{\tilde{f}^2} = \frac{d-4}{d-2}\frac{A_1}{A_2}$ at this fixed point is indeed rather small, so that the approximation is justified a posteriori. For a better approximation one has to solve the equations numerically.

The study presented in [Gies 2003] reveals the presence of such a non-Gaussian UV fixed point for the dimensionless gauge coupling $\tilde{g}$ only for $4 < d < d_{cr}$ where the critical dimension d_{cr} depends on the gauge group. They show that $d_{cr} > 5$ for $N \leq 5$, this seems to point to the possibility that 5-dimensional Yang-Mills theories can be asymptotically safe and renormalizable.

3.3.3 Comments

The nontrivial fixed point that has been found in these calculations could be the basis of asymptotic safety in a spontaneously broken chiral theory. Quadratic divergences are essential in generating the nontrivial fixed point and the cutoff regularization used here is enough to see the emergence of such a fixed point. Moreover, the functional renormalization group techniques allow to go beyond one loop by resumming infinitely many perturbative contributions. Further improvements using these techniques can be achieved by going to higher orders of the derivative expansion.

Within the truncation considered, the presence of the Goldstone bosons does not affect the asymptotic freedom of the gauge fields. On the other hand, the fact that the gauge coupling vanishes in the UV makes the fixed point in the Goldstone boson sector to be the same as in the ungauged case. One expects that the same will be true when the four-derivative terms are added. If this is the case, the fixed point structure of the chiral nonlinear sigma model couplings should be the same as described in [Percacci 2010].

A somewhat worrying aspect of these results, especially if one restricts oneself to the one loop approximation, is that they generally require strong interactions. This follows from the fact that in the beta function of $\tilde{f}$ the loop contribution has to cancel the classical scaling term. Addressing this worry is actually the main reason for using functional renormalization group methods: their validity does not rely on the coupling being small. Of course, one is then making other approximations, namely neglecting higher order terms in the derivative expansion.

If $\tilde{f}_* < 8\pi$ the leading order term of the nonlinear theory that has been studied is the dominant one, and due to the existence of the fixed point chiral perturbation theory is convergent at all energies. In the electroweak version, f is related to the Higgs VEV via the identification $f = 2/\upsilon$. If one follows an renormalization group trajectory towards higher energies one will encounter essentially two distinct regimes. For energies below the mass of the gauge fields, the beta functions are suppressed by the threshold terms. For energies above the mass of the gauge fields the coupling f runs, behaving asymptotically like $1/k$ and giving rise to a nearly scale invariant regime (scale invariance is broken by the running of g, which is however very slow in comparison). The onset of the nearly scale invariant regime depends on the position of the fixed point and occurs earlier for smaller values of $\tilde{f}_*$. For instance, if $\tilde{f}_* = 8\pi$, the scale invariant regime begins at approximately 20 TeV (see Fig. 3.1), whereas if $\tilde{f}_* = 2$, the scale invariant regime begins at approximately 1 TeV.

Chapter 4

Phenomenological applications

Contents

The study presented in Chapter 3 shows that the fixed point of the nonlinear sigma model, whose evidence has been found in [Codello 2009a], persist when one couples the Goldstone bosons to gauge fields. In this case the theory can be nonperturbatively renormalizable. While these results do not properly prove the existence of the fixed point, they are however suggestive because they could have some important implications in particle physics phenomenology. The best known phenomenological application of the model is electroweak chiral perturbation theory, which is the most general parametrization of the Higgs phenomenon in terms of the minimal number of degrees of freedom, the three would-be Goldstone bosons. This theory is described by a set of $SU(2)_L \times U(1)_R$ and CP invariant operators and the lowest order term in the derivative expansion is the well know lagrangian of the gauged nonlinear sigma model (see Appendix C).

In this chapter, the possibility that the electroweak gauged nonlinear sigma model might be asymptotically safe is considered. The first part is devoted to the study of the renormalization group flow of the $SU(2)_L \times U(1)_R$ gauged nonlinear sigma model using functional methods. The computation resembles the one presented in Chapter 3 with some differences that will be highlighted and explained in the first section. After having supported the evidence of the possible UV completion of the electroweak model at the nontrivial fixed point, the general point of view is to assume the existence of such a fixed point and apply the asymptotic safety construction in order to work out the phenomenological consequences of this assumption.

The predictivity of the construction can be tested only when the theory space is enlarged by including higher order operators in the truncation. In the next section of this chapter, dimension four operators, related to the electroweak S and T parameters, are taken into account and the renormalization group flow of the theory is studied by functional methods. The predictions obtained from the asymptotic safety picture are presented and the compatibility of the model with precision measurements is discussed [Fabbrichesi 2011a].

In a realistic model one needs to accommodate also SM fermions coupling them in a chiral invariant way to the Goldstone fields. In this way it is possible to provide a mass for quarks and leptons. In the third section of this chapter, the renormalization group flow of the nonlinear sigma model coupled to fermions is studied. In this case, a one-loop computation shows that the inclusion of fermions drastically modifies the asymptotic properties of the nonlinear sigma model. The modifications one has to provide in order to preserve asymptotic safety of the theory are discussed. In particular, the good UV limit of the theory is ensured by adding to the model four fermion interactions terms [Bazzocchi 2011].

In the last section, the computation of the Goldstone boson scattering amplitude is presented. In this case, the effective action of the theory is obtained by solving the Wetterich equation, using the non-local heat kernel expansion. The integration of the flow leads to divergences that are removed by standard renormalization conditions. The final amplitude is the same of perturbation theory, this shows that the formalism is able to reproduce the known perturbative results.

4.1 $SU(2) \times U(1)$ gauged nonlinear sigma model

In this section the renormalization group flow of the electroweak chiral lagrangian is studied, where the terms considered are restricted to the lowest order in the derivative expansion. The theory is the $SU(2)_L \times U(1)_R$ gauged nonlinear sigma model. The action of this model is a slight generalization of the one introduced in eq. (3.10) specialized to $N = 2$ where the gauge group is enlarged to $SU(2)_L \times U(1)_R$. The corresponding gauge fields are denoted by W^i_μ and B_μ respectively. Using a general parametrization of the Goldstone bosons in terms of (dimensionless) coordinates $\varphi^\alpha(x)$, the Euclidean action reads:

$$\mathcal{S}_0[\varphi, W, B] = \frac{1}{2f^2} \int d^4x \, h_{\alpha\beta}(\varphi) D_\mu \varphi^\alpha D^\mu \varphi^\beta + \frac{1}{4g^2} \int d^4x \, W^i_{\mu\nu} W^{\mu\nu}_i + \frac{1}{4g'^2} \int d^4x \, B_{\mu\nu} B^{\mu\nu} \,, \quad (4.1)$$

where $h_{\alpha\beta}(\varphi)$ is a dimensionless metric of $SU(2)$. As usual, f represents the Goldstone boson coupling with mass dimension -1, while g and g' are the dimensionless gauge couplings. The Goldstone covariant derivative and the gauge field strength tensors are given by:

$$D_\mu \varphi^\alpha = \partial_\mu \varphi^\alpha + W^i_\mu R^\alpha_i(\varphi) - B_\mu L^\alpha_3(\varphi) \,, \quad (4.2)$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + \varepsilon^i_{jl} W^j_\mu W^l_\nu \quad , \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \,, \quad (4.3)$$

where ε^i_{jl} is the three dimensional Levi-Civita symbol. The indices $\alpha, \beta = 1, 2, 3$ run over the target space coordinates while $i, j, l = 1, 2, 3$ are $SU(2)$ Lie-algebra indices. R^α_i and L^α_i are the right- and left-invariant vectorfields on the target space $SU(2)$. In particular, the fields $\mathcal{R}^\alpha_i$ generate the $SU(2)_L$ transformations while L^α_3 is taken as the generator of the $U(1)_R$ transformations. These vectorfields are taken to be orthonormal fields of frames on the group as in eq. (3.9). The action in eq. (4.1) is invariant under local $SU(2)_L \times U(1)_R$ infinitesimal transformations

$$\delta_\varepsilon \varphi^\alpha = -\varepsilon^i_L R^\alpha_i(\varphi) + \varepsilon_R L^\alpha_3(\phi) \quad (4.4)$$

provided that

$$\delta_\varepsilon W^i_\mu = \partial_\mu \varepsilon^i_L + \varepsilon^i{}_{jk} W^j_\mu \varepsilon^k_L \equiv D_\mu \varepsilon^i_L \quad , \quad \delta_\varepsilon B_\mu = \partial_\mu \varepsilon_R \,. \quad (4.5)$$

4.1.1 Background field expansion and gauge fixing

The beta functions have been computed using the background field method as in Chapter 3. The full quantum field $\varphi^\alpha(x)$ is expanded, using normal coordinates $\xi^\alpha(x)$, around nonconstant background field configuration $\bar{\varphi}^\alpha(x)$ as in eq. (2.97). The background field expansions of the geometric objects entering in eq. (4.1) are given by eq. (2.98) and eq. (3.14). The gauge fields split as $W^i_\mu(x) = \bar{W}^i_\mu(x) + w^i_\mu(x)$ and $B_\mu(x) = \bar{B}_\mu(x) + b_\mu(x)$. The background field expansion for the gauge field strength tensors can be read from eq. (3.16) specialized to the $SU(2)$ case. As usual, the action in eq. (4.1) can be expanded in functional Taylor series around the background:

$$\mathcal{S}_0[\varphi, W, B] = \mathcal{S}_0[\bar{\varphi}, \bar{W}, \bar{B}] + \mathcal{S}^{[1]}_0[\bar{\varphi}, \bar{W}, \bar{B}; \xi, w, b] + \mathcal{S}^{[2]}_0[\bar{\varphi}, \bar{W}, \bar{B}; \xi, w, b] + \ldots \quad (4.6)$$

where $\mathcal{S}_0^{[n]}$ is of order n in the fluctuations. The second order piece of the classical action turns out to be:

$$\begin{aligned}
\mathcal{S}_0^{[2]}[\bar\varphi,\bar W,\bar B;\xi,w,b] &= \frac{1}{2f^2}\int d^4x\,\xi^\alpha\Big(-\bar D^2\bar h_{\alpha\beta}-\bar D_\mu\bar\varphi^\varepsilon\bar D^\mu\bar\varphi^\eta\bar R_{\varepsilon\alpha\eta\beta}\Big)\xi^\beta\\
&+\frac{1}{f^2}\int d^4x\,w^i_\mu\Big(\bar h_{\alpha\gamma}\bar D^\mu\bar\varphi^\alpha\bar\nabla_\beta\bar R^\gamma_i+\bar h_{\alpha\beta}\bar R^\alpha_i\bar D^\mu\Big)\xi^\beta\\
&-\frac{1}{f^2}\int d^4x\,b_\mu\Big(\bar h_{\alpha\gamma}\bar D^\mu\bar\varphi^\alpha\bar\nabla_\beta\bar L^\gamma_3+\bar h_{\alpha\beta}\bar L^\alpha_3\bar D^\mu\Big)\xi^\beta\\
&+\frac{1}{2g^2}\int d^4x\,w^i_\mu\Big(-\bar D^2\delta_{ij}\delta^{\mu\nu}+\bar D^\nu\bar D^\mu\delta_{ij}+\frac{g^2}{f^2}\bar h_{\alpha\beta}\bar R^\alpha_i\bar R^\beta_j\delta^{\mu\nu}+\bar W^{l\mu\nu}\varepsilon_{lij}\Big)w^j_\nu\\
&+\frac{1}{2g'^2}\int d^4x\,b_\mu\Big(-\partial^2\delta^{\mu\nu}+\partial^\nu\partial^\mu+\frac{g'^2}{f^2}\bar h_{\alpha\beta}\bar L^\alpha_3\bar L^\beta_3\delta^{\mu\nu}\Big)b_\nu\\
&-\frac{1}{f^2}\int d^4x\,w^i_\mu\,\Big(\bar h_{\alpha\beta}\bar R^\alpha_i\bar L^\beta_3\delta^{\mu\nu}\Big)\,b_\nu\,,
\end{aligned}\tag{4.7}$$

where the bar denotes that the quantities are evaluated on the background. In eq. (4.7) the background covariant derivatives are:

$$\bar D_\mu\xi^\alpha=\bar\nabla_\mu\xi^\alpha+\bar W^i_\mu\bar\nabla_\beta\bar R^\alpha_i\xi^\beta-\bar B_\mu\bar\nabla_\beta\bar L^\alpha_3\xi^\beta\tag{4.8}$$

and

$$\bar D_\mu w^i_\nu=\partial_\mu w^i_\nu+\varepsilon^i_{jk}\bar W^j_\mu w^k_\nu\,.\tag{4.9}$$

The action in eq. (4.7) is a slight generalization of the one in eq. (3.18) in which new cross terms involving the gauge field b_μ are present. Cross terms between the Goldstone bosons and the gauge fields contain nonminimal derivative pieces that can be removed by taking a suitable gauge fixing condition, after having performed an integration by parts as in eq. (3.20). In the last integral of eq. (4.7) there is an explicit mixing between b_μ and the third component of the field w_μ, making use of the orthonormality condition in eq. (3.9) and expanding $h_{\alpha\beta}R^\alpha_iL^\beta_3=\delta_{i3}+\dots$ one can read off the w_3-b mixing mass matrix:

$$\begin{pmatrix} g^2/f^2 & -gg'/f^2\\ -gg'/f^2 & g'^2/f^2\end{pmatrix}.\tag{4.10}$$

The first eigenvalue is $m_A^2=0$ and is identified with the square mass of the photon while the second is $m_Z^2=(g^2+g'^2)/f^2$ and is identified with the square mass of the Z boson. The square mass value for the other gauge fields w_1 and w_2 is $m_W^2=g^2/f^2$.

The gauge fixed action $\mathcal{S}[\bar\varphi,\bar W,\bar B;\xi,w,b]$ is obtained by adding to the original action of eq. (4.6) a background gauge fixing term $\mathcal{S}_{gf}[\bar\varphi,\bar W,\bar B;\xi,w,b]$:

$$\mathcal{S}[\bar\varphi,\bar W,\bar B;\xi,w,b]=\mathcal{S}_0[\bar\varphi,\bar W,\bar B;\xi,w,b]+\mathcal{S}_{gf}[\bar\varphi,\bar W,\bar B;\xi,w,b]\,.\tag{4.11}$$

The explicit form of the gauge fixing action is given by:

$$\mathcal{S}_{gf}[\bar\varphi,\bar W,\bar B;\xi,w,b]=\frac{1}{2g^2}\int d^4x\,\delta_{ij}\chi^i\chi^j+\frac{1}{2g'^2}\int d^4x\,\psi\psi \tag{4.12}$$

where

$$\chi^i=\bar D^\mu w^i_\mu+\frac{g^2}{f^2}\bar h_{\alpha\beta}\bar R^\alpha_i\xi^\beta \qquad , \qquad \psi=\partial^\mu b_\mu-\frac{g'^2}{f^2}\bar h_{\alpha\beta}\bar L^\alpha_3\xi^\beta\,. \tag{4.13}$$

This is the generalization of the 't Hooft-Feynman gauge fixing to the background field method, as discussed in Section 3.1.1. This gauge fixing allows to simplify the mixed terms between the Goldstone bosons and the gauge fluctuations. The resulting quadratic part of the gauge fixed action is:

$$\begin{aligned}
\mathcal{S}^{[2]}[\bar\varphi,\bar W,\bar B;\xi,w,b] \;=\;& \frac{1}{2f^2}\int d^4x\,\xi^\alpha\left(-\bar D^2\bar h_{\alpha\beta}-\bar D_\mu\bar\varphi^\varepsilon\bar D^\mu\bar\varphi^\eta\bar R_{\varepsilon\alpha\eta\beta}+m_W^2\bar h_{\alpha\beta}+m_B^2\bar L^3_\alpha\bar L^3_\beta\right)\xi^\beta\\
&+\frac{2}{f^2}\int d^4x\,w^i_\mu\big(\bar h_{\alpha\gamma}\bar D^\mu\bar\varphi^\alpha\bar\nabla_\beta\bar R^\gamma_i\big)\xi^\beta-\frac{2}{f^2}\int d^4x\,b_\mu\big(\bar h_{\alpha\gamma}\bar D^\mu\bar\varphi^\alpha\bar\nabla_\beta\bar L^\gamma_3\big)\xi^\beta\\
&+\frac{1}{2g^2}\int d^4x\,w^i_\mu\left(-\bar D^2\delta_{ij}\delta^{\mu\nu}+m_W^2\delta_{ij}\delta^{\mu\nu}-2\bar W^{l\mu\nu}\varepsilon_{ilj}\right)w^j_\nu\\
&+\frac{1}{2g'^2}\int d^4x\,b_\mu\left(-\partial^2\delta^{\mu\nu}+m_B^2\delta^{\mu\nu}\right)b_\nu-\frac{1}{f^2}\int d^4x\,w^i_\mu\left(\bar h_{\alpha\beta}\bar R^\alpha_i\bar L^\beta_3\delta^{\mu\nu}\right)b_\nu\,,
\end{aligned} \tag{4.14}$$

where $m_B^2=g'^2/f^2$. In this gauge the operator acting on the gauge fluctuations is a minimal second order operator. The Faddeev-Popov determinant is obtained from the variation of the gauge fixing termsin eq. (4.13) with respect to the infinitesimal gauge transformation, keeping the background fields fixed:

$$\begin{aligned}
\delta_{\varepsilon_L}\chi^i &= \bar D^\mu\delta_{\varepsilon_L}w^i_\mu+\frac{g^2}{f^2}\bar h_{\alpha\beta}\bar R^\alpha_i\delta_{\varepsilon_L}\xi^\beta\,,\\
\delta_{\varepsilon_R}\chi^i &= \frac{g^2}{f^2}\bar h_{\alpha\beta}\bar R^\alpha_i\delta_{\varepsilon_R}\xi^\beta\,,\\
\delta_{\varepsilon_L}\psi &= -\frac{g'^2}{f^2}\bar h_{\alpha\beta}\bar L^\alpha_3\delta_{\varepsilon_L}\xi^\beta\,,\\
\delta_{\varepsilon_R}\psi &= \partial^\mu\delta_{\varepsilon_R}b_\mu-\frac{g'^2}{f^2}\bar h_{\alpha\beta}\bar L^\alpha_3\delta_{\varepsilon_R}\xi^\beta\,.
\end{aligned} \tag{4.15}$$

The variations of the gauge fields can be read directly from eq. (4.5) while the variation of the normal coordinates ξ^α can be worked out using eq. (4.4) together with eq. (2.97) and inverting the series. For the present purposes only the first pieces coming from the variations in eq. (4.15) matter:

$$\delta_{\varepsilon_L}\chi^i = \bar{D}^\mu \bar{D}_\mu \varepsilon_L^i - \frac{g^2}{f^2}\bar{h}_{\alpha\beta}\bar{R}_i^\alpha \bar{R}_j^\beta \varepsilon_L^j + \cdots , \tag{4.16}$$

$$\delta_{\varepsilon_R}\chi^i = \frac{g^2}{f^2}\bar{h}_{\alpha\beta}\bar{R}_i^\alpha \bar{L}_3^\beta \varepsilon_R + \cdots , \tag{4.17}$$

$$\delta_{\varepsilon_L}\psi = \frac{g'^2}{f^2}\bar{h}_{\alpha\beta}\bar{L}_3^\alpha \bar{R}_i^\beta \varepsilon_L^i + \cdots , \tag{4.18}$$

$$\delta_{\varepsilon_R}\psi = \partial^\mu \partial_\mu \varepsilon_R - \frac{g'^2}{f^2}\bar{h}_{\alpha\beta}\bar{L}_3^\alpha L_3^\beta \varepsilon_R + \cdots , \tag{4.19}$$

where the dots stand for terms containing the fields w, b and higher powers of ξ. Using the orthonormality condition of the Killing fields, that the Faddeev-Popov determinant is given by:

$$\det \begin{bmatrix} \frac{\delta\chi}{\delta\varepsilon_L} & \frac{\delta\chi}{\delta\varepsilon_R} \\ \frac{\delta\psi}{\delta\varepsilon_L} & \frac{\delta\psi}{\delta\varepsilon_R} \end{bmatrix} = \det \begin{bmatrix} \bar{D}^2\delta_{ij} - \frac{g^2}{f^2}\delta_{ij} & \frac{gg'}{f^2}\bar{h}_{\alpha\beta}\bar{R}_i^\alpha \bar{L}_3^\beta \\ \frac{gg'}{f^2}\bar{h}_{\alpha\beta}\bar{R}_i^\alpha \bar{L}_3^\beta & \partial^2 - \frac{g'^2}{f^2} \end{bmatrix} \tag{4.20}$$

As usual, one can write the determinant in eq. (4.20) as a functional integral over anticommuting ghost fields (c^i, η):

$$\det \begin{bmatrix} \frac{\delta\chi}{\delta\varepsilon_L} & \frac{\delta\chi}{\delta\varepsilon_R} \\ \frac{\delta\psi}{\delta\varepsilon_L} & \frac{\delta\psi}{\delta\varepsilon_R} \end{bmatrix} = \int \mathcal{D}\bar{c}\mathcal{D}c\mathcal{D}\bar{\eta}\mathcal{D}\eta \, \exp -\mathcal{S}_{gh}[\bar{\varphi}, \bar{W}; \bar{c}, c, \bar{\eta}, \eta] \, . \tag{4.21}$$

The ghost action is given by:

$$\begin{aligned} \mathcal{S}_{gh}[\bar{\varphi}, \bar{W}; \bar{c}, c, \bar{\eta}, \eta] &= \int d^4x \, \bar{c}^i \, (-\bar{D}^2\delta_{ij} + \frac{g^2}{f^2}\delta_{ij}) \, c^j + \int d^dx \, \bar{\eta} \, (-\partial^2 + \frac{g'^2}{f^2}) \, \eta \\ &- \frac{gg'}{f^2}\int d^4x \, \bar{c}^i \, (\bar{h}_{\alpha\beta}\bar{R}_i^\alpha \bar{L}_3^\beta) \, \eta - \frac{gg'}{f^2}\int d^4x \, \bar{\eta} \, (\bar{h}_{\alpha\beta}\bar{R}_i^\alpha \bar{L}_3^\beta) \, c^i \, , \end{aligned} \tag{4.22}$$

where

$$\bar{D}_\mu c^i = \partial_\mu c^i + \varepsilon^i{}_{jl}\bar{W}_\mu^j c^l \, . \tag{4.23}$$

4.1.2 Beta functions

The computation of the beta functions is performed using the exact functional renormalization group equation introduced in Chapter 2. In this case one starts from the background effective average action $\Gamma_k(\varphi, W, B; \theta)$ which is assumed to have the same form of the action $\mathcal{S}[\varphi, W, B; \theta]$, where the bare couplings f, g and g' are replaced by renormalized couplings that depend on the scale k. All the bars from background quantities have been dropped, since no confusion should arise. The functional $\Gamma_k(\varphi, W, B; \theta)$ depends on the background fields φ, W, B and on the classical average fields θ (the variables that are Legendre conjugated to the sources coupled linearly to the quantum field in the path integral definition), where $\theta^T = (\xi^i, w_\mu^i, b_\mu)$

and $\xi^i = R^i_\alpha \xi^\alpha$. From $\Gamma_k[\varphi, W, B;\theta]$, setting $\theta = 0$, it is possible to define a functional which is manifestly both diffeomorphism and gauge invariant, $\Gamma_k[\varphi, W, B] = \Gamma_k[\varphi, W, B; 0]$. It obeys a functional differential equation, which in the present context reads:

$$\frac{\partial}{\partial t}\Gamma_k[\varphi, W, B] = \frac{1}{2}\mathrm{Tr}_\theta\left[\left(\frac{\delta^2\Gamma_k[\varphi, W, B; 0]}{\delta\theta\delta\theta} + \mathcal{R}_k[\varphi, W, B]\right)^{-1}\frac{\partial\mathcal{R}_k[\varphi, W, B]}{\partial t}\right] - \frac{\partial}{\partial t}\log C_k[W]\,, \tag{4.24}$$

where $t = \log(k/k_0)$. From eq. (4.14) one reads the second variation of the effective average action:

$$\Gamma^{[2]}[\varphi, W, B;\theta] = \frac{1}{2}\int d^4x\,\theta^T\Big(\Pi[\varphi, W, B] - E[\varphi, W, B]\Big)\,\theta\,, \tag{4.25}$$

where $\Pi[\varphi, W, B] =$

$$\begin{pmatrix} \frac{1}{f^2}[(-D_\xi^2 + m_W^2)(\mathbf{1}-\mathbf{P}_3)_{ij} + (-D_\xi^2 + m_Z^2)\mathbf{P}_{3ij}] & 0 & 0 \\ 0 & \frac{1}{g^2}(-D_w^2\delta^{\mu\nu}_{ij} + m_W^2\delta^{\mu\nu}_{ij}) & 0 \\ 0 & 0 & \frac{1}{g'^2}(-\partial^2\delta^{\mu\nu} + m_B^2\delta^{\mu\nu}) \end{pmatrix} \tag{4.26}$$

and

$$E[\varphi, W, B] = \begin{pmatrix} \frac{1}{f^2}M_{ij} & \frac{1}{f^2}A^\mu_{ij} & \frac{1}{f^2}B^\mu_j \\ \frac{1}{f^2}A^{T\mu}_{ij} & \frac{2}{g^2}W^{\mu\nu}_{ij} & \frac{1}{f^2}C^{\mu\nu}_i \\ \frac{1}{f^2}B^{T\mu}_j & \frac{1}{f^2}C^{T\mu\nu}_i & 0 \end{pmatrix}. \tag{4.27}$$

The first entry of the matrix in eq. (4.26) has been rewritten introducing the projector $\mathbf{P}_{3ij}$, which is given by:

$$\mathbf{P}_{3ij} = R^\alpha_i R^\beta_j L^3_\alpha L^3_\beta\,. \tag{4.28}$$

The explicit form of the endomorphism matrices that enter in eq. (4.27) is:

$$M_{ij} = R^\alpha_i R^\beta_j D_\mu\varphi^\varepsilon D^\mu\varphi^\eta R_{\varepsilon\alpha\eta\beta} \quad , \quad A^\mu_{ij} = -2h_{\alpha\gamma}D^\mu\varphi^\alpha\nabla_\beta R^\gamma_i R^\beta_j \quad , \quad B^\mu_j = 2h_{\alpha\gamma}D^\mu\varphi^\alpha\nabla_\beta L^\gamma_3 R^\beta_j \tag{4.29}$$

and

$$W^{\mu\nu}_{ij} = W^{l\mu\nu}\varepsilon_{ilj} \qquad , \qquad C^{\mu\nu}_i = h_{\alpha\beta}R^\alpha_i L^\beta_3\delta^{\mu\nu}\,. \tag{4.30}$$

In eq. (4.24) the bosonic cutoff kernel $\mathcal{R}_k[\varphi, W, B]$ is chosen to be

$$\mathcal{R}_k[\varphi, W, B] = \begin{pmatrix} \frac{1}{f^2}R_k(z)(\mathbf{1}-\mathbf{P}_3) + \frac{1}{f^2}R_k(z)\mathbf{P}_3 & 0 & 0 \\ 0 & \frac{1}{g^2}R_k(w) & 0 \\ 0 & 0 & \frac{1}{g'^2}R_k(-\partial^2) \end{pmatrix}. \tag{4.31}$$

In eq. (4.31) $z = -D_\xi^2$ and $w = -D_w^2$, where D_ξ and D_w are defined in eq. (4.8) and eq. (4.9) respectively. The cutoff in the ghost sector is implemented by $C_k[W]$, which is chosen to be

$$C_k[W] = \det \begin{bmatrix} 1 + \frac{R_k(y)}{y+m_W^2} & 0 \\ 0 & 1 + \frac{R_k(-\partial^2)}{-\partial^2+m_B^2} \end{bmatrix} , \tag{4.32}$$

where $y = -D_c^2$ and D_c is defined in eq. (4.23). The cutoff profile functions R_k are taken to be functions only of the background covariant Laplacians (type I). The form of R_k is chosen to be the optimized one of eq. (3.42), which ensures that the integrations over momenta are explicitly calculable. Being constructed with the background Laplacians, this cutoff prescription preserves the background invariance.

The t-derivative of the bosonic cutoff kernel is:

$$\frac{\partial \mathcal{R}_k[\varphi, W, B]}{\partial t} = \begin{pmatrix} \frac{1}{f^2}\partial_t R_k(z)(\mathbf{1} - \mathbf{P}_3) + \frac{1}{f^2}\partial_t R_k(z)\mathbf{P}_3 & 0 & 0 \\ 0 & \frac{1}{g^2}\partial_t R_k(w) & 0 \\ 0 & 0 & \frac{1}{g'^2}\partial_t R_k(-\partial^2) \end{pmatrix} , \tag{4.33}$$

where $\partial_t R_k(z) = 2k^2\theta(k^2 - z)$. Here, the situation is simplified with respect to eq. (3.43) since the contribution coming from the η-terms are neglected and the result will be one-loop.

It is convenient to rewrite the modified inverse bosonic propagator in a more compact form:

$$\frac{\delta^2\Gamma_k[\varphi, W, B; 0]}{\delta\theta\delta\theta} + \mathcal{R}_k[\varphi, W, B] = \Pi_k[\varphi, W, B] - E[\varphi, W, B] , \tag{4.34}$$

where $\Pi_k[\varphi, W, B] = \Pi[\varphi, W, B] + \mathcal{R}_k[\varphi, W, B]$. With the notation introduced above it is possible to rewrite eq. (4.24) as follows:

$$\frac{\partial}{\partial t}\Gamma_k[\varphi, W, B] = \frac{1}{2}\mathrm{Tr}_{\,\theta}\left[(\Pi_k - E)^{-1}\,\partial_t\mathcal{R}_k\right] - \mathrm{Tr}_{\,c}\left[\left(P_k + m_W^2\right)^{-1}\partial_t R_k\right] , \tag{4.35}$$

where $P_k(z) = z + R_k(z)$.

The first trace in eq. (4.35) is evaluated by expanding the argument in powers of $(\Pi_k)^{-1}E$:

$$\frac{1}{2}\mathrm{Tr}_\theta\left[(\Pi_k + E)^{-1}\,\partial_t\mathcal{R}_k\right] = \frac{1}{2}\mathrm{Tr}_\theta\left[\left(\Pi_k^{-1} + \Pi_k^{-1}E\Pi_k^{-1} + \Pi_k^{-1}E\Pi_k^{-1}E\Pi_k^{-1} + \ldots\right)\partial_t\mathcal{R}_k\right] \tag{4.36}$$

In this case, the entire dependence on the couplings is kept in the inverse propagator Π_k through the threshold masses but the trace is expanded in powers of E, which depends on the background fields. The traces in eq. (4.36) are evaluated using heat kernel methods presented in Appendix

B. The first trace gives:

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left[\Pi_k^{-1}\partial_t\mathcal{R}_k\right] &\supset -\frac{1}{(4\pi)^2}\frac{k^2}{k^2+m_W^2}\frac{1}{6}\int d^4x\,\left(\frac{1}{4}W^i_{\mu\nu}W_i^{\mu\nu}+\frac{1}{4}B_{\mu\nu}B^{\mu\nu}\right)\\
&- \frac{1}{(4\pi)^2}\frac{k^2}{k^2+m_W^2}\frac{1}{12}\int d^4x\,W^i_{\mu\nu}B^{\mu\nu}R_i^\alpha L_{3\alpha}\\
&- \frac{1}{(4\pi)^2}\frac{k^2}{k^2+m_W^2}\frac{8}{3}\int d^4x\,\frac{1}{4}W_i^{\mu\nu}W^i_{\mu\nu}
\end{aligned} \tag{4.37}$$

The second trace gives:

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left[\Pi_k^{-1}E\Pi_k^{-1}\partial_t\mathcal{R}_k\right] &\supset \frac{1}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)^2}\frac{1}{2}\int d^4x\,R_{\alpha\beta}D_\mu\varphi^\alpha D^\mu\varphi^\beta\\
&+ \frac{1}{(4\pi)^2}\left[\frac{k^6}{(k^2+m_Z^2)^2}-\frac{k^6}{(k^2+m_W^2)^2}\right]\frac{1}{8}\int d^4x\,(h_{\alpha\beta}-L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta
\end{aligned} \tag{4.38}$$

The third trace gives:

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left[\Pi_k^{-1}E\Pi_k^{-1}E\Pi_k^{-1}\partial_t\mathcal{R}_k\right] &\supset \frac{2g^2/f^2}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)^3}\int d^4x\,h_{\alpha\beta}D_\mu\varphi^\alpha D^\mu\varphi^\beta\\
&+ \frac{1}{2}\frac{g^2/f^2}{(4\pi)^2}\frac{k^6}{k^2+m_W^2}\left[\frac{1}{(k^2+m_Z^2)^2}-\frac{1}{(k^2+m_W^2)^2}\right]\int d^4x\,(h_{\alpha\beta}-L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta\\
&+ \frac{1}{2}\frac{g^2/f^2}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)^2}\left[\frac{1}{(k^2+m_Z^2)}-\frac{1}{(k^2+m_W^2)}\right]\int d^4x\,(h_{\alpha\beta}-L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta\\
&+ \frac{1}{2}\frac{g'^2/f^2}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)^2(k^2+m_B^2)}\int d^4x\,(h_{\alpha\beta}-L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta\\
&+ \frac{1}{2}\frac{g'^2/f^2}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)(k^2+m_B^2)^2}\int d^4x\,(h_{\alpha\beta}-L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta\\
&+ \frac{16}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)^3}\int d^4x\,\frac{1}{4}W^i_{\mu\nu}W_i^{\mu\nu}
\end{aligned} \tag{4.39}$$

The ghost trace gives:

$$\mathrm{Tr}_c\left[\left(P_k+m_W^2\right)^{-1}\partial_t R_k\right] \supset -\frac{1}{(4\pi)^2}\frac{k^2}{k^2+m_W^2}\frac{4}{3}\int d^4x\,\frac{1}{4}W_i^{\mu\nu}W^i_{\mu\nu} \tag{4.40}$$

In order to extract the beta functions, one assumes that the invariant effective average action $\Gamma_k[\varphi,W,B]$ has the same form of the original one, with the bare couplings f, g and g' replaced by renormalized coupling that depend on k:

$$\Gamma_k[\varphi,W,B]=\frac{1}{2f^2}\int d^4x\,h_{\alpha\beta}D_\mu\varphi^\alpha D^\mu\varphi^\beta+\frac{1}{4g^2}\int d^4x\,W^i_{\mu\nu}W_i^{\mu\nu}+\frac{1}{4g'^2}\int d^4x\,B_{\mu\nu}B^{\mu\nu}. \tag{4.41}$$

The system of coupled equations for the beta functions of f, g and g' is obtained by summing up all contributions coming from eq. (4.37), eq. (4.38), eq. (4.39), eq. (4.40) and reading off the coefficients of $(1/2)\int d^4x\, h_{\alpha\beta} D_\mu\varphi^\alpha D^\mu\varphi^\beta$, $(1/4)\int d^4x\, W^i_{\mu\nu}W^{i\mu\nu}$ and $(1/4)\int d^4x\, B_{\mu\nu}B^{\mu\nu}$. Writing the equations in terms of dimensionless quantities, one gets:

$$
\begin{aligned}
\partial_t \tilde{f}^2 &= 2\tilde{f}^2 - \frac{\tilde{f}^2}{(4\pi)^2}\left[\frac{1}{4}\frac{\tilde{f}^2}{(1+\tilde{m}_W^2)^2} + \frac{1}{4}\frac{\tilde{f}^2}{(1+\tilde{m}_Z^2)^2} + \frac{2g^2}{(1+\tilde{m}_W^2)^3}\right.\\
&+ \frac{g^2}{(1+\tilde{m}_W^2)(1+\tilde{m}_Z^2)^2} + \frac{g^2}{(1+\tilde{m}_W^2)^2(1+\tilde{m}_Z^2)}\\
&+ \left.\frac{g'^2}{(1+\tilde{m}_W^2)(1+\tilde{m}_B^2)^2} + \frac{g'^2}{(1+\tilde{m}_W^2)^2(1+\tilde{m}_B^2)}\right] \qquad (4.42)
\end{aligned}
$$

$$
\partial_t g^2 = \frac{1}{(4\pi)^2}\frac{1}{1+\tilde{m}_W^2}\left[\frac{3}{2} - \frac{16}{(1+\tilde{m}_W^2)^2}\right] g^4 \qquad (4.43)
$$

$$
\partial_t g'^2 = \frac{1}{6}\frac{1}{(4\pi)^2}\frac{1}{1+\tilde{m}_W^2} g'^4 \qquad (4.44)
$$

where $\tilde{m}_W^2 = m_W^2/k^2 = g^2/\tilde{f}^2$, $\tilde{m}_Z^2 = m_Z^2/k^2 = (g^2+g'^2)/\tilde{f}^2$, $\tilde{m}_B^2 = m_B^2/k^2 = g'^2/\tilde{f}^2$ and $\tilde{f}^2 = k^2 f^2$ is the dimensionless Goldstone boson coupling. Since the regularization scheme is mass-dependent, one gets automatically the effect of thresholds, which are represented by the factors $1/(1+\tilde{m}^2)$. For $k^2 \gg m^2$ these factors become equal to one, whereas for $k^2 \ll m^2$ the denominators become large and suppress the running, reflecting the decoupling of the corresponding massive field modes.

4.1.3 Results

At this point it is useful to study the limit $k^2 \gg m^2$, expanding the threshold factors for small $\tilde{m}^2$ and retaining the first leading terms, one gets:

$$
\partial_t \tilde{f}^2 = 2\tilde{f}^2 - \frac{1}{2}\frac{\tilde{f}^2}{(4\pi)^2}(\tilde{f}^2 + 6g^2 + 3g'^2) \qquad (4.45)
$$

$$
\partial_t g^2 = -\frac{1}{(4\pi)^2}\frac{29}{2} g^4 \qquad (4.46)
$$

$$
\partial_t g'^2 = \frac{1}{6}\frac{1}{(4\pi)^2} g'^4 \,. \qquad (4.47)
$$

Because of the positive beta function for g', strictly speaking this system does not have a physical acceptable UV fixed point, unless $g' = 0$. However, the running of the gauge couplings is very slow and the Landau pole for g' occurs at trans-Planckian energies. For practical purposes, it is a good approximation to fix them at their experimental values $g = 0.65$ and $g' = 0.35$. In this case the fixed point for $\tilde{f}$ is very slightly modified with respect to the ungauged case:

$$\tilde{f}_*^2 = 2(4\pi)^2 \left(2 - \frac{3g^2 + 3g'^2/2}{(4\pi)^2}\right) . \tag{4.48}$$

4.2 S and T parameters

Consider the $SU(2)_L \times U(1)_R$ gauged nonlinear sigma model action of eq. (4.1). The gauge invariance of the SM demands that the metric $h_{\alpha\beta}$ be invariant under the action of the left- and right-invariant vectorfields, but not necessarily under the $SU(2)_R$ transformations generated by L_1^α and L_2^α. The most general metric of this type is of the form

$$\tilde{h}_{\alpha\beta} = L_\alpha^1 L_\beta^1 + L_\alpha^2 L_\beta^2 + (1 - 2a_0) L_\alpha^3 L_\beta^3 , \tag{4.49}$$

where L_α^i is the basis of left-invariant one-forms dual to L_i^α. If $a_0 < 0$, the geometry of the model corresponds to an elongated three-sphere, while if $0 < a_0 < 1/2$ it corresponds to a squashed three-sphere. In the case $a_0 > 1/2$ the metric would change signature. The parameter a_0 measures the violation of the custodial symmetry $SU(2)_R$ and vanishes in the bare SM Lagrangian. Radiative corrections then induce a small nonvanishing effective value for a_0. It is therefore customary to assume that the metric $h_{\alpha\beta}$ is bi-invariant and to consider the $SU(2)_R$-breaking as due to a separate term in the effective (Euclidean) action:

$$\Delta\mathcal{S}_0[\varphi, W, B] = -\frac{a_0}{f^2} \int d^4x\, D_\mu \varphi^\alpha D^\mu \varphi^\beta L_\alpha^3 L_\beta^3 . \tag{4.50}$$

The full action contains further terms, among these one is usually interested also in

$$\Delta\mathcal{S}_1[\varphi, W, B] = -a_1 \frac{1}{2} \int d^4x\, B^{\mu\nu} W_{\mu\nu}^i R_{i\alpha} L_3^\alpha . \tag{4.51}$$

The operators in eq. (4.50) and eq. (4.51) belong to the complete set of dimension four operators of the electroweak chiral effective lagrangian described in Appendix C. These definitions agree with those of eq. (C.9) and eq. (C.10), except for the rescaling of the gauge fields with the gauge couplings. Note also the minus sign in eq. (4.51) comes from the analytic continuation to the Euclidean space. The running couplings a_0 and a_1 are related to the oblique parameters S and T by [Dobado 2000]

$$S = -16\pi a_1(m_Z) + \frac{1}{6\pi}\left[\frac{5}{12} - \log\left(\frac{m_H}{m_Z}\right)\right] , \tag{4.52}$$

$$T = \frac{2}{\alpha} a_0(m_Z) - \frac{3}{8\pi \cos^2\theta_W}\left[\frac{5}{12} - \log\left(\frac{m_H}{m_Z}\right)\right] . \tag{4.53}$$

The second term on the right hand side corresponds to subtracting the contribution of the Higgs field with mass m_H [Bagger 2000].

4.2.1 Beta functions

In this section the renormalization group flow of the gauge couplings g, g', together with the sigma model coupling f and the parameters a_0 and a_1 is studied. It will be instructive to consider first the ungauged $SU(2) \times U(1)/U(1)$ sigma model, with couplings f and a_0. Quite generally, the beta function of the sigma model is given by the Ricci flow term of eq. (4.38), where $m_W^2 = 0$ and the Ricci tensor is computed using the metric $\tilde{h}_{\alpha\beta}$:

$$\partial_t \left(\frac{1}{f^2} \tilde{h}_{\alpha\beta} \right) = \frac{1}{(4\pi)^2} k^2 \tilde{R}_{\alpha\beta} \tag{4.54}$$

where, as usual, $t = \log k/k_0$. In the basis of the left-invariant vectorfields L_i^α, the Ricci tensor of the metric $\tilde{h}_{\alpha\beta}$ has components

$$R_{11} = R_{22} = \frac{1}{2} + a_0 \quad , \qquad R_{33} = \frac{1}{2} - a_0 \,. \tag{4.55}$$

The beta functions of $\tilde{f}^2 = f^2 k^2$ and a_0 can be obtained by projecting eq. (4.54) in the basis of the left-invariant vectorfields and using eq. (4.55):

$$\partial_t \tilde{f}^2 = 2\tilde{f}^2 - \frac{1}{(4\pi)^2} \tilde{f}^4 \left(\frac{1}{2} + a_0 \right) \tag{4.56}$$

$$\partial_t a_0 = \frac{1}{2} \frac{1}{(4\pi)^2} \tilde{f}^2 a_0 (1 - 2a_0) \,. \tag{4.57}$$

Coming to the gauged case one can first consider the subsystem of the couplings g, g' and f, keeping $a_0 = a_1 = 0$. The detailed computation was presented in Section 4.1 and the beta functions are given in eq. (4.42), eq. (4.43) and eq. (4.44) . As already said, this system does not admit a physically acceptable UV fixed point because of the positive beta function of g'. However, it is a good approximation to treat the gauge couplings as constants. This is reasonable, since their running is very slow and the Landau pole occurs at trans-Planckian energies. It is worth to mention that this $U(1)$ problem could be solved by coupling the system to gravity as shown in [Harst 2011]. From here on, the gauge couplings will be considered fixed to their experimental value $g = 0.65$ and $g' = 0.35$. As in the ungauged case, the beta function of f and a_0 can be extracted from the geometrical beta functional. The running of the metric $\tilde{h}_{\alpha\beta}$ can be obtained by summing up the contributions coming from eq. (4.38) and eq. (4.39)

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_k[\varphi,W,B] \;\supset\;& \frac{1}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)^2}\frac{1}{2}\int d^4x\,\tilde{R}_{\alpha\beta}D_\mu\varphi^\alpha D^\mu\varphi^\beta \\
+\;& \frac{1}{(4\pi)^2}\left[\frac{k^6}{(k^2+m_Z^2)^2}-\frac{k^6}{(k^2+m_W^2)^2}\right]\frac{1}{8}\int d^4x\,(h_{\alpha\beta}-L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta \\
+\;& \frac{g^2/f^2}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)^3}\int d^4x\,(h_{\alpha\beta}+L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta \\
+\;& \frac{1}{2}\frac{g^2/f^2}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)(k^2+m_Z^2)^2}\int d^4x\,(h_{\alpha\beta}-L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta \\
+\;& \frac{1}{2}\frac{g^2/f^2}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)^2(k^2+m_Z^2)}\int d^4x\,(h_{\alpha\beta}-L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta \\
+\;& \frac{1}{2}\frac{g'^2/f^2}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)^2(k^2+m_B^2)}\int d^4x\,(h_{\alpha\beta}-L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta \\
+\;& \frac{1}{2}\frac{g'^2/f^2}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)(k^2+m_B^2)^2}\int d^4x\,(h_{\alpha\beta}-L^3_\alpha L^3_\beta)D_\mu\varphi^\alpha D^\mu\varphi^\beta\,,
\end{aligned} \tag{4.58}$$

where the Ricci flow is computed using the Ricci tensor of the new metric and

$$\frac{\partial}{\partial t}\Gamma_k[\varphi,W,B] \;=\; \frac{1}{2}\int d^4x\,\frac{\partial}{\partial t}\left(\frac{1}{f^2}h_{\alpha\beta}-\frac{2a_0}{f^2}L^3_\alpha L^3_\beta\right)D_\mu\varphi^\alpha D^\mu\varphi^\beta\,. \tag{4.59}$$

The beta functions for $\tilde{f}^2$ and a_0 can be extracted by projecting eq. (4.58) on the basis of the left-invariant vectorfields and using eq. (4.55):

$$\begin{aligned}
\partial_t\tilde{f}^2 \;=\;& 2\tilde{f}^2-\frac{\tilde{f}^2}{(4\pi)^2}\left[\frac{1}{2}\frac{(1+2a_0)\tilde{f}^2}{(1+\tilde{m}_W^2)^2}-\frac{1}{4}\frac{\tilde{f}^2}{(1+\tilde{m}_W^2)^2}+\frac{1}{4}\frac{\tilde{f}^2}{(1+\tilde{m}_Z^2)^2}\right. \\
+\;& \frac{2g^2}{(1+\tilde{m}_W^2)^3}+\frac{g^2}{(1+\tilde{m}_W^2)^2(1+\tilde{m}_Z^2)}+\frac{g^2}{(1+\tilde{m}_W^2)(1+\tilde{m}_Z^2)^2} \\
+\;& \left.\frac{g'^2}{(1+\tilde{m}_W^2)^2(1+\tilde{m}_B^2)}+\frac{g'^2}{(1+\tilde{m}_W^2)(1+\tilde{m}_B^2)^2}\right]
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
\partial_t a_0 \;=\;& \frac{1}{2}\frac{\tilde{f}^2}{(4\pi)^2}\frac{a_0(1-2a_0)}{(1+\tilde{m}_W^2)^2}-\frac{g^2}{(4\pi)^2}\frac{(1+2a_0)}{(1+\tilde{m}_W^2)^3}+\frac{1}{2}\frac{1}{(4\pi)^2}\left[\frac{1}{4}\frac{\tilde{f}^2}{(1+\tilde{m}_Z^2)^2}\right. \\
-\;& \frac{1}{4}\frac{\tilde{f}^2}{(1+\tilde{m}_W^2)^2}+\frac{g^2}{(1+\tilde{m}_W^2)^2(1+\tilde{m}_Z^2)}+\frac{g^2}{(1+\tilde{m}_W^2)(1+\tilde{m}_Z^2)^2} \\
+\;& \left.\frac{g'^2}{(1+\tilde{m}_W^2)^2(1+\tilde{m}_B^2)}+\frac{g'^2}{(1+\tilde{m}_W^2)(1+\tilde{m}_B^2)^2}\right](1-2a_0)\,.
\end{aligned} \tag{4.61}$$

The renormalization group flow of the operator in eq. (4.51) is obtained by Taylor expanding the piece $R_{i\alpha}L_3^\alpha$ around the background, keeping W_μ^i and B_μ as classical fields. In this case the endomorphism part of the second variation is modified by a quantity $\tilde{M}$:

$$\Gamma^{[2]}[\varphi, W, B; \xi] = \frac{1}{2f^2} \int d^4x\, \xi^i \left(-D^2\delta_{ij} - M_{ij} - \tilde{M}_{ij}\right) \xi^j \,, \tag{4.62}$$

where M_{ij} is given in eq. (4.29) and

$$\tilde{M}_{ij} = -a_1 f^2 B_{\mu\nu} W_i^{\mu\nu} (R^\sigma{}_{\alpha\gamma\beta} R^i_\sigma L_3^\gamma + \nabla_\alpha R^i_\sigma \nabla^\sigma L_{3\beta}) R_i^\alpha R_j^\beta \,. \tag{4.63}$$

The trace in eq. (4.38) receives an additional contribution, which in the present context reads:

$$\frac{1}{2}\mathrm{Tr}\left[\Pi_k^{-1} E \Pi_k^{-1} \partial_t \mathcal{R}_k\right] \supset -\frac{1}{2}\frac{a_1 f^2}{(4\pi)^2}\frac{k^6}{(k^2+m_W^2)^2} \int d^4x\, B^{\mu\nu} W^i_{\mu\nu} R_i^\alpha L_{3\alpha} \,. \tag{4.64}$$

The beta function of a_1 is obtained by adding to eq. (4.64) the contribution coming from the trace in eq. (4.37), one gets:

$$\partial_t a_1 = \frac{1}{(4\pi)^2}\frac{1}{1+\tilde{m}_W^2}\left(\frac{1}{6} + \frac{a_1 \tilde{f}^2}{1+\tilde{m}_W^2}\right) \,. \tag{4.65}$$

4.2.2 Results

At this point it is useful to study the system of eq. (4.60), eq. (4.61) and eq. (4.65) for k much larger than all the masses ($g, g' \ll \tilde{f}$). In this case the beta functions simplify to

$$\partial_t \tilde{f}^2 = 2\tilde{f}^2 - \frac{1}{2}\frac{\tilde{f}^2}{(4\pi)^2}\left(\tilde{f}^2(1+2a_0) + 6g^2 + 3g'^2\right) \,, \tag{4.66}$$

$$\partial_t a_0 = \frac{1}{2}\frac{1}{(4\pi)^2}\left(\tilde{f}^2 a_0(1-2a_0) + \frac{3}{2}g'^2\right) \,, \tag{4.67}$$

$$\partial_t a_1 = \frac{1}{(4\pi)^2}\left(\tilde{f}^2 a_1 + \frac{1}{6}\right) \,. \tag{4.68}$$

Terms of order $g^2 a_0$ or $g'^2 a_0$ have been neglected because they are subleading relative to those of order $\tilde{f}^2 a_0$. Note that these beta functions reduce correctly to eq. (4.56) and eq. (4.57) in the ungauged case. The first term in eq. (4.67) corresponds to a self-renormalization of the operator in eq. (4.50). Diagrammatically it corresponds to a quadratically divergent Goldstone boson tadpole and cannot be seen in dimensional regularization. The second term agrees with the results of [Herrero 1994]; it is proportional to g'^2, consistent with the fact that the hypercharge coupling breaks the custodial symmetry. Its effect is to generate a nonzero a_0 even if initially $a_0 = 0$. Also the second term in eq. (4.68) agrees with the one computed in [Herrero 1994],

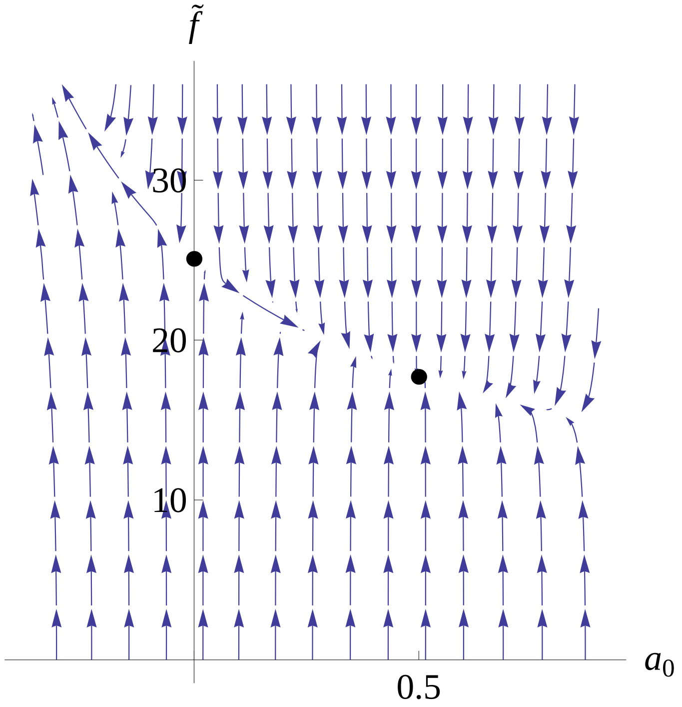


Figure 4.1: Flow in the a_0-$\tilde f$ plane with $a_1 = 0$. The two dots mark the positions of FPI and FPII. Arrows point to increasing energy.

while the first comes from the self-renormalization of the operator in eq. (4.51).

In the ungauged case, the beta functions in eq. (4.56) and eq. (4.57) admit a Gaussian fixed point with $\tilde f_* = 0$ and arbitrary a_0. In addition there exist two nontrivial fixed points: one at $a_{0*} = 0$, $\tilde f_* = 8\pi \approx 25.13$ which is $SU(2)_R$-symmetric and another one with maximally broken $SU(2)_R$ at $a_{0*} = 1/2$, $\tilde f_* = 4\sqrt{2}\pi \approx 17.8$. When the gauge couplings are taken into account, there is no longer a fixed point with $\tilde f = 0$ and the two nontrivial fixed points of the ungauged case turn out to be slightly shifted. The first fixed point FPI occurs at:

$$\tilde f_* = 25.1 \quad , \qquad a_{0*} = -0.000292 \quad , \qquad a_{1*} = -0.000265\,. \tag{4.69}$$

The second fixed point FPII occurs at:

$$\tilde f_* = 17.7 \quad , \qquad a_{0*} = 0.501 \quad , \qquad a_{1*} = -0.000530\,. \tag{4.70}$$

The renormalization group flow is illustrated in Fig. 4.1 and Fig. 4.2. The eigenvalues and eigenvectors of the matrix describing the linearized flow around these fixed points are given in the table 4.1. Recall that negative eigenvalues correspond to UV attractive (relevant) directions. The point FPI has one such direction, that to a good approximation can be identified with the

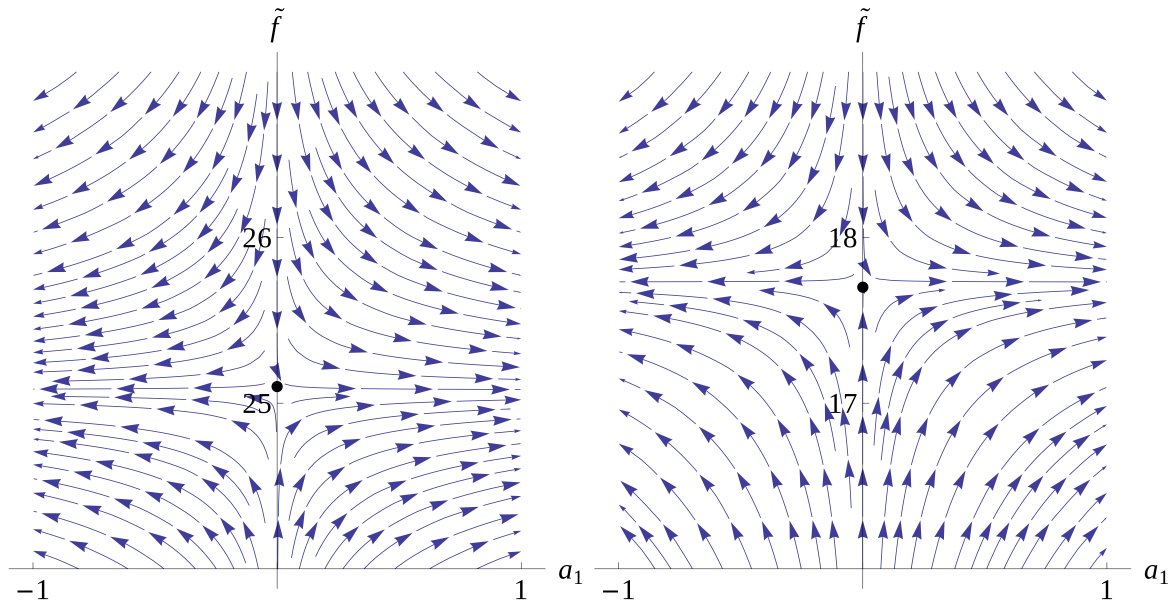


Figure 4.2: The left plot shows the flow in the a_1-$\tilde{f}$ plane with $a_0 = 0$, the dot marks the position of FPI. The right plot shows the flow in the a_1-$\tilde{f}$ plane with $a_0 = 0.5$, the dot marks the position of FPII. Arrows point to increasing energy.

parameter $\tilde{f}$. The point FPII has two relevant directions that lie almost exactly in the a_0-$\tilde{f}$ plane. Within numerical errors one finds a critical trajectory that starts from FPII in the UV approximately in the direction of (minus) its second eigenvector and reaches FPI in the IR from the direction of its second eigenvector. The origin is not a fixed point, but the beta functions become very small there. This almost-fixed point is IR attractive for $\tilde{f}$.

At this point it is useful to study the physics of these fixed points. At $k = m_Z$ one has $\tilde{f} = 2m_Z/v = 0.7415$ and the experimentally allowed values for $a_0(m_Z)$ and $a_1(m_Z)$ are of order 10^{-3}. When one evolves the flow towards higher energies, $\tilde{f}$, a_0 or a_1 will generally diverge. This is, in general, a sign that new physics has to be taken into account. However, there may be trajectories that hit a fixed point in the UV, for them the effective field theory

Fixed point	Eigenvalues	Eigenvectors		
		$\tilde{f}$	a_0	a_1
I	-1.99	1.00	11.6×10^{-6}	14.1×10^{-6}
I	1.99	-0.997	0.0795	-42.2×10^{-6}
I	3.98	0	0	1
II	-1.99	1.00	66.0×10^{-6}	29.9×10^{-6}
II	-0.996	-0.998	0.0563	-40×10^{-6}
II	1.99	0	0	1

Table 4.1: eigenvalues and eigenvectors of the stability matrix at the fixed points.

description actually never breaks down. Such trajectories are said to be asymptotically safe or renormalizable. Requiring that the world be described by a renormalizable trajectory leads to predictions for lowenergy physics. Since FPI has only one relevant direction, there is a single renormalizable trajectory that descends from it towards the origin. Since the beta functions go to zero for $k < m_Z$, the flow is stopped at the scale m_Z (i.e. when $\tilde{f} = 0.7415$) and find, at that scale,

$$a_0(m_Z) = -0.0020, \quad a_1(m_Z) = -0.0032 \;, \tag{4.71}$$

which are 5σ away from the experimental values. The transition takes about four or five e-foldings (a change in scale by a factor e^4-e^5) which means that FPI would be reached at an energy scale of the order of 10 TeV.

The point FPII has two relevant directions and therefore there is a one parameter family of renormalizable trajectories that descend from it. From Fig. 4.1 it is possible to see that for such a trajectory to come close to the origin, it has to be fine tuned to first follow very closely the critical trajectory towards FPI, and hence descend. Going upwards from $k = m_Z$, such a trajectory would take again four or five e-foldings to reach the vicinity of FPI and then another four e-foldings to cross over to FPII, placing the energy scale at which one arrives near FPII at 300-700 TeV. It is clear from Fig. 4.1 that these trajectories will have $a_0(m_Z) > -0.002$. Numerical analysis shows that the locus of endpoints of such trajectories satisfies

$$a_1(m_Z) = -0.00321 - 0.00052\, a_0(m_Z) \tag{4.72}$$

For $a_0 \approx 0.5$ this relation is still true within a few percent.

Using equations (4.52) and (4.53), this translates directly into a linear relation between S and T, which is shown in Fig. 4.3, and constitutes the main result of the analysis. The dot corresponds to the UV critical surface of FPI (4.71), the half-line to the UV critical surface of FPII. Note that the condition of asymptotic safety essentially fixes a_1, and hence S, leaving T arbitrary. Standard model fermions would not change this conclusion, since their contribution is already included in the definition of S and T, but one has to make sure that they do not spoil the fixed point. In the next section the problem of adding fermions will be addressed, in this case the fixed point of $\tilde{f}$ is preserved if four-fermion interactions are added. These interactions only change the beta functions of S and T at higher loops, so one expects the result obtained here to remain valid.

Renormalizable trajectories represent UV complete theories. Within this model there are such trajectories that are in agreement with the experimental data: $S = 0.01 \pm 0.10$ and $T = 0.03 \pm 0.11$ [Nakamura 2010]. They pass near FPI at scales ≈ 10 TeV and then veer towards FPII. There, the custodial symmetry is strongly broken, as witnessed by the large value $a_0 \approx 0.5$. This could be an important (and unexpected) clue about the UV behavior of the theory. In this model the fixed point behavior sets in at energies that are probably too high to make a direct

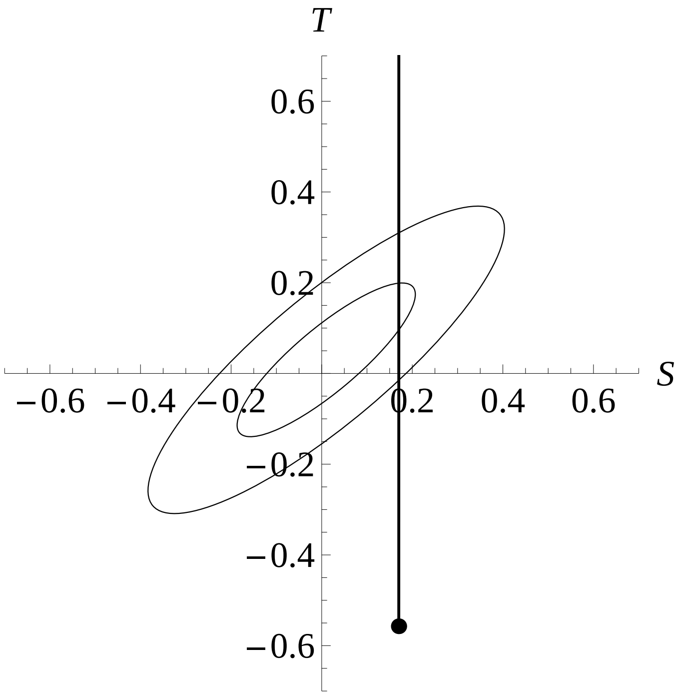


Figure 4.3: The half-line and the dot show the values permitted by asymptotic safety. The ellipses show the 1 and 2 σ experimental bounds with m_H=117GeV [Nakamura 2010].

observation possible at LHC.

4.3 Fermions and Goldstone bosons

This section is devoted to the study of the renormalization group flow of the system obtained by adding fermions to the nonlinear sigma model. Here, the scalar fields are encoded in a $SU(N)$–valued matrix $U = \exp(if\pi^a T_a)$, where π^a are the Goldstone fields, $\mathrm{tr}\, T_a T_b = \delta_{ab}/2$ and f is the Goldstone boson coupling. In the SM case f can be identified with $2/\upsilon$, where $\upsilon = 246\,\mathrm{GeV}$ is the Higgs VEV. The lowest order term of the nonlinear sigma model lagrangian reads:

$$\mathcal{L}_{NL} = -\frac{1}{f^2}\mathrm{Tr}\left(U^\dagger \partial_\mu U U^\dagger \partial^\mu U\right) . \tag{4.73}$$

This model is invariant under separate $SU(N)_L$ and $SU(N)_R$ transformations acting on U by left- and right multiplication respectively.

Consider left- and right-handed fermions $\psi_L^{i\alpha}$ and $\psi_R^{i\alpha}$ carrying the fundamental representation of $SU(N)_L$ and $SU(N)_R$ respectively (corresponding to the indices $i = 1, \ldots, N$), and also the fundamental representation of a color group $SU(N_c)$ (corresponding to the indices $\alpha = 1, \ldots, N_c$). In the real world the latter group is gauged, here it is simply retained as a global symmetry to count fermionic states. Fermions are coupled in a chiral invariant way to the U field by adding to the lagrangian in eq. (4.73) the fermion kinetic term and the (proto)-Yukawa interaction:

$$\mathcal{L}_{\psi^2} = \left(\bar{\psi}_L i\gamma^\mu \partial_\mu \psi_L + \bar{\psi}_R i\gamma^\mu \partial_\mu \psi_R + \mathcal{L}_{Yuk}\right) . \tag{4.74}$$

The Yukawa lagrangian is given by

$$\mathcal{L}_{Yuk} = -\frac{2h_0}{f}\left(\bar{\psi}_L^{i\alpha} U^{ij} \psi_R^{j\alpha} + \text{h.c.}\right), \tag{4.75}$$

where the group indices have been explicitly written out and h_0 is the dimensionless Yukawa coupling. Expanding the matrix U in terms of the π's one can read off all the interaction terms:

$$\begin{aligned}\mathcal{L}_{Yuk} &= -\frac{2h_0}{f}\bar{\psi}\psi - 2ih_0\pi^a\bar{\psi}\gamma_5 T_a\psi - h_0 f\left(\frac{1}{2N}\bar{\psi}\psi\pi^a\pi^a + \frac{1}{2}d_{abc}\pi^a\pi^b\bar{\psi}T_c\psi\right)\\ &+ \frac{ih_0}{3f^2}\left(\frac{1}{2N}\pi^a\pi^a\pi^c\bar{\psi}\gamma_5 T_c\psi + \frac{1}{4N}d_{abc}\pi^a\pi^b\pi^c\bar{\psi}\gamma_5\psi + \frac{1}{4}d_{abe}(if_{ecg} + d_{ecg})\pi^a\pi^b\pi^c\bar{\psi}\gamma_5 T_g\psi\right)\\ &+ \mathcal{O}(\pi^4),\end{aligned} \tag{4.76}$$

where f_{abc} are the structure constants of the group and d_{abc} is the totally symmetric tensor of $SU(N)$. Since the only phenomenologically relevant fermionic contribution comes from the top quark, then h_0 in eq. (4.75) will be viewed as the top Yukawa coupling. Strong, weak and electromagnetic gauge couplings are also neglected (after having checked that they do not alter the conclusions) and for this reason the derivatives in eq. (4.73) and eq. (4.74) are not covariant.

In a more realistic model, the Yukawa interaction should distinguish among the fermion components and there should be more than one family, as in the SM. For example, considering one family of fermions $\psi^t = (u\ d)$ and imposing an $SU(2)_L \times U(1)_R$ symmetry, one is forced to introduce two different Yukawa couplings h_u and h_d for the up- and the down-type fermions. In this case the Yukawa lagrangian would be:

$$\mathcal{L}'_{Yuk} = -\frac{2h_u}{f}\left(\bar{\psi}_L U P_u \psi_R + \text{h.c.}\right) - \frac{2h_d}{f}\left(\bar{\psi}_L U P_d \psi_R + \text{h.c.}\right), \tag{4.77}$$

where $P_u = (1+\sigma_3)/2$ and $P_d = (1-\sigma_3)/2$ are the up- and down-type fermion projectors.

4.3.1 Beta functions

Consider the $SU(N)$ model described by eq. (4.73) and eq. (4.74). The leading terms of the effective average action can be parametrized as

$$\Gamma_k = \int d^4x \left[\frac{Z_\pi(k)}{2}\partial_\mu\pi^a\partial^\mu\pi^a + Z_\psi(k)\bar{\psi}i\gamma^\mu\partial_\mu\psi - 2i\bar{h}(k)\pi^a\bar{\psi}\gamma_5 T_a\psi + \dots\right] \tag{4.78}$$

where the dots represent higher order terms coming from the expansion of eq. (4.73) and eq. (4.75). The quantities $Z_\pi(k)$, $Z_\psi(k)$ and $\bar{h}(k)$ have been computed at one-loop by means of standard diagrammatic technique using a sharp cutoff regulator. The Feynman diagrams that enter in the computation are shown in Fig. 4.4, diagrams (a), (b) and (c) give the vertex corrections to the Yukawa coupling, while diagrams (d), (e) and (f) contribute to scalar and fermion wave function renormalization constants. The one-loop integrals are performed over a momentum shell that goes from some scale k to Λ. Assuming that $\bar{h}(\Lambda) = h_\Lambda$, $Z_\pi(\Lambda) = 1$ and $Z_\psi(\Lambda) = 1$ one obtains:

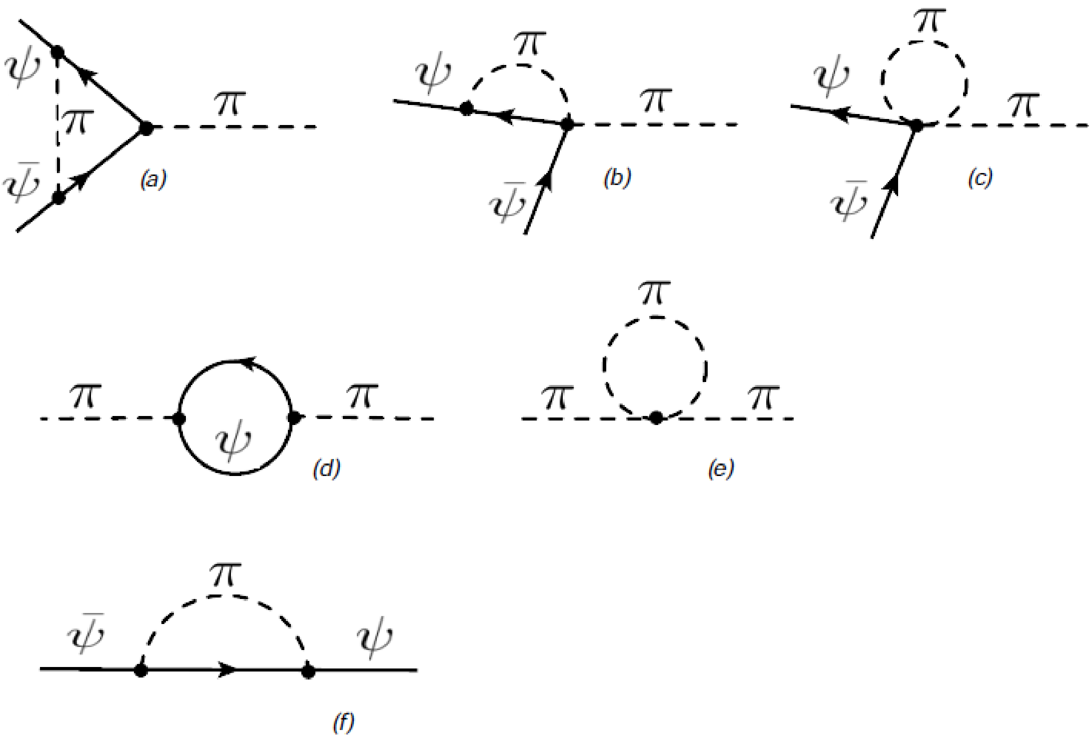


Figure 4.4: One-loop Feynman diagrams entering in the computation of the beta functions of f and h. Diagrams (a), (b) and (c) give the vertex corrections to the Yukawa interaction term, while diagrams (d), (e) and (f) renormalize scalar and fermion propagators.

$$\bar{h}(k) = h_\Lambda \left(1 - \frac{1}{(4\pi)^2} \frac{(2N^2-3)}{3N} \frac{(\Lambda^2 - k^2)}{4} f^2 + \frac{h_\Lambda^2}{(4\pi)^2} \frac{2(N^2-1)}{N} \log \frac{\Lambda^2}{k^2}\right) , \tag{4.79}$$

$$Z_\pi(k) = 1 - \frac{1}{(4\pi)^2} \frac{N}{3} \frac{(\Lambda^2 - k^2)}{4} f^2 + \frac{4N_c h_\Lambda^2}{(4\pi)^2} \log \frac{\Lambda^2}{k^2} , \tag{4.80}$$

$$Z_\psi(k) = 1 + \frac{h_\Lambda^2}{(4\pi)^2} \frac{(N^2-1)}{N} \log \frac{\Lambda^2}{k^2} . \tag{4.81}$$

At this point one defines the renormalized Yukawa coupling $h = \bar{h}\, Z_\pi^{-\frac{1}{2}}\, Z_\psi^{-1}$ and its beta function is computed by taking the derivative of h with respect to the scale k. The beta function of the Goldstone coupling f is obtained by adding to the usual bosonic term (see eq. (3.88)) an additional contribution coming from the fermionic loop which is explicitly given by the term proportional to N_c in eq. (4.80). Then, the one-loop renormalization group equations for the coupling $\tilde{f} = kf$ and h of the $SU(N)$ model turn out to be:

$$\partial_t \tilde{f} = \tilde{f} - \frac{N}{4(4\pi)^2}\tilde{f}^3 + \frac{4N_c}{(4\pi)^2}h^2\tilde{f} \tag{4.82}$$

$$\partial_t h = \frac{1}{(4\pi)^2}\left(4N_c - 2\frac{N^2-1}{N}\right)h^3 + \frac{1}{4(4\pi)^2}\frac{N^2-2}{N}h\tilde{f}^2\,. \tag{4.83}$$

The variable t is, as usual, defined as $\log k/k_0$ where k_0 is a conventional reference energy, which here is taken to be $k_0 = \upsilon$. These beta functions have been obtained by assuming that the mass of the fermions is much smaller than the scale k, this sets a condition $k > h/f$ for the validity of these equations.

4.3.2 Results

The system of equations (4.82)-(4.83) admits a number of possible UV fixed points. There is a formal Gaussian fixed point $\tilde{f} = 0$, $h = 0$ which is outside the domain of the approximation. In the following, the study will be restricted only to renormalization group trajectories for energy scales larger than υ, which corresponds to $\tilde{f} > 2$. There is also a nontrivial fixed point at $h_* = 0$, $\tilde{f}_* = 8\pi/\sqrt{N}$ for which $\tilde{f}$ is a relevant (UV attractive) direction and h is marginally irrelevant. Requiring that the theory lies on a trajectory that reaches it in the UV, implies the triviality of the Yukawa coupling at all scales. This choice is therefore rejected because uninteresting.

A physically interesting trajectory requires nonvanishing h and $\tilde{f}$. If h is treated as a t-independent constant, the β-function for $\tilde{f}$ has a zero at $\tilde{f}_* = 4\sqrt{(4\pi^2 + N_c h^2)/N}$, which is a deformation of the one appearing in the pure bosonic model. The existence of a nontrivial fixed point for the coupled system thus hinges on the existence of a nontrivial zero in the beta function of h. This requires that the first term in the right hand side of eq. (4.83) be negative, which is true for $N > 2N_c$. Unfortunately this condition is not satisfied for the phenomenologically most important case $N = 2$, $N_c = 3$. This is illustrated by the dashed curves in Fig. 4.5, for the initial condition $\tilde{f}_0 = 2$ and $h_0 = 0.7$ at $t = 0$ (thus mimicking the top-quark value). The first term on the right hand side of eq. (4.82) is initially dominant, leading to linear growth of $\tilde{f}$. The second term then grows in absolute value and at some point nearly balances the first one, leading to an approximate fixed point behavior in some range of energies. Eventually h, whose beta function is everywhere positive, becomes large and the third term dominates, leading to a Landau pole and the loss of asymptotic safety. The scale at which destabilization occurs is very

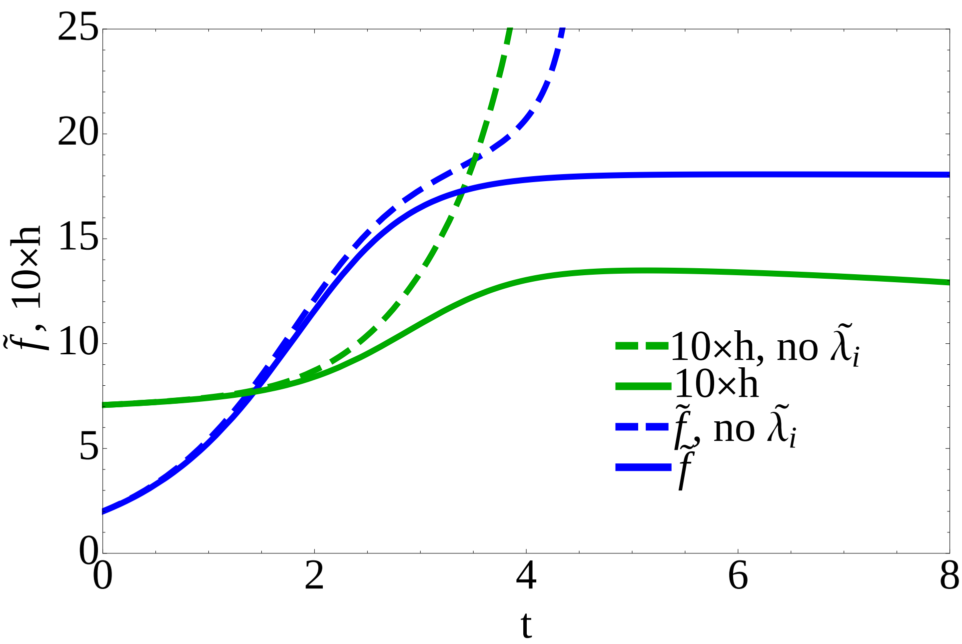


Figure 4.5: Running of $\tilde{f}$ and h (rescaled by a factor of 10) for $N = 2$ and $N_c = 3$. The asymptotic safety behavior of $\tilde{f}$ (blue dashed line curve) is destabilized around $t = 4$ ($\approx$ 22 TeV) due to the increasingly large Yukawa coupling contribution (green dashed line curve) which at about the same scale becomes strongly coupled, that is larger than 2π. The asymptotically safe behavior (continuous lines) is recovered after the introduction of the four-fermion contact interactions, as discussed in Section 4.3.3.

sensitive to the initial conditions and for the Yukawa couplings corresponding to light fermions no destabilization takes place up to very large energies. The conclusion is that the model is not asymptotically safe in the case $N = 2$, $N_c = 3$. For it to be asymptotically safe, either the one loop approximation must break down or else new physical effects must enter in the fermion sector at some energy scale.

It is interesting to compare this behavior to similar models. If the color symmetry was gauged there would be an additional contribution proportional to $g_s^2 h$ to eq. (4.83), which in this case would become

$$\partial_t h = \frac{1}{(4\pi)^2}\left(4N_c - 2\frac{N^2-1}{N}\right)h^3 + \frac{1}{(4\pi)^2}\left(\frac{N^2-2}{4N}\tilde{f}^2 - 3\frac{N_c^2-1}{N_c}g_s^2\right)h\,. \tag{4.84}$$

The new term does not change the behavior shown in Fig. 4.5 because of the asymptotic freedom of the strong sector. No improvements in the UV behavior are obtained even if one considers a different global symmetry group. In the case of the $SU(2)\times U(1)$ model of eq. (4.77) the beta function of the up-type Yukawa fermion h_u turns out to be positive at all scales as well:

$$\partial_t h_u = \frac{2N_c}{(4\pi)^2}h_u^3 + \frac{1}{4(4\pi)^2}h_u\tilde{f}^2\,. \tag{4.85}$$

In the linear sigma model and therefore also in the SM the quadratically divergent term proportional to $\tilde{f}^2 h$ is absent: it is canceled by diagrams containing loops of the Higgs field (see Fig. 4.6). In this case the Yukawa coupling is perturbative up to very high scales [Arason 1992]. A study of the linear version of the model in the context of functional renormalization has been presented in [Jugnickel 1996] for QCD. Another strictly related model is the linear σ-model coupled to one right-handed and N_L left-handed fermions, studied in [Gies 2010a, Gies 2010b]. The Goldstone modes of the nonlinear sigma model are contained in their scalar sector, with the VEV $v = 2/f$ corresponding to the minimum of the scalar potential. They found that the scalar potential and the Yukawa coupling admit a fixed point for $1 \leq N_L \leq 57$. The results obtained here differ due both to the different fermion content and to the non-linear boson-fermion coupling.

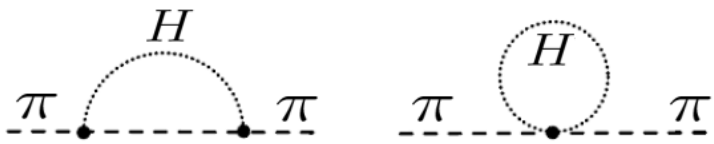


Figure 4.6: Quadratically divergent Higgs diagrams of the linear sigma model. The Goldstone boson and Higgs propagators are represented by broad and narrow dashed lines respectively. Remember that, using a nonlinear representation, the Goldstone-Higgs coupling involves derivatives. In the linear model the divergent parts of these diagrams cancel exactly the one coming from diagrams (c) and (e) of Fig. 4.4.

4.3.3 Four fermion interactions

New physics associated to the SM fermions might restore asymptotic safety and the more natural extension of the model is to include four fermion interactions. In this section, a class of nontrivial UV fixed points with asymptotically free Yukawa couplings, that emerge because of the short-range interactions among fermions, are discussed.

Restricting to the case $N = 2$, the model is enlarged by adding to the lagrangian a complete set of $SU(2)_L \times SU(2)_R$ invariant four fermion operators. Requiring P invariance, all possible chiral invariant operators, up to Fierz reorderings, are given by the following lagrangian:

$$\begin{aligned}
\mathcal{L}_{\psi^4} &= \lambda_1 \left(\bar{\psi}_L^{ia} \psi_R^{ja} \bar{\psi}_R^{jb} \psi_L^{ib} \right) + \lambda_2 \left(\bar{\psi}_L^{ia} \psi_R^{jb} \bar{\psi}_R^{jb} \psi_L^{ia} \right) \\
&+ \lambda_3 \left(\bar{\psi}_L^{ia} \gamma_\mu \psi_L^{ia} \bar{\psi}_L^{jb} \gamma^\mu \psi_L^{jb} + \bar{\psi}_R^{ia} \gamma_\mu \psi_R^{ia} \bar{\psi}_R^{jb} \gamma^\mu \psi_R^{jb} \right) \\
&+ \lambda_4 \left(\bar{\psi}_L^{ia} \gamma_\mu \psi_L^{ib} \bar{\psi}_L^{jb} \gamma^\mu \psi_L^{ja} + \bar{\psi}_R^{ia} \gamma_\mu \psi_R^{ib} \bar{\psi}_R^{jb} \gamma^\mu \psi_R^{ja} \right) . \qquad (4.86)
\end{aligned}$$

The coefficients λ_i have inverse square mass dimension. The lagrangian in eq. (4.86) does not include operators defined by taking the square of the Yukawa term in eq. (4.75) because they are higher order from the point of view of chiral perturbation theory.

Strictly speaking, only the third SM fermion generation requires new physics to emerge at relatively low scales, since four-fermion operators involving the first two generations can be suppressed by much larger scales without spoiling the asymptotic safety scenario. In the following, when experimental bounds will be discussed, the conservative scenario in which (4.86) also the first generation is involved will be considered. One tacitly assumes that the operators in (4.86) are consistent with flavor changing neutral current bounds.

The symmetries imposed on eq. (4.86) make this lagrangian the minimal choice, and the one that will be studied here. More general sets of operators may well be relevant depending on the symmetries of this new sector, the SM group $SU(2) \times U(1)$ being the first instance coming to mind. The analysis is in this case more cumbersome because more operators must be included, but no distinctive features are expected to arise.

The operators in eq. (4.86) are similar to those discussed in top-quark condensation models. In these and other models of composite quarks only operators with vector current structure that is iso- and color singlet are usually considered. Here the full set of operators are taken into account since their couplings mix in the renormalization group evolution equations. A discussion of the four-fermion lagrangian in eq. (4.86), and its fixed points, as a model of chiral symmetry breaking can be found in [Gies 2004]. In their approach it is assumed that when one flows from the fixed point in the UV towards the IR, the couplings $\tilde{\lambda}_i$ become stronger and eventually trigger the formation of a condensate. Here the considered behavior is opposite, the couplings $\tilde{\lambda}_i$ become weaker towards the IR and the breaking of chiral symmetry is due to the Goldstone bosons, which are regarded as fundamental degrees of freedom.

4.3.4 Beta functions

An additional set of diagrams that enter in the computation of the beta function of h has to be taken into account when the four fermion interactions of eq. (4.86) are introduced. These interactions generate a fermion loop correction to the Yukawa vertex, as shown in Fig. 4.7.

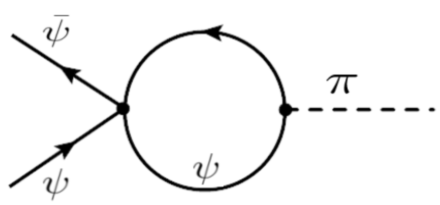


Figure 4.7: Diagram for the one-loop correction to the Yukawa vertex induced by the four fermion interactions.

The new system of coupled beta function equations for the dimensionless variables $\tilde{f}$, h and $\tilde{\lambda}_i = \lambda_i k^2$ turns out to be:

$$
\begin{aligned}
\partial_t \tilde{f} &= \tilde{f} - \frac{1}{32\pi^2}\tilde{f}^3 + \frac{N_c}{4\pi^2} h^2 \tilde{f} \\
\partial_t h &= \frac{1}{16\pi^2}\left[4N_c - 3 + \frac{16}{\tilde{f}^2}(N_c\tilde{\lambda}_1 + \tilde{\lambda}_2)\right] h^3 + \frac{1}{64\pi^2}\left[\tilde{f}^2 - 16(N_c\tilde{\lambda}_1 + \tilde{\lambda}_2)\right] h \\
\partial_t \tilde{\lambda}_1 &= 2\tilde{\lambda}_1 - \frac{1}{4\pi^2}\left[N_c\tilde{\lambda}_1^2 + \frac{3}{2}\tilde{\lambda}_1\tilde{\lambda}_2 - 2\tilde{\lambda}_1\tilde{\lambda}_3 - 4\tilde{\lambda}_1\tilde{\lambda}_4\right] \\
\partial_t \tilde{\lambda}_2 &= 2\tilde{\lambda}_2 + \frac{1}{4\pi^2}\left[\frac{1}{4}\tilde{\lambda}_1^2 + 4\tilde{\lambda}_1\tilde{\lambda}_3 + 2\tilde{\lambda}_1\tilde{\lambda}_4 - \frac{3}{4}\tilde{\lambda}_2^2 + 2(2N_c+1)\tilde{\lambda}_2\tilde{\lambda}_3 + 2(N_c+2)\tilde{\lambda}_2\tilde{\lambda}_4\right] \\
\partial_t \tilde{\lambda}_3 &= 2\tilde{\lambda}_3 + \frac{1}{4\pi^2}\left[\frac{1}{4}\tilde{\lambda}_1\tilde{\lambda}_2 + \frac{N_c}{8}\tilde{\lambda}_2^2 + (2N_c-1)\tilde{\lambda}_3^2 + 2(N_c+2)\tilde{\lambda}_3\tilde{\lambda}_4 - 2\tilde{\lambda}_4^2\right] \\
\partial_t \tilde{\lambda}_4 &= 2\tilde{\lambda}_4 + \frac{1}{4\pi^2}\left[\frac{1}{8}\tilde{\lambda}_1^2 - 4\tilde{\lambda}_3\tilde{\lambda}_4 + (N_c+2)\tilde{\lambda}_4^2\right] ,
\end{aligned}
\tag{4.87}
$$

where the new contributions to β_h, coming from the four fermion interactions, are obtained by computing the diagrams in Fig. 4.7 and the beta functions of the four fermion interaction couplings are taken from [Gies 2004] specialized to the sharp cutoff case.

In the equations for the coefficients $\tilde{\lambda}_i$ contributions coming from the Yukawa terms which are proportional to $h^2\tilde{f}^2$, $h^2\tilde{\lambda}_i$ or $h^2\tilde{\lambda}_i^2/\tilde{f}^2$ have been neglected . These terms are negligible in the UV because the fixed points taken in consideration are those for which the Yukawa coupling approaches zero. Some of them just become comparable to the leading terms for $k \sim \upsilon$, where the flow is stopped.

Notice that only the operators proportional to λ_1 and λ_2 contribute to the beta function of h. The other two operators do not contribute because of their chiral properties. Moreover, it is crucial that the combination $16(N_c\tilde{\lambda}_1 + \tilde{\lambda}_2)$ at the fixed point of $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ be different from zero and larger than $\tilde{f}^2$ because otherwise there would be no UV physically acceptable fixed point.

4.3.5 Results

Here the number of colors is fixed to its phenomenological value $N_c = 3$. The beta functions of the $\tilde{\lambda}_i$ form a closed sub-system. The numerical study of these equations reveals the presence of 16 real fixed points with coordinates given in the first four columns in Table 4.2. The fixed points are listed in order of decreasing trace of the stability matrix $\frac{\partial\beta_{\tilde{\lambda}_i}}{\partial\tilde{\lambda}_j}$, from the most UV-repulsive, the Gaussian fixed point fp0, to the most UV attractive fp4. The coefficients of the operators related to UV-repulsive directions are completely determined by the asymptotic safety condition. Hence, the fixed points with a low number of UV-attractive directions are the most predictive ones, and hence the most phenomenologically appealing. The number in the name of the fixed point is the number of relevant (UV attractive) directions. These values can then be used to find the zeroes of β_h, and these in turn are used to find the zeroes of $\beta_{\tilde{f}}$.

The sixteen fixed points of the complete system for which $h_* = 0$ are taken into account; this requirement implies $\tilde{f}_*^2 = 32\pi^2$. At each of these sixteen fixed points the direction $\tilde{f}$ is always a relevant one, with eigenvalue -0.45; the direction h is also an eigendirection, with eigenvalue

$$\varepsilon_h = \left.\frac{\partial \beta_h}{\partial h}\right|_* = \frac{1}{64\pi^2}\left(\tilde{f}_*^2 - 16(N_c\tilde{\lambda}_{1*} + \tilde{\lambda}_{2*})\right) . \tag{4.88}$$

The numerical values of ε_h are listed in the last column of Table 4.2. The seven fixed points with $\varepsilon_h > 0$ are physically uninteresting since the requirement of flowing to one of them in the UV implies that $h(t) = 0$ at all scales. The other nine fixed points of the fermionic sector for which $\varepsilon_h < 0$ also admit a fixed point with nonzero h, but in this study only those with $h_* = 0$ are considered. For the $\tilde{\lambda}_i$, the condition of flowing to fpnx in the UV yields 4$-$n predictions. The values of $\tilde{f}$ and h remain always free parameters, to be fixed by comparison with the experiment.

The trajectories emerging from the fixed points in the directions of the relevant eigenvectors have been studied numerically . Some of them lead to divergences, others flow to other fixed points. Here the single renormalizable trajectory that ends at fp1c in the UV is considered. This is a natural choice as it is the most predictive (the one with the smallest number of relevant directions) among the fixed points with $\varepsilon_h < 0$.

In order to select the fine-tuned initial conditions in the IR that guarantee asymptotic safety at fp1c, one has first to solve numerically the flow equations of the fermionic subsystem for

Fixed points	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$	ε_h
fp0	0	0	0	0	0.5
fp1a	0	-28.71	-7.18	0	1.22
fp1b	0	0	7.85	-9.51	0.5
fp1c	0	25.61	-4.27	0	-0.15
fp1d	25.80	-1.77	0.19	-1.15	-1.42
fp2a	13.41	20.10	-3.80	-0.24	-1.03
fp2b	20.86	-3.56	7.04	-8.94	-1.00
fp2c	0	-36.55	2.34	-13.92	1.43
fp2d	0	0	-15.79	0	0.5
fp2e	37.17	-37.36	-8.43	-1.65	-1.38
fp2f	-2.92	32.59	4.67	-12.04	-0.10
fp3a	0.	31.67	4.67	-12.06	-0.30
fp3b	19.95	-8.59	-15.27	-0.36	-0.80
fp3c	31.22	-44.52	0.73	-13.38	-0.74
fp3d	-4.87	1.54	-5.42	-20.10	0.83
fp4	0	0	-5.42	-20.13	0.5

Table 4.2: Values of the coefficients $\tilde{\lambda}_{i*}$ for the 16 fixed points discussed in the text. For all the fixed points $\tilde{f}_* = 17.78$ and $h_* = 0$. The fixed point fp1c is boxed.

decreasing t, starting at an initial point $\tilde{\lambda}_{i*} + 10^{-8} v_i$, where v_i is the relevant eigenvector. This trajectory is attracted in the IR towards fp0 after roughly 20 e-foldings, so the four-fermion couplings can be taken arbitrarily small by selecting the IR value of t appropriately. One then shifts t such that this value is zero, in accordance with the convention that $t = 0$ corresponds to the scale $k_0 = v$. Now one picks a trajectory for the whole system by fixing the initial values of the $\tilde{\lambda}_i$ to agree with the ones find by this method, while the initial value of $\tilde{f}$ is $2k_0/v = 2$ and the initial value for h is 0.7. These agree with the initial values of the trajectory discussed in Section 4.3.2. The result is shown in Fig. 4.5 (continuous curves) and Fig. 4.8. In this case, for

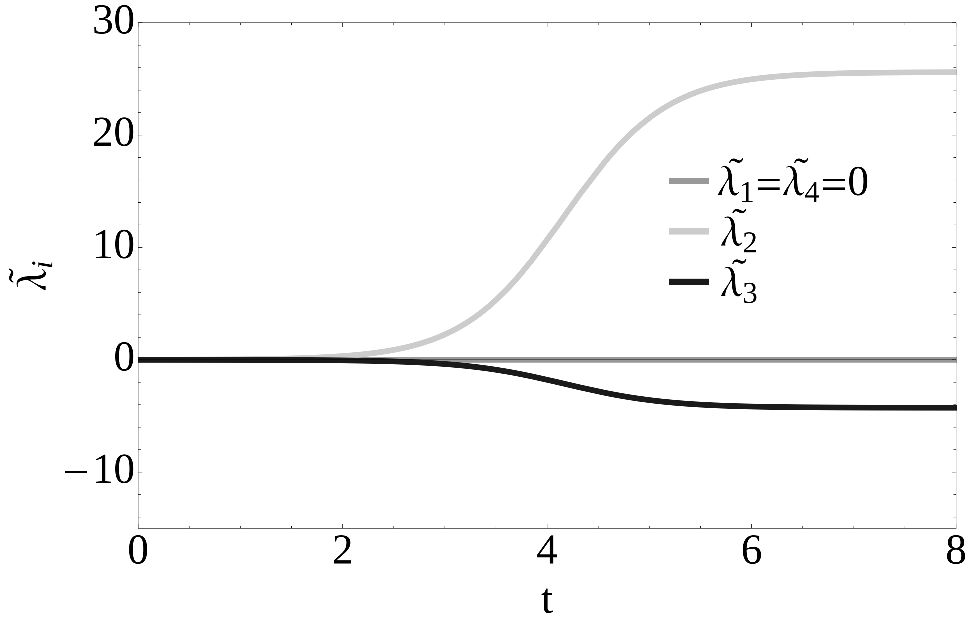


Figure 4.8: Running of the $\tilde{\lambda}_i$ for the fixed point fp1c. $\tilde{\lambda}_1$ and $\tilde{\lambda}_4$ are equal to zero at all energies.

small t the couplings $\tilde{f}$ and h behave as in the model without four fermion interactions, with $\tilde{f}$ and h both increasing. At some point, however, $\tilde{\lambda}_2$ becomes sizable and then the last term in the right hand side of the second equation in (4.87) pulls h towards zero. The trajectory is therefore characterized by a crossover from the IR regime where the fermionic interactions are mainly of Yukawa type, with IR free four-fermion interactions, and the UV regime dominated by the contact interactions, with UV free Yukawa coupling. This is similar to the behavior discussed in [Schwindt 2010], except that here the contact interactions are not bosonized : in the presence of the Goldstone bosons the crossover is automatic.

4.3.6 Experimental constraints

In order to select a realistic trajectory among the various possibilities one needs to compare with experimental results. The model is consistent with electroweak precision measurements, as discussed in Section 4.2, where the values of the parameters S and T are computed for the gauged nonlinear sigma model and the addition of the fermion contact interactions does not modify this result.

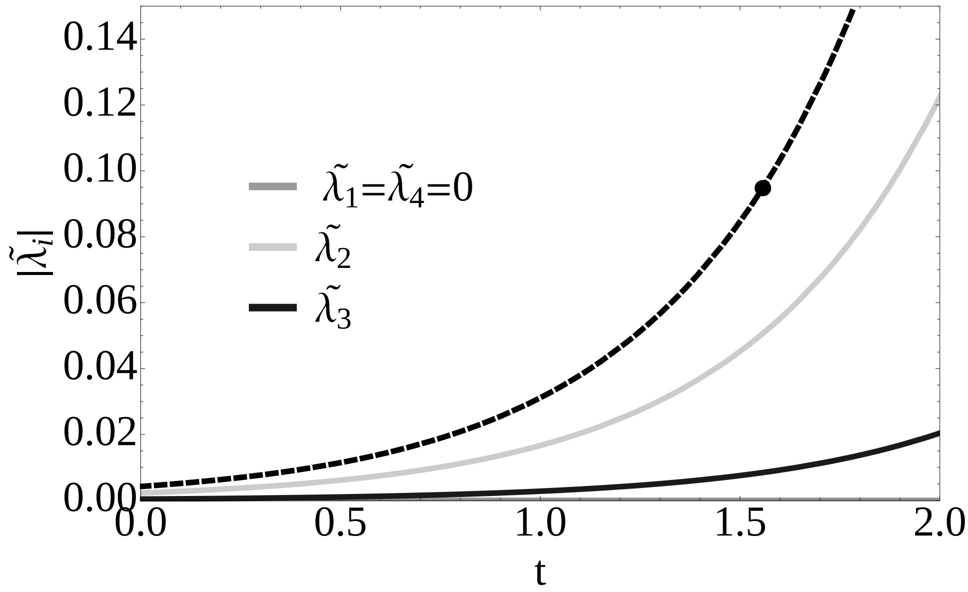


Figure 4.9: A zoom at low energy of Fig. 4.8 for the values (continuous curves) of the absolute values of the four-fermion operator coefficients $\tilde{\lambda}_i$. The dashed curve is the corresponding size for the contact interaction for the bound at $\Lambda = 8.7$ TeV. The dot at $t = 1.55$ on the dashed curve represents the bound at the reference value of the quark-level effective energy of 1.17 TeV. The values of the $\tilde{\lambda}_i$ must lie below the dashed curve for the experimental bound to be satisfied.

Unfortunately the current bounds on contact interactions have been published only for the case in which a single operator, namely the one proportional to λ_3 of eq. (4.86) is present [Eichten 1983]. This is rather unrealistic, given the renormalization group mixing, but no need for a more detailed study was felt necessary.

In this section the aim is to show how the experimental bounds are in principle already able to tell something about the size of the new interactions. Conservatively but rather unrealistically, the current experimental bound on λ_3 is taken and enforced on all coefficients as if they were contributing in the same manner to the partonic cross sections. An additional assumption is that all three generations contribute with identical coefficients $\tilde{\lambda}_i$.

The experimental bound is a lower bound of the so-called contact interaction scale Λ which is related to the coefficient $\tilde{\lambda}_3$ by the identity

$$\tilde{\lambda}_3(k) = \frac{2\pi}{\Lambda^2} k^2 \,. \tag{4.89}$$

A bound on the value of Λ translates into a curve in the $k, \tilde{\lambda}_3$ plane because of the energy dependence in eq. (4.89). As a crude estimate, the experimental bounds are imposed at the quark-level effective energy scale of the LHC, namely $k_{eff} = \sqrt{s}/(2 \times 3) = 1.17$ TeV, where the factor of 2 comes from the sharing of the energy with the gluons and the factor of 3 from the assumed equal energy partition among the three valence quarks. As mentioned before, for simplicity the same constraint to the absolute value of all four coefficients $\tilde{\lambda}_i$ is imposed.

As an example of the constraints it is possible to obtain, one can take the values of $\tilde{\lambda}_i$ on a renormalization group trajectory leading to the fixed point discussed in the previous section, namely `fp1c`, and compare with the most recent published bound [ATL 2012] of $\Lambda = 8.7$ TeV for an integrated luminosity of 4.8 fb^{-1}. Of course, the fixed points with fewer relevant deformations lead to more predictions and therefore will be easier to disprove.

As one can see from Fig. 4.9, the bound is satisfied. In comparing with the experimental data, one may worry whether the power-law running of h and v may imply potentially important mass corrections to the cross sections. Comparing the running of the masses for the solution above one can verify that it does not deviate from the logarithmic one of dimensional regularization for more than 10%, at least below the scale so far explored, namely 600 GeV.

4.4 Goldstone boson scattering

The dynamics of the Goldstone bosons is usually described at low-energy by an effective theory such as the nonlinear sigma model. Consider the case of the chiral $SU(2)_L \times SU(2)_R$-invariant model with action

$$\mathcal{S} = -\frac{1}{f^2} \int d^4x \, \mathrm{Tr}\, U^\dagger \partial_\mu U \, U^\dagger \partial^\mu U \,, \tag{4.90}$$

where U is an $SU(2)$ group element that encodes the Goldstone fields (called also pions) $\vec{\pi} = (\pi^1, \pi^2, \pi^3)$: $U = \exp(f\pi^a T_a)$, $T_a = (i/2)\sigma^a$, $T_a^\dagger = -T_a$, $\mathrm{tr}(T_a T_b) = -(1/2)\delta_{ab}$. The quantity f is the Goldstone boson coupling and is the inverse of the pion decay constant $F_\pi = 2/f$. Expanding the matrix U in terms of the pions, it is possible to rewrite eq. (4.90) as

$$\mathcal{S} = \frac{1}{2} \int d^4x \, h_{ab}(f\vec{\pi}) \partial\pi^a \partial\pi^b \tag{4.91}$$

where

$$h_{ab}(f\vec{\pi}) = \delta_{ab} - \frac{1}{12} f^2 (\vec{\pi}^2 \delta_{ab} - \pi_a \pi_b) + \frac{1}{360} f^4 \vec{\pi}^2 (\vec{\pi}^2 \delta_{ab} - \pi_a \pi_b) + \dots \tag{4.92}$$

Note that the pion fields are canonically normalized and the metric is dimensionless. This theory has derivative couplings and perturbation theory is ill-defined in the UV. A related and phenomenologically even more pressing issue is the high energy behavior of the Goldstone boson scattering amplitude. Any scattering process $\pi^a\pi^b \to \pi^c\pi^d$ is given in terms of a singe amplitude $A(s,t,u)$ which at tree level reads

$$A(s,t,u) = \frac{s}{F_\pi^2}\,, \tag{4.93}$$

where s, t, u are the usual Mandelstam variables. By inspection of eq. (4.93) one obtains a violation of unitarity for $s \approx 8\pi F_\pi^2$, which is usually taken as the cutoff scale for the validity of the theory.

4.4.1 Beta functional

The aim of this section is to reproduce the standard perturbative result for the scattering amplitude using the formalism introduced in Chapter 2. In order to compute the Goldstone boson scattering, one needs to know the effective action of the theory Γ, which, in the present context, is obtained by solving the one-loop Wetterich equation for the average action Γ_k. The procedure to obtain the effective action is the following. One chooses an initial point in theory space, representing the 'bare' action at some high scale Λ ($\Gamma_\Lambda = \mathcal{S}$), and then solves eq. (2.30) for Γ_k. The endpoint of this flow for $k \to 0$ is the ordinary effective action Γ, *i.e.* $\Gamma = \Gamma_0$. This is a way of calculating the effective action by solving a differential equation, rather than performing a functional integral.

In previous calculations, where the renormalization group flow of the nonlinear sigma model was studied, the full field dependence of the effective action has been kept, expanding to second or fourth order in momenta. By contrast in order to compute the scattering amplitude of Goldstone bosons one needs terms of fourth order in the field, but one has to retain the full momentum dependence of Γ. It is convenient to work with dimensionless coordinates $\varphi^a = f\pi^a$ and the computation resembles the one presented in Section 2.3.2. The second variation of the action can be taken from eq. (2.101):

$$\mathcal{S}^{[2]}[\varphi;\xi] = \frac{1}{2f^2}\int d^4x\, \xi^a \left(-\nabla^2 h_{ab} - \mathcal{M}_{ab}\right)\xi^b\,, \tag{4.94}$$

where $\nabla_\mu \xi^a$ is defined in eq. (2.99) and $\mathcal{M}_{ab} = \partial_\mu\varphi^c\partial^\mu\varphi^d R_{acbd}$. In the present context, the beta functional equation to solve is the following:

$$\frac{\partial}{\partial t}\Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left[\left(\frac{\delta^2\mathcal{S}[\varphi;\xi]}{\delta\xi\delta\xi} + R_k\right)^{-1}\frac{\partial R_k}{\partial t}\right]\,. \tag{4.95}$$

The cutoff kernel is chosen to be $R_k(-\nabla^2 - \mathcal{M})$ and its form is the optimized one of eq. (2.12).

In this way the cutoff combines with the quadratic action to produce a function of a single argument:

$$h_k(z) = \frac{\partial_t R_k(z)}{z + R_k(z)} = 2\theta(k^2 - z) \quad , \qquad z = -\nabla^2 - \mathcal{M}\,. \tag{4.96}$$

where in the last step the explicit form of the cutoff function R_k has been used. The beta functional is just the trace of this function

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} h_k(z) = \int_0^\infty ds\, \tilde{h}_k(s) \mathrm{Tr}\; e^{-s(-\nabla^2 - \mathcal{M})}\,, \tag{4.97}$$

where $\tilde{h}_k$ is the inverse Laplace transform of h_k. The full momentum dependence is retained by computing the trace using the non-local heat kernel expansion (see Appendix B). Restricting to fourth order in the fields, one has to evaluate the following terms:

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \frac{1}{(4\pi)^2} \int d^4x \int_0^\infty ds\, \tilde{h}_k(s)\, \mathrm{tr} \left\{ s^{-1} \mathcal{M} + [\mathcal{M} f_{\mathcal{M}}(s\Box) \mathcal{M} + \Omega_{\mu\nu} f_\Omega(s\Box) \Omega^{\mu\nu}] + \cdots \right\}, \tag{4.98}$$

where

$$\Box = -\nabla^2 \quad , \qquad \Omega_{\mu\nu}{}^a{}_b = [\nabla_\mu, \nabla_\nu]^a{}_b = \partial_\mu \varphi^c \partial_\nu \varphi^d R_{cd}{}^a{}_b \tag{4.99}$$

and the structure functions are given by

$$f_{\mathcal{M}}(y) = \frac{1}{2} f(y) \quad , \qquad f_\Omega(y) = -\frac{1}{2y}[f(y) - 1] \quad , \qquad f(y) = \int_0^1 d\xi\, e^{-\xi(1-\xi)y}\,. \tag{4.100}$$

The result, including terms up to four fields, is:

$$\begin{aligned}
\partial_t \Gamma_k[\varphi] = \frac{1}{2} \frac{1}{(4\pi)^2} \int d^4x \Bigg[& k^2\, \partial_\mu \varphi^a \partial^\mu \varphi^b \delta_{ab} - \frac{1}{12} k^2\, (\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd})\, \partial_\mu \varphi^a \partial^\mu \varphi^b \varphi^c \varphi^d \\
+ & \frac{1}{16}\, \partial_\mu \varphi^a \partial^\mu \varphi_a \left(1 - \sqrt{1 - \frac{4k^2}{\Box}} \theta(\Box - 4k^2) \right) \partial_\nu \varphi^b \partial^\nu \varphi_b \\
+ & \frac{1}{16}\, \partial_\mu \varphi^a \partial^\mu \varphi_b \left(1 - \sqrt{1 - \frac{4k^2}{\Box}} \theta(\Box - 4k^2) \right) \partial_\nu \varphi^b \partial^\nu \varphi_a \\
+ & \frac{1}{48}\, \partial^\mu \varphi^b \partial_\nu \varphi_a \left(1 - \left(1 - \frac{4k^2}{\Box} \right)^{3/2} \theta(\Box - 4k^2) \right) \partial_\mu \varphi^a \partial^\nu \varphi_b \\
- & \frac{1}{48}\, \partial^\mu \varphi_a \partial_\nu \varphi^b \left(1 - \left(1 - \frac{4k^2}{\Box} \right)^{3/2} \theta(\Box - 4k^2) \right) \partial_\mu \varphi^a \partial^\nu \varphi_b + O(\varphi^6) \Bigg]\,.
\end{aligned} \tag{4.101}$$

It is important to note that the first two terms appear in the same ratio as in the original action.

This result has already been obtained in Section 2.3.2 and is just a consequence of the fact that the cutoff preserves the $SU(2)\times SU(2)$ invariance of the theory. In addition, the beta functional contains new non-local pieces (form factors) appearing in the last four terms of eq. (4.101). This non-local terms, obtained by using the non-local heat kernel expansion, are crucial for computing the physical amplitude.

4.4.2 Integration of the flow

The effective average action of the theory is obtained by integrating the flow equation (4.101). This yields a divergent result that is regulated by performing the integration from $k = 0$ to $k = \Lambda$:

$$
\begin{aligned}
\Gamma_\Lambda[\varphi] - \Gamma_0[\varphi] = \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x \Bigg[& \frac{\Lambda^2}{2}\,\partial_\mu\varphi^a\partial^\mu\varphi^b\delta_{ab} - \frac{1}{12}\frac{\Lambda^2}{2}\,(\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd})\,\partial_\mu\varphi^a\partial^\mu\varphi^b\varphi^c\varphi^d \\
+ & \frac{1}{16}\,\partial_\mu\varphi^a\partial^\mu\varphi_a\left(1-\frac{1}{2}\log\frac{\Box}{\Lambda^2}\right)\partial_\nu\varphi^b\partial^\nu\varphi_b \\
+ & \frac{1}{16}\,\partial_\mu\varphi^a\partial^\mu\varphi_b\left(1-\frac{1}{2}\log\frac{\Box}{\Lambda^2}\right)\partial_\nu\varphi^b\partial^\nu\varphi_a \\
+ & \frac{1}{16}\,\partial^\mu\varphi^b\partial_\nu\varphi_a\left(\frac{4}{9}-\frac{1}{6}\log\frac{\Box}{\Lambda^2}\right)\partial_\mu\varphi^a\partial^\nu\varphi_b \\
- & \frac{1}{16}\,\partial^\mu\varphi_a\partial_\nu\varphi^b\left(\frac{4}{9}-\frac{1}{6}\log\frac{\Box}{\Lambda^2}\right)\partial_\mu\varphi^a\partial^\nu\varphi_b + O(\varphi^6)\Bigg]\,.
\end{aligned}
\tag{4.102}
$$

The initial condition at scale Λ is chosen to be:

$$
\begin{aligned}
\Gamma_\Lambda[\varphi] &= \frac{1}{2f_\Lambda^2}\int d^4x\, h_{ab}(\varphi)\partial\varphi^a\partial\varphi^b + \frac{\ell_{1\Lambda}}{2}\int d^4x\,\partial_\mu\varphi^a\partial^\nu\varphi_a\partial^\mu\varphi^b\partial_\nu\varphi^b \\
&+ \frac{\ell_{2\Lambda}}{2}\int d^4x\,\partial_\mu\varphi^a\partial^\mu\varphi_a\partial^\nu\varphi^b\partial_\nu\varphi^b\,.
\end{aligned}
\tag{4.103}
$$

The quadratic divergence is removed by redefining

$$
\frac{1}{f_\Lambda^2} = \frac{1}{f^2} + \frac{\Lambda^2}{32\pi^2}\,. \tag{4.104}
$$

The logarithmic divergence is removed by introducing $\mathcal{O}(p^4)$ counterterms with coefficients $\ell_{1\Lambda}$ and $\ell_{2\Lambda}$, which are assumed to be:

$$\begin{aligned}
\ell_{1\Lambda} &= \ell_1(\mu) - \frac{2}{3}\frac{1}{16}\frac{1}{(4\pi)^2}\log\frac{\mu^2}{\Lambda^2} \\
\ell_{2\Lambda} &= \ell_2(\mu) - \frac{1}{3}\frac{1}{16}\frac{1}{(4\pi)^2}\log\frac{\mu^2}{\Lambda^2}\,.
\end{aligned} \tag{4.105}$$

The renormalization conditions of eq. (4.104) and eq. (4.105) yield a finite result for the effective average action that is obtained by solving eq. (4.102) for $\Gamma_0[\varphi] = \Gamma[\varphi]$:

$$\begin{aligned}
\Gamma[\varphi] &= \int d^4x \left[\frac{1}{2f^2}\partial_\mu\varphi^a\partial^\mu\varphi_a - \frac{1}{24f^2}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd})\partial_\mu\varphi^a\partial^\mu\varphi^b\varphi^c\varphi^d\right. \\
&+ \frac{\ell_1(\mu)}{2}\partial_\mu\varphi^a\partial^\nu\varphi_a\partial^\mu\varphi^b\partial_\nu\varphi_b + \frac{\ell_2(\mu)}{2}\int d^4x\,\partial_\mu\varphi^a\partial^\mu\varphi_a\partial^\nu\varphi^b\partial_\nu\varphi_b \\
&- \frac{1}{512\pi^2}\partial_\mu\varphi^a\partial^\mu\varphi_a\left(1 - \frac{1}{2}\log\frac{\Box}{\mu^2}\right)\partial_\nu\varphi^b\partial^\nu\varphi_b \\
&- \frac{1}{512\pi^2}\partial_\mu\varphi^a\partial^\mu\varphi_b\left(1 - \frac{1}{2}\log\frac{\Box}{\mu^2}\right)\partial_\nu\varphi^b\partial^\nu\varphi_a \\
&- \frac{1}{512\pi^2}\partial^\mu\varphi^b\partial_\nu\varphi_a\left(\frac{4}{9} - \frac{1}{6}\log\frac{\Box}{\mu^2}\right)\partial_\mu\varphi^a\partial^\nu\varphi_b \\
&\left.+ \frac{1}{512\pi^2}\partial^\mu\varphi_a\partial_\nu\varphi^b\left(\frac{4}{9} - \frac{1}{6}\log\frac{\Box}{\mu^2}\right)\partial_\mu\varphi^a\partial^\nu\varphi_b + \mathcal{O}(\varphi^6)\right]
\end{aligned} \tag{4.106}$$

Remember that the effective action above is an Euclidean action written in terms of dimensionless fields φ^a. In order to compute the scattering amplitude one has to perform the analytic continuation to Minkowski space and to consider canonically normalized fields $\pi^a = \varphi^a/f$. The resulting effective action is the following:

$$\begin{aligned}
\Gamma[\pi] &= \int d^4x \left[\frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi_a - \frac{f^2}{24}(\delta_{ab}\delta_{cd} - \delta_{ac}\delta_{bd})\partial_\mu\pi^a\partial^\mu\pi^b\pi^c\pi^d\right. \\
&- \frac{\ell_1(\mu)}{2}f^4\partial_\mu\pi^a\partial^\nu\pi_a\partial^\mu\pi^b\partial_\nu\pi_b - \frac{\ell_2(\mu)}{2}\int d^4x\,\partial_\mu\pi^a\partial^\mu\pi_a\partial^\nu\pi^b\partial_\nu\pi_b \\
&+ \frac{1}{512\pi^2}\partial_\mu\pi^a\partial^\mu\pi_a\left(1 - \frac{1}{2}\log\frac{\Box}{\mu^2}\right)\partial_\nu\pi^b\partial^\nu\pi_b \\
&+ \frac{1}{512\pi^2}f^4\partial_\mu\pi^a\partial^\mu\pi_b\left(1 - \frac{1}{2}\log\frac{\Box}{\mu^2}\right)\partial_\nu\pi^b\partial^\nu\pi_a \\
&+ \frac{1}{512\pi^2}f^4\partial^\mu\pi^b\partial_\nu\pi_a\left(\frac{4}{9} - \frac{1}{6}\log\frac{\Box}{\mu^2}\right)\partial_\mu\pi^a\partial^\nu\pi_b \\
&\left.- \frac{1}{512\pi^2}f^4\partial^\mu\pi_a\partial_\nu\pi^b\left(\frac{4}{9} - \frac{1}{6}\log\frac{\Box}{\mu^2}\right)\partial_\mu\pi^a\partial^\nu\pi_b + \mathcal{O}(\pi^6)\right]
\end{aligned} \tag{4.107}$$

In doing this analytic continuation the four derivative terms pick up a minus sign passing from Euclidean to Minkowski space.[1] Notice also that the couplings ℓ_1 and ℓ_2 can be expressed in terms of the coefficients a_4 and a_5 entering in the higher derivative chiral lagrangian of eq. (C.8) by the relations $\ell_1 = -a_4/2$, $\ell_2 = -a_5/2$.

4.4.3 Amplitude

Given an effective action of the form

$$\begin{aligned}
\Gamma[\pi] &= \frac{1}{2}\int d^4x\, \delta_{ab}\partial_\mu\pi^a\partial^\mu\pi^b - B_0\int d^4x\, (\delta_{ab}\delta_{cd}-\delta_{ac}\delta_{bd})\,\partial_\mu\pi^a\partial^\mu\pi^b\pi^c\pi^d \\
&+ B_1\int d^4x\,\partial_\mu\pi^a\partial^\nu\pi_a\partial^\mu\pi^b\partial_\nu\pi_b + B_2\int d^4x\,\partial_\mu\pi^a\partial^\mu\pi_a\partial^\nu\pi^b\partial_\nu\pi_b \\
&+ \int d^4x\,\partial_\mu\pi^a\partial^\mu\pi_a\,[F_{\mathcal{M}}(\Box)]\,\partial_\nu\pi^b\partial^\nu\pi_b \\
&+ \int d^4x\,\partial_\mu\pi^a\partial^\mu\pi_b\,[F_{\mathcal{M}}(\Box)]\,\partial_\nu\pi^b\partial^\nu\pi_a \\
&+ \int d^4x\,\partial^\mu\pi^b\partial_\nu\pi_a\,[F_{\Omega}(\Box)]\,\partial_\mu\pi^a\partial^\nu\pi_b \\
&- \int d^4x\,\partial^\mu\pi_a\partial_\nu\pi^b\,[F_{\Omega}(\Box)]\,\partial_\mu\pi^a\partial^\nu\pi_b + \mathcal{O}(\pi^6)\,,
\end{aligned} \tag{4.108}$$

the pion scattering amplitude $A(s,t,u)$ is given by:

$$\begin{aligned}
A(s,t,u) &= 6B_0 s + B_1(t^2+u^2) + 2B_2 s^2 + 2s^2F_{\mathcal{M}}(-s) + t^2F_{\mathcal{M}}(-t) + u^2F_{\mathcal{M}}(-u) \\
&+ (u^2-s^2)F_{\Omega}(-t) + (t^2-s^2)F_{\Omega}(-u)\,.
\end{aligned} \tag{4.109}$$

Comparing with eq. (4.107), one has that:

$$B_0 = \frac{f^2}{24} \quad , \quad B_1 = -\frac{\ell_1}{2}f^4 \quad , \quad B_2 = -\frac{\ell_2}{2}f^4\,, \tag{4.110}$$

$$F_{\mathcal{M}}(-s) = \frac{f^4}{512\pi^2}\left(1-\frac{1}{2}\log\frac{-s}{\mu^2}\right)\,, \tag{4.111}$$

and

$$F_{\Omega}(-t) = \frac{f^4}{512\pi^2}\left(\frac{4}{9}-\frac{1}{6}\log\frac{-t}{\mu^2}\right)\,. \tag{4.112}$$

[1] The Minkowskian action Γ_M is obtained from the Euclidean action Γ_E via the identification $\Gamma_M = -i\Gamma_E$, after having substituted in Γ_E the space-time metric $\delta_{\mu\nu}$ with $-\eta_{\mu\nu}$.

In this case, one obtains the following well known result for the amplitude [Gasser 1984]:

$$
\begin{aligned}
A(s,t,u) &= \frac{f^2}{4}s - \frac{\ell_1}{2}f^4(t^2+u^2) - \ell_2 f^4 s^2 + \frac{f^4}{512\pi^2}\left(\frac{10s^2+13(t^2+u^2)}{9}\right) \\
&- \frac{1}{3}\frac{f^4}{512\pi^2}\left(2s^2\log\frac{-s}{\mu^2} + t(t-u)\log\frac{-t}{\mu^2} + u(u-t)\log\frac{-u}{\mu^2}\right) . \qquad (4.113)
\end{aligned}
$$

Notice that the result can be expressed in a more familiar form after replacing $f \to 2/F_\pi$, where F_π is the pion decay constant.

4.4.4 Comments

The method used in this section for calculating the effective action involves the solution of a differential equation, rather than the computation of a functional integral. Evaluating the functional traces by means of the non-local heat kernel expansion allows to obtain an effective action which retains the full momentum structure that is useful to properly compute the scattering amplitude.

The issue of divergences presents itself in the course of the solution of the flow equation. In the present context, the integration of eq. (4.101) from $k = 0$ to $k = \Lambda$ produces terms that diverge quadratically and logarithmically as $\Lambda \to \infty$. Therefore, some renormalization conditions have been introduced to obtain a finite result. The computation of the amplitude turns out to be the same as in chiral perturbation theory, thus showing that this formalism is able to reproduce standard results. The structure of the amplitude is such that there is a violation of unitarity for $s \approx 8\pi F_\pi^2$, which is usually taken as the first sign of the breakdown of the theory.

Nonperturbative effects could actually heal the effective field theory of those problems. In particular, if the theory is asymptotically safe then all dimensionless parameters will have finite limits when $k \to \infty$ and one can then also take the energy scale Λ to infinity. The good behavior of the coupling as a function of some external unphysical cutoff k must reflects into a good behavior of the amplitude as a function of the external momenta. As emphasized in [Weinberg 1976, Weinberg 1979a], in discussing asymptotic safety, it would be best to define it directly in terms of observable quantities. Since it is much easier to define the running of the couplings , most of the work so far has concentrated on the computation of the beta function of the Lagrangian coefficients. One might expect that a fixed point for the couplings would translate into a fixed point for physical quantities such as cross sections and decay rates.

In the case of the nonlinear sigma model, the existence of such a fixed point (see Section 2.3.2) means that F_π behaves asymptotically like $k\tilde{F}_{\pi *}$, where $\tilde{F}_{\pi *}$ is a constant real number. A suggestive argument consists in identifying k^2 with s, then the tree level amplitude would stop growing and tend asymptotically to a constant $s/F_\pi^2 \to 1/F_{\pi *}^2$, which could satisfy unitarity conditions.

Here, however, the way the flow equation has been solved does not take into account the information about the UV finiteness of the Goldstone coupling f and the final result is the usual one of perturbation theory. It would be interesting to solve the flow equation nonperturbatively considering, on the right hand side of the beta functional equation, the full scale dependence of the coupling.

Chapter 5

Conclusions

The first part of this thesis has been devoted to the study of the renormalization group of the gauged nonlinear sigma model. The scale dependence of the theory has been computed by means of functional methods which involve the solution of an exact flow equation for the effective average action of the model. The Wetterich equation turned out to be a very useful tool and the obtained results have given some important indications about possible UV completions of the model. The most appealing scenario and the one considered in this thesis is provided by asymptotic safety, which relies on the existence of a nontrivial UV fixed point for the renormalization group. The presence of such a fixed point was already known in the nonlinear sigma model in $d = 2 + \varepsilon$ [Bardeen 1976] and, more recently, evidence about its existence in $d > 2$ has been found in [Codello 2009b]. Further indications about its existence have been obtained by study the model in which gauge fields are coupled to the Goldstone bosons. In the case of the $SU(N)$ gauged nonlinear sigma model, one obtains that the fixed point is preserved after introducing the gauge interactions. In $d = 4$, the asymptotic freedom of the gauge sector makes the fixed point for the Goldstone coupling to be the same as in the ungauged case.

While these results do not prove the existence of the fixed point, they are suggestive and have been taken to justify the phenomenological study presented in the rest of the thesis , in which the one loop results have been assumed to hold in the nonperturbative solution and the consequences of such an assumption have been worked out. The phenomenological application of the model regards the electroweak interactions. The $SU(2) \times U(1)$ gauged nonlinear sigma model provides the most general parametrization of the Higgs phenomenon in terms of a minimal number of degrees of freedom, the three would-be Goldstone bosons. The study of this model reveals that the $U(1)$ coupling does not admit a finite UV limit because its beta function is always positive and the theory is, strictly speaking, not safe. There are two main theoretical solutions of this problem. The first consist in embedding the model in a grand unified theory which is characterized by a single asymptotically free gauge coupling. The second possibility is to invoke some gravity mechanism to prevent the blowing up of the $U(1)$ at finite scale, as shown in [Harst 2011]. However, for phenomenological applications it is a good approximation to consider the gauge couplings fixed to their experimental values, since their running is very slow and the Landau pole for g' occurs at trans-Planckian energies.

The predictivity of the electroweak model has been tested by introducing higher order operators that contribute to the oblique S and T parameters. This model admits two physically viable UV fixed points. The first point is the most appealing one since it has only one relevant direction but the trajectory that flows to it in the UV leads to a prediction about S and T in the IR that is 5σ away from the experimental value. On the other hand, trajectories that hit the second fixed point can lead to values of the parameters that fall within the 2σ ellipse region. This shows that the model can be compatible with precision measurement.

The next step, which is crucial for building a realistic model, consist of coupling fermions to the Goldstone bosons by means of (proto)-Yukawa interactions. In the considered truncation, the model turned out not to admit any physically viable UV fixed point, therefore, the truncation has been extended by adding effective four fermion interactions. In this case, a set of physically acceptable fixed points are present. The study of the most predictive trajectory that hits one of the physically acceptable fixed points gives IR predictions about the four fermion couplings that are in agreement with experimental LHC bounds.

In summary, the study of the minimal electroweak model reveals that an asymptotically safe construction of the electroweak interactions may be possible. However, thanks to the subsequent discovery of an Higgs-like particle at the LHC, this nonlinear sigma model has to be considered too minimal and, to build a realistic model, one has to include the new scalar degree of freedom in the theory. The most general lagrangian that couples the Higgs-like field H to the Goldstone bosons is the following:

$$\mathcal{L}_{NLH} = \frac{\upsilon^2}{4}\mathrm{Tr}\left[D_\mu U D_\mu U^\dagger\right]\left(1 + 2a\frac{H}{\upsilon} + b\frac{H^2}{\upsilon^2} + \cdots\right), \tag{5.1}$$

where a and b are dimensionless couplings that in the SM are equal to one.

This model was introduced as low-energy effective theory describing a strongly interacting light Higgs, but in the spirit of asymptotic safety it can be considered as fundamental. Future investigations have to deal with this kind of models. Therefore, it is important to prove that the fixed point of the theory is preserved by the introduction of the H particle. If the fixed point is present, one expects that asymptotic safety can give predictions about the allowed values for the coefficient a and b of eq. (5.1) that can be tested at the LHC.

Another result of this thesis is the computation of the scattering amplitude for the pure Goldstone model. This is necessary in order to relate the running of the couplings to the UV behavior of physical observables. The results of perturbation theory have been reproduced by means of a functional renormalization group equation. However the computation presented here is only partial and it would be interesting to obtain a nonperturbative improvement of the computation taking into account the UV finiteness of the Goldstone coupling. The expectation is that a fixed point of the coupling should translate into a fixed point for physical quantities. This result could give some insights about the unitarity issue, which is relevant also in the context of quantum gravity where the structure of the tree amplitude is similar.

Appendix A

Functional methods for quantum field theory

This appendix reviews shortly the basic concepts of the functional formulation of quantum field theory. The space-time signature considered is Euclidean, so the framework is actually the one of statistical field theory. However, one assumes that all the computed quantities admit a direct interpretation in terms of Minkowskian field theory, this is achieved by means of Wick rotation to imaginary time. For a general introduction to functional methods in quantum field theory see [Weinberg 1995, Peskin 1995, Zinn-Justin 2002].

The basic object that enters in the quantum field theory formalism is the field $\phi(x)$, which is a map from a d dimensional space-time $\mathcal{M}$ to a Riemannian manifold $\mathcal{N}$. For simplicity, only a single real variable $\phi(x)$ is considered here. Given a classical or bare action $S[\phi]$, the fundamental object in the functional formalism of quantum field theory is the partition function Z. It is defined by:

$$Z = \int \mathcal{D}\phi \, e^{-\mathcal{S}[\phi]}, \tag{A.1}$$

it allows to calculate expectation value of any observable $\mathcal{O}[\phi]$:

$$\langle \mathcal{O}[\phi] \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, \mathcal{O}[\phi] \, e^{-\mathcal{S}[\phi]} \,. \tag{A.2}$$

The physical content of the theory is expressed in terms of n-point correlation or Green's functions, which are expectation values of the product of n fields at different points:

$$G(x_1, x_2, \dots, x_n) = \langle \phi(x_1) \, \phi(x_2) \dots \phi(x_n) \rangle \,. \tag{A.3}$$

To compute the correlation functions in a systematic way it is useful to introduce an auxiliary current $J(x)$ and to define the partition functional $Z[J(x)]$ as follows:

$$Z[J] = \int \mathcal{D}\phi \, e^{-\mathcal{S}[\phi] + \int d^d x \, \phi(x) J(x)} \tag{A.4}$$

In this way it is possible to write the correlation functions as:

$$\langle\phi(x_1)\,\phi(x_2)\ldots\phi(x_n)\rangle = \frac{1}{Z[J]}\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\Bigg|_{J=0} \tag{A.5}$$

Eq. (A.5) says that the partition functional in eq. (A.4) is the generating functional of correlation functions. Usually it is more useful to consider connected correlation functions, these are generated by the following functional:

$$W[J] = \log Z[J]\,. \tag{A.6}$$

Connected correlators are obtained by computing functional derivatives of eq. (A.6). A first derivative gives:

$$\frac{\delta W[J]}{\delta J(x)} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \langle\phi(x)\rangle_J \tag{A.7}$$

which is the vacuum expectation value of the field in presence of the current $J(x)$. Taking a second derivative one gets:

$$\begin{aligned}\frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} &= \frac{1}{Z[J]}\frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)} - \frac{1}{Z[J]^2}\frac{\delta Z[J]}{\delta J(x_1)}\frac{\delta Z[J]}{\delta J(x_2)}\\ &= \langle\phi(x_1)\phi(x_2)\rangle_J - \langle\phi(x_1)\rangle_J\langle\phi(x_2)\rangle_J\,,\end{aligned} \tag{A.8}$$

this corresponds to the connected two-point correlation function, i.e. the propagator. Eqs. (A.7-A.8) show that $W[J]$ is the generating functional of connected correlation or Green's functions, in general one has that the n-point connected correlation function is given by:

$$G_C(x_1, x_2\ldots, x_n) = \langle\phi(x_1)\phi(x_2)\ldots\phi(x_n)\rangle_C = \frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\cdots\frac{\delta}{\delta J(x_n)}W[J]\Bigg|_{J=0}\,. \tag{A.9}$$

In quantum field theory, the natural variable to use is $\varphi(x)$, the vacuum expectation value of the field $\phi(x)$ given in eq. (A.7):

$$\varphi(x) = \langle\phi(x)\rangle_J = \frac{\delta W[J]}{\delta J(x)}\,. \tag{A.10}$$

Here the mean value depends on the source, $\varphi(x) = \varphi_J$. To construct a functional of φ one has to solve eq. (A.10) to obtain $J(x) = J_\varphi$ and take the Legendre transform of the functional $W[J]$. The resulting functional $\Gamma[\varphi]$ is called quantum effective action or just effective action:

$$\Gamma[\varphi] = -W[J] + \int d^d x\varphi(x)J(x)\,. \tag{A.11}$$

Differentiating eq. (A.11) with respect to φ gives:

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = J(x)\,. \tag{A.12}$$

If one sets $J = 0$ in eq. (A.12), one obtains the quantum generalization of the principle of least action $\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)} = 0$. Differentiating eq. (A.10) and eq. (A.12) one gets:

$$G_C(x_1, x_2) = \frac{\delta^2 W[J]}{\delta J(x_2)\delta J(x_1)} = \frac{\delta\varphi(x_1)}{\delta J(x_2)} \qquad \frac{\delta^2\Gamma[\varphi]}{\delta\varphi(x_1)\delta\varphi(x_2)} = \frac{\delta J(x_2)}{\delta\varphi(x_1)} \tag{A.13}$$

this shows that the propagator is the inverse of the Hessian of $\Gamma_k[\varphi]$:

$$\int dx\, G_C(x_1, x)\frac{\delta^2\Gamma[\varphi]}{\delta\varphi(x)\delta\varphi(x_2)} = \delta(x_1 - x_2)\,. \tag{A.14}$$

Using eqs. (A.12), (A.11) and (A.6) in eq. (A.4) it is possible to obtain the integral representation for the effective action:

$$e^{-\Gamma[\varphi]} = \int \mathcal{D}\phi\, e^{-\mathcal{S}[\phi]+\int \frac{\delta\Gamma}{\delta\varphi}(\phi-\varphi)} \tag{A.15}$$

Shifting to the integration variable $\chi = \phi - \varphi$ inside the functional integral in eq. (A.15) leads to:

$$e^{-\Gamma[\varphi]} = \int \mathcal{D}\chi\, e^{-\mathcal{S}[\chi+\varphi]+\int \frac{\delta\Gamma}{\delta\varphi}\chi} \tag{A.16}$$

together with the condition $\langle\chi\rangle = 0$. The solution φ_* of the equation $\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}|_{\varphi_*} = 0$ is the quantum vacuum expectation value of the field, if one inserts it in the integral representation of the effective action in eq. (A.16), it is possible to obtain the following relation for the on-shell effective action:

$$e^{-\Gamma[\varphi_*]} = \int \mathcal{D}\chi\, e^{-\mathcal{S}[\chi+\varphi_*]}\,. \tag{A.17}$$

The relation between the on-shell effective action in eq. (A.17) and the zero-source partition function in eq. (A.1) can be rewritten as follows:

$$\Gamma[\varphi_*] = -\log Z\,. \tag{A.18}$$

this shows how the effective action formalism can be used to calculate the zero-source partition function.

Appendix B

Heat kernel techniques

The aim of this appendix is to present the heat kernel techniques used for the computation of the functional traces. For a review of the more mathematical and geometrical aspects of the heat kernel see [Rosenberg 1997], while for a physicist perspective see [Vassilevich 2003].

Consider a generalized covariant Laplace operator Δ defined by:

$$\Delta = -g_{\mu\nu} D^\mu D^\nu + U = -D^2 + U\,, \tag{B.1}$$

where $g_{\mu\nu}$ is the space-time metric, D_μ is a covariant derivative containing, in general, both the Levi-Civita and the gauge connection and U is a generic endomorphism. Using the Laplace transform formula, the trace of any function $W(\Delta)$ can be rewritten as

$$\mathrm{Tr}\, W(\Delta) = \int_0^\infty ds \tilde{W}(s) \mathrm{Tr}\, e^{-s\Delta} \tag{B.2}$$

where $\tilde{W}(s)$ is the inverse Laplace transform of $W(\Delta)$. The formula in eq. (B.2) shows that in order to compute such a functional trace one just need to know to the desired accuracy the trace of the heat kernel $K(s)$:

$$\mathrm{Tr}\, K(s) = \mathrm{Tr}\, e^{-s\Delta}\,. \tag{B.3}$$

There exist two possible expansions for this trace: the local and the non-local expansion.

B.1 Local heat kernel expansion

The standard asymptotic series expansion for the trace of the heat kernel is given in terms of local curvature polynomials [Vassilevich 2003] reads:

$$\mathrm{Tr}\, K(s) = \frac{1}{(4\pi s)^{d/2}} \sum_{n=0}^{\infty} B_{2n}(\Delta) s^n \tag{B.4}$$

where $B_{2n}(\Delta)$ are the integrated heat kernel coefficients that are related to the unintegrated ones $b_{2n}(\Delta)$ by the following relation:

$$B_{2n}(\Delta) = \int d^d x \sqrt{g}\, \mathrm{tr}\, \mathrm{b_{2n}}(\Delta)\,. \tag{B.5}$$

Note that in eq. (B.4) there is an explicit dependence on the space-time dimension in the prefactor $1/(4\pi s)^{d/2}$, additional dependence on d is generated by the trace operation in eq. (B.5). The quantities $b_{2n}(\Delta)$ have been calculated using various techniques, the first three coefficients are:

$$\begin{aligned} b_0(\Delta) &= \mathbf{1} \\ b_2(\Delta) &= \mathbf{1}\frac{R}{6} - U \\ b_4(\Delta) &= \frac{1}{2}U^2 + \frac{1}{6}D^2U + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu} - \frac{R}{6}U \\ &+ \frac{1}{180}R^2_{\mu\nu\alpha\beta} - \frac{1}{180}R^2_{\mu\nu} + \frac{1}{72}R^2 - \frac{1}{30}D^2R\,, \end{aligned} \tag{B.6}$$

where the space-time curvatures are constructed using the Levi-Civita connection and $\Omega_{\mu\nu}$ is the gauge field strength tensor.

Plugging eq. (B.4) into eq. (B.2) one can express the trace of the heat kernel as a series of the B_{2n} coefficients:

$$\mathrm{Tr}\, W(\Delta) = \frac{1}{(4\pi)^{d/2}}\sum_{n=0}^{\infty}\int_0^{\infty} ds\, s^{n-\frac{d}{2}}\tilde{W}(s)B_{2n}(\Delta) \equiv \frac{1}{(4\pi)^{d/2}}\sum_{n=0}^{\infty} Q_{\frac{d}{2}-n}(W)B_{2n}(\Delta) \tag{B.7}$$

where the 'Q-functionals' or 'threshold functions' are given by:

$$Q_n(W) = \int_0^{\infty} ds\, s^{-n}\tilde{W}(s) = \frac{1}{\Gamma(n)}\int_0^{\infty} dz\, z^{n-1}W(z) \qquad , \qquad n > 0\,. \tag{B.8}$$

B.2 Non-local heat kernel expansion

There exist a more sophisticated version for the heat kernel expansion that retains an an infinite series of terms in a form of non-local structure functions or form factors. This expansion has been developed in [Barvinsky 1987, Barvinsky 1990] and reads as follows:

$$\begin{aligned} \mathrm{Tr}\, e^{-s\Delta} &= \frac{1}{(4\pi s)^{d/2}}\int d^dx\,\sqrt{g}\,\mathrm{tr}\Big\{1 - sU + s\mathbf{1}\frac{R}{6} + s^2\Big[\mathbf{1}R_{\mu\nu}f_{Ric}(s\Box)R^{\mu\nu} + \mathbf{1}Rf_R(s\Box)R \\ &+ Rf_{RU}(s\Box)U + Uf_U(s\Box)U + \Omega_{\mu\nu}f_\Omega(s\Box)\Omega^{\mu\nu}\Big] + \mathcal{O}(\mathcal{R}^3)\cdots\Big\} \end{aligned} \tag{B.9}$$

where $\mathcal{R}$ can be the endomorphism or any of the curvatures and $\Box = -D^2$. The heat kernel structure functions in eq. (B.9) are given by:

$$\begin{aligned}
f_{Ric}(x) &= \frac{1}{6x} + \frac{1}{x^2}[f(x) - 1] \\
f_R(x) &= \frac{1}{32}f(x) + \frac{1}{8x}f(x) - \frac{7}{48x} - \frac{1}{8x^2}[f(x) - 1] \\
f_{RU}(x) &= -\frac{1}{4}f(x) - \frac{1}{2x}[f(x) - 1] \\
f_U(x) &= \frac{1}{2}f(x) \\
f_\Omega(x) &= -\frac{1}{2x}[f(x) - 1] ,
\end{aligned} \tag{B.10}$$

where the basic heat kernel structure function $f(x)$ is defined in terms of the following parameter integral:

$$f(x) = \int_0^1 d\xi e^{-x\xi(1-\xi)} . \tag{B.11}$$

Using in eq. (B.10) the Taylor expansion of the basic structure function

$$f(x) = 1 - \frac{x}{6} + \frac{x^2}{60} + \mathcal{O}(x^3) , \tag{B.12}$$

gives the 'short time' expansion for the structure functions:

$$\begin{aligned}
f_{Ric}(x) &= \frac{1}{60} - \frac{x}{840} + \frac{x^2}{15120} + \mathcal{O}(x^3) \\
f_R(x) &= \frac{1}{120} - \frac{x}{336} + \frac{11x^2}{30240} + \mathcal{O}(x^3) \\
f_{RU}(x) &= -\frac{1}{6} + \frac{x}{30} - \frac{x^2}{280} + \mathcal{O}(x^3) \\
f_U(x) &= \frac{1}{2} - \frac{x}{12} + \frac{x^2}{120} + \mathcal{O}(x^3) \\
f_\Omega(x) &= \frac{1}{12} - \frac{x}{120} + \frac{x^2}{1680} + \mathcal{O}(x^3) .
\end{aligned} \tag{B.13}$$

B.3 Functional traces

In this section, the functional traces of Chapter 3 are explicitly computed using the local heat kernel expansion of eq. (B.7). The arguments of the traces are functions of covariant laplacians of the following form:

$$W(z) = \frac{\partial_t R_k(z) + \eta R_k(z)}{(P_k(z) + m^2)^l} , \tag{B.14}$$

where $P_k(z) = z + R_k(z)$ and the cutoff shape function used is the optimized one:

$$R_k(z) = (k^2 - z)\theta(k^2 - z)\,. \tag{B.15}$$

In this case, the Q-functionals of eq. (B.8) are given by

$$Q_n(W) = \frac{2(n+1)+\eta}{\Gamma(n+2)}\frac{k^{2(n+1)}}{(k^2+m^2)^l}\,. \tag{B.16}$$

Consider the trace in eq. (3.51):

$$\frac{1}{2}\mathrm{Tr}_{\,\theta}\left[\Pi_k^{-1}\partial_t\mathcal{R}_k\right] = \frac{1}{2}\mathrm{Tr}_{\,\xi}\left[\frac{\partial_t R_k(z) + \eta_\xi R_k(z)}{P_k(z)+g^2/f^2}\right] + \frac{1}{2}\mathrm{Tr}_{\,a}\left[\frac{\partial_t R_k(w) + \eta_a R_k(w)}{P_k(w)+g^2/f^2}\right]\,, \tag{B.17}$$

where $z = -D_\xi^2$, $w = -D_a^2$ and the covariant derivatives D_ξ and D_a are defined in eq. (3.19) and eq. (2.37) respectively. The relevant contributions to these traces come from the B_4 coefficients of the heat kernel expansion. For the first trace one gets:

$$\frac{1}{2}\mathrm{Tr}_{\,\xi}\left[\frac{\partial_t R_k + \eta_\xi R_k}{P_k+g^2/f^2}\right] \supset \frac{1}{2}\frac{1}{(4\pi)^{d/2}}Q_{\frac{d}{2}-2}\left(\frac{\partial_t R_k + \eta_\xi R_k}{P_k+g^2/f^2}\right)B_4(z)\,, \tag{B.18}$$

where

$$B_4(z) = \frac{1}{12}\int d^dx\,\mathrm{tr}\left[\Omega^\xi_{\mu\nu}\Omega_\xi^{\mu\nu}\right]. \tag{B.19}$$

The quantity $\Omega^\xi_{\mu\nu}$ is the commutator of the Goldstone covariant derivative:

$$[D_\mu, D_\nu]\xi^\alpha = \Omega_{\mu\nu}{}^\alpha{}_\beta\xi^\beta \qquad , \qquad \Omega^\xi_{\mu\nu}{}^\alpha{}_\beta = D_\mu\varphi^\gamma D_\nu\varphi^\delta R_{\gamma\delta}{}^\alpha{}_\beta + F^i_{\mu\nu}\nabla_\beta K^\alpha_i\,. \tag{B.20}$$

For the second trace one gets:

$$\frac{1}{2}\mathrm{Tr}_{\,a}\left[\frac{\partial_t R_k + \eta_a R_k}{P_k+g^2/f^2}\right] \supset \frac{1}{2}\frac{1}{(4\pi)^{d/2}}Q_{\frac{d}{2}-2}\left(\frac{\partial_t R_k + \eta_a R_k}{P_k+g^2/f^2}\right)B_4(w) \tag{B.21}$$

where

$$B_4(w) = \frac{1}{12}\int d^dx\,\mathrm{tr}\left[\Omega^a_{\mu\nu}\Omega_a^{\mu\nu}\right]\,. \tag{B.22}$$

The quantity $\Omega^a_{\mu\nu}$ is the commutator of the gauge covariant derivative:

$$[D_\mu, D_\nu]a^i_\sigma = \Omega^a_{\mu\nu}{}^{i\rho}{}_{k\sigma}a^k_\rho \qquad , \qquad \Omega^a_{\mu\nu}{}^{i\rho}{}_{k\sigma} = \delta^\rho_\sigma f^i{}_{jk}F^j_{\mu\nu}\,. \tag{B.23}$$

The Q-functionals are then evaluated using the formula in eq. (B.16).

Consider the trace in eq. (3.52), the relevant contribution comes from the B_0 coefficient of the heat kernel expansion:

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}_{\,\theta}\left[\Pi_k^{-1}E\,\Pi_k^{-1}\partial_t\mathcal{R}_k\right] &= \frac{1}{2}\mathrm{Tr}_{\,\xi}\left[\frac{\partial_t R_k(-D_\xi^2)+\eta_\xi R_k(-D_\xi^2)}{(P_k(-D_\xi^2)+g^2/f^2)^2}M\right] \\
&\supset \frac{1}{2}\frac{1}{(4\pi)^{d/2}}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k+\eta_\xi R_k}{(P_k+g^2/f^2)^2}\right)\int d^dx\,\mathrm{tr}M\,, \qquad \text{(B.24)}
\end{aligned}$$

where

$$\mathrm{tr}M = R_{\alpha\beta}D^\mu\varphi^\alpha D_\mu\varphi^\beta\,. \qquad \text{(B.25)}$$

The Q-functional is given by the formula in eq. (B.16) and the following $SU(N)$ relation is used:

$$R_{\alpha\beta} = \frac{N}{4}h_{\alpha\beta}\,. \qquad \text{(B.26)}$$

Consider the trace in eq. (3.53):

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}\,\left[\Pi_k^{-1}E\Pi_k^{-1}E\Pi_k^{-1}\partial_t\mathcal{R}_k\right] &= \frac{1}{2}\mathrm{Tr}_{\,\xi}\left[\frac{g^2}{f^2}\frac{\partial_t R_k+\eta_\xi R_k}{(P_k+g^2/f^2)^3}BB^T\right] \\
&+ \frac{1}{2}\mathrm{Tr}_{\,a}\left[\frac{g^2}{f^2}\frac{\partial_t R_k+\eta_a R_k}{(P_k+g^2/f^2)^3}B^TB\right] \\
&+ \frac{1}{2}\mathrm{Tr}_{\,a}\left[\frac{\partial_t R_k+\eta_a R_k}{(P_k+g^2/f^2)^3}4F^2\right]\,. \qquad \text{(B.27)}
\end{aligned}$$

The relevant contribution comes from the B_0 coefficient of the heat kernel expansion. For the first two traces one gets:

$$\frac{1}{2}\mathrm{Tr}_{\,\xi}\left[\frac{g^2}{f^2}\frac{\partial_t R_k+\eta_\xi R_k}{(P_k+g^2/f^2)^3}BB^T\right] \supset \frac{1}{2}\frac{1}{(4\pi)^{d/2}}\frac{g^2}{f^2}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k+\eta_\xi R_k}{(P_k+g^2/f^2)^3}\right)\int d^dx\,\mathrm{tr}BB^T \qquad \text{(B.28)}$$

$$\frac{1}{2}\mathrm{Tr}_{\,a}\left[\frac{g^2}{f^2}\frac{\partial_t R_k+\eta_a R_k}{(P_k+g^2/f^2)^3}BB^T\right] \supset \frac{1}{2}\frac{1}{(4\pi)^{d/2}}\frac{g^2}{f^2}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k+\eta_a R_k}{(P_k+g^2/f^2)^3}\right)\int d^dx\,\mathrm{tr}B^TB \qquad \text{(B.29)}$$

where

$$\mathrm{tr}BB^T = \mathrm{tr}B^TB = Nh_{\alpha\beta}D^\mu\varphi^\alpha D_\mu\varphi^\beta\,. \qquad \text{(B.30)}$$

For the third trace one gets:

$$\frac{1}{2}\mathrm{Tr}_{\,a}\left[\frac{\partial_t R_k+\eta_a R_k}{(P_k+g^2/f^2)^3}4F^2\right] \supset \frac{1}{2}\frac{1}{(4\pi)^{d/2}}Q_{\frac{d}{2}}\left(\frac{\partial_t R_k+\eta_a R_k}{(P_k+g^2/f^2)^3}\right)4\int d^dx\,\mathrm{tr}F^2 \qquad \text{(B.31)}$$

where

$$\mathrm{tr}F^2 = NF^i_{\mu\nu}F_i^{\mu\nu}\,. \qquad \text{(B.32)}$$

Consider the ghost trace in eq. (3.54), the relevant contribution comes from the B_4 coefficient of the heat kernel expansion:

$$\mathrm{Tr}_c\left[\frac{\partial_t R_k(y)}{P_k(y)+g^2/f^2}\right] \supset \frac{1}{(4\pi)^{d/2}} Q_{\frac{d}{2}-2}\left(\frac{\partial_t R_k}{P_k+g^2/f^2}\right) B_4(y)\,, \tag{B.33}$$

where $y=-D_c^2$ and

$$B_4(y)=\frac{1}{12}\int d^dx\,\mathrm{tr}\left[\Omega^c_{\mu\nu}\Omega_c^{\mu\nu}\right]\,. \tag{B.34}$$

The quantity $\Omega^c_{\mu\nu}$ is the commutator of the ghost covariant derivative

$$[D_\mu, D_\nu]c^i = \Omega^c_{\mu\nu}{}^i{}_j c^j \qquad , \qquad \Omega^c_{\mu\nu}{}^i{}_j = f^i{}_{kj} F^k_{\mu\nu}\,. \tag{B.35}$$

Appendix C

Electroweak chiral lagrangian

Following the notation of [Feruglio 1993, Herrero 1994], the basic building blocks that are used in the construction of the complete $SU(2)_L \times U(1)_R$ invariant electroweak chiral lagrangian are:

$$\begin{aligned} T &= U\sigma_3 U^\dagger \quad , \quad V_\mu = (D_\mu U)U^\dagger\,, \\ D_\mu U &= \partial_\mu U - g\hat{W}_\mu U + g' U \hat{B}_\mu\,, \\ \hat{W}_{\mu\nu} &= \partial_\mu \hat{W}_\nu - \partial_\nu \hat{W}_\mu - g[\hat{W}_\mu, \hat{W}_\nu]\,, \\ \hat{B}_{\mu\nu} &= \partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu \end{aligned} \tag{C.1}$$

where the Goldstone bosons π^a $(a = 1,2,3)$ and the gauge fields are parametrized as

$$\begin{aligned} U &= \exp\left(i\pi^a\sigma_a/\upsilon\right) \quad , \quad \upsilon = 246\,\mathrm{GeV}\,, \\ \hat{W}_\mu &= \frac{-i}{2} W^a_\mu \sigma_a \quad , \quad \hat{B}_\mu = \frac{-i}{2} B_\mu \sigma_3\,. \end{aligned} \tag{C.2}$$

The lowest order terms in the derivative expansion are:

$$\mathcal{L}_{GNL} = \frac{\upsilon^2}{4}\mathrm{Tr}\,[D_\mu U D^\mu U^\dagger] + \mathcal{L}_G \tag{C.3}$$

where $\mathcal{L}_G$ is the kinetic lagrangian for the gauge fields

$$\mathcal{L}_G = \frac{1}{2}\mathrm{Tr}\,[\hat{W}^a_{\mu\nu}\hat{W}^{\mu\nu}_a] + \frac{1}{2}\mathrm{Tr}\,[\hat{B}_{\mu\nu}\hat{B}^{\mu\nu}]\,. \tag{C.4}$$

The following relation holds

$$U^\dagger D_\mu U = -i\frac{\sigma_a}{2} D_\mu\varphi^\alpha L^a_\alpha\,, \tag{C.5}$$

where L^a_α are the left invariant field of frames of eq. (3.9). Using eq. (C.5) it is possible to rewrite the nonlinear sigma model lagrangian as follows:

$$\frac{\upsilon^2}{4}\mathrm{Tr}\,[D_\mu U D^\mu U^\dagger] = \frac{1}{2f^2} h_{\alpha\beta} D_\mu\varphi^\alpha D^\mu\varphi^\beta\,, \tag{C.6}$$

where $f = 2/\upsilon$.

The complete $SU(2)_L \times U(1)_R$ invariant electroweak chiral lagrangian containing the whole set of CP-invariant operators up to dimension four is:

$$\mathcal{L}_{EW\chi} = \mathcal{L}_{GNL} + \sum_i a_i \mathcal{L}_i \tag{C.7}$$

where $\mathcal{L}_{GNL}$ is given in eq. (C.3) and the new operators are :

$$\begin{aligned}
\sum_i a_i \mathcal{L}_i &= a_0 \frac{v^2}{4} [\mathrm{Tr}\,(TV_\mu)]^2 + a_1 \frac{igg'}{2} B_{\mu\nu} \mathrm{Tr}\,(T\hat{W}^{\mu\nu}) \\
&+ a_2 \frac{ig'}{2} B_{\mu\nu} \mathrm{Tr}\,(T\,[V^\mu, V^\nu]) + a_3 g \mathrm{Tr}\,(\hat{W}_{\mu\nu}[V^\mu, V^\nu]) \\
&+ a_4 [\mathrm{Tr}\,(V_\mu V_\nu)]^2 + a_5 [\mathrm{Tr}\,(V_\mu V^\mu)]^2 + a_6 \mathrm{Tr}\,(V_\mu V_\nu) \mathrm{Tr}\,(TV^\mu) \mathrm{Tr}\,(TV^\nu) \\
&+ a_7 \mathrm{Tr}\,(V_\mu V^\mu) Tr(TV_\nu) Tr(TV^\nu) + a_8 \frac{g^2}{4} [\mathrm{Tr}\,(T\hat{W}_{\mu\nu})]^2 \\
&+ a_9 \frac{g}{2} \mathrm{Tr}\,(T\hat{W}_{\mu\nu}) \mathrm{Tr}\,(T[V^\mu, V^\nu]) + a_{10} [\mathrm{Tr}\,(TV_\mu) \mathrm{Tr}\,(TV_\nu)]^2 \\
&+ a_{11} \mathrm{Tr}\,(D_\mu V^\mu)^2 + a_{12} \mathrm{Tr}\,(TD_\mu D_\nu V^\nu) \mathrm{Tr}\,(TV^\mu) + a_{13} \frac{1}{2} [\mathrm{Tr}\,(TD_\mu V_\nu)]^2 \,.
\end{aligned} \tag{C.8}$$

It is possible to translate the first two terms of eq. (C.8) using general coordinates φ^α as follows:

$$a_0 \frac{v^2}{4} [\mathrm{Tr}\,(TV_\mu)]^2 = -\frac{a_0}{f^2} D_\mu \varphi^\alpha D^\mu \varphi^\beta L^3_\alpha L^3_\beta \tag{C.9}$$

and

$$a_1 \frac{igg'}{2} B_{\mu\nu} \mathrm{Tr}\,(T\hat{W}^{\mu\nu}) = a_1 \frac{gg'}{2} B^{\mu\nu} W^i_{\mu\nu} R^\alpha_i L^3_\alpha \,. \tag{C.10}$$

In eq. (C.9) and eq. (C.10) the following relations have been used:

$$\mathrm{Tr}\,(\sigma_3 U^\dagger D_\mu U) = -i D_\mu \varphi^\alpha L^3_\alpha \tag{C.11}$$

and

$$\mathrm{Tr}\,(U\sigma_i U^\dagger \sigma_j) = 2Ad(\varphi)_{ji} = R^\alpha_j L_{\alpha i} \,. \tag{C.12}$$

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