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SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

HOMOLOGY INVARIANTS OF QUADRATIC MAPS

Supervisor
Prof. ANDREI A. AGRACHEV

Candidate
ANTONIO LERARIO

Thesis submitted for the degree of
"Doctor Philosophiæ in Geometry"

Academic Year 2010/2011

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Introduction

Given a real projective algebraic set X we could hope that the equations describing it can give some information on its topology, e.g. on the number of its connected components. Unfortunately in the general case this hope is too vague and there is no direct way to extract such information from the algebraic description of X. Even the problem to decide whether X is empty or not is far from an easy visualization and requires some complicated algebraic machinery.
A first step observation is that as long as we are interested only in the topology of X, we can replace, using some Veronese embedding, the original ambient space with a much bigger $\mathbb{R}\mathrm{P}^n$ and assume that X is cut by *quadratic* equations. The price for this is the increase of the number of equations defining our set; the advantage is that quadratic polynomials are easier to handle and our hope becomes more concrete.
At this point, in a very naive way we can expect that a measurement of the complexity of X is given by the number $k+1$ of quadratic equations we need to cut it in $\mathbb{R}\mathrm{P}^n$. If we define $b(X)$ to be the sum of the Betti numbers[1] of X, the well known Oleinik-Petrovskii-Thom-Milnor inequality would give the following estimate[2]:

$$b(X) \leq O(2(k+1))^{n+1}.$$

This bound seems to contradict our guess: the complexity of the formula is the number of variables n which appears at the exponent.
Surprisingly enough it turns out that the fact that X is defined by quadratic equations allows to interchange the two numbers $n+1$ and $2(k+1)$ and to get the bound:

$$b(X) \leq O(n+1)^{2(k+1)} \tag{1}$$

where now the complexity is the number of quadrics, which appears at the exponent and confirms the genuinity of our naive idea.
The previous phenomenon suggests there is a kind of duality between the number of variables and the number of quadratic equations defining X and this duality appears in the formula (1) where the topology of X is involved. We will see that as we consider finer invariants of the family of quadrics cutting X, the information on its topology becomes richer and richer - the previous bound being given by considering only the *number* of equations.
To fix notations we assume little more generally that X is given by a system of homogeneous quadratic *inequalities*: we consider a polyhedral cone K in $\mathbb{R}^{k+1}$ and a quadratic map $q : \mathbb{R}^{n+1} \to \mathbb{R}^{k+1}$, i.e. a map whose components $q_0, \ldots, q_k$ are real quadratic forms, and we set

$$X = \{[x] \in \mathbb{R}\mathrm{P}^n \,|\, q(x) \in K\}.$$

Notice that the previous definition makes sense because $q(x) = q(-x)$, and the case X is algebraic is obtained by considering the zero cone; by a slight abuse of notations we will write $q^{-1}(K)$ for X.

[1] From now on every homology group is assumed with $\mathbb{Z}_2$ coefficients.

[2] The following bounds are not sharp; we worsened them in order to put in evidence some symmetry.

By composing q with a nonzero covector η in the target space we obtain a quadratic form $\overline{q}(\eta) = \eta q$ and as the covector varies we can reconstruct q itself. We can imagine that the map $\overline{q}$ places linearly the space $(\mathbb{R}^{k+1})^*$ into the the space $\mathcal{Q}(n+1)$ of quadratic forms on $\mathbb{R}^{n+1}$; in classical algebraic geometry the image of this map is called the linear system defined by the quadrics $q_0, \ldots, q_k$. In the case K is the zero cone, i.e. X is algebraic, the common zero locus set of the nonzero elements of the previous linear system is X itself. Alternatively, in the realm of semialgebraic geometry, we can rewrite this fact using inequalities:

$$q^{-1}(0) = \bigcap_{\eta \neq 0} \{\eta q \leq 0\}.$$

In a similar fashion, for a general cone K, is not difficult to show that:

$$q^{-1}(K) = \bigcap_{\eta \in K^\circ \setminus \{0\}} \{\eta q \leq 0\}. \tag{2}$$

The previous equation suggests that it is not the whole linear system the object we should be interested in, but only that part of it which keeps track of the cone K, namely its polar K°. It is natural at this point to consider for every nonzero covector η the simpler invariant we can associate to ηq, namely its positive inertia index. This number, which is usually denoted by $\mathrm{i}^+(\eta q)$, is the maximal dimension of a subspace on which ηq is positive definite; it is clearly invariant by positive multiplication of the form ηq. Thus we are irresistibly led to define the sets

$$\Omega = K^\circ \cap S^k \quad \text{and, for } j \in \mathbb{N}, \quad \Omega^j = \{\omega \in \Omega \,|\, \mathrm{i}^+(\omega q) \geq j\}$$

in the hope that some of their geometric features can give topological information on X.

It is clear that the knowldege of the index function does note give all the richness of the whole quadratic map q; nevertheless the following formula should convince the reader that we are going in the right direction. In fact, setting $\chi(Y)$ for the Euler characteristic of a semialgebraic set Y and $C\Omega$ for the topological space cone of Ω, we have:

$$\chi(X) = \sum_{j=0}^{n} (-1)^{n+j} \chi(C\Omega, \Omega^{j+1}). \tag{3}$$

Example (The bouquet of two cirlces). Consider the map $s : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$s(x) = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2)$$

and the zero cone in $\mathbb{R}^2$. Then $s^{-1}(0)$ is the subset of $\mathbb{RP}^3$ consisting of the rational normal curve and a projective line intersecting at one point; this set is homeomorphic to a bouquet of two circles. Associating to a quadratic form a symmetric matrix by means of a scalar product, the family ηs for $\eta \in \Omega = S^1$ is represented by the matrix:

$$\eta S = \begin{pmatrix} 0 & 0 & 2\eta_0 & \eta_1 \\ 0 & -\eta_0 & -\eta_1 & 0 \\ 2\eta_0 & -\eta_1 & 0 & 0 \\ \eta_1 & 0 & 0 & 0 \end{pmatrix} \qquad \eta = (\eta_0, \eta_1) \in S^1$$

The determinant of this matrix vanishes at the points $\omega = (1, 0)$ and $-\omega = (-1, 0)$; outside of these points the index function must be locally constant. Then it is easy to verify that i^+ equals 2 everywhere except at the only point ω (the positive inertia index of $-\omega s$ is still 2). In this case we have:

$$\Omega^1 = S^1, \quad \Omega^2 = S^1 \backslash \{\omega\}, \quad \Omega^3 = \Omega^4 = \emptyset.$$

Computing formula 3 for this example gives:

$$\chi(s^{-1}(0)) = -(1 - \chi(\Omega^1) - 1 + \chi(\Omega^2)) = \chi(S^1) - \chi(S^1 \backslash \{\omega\}) = -1.$$

This kind of reasoning culminates in the existence of a first quadrant spectral sequence (E_r, d_r) such that

$$E_2^{i,j} = H^i(C\Omega, \Omega^{j+1}) \quad \text{and} \quad E_\infty^* \simeq H_{n-*}(X) \tag{4}$$

Notice that Oleinik-Petrovskii-Thom-Milnor buond applied to the sets Ω^j implies the inequality (1) and the computation of the Euler characteristic of E_2, which equals the Euler characteristic of E_∞, gives equation (3).
This spectral sequence has many interesting properties; as an example consider the problem of computing the rank of the homomorphism induced on the homology by the inclusion ι of X in the ambient space $\mathbb{R}\mathrm{P}^n$. It is remarkable that this information is encoded in the first column of (E_r, d_r); in fact we have:

$$\mathrm{rk}(\iota_*)_k = \dim(E_\infty^{0,n-k}). \tag{5}$$

Example (The bouquet of two circles; continuation). The table of ranks of E_2 for the above example is the following:

$$\mathrm{rk}(E_2) = \begin{array}{|c|c|c} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline \end{array}$$

Since there are no nonzero differentials (by dimensional reasons) then $E_2 = E_\infty$. Thus (4) gives $b_0(s^{-1}(0)) = 1$ and $b_1(s^{-1}(0)) = 2$. On the other hand formula (5) gives $\mathrm{rk}(\iota_*)_1 = 1$, which is confirmed by the fact that $s^{-1}(0)$ contains a projective line.

In the general case the spectral sequence (4) degenerates after $k+2$ steps and the elements of E_2 are only the *candidates* for the homology classes of X. We can take them at a firts approximation, but not all of them are genuine homology classes in X and there are some criteria to decide whether they are or not - checking if they satisfy these criteria is the spirit of computing the differentials of the spectral sequence. In a very precise sense the differentials are the *obstructions* to extend such elements to global classes in $E_\infty = H_{n-*}(X)$.
To understand such obstructions let's do a step back and consider the structure of the set $\mathcal{Q}(n+1)$. Once we fix a scalar product, we can identify it with the space of

symmetric $(n+1)\times(n+1)$ real matrices; in these way we define the *eigenvalues* of a quadratic form p to be those of the corresponding symmetric matrix:

$$\lambda_1(p) \geq \cdots \geq \lambda_{\mathrm{i}^+(p)}(p) > 0 \geq \cdots \geq \lambda_{n+1}(p).$$

If $\mathcal{D}_k$ is the subset of $\mathcal{Q}$ defined by $\{\lambda_k \neq \lambda_{k+1}\}$, then its complement

$$\mathcal{S}_k = \mathcal{Q}\backslash\mathcal{D}_k$$

happens to be a closed pseudomanifold of codimension 2 in $\mathcal{Q}$ and for a film c with boundary in $\mathcal{D}_k$ the linking number of ∂c with $\mathcal{S}_k$ is defined. We denote by $\gamma_{1,k}$ the cohomology class in $H^2(\mathcal{Q},\mathcal{D}_k)$ representing this operation. Now consider the map

$$\overline{p}:\Omega\to\mathcal{Q}$$

defined by restricting the above correspondence $\eta\mapsto\eta q$ to Ω. In a certain sense the pullback $\overline{p}^*\gamma_{1,k}$ is exactly the first of these obstructions we were talking about. To be more precise: for $x\in H^i(C\Omega,\Omega^{j+1})$ the cup product with $\overline{p}^*\gamma_{1,j}$ defines, by restriction, an element in $H^{i+2}(C\Omega,\Omega^j)$ and the homomorphism:

$$x\mapsto (x\smile \overline{p}^*\gamma_{1,j})|_{(C\Omega,\Omega^j)},\quad x\in H^i(C\Omega,\Omega^{j+1}) \tag{6}$$

coincides with the second differential $d_2^{i,j}$ of the previous spectral sequence.

Example (The complex squaring). Consider the quadratic forms

$$q_0(x) = x_0^2 - x_1^2,\quad q_1(x) = 2x_0x_1.$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$ via $(x_0,x_1)\mapsto x_0+ix_1$, the map $q=(q_0,q_1)$ is the complex squaring $z\mapsto z^2$. We easily see that the common zero locus set of q_0 and q_1 in $\mathbb{R}\mathrm{P}^1$ is empty and thus the previous spectral sequence in this case must converge to zero. The image of the linear map $\overline{q}:\mathbb{R}^2\to\mathcal{Q}(2)$ consists of a plane intersecting the set of degenerate forms Z only at the origin; we identify $\mathcal{Q}(2)$ with the space of 2×2 real symmetric matrices. Thus $\overline{q}(S^1)$ is a circle looping around $Z=\{\det = 0\}$ and the index function is constant:

$$\mathrm{i}^+(\omega q) = 1,\quad \omega\in S^1.$$

Thus $\Omega^1 = S^1$ and the table for the ranks of E_2 has the following picture:

$$\mathrm{rk}(E_2) = \begin{array}{|c|c|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline \end{array}$$

The differential $d_2: E_2^{0,1} = H^0(C\Omega,\Omega^2)\to E_2^{2,0} = H^2(C\Omega,\Omega^1)$ must be nonzero, as we know that $E_3 = 0$. In this case the set $\mathcal{S}_1 = \{q\in\mathcal{Q}\,|\,\lambda_1(q)=\lambda_2(q)\}$ equals the set of scalar matrices; we see that $\overline{q}(S^1)$ is linked with this set and thus $\overline{q}^*\gamma_{1,1}\neq 0$.

Unfortunately this is not the end of the story: the description of the second differential is just the first step to the genuine homology E_∞, and higher differentials, as one could expect, seem to be harder to compute.

The set $\mathcal{D}_k$ above defined has the homotopy type of a Grassmannian $G_{k,n+1}$ and if we denote by $w_{1,k}$ the first Stiefel-Whitney class of its tautological bundle and by ∂^* the connecting homomorphism for the long exact sequence of the pair $(\mathcal{Q}, \mathcal{D}_k)$, then we have the following equality:

$$\gamma_{1,k} = \partial^* w_{1,k}.$$

The description of the obstructions as characteristic classes of some vector bundle is more customary in algebraic topology and gives some new perspectives for the computation of higher differentials. As an example in the case the index function is constant $\mathrm{i}^+ \equiv \mu$, there is only one obstruction, i.e. there is only one nonzero differential: it is the last one and equals the cup product with the pullback through $\overline{q}$ of the k-th Stiefel-Whitney class of the tautological bundle over $\mathcal{D}_\mu$.

Example (The Hopf fibration). Consider the quadratic map

$$h : \mathbb{R}^2 \oplus \mathbb{R}^2 \to \mathbb{R}^2 \oplus \mathbb{R}$$

defined, using the previous identification $\mathbb{R}^2 \simeq \mathbb{C}$, by

$$(z, w) \mapsto (2z\overline{w}, |w|^2 - |z|^2).$$

Then it is not difficult to prove that h maps S^3 into S^2 by a Hopf fibration. Hence it follows that

$$\emptyset = h^{-1}(0) \subset \mathbb{R}\mathrm{P}^3.$$

In this case we have $\mathrm{i}^+(\omega h) = 2$ for every $\omega \in \Omega = S^2$. The following table gives the rank for $E_2 = E_3$:

$$\mathrm{rk}(E_2) = \mathrm{rk}(E_3) = \begin{array}{|c|c|c|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \hline \end{array}$$

The class $\overline{h}^* \gamma_{2,2} \in H^2(S^2)$ happens to be nonzero (this fact is related with the fact that the Hopf invariant of h is odd) and thus the cup product with it, which is the differential d_3, is nonzero; the result is that $E_4 = 0$.

Despite these evidences, the general problem of computing higher differentials has not been solved yet.

We should however say that in many interesting cases these higher computations are not necessary: for example the datas for the intersection of two real quadrics in $\mathbb{R}\mathrm{P}^n$ are all encoded in (4) and formulas (6) and (5). This observation leads us to explore the beautiful combinatorics of the index function on a circle. The space of generic[3] linear systems of two quadrics has a kind of algebraic extra structure; this extra structure allows us to label each pencil with a binary array in such a way that performing some rules (i.e. admitted permutations) on its characters corresponds to make generic homotopies of pencils. The combinatorial nature of these ideas leads to a bound on each Betti number of the set of the solutions X of a system of two quadratic inequalities in $\mathbb{R}\mathrm{P}^n$:

$$b_k(X) \leq k + 2, \quad k \geq 0 \tag{7}$$

[3]Generic with respect to a certain nondegeneracy condition

The fact that the bound (7) holds for every pencil of quadrics is a consequence of a very peculiar fact in real algebraic geometry: equations can be made regular[4] using inequalities. To be more precise a single algebraic equation $f = 0$ is equivalent to the pair of inequalities $f \leq 0$ and $f \geq 0$; each of this inequalities can be slightly perturbed to inequalities $f \leq \epsilon$ and $f \geq -\epsilon$ in such a way that they are regular and the homotopy type of the set of the solutions has not changed. To our point of view this fact is the cornerstone of the 'computability' of the topology of semialgebraic sets and plays a central role in all the theory. The same observation in particular applies to complex algebraic set: up to homotopy the study of their topology, even their local topology, is reduced to that of solutions of regular systems of inequalities. This trick works particularly well for example in the case of the intersection of two complex quadrics: here each complex equation is viewed as a pair of real equations and regularization allows to efficiently apply all the previous theory (the fact that the new equations come from complex ones also plays a crucial role). The duality between the parameter space of the linear system and the topology of its base locus appears also in this context; consider, for example, $q_0, \ldots, q_k$ degree two homogeneous polynomials with *complex* coefficients and their zero locus set C in $\mathbb{C}\mathrm{P}^n$. It is natural to consider the following family of susbets of $\mathbb{C}\mathrm{P}^k$ (here $k+1$ is the number of polynomials):

$$Y^j = \{[\alpha_0, \ldots, \alpha_k] \in \mathbb{C}\mathrm{P}^k \,|\, \mathrm{rk}_{\mathbb{C}}(\alpha_0 q_0 + \cdots + \alpha_k q_k) \geq j\}, \quad j \in \mathbb{N}.$$

In terms of these sets, in the same spirit as for (1), we have:

$$b(C) \leq b(\mathbb{C}P^n) + \sum_{j \geq 0} b(Y^{j+1}). \tag{8}$$

The previous formula does not give improvement on the classical bounds on topological complexity (indeed these bounds can be proved using (8) in a similar way as it was done for (1) before). Nevertheless its structure (the set Y^j are the 'critical' points of the linear system) is reminiscent of Morse theory and offers some possible new perspectives.

[4]Any reasonable definiton of regularity works for this purpose

Ackwnoledgements

I am very grateful to my adivsor prof. A. A. Agrachev. He is a wonderful teacher and an inestimable source of mathematics. This work is the natural development of many of his ideas, and it is dedicated to him.

CHAPTER 1

Basic theory

1.1 Vector bundles and characteristic classes

1.1.1 Classification of vector bundles

The material of this section is covered in [19] and [18].
Let $G_{k,n}$ be the Grassmannian of k-dimensional subspaces of $\mathbb{R}^n$. We consider the *tautological bundle* $\tau_{k,n}$ over $G_{k,n}$: it is a vector bundle of real rank k whose fiber $(\tau_{k,n})_W$ over the point $W \in G_{k,n}$ is the vector space W itself and whose vector bundle structure is given by the inclusion $\tau_{k,n} \hookrightarrow G_{k,n} \times \mathbb{R}^n$; we denote by p the projection $\tau_{k,n} \to G_{k,n}$. The inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1} \subset \cdots$ give inclusions $G_{n,k} \subset G_{n,k+1} \subset \cdots$ and we let $G_k = \bigcup_n G_{k,n}$ endowed with the weak topology. The set G_k is called the Grassmann space. The inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1} \subset \cdots$ also give inclusions of vector bundles $\tau_{k,n} \subset \tau_{k,n+1} \subset \cdots$ and we let $\tau_k = \bigcup_n \tau_{k,n}$ endowed with the weak topology; the projection $p : \tau_k \to G_k$ gives a vector bundle structure.
Suppose now we are given a topological space X and a continuous map

$$f : X \to G_k$$

then f defines a vector bundle of rank k over X, which is called the *pull-back bundle* and denoted by $f^*\tau_k$. As a topological space $f^*\tau_k$ is defined by:

$$f^*\tau_k = \{(x, v) \in X \times \tau_k \mid f(x) = p(v)\}$$

and its vector bundle structure is given by the following procedure: if $\psi : p^{-1}(U) \to U \times \mathbb{R}^k$ is a trivialization of τ_k over an open set $U \subset G_k$, then the map $(x, v) \mapsto (x, p_2\psi(v))$, where $p_2 : U \times \mathbb{R}^k \to \mathbb{R}^k$ is the projection on the second factor, is a trivialization over the open set $f^{-1}(U)$. The following proposition is the key result in this context (see [18], Proposition 1.7). We denote by $\overline{f} : f^*(\xi) \to \tau_k$ the map lifting f.

Proposition 1.1.1. *If X is paracompact then the restrictions of a vector bundle $E \to X \times I$ over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.*

In particular applying this result to a homotopy $F : X \times I \to G_{k,n}$ between the maps $f_0 = F|_{X\times\{0\}}$ and $f_1 = F|_{X\times\{1\}}$ we get the following corollary

Corollary 1.1.2. *If X is paracompact and $f_0, f_1 : X \to G_k$ are homotopic maps, then*

$$f_0^*\tau_k \simeq f_1^*\tau_k.$$

Remarkably enough if X is paracompact every rank k vector bundle ξ over X arises in this way, i.e. there exists a continuous map $f : X \to G_k$ such that $\xi = f^*\tau_k$. Combined with the previous corollary, this is exactly the statement of the following Theorem, which gives the classification of rank k vector bundles over paracompact spaces. We use the notations $[X, Y]$ for the set of homotopy classes of maps $f : X \to Y$ and $\mathrm{Vect}^k(X)$ for the set of isomorphism classes of rank k vector bundles over X.

Theorem 1.1.3. *If X is paracompact, then the map $[X, G_k] \to Vect^k(X)$ given by $[f] \mapsto [f^*\tau_k]$ is a bijection.*

We conclude with an observation: for a rank k vector bundle ξ over X, the isomorphism $\xi \simeq f^*\tau_k$ is equivalent to a map

$$g : \xi \to \mathbb{R}^\infty$$

which is a linear injection on each fiber. To see this consider first the natural map $\pi : \tau_k \to \mathbb{R}^\infty$ which embeds each fiber in the ambient space $\mathbb{R}^\infty$; then the composition $g = \pi\overline{f} : f^*(\xi) \to \mathbb{R}^\infty$ is a linear injection on each fiber. Viceversa given a map $g : \xi \to \mathbb{R}^\infty$ which is a linear injection on each fiber, then the map $f : X \to G_k$ defined by $x \mapsto g(\xi_x)$ induces the bundle ξ.

1.1.2 Stiefel-Whitney classes

Consider a rank k vector bundle ξ over the paracompact space X and a map $g : \xi \to \mathbb{R}^\infty$ which is a linear injection on the fibers. If we let $P(\xi)$ be the projectivization of the bundle ξ (it is a fiber bundle over X with fiber $\mathbb{R}\mathrm{P}^{k-1}$) we see that g defines a map

$$P(g) : P(\xi) \to \mathbb{R}\mathrm{P}^\infty$$

which embeds each fiber linearly on a projective subspace. Let $x \in H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}_2)$ be the generator and

$$x^i = P(g)^* y^i \in H^i(P(\xi); \mathbb{Z}_2), \quad i = 0, \dots, k-1$$

Since any two linear injections $\mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^\infty$ are homotopic through linear injections, then the classes $y_i, i = 0, \dots, k-1$, are independent of the choice of g and their restriction to each fiber of $P(\xi)$ generate its cohomology. Thus by Leray-Hirsch theorem, writing $q(x)$ for a polynomial in x, the map $(\alpha, q(x)) \mapsto P(g)^*\alpha \cdot q(x)$ gives an isomorphism of $H^*(X; \mathbb{Z}_2)$-modules

$$H^*(X; \mathbb{Z}_2) \otimes \{1, \dots, x^{k-1}\} = H^*(P(\xi); \mathbb{Z}_2).$$

In particular there exist unique $w_i(\xi) \in H^i(X; \mathbb{Z}_2), i = 1, \dots, k-1$, such that

$$x^k + w_1(\xi) \cdot x^{k-1} + \cdots + w_k(\xi) \cdot 1 = 0$$

Setting $w_0(\xi) = 1$ and $w_j(\xi) = 0$ for $j > k$, the classes $w_0(\xi), w_1(\xi), \dots$ are called the *Stiefel-Whitney classes* of the bundle ξ. The Stiefel-Whitney classes of the tautological bundle τ_k are denoted simply by

$$w_i = w_i(\tau_k) \in H^i(G_k; \mathbb{Z}_2).$$

Consider now the space $(\mathbb{RP}^\infty)^k$ and the bundle $\eta = \tau_1 \oplus \cdots \oplus \tau_1$ over it, where each addendum comes from one copy of $\mathbb{RP}^\infty = G_1$. Since $(\mathbb{RP}^\infty)^k$ is paracompact, then by theorem 1.1.3 there exists a map $\psi : (\mathbb{RP}^\infty)^k \to G_k$ inducing η (indeed viewing $G_k = G_k((\mathbb{R}^\infty)^k)$ the map $(l_1, \ldots, l_k) \mapsto l_1 \times \cdots \times l_k$ is such a map). The induced map ψ^* on the cohomology turns out to be injective; on the other hand $\psi^* w_i$ is the i-th elementary symmetric polynomial in $x_1, \ldots, x_k$, where $x_i \in H^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$ is the generator. Since the elementary symmetric polynomials are algebraically independent this tells that the restriction of ψ^* to the algebra generated by the classes $w_1, \ldots, w_k \in H^*(G_k; \mathbb{Z}_2)$ is also injective. Thus we have the isomorphism of rings

$$H^*(G_k; \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_k]$$

Notice that if $g : \xi \to \mathbb{R}^\infty$ is a linear injection on each fiber and $\overline{h} : h^*\xi \to \xi$ is a bundle map lifting $h : Y \to X$, then $g\overline{h} : h^*\xi \to \mathbb{R}^\infty$ is a linear injection on each fiber and thus

$$h^*(w_i(\xi)) = w_i(h^*(\xi)) \in H^i(Y; \mathbb{Z}_2).$$

In particular given ξ a rank k vector bundle over X and a map $f : X \to G_k$ inducing ξ, then $w_i(\xi) = f^* w_i$. Thus in a certain sense the Stiefel-Whitney classes of a vector bundle ξ over X measure the failure of ξ to be trivial: if ξ is trivial, then it is induced by a constant map $f : X \to G_k$ and thus all its characteristic classes are zero; on the contrary since the Grassmann space is never simply connected, the triviality of f^* is weaker than f being homotopic to a constant map.

1.2 Semialgebraic Geometry

1.2.1 Semialgebraic sets and functions

The material of this section presented without proof is covered in [10].
The family of semialgebraic subsets of $\mathbb{R}^n$ is by definition the smallest family of subsets containing all the sets of the form

$$\{x \in \mathbb{R}^n \mid f(x) > 0\}$$

with $f \in \mathbb{R}[x_1, \ldots, x_n]$, and closed under taking finite intersections, finite unions and complements. Clearly an algebraic set is semialgebraic. Semialgebraic subset of $\mathbb{R}$ are characterized: they are exactly the finite unions of points and of open intervals. If $f : A \to B$ is a map between semialgebraic sets, then f is said to be semialebraic if its graph $\Gamma(f) \subset A \times B$ is semialgebraic.
The most important property of semialgebraic *functions* is that thay can be triangulated, as stated in the following theorem.

Theorem 1.2.1 (Triangulation of semialgebraic functions)**.** *Let S be a closed and bounded semialgebraic subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}$ a continuous semialgebraic function. There exist a finite simplicial complex K in $\mathbb{R}^n$ and a semialgebraic homeomorphism $\Phi : |K| \to S$ such that $f\Phi : |K| \to \mathbb{R}$ is affine on every simplex of K. Moreover, given a finite collection $S_i, \ldots, S_p$ of semialgebraic subsets of S, we can choose K and Φ such that each S_i is union of images by Φ of open simplices of K.*

If S is a semialgebraic set and $B \subset S$ is a compact semialgebraic set then, following [13], we say that $f : S \to [0, \infty)$ is a *rug function* for B in S if f is proper, continuous, semialgebraic and $f^{-1}(0) = B$. The following proposition can be found in [9] (pag. 229, Proposition 9.4.4).

Proposition 1.2.2. *Let $B \subset S$ be compact semialgebraic sets and f be a rug function for B in S. Then there are $\delta > 0$ and a continuous semialgebraic mapping $h : f^{-1}(\delta) \times [0, \delta] \to f^{-1}([0, \delta])$, such that $f(h(x, t)) = t$ for every $(x, t) \in f^{-1}(\delta) \times [0, \delta]$, $h(x, \delta) = x$ for every $x \in f^{-1}(\delta)$, and $h_{|f^{-1}(\delta)\times]0,\delta]}$ is a homeomorphism onto $f^{-1}(]0, \delta])$.*

Proof. By triangulating f we obtain a finite simplicial complex K and a semialgebraic homeomorphism $\phi : |K| \to S$, such that $f \circ \phi$ is affine on every simplex of K and B is union of images of simplices of K. Choose δ so small that for every vertex a of K such that $\phi(a) \notin B$, then $\delta < f(\phi(a))$. Let $x \in f^{-1}(\delta)$, $y = \phi^{-1}(x)$. The point y belongs to a simplex $\sigma = [a_0, \dots, a_d]$ of K. We may assume that $\phi(a_i) \in B$ for $i = 0, \dots, k$, and $\phi(a_i) \notin B$ for $i = k + 1, \dots, d$. Let $(\lambda_0, \dots, \lambda_d)$ be the barycentric coordinates of y in σ. Note that since $f \circ \phi$ is affine on σ, then $\delta = f(x) = f(\phi(y)) = \sum_{i=0}^{d} \lambda_i f(\phi(a_i)) = \sum_{i=k+1}^{d} \lambda_i f(\phi(a_i))$. Hence, if we set $\alpha = \sum_{i=0}^{k} \lambda_i$, we have necessarily $0 < \alpha < 1$. For $t \in [0, \delta]$, we define $h(x, t)$ as the image by ϕ of the point of σ with barycentric coordinates $(\mu_0, \dots, \mu_d)$, where

$$\mu_i = \begin{cases} \frac{t\alpha+\delta-t}{\delta\alpha}\lambda_i & \text{for} \quad i = 0, \dots, k; \\ \frac{t}{\delta}\lambda_i & \text{for} \quad i = k + 1, \dots, d. \end{cases}$$

Then h has the required properties. □

Now we prove a result which describes the structure of some semialgebraic neighborhoods of a semialgebraic compact set.

Proposition 1.2.3. *Let $B \subset S$ be compact semialgebraic sets. Let f be a rug function for B in S. Then there exists δ_f such that for any $\delta' < \delta_f$ there is a semialgebraic retraction*

$$\pi : f^{-1}([0, \delta']) \to B.$$

Proof. First we show that there exists a semialgebraic retraction for small enough semialgebraic neighborhoods. Let $T_\delta = \alpha^{-1}([0, \delta])$ and choose $\delta_f = \delta$ and $\phi : |K| \to S$ as in Proposition 1.2.2. Given $x \in T_\delta$, let $y = \phi^{-1}(x)$. Then y belongs to some simplex $\sigma = [a_0, \dots, a_k]$; let $(\lambda_0, \dots, \lambda_k)$ be its barycentric coordinates with respect to σ. Since $f(x) \le \delta$ then there exist some vertices of σ belonging to $\phi^{-1}(B)$: let $a_0, \dots, a_k$ be these vertices. First notice that $\sum_{i=0}^{k} \lambda_i \neq 0$: if it were zero, then

$$f(x) = f(\phi(y)) = f(\phi(\sum_{i=k+1}^{d} \lambda_i a_i)) = \sum_{i=k+1}^{d} \lambda_i f(\phi(a_i)) > \delta$$

since $f \circ \phi$ is affine; but this contradicts $f(x) \le \delta$.
Now we define $p_\sigma : \phi^{-1}(T_\delta) \cap \sigma \to \phi^{-1}(B)$ by

$$p(x) = p_\sigma(\lambda_0, \dots, \lambda_d) = (\frac{\lambda_0}{\sum_{i=0}^{k} \lambda_i}, \dots, \frac{\lambda_k}{\sum_{i=0}^{k} \lambda_i}).$$

Then p_σ is continuous and semialgebraic and its restriction to $\phi^{-1}(B) \cap \sigma$ is the identity map. Defining $p_{\sigma'}$ in the same way as for p_σ for every simplex σ' we notice that since the $p_{\sigma'}$'s agree on the common faces, then they together define a semialgebraic continuous map $p : \phi^{-1}(T_\delta) \to \phi^{-1}(\delta)$.
Now put $\pi = \phi^{-1}_{|T_\delta} \circ p \circ \phi$: then π is a semialgebraic continuous retraction from T_δ to B; given $\delta' < \delta$ simply compose π with the inclusion $T_{\delta'} \subset T_\delta$ to obtain the required retraction. □

In particular we derive the following corollary.

Corollary 1.2.4. *Let S be a semialgebraic set and $f : S \to [0, \infty)$ be a proper, continuous semialgebraic function. Then for $\epsilon > 0$ small enough the inclusions:*

$$\{f = 0\} \hookrightarrow \{f \leq \epsilon\} \quad \text{and} \quad \{f > \epsilon\} \hookrightarrow \{f \geq \epsilon\} \hookrightarrow \{f > 0\}$$

are homotopy equivalences.

Proof. Let $T = f^{-1}([0, \delta])$ for δ small enough as given by propositions 1.2.2 and 1.2.3. Consider the function

$$g = \pi|_{f^{-1}(\delta)} : \{f = \delta\} \to \{f = 0\}$$

where π is the retraction defined in the proof of proposition 1.2.3. Then propositions 1.2.2 and 1.2.3 combined together prove that T is a mapping cylinder neighborhood of $\{f = 0\}$ in S, i.e. there is a homeomorphism

$$\psi : T \to M_g,$$

where M_g is the mapping cylinder of g, such that $\psi|_{\{f=\delta\} \cup \{f=0\}}$ is the identity map. The conclusion follows from the structure of mapping cylinder neighborhoods. □

1.2.2 Hardt's triviality and the semialgebraic Sard's Lemma

Hardt's triviality theorem exploits the finiteness property of semialgebraic objects and is a cornerstone of semialgebraic geometry.

Theorem 1.2.5 (Hardt's Triviality)**.** *Let S and T be two semialgebraic sets, $f : S \to T$ a continuous semialgebraic mapping, $(S_j)_{j=1,\ldots,q}$ a finite family of semialgebraic subsets of S. There exist a finite partition of T into semialgebraic sets $T = \bigcup_{l=1}^{r} T_l$ and, for each l, a semialgebraic trivialization $\theta_l : T_l \times F_l \to f^{-1}(T_l)$ of f over T_l, compatible with S_j, for $j = 1, \ldots, q$.*

The following is a straightforward corollary of Hardt's triviality.

Proposition 1.2.6. *Let A, B be semialgebraic sets and $g : A \to B$ be a semialgebraic, surjective map. Then g admits a semialgebraic section σ, i.e. a map $\sigma : B \to A$ such that $g(\sigma(b)) = b$ for every $b \in B$.*

Proof. By Hardt's triviality theorem there exists a finite partition

$$B = \coprod_{l=1}^{m} B_l,$$

semialgebraic sets F_l and semialgebraic homeomorphisms $\psi_l : B_l \times F_l \to g^{-1}(B_l)$ for $l = 1, \ldots, m$ such that $g(\psi_l(b, y)) = b$ or every $(b, y) \in B_l \times F_l$. For every $l = 1, \ldots, m$ let $a_l \in F_l$ and define

$$\sigma|_{B_l}(b) = \psi_l(b, a_l).$$

□

Semialgebraic Sard's Lemma strengthen the conclusion that the set of critical values of a smooth map has measure zero to the fact that it is semialgebraic of codimension at least one.

Theorem 1.2.7 (Semialgebraic Sard's Lemma)**.** *Let $f : A \to B$ be a smooth semialgebraic map between two smooth semialgebraic manifolds A and B. Then the set of critical values of f is a semialgebraic subset of B of dimension strictly less than the dimension of B.*

1.3 Space of quadratic forms

1.3.1 Topology

Let V be a real vector space; we denote by $\mathcal{Q}(V)$ the space of all quadratic forms on V. Notice that in the case V is finite dimensional $\mathcal{Q}(V)$ is a vector space of dimension $d(d+1)/2$ where $d = \dim(V)$. Once we fix a scalar product the equation

$$q(x) = \langle x, Qx \rangle, \quad \forall x \in V$$

defines a unique real symmetric $d \times d$ matrix Q and the map $q \mapsto Q$ gives an isomorphism of vector spaces

$$\mathcal{Q}(V) \simeq \mathrm{Sym}(d, \mathbb{R}).$$

We will denote with the symbol $\mathcal{P}(V)$ the space of all symmetric bilinear forms on V; thus given $q \in \mathcal{Q}(V)$ the equation

$$2p(x, y) = q(x + y) - q(x) - q(y), \quad \forall x, y \in V$$

defines a a bilinear p form called the polarization of q and gives an isomorphism of vector spaces

$$\mathcal{Q}(V) \simeq \mathcal{P}(V).$$

We will sometimes use these isomorphisms and denote with capital letters symmetric matrices associated to quadratic forms and with the same letter their polarization by distinguish the latter two on the number of their arguments; in the case $V = \mathbb{R}^d$ we will use the shortened notation $\mathcal{Q}(d)$ for the space of quadratic forms on it. In many cases, when it will be clear from the context which is the space V in question,

we will omit its symbol and simply write $\mathcal{Q}$ for the space $\mathcal{Q}(V)$; a similar remark applies for many other objects we are going to define and depending on V.
The positive inertia index $\mathrm{i}^+(q)$ of a quadratic form is defined by

$$\mathrm{i}^+(q) = \max\{\dim(W) \mid W \text{ is a subspace of } V \text{ and } q|_W > 0\}$$

and analogously its negative inertia index is given by $\mathrm{i}^-(q) = \mathrm{i}^+(-q)$. The kernel of a quadratic forms $q \in \mathcal{Q}(V)$ is defined by

$$\ker(q) = \{v \in V \mid q(x, v) = 0 \quad \forall x \in V\}$$

it is a vector subspace of V. The rank of q is defined by $\mathrm{rk}(q) = d - \dim\ker(q)$.
Sylvester's law of inertia asserts that every quadratic forms q admits coordinates $x_1, \ldots, x_a$, $y_1, \ldots, y_b$, $z_1, \ldots, z_c$ for which

$$q(x, y, z) = x_1^2 + \cdots + x_a^2 - y_1^2 - \cdots - y_b^2$$

where $a = \mathrm{i}^+(q), b = \mathrm{i}^-(q)$ and $c = \dim\ker(q)$; alternatively this tells that there exists a matrix $M \in \mathrm{Gl}(d, \mathbb{R})$ such that M^TQM is diagonal with the obvious diagonal elements.
For every $k = 0, \ldots, \dim(V)$ we define the set

$$Z_k(V) = \{q \in \mathcal{Q}(V) \mid \dim\ker(q) = k\}.$$

We easily see that $Z(V) = \mathrm{Cl}(Z_1(V))$ is the set of degenerate quadratic forms, i.e.

$$Z(V) = \{q \in \mathcal{Q}(V) \mid \ker(q) \neq 0\}.$$

It is an algebraic hypersurface of $\mathcal{Q}(V)$ which under the isomorphism $\mathcal{Q}(V) \simeq \mathrm{Sym}(d, \mathbb{R})$ is given by $Z(V) = \{Q \in \mathrm{Sym}(d, \mathbb{R}) \mid \det(Q) = 0\}$; moreover since its equation are homogeneous $Z(V)$ is a cone.
For example, in the case $d = 2$ we have $\mathcal{Q}(V) \simeq \mathbb{R}^3$ and in coordinates $\mathrm{Sym}(2, \mathbb{R}) = \{\left(\begin{smallmatrix} x+y & z \\ z & x-y \end{smallmatrix}\right) \mid x, y, z \in \mathbb{R}\}$ the set $Z(V)$ is given by the equation $x^2 = y^2 + z^2$; notice that in this case the set of singular points of $Z(V)$ reduces to the origin, which has codimension 3 in $\mathcal{Q}(V)$. This phenomenon is typical, as described by the following proposition.

Proposition 1.3.1. *Let $q_0 \in \mathcal{Q}(V)$ be a quadratic map and let V_0 be its kernel. Then there exists a neighborhood U_{q_0} of q_0 and a smooth semialgebraic map $\phi : U_{q_0} \to \mathcal{Q}(V_0)$ such that: 1) $\phi(q_0) = 0$; 2) $\mathrm{i}^-(q) = \mathrm{i}^-(q_0) + \mathrm{i}^-(\phi(q))$; 3) $\dim\ker(q) = \dim\ker(\phi(q))$; 4) for every $p \in \mathcal{Q}$ we have $d\phi_{q_0}(p) = p|_V$.*

Proof. Let γ be a closed semialgebraic contour in the complex plane separating the non zero eigenvalues of q_0 from the origin. For any q such that the corresponding operator does not have eigenvalues on γ we define π_q to be the orthogonal projection onto the invariant subspace $V_\gamma(q)$ of the operator Q corresponding to the eigenvalues which lie inside the contour - formally speaking we have to consider the semialgebraic set S of the pairs (q, L) where L is a linear map from $\mathbb{R}^{n+1}$ to $V_\gamma(q)$ - and the correspondence $q \mapsto \pi_q$ is semialgebraic. Notice that in particular $\pi_{q_0}|_{V_0} = \mathrm{id}_{V_0}$. Then the correspondence $q \mapsto \Phi(q) = q \circ \pi_q|_V$ is semialgebraic and satisfyies the required properties. $\square$

Thus for every $k = 0, \dots, \dim(V)$ we have that $Z_k(V)$ is a smooth semialgebraic subset of $\mathcal{Q}(V)$ of codimension $k(k+1)/2$. The closure of $Z_k(V)$ is $\mathrm{Cl}(Z_k(V)) = \cup_{j \geq k} Z_j(V) = \{q \in \mathcal{Q}(V) \,|\, \dim \ker q \geq k\}$ and its singular locus is $\mathrm{Cl}(Z_{k+1}(V))$; in particular we get that the singular locus of $Z(V)$ has codimension 3 in $\mathcal{Q}(V)$.

1.3.2 Geometry of the index function

In terms of the index functions $\mathrm{i}^+, \mathrm{i}^- : \mathcal{Q}(V) \to \mathbb{N}$ above defined, we define the two family of subsets of $\mathcal{Q}(V)$

$$\mathcal{Q}_k(V) = \{q \,|\, \mathrm{i}^+(q) \leq k\} \quad \text{and} \quad \mathcal{Q}^k(V) = \{q \,|\, \mathrm{i}^+(q) \geq k\}.$$

Notice that $\mathcal{Q}_k(V)$ is a closed subset of $\mathcal{Q}(V)$ whereas $\mathcal{Q}^k(V)$ is open. The set $\mathcal{Q}_0(V)$ is the set of nonnegative quadratic forms and it is a convex closed cone in $\mathcal{Q}(V)$; to describe the topology of the sets $\mathcal{Q}_k(V)$, $k \geq 0$ we use the following trick. We fix a scalar product and for any $q \in \mathcal{Q}(V) = \mathcal{Q}(d)$ we consider the corresponding symmetric matrices Q; we order its eigenvalues in decreasing way:

$$\lambda_1(Q) \geq \cdots \geq \lambda_{\mathrm{i}^+(q)}(Q) > 0 \geq \lambda_{\mathrm{i}^+(q)+1}(Q) \geq \cdots \lambda_d(Q).$$

Once the scalar product is fixed we will refer to the eigenvalues of q by actually meaning the eigenvalues of Q. The λ_i are continuous (but not smooth) semialgebraic functions on $\mathcal{Q}(d)$. If I denotes the identity matrix, then the map $Q \mapsto Q + (\lambda_{k+1}(Q) - \lambda_1(Q))I$ defines a semialgebraic homeomorphism of $\mathcal{Q}(d)$ onto itself, carrying $\mathcal{Q}_k(d)$ onto $\mathcal{Q}_0(d)$: thus the sets $\mathcal{Q}_k(d)$ are all homeomorphic.
We define also the family of subset of $\mathcal{Q}(V)$

$$\mathcal{D}_k(V) = \{q \in \mathcal{Q}(V) \,|\, \lambda_k(q) \neq \lambda_{k+1}(q)\}.$$

Notice that $\mathcal{Q}^k(V) \backslash \mathcal{Q}^{k+1}(V)$ is a subset of $\mathcal{D}_k(V)$ for every possible choice of the scalar product in V. Let us fix the dimension of $V = \mathbb{R}^d$; on the space $\mathcal{D}_k$ (we omit for brevity the symbol V in parenthesis) is naturally defined the vector bundle

$$\mathbb{R}^k \hookrightarrow \Lambda_k^+ \longrightarrow \mathcal{D}_k$$

whose fiber over the point $q \in \mathcal{D}_k$ is the vector space $(\Lambda_k^+)_q = \mathrm{span}\{x \in V \,|\, Qx = \lambda_i x, 1 \leq i \leq k\}$ and whose vector bundle structure is given by its inclusion in $\mathcal{D}_k \times V$. Similarly the vector bundle $\mathbb{R}^{d-k} \hookrightarrow \Lambda_k^- \to \mathcal{D}_k$ has fiber over the point $q \in \mathcal{D}_k$ the vector space $(\Lambda_k^-)_q = \mathrm{span}\{x \in V \,|\, Qx = \lambda_i x, k+1 \leq i \leq d\}$ and vector bundle structure given by its inclusion in $\mathcal{D}_k \times V$. In the sequel we will need for $q \in \mathcal{D}_k$ the projective spaces:

$$P_k^+(q) = \text{projectivization of}\,(\Lambda_k^+)_q \quad \text{and} \quad P_k^-(q) = \text{projectivization of}\,(\Lambda_k^-)_q.$$

For a given $q \in \mathcal{Q}$ with $\mathrm{i}^-(q) = i$ (which implies $q \in D_{d-i}$) we will use the simplified notation

$$P^+(q) \doteq P_{d-i}^+(q) \quad \text{and} \quad P^-(q) \doteq P_{d-i}^-(q).$$

(even if $q \in \mathcal{D}_{n+1-i}$ for every metric still there is dependence on the metric for these spaces, but we omit it for brevity of notations; the reader should pay attention). Notice that $q|_{P^-(q)} < 0$ whereas $q|_{P^+(q)} \geq 0$, i.e. $P^+(q)$ contains also $\mathbb{P}(\ker q)$. The following picture may help the reader:

$$\underbrace{\lambda_1(q) \geq \cdots \geq \lambda_{d-\mathrm{i}^-(q)}(q)}_{P^+(q)} \geq 0 > \underbrace{\lambda_{d+1-\mathrm{i}^-(q)}(q) \geq \cdots \geq \lambda_d(q)}_{P^-(q)}$$

We set $w_{i,k}^-$ for the i-th Stiefel-Whitney classes of the bundle Λ_k^-; on the other hand since our results will be stated in terms of the bundle Λ_k^+ we simplify the notation for its characteristic classes and we simply denote by $w_{i,k}$ its i-th stiefel-Whitney class:

$$w_{i,k} = w_i(\Lambda_k^+) \in H^i(\mathcal{D}_k; \mathbb{Z}_2).$$

Notice that $\Lambda_k^+ \oplus \Lambda_k^- = \mathcal{D}_k \times \mathbb{R}^d$ and thus Whitney product formula holds for their total Stiefel-Whitney classes: $w(\Lambda_k^+)w(\Lambda_k^-) = 1$. In particular this implies $w_{1,k} = w_{1,k}^-$.

Proposition 1.3.2 (Agrachev). *For any two real numbers $\alpha_1 > \alpha_2$ the set*

$$\mathcal{R}_k = \{q \in \mathcal{Q}(n) \,|\, \lambda_1(q) = \cdots = \lambda_k(q) = \alpha_1 > \alpha_2 = \lambda_{k+1}(q) = \cdots = \lambda_{n+1}(q)\}$$

is homeomorphic to the Grassmannian $G_{k,n}$ of k-planes in $\mathbb{R}^n$; moreover $\mathcal{R}_k$ is a deformation retract of $\mathcal{D}_k$.

Proof. The homeomorphism between $\mathcal{R}_k$ and the Grassmannian $G_{k,n}$ is given simply by associating to each $q \in \mathcal{R}_k$ the eigenspace of Q associated to the eigenvalue α. Since the symmetric matrix Q is determined uniquely by its eigenvalues and the invariant subspaces corresponding to these eigenvalues, we define the deformation retraction by sending the pair $(q,t) \in \mathcal{D}_k \times [0,1]$ to the quadratic map q_t whose i-th eigenvalue is $t\alpha_1 + (1-t)\lambda_i(q)$ for $i = 1, \ldots, k$ and $t\alpha_2 + (1-t)\lambda_i(q)$ for $i = k+1, \ldots, d$ and whose invariant subspaces stay fixed. □

We consider now the complement of $\mathcal{D}_k$, namely the closed set:

$$\mathcal{S}_k = \mathcal{Q} \backslash D_k.$$

It follows from proposition 1.3.1 that $\mathcal{S}_k$ is a pseudomanifold of codimension 2 in $\mathcal{Q}$; it is then possible to define, for a given 1-cycle c in $\mathcal{D}_k$ the linking number of c with $\mathcal{S}_k$. Using this linking number we can give an alternative description of the class $w_{1,k}$.

Proposition 1.3.3. *The value of $w_{1,k}$ on $[c] \in H_1(\mathcal{D}_k; \mathbb{Z}_2)$ equals the linking number* $\mathrm{lk}(c, \mathcal{S}_k)$ *of c with $\mathcal{S}_k$.*

Proof. Clearly the restriction of the bundle Λ_k^+ is the pullback of the tautological bundle $\tau_{k,n}$ of the Grassmannian $G_{k,n}$ under the previous homeomorphism; hence $w_{1,k}|_{\mathcal{R}_k}$ is nonzero and since $\mathcal{D}_k$ deformation retracts onto $\mathcal{R}_k$ then $w_{1,k}$ is the only nonzero class in $H^1(\mathcal{D}_k; \mathbb{Z}_2)$; since by Alexander-Pontryagin duality the class determined by linking number with $\mathcal{S}_k$ is also nonzero, then these two classes are equal. □

We stress that the definition of the previous characteristic classes depends on the scalar product we fixed on V; on the other side since $\mathcal{Q}^k(V)\backslash\mathcal{Q}^{k+1}(V)$ is contained in $\mathcal{D}_k(V)$ for *every* choice of scalar product and since the space of all scalar product is connected, then the restrictions of the classes $w_{1,k}$ to $\mathcal{Q}^k(V)\backslash\mathcal{Q}^{k+1}(V)$ *do not* depend on the scalar product.

1.3.3 Families of quadratic forms

If V and W are vector spaces we define

$$\mathcal{Q}(V,W)=\{p:V\to W\,|\,\eta p\in\mathcal{Q}(V)\quad\forall\eta\in W^*\}.$$

In the case $V=\mathbb{R}^n$ and $W=\mathbb{R}^k$ we will simply write $\mathcal{Q}(n,k)$ for $\mathcal{Q}(\mathbb{R}^n,\mathbb{R}^k)$.
Given $p\in\mathcal{Q}(V,W)$ the correspondence $\eta\mapsto\eta p$ gives by definition a linear map

$$\overline{p}:W^*\to\mathcal{Q}(V)$$

If we are given $q_0,\ldots,q_k\in\mathcal{Q}(V)$ then $x\mapsto(q_0,\ldots,q_x)$ defines a quadratic map $q:V\to\mathbb{R}^{k+1}$, i.e. an element of $\mathcal{Q}(V,\mathbb{R}^{k+1})$ and the map $\overline{q}:(\mathbb{R}^{k+1})^*\to\mathcal{Q}(V)$ is given by $\eta=(\eta_0,\ldots,\eta_k)\mapsto\eta_0q_0+\cdots+\eta_kq_k$. The image of $\overline{q}$ is a linear subspace of $\mathcal{Q}(V)$ which is called the linear system of $q_0,\ldots,q_k$; the arrangement of this linear space with respect to the set $Z(V)$ of degenerate forms will be the main ingredient of our theory.
More generally if A is a semialgebraic set we can consider semialgebraic continuous maps

$$f:A\to\mathcal{Q}(V)$$

For such a map we define the following two family of subsets of A:

$$A_k=\{a\in A\,|\,\mathrm{i}^-(f(a))\le k\}\quad\text{and}\quad A^k=\{a\in A\,|\,\mathrm{i}^+(f(a))\ge k\}.$$

We will often use also the following auxiliary construction: given a positive definite form $p\in\mathcal{Q}(V)$ we consider for $\epsilon>0$ the sets

$$A_k(\epsilon)=\{a\in A\,|\,\mathrm{i}^-(f(a)-\epsilon p)\le k\}.$$

We have the following lemma.

Lemma 1.3.4. *Let $f:A\to\mathcal{Q}(V)$ be a semialgebraic continuous map and $dim(V)=n+1$. For every $j\in\mathbb{N}$ we have $A^{j+1}=\bigcup_{\epsilon>0}A_{n-j}(\epsilon)$; moreover every compact subset of A^{j+1} is contained in some $A_{n-j}(\epsilon)$ and in particular*

$$\varinjlim_{\epsilon}\{H_*(A_{n-j}(\epsilon))\}=H_*(A^{j+1}).$$

Proof. Let $a\in\bigcup_{\epsilon>0}A_{n-j}(\epsilon)$; then there exists $\overline{\epsilon}$ such that $a\in A_{n-j}(\epsilon)$ for every $\epsilon<\overline{\epsilon}$. Since for ϵ small enough

$$\mathrm{i}^-(\epsilon)(f(a))=\mathrm{i}^-(f(a))+\dim(\ker f(\omega))$$

then it follows that

$$\mathrm{i}^+(f(a)) = n+1-\mathrm{i}^-(f(a)) - \dim(\ker f(a)) \geq j+1.$$

Viceversa if $a \in A^{j+1}$ the previous inequality proves $a \in A_{n-j}(\epsilon)$ for ϵ small enough, i.e. $a \in \bigcup_{\epsilon>0} A_{n-j}(\epsilon)$.
Moreover if $a \in A_{n-j}(\epsilon)$ then, eventually choosing a smaller ϵ, we may assume ϵ properly separates the spectrum of $f(a)$ and thus, by continuity of the map f, there exists U open neighborhood of a such that ϵ properly separates also the spectrum of $f(a')$ for every $a' \in U$ (see [20] for a detailed discussion of the regularity of the eigenvalues of a family of symmetric matrices). Hence $a \in A_{n-j}(\epsilon)$ for every $a' \in U$. From this consideration it easily follows that each compact set in A^{j+1} is contained in some $A_{n-j}(\epsilon)$ and thus

$$\varinjlim_{\epsilon}\{H_*(A_{n-j}(\epsilon))\} = H_*(A^{j+1}).$$

□

The map $f : A \to \mathcal{Q}$ is used to pullback the characteristic classes previously defined to the above families of subsets of A. Specifically we define the family of subsets of A (with their corresponding characteristic classes):

$$D_k = f^{-1}(\mathcal{D}_k), \quad f^*w_{i,k} = w_i(f^*\Lambda_k^+) \in H^i(D_k;\mathbb{Z}_2).$$

We notice also the following fact: the set $D_k \cup A^{k+1}$ contains the set A^k. Indeed if ω is in A^k then either $\mathrm{i}^+(f(\omega)) = k$ or $\mathrm{i}^+(f(\omega)) \geq k+1$: in the first case certainly $\lambda_k(f(\omega)) > 0 \geq \lambda_{k+1}(f(\omega))$ and thus ω belongs to D_k, in the second case ω is in Ω^{k+1}. Thus if $x \in H^i(A, A^{k+1})$ we can consider the cup product $x \smile \partial^* f^* w_{1,k} \in H^{i+2}(A, A^{k+1} \cup D_k; \mathbb{Z}_2)$, where

$$\partial^* : H^1(D_k;\mathbb{Z}_2) \to H^2(A, D_k;\mathbb{Z}_2)$$

is the connecting homomorphism in the long exact sequence of the pair (A, D_k). Because of the previous observation, we can restrict the previous cup product to $H^{i+2}(A, A^k)$ and define for $x \in H^i(A, A^{k+1})$ the class

$$(x \smile f^*\gamma_{1,k})|_{(A,A^k)} \in H^{i+2}(A, A^k), \quad \gamma_{1,k} = \partial^* w_{1,k}.$$

Since the restriction of the classes $f^*w_{1,k}$ on $A^k \backslash A^{k+1}$ do not depend on the scalar product we used to define them, the previous correspondence $x \mapsto (x \smile f^*\gamma_{1,k})|_{(A,A^k)}$ also do not depends on the scalar product; it will play a central role in the sequel. Sometimes, when it will be clear from the context, to shorten notations we will write simply $w_{i,k}$ and $\gamma_{i,k}$ for the pull-back via $\overline{p}^*$ of the previous classes.
Let now Ω be a closed semialgebraic subset of S^1 and $f : \Omega \to \mathcal{Q}(\mathbb{R}^{n+1})$ be a semialgebraic map. Consider the semialgebraic set

$$C = \{(\omega, [x]) \in \Omega \times \mathbb{RP}^n \mid f(\omega)(x) \geq 0\}.$$

Since the projection $p_1 : C \to \Omega$ is a semialgebraic map, then by Hardt's triviality theorem there exixts a finite semialgebraic partition $\Omega = \coprod S_l$ such that p_1 is trivial over each S_l. The semialgebraic subsets of Ω are union of points and intervals (arcs); thus there exist a finite number of points $\{\omega_\alpha\}_{\alpha\in A}$ and a finite number of open arcs $\{I_{\alpha\beta}\}_{\alpha,\beta\in A}$ such that C is the disjoint union of the inverse image under p_1 of them; moreover p_1 is trivial over each of these subsets of Ω. For each $\omega\in\Omega$ we define the number

$$a(\omega)=n-\mathrm{i}^-(f(\omega))$$

and notice that $\Omega_{n-k}=\{\omega\in\Omega\,|\,a(\omega)\geq k\}$. Using the notations introduced above we clearly have that $p_1^{-1}(\eta)$ deformation retracts to $P^+(f(\eta))\simeq\mathbb{R}\mathrm{P}^{a(\eta)}$. Consider now the topological space

$$S=\{(\omega,[x])\in\Omega\times\mathbb{R}\mathrm{P}^n\,|\,[x]\in P^+(\omega)\}.$$

Lemma 1.3.5. *The inclusion $S\hookrightarrow C$ is a homotopy equivalence; indeed C deformation retracts to S.*

Proof. For every $\alpha\in A$ let U_α be a closed neighborhood of ω_α such that the inclusion $P^+(\omega_\alpha)\hookrightarrow C|_{U_\alpha}$ is a homotopy equivalence (such a neighborhood exists by lemma 1.2.4 applied to the function $f:(\omega,[x])\mapsto\mathrm{dist}(\omega,\omega_\alpha)$ and noticing that the inclusion $P^+(\omega_\alpha)\hookrightarrow\{f=0\}$ is a homotopy equivalence). If U_α is sufficiently small, then $S|_{U_\alpha}$ deformation retracts to $P^+(\omega_\alpha)$: since the eigenvalues of $f(\omega)$ depend continuously on ω and $d(\omega_\alpha)\geq d(\omega)$ for ω sufficiently close to ω_α, the deformation retraction is performed simply by sending each $P^+(\omega)$ to $\lim_{\omega\to\omega_\alpha}P^+(\omega)\subseteq P^+(\omega_\alpha)$. Now we have that $P^+(\omega_\alpha)\hookrightarrow S|_{U_\alpha}$ and $P^+(\omega_\alpha)\hookrightarrow C|_{U_\alpha}$ are both homotopy equivalences; since the second one is the composition $P^+(\omega_\alpha)\hookrightarrow S|_{U_\alpha}\hookrightarrow C|_{U_\alpha}$ then $S|_{U_\alpha}\hookrightarrow C|_{U_\alpha}$ also is a homotopy equivalence. Since $(C|_{U_\alpha},S|_{U_\alpha})$ is a CW-pair, then the previous homotopy equivalence implies $C|_{U_\alpha}$ deformation retracts to $S|_{U_\alpha}$ (see [17]).
Let now $W=(\cup_\alpha V_\alpha)$; since $C|_{W^c}$ is a locally trivial fibration, then clearly it deformation retracts to $S|_{W^c}$; since each V_α is closed, then C deformation retracts to $C|_W\cup S|_{W^c}$. Since the deformation retraction of each $C|_{U_\alpha}$ fixes $S|_{U_\alpha}$ and $\mathrm{Cl}(S|_{W^c})\cap C|_W\subseteq S$ then all this deformation retractions matches together to give de desired deformation retraction of C to S. $\square$

We can easily derive the following corollary which describes the cohomology of C.

Corollary 1.3.6. $H^k(C;\mathbb{Z}_2)\simeq H_0(\Omega_{n-k};\mathbb{Z}_2)\oplus H_1(\Omega_{n-k+1};\mathbb{Z}_2)$.

Proof. We only give a sketch: the rigorous details are left to the reader. Notice that a spectral sequence argument, as discussed later, immediately gives the result; however we prefer not to introduce the machinery of spectral sequences in this chapter.
We can give a cellular structure to S in the following way: for every ω_α such that $a(\omega_\alpha)\geq k$ we place a k-dimensional cell e^k_α representing a k-dimensional cell of $P^+(\omega_\alpha)$; for every arc $I_{\alpha\beta}$ such that $a(\omega)\geq k-1$ for every $\omega\in I_{\alpha\beta}$ we place another k-dimensional cell $e^k_{\alpha\beta}$ representing a k dimensional cell of $S|_{I_{\alpha\beta}}$. In this way, working with $\mathbb{Z}_2$ coefficients we have:

$$\partial e^k_\alpha=0\quad\text{and}\quad\partial e^k_{\alpha\beta}=e^{k-1}_\alpha+e^{k-1}_\beta$$

and the statement follows now from cellular homology (see [17]) and Leray-Hirsch theorem. $\square$

1.4 Systems of two quadratic inequalities

1.4.1 Homogeneous case

Let q_0, q_1 be in $\mathcal{Q}(\mathbb{R}^{n+1})$ and consider one of the three following systems:

$$\begin{cases} q_0(x) = 0 \\ q_1(x) = 0 \end{cases} \quad \text{or} \quad \begin{cases} q_0(x) = 0 \\ q_1(x) \leq 0 \end{cases} \quad \text{or} \quad \begin{cases} q_0(x) \leq 0 \\ q_1(x) \leq 0 \end{cases}$$

If we let $q \in \mathcal{Q}(\mathbb{R}^{n+1}, \mathbb{R}^2)$ be the map defined by (q_0, q_1) we have that the sets of the solutions of the previous systems equals $q^{-1}(K) \subseteq \mathbb{R}^2$ for

$$K = \{0\}, \quad K = \{x_0 \leq 0, x_1 = 0\}, \quad K = \{x_0 \leq 0, x_1 \leq 0\}$$

The matter of this section will be the study of the cohomology of the two sets

$$Y = q^{-1}(K) \cap S^n \quad \text{and} \quad X = p(Y)$$

for a general convex polyhedral cone $K \subseteq \mathbb{R}^2$ (here we denoted by $p : S^n \to \mathbb{R}\mathrm{P}^n$ the covering map; notice that since q is homogeneous of degree two, then $X = \{[x] \in \mathbb{R}\mathrm{P}^n \,|\, q(x) \in K\}$) and the three previous cases will be included as particular ones. Indeed if $K = \{\eta_1 \leq 0, \eta_2 \leq 0, \eta_3 \leq 0,\ \eta_i \in (\mathbb{R}^2)^*, i = 1, 2, 3\}$ (this is the general form of a polyhedral cone in $\mathbb{R}^2$), then $q^{-1}(K)$ corresponds to the set of the solution of the system $\{\eta_i q \leq 0,\ i = 1, 2, 3\}$.
We start by proving the following theorem.

Theorem 1.4.1. *$b_k(\mathbb{R}\mathrm{P}^n \backslash X) = b_0(\Omega^{k+1}) + b_1(\Omega^k)$ for every $k \in \mathbb{N}$.*

Proof. Consider the set $B = \{(\omega, [x]) \in \Omega \times \mathbb{R}\mathrm{P}^n \,|\, \omega q(x) > 0\}$ and the projection p_2 to the second factor. This projection is easily seen to be a homotopy equivalence (the fibers are contractible; a more precise proof will be given later in proposition 3.3.1) and its image is $\mathbb{R}\mathrm{P}^n \backslash X$ (this follows from $K^{\circ\circ} = K$). Letting $a : B \to [0, \infty)$ be the semialgebraic function $(\omega, [x]) \mapsto \omega q(x)$, then by lemma 1.2.4 there is $\epsilon > 0$ such that the inclusion $C(\epsilon) = \{a \geq \epsilon\} \hookrightarrow B$ is a homotopy equivalence. On the other side the set $C(\epsilon)$ admits the following description once we fix a positive definite form $p \in \mathcal{Q}(\mathbb{R}^{n+1})$:

$$C(\epsilon) = \{(\omega, [x]) \in \Omega \times \mathbb{R}\mathrm{P}^n \,|\, f_\epsilon(\omega)(x) \geq 0\}$$

where $f_\epsilon : \Omega \to \mathcal{Q}(\mathbb{R}^{n+1})$ is given by $\omega \mapsto \omega q - \epsilon p$. Corollary 1.3.6 implies now for $\epsilon > 0$ small enough

$$H^k(C) = H_0(\Omega_{n-k}(\epsilon)) \oplus H_1(\Omega_{n-k+1}(\epsilon))$$

and the conclusion follows from lemma 1.3.4. □

Using a similar argument one can prove the following formula, which relates the cohomology of Y to that of the sets $\Omega^{k+1}, k = 0, \ldots, n$.

Theorem 1.4.2. *For $k < n - 2$ the following formula holds:*

$$\tilde{b}_k(Y) = \tilde{b}_{n-k-1}(S^n \backslash Y) = b_0(\Omega^{n-k}, \Omega^{n-k+1}) + b_1(\Omega^{n-k-1}, \Omega^{n-k}).$$

In the spherical case our computations gives the cohomology of $S^n\backslash Y$ and Alexander duality directly gives that of Y. In the case of the projective solutions to reconstruct the cohomology of X from that of $\mathbb{R}\mathrm{P}^n\backslash X$ we have to compute also the map induced by the inclusion $c : \mathbb{R}\mathrm{P}^n\backslash X \to \mathbb{R}\mathrm{P}^n$ on the cohomology.

Proposition 1.4.3. *Set* $\mu = \max_{\omega\in\Omega} \mathrm{i}^+(\omega)$. *Then for* $k \leq \mu - 1$

$$H^k(\mathbb{R}\mathrm{P}^n) \xrightarrow{c^*} H^k(\mathbb{R}\mathrm{P}^n\backslash X)$$

is injective and for $k \geq \mu + 1$ *is zero.*

Notice that the case $k = \mu$ is excluded from this statement: it deserves a special treatment.

Proof. Consider the commutative diagram of maps

$$\begin{array}{ccc} B & \xrightarrow{\quad i \quad} & \Omega \times \mathbb{R}\mathrm{P}^n \\ {\scriptstyle p_2|_B}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathbb{R}\mathrm{P}^n\backslash X & \xrightarrow{\quad c \quad} & \mathbb{R}\mathrm{P}^n \end{array}$$

Since $p_2|_B$ is a homotopy equivalence, then $c^* = i^* \circ p_2^*$. If $k \leq \mu - 1$, then $\Omega^{k+1} \neq \emptyset$; thus let $\eta \in \Omega^{k+1}$. Then $p_1^{-1}(\eta) \cap B = \{\eta\} \times P^{d_\eta}$, where P^{d_η} is a projective space of dimension $d_\eta = \mathrm{i}^+(\eta) - 1 \geq k$; in particular the inclusion $P^{d_\eta} \xrightarrow{i_\eta} \mathbb{R}\mathrm{P}^n$ induces isomorphism on the k-th cohomology group. The following factorization of i_η^* concludes the proof of the first part (all the maps are the natural ones):

$$\begin{array}{ccc} H^k(\mathbb{R}\mathrm{P}^n) & \xrightarrow{\ i_\eta^*\ } & H^k(P^{d_\eta}) \\ \downarrow & & \uparrow \\ H^k(\Omega \times \mathbb{R}\mathrm{P}^n) & \longrightarrow & H^k(B) \end{array}$$

For the second statement simply observe that for $k \geq \mu + 1$ we have $\Omega^k = \emptyset$ and thus

$$H^k(\mathbb{R}\mathrm{P}^n\backslash X) \simeq H^0(\Omega^{k+1}) \oplus H^1(\Omega^k) = 0.$$

□

It remains to study $H^\mu(\mathbb{R}\mathrm{P}^n\backslash X) \to H^\mu(\mathbb{R}\mathrm{P}^n)$. For this purpose we introduce the bundle $L_\mu \to \Omega^\mu$ whose fiber at the point $\eta \in \Omega^\mu$ equals $\mathrm{span}\{x \in \mathbb{R}^{n+1} \,|\, \exists\lambda > 0 \text{ s.t. } (\eta Q)x = \lambda x\}$ and whose vector bundle structure is given by its inclusion in $\Omega^\mu \times \mathbb{R}^{n+1}$. Notice that this vector bundle coincide by definition with $\overline{q}^*\Lambda_k^+$. Recall that we defined $\overline{q}^* w_{1,\mu} \in H^1(\Omega^\mu)$ to be the first Stiefel-Whitney class of L_μ. We have the following result.

Proposition 1.4.4. *$rk(c^*)_\mu = 0 \quad \Longleftrightarrow \quad \overline{q}^* w_{1,\mu} = 0.$*

Proof. In the case $\Omega^\mu \neq S^1$, then clearly $\overline{q}^* w_{1,\mu}$ is zero and also $\mathrm{rk}(c^*)_\mu$ is zero since $H^\mu(\mathbb{R}\mathrm{P}^n \backslash X) = 0$. If $\Omega^\mu = S^1$, then i^+ is constant and we consider the projectivization $P(L_\mu)$ of the bundle L_μ. In this case it is easily seen that the inclusion

$$P(L_\mu) \overset{\lambda}{\hookrightarrow} B$$

is a homotopy equivalence and, since $\mathrm{rk}(c^*) = \mathrm{rk}(i^* \circ p_2^*)$ we have $\mathrm{rk}(c^*) = \mathrm{rk}(\lambda^* \circ i^* \circ p_2^*)$. Let us call l the map $p_2 \circ i \circ \lambda$; then $l : P(L_\mu) \to \mathbb{R}\mathrm{P}^n$ is a map which is linear on the fibres and if $y \in H^1(\mathbb{R}\mathrm{P}^n)$ is the generator, we have by Leray-Hirsch

$$H^*(P(L_\mu)) \simeq H^*(S^1) \otimes \{1, l^*y, \dots, l^*y^{\mu-1}\}.$$

By the Whitney formula we get

$$l^*y^\mu = w_1(L_\mu) \cdot (l^*y)^{\mu-1}$$

which proves $(c^*)_\mu$ is zero iff $w_1(L_\mu) = \overline{q}^* w_{1,\mu} = 0$. □

Collecting together Theorem 1.4.1 and the previous two propositions allows us to split the long exact sequence of the pair $(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \backslash X)$ and, since $H_*(X) \simeq H^{n-*}(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \backslash X)$, to compute the Betti numbers of X.
We first define the table $E = (e_{i,j})_{i,j \in \mathbb{Z}}$ with $e_{i,j} \in \mathbb{N}$, and whose nonzero part $E' = \{e_{i,j} \,|\, 0 \leq i \leq 2,\, 0 \leq j \leq n\}$ is the following table:

$$E' = \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline \vdots & \vdots & \vdots \\ \hline 1 & 0 & 0 \\ \hline c & 0 & 0 \\ \hline 0 & b_0(\Omega^\mu) - 1 & d \\ \hline \vdots & \vdots & \vdots \\ \hline 0 & b_0(\Omega^1) - 1 & b_1(\Omega^1) \\ \hline \end{array}$$

where $c = e_{0,\mu}$ and we have $(c, d) = (1, b_1(\Omega^\mu))$ if $w_{1,\mu} = 0$ and $(c, d) = (0, 0)$ otherwise.

Theorem 1.4.5. *If $\mu = n+1$ then X is empty; in the contrary case for every $k \in \mathbb{Z}$ the following formula holds:*

$$b_k(X) = e_{0,n-k} + e_{1,n-k-1} + e_{2,n-k-2}.$$

Moreover if $j : X \to \mathbb{R}\mathrm{P}^n$ is the inclusion map and j_* is the map induced on homology, then

$$e_{0,n-k} = \mathrm{rk}(j_*)_k.$$

The last statement follows from the formula

$$b_{n-k}(\mathbb{R}\mathrm{P}^n) = \mathrm{rk}(c^*)_{n-k} + \mathrm{rk}(j_*)_k.$$

A direct proof of the previous theorems will be given later, once we will have more powerful instruments, i.e. spectral sequences, at our disposal.

Remark 1. The previous theorem raises the question when can happen $w_{1,\mu} \neq 0$. Since $\mu = \max \mathrm{i}^+$, then clearly $\Omega = S^1$ and $\mathrm{i}^+ \equiv \mu$. Moreover since $\mu = \mathrm{i}^+(\eta) = n+1-\ker(\eta Q) - \mathrm{i}^+(-\eta) = n+1-\ker(\eta Q) - \mu$ it follows $\mu \leq [\frac{n+1}{2}]$.
It is interesting to classify pairs of quadratic forms (q_0, q_1) such that i^+ is constant; this classification follows from a general theorem on the classification up to congruence of pencils of real symmetric matrices (see [25]).

1.4.2 Quadratic maps to the plane: convexity properties

We discuss here some applications of the previous results; in particular we will see that the image of a quadratic map $q \in \mathcal{Q}(n+1, 2)$ has some convexity properties both in the case of the whole map q and its restriction to the unit sphere S^n. The material of this section is classical; for a reference the reader can see [4].

Theorem 1.4.6 (Calabi). *Let $q = (q_0, q_1)$ be in $\mathcal{Q}(n+1, 2)$ and $n+1 \geq 3$. If $q(S^n)$ does not contain the zero, then there exists a real linear combination $\omega q_0 + \omega_1 q_1$ which is positive definite.*

Proof. The hypothesis is equivalent to $n+1 \geq 3$ and $X = \{\overline{x} \in \mathbb{R}\mathrm{P}^n \,|\, q_0(x) = 0 = q_1(x)\} = \emptyset$ and the thesis to $\Omega^{n+1} \neq \emptyset$.
First notice that for every $k \geq 2$ we have $b_1(\Omega^k) = 0$: if it was the contrary, then $b_0(\Omega^k) = 1 = b_1(\Omega^{k-1})$ and Theorem 1.4.1 would give $b_{k-1}(\mathbb{R}\mathrm{P}^n \backslash X) = b_{k-1}(\mathbb{R}\mathrm{P}^n) = b_0(\Omega^k) + b_1(\Omega^{k-1}) = 2$, which is absurd. Thus if $n+1 > 2$ we have

$$1 = b_n(\mathbb{R}\mathrm{P}^n) = b_n(\mathbb{R}\mathrm{P}^n \backslash X) = b_0(\Omega^{n+1}) + b_1(\Omega^n) = b_0(\Omega^{n+1})$$

which implies $\Omega^{n+1} \neq \emptyset$. $\square$

Thus the previous theorem states that for $n+1 \geq 3$

$$X = \emptyset \Rightarrow \Omega^{n+1} \neq \emptyset.$$

Also the contrary is true, with no restriction on n : if $X \neq \emptyset$ then $0 = b_n(\mathbb{R}\mathrm{P}^n \backslash X) = b_0(\Omega^{n+1}) + b_1(\Omega^n)$ which implies $\Omega^n \neq S^1$ and $\Omega^{n+1} = \emptyset$. Thus we have the following corollary.

Corollary 1.4.7. *If $n+1 \geq 3$, then $X = \emptyset \iff \Omega^{n+1} \neq \emptyset$.*

Using the previous we can prove the well known quadratic convexity theorem.

Theorem 1.4.8. *If $n+1 \geq 3$ and $q : \mathbb{R}^{n+1} \to \mathbb{R}^2$ is defined by $x \mapsto (q_0(x), q_1(x))$, where q_0, q_1 are real quadratic forms, then*

$$q(S^n) \subset \mathbb{R}^2 \quad \textit{is a convex set.}$$

Proof. First observe that if $S^n = \{g(x) = 1\}$ with g quadratic form, then for a given $c = (c_0, c_1)$ we have $S^n \cap q^{-1}(c) \neq \emptyset$ iff $S^n \cap q_c^{-1}(0) \neq \emptyset$ iff $X(q_c) = \emptyset$, where q_c is the quadratic map whose components are $(q_0 - c_0 g, q_1 - c_1 g)$ and $X(q_c) = \{\overline{x} \in \mathbb{R}\mathrm{P}^n \,|\, q_c(x) = 0\}$. Thus by Corollary 1.4.7 we have $X(q_c) \neq \emptyset$ iff $\Omega^{n+1}(q_c) = \emptyset$ (here $n+1 \geq 3$).
Let now $a = (a_0, a_1)$ and $b = (b_0, b_1)$ be such that $X(q_a) \neq \emptyset \neq X(q_b)$ and suppose

there exists $T \in [0,1]$ such that $aT + (1-T)b \notin q(S^n)$. Then by Corollary 1.4.7 there exists $\eta \in \mathbb{R}^2$ such that

$$\eta Q - \langle \eta, aT + (1-T)b \rangle I > 0.$$

Assume $\langle \eta, a-b \rangle \geq 0$, otherwise switch the role of a and b. We have $0 < \eta Q - \langle \eta, aT + (1-T)b \rangle I = \eta Q + \langle \eta, T(b-a) \rangle I - \langle \eta, b \rangle I \leq \eta Q - \langle \eta, b \rangle I$. Thus we got $\eta Q - \langle \eta, b \rangle I > 0$, which implies $\Omega^{n+1}(q_b) \neq \emptyset$, but this is impossible by corollary 1.4.7 since $X(q_b) \neq \emptyset$. Hence for every $t \in [0,1]$ we have $at + (1-t)b \in q(S^n)$. $\square$

The conclusion of the previous theorems are false if $n+1 = 2$: pick $q_0(x,y) = x^2 - y^2$ and $q_1(x,y) = 2xy$, then $q_0(x) = q_1(x) = 0$ implies $x = 0$ but any real linear combination of q_0 and q_1 is sign indefinite. Moreover $q(S^1) = S^1$ which of course is not a convex subset of $\mathbb{R}^2$.

Corollary 1.4.9. *If $q : \mathbb{R}^{n+1} \to \mathbb{R}^2$ has homogeneous quadratic components, then $q(\mathbb{R}^{n+1})$ is closed and convex.*

Proof. Since $q(\mathbb{R}^{n+1})$ is the positive cone over $q(S^n)$, then it is closed and convex. $\square$

The previous proof works only for $n+1 \geq 3$, but the theorem is actually true with no restriction on n. The number of quadratic forms is indeed important, as the following example shows: let $q : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $(x_0, x_1, x_2) \mapsto (x_0x_1, x_0x_2, x_1x_2)$; then the image of $\mathbb{R}^3$ under q consists of the four hortants $\{x_0 \geq 0, x_1 \geq 0, x_2 \geq 0\}$, $\{x_0 \leq 0, x_1 \leq 0, x_2 \geq 0\}$, $\{x_0 \leq 0, x_1 \geq 0, x_2 \leq 0\}$, $\{x_0 \geq 0, x_1 \leq 0, x_2 \leq 0\}$.

1.4.3 Level sets of quadratic maps: nonemptyness conditions

Consider the smooth map

$$\nu : \mathbb{R}^{n+1} \to \mathcal{Q}(\mathbb{R}^{n+1})$$

which is given by $(x_0, \ldots, x_n) \mapsto \frac{1}{2}\sum_{i,j} x_i x_j$. The map ν is called the degree two Veronese map (actually the standard definition considers the map $\tilde{\nu} : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^N$ induced by ν, where $N = \frac{(n+1)(n+2)}{2} - 1$). The image $\mathcal{V}$ of ν is homeomorphic to $\mathbb{R}^{n+1}$ and named the Veronese surface (even though in general is not a surface):

$$\mathcal{V} = \nu(\mathbb{R}^{n+1})$$

The geometric interesting property of this Veronese map is that it transforms the geometry of intersection of quadrics in $\mathbb{R}^{n+1}$ in the geometry of intersection of linear spaces with $\mathcal{V}$ in $\mathcal{Q}(\mathbb{R}^{n+1})$. Giving coordinates $\{z_{ij} = x_i x_j\}_{0 \leq i \leq j \leq n}$ in $\mathcal{Q}(\mathbb{R}^{n+1})$, we have that if $A \subset \mathbb{R}^{n+1}$ is an algebraic set cut by quadratic equations

$$A = \{q_l(x) = \frac{1}{2}\sum_{i,j} a_{ij}^l x_i x_j = 0, \quad l = 0, \ldots, k\}$$

and $W_{q,0} \subset \mathcal{Q}(\mathbb{R}^{n+1})$ is the linear space given by the equations

$$\sum_{i,j} a_{ij} z_{ij} = 0, \quad l = 0, \ldots, k$$

then ν gives a homeomorphism:

$$A = \mathcal{V} \cap W_{q,0}.$$

In a similar fashion we want to study now level sets of quadratic maps $q = (q_0, \dots, q_k) \in \mathcal{Q}(\mathbb{R}^{n+1}, \mathbb{R}^{k+1})$, the previous case being the description of $q^{-1}(0)$. If we fix a scalar product on $\mathbb{R}^{n+1}$ we can consider the identification $\mathcal{Q}(V) = \mathrm{Sym}(n+1, \mathbb{R})$ and endow this space with the scalar product given by

$$\langle Q_1, Q_2 \rangle = \mathrm{tr}(Q_1 Q_2), \quad Q_1, Q_2 \in \mathrm{Sym}(n+1, \mathbb{R}).$$

If we still denote by $\nu(x)$ the symmetric matrix corresponding to $\nu(x)$, we have that $q_i(x) = b_i$ if and only if $\langle Q_i, \nu(x) \rangle = b_i$ for $i = 0, \dots, k$ as it is easily verified by short computations. The equations

$$\langle Q_i, Y \rangle = b_i, \quad i = 0, \dots, k, \quad Y \in \mathrm{Sym}(n+1, \mathbb{R})$$

define an affine space $W_{q,b} \subset \mathrm{Sym}(n+1, \mathbb{R})$ and the previous observation gives

$$q^{-1}(b) = W_{q,b} \cap \mathcal{V}, \quad b = (b_0, \dots, b_k).$$

Observe that $\mathcal{V} \subset \mathrm{Sym}(n+1, \mathbb{R})$ consists exactly of positive semidefinite matrices of rank less or equal than one:

$$\mathcal{V} = \{Q \in \mathrm{Sym}(n+1, \mathbb{R}) \,|\, Q \geq 0 \quad \text{and} \quad \mathrm{rk}(Q) \leq 1\}$$

Indeed if Q is such a matrix, then by Sylvester's law of inertia there exists $M \in \mathrm{GL}(n+1, \mathbb{R})$ such that $M^T Q M$ is diagonal with the first element of the diagonal equal to one the others equal to zero; thus $Q = \nu(Me_1)$. For example in the case $n+1 = 2$ we have that $\mathcal{V}$ coincide with the positive half of the cone Z of degenerate matrices.
Thus to check nonemptyness of level sets of quadratic maps is equivalent to check emptyness of the intersection of affine subspaces of $\mathcal{Q}(\mathbb{R}^{n+1})$ with the Veronese surface:

$$q^{-1}(b) \neq \emptyset \iff W_{q,b} \cap \mathcal{V} \neq \emptyset.$$

A weaker formulation of this problem is that of finding a positive definite matrix Y belonging to $W_{q,b}$; if such a matrix can be find of rank less or equal than one, then $q^{-1}(b) \neq \emptyset$. It turns out that in the case of $k+1 = 2$ to find a positive semidefinite solution to the previous equations is equivalent to find a positive semidefinite of rank less or equal than one. This result is a direct corollary of the following general theorem (see [4]).

Theorem 1.4.10. *Let $A \subset Sym(d, \mathbb{R})$ be an affine subspace of codimension strictly less than $c(r) = (r+1)(r+2)/2$ for some nonegative r. If there exists $Y \geq 0$ in A, then there exists $X \in A$ such that $X \geq 0$ and $rk(X) \leq r$.*

In the case $r = 1$ we have $c(r) = 3$ and thus if we have a positive semidefinite solution, then we have one positive semidefinite of rank less or equal then one.
The elementary condition we give to check emptyness in terms of the previous theory is the following.

Theorem 1.4.11. *Let* $q \in \mathcal{Q}(n+1, 2)$ *and* $b = (b_0, b_1) \in \mathbb{R}^2$. *Set* $\Omega(b) = (\{t^2 b\}_{t \in \mathbb{R}})^\circ$. *Then*

$$q^{-1}(b) = \emptyset \quad \textit{if and only if} \quad \max_{\eta \in \Omega(b)} \mathrm{i}^+(\eta) = n + 1.$$

Proof. Consider the set $X = \{[x] \in \mathbb{R}\mathrm{P}^n \,|\, q(x) = t^2 b\}$; then

$$q^{-1}(b) = \emptyset \quad \text{if and only if} \quad X = \emptyset.$$

Indeed if $q(x) = b$, then since $b \neq 0$ also $x \neq 0$ and $[x] \in X$; if $[x] \in X$ then $q(x) = t^2 b$ and thus $q(x/t^2) = b$. By theorem 1.4.1

$$X = \emptyset \quad \text{if and only if} \quad H^0(\Omega(b)^{n+1}) = H^n(\mathbb{R}\mathrm{P}^n \backslash X) = \mathbb{Z}_2$$

from which the conclusion follows. $\square$

CHAPTER 2

Nondegeneracy conditions

2.1 Convex sets

2.1.1 Tangent space and transversality

A convex subset K of $\mathbb{R}^n$ is a subset which verifies $tx + (1-t)y \in K$ for every $t \in I$ and $x, y \in K$. The polar $K^\circ \subset (\mathbb{R}^n)^*$ is defined by

$$K^\circ = \{\eta \in (\mathbb{R}^n)^* \mid \eta(x) \leq 0 \text{ for every } x \in K\}.$$

Small convex sets have a kind of stability condition with respect to diffeomorphism; we prove it here it for future reference. Recall that for a given convex function a and $c \in \mathbb{R}$ the set $\{a < c\}$ is convex.

Lemma 2.1.1. *Let $a : \mathbb{R}^n \to [0, \infty)$ be a proper convex smooth function and $x_0 \in \mathbb{R}^n$ such that $a(x_0) = 0$, $da_{x_0} \equiv 0$ and the Hessian $He(a)_{x_0}$ of a at x_0 is positive definite. Let also $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Then there exists $\overline{\epsilon} > 0$ such that for every $\epsilon < \overline{\epsilon}$*

$$\psi(\{a < \epsilon\}) \quad \textit{is convex.}$$

Proof. Let ϕ be the inverse of ψ, $y_0 = \psi(x_0)$ and $\hat{a} \doteq a \circ \phi$. Then the set $\psi(\{a < \epsilon\})$ equals $\{\hat{a} < \epsilon\}$. Since $da_{x_0} \equiv 0$, then

$$\mathrm{He}(\hat{a})_{y_0} = {}^tJ\phi_{y_0} \mathrm{He}(a)_{x_0} J\phi_{y_0} > 0$$

and thus, by continuity of the map $y \mapsto \mathrm{He}(\hat{a})_y$, the function $\hat{a}$ is convex on $B(y_0, \epsilon')$ for sufficiently small ϵ'; hence for every $c > 0$ the set $\{\hat{a}|_{B(y_0,\epsilon')} < c\}$ is convex. Since a is proper, then there exists ϵ such that $\{y \mid a(\phi(y)) < \epsilon\} \subset B(y_0, \epsilon')$. Thus $\{\hat{a} < \epsilon\} = \{\hat{a}_{|B(y_0,\epsilon')} < \epsilon\}$ is convex. ☐

Consider a family of functions $a_w : x \mapsto a(x + x_0 - w), w \in W \subset \mathbb{R}^n$ with compact closure, with a satisfying the conditions of the previous lemma. Since $\mathrm{He}(a_w)_x = \mathrm{He}(a)_x$, then the exstimate on $\mathrm{He}(a_w)_w$ can be made uniform on W. In particular taking $a(x) = |x|^2$ we derive the following corollary.

Corollary 2.1.2. *Let U be an open subset of $\mathbb{R}^n$ and $\psi : U \to \mathbb{R}^n$ be a diffeomorphism onto its image. For every $x \in U$ there exists $\delta_c(x) > 0$ such that for every $B(y, r) \subset B(x, 3\delta_c(x))$ with $r < \delta_c(x)$*

$$\psi(B(y, r)) \quad \textit{is convex.}$$

Moreover if ψ is semialgebraic, then the function $x \mapsto \delta_c(x)$ can be chosen semialgebraic.

Proof. The first part follows immediately from Lemma 2.1.1 and the previous remark.
In the case ψ is semialgebraic, then the condition for $\delta_c(x)$ to satisfy the requirements of the previous corollary is a semialgebraic condition (according to Lemma 2.1.1 it is given by semialgebraic inequalities); thus the set

$$S = \{(x, \delta) \in U \times (0, \infty) \mid \delta \text{ satisfies the condition of corollary 2.1.2}\}$$

is semialgebraic. Consider thus the semialgebraic function $g : S \to U$ given by the restriction of the projection on the first factor. Then the first part of the proof tells that g is surjective; proposition 1.2.6 ensures there exists a semialgebraic section $x \mapsto (x, \delta_c(x))$ of g and the function δ_c is thus semialgebraic. □

We define now the tangent space to a convex set; this definition applies also in the case we have a set $\Omega \subset \mathbb{R}^{k+1}$ diffeomorphic to a convex set, using the diffeomorphism to define it.

Definition 2.1.3. *Let $K \subset \mathbb{R}^{k+1}$ be a convex set and $y \in K$. We define the tangent space to K at y by:*

$$T_yK = cone(K - y)$$

where $cone(K - y) = \{v \in \mathbb{R}^{k+1} \,:\, v = t(x - y) \text{ with } t > 0 \text{ and } x \in K\}$.

All the definitions concerning smooth maps can be extended to the case of convex sets (see [3]). For $\Omega = K^\circ \cap S^k$, with K° a convex cone, and $\omega \in \Omega$ we define:

$$T_\omega\Omega = T_\omega K \cap T_\omega S^k.$$

We will say that a map $K \to M$, where M is a smooth manifold, is a smooth map if it extends to a smooth map on an open neighborhood of K in $\mathbb{R}^{k+1}$; we will say that $f : \Omega \to M$ is smooth if it extends to a smooth map on K. If K is a convex with empty interior and x is a point in the interior of K relative to the linear space generated by K we still call x an interior point of K. With this in mind the tangent space to an interior point of K is the standard tangent space, while much richer is the structure of the tangent space to a boundary point: in analogy with the smooth case, T_yK is the cone which *best* approximates the convex K ate the point y.

Definition 2.1.4. *Let $f : M \to \mathbb{R}^n$ be a smooth map and $K \subset \mathbb{R}^n$ be a convex set. We say that f is transversal to K if for every $x \in M$ such that $f(x) \in K$ we have*

$$df_x(T_xM) + T_{f(x)}K = \mathbb{R}^n.$$

This condition is analogous to the standard one and if K is a smooth submanifold of $\mathbb{R}^n$ it is indeed equivalent; for this reason we will use the notation $f \pitchfork K$ for a map transversal to K. Roughly speaking this condition requires transversality on the interior points of K and on the points such that $f(x) \in \partial K$ it requires that the image of df_x is not contained in any supporting hyperplane for K at $f(x)$.
In analogy with the smooth case, we have the following proposition.

Proposition 2.1.5. *Let $f : M \to \mathbb{R}^n$ be a smooth map and $K \subset \mathbb{R}^n$ be a closed convex susbset. If f is transversal to K, then $f^{-1}(K)$ is a topological submanifold with boundary $\partial f^{-1}(K) = f^{-1}(\partial K)$ of M.*

Proof. If the interior of K is empty, then f is still transversal to a small open neighborhood U of K in the space generated by K and thus $f^{-1}(U)$ is a smooth submanifold of M; in this case we replace M with this submanifold. Thus we may assume the interior of K is nonempty. Let $x_0 \in \text{int}(K)$ and consider the vector field $v(x) = x - x_0$ on $\mathbb{R}^n$. This vector field is pointing inward at each point $x \in \partial K$:

$$-v(x) \in \text{int}(T_x K), \quad x \in \partial K.$$

The transversality condition thus implies there exist two vector fields $w : M \to TM$ and $r : M \to T\mathbb{R}^n$ along f, with $r(m) \in T_{f(m)}K$ if $f(m) \in K$ and such that

$$v(f(y)) = df_y w(y) + r(y) \quad \text{for every } y \text{ such that } f(y) \in K.$$

Now if $f(y) \in K$ then $-df_y w(y) = df_y(-w(y)) \in \text{int}(T_f(y)K)$; (these vector fields are first built locally and then glued together with a partition of unity). The integral curves of $-w$ define a collaring of the set $f^{-1}(\partial K)$ and allow us to represent a neighborhood U_y of an arbitrary point $y \in f^{-1}(\partial K)$ as the product of $f^{-1}(\partial K) \cap U_y$ times an interval. Since $f^{-1}(\text{int}(K))$ is a smooth submanifold of M, the conclusion follows. □

2.1.2 Thom's Isotopy lemma

In this section we prove a result analogous to Thom's isotopy lemma: if we perform a homotopy of maps $f_t : M \to K$ all tranversal to K, then the topology of $f_t^{-1}(K)$ stays invariate.

Lemma 2.1.6. *Let M be a compact manifold, $K \subset \mathbb{R}^n$ a closed convex subset with $\partial K \neq \emptyset$. If $f_t : M \to \mathbb{R}^n, t \in [0,1]$, is a smooth homotopy such that $f_t \pitchfork K$ for $t \in [0,1]$, then there exists a homeomorphism $F : M \to M$ such that $F({f_0}^{-1}(K)) = {f_1}^{-1}(K)$.*

Proof. Thanks to proposition 2.1.5 we have that $B_0 \doteq {f_0}^{-1}(K)$ is a topological submanifold of M. Suppose ${f_0}^{-1}(\partial K) \neq \emptyset$, otherwise the result follows from the standard isotopy lemma.
Following the proof of proposition of 2.1.5 we have a vector field w on M such that the condition $f(x) \in \partial K$ implies $(df_0)_x w(x) \in \text{int}(T_{f(x)}K)$ and the integral curves of this vector field define a collaring $\mathcal{C}(\partial B_0)$ of ∂B_0 in M :

$$\tau : \partial B_0 \times [0, \infty) \xrightarrow{\simeq} \mathcal{C}(\partial B_0).$$

Fix now $a > 0$ and consider the function $\alpha : \partial B_0 \to [0, \infty)$ defined by

$$\alpha(x) = \text{dist}(f_0(\tau(x, a)), \partial K).$$

Since M is compact then also ∂B_0 is compact and α, which is continuous, has a minimum c; moreover since dfw is pointing inward along ∂K, we have $c \neq 0$.
Consider now a new convex $\tilde{K}$ with smooth boundary, approximating K from the interior, and such that for everty $t \in I$ we still have $f_t \pitchfork \tilde{K}$, dfw is still pointing inward along $\partial\tilde{K}$ and it verifies

$$\max_{k \in \partial K, \tilde{k} \in \partial \tilde{K}} \{\text{dist}(k, \partial\tilde{K}), \text{dist}(\tilde{k}, \partial K)\} \leq c/2.$$

Since $\partial\tilde{K}$ separates $\mathbb{R}^n$ in two connected components, the previous conditions on $\tilde{K}$ guarantee that for every x in ∂B_0 the two points $f(x)$ and $f(\tau(x,a))$ lie in two different components. Thus for every $x \in \partial B_0$ there exists $t_x \in [0,a)$ such that

$$f(\tau(x,t_x)) = f(\gamma_x(t_x)) \in \partial\tilde{K}$$

where γ_x is the integral curve of w going out from x. Moreover such t_x is unique: the hypotheses on w imply that the curve $f\gamma_x$ must be pointing inward at each point of $\partial\tilde{K}$.
Since t_x depends continuously on x, then the map $\sigma : \partial B_0 \to \mathcal{C}(\partial B_0)$ defined by

$$\sigma(x) = \tau(x,t_x),$$

gives a section of $\mathcal{C}(\partial B_0)$ whose image is ${f_0}^{-1}(\partial\tilde{K})$. This implies there exists a homeomorphism $\tilde{F_0} : M \to M$ such that

$$\tilde{F_0}({f_0}^{-1}(K)) = {f_0}^{-1}(\tilde{K}).$$

Applying the same reasoning to f_1 we will get a homeomorphism $\tilde{F_1} : M \to M$ such that $\tilde{F_1}({f_1}^{-1}(K)) = {f_1}^{-1}(\tilde{K})$.
Thom's isotopy lemma for the convex $\tilde{K}$ gives an isotopy $F_t : M \to M$ such that for every t it holds $F_t({f_0}^{-1}(\tilde{K})) = {f_t}^{-1}(\tilde{K})$ and the map

$$F = \tilde{F}_1^{-1}F_1\tilde{F_0}$$

defines the required homeomorphism. □

2.2 Systems of quadratic inequalities

2.2.1 Systems of algebraic inequalities

In this section we consider in great generality systems of polynomials (quadratic) inequalities. By such a system we mean a collection of inequalities

$$\begin{cases} q_1(x) \le 0 \\ \quad\vdots \\ q_k(x) \le 0 \end{cases}$$

with $q_i \in \mathbb{R}[x_1,\ldots,x_n]$. The set of the solutions of the previous system is a semialgebraic subset A of $\mathbb{R}^n$:

$$A = \{x \in \mathbb{R}^n \,|\, q_i(x) \le 0 \quad i = 0,\ldots,k\}.$$

In the case the polynomials q_i are homogeneous this set is contractible: if x is in A then for every $t > 0$ also tx is in A; moreover in this case A is the cone over its intersection Y with the sphere S^{n-1}. In the case each q_i has *even* degree, then it is also defined $X = \{[x] \in \mathbb{RP}^{n-1} \,|\, q_i(x) \le 0 \quad i = 0,\ldots,k\}$. Denoting by $p : S^{n-1} \to \mathbb{RP}^{n-1}$ the covering map we have

$$Y = A \cap S^{n-1}, \quad X = p(Y).$$

If now we perturb a little the inequalities describing Y the homotopy type of the corrsponding set of the solutions on the sphere does not change, as stated in the following lemma.

Lemma 2.2.1. *Let $Y \subset S^{n-1}$ be the set of the solutions of the system:*

$$\begin{cases} q_1(x) \leq 0 \\ \vdots \\ q_k(x) \leq 0 \end{cases}$$

where the q_i's are polynomial functions.
Then there exists $\delta > 0$ such that for every $\epsilon_1, \ldots, \epsilon_k \in [0, \delta]$ the inclusion of Y in the set Y_ϵ of the solutions on the sphere of the system:

$$\begin{cases} q_1(x) \leq \epsilon_1 \\ \vdots \\ q_k(x) \leq \epsilon_k \end{cases}$$

is a homotopy equivalence.

Proof. Fix an index $i \in \{1, \ldots, k\}$ and consider the semialgebraic set $\hat{Y}_i \subset S^{n-1}$ defined by all inequalities defining Y but the i-th one, and the function $\alpha_i : \hat{Y}_i \to \mathbb{R}$ defined by:

$$\alpha_i = \max\{0, q_i\}.$$

Then $Y = \alpha_i^{-1}(0)$ and, since $\hat{Y}_i$ is semialgebraic and compact, α_i is a rug function for Y in $\hat{X}_i$. Then by corollary 1.2.4 there exists a positive δ_i such that for every $\delta_i \geq \epsilon_i > 0$ the set Y is homotopy equivalent to the subset $Y_i \subset S^{n-1}$ of solutions of the system:

$$\begin{cases} q_1(x) \leq 0 \\ \vdots \\ q_i(x) \leq \epsilon_i \\ \vdots \\ q_k(x) \leq 0 \end{cases}$$

Now replace Y with Y_i (notice that Y_i is semialgebraic and compact) and repeat the argument for one of the remaining indexes. Iterating for each index and putting $\delta = \min\{\delta_1, \ldots, \delta_k\}$ gives the result. □

Remark 2. The previous statement is valid also if the q_i are simply continuous semialgebraic functions. In the case each q_i is homogeneous of degree two, then the result holds also for the set of the solutions of the previous system on the projective space $\mathbb{RP}^{n-1}$.

Remark 3. Notice that an equation like $p(x) = 0$ is equivalent to the couple of equations $p(x) \leq 0$ and $-p(x) \leq 0$. Hence the result of perturbing one single *equation* are *two* inequalities.

The previous setting admits also a dual description. Namely, consider

$$f : \mathbb{R}^n \to \mathbb{R}^k \quad \text{and} \quad K \subset \mathbb{R}^k$$

where f is a polynomial mapping, i.e. the components $(f_1, \ldots, f_k)$ are polynomials, and K is a polyhedral cone. Thus K admits a descripition as the set of $y \in \mathbb{R}^k$ such that $\eta_j(y) \leq 0$ for certain covectors $\eta_j \in (\mathbb{R}^{k+1})^*$, $j = 1, \dots, l$. Using the η_i we can define the functions $q_i = \eta_i f$, $i = 1, \dots, k$,: they are polynomials and by definition

$$A = f^{-1}(K).$$

Using this convention we will often refer to a pair (q, K), with q a polynomial map and K a polyhedral cone, as a *system of polynomial inequalities*.
In the case the polynomials f_i are not homogeneous we can consider their homogenization ${}^h f_i(x_0, \dots, x_n) = x_0^{\deg f_i} f_i(x_1, \dots, x_n)$ and the resulting homogenization ${}^h f$ of the map f. If the f_i were of even degree, then the ${}^h f_i$ also are of even degree and it is defined $X = \{[x] \in \mathbb{R}\mathrm{P}^n \,|\, {}^h f_i(x) \in K \quad i = 1, \dots, k\}$. Identifying $\mathbb{R}^n$ with $\{x_0 \neq 0\} \subset \mathbb{R}\mathrm{P}^n$ and setting X_0 for $X \cap \{x_0 = 0\}$ we have the homeomorphism $A = X \backslash X_0$ (notice that in the case the f_i were already homogeneous then $X_0 = \{x_0 = 0\}$ and $A = X \backslash \mathbb{R}\mathrm{P}^{n-1}$).

2.2.2 The quadratic case

Since we will be primarily interest in the case the polynomials q_i are quadratic, we deserve to them a special section. In this case a crucial observation is that perturbing *homogeneous quadratic* inequalities on the sphere or on the projective space still gives *homogeneous quadratic* inequalities. The reason for this is the following: suppose we fix a positive definite form $p \in \mathcal{Q}(\mathbb{R}^{n+1})$ in such a way that the sphere is the *unit sphere* with respect to this form. Then each inequality of the kind $q \leq \epsilon$ with $q \in \mathcal{Q}(\mathbb{R}^{n+1})$ and ϵ a real number can be written on this sphere as $q \leq \epsilon p$ and this last inequality is *homogeneous* and *quadratic* and the same reasoning hold on the projective space.
In this case if we consider quadratic polynomials q_i (not necessarily homogeneous) and a polyhedral cone $K \subset \mathbb{R}^k$ a kind of Alexander duality holds for $A = f^{-1}(K) \subset \mathbb{R}^n \subset \mathbb{R}\mathrm{P}^n$ and $\mathbb{R}\mathrm{P}^n \backslash A$.

Lemma 2.2.2. *Let $q : \mathbb{R}^n \to \mathbb{R}^k$ be a quadratic map and $K \subset \mathbb{R}^k$ be a polyhedral cone. Then, using $\mathbb{Z}_2$ coefficients, the following formula holds for $A = q^{-1}(K)$:*

$$\tilde{b}_i(A) = b_{n-i-1}(\mathbb{R}\mathrm{P}^n \backslash A) - b_{n-i-1}(\mathbb{R}\mathrm{P}^n), \quad i \geq 0$$

Proof. Consider the homogenization ${}^h q : \mathbb{R}^{n+1} \to \mathbb{R}^k$ of the map q and the sets $X = \{[x] \in \mathbb{R}\mathrm{P}^n \,|\, {}^h q(x) \in K\}$ and $X_0 = X \cap \{x_0 = 0\}$ in such a way that $A = X \backslash X_0$. Embed semialgebraically $\mathbb{R}\mathrm{P}^n$ into some $\mathbb{R}^m$ and consider the semialgebraic function $\alpha : X \to [0, \infty)$ defined by the distance from $\{x_0 = 0\}$. Since X is compact we have that α is a rug function for $X_0 = \alpha^{-1}(0)$ in X; thus if we set U_ϵ for the preimage $\alpha^{-1}[0, \epsilon)$ we have by corollary 1.2.4 that the inclusion

$$X \backslash U_\epsilon \hookrightarrow X \backslash X_0$$

is a homotopy equivalence for $\epsilon > 0$ small enough. In particular, since $X \backslash U_\epsilon$ is closed, we can use Alexander duality and get

$$H^i(A; \mathbb{Z}_2) \simeq H_{n-i}(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \backslash (X \backslash U_\epsilon); \mathbb{Z}_2).$$

Since A is affine, the homomorphism $(i^*)_{n-i}$ induced by its inclusion in $\mathbb{R}\mathrm{P}^n$ on the cohomologies is injective for $i = n$ and zero otherwise. Hence by naturality of Alexander duality, the connecting homomorphism

$$\partial : H_i(\mathbb{R}\mathrm{P}^n; \mathbb{Z}_2) \to H_i(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \backslash A; \mathbb{Z}_2)$$

is injective for $i = n$ and zero otherwise. This observation allows us to split the long exact sequence of the pair $(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \backslash A)$ and to get the desired result (the asymmetry at $i = n$ disappears if we take reduced Betti numbers for A). $\square$

2.2.3 Regular systems

We introduce in this section the class of *regular* systems of quadratic inequalities. We use the description of a system as a pair (q, K) where $q \in \mathcal{Q}(n, k)$ and $K \subset \mathbb{R}^k$ is a convex polyhedral cone; thus if K is defined by the inequalities $\eta_i \leq 0$ for certain $\eta_0, \ldots, \eta_l$ linear functionals on $\mathbb{R}^k$, the system is that defined by the inequalities $\eta_i q \leq 0$ for $i = 0, \ldots, l$. The definition of regularity is the following.

Definition 2.2.3 (Regular system). *Let $q \in \mathcal{Q}(V, W)$ be a quadratic map and $K \subset W$ be a convex polyhedral cone. We say that the system (q, K) is* degenerate *(or equivalently that the map q is* degenerate with respect to K*) if there exists a nonzero x in V and a nonzero η in K° such that $q(x) \in K$ and $\eta Q x = 0$. We say that the system is* nondegenerate *(or that the map q is* nondegenerate with respect to K*) if the previous condition is not verified.*

We can reformulate the previous and say that q is nondegenerate with respect to K if for every nonzero x in V and every nonzero ω in K° such that $q(x) \in K$ then the composition ωdq_x is not identically zero. Given a polyhedral cone K in a vector space W, we define the set

$$\mathcal{Q}(V, W; K) = \{q \in \mathcal{Q}(V, W) \,|\, q \text{ is nondegenerate with respect to } K\}.$$

If we consider the restriction of a map $q \in \mathcal{Q}(V, W; K)$ to the unit sphere we get a smooth map which is transversal to K. Using the above notations this means that for every point x in S^n the image of the differential $d\tilde{q}_x$ is not contained in any supporting hyperplane for K at $q(x)$. Indeed suppose on the contrary that the image of $d\tilde{q}_x$ is contained in a supporting hyperplane of K at $q(x)$: this means there exists a nonzero η in K° such that $\eta d\tilde{q}_x = 0$ and also $\eta q(x) = 0$. Since the image of dq_x equals $\mathrm{im}(d\tilde{q}_x) + \mathrm{span}\{q(x)\}$, then such a covector η vanish on the whole image of dq_x, hence q would be degenerate with respect to K. Actually it turns out that for $q \in \mathcal{Q}(V, W)$ the conditions $\tilde{q}$ is transversal to K and $q \in \mathcal{Q}(V, W; K)$ are equivalent, as stated in the following proposition.

Proposition 2.2.4. *Let $K \subset \mathbb{R}^k$ be a convex polyhedral cone and $q \in \mathcal{Q}(n+1, k)$ be a quadratic map. Then q is nondegenerate with respect to K if and only if $q|_{S^n}$ is tranvsersal to K.*

Proof. It remains to prove that if $q|_{S^n}$ is transversal to K then q is nondegenerate with respect to K. Suppose $x \in \mathbb{R}^{n+1} \backslash \{0\}$ is a point of degeneracy for q, i.e. $q(x) \in K$ and there exists $\eta \in K^\circ \backslash \{0\}$ such that $\omega dq_x = 0$; then

$$\mathrm{im}(dq_x) + K \neq \mathbb{R}^k$$

otherwise let $v \in \mathbb{R}^k$ such that $\eta(v) > 0$ and write $v = dq_x w + k$ for some $w \in \mathbb{R}^{n+1}$ and $k \in K$; then it should be $0 < \eta(v) = \eta dq_x w + \eta(k) \leq 0$ which is absurd. Since $x = u\|x\|$, with $u \in S^n$, then $\tilde{q}(u) = \|x\|^{-2} q(x) \in K$, since K is a cone. Moreover

$$\operatorname{im}(dq_x) + K = \operatorname{im}(d\tilde{q}_u) + \operatorname{span}\{\tilde{q}(u)\} + K = \operatorname{im}(d\tilde{q}_u) + T_{\tilde{q}(u)} K \neq \mathbb{R}^k$$

which shows that u is a point of tangency for $\tilde{q}$. This proves the lemma. $\square$

2.2.4 Regularization without changing homotopy type

In this section we prove that a system can be slightly perturbed in such a way that it becomes regular without changing the homotopy type of the set of the solutions. To this we introduce the following definition of nondegeneracy of a smooth map $f : \Omega \to \mathcal{Q}(V)$.

Definition 2.2.5. *Let Ω be a convex cone (or a set diffeomorphic to a convex cone) and $f : \Omega \to \mathcal{Q}(V)$ be a smooth map. We say that f is* degenerate at $\omega_0 \in \Omega$ *if there exists a nonzero x in* $\ker f(\omega_0)$ *such that for every $v \in T_{\omega_0}\Omega$ we have $(df_{\omega_0} v)(x, x) \leq 0$; in the contrary case we say that f is* nondegenerate at ω_0. *We say that f is* nondegenerate *if it is nondegenerate at each point $\omega \in \Omega$.*

For a given $q \in \mathcal{Q}(V, W)$ we defined the map $\overline{q} : W^* \to \mathcal{Q}(V)$ by the correspondence $\eta \mapsto \eta q$. If K is a convex polyhedral cone in W then the two nondgeneracy conditions, the one on q and that on $\overline{q}|_{K^\circ}$, are indeed equivalent, as shown in the following lemma.

Lemma 2.2.6. *If $K \subset W$ is a polyhedral cone and $q \in \mathcal{Q}(V, W)$ then q is nondegenerate with respect to K if and only if $\overline{q}|_{K^\circ}$ is nondegenerate.*

Proof. Suppose $\overline{q}|_{K^\circ}$ is degenerate; let $\eta \neq 0$ be in K° and $x \neq 0$ be in $\ker \overline{q}\eta$ such that for every v in $T_\eta K^\circ$ we have $(d\overline{q}_\eta v)(x) \leq 0$. Then writing v as $\omega - t^2\eta$ with $\omega \in K^\circ$ we have $\omega q(x) = (\omega - \eta) q(x) = (d\overline{q}_\eta(\omega - t^2\eta))(x) \leq 0$ for every $\omega \in K^\circ$ (where we have used the fact that $x \in \ker f(\eta)$). This tells that $q(x) \in K$ and thus that q is degenerate with respect to K ($\eta dq_x = 0$ since $x \in \ker \overline{q}\eta$).
On the contrary suppose q is degenerate with respect to K. Thus there exists a nonzero x and a nonzero $\eta \in K^\circ$ such that $q(x) \in K$ and $\eta dq_x = 0$. Since $q(x) \in K$ then for every $\omega \in K^\circ$ we have $\omega q(x) \leq 0$. Writing $v \in T_\eta K^\circ$ as $\omega - t^2\eta$ we have $(d\overline{q}_\eta v)(x) = (d\overline{q}_\eta(\omega - t^2\eta))(x) = \omega q(x) \leq 0$; since $\eta dq_x = 0$ then $x \in \ker \eta q$ and thus $\overline{q}|_{K^\circ}$ is degenerate. $\square$

Before proving the main result of this section, we will prove two technical lemmata. We recall that we stratified the set of singular forms $Z = \coprod Z_j$ (we omit for brevity of notations the symbol V in parenthesis); this stratification turns out to be smooth and semialgebraic (the name Nash is used for such stratifications). This stratification can be obtained by considering the set $S = \{(x, q) \in V \times Z \mid x \in \ker q\}$ and the map $p : S \to Z$ which is the restriction of the projection on the second factor: a trivialization for p gives a substratification of $Z = \coprod Z_j$.

Lemma 2.2.7. *Let r be a singular form and suppose $r \in Z_j$ for some stratum of Z as above. Then for every $q \in T_r Z_j$ and $x_0 \in ker(r)$ we have $q(x_0, x_0) = 0$.*

Proof. Let $r : I \to Z_j$ be a smooth curve such that $r(0) = r$ and $\dot{r}(0) = q$. By the triviality of p over Z_j it follows that there exists $x : I \to \mathbb{R}^{n+1}$ such that $x(0) = x_0$ and $x(t) \in \ker(r(t))$ for every $t \in I$. This implies $r(t)(x(t), x(t)) \equiv 0$ and deriving we get

$$0 = \dot{r}(0)(x(0), x(0)) + 2r(0)(x(0), \dot{x}(0)) = q(x_0, x_0).$$

□

Lemma 2.2.8. *Let $\Omega = \coprod Y_i$ be a finite partiton with each Y_i smooth and semialgebraic, $f : \Omega \to \mathcal{Q}(V)$ be a semialgebraic smooth map and $Z(V) = \coprod Z_j$ as above. Suppose that for every Y_i the map $f|_{Y_i}$ is transversal to all strata of $Z(V)$. Then f is nondegenerate.*

Proof. Let $\omega_0 \in \Omega$ and $x \in \ker(f(\omega_0))\backslash\{0\}$; we must prove that there exists $v \in T_{\omega_0}\Omega$ such that $(df_{\omega_0}v)(x, x) > 0$. Let Y_i such that $\omega_0 \in Y_i$. Then $T_{\omega_0}Y_i \subset T_{\omega_0}\Omega$; suppose $f(\omega_0) \in Z_j$. Since $f|_{Y_i}$ is transversal to Z_j, then

$$\mathrm{im}(df|_{Y_i})_{\omega_0} + T_{f(\omega_0)}Z_j = \mathcal{Q}(V).$$

Thus let $q^+ \in \mathcal{Q}(V)$ be a positive definite form, $v \in T_{\omega_0}Y_i$ and $\dot{r} \in T_{f(\omega_0)}Z_j$ such that

$$df_{\omega_0}v + \dot{r} = q^+.$$

Since $x \in \ker(f(\omega_0))\backslash\{0\}$, then the previous lemma implies $\dot{r}(x, x) = 0$, and plugging in the previous equation we get

$$(df_{\omega_0}v)(x, x) = (df_{\omega_0}v)(x, x) + \dot{r}(x, x) = q^+(x, x) > 0.$$

□

We are ready now to prove the main result of this section.

Theorem 2.2.9. *Let $f : \Omega \to \mathcal{Q}(V)$ be a semialgebraic smooth map. Then there exists a definite positive form $q_0 \in \mathcal{Q}(V)$ such that for every $\epsilon > 0$ sufficiently small the map $f_\epsilon : \Omega \to \mathcal{Q}(V)$ defined by*

$$\omega \mapsto f(\omega) - \epsilon q_0$$

is nondegenerate.

Proof. Let $\Omega = \coprod Y_i$ and $Z = \coprod Z_j$ be as above. For every Y_i consider the map $F_i : Y_i \times \mathcal{Q}^+ \to \mathcal{Q}$ (we denote by $\mathcal{Q}^+(V)$ the set of positive definite forms) defined by

$$(\omega, q_0) \mapsto f(\omega) - q_0.$$

Since $\mathcal{Q}^+$ is open in $\mathcal{Q}$, then F_i is a submersion and $F_i^{-1}(Z)$ is Nash-stratified by $\coprod F_i^{-1}(Z_j)$. Then for $q_0 \in \mathcal{Q}^+$ the evaluation map $\omega \mapsto f(\omega) - q_0$ is transversal to all strata of Z if and only if q_0 is a regular value for the restriction of the second factor projection $\pi_i : Y_i \times \mathcal{Q}^+ \to \mathcal{Q}^+$ to each stratum of $F_i^{-1}(Z) = \coprod F_i^{-1}(Z_j)$. Thus let $\pi_{ij} = (\pi_i)|_{F_i^{-1}(Z_j)} : F_i^{-1}(Z_j) \to \mathcal{Q}^+$; since all datas are smooth semialgebraic, then by semialgebraic Sard's Lemma, the set $\Sigma_{ij} = \{\hat{q} \in \mathcal{Q}^+ \mid \hat{q} \text{ is a critical value of } \pi_{ij}\}$ is a semialgebraic subset of $\mathcal{Q}^+$ of dimension $\dim(\Sigma_{ij}) < \dim(\mathcal{Q}^+)$. Hence $\Sigma = \cup_{i,j}\Sigma_{ij}$ also is a semialgebraic subset of $\mathcal{Q}^+$ of dimension $\dim(\Sigma) < \dim(\mathcal{Q}^+)$ and for every $q_0 \in \mathcal{Q}^+ \backslash \Sigma$ and for every i, j the restriction of $\omega \mapsto f(\omega) - q_0$ to Y_i is transversal to Z_j. Thus by the previous lemma $f - q_0$ is nondegenerate. Since Σ is semialgebraic of codimension at least one, then there exists $q_0 \in \mathcal{Q}^+ \backslash \Sigma$ such that $\{tq_0\}_{t>0}$ intersects Σ in a finite number of points, i.e. for every $\epsilon > 0$ sufficiently small $\epsilon q_0 \in \mathcal{Q}^+ \backslash \Sigma$. The conclusion follows. □

As a corollary we immediately get that the following.

Corollary 2.2.10. *Every system of quadratic inequalities can be regularized without changing the homotopy type of the set of its spherical or projective solutions.*

Proof. Given a system of quadratic inequalities (q, K), we apply the previous theorem to the map $\overline{q}$ and the convex set K°. Since nondegeneracy of $\overline{q}|_{K^\circ}$ is equivalent to nondegeneracy of the systems, then for $\epsilon > 0$ small enough lemma 2.2.1 gives the result. □

We conclude this section by stating a property of nondegenerate maps that will be useful in the sequel. Let $f : \Omega \to \mathcal{Q}(V)$ be a smooth map. We define, for every $U \subset \Omega$ the set

$$B_f(U) = \{(\omega, x) \in U \times \mathbb{R}\mathrm{P}^n \mid f(\omega)(x) > 0\}.$$

Lemma 2.2.11. *Let $f : \Omega \to \mathcal{Q}(V)$ be a smooth* nondegenerate *map. Then there exists $\delta_1 : \Omega \to (0, +\infty)$ such that for every $\omega \in \Omega$, for every $U_1 \subset U_2$ closed convex neighborhoods of ω with diam$(U_2) < \delta_1(\omega)$ and for every $\eta \in U_1$ such that* $\mathrm{i}^-(f(\eta)) = \mathrm{i}^-(f(\omega))$ *and* $\det(f(\eta)) \neq 0$ *the inclusions*

$$(\eta, P^+(f(\eta))) \hookrightarrow B_f(U_1) \hookrightarrow B_f(U_2)$$

are homotopy equivalences.
Moreover in the case f is semialgebraic, then the function δ_1 can be chosen to be semialgebraic (but in general not continuous).

Proof. The existence of δ_1 is the statement of Lemma 8 of [3]. The fact that δ_1 can be chosen to be semialgebraic if f is semialgebraic follows directly from the proof of Lemma 7 of [3]: in fact the set S of pairs $(\omega, \delta) \in \Omega \times (0, \infty)$ such that δ_1 satisfies the requirement of the lemma is semialgebraic (it is given by a formula with semialgebraic inequalities). Lemma 8 of [3] tells that the projection on the first factor $g|_S : S \to \Omega$ is surjective and, arguing as in lemma 2.1.2, proposition 1.2.6 gives the semialgebraicity. □

2.3 K-homotopy classes

2.3.1 K-homotopies

Consider now a polyhedral cone $K \subset W$. The set $\mathcal{Q}(V, W; K)$ has many connected components and given two maps q_0 and q_1 in the same component we can join them by a path q_t all made of non degenerate maps. In particular this path defines an homotopy between $q_0|_{S^n}$ and $q_1|_{S^n}$; since at each time of the homotopy the map q_t is

nondegenerate with respect to K, then by lemma 2.2.9 we have $q_t|_{S^n}$ is transversal to K for every time and by lemma 2.1.6 we have that the set of the spherical (projective) solutions of (q_0, K) is homeomorphic to the set of the spherical (projective) solutions of (q_1, K). In the case q_0 and q_1 are in the same connected component of $\mathcal{Q}(V, W; K)$ we will say that they are K-homotopic.
The goal of classifications of K-homotopy classes is that of find a label to name each connected component of $\mathcal{Q}(n, m; K)$. Here we will study two cases in which this label is particularly simple; the first is the case $K = \{x_0 \leq 0, x_1 \leq 0\} \subset \mathbb{R}^2$ where to each $q \in \mathcal{Q}(n, 2; K)$ we associate a partioned binary array and some rules to transform one array into the other such that two maps are K-homotopic if and only if their arrays can be transformed one into the other by mean of these rules; the second is the case $K = \{0\} \subset \mathbb{R}^3$ and $q : \mathbb{R}^3 \to \mathbb{R}^3$ where it is possible to associate a modulo two invariant of the map.
The classification of K-homotopy classes for $K = \{0\} \subset \mathbb{R}^2$ is studied in [1].

2.3.2 Two quadrics

In this section the cone K is assumed to be the subset of $\mathbb{R}^2$ defined by $x_0 \leq 0$ and $x_1 \leq 0$. It is a quadrant in the plane and indeed any system of quadratic inequalities (q, K') with $K' \subset \mathbb{R}^2$ homeomorphic to K is equivalent to the system (q, K) by a linear change of variable; thus in this section we are studying regular homotopy classes of systems of two *independent* quadratic inequalities.
We begin by studying K-homotopy classes in $\mathcal{Q}(2, 2; K)$.

Lemma 2.3.1. *If $q \in \mathcal{Q}(2, 2; K)$ then it is K-homotopic to $p : \mathbb{R}^2 \to \mathbb{R}^2 \simeq \mathbb{C}$*

$$(x_0, x_1) \xrightarrow{p} x_0^2 e^{i\theta_0} + x_1^2 e^{i\theta_1}$$

such that $\theta_0 \neq \pm\theta_1$ and $\theta_1, \theta_0 \neq k\pi/2$, $k \in \mathbb{Z}$.

Proof. Consider the following equation for $[\omega] \in \mathbb{R}\mathrm{P}^1$:

$$\det(\omega p) = 0$$

and let $\Delta : \mathcal{Q}(2, 2) \to \mathbb{R}$ its discriminant. Then Δ is a polynomial function not identically zero and $\{\Delta(p) = 0\}$ is a proper algebraic subset of $\mathcal{Q}(2, 2)$; since $\mathcal{Q}(2, 2; K)$ is open, we may assume $\Delta(q) \neq 0$.
If $\Delta(q) > 0$ then there are two noncollinear roots $[\omega_0]$ and $[\omega_1]$ in $\mathbb{R}\mathrm{P}^1$.
This means that the image of $q^* : \omega \mapsto \omega q$ intersects the set Z of degenerate quadratic forms in two distinct lines.
Since $\det(\omega_j Q) = 0$, for $j = 0, 1$, then there exist x_0 and x_1 in $\mathbb{R}^2$ different from zero and such that

$$x_0^T(\omega_0 Q) = 0, \qquad x_1^T(\omega_1 Q) = 0.$$

Moreover, since ω_0 and ω_1 are linearly independent, then

$$\omega_0(x_0^T Q x_1) = \omega_1(x_0^T Q x_1) = 0$$

It follows that

$$x_0^T Q x_1 = 0.$$

Moreover if x_0 and x_1 were collinear, then writing $\eta \in K^\circ\backslash\{0\}$ as a linear combination of ω_0 and ω_1,

$$\eta = c_0\omega_0 + c_1\omega_1,$$

we would have $q(x_0) = 0 \in K$, $x_0 \neq 0$, and

$$x_0^T(\eta Q) = x_0^T(c_0\omega_0 + c_1\omega_1)Q = x_0^T(c_0\omega_0 Q) + x_0^T(c_1\omega_1 Q) = 0$$

against the nondegeneracy hypothesis on q.
The condition $x_j^T(\omega_j Q) = 0$ tells that the quadratic form $\omega_j q$ restricted to $\{\lambda x_j\}$ is zero; nevertheless $\omega_j q$ is not identically zero: if for example it was $(\omega_1 q)(x) = 0$ for every $x \in \mathbb{R}^2$, then in coordinates (ω_0, ω_1) we would have

$$q(x) = q(x_0, x_1) = (a{x_0}^2, 0), \qquad Jq(x_0, x_1) = \begin{pmatrix} 2ax_0 & 0 \\ 0 & 0 \end{pmatrix}$$

and for every $\lambda \neq 0$ we would have $q(0, \lambda) = 0 \in K$ and $Jq(0, \lambda) \equiv 0$, against the nondegeneracy assumption.
Thus $q : \mathbb{R}^2 \to \mathbb{R}^2 \simeq \mathbb{C}$ is of the form

$$q(x) = \langle a_0, x\rangle^2 e^{i\theta_0} + \langle a_1, x\rangle^2 e^{i\theta_1} \tag{2.1}$$

with $\theta_0 \neq \pm\theta_1$ (since $\Delta(q) \neq 0$) and $a_0, a_1 \in \mathbb{R}^2$ such that

$$\langle a_0, x_1\rangle = \langle a_1, x_0\rangle = 0 \quad \text{and} \quad q(\lambda x_j) = \lambda^2\langle a_j, x_j\rangle^2 e^{i\theta_j} \qquad \text{for } j = 0, 1.$$

The nondegeracy condition implies none of $e^{i\theta_0}$ and $e^{i\theta 1}$ is a generator of K and thus slightly perturbing we obtain $e^{i\theta_j} \neq k\pi/2$, $k \in \mathbb{Z}$. We can clearly change a_0 and a_1 trough K-homotopies as to arrive to p.
If $\Delta(q) < 0$ there are no real roots of the previous equation: $[\omega_0], [\omega_1] \in \mathbb{CP}^1$; moreover since the coefficients of the equation are real, then $[\omega_0] = [\overline{\omega_1}]$. In this case the non existence of real roots guarantees automatically nondegeneracy. We exhibit now a K-homotopy between q and a map with positive discriminant. First notice that we have $\det(\omega q) \neq 0$ for every $\omega \neq 0$ and thus $dq_{|\mathbb{R}^2\backslash\{0\}}$ is surjective; moreover for every $\eta \neq 0$ we have $\mathrm{i}^+(\eta q) = 1$. Thus let $\eta \in \mathrm{int}(K)$ and $e^\perp$ orthogonal to e. In coordinates $(e, e^\perp)$ we have

$$q(x) = (\langle e, q(x)\rangle, \langle e^\perp, q(x)\rangle).$$

Diagonalizing the first component we find a basis (y_0, y_1) of $\mathbb{R}^2$ such that in coordinates we have

$$q(x) = ({x_0}^2 - {x_1}^2, a{x_0}^2 + b{x_1}^2 + cx_0x_1).$$

We define the homotopy q_t through the equation:

$$q_t(x) = ({x_0}^2 - {x_1}^2, t(a{x_0}^2 + b{x_1}^2 + cx_0x_1)).$$

Naturally we have

$$Jq_t(x_0, x_1) = \begin{pmatrix} 2x_0 & 2tax_0 + tcx_1 \\ -2x_1 & 2tbx_1 + tcx_0 \end{pmatrix}$$

$$\det(Jq_t(x_0,x_1)) = t\det(Jq_1(x_0,x_1))$$

and thus q_t is nondegenerate for every t: for $t\neq 0$ the differential of $q_{t|\mathbb{R}^2\setminus\{0\}}$ is surjective; for $t=0$ we have $\Delta(q_0)=0$ but the choice of e guarantees nondegeneracy. Thus after this homotopy q will be of the form

$$q_0(x) = \langle a_0, x\rangle^2 e^{i\theta} + \langle b_0, x\rangle^2 e^{-i\theta},$$

with $e^{i\theta}=\lambda^2 e$ and a_0 and b_0 nonzero; a small rotation of one of the two vectors $e^{i\theta}$ or $e^{-i\theta}$ gives the K-homotopy between q_0 and a map with positive discriminant, to which the previous part applies. $\square$

Using the previous lemma we can attach to each $q\in\mathcal{Q}(2,2;K)$ a word $s(q)$ of three characters from the sets $\{\omega,\hat{\omega},z\}$ in the following way. Let p be given by the previous lemma, fix the orientation $(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right),\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$ on $\mathbb{R}^2$ and let $\omega_j=-\hat{\omega}_j\in S^1$, $j=0,1$ be such that

$$\langle e^{i\theta_j},\omega_j\rangle = \langle e^{i\theta_j},\hat{\omega}_j\rangle = 0 \quad \text{and} \quad (e^{i\theta_j},\omega_j) \text{ is positively oriented}$$

Notice that by assumption on K we have $K=\mathrm{cone}\{z=\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), w=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$. The previous lemma implies no $\omega_j,\hat{\omega}_j$, $j=0,1$ belongs to $\{z,w,-w\}$. Thus on the arc joining $-w$ to w clockwise there is one among $\{\omega_0,\hat{\omega}_0\}$, one among $\{\omega_1,\hat{\omega}_1\}$ and z. We define $s(q)$ to be the word obtained writing the letters of the points we meet going from $-w$ to w clockwise without indices. Twelve possibilities can happen and we partition them into four disjoint subsets (the reason for this partition will become clear in a while):

(1) $[\omega\omega z]=\{\omega\omega z\}$

(2) $[\omega\hat{\omega}z]=\{\omega\hat{\omega}z, \omega z\hat{\omega}, \hat{\omega}\omega z, \omega z\omega\}$

(3) $[\hat{\omega}z\omega]=\{\hat{\omega}z\omega, z\omega\omega, z\hat{\omega}\omega, z\hat{\omega}\hat{\omega}, \hat{\omega}z\hat{\omega}, \hat{\omega}\hat{\omega}z\}$

(4) $[z\omega\hat{\omega}]=\{z\omega\hat{\omega}\}$.

For a given $q\in\mathcal{Q}(2,2;K)$ we define

$$\sigma(q)=[s(q)]$$

and prove the following result, which classify K-homotopy classes (i.e. connected components) of $\mathcal{Q}(2,2;K)$.

Theorem 2.3.2. *Two maps $q_0,q_1\in\mathcal{Q}(2,2;K)$ are K-homotopic if and only if*

$$\sigma(q_0)=\sigma(q_1).$$

In particular $P(2,2;K)$ has four connected components.

Proof. Notice first that the four cases we described correspond to the following situation:

(1) : both $e^{i\theta_0}$ and $e^{i\theta_1}$ belong to $\mathrm{int}(K)$;

(2) : one among $e^{i\theta_0}$ and $e^{i\theta_1}$ belongs to $\mathrm{int}(K)$ and the other does not;

(3) : both $e^{i\theta_0}$ and $e^{i\theta_1}$ do not belong to K and $p(\mathbb{R}^2) \cap K = \{0\}$;

(4) : both $e^{i\theta_0}$ and $e^{i\theta_1}$ do not belong to K and $p(\mathbb{R}^2) \supset K$.

Clearly if $\sigma(q_0) = \sigma(q_1)$ then q_0 and q_1 are K-homotopic: first make a homotopy from q_0 to p_0 given by the lemma. Then rotating the vectors $e^{i\theta_0}$ and $e^{i\theta_1}$ gives a homotopy between p_0 and p_1, where p_1 comes from the lemma applied to q_1; this homotopy is a K-homotopy because $\sigma(q_0) = \sigma(q_1)$ (the reader can check it simply drawing a picture). Finally perform the homotopy from p_1 to q_1.
On the contrary if q_0 and q_1 are K-homotopic, then also p_0 and p_1 are K-homotopic. If $\sigma(q_0) \neq \sigma(q_1)$, then the homotopy joining p_0 and p_1 must have zero discriminant at a certain point p_s, $s \in [0,1]$. Let $\overline{p}_s : \mathbb{R}^2 \to \mathcal{Q}(\mathbb{R}^2) \simeq \mathbb{R}^3$ be the map $\omega \mapsto \omega p_s$; then $\overline{p}_s$ is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$. Since the set of linear maps $L : \mathbb{R}^2 \to \mathbb{R}^3$ with rank less than or equal to one has codimension two, then we may assume $\mathrm{rk}(\overline{p}_s) = 2$: if there is a K-homotopy p_t between p_0 and p_1 then there also is a K-homotopy avoiding the codimension two set of maps with not maximal rank.
The nondegeneracy condition of p_s traduced in the nondegeneracy for the linear map $\overline{p}_s$ is:

$$\forall \eta \in K^\circ \backslash \{0\},\, \forall y \in \ker(\overline{p}_s \eta)\backslash\{0\} \quad \exists v \in T_\eta K^\circ \quad \text{s.t} \quad (d_\eta \overline{p}_s v)(y) > 0.$$

Thus if we set $Z = \{q \in \mathcal{Q}(\mathbb{R}^2) \mid \det(q) = 0\}$, then we have $\overline{p}_s(\mathbb{R}^2)$ intersects Z in a line l. Now in principle three possibilities can happen: (i) $\overline{p}_s(K^\circ) \cap l \subset \mathrm{int}(K^\circ)$, in which case p_s would be degenerate with respect to K; (ii) $\overline{p}_s(K^\circ) \cap l \subset \partial K^\circ$, a case which has codimension at least two and thus that can be avoided; (iii) $\overline{p}_s(K^\circ) \cap l = \{0\}$, in which case p_s is nondegenerate with respect to K.
Thus if the discriminant of p_s vanishes performing a K-homotopy between maps p_0, p_1 with positive discriminant, then it can happen only in the described way and thus, recalling the proof of lemma 2.3.1, we have $\sigma(p_0) = \sigma(p_1)$ which concludes the proof. □

We move now to the classification of K-homotopy classes in $\mathcal{Q}(m, 2; K)$. We adopt the following convention: if $q_1, \ldots, q_k$ are quadratic maps with $q_j \in \mathcal{Q}(n_j, 2)$ for $j = 1, \ldots, k$, we define the quadratic map $q_1 \oplus \cdots \oplus q_k = q \in \mathcal{Q}(n_1 + \cdots + n_k, 2)$ by the formula

$$q(x) = q(x_1, \ldots, x_k) = \sum_{j=1}^{k} q_j(x_j).$$

The following is a classical result.

Lemma 2.3.3. *Let q in $\mathcal{Q}(m, 2)$ such that $\Delta(q) \neq 0$. Then there exist $q_j \in \mathcal{Q}(2, 2)$ for $j = 1, \ldots, l$ and $p_k \in \mathcal{Q}(1, 2)$ for $k = 1, \ldots, b$ such that $2l + b = m$ and*

$$q = (\mathop{\oplus}_{j=1}^{l} q_j) \oplus (\mathop{\oplus}_{k=1}^{b} p_k).$$

Proof. See [3] □

In particular lemma 2.3.3 implies that if $q \in \mathcal{Q}(m, 2; K)$ then each q_j must belong to $\mathcal{Q}(2, 2; K)$ and each p_k to $\mathcal{Q}(1, 2; K)$.
For our purpose we need the following lemma.

Lemma 2.3.4. *Let $q_0 \in \mathcal{Q}(m, 2; K)$ such that $q_0 = s_0 \oplus r$ with $r \in \mathcal{Q}(m-2, 2; K)$, $s \in \mathcal{Q}(2, 2; K)$ and $\Delta(s_0) < 0$.*
Then q_0 is K-homotopic to a map $q_1 = s_1 \oplus r$ such that $\sigma(s_1) = [\omega\hat{\omega}z]$ and $\Delta(s_1) > 0$.

Proof. Consider the K-homotopy s_t we built when we proved that $\Delta(s_0) < 0$ then $\sigma(s_0) = [\omega\hat{\omega}z]$ and stop this homotopy once we reach a map $\tilde{s} = s_T$ with zero discriminant. Thus suppose we have a K-homotopy s_t between s_0 and $\tilde{s}$. We define $q_t = s_t \oplus r$; then we have $q_t(x) = (x^T Q_1(t)x, x^T Q_2(t)x)$ with

$$Q_j(t) = \begin{pmatrix} S_j(t) & 0 \\ 0 & R_j \end{pmatrix} \qquad j = 1, 2$$

If $w = (w_1, w_2)$, then

$$wQ(t) = w_1 Q_1(t) + w_2 Q_2(t) = \begin{pmatrix} w_1 S_1(t) + w_2 S_2(t) & 0 \\ 0 & w_1 R_1 + w_2 R_2 \end{pmatrix}$$

Suppose there exists $\tau \in (0, T)$ such that q_τ is degenerate with respect to K; then there would exist a nonzero vector $x = (x_s, x_r) \in \mathbb{R}^2 \times \mathbb{R}^{m-2}$ and a covector $w \in K^\circ \backslash \{0\}$ such that $q_\tau(x) \in K$ and $w(dq_\tau)_x \equiv 0$. Since $(dq_\tau)_x = x^T(\omega Q(\tau))$ then $x_s = 0$, because for $x_s \neq 0$ the linear map $w(ds_\tau)_{x_s} = {x_s}^T(w_1 S_1(\tau) + w_2 S_2(\tau))$ is nonzero; thus $r(x_r) = q(x) \in K$ and ${x_r}^T(w_1 R_1 + w_2 R_2) = 0$ against the fact that r is nondegenerate with respect to K. Thus we showed that for $t \neq T$ the map q_t is nondegenerate with respect to K.
On the other side for $t = T$ we have $(dq_T)_{(x_s, x_r)} = (ds_T)_{x_s} P_s + dr_{x_r} P_r$, where P_s and P_r are the projections on the subspace respectively of the first 2 coordinates and the remaining $m - 2$.
Thus suppose $(x_s, x_r) \neq (0, 0)$ and $q_T(x_s, x_r) \in K$. Then two cases can happen: $x_s \neq 0$ and $x_s = 0$. If $x_s \neq 0$ then no supporting hyperplane for K contains the image of the differential $(dq_T)_{(x_s, x_r)}$ because no supporting hyperplane for K contains the image of the differential $(ds_T)_{x_s}$; if $x_s = 0$ then since r is nondegenerate with respect to K, then no supporting hyperplane of K contains the image of the differential $(dq_T)_{(0, x_r)} = dr_{x_r}$. Thus in both cases q_T is nondegenerate with respect to K.
Let now $\{s_n\}_{n>1} \subset \mathcal{Q}(2, 2; K)$ be a sequence of maps such that for every n we have $\sigma(s_n) = [\omega\hat{\omega}z]$, $\Delta(s_n) > 0$ and $s_n \to s_T$.
If we define $q_n = s_n \oplus r$, then clearly $q_n \to q_T$. Since $\mathcal{Q}(m, 2; K)$ is open in $\mathcal{Q}(n, 2)$ and q_T is nondegenerate with respect to K, then there exists $\overline{n}$ such that $q_{\overline{n}}$ is nondegenerate with respect to K and $q_{\overline{n}}$ is K-homotopic to q_T.
Let finally $s_1 = s_{\overline{n}}$, $q_1 = s_1 \oplus r = q_{\overline{n}}$ and q_t be the composition of the two K-homotopies from q_0 to q_T and from q_T to $q_{\overline{n}}$. Then $\sigma(s_1) = [\omega\hat{\omega}z]$, $\Delta(s_1) > 0$ and q_t is the required K-homotopy. □

We describe now a procedure to associate to each $q \in \mathcal{Q}(m, 2; K)$ a word of $m+1$ letters on the set of characters $\{\omega, \hat{\omega}, z\}$.

Again let $\Delta : \mathcal{Q}(m,2) \to \mathbb{R}$ the discriminant of the equation $\det(\omega p) = 0$: it is a polynomial function and $\{\Delta(p) = 0\}$ is a proper algebraic set; hence q is K-homotopic to q' with $\Delta(q') \neq 0$. Applying lemma 2.3.3 we get that q is K-homotopic to a map of the form $(\overset{l}{\underset{j=1}{\oplus}} q_j) \oplus (\overset{b}{\underset{k=1}{\oplus}} p_k)$ with each q_j and each p_k nondegenerate with respect to K. Lemma 2.3.4 allows now to change each q_j with $\Delta(q_j) < 0$ in a q_j' with $\Delta(q_j') > 0$ without losing nondegeneracy w.r.t. K. Thus there exist $e^{i\theta_1}, \dots, e^{i\theta_m}$ such that q, up to K-homotopies, is of the form:

$$q(x) = q(x_1, \dots, x_m) = \sum_{j=1}^{m} e^{i\theta_j} {x_j}^2$$

Slightly perturbing the $e^{i\theta_j}$'s (which does not affect nondegeneracy w.r.t. K) we may assume $\theta_i \neq \pm\theta_j$ for $i \neq j$ and $\theta_j \neq k\pi/2$ for $k \in \mathbb{Z}$ and $j = 1, \dots, m$. Fix now the orientation $(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))$ on $\mathbb{R}^2$ and let $\omega_j = -\hat{\omega}_j \in S^1$, $j = 1, \dots, m$ be such that

$$\langle e^{i\theta_j}, \omega_j \rangle = \langle e^{i\theta_j}, \hat{\omega}_j \rangle = 0 \quad \text{and} \quad (e^{i\theta_j}, \omega_j) \text{ is positively oriented}$$

Exactly as we did for the case $q \in \mathcal{Q}(2,2;K)$ we associate now to $q \in \mathcal{Q}(m,2;K)$ the word $s(q)$ obtained by writing the characters of the point we meet going clockwise on S^1 from $-\hat{z}$ to $\hat{z}$ (omitting the indices). A lot of possibilities can happen now and we introduce the following rules to change one word into another:

(A) $s_1\hat{\omega}zs_2 = s_1z\hat{\omega}s_2$: we can commute $\hat{\omega}$ and z;

(B) $s\omega = \hat{\omega}s$ for every word s with characters in $\{z, \hat{\omega}, \omega\}$: if $\hat{\omega}$ is the last character of one word, we can cancel it and place $\hat{\omega}$ at the beginning of the word as the first character;

(C) $s_1\hat{\omega}s_2\omega s_3zs_4 = s_1\omega s_2\hat{\omega}s_3zs_4$ for every choice of words s_1, s_2, s_3, s_4 with characters in $\{\omega, \hat{\omega}\}$: we can commute $\hat{\omega}$ and ω to the left of z.

We will see that each rule correspond to a precise K-homotopy between two quadratic maps and that the previous are exactly the K-homotopies we can perform. In view of this idea we give the following definition.

Definition 2.3.5. *We define $\mathcal{S}(m,2;K)$ to be the set of equivalence classes of words of maps $q \in \mathcal{Q}(m,2;K)$ under the relation that two words are equivalent if and only if we can change one into the other with the previous rules. We let $\sigma(q)$ be the class of $s(q)$ in $\mathcal{S}(m,2;K)$.*

Before proving the main theorem of this section, we first prove one useful lemma. If $q \in \mathcal{Q}(m,2)$ is given by

$$q(x) = q(x_1, \dots, x_n) = \sum_{j=1}^{n} e^{i\theta_j} {x_j}^2$$

then for every pair of distinct indices (a,b) we define $q_{ab} \in \mathcal{Q}(2,2)$ by

$$q_{ab}(x_a, x_b) = e^{i\theta_a} {x_a}^2 + e^{i\theta_b} {x_b}^2.$$

Lemma 2.3.6. *Let $q \in \mathcal{Q}(m,2)$ be defined by*

$$q(x) = q(x_1, \dots, x_m) = \sum_{j=1}^{m} e^{i\theta_j} {x_j}^2.$$

Then q is nondegenerate with respect to K if and only if q_{ab} is nondgenerate w.r.t. K for every pair of distinct indices (a,b).

Proof. Clearly if q is nondegenerate w.r.t. K then for every pair (a,b) of distinct indices q_{ab} is nondegenerate w.r.t. K.
Viceversa suppose q is degenerate w.r.t. K and let us prove that there exists a pair of distinct indices (a,b) such that q_{ab} is degenerate w.r.t. K.
Degeneracy of q implies that there exists a nonzero vector $x = (x_1, \dots, x_m)$ and a covector $\omega \in K^{\circ} \backslash \{0\}$ such $q(x) \in K$ and $\omega dq_x \equiv 0$.
If all the components of x but x_j were zero, then for every $l \neq j$ we have q_{lj} degenerate w.r.t. K.
If x has $k > 1$ nonzero components, the first k for example, then since

$$dq_x = \sum_{j=1}^{k} 2x_j e^{i\theta_j} dx_j$$

all the vectors $e^{i\theta_1}, \dots, e^{i\theta_k}$ must be collinear, otherwise the rank of dq_x would be 2 (against the fact that there exists $\omega \in K^{\circ} \backslash \{0\}$ such that $\omega dq_x \equiv 0$).
If $e^{i\theta_1} = e^{i\theta_2} = \cdots = e^{i\theta_k}$ then it must be $e^{i\theta_1} \in \partial K$ and thus q_{12} is degenerate w.r.t. K; if among $e^{i\theta_1}, \dots, e^{i\theta_k}$ there are two vectors with different signs, for example $e^{i\theta_1}$ and $e^{i\theta_2}$, then nondegeneracy of q implies no one among $e^{i\theta_1}, \dots, e^{i\theta_k}$ belongs to $\mathrm{int}(K)$; thus either one among them coincides with one generator of the cone K or $q(x) = 0$ and thus $q_{12}(x_1, x_2) = 0 \in K$: in both cases q_{12} is degenerate w.r.t. K.
$\square$

Everything is ready now for the proof of the following theorem, which classifies K-homotopy classes of $\mathcal{Q}(m,2;K)$.

Theorem 2.3.7. *The set $\mathcal{S}(m,2;K)$ calssifies K-homotopy classes of $\mathcal{Q}(m,2;K)$: two maps $q_0, q_1 \in \mathcal{Q}(m,2;K)$ are K-homotopic if and only if*

$$[s(q_0)] = [s(q_1)].$$

Moreover the sequence of rules we have to apply to change $s(q_0)$ to $s(q_1)$ describes one possible K-homotopy.

Proof. Thanks to the previous lemma 2.3.6 if $q \in \mathcal{Q}(m,2;K)$ and we perform a rotation of the $e^{i\theta_j}$'s such that for every pair of distinct indices (a,b) the map q_{ab} is nondegenerate, then the result is a K-homotopy. Thus every rule corresponds to a precise K-homotopy and $\sigma(q_0) = \sigma(q_1)$ implies q_0 and q_1 are K-homotopic. Moreover from the proof of lemma 2.3.4 it follows that if $q = r \oplus s$ with $s \in \mathcal{Q}(2,2)$ and $\Delta(s) < 0$ then q is nondegenerate w.r.t. K if and only if r is; thus iterating the reasoning, if $q = v_1 \oplus \cdots \oplus v_k \oplus s_1 \oplus \cdots \oplus s_l$ with the v_j's representing maps in $\mathcal{Q}(1,2;K)$ and the s_j's maps in $\mathcal{Q}(2,2;K)$ with negative discriminant, then q is nondegenerate w.r.t. K if and only if $v_1 \oplus \cdots \oplus v_k$ is nondegenerate w.r.t. K. Moreover if

$$s(v_1 \oplus \cdots \oplus v_k) = u_1 z u_2$$

with u_1 and u_2 words in $\{\omega, \hat{\omega}\}$, then we have

$$\sigma(q) = [s(v_1 \oplus \cdots \oplus v_k \oplus s_1 \oplus \cdots \oplus s_l)] = [(\omega\hat{\omega})^l u_1 z u_2]$$

where $(\omega\hat{\omega})^l$ we mean the word $\omega\hat{\omega}$ repeated l times.
We prove now that if q_0 and q_1 are K-homotopic, then $\sigma(q_0) = \sigma(q_1)$.
First notice we may assume q_0 and q_1 are in the form given by lemma 2.3.3. As before we may suppose the K-homotopy is generic (i.e. we can avoid sets of codimension grater or equal to two). To a given $q \in \mathcal{Q}(m,2)$ we can associate a linear map $\overline{q} : \mathbb{R}^2 \to \mathcal{Q}(\mathbb{R}^m) \simeq \mathbb{R}^{\frac{m(m+1)}{2}}$ by the correspondence $\omega \mapsto \omega q$. The set of linear maps $L : \mathbb{R}^2 \to \mathcal{Q}(\mathbb{R}^m)$ with rank less or equal to one is an algebraic subset of codimension greater than one, hence it can be avoided (i.e. if there is a K-homotopy q_t then there is also one with $\mathrm{rk}(q_t^*) = 2$ for every t). The set of linear maps $L : \mathbb{R}^2 \to \mathcal{Q}(\mathbb{R}^m)$ such that the image of L is tangent to $Z = \{q \in \mathcal{Q}(\mathbb{R}^m) \mid \det(q) = 0\}$ in at least two distinct lines has codimension greater than one, hence can be avoided: generically a K-homotopy will meet $\{\Delta(q) = 0\}$ only a finite number of time and in these cases only two roots of $\det(\omega q) = 0$ will coincide.
Let A be the set of maps in $\mathcal{Q}(m,2;K)$ with exactly two equal roots of the equation $\det(\omega q) = 0$. Thus let q_t be a generic K-homotopy (in particular $\Delta(q_1) \neq 0$ and $\Delta(q_2) \neq 0$).
It is sufficient to show that each time we meet A the class of the word does not change. Suppose $q_{t_1} = v_1 \oplus \cdots \oplus v_k \oplus s_1 \oplus \cdots \oplus s_l$ for $t_1 < T$, $q_T \in A$ and $q_{t_2} = v'_1 \oplus \cdots \oplus v'_a \oplus s'_1 \oplus \cdots \oplus s'_b$ for $t_2 > T$, where the v_j's and the v'_j's represent maps in $\mathcal{Q}(1,2;K)$ and the remaining s'_j's and s_j's are in $\mathcal{Q}(2,2;K)$ and have negative discriminant. We adopt the convention that if there are no maps of a certain type, then the corresponding number in $\{k,l,a,b\}$ is zero. Assume between t_1 and t_2 the discriminant of q_t vanishes only at T.
When q_t meet A two roots happen to coincide. These could be real before T and real after, or real before T e complex after or viceversa.
In the first case $q_{t_2} = v'_1 \oplus \cdots \oplus v'_k \oplus s'_1 \oplus \cdots \oplus s'_l$ and $\sigma(v_1 \oplus \cdots \oplus v_k) = \sigma(v'_1 \oplus \cdots \oplus v'_k)$ (we simply performed a rule); thus recalling what we stressed at the beginning of the proof, we have $\sigma(q_{t_1}) = \sigma(q_{t_2})$.
In the second case two real roots became complex (switching t_1 and t_2 we get the other case): then it must be $\sigma(q_{t_1}) = [(\omega\hat{\omega})^l u_1 z u_2]$ with $l > 1$ and $[u_1 z u_2] = \sigma(v_1 \oplus \cdots \oplus v_k)$. In this case $q_{t_2} = v'_1 \oplus \cdots \oplus v'_{k-2} \oplus s'_1 \oplus \cdots \oplus s'_{l+1}$ and thus $\sigma(q_{t_2}) = [(\omega\hat{\omega})^{l+1} u'_1 z u'_2]$ with $\sigma(v'_1 \oplus \cdots \oplus v'_{k-2}) = [u'_1 z u'_2]$. On the other side, assuming the last two roots became complex, then because of nondegeneracy they could have done it only in the way we previously described. Moreover from lemma 2.3.6 it follows that the K-homotopy between q_{t_1} e q_{t_2} induces a K-homotopy between $v_1 \oplus \cdots \oplus v_{k-2}$ and $v'_1 \oplus \cdots \oplus v'_{k-2}$. Since during this last homotopy the discriminant never vanishes, then $\sigma(v_1 \oplus \cdots \oplus v_{k-2}) = \sigma(v'_1 \oplus \cdots \oplus v'_{k-2})$ and thus $\sigma(q_{t_1}) = \sigma(q_{t_2})$. This concludes the proof. $\square$

We can choose a canonical representative for $[s(q)] \in \mathcal{S}(n,2;K)$ and adopt the convention that x^r, with $r \in \mathbb{N}$, means that the character x is repeated r times. In this way we have that each $q \in \mathcal{Q}(m,2;K)$ is K-homotopic to a map q' of the form:

$$s(q') = \omega^a \hat{\omega}^b z \omega^{c_1} \hat{\omega}^{d_1} \cdots \omega^{c_r} \hat{\omega}^{d_r}$$

with $a + b + \sum c_j + \sum d_j = m$.
We notice also the following, which immediately follows from the definitions.

Corollary 2.3.8. *If $\eta \in K^\circ \cap S^1$ and $\eta \neq \omega, \neq \hat{\omega}$:*

$$\omega^a \hat{\omega}^b z \omega^{c_1} \hat{\omega}^{d_1} \cdots (\eta) \cdots \omega^{c_r} \hat{\omega}^{d_r}$$

then we have

$$\mathrm{i}^+(\eta) = \hat{\lambda}(\eta) + \rho(\eta),$$

where $\hat{\lambda}(\eta)$ is the number of $\hat{\omega}$ in $s(q)$ on the left of η and $\rho(\eta)$ is the number of ω in $s(q)$ to the right of η.

2.3.3 Three quadrics in the projective plane

The beautiful subject of this section is due to Agrachev and is developed in [1].
We study K-homotopy classes of maps q in $\mathcal{Q}(3,3;K)$ for $K = \{0\}$. It is customary to call homotopies that are nondegenerate with respect to the zero cone *rigid isotopies.* In this case nondegeneracy of the map q with respect to $\{0\}$ is equivalent to $q^{-1}(0) = 0$, i.e. the set $Y(q)$ of spherical solutions of the system $(q, \{0\})$ must be empty. If we were in the complex case, then the set of triples of complex homogeneous polynomials of degree two such that their common zero locus in the complex projective plane is empty will have only one connected component. Indeed its complement has complex codimension at least one, which means that its real codimension is at least two: thus it cannot separate the space of triples and we have only one component. In our case the real codimension of the singular triples is at least one, hence it can separate the space of real triples, namely $\mathcal{Q}(3,3)$. To find an invariant of a connected component, for every $q \in \mathcal{Q}(3,3;\{0\})$ we consider the map

$$\hat{q} : \mathbb{R}\mathrm{P}^2 \to S^2, \quad [x] \mapsto q(x)/\|q(x)\|$$

Notice that the previous setting makes sense since $Y(q) = \emptyset$ and is well defined since $q(x) = q(-x)$. A well known theorem of Hopf states that two maps f_0, f_1 from a nonorientable manifold M of dimension m to the sphere S^m are homotopic if and only if they have the same modulo two degrees. Thus in particular if q_0 and q_1 are in the same connected component of $\mathcal{Q}(3,3;\{0\})$ then they are homotopic and their modulo two degrees coincide. We start by proving the following.

Proposition 2.3.9. *For a generic $q \in \mathcal{Q}(3,3)$ the condition deg$(\hat{q}) = 0$ implies $\hat{q}(\mathbb{R}\mathrm{P}^2) \neq S^2$.*

Proof. For any point $y \in S^2$ in general position the sets $\hat{q}^{-1}(y)$ and $\hat{q}^{-1}(-y)$ must consist of an even number of points. At the same time the set $\hat{q}^{-1}(y) \cup \hat{q}^{-1}(-y)$ is a transverse intersection of two quadrics in $\mathbb{R}\mathrm{P}^2$ and so by Bezout's theorem its either empty or consists of two or four points. If at least one of the sets $\hat{q}^{-1}(y)$ or $\hat{q}^{-1}(-y)$ is empty, there is nothing more to prove. There remains the case in which both $\hat{q}^{-1}(y)$ and $\hat{q}^{-1}(-y)$ consist of two points, i.e. for any point y on S^2 in general position the preimage of y has two points. Since the map $\hat{q} : \mathbb{R}\mathrm{P}^2 \to S^2$ cannot be a cover, then it must have folds and the preimage of one point on one side of the fold must have two points more than the preimage of a point on the other side of the fold. This contradiction completes the proof. □

We also prove the folowing lemma, which gives a condition for two quadratic maps to be rigid isotopic.

Lemma 2.3.10. *Let q_0 and q_1 be in $\mathcal{Q}(n, k; \{0\})$ with $k \geq 2$. If $q_0(\mathbb{R}^n) \neq \mathbb{R}^k$ and $q_1(\mathbb{R}^n) \neq \mathbb{R}^k$, then q_0 and q_1 are rigidly isotopic.*

Proof. Let $l_i \in \mathbb{R}^k \backslash q_i(\mathbb{R}^n)$ for $i = 0, 1$; clearly for $t > 0$ we have $tl_i \in \mathbb{R}^k \backslash q_i(\mathbb{R}^n)$. Let $p \cdot l_i$ denote the quadratic map $x \mapsto \|x\|^2 l_i$. The family $(1-t)q_i - tp \cdot l_i$ defines a rigid isotopy between q_i and $-p \cdot l_i$: indeed if x is nonzero then $(1-t)q_i(x) - tp \cdot l_i \neq 0$, otherwise $q_i(x)$ would belong to the half line spanned by l_i which is not contained in the image of q_i. On the other hand the maps $p \cdot l_0$ and $p \cdot l_1$ are obviously rigidly isotopic. □

Everything is ready now for the proof of the main theorem of this section.

Theorem 2.3.11. *Two maps q_0 and q_1 in $\mathcal{Q}(3, 3; \{0\})$ such that the modulo two degrees of $\hat{q}_0$ and $\hat{q}_1$ coincide and are equal to zero are rigidly isotopic.*

Proof. As already noticed if q_0 and q_1 are rigidly isotopic, then by Hopf's theorem their degree modulo two is the same. On the contrary if the degrees of the associated map are zero, then by slightly perturbing them (which does not affect their rigid isotopy class since $\mathcal{Q}(3, 3; \{0\})$ is open) we may assume by proposition 2.3.9 that $q_i(\mathbb{R}^3) \neq \mathbb{R}^3$ for $i = 0, 1$. The previous lemma 2.3.10 tells now q_0 and q_1 are rigidly isotopic. □

CHAPTER 3

Spectral sequences

Here we fix some notations and make some remarks concerning spectral sequences which will be useful in the sequel. We always make use of $\mathbb{Z}_2$ coefficients, in order to avoid sign problems; the following results still hold for $\mathbb{Z}$ coefficients, but sign must be put appropriately. All the introductory material we present here is covered (up to some small modifications) in [11], to which the reader is referred for more precise details.

3.1 Mayer-Vietoris spectral sequences

We begin with the following.

Lemma 3.1.1. *Let (C_*, ∂_*) be an acyclic free chain complex and (D_*, ∂_*^D) be an acyclic subcomplex. Then there exists a chain homotopy*

$$K_* : C_* \to C_{*+1}$$

such that $\partial_{*+1}K_* + K_{*-1}\partial_* = I_*$ and $K_*(D_*) \subset (D_{*+1})$.

Proof. By taking a right inverse s_{q-1}^D of ∂_q^D, which exists since D_{q-1} and hence Z_{q-1}^D are free, a chain contraction K_q^D for D is defined by: $K_q^D = s_q^D(I_q - s_{q-1}^D\partial_q^D)$. Since Z_q is free, then it is possible to extend s_{q-1}^D to a right inverse s_{q-1} of ∂_q:

$$s_{q-1} : Z_{q-1} = B_{q-1} \to C_q.$$

Then by setting

$$K_q = s_q(I_q - s_{q-1}\partial_q)$$

we obtain a chain contraction for the complex (C_*, ∂_*) which restricts to a chain contraction for the subcomplex (D_*, ∂_*^D). □

Let now X be a topological space and Y be a subspace. Consider an open cover $\mathcal{U} = \{V_\alpha\}_{\alpha\in A}$ for X; we assume A to be ordered. For every $\alpha_0, \dots, \alpha_p \in A$ we define $V_{\alpha_0\cdots\alpha_p}$ to be $V_{\alpha_0} \cap \cdots \cap V_{\alpha_p}$ (sometimes we will use the shortened notations $\overline{\alpha}$ for $(\alpha_0, \dots, \alpha_p)$ and $V_{\overline{\alpha}}$ for $V_{\alpha_0\cdots\alpha_p}$). The Mayer-Vietoris bicomplex $E_0^{*,*}(Y,\mathcal{U})$ for the pair (X, Y) relative to the cover $\mathcal{U}$ is defined by

$$E_0^{p,q}(Y,\mathcal{U}) = \check{C}^p(\mathcal{U},\mathcal{U}\cap Y; C^q) = \prod_{\alpha_0<\cdots<\alpha_p} C^q(V_{\alpha_0\cdots\alpha_p}, V_{\alpha_0\cdots\alpha_p} \cap Y).$$

This bicomplex is endowed with two differentials: $d : E_0^{p,q}(Y,\mathcal{U}) \to E_0^{p,q+1}(Y,\mathcal{U})$ and $\delta : E_0^{p,q}(Y,\mathcal{U}) \to E_0^{p+1,q}(Y,\mathcal{U})$ defined for $\eta = (\eta_{\alpha_0\cdots\alpha_p}) \in E_0^{p,q}(Y,\mathcal{U})$ by:

$$(d\eta)_{\alpha_0\cdots\alpha_p} = d\eta_{\alpha_0\cdots\alpha_p} \quad \text{and} \quad (\delta\eta)_{\alpha_0\cdots\alpha_{p+1}} = \sum_{i=0}^{p} \eta_{\alpha_0\cdots\check{\alpha}_i\cdots\alpha_p}|_{V_{\alpha_0\cdots\alpha_p}}.$$

By the Mayer-Vietoris principle, each row of the augmented chain complex of $E_0^{*,*}(Y,\mathcal{U})$ is exact, i.e. for each $q \geq 0$ the chain complex

$$0 \to C^q_{\mathcal{U}}(X,Y) \to \check{C}^0(\mathcal{U},\mathcal{U}\cap Y, C^q) \to \cdots$$

is acyclic - we recall that $(C^*_{\mathcal{U}}(X,Y), d)$ is defined to be the complex of $\mathcal{U}$-small singular cochains and that the following isomorphism holds:

$$H_d(C^*_{\mathcal{U}}(X,Y)) \simeq H^*(X,Y).$$

From this it follows that

$$H^*(X,Y) \simeq H^*_D(E_0^{*,*}(Y,\mathcal{U}))$$

where $H^*_D(E_0^{*,*}(Y,\mathcal{U}))$ is the cohomology of the complex $E_0^{*,*}(Y,\mathcal{U})$ with differential $D = d + \delta$. We also recall that

$$r^* : C^*_{\mathcal{U}}(X,Y) \to \check{C}^*(\mathcal{U},\mathcal{U}\cap Y, C^*)$$

induces isomorphisms on cohomologies; if we take a chain contraction K for the Mayer-Vietoris rows of the pair (X,Y), then we can define a homotopy inverse f to r^* by the following procedure. If $c = \sum_{i=0}^n c_i$ and $Dc = \sum_{i=0}^{n+1} b_i$ then we set

$$f(c) = \sum_{i=0}^{n} (dK)^i c_i + \sum_{i=0}^{n+1} K(dK)^{i-1} b_i.$$

We define now $E_1^{*,*}(Y,\mathcal{U}) = H_d(E_0(Y,\mathcal{U}))^{*,*}$. The bicomplex $E_1^{*,*}(Y,\mathcal{U})$ is naturally endowed with a differential $d_1(\mathcal{U})$ defined in the following way: let $\eta \in E_0^{p,q}(Y,\mathcal{U})$ be such that $d\eta = 0$, i.e. η defines a class denoted by $[\eta]_1$ in $E_1^{p,q}(Y,\mathcal{U})$; then $d_1(\mathcal{U})[\eta]_1$ is defined to be $[\delta\eta]_1 \in E_1^{p+1,q}(Y,\mathcal{U})$.
In general we say that an element $\eta \in E_0^{p,q}(Y,\mathcal{U})$ such that $d\eta = 0$ can be extended to a *zig-zag* of lenght r if there exist $\eta_i \in E_0^{p+i,q-i}(Y,\mathcal{U})$ for for $i = 0,\ldots,r-1$ such that $\eta_0 = \eta$ and $\delta\eta_i = d\eta_{i+1}$ for every $i = 0,\ldots,r-2$ (notice that this is a necessary condition for η to define a class in $H_D(E_0(Y,\mathcal{U}))$).
Thus $\eta \in E_0^{p,q}(Y,\mathcal{U})$ can be extended to a zig-zag of lenght 1 if and only if $d\eta = 0$, i.e. η defines a class $[\eta]_1 \in E_1^{p,q}(Y,\mathcal{U})$. We define inductively $E_r(Y,\mathcal{U})$ from $E_{r-1}(Y,\mathcal{U})$ in the following way:

$$E_r(Y,\mathcal{U}) = H_{d_{r-1}(\mathcal{U})}(E_{r-1}(Y,\mathcal{U}))$$

and if $\eta \in E_0^{p,q}(Y,\mathcal{U})$ is such that its class is defined in $E_r^{p,q}(Y,\mathcal{U})$ we denote it by $[\eta]_r$; moreover we define the differential $d_r(\mathcal{U}) : E_r(Y,\mathcal{U}) \to E_r(Y,\mathcal{U})$ by the formula:

$$d_r(\mathcal{U})[\eta]_r = [\delta\eta_{r-1}]_r,$$

where $\eta_0, \dots, \eta_{r-1}$ is a zig-zag of lenght r extending η (the fact that this zig-zag exists is ensured by the fact that the class $[\eta]_r$ is defined and similarly for the fact that $[\delta\eta_{r-1}]_r$ is defined). If $\eta \in E_0^{p,q}(Y,\mathcal{U})$ defines a class $[\eta]_r \in E_r^{p,q}(Y,\mathcal{U})$, then it is said to *survive to* $E_r(Y,\mathcal{U})$. If this inductive procedure stabilize, i.e. if we have $E_r(Y,\mathcal{U}) = E_{r+l}(Y,\mathcal{U})$ for some $r \geq 0$ and for every $l \geq 0$, then we denote by $E_\infty(Y,\mathcal{U})$ this stable value. It is a remarkable fact that in this case, setting $E_r^*(Y,\mathcal{U}) = \oplus_{p+q=*} E_r^{p,q}(Y,\mathcal{U})$, we have for every $l \in \mathbb{Z}$:

$$E_\infty^l(Y,\mathcal{U}) \simeq H_D^l(E_0^{*,*}(Y,\mathcal{U})) \simeq H^l(X,Y)$$

but these isomorphisms are not canonical, i.e in general they only tell that the dimensions of the vector spaces coincide.
The sequence of vector spaces with differentials $(E_r(Y,\mathcal{U}), d_r(\mathcal{U}))_{r\geq 0}$ is called the *Mayer-Vietoris spectral sequence relative to* $\mathcal{U}$ and the fact that the previous isomorphism holds translates the sentence that *the spectral sequence converges to* $H_D^*(E_0^{*,*}(Y,\mathcal{U}))$.
We recall also that the bicomplex $E_0^{*,*}(Y,\mathcal{U})$ is endowed with a $\mathbb{Z}_2$-bilinear product $E_0^{p,q}(Y,\mathcal{U}) \times E_0^{r,s}(Y,\mathcal{U}) \to E_0^{p+r,q+s}(Y,\mathcal{U})$ defined for $\eta \in E_0^{p,q}(Y,\mathcal{U}), \psi \in E_0^{r,s}(Y,\mathcal{U})$ by:

$$(\eta \cdot \psi)_{\alpha_0\cdots\alpha_{p+r}} = \eta_{\alpha_0\cdots\alpha_p}|_{V_{\alpha_0\cdots\alpha_{p+r}}} \smile \psi_{\alpha_p\cdots\alpha_{p+r}}|_{V_{\alpha_0\cdots\alpha_{p+r}}},$$

where on the right hand side we perform the usual cup product. The differentials D, d, δ are derivations with respect to this product (we are using $\mathbb{Z}_2$ coefficients and no signs are appearing) and each $E_r(Y,\mathcal{U})$ inherits a product structure from $E_{r-1}(Y,\mathcal{U})$; it is worth noticing that the product structure of $E_\infty^*(Y,\mathcal{U})$ is different from that of $H_D^*(E_0^{*,*}(Y,\mathcal{U}))$.

3.1.1 Leray's spectral sequences

In the case we have a continuous map $f : X \to \Omega$ and an open cover $\mathcal{W}$ of Ω we have that $f^{-1}\mathcal{W}$ is an open cover of X. Setting $\mathcal{U} = f^{-1}\mathcal{W}$ in the previous constuction, the corresponding spectral sequence is named the *relative Leray's spectral sequence of f with respect to the cover* $\mathcal{W}$ (the case $Y = \emptyset$ correspond to the usual Leray's construction as presented in [11].)
If we take the direct limit over all the open covers of Ω (with the natural restriction homomorphisms) we get what is called the *(relative) Leray's spectral sequence of the map f*:

$$(E_r(Y), d_r) = \varinjlim_{\mathcal{U}=f^{-1}\mathcal{W}} \{(E_r(Y,\mathcal{U}), d_r(\mathcal{U}))\}$$

The following result can be stated in much more generality (see [15]), but for our purpose the following version is sufficient.

Theorem 3.1.2. *Let $Y \subset X$ and Ω be semialgebraic sets and $f : X \to \Omega$ be a continuous, semialgebraic map. Then the Leray's spectral sequence of f converges to $H^*(X,Y)$ and the following holds:*

$$E_2^{p,q}(Y) \simeq \check{H}^p(\Omega, \mathcal{F}^q)$$

where $\mathcal{F}^q$ is the sheaf generated by the presheaf $V \mapsto H^q(f^{-1}(V), f^{-1}(V) \cap Y)$.

3.1.2 Some more properties

If we let $Z \subset Y$ be a subspace, then $E_0^{*,*}(Y)$ is naturally included in the Mayer-Vietoris bicomplex $E_0^{*,*}(Z)$ for the pair (X, Z) relative to the cover $\mathcal{U}$ (here we omit to write the $\mathcal{U}$ to avoid heavy notations):

$$i_0 : E_0^{*,*}(Y) \hookrightarrow E_0^{*,*}(Z).$$

Since i_0 obviously commutes with the total differentials, then it induces a morphism of spectral sequence, and thus a map

$$i_0^* : H_D^*(E_0(Y)) \to H_D^*(E_0(Z)).$$

At the same time the inclusion $j : (X, Z) \hookrightarrow (X, Y)$ induces a map

$$j^* : H^*(X, Y) \to H^*(X, Z).$$

With the previous notations we prove the following lemma.

Lemma 3.1.3. *There are group isomorphisms $f_Y^* : H_D^*(E_0(Y)) \to H^*(X, Y)$ and $f_Z^* : H_D^*(E_0(Z)) \to H^*(X, Z)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} H_D^*(E_0(Y)) & \xrightarrow{i_0^*} & H_D^*(E_0(Z)) \\ {\scriptstyle f_Y^*}\downarrow & & \downarrow{\scriptstyle f_Z^*} \\ H^*(X, Y) & \xrightarrow{j^*} & H^*(X, Z) \end{array}$$

Proof. The augmented Mayer-Vietoris complex for the pair (X, Y) relative to $\mathcal{U}$ is a subcomplex of the augmented Mayer-Vietoris complex for the pair (X, Z) relative to $\mathcal{U}$. Thus by Lemma 3.1.1 for every $q \geq 0$ there exists a chain contraction K_Z for the complex

$$0 \to C_{\mathcal{U}}^q(X, Z) \to \check{C}^0(\mathcal{U}, \mathcal{U} \cap Z, C^q) \to \cdots$$

which restricts to a chain contraction K_Y for the complex

$$0 \to C_{\mathcal{U}}^q(X, Y) \to \check{C}^0(\mathcal{U}, \mathcal{U} \cap Y, C^q) \to \cdots$$

We define f_Y and f_Z with the above construction and we take f_Y^* and f_Z^* to be the induced maps in cohomology. Then f_Z restricted to $E_0^{*,*}(Y)$ coincides with f_Y and since j^* is induced by the inclusion $j^\natural : C_{\mathcal{U}}^q(X, Y) \to C_{\mathcal{U}}^q(X, Z)$, then the conclusion follows. $\square$

Remark 4. Notice that $i_0 : E_0^{*,*}(Y) \to E_0^{*,*}(Z)$ induces maps of spectral sequences respecting the bigradings $(i_r)_{a,b} : E_r^{a,b}(Y) \to E_r^{a,b}(Z)$ and thus also a map $i_\infty : E_\infty(Y) \to E_\infty(Z)$. Even tough $E_\infty(Y) \simeq H^*(X, Y)$ and $E_\infty(Z) \simeq H^*(X, Z)$, in general i_∞ does not equal j_* (neither their ranks do); the same considerations hold for the more general case of a map of pairs $f : (X, Y) \to (X', Y')$.

We recall also the following fact. Given a first quadrant bicomplex $E_0^{*,*}$ with total differential $D = d+\delta$ and associated convergent spectral sequence $(E_r, d_r)_{r\geq 0}$, then

$$E_\infty^* \simeq H_D^*(E_0)$$

and there is a *canonical* homomorphism

$$p_E : H_D^*(E_0) \to E_\infty^{0,*}$$

constructed as follows. Let $[\psi]_D \in H_D^k(E_0)$; then there exists $\psi_i \in E_0^{i,k-i}$ for $i = 0, \ldots, k$ such that $D(\psi_0 + \cdots + \psi_k) = 0$ and

$$[\psi]_D = [\psi_0 + \cdots + \psi_k]_D.$$

By definition of the differentials $d_r, r \geq 0$, the element ψ_0 survives to E_∞. We check that the correspondence

$$p_E : [\psi]_D \mapsto [\psi_0]_\infty$$

is well defined: since $\psi_0 \in E_0^{0,k}$ and $E_0^{i,j} = 0$ for $i < 0$, then $[\psi_0]_\infty = [\psi_0']_\infty$ if and only if ψ_0 and ψ_0' survive to E_∞ and $[\psi_0]_1 = [\psi_0']_1$; if $\psi = \psi' + D\phi$, then $\psi_0 = \psi_0' + d\phi_0$ and thus $[\psi_0]_1 = [\psi_0']_1$.

3.2 Homogeneous spherical case

Let $q \in \mathcal{Q}(n+1, 2)$ and $K \subset \mathbb{R}^2$ be a closed polyhedral cone. Recall that we defined the set of spherical solution of the system (q, K) to be

$$Y = \{x \in S^n \,|\, q(x) \in K\}.$$

We recall that the map q defines a map $\overline{q} : \Omega \to \mathcal{Q}(\mathbb{R}^{n+1})$, where Ω is the intersection of the polar cone K° with the unit sphere. Using the above notations for a family of quadratic forms we set

$$\Omega^j = \{\omega \in \Omega \,|\, \mathrm{i}^+(\omega q) \geq j\}.$$

In the first chapter we stated (without a proof) the following formula for the $\mathbb{Z}_2$-Betti numbers of Y :

$$\tilde{b}_k(Y) = \tilde{b}_{n-k-1}(S^n \backslash Y) = b_0(\Omega^{n-k}, \Omega^{n-k+1}) + b_1(\Omega^{n-k-1}, \Omega^{n-k}), \quad k < n-2$$

Here, using the technique of spectral sequence, we give a direct proof of the previous formula.

Proof. As already noticed the first equality follows from Alexander duality. For the second consider the set

$$B = \{(\omega, x) \in \Omega \times S^n \,|\, (\omega q)(x) > 0\}.$$

The projection $p_2 : B \to S^n$ gives a homotopy equivalence $B \sim p_2(B) = S^n \backslash Y$ (the fibers are contractible). On the other corollary 1.2.4 guarantess that for $\epsilon > 0$ sufficiently small the inclusion

$$B(\epsilon) = \{(\omega, x) \in \Omega \times S^n \,|\, (\omega q)(x) \geq \epsilon\} \hookrightarrow B$$

is a homotopy equivalence. Consider $\pi = p_1|_{B(\epsilon)} : B(\epsilon) \to \Omega$ and the Leray spectral sequence associated to it:

$$(E_r(\epsilon), d_r) \Rightarrow H^*(B(\epsilon); \mathbb{Z}_2), \ E_2(\epsilon)^{i,j} = \check{H}^i(\Omega, \mathcal{F}^j(\epsilon)),$$

where $\mathcal{F}^j(\epsilon)$ is the sheaf associated to the presheaf $V \mapsto \tilde{H}^j(\pi^{-1}(V))$. Since $B(\epsilon)$ and Ω are locally compact and π is proper ($B(\epsilon)$ is compact) then the following isomorphism holds for the stalk of $\mathcal{F}^j(\epsilon)$ at each point $\omega \in \Omega$:

$$\mathcal{F}^j(\epsilon)_\omega \simeq \tilde{H}^j(\pi^{-1}(\omega)).$$

Let $g \in \mathbb{R}[x_0, \ldots, x_n]_{(2)}$ such that $S^n = \{g(x) = 1\}$, then $\pi^{-1}(\omega) \simeq \{x \in S^n \,|\, (\omega q - \epsilon g)(x) \geq 0\}$ has the homotopy type of a sphere of dimension $n - \mathrm{ind}^-(\omega q - \epsilon g)$; thus if we set $\mathrm{i}^-(\epsilon)$ for the function $\omega \mapsto \mathrm{ind}^-(\omega q - \epsilon g)$, we have that for $j > 0$ the sheaf $\mathcal{F}^j(\epsilon)$ is locally constant with stalk $\mathbb{Z}_2$ on the set $\Omega_{n-j}(\epsilon) \backslash \Omega_{n-j-1}(\epsilon)$, where $\Omega_{n-j}(\epsilon) = \{\mathrm{i}^-(\epsilon) \leq n - j\}$, and zero on its complement. Since $\Omega_{n-j-1}(\epsilon)$ is closed in $\Omega_{n-j}(\epsilon)$, we have for $j > 0$:

$$\check{H}^i(\Omega, \mathcal{F}^j(\epsilon)) = \check{H}^i(\Omega_{n-j}(\epsilon), \Omega_{n-j-1}(\epsilon)).$$

Since the sets $\{\Omega_{n-j}(\epsilon)\}_{j \in \mathbb{N}}$ are CW-subcomplex of the one-dimensional complex S^1, then $E_2^{i,j}(\epsilon) = 0$ for $i \geq 2$ (we can take triple intersections of open sets in the cover to be empty) and the Leray spectral sequence of π degenerates at $E_2(\epsilon)$. By semialgebraic triviality the topology of $\Omega_{n-j}(\epsilon)$ is definitely constant in ϵ and form small ϵ we have

$$E_2^{i,j}(\epsilon) \simeq \varinjlim \{\check{H}^i(\Omega_{n-j}(\epsilon), \Omega_{n-j-1}(\epsilon))\}, \quad j > 0.$$

Lemma 1.3.4 implies $E_2^{i,j}(\epsilon) \simeq \check{H}^i(\Omega^{j+1}, \Omega^{j+2})$ and the conclusion follows. □

Remark 5. In the case of more than two quadrics, the same argument yields a spectral sequence $(E_r, d_r)_{r \geq 0}$ converging to the cohomology of Y such that for $j > 0$:

$$E_2^{i,j} = H^i(\Omega^{j+1}, \Omega^{j+2}).$$

The anomaly at $j = 0$ is due to the fact that there is no canonical choice of the generator of $H^0(S^0) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

3.3 The main spectral sequence

From now on the object of our interest will be the set of the solutions of *projective* solutions of a system of quadratic inequalities. Namely we consider a quadratic map $p \in \mathcal{Q}(n+1, k+1)$ and polyhedral cone $K \subset \mathbb{R}^{k+1}$ and we define the set

$$X = \{[x] \in \mathbb{R}\mathrm{P}^n \,|\, p(x) \in K\}.$$

Using our standard notation we define the map $p^* : \Omega \to \mathcal{Q}(\mathbb{R}^{n+1})$ and the sets

$$\Omega^j = \{\omega \in \Omega \,|\, \mathrm{i}^+(\omega p) \geq j\}.$$

In the previous section we introduced many times the correspondence space

$$B = \{(\omega, [x]) \in \Omega \times \mathbb{R}\mathrm{P}^n \,|\, (\omega p)(x) > 0\}$$

and we stated some of its properties, the most important of which is that it is homotopy equivalent to $\mathbb{R}\mathrm{P}^n \backslash X$. Here we investigate deeply the topology of B to get result on the topology of X. We have that $B \subset \Omega \times \mathbb{R}\mathrm{P}^n$ and we call β_l and β_r the restrictions to B of the projection on the first and the second factor.

Lemma 3.3.1. *The projection β_r on the second factor defines a homotopy equivalence between B and $\mathbb{R}\mathrm{P}^n \backslash X = \beta_r(B)$.*

Proof. The equality $\beta_r(B) = \mathbb{R}\mathrm{P}^n \backslash X$ follows from $(K^\circ)^\circ = K$. For every $x \in \mathbb{R}\mathrm{P}^n$ the set $\beta_r^{-1}(x)$ is the intersection of the set $\Omega \times \{x\}$ with an open half space in $(\mathbb{R}^{k+1})^* \times \{x\}$. Let (ω_x, x) be the center of gravity of the set $\beta_r^{-1}(x)$. It is easy to see that ω_x depends continuosly on $x \in \beta_r(B)$. Further it follows form convexity considerations that $(\omega_x / \|\omega_x\|, x) \in B$ and for any $(\omega, x) \in B$ the arc $(\frac{t\omega_x + (1-t)\omega_x}{\|t\omega_x + (1-t)\omega_x\|}, x)$, $0 \leq t \leq 1$ lies entirely in B. It is clear that $x \mapsto (\omega_x / \|\omega_x\|, x)$, $x \in \beta_r(B)$ is a homotopy inverse to β_r. $\square$

We first construct a slightly more general spectral sequence (F_r, d_r) converging to $H^*(\Omega \times \mathbb{R}\mathrm{P}^n, B)$ which in general is *not* isomorphic to $H_{n-*}(X)$. The required spectral sequence (E_r, d_r) arises by applying the following Theorem to a modification $(\hat{q}, \hat{K})$ of the pair (q, K) such that $H^*(\hat{\Omega} \times \mathbb{R}\mathrm{P}^n, \hat{B}) \simeq H_{n-*}(X)$.

Theorem 3.3.2. *There exists first quadrant cohomology spectral sequence (F_r, d_r) converging to $H^*(\Omega \times \mathbb{R}\mathrm{P}^n, B; \mathbb{Z}_2)$ such that for every $i, j \geq 0$*

$$F_2^{i,j} = H^i(\Omega, \Omega^{j+1}; \mathbb{Z}_2).$$

Proof. Fix a positive definite form and consider the well defined function $\alpha : \Omega \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}$ defined by $(\omega, x) \mapsto (\omega p)(x)$. The function α is continuos, proper ($\Omega \times \mathbb{R}\mathrm{P}^n$ is compact), semialgebraic and $B = \{\alpha > 0\}$. By corollary 1.2.4, there exists $\epsilon > 0$ such that the inclusion:

$$B(\epsilon) = \{\alpha > \epsilon\} \hookrightarrow B$$

is a homotopy equivalence.
Consider the projection $\beta_l(\epsilon) : B(\epsilon) \to \Omega$ on the first factor; then by theorem 3.1.2 there exists a cohomology spectral sequence $(F_r(\epsilon), d_r(\epsilon))$ converging to the cohomology group $H^*(\Omega \times \mathbb{R}\mathrm{P}^n, B(\epsilon); \mathbb{Z}_2) \simeq H^*(\Omega \times \mathbb{R}\mathrm{P}^n, B; \mathbb{Z}_2)$ such that:

$$F_2^{i,j}(\epsilon) = \check{H}^i(\Omega, \mathcal{F}^j(\epsilon))$$

where $\mathcal{F}^j(\epsilon)$ si the sheaf generated by the presheaf $V \mapsto H^j(V \times \mathbb{R}\mathrm{P}^n, \beta_l(\epsilon)^{-1}(V); \mathbb{Z}_2)$. Let now ω be in Ω; then for the stalk $(\mathcal{F}_j(\epsilon))_\omega = \varinjlim_{\omega \in V} \mathcal{F}_j(\epsilon)(V)$ we have from Lemma 2.2.11

$$(\mathcal{F}_j(\epsilon))_\omega \simeq H^j(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^{n-\mathrm{ind}^-(\omega p - \epsilon q_0)})$$

Hence if we set $\mathrm{i}^-(\epsilon)$ for the function $\omega \mapsto \mathrm{ind}^-(\omega p - \epsilon q_0)$, the following holds:

$$(\mathcal{F}^j(\epsilon))_\omega = \begin{cases} \mathbb{Z}_2 & \text{if } \mathrm{i}^-(\epsilon)(\omega) > n - j; \\ 0 & \text{otherwise} \end{cases}$$

Thus the sheaf $\mathcal{F}^j(\epsilon)$ is zero on the closed set $\Omega_{n-j}(\epsilon) = \{\mathrm{i}^-(\epsilon) \leq n-j\}$ and is locally constant with stalk $\mathbb{Z}_2$ on its complement; hence:

$$F_2^{i,j}(\epsilon) = \check{H}^i(\Omega, \mathcal{F}^j(\epsilon)) = \check{H}^i(\Omega, \Omega_{n-j}(\epsilon); \mathbb{Z}_2).$$

Consider now, for $\epsilon > 0$, the complex $(F_0(\epsilon), D(\epsilon) = d + \delta)$. Then for $\epsilon_1 < \epsilon_2$ the inclusion $C(\epsilon_2) \hookrightarrow C(\epsilon_1)$ defines a morphism of filtered differential graded modules $i_0(\epsilon_1, \epsilon_2) : (F_0(\epsilon_1), D(\epsilon_1)) \to (F_0(\epsilon_2), D(\epsilon_2))$ turning $\{(F_0(\epsilon), D(\epsilon))\}_{\epsilon>0}$ into an inverse system and thus $\{(F_r(\epsilon), d_r(\epsilon))\}_{\epsilon>0}$ into an inverse system of spectral sequences. We define

$$(F_r, d_r) = \varprojlim_{\epsilon}\{(F_r(\epsilon), d_r(\epsilon))\}.$$

We examine $i_2(\epsilon_1, \epsilon_2) : F_2^{i,j}(\epsilon_1) \to F_2^{i,j}(\epsilon_2)$; it is readily verified that for $i, j \geq 0$ the map $i_2(\epsilon_1, \epsilon_2)_{i,j} : F_2^{i,j}(\epsilon_1) \to F_2^{i,j}(\epsilon_2)$ equals the map

$$i^*(\epsilon_1, \epsilon_2) : \check{H}^i(\Omega, \Omega_{n-j}(\epsilon_1)) \to \check{H}^i(\Omega, \Omega_{n-j}(\epsilon_2))$$

given by the inclusion of pairs $(\Omega, \Omega_{n-j}(\epsilon_2)) \hookrightarrow (\Omega, \Omega_{n-j}(\epsilon_1))$. By semialgebraicity $i^*(\epsilon_1, \epsilon_2)$ is an isomorphism for small ϵ_1, ϵ_2, hence $i_2(\epsilon_1, \epsilon_2)$ is definitely an isomorphism and thus $i_\infty(\epsilon_1, \epsilon_2)$ and $i_0^*(\epsilon_1, \epsilon_2) : H_D^*(F_0(\epsilon_1)) \to H_D^*(F_0(\epsilon_2))$ are definitely isomorphisms. Thus we have

$$F_2^{i,j} \simeq \varprojlim\{H^i(\Omega, \Omega_{n-j}(\epsilon); \mathbb{Z}_2)\}.$$

Lemma 1.3.4 gives $\varinjlim\{H_*(\Omega, \Omega_{n-j}(\epsilon))\} \simeq H_*(\Omega, \Omega^{j+1})$ (using the long exact sequences of pairs). The chain of isomorphisms

$$\varprojlim\{H^i(\Omega, \Omega_{n-j}(\epsilon); \mathbb{Z}_2)\} \simeq (\varinjlim\{H_i(\Omega, \Omega_{n-j}(\epsilon); \mathbb{Z}_2)\})^* = (H_i(\Omega, \Omega^{j+1}; \mathbb{Z}_2))^*$$

finally gives

$$F_2^{i,j} = H^i(\Omega, \Omega^{j+1}; \mathbb{Z}_2).$$

$\square$

Remark 6. Lemma 2.2.11 is not really needed to construct the spectral sequence (we have not used it for the spherical case). If we consider $C(\epsilon) = \{(\omega, x) \in \Omega \times \mathbb{R}\mathrm{P}^n : (\omega p)(x) \geq 0)\}$ then by lemma 1.2.4 the inclusion $C(\epsilon) \hookrightarrow B$ is a homotopy equivalence for ϵ small enough. Consider the projection $\beta_l(\epsilon) : C(\epsilon) \to \Omega$ on the first factor; then by theorem 3.1.2 there exists a cohomology spectral sequence $(F_r(\epsilon), d_r(\epsilon))$ converging to the cohomology group $H^*(\Omega \times \mathbb{R}\mathrm{P}^n, C(\epsilon); \mathbb{Z}_2) \simeq H^*(\Omega \times \mathbb{R}\mathrm{P}^n, B; \mathbb{Z}_2)$ such that:

$$F_2^{i,j}(\epsilon) = \check{H}^i(\Omega, \mathcal{F}^j(\epsilon))$$

where $\mathcal{F}^j(\epsilon)$ si the sheaf generated by the presheaf $V \mapsto H^j(V \times \mathbb{R}\mathrm{P}^n, \beta_l(\epsilon)^{-1}(V); \mathbb{Z}_2)$. Since $C(\epsilon)$ and Ω are locally compact and $\beta_l(\epsilon)$ is proper ($C(\epsilon)$ is compact), then the following isomorphism holds for the stalk of $\mathcal{F}^j(\epsilon)$ at each $\omega \in \Omega$ (see [15], Remark 4.17.1, p. 202):

$$(\mathcal{F}^j(\epsilon))_\omega \simeq H^j(\{\omega\} \times \mathbb{R}\mathrm{P}^n, \beta_l(\epsilon)^{-1}(\omega); \mathbb{Z}_2).$$

The set $\beta_l(\epsilon)^{-1}(\omega) = \{x \in \mathbb{R}\mathrm{P}^n \,|\, (\omega p)(x) \geq \epsilon\} = \{x \in \mathbb{R}\mathrm{P}^n \,|\, (\omega p - \epsilon q_0)(x) \geq 0\}$ has the homotopy type of a projective space of dimension $n - \mathrm{ind}^-(\omega p - \epsilon q_0)$ and it follows that, as above, $F_2^{i,j}(\epsilon) \simeq H^i(\Omega, \Omega_{n-j}(\epsilon))$. Letting ϵ be small enough, Lemma 1.3.4 gives as before

$$\varprojlim_{\epsilon}\{F_2^{i,j}(\epsilon)\} = H^i(\Omega, \Omega^{j+1}).$$

It is possible to show that actually the two spectral sequences agree, but we prefer the previous approach because it is more practical for computations.

Remark 7. In the case $K \neq -K$, i.e. $\Omega \neq S^l$, then (E_r, d_r) converges to $H_{n-*}(X, \mathbb{Z}_2)$. This follows by comparing the two cohomology long exact sequences of the pairs $(\Omega \times \mathbb{R}\mathrm{P}^n, B)$ and $(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \backslash X)$ via the map β_r. In this case $\beta_r : \Omega \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^n$ is a homotopy equivalence and the Five Lemma and Lemma 3.3.1 together give

$$H^*(\Omega \times \mathbb{R}\mathrm{P}^n, B) \simeq H^*(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \backslash X) \simeq H_{n-*}(X)$$

the last isomorphism being given by Alexander-Pontryagin Duality.

Theorem 3.3.3 (The spectral sequence)**.** *There exists a cohomology spectral sequence of the first quadrant (E_r, d_r) converging to $H_{n-*}(X; \mathbb{Z}_2)$ such that*

$$E_2^{i,j} = H^i(C\Omega, \Omega^{j+1}; \mathbb{Z}_2).$$

Proof. Keeping in mind the previous remark, we work the general case (i.e. also the case $K = \{0\}$). We replace K with $\hat{K} = (-\infty, 0] \times K$, the map p with the map $\hat{p} : \mathbb{R}^{n+1} \to \mathbb{R}^{k+2}$ defined by $\hat{p} = (-q_0, p)$, where q_0 is a positive definite form and Ω with

$$\hat{\Omega} = \hat{K}^\circ \cap S^{k+1}.$$

We also define

$$\hat{\Omega}^{j+1} = \{(\eta, \omega) \in \hat{\Omega} \,|\, \mathrm{ind}^+(\omega p - \eta q_0) \geq j+1\}.$$

Then, by construction,

$$\hat{p}^{-1}(\hat{K}) = p^{-1}(K) = X.$$

Applying Theorem 3.3.2 to the pair $(\hat{p}, \hat{K})$, with the previous remark in mind, we get a spectral sequence $(\hat{E}_r, \hat{d}_r)$ converging to $H_{n-*}(X; \mathbb{Z}_2)$ with

$$\hat{E}_2^{i,j} = H^i(\hat{\Omega}, \hat{\Omega}^{j+1}; \mathbb{Z}_2).$$

We identify Ω^{j+1} with $\hat{\Omega}^{j+1} \cap \{\eta = 0\}$ and we claim that the inclusion of pairs $(\hat{\Omega}, \Omega^{j+1}) \hookrightarrow (\hat{\Omega}, \hat{\Omega}^{j+1})$ induces an isomorphism in cohomology. This follows from the fact that $\hat{\Omega}^{j+1}$ deformation retracts onto Ω^{j+1} along the meridians (the deformation retraction is defined since $j \geq 0$ and $\mathrm{i}^+(1, 0, \ldots, 0) = 0$, thus the "north pole" of S^{k+1} does not belong to any of the $\hat{\Omega}^{j+1}$). If $\eta_1 \leq \eta_2$ then $\mathrm{ind}^+(\omega p - \eta_1 q_0) \geq \mathrm{ind}^+(\omega p - \eta_2 q_0)$: thus if $(\eta, \omega) \in \hat{\Omega}^{j+1}$ then all the points on the meridian arc connecting (η, ω) with $\Omega = \hat{\Omega} \cap \{\eta = 0\}$ belong to $\hat{\Omega}^{j+1}$.
Noticing that $(\hat{\Omega}, \Omega^{j+1}) \approx (C\Omega, \Omega^{j+1})$, where $C\Omega$ stands for the topological space cone of Ω, concludes the proof. $\square$

Corollary 3.3.4. *Let $\mu = \max_{\eta \in \Omega} \mathrm{i}^+(\eta)$, and $0 \leq b \leq n - \mu - k$ then*

$$H_b(X) = \mathbb{Z}_2.$$

In particular if $n \geq \mu + k$ then X is nonempty.

Proof. Simply observe that the group $E_2^{0,n-b}$ equals $\mathbb{Z}_2$ for $0 \leq b \leq n-\mu-k$ and that all the differentials $d_r : E_r^{0,n-b} \to E_r^{r,n-b+r-1}$ for $r \geq 0$ are zero, since they take values in zero elements. Hence

$$\mathbb{Z}_2 = E_\infty^{0,n-b} = H_b(X).$$

$\square$

We can also derive the following formula, which gives the Euler characteristic of X.

Corollary 3.3.5 (Euler characteristic formula)**.**

$$\chi(X) = \sum_{j=0}^{n} (-1)^{n+j} \chi(C\Omega, \Omega^{j+1})$$

Proof. It is a direct consequence of theorem 3.3.3 and the fact that in a spectral sequence each term is the homology of its predecessor. $\square$

3.4 The second differential

3.4.1 Preliminaries

We continue in this section the discussion on the properties of a given smooth (semi-algebraic) map

$$f : \Omega \to \mathcal{Q}(V)$$

where we assume Ω is diffeomorphic to a convex set (in the case of our major interest we will have $\Omega = K^\circ \cap S^k$. Recall that we have defined for every subset U of Ω the correspondence space

$$B_f(U) = \{(\omega, x) \in U \times \mathbb{RP}^n \,|\, f(\omega)(x) > 0\}.$$

Let now $M(\omega) < 0$ be a number such that

$$\lambda_{n+2-\mathrm{i}^-(f(\omega))}(f(\omega)) < M(\omega)$$

(notice that by definition $\lambda_{n+2-\mathrm{i}^-(\omega)}(f(\omega))$ is the biggest negative eigenvalue of $f(\omega)$). Then by continuity there exists $\delta_2''(\omega)$ such that for every neighborhood V of ω with $\mathrm{diam}(V) < \delta_2''(\omega)$ and for every $\eta \in V$

$$\lambda_{n+2-\mathrm{i}^-(f(\omega))}(f(\eta)) < M(\omega).$$

Thus for every neighborhood U of ω with $\mathrm{diam}(U) < \delta_2''(\omega)$ we define:

$$P^-(\omega, U) = \{x \in \mathbb{RP}^n \,|\, \text{there exists } \eta \in U \text{ s.t. } x \in P^-_{n+1-\mathrm{i}^-(f(\omega))}(f(\eta))\}.$$

For $x, y \in \Omega$ we denote by $\mathrm{dist}(x, y)$ their euclidean distance and for $r > 0$ we set $B(x, r) = \{\omega \in \Omega \,|\, \mathrm{dist}(x, \omega) \leq r\}$. We claim the following.

Lemma 3.4.1. *For every $\omega \in \Omega$ there exists $0 < \delta_2'(\omega) < \delta_2''(\omega)$ such that for every neighborhood of ω with diam$(V) < \delta_2'(\omega)$*

$$\mathrm{Cl}(P^-(\omega, V)) \subseteq \mathbb{R}\mathrm{P}^n \backslash \{f(\omega)(x) \geq 0\}.$$

Proof. By absurd suppose for every $k \in \mathbb{N}$ the two sets $\mathrm{Cl}(P^-(\omega, B(\omega, 1/k)))$ and $\{f(\omega)(x) \geq 0\}$ intersect. Then for every $k \in \mathbb{N}$ there exists a sequence $x_k^l \to x_k$ such that for every x_k^l there exists $\omega_k^l \in B(\omega, 1/k)$ such that $x_k^l \in P^-_{n+1-\mathrm{i}^-(\omega)}(f(\omega_k^l))$ and $f(\omega)(x_k) \geq 0$.
Then it follows that $f(\omega_k^l)(x_k^l) < M(\omega)$ and, by extracting convergent subsequences, that

$$0 \leq \lim_{k \to \infty} f(\omega)(x_k) = \lim_{k \to \infty} f(\omega_k)(x_k) \leq M(\omega)$$

which is absurd since $M(\omega) < 0$ by definition. □

Lemma 3.4.2. *For every $\omega \in \Omega$ there exists $0 < \delta_2(\omega) < \delta_2''(\omega)$ such that for every neighborhood V of ω with diam$(V) < \delta_2(\omega)$ the following holds:*

$$\mathrm{Cl}(P^-(\omega, V)) \subset \mathbb{R}\mathrm{P}^n \backslash \beta_r(B_f(V)).$$

Moreover in the case f is semialgebraic, then $\omega \mapsto \delta_2(\omega)$ can be chosen semialgebraic.

Proof. Let W be a neighborhood of ω with $\mathrm{diam}(W) < \delta_2'(\omega)$. Then the two compact sets $\mathrm{Cl}(P^-(\omega, W))$ and $\{f(\omega)(x) \geq 0\}$ do not intersect by the previous Lemma. Consider the continuous function $a : \mathrm{Cl}(W) \times \mathbb{R}\mathrm{P}^n \to \mathbb{R}$ defined by $a(\eta, x) = f(\eta)(x)$ and a neighborhood U of $\{f(\omega)(x) \geq 0\}$ in $\mathbb{R}\mathrm{P}^n$ disjoint form $\mathrm{Cl}(P^-(\omega, W))$. Then $\beta_r^{-1}(U) \cap \{a \geq 0\}$ is an open neighborhood of $\{\omega\} \times \{f(\omega)(x) \geq 0\}$ in $\{a \geq 0\}$. Consider now $b : \{a \geq 0\} \to \mathbb{R}$ defined by $(\eta, x) \mapsto d(\eta, \omega)$. Then, since $\{a \geq 0\}$ is compact, the family $\{b^{-1}[0, \delta)\}_{\delta > 0}$ is a fundamental system of neighborhoods of $b^{-1}(0) = \{\omega\} \times \{f(\omega)(x) \geq 0\}$ in $\{a \geq 0\}$. Thus there exists $\overline{\delta}$ such that $b^{-1}[0, \overline{\delta}) \subset \beta_r^{-1}(U) \cap \{a \geq 0\}$. Hence any $\delta_2(\omega)$ such that $B(\omega, 3\delta_2(\omega)) \subset B(\omega, \overline{\delta}) \cap W$ satisfies the requirement, since every neighborhood V of ω with $\mathrm{diam}(V) < \delta_2(\omega)$ is contained in $B(\omega, 3\delta_2(\omega))$ and

$$\begin{aligned}\mathrm{Cl}(P^-(\omega, B(\omega, 3\delta_2(\omega)))) &\subset \mathrm{Cl}(P^-(\omega, W)) \\ &\subset \mathbb{R}\mathrm{P}^n \backslash \beta_r(\{a \geq 0\}) \subset \mathbb{R}\mathrm{P}^n \backslash \beta_r(B_f(B(\omega, 3\delta_2(\omega)))).\end{aligned}$$

Suppose now that f is semialgebraic. Then the set $S = \{(\omega, \delta) \in \Omega \times (0, \infty) \,|\, \forall r < 2\delta, \forall x \in \mathrm{Cl}(P^-(\omega, B(\omega, r))) \,|\, x \in \mathbb{R}\mathrm{P}^n \backslash \beta_r(B_f(B(\omega, r)))\}$ is semialgebraic too. Let $g : S \to \Omega$ be the restriction of the projection on the first factor; then g is semialgebraic and by the previous part of the proof it is surjective (for every $\omega \in \Omega$ there exists a δ satisfying the query). Proposition 1.2.6 implies that g has a semialgebraic section $\omega \mapsto (\omega, \delta_2(\omega))$ and δ_2 is the required semialgebraic function. □

3.4.2 Construction of regular covers

The aim of this section is to detect a family of covers of Ω, cofinal in the family of all covers, for which the direct limit map for our spectral sequence will be an isomorphism and such that they will be practical for computations.

Lemma 3.4.3. *Let $f : \Omega \to \mathcal{Q}(V)$ be a smooth map transversal to all strata of $Z(V) = \coprod Z_j$. For every $\omega \in \Omega$ let $U_{f(\omega)}$ and $\phi : U_{f(\omega)} \to \mathcal{Q}(ker(f(\omega))$ be defined by setting $q_0 = f(\omega)$ in proposition 1.3.1. Then there exists $\delta_3'(\omega) > 0$ and $\psi : B(\omega, \delta_3'(\omega)) \to \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l$, where $l + dim(\mathcal{Q}(ker(f(\omega))) = dim(\Omega)$, such that ψ is a diffeomorphism onto its image and the following diagram is commutative:*

$$\begin{tikzcd} B(\omega, \delta_3'(\omega)) \arrow[rr, "\psi"] \arrow[dr, "\phi \circ f"'] & & \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l \arrow[dl, "p_1"] \\ & \mathcal{Q}(\ker f(\omega)) & \end{tikzcd}$$

Moreover if f is semialgebraic then $\omega \mapsto \delta_3'(\omega)$ can be chosen to be semialgebraic.

Proof. If $\det(f(\omega)) \neq 0$ then let $\delta_3'(\omega) > 0$ be such that $f(B(\omega, \delta_3'(\omega))) \cap Z = \emptyset$; in the contrary case let $f(\omega) \in Z_j$ for some j. Consider $\phi : U_{f(\omega)} \to \mathcal{Q}(\ker f(\omega))$ the map given by the previous proposition. Since $d\phi_{f(\omega)}p = p_{|\ker f(\omega)}$ then $d\phi_{f(\omega)}$ is surjective. On the other hand by transversality of f to Z_j we have:

$$\mathrm{im}(df_\omega) + T_{f(\omega)}Z_j = \mathcal{Q}$$

Since $\phi(Z_j) = \{0\}$ (notice that this condition implies $(d\phi_{f(\omega)})|_{T_{f(\omega)}Z_j} = 0$) then

$$\mathcal{Q}(\ker f(\omega)) = \mathrm{im}(d\phi_{f(\omega)}) = \mathrm{im}(d(\phi \circ f)_\omega)$$

which tells $\phi \circ f$ is a submersion at ω. Thus by the rank theorem there exists U_ω and a diffeomorphism onto its image $\psi : U_\omega \to \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l$ such that $p_1 \circ \psi = \phi \circ f$. Taking $\delta_3'(\omega) > 0$ such that $B(\omega, \delta_3'(\omega)) \subset U_\omega$ concludes the proof.
In the case f is semialgebraic, then the set

$$S = \{(\omega, \delta) \in \Omega \times (0, \infty) \, : \, \psi|_{B(\omega,\delta)} \text{ is a diffeomorphism}\}$$

is semialgebraic too (by semialgebraic rank theorem ψ is semialgebraic (see [10]) and the condition to be a diffeomorphism is a semialgebraic condition on its Jacobian). By the previous part of the proof we have that the restriction $g|_S$ of the projection on the first factor is surjective and the semialgebraic choice for δ_3' follows (as in the proof of lemma 3.4.2) from proposition 1.2.6. □

Corollary 3.4.4. *Under the assumption of lemma 3.4.3, for every $\omega \in \Omega$ there exists $\delta_3(\omega) > 0$ such that for every $B(\omega', r) \subset B(\omega, 3\delta_3(\omega))$ with $r < \delta_3(\omega)$ then*

$$\psi(B(\omega', r)) \quad \textit{is convex.}$$

In particular if $\omega \in B(\omega_k, r_k)$ for some $\omega_0, \ldots, \omega_i \in \Omega$ and $r_0, \ldots, r_i < \delta_3(\omega)$, then for every $j \in \mathbb{N}$ the space

$$\{\eta \in \Omega \,|\, \mathrm{i}^-(f(\eta)) \leq n - j\} \cap (\bigcap_{k=0}^{i} B(\omega_k, r_k)) \quad \textit{is acyclic.}$$

Moreover if f is semialgebraic, then δ_3 can be chosen semialgebraic.

Proof. The first part of the statement follows by applying lemma 3.4.3 and corollary 2.1.2 to $\psi : U_\omega \to \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l$.
For the second part notice that by Proposition 1.3.1 we have for every $\eta \in U_\omega$ (using the above notations):

$$\mathrm{i}^-(f(\eta)) = \mathrm{i}^-(f(\omega)) + \mathrm{i}^-(p_1(\psi(\eta))).$$

This implies that, setting as above $\Omega_{n-j}(f) \doteq \{\eta \in \Omega \,|\, \mathrm{i}^-(f(\eta)) \leq n-j\}$,

$$\psi(U_\omega \cap \Omega_{n-j}(f)) \subseteq \mathcal{Q}_{n-j}(\ker f(\omega)) \times \mathbb{R}^l,$$

where $\mathcal{Q}_{n-j}(\ker(f(\omega))) = \{q \in \mathcal{Q}(\ker f(\omega)) \,|\, \mathrm{i}^-(q) \leq n-j\}$. Since for each $k = 0, \ldots, i$ the set $\psi(B(\omega_k, r_k))$ is convex, then

$$\bigcap_{k=0}^{i} \psi(B(\omega_k, r_k)) \quad \text{is convex}$$

and by hypothesis it contains $\psi(\omega)$. Since $\mathcal{Q}_{n-j}(\ker f(\omega)) \times \mathbb{R}^l$ (if nonempty) has linear conical structure with respect to $\psi(\omega)$, then

$$\psi(\Omega_{n-j}(f)) \cap \bigcap_{k=0}^{i} \psi(B(\omega_k, r_k)) \quad \text{is acyclic}$$

and since $\psi : \bigcap_k B(\omega_k, r_k) \subset U_\omega \to \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l$ is a homeomorphism onto its image the conclusion follows.
In the case f is semialgebraic, then ψ is semialgebraic and we let δ_3' and δ_c be given by Lemma refcomm and Corollary 2.1.2 respectively. As both δ_3' and δ_c can be chosen semialgebraic, then the same holds true for $\delta_3 = \min\{\delta_3', \delta_c\}$.

□

Let now $f : \Omega \to \mathcal{Q}(V)$ be smooth, semialgebraic and transversal to all strata of $Z(V) = \coprod Z_j$. Then we define $\delta : \Omega \to (0, \infty)$ by

$$\delta(\omega) = \min\{\delta_1(\omega), \delta_2(\omega), \delta_3(\omega)\}.$$

By construction δ can be chosen to be semialgebraic. Under this assumption we prove the following.

Lemma 3.4.5. *Let $\mathcal{W}$ be an open cover of Ω and f and δ as above. Then there exists a locally finite refinement $\mathcal{U} = \{V_\alpha = B(x_\alpha, \delta_\alpha), x_\alpha \in \Omega\}_{\alpha \in A}$ satisfying the following conditions: (i) for every multi-index $\bar{\alpha} = (\alpha_0 \cdots \alpha_i)$ with $V_{\bar{\alpha}} \neq \emptyset$ there exists $\omega_{\bar{\alpha}} \in V_{\bar{\alpha}}$ such that for every $k = 0, \ldots, i$ the following holds:*

$$B(x_{\alpha_k}, \delta_{\alpha_k}) \subset B(\omega_{\bar{\alpha}}, \delta(\omega_{\bar{\alpha}}));$$

for every $\bar{\alpha}$ multi-index we let $n_{\bar{\alpha}}$ be the minimum of $\mathrm{i}^- \circ f$ over $V_{\bar{\alpha}} \neq \emptyset$, then the cover $\mathcal{U}$ can be chosen as to satisfy (ii):

$$n_{\alpha_0 \cdots \alpha_i} = \max\{n_{\alpha_0}, \ldots, n_{\alpha_i}\}.$$

Proof. We first set some notations. Let $\mathcal{N} = \coprod_{i=1}^{l} N_i \subset \Omega$ be a finite family of disjoint smooth submanifold such that $\delta_{|\mathcal{N}}$ is continuous. For $i = 1, \ldots, l$ let also $N_i' \subset N_i$ be a compact subset and define $\mathcal{N}' = \coprod N_i'$.
Then there exists $\epsilon(\mathcal{N}, \mathcal{N}') > 0$ such that for $i \neq j$ the two sets $\{x \in \Omega \mid d(x, N_i') < \epsilon(\mathcal{N}, \mathcal{N}')\}$ and $\{x \in \Omega \mid d(x, N_j') < \epsilon(\mathcal{N}, \mathcal{N}')\}$ are disjoint.
Let $\mathcal{W}_{\mathcal{N}'}$ be the cover $\{W \cap \mathcal{N}' \mid W \in \mathcal{W}\}$ and $\lambda_{\mathcal{N}'} > 0$ be its Lebesgue number.
Finally let $\delta'_{\mathcal{N}'} = \min_{\eta \in \mathcal{N}'} 3\delta(\eta) > 0$ which exists since $\delta_{|\mathcal{N}}$ is continuos and $\mathcal{N}'$ is compact.
We define $\delta(\mathcal{N}, \mathcal{N}') > 0$ to be any number such that

$$\delta(\mathcal{N}, \mathcal{N}') < \min\{\epsilon(\mathcal{N}, \mathcal{N}'), \lambda_{\mathcal{N}'}, \delta'_{\mathcal{N}'}\}.$$

We construct now the desired cover. Let $h : |K| \to \Omega$ be a smooth semialgebraic triangulation (i.e. smooth one each simplex) of Ω respecting the semialgebraic sets $\{\omega \in \Omega \mid \mathrm{i}^-(f(\omega)) = k\}_{k \in \mathbb{N}}$ and such that δ is continuous on each simplex (see [10]). Thus $\Omega = \coprod S_i$, where $i = 0, \ldots, k$ and S_i is the image under h of the i-th skeleton of the complex K.
Let $S_0 = \{x_0, \ldots, x_v\}$ and define

$$\mathcal{U}_0 \doteq \{B(x_i, \delta(S_0, S_0)), i = 0, \ldots, v\}$$

and $T_0 = \cup_i B(x_i, \delta(S_0, S_0))$.
Now proceed inductively: first set $S_i = \coprod_{\sigma_{i,j} \in K_i} h(\sigma_{i,j})$ and $S_i' = \coprod h(\sigma_{i,j}) \backslash T_{i-1}$. Then let $\mathcal{U}_i = \{B(x_i^j, \delta_i) \mid x_i^j \in S_i' \text{ and } \delta_i < \delta(S_i, S_i')\}$ be such that $\mathcal{U}_i$ and $\mathcal{U}_i \cap S_i'$ have the same combinatorics; let also T_i be defined by

$$T_i = \cup_{V \in \mathcal{U}_i} V.$$

With the previous settings we finally define

$$\mathcal{U} \doteq \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_k.$$

Then $\mathcal{U}$ verifies by construction the requirements and this concludes the proof. $\square$

Definition 3.4.6. *Let $f : \Omega \to \mathcal{Q}(V)$ be a smooth semialgebraic map transverse to all strata of $Z = \coprod Z_j$ and δ the semialgebraic function min$\{\delta_1, \delta_2, \delta_3\}$ where δ_1, δ_2 and δ_3 are given by Lemma 3.4.2, Lemma 2.2.11 and Corollary 3.4.4. Let $\mathcal{W}$ the open cover of Ω defined by*

$$\mathcal{W} = \{V_\alpha = B(x_\alpha, \delta_\alpha)\}_{\alpha \in A}$$

for certain $x_\alpha \in \Omega$ and $\delta_\alpha > 0$, $\alpha \in A$. Then $\mathcal{W}$ will be called an f-regular cover of Ω if it satisfies conditions (i) and (ii) of Lemma 3.4.5.

In particular Lemma 3.4.5 tells that the set of f-regular covers is cofinal in the set of all covers of Ω.

3.4.3 Computations

Suppose that a scalar product on $\mathbb{R}^{n+1}$ has been fixed and let as above $w_{1,j}$ be the first Stiefel-Whitney class of $\Lambda_j^+ \to \mathcal{D}_j$. We recall that we set ∂^* for the connecting homomorphism of the pair and we defined the class

$$\gamma_{1,j} = \partial^* w_{1,j} \in H^2(\mathcal{Q}(\mathbb{R}^{n+1}), \mathcal{D}_j).$$

Letting $\overline{p} : \Omega \to \mathcal{Q}(\mathbb{R}^{n+1})$ be the map defined by $\omega \mapsto \omega p$, then $\Omega^{k+1} = \overline{p}^{-1}(\mathcal{Q}^{k+1})$ and we noticed that given $x \in H^i(\Omega, \Omega^{j+1})$ then the product $(x \smile \overline{p}^* \gamma_{1,j})|_{(\Omega,\Omega^{j+1})}$ does not depend on the choice of the scalar product; indeed it gives the second differential for the spectral sequence of Theorem 3.3.2.

Theorem 3.4.7. *Let $(F_r, d_r)_{r\geq 0}$ be the spectral sequence of theorem 3.3.2. Then for every $i, j \geq 0$ the differential $d_2 : F_2^{i,j} \to F_2^{i+2,j-1}$ is given by:*

$$d_2(x) = (x \smile \overline{p}^* \gamma_{1,j})|_{(\Omega,\Omega^j)}.$$

Proof. We fix at the very beginning a scalar product g; for this proof we will use in the notations for the various objects their dependence on g.
Recall from Theorem 3.3.2 that we have defined (F_r, d_r) by:

$$(F_r, d_r) = \varprojlim_{\epsilon} \varinjlim_{\beta_l^{-1}\mathcal{W}} \{(F_r(\epsilon, \mathcal{U}), d_r(\epsilon, \mathcal{U}))\}$$

where the $(\epsilon, \mathcal{U})$-pair is the relative Leray's spectral sequence for the pair $(\Omega \times \mathbb{RP}^n, B(\epsilon))$, the map β_l and the cover $\mathcal{U} = \beta_l^{-1}\mathcal{W}$ (the direct limit ranges over all covers of Ω). The set $B(\epsilon)$ was defined using the function $\alpha : \Omega \times \mathbb{RP}^n \to \mathbb{R}$, $\alpha(\omega, x) = (\omega p)(x)/q_0(x)$, where q_0 is a positive definite form, as $B(\epsilon) = \{\alpha > \epsilon\}$. By lemma 2.2.9 we may assume q_0 is such that the map:

$$f_\epsilon : \omega \mapsto \omega p - \epsilon q_0$$

is nondegenerate (and also can be made transversal to Z and $\mathcal{Q} \backslash \mathcal{D}^g$, where $\mathcal{D}^g = \cap_j \mathcal{D}_j^g$). In this way Lemma 3.4.5 ensures the existence of an f_ϵ-regular cover of Ω:

$$\mathcal{W} = \{V_\alpha = B(x_\alpha, \delta_\alpha)\}_{\alpha \in A}$$

Plan of the proof. The proof is long and we subdivide it in three parts. In the first part we introduce some auxiliary materials. In the second part we compute for ϵ small and $\mathcal{U} = \beta_l^{-1}\mathcal{W}$ the differential $d_2(\epsilon, \mathcal{U})$. Since $\mathcal{W}$ is f_ϵ-regular, then by Lemma 3.4.4, it is acyclic for each $\mathcal{F}^j(\epsilon)$ and thus the limit map gives for every $i, j \in \mathbb{Z}$ isomorphisms:

$$F_2^{i,j}(\epsilon, \mathcal{U}) \simeq F_2^{i,j}(\epsilon).$$

Under this isomorphism the differential $d_2(\epsilon, \mathcal{U})$ happens to be given by:

$$x \mapsto (x \smile f_\epsilon^* \gamma_{1,j}^g)|_{(\Omega,\Omega_{n-j+1}(\epsilon))}$$

Thus under the limit map the second differential is given by the previous formula for every f_ϵ-regular cover; since the set of such covers is cofinal in all covers of Ω,

then the previous is actually the expression for $d_2(\epsilon)$. In the last part we perform the ϵ-limit and get the expression for d_2.
We stress that the definition of our spectral sequence using direct and inverse limits is somehow formal: both limits are attained for ϵ small enough and $\mathcal{W}$ a f_ϵ-regular cover.

Auxiliary material. Let $K_0^{*,*} = K_0^{*,*}(\mathcal{U})$ be the Kunneth bicomplex associated to the map $\beta_l : \Omega \times \mathbb{R}\mathrm{P}^n \to \Omega$ with respect to $\mathcal{U}$. Notice that $F_0^{*,*}(\epsilon, \mathcal{U})$ is a subcomplex of $K_0^{*,*}$ and we denote by δ_F, d_F and δ_K, d_K the respective bicomplex differentials (the first two are the restriction to $F_0^{*,*}$ of the second two).
For every $\omega \in \Omega$ and $\epsilon > 0$ we let $\mathrm{i}^-(\epsilon)(\omega) = \mathrm{ind}^-(\omega p - \epsilon q_0)$ and for every multi-index $\bar{\alpha} = (\alpha_0, \ldots, \alpha_i)$ such that $V_{\bar{\alpha}} \neq \emptyset$ we let $n_{\bar{\alpha}}$ be the minimum of $\mathrm{i}^-(\epsilon)$ over $V_{\bar{\alpha}}$. We take an order on the index set A such that

$$\alpha \leq \beta \quad \text{implies} \quad n_\alpha \leq n_\beta.$$

In this way, by Lemma 3.4.5, for every multi-index $\bar{\alpha} = (\alpha_0, \ldots, \alpha_i)$ such that $V_{\bar{\alpha}} \neq \emptyset$ we have that $n_{\bar{\alpha}} = n_{\alpha_i}$. For every multi-index $\bar{\alpha}$ such that $V_{\bar{\alpha}} \neq \emptyset$ let $\omega_{\bar{\alpha}}$ be given by Lemma 3.4.5, $\mathrm{i}^-(\epsilon)(\omega_{\bar{\alpha}}) = n_{\bar{\alpha}}$, and we let $\eta_{\bar{\alpha}} \in V_{\bar{\alpha}}$ be such that $\det(f_\epsilon(\eta_{\bar{\alpha}})) \neq 0$, $\mathrm{i}^-(\epsilon)(\eta_{\bar{\alpha}}) = n_{\bar{\alpha}}$ and $f_\epsilon(\eta_{\bar{\alpha}}) \in \mathcal{D}^g$ (such $\eta_{\bar{\alpha}}$ always exists, and by transversality of the map f_ϵ to Z and to $\mathcal{Q}\backslash\mathcal{D}^g$, which have respectively codimension one and two, there are plenty of them).
For every $0 \leq j \leq n$ and $\alpha \in A$ we define

$$N(\alpha, j) = (P_j^-)^g(f_\epsilon(\eta_\alpha))$$

where the g on $(P_j^-)^g$ denotes the dependence on the fixed scalar product. Moreover we let $\nu(\alpha, j) \in C^j(\mathbb{R}\mathrm{P}^n)$ be the cochain defined by the intersection number with $N(\alpha, j)$. This cochain is defined only on singular chains that are transverse to $N(\alpha, j)$, but since such chains define the same homology groups as the singular ones we may restrict to them. The reader that feels uncomfortable with this assumption may prefer to use from the very beginning triangulations of all the topological spaces we introduced (everything is semialgebraic) and a bicomplex with simplicial cochains instead of singular cochains; then using dual cell decompositions the above cochains happen to be everywhere defined. This procedure will end up with an isomorphic spectral sequence, but it is remarkably more cumbersome.
We define a cochain $\psi^{0,j} \in K_0^{0,j}$ by

$$\psi^{0,j}(\alpha) = \beta_r^* \nu(\alpha, j).$$

Notice that if $n - n_\alpha + 1 \leq j \leq n$ then , by Lemma 3.4.2, $N(\alpha, j) \subset \mathbb{R}\mathrm{P}^n \backslash \beta_r(B_\alpha(\epsilon))$ and thus $\nu(\alpha, j) \in C^j(\mathbb{R}\mathrm{P}^n, \beta_r(B_\alpha(\epsilon))$. Hence

$$n - n_\alpha + 1 \leq j \leq n \quad \text{implies} \quad \psi^{0,j}(\alpha) \in C^j(V_\alpha \times \mathbb{R}\mathrm{P}^n, B_\alpha(\epsilon)) \tag{3.1}$$

Moreover $N(\alpha, n - n_\alpha + 1)$ is a $(n_\alpha - 1)$-dimensional projective space contained in $\mathbb{R}\mathrm{P}^n \backslash \beta_r(B_{\alpha_0 \ldots \alpha_i \alpha}(\epsilon))$ for every $(\alpha_0, \ldots, \alpha_i)$; thus by Lemma 2.2.11 if $n - n_\alpha + 1 \leq$

$j \le n$ then the cohomology class of $\nu(\alpha, j)$ generates $H^j(\mathbb{RP}^n, \beta_r(B_\alpha(\epsilon)))$. Hence it follows that for every $\overline{\alpha} = (\alpha_0 \cdots \alpha_i \alpha)$ such that $V_{\overline{\alpha}} \neq \emptyset$

$$n - n_\alpha + 1 \le j \le n \quad \text{implies} \quad [\psi^{0,j}(\alpha)_{|\overline{\alpha}}] \text{ generates } H^j(V_{\overline{\alpha}} \times \mathbb{RP}^n, B_{\overline{\alpha}}(\epsilon)) = \mathbb{Z}_2 \tag{3.2}$$

For every $\alpha_0, \alpha_1 \in A$ such that $V_{\alpha_0\alpha_1} \neq \emptyset$ we consider a curve $c_{\alpha_0\alpha_1} : I \to V_{\alpha_0} \cup V_{\alpha_1}$ such that $c_{\alpha_0\alpha_1}(i) = \eta_{\alpha_i}$, $i = 0, 1$; since $\Omega \backslash f_\epsilon^{-1}(\mathcal{D}^g)$ has codimension two in Ω, then we may choose $c_{\alpha_0\alpha_1}$ such that for every $t \in I$ we have $f_\epsilon(c_{\alpha_0\alpha_1}(t)) \in \mathcal{D}^g$. Consider the $\mathbb{R}^{n-j+1}$-bundle $L_j^g(\alpha_0\alpha_1) = c_{\alpha_0\alpha_1}^* f_\epsilon^*(\Lambda_j^-)^g$ over I and its projectivization $P(L_j^g(\alpha_0\alpha_1))$. Then the natural map

$$P(L_j^g(\alpha_0\alpha_1)) \to \mathbb{RP}^n$$

defines a $(n-j+1)$-chain $T(\alpha_0\alpha_1, j-1)$ in $\mathbb{RP}^n$. Let $\tau(\alpha_0\alpha_1, j-1)$ be the $j-1$-cochain defined by the intersection number with $T(\alpha_0\alpha_1, j-1)$. Notice that this cochain is defined only on singular chains that are transverse to $T(\alpha_0\alpha_1, j-1)$ and the same consideration we made above for the definition of $\nu(\alpha, j)$ applies here. Thus we define $\theta^{1,j-1} \in K_0^{1,j-1}$ by setting for every α_0, α_1 with $V_{\alpha_0\alpha_1} \neq \emptyset$

$$\theta^{1,j-1}(\alpha_0\alpha_1) = \beta_r^* \tau(\alpha_0\alpha_1, j-1).$$

Notice that $\partial T(\alpha_0\alpha_1, j-1) = N(\alpha_0, j) + N(\alpha_1, j)$, hence $d\tau(\alpha_0\alpha_1, j-1) = \nu(\alpha_0, j) + \nu(\alpha_1, j)$; it follows that

$$\delta_K \psi^{0,j} = d_K \theta^{1,j-1}. \tag{3.3}$$

Moreover by construction if $n - n_{\alpha_0} + 1 \le j \le n$ and $n - n_{\alpha_1} + 1 \le j \le n$, which implies $n - n_{\alpha_0\alpha_1} + 1 \le j \le n$, then

$$\theta^{1,j-1}(\alpha_0\alpha_1) \in C^{j-1}(V_{\alpha_0\alpha_1} \times \mathbb{RP}^n, B_{\alpha_0\alpha_1}(\epsilon)). \tag{3.4}$$

We compute now $\delta_K \theta^{1,j-1}$. Let $(\alpha_0\alpha_1\alpha_2) = \bar{\alpha}$ be such that $V_{\bar{\alpha}} \neq \emptyset$. Then the curves $c_{\alpha_0\alpha_1}, c_{\alpha_1\alpha_2}$ and $c_{\alpha_2\alpha_0}$ define a map $\sigma_{\alpha_0\alpha_1\alpha_2} : S^1 \to \Omega$ and we have the bundle $L_j^g(\alpha_0\alpha_1\alpha_2) = \sigma_{\alpha_0\alpha_1\alpha_2}^* f_\epsilon^*(\Lambda_j^-)^g$ and its projectivization $P(L_j^g(\alpha_0\alpha_1\alpha_2))$ over S^1. The natural map

$$P(L_j^g(\alpha_0\alpha_1\alpha_2)) \to \mathbb{RP}^n$$

defines a $(n-j+1)$-cochain whose pullback under β_r^* by construction equals the cochain $\delta_K \theta^{1,j-1}(\alpha_0\alpha_1\alpha_2)$. Thus by definition of Stiefel-Whitney classes we have:

$$\delta_K \theta^{1,j-1}(\alpha_0\alpha_1\alpha_2) = w_1(\partial(\alpha_0\alpha_1\alpha_2))(\psi^{0,j-1}(\alpha_2)_{|\alpha_0\alpha_1\alpha_2}) + dr^{2,j-1}(\alpha_0\alpha_1\alpha_2) \tag{3.5}$$

where $w_1(\partial(\alpha_0\alpha_1\alpha_2)) = w_1(L_j^g(\alpha_0\alpha_1\alpha_2))$. Let now $\xi^i \in F_1^{i,j}(\epsilon, \mathcal{U})$; we define $\xi^{i,0} \in K_0^{i,0}$ by

$$\xi^{i,0}(\alpha_0 \ldots \alpha_i) \equiv \xi^i(\alpha_0 \ldots \alpha_i)$$

i.e. the values of $\xi^{i,0}(\alpha_0 \ldots \alpha_i)$ on every 0-chain equals $\xi^i(\alpha_0 \ldots \alpha_i) \in \mathbb{Z}_2$. Notice that by construction $d_K \xi^{i,0} = 0$ and that

$$d_1 \xi^i = 0 \quad \text{implies} \quad \delta_K \xi^{i,0} = 0. \tag{3.6}$$

The computation of $d_2(\epsilon,\mathcal{U})$. Pick $x \in F_2^{i,j}(\epsilon,\mathcal{U}) \simeq F_2^{i,j}(\epsilon)$ and $\xi^i \in F_1^{i,j}(\epsilon,\mathcal{U})$ such that $d_1\xi^1 = 0$ and $x = [\xi^i]_2$. According to the definition of $d_2(\epsilon,\mathcal{U})$, to compute it on x we must find in $F_0(\epsilon,\mathcal{U})$ a zig-zag:

$$\begin{tikzcd} 0 & \\ \eta_0 \arrow[u, "d_F"] \arrow[r, "\delta_F"] & \cdot \\ & \eta_1 \arrow[u, "d_F"'] \end{tikzcd}$$

such that $[\eta_0]_2 = x$. This will give

$$d_2(\epsilon,\mathcal{U})x = [\delta\eta_1]_2.$$

We claim that $\eta_0 = \xi^{i,0} \cdot \psi^{0,j}$, $\eta_1 = \xi^{i,0} \cdot \theta^{1,j-1}$ is such a zig-zag. First notice that since $\xi^i \in F_1^{i,j}(\epsilon,\mathcal{U})$, then (3.1) implies $\xi^{i,0} \cdot \psi^{0,j} \in F_0^{i,j}(\epsilon,\mathcal{U})$. Moreover by (3.2) it follows that $[\xi^{i,0} \cdots \psi^{0,j}]_1 = \xi^i$ and thus

$$[\xi^{i,0} \cdot \psi^{0,j}]_2 = x.$$

We calculate now:

$$\begin{aligned} \delta_F(\xi^{i,0} \cdot \psi^{0,j}) &= \delta_K(\xi^{i,0} \cdot \psi^{0,j}) = \xi^{i,0} \cdot \delta_K\psi^{0,j} = \xi^{i,0} \cdot d_K\theta^{1,j-1} \\ &= d_K(\xi^{i,0} \cdot \theta^{1,j-1}) = d_F(\xi^{i,0} \cdot \theta^{1,j-1}). \end{aligned}$$

The first equality comes from $F_0^{i,j}(\epsilon,\mathcal{U}) \subset K_0^{i,j}$; the second from $d_1\xi^i = 0$; the third from (3.3); the fourth from (3.6); the last by $\xi^{i,0} \cdot \theta^{1,j-1} \in F_0^{i+1,j-1}(\epsilon,\mathcal{U})$, which is a direct consequence of (3.4). Thus the chosen pair is such a required zig-zag and we can finally compute $d_2(\epsilon,\mathcal{U})(x) = [\delta_F(\xi^{i,0} \cdot \theta^{1,j-1})]_2$. We have:

$$\delta_F(\xi^{i,0} \cdot \theta^{1,j-1}) = \delta_K(\xi^{i,0} \cdot \theta^{1,j-1}) = \xi^{i,0} \cdot \delta_K\theta^{1,j-1}$$

and thus by (3.5) we derive:

$$[\delta_F(\xi^{i,0} \cdot \theta^{1,j-1})]_1(\alpha_0 \cdots \alpha_{i+2}) = \xi^i(\alpha_0 \cdots \alpha_i)w_1(\partial(\alpha_i\alpha_{i+1}\alpha_{i+2})).$$

This gives the description of $[\delta_F(\xi^{i,0} \cdot \theta^{1,j-1})]_1$ as the cochain representing (using $F_2(\epsilon,\mathcal{U}) \simeq F_2(\epsilon)$) the cohomology class $(x \smile f_\epsilon^*\gamma_{1,j}^g)|_{(\Omega,\Omega_{n-j+1}(\epsilon))}$ (here we are using the fact that the sum of the bundles Λ_k^+ and Λ_k^- is trivial and $w_{1,j}^g = w_1(\Lambda_j^-)$). This ends the first part of the proof.

Perfoming the limits. We proceed now with the second part of the proof. Consider the following sequences of maps:

$$H^2(\mathcal{Q},\mathcal{D}_j^g) \xrightarrow{f_\epsilon^*} H^2(\Omega, D_j^g(\epsilon)) \xrightarrow{r_\epsilon^*} H^2(\Omega,\Omega_{n-j+1}(\epsilon)\backslash\Omega_{n-j}(\epsilon)).$$

Notice that $r_\epsilon^* f_\epsilon^*\gamma_{1,j}^g$ does not depend on g and thus the differential $d_2(\epsilon)$ is given by

$$x \mapsto (x \smile f_\epsilon^*\gamma_{1,j}^g)|_{(\Omega,\Omega_{n-j+1}(\epsilon))}$$

for *any* g. Let now $g = q_0$; then in this case $D_j^{q_0} = D_j^{q_0}(\epsilon)$ and $f^* = f_\epsilon^*$. Consider the following commutative diagram of inclusions:

$$\begin{tikzcd}
(\Omega, \Omega^j) \arrow[r, "\iota"] & (\Omega, \Omega^{j+1} \cup D_j^{q_0}) \\
(\Omega, \Omega_{n-j+1}(\epsilon)) \arrow[u, "\rho(\epsilon)"] \arrow[r, "\iota(\epsilon)"] & (\Omega, \Omega_{n-j}(\epsilon) \cup D_j^{q_0}) \arrow[u, "\hat{\rho}(\epsilon)"']
\end{tikzcd}$$

Then, using $\rho(\epsilon)$ also for the inclusion $(\Omega, \Omega_{n-j}(\epsilon)) \hookrightarrow (\Omega, \Omega^{j+1})$, we have for $x \in H^i(\Omega, \Omega^{j+1})$ the following chain of equalities:

$$\begin{aligned}
\rho(\epsilon)^*((x \smile \overline{p}^*\gamma_{1,j})|_{(\Omega,\Omega^j)}) &= \rho(\epsilon)^*\iota^*(x \smile f^*\gamma_{1,j}^{q_0}) = \iota(\epsilon)^*\hat{\rho}(\epsilon)^*(x \smile f^*\gamma_{1,j}^{q_0}) \\
&= \iota(\epsilon)^*(\rho(\epsilon)^*x \smile f_\epsilon^*\gamma_{1,j}^{q_0}) = d_2(\epsilon)(\rho(\epsilon)^*x).
\end{aligned}$$

This proves that the following diagram is commutative:

$$\begin{tikzcd}
H^i(\Omega, \Omega^{j+1}) \arrow[r, "(\cdot \smile \overline{p}^*\gamma_j)|_{(\Omega,\Omega^j)}"] \arrow[d, "\rho(\epsilon)^*"'] & H^{i+2}(\Omega, \Omega^j) \arrow[d, "\rho(\epsilon)^*"] \\
H^i(\Omega, \Omega_{n-j}(\epsilon)) \arrow[r, "d_2(\epsilon)"] & H^{i+2}(\Omega, \Omega_{n-j+1}(\epsilon))
\end{tikzcd}$$

From this the conclusion follows. $\square$

We are now ready to prove the statement concerning the second differential of the spectral sequence of theorem 3.3.3. The only difference from the previous spectral sequence is that the class $\gamma_{1,j}$ in this case is pulled-back via $\overline{p}$ to the whole $(C\Omega, D_j)$, where $C\Omega = K^\circ \cap B^{k+1}$ and B^{k+1} is the ball in $\mathbb{R}^{k+1}$ whose boundary is S^k.

Theorem 3.4.8 (The second differential). *For every $i, j \geq 0$ the differential $d_2 : E_2^{i,j} \to E_2^{i+2,j-1}$ is given by:*

$$d_2(x) = (x \smile \overline{p}^*\gamma_{1,j})|_{(C\Omega,\Omega^j)}.$$

Proof. We replace now K with $\hat{K} = (-\infty, 0] \times K$, the map p with the map $\hat{p} = (q_0, p) : \mathbb{R}^{n+1} \to \mathbb{R}^{k+2}$, where $q_0 \in \mathcal{Q}^+$, and we apply the previoius Theorem to $(\hat{p}, \hat{K})$. As for theorem 3.3.3 we use the deformation retraction $(\hat{\Omega}, \hat{\Omega}^{j+1}) \to (\hat{\Omega}, \Omega^{j+1}) = (C\Omega, \Omega^{j+1})$. Notice that we have also the deformation retraction

$$r : (\hat{\Omega}, \hat{D}_j) \to (\hat{\Omega}, D_j)$$

where D_j is identified with $\hat{D}_j \cap \{\eta = 0\}$: by definition $\omega \in D_j$ if and only if $(\eta, \omega) \in \hat{D}_j$ and for every $0 < j < n+1$ we have $(1, 0, \ldots, 0) \notin D_j$ since all the eigenvalues of $\langle (1, 0, \ldots, 0), \hat{p} \rangle = -q_0$ with respect to q_0 coincide. Then by naturality the conclusion follows.

$\square$

3.5 Projective inclusion

In this section we study the image of the homology of X under the inclusion map

$$\iota : X \to \mathbb{R}\mathrm{P}^n.$$

Using the above notations, we define $\hat{B} = \{(\hat{\omega}, x) \in \hat{\Omega} \times \mathbb{R}\mathrm{P}^n \mid (\hat{\omega}\hat{p})(x) > 0\}$ and we call (E_r, d_r) the spectral sequence of theorem 3.3.3 converging to $H^*(\hat{\Omega} \times \mathbb{R}\mathrm{P}^n, \hat{B})$. Moreover we let $K_0^{*,*}$ be the Leray bicomplex for the map $\hat{\Omega} \times \mathbb{R}\mathrm{P}^n \to \hat{\Omega}$ (it equals the Kunneth bicomplex for $\hat{\Omega} \times \mathbb{R}\mathrm{P}^n$). Thus there is a morphism of spectral sequence $(i_r : E_r \to K_r)_{r \geq 0}$ induced by the inclusion $j : (\hat{\Omega} \times \mathbb{R}\mathrm{P}^n, \emptyset) \to (\hat{\Omega} \times \mathbb{R}\mathrm{P}^n, B)$. With the above notations we prove the following theorem which gives the rank of the homomorphism

$$\iota_* : H_*(X) \to H_*(\mathbb{R}\mathrm{P}^n).$$

Theorem 3.5.1. *For every $b \in \mathbb{Z}$ the following holds:*

$$\mathrm{rk}(\iota_*)_b = \mathrm{rk}(i_\infty)_{0,n-b}.$$

Moreover the map $(i_\infty)_{0,n-b} : E_\infty^{0,n-b} \to K_\infty^{0,n-b} = \mathbb{Z}_2$ is an isomorphism onto its image.

Proof. First we look at the following commutative diagram of maps

$$\begin{tikzcd}
H^{n-b}(\hat{\Omega} \times \mathbb{R}\mathrm{P}^n, \hat{B}) \arrow[r, "(j^*)_{n-b}"] & H^{n-b}(\hat{\Omega} \times \mathbb{R}\mathrm{P}^n) \\
H^{n-b}(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \backslash X) \arrow[u, "\beta_l^*"] \arrow[r, "(j'^*)_{n-b}"] & H^{n-b}(\mathbb{R}\mathrm{P}^n) \arrow[u, "\beta_l^*"'] \\
H_b(X) \arrow[u, "P^*"] \arrow[r, "(\iota_*)_b"] & H_b(\mathbb{R}\mathrm{P}^n) \arrow[u, "P^*"']
\end{tikzcd}$$

where the maps ι_*, j^* and j'^* are those induced by inclusions and the P^*'s are Poincaré duality isomorphisms; commutativity follows from naturality of Poincaré duality. Since $\hat{\Omega} \approx C\Omega$, then it is contractible and $\beta_l : (\hat{\Omega} \times \mathbb{R}\mathrm{P}^n, \hat{B}) \to (\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \backslash X)$ is a homotopy equivalence; hence all the vertical arrows are isomorphisms. Thus we identify $(\iota_*)_b$ with $(j^*)_{n-b}$.
Let now $\epsilon > 0$ be such that $\hat{B}(\epsilon) \hookrightarrow \hat{B}$ is a homotopy equivalence, where $\hat{B}(\epsilon) = \{(\hat{\omega}, x) \in \hat{\Omega} \times \mathbb{R}\mathrm{P}^n \mid (\hat{\omega}\hat{p})(x) > \epsilon\}$ (such ϵ exists by Lemma 1.2.4). Then the inclusion of pairs

$$(\hat{\Omega} \times \mathbb{R}\mathrm{P}^n, \hat{B}(\epsilon)) \xrightarrow{\hat{j}(\epsilon)} (\hat{\Omega} \times \mathbb{R}\mathrm{P}^n, \hat{B})$$

also is a homotopy equivalence and the inclusion $(\hat{\Omega} \times \mathbb{R}\mathrm{P}^n, \emptyset) \xrightarrow{j} (\hat{\Omega} \times \mathbb{R}\mathrm{P}^n, \hat{B})$ factors trough:

$$\begin{tikzcd}
(\hat{\Omega} \times \mathbb{RP}^n, \emptyset) \arrow[rr, "j"] \arrow[dr, "j(\epsilon)"'] & & (\hat{\Omega} \times \mathbb{RP}^n, \hat{B}) \\
& (\hat{\Omega} \times \mathbb{RP}^n, \hat{B}(\epsilon)) \arrow[ur, "\hat{j}(\epsilon)"'] &
\end{tikzcd}$$

Since $\hat{j}(\epsilon)$ is a homotopy equivalence, it follows that:

$$\mathrm{rk}(j^*)_{n-b} = \mathrm{rk}(j(\epsilon)^*)_{n-b}.$$

Let now $\mathcal{W}$ be any cover of $\hat{\Omega}$ and $\mathcal{U} = \beta_l^{-1}\mathcal{W}$. Consider the Leray-Mayer-Vietoris bicomplexes $\hat{F}^{*,*}(\epsilon,\mathcal{U})$ and $K_0^{*,*}(\mathcal{U})$ with their respective associated spectral sequences; since $i_0(\epsilon,\mathcal{U}) : \hat{F}_0^{*,*}(\epsilon,\mathcal{U}) \hookrightarrow K_0^{*,*}(\mathcal{U})$ there is a morphism of respective spectral sequences. Moreover by Mayer-Vietoris argument, the spectral sequence $(\hat{F}_r(\epsilon,\mathcal{U}), \hat{d}_r(\epsilon,\mathcal{U}))_{r\geq 0}$ converges to $H^*(\hat{\Omega} \times \mathbb{RP}^n, \hat{B}(\epsilon))$ and $(K_r(\mathcal{U}), d_r(\mathcal{U}))_{r\geq 0}$ converges to $H^*(\hat{\Omega} \times \mathbb{RP}^n, \emptyset)$. We look now at the following commutative diagram:

$$\begin{tikzcd}
E_\infty^{0,n-b}(\epsilon,\mathcal{U}) \arrow[r, "(i_\infty(\epsilon,\mathcal{U}))_{0,n-b}"] & K_\infty^{0,n-b}(\mathcal{U}) \\
H_D^{n-b}(E_0(\epsilon,\mathcal{U})) \arrow[u, "p_E(\epsilon,\mathcal{U})"] \arrow[r, "(i_0^*(\epsilon,\mathcal{U}))_{n-b}"] & H_D^{n-b}(K_0(\mathcal{U})) \arrow[u, "p_K"'] \\
H^{n-b}(\hat{\Omega} \times \mathbb{RP}^n, \hat{B}) \arrow[u, "(f_E^*)^{-1}"] \arrow[r, "(j^*)_{n-b}"] & H^{n-b}(\hat{\Omega} \times \mathbb{RP}^n) \arrow[u, "(f_K^*)^{-1}"']
\end{tikzcd}$$

The upper square is commutative, since if we let $\psi = \psi_0 + \cdots + \psi_{n-b} \in E_0^{n-b}$ with $D\psi = 0$, then (avoiding the $(\epsilon,\mathcal{U})$-notations, but only for the next formula):

$$p_K(i_0^*)_{n-b}[\psi]_E = p_K[\psi]_K = [\psi_0]_{\infty,K} = (i_\infty)_{0,n-b}[\psi_0]_{\infty,E} = (i_\infty)_{0,n-b}p_E[\psi]_E.$$

The lower square is the one coming from Lemma 3.1.3 with the vertical arrows inverted, hence it is commutative.
Since $K_\infty(\mathcal{U}) = K_2(\mathcal{U})$ has only one column (the first), then $p_K(\mathcal{U}) : H_D^{n-b}(K_0(\mathcal{U})) \to K_\infty^{0,n-b}(\mathcal{U})$ is an isomorphism, hence for $0 \leq b \leq n$ and using the above identifications we can identify the map $(j^*)_{n-b}$ with

$$(i_\infty(\epsilon,\mathcal{U}))_{0,n-b}(p_E(\epsilon,\mathcal{U}))_{n-b} : H_D^{n-b}(E_0(\epsilon,\mathcal{U})) \to \mathbb{Z}_2.$$

Since $(p_E(\epsilon,\mathcal{U}))_{n-b}$ is surjective, then:

$$\mathrm{rk}(j^*)_{n-b} = \mathrm{rk}(i_\infty(\epsilon,\mathcal{U}))_{0,n-b}.$$

By Corollary 3.4.4 and Lemma 3.4.5 there exists a family $\mathcal{C}$ of covers which is cofinal in the family of all covers such that for every $\mathcal{U} \in \mathcal{C}$ the natural map $\hat{F}_2^{i,j}(\epsilon,\mathcal{U}) \to$

$\hat{F}_2^{i,j}(\epsilon)$ is an isomorphism. It follows that $\mathrm{rk}(i_\infty(\epsilon,\mathcal{U})_{0,n-b}) = \mathrm{rk}(i_\infty(\epsilon))_{0,n-b}$, and thus by semialgebraicity we have

$$\mathrm{rk}(i_\infty(\epsilon))_{0,n-b} = \mathrm{rk}(i_\infty)_{0,n-b}.$$

It remains to study the map $(i_\infty)_{0,n-b} : E_\infty^{0,n-b} \to K_\infty^{0,n-b} = K_2^{0,n-b}$.
If $E_\infty^{0,n-b}$ is zero, then $(i_\infty)_{0,n-b}$ is obviously an isomorphism onto its image.
If $E_\infty^{0,n-b}$ is not zero then, since $E_2^{0,n-b} = H^0(C\Omega, \Omega^{n-b+1})$, it must be $\Omega^{n-b+1} = \emptyset$ and thus $\hat{\Omega}^{n-b+1} = \emptyset$ and

$$E_\infty^{0,n-b} = E_2^{0,n-b} = \mathbb{Z}_2.$$

From this it follows that

$$i_\infty^{0,n-b} = i_2^{0,n-b}.$$

By the definition of the two spectral sequences as direct limits, for ϵ sufficiently small and $\mathcal{U}$ an f_ϵ regular cover, we see that $i_2(\epsilon,\mathcal{U})^{0,n-b}$ is the identity and thus also $i_2^{0,n-b} : H^0(\hat{\Omega},\emptyset) \to H^0(\hat{\Omega}) \otimes H^{n-b}(\mathbb{R}\mathrm{P}^n)$ is the identity and then the conclusion follows. □

Remark 8. Since here we do not need the cover to be convex, the existence of the family $\mathcal{C}$ follows from easier consideration. Let $h : \hat{\Omega} \to |K| \subset \mathbb{R}^N$ be a triangulation respecting the filtration $\{\hat{\Omega}_j\}_{j=0}^{n+2}$, and $\mathcal{W}$ be a cover of $\hat{\Omega}$. Let $\mathcal{V}'$ be a *convex* cover of $|K|$ refining $h(\mathcal{W})$ and such that for every $U' \in \mathcal{V}'$ the intersection $h(\hat{\Omega}_j) \cap U'$ is contractible for every j (the existence of such a $\mathcal{V}'$ follows from the fact that $h(\hat{\Omega}_j)$ is a subcomplex of $|K|$). Then the cover $\mathcal{V} = h^{-1}(\mathcal{V}')$ refines $\mathcal{W}$ and since for every j and $U \in \mathcal{V}$ the intersection $\hat{\Omega}_j \cap U$ is contractible, then the natural map $\hat{F}_2^{i,j}(\epsilon, \beta_l^{-1}\mathcal{V}) \to \hat{F}_2^{i,j}(\epsilon)$ is an isomorphism.

We can immediately derive the following elementary corollary

Corollary 3.5.2. *If $b > n - \mu$ then $(j_*)_b = 0$.*

Proof. Since $n - b < \mu$ then $\Omega^{n-b+1} \neq \emptyset$. This gives $E_2^{0,n-b} = 0$ and thus applying the previous theorem the conclusion follows. □

3.6 Hyperplane sections

We consider here the following problem: given $X \subset \mathbb{R}\mathrm{P}^n$ defined by quadratic inequalities and V a codimension one subspace of $\mathbb{R}^{n+1}$ with projectivization $\bar{V} \subset \mathbb{R}\mathrm{P}^n$, determine the homology of $(X, X \cap \bar{V})$.
Thus let $p : \mathbb{R}^{n+1} \to \mathbb{R}^{k+1} \supseteq K$ be homogeneous quadratic and $X = p^{-1}(K) \subset \mathbb{R}\mathrm{P}^n$.
Let h be a degree one homogeneous polynomial such that

$$V = \{h = 0\} = \{h^2 = 0\}.$$

We can consider the function $\mathrm{i}_V^+ : \Omega \to \mathbb{N}$ defined by

$$\mathrm{i}_V^+(\omega) = \mathrm{i}^+(\omega p|_V)$$

and we try describe the homology of $(X, X\cap\bar V)$ only in terms of i^+ and i^+_V.
We introduce the quadratic map $p_h:\mathbb{R}^{n+1}\to\mathbb{R}^{k+2}$ defined by

$$p_h \doteq (p, h^2).$$

Then we have the following equalities:

$$X = p_h^{-1}(K\times\mathbb{R}) \quad\text{and}\quad X\cap\bar V = p_h^{-1}(K\times(-\infty,0]).$$

We consider $\hat\Omega = (K\times(-\infty,0])^\circ\cap S^{k+1}$, and the function $\mathrm{i}^+_h:\mathbb{R}^{k+1}\times\mathbb{R}\to\mathbb{N}$ defined by

$$\mathrm{i}^+_h(\omega,t) = \mathrm{i}^+(\bar p_h(\omega,t)) = \mathrm{i}^+(\omega p + th^2), \quad (\omega,t)\in\mathbb{R}^{k+1}\times\mathbb{R}.$$

For the moment we define, for $j\in\mathbb{Z}$ the set

$$\hat\Omega^{j+1} = \{\eta\in\hat\Omega\,|\,\mathrm{i}^+_h(\eta)\geq j+1\}$$

and we identify Ω with $\{(\omega,t)\in\hat\Omega\,|\,t=0\}$.
With the previous notations we prove the following.

Lemma 3.6.1. *There exists a cohomology spectral sequence (G_r,d_r) of the first quadrant converging to $H_{n-*}(X, X\cap\bar V)$ such that*

$$G_2^{i,j} = H^i(\hat\Omega^{j+1},\Omega^{j+1}).$$

Proof. Consider for $\epsilon>0$ the sets $C_h(\epsilon) = \{(\eta,x)\in\hat\Omega\times\mathbb{R}\mathrm{P}^n\,|\,(\eta p_h)(x)\geq\epsilon\}$ and $C(\epsilon) = C_h(\epsilon)\cap\Omega\times\mathbb{R}\mathrm{P}^n$. By Lemma 1.2.4 for small ϵ the inclusion

$$(C_h(\epsilon), C(\epsilon))\hookrightarrow(B_h,B)$$

is a homotopy equivalence (here B_h stands for $\{(\eta,x)\in\hat\Omega\times\mathbb{R}\mathrm{P}^n\,|\,(\eta p_h)(x)>0\}$ and B for $B_h\cap\Omega\times\mathbb{R}\mathrm{P}^n$).
Consider the projection $\beta_r:\hat\Omega\times\mathbb{R}\mathrm{P}^n\to\mathbb{R}\mathrm{P}^n$; then $\beta_r(B_h) = \mathbb{R}\mathrm{P}^n\backslash(X\cap H)$ and $\beta_r(B) = \mathbb{R}\mathrm{P}^n\backslash X$; moreover by Lemma 3.3.1 the previous are homotopy equivalences. Hence it follows:

$$H^*(C_h(\epsilon),C(\epsilon))\simeq H^*(B_h,B)\simeq H^*(\mathbb{R}\mathrm{P}^n\backslash(X\cap H),\mathbb{R}\mathrm{P}^n\backslash X)\simeq H_{n-*}(X,X\cap H)$$

where the last isomorphism is given by Alexander-Pontryagin Duality. Consider now $\beta_l: C_h(\epsilon)\to\hat\Omega$. Then by Theorem 3.1.2 there is a cohomology spectral sequence $(G_r(\epsilon), d_r(\epsilon))$ converging to $H^*(C_h(\epsilon),C(\epsilon))$ such that

$$G_2^{i,j} = \check H^i(\hat\Omega,\mathcal{G}^j(\epsilon))$$

where $\mathcal{G}^j(\epsilon)$ is a sheaf such that for $\eta\in\hat\Omega$

$$(\mathcal{G}^j(\epsilon))_\eta = H^j(\beta_l^{-1}(\eta)\cap C_h(\epsilon),\beta_l^{-1}(\eta)\cap C(\epsilon))$$

(here, reasoning as in Remark 6, we are using the fact that both $C_h(\epsilon)$ and $C(\epsilon)$ are compact). We use now $\mathrm{i}^-_h(\epsilon):\hat\Omega\to\mathbb{N}$ for the function $\eta\mapsto\mathrm{i}^-(\eta p_h-\epsilon g)$ where g is an arbitrary positive definite form, and we set $\hat\Omega_{n-j}(\epsilon) = \{\mathrm{i}^-_h(\epsilon)\leq n-j\}$. If $\eta\notin\Omega$, then $(\beta_l^{-1}(\eta)\cap C_h(\epsilon),\beta_l^{-1}(\eta)\cap C(\epsilon))\simeq(\mathbb{R}\mathrm{P}^{n-\mathrm{i}_h^-(\epsilon)(\eta)},\emptyset)$; on the contrary if $\eta\in\Omega$ then $(\beta_l^{-1}(\eta)\cap C_h(\epsilon),\beta_l^{-1}(\eta)\cap C(\epsilon))=(\mathbb{R}\mathrm{P}^{n-\mathrm{i}_h^-(\epsilon)(\eta)},\mathbb{R}\mathrm{P}^{n-\mathrm{i}_h^-(\epsilon)(\eta)})$. Since Ω is closed in $\hat{\Omega}$, it follows that

$$G_2^{j,j}(\epsilon)=\check{H}^i(\hat{\Omega}_{n-j}(\epsilon),\Omega_{n-j}(\epsilon)).$$

We define now

$$(G_r,d_r)=\varprojlim_{\epsilon}\{(G_r(\epsilon),d_r(\epsilon))\}$$

and Lemma 1.3.4 finally gives

$$G_2^{i,j}=H^i(\hat{\Omega}^{j+1},\Omega^{j+1}).$$

□

We are ready now for the proof of the main theorem of this section; we define for $j>0$ the following set:

$$\Omega_V^j=\{\omega\in\Omega:\mathrm{i}^+\,(\omega p|_V)\geq j\}.$$

Theorem 3.6.2. *There exists a cohomology spectral sequence (G_r,d_r) of the first quadrant converging to $H_{n-*}(X,X\cap\bar{V})$ such that*

$$G_2^{i,j}=H^i(\Omega_V^j,\Omega^{j+1}),\ j>0,\quad G_2^{i,0}=H^i(C\Omega,\Omega^1).$$

Proof. Take the spectral sequence (G_r,d_r) to be that of lemma 3.6.1; then it remains to prove that $G_2^{i,j}$ is isomorphic to the group described in the statement.
In the case $j=0$ we have that $\hat{\Omega}^1$ contains $(0,\ldots,0,1)$ and, since $t_1\leq t_2$ implies $\mathrm{i}_h(\omega,t_1)\leq\mathrm{i}_h^+(\omega,t_2)$, the set $\hat{\Omega}^1$ is contractible. Thus, using the long exact sequences of the pairs, we see that for every $i\geq 0$ the following holds:

$$G_2^{i,0}=H^i(\hat{\Omega}^1,\Omega^1)\simeq H^i(C\Omega,\Omega^1).$$

We study now the case $j>0$.
We identify $\hat{\Omega}\backslash\{(0,\ldots,0,1)\}$ with $\Omega\times[0,\infty)$ via the index preserving homeomorphism

$$(\omega,t)\mapsto(\omega,t)/\|\omega\|.$$

Thus, under the above identification, we have for $j>0$

$$\hat{\Omega}^{j+1}=\{(\omega,t)\in\Omega\times[0,\infty)\,|\,\mathrm{i}_h^+(\omega,t)\geq j+1\}$$

and letting $\pi:\Omega\times[0,\infty)$ be the projection onto the first factor, we see that

$$\pi(\hat{\Omega}^{j+1})=\{\omega\,|\,\exists t>0\ \text{ s.t. }\ \mathrm{i}_h^+(\omega,t)\geq j+1\}.$$

We prove that $\pi:\hat{\Omega}^{j+1}\to\pi(\hat{\Omega}^{j+1})$ is a homotopy equivalence. Let $\omega\in\pi(\hat{\Omega}^{j+1})$, then there exists $t_\omega>0$ such that $(\omega,t_\omega)\in\hat{\Omega}^{j+1}$. Since $\hat{\Omega}^{j+1}$ is open, then there exists an open neighboroud $U_\omega\times(t_1,t_2)$ of (ω,t) in $\hat{\Omega}^{j+1}$; in particular for every $\eta\in U_\omega$ we have $(\eta,t_\omega)\in\hat{\Omega}^{j+1}$ and $\sigma_\omega:\eta\mapsto(\eta,t_\omega)$ is a section of π over U_ω. Collating together the different σ_ω for $\omega\in\pi(\hat{\Omega}^{j+1})$, with the help of a partition of unity, we get a section $\sigma : \pi(\hat{\Omega}^{j+1}) \to \hat{\Omega}^{j+1}$ of π. Since for every $\omega \in \pi(\hat{\Omega}^{j+1})$ the set $\{t \geq 0 \,|\, (\omega, t) \in \hat{\Omega}^{j+1}\}$ is an interval, a straight line homotopy gives the homotopy between $\sigma \circ \pi$ and the identity on $\hat{\Omega}^{j+1}$. This implies $\pi : \hat{\Omega}^{j+1} \to \pi(\hat{\Omega}^{j+1})$ is a homotopy equivalence. Using the five lemma and the naturality of the commutative diagrams of the long exact sequences of pairs given by $\pi : (\hat{\Omega}^{j+1}, \Omega^{j+1}) \to (\pi(\hat{\Omega}^{j+1}), \Omega^{j+1})$ we get ($\pi_{|\Omega^{j+1}} = \mathrm{Id}_{|\Omega^{j+1}}$):

$$G_2^{i,j} = H^i(\hat{\Omega}^{j+1}, \Omega^{j+1}) \simeq H^i(\pi(\hat{\Omega}^{j+1}), \Omega^{j+1}).$$

It remains to prove that for $j > 0$

$$\pi(\hat{\Omega}^{j+1}) = \Omega_V^j.$$

First suppose that $(\omega, t) \in \hat{\Omega}^{j+1}$. Then there exists a subspace W^{j+1} of dimension at least $j+1$ such that $\bar{p}(\omega, t)|_{W^{j+1}} > 0$. Then

$$\omega p|_{W^{j+1} \cap V} = \bar{p}(\omega, t)|_{W^{j+1} \cap V} > 0$$

and by Grassmann formula

$$\dim(W^{j+1} \cap V) = \dim(W^{j+1}) + \dim(V) - \dim(W^{j+1} + V) \geq j$$

which implies $\mathrm{i}_V^+(\omega) \geq j$, i.e. $\pi(\omega, t) \in \Omega_V^j$. Thus

$$\pi(\hat{\Omega}^{j+1}) \subset \Omega_V^j.$$

Now let ω be in Ω_V^j; we prove that there exists $t > 0$ such that $\mathrm{i}_h^+(\omega, t) \geq j + 1$. Since $\omega \in \Omega_V^j$ then there exists a subspace $V^j \subset V$ of dimension at least j such that

$$\omega p|_{V^j} > 0.$$

Fix a scalar product on $\mathbb{R}^{n+1}$ and let $e \in \mathbb{R}^{n+1}$ be such that $V^\perp = span\{e\}$; consider the space $W = \{\lambda e\}_{\lambda \in \mathbb{R}} + V^j$, whose dimension is at least $j+1$ since $e \perp V^j \subset V$. Then the matrix for $\bar{p}_h(\omega, t)|_W$ with respect to the fixed scalar product has the form:

$$Q_W(\omega, t) = \begin{pmatrix} \omega a_0 + t & {}^t\omega a \\ \omega a & \omega Q_{V^j} \end{pmatrix}$$

where ωQ_{V^j} is the matrix for $\bar{p}(\omega, t)|_{V^j} = \omega p|_{V^j}$. Since $\omega p|_{V^j} > 0$ we have that for $t > 0$ big enough $\det(Q_W(\omega, t)) = t \det(\omega Q_{V^j}) + \det\begin{pmatrix} \omega a_0 & \omega a \\ \omega a & \omega Q_{V^j} \end{pmatrix}$ has the same sign of $\det(\omega Q_{V^j}) > 0$. For such a t we have

$$\bar{p}_h(\omega, t)|_W > 0$$

and since $\dim(W) \geq j + 1$ this implies $(\omega, t) \in \hat{\Omega}^{j+1}$ and $\omega \in \pi(\hat{\Omega}^{j+1})$. Thus

$$\Omega_V^j \subset \pi(\hat{\Omega}^{j+1})$$

and this concludes the proof. □

3.7 Higher differentials

Let $X \subset \mathbb{R}\mathrm{P}^n$ be a compact, locally contractible subset and consider the two inclusions:

$$X \xrightarrow{j} \mathbb{R}\mathrm{P}^n \quad \text{and} \quad \mathbb{R}\mathrm{P}^n \backslash X \xrightarrow{c} \mathbb{R}\mathrm{P}^n.$$

We recall the existence for every $k \in \mathbb{Z}$ of the following exact sequence, which is a direct consequence of Alexander-Pontryagin Duality:

$$0 \to \ker(c_*) \to H_k(\mathbb{R}\mathrm{P}^n \backslash X) \xrightarrow{c_*} H_k(\mathbb{R}\mathrm{P}^n) \simeq H^{n-k}(\mathbb{R}\mathrm{P}^n) \xrightarrow{i^*} H^{n-k}(X) \to \mathrm{coker}(i^*) \to 0$$

In particular we have the following equality for the k-th $\mathbb{Z}_2$-Betti number of $\mathbb{R}\mathrm{P}^n$:

$$b_k(\mathbb{R}\mathrm{P}^n) = \mathrm{rk}(c^*)_k + \mathrm{rk}(j_*)_{n-k} \tag{3.7}$$

Consider now $p : \mathbb{R}^{n+1} \to \mathbb{R}^{k+1} \supseteq K$ such that

$$\mathrm{i}^+(\overline{p}\eta) = \mu \quad \forall \eta \in \Omega.$$

Then in this case $\Omega^1 = \cdots = \Omega^\mu = \Omega$ and $\Omega^{\mu+1} = \cdots = \Omega^{n+1} = \emptyset$. For *any* scalar product g on $\mathbb{R}^{n+1}$ we have $D_\mu = \Omega^\mu = \Omega$ and we denoted by $\overline{p}^* w_{k,\mu}$ the k-th Stiefel-Whitney class of the $\mathbb{R}^\mu$-bundle $\overline{p}^*\Lambda_j^+ \to \Omega$ (notice that this class does not depend on g). As above we set $\gamma_{k,\mu} = \partial^* w_{k,\mu} \in H^{k+1}(\mathcal{Q}, \mathcal{D}_\mu)$; thus the class $\overline{p}^*\gamma_{k,\mu}$ belongs to $H^{k+1}(C\Omega, \Omega) \simeq H^k(\Omega)$.
Letting (E_r, d_r) be the spectral sequence of theorem 3.3.3 converging to $H_{n-*}(X)$, where as usual $X = p^{-1}(K) \subseteq \mathbb{R}\mathrm{P}^n$, we have that (E_r, d_r) degenerates at $(k+2)$-th step and $E_2 = \cdots = E_{k+1}$. Moreover E_{k+1} has entries only in the 0-th and the $(k+1)$-th column:

$$E_{k+1}^{a,b} = \begin{cases} \mathbb{Z}_2 & \text{if } a = 0 \text{ and } \mu \le b \le n \text{ or} \\ & \quad a = k+1 \text{ and } 0 \le b < \mu \\ 0 & \qquad \text{otherwise} \end{cases}$$

Thus the only possible nonzero differential is d_{k+1}, for which we prove the following.

Theorem 3.7.1. *Suppose* $\mathrm{i}^+ \equiv \mu$. *Then* $E_2 = \cdots = E_{k+1}$ *and the only possible nonzero differential is* $d_{k+1} : E_{k+1}^{0,b} \to E_{k+1}^{k+1,b-k}$ *for* $\mu \le b \le n$ *and it is given by:*

$$d_{k+1}(x) = x \smile \overline{p}^*\gamma_{k,\mu}$$

Remark 9. Notice that $\gamma_{k,\mu}$ and x are nothing but numbers modulo 2, thus since $E_{k+1}^{0,b} = \mathbb{Z}_2 = E_{k+1}^{k+1,b-k}$ the element $d_{k+1}(x)$ is nothing but the product $x\gamma_{k,\mu}$.

Proof. By theorem 3.5.1 we have that $d_{k+1} : E_{k+1}^{0,b} \to E_{k+1}^{k+1,b-k}$ is identically zero if and only if $\mathrm{rk}(j_*)_{n-b} = 1$ and formula (3.7) implies

$$(d_{k+1})_{0,b} \equiv 0 \quad \text{iff} \quad \mathrm{rk}(c^*)_b = 0$$

where c^* is the map induced by $c : \mathbb{RP}^n \backslash X \hookrightarrow \mathbb{RP}^n$. Consider now the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "\iota"] \arrow[d, "\beta_{r|B}"'] \arrow[dr, "\beta_r \circ \iota"] & \Omega \times \mathbb{RP}^n \arrow[d, "\beta_r"] \\ \mathbb{RP}^n \backslash X \arrow[r, "c"] & \mathbb{RP}^n \end{tikzcd}$$

Since $\beta_{r|B}$ is a homotopy equivalence, then

$$\mathrm{rk}(c^*)_b = \mathrm{rk}(\iota^* \beta_r^*)_b.$$

Let $\mathbb{RP}^{\mu-1} \hookrightarrow P(\bar{p}^* \Lambda_\mu) \to \Omega$ be the projectivization of the bundle $\mathbb{R}^\mu \hookrightarrow \bar{p}^* \Lambda_\mu \to \Omega$. It is easily seen that the inclusion

$$P(\bar{p}^* \Lambda_\mu) \hookrightarrow B$$

is a homotopy equivalence. From this, letting $l : P(\bar{p}^* \Lambda_\mu) \to \mathbb{RP}^n$ be the restriction of $\beta_r \circ \iota$ to $P(\bar{p}^* \Lambda_\mu)$, it follows that:

$$\mathrm{rk}(c^*)_b = \mathrm{rk}(l^*)_b.$$

Let $y \in H^1(\mathbb{RP}^n)$ be the generator; since l is a linear embedding on each fiber, then by Leray-Hirsch, it follows that

$$H^*(P(\bar{p}^* \Lambda_\mu)) = H^*(\Omega) \otimes \{1, l^* y, \ldots, (l^* y)^{\mu-1}\}.$$

Thus for $\mu \leq b \leq n$ we have:

$$\begin{aligned} l^* y^b =& (l^* y)^b = (l^* y)^\mu \smile (l^* y)^{b-\mu} \\ =& \beta_l^* \overline{p}^* w_{k,\mu} \smile (l^{*y})^{\mu-k} \smile (l^* y)^{b-\mu} \\ =& \beta_l^* \overline{p}^* w_{k,\mu} \smile (l^* y)^{b-k}. \end{aligned}$$

Thus $(d_{k+1})_{0,b}$ is zero if and only if $\overline{p}^* w_{k,\mu} = 0$ and by looking at the definition of $\overline{p}^* \gamma_{k,\mu}$ we see that

$$d_{k+1}(x) = x \smile \overline{p}^* \gamma_{k,\mu}.$$

□

Notice also that in the case $\mathrm{i}^+ \equiv \mu$ if we take $l_v^+ = \{t^2 v\}_{t \in \mathbb{R}}$, then we can easily calculate the homology of $X_{l_v^+} = \{x \in \mathbb{P}^n : p(x) \in l_v^+\}$ (the preimage of a half line): using theorem 3.3.3 we immediatly see that $E_2 = E_\infty$ which implies $H_*(X_{l_v^+}) \simeq H_*(\mathbb{P}^{n-\mu})$.

Example 1 (see [22]). For $a = 1, 2, 4, 8$ consider the isomorphism $\mathbb{R}^a \simeq A$ where A denotes respectively $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Consider the quadratic map

$$h_a : \mathbb{R}^a \oplus \mathbb{R}^a \to \mathbb{R}^a \oplus \mathbb{R}$$

defined, using the previous identification $\mathbb{R}^a \simeq A$, by

$$(z,w) \mapsto (2z\overline{w}, |w|^2 - |z|^2).$$

Then it is not difficult to prove that h_a maps S^{2a-1} into S^a by a Hopf fibration. Hence it follows that

$$\emptyset = K_a \doteq h_a^{-1}(0) \subset \mathbb{R}\mathrm{P}^{2a-1}.$$

In each case we have $\mathrm{i}^+(\omega h_a) = a$ for every $\omega \in \Omega = S^a$. Using Theorem 3.7.1, since $K_a = \emptyset$ then d_{a+1} must be an isomorphism, hence

$$0 \neq w_{a,a} = w_a(\bar{h}_a^*\Lambda_a) \in H^a(S^a).$$

For example in the case $a = 2$ we have the standard Hopf fibration $h_2|_{S^3} : S^3 \to S^2$ and the table for $E_2 = E_3$ is:

$$\begin{array}{|c|c|c|c}
\mathbb{Z}_2 & 0 & 0 & 0 \\
\mathbb{Z}_2 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}_2 \\
0 & 0 & 0 & \mathbb{Z}_2 \\
\hline
\end{array}$$

The bundle $\mathbb{R}^2 \hookrightarrow \bar{h}_a^*\Lambda_2 \to S^2$ has total Stiefel-Whitney class

$$w(\bar{h}_a^*\Lambda_2) = 1 + w_{2,2}, \quad w_{2,2} \neq 0$$

and the differential d_3 is an isomorphism.
Notice that for $a = 1, 2, 4, 8$ we have $\ker(\omega h_a) = 0$ for every $\omega \in \Omega$. It is an interesting fact that the contrary also is true.

Fact 1. if $p : \mathbb{R}^m \to \mathbb{R}^l$ is such that $\ker(\omega p) = \{0\}$ for every $\omega \in S^l$ and $p|_{S^{m-1}} : S^{m-1} \to S^{l-1}$ then, up to orthonormal change of coordinates $p = h_a$ for some $a \in \{1, 2, 4, 8\}$.

Proof. First observe that $\mathrm{i}^+ \equiv c$ for a constant c and that $m = 2c$. Then, since p maps the sphere S^{2c-1} to the sphere S^{l-1}, we have

$$\emptyset = p^{-1}(\{0\}) \subset \mathbb{R}\mathrm{P}^{2c-1}.$$

Thus Theorem 3.7.1 implies that the differential d_l must be an isomorphism and this forces $l = c + 1$. Moreover the condition $\ker(\omega p) = \{0\}$ for every $\omega \in S^{c-1}$ says also $p_{|S^{2c-1}} : S^{2c-1} \to S^c$ is a submersion. It is a well-known result (see [26]) that the preimage of a point trough a quadratic map between spheres is a sphere, and thus $p_{|S^{2c-1}}$ is the projection of a sphere-bundle between spheres, hence it must be a Hopf fibration. $\square$

The situation in the case $\{\omega \in S^{l-1} \mid \ker(\omega p) \neq 0\} = \emptyset$ with only the assumption $X = \emptyset$ (which is weaker than $p(S^{m-1}) \subset S^{l-1}$) is more delicate.

Example 2. For $i = 1, \dots, l$ let $p_i : \mathbb{R}^{n_i} \to \mathbb{R}^{k+1}$ be a quadratic map and set $N = \sum_i n_i$. Define the map

$$\oplus_i p_i : \mathbb{R}^N \to \mathbb{R}^{k+1}$$

by the formula

$$(x_1, \ldots, x_l) \mapsto \sum_{i=1}^{l} p_i(x_i) \quad x_i \in \mathbb{R}^{n_i}.$$

Then for every $\omega \in S^k$ we have

$$\mathrm{i}^+(\omega(\oplus_i p_i)) = \sum_{i=1}^{l} \mathrm{i}^+(\omega p_i).$$

In particular if each p_i has constant positive index function with constant value μ_i, then $\oplus_i p_i$ has also constant positive index function with constant value $\sum_i \mu_i$. Generalizing the previous example, we consider now for $a = 1, 2, 4, 8$ the map $h_a : \mathbb{R}^{2a} \to \mathbb{R}^{a+1}$ defined above and we take for $n \in \mathbb{N}$ the map

$$n \cdot h_a \doteq \oplus_{i=1}^{n} h_a : \mathbb{R}^{2an} \to \mathbb{R}^{a+1}.$$

In coordinate the map $n \cdot h_a$ is written by:

$$(w, z) \mapsto (2\langle z, w\rangle, \|w\|^2 - \|z\|^2), \quad w, z \in A^n.$$

Then for this map we have

$$\mathrm{i}^+ \equiv na, \quad \text{and} \quad (n \cdot \bar{h}_a)^* \Lambda_{na} = n(\bar{h}_a^* \Lambda_a) = \underbrace{\bar{h}_a^* \Lambda_a \oplus \cdots \oplus \bar{h}_a^* \Lambda_a}_{n}$$

The solution of $\{n \cdot h_a = 0\}$ on the sphere S^{2a-1} is diffeomorphic to the Stiefel manifold of 2-frames in A^n, and it is a double cover of

$$n \cdot K_a \doteq \{n \cdot h_a = 0\} \subset \mathbb{RP}^{2na-1}.$$

We can proceed now to the calculation of the $\mathbb{Z}_2$-cohomology of $n \cdot K_a$, using Theorem 3.7.1: we only need to compute d_{a+1}, i.e. $w_a(n\bar{h}_a^* \Lambda_a)$. Since $w_a(\bar{h}_a^* \Lambda_a) = w_{a,a} \neq 0$, and $w_k(\bar{h}_a^* \Lambda_a) = 0$ for $k \neq 0, k \neq a$, then we have

$$w_a(n\bar{h}_a^* \Lambda_a) = n \bmod 2 \in \mathbb{Z}_2 = H^a(S^a).$$

CHAPTER 4

Complex theory

4.1 Real projective sets and complex projective sets

We start by considering the bundle

$$S^1 \to \mathbb{R}\mathrm{P}^{2n+1} \xrightarrow{\pi} \mathbb{C}\mathrm{P}^n$$

where the map π is given by $[x_0, y_0, \dots, x_n, y_n] \mapsto [x_0 + iy_0, \dots, x_n + iy_n]$. The fiber of π over a point $[v] \in \mathbb{C}\mathrm{P}^n$ equals the projectivization of the two dimensional real vector space $\mathrm{span}_{\mathbb{R}}\{v, iv\} \subset \mathbb{C}^{n+1} \simeq \mathbb{R}^{2n+2}$. Thus $\mathbb{R}\mathrm{P}^{2n+1}$ is the total space of the projectivization of the tautological bundle $O(-1) \to \mathbb{C}\mathrm{P}^n$ view as a rank two real vector bundle. Applying Leray-Hirsch we get a cohomology class $x \in H^1(\mathbb{R}\mathrm{P}^{2n+1}; \mathbb{Z}_2)$, which restricts to a generator of the cohomology of each fiber, such that the map $\alpha \otimes p(x) \mapsto \pi^*\alpha \smile p(x)$, where $p \in \mathbb{Z}_2[x]/(x^2)$ and $\alpha \in H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, gives an isomorphism of $H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_2)$-modules

$$H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_2) \otimes \{1, x\} \simeq H^*(\mathbb{R}\mathrm{P}^{2n+1}; \mathbb{Z}_2).$$

In particular this tells that π^* is injective with image the even dimensional cohomology (recall that $|\alpha| = 2$).
The following geometric description of the map π also gives an alternative proof of the previous statement. Consider the restriction of π to $\{[x_0, y_0, x_1, 0, \dots, 0]\} \simeq \mathbb{R}\mathrm{P}^2$: we see that it maps $\mathbb{R}\mathrm{P}^2$ to $\{[z_0, z_1, 0, \dots, 0]\} \simeq \mathbb{C}\mathrm{P}^1$ trough a homeomoprhism $\{x_1 \neq 0\} \simeq \{z_1 \neq 0\}$ and by collapsing the line at infinity $\{x_1 = 0\}$ to the point $[1, 0, \dots, 0]$. It follows that the modulo 2 degree of $\pi|_{\mathbb{R}\mathrm{P}^2}$ is one. Using the isomorphism $H^*(\mathbb{R}\mathrm{P}^{2n+1}; \mathbb{Z}_2) \simeq \mathbb{Z}_2[\beta]/(\beta^{2n+2})$, $|\beta| = 1$, we see that

$$\pi^* : H^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_2) \to H^*(\mathbb{R}\mathrm{P}^{2n+1}; \mathbb{Z}_2)$$

is given by $\alpha \mapsto \beta^2$, where $\beta|_{\mathbb{R}\mathrm{P}^2}$ generates $H^1(\mathbb{R}\mathrm{P}^2; \mathbb{Z}_2)$. If we consider the Gysin sequence with $\mathbb{Z}_2$ coefficients for π, then the injectivity of π^* implies that for every j the following portion of the sequence is exact

$$0 \to H^j(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_2) \xrightarrow{\pi^*} H^j(\mathbb{R}\mathrm{P}^{2n+1}; \mathbb{Z}_2) \to H^{j-1}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_2) \xrightarrow{\smile e} 0$$

where $e = e(\pi)$ is the modulo 2 euler class of π, which of course turns out to be zero.
Let now $I \subset \mathbb{C}[z_0, \dots, z_n]$ be a homogeneous ideal; we will denote by $C = C(I)$ its zero locus in $\mathbb{C}\mathrm{P}^n$. If we restrict the bundle $O(-1) \to \mathbb{C}\mathrm{P}^n$ to C we get a bundle:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & O(-1) \arrow[d] \\ C \arrow[r] & \mathbb{C}\mathrm{P}^n \end{tikzcd}$$

and if we consider the previous as rank two real vector bundles and take their projectivization we get:

$$\begin{tikzcd} \mathbb{R}\mathrm{P}^1 \arrow[r] & R \arrow[r, "i_R"] \arrow[d, "\pi|_R"'] & \mathbb{R}\mathrm{P}^{2n+1} \arrow[d, "\pi"] \\ & C \arrow[r, "i_C"] & \mathbb{C}\mathrm{P}^n \end{tikzcd}$$

where i_R and i_C are the inclusion maps.
It is clear that R is an algebraic subset of $\mathbb{R}\mathrm{P}^{2n+1}$ whose equations are given by considering each polynomial $f \in I$ as a pair of polynomials $f^a = \mathrm{Re}(f), f^b = \mathrm{Im}(f) \in \mathbb{R}[x_0, y_0, \ldots, x_n, y_n]$. Applying Leray-Hirsch to $\pi|_R$, or the identity $e(\pi|_R) = e(\pi)|_C = 0$, we get the isomoprhism of $H^*(C; \mathbb{Z}_2)$-modules:

$$H^*(C; \mathbb{Z}_2) \otimes \{1, x|_R\} \simeq H^*(R, \mathbb{Z}_2).$$

The previous isomorphism allows us to compute $\mathbb{Z}_2$-Betti numbers of C once those of R are known, via the following formula:

$$b_j(C; \mathbb{Z}_2) = \sum_{k=0}^{j} (-1)^k b_{j-k}(R; \mathbb{Z}_2).$$

We have the following equalities for the Stiefel-Whitney classes of E, which come from the fact that E is the realification of a complex bundle: $w_{2k}(E) = c_k(E) \bmod 2$, where c_k is the k-th Chern class of E seen as a complex bundle, and $w_{2k+1}(E) = 0$. Since E has real rank two we have:

$$w_2(E) = i_C^* z \quad \text{and} \quad w_i(E) = 0,\ i \neq 0, 2,$$

where z is the generator of $H^2(\mathbb{C}\mathrm{P}^n, \mathbb{Z}_2)$ and we have used the equalities $w_2(E) = c_1(E) = i_C^* c_1(O(-1)) = i_C^* z$.
The following lemma relates the homomorphisms i_C^* and i_R^*.

Lemma 4.1.1. *There exists an odd r such that $(i_R^*)_k : H^k(\mathbb{R}\mathrm{P}^{2n+1}; \mathbb{Z}_2) \to H^k(R; \mathbb{Z}_2)$ is injective for $k \leq r$ and zero for $k > r$. Moreover for every k we have*

$$rk(i_C^*)_{2k} = rk(i_R^*)_{2k} = rk(i_R^*)_{2k+1}.$$

Proof. Let a be such that $(i_R^*)_a \equiv 0$; then using the cup product structure of $H^*(\mathbb{R}\mathrm{P}^{2n+1}; \mathbb{Z}_2) = \mathbb{Z}_2[\beta]/(\beta^{2n+2})$, we have

$$i_R^* \beta^{a+k} = i_R^* \beta^a \smile i_R^* \beta^k = 0.$$

For the second part of the statement notice that $R = P(E) \xrightarrow{i_R} \mathbb{R}\mathrm{P}^{2n+1}$ is linear on the fibres and thus, letting $y = i_R^* \beta$ we have $y^2 = (w_2(E) + w_1(E)y) = w_2(E)y$ (since $w_1(E)$ is zero), where we interpret $w_i(E)$ as a class on R via $\pi|_R^*$. It follows that

$$y^{2k} = w_2(E)^k \quad \text{and} \quad y^{2k+1} = w_2(E)^k y.$$

On the other hand, since $w_2(E) = i_C^* z$, then the conclusion follows. $\square$

4.2 The spectral sequence in the complex case

In this section we study the topology of R in the case C is cut by quadrics, i.e.

$$C = V_{\mathbb{C}\mathrm{P}^n}(q_0, \ldots, q_l), \quad q_0, \ldots, q_l \in \mathbb{C}[z_0, \ldots, z_n]_{(2)}$$

For a given $q \in \mathbb{C}[z_0, \ldots, z_n]_{(2)}$, $q(z) = z^T Q z$ with $Q = A - iB$ and $A, B \in \mathrm{Sym}(n+1, \mathbb{R})$ we define the symmetric matrix

$$P = \left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right).$$

We set $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ (it is a $(n+2) \times (n+2)$ matrix) and given $q_0, \ldots, q_l \in \mathbb{C}[z_0, \ldots, z_n]_{(2)}$ we define $p : S^{2l+1} \to \mathrm{Sym}(2n+2, \mathbb{R})$ by

$$(a_0, b_0, \ldots, a_l, b_l) \overset{p}{\mapsto} a_0 P_0 - b_0 J P_0 + \cdots + a_l P_l - b_l J P_l.$$

For every polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]$ recall that we have defined the polynomials $f^a, f^b \in \mathbb{R}[x_0, y_0, \ldots, x_n, y_n]$ by

$$f^a(x, y) = \mathrm{Re}(f)(x + iy), \qquad f^b(x, y) = \mathrm{Im}(f)(x + iy).$$

Thus if $C = V_{\mathbb{C}\mathrm{P}^n}(q_0, \ldots, q_l)$, we have

$$R = V_{\mathbb{R}\mathrm{P}^{2n+1}}(q_0^a, q_0^b, \ldots, q_l^a, q_l^b)$$

We easily see that $\mathrm{i}^+(a_0 q_0^a + b_0 q_0^b + \cdots + a_l q_l^a + b_l q_l^b) = \mathrm{i}^+(p(a_0, b_0, \ldots, b_l, q_l))$: this is simply because P_j and $-JP_j$ are the symmetric matrices associated respectively to the quadratic forms q_j^a and q_j^b.
Following the previous chapters for every $j \in \mathbb{N}$ we define

$$\Omega^j = \{\alpha \in S^{2l+1} \,|\, \mathrm{i}^+(p(\alpha)) \geq j\}$$

and if we let B be the unit ball in $\mathbb{R}^{2l+2}$, $\partial B = S^{2l+1}$ we recall the existence of a first quadrant spectral sequence $(E_r, d_r)_{r \geq 0}$ such that:

$$(E_r, d_r) \Rightarrow H_{2n+1-*}(R; \mathbb{Z}_2), \quad E_2^{i,j} = H^i(B, \Omega^{j+1}; \mathbb{Z}_2).$$

For $j \in \mathbb{N}$ if we let $P^j \subset S^{2l+1} \subset \mathbb{C}^{l+1}$ be defined by

$$P^j = \{(\alpha_0, \ldots, \alpha_l) \in S^{2l+1} \,|\, \mathrm{rk}_{\mathbb{C}}(\alpha_0 q_0 + \ldots + \alpha_l q_l) \geq j\}$$

we can rewrite theorem 3.3.3 in the following more natural way.

Theorem 4.2.1. *There exists a cohomology spectral sequence of the first quadrant (E_r, d_r), converging to $H_{2n+1-*}(R, \mathbb{Z}_2)$ such that $E_2^{i,j} = H^i(B, P^{j+1}; \mathbb{Z}_2)$.*

Proof. We will prove that for every j the two sets P^{j+1} and Ω^{j+1} are homeomorphic, and in fact if $\tau : \mathbb{C}^{l+1} \to \mathbb{C}^{l+1}$ denotes complex coniugation that we have

$$\tau(P^{j+1}) = \Omega^{j+1}.$$

If we use the matrix notation for each q_j we have $q_j(z) = z^T Q_j z$ for $Q_j \in \mathrm{Sym}(n+1, \mathbb{C})$ and writing $Q_j = A_j - iB_j$ with $A_j, B_j \in \mathrm{Sym}(n+1, \mathbb{R})$

$$q_j^a(x,y) = \langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \begin{pmatrix} A_j & B_j \\ B_j & -A_j \end{pmatrix} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \rangle, \quad q_j^b(x,y) = \langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \begin{pmatrix} -B_j & A_j \\ A_j & B_j \end{pmatrix} \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \rangle.$$

In particular notice that the matrix associated to the real quadratic form $a_0 q_0^a + b_0 q_0^b + \cdots + a_l q_l^a + b_l q_l^b$ is of the form

$$M = \left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right)$$

for $A, B \in \mathrm{Sym}(n+1, \mathbb{R})$. If λ is an eigenvalue of M and V_λ is the corresponding eigenspace, then the map $(u,v) \mapsto (-v,u)$ gives an isomorphism $V_\lambda \simeq V_{-\lambda}$. This implies

$$2\mathrm{i}^+(M) = \mathrm{rk}_{\mathbb{R}}(M).$$

On the other side it is easy to show that

$$\mathrm{rk}_{\mathbb{R}} \left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right) = 2\mathrm{rk}_{\mathbb{C}}(A - iB)$$

in fact the map $(u,v) \mapsto u + iv$ gives an isomorphism of real vector space $\ker(M) \simeq \ker(A - iB)$. Comparing now the matrices associated to $a_0 q_0^a + b_0 q_0^b + \cdots + a_l q_l^a + b_l q_l^b$ and to $(a_0 - ib_0)q_0 + \ldots + (a_l - ib_l)q_l$ we get the result.

$\square$

Remark 10. Even more natural than the sets $\{P^j\}_{j \in \mathbb{N}}$ are the sets

$$Y^j = \{[\alpha] \in \mathbb{C}\mathrm{P}^l,\, \alpha \in S^{2l+1} |\, \mathrm{rk}(p(\alpha)) \geq j\}.$$

If we consider the hopf bundle $S^1 \to S^{2l+1} \xrightarrow{h} \mathbb{C}\mathrm{P}^l$ we see that $h(P^j) = Y^j$ and thus

$$P^j \neq S^{2l+1} \Rightarrow H^*(P^j) = H^*(Y^j) \otimes H^*(S^1).$$

In this way we see that it is possible to express all the data for E_2 of the previous spectral sequence only in terms of the linear system $\mathbb{P}(\mathrm{span}(q_0, \ldots, q_l)) \subset \mathbb{P}(\mathbb{C}[z_0, \ldots, z_n]_{(2)})$.

We focus now on the second differential of our spectral sequence; for convenience of the reader we recall the construction made in the previous chapters.
For each $P \in \mathrm{Sym}(2n+2, \mathbb{R})$ we ordered the eigenvalues of P in increasing way:

$$\lambda_1(P) \geq \cdots \geq \lambda_{2n+2}(P)$$

and we defined

$$D_j = \{\alpha \in S^{2l+1} \,|\, \lambda_j(p(\alpha)) \neq \lambda_{j+1}(p(\alpha))\}.$$

Then there is a naturally defined bundle $\mathbb{R}^j \to L_j \to D_j$ whose fiber over a point $\alpha \in D_j$ equals $(L_j)_\alpha = \mathrm{span}\{v \in \mathbb{R}^{2n+2} \,|\, p(\alpha)v = \lambda_i v,\, i = 1, \ldots, j\}$ and whose vector bundle structure is given by the inclusion $L_j \hookrightarrow D_j \times \mathbb{R}^{2n+2}$. We defined $\overline{p}^* w_{1,j} \in H^1(D_j)$ to be the first Stiefel-Whitney class of L_j and

$$\overline{p}^* \gamma_{1,j} = \partial^* \overline{p}^* w_{1,j} \in H^2(B, D_j)$$

where $\partial^* : H^1(D_j) \to H^2(B, D_j)$ is the connecting isomorphism. With this notation theorem 3.4.8 gives the following description of $d_2 : H^i(B, \Omega^{j+1}) \to H^{i+2}(B, \Omega^j)$:

$$d_2(x) = (x \smile \overline{p}^* \gamma_{1,j})|_{(B,\Omega^j)}$$

If we let $[\alpha_0, \dots, \alpha_l] = [a_0 + ib_0, \dots, a_l + ib_l] \in \mathbb{C}\mathrm{P}^l$ such that $\alpha = (a_0, b_0, \dots, a_l, b_l) \in S^{2l+1}$ then $p|_{h^{-1}[\alpha_0,\dots,\alpha_l]} : S^1 \to \mathrm{Sym}(2n+2, \mathbb{R})$ equals

$$\theta \mapsto \left(\begin{smallmatrix} I\cos\theta & -I\sin\theta \\ I\sin\theta & I\cos\theta \end{smallmatrix}\right) p(\alpha)$$

as one can easily check. The following lemma is the main ingredient for the explicit computations of d_2.

Lemma 4.2.2. *Let $A, B, I \in \mathrm{Sym}(n+1, \mathbb{R})$, with I the identity matrix, $R(\theta) = \left(\begin{smallmatrix} I\cos\theta & -I\sin\theta \\ I\sin\theta & I\cos\theta \end{smallmatrix}\right)$ and $M = \left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right)$. Let $c : S^1 \to \mathrm{Sym}(2n+2, \mathbb{R})$ be defined by*

$$\theta \mapsto R(\theta)M.$$

Consider the bundle c^*L over S^1 whose fibre at the point $\theta \in S^1$ is

$$(c^*L)_\theta = \mathrm{span}\{w \in \mathbb{R}^{2n+2} \mid \exists \lambda > 0 \mid c(\theta)w = \lambda w\}$$

and whose vector bundle structure is given by its inclusion in $S^1 \times \mathbb{R}^{2n+2}$. Then the following holds for the first Stiefel-Whitney class of c^*L :

$$w_1(c^*L) = \mathrm{rk}_{\mathbb{C}}(A - iB) \bmod 2.$$

Proof. First notice that if $w = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$ is an eigenvector of $\left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right)$ for the eigenvalue λ, then $Jw = \left(\begin{smallmatrix} v \\ -u \end{smallmatrix}\right)$ is an eigenvector for the eigenvalue $-\lambda$. It follows that there exists a basis $\{w_1, Jw_1, \dots, w_{n+1}, Jw_{n+1}\}$ of $\mathbb{R}^{2n+2}$ of eigenvectors of $\left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right)$ such that $\left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right) w_j = \lambda_j w_j$ with $\lambda_j \geq 0$. Let now $W_j = \mathrm{span}\{w_j, Jw_j\}$. Then W_j is $R(\theta)$-invariant: $R(\theta)w_j = \cos\theta w_j - \sin\theta Jw_j$ and $R(\theta)Jw_j = \sin\theta w_j + \cos\theta Jw_j$. Thus, using the above basis, we see that $R(\theta)$ is congruent to

$$M^T R(\theta) M = \mathrm{diag}(D_1(\theta), \dots, D_{n+1}(\theta)), \quad D_j(\theta) = \lambda_j \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$$

If $c_j : \theta \mapsto D_j(\theta)$, then clearly we have the splitting $c^*L = c_1^*L \oplus \cdots \oplus c_{n+1}^*L$. Since $w_j(c^*L) = 0$ if and only if $\lambda_j = 0$, then

$$w_1(c^*L) = \frac{1}{2}\mathrm{rk}_{\mathbb{R}} \left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right) = \mathrm{rk}_{\mathbb{C}}(A - iB)$$

where the last equality comes from the proof of Theorem 4.2.1. □

Corollary 4.2.3 (The cohomology of one single quadric)**.** *Let $q \in \mathbb{C}[z_0, \dots, z_n]_{(2)}$ be a quadratic form with $\mathrm{rk}(q) = \rho > 0$ and*

$$C = V(q) \subset \mathbb{C}\mathrm{P}^n$$

Then the Betti numbers of C are:

$$\rho \text{ even:} \quad b_j(C) = \begin{cases} 0 & \text{if } j \text{ is odd;} \\ 1 & \text{if } j \text{ is even, } 0 \leq j \leq 2n-2,\ j \neq 2n-\rho \\ 2 & \text{if } j = 2n-\rho \end{cases}$$

$$\rho \text{ odd:} \quad b_j(C) = \begin{cases} 0 & \text{if } j \text{ is odd;} \\ 1 & \text{if } j \text{ is even, } 0 \leq j \leq 2n-2 \end{cases}$$

Proof. We first compute $H^*(R)$ using Theorem 4.2.1: in this case if $Q = A - iB$, then R is the intersection of the two quadrics defined by the symmetric matrices $P = \left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right)$ and $-JP = \left(\begin{smallmatrix} B & A \\ A & -B \end{smallmatrix}\right)$ and $p : S^1 \to \mathrm{Sym}(2n+2, \mathbb{R})$ equals $\theta \mapsto R_\theta \left(\begin{smallmatrix} A & B \\ B & -A \end{smallmatrix}\right)$. The function i^+ has constant value ρ and thus the E_2 table for R has the following picture:

$$\begin{array}{c|c|c|c|}
2n+1 & \mathbb{Z}_2 & 0 & 0 \\
\hline
 & \vdots & \vdots & \vdots \\
\rho & \mathbb{Z}_2 & 0 & 0 \\
\hline
 & 0 & 0 & \mathbb{Z}_2 \\
 & \vdots & \vdots & \vdots \\
 & 0 & 0 & \mathbb{Z}_2 \\
\hline
\end{array}$$

The only (possibly) nonzero differential is

$$d_2 : E_2^{0,\rho} \to E_2^{2,\rho-1}$$

which by the previous discussion equals $1 \mapsto \partial^* w_1(p^* L)$. Lemma 4.2.2 implies now

$$d_2(1) = \rho \bmod 2$$

Thus if ρ is even $E_2 = E_\infty$ and if ρ is odd the $(\rho-1)$-th and the ρ-th row of $E_3 = E_\infty$ are zero. Applying the formula $b_j(C;\mathbb{Z}_2) = \sum_{k=0}^{j} (-1)^k b_{j-k}(R;\mathbb{Z}_2)$ gives the result. □

Using the previous spectral sequence we can easily compute the rank of the map induced on the $\mathbb{Z}_2$-cohomology by the inclusion

$$i_C : C \hookrightarrow \mathbb{C}\mathrm{P}^n.$$

We recall that from theorem 3.5.1 we have $\dim(E_\infty^{0,2n+1-k}) = \mathrm{rk}(i_R^*)_k$; thus applying Lemma 4.1.1 we get the following.

Theorem 4.2.4. *For every k we have*

$$\mathrm{rk}(i_C^*)_{2k} = \dim E_\infty^{0,2n+1-2k}$$

and the zeroth column of E_∞ must be the following:

$$E_\infty^{0,*} = \begin{array}{|c|}
\hline
\mathbb{Z}_2 \\
\vdots \\
\mathbb{Z}_2 \\
\hline
0 \\
\vdots \\
0 \\
\hline
\end{array}$$

where the number of $\mathbb{Z}_2$ summand is an even number $r+1$, and r is that given by Lemma 4.1.1.

Notice in particular that $E_\infty^{0,2a} = \mathbb{Z}_2$ iff $E_\infty^{0,2a+1} = \mathbb{Z}_2$.

4.3 The intersection of two quadrics

We apply here the previous result to compute the cohomology of the intersection of two complex quadrics:

$$C = V(q_0, q_1) \subset \mathbb{C}\mathrm{P}^n.$$

As it can be expected, here all the data depends only on the complex pencil of quadrics $\{\alpha q\}_{[\alpha]\in\mathbb{C}\mathrm{P}^1}$. For example, assuming the pencil has $[\alpha_1], \dots, [\alpha_l]$ singular points (if $l < n+1$ then C is not a complete intersection), then all we need to compute the second differential for the spectral sequence converging to $H_{2n+1-*}(R)$ is the knowledge of the rank of $\alpha_i q$ and the multiplicity of $[\alpha_i]$ as a zero of $\det(\alpha q) = 0$ for $i = 1, \dots, l$.
More precisely we start defining

$$\Sigma_j = \{[\alpha] \in \mathbb{C}\mathrm{P}^1 \,|\, \mathrm{rk}(\alpha_0 q_0 + \alpha_1 q_1) \leq j-1\}.$$

For $j \leq \mu = \max \mathrm{i}^+$ we see that Σ_j consists of a finite number of points (it is a proper algebraic subset) $\Sigma_j = \{[\alpha_1], \dots [\alpha_{\sigma_j}]\}$, where we have set

$$\sigma_j = \mathrm{card}(\Sigma_j), \quad j \leq \mu.$$

The discussion of the previous sections implies that for every $[\alpha] \in \mathbb{C}\mathrm{P}^1$ the function i^+ is constant on the circle $h^{-1}[\alpha] \subset S^3$ with value

$$\mathrm{i}^+|_{h^{-1}[\alpha]} \equiv \mathrm{rk}(\alpha_0 q_0 + \alpha_1 q_1) = \rho([\alpha]).$$

Thus it is defined the bundle $\mathbb{R}^{\rho([\alpha])} \to L_{[\alpha]} \to h^{-1}[\alpha]$ of positive eigenspace of $p|_{h^{-1}[\alpha]}$ and Lemma 4.2.2 implies

$$w_1(L_{[\alpha]}) = \rho([\alpha]) \bmod 2.$$

Fore every $[\alpha] \in \mathbb{C}\mathrm{P}^1$ we let $m_{[\alpha]}$ be the multiplicity of $[\alpha]$ as a solution of $\det(\alpha_0 Q_0 + \alpha_1 Q_1) = 0$; notice that in general $n + 1 - \rho([\alpha]) \neq m_{[\alpha]}$.
For every $j \in \mathbb{N}$ we see that

$$\Omega^{j+1} = S^3 \backslash h^{-1}(\Sigma_{j+1})$$

If we let ν be the minimum of i^+ over S^3, we see that for $i > 0$ and $\nu + 1 \leq j + 1 \leq \mu$

$$E_2^{i,j} = H^i(B, S^3 \backslash h^{-1}(\Sigma_{j+1})) \simeq \tilde{H}_{3-i}(h^{-1}(\Sigma_{j+1})) = \begin{cases} 0 & \text{if } i \neq 2, 3; \\ \mathbb{Z}_2^{\sigma_{j+1}} & \text{if } i = 2 \\ \mathbb{Z}_2^{\sigma_{j+1}-1} & \text{if } i = 3 \end{cases}$$

This gives the following picture for the table of ranks of E_2 :

$2n+1$	1	0			
	$\vdots$	$\vdots$			
	1	0			
μ	1	0			
$\mu-1$	0	0	σ_μ	$\sigma_\mu - 1$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ν	0	0	$\sigma_{\nu+1}$	$\sigma_{\nu+1}-1$	0
	0	0	0	0	1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	0	0	0	0	1

We proceed now with the computation of the second differential; the only two possibly nonzero differential are $d_2^{0,\mu}$ and $d_2^{2,\nu}$, for which the following theorem holds; for an integer m we let $\overline{m} \in \mathbb{Z}_2$ be its residue modulo 2.

Theorem 4.3.1. *The following formula holds for the differential* $d_2^{2,\nu} : \mathbb{Z}_2^{\sigma_{\nu+1}} \to \mathbb{Z}_2$

$$d_2^{2,\nu}(x_1, \dots, x_{\sigma_{\nu+1}}) = \overline{\nu} \sum_{k=1}^{\sigma_{\nu+1}} x_k.$$

Moreover in the case $\mu = n+1$, *we also have the following explicit expression for* $d_2^{0,n+1} : \mathbb{Z}_2 \to \mathbb{Z}_2^{\sigma_{n+1}}$

$$d_2^{0,n+1}(x) = x(\overline{m}_1, \dots, \overline{m}_{\sigma_\mu})$$

where $m_k = m_{[\alpha_k]}$.

Proof. We start with

$$d_2 : E_2^{2,\nu} \simeq \tilde{H}^1(h^{-1}(\Sigma_{\nu+1})) \to E_2^{4,\nu-1} = \mathbb{Z}_2$$

which is given by $x \mapsto (x \smile \gamma_{1,\nu})|_{(B,\Omega^\nu)}$. In order to do that we choose a small neighborhood $U(\epsilon)$ of $\Sigma_{\nu+1} = \{[\beta_1], \dots, [\beta_{\sigma_{\nu+1}}]\}$ and we define $C(\epsilon) = h^{-1}(U(\epsilon))$. If we set $\gamma_{1,\nu}(\epsilon) = \gamma_{1,\nu}|_{(B,C(\epsilon))}$, then since $C(\epsilon) \cup \Omega^{\nu+1} = \Omega^\nu = S^3$,

$$d_2^{2,\nu}(x) = (x \smile \gamma_{1,\nu}(\epsilon))|_{(B,S^3)}.$$

We let $\partial^* c_1, \dots, \partial^* c_{\sigma_{\nu+1}}$ be the generators of $H^2(B, C(\epsilon)) \overset{\partial^*}{\simeq} H^1(C(\epsilon))$, where c_k is the dual of $h^{-1}[\beta_k]$, $k = 1, \dots, \sigma_{\nu+1}$. Lemma 4.2.2 implies now that $w_1(L_{[\beta_i]}) = \nu \bmod 2$ because $\nu = \min \mathrm{i}^+ = \mathrm{rk}(p(\beta_k))$ for every $k = 1, \dots, \sigma_{\nu+1}$. It follows that

$$\gamma_{1,\nu}(\epsilon) = \overline{\nu} \sum_{k=1}^{\sigma_{\nu+1}} \partial^* c_k.$$

If we let now $\partial^* g_1, \dots, \partial^* g_{\sigma_{\nu+1}}$ be the generators of $H^2(B, \Omega^{\nu+1}) \overset{\partial^*}{\simeq} H^1(\Omega^{\nu+1})$, where $g_k = \mathrm{lk}(\,\cdot\,, h^{-1}[\beta_k])$, $k = 1, \dots, \sigma_{\nu+1}$, we have the following formula

$$d_2^{2,\nu}(x) = \overline{\nu} \sum_{k=1}^{\sigma_{\nu+1}} x^k \qquad x = \sum_{k=1}^{\sigma_{\nu+1}} x^k \partial^* g_k.$$

We assume now that $\mu = n+1$ and we compute

$$d_2 : E_2^{0,n+1} \simeq \mathbb{Z}_2 \to E_2^{2,n} \simeq \tilde{H}_1(h^{-1}(\Sigma_{n+1})).$$

Consider thus $\Sigma_{n+1} = \{[\alpha_1], \ldots, [\alpha_{\sigma_{n+1}}]\}$ and let $f_1, \ldots, f_{\sigma_{n+1}}$ be the generators of $\tilde{H}^1(S^3 \backslash h^{-1}(\Sigma_{n+1}))$:

$$f_k(c) = \mathrm{lk}(c, h^{-1}[\alpha_k]) \quad \forall c \in \tilde{H}_1(S^3 \backslash h^{-1}(\Sigma_{n+1})).$$

In this way we have

$$H^2(B, \Omega^{n+1}) = \langle \partial^* f_1, \ldots, \partial^* f_{\sigma_{n+1}} \rangle.$$

By proposition 1.3.3 we have

$$w_{1,n+1} = p^* \mathrm{lk}(\,\cdot\,, \{\lambda_{n+1} = \lambda_{n+2}\})$$

In our case $p^{-1}\{\lambda_{n+1} = \lambda_{n+2}\} = h^{-1}(\Sigma_{n+1})$: if $\alpha \notin h^{-1}(\Sigma_{n+1})$, then $\mathrm{rk}(p(\alpha)) = n+1$ and thus $\mathrm{i}^+(p(\alpha)) = n+1$ and $\lambda_{n+1}(p(\alpha)) > \lambda_{n+2}(p(\alpha))$; on the contrary if $\alpha \in h^{-1}(\Sigma_{n+1})$, then $\mathrm{rk}(p(\alpha)) \leq n$ and $\lambda_{n+1}(p(\alpha)) = \lambda_{n+2}(p(\alpha)) = 0$. Since $\gamma_{1,n+1} = \partial^* w_{1,n+1}$, then we have

$$d_2^{0,n+1}(1) = \gamma_{1,n+1} = \sum_{k=1}^{\sigma_{n+1}} \overline{m}_k \partial^* f_k$$

where $m_k = m_{[\alpha_k]}$ comes from the fact that we are taking the pull-back of the class $\mathrm{lk}(\,\cdot\,, \{\lambda_{n+1} = \lambda_{n+2}\})$ through p and multiplicities have to be taken into account.

□

Remark 11. Notice that if $\mu = n+1$ and $\nu = n$, then

$$\overline{n} \sum_i \overline{m_i} = d_2^{2,n} \circ d_2^{0,n+1}(1) = \overline{n(n+1)} = 0.$$

Remark 12. Consider the bundle $\mathbb{R}^\mu \to L_\mu \to D_\mu$ as defined in the second section and its projectivization $\mathbb{R}\mathrm{P}^{\mu-1} \to P_\mu \xrightarrow{\pi} D_\mu$. Since $L_\mu \subset D_\mu \times \mathbb{R}^{2n+2}$, then $P_\mu \subset D_\mu \times \mathbb{R}\mathrm{P}^{2n+1}$ and the restriction of the projection on the second factor

$$l : P_\mu \to \mathbb{R}\mathrm{P}^{2n+1}$$

is a map which is a linear embedding on the fibres. It is not difficult to prove that for this map we have $\mathrm{rk}(l^*)_k \leq 1 - \mathrm{rk}(i_{R^*})_{2n+1-k}$ (see 3.7.1). Thus by theorem 3.5.1 we have the following implication:

$$\mathrm{rk}(l^*)_k = 1 \Rightarrow E_\infty^{0,k} = 0.$$

Using the fact that l is linear on the fibres, we can compute $l^* x^\mu$ where x is the generator of $H^1(\mathbb{R}\mathrm{P}^{2n+1}; \mathbb{Z}_2)$ and $l^* x = y$:

$$y^\mu = (w_1(L_\mu) y + w_2(L_\mu)) y^{\mu-2}$$

where we interpret $w_i(L_\mu)$ as a class on P_μ via π^*. Thus we see that

$$(w_1(L_\mu) \neq 0 \;\text{ or }\; w_2(L_\mu) \neq 0) \Rightarrow E_\infty^{0,\mu} = 0.$$

Applying the same reasoning and computing y^k for $k \geq \mu + 1$ we get similar conditions for the vanishing of $E_\infty^{0,k}$. Notice in particular that $w_1(L_\mu) = w_{1,\mu}$, hence if it is nonzero $d_2^{0,\mu}$ also is nonzero and $E_3^{0,\mu} = 0$, which consequently gives $E_\infty^{0,\mu} = 0$. Such considerations suggest that higher differential $d_r^{0,*}$ for (E_r, d_r) are closely related to higher characteristic classes.

We get as a corollary of the previous theorem the following well known fact from plane geometry.

Corollary 4.3.2. *The intersection of two quadrics in $\mathbb{C}\mathrm{P}^2$ consists of four points if and only if the associated pencil has exactly three singular elements.*

Proof. Notice that a for a pencil of quadrics in $\mathbb{C}\mathrm{P}^2$ generated by Q_0, Q_1 the following four possibilities can happen for

$$\{[\alpha_0, \alpha_1] \in \mathbb{C}\mathrm{P}^1 \mid \det(\alpha_0 Q_0 + \alpha_1 Q_1) = 0\} = \begin{cases} \mathbb{C}\mathrm{P}^1 & (\infty) \\ \text{one point} & (1) \\ \text{two points} & (2) \\ \text{three points} & (3) \end{cases}$$

The general table for the ranks of $E_2(R)$ has the following picture:

1	0	0	0	0
1	0	0	0	0
1	0	0	0	0
a	0	c	c'	0
b	0	d	d'	f
0	0	e	e'	g

Now $b_0(C) = b_0(R) \leq 1 + c' + f$ and

(∞) : $a = 1$, $c = c' = f = 0$ and $b_0(C) = 1$.

$(1), (2)$: $a = b = 0$, $c' = c - 1 \leq 1$, $f \leq 1$ and $b_0(C) \leq 3$.

(3) : $a = b = 0$, $c = 3$, $c' = 2$, $f = 1$ and by Theorem 4.3.1 $d_2^{2,2}$ is identically zero ($\nu = 2$ is even); also, since $E_\infty^{0,5} = \mathbb{Z}_2$, Theorem 4.2.4 implies $E_\infty^{0,4} = \mathbb{Z}_2$ (the number of $\mathbb{Z}_2$ summands in $E_\infty^{0,*}$ is even); thus $d_2^{0,4} = d_3^{0,4} = d_4^{0,4} \equiv 0$ and $b_0(C) = 4$.

It follows that

$$b_0(C) = 4 \iff (3).$$

□

Example 3 (The complete intersection of two quadrics). We recall from [24] that the condition for $C = V(q_0, q_1)$ to be a complete intersection is equivalent to have $\mu = n+1$, $\sigma_\mu = n+1$ and $\nu = n$. In other words the equation $\det(\alpha_0 Q_0 + \alpha_1 Q_1) = 0$

must have $n+1$ distinct roots and at each root $[\alpha_0, \alpha_1]$ the pencil must by simply degenerate, i.e. the rank of $\alpha_0 Q_0 + \alpha_1 Q_1$ must be n (notice in particular that for the case $n = 2$ we have the above result).
Thus the table for the rank of E_2 is the following:

$$\operatorname{rk}(E_2) = \begin{array}{r|c|c|c|c|c|}
2n+1 & 1 & 0 & & & \\
 & \vdots & \vdots & & & \\
 & 1 & 0 & & & \\
n+1 & 1 & 0 & & & \\
\hline
n & 0 & 0 & n+1 & n & 0 \\
 & 0 & 0 & 0 & 0 & 1 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

We distinguish the two cases n even and n odd.

(n even) : In this case, by Theorem 4.3.1, $d_2^{0,n+1}$ is injective and $d_2^{2,n}$ is zero. Hence the table for the rank of E_3 is the following:

$$\operatorname{rk}(E_3) = \begin{array}{r|c|c|c|c|c|}
2n+1 & 1 & 0 & & & \\
 & \vdots & \vdots & & & \\
 & 1 & 0 & & & \\
 & 1 & 0 & 0 & 0 & 0 \\
n+1 & 0 & 0 & 0 & 0 & 0 \\
\hline
n & 0 & 0 & n & n & 0 \\
 & 0 & 0 & 0 & 0 & 1 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

Since $d_3^{0,n+3} = d_4^{0,n+3} = 0$ then $E_\infty^{0,n+3} = \mathbb{Z}_2$; since n is even, then by Theorem 4.2.4 we have $E_\infty^{0,n+2} = E_\infty^{0,n+3} = \mathbb{Z}_2$ and thus $d_3^{0,n+2} = d_4^{0,n+2} = 0$. This implies

$$E_3 = E_\infty.$$

Thus the $\mathbb{Z}_2$-Betti numbers of R are:

$$b_j(R) = \begin{cases} 1 & \text{if } j \neq n-2, n-1,\ 0 \leq j \leq 2n-1; \\ n+2 & \text{if } j = n-2, n-1 \end{cases}$$

Consequently the $\mathbb{Z}_2$-Betti numbers of C are:

$$b_j(C) = \begin{cases} 0 & \text{if } j \text{ is odd}; \\ 1 & \text{if } j \text{ is even, } j \neq n-2 \text{ and } 0 \leq j \leq 2n-2 \\ n+2 & \text{if } j = n-2 \end{cases}$$

(n odd) : in this case, by Theorem 4.3.1, $d_2^{0,n+1}$ is injective and $d_2^{2,n}$ is surjective. Thus the table for the rank of E_3 is the following:

$$\mathrm{rk}(E_3) = \begin{array}{c|c|c|c|c|c|}
2n+1 & 1 & 0 & & & \\
 & \vdots & \vdots & & & \\
 & 1 & 0 & & & \\
 & 1 & 0 & 0 & 0 & 0 \\
n+1 & 0 & 0 & 0 & 0 & 0 \\
\hline
n & 0 & 0 & n-1 & n & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 1 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

Since $E_\infty^{0,n+1} = 0$ and n is odd, then by Theorem 4.2.4 we have $E_\infty^{0,n+2} = 0$, thus $d_3^{0,n+2}$ must be injective and the table of rank of $E_4 = E_\infty$ must be the following:

$$\mathrm{rk}(E_4) = \mathrm{rk}(E_\infty) = \begin{array}{c|c|c|c|c|c|}
2n+1 & 1 & 0 & & & \\
 & \vdots & \vdots & & & \\
 & 1 & 0 & & & \\
 & 0 & 0 & 0 & 0 & 0 \\
n+1 & 0 & 0 & 0 & 0 & 0 \\
\hline
n & 0 & 0 & n-1 & n-1 & 0 \\
 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 1 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}$$

Thus the $\mathbb{Z}_2$-Betti numbers of R are:

$$b_j(R) = \begin{cases} 1 & \text{if } j \neq n-2, n-1,\ 0 \leq j \leq 2n-1; \\ n & \text{if } j = n-2, n-1 \end{cases}$$

Consequently the $\mathbb{Z}_2$-Betti numbers of C are:

$$b_j(C) = \begin{cases} 0 & \text{if } j \text{ is odd and } j \neq n-2; \\ 1 & \text{if } j \text{ is even, } 0 \leq j \leq 2n-2 \\ n-1 & \text{if } j = n-2 \end{cases}$$

Thus the complete intersection of two quadrics C in $\mathbb{C}\mathrm{P}^n$ has complex dimension $m = n-2$ and its m-th Betti number is $m+4$ if m is even and $m+1$ if m is odd.

Example 4. Consider the two quadrics

$$q_0(z_0, z_1, z_2, z_3) = z_0 z_2 - z_1^2 \quad \text{and} \quad q_1(z_0, z_1, z_2, z_3) = z_0 z_3 - z_1 z_2.$$

Then $\det(\alpha_0 Q_0 + \alpha_1 Q_1) = \alpha_1^4$ and $\mathrm{rk}(\alpha_0 Q_0 + \alpha_1 Q_1) \equiv 4$ except at the point $[1,0]$ where we have $\mathrm{rk}(Q_0) = 3$. Notice in this case that $\mathrm{rk}(p([1,0])) \neq n+1-m_{[\alpha]} =$

$4 - m_{[\alpha]} = 0$. The table for the rank of E_2 has the following picture:

$$\mathrm{rk}(E_2) = \begin{array}{|c|c|c|c|c|}\hline 1&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ \hline\end{array}$$

Since $\mu = 4 = n + 1$, then we can use the previous formula for $d_2^{0,\mu}$ and since we have $m_{[1,0]} = 4$ it follows $d_2^{0,4} \equiv 0$. On the other hand $d_2^{2,4}$ is multiplication by $\nu = 3 \bmod 2$, hence it is an isomorphism. Hence the table for the rank of E_3 has the following picture:

$$\mathrm{rk}(E_3) = \begin{array}{|c|c|c|c|c|}\hline 1&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&1\\ \hline\end{array}$$

Since $d_3^{0,5} = d_4^{0,5} \equiv 0$, then $E_\infty^{0,5} = \mathbb{Z}_2$ and Theorem 4.2.4 implies also $E_\infty^{0,4} = \mathbb{Z}_2$. Thus $E_3 = E_4 = E_\infty$. Hence, for the only possible nonzero Betti numbers of R we have $b_0(R) = b_1(R) = 1$, $b_2(R) = b_3(R) = 2$. This implies the following for the Betti numbers of C :

$$b_0(C) = 1,\, b_2(C) = 2 \quad \text{and} \quad b_i(C) = 0,\, i \neq 0, 2.$$

Using Theorem 4.2.4 we see that $(i_C^*)_0$ and $(i_C)_2^*$ are injective.
Looking directly at the equations for C we see that it equals the union of the skew-cubic and a (complex projective) line meeting at one point; thus topologically $C \sim S^2 \vee S^2$.

Example 5. Consider the two quadrics

$$q_0(z_0, z_1, z_2) = z_0^2 - z_1^2 \quad \text{and} \quad q_1(z_0, z_1, z_2) = 2z_0(z_1 + z_2).$$

We have $\det(\alpha_0 Q_0 + \alpha_1 Q_1) \equiv 0$ and $\mathrm{rk}(\alpha_0 Q_0 + \alpha_1 Q_1) = 2$ for every $[\alpha_0, \alpha_1] \in \mathbb{CP}^1$. Thus the table for the rank of E_2 is:

$$\mathrm{rk}(E_2) = \begin{array}{|c|c|c|c|c|}\hline 1&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 0&0&0&0&1\\ 0&0&0&0&1\\ \hline\end{array}$$

By dimensional reasons, the only possibly nonzero differential is d_4. Since $d_4^{0,2} = 0$, then $E_\infty^{0,2} = \mathbb{Z}_2$ and by Theorem 4.2.4 also $E_\infty^{0,3} = \mathbb{Z}_2$. Since $E_\infty^{0,3} = \mathbb{Z}_2$, then $d_4^{0,3} = 0$; hence $E_\infty^{4,0} = \mathbb{Z}_2$. On the other side we have $d_4^{0,5} = 0$ and hence $E_\infty^{0,5} = \mathbb{Z}_2$; Theorem 4.2.4 implies $E_\infty^{0,4} = \mathbb{Z}_2$. Since $E_\infty^{0,4} = \mathbb{Z}_2$, then $d_4^{0,4} = 0$ and $E_\infty^{4,1} = \mathbb{Z}_2$.
All this tells us that $E_\infty = E_2$. The only possible nonzero Betti numbers of R are $b_0(R) = b_1(R) = 2$, $b_2(R) = b_3(R) = 1$. This implies the following for the Betti numbers of C :

$$b_0(C) = 2,\ b_2(C) = 1 \quad \text{and} \quad b_i(C) = 0,\ i \neq 0, 2.$$

Looking directly at the equations of C we see that it equals the union of the point $[1, 1, 0]$ and the complex projective line $\{z_0 + z_1 = 0\}$.

CHAPTER 5

Applications and examples

5.1 An approach to complexity

5.1.1 Topological view on complexity results

Consider a smooth compact algebraic set X defined by the equation $f = 0$ in $\mathbb{R}^n$, where f is a degree d polynomial:

$$X = \{f = 0\}, \quad \deg(f) = d.$$

Then for almost every line in $\mathbb{R}^n$ the orthogonal projection of X to this line will be a Morse function. Thus, eventually after performing a linear change of variable, we may assume that the x_1 coordinate is a Morse function for X. The set of critical points of this Morse function is the algebraic set defined by the equations $f = \partial f/\partial x_2 = \ldots = \partial f/\partial x_n = 0$: thus it consists of at most $d(d-1)^{n-1}$ points. If we denote with $b(X)$ the sum of the Betti numbers of X, then standard Morse theory gives the following bound, due to Milnor [21]:

$$b(X) \leq d(d-1)^{n-1}$$

The number $b(X)$ is sometimes called the *topological complexity* of X.
If we carefully observe the proof of Morse inequalities we see that if we want to estimate the complexity of a set defined by one single *inequality* $f \leq 0$, then this can be bounded with the half of the complexity of the smooth variety defined by the zeroes of f :

$$b(\{f \leq 0\}) \leq \frac{1}{2} b(\{f = 0\}).$$

If we plug-in some basic semialgebraic geometry we can estimate also the complexity of algebraic sets (not necessarily smooth) defined by polynomials of degree at most d. Here it goes as follows: suppose S is given by $f_1 = \ldots, f_s = 0$ in $\mathbb{R}^n$, where each f_i has degree at most d. Then the previous system of equations is equivalent to the single equation $g = f_1^2 + \cdots + f_s^2 = 0$. By corollary 1.2.4 there exists $\epsilon > 0$ such that the inclusion $\{g = 0\} \hookrightarrow \{g \leq \epsilon\}$ is a homotopy equivalence; moreover we may assume also the previous inequality is regular, i.e. $g = \epsilon$ is smooth. Then the previous estimate gives

$$b(S) \leq d(2d-1)^{n-1}.$$

The previous argument is not exact at all (some modifications are needed in the non-compact case), but is sufficient to give an idea of the general method. The general statement goes under the name of Oleinik-Petrovskii-Thom-Milnor bound and is the following.

Theorem 5.1.1 (Oleinik-Petrovskii-Thom-Milnor)**.** *Let $f_1, \ldots, f_s$ be polynomials in $\mathbb{R}[x_1, \ldots, x_n]$ of degree at most d and consider the semialgebraic set*

$$X = \{f_1 \leq 0, \ldots, f_s \leq 0\}.$$

Then the following bound holds for the complexity of X :

$$b(X) \leq O(sd)^n.$$

The previous bound is thus *exponential* in the number n of variables. In the case S is defined by k *quadratic* inequalities in $\mathbb{R}^n$, it gives

$$b(S) \leq O(2k)^n \leq O(2(k+1))^{n+1}.$$

In the previous chain we added the inequality to see more symmetry in comparison with next result, which concerns bound for semialgebraic sets defined by system of quadratic inequalities. In fact in this case the introduction of the correspondence space allows us to switch the role of the number of variables and the number of inequalities, as the following theorem shows.

Theorem 5.1.2. *Let S be a semialgebraic susbet of $\mathbb{R}^n$ defined by k* quadratic *inequalities. Then the following estimate holds:*

$$b(S) \leq O(n+1)^{2(k+1)}$$

Proof. First notice that we can construct a semialgebraic set S' in $\mathbb{R}\mathrm{P}^n$ defined by homogeneous inequalities (one more than those defining S) and which is homotopy equivalent to S. To do this we take the homogeneization ${}^h f_i \leq 0$ of the inequalities defining S and add the inequality $x_0^2 \geq \epsilon$ for $\epsilon > 0$; these inequalities define S'. It follows from corollary 1.2.4 that for ϵ small enough S and S' are homotopy equivalent. Moreover the inequality $x_0^2 \geq \epsilon$ is equivalent to $x_0^2 \geq \epsilon(x_0^2 + \cdots + x_n^2)$; thus S' is defined by a system of homogeneous quadratic inequalities and we can apply our theory. We let (E_r, d_r) be the spectral sequence of theorem 3.3.3 converging to $H_{n-*}(S'; \mathbb{Z}_2) = H_{n-*}(S; \mathbb{Z}_2)$. By the general theory of spectral sequence and universal coefficients theorem we have that

$$b(S) \leq b(S; \mathbb{Z}_2) \leq \mathrm{rk}(E_2)$$

and the previous rank is estimated by the topological complexity of the sets Ω^{j+1} since $b(C\Omega, \Omega^{j+1}) \leq 1 + b(\Omega^{j+1})$. Consider the polynomial $\det(\omega Q - tI) = a_0(\omega) + \cdots + a_n(\omega)t^n + t^{n+1}$; then by Descartes' rule of signs the positive inertia index of ωQ is given by the sign variation in the sequence $(a_0(\omega), \ldots, a_n(\omega))$. Thus the sets Ω^{j+1} are defined on the sphere S^k by quantifier-free formulas whose atoms are polynomials in $k+1$ variables and of degree less than $n+1$. For such sets we have the estimate, proved by Basu in [6]: $b(\Omega^{j+1}) \leq O(n+1)^{2k+1}$ (this estimate was later improved by the same author, but for our purpose it suffices). Putting all together we get:

$$b(S) \leq \sum_{j=0}^{n} b(C\Omega, \Omega^{j+1}) \leq n+1 + \sum_{j=0}^{n} b(\Omega^{j+1}) \leq O(n+1)^{2k+2}$$

where we see that the number of variables and the number of equations are switched. $\square$

Notice that, as stated in the proof, by universal coefficients theorem we always have $b(X;\mathbb{Z}) \leq b(X;\mathbb{Z}_2)$; thus in general any estimate for $\mathbb{Z}_2$ coefficients is valid also for integer coefficients.
One should expect that in the complex projective case a similar result holds; we formulate it in the following form. Recall that given $q_0, \ldots, q_k$ homogeneous degree two polynomials with *complex* coefficients we can consider the following family of susbets of $\mathbb{C}\mathrm{P}^k$ (here $k+1$ is the number of polynomials):

$$Y^j = \{[\alpha_0, \ldots, \alpha_k] \in \mathbb{C}\mathrm{P}^k \,|\, \mathrm{rk}(\alpha_0 q_0 + \cdots + \alpha_k q_k) \geq j\}, \quad j \in \mathbb{N}$$

Theorem 5.1.3. *If C is the common zero locus in $\mathbb{C}\mathrm{P}^n$ of the polynomials $q_0, \ldots, q_k \in \mathbb{C}[x_0, \ldots, x_n]_{(2)}$, then using the above notation for the sets Y^j, the following estimate holds:*

$$b(C) \leq b(\mathbb{C}\mathrm{P}^n) + \sum_{j\geq 0} b(Y^{j+1}).$$

Proof. Using the notation of the previous chapter, there exists a real algebraic set $R \subset \mathbb{R}\mathrm{P}^{2n+1}$ which is the total space of a S^1-bundle with even Euler characteristic over C. It follows that the $\mathbb{Z}_2$-Poincaré polynomials p_C of C and p_R of R are related by $p_R(t) = (1+t)p_C(t)$. The previous equation in particular gives

$$b(R) = p_R(1) = 2p_C(1) = 2b(C).$$

We estimate $b(R)$ using the spectral sequence of theorem 4.2.1: we have

$$b(R) \leq \sum_{j=0}^{2n+2} b(B, P^{j+1}).$$

In the case P^{j+1} is not empty, i.e. in the case $j+1 \leq \mu$ we have $b(B, P^{j+1}) \leq b(P^{j+1}) - 1$; in the case P^{j+1} is empty we have $b(B, P^{j+1}) = 1$. Thus we have

$$b(R) \leq 2n + 2 - \mu + \sum_{j=0}^{\mu-1}(b(P^{j+1}) - 1) \leq 2n + 2 + \sum_{j\geq 0} b(P^{j+1})$$

We saw that by definition of the set P^{j+1}, in the case it is not the whole sphere S^{2k+1}, then there is a homeomorphism $P^{j+1} = S^1 \times Y^{j+1}$ which gives $b(P^{j+1}) = 2b(Y^{j+1})$; on the other hand in the case $P^{j+1} = S^{2k+1}$, then $Y^{j+1} = \mathbb{C}\mathrm{P}^k$ and $b(P^{j+1}) \leq 2b(Y^{j+1})$. This gives the inequality:

$$b(R) \leq 2(n + 1 + \sum_{j\geq 0} b(Y^{j+1}))$$

which, using $b(R) = 2b(C)$, directly gives

$$b(C) \leq b(\mathbb{C}\mathrm{P}^n) + \sum_{j\geq 0} b(Y^{j+1}).$$

□

5.1.2 General bounds for systems of two quadratic inequalities

In this section we give explicit estimates for each Betti number of the set of the solutions of a system of two quadratic inequalities Y on the sphere and X in the projective space. The techniques we are going to use are those introduced to prove theorem 2.3.7: the idea is that of reduce the study of possibilities for the topology of Y to the combinatorics of words as above. We first give some estimates in the nondegenerate case and then we derive the estimates in the general case. We start with the following proposition.

Proposition 5.1.4. *Let $q \in \mathcal{Q}(n+1,2)$ be non degenerate with respect to $K = \{x_0 \leq 0,\, x_1 \leq 0\}$ and $X = \{[x] \in \mathbb{R}\mathrm{P}^n \,|\, q_0(x) \leq 0,\, q_1(x) \leq 0\}$. Then for every $k \in \mathbb{N}$ we have*

$$b_k(X) \leq k+2.$$

Moreover, in the case $b_k(X) = k+2$, then $\mathrm{rk}(j_* : H_k(X) \to H_k(\mathbb{R}\mathrm{P}^n)) = 1$.

Proof. We start by proving the inequality

$$b_0(\Omega^{n-k}) \leq k+2$$

for the canonical representative q' of a map $q \in \mathcal{Q}(n+1,2;K)$.
Assumnig $b_0(\Omega^{n-k}) \geq 2$, we have that there exist $\eta_1, \eta_2 \in \Omega$ such that $\mathrm{i}^+(\eta_1) = \mathrm{i}^+(\eta_2) = n-k$ and the index function decreases and increases at least once between them; thus the word $s(q')$ must contain the string of characters $(\hat{\omega}\omega)^r$ for a certain $r > 0$ between η_1 and η_2. Since we are searching for the maximum of $b_0(\Omega^{n-k})$ this implies that the word for q' must be the following:

$$s(q') = \omega^a z \hat{\omega}^b (\omega\hat{\omega})^r (\eta_2)$$

for certain $a, b, r \geq 0$, where the η_2 in parenthesis indicates the position of η_2 on Ω. In particular we may assume $a = 0$ and since $\mathrm{i}^+(\eta_2) = n-k$ we have $b+r = n-k$. On the other hand $b+2r = n+1$; combined together we get $r = k+1$ and $b = n-2k+1$. For such a choice we see that $b_0(\Omega^{n-k}) = k+2$ and the inequality for every other q' follows. Now, using Theorem 1.4.5 we have that

$$b_k(X) = e_{0,n-k} + b_0(\Omega^{n-k}) - 1 \leq b_0(\Omega^{n-k}) \leq k+2$$

where $e_{0,n-k} = \mathrm{rk}(j_*)_k$; finally notice that in the first inequality have equality if and only if $\mathrm{rk}(j_*)_k = 1$.

□

As a corollary, using the transfer exact sequence with $\mathbb{Z}_2$ coefficients for the double covering $p : Y \to X$ (see [17]) we have the following.

Proposition 5.1.5. *Let $q :\in \mathcal{Q}(n+1,2)$ be non degenererate with respect to $K = \{x_0 \leq 0,\, x_1 \leq 0\}$ and $Y = q^{-1}(K) \cap S^n$. Then for every $k \in \mathbb{N}$ we have*

$$b_k(Y) \leq 2k+4.$$

The following lemma allows to remove the hypothesis of nondegeneracy with respect to K, at least for the case of inequalities.

Lemma 5.1.6. *Consider $q \in \mathcal{Q}(n+1,2)$ and $K = \{x_0 \leq 0,\, x_1 \leq 0\} \subset \mathbb{R}^2$; then there exists $q' \in \mathcal{Q}(n+1,2;K)$ such that $Y(q) = q^{-1}(K) \cap S^n$ has the same homotopy type of $Y(q') = {q'}^{-1}(K) \cap S^n$. The same result holds true for $X(q) = p(Y(q))$ and $X(q') = p(Y(q'))$ defined as above (p is the covering map).*

Proof. If $q = (q_0, q_1)$ then $Y(q) = q^{-1}(K) \cap S^n$ coincides with the set of solutions of the following system:

$$\begin{cases} q_0(x) \leq 0 \\ q_1(x) \leq 0 \\ {x_1}^2 + \cdots + {x_{n+1}}^2 = 1 \end{cases}$$

By semialgebraicity the set of solutions of the previous system is a deformation retract, for small $\epsilon_0 > 0$ and $\epsilon_1 > 0$, of the set $Y_\epsilon(q)$ of the solutions of the following one:

$$\begin{cases} q_0(x) \leq \epsilon_0 \\ q_1(x) \leq \epsilon_1 \\ {x_1}^2 + \cdots + {x_{n+1}}^2 = 1 \end{cases}$$

In other words $Y(q)$ has the same homotopy type of $Y_\epsilon(q)$.
To conclude the proof it is sufficient to show that there exists $q_\epsilon = q' \in \mathcal{Q}(n+1,2;K)$ such that $Y(q_\epsilon) = Y_\epsilon(q)$.
Thanks to Sard's Lemma we choose two real numbers ϵ_0 and ϵ_1 such that (ϵ_0, ϵ_1) is a regular value of q, such that ϵ_i is not an eigenvalue of Q_i, $i = 0, 1$ and such that $Y(q_\epsilon)$ and $Y(q)$ are homotopically equivalent (this last condition is satisfiable since the set of (ϵ_0, ϵ_1) satisfying the first two conditions is the complement of a one-dimensional semialgebraic set). In this way the quadratic map q_ϵ, defined by

$$q_\epsilon(x) = (q_0(x) - \epsilon_0\|x\|^2, q_1(x) - \epsilon_1\|x\|^2)$$

is nondegenerate with respect to K.
The condition (ϵ_0, ϵ_1) is a regular value of q guarantees nondegeneracy at $\{0\}$, while the condition that ϵ_i is not an eigenvalue of Q_i, for $i = 0, 1$, guarantees nondegeneracy at ∂K.
The set $Y(q_\epsilon)$ coincides with the set of the solutions of

$$\begin{cases} q_0(x) - \epsilon_0\|x\|^2 \leq 0 \\ q_1(x) - \epsilon_1\|x\|^2 \leq 0 \\ {x_1}^2 + \cdots + {x_{n+1}}^2 = 1 \end{cases}$$

and thus with the set $Y_\epsilon(q)$.
The proof works the same in the projective case. □

In particular the previous lemma tells that for a general $q \in \mathcal{Q}(n+1,2)$ and $K = \{x_0 \leq 0,\, x_1 \leq 0\}$ we still have the estimates of the previous section.

Corollary 5.1.7. *If $q \in \mathcal{Q}(n+1,2)$, $K = \{x_0 \leq 0,\, x_1 \leq 0\}$ and $Y = q^{-1}(K \cap S^n)$, $p(Y) \subset \mathbb{R}\mathrm{P}^n$, then we have*

$$b_k(Y) \leq 2k + 4 \quad \textit{and} \quad b_k(p(Y)) \leq k + 2.$$

5.2 Level sets of quadratic maps: topology

In this section we discuss more closely the topology of the level sets of a homogeneous quadratic map. We start with the following observation, which was already used in the proof of theorem 5.1.2. In the case we are given a semialgebraic subset A in $\mathbb{R}^n$ defined by inequalities involving polynomials of degree two (the presence of degree one polynomials reduce to this case by restricting to affine subspaces), then we can find a semialgebraic subset A' in $\mathbb{R}\mathrm{P}^n$ such that the inclusion of A in A' is a homotopy equivalence and A' is defined by quadratic inequalities in $\mathbb{R}\mathrm{P}^n$. Consider first the projective closure $\overline{A}$ of A in $\mathbb{R}\mathrm{P}^n$, which amounts to consider the system of quadratic inequalities in $\mathbb{R}\mathrm{P}^n$ defined by the homoegenization of the polynomials defining A. Then $\overline{A}$ is obtained from A by adding the set of the solutions of a system of quadratic inequalities at infinity, namely on the hyperplane $\{x_0 = 0\}$, where x_0 is the new variable we added by homogenization (the restriction of the homogenization of the system to this hyperplane is clearly homogeneous). Consider now the inequality

$$l_\epsilon(x_0, \ldots, x_n) = \epsilon(x_1^2 + \cdots + x_n^2) - x_0^2 \leq 0.$$

Notice that this is not exactly the inequality used in theorem 5.1.2, but is preferrable to use it for computations. We want to show that for $\epsilon > 0$ small enough A and $\overline{A} \cap \{l_\epsilon \leq 0\}$ are homotopy equivalent. Notice first that $l_\epsilon = 0$ has no solutions with $x_0 = 0$: in fact in this case it must be $x_1^2 + \cdots + x_n^2 \leq 0$ which implies $x_1 = \cdots = x_n = 0$, but this is impossible on $\mathbb{R}\mathrm{P}^n$. Thus on the projective space the inequalitty $l_\epsilon \leq 0$ is equivalent to the one $x_0^{-2}(x_1^2 + \cdots + x_n^2) \leq R$ where $R = \epsilon^{-1}$. In non-homogeneous coordinates on $\mathbb{R}^n$ we can rewrite the last inequality as $y_1^2 + \cdots + y_n^2 \leq R$, hence in particular:

$$\overline{A} \cap \{l_\epsilon \leq 0\} = A \cap \{\|y\|^2 \leq R\}.$$

Consider now the semialgebraic map $\psi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ defined by sending x to $x\|x\|^{-2}$, and the semialgebraic set $S = \psi(A) \cup \{0\}$. Then ψ maps $A \cap \{\|y\|^2 \leq R\}$ to $S \cap \{\|y\|^2 \leq R^{-1}\}$ and the conclusion follows from corollary 1.2.4 applied to the norm function on S.

In summary this argument shows that A is homotopy equivalent to the set $A(\epsilon)$ defined in the projective space by homoegenization of the inequalities defining A and by adding the inequality $l_\epsilon \leq 0$ for $\epsilon > 0$ small enough.

We apply the previous discussion to the study of a level set of a quadratic map to the plane. Namely consider $q = (q_0, q_1) \in \mathcal{Q}(n, 2)$ and the preimage A_c of $c = (c_0, c_1)$ under q. In view of the previous discussion we are led to consider $\hat{q}_\epsilon \in \mathcal{Q}(n+1, 3)$ whose components are $({}^h q_0, {}^h q_1, l_\epsilon)$ and the cone $\hat{K} = \mathbb{R} \times (-\infty, 0] \subset \mathbb{R}^3$. Letting $\hat{\Omega}$ be the cone $\hat{K}^\circ \cap S^2$ in the space $\mathbb{R}^2 \times \mathbb{R}$, theorem 3.3.3 gives for small ϵ a spectral sequence $(\hat{E}_r(\epsilon), \hat{d}_r(\epsilon))$ such that

$$(\hat{E}_r(\epsilon), \hat{d}_r(\epsilon)) \Rightarrow H_{n-*}(A(\epsilon)) \simeq H_{n-*}(A) \quad \text{and} \quad \hat{E}_2^{i,j}(\epsilon) = H^i(C\hat{\Omega}, \hat{\Omega}^{j+1}(\epsilon)),$$

where now the index filtration is given by the sets

$$\hat{\Omega}^{j+1}(\epsilon) = \{(\omega, t) \in \hat{\Omega} \,|\, \mathrm{i}^+(\omega^h q + t l_\epsilon) \geq j + 1\}.$$

Using the previous spectral sequence we prove the following theorem, which describes the homological structure of A_c for any c in $\mathbb{R}^2$.

Theorem 5.2.1. *Let q be a map in $\mathcal{Q}(n,2)$, c be a point in $\mathbb{R}^2$ and set $A_c = q^{-1}(c)$. We define the following family of subsets of S^1:*

$$C_k = \{\omega \in S^1 \,|\, \langle \omega, c\rangle < 0 \text{ and } \mathrm{i}^-(\omega q) \leq k\}$$

and the number

$$\nu_c = \min\{\mathrm{i}^-(\omega q) \,|\, \omega \in S^1,\, \langle \omega, c\rangle < 0\}.$$

Then we have

(i) $A_c = \emptyset \iff \nu_c = 0$;

(ii) if $\nu_c \neq 0$, *then* $\tilde{b}_k(A_c) = b_0(C_{k+1}, C_k) + b_1(C_{k+2}, C_{k+1})$ *for* $0 \leq k \leq n$.

Proof. First notice that the condition $\nu_c = 0$ is equivalent to

$$\exists \eta \in q(\mathbb{R}^n)^\circ \quad \text{s.t.} \quad \langle \eta, c\rangle > 0.$$

Suppose $A_c = \emptyset$; then if $\nu_c \neq 0$ we would have $\forall \eta \in q(\mathbb{R}^n)^\circ$ the inequality $\langle \eta, c\rangle \leq 0$, namely $c \in q(\mathbb{R}^n)^{\circ\circ} = q(\mathbb{R}^n)$ which is absurd - remeber that $q(\mathbb{R}^n)$ is a closed convex (polyhedral) cone by corollary 1.4.9. On the contrary if $A_c \neq \emptyset$, then $c \in q(\mathbb{R}^n)$ and hence $\{tc\}^\circ_{t\geq 0} \supset q(\mathbb{R}^n)^\circ$; thus $\langle \eta, c\rangle \leq 0$ for every $\eta \in q(\mathbb{R}^n)^\circ$. This proves part (i).
For part (ii) we are substantially going to prove that

$$\hat{E}_2^{i,j}(\epsilon) = H^i(C\hat{\Omega}, \hat{\Omega}^{j+1}(\epsilon)) \simeq H^i(C_{n-j+1}, C_{n-j})$$

for small ϵ and that if $A_c \neq \emptyset$, then $\hat{E}_2(\epsilon) \simeq \hat{E}_\infty(\epsilon)$.
Notice also that for $i \geq 1$ we have

$$H^i(C\hat{\Omega}, \hat{\Omega}^{j+1}(\epsilon)) \simeq \tilde{H}^{i-1}(\hat{\Omega}^{j+1}(\epsilon)).$$

Set $\hat{\Omega}_\geq(\epsilon) = \{(\omega,t) \in \hat{\Omega} \,|\, \langle (c,\epsilon), (\omega,t)\rangle \geq 0\}$ and $\hat{\Omega}_\leq(\epsilon) = \{(\omega,t) \in \hat{\Omega} \,|\, \langle (c,\epsilon), (\omega,t)\rangle \leq 0\}$. Notice that if $(\omega,t) \in \hat{\Omega}^k(\epsilon) \cap \hat{\Omega}_\geq(\epsilon)$ for $k \leq n$ then for every $t' \geq t$ we have $(\omega,t') \in \hat{\Omega}^k \cap \hat{\Omega}_\geq(\epsilon)$. Define $\Omega(\epsilon) = \partial\hat{\Omega}_\geq(\epsilon) \sim \Omega$ and $\Omega^k(\epsilon) = \{(\omega,t) \in \Omega(\epsilon) \,|\, \mathrm{i}^+(\omega,t) \geq k\}$. Then since $\mathrm{i}^+(0,1) = n$, for $k \leq n$ we have

$$\hat{\Omega}^k(\epsilon) \sim (\hat{\Omega}^k(\epsilon) \cup \Omega(\epsilon)) \cup C\Omega^k(\epsilon).$$

Thus we derive the following chain of isomorphisms:

$$\tilde{H}^*(\hat{\Omega}^k(\epsilon)) \simeq \tilde{H}^*((\hat{\Omega}^k(\epsilon) \cup \Omega(\epsilon)) \cup C\Omega^k(\epsilon)) \simeq H^*(\hat{\Omega}^k(\epsilon) \cup \Omega(\epsilon)), \Omega^k(\epsilon)).$$

We define now the set $\Omega^k_\geq(\epsilon) = \Omega^k(\epsilon) \cap \hat{\Omega}_\geq(\epsilon) \subset \Omega^k(\epsilon)$ and notice that its closure is contained in the interior of $\Omega^k(\epsilon)$; thus we can apply the excision theorem and get:

$$\tilde{H}^*(\hat{\Omega}^k(\epsilon)) \simeq H^*((\hat{\Omega}^k_\leq(\epsilon) \cup \Omega^k(\epsilon)) \backslash \Omega^k_\geq(\epsilon), \Omega^k(\epsilon) \backslash \Omega^k_\geq(\epsilon))$$

If we denote by $\tilde{\Omega}^k(\epsilon)$ the set $\hat{\Omega}^k_\leq(\epsilon) \backslash \{\langle \omega, c\rangle \geq 0\}$ we finally have the isomorphism:

$$\tilde{H}^*(\hat{\Omega}^k(\epsilon)) \simeq H^*(\tilde{\Omega}^k(\epsilon), \tilde{\Omega}^k(\epsilon) \cup \partial\hat{\Omega}_\geq(\epsilon)).$$

Now we consider the set $C = \{\langle \omega, c\rangle < 0\}$ and the function $\theta_\epsilon : C \to \mathbb{N}$ defined by

$$\omega \mapsto \mathrm{i}^+(\omega Q - \langle c, \omega\rangle \epsilon I).$$

We call $C^k(\epsilon)$ the set $\{\theta_\epsilon \geq k\}$ and notice that for ϵ small we have isomorphisms:

$$H^*(\tilde{\Omega}^k(\epsilon), \tilde{\Omega}^k(\epsilon) \cup \partial\hat{\Omega}_{\geq}(\epsilon)) \simeq H^*(C^{k-1}(\epsilon), C^k(\epsilon)).$$

Since for $\epsilon_1 \leq \epsilon_2$ we have $C^{k-1}(\epsilon_1) \subset C^{k-1}(\epsilon_2)$, then for small $\epsilon > 0$

$$\check{H}^*(C^{k-1}(\epsilon), C^k(\epsilon)) \simeq \varinjlim_{\epsilon}\{\check{H}^*(C_{k-1}(\epsilon), C^k(\epsilon))\} \simeq \check{H}^*(\bigcap_\epsilon C^{k-1}(\epsilon), \bigcap_\epsilon C^k(\epsilon)).$$

Moreover $\bigcap_\epsilon C^k(\epsilon) = \{\omega \in C \,|\, \mathrm{i}^-(\omega) \leq n-k\}$ (notice that $\mathrm{i}^-(\omega) = \mathrm{i}^+(-\omega)$) and thus setting $C_l = \{\omega \in C \,|\, \mathrm{i}^-(\omega) \leq l\}$ we finally end up with

$$\hat{E}_2^{i,j}(\epsilon) \simeq H^{i-1}(C_{n-j+1}, C_{n-j}) \quad i \geq 1,\ \epsilon > 0 \text{ small.}$$

We have $\max_{\hat{\Omega}} \mathrm{i}^+ \geq n$ and thus $\hat{E}_2^{0,j}(\epsilon) = 0$ for $j \leq n-1$ and small ϵ; on the other side if $A_c \neq \emptyset$ then by theorem 4.2.4 we must have $\hat{E}_2^{0,n}(\epsilon) = \mathbb{Z}_2$ for small ϵ and the only possibly nonzero differential is $\hat{d}_2(\epsilon)^{0,n} : \mathbb{Z}_2 \to \hat{E}_2^{2,n-1}$. Since $A_c \neq \emptyset$, then $C_0 = \emptyset$ and thus $\hat{E}_2^{2,n-1} = H^1(C_1, C_0) = 0$ and $\hat{E}_2^*(\epsilon) = \hat{E}_\infty^* \simeq H_{n-*}(A)$. This concludes the proof of part (ii). □

As an easy corollary we get the following for $q \in \mathcal{Q}(n, 2)$.

Corollary 5.2.2. $q(\mathbb{R}^n) = \{tv \,|\, v \in -\Omega_0^\circ,\ t \geq 0\}$.

Proof. By property (i) of theorem 5.2.1 we have

$$q(\mathbb{R}^n) = \{c \in \mathbb{R}^n \,|\, \nu_c \neq 0\} = \{c \in \mathbb{R}^n \,|\, c \in -\{\mathrm{i}^- \leq 0\}^\circ\}$$

where clearly $\{\mathrm{i}^- \leq 0\}$ is a convex cone. □

Remark 13. The statement of the previous theorem still holds for systems of *inequalities*: if $A = \{q_0 \leq c_0, q_1 \leq c_1\}$ then $A = q_c^{-1}(K)$ for $q_c = (q_0 - c_0, q_1 - c_1)$ and K a certain cone and the result is the same by setting $C_k = \{\omega \in \Omega = K^\circ \cap S^1 \,|\, \langle \omega, c\rangle < 0,\ \mathrm{i}^+(-\omega) \leq k\}$.

5.3 Infinite dimensional case

We consider here the case H is a Hilbert space and q_0, q_1 are continuous quadratic forms on on H :

$$q_i(x) = \langle x, Q_i x\rangle \quad Q_i \text{ is linear, continuous and selfadjoint.}$$

In this case we easily prove the following generalization of theorem 1.4.9.

Theorem 5.3.1. *Let q_0, q_1 be two quadratic forms on H and $q : H \to \mathbb{R}^2$ the map $x \mapsto (q_0(x), q_1(x))$. Then $q(H)$ is a convex subset of $\mathbb{R}^2$, but not necessarily closed.*

Proof. Let $a = q(\alpha)$ and $b = q(\beta)$ be in the image of q. Consider $V = \mathrm{span}(\alpha, \beta)$; then $q|_V(V)$ is convex by theorem 1.4.9 and thus for every $t \in [0,1]$ there exists $v_t \in V \subset H$ such that $ta + (1-t)b = q|_V(v_t) = q(v_t)$. □

If $c = (c_0, c_1) \in \mathbb{R}^2$ we are interested in the set

$$A_c = \{x \in H \,|\, q_0(x) = c_0,\ q_1(x) = c_1\}$$

with its induced topology from $A_c \subset H$. Without any regularity assumption the set A_c can be very wild, but we can however attach to it some algebraic invariant, namely

$$\mathcal{H}_*(A_c) \doteq \varinjlim_{V \in \mathcal{F}} \{\tilde{H}_*(A_c \cap V)\},$$

where $\mathcal{F} = \{V \subset H \,|\, V \text{ finite dimensional subspace of } H\}$, and then give conditions for which $\mathcal{H}_*(A_c)$ coincides with $\tilde{H}_*(A_c)$. We recall the definition of positive inertia index for a quadratic form q on H :

$$\mathrm{ind}^+(q) = \max\{\dim(V) \,|\, V \subset H,\ q|_V > 0\}$$

and we define also, using the notation of the previous section,

$$\mathcal{C} = \{\omega \in C \,|\, \mathrm{i}^+(-\omega) < \infty\} \quad \text{and} \quad \mathcal{C}_k = C_k \cap \mathcal{C}.$$

The set $\mathcal{C}$ happens to be a convex subset of C, but the subsets $\mathcal{C}_k$ are not in general euclidean neighborhood retracts and thus their Cech cohomology may not coincide with their singular cohomology.

Lemma 5.3.2. *If $A_c = \emptyset$ then $\mathcal{H}_*(A_c) = 0$. If $A_c \neq \emptyset$ then*

$$\mathcal{H}_k(A_c) = \check{H}^0(\mathcal{C}_{k+1}, \mathcal{C}_k) \oplus \check{H}^1(\mathcal{C}_{k+2}, \mathcal{C}_{k+1})$$

Proof. If $A_c = \emptyset$ then clearly for every $V \subset H$ we have $A_c \cap V = \emptyset$ and $H_*(A_c \cap V) = 0$ which implies $\mathcal{H}_*(A_c) = 0$.
On the contrary if $A_c \neq \emptyset$, then setting $C_k(W) = \{\omega \in C \,|\, \mathrm{ind}^-(\omega q|_W) \leq k\}$ for $V \subset W$ subspaces we have $C_k(W) \overset{i_V^W}{\hookrightarrow} C_k(V)$. We refer to [3] for the proof that $H_*(A_c \cap V) \to H_*(A_c \cap W)$ induces on the graded complex associated to spectral sequence of theorem 4.2.4 the maps

$$H^*(C_{k+1}(V), C_k(V)) \xrightarrow{(i_V^W)^*} H^*(C_{k+1}(W), C_k(W)).$$

It follows from the properties of Cech cohomology that

$$\varinjlim \{H^*(C_{k+1}(V), C_k(V))\} = \check{H}^*(\bigcap_{V \in \mathcal{F}} C_{k+1}(V), \bigcap_{V \in \mathcal{F}} C_k(V))$$

and since $\bigcap_{V \in \mathcal{F}} C_k(V) = \mathcal{C}_k$ then the conclusion follows. □

Notice that the proof of part (i) of theorem 5.2.1 here does not apply, because in general $q(H)$ is not closed and hence $q(H)^{\circ\circ}$ can be different from $q(H)$. The following proposition gives a sufficient condition for $\mathcal{H}_*(A_c) \simeq \tilde{H}_*(A_c)$.

Proposition 5.3.3. *Suppose $c = (c_0, c_1) \in \mathbb{R}^2$ is a regular value for the homogeneous quadratic map $q : H \to \mathbb{R}^2$. Then*

$$\mathcal{H}_*(A_c) \simeq \tilde{H}_*(A_c).$$

Proof. We give only a sketch; for details the reader is advised to see [3]. If c is a regular value, then A_c is a Hilbert submanifold of H and has a tubular neighborhood U_c. Thus $\tilde{H}_*(U_c) \simeq \tilde{H}_*(A_c)$ and any singular chain in A_c can be turned in a chain lying in a finite dimensional subspace of H without leaving U_c. The conclusion follows. □

In the case $c = 0$, then A_0 is contractible and is possible to study the topology of $A_0 \cap \{x \in H \mid \|x\| = 1\}$ in a similar way; for a precise treatment in the nondegenerate case the reader is referred again to [3].

5.4 Examples

We collect in this section a series of examples, which should give an idea of the method to effectively make computations using the previous theorems. We start with the most simple case, i.e. that of a single quadric in $\mathbb{RP}^n$. Let $q \in \mathcal{Q}$ be a quadratic form on $\mathbb{R}^{n+1}$ with signature (a, b) with $a \leq b$ (otherwise we can replace q with $-q$) and $a + b = \mathrm{rk}(q) \leq n + 1$. Consider

$$X_{a,b} = \{q = 0\} \subset \mathbb{RP}^n.$$

For example, in the case q is nondegenerate (i.e. $a + b = n + 1$) then $X_{a,b}$ is smooth and $S^{a-1} \times S^{b-1}$ is a double cover of it.
Define the two vectors $h^-(X_{a,b}), h^+(X_{a,b}) \in \mathbb{N}^n$ by:

$$h^-(X_{a,b}) = (\underbrace{1, \ldots, 1}_{n+1-b}, 0, \ldots, 0), \quad h^+(X_{a,b}) = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{a}).$$

Then a straightforward application of theorem 3.3.3 gives the following identity for the array whose components are the $\mathbb{Z}_2$-Betti numbers of $X_{a,b}$:

$$(b_0(X_{a,b}), \ldots, b_n(X_{a,b})) = h^-(X_{a,b}) + h^+(X_{a,b}).$$

Moreover if we let $j : X_{a,b} \to \mathbb{P}^n$ be the inclusion, then theorem 3.5.1 gives the following:

$$(\mathrm{rk}(j_*)_0, \ldots, \mathrm{rk}(j_*)_n) = h^-(X_{a,b}).$$

Example 6. We compute the cohomology of the real grassmannian $G_{2,4}$. Using the Plucker embedding we realize it as the algebraic subset of $\mathbb{RP}^5$ cut by the single quadratic equation:

$$q(z) = z_0 z_5 - z_1 z_4 + z_2 z_3 = 0$$

Thus in this case $q \in \mathcal{Q}(6)$ and its signature is $(3, 3)$; in particular, using the above notations we have $G_{2,4} = X_{3,3}$ and:

$$h^-(G_{2,4}) = (1, 1, 1, 0, 0), \quad h^+(G_{2,4}) = (0, 0, 1, 1, 1).$$

Notice in particular that $h^-(G_{2,4})$ gives the vector of the ranks of the map induced by the Plucker emebedding i in $\mathbb{R}\mathrm{P}^5$: let's check this fact using some elementary algebraic geometry. We let y be the generator of $H^1(\mathbb{R}\mathrm{P}^5;\mathbb{Z}_2)$; then y is Poincaré dual to a hyperplane. The intersection of $i(G_{2,4})\subset\mathbb{R}\mathrm{P}^5$ with a generic hyperplane is in $\mathbb{R}\mathrm{P}^5$ is easily seen to be the schubert cycle σ_1, hence its Poincaré dual f_1 (this cohomology class equals $w_1(\tau_{2,4})$) satisfies

$$i^*y=f_1$$

and $\mathrm{rk}(i^*)_1=1$, since f_1 is non zero in $H^1(G(2,4);\mathbb{Z}_2)$. Now, using Pieri's formula, we have that $\sigma_1\cdot\sigma_1=\sigma_2+\sigma_{1,1}$, which is nonzero and gives $\mathrm{rk}(i^*)_2=1$. Using again Pieri's formula we have that $(\sigma_1)^3=\sigma_3+2\sigma_{2,1}+\sigma_{1,1,1}$; thus the Poincaré dual f_1^3 of σ_1^3 is zero in $H^3(G(2,4);\mathbb{Z}_2)$ and this gives $\mathrm{rk}(i^*)_3=0$.

More interesting is the case of two quadrics. In the case $q=(q_0,q_1)$ and i^+ not constant, then the spectral sequence of theorem 3.3.3 degenerates at the second step and $E_2=E_\infty$. In the case of constant positive index we can use Theorem 3.7.1 to find $H_*(p^{-1}(K))$ (notice that $K\neq\{0\}$ again implies $E_2=E_\infty$.)

Example 7. Consider the two quadratic forms

$$q_0(x)=2x_0x_1-x_1^2,\quad q_1(x)=x_1^2+x_2^2-x_3^2.$$

Their common zero locus set X in $\mathbb{R}\mathrm{P}^3$ consists of two lines intersecting at one point and one circle intersecting each line in one point. Thus X is homotopy equivalent to a bouquet of four circles. The table for the ranks of E_2 in this case is the following:

$$\mathrm{rk}(E_2)=\begin{array}{|c|c|c}1&0&0\\1&0&0\\0&2&0\\0&0&1\\\hline\end{array}$$

We see that there is no differential, hence $E_\infty=E_3=E_2$. The homomorphism $(i^*)_1$ induced on the cohomology by the inclusion $i:X\to\mathbb{R}\mathrm{P}^3$ is injective (X contains a line).

Example 8. Consider the two quadratic forms:

$$q_0(x)=x_0^2+x_1^2,\quad q_1(x)=x_3^2-x_4^2.$$

Their common zero locus set X in $\mathbb{R}\mathrm{P}^3$ consists of two points. The table for the rank of E_2 in this case is the following:

$$\mathrm{rk}(E_2)=\begin{array}{|c|c|c}1&0&0\\0&1&0\\0&0&0\\0&0&0\\\hline\end{array}$$

Since that there is no non-vanishing second differential we have $E_\infty=E_3=E_2$.

Clearly if the number of quadrics increases, then the computations for the differentials became more difficult, whereas those for the second term of the spectral sequence are still relatively simple.

Example 9. Consider the map $p \in \mathcal{Q}(4,3)$ given by

$$p(x_0, x_1, x_2, x_3) = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2).$$

Then $C = \{p = 0\} \subset \mathbb{R}\mathrm{P}^3$ is the rational normal curve, the so called twisted cubic. In this case $\Omega = S^2$ and the set $\{\omega \in \Omega \,|\, \ker(\omega p) \neq 0\}$ consists of two disjoint ovals in S^2, bounding two disks B_1, B_2. Then S^2 is the disjoint union of the sets $\mathrm{Int}(B_1), \partial B_1, R, \partial B_2, \mathrm{Int}(B_2)$, on which the function i^+ is constant with value respectively $2, 1, 2, 2, 2$. Then

$$\Omega^1 = S^2, \quad \Omega^2 = S^2 \backslash \partial B_1, \quad \Omega^3 = \emptyset$$

and the table for the ranks of the second term of the spectral sequence of theorem 3.3.3 converging to $H_{3-*}(C)$ is the following:

$$\mathrm{rk}(E_2) = \begin{array}{|c|c|c|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \end{array}$$

The differential $d_2 : E_2^{1,1} \to E_2^{3,0}$ is an isomorphism; hence $E_3 = E_\infty$ has the following picture:

$$\mathrm{rk}(E_3) = \begin{array}{|c|c|c|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \end{array}$$

From the previous, using theorem 3.5.1, we see that $j_* : H_1(C) \to H_1(\mathbb{P}^3)$ is an isomorphism (we can check this fact also by noticing that, since C is a curve of degree 3, then the intersection number of C with a generic hyperplane $H \subset \mathbb{P}^3$ is odd).

Example 10. Consider the quadratic map $q \in \mathcal{Q}(3,3)$ defined by:

$$q(x_0, x_1, x_2) = (x_0x_1, x_0x_2, x_1x_2)$$

and the convex cone $K = \{0\}$. Then we have:

$$\Omega = \Omega^1 = S^2, \quad \Omega^2 = \{\omega \in S^2 \,|\, \omega_0\omega_1\omega_2 < 0\}, \quad \Omega^3 = \emptyset.$$

In this case the table for the ranks of E_2 has the following picture:

$$\mathrm{rk}(E_2) = \begin{array}{|cccc} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \end{array}$$

The element $E_\infty^{1,1}$ cannot be zero: in fact $(E_r^{1,1})_{r\geq 2}$ can lose rank only because of d_2 and in this case it can decrease at most by one. This implies the set $\{q = 0\}$ in $\mathbb{R}\mathrm{P}^2$ is nonempty. On the other side, because of theorem 3.5.1, nonemptyness of $\{q = 0\}$ implies $E_\infty^{0,2} = \mathbb{Z}_2$. Since the term $E_r^{3,0}$ must become zero at a certain step, then the differential $d_2 : E_2^{1,1} \to E_2^{3,0}$ is nonzero. Hence $\mathrm{rk}(E_\infty) = 3$ and $\{q = 0\}$ consists of 3 points.

There are some cases in which the problem of describing the index function can be reduced to a simpler problem; this is the case of a quadratic map defined by a bilinear one. We start noticing the following.

Lemma 5.4.1. *Let L be a $n \times n$ real matrix and Q_L be the symmetric $2n \times 2n$ matrix defined by:*

$$Q_L = \begin{pmatrix} 0 & L \\ {}^tL & 0 \end{pmatrix}$$

Then, setting q_L for the quadratic form defined by $x \mapsto \langle x, Q_L x\rangle$ we have:

$$\mathrm{i}^+(q_L) = rk(L).$$

Proof. Let $x = (z, w) \in \mathbb{R}^{2n} \simeq \mathbb{R}^n \oplus \mathbb{R}^n$; then $Q_L \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} Lw \\ {}^tLz \end{smallmatrix}\right)$. Hence $\ker Q_L = \ker {}^tL \oplus \ker L$ and

$$\dim(\ker Q_L) = 2\dim(\ker L).$$

Consider now the characteristic polynomial f of Q_L:

$$f(t) = \det(Q_L - tI) = \det(t^2 I - {}^tLL) = (-1)^n \det({}^tLL - t^2 I) = (-1)^n g(t^2)$$

where g is the characteristic polynomial of ${}^tLL$. Let now $\lambda \in \mathbb{R}$ be such that $g(\lambda) = 0$; since ${}^tLL \geq 0$, then $\lambda \geq 0$ and $f(\pm\sqrt{\lambda}) = 0$. Since Q_L is diagonalizable, then for each one of its eigenvalues algebraic and geometric multiplicity coincide, hence

$$\mathrm{i}^+(q_L) = \mathrm{i}^-(q_L) = \frac{1}{2}\mathrm{rk}(Q_L).$$

It follows that

$$\mathrm{i}^+(q_L) = \frac{1}{2}(2n - \dim(\ker Q_L)) = \mathrm{rk}(L).$$

□

In particular if $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{k+1}$ is a bilinear antisymmetric map whose components are defined by

$$(x, y) \mapsto \langle (x, y), \left(\begin{smallmatrix} 0 & B_i \\ {}^tB_i & 0 \end{smallmatrix}\right) (x, y)\rangle$$

for certain real squared matrices B_i, $i = 1, \ldots, k+1$, then we can consider the quadratic map

$$p_b : \mathbb{R}^{2n} \to \mathbb{R}^{k+1}$$

defined by $(x, y) \mapsto b(x, y)$. In this case we define for $\omega \in S^k$ the matrix ωB by

$$\omega B = \omega_1 B_1 + \cdots + \omega_{k+1} B_{k+1}.$$

By lemma 5.4.1 we have

$$\mathrm{i}^+(\omega p_b) = \mathrm{rk}(\omega B).$$

Example 11. Let $\mathbb{R}^8$ be identified with the space of pairs of 2×2 real matrices. We apply the previous consideration to describe the topology of

$$\Gamma = \{(X, Y) \in \mathbb{R}^8 \, : \, [X, Y] = 0\}.$$

Since the equation for Γ are homogeneous, it is a cone, and we can study the homology of its projectivization

$$\mathbb{P}(\Gamma) \subset \mathbb{R}\mathrm{P}^7.$$

If we define $V = \{(X, Y) \in \mathbb{R}^8 \, : \, \mathrm{tr}(X) = \mathrm{tr}(Y) = 0\}$ and $\Gamma_V = \Gamma \cap V$, then it is readily seen that

$$\Gamma = \Gamma_V \oplus \mathbb{R}^2.$$

We proceed first to the computation of $H_*(\mathbb{P}(\Gamma_V))$ using the above theorems.
In coordinates $(X, Y) = \left(\left(\begin{smallmatrix} x & y \\ z & -x \end{smallmatrix}\right), \left(\begin{smallmatrix} w & t \\ s & -w \end{smallmatrix}\right)\right)$ we have

$$\{[X, Y] = 0\} \cap V = \{tz - ys = xt - yw = sx - wz = 0\}.$$

Consider the following matrices

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and the bilinear map $b : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ whose components are $(x, y) \mapsto \langle x, B_i y \rangle$. Then $p_b : V \to \mathbb{R}^3$ equals the quadratic map defined by $(X, Y) \mapsto [X, Y]$ (we are using the above notations for the quadratic map p_b defined by a bilinear map b). It follows that

$$\Gamma_V = V \cap \Gamma = \{p_b = 0\}.$$

Using ωB for the matrix $\omega_1 B_1 + \omega_2 B_2 + \omega_3 B_3$, then by the previous fact we have

$$\mathrm{i}^+(\omega p_b) = \mathrm{rk}(\omega B) \;\; \forall \omega \in S^2.$$

Let ωQ_b the symmetric matrix associated to ωp_b by the rule $(\omega p_b)(x) = \langle x, \omega Q_b x \rangle$. Then

$$\omega Q_b = \begin{pmatrix} 0 & \omega B \\ {}^t\omega B & 0 \end{pmatrix}$$

The matrix ωB, for $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$ has the following form:

$$\begin{pmatrix} 0 & \omega_3 & \omega_2 \\ -\omega_2 & -\omega_1 & 0 \\ -\omega_3 & 0 & \omega_1 \end{pmatrix}$$

and we immediatly see that $\mathrm{rk}(\omega B) = 2$ for $\omega \neq 0$; this gives

$$\mathrm{i}^+(\omega p_b) = 2 \quad \forall \omega \in S^2.$$

Since $\mathrm{i}^+ \equiv 2$, we can apply Theorem 3.7.1; letting (E_r, d_r) be the spectral sequence of theorem 3.3.3 converging to $H_{n-*}(\mathbb{P}(\Gamma_V))$, we have the following picture for $E_2 = E_3$:

$$\mathrm{rk}(E_2) = \mathrm{rk}(E_3) = \begin{array}{|c|c|c|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \hline \end{array}$$

Consider the section $\sigma : S^2 \to S^2 \times \mathbb{R}^6$ defined for $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$ by:

$$\sigma(\omega) = (\omega_2, 0, \omega_1, -\omega_1\omega_3, \omega_2\omega_3, \omega_1^2 + \omega_2^2).$$

Since for every $\omega \in S^2$

$$(\omega Q_b)\sigma(\omega) = \sigma(\omega)$$

then it follows that σ is a section of the bundle $\bar{p}_b^*\Lambda_2$. The index sum of the zeroes of σ (which occur only at $(0,0,1), (0,0,-1) \in S^2$) is even, thus the euler class e of $\bar{p}_b^*\Lambda_2$ is even. This implies

$$w_2(\bar{p}_b^*\Lambda_2) = e \bmod 2 = 0.$$

Thus by Theorem 3.7.1 we have $d_3 \equiv 0$ and $E_2 = E_3 = E_\infty$. It follows that the only nonzero homology groups of $\mathbb{P}(\Gamma_V)$ are:

$$H_0(\mathbb{P}(\Gamma_V)) = H_3(\mathbb{P}(\Gamma_V)) = \mathbb{Z}_2 \quad \text{and} \quad H_1(\mathbb{P}(\Gamma_V)) = H_2(\mathbb{P}(\Gamma_V)) = (\mathbb{Z}_2)^2.$$

Actually since the equations for Γ_V are given by the vanishing of the minors of the matrix $\left(\begin{smallmatrix} x & z & y \\ w & s & t \end{smallmatrix}\right)$, then Γ_V is the Segre variety $\Sigma_{2,1} \simeq \mathbb{R}\mathrm{P}^1 \times \mathbb{R}\mathrm{P}^2$.

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