Scuola Internazionale Superiore di Studi Avanzati
Area of Mathematics
Ph.D. in Geometry

Biregular and Birational Geometry of Algebraic Varieties

Alex Massarenti
July 5, 2013

Advisor:
Prof. Massimiliano Mella

Internal Advisor:
Prof. Barbara Fantechi

Submitted in partial fulfillment
of the requirements for the degree
of Doctor of Philosophy

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INTRODUCTION

Every area of mathematics is characterized by a guiding problem. In algebraic geometry such problem is the classification of algebraic varieties. In its strongest form it means to classify varieties up to biregular morphisms. However, birationally equivalent varieties share many interesting properties. Therefore for any birational equivalence class it is natural to work out a variety, which is the simplest in a suitable sense, and then study these varieties. This is the aim of birational geometry. In the first part of this thesis we deal with the biregular geometry of moduli spaces of curves, and in particular with their biregular automorphisms. However, in doing this we will consider some aspects of their birational geometry. The second part is devoted to the birational geometry of varieties of sums of powers and to some related problems which will lead us to computational geometry and geometric complexity theory.

Part i is devoted to moduli spaces of curves, their fibrations and their automorphisms. The search for an object parametrizing n-pointed genus g smooth curves is a very classical problem in algebraic geometry. In [DM] *P. Deligne* and *D. Mumford* proved that there exists an irreducible scheme $M_{g,n}$ coarsely representing the moduli functor of n-pointed genus g smooth curves. Furthermore they provided a compactification $\overline{M}_{g,n}$ of $M_{g,n}$ adding Deligne-Mumford stable curves as boundary points and pointed out that the obstructions to represent the moduli functor of Deligne-Mumford stable curves in the category of schemes came from automorphisms of the curves. However this moduli functor can be represented in the category of algebraic stacks, indeed there exists a smooth Deligne-Mumford algebraic stack $\overline{\mathcal{M}}_{g,n}$ parametrizing Deligne-Mumford stable curves.

In Chapter 1 we recall some well known facts about the moduli space $\overline{M}_{g,n}$ and the stack $\overline{\mathcal{M}}_{g,n}$. These two geometric objects have been among the most studied objects in algebraic geometry for several decades. Despite this, many natural questions about their biregular and birational geometry remain unanswered.

Chapter 2 is devoted to the computation of the automorphism groups of $\overline{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$. These results appeared in [Ma]. The biregular automorphisms of the moduli space $M_{g,n}$ of n-pointed genus g-stable curves and of its Deligne-Mumford compactification $\overline{M}_{g,n}$ has been studied in a series of papers, for instance [BM1] and [Ro].
Recently, in [BM1] and [BM2], *A. Bruno* and *M. Mella* studied the fibrations of $\overline{M}_{0,n}$ using its description as the closure of the subscheme of the Hilbert scheme parametrizing rational normal curves passing through n points in linear general position in $\mathbb{P}^{n-2}$ given by *M. Kapranov* in [Ka]. It was expected that the only possible biregular automorphisms of $\overline{M}_{0,n}$ were the ones associated to a permutation of the markings. Indeed *Bruno* and *Mella* as a consequence of their theorem on fibrations derive that the automorphism group of $\overline{M}_{0,n}$ is the symmetric group S_n for any $n \geqslant 5$ [BM2, Theorem 4.3].
The aim of this work is to extend [BM2, Theorem 4.3] to arbitrary values of g, n and to the stack $\overline{\mathcal{M}}_{g,n}$. Our main result can be stated as follows.

Theorem. *Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stack parametrizing Deligne-Mumford stable n-pointed genus g curves, and let $\overline{M}_{g,n}$ be its coarse moduli space. If $2g-2+n \geqslant 3$ then*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \cong \mathrm{Aut}(\overline{M}_{g,n}) \cong S_n$$

the symmetric group on n *elements. For* $2g-2+n<3$ *we have the following special behavior:*

- $\mathrm{Aut}(\overline{M}_{1,2})\cong(\mathbb{C}^*)^2$ *while* $\mathrm{Aut}(\overline{\mathcal{M}}_{1,2})$ *is trivial,*
- $\mathrm{Aut}(\overline{M}_{0,4})\cong\mathrm{Aut}(\overline{\mathcal{M}}_{0,4})\cong\mathrm{Aut}(\overline{M}_{1,1})\cong PGL(2)$ *while* $\mathrm{Aut}(\overline{\mathcal{M}}_{1,1})\cong\mathbb{C}^*$,
- $\mathrm{Aut}(\overline{M}_{g})$ *and* $\mathrm{Aut}(\overline{\mathcal{M}}_{g})$ *are trivial for any* $g\geqslant 2$.

These issues have been investigated in the Teichmüller-theoretic literature on the automorphisms of moduli spaces $M_{g,n}$ developed in a series of papers by *H.L. Royden*, *C. J. Earle*, *I. Kra*, *M. Korkmaz*, [Ro], [EK] and [Kor]. A fundamental result, proved by *Royden* in [Ro], states that the moduli space $M_{g,n}^{un}$ of genus g smooth curves marked by n unordered points has no non-trivial automorphisms if $2g-2+n\geqslant 3$, which is exactly our bound.
Note that in the cases $g=n=1$ and $g=1,n=2$ the automorphism group of the stack differs from that of the moduli space. This is particularly evident for $\overline{M}_{1,1}$. It is well known that $\overline{M}_{1,1}\cong\mathbb{P}^1$ and $\overline{\mathcal{M}}_{1,1}\cong\mathbb{P}(4,6)$. Clearly $\mathbb{P}^1\cong\mathbb{P}(4,6)$ as varieties, however they are not isomorphic as stacks, indeed $\mathbb{P}(4,6)$ has two stacky points with stabilizers $\mathbb{Z}_4$ and $\mathbb{Z}_6$. These two points are fixed by any automorphism of $\mathbb{P}(4,6)$ while they are indistinguishable from any other point on the coarse moduli space $\overline{M}_{1,1}$.
The proof of the main Theorem is essentially divided into two parts: the cases $2g-2+n\geqslant 3$ and $2g-2+n<3$.
When $2g-2+n\geqslant 3$ the main tool is [GKM, Theorem 0.9] in which *A. Gibney*, *S. Keel* and *I. Morrison* give an explicit description of the fibrations $\overline{M}_{g,n}\to X$ of $\overline{M}_{g,n}$ on a projective variety X in the case $g\geqslant 1$. This result, combined with the triviality of the automorphism group of the generic curve of genus $g\geqslant 3$, let us to prove that the automorphism group of $\overline{M}_{g,1}$ is trivial for any $g\geqslant 3$. Since every genus 2 curve is hyperelliptic and has a non trivial automorphism, the hyperelliptic involution, the argument used in the case $g\geqslant 3$ completely fails. So we adopt a different strategy: first we prove that any automorphism of $\overline{M}_{2,1}$ preserves the boundary and then we apply a famous theorem of *H. L. Royden* [Moc, Theorem 6.1] to conclude that $\mathrm{Aut}(\overline{M}_{2,1})$ is trivial.
Then, applying [GKM, Theorem 0.9] we construct a morphism of groups between $\mathrm{Aut}(\overline{M}_{g,n})$ and S_n. Finally we generalize *Bruno* and *Mella*'s result proving that $\mathrm{Aut}(\overline{M}_{g,n})$ is indeed isomorphic to S_n when $2g-2+n\geqslant 3$.
When $2g-2+n<3$ a case by case analysis is needed. In particular the case $g=1,n=2$ requires an explicit description of the moduli space $\overline{M}_{1,2}$. Carefully analyzing the geometry of this surface we prove that $\overline{M}_{1,2}$ is isomorphic to a weighted blow up of $\mathbb{P}(1,2,3)$ in the point $[1:0:0]$, in particular $\overline{M}_{1,2}$ is toric. From this we derive that $\mathrm{Aut}(\overline{M}_{1,2})$ is isomorphic to $(\mathbb{C}^*)^2$.
Finally we consider the moduli stack $\overline{\mathcal{M}}_{g,n}$. The canonical map $\overline{\mathcal{M}}_{g,n}\to\overline{M}_{g,n}$ induces a morphism of groups $\mathrm{Aut}(\overline{\mathcal{M}}_{g,n})\to\mathrm{Aut}(\overline{M}_{g,n})$. Since this morphism is injective as soon as the general n-pointed genus g curve is automorphisms free, we easily derive that the automorphism group of the stack $\overline{\mathcal{M}}_{g,n}$ is isomorphic to S_n if $2g-2+n\geqslant 3$. Then we show that $\mathrm{Aut}(\overline{\mathcal{M}}_{1,2})$ is trivial using the fact that the canonical divisor of $\overline{\mathcal{M}}_{1,2}$ is a multiple of a boundary divisor.

In Chapter 3 we extend the techniques of Chapter 2 to moduli spaces of weighted pointed curves. These results appeared in [MM2]. In [Has] *B. Hassett* introduced new compactifications $\overline{\mathcal{M}}_{g,A[n]}$ of the moduli stack $\mathcal{M}_{g,n}$ and $\overline{M}_{g,A[n]}$ for the coarse moduli space $M_{g,n}$, by assigning rational weights $A=(a_1,...,a_n)$, $0<a_i\leqslant 1$ to the markings. In genus zero some of these spaces appear as intermediate steps of the blow-up construction of $\overline{M}_{0,n}$ developed by *M. Kapranov* in [Ka], while in higher genus they may be related to the Log minimal model program on $\overline{M}_{g,n}$.

We deal with fibrations and automorphisms of these Hassett's spaces. Our approach consists in extending some techniques introduced in Chapter 2, [BM1] and [BM2] to study fiber type morphisms from Hassett's spaces and then apply this knowledge to compute their automorphism groups.
In [BM1] and [BM2], *A. Bruno* and *M. Mella*, thanks to Kapranov's works [Ka], managed to translate issues on the moduli space $\overline{M}_{0,n}$ in terms of classical projective geometry of $\mathbb{P}^{n-3}$. Studying linear systems on $\mathbb{P}^{n-3}$ with particular base loci they derived a theorem on the fibrations of $\overline{M}_{0,n}$.

Theorem. *[BM2, Theorem 1] Let $f : \overline{M}_{0,n} \to \overline{M}_{0,r}$ be a dominant morphism with connected fibers. Then f factors through a forgetful map.*

Via this theorem on fibrations they construct a morphism of groups between $\mathrm{Aut}(\overline{M}_{g,n})$ and S_n, the symmetric group on n elements, and prove the following theorem:

Theorem. *[BM2, Theorem 3] The automorphism group of $\overline{M}_{0,n}$ is isomorphic to S_n for any $n \geqslant 5$.*

As already noticed some of the Hassett's spaces are partial resolutions of Kapranov's blow-ups. The main novelty is that not all forgetful maps are well defined as morphisms. Nonetheless we are able to control this problem and derive a weighted version of the fibration theorem. This allows us to compute the automorphisms of all intermediate steps of Kapranov's construction, see Construction 3.0.11 for the details.

Theorem. *The automorphism groups of the Hassett's spaces appearing in Construction 3.0.11 are given by*

- $\mathrm{Aut}(\overline{M}_{0,A_{r,s}[n]}) \cong (\mathbb{C}^*)^{n-3} \times S_{n-2}$, if $r = 1$, $s < n-3$,
- $\mathrm{Aut}(\overline{M}_{0,A_{r,s}[n]}) \cong (\mathbb{C}^*)^{n-3} \times S_{n-2} \times S_2$, if $r = 1$, $s = n-3$,
- $\mathrm{Aut}(\overline{M}_{0,A_{r,s}[n]}) \cong S_n$, if $r \geqslant 2$.

In particular the Hassett's space $\overline{M}_{A_{1,n-3}[n]}$, that is $\mathbb{P}^{n-3}$ blown-up at all the linear spaces of codimension at least two spanned by subsets of $n-2$ points in linear general position, is the Losev-Manin's moduli space $\overline{L}_{n-2}$ introduced by *A. Losev* and *Y. Manin* in [LM], see [Has, Section 6.4].
In higher genus we approach the same problem. This time the fibration theorem is inherited by [GKM, Theorem 0.9]. Concerning the automorphisms, for Hassett's spaces the situation is a bit more complicated than for $\overline{M}_{g,n}$ because a permutation of the markings may not define an automorphism of the Hassett's space $\overline{M}_{g,A[n]}$. Indeed in order to define an automorphism permutations have to preserve the weight data in a suitable sense, see Definition 3.2.11. We denote by $\mathcal{A}_{A[n]}$ the subgroup of S_n of permutations inducing automorphisms of $\overline{M}_{g,A[n]}$ and $\overline{\mathcal{M}}_{g,A[n]}$. In Theorems 3.2.16 and 3.2.19 we prove the following statement:

Theorem. *Let $\overline{\mathcal{M}}_{g,A[n]}$ be the Hassett's moduli stack parametrizing weighted n-pointed genus g stable curves, and let $\overline{M}_{g,A[n]}$ be its coarse moduli space. If $g \geqslant 1$ and $2g-2+n \geqslant 3$ then*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,A[n]}) \cong \mathrm{Aut}(\overline{M}_{g,A[n]}) \cong \mathcal{A}_{A[n]}.$$

Furthermore

- $\mathrm{Aut}(\overline{M}_{1,A[2]}) \cong (\mathbb{C}^*)^2$ *while* $\mathrm{Aut}(\overline{\mathcal{M}}_{1,A[2]})$ *is trivial,*
- $\mathrm{Aut}(\overline{M}_{1,A[1]}) \cong \mathrm{PGL}(2)$ *while* $\mathrm{Aut}(\overline{\mathcal{M}}_{1,A[1]}) \cong \mathbb{C}^*$.

Note that this Theorem is exactly the weighted analogue of the main result of Chapter 2.

Chapter 4 collects some conjectures on fibrations and automorphisms of the moduli spaces of stable maps. In symplectic topology and algebraic geometry, *Gromov-Witten invariants* are rational numbers that, in certain situations, count holomorphic curves. The Gromov-Witten invariants may be packaged as a homology or cohomology class, or as the deformed cup product of quantum cohomology. These invariants have been used to distinguish symplectic manifolds that were previously indistinguishable. They also play a crucial role in string theory. They are named for *M. Gromov* and *E. Witten*.

Gromov-Witten invariants are of interest in string theory. In this theory the elementary particles are made of tiny strings. A string traces out a surface in the spacetime, called the worldsheet of the string. The moduli space of such parametrized surfaces, at least a priori, is infinite-dimensional; no appropriate measure on this space is known, and thus the path integrals of the theory lack a rigorous definition.

However in a variation known as *closed A model topological string theory* there are six spacetime dimensions, which constitute a symplectic manifold, and it turns out that the worldsheets are necessarily parametrized by pseudoholomorphic curves, whose moduli spaces are only finite-dimensional. Gromov-Witten invariants, as integrals over these moduli spaces, are then path integrals of the theory.

The appropriate moduli spaces were introduced by *M. Kontsevich* in [Kh], these spaces are denoted by $\overline{M}_{g,n}(X,\beta)$ where X is a projective scheme, and parametrize holomorphic maps from n-pointed genus g curves, whose images have homology class β, to X. If X is a homogeneous variety the $\overline{M}_{0,n}(X,\beta)$ is a normal, projective variety of pure dimension. Furthermore if $X = \mathbb{P}^N$ then $\overline{M}_{0,n}(\mathbb{P}^N,d)$ is irreducible. On the other hand when $g \geqslant 1$, and even when $g = 0$ for most schemes $X \neq \mathbb{P}^N$ the space $\overline{M}_{g,n}(X,\beta)$ may have many components of dimension greater than expected. To overcome this gap and give a rigorous definition of Gromov-Witten invariants *J. Li, G. Tian* in [LT1], [LT2], and *K. Behrend, B. Fantechi* in [BF] introduce the notions of *virtual fundamental class* and *virtual dimension*.

Recently *F. Poma* in [Po], using intersection theory on Artin stacks developed by *A. Kresch* in [Kr], constructed a perfect obstruction theory leading to a virtual class and then to a rigorous definition of Gromov-Witten invariants in positive and mixed characteristic, satisfying the axioms of Gromov-Witten invariants given by *M. Kontsevich* and *Y. Manin* in [KhM], and the *WDVV* equations.

The Gromov-Witten potential, which is a function encoding the information carried by Gromov-Witten invariants, satisfies *WDVV* equations. This is equivalent to the associativity of the quantum product. As a consequence it turns out that the quantum cohomology ring QH^*X is a supercommutative algebra, and the complex cohomology $H^*(X,\mathbb{C})$ has a structure of Frobenius manifold. For these reasons, the moduli spaces of stable maps play a key role both in geometry and in theoretical physics.

By virtue of the results obtained in Chapters 2 and 3 I believe that in most cases the automorphisms of a moduli space parametrizing curves, and perhaps those of moduli spaces in general, are just modular automorphisms, that is automorphisms that derive from the nature of the parametrized objects. My belief is also supported by the calculation of the automorphisms of moduli spaces of vector bundles over a curve in [BGM].

In Chapter 4 we consider the space $\overline{M}_{0,n}(\mathbb{P}^N,d)$. After giving some evidence on what its automorphisms should be by observing that S_n and $\mathrm{Aut}(\mathbb{P}^N)$ act naturally on $\overline{M}_{0,n}(\mathbb{P}^N,d)$ we conjecture that:

Conjecture. *For any* $n \geqslant 5$ *we have*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N,d)) \cong \mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^N,d)) \cong S_n \times PGL(N+1).$$

By the way, such conjecture would fit in a more general theory of a modular nature of the automorphisms of varieties admitting a modular interpretation.

Part ii is devoted to Varieties of Sums of Powers and to some related topics. In 1770 *E. Waring* stated that every integer is a sum of at most 9 positive cubes. Later on *C.G.J. Jacobi* and others considered the problem of finding all the decompositions of a given number into sums of cubes, [Di]. Since then many problems related to additive decomposition have been named after Waring.

For instance a variation on the Waring problem asked which is the minimum positive integer h such that the generic polynomial of degree d on $\mathbb{P}^n$ admits a decomposition as a sum of h powers of linear forms. In 1995 *J. Alexander* and *A. Hirshowitz* [AH] completely solved this problem over an algebraically closed field of characteristic zero. They proved that the minimum integer h is the expected one $h = \lfloor \frac{1}{n+1}\binom{n+d}{d} \rfloor$, except in the following cases: $d = 2$, for any n, h such that $2 \leqslant h \leqslant n$; $d = 4$, $n = 2$, $h = 5$; $d = 4$, $n = 3$, $h = 9$; $d = 3$, $n = 4$, $h = 7$; $d = 4$, $n = 4$, $h = 14$.

The set up we are interested in is that of homogeneous polynomials over the complex field. Let $F \in k[x_0, \dots, x_n]_d$ be a general homogeneous polynomial of degree d. The additive decomposition we are looking for is

$$F = L_1^d + \dots + L_h^d,$$

where $L_i \in k[x_0, \dots, x_n]_1$ are linear forms. The problem is a classical one. The first results are due to *J.J. Sylvester*, [Sy] and then to *D. Hilbert*, [Hi], *H.W. Richmond*, [Ri], *F. Palatini*, [Pa], and many others. In the old times the attention was essentially focused on studying the cases in which the above decomposition is unique. When this happens the unique decomposition gives a canonical form of a general polynomial. As widely expected the canonical form very seldom exists [Me2] [Me1].

The set of additive decompositions of a given general polynomial is usually compactified in $\mathrm{Hilb}((\mathbb{P}^n)^*)$ and is called the *Variety of Sums of Powers*, VSP for short, see Definition 5.0.6 for the precise statement. The interest in these special varieties increased greatly after *S. Mukai* [Mu1] gave a description of the Fano 3-fold V_{22} as a VSP of quartic polynomials in three variables. Since then different authors have exploited the area and generalized Mukai's techniques to other polynomials, [DK], [RS], [IR1], [IR2], [TZ]. See [Do] for a very nice survey. The known cases are not many and, to the best of our knowledge, this is the state of the art.

d	n	h	$\mathrm{VSP}(F_d, h)$	Reference
$2h-1$	1	h	1 point	Sylvester[Sy]
2	2	3	quintic Fano 3 − fold	Mukai[Mu1]
3	2	4	$\mathbb{P}^2$	Dolgachev and Kanev[DK]
4	2	6	Fano 3 − fold V_{22}	Mukai[Mu1]
5	2	7	1 point	Hilbert, [Hi], Richmond, [Ri], Palatini, [Pa]
6	2	10	K3 surface of genus 20	Mukai[Mu2]
7	2	12	5 points	Dixon and Stuart[Dx]
8	2	15	16 points	Mukai[Mu2]
2	3	4	$\mathbb{G}(1,4)$	Ranestad and Schreyer[RS]
3	3	5	1 point	Sylvester's Pentahedral Theorem[Sy]
3	4	8	$\mathcal{W}$	Ranestad and Schreyer[RS]
3	5	10	$\mathcal{S}$	Iliev and Ranestad[IR1]

where $\mathcal{W}$ is the 5-dimensional variety parametrizing lines in the linear complete intersection $\mathbb{P}^{10} \cap \mathbb{OG}(5,10) \subseteq \mathbb{P}^{15}$ of the 10-dimensional orthogonal Grassmannian $\mathbb{OG}(5,10)$, and $\mathcal{S}$ is a smooth symplectic 4-fold obtained as a deformation of the Hilbert square of a polarized K3 surface of genus eight.

Chapter 5 contains the results of [MM1]. In this chapter we aim to understand a general birational behavior of VSP. To do this we prefer to adopt a different compactification. This approach is probably less efficient than the usual one to study the biregular nature of VSP. On the other hand it allows to study birational properties in an easier way.
Let $F \in k[x_0,\dots,x_n]_d$ be a general homogeneous polynomial of degree d and $V = V_{d,n} \subset \mathbb{P}^N = \mathbb{P}(k[x_0,\dots,x_n]_d)$ the Veronese variety. A general additive decomposition into h linear factors

$$F = \sum_1^h L_i^d$$

is associated to an h-secant linear space of dimension $h-1$ to the Veronese $V \subset \mathbb{P}^N$. In this way we can realize the set of additive decompositions into $\mathbb{G}(h-1,N)$ and consider the closure there. This compactification is expected to be more singular than the one into the Hilbert scheme, and it is well defined only for $h < N-n$. See Remark 5.0.13 for a brief comparison with VSP. On the other hand we may use projective techniques and this yields several interesting results about the birational nature of VSP's.

Theorem. *Assume that F is a general quadratic polynomial in $n+1$ variables. Then the irreducible components of $VSP(F,h)$ are unirational for any h and rational for $h = n+1$.*

This theorem cannot be extended to higher degrees. For instance think about the mentioned examples of either *S. Mukai* or *A. Iliev* and *K. Ranestad*. On the other hand rational connectedness should be the general pattern for this class of varieties. In this direction the main result in Chapter 5 is the rational connectedness of infinitely many VSP with arbitrarily high degree and number of variable.

Theorem. *Assume that for some positive integer $0 < k < n$ the number $\frac{\binom{d+n}{n}-1}{k+1}$ is an integer. Then the irreducible components of $VSP(F,h)$ are rationally connected for $F \in k[x_0,\dots,x_n]_d$ general and $h \geqslant \frac{\binom{n+d}{n}-1}{k+1}$.*

The common kernel of these theorems is Theorem 5.1.1 which, under suitable assumption, connects $VSP(F,h)$ with chains of $VSP(F,h-1)$. In this way we reduce the rational connectedness computations to special values of h where the compactification in the Grassmannian variety is well defined.

In Chapter 6 we extend the definition of VSP replacing the Veronese variety V with an arbitrary non-degenerate variety $X \subset \mathbb{P}^N$. We denote these varieties by $VSP_H^X(h)$. In Proposition 6.1.4 we prove a rationality result on $VSP_H^X(h)$ when $X \subset \mathbb{P}^N$ is a variety of minimal degree. Then, in Theorem 6.3.3, we generalize Theorem 5.3.1 replacing the Veronese variety with an arbitrary unirational variety.

In Chapter 7 we consider the problem of finding explicit decompositions of homogeneous polynomials as sums of powers of linear forms. Polynomials often appear in issues of applied mathematics, for instance in signal theory [CM], algebraic complexity theory [BCS], coding and information theory [Ro]. For applied sciences is interesting to determine:

- *whether a polynomial admits a decomposition into a number of forms,*

- *and eventually to calculate explicitly the decomposition.*

We first focus on the case $\mathcal{S}ec_h(V_d^n) = \mathbb{P}^N$. Using apolarity we give an effective method to reconstruct the decompositions in a number of cases (construction 7.1.1). Then we concentrate on cases where the decomposition is unique; as the above table shows, if $\mathcal{S}ec_h(V_d^n) = \mathbb{P}^N$, these are very few. In each case we give an algorithm to calculate the decomposition 7.1.6, 7.1.9, 7.1.12, and provide examples using symbolic calculus software such as *MacAulay2* [Mc2] and *MatLab*. Furthermore we use *Bertini* [Be] to solve systems of polynomial equations of high computational complexity. All scripts are listed in Appendix 7.2.4.
Then we focus our attention on the case $\mathcal{S}ec_h(V_d^n) \subsetneqq \mathbb{P}^N$ and adopt the philosophy dictated by the following trivial but crucial observation:
If $F = \sum_{i=1}^h \lambda_i L_i^d$ *then its partial derivatives of order* l *lie in the linear space* $\langle L_1^{d-l}, \dots, L_h^{d-l} \rangle$ *for any* $l = 1, \dots, d-1$.
In the case $n = 2$ we prove that, in order to establish if a homogeneous polynomial $F \in k[x_0, x_1]_d$ admits a decomposition as sum of h powers, it is enough to verify that $\dim(H_\partial) = h - 1$, where H_∂ is the linear space spanned by the partial derivatives of order $d - h$ of F. Furthermore, if $\dim(H_\partial) = h - 1$ we get a method to write the linear forms related to F 7.2.9. Finally trying to extend the method in higher dimension we compute the dimension of the linear space of polynomials whose $(d-1)$-derivatives lie in general linear subspace $H \subset (\mathbb{P}^N)^*$, this space is also called the $(d-1)$-th prolongation of H. Consequently we find the formula for the dimension of $\mathcal{S}ec_h(V_2^n)$, and the secant defect of V_2^n. Furthermore we obtain a criterion to determine whether a polynomial admits a decomposition in the cases $d = 2$ and $d = 3, h = 2$.

Chapter 8 is devoted to the study of a particular tensor, namely the matrix multiplication tensor. Homogeneous polynomials are symmetric tensors and in Chapter 7 we considered their decompositions as sums of linear forms, that is as sums of rank one symmetric tensors. Similarly in Chapter 8 we study the matrix multiplication tensor in order to give a lower bound on its rank. These last results appeared in [MR].
The multiplication of two matrices is one of the most important operations in mathematics and applied sciences. To determine the complexity of matrix multiplication is a major open question in algebraic complexity theory. Recall that the matrix multiplication $M_{n,l,m}$ is defined as the bilinear map

$$\begin{array}{rccc} M_{n,l,m}: & Mat_{n\times l}(\mathbb{C}) \times Mat_{l\times m}(\mathbb{C}) & \to & Mat_{n\times m}(\mathbb{C}) \\ & (X, Y) & \mapsto & XY, \end{array}$$

where $Mat_{n\times l}(\mathbb{C})$ is the vector space of $n \times l$ complex matrices. A measure of the complexity of matrix multiplication, and of tensors in general, is the *rank*. For the bilinear map $M_{n,l,m}$ this is the smallest natural number r such that there exist $a_1, \dots, a_r \in Mat_{n\times l}(\mathbb{C})^*$, $b_1, \dots, b_r \in Mat_{l\times m}(\mathbb{C})^*$ and $c_1, \dots, c_r \in Mat_{n\times m}(\mathbb{C})$ decomposing $M_{n,l,m}(X, Y)$ as

$$M_{n,l,m}(X, Y) = \sum_{i=1}^{r} a_i(X) b_i(X) c_i$$

for any $X \in Mat_{n\times l}(\mathbb{C})$ and $Y \in Mat_{l\times m}(\mathbb{C})$.
In the case of square matrices the standard algorithm gives an expression of the form $M_{n,n,n}(X, Y) = \sum_{i=1}^{n^3} a_i(X) b_i(X) c_i$. However *V. Strassen* showed that such algorithm is not optimal [S].
We are concerned with lower bounds on the rank of matrix multiplication. The first lower

bound $\frac{3}{2}n^2$ was proved by *V. Strassen* [S1] and then improved by *M. Bläser* [Bl], who found the lower bound $\frac{5}{2}n^2 - 3n$.
Recently *J.M. Landsberg* [La1], building on work with *G. Ottaviani* [LO1], found the new lower bound $3n^2 - 4n^{\frac{3}{2}} - n$. The core of Landsberg's argument is the proof of the Key Lemma [La1, Lemma 4.3]. We improve the Key Lemma and in Theorem 8.2.4 we obtain new lower bounds for matrix multiplication.
Our strategy is the following. We prove Lemma 8.2.2, which is the improved version of [La1, Lemma 4.3], using the classical identities for determinants of Lemma 8.0.30 and Lemma 8.0.31, to lower the degree of the equations that give the lower bound for border rank for matrix multiplication. Then we exploit this lower degree as Bläser and Landsberg did.

Part I

AUTOMORPHISMS OF MODULI SPACES OF CURVES

1

A BRIEF SURVEY ON MODULI OF CURVES

To fix the ideas, we work over an algebraically closed field k. Consider a class of objects $\mathcal{M}$ over k, for instance the class of closed subschemes of $\mathbb{P}^n$ with fixed Hilbert Polynomial, the class of curves of genus g over k, the class of vector bundles of given rank and Chern classes over a fixed scheme, and so on. We wish to classify the objects in $\mathcal{M}$.
The first step is to give a rule to determine when two objects of $\mathcal{M}$ are the same (usually isomorphic) and then to give the elements of $\mathcal{M}$ up to isomorphism. This determines $\mathcal{M}$ as a set. Now we want to put a natural structure of variety or scheme on $\mathcal{M}$. In other words we are looking for a scheme M whose closed points are in a one-to-one correspondence with the elements of $\mathcal{M}$, and whose scheme structure describes the variations of elements in $\mathcal{M}$, more precisely how they behave in families.

Definition 1.0.1. A family of elements of $\mathcal{M}$, over the parameter scheme S of finite type over k, is a scheme $X \to S$ flat over S, whose fibers at closed points are elements of $\mathcal{M}$.

The first request on M, to be a Moduli Space for the class $\mathcal{M}$, is that for any family $X \to S$ of objects of $\mathcal{M}$ there exists a morphism $\varphi : S \to M$ such that for any closed point $s \in S$, the image $f(s) \in \mathcal{M}$ corresponds to the isomorphism class of the fiber $X_s = \varphi^{-1}(s)$ in $\mathcal{M}$.
Furthermore we want the assignment of the morphism φ to be functorial. To explain the last sentence consider the functor $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$, that assigns to S the set $\mathcal{F}(S)$ of families $X \to S$ of elements of $\mathcal{M}$ parametrized by S. If $S^{'} \to S$ is a morphism, for any family $X \to S$ we can consider the fiber product $X \times_S S^{'} \to S^{'}$, that is a family over $S^{'}$. In this way the morphism $S^{'} \to S$ gives rise to a map of set $\mathcal{F}(S) \to \mathcal{F}(S^{'})$, and $\mathcal{F}$ becomes a controvariant functor.
In this language to assign a morphism $\varphi : S \to M$ to any family $X \to S$ with the required properties, means to give a functorial morphism $\alpha : \mathcal{F} \to Hom(-, M)$.
Finally we want to make M unique with the above properties. So we require that if N is any other scheme, and $\beta : \mathcal{F} \to Hom(-, N)$ is a functorial morphism, then there exists a unique morphism $e : M \to N$ such that $\beta = h_e \circ \alpha$, where $h_e : Hom(-, M) \to Hom(-, N)$ is the induced map on associated functors.

Definition 1.0.2. We define a *coarse moduli space* for the family $\mathcal{M}$ to be a scheme M over k, with a morphism of functors $\alpha : \mathcal{F} \to Hom(-, M)$ such that

- the induced map $\mathcal{F}(Spec(k)) \to Hom(Spec(k), M)$ is bijective i.e. there is a one-to-one correspondence with isomorphism classes of elements of $\mathcal{M}$ and closed points of M,
- α is universal in the sense explained above.

We define a *tautological family* for $\mathcal{M}$ to be a family $X \to M$ such that for each closed point $m \in M$, the fiber X_m is the element of $\mathcal{M}$ corresponding to m by the bijection $\mathcal{F}(Spec(k)) \to Hom(Spec(k), M)$ above.

A *jump phenomenon* for $\mathcal{M}$ is a family $X \to S$, where S is an integral scheme of dimension at least one, such that all fibers X_s for $s \in S$ are isomorphic except for one X_{s_0} that is different. In this case the corresponding morphism $S \to M$ have to map s_0 to a point and all other closed points of S to another point, but this is not possible for a morphism of schemes, so a coarse moduli space for $\mathcal{M}$ fails to exist.

Example 1.0.3. Consider the family $y^2 = x^3 + t^2x + t^3$ over the t-line. Then for any $t \neq 0$ we get smooth elliptic curves all with the same j-invariant

$$j = 12^3 \cdot \frac{4t^6}{4t^6 + 27t^6} = 12^3 \cdot \frac{4}{31},$$

and hence all isomorphic. But for $t = 0$ we get the cusp $y^2 = x^3$. This is a jump phenomenon, so the cuspidal curve cannot belong to a class having a coarse moduli space.

Definition 1.0.4. Let $\mathcal{F}$ be the functor associated to the moduli problem $\mathcal{M}$. If $\mathcal{F}$ is isomorphic to a functor of the form $\mathrm{Hom}(-, M)$, then we say that $\mathcal{F}$ is representable, and we call M a *fine moduli space* for $\mathcal{M}$.

Let $\alpha : \mathcal{F} \to \mathrm{Hom}(-, M)$ be an isomorphism. In particular $\mathcal{F}(M) \to \mathrm{Hom}(M, M)$ is an isomorphism, and there is a unique family $X_{\mathcal{U}} \to M$ corresponding to the identity map $\mathrm{Id}_M \in \mathrm{Hom}(M, M)$. The family $X_{\mathcal{U}}$ is called the *universal family* of the fine moduli space M. Note that for any family $X \to S$ there exists an unique morphism $S \to M$, such that $X \to S$ is obtained by base extension from the universal family. Conversely, if there is a scheme M and a family $X_{\mathcal{U}}$ with the above properties then $\mathcal{F}$ is represented by M.

Remark 1.0.5. If M is a fine moduli space for $\mathcal{M}$ then it is also a coarse moduli space, furthermore the universal family $X_{\mathcal{U}} \to M$ is a tautological family.

A benefit of having a fine moduli space is that we can study it using infinitesimal methods.

Proposition 1.0.6. *Let* M *be a fine moduli space for the moduli problem* $\mathcal{M}$, *and let* $X_0 \in \mathcal{M}$ *be an element corresponding to a point* $x_0 \in M$. *The Zariski tangent space* $T_{x_0}M$ *is in one-to-one correspondence with the set of families* $X \to D$ *over the dual numbers* $D = k[\epsilon]/(\epsilon^2)$, *whose closed fibers are isomorphic to* X_0.

Proof. We know that to give a morphism $f : \mathrm{Spec}(D) \to M$ is equivalent to give a closed point $x_0 \in M$ and a tangent direction $v \in T_{x_0}M$. But a morphism $f : \mathrm{Spec}(D) \to M$ corresponds to a unique family $X \to \mathrm{Spec}(D)$ whose closed fibers are isomorphic to $X_0 \in \mathcal{M}$ corresponding to the point $x_0 \in M$, where $x_0 = f((\mathrm{Spec}(D))_{red})$. ☐

Let $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$ be the functor associated to the moduli problem $\mathcal{M}$. Suppose that $\mathcal{F}$ is representable, and let M be the corresponding fine moduli space. For any local Artin k-algebra A we have that $\mathrm{Spec}(A)$ is a fat point and $(\mathrm{Spec}(A))_{red}$ is a single point. For any $x_0 \in M$ we can define the infinitesimal deformation functor of $\mathcal{F}$ as the functor $\mathfrak{Art} \to \mathfrak{Sets}$ that sends A in the set of morphisms $f : \mathrm{Spec}(A) \to M$ such that $f((\mathrm{Spec}(A))_{red}) = x_0$. Clearly studying this functor we get information on the geometry of M in a neighborhood of x_0.

Recall that a pro-object is an inverse limit of objects in $\mathfrak{Art}$, the category of Artin local algebras over a field k. If $\mathcal{F} : \mathfrak{Art} \to \mathfrak{Sets}$ is a deformation functor we say that $\mathcal{F}$ is pro-representable if it is isomorphic to $\mathrm{Hom}(-, R)$ for some pro-object R.

Proposition 1.0.7. *Let* $\mathcal{F}$ *be the functor associated to the moduli problem* $\mathcal{M}$, *and* $X_0 \in \mathcal{M}$. *Consider the functor* $\mathcal{F}_0$ *that to each local Artin ring* A *over* k *assigns the set of families of* $\mathcal{M}$ *over* $\mathrm{Spec}(A)$ *whose closed fiber is isomorphic to* X_0. *If* $\mathcal{M}$ *has a fine moduli space, then the functor* $\mathcal{F}_0$ *is pro-representable.*

Proof. Let M be a fine moduli scheme for $\mathcal{M}$, and let $x_0 \in M$ corresponds to $X_0 \in \mathcal{M}$. Let $\mathcal{O}_{M,x_0}$ be the local ring of M at x_0 and $\mathfrak{M}_{x_0}$ its maximal ideal. The natural homomorphisms

$$... \to \mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0}^3 \to \mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0}^2 \to \mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0},$$

make $(\mathcal{O}_{M,x_0}/\mathfrak{M}^n_{x_0})$ into an inverse system of rings. The inverse limit $\varprojlim \mathcal{O}_{M,x_0}/\mathfrak{M}^n_{x_0}$ is denoted by $\hat{\mathcal{O}}_{M,x_0}$, and is called the completion of $\mathcal{O}_{M,x_0}$ with respect to $\mathfrak{M}_{x_0}$ or the $\mathfrak{M}_{x_0}$-adic completion of $\mathcal{O}_{M,x_0}$.
Since M is a fine moduli space, each element of $\mathcal{F}_0(A)$ corresponds to a unique morphism $\mathrm{Spec}(A) \to M$ that maps $(\mathrm{Spec}(A)_{red}) = \mathrm{Spec}(k)$ at x_0. Such morphism corresponds to a ring homomorphism $\hat{\mathcal{O}}_{M,x_0} \to A$. We conclude that the functor $\mathcal{F}_0$ is pro-representable and that it is represented by the pro-object $\hat{\mathcal{O}}_{M,x_0}$, $\mathfrak{M}_{x_0}$-adic completion of $\mathcal{O}_{M,x_0}$. □

Definition 1.0.8. A controvariant functor $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$ is a *sheaf for the Zariski topology*, if for every scheme S and every $\{\mathcal{U}_i\}$ open covering of S, the diagram

$$\mathcal{F}(S) \to \prod \mathcal{F}(\mathcal{U}_i) \rightrightarrows \prod \mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j)$$

is exact. This means that:

- given $x, y \in \mathcal{F}(S)$ whose restriction to $\mathcal{F}(\mathcal{U}_i)$ are equal for all i, then $x = y$,
- given a collection of elements $x_i \in \mathcal{F}(\mathcal{U}_i)$ for each i, such that for each i, j, the restrictions of x_i, x_j to $\mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j)$ are equal, then there exists an element $x \in \mathcal{F}(S)$ whose restriction to each $\mathcal{F}(\mathcal{U}_i)$ is x_i.

Proposition 1.0.9. *If the moduli problem $\mathcal{M}$ has a fine moduli space, then the associated functor $\mathcal{F}$ is a sheaf in the Zariski topology.*

Proof. Since $\mathcal{M}$ has a fine moduli space, for any scheme S we have $\mathcal{F}(S) = \mathrm{Hom}(S, M)$. Furthermore morphisms of schemes are determined locally, and can be glued if they are given locally and are compatible on overlaps. □

Remark 1.0.10. Using Grothendieck's theory of *descent* one can show that a representable functor is a sheaf for the faithfully flat quasi-compact topology, and hence also for the étale topology.

Examples of Moduli Spaces

We will give some examples of representable functors.

Example 1.0.11. (Grassmannians) Let V be a k-vector space of dimension n, and let $r \leqslant n$ be a fixed integer. Consider the controvariant functor $\mathrm{Gr} : \mathfrak{Sch} \to \mathfrak{Sets}$ defined as follows

- For any scheme S, $\mathrm{Gr}(S)$ is the set of rank r vector subbundle of the trivial bundle $S \times V$.
- If $f : S \to S^{'}$ is a morphism of schemes, and $E_{S^{'}}$ is a rank r subbundle of $S^{'} \times V$, we define
$$\mathrm{Gr}(f)(E_{S'}) = f^*(E_{S'}) = (f \times \mathrm{Id}_V)^{-1}(E_{S'}).$$

Note that for $S = \mathrm{Spec}(k)$ we have that $\mathrm{Gr}(\mathrm{Spec}(k))$ is the set of rank r subbundle of $\mathrm{Spec}(k) \times V = V$ i.e. the set of r-dimensional subspace of V, that is the Grassmannian $\mathrm{Gr}(r, V)$.
If $E \in \mathrm{Gr}(S)$ is a rank r subbundle of $S \times V$, we can construct a morphism $f_E : S \to \mathrm{Gr}(r, V)$ defined by $s \mapsto E_s$, where E_s is the fiber of E over $s \in S$. In this way we get a map

$$\varphi(S) : \mathrm{Gr}(S) \to \mathrm{Hom}(S, \mathrm{Gr}(r, V)),\ E \mapsto f_E.$$

The collection $\{\varphi(S)\}$ gives a functorial isomorphism between Gr and $\mathrm{Hom}(-, \mathrm{Gr}(r, V))$. Then the functor Gr is representable and the Grassmannian $\mathrm{Gr}(r, V)$ is the corresponding fine moduli space. The universal family corresponding to the identity map $\mathrm{Id}_{\mathrm{Gr}(r,V)} \in \mathrm{Hom}(\mathrm{Gr}(r,V), \mathrm{Gr}(r,V))$ is clearly the universal bundle on $\mathrm{Gr}(r,V)$ given by $\{(W,v) \,|\, v \in W\} \subseteq \mathrm{Gr}(r,V) \times V$.

Example 1.0.12. (Hilbert Scheme) Let $P \in \mathbb{Q}[z]$ be a fixed polynomial. For any S scheme over k consider $\mathbb{P}^N_S = \mathbb{P}^N \times_k S$, and the functor

$$\mathrm{Hilb}^N_P : \mathfrak{Sch} \to \mathfrak{Sets},$$

that maps S in the set of subschemes $Y \subseteq \mathbb{P}^N_S$ such that the projection $\pi : Y \to S$ is flat, and for any $s \in S$ the fiber $\pi^{-1}(s)$ is a subscheme of $\mathbb{P}^N$ with Hilbert polynomial P. The functor Hilb^N_P is representable by a scheme $\mathrm{Hilb}_P(\mathbb{P}^N)$ projective over k and called the Hilbert Scheme.

To any closed subscheme $Y \subseteq \mathbb{P}^N$ we can associate its structure sheaf $\mathcal{O}_Y$, its ideal sheaf $\mathcal{I}_Y$, and the structure sequence

$$0 \mapsto \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_Y \mapsto 0.$$

Then we can regard the Hilbert scheme as the space parametrizing all the quotients $\mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_Y$, with Hilbert polynomial P.

Example 1.0.13. (Grothendieck's Quot Scheme) As a generalization of the discussion above consider a fixed coherent sheaf $\mathcal{E}$ on $\mathbb{P}^N$. The scheme parametrizing all the quotients $\mathcal{E} \to \mathcal{F} \mapsto 0$ with Hilbert polynomial P is called the Quot Scheme. Grothendieck showed that the local deformation functor of the Quot functor is pro-representable and that the Quot functor is representable by a projective scheme.

Example 1.0.14. (Picard Scheme) Let X be a scheme of finite type over an algebraically closed field k and let $x \in X$ be a fixed point. Consider the functor

$$\mathrm{Pic}_{X,x} : \mathfrak{Sch} \to \mathfrak{Sets},$$

that associates to S the group of all invertible shaves $\mathcal{L}$ on $X \times S$, with a fixed isomorphism $\mathcal{L}_{|x} \times S \cong \mathcal{O}_S$.
If X is integral and projective, then this functor is representable by a separated scheme, locally of finite type over k, called the Picard Scheme of X.

Example 1.0.15. (Hilbert-Flag Scheme) Consider a functor that associates to each scheme S a flag $Y_1 \subseteq Y_2 \subseteq ... \subseteq Y_k \subseteq \mathbb{P}^N_S$ of closed subscheme, all flat over S and where the fibers if Y_j have a fixed Hilbert Polynomial P_j for any $j = 1, ..., k$. This functor is representable by a scheme, projective over k, called the Hilbert-Flag Scheme.

1.1 GIT CONSTRUCTION OF $\overline{M}_g$

The aim of Geometric invariant theory is to solve the problem of constructing quotient in the framework of algebraic geometry. In this section we collect the main results of this theory, which are fundamental for the construction of moduli spaces. For a detailed discussion see [MFK], and for a complete and very readable treatment see [Do].
We concentrate on the special case of projective schemes and reductive groups. So let Z be a projective scheme and let G be a reductive group acting on Z. Consider an embedding $Z \to \mathbb{P}^r = \mathbb{P}(V)$ given by a line bundle $\mathcal{L}$ on Z, so that $Z = \mathrm{Proj}(S)$ for some graded ring S finitely generated over k. When the action of G on Z can be lifted to an action on V we

say that there exists a G-linearization of $\mathcal{L}$, or that G acts linearly with respect to the given embedding. In this case G acts on S and the subring

$$S^G = \{s \in S \mid gs = s \, \forall\, g \in G\} \subseteq S,$$

is called the ring of invariants of S with respect to the action of G. A fundamental theorem in geometric invariant theory ensures that if G is reductive then S^G is a graded algebra, finitely generated over k. In particular for affine schemes we have the following.

Theorem 1.1.1. *(Nagata) Let G be a geometrically reductive algebraic group acting rationally on an affine scheme* $\mathrm{Spec}(A)$. *Then* A^G *is a finitely generated* k*-algebra.*

The inclusion $S^G \hookrightarrow S$ induces a rational map

$$\pi : \mathrm{Proj}(S) = Z \dashrightarrow Q := \mathrm{Proj}(S^G),\ z \mapsto (f_0(z), \dots, f_h(z)),$$

where the f_i's are generators of S^G. The open subset

$$Z^{ss} := \{z \in Z \mid f(z) \neq 0 \text{ for some homogeneous nonconstant} f \in S^G\},$$

that is the locus where π is regular, is called the locus of semi-stable points with respect to the action of G. Now it seems natural to view Q as the quotient of Z^{ss} modulo G. However the fibers of π may fail to be equal to the orbits of G, indeed it may happen that there are non-closed orbits and in this case the closed points of Q will not be in bijective correspondence with the orbits of G. Let M_G be the maximum among the dimensions of all G-orbits in Z^{ss}, this discussion leads us to the following definition

$$Z^s := \{z \in Z^{ss} \mid \overline{O_G(z)} \cap Z^{ss} = O_G(z) \text{ and } \dim(O_G(z)) = M_G\}.$$

The subset Z^s is called the set of stable points with respect to the action of G. We expect that the fibers of $\pi_{|Z^s}$ are equal to orbits of G.

Theorem 1.1.2. *(Fundamental Theorem of GIT) Let G be a reductive group acting linearly on a projective scheme* $Z = \mathrm{Proj}(S)$. *The quotient* $Q := \mathrm{Proj}(S^G)$ *is a projective scheme and the morphism*

$$\pi : Z^{ss} \to Q$$

satisfies the following properties:

- *For every* $x, y \in Z^{ss}$, $\pi(x) = \pi(y)$ *if and only if* $\overline{O_G(x)} \cap \overline{O_G(y)} \cap Z^{ss} \neq \emptyset$.
- *(Universal property) If there exists a scheme* Q' *with a* G*-invariant morphism* $\pi' : Z^{ss} \to Q'$, *then there exists a unique morphism* $\varphi : Q \to Q'$ *such that* $\pi' = \psi \circ \pi$.
- *For every* $x, y \in Z^s$, $\pi(x) = \pi(y)$ *if and only if* $O_G(x) = O_G(y)$.

A quotient satisfying the first and the second properties of Theorem 1.1.2 is called a *categorical quotient* and denoted by $Z//G$. If in addition the quotient satisfies the third property then it is called a *geometric quotient* and denoted by Z/G.
The most efficient tool to check stability is probably the so called *numerical criterion for stability*. This criterion reduces the study of the action of a reductive group G to the study of the action of its one-parameter subgroups. Let G be a reductive group acting linearly on $\mathbb{P}(V)$ and let $Z \subset \mathbb{P}(V)$ be a G-invariant subscheme. If G_m denotes k^* with is multiplicative structure and

$$\lambda : G_m \to G$$

is a one-parameter subgroup of G, there exist a basis $\{v_0, ..., v_r\}$ of V and integers $\{w_0, ..., w_r\}$ such that the action of λ on V is given by

$$\lambda(t)v_i = t^{w_i} v_i \; \forall \, t \in G_m, \; 0 \leqslant i \leqslant r.$$

If $v = \sum_{i=0}^r \alpha_i v_i$ the integers n_j such that the α_j do not vanish are called the λ-weights of v. We denote by $z \in Z$ the point corresponding to the vector $v_z \in V$.

Theorem 1.1.3. *(Hilbert-Mumford) The point $z \in Z$ is semi-stable if and only if for any one-parameter subgroup λ of G the λ-weights of v_z are not all positive.*
The point $z \in Z$ is stable if and only if for any one-parameter subgroup λ of G the vector v_z has both positive and negative λ-weights.
The point $z \in Z$ is unstable if and only if there exists a one-parameter subgroup λ of G such that the λ-weights of v_z are all positive.

Construction of $\overline{M}_g$

Fix integers $d \gg 0$, $g \geqslant 3$ and $N = d - g$. Let $\text{Hilb}_N^{P(x)}$ be the Hilbert scheme finely parametrizing the close subschemes of $\mathbb{P}^N$ with Hilbert polynomial $P(x) = dx - g + 1$. There exists a universal family $\mathcal{H}$ with a tautological polarization $\mathcal{L}$

$$\mathcal{L} \to \mathcal{H} \xrightarrow{\pi} \text{Hilb}_N^{P(x)},$$

such that the fiber $X_h := \pi^{-1}(h)$ is isomorphic to the subscheme of $\mathbb{P}^N$ corresponding to $h \in \text{Hilb}_N^{P(x)}$, and $L_h := \mathcal{L}_{|X_h}$ is isomorphic to the line bundle giving the embedding of X_h in $\mathbb{P}^N$.
Let $X \subset \mathbb{P}^N$ be a curve, we want to construct its Hilbert point in $\text{Hilb}_N^{P(x)}$, and consider the exact sequence

$$0 \mapsto \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \mapsto 0.$$

By a theorem due to *J. P. Serre*, for $m >> 0$, we get the following exact sequence in cohomology

$$0 \mapsto H^0(\mathbb{P}^N, \mathcal{I}_X(m)) \to H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(X, \mathcal{O}_X(m)) \mapsto 0.$$

Furthermore it can be proven that there exists an integer $\overline{m}$ such that for any $m \geqslant \overline{m}$ and for any subscheme of $\mathbb{P}^N$ having Hilbert polynomial $P(x)$ the above sequence is exact. This means that the degree m part of the ideal of X, that is $H^0(\mathbb{P}^N, \mathcal{I}_X(m))$, uniquely determines X. We can associate to X a point in the Grassmannian parametrizing $P(m)$-dimensional quotients of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$ and this correspondence is injective. For any $m \geqslant \overline{m}$ we get an embedding

$$\varphi_m : \text{Hilb}_N^{P(x)} \to \mathbb{P}(\bigwedge^{P(m)} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))).$$

We have an action of $SL(N+1)$ on $\mathbb{P}(\bigwedge^{P(m)} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)))$ and any embedding φ_m determines a linearization of the action of $SL(N+1)$ on $\text{Hilb}_N^{P(x)}$. Our aim is to construct $\overline{M}_g$ as a quotient of a suitable subscheme of $\text{Hilb}_N^{P(x)}$.
Translating the *Hilbert-Mumford criterion* 1.1.3 in this setting one gets the following theorem:

Theorem 1.1.4. *If $d \geqslant 20(g-1)$ then there are infinitely many linearizations of the action of $SL(N+1)$ on $\text{Hilb}_N^{P(x)}$ such that*

- *(Mumford-Gieseker) if $X \subset \mathbb{P}^N$ is a smooth, connected, non-degenerate curve of genus g and degree d, then its Hilbert point is stable,*

- *(Gieseker) if* $h \in \mathrm{Hilb}_N^{P(x)}$ *is a* $SL(N+1)$*-semi-stable point then all connected component of* X_h *are Deligne-Mumford semi-stable curves.*

Consider now the case $d = r(2g-2)$ for an integer r and fix once and for all an integer m such that Gieseker-Mumford theorem holds. Consider the following subset of $\mathrm{Hilb}_N^{p(m)\ ss}$

$$H = \{h \in \mathrm{Hilb}_N^{p(m)\ ss} \mid \mathcal{L}_{|X_h} \cong \omega_{X_h}^{\otimes r} \text{ and the curve is connected}\}.$$

The $SL(N+1)$-invariant set H parametrizes only DM-stable curves by Gieseker's theorem. In fact, for $r \geqslant 3$ the dualizing sheaf $\omega_X^{\otimes r}$ is very ample on DM-stable curves and it contracts exactly the destabilizing components of a DM-semi-stable curve.
Finally one can prove that H consists only of $SL(N+1)$-stable points, that it is a closed subscheme of $\mathrm{Hilb}_N^{p(m)\ ss}$ and that the r-th projective canonical model of any stable curve of genus g is an H. At this point it is natural to construct the moduli space of genus g stable curves as the GIT quotient

$$\overline{M}_g := H/SL(N+1).$$

1.2 THE STACK $\overline{\mathcal{M}}_{g,n}$

The study of moduli problems introduces a new kind of objects: the so called moduli stacks. We have seen that a moduli problem gives rise to a functor, if the functor is representable we have a fine moduli space, that is a scheme. Sometimes, if it is not representable one can find a coarse moduli space, which parametrizes the isomorphism classes of our objects over a field, but does not describe all the possible families of objects. It happens that the functor related to a moduli problem is not representable by a scheme. We search for a sort of generalized scheme.
A scheme is constructed out of affine schemes by gluing the isomorphism defined on Zariski open subset. In the same spirit consider a collection of schemes $\{X_i\}$, and for each i, j étale morphisms $Y_{i,j} \to X_i$, $Y_{j,i} \to X_j$ and isomorphisms $\varphi_{i,j} : Y_{i,j} \to Y_{j,i}$, satisfying a cocycle condition for each i, j, k. We glue together the X_i along the $\varphi_{i,j}$. This quotient may not exist in the category of schemes, but it is an *algebraic space*.
Instead of the functor $\mathcal{F}$, which sends any scheme S in the set of isomorphism classes of families $X \to S$, consider a new object $\mathcal{F}$, which to each scheme S assigns the category $\mathcal{F}(S)$ of families and isomorphisms between such families. This object is called a fibered category over the category of schemes. The sheaf axioms for the functor $\mathcal{F}$ are replaced by the *stack axioms* for the fibered category $\mathcal{F}$, which are the following. For any scheme S and any étale covering $\{U_i \to S\}$, consider

$$\mathcal{F}(S) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_S U_j) \rightrightarrows \prod \mathcal{F}(U_i \times_S U_j \times_S U_k).$$

- The fact that the first arrow is injective means that if $a, b \in \mathcal{F}(S)$ and if a_i, b_i are their restriction on $\mathcal{F}(U_i)$, and there is an isomorphism $\varphi_i : a_i \to b_i$ such that for each i, j the isomorphisms φ_i, φ_j restrict to the same isomorphism of $a_{i,j}$ and $b_{i,j}$ on $U_i \times_S U_j$, then there is a unique isomorphism φ inducing φ_i on each U_i.

- The fact that the sequence is exact at the first middle term means that if we give objects $a_i \in \mathcal{F}(U_i)$ for each i and isomorphisms $\varphi_{i,j} : a_i \to a_j$ on $U_i \times_S U_j$ satisfying a cocycle condition on each $U_i \times_S U_j \times_S U_k$, then there exists a unique object $a \in \mathcal{F}(S)$ restricting to each a_i on U_i.

A *Deligne-Mumford stack* is a fibered category $\mathcal{F}$ satisfying the stack axioms, and such that there exists a scheme X and a surjective étale morphism $\mathrm{Hom}(-, X) \to \mathcal{F}$. An *Artin stack* is a

fibered category $\mathcal{F}$ satisfying the stack axioms, and such that there exists a scheme X and a surjective smooth morphism $\mathrm{Hom}(-, X) \to \mathcal{F}$.
The moduli space of curves $\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack for any $g \geqslant 2$. In the paper *The irreducibility of the space of curves of given genus* [DM], Deligne and Mumford introduced stacks for the first time, they compactified the stack $\mathcal{M}_g$ adding stable curves, and they proved its irreducibility in any characteristic.

We define a family of pointed curves of genus g parametrized by a scheme S as an object

$$\begin{tikzcd} C \arrow[d, "\pi"'] \\ S \arrow[u, bend right, "\sigma_1,\dots,\sigma_n"'] \end{tikzcd}$$

where π is a flat and proper morphism, σ_i is a section of π for any $i = 1, \dots, n$, $C_s = \pi^{-1}(s)$ is a nodal connected curve of arithmetic genus g and $\sigma_i(s)$ are distinct smooth points for any $s \in S(k)$.
A morphism between two families $C \to S$, $C' \to S$ over S is a morphism of schemes $\varphi : C \to C'$ such that the following diagrams

$$\begin{tikzcd} C \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & C' \arrow[dl, "\pi'"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\varphi"] & & C' \\ & S \arrow[ul, "\sigma_i"] \arrow[ur, "\sigma_i'"'] & \end{tikzcd}$$

commute. We consider the pseudofunctor

$$\mathfrak{M}_{g,n} : \mathfrak{Sch} \longrightarrow \mathfrak{Groupoids}$$

mapping a scheme S to the groupoid $\mathfrak{M}_{g,n}(S)$ whose objects are the families parametrized by S and whose morphisms are the isomorphisms between these families. A curve $(C, x_1, \dots, x_n) \in \mathrm{Obj}(\mathfrak{M}_{g,n}(\mathrm{Spec}(k)))$ is called a *pre-stable genus g curve*. We denote by $\mathfrak{M}_{g,n}$ the stack associated to this pseudofunctor.

Remark 1.2.1. The stack $\mathfrak{M}_{g,n}$ is never a DM-algebraic stack. It contains points representing curves with automorphism groups of positive dimension. Take a smooth curve $(C, x_1, \dots, x_n) \in \mathrm{Obj}(\mathfrak{M}_{g,n}(\mathrm{Spec}(k)))$ and consider $(C', x_1', \dots, x_n')$ where $C' := C \cup \mathbb{P}^1$, $x_i' := x_i$ for $i < n$ and $x_n' := \infty \in \mathbb{P}^1$. Then C' is a nodal connected curve of arithmetic genus $p_a(C') = g$, but $\dim(\mathrm{Aut}(C')) = 1$.

Definition 1.2.2. A pre-stable genus g curve $(C, x_1, \dots, x_n)$ with n marked points is called *stable* if one of the following equivalent conditions are satisfied

- $\mathrm{Aut}(C, x_1, \dots, x_n)$ is étale;
- $\mathrm{Aut}(C, x_1, \dots, x_n)$ is finite;
- Let $\tilde{C} \to C$ be the normalization of C. For any irreducible component $\tilde{C_i}$ of $\tilde{C}$ the inequality $2g(\tilde{C_i}) - 2 + n_i > 0$ holds, where n_i is the number of special points on $\tilde{C_i}$, that are points mapped to a node or to a marked point on C.

We define $\overline{\mathcal{M}}_{g,n}$ in the same way of the stack $\mathfrak{M}_{g,n}$ but adding the stability condition on the fibers. Clearly we have a natural morphism $\overline{\mathcal{M}}_{g,n} \to \mathfrak{M}_{g,n}$ and if $2g-2+n>0$ there is a morphism $\mathcal{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. Both these morphisms are open embeddings.
On the other hand we can construct a category fibered in groupoids in the following way. Let $g, n \in \mathbb{Z}$ such that $g, n \geqslant 0$ and $2g-2+n>0$. We define a category $\mathfrak{M}_{g,n}$ over the category of schemes in the following way. Obj($\mathfrak{M}_{g,n}$) consists of families

$$\begin{tikzcd}
C \arrow[d, shift right, "\pi"'] \\
S \arrow[u, bend right, "\sigma_1,\ldots,\sigma_n"']
\end{tikzcd}$$

where π is a flat and proper morphism, σ_i is a section of π for any $i = 1, \ldots, n$, $C_s = \pi^{-1}(s)$ is a smooth connected curve of genus g and $\sigma_i(s)$ are distinct smooth points for any $s \in S(k)$. A morphism between two objects $C \to S$ and $C^{'} \to S^{'}$ is a couple $(\overline{f}, f)$ where $\overline{f} : C \to C^{'}$ and $f : S \to S^{'}$ are morphisms of schemes and the following diagrams

$$\begin{tikzcd}
C \arrow[r, "\overline{f}"] \arrow[d, "\pi"'] & C^{'} \arrow[d, "\pi^{'}"] \\
S \arrow[r, "f"'] & S^{'}
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[r, "\overline{f}"] & C^{'} \\
S \arrow[r, "f"'] \arrow[u, "\sigma_i"] & S^{'} \arrow[u, "\sigma_i{'}"']
\end{tikzcd}$$

commute. This category is called the *category of* n*-pointed genus* g *smooth curves*. The category $\mathfrak{M}_{g,n}$ is a category fibered in groupoids over the category of schemes and this remains true even if the inequality $2g-2+n>0$ does not hold. One can prove that in this category morphisms are a sheaf and that every descend datum is effective.

Theorem 1.2.3. *The category fibered in groupoids* $\mathfrak{M}_{g,n}$ *is a stack.*

Proof. Consider a scheme S and two families ξ and $\xi^{'}$

$$\begin{tikzcd}
C \arrow[d, shift right, "\pi"'] \\
S \arrow[u, bend right, "\sigma_1,\ldots,\sigma_n"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C^{'} \arrow[d, shift right, "\pi^{'}"'] \\
S \arrow[u, bend right, "\sigma_1^{'},\ldots,\sigma_n^{'}"']
\end{tikzcd}$$

parametrized by S. We define a functor

$$F : \mathfrak{Sch}/S \longrightarrow \mathfrak{Sets}$$

sending $f : X \to S$ to $\mathrm{Mor}(f^*\xi, f^*\xi^{'})$. By applying the universal property of the fiber product we get the following diagrams

$$\begin{tikzcd}
X \arrow[rrd, bend left, "\sigma_i \circ f"] \arrow[rd, "\sigma_{i,X}"] \arrow[rdd, bend right, "Id_X"'] & & \\
& C_X := C \times_S X \arrow[r] \arrow[d, "\pi_X"'] & C \arrow[d, "\pi"] \\
& X \arrow[r, "f"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rrd, bend left, "\sigma_i^{'} \circ f"] \arrow[rd, "\sigma_{i,X}^{'}"] \arrow[rdd, bend right, "Id_X"'] & & \\
& C_X^{'} := C^{'} \times_S X \arrow[r] \arrow[d, "\pi_X^{'}"'] & C^{'} \arrow[d, "\pi^{'}"] \\
& X \arrow[r, "f"] & S
\end{tikzcd}$$

To give a morphism $f^*\xi \to f^*\xi^{'}$ is equivalent to giving a morphism $\tilde{f} : C_X \to C^{'}_X$ such that $\sigma_{i,X} = \sigma^{'}_{i,X} \circ \tilde{f}$, $\pi_X = \pi^{'}_X \circ \tilde{f}$, and $\tilde{f}$ makes the diagram over the identity cartesian. That is $\tilde{f}$ is an isomorphism. Now, let $\{X_i \to X\}$ be an étale cover, and consider isomorphisms $\tilde{f}_i : C_{X_i} \to C^{'}_{X_i}$ such that $\tilde{f}_{i|C_{X_{i,j}}}$ and $\tilde{f}_{j|C_{X_{i,j}}}$ are naturally isomorphic. Since $\{C_{x_i} \to C_X\}$ is an étale cover and morphisms form a sheaf in the étale topology, the $\tilde{f}_i$ glue to a morphism $\tilde{f} : C_X \to C^{'}_X$. The morphism $\tilde{f}$ commutes with $\pi_X, \sigma_{X,i}, \pi^{'}_X, \sigma^{'}_{X,i}$, since this is true for the $\tilde{f}_i$ and morphisms are a sheaf in the étale topology. Furthermore we can define $\tilde{g}^{-1}$ étale locally and then glue. This proves that morphisms are a sheaf.

Now, let S be a scheme, $\{S_i \to S\}$ an étale cover, ξ_i objects $C_i \to S_i$, and $\varphi_{i,j} : C_{i|S_{i,j}} \to C_{j|S_{i,j}}$ isomorphisms. Using the $\varphi_{i,j}$ we can glue the ξ_i to a global ξ over S, by descent theory we obtain a morphism $\pi : C \to S$. To construct the sections consider the composition

$$\begin{tikzcd} S_i \arrow[r, "\sigma_{S_{i,j}}"] & C_i \arrow[r] & C \end{tikzcd}$$

which agree locally and glue to define global sections $\sigma_{i,S} : S \to C$. Since $\{S_i \to S\}$ is an étale cover, and the ground field is algebraically closed, any morphism $\mathrm{Spec}(K) \to S$ factors through at least one of the $S_i \to S$. Then the fibers of π are genus g connected curves. Finally, since smoothness and properness are local in the target even in the Zariski topology the morphism π is smooth and proper. This proves that every descent datum is effective. ☐

Lemma 1.2.4. *Let* $(C, \{x_1, ..., x_n\})$ *be a* n*-pointed genus* g *pre-stable curve. The sheaf* $\omega_C(x_1 + ... + x_n)$ *is ample if and only if* $(C, \{x_1, ..., x_n\})$ *is stable.*

Proof. An invertible sheaf $\mathcal{L}$ on a proper curve C is ample if and only if it has positive degree on every irreducible component of C. Let C_i be an irreducible component of C. We have $\deg(\omega_C(x_1 + ... + x_n)_{|C_i|}) = \deg(\omega_{C|C_i}) + m_{C_i} = \deg(\omega_{C_i}) + \sharp(C_i \cap C_i^c) + m_{C_i} = 2p_a(C_i) - 2 + \sharp(C_i \cap C_i^c) + m_{C_i} = 2p_a(C_i) - 2 + n_{C_i}$, where m_{C_i}, n_{C_i} are respectively the number of marked and special points on C_i. Now, $\deg(\omega_C(x_1 + ... + x_n)_{|C_i|}) > 0$ for any i if and only if $2p_a(C_i) - 2 + n_{C_i} > 0$ for any i if only if $(C, \{x_1, ..., x_n\})$ is stable. ☐

Definition 1.2.5. Let X be a scheme, and G be a group scheme acting on X. The quotient stack $[X/G]$ is defined as the category whose objects are of the type

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \\ S & \end{tikzcd}$$

where $P \to S$ is a principal G-bundle, $P \to X$ is a G-equivariant morphism, and whose morphisms are isomorphisms of principal G-bundle commuting with maps to X.

Let $\pi : C \to S$ be a family of stable curves of genus g. By Lemma 1.2.4 the relative dualizing sheaf $\omega_{C/S}$ is relatively ample. The r-th power $\omega_{C/S}^{\otimes r}$ is relatively ample, and $\pi_*\omega_{C/S}^{\otimes r}$ is locally free of rank $N + 1 = h^0(\omega_{C/S}^{\otimes r}) = (2r - 1)(g - 1)$ on S. Therefore any genus g stable curve can be embedded in $\mathbb{P}^N$ using the sections of $\omega_{C/S}^{\otimes r}$. The Hilbert polynomial of such a curve is determined by $\deg(P) = 1, P(0) = 1 - g, P(1) = \chi(\omega_{C/S}^{\otimes r})$. We can write $P(z) = Az + B$, then $P(0) = B = 1 - g$, and $P(1) = A = \chi(\omega_{C/S}^{\otimes r})$. Then

$$P(z) = (2rz - 1)(g - 1).$$

Let $\mathrm{Hilb}^P(\mathbb{P}^N)$ be the Hilbert scheme parametrizing subschemes of $\mathbb{P}^N$ with Hilbert polynomial P. There is a closed subscheme H of $\mathrm{Hilb}^P(\mathbb{P}^N)$ parametrizing m-canonically embedded stable curves. To give a morphism $S \to H$ is equivalent to give a closed subscheme $i : C \hookrightarrow \mathbb{P}^N \times S$ such that the projection $\pi : C \to S$ is a family of genus g stable curves, and there exists an isomorphism $\varphi : \mathbb{P}(\pi_* \omega_{C/S}^{\otimes r}) \to \mathbb{P}^N \times S$ making the diagram

$$\begin{tikzcd}
C \arrow[rr, "\varphi"] \arrow[dr, "i"'] & & \mathbb{P}(\pi_* \omega_{C/S}^{\otimes r}) \arrow[dl] \\
& \mathbb{P}^N \times S &
\end{tikzcd}$$

commutative. Finally there is a natural action of $\mathrm{Aut}(\mathbb{P}^N) = PGL(N+1)$ on H given by

$$PGL(N+1) \times H \to H, \ (\sigma, \alpha : C \hookrightarrow \mathbb{P}^N \times S) \mapsto (\sigma^{-1} \circ \alpha : C \hookrightarrow \mathbb{P}^N \times S).$$

Theorem 1.2.6. *For* $g \geqslant 2$ *there is an equivalence of stacks*

$$\overline{\mathcal{M}}_g \cong [H/PGL(N+1)].$$

Proof. Let $\pi : C \to S$ be a family of genus g stable curves. We have a canonical projective bundle $P_\pi := \mathbb{P}(\pi_* \omega_{C/S}^{\otimes r}) \to S$. Let $E := \mathrm{Isom}_S(P_\pi, \mathbb{P}^N_S)$ be the S-scheme parametrizing isomorphisms from P_π to $\mathbb{P}^N_S$. The group $PGL(N+1)$ acts on E by

$$PGL(N+1) \times E \to E, \ (\sigma, \varphi) \mapsto \sigma^{-1} \circ \varphi.$$

and E is a $PGL(N+1)$-principal bundle. Now, consider the pull-back

$$\begin{tikzcd}
C_E = C \times_S E \arrow[r, "\pi_E"] \arrow[d] & E \arrow[d] \\
C \arrow[r, "\pi"] & S
\end{tikzcd}$$

since the projection $E \times_S E \to E$ has a section $\Delta : E \to E \times E$, the $\mathbb{P}^N$-bundle $P_{\pi_E} := \mathbb{P}(\pi_{E*} \omega_{C_E/E}^{\otimes m})$ is trivial, and we have an isomorphism $\xi_E : \mathbb{P}_{\pi_E} \to \mathbb{P}^N_S \times_S E$. Let $i_E : C_E \to \mathbb{P}_{\pi_E}$ be the canonical embedding, the composition $\xi_E \circ \grave{\imath}_E : C_E \to \mathbb{P}^N_S \times_S E$ gives a family of stable curves in $\mathbb{P}^N$, corresponding to a morphism $f_\pi : E \to H$, which clearly is $PGL(N+1)$-equivariant.
Now, consider a morphism

$$\begin{tikzcd}
C' \arrow[r, "\varphi"] \arrow[d, "\pi'"'] & C \arrow[d, "\pi"] \\
S' \arrow[r, "\psi"] & S
\end{tikzcd}$$

in $\overline{\mathcal{M}}_g$. We have a canonical isomorphism $\pi'_* \omega_{C'/S'} \cong \varphi^* \pi_* \omega_{C/S}$ and two cartesian squares

$$\begin{tikzcd}
\mathbb{P}(\omega_{C'/S'}^{\otimes m}) \arrow[r] \arrow[d] & \mathbb{P}(\omega_{C/S}^{\otimes m}) \arrow[d] & E' \arrow[r, "f_{\varphi'}"] \arrow[d] & E \arrow[d] \\
S' \arrow[r, "\psi"] & S & S' \arrow[r, "\varphi"] & S
\end{tikzcd}$$

where $f_{\varphi'}$ is compatible with f_π and $f_{\pi'}$. Then we get the following:

- an objects $\pi : C \to S$ to

$$\begin{tikzcd}
E \arrow[r, "f_{\pi}"] \arrow[d] & H \\
S &
\end{tikzcd}$$

- a morphism

$$\begin{tikzcd}
C' \arrow[r, "\varphi"] \arrow[d, "\pi'"'] & C \arrow[d, "\pi"] \\
S' \arrow[r, "\psi"] & S
\end{tikzcd}$$

  to a morphism

$$\begin{tikzcd}
E' \arrow[r, "f_{\varphi'}"] \arrow[d] & E \arrow[d] \\
S' \arrow[r, "\varphi"] & S
\end{tikzcd}$$

This defines a morphism of stacks

$$F : \overline{\mathcal{M}}_g \to [H/PGL(N+1)].$$

On the other hand given a morphism $S \to H$ we have a corresponding family $\pi_S : C \to S$ of genus g stable curves embedded in $\mathbb{P}^N_S$. By forgetting the embedding $C \hookrightarrow \mathbb{P}^N_S$ we obtain an object in $\overline{\mathcal{M}}_g$, furthermore morphisms in the same $PGL(N+1)$-orbit are sent to the same object of $\overline{\mathcal{M}}_g$. So we get a morphism

$$G : [H/PGL(N+1)] \to \overline{\mathcal{M}}_g.$$

Take an object $\xi := (E'/S \to H)$ in $[H/PGL(N+1)]$, and let $\tilde{\pi}_{E'} : C' \to E'$ be the family induced by the $PGL(N+1)$-equivariant morphism $E' \to H$. If $\mathcal{H} \to H$ is the universal family then $\tilde{\pi}_{E'} : C' \to E'$ is the pull-back of $\mathcal{H} \to H$ by the morphism $E' \to H$. Furthermore if $E \to E'$ we can consider the pull-back $\tilde{C}_E \to E$ and the following diagram

$$\begin{tikzcd}
\tilde{C}_E \arrow[r] \arrow[d] & \tilde{C} \arrow[r] \arrow[d, "\tilde{\pi}_{E'}"] & \mathcal{H} \arrow[d] \\
E \arrow[r] & E' \arrow[r] \arrow[d] & H \\
 & S &
\end{tikzcd}$$

The scheme $\tilde{C}_E$ carries a natural $PGL(N+1)$-action. By descent theory $C = \tilde{C}_E/PGL(N+1)$ exists as a scheme, and there is a morphism $\pi : C \to S$ such that the base extension $\pi_{E'} : C \times_S E' \to E'$ is exactly $\tilde{\pi}_{E'} : \tilde{C} \to E'$:

$$\begin{tikzcd}
\tilde{C} \arrow[r, "\sim"] \arrow[dr, "\tilde{\pi}_{E'}"'] & C \times_S E' \arrow[r] \arrow[d, "\pi_{E'}"] & C \arrow[d, "\pi"] \\
 & E' \arrow[r] & S
\end{tikzcd}$$

The family $\pi : C \to S$ is exactly $G(\xi) \in \overline{\mathcal{M}}_g$. If $E = \mathrm{Isom}_S(P_\pi, \mathbb{P}^N_S)$ where $P_\pi = \mathbb{P}(\pi_* \omega^{\otimes m}_{C/S})$ we get that $F \circ G(\xi)$ is isomorphic to ξ, that is $F \circ G \cong \mathrm{Id}$. Finally, from the construction it is clear that $G \circ F \cong \mathrm{Id}$. ☐

Proposition 1.2.7. *For any* $g \geqslant 2$ *the stack* $\overline{\mathcal{M}}_g$ *is a Deligne-Mumford stack.*

Proof. Since a genus $g \geqslant 2$ stable curve over an algebraically closed field has a finite and reduced automorphism group the stabilizers of the geometric points of $\overline{\mathcal{M}}_g$ are finite and reduced. So $\overline{\mathcal{M}}_g$ is a DM stack. ☐

1.3 DETAILS ON ALGEBRAIC CURVES

In this section we recall some well known results on algebraic curves and their automorphisms. Finally, using deformation theory we prove that $\overline{\mathcal{M}}_g$ is as smooth stack.

Curves of Genus Zero

There is only one smooth curve of genus $g = 0$ over an algebraically closed field k, namely $\mathbb{P}^1_k$. A family of curves of genus zero over a scheme S is a scheme X, smooth and projective over S, whose fibers are curves of genus zero.

Proposition 1.3.1. *The space* $M = \mathrm{Spec}(k)$ *is a coarse moduli scheme for curves of genus zero. Furthermore it has a tautological family.*

Proof. The set $\mathrm{Hom}(\mathrm{Spec}(k), \mathrm{Spec}(k))$ consists of a single element and clearly is in a one-to-one correspondence with the set of families over $\mathrm{Spec}(k)$ that consists of the family $\mathbb{P}^1_k \to \mathrm{Spec}(k)$. Clearly $\mathbb{P}^1_k \to \mathrm{Spec}(k)$ is a tautological family. If $X \to S$ is a family there is a unique morphism $S \to M = \mathrm{Spec}(k)$, in this way we get the functorial morphism $\alpha : \mathcal{F} \to \mathrm{Hom}(-, M)$.
Now suppose that $\beta : \mathcal{F} \to \mathrm{Hom}(-, N)$ is another morphism of functors. In particular the family $\mathbb{P}^1_k \to M$ determines a morphism $e \in \mathrm{Hom}(M, N)$. Let $X \to S$ a family over a scheme S of finite type over k. For any closed point $s \in S$ the fiber is $X_s \cong \mathbb{P}^1$, then any closed point s goes to the point $n = e(M) \in N$. Now the restriction of the family on S to an Artin closed subscheme of S is trivial, so factor through $\mathrm{Spec}(k)$. We conclude that the morphism β factors through α. ☐

Clearly the tautological family is $\mathbb{P}^1 \to \mathrm{Spec}(k)$, that is the unique family over $M = \mathrm{Spec}(k)$. Suppose $M = \mathrm{Spec}(k)$ to be a fine moduli space for the curves of genus zero. Then the universal family is $\mathbb{P}^1 \to \mathrm{Spec}(k)$. Since any other family is obtained by base extension from the universal family it must be trivial i.e. of the form $\mathbb{P}^1 \times_k S \to S$. But the ruled surfaces provide an example of non trivial families of curves of genus zero.
Consider for instance the blow up $Bl_p\mathbb{P}^2$ of $\mathbb{P}^2$ is a point p. The projection $\pi : Bl_p\mathbb{P}^2 \to \mathbb{P}^1$ makes $Bl_p\mathbb{P}^2$ into a ruled surface, but it is not a product. Note that $\mathrm{Pic}(Bl_p\mathbb{P}^2) = \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, but on $Bl_p\mathbb{P}^2$ we have a (-1)-curve, the exceptional divisor. Suppose that there is a (-1)-curve $C = (a, b)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We have $C^2 = (aL + bR)(aL + bR) = 2ab = -1$, a contradiction.

Definition 1.3.2. A *pointed* curve of genus zero over k is a curve of genus zero with a choice of a k-rational point. A family of pointed curves of genus zero is a flat family $X \xrightarrow{\pi} S$, whose geometric fibers are curves of genus zero, with a section $\sigma : S \to X$.

The fact that $\sigma : S \to X$ is a section means that $\pi \circ \sigma = \mathrm{Id}_S$. Then for any point $s \in S$ the image $\sigma(s)$ is a point of the fiber $X_s \cong \mathbb{P}^1$ over s. The section σ is sometimes called an S-point

of X.
A way to obtain a fine moduli space for the curves of genus zero is to rigidify the curves by taking three distinct points. We know that there is a unique automorphism of $\mathbb{P}^1$ that fixed three distinct points, namely the identity. Consider the families of curves of genus zero with three marked points i.e. the families of $X \to S$, whose fibers are curves of genus zero, with three sections $\sigma_1, \sigma_2, \sigma_3 : S \to X$, such that on each fiber the sections have distinct support. Since a curve X of genus zero with three marked points is rigid i.e. $\mathrm{Aut}(X) = \{\mathrm{Id}_X\}$, the corresponding functor is representable by $M = \mathrm{Spec}(k)$ and the universal family is $\mathbb{P}^1 \to \mathrm{Spec}(k)$ with three distinct points, say $[0:1], [1:0], [1:1]$.

Grothendieck Spectral Sequence

We begin recalling the notion of *five terms exact sequence* or *exact sequence of low degree terms* associated to a spectral sequence. Let

$$E_2^{h,k} \Longrightarrow H^n(A)$$

be a spectral sequence whose terms are non trivial only for $h, k \geqslant 0$. Then this is an exact sequence

$$0 \mapsto E_2^{1,0} \to H^1(A) \to E_2^{0,1} \to E_2^{2,0} \to H^2(A).$$

The *Grothendieck spectral sequence* is an algebraic tool to express the derived functors of a composition of functors $\mathcal{G} \circ \mathcal{F}$ in terms of the derived functors of $\mathcal{F}$ and $\mathcal{G}$.
Let $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ and $\mathcal{G} : \mathcal{C}_2 \to \mathcal{C}_3$ be two additive covariant functors between abelian categories. Suppose that $\mathcal{G}$ is left exact and that $\mathcal{F}$ takes injective objects of $\mathcal{C}_1$ in $\mathcal{G}$-acyclic objects of $\mathcal{C}_2$. Then there exists a spectral sequence for any object A of $\mathcal{C}_1$

$$E_2^{h,k} = (R^h\mathcal{G} \circ R^k\mathcal{F})(A) \Longrightarrow R^{h+k}(\mathcal{G} \circ \mathcal{F})(A).$$

The corresponding exact sequence of low degrees is the following

$$0 \mapsto R^1\mathcal{G}(\mathcal{F}(A)) \to R^1(\mathcal{G}\mathcal{F}(A)) \to \mathcal{G}(R^1\mathcal{F}(A)) \to R^2\mathcal{G}(\mathcal{F}(A)) \to R^2(\mathcal{G}\mathcal{F})(A).$$

As a special case of the Grothendieck spectral sequence we get the *Leray spectral sequence*. Let $f : X \to Y$ be a continuous map between topological spaces. We take $\mathcal{C}_1 = \mathfrak{Ab}(X)$ and $\mathcal{C}_2 = \mathfrak{Ab}(Y)$ to be the categories of sheaves of abelian groups over X and Y respectively. Then we take $\mathcal{F}$ to be the direct image functor $f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ and $\mathcal{G} = \Gamma_Y : \mathfrak{Ab}(Y) \to \mathfrak{Ab}$ to be the global section functor, where $\mathfrak{Ab}$ is the category of abelian groups. Note that

$$\Gamma_Y \circ f_* = \Gamma_X : \mathfrak{Ab}(X) \to \mathfrak{Ab}$$

is the global section functor on X. By Grothendieck's spectral sequence we know that $(R^h\Gamma_Y \circ R^k f_*)(\mathcal{E}) \Longrightarrow R^{h+k}(\Gamma_Y \circ f_*)(\mathcal{E}) = R^{h+k}\Gamma_X(\mathcal{E})$ for any $\mathcal{E} \in \mathfrak{Ab}(X)$, that is

$$H^h(Y, R^k f_*\mathcal{E}) \Longrightarrow H^{h+k}(X, \mathcal{E}).$$

The exact sequence of low degrees looks like

$$0 \mapsto H^1(Y, f_*\mathcal{E}) \to H^1(X, \mathcal{E}) \to H^0(Y, R^1 f_*\mathcal{E}) \to H^2(Y, f_*\mathcal{E}) \to H^2(X, \mathcal{E}).$$

Finally we work out the *spectral sequence of Ext functors*. Let $\mathcal{E} \in \mathfrak{Coh}(X)$ be a coherent sheaf on a scheme X. Consider the functor

$$\mathcal{H}om(\mathcal{E}, -) : \mathfrak{Coh}(X) \to \mathfrak{Coh}(X),\ \mathcal{Q} \mapsto \mathcal{H}om(\mathcal{E}, \mathcal{Q}),$$

and the global section functor

$$\Gamma_X : \mathfrak{Coh}(X) \to \mathfrak{Ab},\ \mathcal{Q} \mapsto \Gamma_X(\mathcal{Q}).$$

Note that $\Gamma_X \circ \mathcal{H}om(\mathcal{E}, -) = \mathrm{Hom}(\mathcal{E}, -)$. By Grothendieck spectral sequence we have $(R^h\Gamma_X \circ R^k\mathcal{H}om(\mathcal{E}, -))(\mathcal{Q}) \Longrightarrow R^{h+k}(\mathrm{Hom}(\mathcal{E}, -)(\mathcal{Q})$ for any $\mathcal{Q} \in \mathfrak{Coh}(X)$, that is

$$H^h(X, \mathcal{E}xt^k(\mathcal{E}, \mathcal{Q})) \Longrightarrow \mathrm{Ext}^{h+k}(\mathcal{E}, \mathcal{Q}).$$

The corresponding sequence of low degrees is

$$0 \mapsto H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{Q})) \to \mathrm{Ext}^1(\mathcal{E}, \mathcal{Q}) \to H^0(X, \mathcal{E}xt^1(\mathcal{E}, \mathcal{Q})) \to H^2(X, \mathcal{H}om(\mathcal{E}, \mathcal{Q})) \to \mathrm{Ext}^2(\mathcal{E}, \mathcal{Q}).$$

Deformations of Schemes

Let X be a smooth scheme of finite type over k. We define the deformation functor $\mathrm{Def}_X : \mathfrak{Art} \to \mathfrak{Sets}$ of X sending an Artin ring A to the set of couples $(X_A \overset{\pi_A}{\to} \mathrm{Spec}(A), \varphi)$ modulo isomorphism, where π_A is a smooth morphism, $\varphi : X \to X_0$ is an isomorphism, X_0 is defined by the cartesian diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_A \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(A)
\end{tikzcd}$$

and (X_A, φ), (X'_A, φ') are isomorphic if there is an isomorphism $\alpha : X_A \to X'_A$ such that the diagram

$$\begin{tikzcd}
X_A \arrow[rr, "\alpha"] \arrow[dr, "\pi_A"'] & & X'_A \arrow[dl, "\pi'_A"] \\
& \mathrm{Spec}(A) &
\end{tikzcd}$$

commutes and $\varphi' = \alpha \circ \varphi$.

Theorem 1.3.3. *For any semi-small exact sequence* $0 \mapsto I \to A \to B \mapsto 0$ *in* $\mathfrak{Art}$, *let* $T^i\mathrm{Def}_X = H^i(X, T_X)$, *then*

1. *there exists a functorial exact sequence*
$$T^1\mathrm{Def}_X \otimes I \to \mathrm{Def}_X(A) \to \mathrm{Def}_X(B) \to T^2\mathrm{Def}_X \otimes I;$$
2. *for any* $(X_A, \pi_A, \varphi) \in \mathrm{Def}_X(A)$, *let* $G = \mathrm{Stab}(X_A) \subseteq T^1\mathrm{Def}_X \otimes I$, *we have a functorial exact sequence*
$$0 \mapsto T^0\mathrm{Def}_X \otimes I \to \mathrm{Aut}(X_A) \to \mathrm{Aut}(X_B) \to G \mapsto 0.$$

Now let X be any scheme over k. Consider the exact sequence of low degree for Ext functors with sheaves Ω_X and $\mathcal{O}_X$. We have

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \to H^2(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)).$$

The set of deformations of X over the dual numbers $D = \frac{k[\epsilon]}{\epsilon^2}$ is in one-to-one correspondence with the group $\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)$. Then we get the sequence

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to \mathrm{Def}_X(D) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \to H^2(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)).$$

Let X be a smooth scheme and let Y be a closed subscheme with ideal sheaf $\mathcal{I}$. We have an exact sequence of sheaves

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0,$$

where the first map is the differential. Furthermore Y is smooth if and only if

- Ω_Y is locally free,
- the sequence is also exact on the left

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

In this case the sheaf $\mathcal{I}$ is locally generated by $\mathrm{Codim}(Y, X)$ elements, and its is locally free of rank $\mathrm{Codim}(Y, X)$ on Y.

Remark 1.3.4. Let $Y \subseteq X$ be an hypersurface not necessarily smooth. We can associate to Y a Cartier divisor $\{(\mathcal{U}_i, f_i)\}$, and the ideal sheaf $\mathcal{I}$ is locally generated by f_i on $\mathcal{U}_i$. Furthermore $\mathcal{O}_X(Y)$ is the sheaf locally generated by f_i^{-1} on $\mathcal{U}_i$. We conclude that $\mathcal{O}_X(-Y) \cong \mathcal{I}$ is locally free. If $Y \subseteq X$ is a reduced hypersurface, then $\mathcal{I}$ is locally free of rank one. We have the differential $d : \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y$, if f is a local generator of $\mathcal{I}$ then df is a local generator of $\mathrm{Im}(d)$, since Y is reduced then $df \neq 0$, $\mathrm{Im}(d)$ is locally free of rank one, and the map d is injective. So we have again an exact sequence

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

Let $f = f(x_1, ..., x_n)$, with $n = \dim(X)$, be a local equation for Y in X. Then $df = \frac{\partial f}{\partial x_1} dx_1 + ... + \frac{\partial f}{\partial x_n}$. Since Y is reduced the differential is injective, furthermore $\mathcal{I}/\mathcal{I}^2$ is locally free of rank one and $\Omega_X \otimes \mathcal{O}_Y$ is locally free of rank n. Applying $\mathrm{Hom}(-, \mathcal{O}_Y)$ to the sequence

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0,$$

we obtain

$$0 \mapsto \mathrm{Hom}(\Omega_Y, \mathcal{O}_Y) \to \mathrm{Hom}(\Omega_{X|Y}, \mathcal{O}_Y) \to \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \to \mathrm{Ext}^1(\Omega_Y, \mathcal{O}_Y) \to \mathrm{Ext}^1(\Omega_{X|Y}, \mathcal{O}_Y).$$

Remark 1.3.5. Let X be a noetherian scheme such that any coherent sheaf on X is quotient of a locally free sheaf i.e. Coh(X) has enough locally free objects. We define the homological dimension of $\mathcal{F} \in \mathrm{Coh}(X)$, denoted by $\mathrm{hd}(\mathcal{F})$, to be the least length of a locally free resolution of $\mathcal{F}$ or ∞ if there is no finite one. Clearly $\mathcal{F}$ is locally free if and only if $\mathrm{hd}(\mathcal{F}) = 1$ if and only if $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$ far any $\mathcal{G} \in \mathrm{Mod}(X)$. Furthermore $\mathrm{hd}(\mathcal{F}) \leqslant n$ if and only if $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = 0$ for any $i > n$ and $\mathcal{G} \in \mathrm{Mod}(X)$. Finally $\mathrm{hd}(\mathcal{F}) = \mathrm{Sup}_{x \in X}(\mathrm{pd}_{\mathcal{O}_x} \mathcal{F}_x)$, where pd is the projective dimension.

In our case $\Omega_{X|Y}$ is locally free, and by the preceding remark $\mathrm{Ext}^1(\Omega_{X|Y}, \mathcal{O}_Y) = 0$. Then we get the exact sequence

$$0 \mapsto \mathrm{Hom}(\Omega_Y, \mathcal{O}_Y) \to \mathrm{Hom}(\Omega_{X|Y}, \mathcal{O}_Y) \to \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \to \mathrm{Ext}^1(\Omega_Y, \mathcal{O}_Y) \mapsto 0.$$

Consider now the special case $X = \mathbb{A}^n$ and $Y = \mathrm{Spec}(A)$, where $A = k[x_1, ..., x_n]/(f)$. The map $\mathrm{Hom}(\Omega_{\mathbb{A}^n|Y}, \mathcal{O}_Y) \to \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is the transpose of the differential $d : \mathcal{I}/\mathcal{I}^2 \to$

$\Omega_{\mathbb{A}^n|Y}$. Furthermore $\mathrm{Hom}(\Omega_{\mathbb{A}^n|Y}, \mathcal{O}_Y) \cong A^n$ and $\mathrm{Hom}(\mathcal{I}/\mathcal{I}^2) \cong A$. We can write the map $\mathrm{Hom}(\Omega_{\mathbb{A}^n|Y}, \mathcal{O}_Y) \to \mathrm{Hom}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ as

$$\varphi : A^n \to A,\ (\alpha_1, ..., \alpha_n) \mapsto \alpha_1 \frac{\partial f}{\partial x_1} + ... + \alpha_n \frac{\partial f}{\partial x_n}.$$

We rewrite our exact sequence as

$$0 \mapsto \mathrm{Hom}(\Omega_Y, \mathcal{O}_Y) \to A^n \to A \to \mathrm{Ext}^1(\Omega_Y, \mathcal{O}_Y) \mapsto 0.$$

Then $\mathrm{Im}(\varphi) = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}) \subseteq A$, and $\mathrm{Ext}^1(\Omega_Y, \mathcal{O}_Y) \cong A/(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$.
Now let $Y = C \subseteq \mathbb{A}^2$ be a nodal curve. In an étale neighborhood of the node we can assume $C = \mathrm{Spec}(A)$, where $A = k[x, y]/(xy)$. From the preceding discussion we get $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \cong A/(x, y) \cong k$. So $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)_p = 0$ if p is a smooth point of C and $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)_p = k$ if $p \in \mathrm{Sing}(C)$. Furthermore

$$\mathcal{E}xt^1(\Omega_C, \mathcal{O}_X) \cong \sum_{p \in \mathrm{Sing}(C)} \mathcal{O}_p.$$

Curves of Genus One

An *elliptic curve* over an algebraically closed field is a smooth projective curve of genus one. Let X be an elliptic curve and let $P \in X$ be a point, consider the linear system $|2P|$ on X. Since the curve is not rational $|2P|$ has no base points, and since $\deg(K - 2P) = 2g - 2 - 2 = -2 < 0$ the divisor $|2P|$ is non-special i.e. $h^0(K - 2P) = 0$. By Riemann-Roch theorem $h^0(2P) = \deg(2P) - g + 1 = 2$. Then the linear system $|2P|$ defines a morphism $f : X \to \mathbb{P}^1$ of degree 2 on $\mathbb{P}^1$. Now by Riemann-Hurwitz theorem we have

$$2g - 2 = \deg(f)(2g_{\mathbb{P}^1} - 2) + \deg(R_f),$$

then $\deg(R_f) = 2 \cdot \deg(f) = 4$, and f is ramified in four points and clearly P is one of them. If x_1, x_2, x_3, ∞ are the four branch points in $\mathbb{P}^1$, then there is a unique automorphism of $\mathbb{P}^1$ sending x_1 to 0, x_1 to 1, and leaving ∞ fixed, namely $y = \frac{x - x_1}{x_2 - x_1}$. After this change of coordinates we can assume that f is branched over $0, 1, \lambda, \infty \in \mathbb{P}^1$, whit $\lambda \in k$, $\lambda \neq 0, 1$.
We define the j-invariant of the elliptic curve X by

$$j = j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

It is well known that over an algebraically closed field k with $\mathrm{char}(k) \neq 2$ the scalar $j(X)$ depends only on X. Furthermore two elliptic curves X, X' are isomorphic if and only if $j(X) = j(X')$, and every element of k is the j-invariant of some elliptic curve. Then there is a one-to-one correspondence with the set of elliptic curves up to isomorphism and $\mathbb{A}^1_k$ given by $X \mapsto j(X)$.

Definition 1.3.6. A family of elliptic curves over a scheme S is a flat morphism of schemes $X \to S$ whose fibers are smooth curves of genus one, with a section $\sigma : S \to X$. In particular, an elliptic curve is a smooth curve C of genus one with a rational point $P \in C$.

Consider the functor $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$ where $\mathcal{F}(S)$ is the set of families of elliptic curves over S modulo isomorphism. One can prove that $\mathcal{F}$ does not have a fine moduli space, but the affine line $\mathbb{A}^1_k$ is a coarse moduli space for $\mathcal{F}$.

Now a natural question is how to compactify this coarse moduli space to obtain a complete moduli space. In addition to elliptic curves we admit also irreducible nodal curve of arithmetic genus $p_a = 1$ with a fixed nonsingular point. We consider families $X \to S$ whose fibers are elliptic curves or pointed nodal curve, then taking $j(C) = \infty$ for the nodal curve the projective line $\mathbb{P}^1$ becomes a coarse moduli space.

Let C be a reduced, irreducible curve with $p_a = 1$ and such that $Sing(C)$ is a node. Such a curve can be embedded in $\mathbb{P}^2$ as the nodal cubic $C = Z(y^2z - x^3 + x^2z)$. Consider the low degrees exact sequence for Ext functors,

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_C, \mathcal{O}_C)) \to Ext^1(\Omega_C, \mathcal{O}_C) \to H^0(X, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) \to H^2(X, \mathcal{H}om(\Omega_C, \mathcal{O}_C)).$$

Since $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$ is concentrated at the singular point of C we know that $H^0(X, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))$ is a 1-dimensional k-vector space. Now we consider the sheaf $\mathcal{H}om(\Omega_C, \mathcal{C}) = T_C$.

Recall that if X is a smooth variety and $Y \subseteq X$ is a closed irreducible subscheme defined by the sheaf of ideals $\mathcal{I}$, then there is an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

Furthermore Y is smooth if and only if

- the sheaf Ω_Y is locally free, and
- the sequence above is also exact on the left

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

Consider the sequence for a general subscheme Y and apply the functor $\mathcal{H}om(-, \mathcal{O}_Y)$. We obtain

$$0 \mapsto T_Y \to T_{X|Y} \to N_{Y/X} \to \mathcal{E}xt^1(\Omega_Y, \mathcal{O}_Y) \mapsto 0.$$

For our nodal curve C in $\mathbb{P}^2$ we have

$$0 \mapsto T_C \to T_{\mathbb{P}^2|C} \to N_{C/\mathbb{P}^2} \to \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C) \mapsto 0.$$

We know that $N_{C/\mathbb{P}^2} = \mathcal{O}_C(C) = \mathcal{O}_C(3)$, let D be the divisor associated to $\mathcal{O}_C(3)$. Since C is a local complete intersection the dualizing sheaf ω° is an invertible sheaf. We define the canonical divisor as the divisor corresponding to ω° with support in C_{reg}. Since there are no regular differentials on C we have $deg(K - D) < 0$. By Riemann-Roch theorem for singular curves we get

$$h^0(N_{C/\mathbb{P}^2}) = deg(D) + 1 - p_a = 9 + 1 - 1 = 9.$$

Consider now the Euler sequence

$$0 \mapsto \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \to T_{\mathbb{P}^2} \mapsto 0.$$

Tensorizing by $\mathcal{O}_C$ we get

$$0 \mapsto \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus 3} \to T_{\mathbb{P}^2|C} \mapsto 0.$$

Using the dualizing sheaf $\omega_C^\circ \cong \mathcal{O}_C$, and Serre duality we get $h^1(\mathcal{O}_C(1)) = h^0(\mathcal{O}_C(-1)) = 0$. The cohomology sequence looks like

$$0 \mapsto H^0(C, \mathcal{O}_C) \to H^0(C, \mathcal{O}_C(1)^{\oplus 3}) \to H^0(C, T_{\mathbb{P}^2|C}) \to H^1(C, \mathcal{O}_C) \mapsto 0,$$

so $h^0(T_{\mathbb{P}^2|C}) = 9$. Furthermore the map $H^0(C, N_{C/\mathbb{P}^2}) \to H^0(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))$ is surjective since the former parametrizes the embedded deformations of C as a subscheme of $\mathbb{P}^2$ and the latter parametrizes the abstract deformations of the node. We conclude that $h^0(T_C) > 0$. Let $\sigma \in H^0(C, T_C)$ be a nonzero section, we have an exact sequence $0 \mapsto \mathcal{O}_C \overset{\sigma}{\to} T_C \to R \mapsto 0$. The cokernel R is not zero, because T_C is not locally free. Then $\check{T}_C$ is a proper subsheaf of $\mathcal{O}_C$, using the dualizing sheaf $\omega_C^\circ \cong \mathcal{O}_C$ and Serre duality we get $h^1(T_C) = h^0(\check{T}_C) = 0$. We conclude that $\mathrm{Def}(C)$ is one-dimensional.

Automorphisms of Curves

The only curve of genus one is $\mathbb{P}^1$, and its automorphism group is $\mathrm{PGL}(2)$ which is an open subset of $\mathbb{P}^3$. If we choose one or two marked points in $\mathbb{P}^1$ the automorphism group remains infinite of dimension two and one respectively. However a well known theorem in projective geometry asserts that if we fix three marked points the automorphism group is trivial.
We will see that an elliptic curve has infinitely many automorphisms, but if we choose a marked point then its automorphism group is finite. Finally we will prove that any curve X of genus $g \geqslant 2$ has finitely many automorphisms, and we will give a bound on the cardinality on $\mathrm{Aut}(X)$.
Recall that an elliptic curve X has a group structure, more precisely if we fix a point on X then we get a bijective correspondence between the points of X and the divisors of degree zero in $\mathrm{Cl}^0(X)$, so any translation $X \times X \to X$ gives an automorphism of X. Clearly if we choose a marked point $p \in X$, then the only possible translation is the identity, in this way the automorphism group becomes finite.

Proposition 1.3.7. *Let E be an elliptic curve over k with a marked point. The automorphism group $\mathrm{Aut}(E)$ is a finite group of order dividing 24. More precisely*

- *if* $j(E) \neq 0, 1728$, *then* $|\mathrm{Aut}(E)| = 2$,
- *if* $j(E) = 1728$ *and* $\mathrm{chat}(k) \neq 2, 3$, *then* $|\mathrm{Aut}(E)| = 4$,
- *if* $j(E) = 0$ *and* $\mathrm{chat}(k) \neq 2, 3$, *then* $|\mathrm{Aut}(E)| = 6$,
- *if* $j(E) = 0, 1728$ *and* $\mathrm{chat}(k) = 3$, *then* $|\mathrm{Aut}(E)| = 12$,
- *if* $j(E) = 0, 1728$ *and* $\mathrm{chat}(k) = 2$, *then* $|\mathrm{Aut}(E)| = 24$.

Proof. We consider the case $\mathrm{char}(k) \neq 2, 3$. Then E can be realized as a plane smooth cubic and can be written in Weierstrass form

$$y^2 = x^3 + \alpha x + \beta,$$

furthermore every automorphism of E is of the form

$$x = u^2 x', \; y = u^3 y',$$

for some $u \in \underline{k}^*$. Such a substitution will give an automorphism if and only if

$$u^{-4}\alpha = \alpha, \; u^{-6}\beta = \beta.$$

If $\alpha \cdot \beta = 0$ then $j(E) \neq 0, 1728$, the only possibilities are $u = \pm 1$. If $\beta = 0$ then $j(E) = 1728$, and u satisfies $u^4 = 1$, so $\mathrm{Aut}(E)$ is cyclic of order 4. If $\alpha = 0$ then $j(E) = 0$, and u satisfies $u^6 = 1$, so $\mathrm{Aut}(E)$ is cyclic of order 6. $\square$

Proposition 1.3.8. *Any smooth curve* X *of genus* $g \geqslant 2$ *has finitely many automorphisms.*

Before proving the proposition we recall some general facts about canonically embedded varieties.

Remark 1.3.9. (*Canonically Embedded Varieties*) Let $f : X \to Y$ be a dominant morphism between smooth varieties. The pullback $f^* : f^*\Omega_Y \to \Omega_X$ defines a canonical morphisms between the cotangent sheaves, and since pullback commutes with maximal exterior powers we get a canonical morphism $f^* : f^*\omega_Y \to \omega_X$ of the canonical sheaves. In particular if $X = Y$ and $f \in \mathrm{Aut}(X)$, since $f^*\omega_X \cong \omega_X$, we get an automorphism f^* of ω_X. Then an automorphism of X induces an automorphism of ω_X, and an automorphism on the vector space of the its global section $H^0(X, \omega_X)$.
Suppose now that ω_X is ample, then $\omega_X^{\otimes n}$ is very ample for some $n \geqslant 0$. Any automorphism of X induces also an automorphism of $\omega_X^{\otimes n}$. Let $\varphi : X \to \mathbb{P}(H^0(X, \omega_X^{\otimes n})^*)$ be the corresponding embedding. Then we have an action of $\mathrm{Aut}(X)$ on $\mathbb{P}(H^0(X, \omega_X^{\otimes n})^*)$, and any $f \in \mathrm{Aut}(X)$ induces an automorphism of $\mathbb{P}(H^0(X, \omega_X^{\otimes n})^*) = \mathbb{P}^N$. We have seen that if X has ample canonical sheaf then $\mathrm{Aut}(X)$ is a closed algebraic subgroup of $PGL(N+1)$. Clearly the same argument works if X has ample anticanonical sheaf.

Proof. Recall that if $f : X \to Y$ is a morphism of schemes, with X separated and Y smooth, and Def_f is the deformation functor of f, then $T^1 Def_f = H^0(X, f^*T_Y)$. In particular for $f = Id_X : X \to X$ we get $T^1_{Id_X} Def_{Id_X} = T_{Id_X}\mathrm{Aut}(X) = H^0(X, T_X)$, and $h^0(X, T_X) = 0$ since X is a curve of genus $g \geqslant 2$. The curve X has canonical ample sheaf, and by the preceding remark we can embed $\mathrm{Aut}(X)$ in $PGL(N+1) \subseteq \mathbb{P}^{(N+1)^2-1}$ as closed subscheme. Since the tangent space of $\mathrm{Aut}(X)$ has dimension zero we conclude that $\mathrm{Aut}(X)$ is a finite set of points. $\square$

In the following proposition we give a bound on the number of automorphisms of a curve of genus $g \geqslant 2$.

Proposition 1.3.10. *Let* X *be a projective curve of genus* $g \geqslant 2$, *then the group* $\mathrm{Aut}(X)$ *is finite and* $|\mathrm{Aut}(X)| \leqslant 84(g-1)$.

Proof. Let $W(X)$ be the set of Weierstrass points of X, we know that $W(X)$ is finite. If $\varphi \in \mathrm{Aut}(X)$ is a non trivial automorphism then φ has at most $2g+2$ fixed points. Since the set of Weierstrass points is fixed by the group $\mathrm{Aut}(X)$ we have a morphism

$$F : \mathrm{Aut}(X) \to \mathrm{Perm}(W(X)),$$

where $\mathrm{Perm}(W(X))$ is the group of permutations of $W(X)$. If X is non hyperelliptic there are more than $2g+2$ Weierstrass points on X and there is a unique automorphism that leaves more that $2g+2$ points fixed, the identity. So $\ker(F) = \{Id_X\}$.
If X is hyperelliptic then any automorphism in the subgroup (J) generated by the involution $J : X \to X$ fixes the Weierstrass points, but since $J^2 = Id_X$ this subgroup is finite. We conclude that F is a morphism of $\mathrm{Aut}(X)$ into a finite group and with finite kernel, then the group $\mathrm{Aut}(X)$ is finite.
Let $G = \mathrm{Aut}(X)$ and $|G| = n$, consider the projection $\pi : X \to X/G$. For any $\overline{x} \in X/G$ we have $\pi^{-1}(\overline{x}) = \{x \in X \,|\, \pi(x) = \overline{x}\} = \{x \in X \,|\, \exists\, g \in G,\ g(x) = \overline{x}\} = \{g^{-1}(\overline{x}),\ g \in G\}$, then π is a morphism of degree n. The map π is branched only at fixed point of G. Let $P_1, ..., P_s$ be a maximal sets of ramification points of X lying over distinct points of X/G, and let r_i be the index of ramification of P_i. Recall that if $P \in X$ is a ramification point, and r is its ramification index, then the fiber $\pi^{-1}(\pi(P))$ consists of exactly $\frac{n}{r}$ points, each having ramification index r,

essentially because X is a covering space for X/G. So in the fiber of any P_j there are $\frac{n}{r_j}$ points each with ramification index r_j. Then the degree of the ramification divisor is

$$\deg(R_\pi) = \sum_{j=1}^{s} (r_j - 1)\frac{n}{r_j} = n\sum_{j=1}^{s}(1 - \frac{1}{r_j}).$$

By Riemann-Hurwitz formula we get $2g - 2 = n(2\alpha - 2) + n\sum_{j=1}^{s}(1 - \frac{1}{r_j})$, where α is the genus of X/G. Then

$$\frac{2g-2}{n} = 2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j}).$$

Note that since $r_j \geqslant 2$ we have $\frac{1}{2} \leqslant 1 - \frac{1}{r_j} < 1$. Since we may assume $n > 1$ it is clear that $g > \alpha$. Now we have to analyze the expression $2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j})$.

- If $\alpha \geqslant 2$ we obtain $2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j}) \geqslant 2 - \sum_{j=1}^{s}(1 - \frac{1}{r_j}) \geqslant 2$, so $\frac{2g-2}{n} \geqslant 2$ and
$$n \leqslant g - 1.$$
- If $\alpha = 1$ then $2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j}) = \sum_{j=1}^{s}(1 - \frac{1}{r_j}) \geqslant \frac{1}{2}$, so $\frac{2g-2}{n} \geqslant \frac{1}{2}$ and
$$n \leqslant 4(g - 1).$$
- If $\alpha = 0$ then $2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j}) = \sum_{j=1}^{s}(1 - \frac{1}{r_j}) - 2$. Since $\sum_{j=1}^{s}(1 - \frac{1}{r_j}) - 2 > 0$ and $1 - \frac{1}{r_j} < 1$, we conclude that $s \geqslant 3$.
  - If $s \geqslant 5$, then $\sum_{j=1}^{s}(1 - \frac{1}{r_j}) - 2 \geqslant \frac{1}{2}$, so $\frac{2g-2}{n} \geqslant \frac{1}{2}$ and
  $$n \leqslant 4(g - 1).$$
  - If $r = 4$ then the r_j cannot be all equal to 2, otherwise we would have $\frac{2g-2}{n} = 0$, so $g = 1$. Then at least one is $\geqslant 3$ and gives$\sum_{j=1}^{s}(1 - \frac{1}{r_j}) - 2 \geqslant 3(1 - \frac{1}{2}) + (1 - \frac{1}{3}) - 2 = \frac{1}{6}$, so $\frac{2g-2}{n} \geqslant \frac{1}{6}$ and
  $$n \leqslant 12(g - 1).$$
  - In the case $s = 3$ we can assume without loss of generality $2 \leqslant r_1 \leqslant r_2 \leqslant r_3$. We have $r_3 > 3$ otherwise $\sum_{j=1}^{s}(1 - \frac{1}{r_j}) - 2 < 0$. Then $r_2 \geqslant 3$.
  If $r_3 \geqslant 7$ then $n \leqslant 84(g - 1)$.
  If $r_3 = 6$ and $r_1 = 2$ then $r_2 \geqslant 4$ and $n \leqslant 24(g - 1)$.
  If $r_3 = 6$ and $r_1 \geqslant 3$ then $n \leqslant 12(g - 1)$.
  If $r_3 = 5$ and $r_1 = 2$ then $r_2 \geqslant 4$ and $n \leqslant 40(g - 1)$.
  If $r_3 = 5$ and $r_1 \geqslant 3$ then $n \leqslant 15(g - 1)$.
  If $r_3 = 4$ then $r_1 \geqslant 3$ and $n \leqslant 24(g - 1)$.

□

To compactify the coarse moduli space M_g Deligne and Mumford introduces *stable* curves. We have seen that $T_{Id_X}\operatorname{Aut}(X) = H^0(X, T_X)$, an element of this space is called an *infinitesimal automorphism*.

Definition 1.3.11. A reduced, connected, projective curve X, having at most nodes as singularities is said to be stable if $H^0(X, T_X) = 0$, i.e. X has no infinitesimal automorphisms.

Clearly for a curve X of genus $g \geqslant 2$ the following are equivalent,

- X has no infinitesimal automorphisms,
- $H^0(X, T_X) = 0$,
- $\mathrm{Aut}(X)$ is finite.

By the preceding discussion any smooth curve of genus $g \geqslant 2$ is stable.

Consider the local infinitesimal deformation functor of $\mathcal{F}$ for a stable curve X of genus $g \geqslant 2$,

$$\mathrm{Def}_X : \mathfrak{Art} \to \mathfrak{Sets},$$

which associates to any Artin local algebra A the set of isomorphism classes $\Upsilon \to \mathrm{Spec}(A)$ of families of curves of genus g over $\mathrm{Spec}(A)$, with a fixed isomorphism $\Upsilon_0 \to X$, where $\Upsilon_0 \to \mathrm{Spec}(k)$ is the central fiber of Υ. Note that the isomorphism $\Upsilon_0 \to X$ is not unique, indeed we can recover any other isomorphism composing with an automorphism of X, and the set of such isomorphisms is a principal homogeneous space under the action of $\mathrm{Aut}(X)$. The following remark will be important in order to prove that $\overline{\mathcal{M}}_g$ is smooth.

Remark 1.3.12. Let X be a proper scheme and let Def_X be its deformation functor. Then $T^i_{\mathrm{Def}_X} = \mathrm{Ext}^i(L^\bullet_X, \mathcal{O}_X)$, where $L^\bullet_X$ is the cotangent complex of X. If X has only local complete intersection singularities the $L^\bullet_X$ coincides with Ω_X in degree zero. Recall that from the spectral sequence of Ext groups we have

$$H^q(X, \mathcal{E}xt^p(\Omega_X, \mathcal{O}_X)) \Rightarrow \mathrm{Ext}^{p+q}(\Omega_X, \mathcal{O}_X).$$

Consider the special case where $X = C$ is a nodal curve and $p + q = 2$. Then

- $H^0(C, \mathcal{E}xt^2(\Omega_C, \mathcal{O}_C)) = 0$ because Ω_C admits a locally free resolution of length one. Indeed take an embedding $C \to Y$ of Y in a smooth surface, then we have an exact sequence
$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_Y \otimes \mathcal{O}_C \to \Omega_C \mapsto 0.$$
- $H^1(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) = 0$ because $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$ is supported on $\mathrm{Sing}(C)$ which is zero dimensional.
- $H^2(C, \mathcal{H}om(\Omega_C, \mathcal{O}_C)) = 0$ because $\dim(C) = 1$.

We conclude that $\mathrm{Ext}^2(\Omega_C, \mathcal{O}_C) = T^2\mathrm{Def}_C = 0$.

Theorem 1.3.13. *(Smoothness of $\overline{\mathcal{M}}_g$) Let X be a stable curve of arithmetic genus $g \geqslant 2$. Then the functor of local infinitesimal deformations Def_X of X is pro-representable by a regular local ring of dimension $3g - 3$. In other words $\overline{\mathcal{M}}_g$ is a smooth Deligne-Mumford stack of dimension*

$$\dim(\overline{\mathcal{M}}_g) = 3g - 3.$$

Proof. The functor Def_X is pro-representable since X is projective and does not have infinitesimal automorphism. Furthermore $T^2\mathrm{Def}_X = H^2(X, T_X) = 0$ since $\dim(X) = 1$, then there are no obstructions to deforming X and the local ring representing Def_X is regular. Furthermore from remark 1.3.12 we get $\mathrm{Ext}^2(\Omega_X, \mathcal{O}_X) = T^2\mathrm{Def}_X = 0$ for a nodal curve. Then in any case the deformation functor of X is unobstructed. So far we have proved that $\overline{\mathcal{M}}_g$ is a smooth DM stack. To compute its dimension we distinguish two cases.

- If X is a smooth curve, and $0 \mapsto I \to A \to B \mapsto 0$ is a semi-small exact sequence in $\mathfrak{Art}$, then there is a functorial exact sequence

  $$H^1(X, T_X) \otimes I \to \mathrm{Def}_X(A) \to \mathrm{Def}_X(B) \to H^2(X, T_X) \otimes I.$$

  On a curve $T_X = \omega_X^{\vee}$, where ω_X is the canonical sheaf of X. Then $\deg(T_X) = 2-2g$, and since $h^0(T_X) == 0$, by Riemann-Roch theorem we get $h^0(T_X) - h^1(T_X) = 2-2g-g+1 = 3-3g$, and $h^1(T_X) = 3g-3$. We conclude that in a point $x \in \overline{\mathcal{M}}_g$ corresponding to the isomorphism class of a smooth curve X, the tangent space $T_x\overline{\mathcal{M}}_g$ has dimension $3g-3$.

- Now consider the case where X is a stable nodal curve. We have a sequence

  $$0 \mapsto H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \mapsto 0,$$

  there being no H^2 on a curve. We denote by δ the number of nodes in X. Since the sheaf Ω_X is locally free on the smooth locus of X, the sheaf $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))$ is just k at each node, then $\dim(H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))) = \delta$. The curve X is l.c.i, then the dualizing sheaf ω_X is an invertible sheaf, and since $\omega_X \cong \Omega_X$ on the open set of regular points, we have an injective morphism $\omega_X^{\vee} \to \mathcal{H}om(\Omega_X, \mathcal{O}_X)$, and an exact sequence

  $$0 \mapsto \omega_X^{\vee} \to \mathcal{H}om(\Omega_X, \mathcal{O}_X) \to \mathcal{O}_Z \mapsto 0,$$

  where $Z = \mathrm{Sing}(X)$. Since X is stable $h^0(\mathcal{H}om(\Omega_X, \mathcal{O}_X)) = 0$, by the cohomology exact sequence we get $h^0(\omega_X^{\vee}) = 0$, and

  $$0 \mapsto H^0(X, \mathcal{O}_Z) \to H^1(X, \omega_X^{\vee}) \to H^1(\mathcal{H}om(\Omega_X, \mathcal{O}_X)) \mapsto 0.$$

  By Riemann-Roch for singular curves we get $h^1(\omega_X^{\vee}) = 3g-3$, and since $h^0(\mathcal{O}_Z) = \delta$ we get $h^1(\mathcal{H}om(\Omega_X, \mathcal{O}_X)) = 3g-3-\delta$. Finally

  $$\dim(\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X)) = h^1(T_X) + h^0(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = 3g-3-\delta+\delta = 3g-3.$$

We conclude that any point of $\overline{\mathcal{M}}_g$ is smooth and $\overline{\mathcal{M}}_g$ is a smooth stack of dimension $3g-3$. □

Remark 1.3.14. Theorems 1.2.6 and 1.3.13 hold also for $n > 0$. That is $\overline{\mathcal{M}}_{g,n}$ is a smooth DM-stack of dimension $3g-3+n$ for any g, n such that $2g-2+n > 0$. The notation is more convoluted but the proofs work exactly in the same way.

2

THE AUTOMORPHISM GROUP OF $\overline{M}_{g,n}$

We work over the field of complex numbers. Let us begin whit some preliminaries on $\overline{M}_{g,n}$ and the moduli stack $\overline{\mathcal{M}}_{g,n}$.

Nodal curves

The arithmetic genus g of a connected curve C is defined as $g = h^1(C, \mathcal{O}_C)$. Suppose that C has at most nodal singularities. Let $C = \bigcup_{i=1}^{\gamma} C_i$ be the irreducible components decomposition of C, and set $\delta := \sharp \operatorname{Sing}(C)$. Let

$$\nu : \overline{C} = \bigsqcup_{i=1}^{\gamma} \overline{C}_i \to C$$

be the normalization of C. The associated morphism $\mathcal{O}_C \hookrightarrow \mathcal{O}_{\overline{C}}$ on the structure sheaves yield the following sequence in cohomology

$$0 \mapsto H^0(C, \mathcal{O}_C) \to H^0(\overline{C}, \mathcal{O}_{\overline{C}}) \to \mathbb{C}^{\delta} \to H^1(C, \mathcal{O}_C) \to H^1(\overline{C}, \mathcal{O}_{\overline{C}}) \mapsto 0.$$

We get a formula for the arithmetic genus g of C

$$g = h^1(\overline{C}, \mathcal{O}_{\overline{C}}) + \delta - \gamma + 1 = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$$

where $g_i = h^1(\overline{C}_i, \mathcal{O}_{\overline{C}_i})$ is the geometric genus of C_i.

Definition 2.0.15. A *stable n-pointed curve* is a complete connected curve C that has at most nodal singularities, with an ordered collection $x_1, ..., x_n \in C$ of distinct smooth points of C, such that $(C, x_1, ..., x_n)$ has finitely many automorphisms.

This finiteness condition is equivalent to say that every rational component of the normalization of C has at least three points lying over singular or marked points of C.
As we saw in Chapter 1 moduli spaces of smooth algebraic curves have been defined and then compactified adding stable curves by *Deligne* and *Mumford* in [DM]. Furthermore *Deligne* and *Mumford* proved that, if $2g-2+n > 0$, there exists a coarse moduli space $\overline{M}_{g,n}$ parametrizing isomorphism classes of n-pointed stable curves of arithmetic genus g, and this space is an irreducible projective variety of dimension $3g-3+n$.

Boundary of $\overline{M}_{g,n}$ and dual modular graphs

The points in the boundary $\partial\overline{M}_{g,n}$ of the moduli space $\overline{M}_{g,n}$ represent isomorphisms classes of singular pointed stable curves. The geometry of such curves is encoded in a graph, called dual modular graph. The boundary has a stratification whose loci, called strata, parametrize curves of a certain topological type and with a fixed configuration of the marked points.
Each nodal curve has an associated graph. This allows to represent nodal curves in a very simple way and translate some issues related to nodal curves in the language of graph theory. Let C be a connected nodal curve with γ irreducible components and δ nodes. The dual

graph Γ_C of C is the graph whose vertexes represent the irreducible components of C and whose edges represent nodes lying on two components.
More precisely, each irreducible component is represented by a vertex labeled by two numbers: the genus and the number of marked points of the component. An edge connecting two vertex means that the two corresponding components intersect in the node corresponding to the edge. A loop on a vertex means that the corresponding component has a self-intersection. Recently, *S. Maggiolo* and *N. Pagani* developed a software that generates all stable dual graphs for prescribed values of g, n whose detailed description can be found in [MP].
We denote by Δ_{irr} the locus in $\overline{M}_{g,n}$ parametrizing irreducible nodal curves with n marked points, and by $\Delta_{i,P}$ the locus of curves with a node which divides the curve into a component of genus i containing the points indexed by P and a component of genus $g-i$ containing the remaining points.
The closures of the loci Δ_{irr} and $\Delta_{i,P}$ are the irreducible components of the boundary $\partial\overline{M}_{g,n}$ [Mor, Proposition 1.21].

Forgetful morphisms

For any $i = 1, ..., n$ there is a canonical forgetful morphism

$$\pi_i : \overline{M}_{g,n} \to \overline{M}_{g,n-1}$$

forgetting the i-th marked point. If $g > 2$ and $[C, x_1, ..., \hat{x_i}, ..., x_n] \in \overline{M}_{g,n-1}$ is a general point the fiber

$$\pi_i^{-1}([C, x_1, ..., \hat{x_i}, ..., x_n]) \cong C$$

is isomorphic to C and π_i plays the role of the universal curve. Note that if $n \geqslant 2$ the fiber $\pi_i^{-1}([C, x_1, ..., \hat{x_i}, ..., x_n])$ always intersects the boundary of $\overline{M}_{g,n}$, in fact the points of the fiber corresponding to marked points represent singular curves with two irreducible components: C itself and a $\mathbb{P}^1$ with two marked points and intersecting C in a point. In the same way for any $I \subseteq \{1, ..., n\}$ we have a forgetful map $\pi_I : \overline{M}_{g,n} \to \overline{M}_{g,n-|I|}$. The map π_i has sections $s_{i,j} : \overline{M}_{g,n-1} \to \overline{M}_{g,n}$ defined by sending the point $[C, x_1, ..., \hat{x_i}, ..., x_n]$ to the isomorphism class of the n-pointed genus g curve obtained by attaching at $x_j \in C$ a $\mathbb{P}^1$ with two marked points labeled by x_i and x_j.

The universal curve

The moduli space $\overline{M}_{g,1}$ with the forgetful morphism $\pi : \overline{M}_{g,1} \to \overline{M}_g$ at first glance seems to play the role of the universal curve over $\overline{M}_g$. However, on closer examination one realizes that $\pi^{-1}([C]) \cong C$ if and only if $[C] \in \overline{M}_g^0$ the locus of automorphisms-free curves. It is well known that the set-theoretic fiber of $\pi : \overline{M}_{g,1} \to \overline{M}_g$ over $[C] \in \overline{M}_g$ is the quotient $C/\operatorname{Aut}(C)$. For example over an open subset of $\overline{M}_2$ the fibration $\pi : \overline{M}_{2,1} \to \overline{M}_2$ is a $\mathbb{P}^1$-bundle and this is true even scheme-theoretically.

Remark 2.0.16. The situation is different if instead of considering the moduli space $\overline{M}_{g,1}$ we consider the Deligne-Mumford moduli stack $\overline{\mathcal{M}}_{g,1}$. In fact, in this case the fiber $\pi^{-1}([C])$ is isomorphic to C and via the morphism $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ the stack $\overline{\mathcal{M}}_{g,1}$ plays the role of the universal curve over $\overline{\mathcal{M}}_g$.

Divisor classes on $\overline{\mathcal{M}}_{g,n}$

Let us briefly recall the definitions of classes λ and ψ_i on $\overline{\mathcal{M}}_{g,n}$. Consider the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgetting one of the marked points and its sections $\sigma_1, ..., \sigma_n$:

$\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf of the morphism π. The Hodge class is defined as

$$\lambda := c_1(\pi_*(\omega_\pi)).$$

The classes ψ_i are defined as

$$\psi_i := \sigma_i^*(c_1(\omega_\pi))$$

for any $i = 1, ..., n$. Finally we denote by δ_{irr} and $\delta_{i,P}$ the boundary classes on $\overline{\mathcal{M}}_{g,n}$.

Cyclic quotient singularities

Any cyclic quotient singularity is of the form $\mathbb{A}^n/\mu_r$, where μ_r is the group of r-roots of unit. The action $\mu_r \curvearrowright \mathbb{A}^n$ can be diagonalized, and then written in the form

$$\mu_r \times \mathbb{A}^n \to \mathbb{A}^n,\ (\epsilon, x_1, ..., x_n) \mapsto (\epsilon^{a_1}x_1, ..., \epsilon^{a_n}x_n),$$

for some $a_1, ..., a_r \in \mathbb{Z}/\mathbb{Z}_r$. The singularity is thus determined by the numbers $r, a_1, ..., a_n$. Following the notation set by *M. Reid* in [Re], we denote by $\frac{1}{r}(a_1, ..., a_n)$ this type of singularity.

Fibrations of $\overline{M}_{g,n}$

The following result by *A. Gibney, S. Keel* and *I. Morrison* gives an explicit description of the fibrations $\overline{M}_{g,n} \to X$ of $\overline{M}_{g,n}$ on a projective variety X in the case $g \geqslant 1$. We denote by N the set $\{1, ..., n\}$ of the markings, if $S \subset N$ then S^c denotes its complement.

Theorem 2.0.17. *(Gibney - Keel - Morrison) Let* $D \in \mathrm{Pic}(\overline{M}_{g,n})$ *be a nef divisor.*

- *If* $g \geqslant 2$ *either* D *is the pull-back of a nef divisor on* $\overline{M}_{g,n-1}$ *via one of the forgetful morphisms or* D *is big and the exceptional locus of* D *is contained in* $\partial\overline{M}_{g,n}$.
- *If* $g = 1$ *either* D *is the tensor product of pull-backs of nef divisors on* $\overline{M}_{1,S}$ *and* $\overline{M}_{1,S^c}$ *via the tautological projection for some subset* $S \subseteq N$ *or* D *is big and the exceptional locus of* D *is contained in* $\partial\overline{M}_{g,n}$.

The above theorem will be crucial to determine the automorphism group of $\overline{M}_{g,n}$, and can be found in [GKM, Theorem 0.9]. An immediate consequence of 2.0.17 is that for $g \geqslant 2$ any fibration of $\overline{M}_{g,n}$ to a projective variety factors through a projection to some $\overline{M}_{g,i}$ with $i < n$, while $\overline{M}_g$ has no non-trivial fibrations. This last fact had already been shown by *A. Gibney* in her Ph.D. Thesis [Gib].
Such a clear description of the fibrations of $\overline{M}_{g,n}$ is no longer true for $g = 1$, an explicit counterexample to this fact was given by *R. Pandharipande* and can be found in [BM2, Example A.2], see also [Pan] for similar constructions. However, if we consider the fibrations of the type

$$\overline{M}_{1,n} \xrightarrow{\varphi} \overline{M}_{1,n} \xrightarrow{\pi_i} \overline{M}_{1,n-1}$$

where φ is an automorphism of $\overline{M}_{1,n}$, thanks to the second part of Theorem 2.0.17 we can prove the following lemma.

Lemma 2.0.18. *Let* φ *be an automorphism of* $\overline{M}_{1,n}$. *Any fibration of the type* $\pi_i \circ \varphi$ *factorizes through a forgetful morphism* $\pi_j : \overline{M}_{1,n} \to \overline{M}_{1,n-1}$.

Proof. By the second part of Theorem 2.0.17 the fibration $\pi_i \circ \varphi$ factorizes through a product of forgetful morphisms $\pi_{S^c} \times \pi_S : \overline{M}_{1,n} \to \overline{M}_{1,S} \times_{\overline{M}_{1,1}} \overline{M}_{1,S^c}$ and we have a commutative diagram

$$\begin{tikzcd}
\overline{M}_{1,n} \arrow[r, "\varphi"] \arrow[d, "\pi_{S^c} \times \pi_S"'] & \overline{M}_{1,n} \arrow[d, "\pi_i"] \\
\overline{M}_{1,S} \times_{\overline{M}_{1,1}} \overline{M}_{1,S^c} \arrow[r, "\overline{\varphi}"] & \overline{M}_{1,n-1}
\end{tikzcd}$$

The fibers of π_i and $\pi_{S^c} \times \pi_S$ are both 1-dimensional. Furthermore φ maps the fiber of $\pi_{S^c} \times \pi_S$ over $([C, x_{a_1}, ..., x_{a_s}], [C, x_{b_1}, ..., x_{b_{n-s}}])$ to $\pi_i^{-1}(\overline{\varphi}([C, x_{a_1}, ..., x_{a_s}], [C, x_{b_1}, ..., x_{b_{n-s}}]))$. Take a point $[C, x_1, ..., x_{n-1}] \in \overline{M}_{1,n-1}$, the fiber $\pi_i^{-1}([C, x_1, ..., x_{n-1}])$ is mapped isomorphically to a fiber Γ of $\pi_{S^c} \times \pi_S$ which is contracted to a point $y = (\pi_{S^c} \times \pi_S)(\Gamma)$. The map

$$\overline{\psi} : \overline{M}_{1,n-1} \to \overline{M}_{1,S} \times_{\overline{M}_{1,1}} \overline{M}_{1,S^c}, \ [C, x_1, ..., x_{n-1}] \mapsto y,$$

is clearly the inverse of $\overline{\varphi}$. So $\overline{\varphi}$ defines a bijective morphism between $\overline{M}_{1,S} \times_{\overline{M}_{1,1}} \overline{M}_{1,S^c}$ and $\overline{M}_{1,n-1}$, and since $\overline{M}_{1,n-1}$ is normal $\overline{\varphi}$ is an isomorphism. This forces $S = \{j\}$, $S^c = \{1, ..., \bar{j}, ..., n\}$. So we reduce to the commutative diagram

$$\begin{tikzcd}
\overline{M}_{1,n} \arrow[r, "\varphi"] \arrow[d, "\pi_{S^c} \times \pi_j"'] & \overline{M}_{1,n} \arrow[d, "\pi_i"] \\
\overline{M}_{1,1} \times_{\overline{M}_{1,1}} \overline{M}_{1,n-1} \arrow[r, "\overline{\varphi}"] & \overline{M}_{1,n-1}
\end{tikzcd}$$

and $\pi_i \circ \varphi$ factorizes through the forgetful morphism π_j. $\square$

2.1 THE MODULI SPACE OF 2-POINTED ELLIPTIC CURVES

Let (C, p) be a nodal elliptic curve. Then there exists $(a, b) \in \mathbb{A}^2 \setminus (0, 0)$ such that (C, p) is isomorphic to $(C^{'}, [0 : 1 : 0])$, where

$$C^{'} = Z(zy^2 - x^3 - axz^2 - bz^3) \subset \mathbb{P}^2.$$

This representation is called *Weierstrass representation* of the elliptic curve. Consider now the 4-fold

$$X := Z(zy^2 - x^3 - axz^2 - bz^3) \subset \mathbb{A}^3_0 \times \mathbb{A}^2_0.$$

There is an action of $\mathbb{C}^* \times \mathbb{C}^* \curvearrowright X$ given by

$$\mathbb{C}^* \times \mathbb{C}^* \times X \to X, \ ((\lambda, \xi), (x, y, z, a, b)) \mapsto (\xi\lambda^2 x, \xi\lambda^3 y, \xi z, \lambda^4 a, \lambda^6 b).$$

The moduli stack $\overline{\mathcal{M}}_{1,1}$ is the quotient stack $[\mathbb{A}^2 \setminus (0, 0)/\mathbb{C}^*] \cong \mathbb{P}(4, 6)$ and the moduli space $\overline{M}_{1,1}$ is the quotient $\mathbb{A}^2 \setminus (0, 0)/\mathbb{C}^* \cong \mathbb{P}^1$. There are two points of $\overline{\mathcal{M}}_{1,1}$ that are stabilized by the action of μ_4 and μ_6 respectively. These are classes of curves whose Weierstrass representations can be chosen respectively as:

$$C_4 := \{y^2 z = x^3 + xz^2\} \subset \mathbb{P}^2,$$

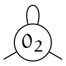
$$C_6 := \{y^2 z = x^3 + z^3\} \subset \mathbb{P}^2.$$

Now, $\overline{\mathcal{M}}_{1,2}$ is the universal curve over $\overline{\mathcal{M}}_{1,1}$, so $\overline{\mathcal{M}}_{1,2} = [X/\mathbb{C}^* \times \mathbb{C}^*]$ and $\overline{M}_{1,2} = X/\mathbb{C}^* \times \mathbb{C}^*$. In order to determine the singularities of $\overline{M}_{1,2}$ we have to analyze carefully the action $\mathbb{C}^* \times \mathbb{C}^* \curvearrowright X$.
Since $\overline{\mathcal{M}}_{1,2}$ is a smooth Deligne-Mumford stack the coarse moduli space $\overline{M}_{1,2}$ will have finite quotient singularities at the places where the automorphism groups jump. Let (C, p) be a elliptic curve over $\mathbb{C}$, it is well known that

- $|\operatorname{Aut}(C, p)| = 2$ if $j(C) \neq 0, 1728$,
- $|\operatorname{Aut}(C, p)| = 4$ if $j(C) = 1728$,
- $|\operatorname{Aut}(C, p)| = 6$ if $j(C) \neq 0$.

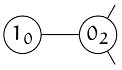
Adding a marked point will kill some automorphisms. We expect that points of type (C, p, q) with $|\operatorname{Aut}(C, p)| = 2$ will have trivial automorphism group. Automorphisms will jump on the points (C, p, q) with $|\operatorname{Aut}(C, p)| = 4, 6$. To understand the behavior of the boundary $\partial\overline{M}_{1,2}$ we have to observe the following possible degenerations.

- The divisor Δ_{irr} whose general point is a curve with dual graph

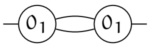
  and so automorphisms free.

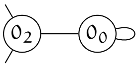
- The divisor $\Delta_{0,2}$ whose general point is a curve with dual graph

  and so with two automorphisms coming from the elliptic involution. Here we expect to get two singular points when the number of automorphisms of the elliptic curve jumps to 4 and 6.

- Two further degenerations in codimension two with the following dual graphs.

  Here the automorphism group remains of order two, so we do not expect to have singularities.

Proposition 2.1.1. *The moduli space $\overline{M}_{1,2}$ is a rational surface with four singular points. Two singular points lie in $M_{1,2}$, and are:*

- *a singularity of type $\frac{1}{4}(2,3)$ representing an elliptic curve of Weierstrass representation C_4 with marked points $[0:1:0]$ and $[0:0:1]$;*
- *a singularity of type $\frac{1}{3}(2,4)$ representing an elliptic curve of Weierstrass representation C_6 with marked points $[0:1:0]$ and $[0:1:1]$.*

The remaining two singular points lie on the boundary divisor $\Delta_{0,2}$, and are:

- *a singularity of type $\frac{1}{6}(2,4)$ representing a reducible curve whose irreducible components are an elliptic curve of type C_6 and a smooth rational curve connected by a node;*

- *a singularity of type $\frac{1}{4}(2,6)$ representing a reducible curve whose irreducible components are an elliptic curve of type C_4 and a smooth rational curve connected by a node.*

Proof. The rationality of $\overline{M}_{1,2}$ follows from the fact that the forgetful map $\overline{M}_{1,2} \to \overline{M}_{1,1}$ realizes $\overline{M}_{1,2}$ as a ruled surface over $\mathbb{P}^1$.
To compute the singularities we study the action on X. Note that on X, $z = 0 \Rightarrow x = 0 \Rightarrow y \neq 0$. So X is covered by the charts $\{z \neq 0\}$ and $\{y \neq 0\}$.
Consider first the chart $\{z \neq 0\}$. On this chart X is given by $\{y^2 = x^3 + ax + b\}$ so $b = y^2 - x^3 - ax$. We can take (x, y, a) as coordinates, and the action of $\mathbb{C}^* \times \mathbb{C}^*$ is given by $(\lambda, x, y, a) \mapsto (\lambda^2 x, \lambda^3 y, \lambda^4 a)$. The point $(0, 0, 0)$ is stabilized by $\mathbb{C}^* \times \mathbb{C}^*$, so does not produce any singularity. Since $(2, 3) = (3, 4) = 1$ the points (x, y, a) such that $xy \neq 0$ or $ya \neq 0$ have trivial stabilizer.
If $y = 0$ the action is given by $(\lambda, x, a) \mapsto (\lambda^2 x, \lambda^4 a)$. We distinguish two cases.

- If $x = 0$ then $a \neq 0$, the stabilizer is μ_4. So on the chart $a \neq 0$ we have a singularity of type $\frac{1}{4}(2, 3)$. Note that $x = y = 0$ implies $b = 0$. The singular point corresponds to a smooth elliptic curve of Weierstrass form C_4 and whose second marked point is $[0 : 0 : 1]$.

- If $x \neq 0$ then the stabilizer is μ_2 and on this chart we find points of type $\frac{1}{2}(1, 0)$ and these are smooth points.

If $y \neq 0$, then $\lambda^3 = 1$ and we get a singularity of type $\frac{1}{3}(2, 4)$, that is a A_2 singularity, in the point $a = x = 0$. This is a curve of type C_6 where we mark the point $[0 : 1 : 1]$. In $\overline{M}_{1,2}$ the singular point we found represents a smooth elliptic curve of Weierstrass form C_6 and whose second marked point is $[0 : 1 : 1]$.
Consider now the locus $\{z = 0\}$. We can take $y = 1$ and X is given by $\{z = x^3 + axz^2 + bz^3\}$. We are interested in a neighborhood of $x = z = 0$. Let $f(x, z, a, b) = z - x^3 - axz^2 - bz^3$ be the polynomial defining X. Since $\frac{\partial f}{\partial z}_{|z=0} \neq 0$ we can chose (x, a, b) as local coordinates. The action is given by $(\lambda, x, a, b) \mapsto (\lambda^2 x, \lambda^4 a, \lambda^6 b)$. If $x \neq 0$ the stabilizer is trivial. If $x = 0$ and $ab \neq 0$ the stabilizer is μ_2 and does not produce any singularity. We get the following two singular points.

- If $a = 0, b \neq 0$ then we have a singular point of type $\frac{1}{6}(2, 4)$. In this case we get an elliptic curve of type C_6 where we are taking the second marked point equal to the first $[0 : 1 : 0]$. So this singular point is a point on the boundary divisor $\Delta_{0,2}$ representing a reducible curve whose irreducible components are an elliptic curve of type C_6 and a smooth rational curve connected by a node.

- If $a \neq 0, b = 0$ we get a singular point of type $\frac{1}{4}(2, 6)$. We have an elliptic curve of type C_4 where the second marked point coincides with the first $[0 : 1 : 0]$. This singular point is a point on the boundary divisor $\Delta_{0,2}$ representing a reducible curve whose irreducible components are an elliptic curve of type C_4 and a smooth rational curve connected by a node.

These two points are the only singularities on the divisor $\Delta_{0,2}$. □

The rational Picard group of $\overline{M}_{1,2}$ is freely generated by the two boundary divisors [Be, Theorem 3.1.1]. The divisors Δ_{irr} and $\Delta_{0,2}$ are both smooth, rational curves. The boundary divisor Δ_{irr} has zero self intersection while $\Delta_{0,2}$ has negative self intersection. In [Sm] *D.I. Smyth* proves that on $\overline{M}_{1,2}$ there exists a birational morphisms contracting $\Delta_{0,2}$. In the following we give a precise description of this contraction. Let us briefly recall the structure of a weighted blow up.

Remark 2.1.2. Let $\pi_\omega : Y \to \mathbb{C}^2$ be the weighted blow up of $\mathbb{C}^2$ at the origin with weight $\omega = (\omega_1, \omega_2)$,

$$Y = \{((x,y),[u:v]) \in \mathbb{C}^2 \times \mathbb{P}(\omega_1,\omega_2) \mid (x,y) \in \overline{[u:v]}\}.$$

Then Y is given by the equation $x^{\omega_1}v - y^{\omega_2}u$ in $\mathbb{C}^2 \times \mathbb{P}(\omega_1,\omega_2)$. The blow up surface Y is covered by two chart.

- On the chart $v = 1$ we have $x^{\omega_1} = y^{\omega_2}u$ and $\lambda^{\omega_2} = 1$. The action of $\mathbb{C}^*$ is given by $\lambda \cdot (y,u) = (\lambda^{\omega_2}y, \lambda^{\omega_1}u)$, so the point $x = y = u = 0$ is a cyclic quotient singularity of type $\frac{1}{\omega_2}(\omega_1,\omega_2)$.
- On the chart $u = 1$ we have $y^{\omega_2} = x^{\omega_1}v$ and $\lambda^{\omega_1} = 1$. The action of $\mathbb{C}^*$ is given by $\lambda \cdot (x,v) = (\lambda^{\omega_1}x, \lambda^{\omega_2}v)$, so the point $x = y = v = 0$ is a cyclic quotient singularity of type $\frac{1}{\omega_1}(\omega_1,\omega_2)$.

The singular points of Y are cyclic quotient singularities located at the exceptional divisor. Actually they coincide with the origins of the two charts.

Theorem 2.1.3. *The moduli space $\overline{M}_{1,2}$ is isomorphic to a weighted blow up of the weighted projective plane $\mathbb{P}(1,2,3)$ in its smooth point $[1:0:0]$. In particular $\overline{M}_{1,2}$ is a toric variety.*

Proof. Recall the description of $\overline{M}_{1,2}$ given at the beginning of this section. On the chart $\mathcal{U}_z := \{z \neq 0\}$ we define a morphism

$$f_{\mathcal{U}_z} : \mathcal{U}_z \to \mathbb{P}(1,2,3),\ (x,y,z,a,b) \mapsto (x, az^2, bz^3).$$

Note that the action of $\mathbb{C}^* \times \mathbb{C}^*$ on this triple is given by $(\xi\lambda^2, \xi^2\lambda^4, \xi^3\lambda^6)$, and $f_{\mathcal{U}_z}$ is indeed a well defined morphism to $\mathbb{P}(1,2,3)$.
On the open set $\{z \neq 0\}$ we can set $z = 1$ and ignore the action of ξ. If we forget y we can derive it up to a sign and this corresponds to the action of $\lambda = -1$.
Note that the morphism $f_{\mathcal{U}_z}$ maps the two singular point in $M_{1,2}$ we found in Proposition 2.1.1 in the points $[0:1:0], [0:0:1] \in \mathbb{P}(1,2,3)$, which are the only singularities of the weighted projective plane and of the same type of the singularities on $M_{1,2}$.
On $\mathcal{U}_y := \{y \neq 0\}$ the equation of $\overline{M}_{1,2}$ is $z = x^3 + axz^2 + bz^3$. So, as explained in the proof of Proposition 2.1.1 x is a local parameter near $z = 0$. We can consider the morphism

$$f_{\mathcal{U}_y}(x,y,z,a,b) = \left(1, a\left(\frac{x^2+az^2}{1-bz^2}\right)^2, b\left(\frac{x^2+az^2}{1-bz^2}\right)^3\right).$$

From this formulation it is clear that $f_{\mathcal{U}_y}$ is defined even on the locus $\{x = 0\}$ and the divisor $\Delta_{0,2} = \{x = z = 0\}$ is contracted in the smooth point $[1:0:0]$ of $\mathbb{P}(1,2,3)$.
On $\mathcal{U}_z \cap \mathcal{U}_y$ we have $\frac{z}{x} = \frac{x^2+az^2}{1-bz^2}$ and $f_{\mathcal{U}_z} = f_{\mathcal{U}_y}$, so $f_{\mathcal{U}_z}, f_{\mathcal{U}_y}$ glue to a morphism

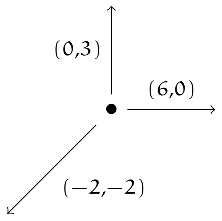
$$f : \overline{M}_{1,2} \to \mathbb{P}(1,2,3).$$

Then f is a blow up of $\mathbb{P}(1,2,3)$ in $[1:0:0]$ and $\Delta_{0,2}$ is the corresponding exceptional divisor. By Proposition 2.1.1 there are two singular points of type $\frac{1}{6}(2,4), \frac{1}{4}(2,6)$ on $\Delta_{0,2}$, and by Remark 2.1.2 the only way to obtain these two singularities is to perform a weighted blow up in $[1:0:0]$. □

Remark 2.1.4. The weighted projective space $\mathbb{P}(a_0, ..., a_n)$ is defined by

$$\mathbb{P}(a_0, ..., a_n) = \mathbb{P}(S),$$

where $a_0, ..., a_n$ are positive integers and S is the graded polynomial ring $k[x_0, ..., x_n]$, graded by $\deg(x_i) = a_i$.
Consider the set of vectors $V = \{e_1, ..., e_n, e_0 = -e_1 - ... - e_n\}$ in $\mathbb{R}^n$ and the fan whose cones are generated by proper subset of V in the lattice generated by $\frac{1}{a_1} e_i$ for $i = 0, ..., n$. The toric variety associated to this fan is $\mathbb{P}(a_0, ..., a_n)$. For what follows it is particularly interesting the fan of $\mathbb{P}(1, 2, 3)$:



Note that $(6, 0) + (0, 3) = 2(3, 1)$ and $(6, 0) + (-2, -2) = 2(2, -1)$. These points correspond to the two singular points of $\mathbb{P}(1, 2, 3)$. For a detailed toric description of the weighted projective space see [Ji, Section 3].

2.2 AUTOMORPHISMS OF $\overline{M}_{g,n}$

Our aim is to proceed by induction on n. The first step of induction is Proposition 2.2.5. In our argument the key fact is that the generic curve of genus $g > 2$ is automorphisms free. This is no longer true if $g = 2$ since every genus 2 curve is hyperelliptic and has a non trivial automorphism: the hyperelliptic involution. So we adopt a different strategy. First we prove that any automorphism of $\overline{M}_{2,1}$ preserves the boundary and then we apply a famous theorem of *H. L. Royden* which implies that $M_{g,n}^{un}$ (the moduli space of smooth genus g curves with unordered marked points) admits no non-trivial automorphisms or unramified correspondences for $2g - 2 + n \geqslant 3$ [Moc, Theorem 6.1]. In the case $g = 1$ the following observations will be crucial.

Remark 2.2.1. Let $[C, x_1, x_2]$ be a two pointed elliptic curve and let x_1 be the origin of the group law on C. Let $\tau : C \to C$ be the translation mapping x_2 in x_1, and let η be the elliptic involution. Then $\eta \circ \tau : C \to C$ is an automorphism of C switching x_1 and x_2. Then $[C, x_1, x_2] = [C, x_2, x_1]$ and $\overline{M}_{1,2} \cong \overline{M}_{1,2}^{un}$.

Lemma 2.2.2. *Any automorphism of $\overline{M}_{1,2}$ and $\overline{M}_{1,3}$ preserves the divisor $\Delta_{0,2}$.*

Proof. By Theorem 2.1.3 the divisor $\Delta_{0,2} \subset \overline{M}_{1,2}$ is the only contractible, smooth, rational curve in $\overline{M}_{1,2}$. Then it is stabilized by any automorphism.
By $\Delta_{0,2} \subset \overline{M}_{1,3}$ we mean the divisor parametrizing reducible curve $\mathbb{P}^1 \cup E$, where E is an elliptic tail, with two marked points on the rational tail and the remaining point is free. Let φ be an automorphism of $\overline{M}_{1,3}$ such that $\varphi(\Delta_{0,2}) \nsubseteq \Delta_{0,2}$ then composing φ with a morphism forgetting a marked point and considering the associated commutative diagram

$$\begin{tikzcd}
\overline{M}_{1,3} \arrow[r, "\varphi"] \arrow[d, "\pi_j"'] & \overline{M}_{1,3} \arrow[d, "\pi_i"] \\
\overline{M}_{1,2} \arrow[r, "\overline{\varphi}"] & \overline{M}_{1,2}
\end{tikzcd}$$

we get an automorphism $\overline{\varphi}$ of $\overline{M}_{1,2}$ which does not preserve $\Delta_{0,2}$. □

Lemma 2.2.3. *[GKM, Corollary 0.12] Any automorphism of $\overline{M}_g$ preserves the boundary.*

Proof. Let λ be the Hodge class on $\overline{M}_g$. It is known that λ induces a birational morphism $f : \overline{M}_g \to X$ on a projective variety whose exceptional locus is the boundary $\partial\overline{M}_g$ [Ru]. Assume that there exists an automorphism $\varphi : \overline{M}_g \to \overline{M}_g$ which does not preserve the boundary. Then there is a point $[C] \in \partial\overline{M}_g$ such that $\varphi([C]) = [C^{'}] \in M_g$.
Now $f \circ \varphi$ is a birational morphism whose exceptional locus is $\varphi^{-1}(\partial\overline{M}_g)$, and by the assumption on φ we have $\varphi^{-1}(\partial\overline{M}_g) \cap M_g \neq \emptyset$. So we construct a big line bundle on $\overline{M}_g$ whose exceptional locus is not contained in the boundary and this contradicts Theorem 2.0.17. ☐

Proposition 2.2.4. *For any $g \geqslant 2$ the only automorphism of $\overline{M}_g$ is the identity.*

Proof. Let φ be an automorphism of $\overline{M}_g$. By Lemma 2.2.3 φ restricts to an automorphisms $\varphi_{|M_g}$ of M_g. If $g \geqslant 3$ by Royden's theorem [Moc, Theorem 6.1] $\varphi_{|M_g}$ is the identity, then $\varphi = Id_{\overline{M}_g}$.
If $g = 2$ the canonical divisor K_C of a smooth genus 2 curve induces a degree 2 morphism on $\mathbb{P}^1$ branched in 6 points. So we have a morphism

$$f : M_2 \to M_{0,6}/S_6 \cong M_{0,6}^{un}$$

and since from a 6-pointed smooth rational curve we can reconstruct the corresponding genus 2 curve f is indeed an isomorphism. Then φ induces an automorphism $\tilde{\varphi}$ of $M_{0,6}^{un}$, again by [Moc, Theorem 6.1] we have $\tilde{\varphi} = Id_{M_{0,6}^{un}}$ and therefore $\varphi = Id_{\overline{M}_2}$. ☐

Proposition 2.2.5. *For any $g \geqslant 2$ the only automorphism of $\overline{M}_{g,1}$ is the identity. Furthermore* $Aut(\overline{M}_{1,3}) \cong S_3$.

Proof. Let $\varphi : \overline{M}_{g,1} \to \overline{M}_{g,1}$ be an automorphism. By Theorem 2.0.17 the fibration

$$\pi_1 \circ \varphi : \overline{M}_{g,1} \to \overline{M}_g$$

factors through a forgetful morphism which is necessarily π_1. We have a commutative diagram

$$\begin{tikzcd}
\overline{M}_{g,1} \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & \overline{M}_{g,1} \arrow[d, "\pi_1"] \\
\overline{M}_g \arrow[r, "\overline{\varphi}"] & \overline{M}_g
\end{tikzcd}$$

so the morphism φ maps the fiber of π_1 over $[C]$ to the fiber of π_1 over $[C^{'}] := \overline{\varphi}([C])$. Now we distinguish two cases.

- If $g > 2$ then $\pi_1^{-1}([C])$ is a smooth genus g curve, so it is automorphisms-free. Let $[C], [C^{'}] \in \overline{M}_g$ be two general points, then $\pi_1^{-1}([C]) \cong C$, $\pi_1^{-1}([C^{'}]) \cong C^{'}$ and

  $$\varphi_{|\pi_1^{-1}([C])} : C \to C^{'}$$

  is an isomorphism. So $C^{'} \cong C$, $[C^{'}] := \overline{\varphi}([C]) = [C]$ and $\overline{\varphi} = Id_{\overline{M}_g}$. We are thus reduced to a commutative triangle

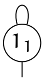
  $$\begin{tikzcd}
  \overline{M}_{g,1} \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & \overline{M}_{g,1} \arrow[dl, "\pi_1"] \\
  & \overline{M}_g &
  \end{tikzcd}$$

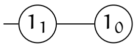
and for any $[C] \in \overline{M}_g$ the restriction of φ to the fiber of π_1 defines an automorphism of the fiber. Since $g > 2$ we conclude that φ is the identity on the general fiber of π_1 so it has to be the identity on $\overline{M}_{g,1}$.

- Consider now the case $g = 2$. Let $\varphi : \overline{M}_{2,1} \to \overline{M}_{2,1}$ be an automorphism. As usual we have a commutative diagram

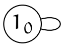
$$\begin{tikzcd}
\overline{M}_{2,1} \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & \overline{M}_{2,1} \arrow[d, "\pi_1"] \\
\overline{M}_2 \arrow[r, "\overline{\varphi}"] & \overline{M}_2
\end{tikzcd}$$

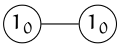
  The boundary of $\overline{M}_{2,1}$ has two codimension one components parametrizing curves whose dual graphs are





  Similarly the boundary of $\overline{M}_2$ has two irreducible components parametrizing curves with dual graphs





Clearly $\pi_1(\Delta_{irr,1}) = \Delta_{irr}$ and $\pi_1(\Delta_{1,1}) = \Delta_1$. Suppose that φ maps either the class of a nodal curve or the class of the union of two elliptic curves to the class of smooth genus 2 curve then $\overline{\varphi}$ has to do the same, and this contradicts Lemma 2.2.3.
Then φ maps an open subset of $\partial\overline{M}_{1,2}$ to an open subset of $\partial\overline{M}_{1,2}$ and both these open sets has to intersect the irreducible components of $\partial\overline{M}_{1,2}$. Now the continuity of φ is enough to conclude that φ preserves the boundary of $\overline{M}_{2,1}$.
Then φ restrict to an automorphism $M_{2,1} \to M_{2,1}$. By [Moc, Theorem 6.1] the only automorphism of $M_{2,1}$ is the identity. Finally $\varphi_{|M_{2,1}} = Id_{M_{2,1}}$ implies $\varphi = Id_{\overline{M}_{2,1}}$.
Consider now the case $g = 1, n = 3$. By Lemma 2.0.18 there exists a factorization $\pi_i \circ \varphi^{-1} = \overline{\varphi^{-1}} \circ \pi_{j_i}$, furthermore by Lemma 2.2.8 this factorization is unique. So we have a well defined morphism

$$\chi : Aut(\overline{M}_{1,3}) \to S_3,\ \varphi \mapsto \sigma_\varphi$$

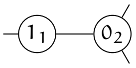
where

$$\sigma_\varphi : \{1,2,3\} \to \{1,2,3\},\ i \mapsto j_i.$$

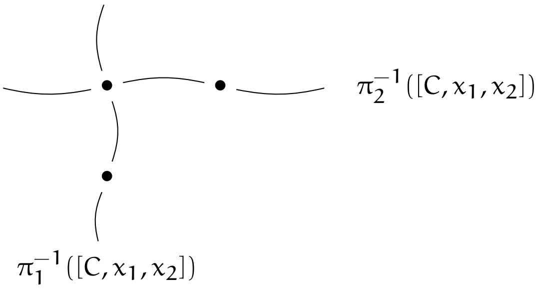
Let φ be an automorphism of $\overline{M}_{1,3}$ inducing the trivial permutation. Then φ^{-1} induces the trivial permutation as well and we have three commutative diagrams

$$\begin{tikzcd}
\overline{M}_{1,3} \arrow[r, "\varphi"] \arrow[d, "\pi_i"'] & \overline{M}_{1,3} \arrow[d, "\pi_i"] \\
\overline{M}_{1,2} \arrow[r, "\overline{\varphi}"] & \overline{M}_{1,2}
\end{tikzcd}$$

Let $[C, x_1, x_2] \in \overline{M}_{1,2}$ be a general point. The fiber $\pi_i^{-1}([C, x_1, x_2])$ intersects the boundary divisors $\Delta_{0,2} \subset \overline{M}_{1,3}$ in two points corresponding to curves with the following dual graph



The two points in $\pi_i^{-1}([C, x_1, x_2]) \cap \Delta_{0,2}$ can be identified with x_1, x_2. Now let $[C^{'}, x_1^{'}, x_2^{'}]$ be the image of $[C, x_1, x_2]$ via $\overline{\varphi}$. Similarly $\pi_i^{-1}([C^{'}, x_1^{'}, x_2^{'}]) \cap \Delta_{0,2} = \{x_1^{'}, x_2^{'}\}$. By Lemma 2.2.2 we have $\varphi(\pi_i^{-1}([C, x_1, x_2]) \cap \Delta_{0,2}) = \pi_i^{-1}([C^{'}, x_1^{'}, x_2^{'}]) \cap \Delta_{0,2}$ and by Remark 2.2.1 $[C^{'}, x_1^{'}, x_2^{'}] = [C, x_1, x_2]$ and $\overline{\varphi}$ has to be identity.
So φ restrict to an automorphism of the elliptic curve $\pi_1^{-1}([C, x_1, x_2]) \cong C$ mapping the set $\{x_1, x_2\}$ into itself. On the other hand φ restricts to an automorphism of the elliptic curve $\pi_2^{-1}([C, x_1, x_2]) \cong C$ with the same property. Note that $\pi_2^{-1}([C, x_1, x_2]) \cap \pi_1^{-1}([C, x_1, x_2]) = \{x_1\}$. The situation is resumed in the following picture:



Combining these two facts we have that φ restricts to an automorphism of $\pi_1^{-1}([C, x_1, x_2]) \cong C$ fixing x_1 and x_2. Since C is a general elliptic curve we have that $\varphi_{|\pi_1^{-1}([C, x_1, x_2])}$ is the identity, and since $[C, x_1, x_2] \in \overline{M}_{1,2}$ is general we conclude that $\varphi = Id_{\overline{M}_{1,3}}$. $\square$

The arguments used in the cases $g \geqslant 2$ and $g = 1, n \geqslant 3$ completely fail in the case $g = 1, n = 2$. However, Theorem 2.1.3 provides a very explicit description of $\overline{M}_{1,2}$ which allows us to describe its automorphism group. Since $\overline{M}_{1,2}$ is a toric surface we know that $(\mathbb{C}^*)^2 \subseteq \mathrm{Aut}(\overline{M}_{1,2})$.

Remark 2.2.6. The automorphisms of $\mathbb{P}(a_0, ..., a_n)$ are the automorphisms of the graded k-algebra $S = k[x_0, ..., x_n]$. In particular the automorphisms of $\mathbb{P}(1, 2, 3)$ are of the form

$$\begin{aligned} x_0 &\mapsto \alpha_0 x_0, \\ x_1 &\mapsto \alpha_1 x_0^2 + \beta_1 x_1, \\ x_2 &\mapsto \alpha_2 x_0^3 + \beta_2 x_0 x_1 + \gamma_2 x_2, \end{aligned}$$

and the the automorphisms of $\mathbb{P}(1, 2, 3)$ fixing $[1 : 0 : 0]$ are of the form

$$\begin{aligned} x_0 &\mapsto \alpha_0 x_0, \\ x_1 &\mapsto \beta_1 x_1, \\ x_2 &\mapsto \beta_2 x_0 x_1 + \gamma_2 x_2, \end{aligned}$$

with $\alpha_0, \beta_1, \gamma_2 \in k^*$ and $\beta_2 \in k$. The composition law in this group is given by

$$(\alpha_0, \beta_1, \beta_2, \gamma_2) * (\alpha_0^{'}, \beta_1^{'}, \beta_2^{'}, \gamma_2^{'}) = (\alpha_0 \alpha_0^{'}, \beta_1 \beta_1^{'}, \alpha_0 \beta_1 \beta_2^{'} + \beta_2 \gamma_2^{'}, \gamma_2 \gamma_2^{'}).$$

This remark highlights why the automorphisms of the coarse moduli space $\overline{M}_{g,n}$ in general should be different from the automorphisms of the stack $\overline{\mathcal{M}}_{g,n}$. It is well known that $\overline{M}_{1,1} \cong \mathbb{P}^1$ and $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}(4, 6)$. Clearly $\mathbb{P}^1 \cong \mathbb{P}(4, 6)$ as varieties, however they are not isomorphic as stacks, indeed $\mathbb{P}(4, 6)$ has two stacky points with stabilizers $\mathbb{Z}_4$ and $\mathbb{Z}_6$. These two points are fixed by any automorphism of $\mathbb{P}(4, 6)$ while they are indistinguishable from any other point on the coarse moduli space $\overline{M}_{1,1}$. By the previous description the automorphisms of $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}(4,6)$ are of the form

$$\begin{aligned} x_0 &\mapsto \alpha_0 x_0, \\ x_1 &\mapsto \beta_1 x_1, \end{aligned}$$

with $\alpha_0, \alpha_1 \in k^*$.

Proposition 2.2.7. *The automorphism group of* $\overline{M}_{1,2}$ *is isomorphic to* $(\mathbb{C}^*)^2$.

Proof. By Theorem 2.1.3 $\overline{M}_{1,2}$ is a weighted blow up of $\mathbb{P}(1,2,3)$ in $[1:0:0]$. Let φ be an automorphism of $\overline{M}_{1,2}$. Then we have a commutative diagram

$$\begin{tikzcd} \overline{M}_{1,2} \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & \overline{M}_{1,2} \arrow[d, "\pi_1"] \\ \overline{M}_{1,1} \arrow[r, "\overline{\varphi}"] & \overline{M}_{1,1} \end{tikzcd}$$

and φ has to map fibers of π_1 on fibers of π_1. Let $f : \overline{M}_{1,2} \to \mathbb{P}(1,2,3)$ be the contraction described in Theorem 2.1.3. Let $p_4, p_6 \in \Delta_{0,2}$ be the two singular points on the exceptional divisor, and let $q_4, q_6 \in M_{1,2}$ be the other two singular points. Since $\Delta_{0,2}$ is the only rational contractible curve in $\overline{M}_{1,2}$ it has to be stabilized by φ, furthermore $\varphi(p_4) = p_4$ and $\varphi(p_6) = p_6$. Let F_6 be the fiber of π_1 trough p_6, q_6 and let F_4 be the fiber of π_1 trough p_4, q_4. Since $\varphi(q_4) = q_4$ and $\varphi(q_6) = q_6$ we get $\varphi(F_4) = F_4$ and $\varphi(F_6) = F_6$.

We denote by $L_6 := f(F_6), L_4 := f(F_4)$ the images via f of F_6 and F_4 respectively. The automorphism φ induces via f an automorphism $\tilde{\varphi}$ of $\mathbb{P}(1,2,3)$ fixing $[1:0:0]$ and stabilizing L_6, L_4. Let G be the group

$$G := \{g \in \mathrm{Aut}(\mathbb{P}(1,2,3)) \mid g([1:0:0]) = [1:0:0],\ g(L_4) = L_4,\ g(L_6) = L_6\},$$

and consider the morphism of groups

$$\chi : \mathrm{Aut}(\overline{M}_{1,2}) \to G,\ \varphi \mapsto \tilde{\varphi}.$$

Clearly χ is injective.

Let x_0, x_1, x_2 be the coordinates on $\mathbb{P}(1,2,3)$. Note that the fiber F_6 corresponding to the Weierstrass curve C_6 and the fiber F_4 corresponding to the Weierstrass curve C_4 are mapped by f in the curves $L_6 = \{x_1 = 0\}$ and $L_4 = \{x_2 = 0\}$. By Remark 2.2.6 the automorphisms of $\mathbb{P}(1,2,3)$ fixing $[1:0:0]$ are of the form

$$\begin{aligned} x_0 &\mapsto \alpha_0 x_0, \\ x_1 &\mapsto \beta_1 x_1, \\ x_2 &\mapsto \beta_2 x_0 x_1 + \gamma_2 x_2, \end{aligned}$$

and forcing an automorphism to stabilize L_4 and L_6 gives $\beta_2 = 0$. Then the automorphisms in G are of the form

$$\begin{aligned} x_0 &\mapsto \alpha_0 x_0, \\ x_1 &\mapsto \beta_1 x_1, \\ x_2 &\mapsto \gamma_2 x_2, \end{aligned}$$

where $\alpha_0, \beta_1, \gamma_2 \in \mathbb{C}^*$, so $G \cong (\mathbb{C}^*)^2$. The automorphism $\tilde{\varphi}(x_0, x_1, x_2) = (\alpha_0 x_0, \beta_1 x_1, \gamma_2 x_2)$ is $\chi(\varphi)$ where φ is the automorphism of $\overline{M}_{1,2}$ acting as $\varphi(x, y, a, b) = (\alpha_0 x, \beta_1 a, \gamma_2 b)$. Consider the fibration $\overline{M}_{1,2} \to \overline{M}_{1,1}$. The automorphism φ acts on the couple (a, b) as an automorphism of $\overline{M}_{1,1} \cong \mathbb{P}^1$ and multiplying by α_0 on the fibers. So χ is surjective. ☐

In order to proceed by induction on n we need the following lemma.

Lemma 2.2.8. *Let $\varphi : \overline{M}_{g,n} \to \overline{M}_{g,n}$ be an automorphism. For any $j = 1, ..., n$ there exists a commutative diagram*

$$\begin{tikzcd}
\overline{M}_{g,n} \arrow[r, "\varphi"] \arrow[d, "\pi_i"'] & \overline{M}_{g,n} \arrow[d, "\pi_j"] \\
\overline{M}_{g,n-1} \arrow[r, "\overline{\varphi}"] & \overline{M}_{g,n-1}
\end{tikzcd}$$

- *The morphism $\overline{\varphi}$ is an automorphism of $\overline{M}_{g,n-1}$;*
- *the factorization of $\pi_j \circ \varphi$ is unique for any $j = 1, ..., n$.*

Proof. The existence of such a diagram is ensured by Theorem 2.0.17 and Lemma 2.0.18. Let $[C, x_1, ..., x_{n-1}] \in \overline{M}_{g,n-1}$ be a point, the automorphism φ^{-1} maps isomorphically the fiber of π_j over $[C, x_1, ..., x_{n-1}]$ to a fiber F of π_i, so $\pi_i(F) = [C', x'_1, ..., x'_{n-1}]$ is a point. Define $\overline{\psi} : \overline{M}_{g,n-1} \to \overline{M}_{g,n-1}$ as $\overline{\psi}([C, x_1, ..., x_{n-1}]) = [C', x'_1, ..., x'_{n-1}]$. Clearly $\overline{\psi}$ is the inverse of $\overline{\varphi}$.

Suppose that $\pi_j \circ \varphi$ admits two factorizations $\overline{\varphi}_1 \circ \pi_i$ and $\overline{\varphi}_2 \circ \pi_h$. Then the equality $\overline{\varphi}_1 \circ \pi_i([C, x_1, ..., x_n]) = \overline{\varphi}_2 \circ \pi_h([C, x_1, ..., x_n])$ for any $[C, x_1, ..., x_n] \in \overline{M}_{g,n}$ implies

$$\overline{\varphi}_1([C, y_1, ..., y_{n-1}]) = \overline{\varphi}_2([C, y_1, ..., y_{n-1}])$$

for any $[C, y_1, ..., y_{n-1}] \in \overline{M}_{g,n-1}$. Now $\overline{\varphi}_1 = \overline{\varphi}_2$ implies $\overline{\varphi}_1 \circ \pi_i = \overline{\varphi}_1 \circ \pi_h$ and since $\overline{\varphi}_1$ is an isomorphism we have $\pi_i = \pi_h$. ☐

At this point we can prove the general theorem by induction on n.

Theorem 2.2.9. *The automorphism group of $\overline{M}_{g,n}$ is isomorphic to the symmetric group on n elements S_n*

$$\mathrm{Aut}(\overline{M}_{g,n}) \cong S_n$$

for any g, n such that $2g - 2 + n \geqslant 3$.

Proof. Proposition 2.2.5 gives the cases $g \geqslant 2, n = 1$ and $g = 1, n = 3$. We proceed by induction on n. Let φ be an automorphism of $\overline{M}_{g,n}$, consider the composition $\pi_i \circ \varphi^{-1}$. By Theorem 2.0.17 there exists a factorization $\pi_i \circ \varphi^{-1} = \overline{\varphi^{-1}} \circ \pi_{j_i}$, furthermore by Lemma 2.2.8 this factorization is unique. So we have a well defined map

$$\chi : \mathrm{Aut}(\overline{M}_{g,n}) \to S_n,\ \varphi \mapsto \sigma_\varphi$$

where

$$\sigma_\varphi : \{1, ..., n\} \to \{1, ..., n\},\ i \mapsto j_i.$$

In order to prove that σ_φ is actually a permutation we prove that it is injective. Suppose to have $\sigma_\varphi(i) = j_i = \sigma_\varphi(h)$. This means that φ^{-1} defines an isomorphism between the fibers of π_{j_i} and π_i, but also between the fibers of π_{j_i} and π_h. This forces $\pi_i = \pi_h$.

We now prove that the map χ is a morphism of groups. Let $\varphi, \psi \in \overline{M}_{g,n}$ be two automorphisms. The fibration $\pi_i \circ \psi^{-1}$ factorizes through π_{j_i} and similarly $\pi_{j_i} \circ \varphi^{-1}$ factorizes though π_{h_i}. By uniqueness of the factorization $\pi_i \circ (\psi^{-1} \circ \varphi^{-1})$ factorizes through π_{h_i} also. The situation is resumed in the following commutative diagram

$$\begin{tikzcd}
\overline{M}_{g,n} \arrow[r, "\varphi^{-1}"] \arrow[d, "\pi_{h_i}"'] & \overline{M}_{g,n} \arrow[r, "\psi^{-1}"] \arrow[d, "\pi_{j_i}"] & \overline{M}_{g,n} \arrow[d, "\pi_i"] \\
\overline{M}_{g,n-1} \arrow[r, "\overline{\varphi^{-1}}"] \arrow[rr, bend right, "\overline{(\varphi\circ\psi)^{-1}}"'] & \overline{M}_{g,n-1} \arrow[r, "\overline{\psi^{-1}}"] & \overline{M}_{g,n-1}
\end{tikzcd}$$

This means that $\sigma_{\psi}(i) = j_i$, $\sigma_{\varphi}(j_i) = h_i$ and $\sigma_{\varphi\circ\psi}(i) = h_i$. Then $\sigma_{\varphi\circ\psi}(i) = \sigma_{\varphi}(j_i) = \sigma_{\varphi}(\sigma_{\psi}(i))$, that is $\chi(\varphi \circ \psi) = \chi(\varphi) \circ \chi(\psi)$.
Since any permutation of the marked points induces an automorphism of $\overline{M}_{g,n}$ the morphism χ is surjective. Now we compute its kernel.
Let $\varphi \in Aut(\overline{M}_{g,n})$ be an automorphism such that $\chi(\varphi)$ is the identity, that is for any $i = 1, ..., n$ the fibration $\pi_i \circ \varphi^{-1}$, and the fibration $\pi_i \circ \varphi$ as well, factor through π_i and we have n commutative diagrams

$$\begin{tikzcd}
\overline{M}_{g,n} \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & \overline{M}_{g,n} \arrow[d, "\pi_1"] \\
\overline{M}_{g,n-1} \arrow[r, "\overline{\varphi}_1"'] & \overline{M}_{g,n-1}
\end{tikzcd}
\quad \dots \quad
\begin{tikzcd}
\overline{M}_{g,n} \arrow[r, "\varphi"] \arrow[d, "\pi_n"'] & \overline{M}_{g,n} \arrow[d, "\pi_n"] \\
\overline{M}_{g,n-1} \arrow[r, "\overline{\varphi}_n"'] & \overline{M}_{g,n-1}
\end{tikzcd}$$

By Lemma 2.2.8 the morphisms $\overline{\varphi}_i$ are automorphisms of $\overline{M}_{g,n-1}$ and by induction hypothesis $\overline{\varphi}_1, ..., \overline{\varphi}_n$ act on $\overline{M}_{g,n-1}$ as permutations.
The action of $\overline{\varphi}_i$ on the marked points $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n$ has to lift to the same automorphism φ for any $i = 1, ..., n$. So the actions of $\overline{\varphi}_1, ..., \overline{\varphi}_n$ have to be compatible and this implies $\overline{\varphi}_i = Id_{\overline{M}_{g,n-1}}$ for any $i = 1, ..., n$. We distinguish two cases.

- Assume $g \geqslant 3$. It is enough to observe that φ restricts to an automorphism of the fibers of π_1. Then φ restricts to the identity on the general fiber of π_1, so $\varphi = Id_{\overline{M}_{g,n}}$.
- Assume $g = 1, 2$. Note that φ restricts to an automorphism of the fibers of π_1 and π_2. So φ defines an automorphism of the fiber of π_1 with at least two fixed points in the case $g = 1, n \geqslant 3$ and one fixed point in the case $g = 2, n \geqslant 2$. Since the general 2-pointed genus 1 curve and the general 1-pointed genus 2 curves have no non trivial automorphisms we conclude as before that φ restricts to the identity on the general fiber of π_1, so $\varphi = Id_{\overline{M}_{g,n}}$.

This proves that χ is injective and defines an isomorphism between $Aut(\overline{M}_{g,n})$ and S_n. ☐

We want to use the techniques developed in this section to recover [BM2, Theorem 4.3]. The moduli spaces $\overline{M}_{0,4}$ is isomorphic to the projective line $\mathbb{P}^1$ while $\overline{M}_{0,5}$ is the blow-up of $\mathbb{P}^2$ in four points in general position. The following is well known but we want to give a proof following the argument used in Proposition 2.2.5.

Proposition 2.2.10. *The automorphism group of* $\overline{M}_{0,5}$ *is isomorphic to* S_5.

Proof. It is well known that any fibration $\overline{M}_{0,5} \to \overline{M}_{0,4}$ factorizes through a forgetful morphism, see for instance [BM2]. This yields a surjective morphism of groups

$$\chi : Aut(\overline{M}_{0,5}) \to S_5$$

exactly as in Theorem 2.2.9. Let φ be an automorphism of $\overline{M}_{0,5}$ inducing the trivial permutation. Then φ^{-1} induces the trivial permutation as well and we get five commutative diagrams

$$\begin{tikzcd}
\overline{M}_{0,5} \arrow[r, "\varphi"] \arrow[d, "\pi_i"'] & \overline{M}_{0,5} \arrow[d, "\pi_i"] \\
\overline{M}_{0,4} \arrow[r, "\overline{\varphi}_i"] & \overline{M}_{0,4}
\end{tikzcd}$$

for $i = 1, ..., 5$. The fiber of π_i on $[C, x_1, ..., x_4] \in \overline{M}_{0,4}$ intersects the boundary $\partial\overline{M}_{0,4}$ in four points corresponding to $x_1, ..., x_4$. Consider $[C', x_1', ..., x_4'] := \overline{\varphi}_{i|[C,x_1,...,x_4]}([C, x_1, ..., x_4])$. The points in $\pi_i^{-1}([C, x_1, ..., x_4]) \cap \partial\overline{M}_{0,4}$ and in $\pi_i^{-1}([C', x_1', ..., x_4']) \cap \partial\overline{M}_{0,4}$ lie on (-1)-curves, so the automorphism φ maps the fiber of π_i over $[C, x_1, ..., x_4]$ to the fiber of π_i over $[C', x_1', ..., x_4^i]$ sending the set $\{x_1, ..., x_4\}$ to the set $\{x_1', ..., x_4'\}$. Then $\overline{\varphi}_1, ..., \overline{\varphi}_5$ act as permutations of the marking and since they come from the same automorphism φ they have to be compatible. This forces $\overline{\varphi}_1 = ... = \overline{\varphi}_5 = Id_{\overline{M}_{0,4}}$.
Let $[C, x_1, ..., x_4] \in \overline{M}_{0,4}$ be a general point. The automorphism φ restricts to an automorphism of the fiber $\pi_1^{-1}([C, x_1, ..., x_4]) \cong \mathbb{P}^1$ stabilizing the subscheme $\{x_1, ..., x_4\} \subset \pi_1^{-1}([C, x_1, ..., x_4])$. Since $x_1, ..., x_4$ are general points of C they have a cross-ratio different from the cross-ratio of each permutation. This means that $\varphi_{|C}$ is an automorphism of $\mathbb{P}^1$ fixing four points. So φ restricts to the identity on the general fiber of π_1 and this forces $\varphi = Id_{\overline{M}_{0,5}}$. $\square$

Remark 2.2.11. The moduli space $\overline{M}_{0,5}$ is isomorphic to a Del Pezzo surface of degree 5, by Proposition 2.2.10 we recover that the automorphism group of such a surface is S_5. For a direct proof of this classical fact which does not use the theory of moduli spaces see [DI, Section 3].

Now with the same argument of Theorem 2.2.9 we can prove the following:

Theorem 2.2.12. *The automorphism group of $\overline{M}_{0,n}$ is isomorphic to the symmetric group on n elements S_n*

$$Aut(\overline{M}_{0,n}) \cong S_n$$

for any $n \geqslant 5$.

Proof. The step zero of the induction is Proposition 2.2.10. As usual we have a surjective morphism of groups

$$\chi : \overline{M}_{0,n} \to S_n.$$

Proceeding as in the proof of Theorem 2.2.9 we get that an automorphism φ inducing the trivial permutation has to restrict to an automorphism of the fiber of $\pi_i : \overline{M}_{0,n} \to \overline{M}_{0,n-1}$ fixing $k \geqslant 4$ points. So it has to be the identity on the general fiber of π_i, and therefore also on $\overline{M}_{0,n}$. $\square$

In [GKM, Corollary 0.12] *Gibney*, *Keel* and *Morrison* proved that any automorphism of $\overline{M}_g$ must preserve the boundary.
From Theorem 2.2.9 follows immediately that the boundary of $\overline{M}_{g,n}$ has a good behavior under the action of $Aut(\overline{M}_{g,n})$. The result is even stronger than the preservation of the boundary.

Corollary 2.2.13. *If $2g - 2 + n \geqslant 3$ any automorphism of $\overline{M}_{g,n}$ must preserve all strata of the boundary.*

Proof. Since any automorphism is a permutation the class of a pointed curve $[C, x_1, ..., x_n]$ is mapped by an automorphism in a class $[C', x'_1, ..., x'_n]$ representing a pointed curve of the same topological type of the pointed curve C. ☐

2.3 AUTOMORPHISMS OF $\overline{\mathcal{M}}_{g,n}$

Let $\mathcal{X}$ be an algebraic stack over $\mathbb{C}$. A coarse moduli space for $\mathcal{X}$ over $\mathbb{C}$ is a morphism $\pi : \mathcal{X} \to X$, where X is an algebraic space over $\mathbb{C}$ such that

- the morphism π is universal for morphisms to algebraic spaces,
- π induces a bijection between $|\mathcal{X}|$ and the closed points of X, where $|\mathcal{X}|$ denotes the set of isomorphism classes in $\mathcal{X}$.

Remark 2.3.1. If $\mathcal{X}$ admits a coarse moduli space $\pi : \mathcal{X} \to X$ then this is unique up to unique isomorphism.

A separated algebraic stack has a coarse moduli space which is a separated algebraic space [KM, Corollary 1.3].

Let $\mathcal{X}$ be a separated stack admitting a scheme X as coarse moduli space $\pi : \mathcal{X} \to X$. The map π is universal for morphisms in schemes, that is for any morphism $f : \mathcal{X} \to Y$, with Y scheme, there exists a unique morphisms of schemes $g : X \to Y$ such that the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "\pi"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

commutes. Now, let $\varphi : \mathcal{X} \to \mathcal{X}$ be an automorphism of the stack $\mathcal{X}$, and consider $\pi \circ \varphi : \mathcal{X} \to X$. Then these exists a unique $\tilde{\varphi}$ such that the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\varphi"] \arrow[d, "\pi"'] & \mathcal{X} \arrow[d, "\pi"] \\
X \arrow[r, "\tilde{\varphi}"] & X
\end{tikzcd}$$

commutes. By uniqueness we have $(\tilde{\varphi})^{-1} = \widetilde{\varphi^{-1}}$. So $\tilde{\varphi}$ is an automorphisms of X, and we get a morphism of groups

$$\mathrm{Aut}(\mathcal{X}) \to \mathrm{Aut}(X),\ \varphi \mapsto \tilde{\varphi}.$$

Remark 2.3.2. Even if $\mathcal{X}$ is a Deligne-Mumford stack with trivial generic stabilizer the above morphism of groups is not necessarily injective. As instance in [ACV, Proposition 7.1.1] *D. Abramovich*, *A. Corti* and *A. Vistoli* consider a twisted curve $\mathcal{C}$ over an algebraically closed field and its coarse moduli space C. They prove that for any node $x \in C$ the stabilizer of a geometric point of $\mathcal{C}$ over x contributes to the automorphism group of $\mathcal{C}$ over C.

However since $\overline{\mathcal{M}}_{g,n}$ is a normal, Deligne-Mumford stack, as soon as its general point has trivial stabilizer, the morphism

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \to \mathrm{Aut}(\overline{M}_{g,n})$$

is injective. Our next goal is to prove this last statement.

Lemma 2.3.3. *Let $f : X \to Y$ be a finite morphism from a scheme X to an irreducible normal variety Y, let $U \subseteq Y$ be an open dense subscheme of Y, and let $s : U \to X$ be a section of f over U. Then $s : U \to X$ extends to a section $\overline{s} : Y \to X$.*

Proof. Consider the fiber product

$$\begin{tikzcd}
V := U \times_Y X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X \arrow[d, "f"] \\
U \arrow[r] & Y
\end{tikzcd}$$

and let V_s be the closure of $(\mathrm{Id}_U \times s)(U)$ in V. Now $\mathrm{Id}_U \times s : U \to V_s$ and $\pi_{1|V_s} : V_s \to U$ are birational. Since π_2 is an open embedding we have that $\pi_{2|V_s}$ is dominant. Let Z be the closure of $\pi_2(V_s)$ in X. Then $f_{|Z} : Z \to Y$ is birational and quasi-finite. Since Y is an irreducible normal variety the Zariski main theorem implies that $f_{|Z}$ is an isomorphism. The inverse $(f_{|Z})^{-1}$ is the section $\overline{s}$ we were looking for. ☐

Proposition 2.3.4. *[FMN, Proposition A.1] Let $\mathcal{X}, \mathcal{Y}$ be Deligne-Mumford stacks, let $f_1, f_2 : \mathcal{X} \to \mathcal{Y}$ be morphisms of stacks, and let $i : \mathcal{U} \hookrightarrow \mathcal{X}$ be a dominant open immersion. Assume $\mathcal{X}$ normal and $\mathcal{Y}$ separated. If there is a 2-arrow $\alpha : f_1 \circ i \Longrightarrow f_2 \circ i$ then there exists a unique 2-arrow $\overline{\alpha} : f_1 \Longrightarrow f_2$ such that $\overline{\alpha} * \mathrm{Id}_i = \alpha$.*

Proof. Since $\mathcal{X}$ is a normal Deligne-Mumford stack there exists an affine étale chart of $\mathcal{X}$ which is a disjoint union of affine irreducible normal schemes. So we can assume that $\mathcal{X}$ is an affine irreducible normal scheme X. We denote by U the dense open subscheme $\mathcal{U}$ in X.
Now consider the morphism $(f_1 \times f_2) : X \to \mathcal{Y} \times \mathcal{Y}$, the diagonal morphism $\Delta_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ and their fiber product:

$$\begin{tikzcd}
Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{Y} \arrow[d, "\Delta_{\mathcal{Y}}"] \\
X \arrow[r, "f_1 \times f_2"] & \mathcal{Y} \times \mathcal{Y}
\end{tikzcd}$$

note that since $\mathcal{Y}$ is separated $\Delta_{\mathcal{Y}}$ is proper, then $\Delta_{\mathcal{Y}}$ is finite and Z is a scheme. Similarly we can consider the fiber product of $\pi_1 : Z \to X$, $i : U \hookrightarrow X$ and summing up the situation in the following diagram.

$$\begin{tikzcd}
U \arrow[rr, "i"] \arrow[dr, "F"] \arrow[ddr, bend right, "\mathrm{Id}_U"'] & & X \arrow[dr, dashed, "G"] \arrow[drr, bend left, "f_1"] \arrow[ddr, bend right, "\mathrm{Id}_X"'] & & \\
& V \arrow[rr, "p_2"] \arrow[d, "p_1"'] & & Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathcal{Y} \arrow[d, "\Delta_{\mathcal{Y}}"] \\
& U \arrow[rr, "i"] & & X \arrow[r, "f_1 \times f_2"] & \mathcal{Y} \times \mathcal{Y}
\end{tikzcd}$$

Now recall that we have a 2-arrow $\alpha : f_1 \circ i \Longrightarrow \varphi_2 \circ i$, by the universal property of the fiber product there exists a morphism $F : U \to V$. The existence of a 2-arrow $\overline{\alpha} : f_1 \Longrightarrow f_2$ such that $\overline{\alpha} * \mathrm{Id}_i = \alpha$ is now equivalent to the existence of a morphism $G : X \to Z$ such that $\pi_1 \circ G = \mathrm{Id}_X$ and $G \circ i = p_2 \circ F$.
Since $\Delta_{\mathcal{Y}}$ is finite and Z is a scheme we have that $\pi_1 : Z \to X$ is finite and $p_2 \circ F : U \to Z$ is a section of π_1 over U. Now X is an irreducible normal scheme and by Lemma 2.3.3 the

section $p_2 \circ F : U \to Z$ can be extended uniquely to a section $G : X \to X$ which is exactly the morphism we were looking for.
It remains to prove the uniqueness. Assume that $\mathcal{X}$ is a scheme X and $\mathcal{Y}$ is a global quotient $[Z/G]$ where G is a separated group scheme. The morphism $f_i : X \to [Z/G]$ is given by a G-principal bundle $\pi_i : P_i \to X$ and a G-equivariant morphism $P_i \to Z$ for $i = 1, 2$. Suppose that $\alpha, \beta : P_1 \to P_2$ are morphisms such that $\alpha_{|\pi_1^{-1}(U)} = \beta_{|\pi_2^{-1}(U)}$. Since G is separated we have that π_i is separated, so $\alpha = \beta$.
Now remove the assumption that $\mathcal{X}$ is a scheme but still consider the case $\mathcal{Y} = [Z/G]$. Let X be an étale atlas of $\mathcal{X}$. By the first part of the proof we have that $\alpha_{|X} = \beta_{|X}$, since $\mathrm{Mor}(f_1, f_2)$ is a sheaf on $\mathcal{X}$ we have that $\alpha = \beta$.
Finally if $\mathcal{Y}$ is not a global quotient we cover it by global quotients and conclude using the fact that $\mathrm{Mor}(f_1, f_2)$ is a sheaf on $\mathcal{X}$. ☐

Proposition 2.3.5. *The morphism of groups*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \to \mathrm{Aut}(\overline{M}_{g,n})$$

is injective as soon as the general n-pointed genus g curve has no non trivial automorphisms.

Proof. In Proposition 2.3.4 take $\mathcal{X} = \mathcal{Y} = \overline{\mathcal{M}}_{g,n}$. Since we consider the case when the general n-pointed genus g curve has no non trivial automorphisms there is a dense open subscheme $U \subset \overline{M}_{g,n}$ where the canonical map $\overline{\mathcal{M}}_{g,n} \to \overline{M}_{g,n}$ is an isomorphism. Note that $\overline{\mathcal{M}}_{g,n}$ is an irreducible normal and separated Deligne-Mumford stack, so the hypothesis of Proposition 2.3.4 are satisfied.
Let $f : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be an automorphism inducing the identity on the coarse moduli space $\overline{M}_{g,n}$, then there is a 2-arrow $\alpha : f_{|U} \implies Id_U$. By Proposition 2.3.4 there exists a unique 2-arrow $\overline{\alpha} : f \implies Id_{\overline{\mathcal{M}}_{g,n}}$ extending α. We conclude that $\overline{\alpha}$ is an isomorphism and f is isomorphic to the identity of $\overline{\mathcal{M}}_{g,n}$. ☐

Theorem 2.3.6. *The automorphism group of the stack $\overline{\mathcal{M}}_{g,n}$ is isomorphic to the symmetric group on n elements S_n*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \cong S_n$$

for any g, n such that $2g - 2 + n \geqslant 3$. Furthermore $\mathrm{Aut}(\overline{\mathcal{M}}_g)$ is trivial for any $g \geqslant 2$.

Proof. For any g, n in our range the general point of $\overline{\mathcal{M}}_{g,n}$ has trivial automorphism group. So by Proposition 3.2.18 the morphism of groups

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \to \mathrm{Aut}(\overline{M}_{g,n})$$

is injective. By Theorem 2.2.9 and [BM2, Theorem 4.3] we know that $\mathrm{Aut}(\overline{M}_{g,n}) \cong S_n$ for the values of g and n we are considering. Since any permutation of the marked points in an automorphism of $\overline{\mathcal{M}}_{g,n}$ we conclude that

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,n}) \cong \mathrm{Aut}(\overline{M}_{g,n}) \cong S_n.$$

Since the general curve of genus $g \geqslant 3$ is automorphisms free the morphism

$$\mathrm{Aut}(\overline{\mathcal{M}}_g) \to \mathrm{Aut}(\overline{M}_g)$$

is injective. We conclude by Proposition 2.2.4. In the case $g = 2$ consider the fiber product

$$\begin{tikzcd}
\overline{\mathcal{M}}_{2,1} \times_{\overline{\mathcal{M}}_2} \overline{\mathcal{M}}_2 \cong \overline{\mathcal{M}}_{2,1} \arrow[r, "\psi"] \arrow[d] & \overline{\mathcal{M}}_{2,1} \arrow[d, "\pi_1"] \\
\overline{\mathcal{M}}_2 \arrow[r, "\varphi"] & \overline{\mathcal{M}}_2
\end{tikzcd}$$

where $\varphi \in \mathrm{Aut}(\overline{\mathcal{M}}_2)$. Since φ is an automorphism ψ also is an automorphism. By the previous part of the proof we know that $\mathrm{Aut}(\overline{\mathcal{M}}_{2,1}) \cong \mathrm{Aut}(\overline{M}_{2,1})$ is trivial. So $\psi = \mathrm{Id}_{\overline{\mathcal{M}}_{2,1}}$ and therefore $\varphi = \mathrm{Id}_{\overline{\mathcal{M}}_2}$. □

As we saw in Proposition 2.2.7 the case $g = 1, n = 2$ is pathological from the point of view of the automorphisms. Since $\mathrm{Aut}(\overline{M}_{1,2}) \cong (\mathbb{C}^*)^2$ the injectivity of the morphism $\mathrm{Aut}(\overline{\mathcal{M}}_{1,2}) \to \mathrm{Aut}(\overline{M}_{1,2})$ does not say to much on $\mathrm{Aut}(\overline{\mathcal{M}}_{1,2})$. Since all the automorphisms of $\overline{M}_{1,2}$ are toric we expect them to disappear on the stack. In the following proposition we prove that $\mathrm{Aut}(\overline{\mathcal{M}}_{1,2})$ is trivial exploiting the particular form of its canonical divisor.

Proposition 2.3.7. *The only automorphism of the moduli stack $\overline{\mathcal{M}}_{1,2}$ is the identity.*

Proof. An application of the Grothendieck-Riemann-Roch theorem [HM, Section 3E] gives the following formula for the canonical class of $\overline{\mathcal{M}}_{1,2}$

$$K_{\overline{\mathcal{M}}_{1,2}} = 13\lambda - 2\delta + \psi \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{1,2}).$$

The Picard group $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{1,2})$ is freely generated by λ and the boundary classes, furthermore the following relations hold [AC, Theorem 2.2]:

$$\delta_{irr} = 12\lambda,\ \psi = 2\lambda + 2\delta_{0,2}.$$

We can write the canonical class in terms of the boundary divisors as

$$K_{\overline{\mathcal{M}}_{1,2}} = \frac{13}{12}\delta_{irr} - 2\delta_{irr} - 2\delta_{0,2} + \frac{2}{12}\delta_{irr} + 2\delta_{0,2} = -\frac{3}{4}\delta_{irr}.$$

Note that δ_{irr} is a fiber of the forgetful morphism $\pi_1 : \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{1,1}$. Any automorphism φ of $\overline{\mathcal{M}}_{1,2}$ preserves the canonical bundle, that is $\varphi^* K_{\overline{\mathcal{M}}_{1,2}} = K_{\overline{\mathcal{M}}_{1,2}}$ in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{1,2})$. Since $K_{\overline{\mathcal{M}}_{1,2}}$ is a multiple of the fiber δ_{irr} the fibration $\pi_1 \circ \varphi$ factorizes through π_1 (recall that by Remark 2.2.1 on $\overline{\mathcal{M}}_{1,2}$ the forgetful morphisms induce the same fibration). So we have the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathcal{M}}_{1,2} \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & \overline{\mathcal{M}}_{1,2} \arrow[d, "\pi_1"] \\
\overline{\mathcal{M}}_{1,1} \arrow[r, "\overline{\varphi}"] & \overline{\mathcal{M}}_{1,1}
\end{tikzcd}$$

Let $[C,p] \in \overline{\mathcal{M}}_{1,1}$ be a general point and let $[C',p'] = \overline{\varphi}([C,p])$ be its image. Then $\alpha := \varphi_{|\pi_1^{-1}([C,p])}$ defines an isomorphism between C and C'. If $q' = \alpha(p)$ then there exists an automorphism τ' of C' mapping q' to p'. So $\tau' \circ \alpha$ is an isomorphism between C and C' mapping p to p'. This means that $[C,p] = [C',p']$, $\overline{\varphi}$ is the identity and φ restricts to an automorphism of the fiber of π_1, furthermore by Lemma 2.2.2 has to preserve the boundary divisor $\delta_{0,2}$. The general fiber of π_1 is a general elliptic curve, so it has only two automorphisms. Clearly both these automorphisms act trivially on $\overline{\mathcal{M}}_{1,2}$, so $\varphi = \mathrm{Id}_{\overline{\mathcal{M}}_{1,2}}$. □

3

HASSETT'S MODULI SPACES

We work over an algebraically closed field of characteristic zero. We introduce Hassett's moduli spaces and their relations with the Kapranov's realizations of $\overline{M}_{0,n}$. Let S be a Noetherian scheme and g, n two non-negative integers. A family of nodal curves of genus g with n marked points over S consists of a flat proper morphism $\pi : C \to S$ whose geometric fibers are nodal connected curves of arithmetic genus g, and sections $s_1, ..., s_n$ of π. A collection of input data $(g, A) := (g, a_1, ..., a_n)$ consists of an integer $g \geqslant 0$ and the weight data: an element $(a_1, ..., a_n) \in \mathbb{Q}^n$ such that $0 < a_i \leqslant 1$ for $i = 1, ..., n$, and

$$2g - 2 + \sum_{i=1}^{n} a_i > 0.$$

Definition 3.0.8. A family of nodal curves with marked points $\pi : (C, s_1, ..., s_n) \to S$ is stable of type (g, A) if

- the sections $s_1, ..., s_n$ lie in the smooth locus of π, and for any subset $\{s_{i_1}, ..., s_{i_r}\}$ with non-empty intersection we have $a_{i_1} + ... + a_{i_r} \leqslant 1$,
- $K_\pi + \sum_{i=1}^{n} a_i s_i$ is π-relatively ample.

B. Hassett in [Has, Theorem 2.1] proved that given a collection (g, A) of input data, there exists a connected Deligne-Mumford stack $\overline{\mathcal{M}}_{g,A[n]}$, smooth and proper over $\mathbb{Z}$, representing the moduli problem of pointed stable curves of type (g, A). The corresponding coarse moduli scheme $\overline{M}_{g,A[n]}$ is projective over $\mathbb{Z}$.
Furthermore by [Has, Theorem 3.8] a weighted pointed stable curve admits no infinitesimal automorphisms and its infinitesimal deformation space is unobstructed of dimension $3g - 3 + n$. Then $\overline{\mathcal{M}}_{g,A[n]}$ is a smooth Deligne-Mumford stack of dimension $3g - 3 + n$.

Remark 3.0.9. Since $\overline{\mathcal{M}}_{g,A[n]}$ is smooth as a Deligne-Mumford stack the coarse moduli space $\overline{M}_{g,A[n]}$ has finite quotient singularities, that is étale locally it is isomorphic to a quotient of a smooth scheme by a finite group. In particular $\overline{M}_{g,A[n]}$ is normal.

Fixed g, n, consider two collections of weight data $A[n], B[n]$ such that $a_i \geqslant b_i$ for any $i = 1, ..., n$. Then there exists a birational *reduction morphism*

$$\rho_{B[n],A[n]} : \overline{M}_{g,A[n]} \to \overline{M}_{g,B[n]}$$

associating to a curve $[C, s_1, ..., s_n] \in \overline{M}_{g,A[n]}$ the curve $\rho_{B[n],A[n]}([C, s_1, ..., s_n])$ obtained by collapsing components of C along which $K_C + b_1 s_1 + ... + b_n s_n$ fails to be ample.
Furthermore, for any g consider a collection of weight data $A[n] = (a_1, ..., a_n)$ and a subset $A[r] := (a_{i_1}, ..., a_{i_r}) \subset A$ such that $2g - 2 + a_{i_1} + ... + a_{i_r} > 0$. Then there exists a *forgetful morphism*

$$\pi_{A[n],A[r]} : \overline{M}_{g,A[n]} \to \overline{M}_{g,A[r]}$$

associating to a curve $[C, s_1, ..., s_n] \in \overline{M}_{g,A[n]}$ the curve $\pi_{A[n],A[r]}([C, s_1, ..., s_n])$ obtained by collapsing components of C along which $K_C + a_{i_1} s_{i_1} + ... + a_{i_r} s_{i_r}$ fails to be ample.
For the details see [Has, Section 4].
In the following we will be especially interested in the boundary of $\overline{M}_{g,A[n]}$. The boundary

of $\overline{M}_{g,A[n]}$, as for $\overline{M}_{g,n}$, has a stratification whose loci, called strata, parametrize curves of a certain topological type and with a fixed configuration of the marked points.
We denote by Δ_{irr} the locus in $\overline{M}_{g,A[n]}$ parametrizing irreducible nodal curves with n marked points, and by $\Delta_{i,P}$ the locus of curves with a node which divides the curve into a component of genus i containing the points indexed by P and a component of genus $g-i$ containing the remaining points.

Kapranov's blow-up constructions

We follow [Ka]. Let $(C, x_1, ..., x_n)$ be a genus zero n-pointed stable curve. The dualizing sheaf ω_C of C is invertible, see [Kn]. By [Kn, Corollaries 1.10 and 1.11] the sheaf $\omega_C(x_1 + ... + x_n)$ is very ample and has $n-1$ independent sections. Then it defines an embedding $\varphi : C \to \mathbb{P}^{n-2}$. In particular if $C \cong \mathbb{P}^1$ then $\deg(\omega_C(x_1 + ... + x_n)) = n-2$, $\omega_C(x_1 + ... + x_n) \cong \varphi^*\mathcal{O}_{\mathbb{P}^{n-2}}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n-2)$, and $\varphi(C)$ is a degree $n-2$ rational normal curve in $\mathbb{P}^{n-2}$. By [Ka, Lemma 1.4] if $(C, x_1, ..., x_n)$ is stable the points $p_i = \varphi(x_i)$ are in linear general position in $\mathbb{P}^{n-2}$.
This fact combined with a careful analysis of limits in $\overline{M}_{0,n}$ of 1-parameter families in $M_{0,n}$ led *M. Kapranov* to prove the following theorem:

Theorem 3.0.10. *[Ka, Theorem 0.1] Let $p_1, ..., p_n \in \mathbb{P}^{n-2}$ be n points in linear general position, and let $V_0(p_1, ..., p_n)$ be the scheme parametrizing rational normal curves through $p_1, ..., p_n$. Consider $V_0(p_1, ..., p_n)$ as a subscheme of the Hilbert scheme $\mathcal{H}$ parametrizing subschemes of $\mathbb{P}^{n-2}$. Then*

- $V_0(p_1, ..., p_n) \cong M_{0,n}$.
- *Let $V(p_1, ..., p_n)$ be the closure of $V_0(p_1, ..., p_n)$ in $\mathcal{H}$. Then $V(p_1, ..., p_n) \cong \overline{M}_{0,n}$.*

Kapranov's construction allows to translate many issues of $\overline{M}_{0,n}$ into statements on linear systems on $\mathbb{P}^{n-3}$. Consider a general line $L_i \subset \mathbb{P}^{n-2}$ through p_i. There is a unique rational normal curve C_{L_i} through $p_1, ..., p_n$ and with tangent direction L_i in p_i. Let $[C, x_1, ..., x_n] \in \overline{M}_{0,n}$ be a stable curve and let $\Gamma \in V_0(p_1, ..., p_n)$ be the corresponding curve. Since $p_i \in \Gamma$ is a smooth point considering the tangent line $T_{p_i}\Gamma$, with some work [Ka], we get a morphism

$$f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}, \ [C, x_1, ..., x_n] \mapsto T_{p_i}\Gamma.$$

Furthermore f_i is birational and it defines an isomorphism on $M_{0,n}$. The birational maps $f_j \circ f_i^{-1}$

$$\begin{tikzcd}
& \overline{M}_{0,n} \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\
\mathbb{P}^{n-3} \arrow[rr, dashed, "f_j \circ f_i^{-1}"] & & \mathbb{P}^{n-3}
\end{tikzcd}$$

are standard Cremona transformations of $\mathbb{P}^{n-3}$ [Ka, Proposition 2.12]. For any $i = 1, ..., n$ the class Ψ_i is the line bundle on $\overline{M}_{0,n}$ whose fiber on $[C, x_1, ..., x_n]$ is the tangent line $T_{p_i}C$. From the previous description we see that the line bundle Ψ_i induces the birational morphism $f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$, that is $\Psi_i = f_i^*\mathcal{O}_{\mathbb{P}^{n-3}}(1)$. In [Ka] Kapranov proved that Ψ_i is big and globally generated, and that the birational morphism f_i is an iterated blow-up of the projections from p_i of the points $p_1, ..., \hat{p_i}, ...p_n$ and of all strict transforms of the linear spaces they generate, in order of increasing dimension.

Construction 3.0.11. [Ka] More precisely, fixed $(n-1)$-points $p_1, ..., p_{n-1} \in \mathbb{P}^{n-3}$ in linear general position:

(1) Blow-up the points $p_1,\dots,p_{n-2}$, then the lines $\langle p_i, p_j \rangle$ for $i,j = 1,\dots,n-2$,..., the $(n-5)$-planes spanned by $n-4$ of these points.

(2) Blow-up p_{n-1}, the lines spanned by pairs of points including p_{n-1} but not p_{n-2},..., the $(n-5)$-planes spanned by $n-4$ of these points including p_{n-1} but not p_{n-2}.

$\vdots$

(r) Blow-up the linear spaces spanned by subsets $\{p_{n-1}, p_{n-2},\dots,p_{n-r+1}\}$ so that the order of the blow-ups in compatible by the partial order on the subsets given by inclusion, the $(r-1)$-planes spanned by r of these points including $p_{n-1}, p_{n-2},\dots,p_{n-r+1}$ but not p_{n-r},..., the $(n-5)$-planes spanned by $n-4$ of these points including $p_{n-1}, p_{n-2},\dots,p_{n-r+1}$ but not p_{n-r}.

$\vdots$

$(n-3)$ Blow-up the linear spaces spanned by subsets $\{p_{n-1}, p_{n-2},\dots,p_4\}$.

The composition of these blow-ups is the morphism $f_n : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ induced by the psi-class Ψ_n. Identifying $\overline{M}_{0,n}$ with $V(p_1,\dots,p_n)$, and fixing a general $(n-3)$-plane $H \subset \mathbb{P}^{n-2}$, the morphism f_n associates to a curve $C \in V(p_1,\dots,p_n)$ the point $T_{p_n} C \cap H$.
We denote by $W_{r,s}[n]$ the variety obtained at the r-th step once we finish blowing-up the subspaces spanned by subsets S with $|S| \leqslant s+r-2$, and by $W_r[n]$ the variety produced at the r-th step. In particular $W_{1,1}[n] = \mathbb{P}^{n-3}$ and $W_{n-3}[n] = \overline{M}_{0,n}$.

In [Has, Section 6.1] Hassett interprets the intermediate steps of Construction 3.0.11 as moduli spaces of weighted rational curves. Consider the weight data

$$A_{r,s}[n] := (\underbrace{1/(n-r-1),\dots,1/(n-r-1)}_{(n-r-1)\text{ times}}, s/(n-r-1), \underbrace{1,\dots,1}_{r\text{ times}})$$

for $r = 1,\dots,n-3$ and $s = 1,\dots,n-r-2$. Then $W_{r,s}[n] \cong \overline{M}_{0,A_{r,s}[n]}$, and the Kapranov's map $f_n : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ factorizes as a composition of reduction morphisms

$$\begin{aligned} &\rho_{A_{r,s-1}[n],A_{r,s}[n]} : \overline{M}_{0,A_{r,s}[n]} \to \overline{M}_{0,A_{r,s-1}[n]}, \; s = 2,\dots,n-r-2, \\ &\rho_{A_{r,n-r-2}[n],A_{r+1,1}[n]} : \overline{M}_{0,A_{r+1,1}[n]} \to \overline{M}_{0,A_{r,n-r-2}[n]}. \end{aligned}$$

Remark 3.0.12. The Hassett's space $\overline{M}_{A_{1,n-3}[n]}$, that is $\mathbb{P}^{n-3}$ blown-up at all the linear spaces of codimension at least two spanned by subsets of $n-2$ points in linear general position, is the Losev-Manin's moduli space $\overline{L}_{n-2}$ introduced by *A. Losev* and *Y. Manin* in [LM], see [Has, Section 6.4]. The space $\overline{L}_{n-2}$ parametrizes $(n-2)$-pointed chains of projective lines $(C, x_0, x_\infty, x_1,\dots,x_{n-2})$ where:

- C is a chain of smooth rational curves with two fixed points x_0, x_∞ on the extremal components,
- $x_1,\dots,x_{n-2}$ are smooth marked points different from x_0, x_∞ but non necessarily distinct,
- there is at least one marked point on each component.

By [LM, Theorem 2.2] there exists a smooth, separated, irreducible, proper scheme representing this moduli problem. Note that after the choice of two marked points in $\overline{M}_{0,n}$ playing the role of x_0, x_∞ we get a birational morphism $\overline{M}_{0,n} \to \overline{L}_{n-2}$ which is nothing but a reduction morphism.
For example $\overline{L}_1$ is a point parametrizing a $\mathbb{P}^1$ with two fixed points and a free point, $\overline{L}_2 \cong \mathbb{P}^1$, and $\overline{L}_3$ is $\mathbb{P}^2$ blown-up at three points in general position, that is a Del Pezzo surface of degree six, see [Has, Section 6.4] for further generalizations.

We develop in some details the simplest case in genus zero.

Example 3.0.13. Let $n = 5$, and fix $p_1, ..., p_4 \in \mathbb{P}^2$ points in general position. The first step consists in blowing-up p_1, p_2, p_3, and in the second step we blow up p_4.
The Kapranov's map $f_5 : \overline{M}_{0,5} \to \mathbb{P}^2$ is the projection from $p_5 \in \mathbb{P}^3$. At the step $r = 1, s = 1$ we get $W_{1,1}[5] = \mathbb{P}^2$ and the weights are

$$A_{1,1}[5] := (1/3, 1/3, 1/3, 1/3, 1).$$

While for $r = 2, s = 1$ we get $W_{2,1}[5] = W_2[n] \cong \overline{M}_{0,5}$, indeed in this case the weight data are

$$A_{2,1}[5] := (1/2, 1/2, 1/2, 1, 1).$$

Note that as long as all the weights are strictly greater than 1/3, the Hassett's space is isomorphic to $\overline{M}_{0,n}$ because at most two points can collide, so the only components that get contracted are rational tail components with exactly two marked points. Since these have exactly three special points they have no moduli and contracting them does not affect the coarse moduli space even though it does change the universal curve, see also [Has, Corollary 4.7]. In our case $\overline{M}_{0,A_{2,1}[5]} \cong \overline{M}_{0,5}$.
We have only one intermediate step, namely $r = 1, s = 2$. The moduli space $W_{1,2}[5] \cong \overline{M}_{0,A_{1,2}[5]}$ parametrizes weighted pointed curves with weight data

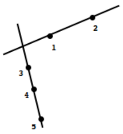
$$A_{1,2}[5] := (1/3, 1/3, 1/3, 2/3, 1).$$

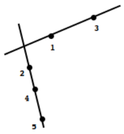
This means that the point p_5 is allowed to collide with p_1, p_2, p_3 but not with p_4 which has not yet been blown-up. The Kapranov's map $f_5 : \overline{M}_{0,5} \to \mathbb{P}^2$ factorizes as

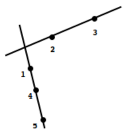
$$\begin{tikzcd}
\overline{M}_{0,5} \cong \overline{M}_{0,A_{2,1}[5]} \arrow[dd, "f_5"'] \arrow[dr, "\rho_1"] & \\
& \overline{M}_{0,A_{1,2}[5]} \arrow[dl, "\rho_2"] \\
\mathbb{P}^2 \cong \overline{M}_{0,A_{1,1}[5]} &
\end{tikzcd}$$

where ρ_1, ρ_2 are the corresponding reduction morphisms. Let us analyze these two morphisms.

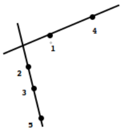
- Given $(C, s_1, ..., s_5) \in \overline{M}_{0,A_{2,1}[5]}$ the curve $\rho_1(C, s_1, ..., s_5)$ is obtained by collapsing components of C along which $K_C + \frac{1}{3}s_1 + \frac{1}{3}s_2 + \frac{1}{3}s_3 + \frac{2}{3}s_4 + s_5$ fails to be ample. So it contracts the 2-pointed components of the following curves:

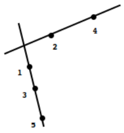
  

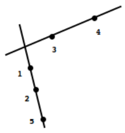
  

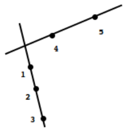
  along which $K_C + \frac{1}{3}s_1 + \frac{1}{3}s_2 + \frac{1}{3}s_3 + \frac{2}{3}s_4 + s_5$ is anti-ample, and the 2-pointed components of the following curves:

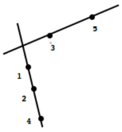
  

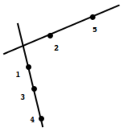
along which $K_C + \frac{1}{3}s_1 + \frac{1}{3}s_2 + \frac{1}{3}s_3 + \frac{2}{3}s_4 + s_5$ is nef but not ample. However all the contracted components have exactly three special points, and therefore they does not have moduli. This affects only the universal curve but not the coarse moduli space. Finally $K_C + \frac{1}{3}s_1 + \frac{1}{3}s_2 + \frac{1}{3}s_3 + \frac{2}{3}s_4 + s_5$ is nef but not ample on the 3-pointed component of the curve

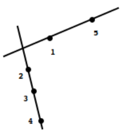


In fact this corresponds to the contraction of the divisor $E_{5,4} = f_5^{-1}(p_4)$.

- The morphism ρ_2 contracts the 3-pointed components of the curves







along which $K_C + \frac{1}{3}s_1 + \frac{1}{3}s_2 + \frac{1}{3}s_3 + \frac{1}{3}s_4 + s_5$ has degree zero. This corresponds to the contractions of the divisors $E_{5,3} = f_5^{-1}(p_3)$, $E_{5,2} = f_5^{-1}(p_2)$ and $E_{5,1} = f_5^{-1}(p_1)$.

There are many other factorizations of the morphisms $f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ as compositions of reduction morphisms. Another example is the following construction due to Kapranov [Ka].

Construction 3.0.14. Fixed $(n-1)$-points $p_1, ..., p_{n-1} \in \mathbb{P}^{n-3}$ in linear general position:

(1) Blow-up the points $p_1, ..., p_{n-1}$,

(2) Blow-up the strict transforms of the lines $\langle p_{i_1}, p_{i_2}\rangle$, $i_1, i_2 = 1, ..., n-1$,

$\vdots$

(k) Blow-up the strict transforms of the $(k-1)$-planes $\langle p_{i_1}, ..., p_{i_k}\rangle$, $i_1, ..., i_k = 1, ..., n-1$,

$\vdots$

$(n-4)$ Blow-up the strict transforms of the $(n-5)$-planes $\langle p_{i_1}, ..., p_{i_{n-4}}\rangle$, $i_1, ..., i_{n-4} = 1, ..., n-1$.

Now, consider the Hassett's spaces $X_k[n] := \overline{M}_{0,A[n]}$ for $k = 1, ..., n-4$, such that

- $a_1 + a_n > 1$ for $i = 1, ..., n-1$,
- $a_{i_1} + ... + a_{i_h} \leqslant 1$ for each $\{i_1, ..., i_h\} \subset \{1, ..., n-1\}$ with $r \leqslant n-k-2$,
- $a_{i_1} + ... + a_{i_h} > 1$ for each $\{i_1, ..., i_h\} \subset \{1, ..., n-1\}$ with $r > n-k-2$.

Then $X_k[n]$ is isomorphic to the variety obtained at the step k of the blow-up construction, see [Has, Section 6.2] for the details.

This section is devoted to study fiber type morphisms of Hassett's moduli spaces. The results are based on and generalize Bruno-Mella type argument [BM2] for genus zero, and [GKM, Theorem 0.9] on fibrations of $\overline{M}_{g,n}$.
Let us start with the genus zero case. In what follows we adapt the proofs and results of [BM2] to this generalized setting. For this purpose we restrict ourselves to the Hassett's spaces satisfying the following definition.

Definition 3.1.1. We say that a Hassett's moduli space $\overline{M}_{0,A[n]}$ *factors Kapranov* if there exists a morphism ρ_2 that makes the following diagram commutative

$$\begin{tikzcd}
\overline{M}_{0,n} \arrow[d, "\rho_1"'] \arrow[dr, "f_i"] & \\
\overline{M}_{0,A[n]} \arrow[r, "\rho_2"] & \mathbb{P}^{n-3}
\end{tikzcd}$$

where f_i is a Kapranov's map and ρ_1 is a reduction. We call such a ρ_2 a *Kapranov factorization*. Note that if a Hassett's moduli space $\overline{M}_{0,A[n]}$ factors a Kapranov's map f_i then it factors any other Kapranov's map f_j.

Remark 3.1.2. There are Hassett's spaces that do not factor Kapranov. For instance consider the Hassett's spaces appearing in [Has, Section 6.3]. The space $\overline{M}_{0,A[5]}$ with

$$A[5] = (1-2\epsilon, 1-2\epsilon, 1-2\epsilon, \epsilon, \epsilon)$$

where ϵ is an arbitrarily small positive rational number, is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Therefore $\overline{M}_{0,A[5]}$ does not admit any birational morphism on $\mathbb{P}^2$. Note that the forgetful morphisms forgetting the fourth and the fifth point correspond to the natural projections from $\mathbb{P}^1 \times \mathbb{P}^1$. Let us stress that these are the only morphisms of these moduli spaces and no birational reduction is allowed.
Furthermore, note that the Hassett's spaces appearing in Constructions 3.0.11 and 3.0.14 factor Kapranov by construction.

Lemma 3.1.3. *Let $\overline{M}_{0,A[n]}$ be a Hassett's space that factors Kapranov and $\pi^H_{i_1,\dots,i_{n-r}} : \overline{M}_{0,A[n]} \to \overline{M}_{0,A[r]}$ be a forgetful morphism, where $A[r]$ is the weight data associated to the indexes $\{1,\dots,n\} \setminus \{i_1,\dots,i_{n-r}\}$. Then $\overline{M}_{0,A[r]}$ factors Kapranov as well.*

Proof. Consider the following diagram

$$\begin{tikzcd}
\overline{M}_{0,n} \arrow[rrr, "\pi_{i_1,\dots,i_{n-r}}"] \arrow[dr, "\rho_1"] \arrow[ddd, "f_k"'] & & & \overline{M}_{0,r} \arrow[dl, "\rho_1'"'] \arrow[ddd, "f_k'"] \\
& \overline{M}_{0,A[n]} \arrow[r, "\pi^H_{i_1,\dots,i_{n-r}}"] \arrow[d, "\mu"'] \arrow[ddl, "\rho_2"'] & \overline{M}_{0,A[r]} \arrow[l, bend left, "s_{i_1,\dots,i_{n-r},j}"] \arrow[ddr, "\rho_2'"] & \\
& X_H \arrow[dl, "p"] \arrow[drr, "\nu"] & & \\
\mathbb{P}^{n-3} \arrow[rrr, dashed, "\pi_H"] & & & \mathbb{P}^{r-3}
\end{tikzcd}$$

where $\pi_{i_1,...,i_{n-r}}$ is the forgetful morphism on $\overline{M}_{0,n}$ corresponding to $\pi^H_{i_1,...,i_{n-r}}$, $f^{'}_k$ is the Kapranov's map corresponding to f_k with $k \notin \{i_1, ..., i_{n-r}\}$, and π_H is the projection from the linear space $H = \langle p_1, ..., p_{n-r}\rangle$ induced by $\pi^H_{i_1,...,i_{n-r}}$. Furthermore, let X_H be the blow-up of $\mathbb{P}^{n-3}$ along H. We want to define $\rho^{'}_1$ and $\rho^{'}_2$.
The birational morphism $\rho^{'}_1$ is simply the reduction morphism induced by ρ_1 on $\overline{M}_{0,r}$. Now, consider a section $s_{i_1,...,i_{n-r},j} : \overline{M}_{0,A[r]} \to \overline{M}_{0,A[n]}$ of $\pi^H_{i_1,...,i_{n-r}}$, with $j \neq k$, associating to $[C, x_1, ..., x_r]$ the isomorphism class of the stable curve obtained by adding at x_j a smooth rational curve with $n - r + 1$ marked points, labeled by $x_j, x_{i_1}, ..., x_{i_{n-r}}$. Since $j \neq k$ the image of $s_{i_1,...,i_{n-r},j}$ is not contained in the exceptional locus of μ, and we have a birational morphism $\rho^{'}_2 := \nu \circ \mu \circ s_{i_1,...,i_{n-r},j}$. Clearly $f^{'}_k = \rho^{'}_2 \circ \rho^{'}_1$. □

Proposition 3.1.4. *Assume that $\overline{M}_{0,A[n]}$ factors Kapranov. Then any dominant morphism with connected fibers $f : \overline{M}_{0,A[n]} \to \overline{M}_{0,4} \cong \mathbb{P}^1$ factors through a forgetful map.*

Proof. Let $f : \overline{M}_{0,A[n]} \to \overline{M}_{0,4} \cong \mathbb{P}^1$ be a dominant morphism and $\rho_1 : \overline{M}_{0,n} \to \overline{M}_{0,A[n]}$ a reduction morphism. The composition $f \circ \rho_1 : \overline{M}_{0,n} \to \mathbb{P}^1$ is a dominant morphism with connected fibers. By [BM2, Theorem 3.7] $f \circ \rho_1$ factorizes through a forgetful map $\pi := \pi_{i_1,...,i_{n-4}}$ and by hypothesis we may choose a Kapranov's map f_j yielding a factorization as follows

$$\begin{tikzcd}
 & \overline{M}_{0,n} \arrow[d, "\rho_1"] \arrow[dr, "f_j"] \arrow[ddl, "\pi"'] & \\
 & \overline{M}_{0,A[n]} \arrow[r, "\rho_2"] \arrow[d, "f"] \arrow[dl, "\pi^H"'] & \mathbb{P}^{n-3} \arrow[d, dashed, "\tilde{\pi}"] \\
\overline{M}_{0,4} \arrow[r, "\varphi"] & \overline{M}_{0,4} \cong \mathbb{P}^1 & \mathbb{P}^1 \arrow[l]
\end{tikzcd}$$

where $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ and $\tilde{\pi}$ is a linear projection from a codimension two linear space. This yields that the base locus of $\tilde{\pi}$ is resolved by the morphism ρ_2. So the forgetful map π is defined also on $\overline{M}_{0,A[n]}$ and gives rise to the following diagram

$$\begin{tikzcd}
 & \overline{M}_{0,n} \arrow[d, "\rho_1"] \arrow[ddl, "\pi"'] \\
 & \overline{M}_{0,A[n]} \arrow[d, "f"] \arrow[dl, "\pi^H"'] \\
\overline{M}_{0,4} \arrow[r, "\varphi"] & \overline{M}_{0,4} \cong \mathbb{P}^1
\end{tikzcd}$$

where $\pi^H := \pi^H_{i_1,...,i_{n-4}}$. On $M_{0,A[n]}$ the fibration f coincides with $\varphi \circ \pi^H_{i_1,...,i_{n-4}}$, and since $M_{0,A[n]}$ is an open dense subset of $\overline{M}_{0,A[n]}$ we have $f = \varphi \circ \pi^H_{i_1,...,i_{n-4}}$. □

Assume that $\overline{M}_{0,A[n]}$ factors Kapranov. Then a forgetful morphism $\pi_{i_1,...,i_{n-r}} : \overline{M}_{0,A[n]} \to \overline{M}_{0,A[r]}$ induces a linear projection $\pi_H : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{r-3}$, where $H = \langle p_1, ..., p_{n-r} \rangle$ is the span of $p_1, ..., p_{n-r}$ in the Kapranov's description of $\overline{M}_{0,n}$. We want to prove that a sort of converse is also true.

Proposition 3.1.5. *Assume that* $\overline{M}_{0,A[n]}$ *factors Kapranov. Let* $f : \overline{M}_{0,A[n]} \to X$ *be a surjective morphism on a projective variety. Let* $D \in \text{Pic}(X)$ *be a base point free divisor and* $\mathcal{L}_i = \rho_*(f^*(D))$, *where* $\rho_n : \overline{M}_{0,A[n]} \to \mathbb{P}^{n-3}$ *is a reduction morphism. If* $\text{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i$ *for some* j *then* f *factors through the forgetful map* $\pi_j : \overline{M}_{0,A[n]} \to \overline{M}_{0,A[n-1]}$.
Assume that $\overline{M}_{0,B[r]}$ *factors Kapranov. Let* $f : \overline{M}_{0,A[n]} \to \overline{M}_{0,B[r]}$ *be a surjective morphism and* $\pi : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{r-3}$ *the induced map on the projective spaces. We have the following commutative diagram*

$$\begin{tikzcd}
\overline{M}_{0,A[n]} \arrow[r, "f"] \arrow[d, "\rho_n"'] & \overline{M}_{0,B[r]} \arrow[d, "\rho_r"] \\
\mathbb{P}^{n-3} \arrow[r, dashed, "\pi"] & \mathbb{P}^{r-3}
\end{tikzcd}$$

where ρ_n *and* ρ_r *are Kapranov factorizations. Let* $\mathcal{L}_i = \rho_{n*}(f^*(\rho_{r*}^{-1}(\mathcal{O}(1))))$ *and assume*

$$\mathcal{L}_i = |\mathcal{O}_{\mathbb{P}^{n-3}}(1) \otimes \mathcal{I}_{\langle p_{i_1},...,p_{i_s} \rangle}|,$$

then $s = n - r$ *and* f *factorizes via the forgetful map* $\pi_{i_1,...,i_{n-r}} : \overline{M}_{0,A[n]} \to \overline{M}_{0,A[r]}$.

Proof. If $\pi_j : \overline{M}_{0,A[n]} \to \overline{M}_{0,A[n-1]}$ is a forgetful morphism, then the fibers of π_j are mapped by a reduction morphism $\rho_n : \overline{M}_{0,A[n]} \to \mathbb{P}^{n-3}$ to lines through p_j. The general element in the linear system $|\mathcal{L}_i|$ restricts on a line through p_j to a divisor of degree $\deg \mathcal{L}_i - \text{mult}_{p_j} \mathcal{L}_i$. Since $\text{mult}_{p_j} \mathcal{L}_i = \deg \mathcal{L}_i$ we have that $\mathcal{L}_i$ is numerically trivial on lines through p_j. Then $f^*(D)$ is base point free and numerically trivial on every fiber of π_j. Furthermore $\text{Pic}(\overline{M}_{0,A[n]}/\overline{M}_{0,A[n-1]}) = \text{Num}(\overline{M}_{0,A[n]}/\overline{M}_{0,A[n-1]})$, then $f^*(D)$ is π_j-trivial. We conclude that f contracts fibers of π_j.
Consider the morphism $s_{j,h} : \overline{M}_{0,A[n-1]} \to \overline{M}_{0,A[n]}$ mapping $[(C, x_1, ..., \hat{x}_j, ..., x_n)]$ to the isomorphism class of the n-pointed stable curve obtained by attaching at x_j a $\mathbb{P}^1$ marked with two points with labels x_j and x_h. Then $s_{j,h}$ is a section of π_j, the morphism $g := f \circ s_{j,h}$ makes the diagram

$$\begin{tikzcd}
\overline{M}_{0,A[n]} \arrow[rr, "f"] \arrow[dr, "\pi_j"] & & X \\
& \overline{M}_{0,A[n-1]} \arrow[ul, bend left, "s_{j,h}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutative, and f factorizes through π_j.
Now, assume $\mathcal{L}_i = |\mathcal{O}_{\mathbb{P}^{n-3}}(1) \otimes \mathcal{I}_{\langle p_{i_1},...,p_{i_s} \rangle}|$. For any p_{i_j} we have $\text{mult}_{p_{i_j}} \mathcal{L}_i = \deg \mathcal{L}_i$. By the first statement f factors through π_{i_k} for any $k \in \{i_i, \ldots, i_s\}$. The generic fiber of f has dimension $n - r$, therefore $s = n - r$ and f factors through $\pi_{i_1,...,i_{n-r}} : \overline{M}_{0,A[n]} \to \overline{M}_{0,A[r]}$. □

The following is the statement we were looking for in the genus zero case.

Theorem 3.1.6. *Assume that* $\overline{M}_{0,A[n]}$ *and* $\overline{M}_{0,B[r]}$ *factor Kapranov. Let* $f : \overline{M}_{0,A[n]} \to \overline{M}_{0,B[r]}$ *be a dominant morphism with connected fibers. Then* f *factors through a forgetful map* $\pi_I : \overline{M}_{0,A[n]} \to \overline{M}_{0,A[r]}$.

Proof. We proceed by induction on $\dim \overline{M}_{0,B[r]}$. Let $\rho_r : \overline{M}_{0,B[r]} \to \mathbb{P}^{r-3}$ be a Kapranov factorization, and consider a forgetful map $\pi_{r-1} : \overline{M}_{0,B[r]} \to \overline{M}_{0,B[r-1]}$. We denote by $E_{i,j}$ the image of the section $s_{i,j} : \overline{M}_{0,B[r-1]} \to \overline{M}_{0,B[r]}$, note that $E_{i,j}$ is the divisor parametrizing reducible curves $C_1 \cup C_2$, where C_1 is a smooth rational curve with $r-2$ marked points, and C_2 is a smooth rational curve with two marked points labeled by x_i, x_j.
The first induction step is Proposition 3.1.4. By Lemma 3.1.3 the space $\overline{M}_{0,B[r-1]}$ factors Kapranov. So we may consider a Kapranov factorization $\rho_{r-1} : \overline{M}_{0,B[r-1]} \to \mathbb{P}^{r-4}$, and the linear projection $\pi : \mathbb{P}^{r-3} \dashrightarrow \mathbb{P}^{r-4}$ induced by $|\mathcal{O}_{\mathbb{P}^{r-3}}(1) \otimes \mathcal{I}_{p_{r-1}}|$. The morphism $\pi_{r-1} \circ f$ is dominant and with connected fibers, hence we may apply the induction hypothesis to it. So we can choose a Kapranov factorization $\rho_n : \overline{M}_{0,A[n]} \to \mathbb{P}^{n-3}$ such that

$$\rho_{n*}((\rho_r \circ f)_*^{-1}(|\mathcal{O}_{\mathbb{P}^{r-3}}(1) \otimes \mathcal{I}_{p_{r-1}}|)) \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|. \tag{3.1.1}$$

We may assume, without loss of generality, that $\rho_r^{-1}(p_{r-1}) = E_{r,r-1}$. Let us summarize the situation in the following commutative diagram

$$\begin{tikzcd}
\overline{M}_{0,A[n]} \arrow[r, "f"] \arrow[d, "\rho_n"'] & \overline{M}_{0,B[r]} \arrow[r, "\pi_{r-1}"] \arrow[d, "\rho_r"'] & \overline{M}_{0,B[r-1]} \arrow[d, "\rho_{r-1}"] \\
\mathbb{P}^{n-3} \arrow[r, dashed, "\beta"] \arrow[rr, dashed, bend right, "\alpha"'] & \mathbb{P}^{r-3} \arrow[r, dashed, "\pi"] & \mathbb{P}^{r-4}
\end{tikzcd}$$

where $\beta = \rho_r \circ f \circ \rho_n^{-1}$, and $\alpha = \pi \circ \beta$ is a linear projection. By Proposition 3.1.5 to conclude it is enough to show that $\rho_{n*}((\rho_r \circ f)^*(\mathcal{O}_{\mathbb{P}^{r-3}}(1))) \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|$. Hence, by equation (3.1.1), it is enough to show that $f^*(E_{r,r-1})$ is contracted by ρ_n.
Let $\mathcal{L}$ be the line bundle on $\mathbb{P}^{n-3}$ inducing the map

$$\alpha = \rho_{r-1} \circ \pi_{r-1} \circ f \circ \rho_n^{-1}.$$

By induction hypothesis we may assume $\mathcal{L} = |\mathcal{O}_{\mathbb{P}^{n-3}}(1) \otimes \mathcal{I}_P|$, where $P = \langle p_{r-1}, ..., p_{n-1} \rangle$, and $\alpha(p_j) = p_j$, $\pi(p_j) = p_j$ for $j < r-1$.
For any $E_{j,r} \neq E_{r-1,r}$ the map $\pi_{r-1|E_{j,r}} : \overline{M}_{0,B[r-1]} \to \overline{M}_{0,B[r-1]}$ is a forgetful map onto $\overline{M}_{0,B[r-2]}$. Then for any $E_{i,r} \subset \overline{M}_{0,B[r]}$, with $i < r$, we have

$$f^*(E_{i,r}) = (\pi_{r-1} \circ f)^*(E_{i,r-1}) = E_{i,n},$$

so $f^*(E_{i,r})$ is contracted by ρ_n for any $i < r-1$.
Fixed a reduction morphism $\rho_n : \overline{M}_{0,A[n]} \to \mathbb{P}^{n-3}$, consider a forgetful morphism $\pi_i : \overline{M}_{0,B[r]} \to \overline{M}_{0,B[r-1]}$ with $i < r$. To any such forgetful morphism we associate a Kapranov factorization $\rho_{n,i} : \overline{M}_{0,A[n]} \to \mathbb{P}^{n-3}$ such that $f^*(E_{j,r}) = E_{j,i}$ for $i \neq j$. However the divisor $E_{i,j}$ is contracted to a point only by the Kapranov factorizations $\rho_{n,i}, \rho_{n,j}$ factoring f_i, f_j respectively. Then the image of $E_{i,r}$ via $\rho_{n*} \circ f^*$ does not depend on the map π_i, so $\rho_{n*} \circ f^*$ is a point for any forgetful morphism $\pi_i : \overline{M}_{0,B[r]} \to \overline{M}_{0,B[r-1]}$, and

$$\rho_n^*(\mathcal{O}_{\mathbb{P}^{r-3}}(1)) = \rho_{r*}^{-1}(|\mathcal{O}_{\mathbb{P}^{r-3}}(1) \otimes \mathcal{I}_{p_{r-1}}|) + E_{r-1,r}.$$

Then, if $\mathcal{E}$ is the line bundle on $\mathbb{P}^{n-3}$ inducing α, we get

$$\mathcal{E} = \rho_{n*}((\rho_r \circ f)^*(\mathcal{O}_{\mathbb{P}^{r-3}}(1))) = \rho_{n*}((\rho_r \circ f)^*(|\mathcal{O}_{\mathbb{P}^{r-3}}(1) \otimes \mathcal{I}_{p_{r-1}}|)) \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(1)|.$$

So α is induced by a linear system of hyperplanes, that is α is a linear projection, and by Proposition 3.1.5 we conclude. $\square$

Next we concentrate on higher genera. If $g \geqslant 1$ then all forgetful morphisms are always well defined. Therefore the following is just a simple adaptation of [GKM, Theorem 0.9].

Proposition 3.1.7. *Let* $f : \overline{M}_{g,A[n]} \to X$ *be a dominant morphism with connected fibers.*

- *If* $g \geqslant 2$ *either* f *is of fiber type and factorizes through a forgetful morphism* $\pi_I : \overline{M}_{g,A[n]} \to \overline{M}_{g,A[r]}$*, or* f *is birational and* $\mathrm{Exc}(f) \subseteq \partial\overline{M}_{g,A[n]}$.
- *If* $g = 1$ *either* f *is of fiber type and factorizes through a product* $\pi_S^H \times \pi_{S^c}^H : \overline{M}_{1,A[n]} \to \overline{M}_{1,A[i]} \times_{\overline{M}_{1,A[m]}} \overline{M}_{1,A[n-i]}$ *for some subset* S *of the markings, or* f *is birational and* $\mathrm{Exc}(f) \subseteq \partial\overline{M}_{1,A[n]}$.

Proof. By [Has, Theorem 4.1] any Hassett's moduli space $\overline{M}_{g,A[n]}$ receives a birational reduction morphism $\rho_n : \overline{M}_{g,n} \to \overline{M}_{g,A[n]}$ restricting to the identity on $M_{g,n}$. The composition $f \circ \rho_n : \overline{M}_{g,n} \to X$ gives a fibration of $\overline{M}_{g,n}$ to a projective variety.
If f is of fiber type by [GKM, Theorem 0.9] the morphism $f \circ \rho_n$ factorizes through a forgetful map $\pi_i : \overline{M}_{g,n} \to \overline{M}_{g,i}$, with $i < n$, and a morphism $\alpha : \overline{M}_{g,i} \to X$. Considering the corresponding forgetful map $\pi_i^H : \overline{M}_{g,A[n]} \to \overline{M}_{g,A[i]}$ on the Hassett's spaces, and another birational morphism $\rho_i : \overline{M}_{g,i} \to \overline{M}_{g,A[i]}$ restricting to the identity on $M_{g,i}$, we get the following commutative diagram:

$$\begin{tikzcd}
\overline{M}_{g,n} \arrow[r, "\pi_i"] \arrow[d, "\rho_n"'] & \overline{M}_{g,i} \arrow[d, "\rho_i"] \arrow[ddr, bend left, "\alpha"] & \\
\overline{M}_{g,A[n]} \arrow[r, "\pi_i^H"] \arrow[drr, bend right, "f"'] & \overline{M}_{g,A[i]} \arrow[dr, "\alpha^H"] & \\
 & & X
\end{tikzcd}$$

Note that $\rho_i \circ \pi_i$ and $\pi_i^H \circ \rho_n$ are defined on $\overline{M}_{g,n}$ and coincide on $M_{g,n}$. Since $\overline{M}_{g,n}$ is separated we have $\rho_i \circ \pi_i = \pi_i^H \circ \rho_n$. Let $s : \overline{M}_{g,A[i]} \to \overline{M}_{g,A[n]}$ be a section of π_i^H. We define $\alpha^H := f \circ s$. Clearly α^H coincides with α on $M_{g,A[i]}$, and $\alpha^H \circ \pi_i^H = f$.
Now, assume that f is birational. If $\mathrm{Exc}(f) \cap \partial\overline{M}_{g,A[n]} \neq \emptyset$ then $\mathrm{Exc}(f \circ \rho_n) \cap \partial\overline{M}_{g,n} \neq \emptyset$. This contradicts [GKM, Theorem 0.9]. So $\mathrm{Exc}(f) \subseteq \overline{M}_{g,A[n]}$.
Let us consider the case $g = 1$. If f is of fiber type, by the second part of [GKM, Theorem 0.9], the fibration $f \circ \rho_n$ factors through $\pi_I \times \pi_{I^c}$. Our aim is the define a morphism α^H completing the following commutative diagram

$$\begin{tikzcd}
\overline{M}_{1,n} \arrow[r, "\pi_I \times \pi_{I^c}"] \arrow[d, "\rho_n"'] & \overline{M}_{1,S} \times_{\overline{M}_{1,m}} \overline{M}_{1,S^c} \arrow[d, "\rho_i \times \rho_{i^c}"] \arrow[ddr, bend left, "\alpha"] & \\
\overline{M}_{1,A[n]} \arrow[r, "\pi_I^H \times \pi_{I^c}^H"] \arrow[drr, bend right, "f"'] & \overline{M}_{1,A[i]} \times_{\overline{M}_{1,A[m]}} \overline{M}_{1,A[n-i]} \arrow[dr, "\alpha^H"] & \\
 & & X
\end{tikzcd}$$

As before we consider two sections s, s' of π_I^H and $\pi_{I^c}^H$ respectively and define $\alpha^H := f \circ (s \times s')$.

If f is birational and $\mathrm{Exc}(f)\cap\partial\overline{M}_{1,A[n]}\neq\emptyset$ then $\mathrm{Exc}(f\circ\rho_n)\cap\partial\overline{M}_{1,n}\neq\emptyset$. Again this contradicts the second part of [GKM, Theorem 0.9]. So $\mathrm{Exc}(f)\subseteq M_{1,A[n]}$. □

The case $g=1$ is not as neat as the others. Luckily enough in the special case we are interested in something better can be said. If we consider the fibrations of the type

$$\overline{M}_{1,A[n]}\xrightarrow{\varphi}\overline{M}_{1,A[n]}\xrightarrow{\pi_i}\overline{M}_{1,A[n-1]}$$

where φ is an automorphism of $\overline{M}_{1,A[n]}$, thanks to the second part of Proposition 3.1.7 we can prove the following lemma.

Lemma 3.1.8. *Let φ be an automorphism of $\overline{M}_{1,A[n]}$. Any fibration of the type $\pi_i\circ\varphi$ factorizes through a forgetful morphism $\pi_j:\overline{M}_{1,A[n]}\rightarrow\overline{M}_{1,A[n-1]}$.*

Proof. By the second part of Theorem 3.1.7 the fibration $\pi_i\circ\varphi$ factorizes through a product of forgetful morphisms $\pi_{S^c}\times\pi_S:\overline{M}_{1,A[n]}\rightarrow\overline{M}_{1,A[i]}\times_{\overline{M}_{1,A[1]}}\overline{M}_{1,A[n-i]}$ and we have a commutative diagram

$$\begin{tikzcd}
\overline{M}_{1,A[n]} \arrow[r, "\varphi"] \arrow[d, "\pi_{S^c}\times\pi_S"'] & \overline{M}_{1,A[n]} \arrow[d, "\pi_i"] \\
\overline{M}_{1,A[i]}\times_{\overline{M}_{1,A[1]}}\overline{M}_{1,A[n-i]} \arrow[r, "\overline{\varphi}"] & \overline{M}_{1,A[n-1]}
\end{tikzcd}$$

The fibers of π_i and $\pi_{S^c}\times\pi_S$ are both 1-dimensional. Furthermore φ maps the fiber of $\pi_{S^c}\times\pi_S$ over $([C,x_{a_1},...,x_{a_i}],[C,x_{b_1},...,x_{b_{n-i}}])$ to $\pi_i^{-1}(\overline{\varphi}([C,x_{a_1},...,x_{a_i}],[C,x_{b_1},...,x_{b_{n-i}}]))$. Take a point $[C,x_1,...,x_{n-1}]\in\overline{M}_{1,A[n-1]}$, the fiber $\pi_i^{-1}([C,x_1,...,x_{n-1}])$ is mapped isomorphically to a fiber Γ of $\pi_{S^c}\times\pi_S$ which is contracted to a point $y=(\pi_{S^c}\times\pi_S)(\Gamma)$. The map

$$\overline{\psi}:\overline{M}_{1,A[n-1]}\rightarrow\overline{M}_{1,A[i]}\times_{\overline{M}_{1,A[1]}}\overline{M}_{1,A[n-i]},\ [C,x_1,...,x_{n-1}]\mapsto y,$$

is the inverse of $\overline{\varphi}$ which defines a bijective morphism between $\overline{M}_{1,A[i]}\times_{\overline{M}_{1,A[1]}}\overline{M}_{1,A[n-i]}$ and $\overline{M}_{1,A[n-1]}$, since by Remark 3.0.9 $\overline{M}_{1,A[n-1]}$ is normal $\overline{\varphi}$ is an isomorphism. This forces $S=\{j\}$, $S^c=\{1,...,\hat{j},...,n\}$. So we reduce to the commutative diagram

$$\begin{tikzcd}
\overline{M}_{1,A[n]} \arrow[r, "\varphi"] \arrow[d, "\pi_{S^c}\times\pi_j"'] & \overline{M}_{1,A[n]} \arrow[d, "\pi_i"] \\
\overline{M}_{1,A[1]}\times_{\overline{M}_{1,A[1]}}\overline{M}_{1,A[n-1]} \arrow[r, "\overline{\varphi}"] & \overline{M}_{1,A[n-1]}
\end{tikzcd}$$

and $\pi_i\circ\varphi$ factorizes through the forgetful morphism π_j. □

3.2 AUTOMORPHISMS OF $\overline{M}_{g,A[n]}$ AND $\overline{\mathcal{M}}_{g,A[n]}$

Let $\varphi:\overline{M}_{g,A[n]}\rightarrow\overline{M}_{g,A[n]}$ be an automorphism and $\pi_i:\overline{M}_{g,A[n]}\rightarrow\overline{M}_{g,A[n-1]}$ a forgetful morphism. We stress that in the case $g=0$ we consider only the Hassett's spaces of Definition 3.1.1, so by Lemma 3.1.3 if $\overline{M}_{0,A[n]}$ factors Kapranov then $\overline{M}_{0,A[n-1]}$ factors

Kapranov as well, and we can apply Theorem 3.1.6. Then, by Theorem 3.1.6, Proposition 3.1.7 and Lemma 3.1.8, we have the following diagram

$$\begin{tikzcd}
\overline{M}_{g,A[n]} \arrow[r, "\varphi^{-1}"] \arrow[d, "\pi_{j_i}"'] & \overline{M}_{g,A[n]} \arrow[d, "\pi_i"] \\
\overline{M}_{g,A[n-1]} \arrow[r, "\tilde{\varphi}"] & \overline{M}_{g,A[n-1]}
\end{tikzcd}$$

where π_{j_i} is again forgetful map. This allows us to associate to an automorphism a permutation in S_r, where r is the number of well defined forgetful maps, and to define a morphism of group

$$\chi : \mathrm{Aut}(\overline{M}_{g,A[n]}) \to S_r, \ \varphi \mapsto \sigma_\varphi$$

where

$$\sigma_\varphi : \{1, ..., r\} \to \{1, ..., r\}, \ i \mapsto j_i.$$

Note that in order to have a morphism of groups we have to consider φ^{-1} instead of φ. This section is devoted to study the image and the kernel of χ.
First we consider the genus zero case and in particular the spaces that naturally appears as factorizations of the Kapranov's construction of $\overline{M}_{0,n}$. Recall that the weights of the Hassett's space appearing at the step (r, s) of Construction 3.0.11 are given by:

$$A_{r,s}[n] := (\underbrace{1/(n-r-1), ..., 1/(n-r-1)}_{(n-r-1) \text{ times}}, s/(n-r-1), \underbrace{1, ..., 1}_{r \text{ times}})$$

for $r = 1, ..., n-3$ and $s = 1, ..., n-r-2$. In particular, if $r = 1$ we have

$$(\underbrace{1/(n-2), ..., 1/(n-2)}_{(n-2)-\text{times}}, s/(n-2), 1).$$

Since $2g-2+\frac{n-2}{n-2}+\frac{s}{n-2} < 0$ and $2g-2+\frac{n-2}{n-2}+1 = 0$, by [Has, Theorem 4.3] the forgetful maps π_n and π_{n-1} are not well defined.
If $r \geqslant 2$ we have $2g-2+\frac{n-r-1}{n-r-1}+\frac{s}{n-r-1}+(r-1) > 0$ and by [Has, Theorem 4.3] all the forgetful morphisms are well defined. This means that we have a morphism of groups from $\mathrm{Aut}(\overline{M}_{0,A_{r,s}[n]})$ to S_{n-2} if $r = 1$, and to S_n if $r \geqslant 2$.
We describe in detail the case $n = 5$ and the case $n = 6$ where all issues appear.

Proposition 3.2.1. *The automorphism group of* $\overline{M}_{0,A_{1,2}[5]}$ *is isomorphic to* $(\mathbb{C}^*)^2 \times S_3 \times S_2$.

Proof. Recall that at the step $r = 1, s = 2$ only three points has been blown-up. We have only three forgetful morphisms. By the factorization property in Theorem 3.1.6 we get a surjective morphism of groups

$$\chi : \mathrm{Aut}(\overline{M}_{0,A_{1,2}[5]}) \to S_3.$$

Now, consider an automorphism φ of $\overline{M}_{0,A_{1,2}[5]}$ inducing the trivial permutation. Then φ induces a birational transformation $\varphi_{\mathcal{H}} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ fixing p_1, p_2, p_3 and stabilizing the lines through p_i, $i = 1, 2, 3$.
Let $|\mathcal{H}| \subseteq |\mathcal{O}_{\mathbb{P}^2}(d)|$ be the linear system associated to $\varphi_{\mathcal{H}}$. If L_i is a line through p_i we have

$$\deg(\varphi_{\mathcal{H}}(L_i)) = d - \mathrm{mult}_{p_i} \mathcal{H} = 1,$$

So $\mathrm{mult}_{p_i} \mathcal{H} = d-1$. Since the linear system $|\mathcal{H}|$ does not have fixed component the inequality $2(d-1) \leqslant d$ holds, and we get $d \leqslant 2$.

If $d=1$ the birational map $\varphi_{\mathcal{H}}$ is an automorphism of $\mathbb{P}^2$ fixing p_1, p_2, p_3. These correspond to diagonal, non-singular matrices.
If $d=2$ then $|\mathcal{H}|$ is the linear system of conics with three base points and $\varphi_{\mathcal{H}}$ is the standard Cremona transformation of $\mathbb{P}^2$.
Therefore $\ker(\chi) = (\mathbb{C}^*)^2 \times S_2$ and from the splitting exact sequence of groups

$$0 \mapsto (\mathbb{C}^*)^2 \times S_2 \to \operatorname{Aut}(\overline{M}_{0,A_{1,2}[5]}) \to S_3 \mapsto 0.$$

we get $\operatorname{Aut}(\overline{M}_{0,A_{1,2}[5]}) \cong (\mathbb{C}^*)^2 \times S_3 \times S_2$. □

Now, let us consider the case $n=6$. Construction 3.0.11 is as follows:

- $r=1$, $s=1$, gives $\mathbb{P}^3$,
- $r=1$, $s=2$, we blow-up the points $p_1, ..., p_4 \in \mathbb{P}^3$ and get the Hassett's space with weights $A_{1,2}[6] := (1/4, 1/4, 1/4, 1/4, 1/2, 1)$,
- $r=1$, $s=3$, we blow-up the lines $\langle p_i, p_j \rangle$, $i,j = 1, ..., 4$, and get the Hassett's space with weights $A_{1,3}[6] := (1/4, 1/4, 1/4, 1/4, 3/4, 1)$,
- $r=2$, $s=1$, we blow-up the point p_5, and get the Hassett's space with weights $A_{2,1}[6] := (1/3, 1/3, 1/3, 1/3, 1, 1)$,
- $r=2$, $s=2$, we blow-up the lines $\langle p_i, p_5 \rangle$, $i,j = 1, ..., 3$, and get the Hassett's space with weights $A_{2,2}[6] := (1/3, 1/3, 1/3, 2/3, 1, 1)$,
- $r=3$, $s=1$, we blow-up the line $\langle p_4, p_5 \rangle$ and get the Hassett's space with weights $A_{3,1}[6] := (1/2, 1/2, 1/2, 1, 1, 1)$, that is $\overline{M}_{0,6}$.

Proposition 3.2.2. *If $n=6$ the automorphism groups of the Hassett's spaces appearing in Construction 3.0.11 are given by*

- $\operatorname{Aut}(\overline{M}_{0,A_{r,s}[6]}) \cong (\mathbb{C}^*)^3 \times S_4$, if $r=1$, $1<s<3$,
- $\operatorname{Aut}(\overline{M}_{0,A_{r,s}[6]}) \cong (\mathbb{C}^*)^3 \times S_4 \times S_2$, if $r=1$, $s=3$,
- $\operatorname{Aut}(\overline{M}_{0,A_{r,s}[6]}) \cong S_6$, if $r \geqslant 2$.

Proof. If $r=1$, we have a surjective morphism of groups

$$\chi : \operatorname{Aut}(\overline{M}_{0,A_{r,s}[6]}) \to S_4.$$

An automorphism φ of $\overline{M}_{0,A_{r,s}[6]}$ whose image in S_4 is the identity induces a birational transformation $\varphi_{\mathcal{H}} : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ fixing p_1, p_2, p_3, p_4 and stabilizing the lines through p_i, $i = 1,2,3,4$. Let $|\mathcal{H}| \subseteq |\mathcal{O}_{\mathbb{P}^3}(d)|$ be the linear system associated to $\varphi_{\mathcal{H}}$. If L_i is a line through p_i we have

$$\deg(\varphi_{\mathcal{H}}(L_i)) = d - \operatorname{mult}_{p_i} \mathcal{H} = 1.$$

This yields

$$\operatorname{mult}_{p_i} \mathcal{H} = d-1,\ \operatorname{mult}_{\langle p_i, p_j \rangle} \mathcal{H} \geqslant d-2, \text{and } \operatorname{mult}_{\langle p_i, p_j, p_k \rangle} \mathcal{H} \geqslant d-3. \tag{3.2.1}$$

The linear system $\mathcal{H}$ does not have fixed components therefore $d \leqslant 3$ and in equation (3.2.1) all inequalities are equalities. If $d=1$ then $\varphi_{\mathcal{H}}$ is an automorphism of $\mathbb{P}^3$ fixing p_1, p_2, p_3, p_4.

These correspond to diagonal, non-singular matrices.
If $d \neq 1$, again by Theorem 3.1.6, we have the following commutative diagram

$$\begin{tikzcd}
\overline{M}_{0,A[n]} \arrow[r, "\varphi^{-1}"] \arrow[d, "\pi_{j_i,j_2}"'] & \overline{M}_{0,A[n]} \arrow[d, "\pi_{i_1,i_2}"] \\
\overline{M}_{0,A[n-2]} \arrow[r, "\tilde{\varphi}"] & \overline{M}_{0,A[n-2]}
\end{tikzcd}$$

Therefore $\varphi_{\mathcal{H}}$ induces a Cremona transformation on the general plane containing the line $\langle p_1, p_2\rangle$. So on such a general plane the linear system $\mathcal{H}$ needs a third base point, outside $\langle p_1, p_2\rangle$. This means that in $\mathbb{P}^3$ a codimension two linear space has to be blown-up. So $s = d = 3$ and $\varphi_{\mathcal{H}}$ is the standard Cremona transformation of $\mathbb{P}^3$. We conclude that $\ker(\chi) = (\mathbb{C}^*)^3$ if $s < 3$, and $\ker(\chi) = (\mathbb{C}^*)^3 \times S_2$ if $s = 3$.
When $r \geqslant 2$ the fifth point p_5 has been blown-up. We have all the forgetful morphisms and a surjective morphism of groups

$$\chi : \mathrm{Aut}(\overline{M}_{0,A_{r,s}[6]}) \to S_6.$$

An automorphism corresponding to the trivial permutation induces a birational transformation $\varphi_{\mathcal{H}}$ of $\mathbb{P}^3$ fixing $p_1, ..., p_5$, stabilizing the lines through p_i, $i = 1, ..., 5$, but now it has the additional constraint to stabilize the twisted cubics C through $p_1, ..., p_5$. By the equality

$$\deg(\varphi_{\mathcal{H}}(C)) = 3d - \mathrm{mult}_{p_i} \mathcal{H} = 3d - 5(d-1) = 3,$$

we conclude that $d = 1$ and $\varphi_{\mathcal{H}}$ is an automorphism of $\mathbb{P}^3$ fixing five points in linear general position, so it is forced to be the identity. ☐

Now, let us consider the general case. The following lemma generalizes the ideas in the proof of Proposition 3.2.2 and leads us to control the degree and type of linear systems involved in the computation of the automorphisms of the spaces appearing in Construction 3.0.11.

Lemma 3.2.3. *Let $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^{n-3}}(d)|$ be a linear system and $\{p_1, \dots, p_a\} \subset \mathbb{P}^{n-3}$ a collection of points. Assume that $\mathrm{mult}_{p_i} \mathcal{H} = d - 1$, for $i = 1, \dots, a$. Let $L_{i_1,\dots,i_h} = \langle p_{i_1}, \dots, p_{i_h}\rangle$ be the linear span of h points in $\{p_1, \dots, p_a\}$, then*

$$\mathrm{mult}_{L_{i_1,\dots,i_h}} \mathcal{H} \geqslant d - h.$$

Assume further that $\mathcal{H}$ does not have fixed components, $a = n - 2$ and the rational map, say $\varphi_{\mathcal{H}}$, induced by $\mathcal{H}$ lifts to an automorphism of $\overline{M}_{A_{1,s}[n]}$ that preserves the forgetful maps onto $\overline{M}_{A_{1,s}[n-1]}$. Then

$$\mathrm{mult}_{L_{i_1,\dots,i_h}} \mathcal{H} = d - h,$$

$s = d = n - 3$, and $\varphi_{\mathcal{H}}$ is the standard Cremona transformation centered at $\{p_1, \dots, p_{n-2}\}$.

Proof. The first statement is meaningful only for $h < d$. We prove it by a double induction on d and h. The initial case $d = 2$ and $a = 1$ is immediate. Let us consider $\Pi := L_{p_{i_1},\dots,p_{i_h}}$ and $L_j = \langle p_{i_1}, \dots, \hat{p}_{i_j}, ..., p_{i_h}\rangle$ the linear span of $h - 1$ points in $\{p_1, \dots, p_h\}$. Then by induction hypothesis

$$\mathrm{mult}_{L_j} \mathcal{H}_{|\Pi} \geqslant d - (h - 1),$$

and L_j is a divisor in Π. By assumption $d > h$ hence $d(h-1) > h(h-1)$ and

$$h(d - (h - 1)) > d.$$

This yields $\Pi \subset \mathrm{Bl}\,\mathcal{H}$. Let A be a general linear space of dimension h containing Π. Then we may decompose $\mathcal{H}_{|A} = \Pi + \mathcal{H}_1$ with $\mathcal{H}_1 \subset |\mathcal{O}(d-1)|$ and

$$\operatorname{mult}_{L_j} \mathcal{H}_1 \geqslant d-1-(h-1). \tag{3.2.2}$$

Arguing as above this forces $\Pi \subset \mathcal{H}_1$ as long as $h(d-1-(h-1)) > d-1$, that is $d-1 > h$, and recursively gives the first statement.
Assume that the map $\varphi_{\mathcal{H}}$ lifts to an automorphism that preserves the forgetful maps onto $\overline{M}_{A_{1,s}[n-1]}$. This forces some immediate consequences:

i) $a = n-2$ and the points p_i are in general position,

ii) the scheme theoretic base locus of $\mathcal{H}$ is the span of all subsets of at most $s-1$ points.

Since $L_{p_{i_1},\dots,p_{i_s}} \not\subset \mathrm{Bl}(\mathcal{H})$ equation (3.2.2) yields

$$s \geqslant d. \tag{3.2.3}$$

Furthermore the hyperplane $H = \langle p_{i_1}, \dots, p_{i_{n-3}} \rangle$ contains $(n-3)$ codimension two linear spaces of the form L_j, each of multiplicity $d-(n-4)$ for the linear system $\mathcal{H}$. The linear system $\mathcal{H}$ does not have fixed components hence $(n-3)(d-n+4) \leqslant d$ and we get

$$d \leqslant n-3.$$

Claim 1. $\mathrm{Bl}\,\mathcal{H} \not\supset L_{i_1,\dots,i_d}$.

Proof. Assume that $\mathrm{Bl}\,\mathcal{H} \supset L_{i_1,\dots,i_d}$ then the restriction $\mathcal{H}_{|L_{i_1,\dots,i_{d+1}}}$ contains a fixed divisor of degree $d+1$ and $L_{i_1,\dots,i_{d+1}} \subset \mathrm{Bl}\,\mathcal{H}$. A recursive argument then shows that $\mathrm{Bl}\,\mathcal{H}$ has to contain all the linear spaces spanned by the $n-2$ points yielding a contradiction. ☐

The claim together with ii) and equation (3.2.3) yield

$$s = d,$$

and

$$\operatorname{mult}_{L_{i_1,\dots,i_{d-1}}} \mathcal{H} = d-(d-1) = 1.$$

Then, recursively this forces the equality in equation (3.2.2) for any value of h. To conclude let us consider the commutative diagram

$$\begin{tikzcd}
\overline{M}_{A_{1,s}[n]} \arrow[r, "\varphi"] \arrow[d] & \overline{M}_{A_{1,s}[n]} \arrow[d] \\
\mathbb{P}^{n-3} \arrow[r, dashed, "\varphi_{\mathcal{H}}"] \arrow[d, dashed] & \mathbb{P}^{n-3} \arrow[d, dashed] \\
\mathbb{P}^{n-5} \arrow[r, dashed] & \mathbb{P}^{n-5}
\end{tikzcd}$$

By Theorem 3.1.6 we know that φ composed with a forgetful map onto $\overline{M}_{A_{1,s}[n-2]}$ is again a forgetful map. This forces the map $\varphi_{\mathcal{H}}$ to induce a Cremona transformation on the general plane containing $\{p_{i_1}, p_{i_2}\}$. Let Π be a general plane containing $\{p_{i_1}, p_{i_2}\}$. Then the mobile part of $\mathcal{H}_{|\Pi}$ is a linear system of conics with two simple base points in p_{i_1} and p_{i_2}. This forces the presence of a further base point to produce a Cremona transformation. Therefore a codimension two linear space has to be blown-up. This shows that $s = d = n-3$. To conclude we observe that the linear system of forms of degree $n-3$ in $\mathbb{P}^{n-3}$ having the assigned base locus has dimension $n-2$ and gives rise to the standard Cremona transformation. ☐

Theorem 3.2.4. *The automorphism groups of the Hassett's spaces appearing in Construction 3.0.11 are given by*

- $\mathrm{Aut}(\overline{M}_{0,A_{r,s}[n]}) \cong (\mathbb{C}^*)^{n-3} \times S_{n-2}$, if $r = 1$, $1 < s < n-3$,
- $\mathrm{Aut}(\overline{M}_{0,A_{r,s}[n]}) \cong (\mathbb{C}^*)^{n-3} \times S_{n-2} \times S_2$, if $r = 1$, $s = n-3$,
- $\mathrm{Aut}(\overline{M}_{0,A_{r,s}[n]}) \cong S_n$, if $r \geqslant 2$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\overline{M}_{A_{1,s}[n]} \arrow[r, "\varphi"] \arrow[d, "f"'] & \overline{M}_{A_{1,s}[n]} \arrow[d, "f"] \\
\mathbb{P}^{n-3} \arrow[r, dashed, "\varphi_{\mathcal{H}}"] & \mathbb{P}^{n-3}
\end{tikzcd}$$

where f is a Kapranov factorization. If $r = 1$ we have $n-2$ forgetful morphisms and a surjective morphism of groups

$$\chi : \mathrm{Aut}(\overline{M}_{0,A_{1,s}[n]}) \to S_{n-2}.$$

Let φ be an automorphism of $\overline{M}_{0,A_{r,s}[n]}$ such that $\chi(\varphi)$ is the identity. Then φ preserves the forgetful maps onto $\overline{M}_{A_{1,s}[n-1]}$ and the birational map $\varphi_{\mathcal{H}}$ induced by φ stabilizes lines through $p_1, ..., p_{n-2}$.

Let $|\mathcal{H}| \subseteq |\mathcal{O}_{\mathbb{P}^{n-3}}(d)|$ be the linear system associated to $\varphi_{\mathcal{H}}$. If L_i is a line through p_i we have

$$\deg(\varphi_{\mathcal{H}}(L_i)) = d - \mathrm{mult}_{p_i} \mathcal{H} = 1.$$

So $\mathrm{mult}_{p_i} \mathcal{H} = d - 1$.

If $s < n-3$, by Lemma 3.2.3, the linear system $\mathcal{H}$ is free from base points and $d = 1$. Then the kernel of χ consists of biregular automorphisms of $\mathbb{P}^{n-3}$ fixing $n-2$ points in general position, so $\ker(\chi) = (\mathbb{C}^*)^{n-3}$ and $\mathrm{Aut}(\overline{M}_{0,A_{1,s}[n]}) \cong (\mathbb{C}^*)^{n-3} \times S_{n-2}$.

If $s = n-3$, by Lemma 3.2.3, the only linear system with base points is associated to the standard Cremona transformation of $\mathbb{P}^{n-3}$. This gives $\ker(\chi) = (\mathbb{C}^*)^{n-3} \times S_2$ and $\mathrm{Aut}(\overline{M}_{0,A_{1,s}[n]}) \cong (\mathbb{C}^*)^{n-3} \times S_{n-2} \times S_2$.

When $r \geqslant 2$ the last point p_{n-1} has been blown-up and again by Lemma 3.1.3 we have a surjective morphism of groups

$$\chi : \mathrm{Aut}(\overline{M}_{0,A_{r,s}[n]}) \to S_n.$$

Any automorphism φ preserving the forgetful maps onto $\overline{M}_{A_{r,s}[n-1]}$ preserves the lines L_i through p_i and the rational normal curves C through $p_1, ..., p_{n-1}$. The equalities

$$\begin{aligned} &\deg(\varphi_{\mathcal{H}}(L_i)) = d - \mathrm{mult}_{p_i} \mathcal{H} = 1, \\ &\deg(\varphi_{\mathcal{H}}(C)) = (n-3)d - \textstyle\sum_{i=1}^{n-1} \mathrm{mult}_{p_i} \mathcal{H} = n-3. \end{aligned} \tag{3.2.4}$$

yield $d = 1$. So $\varphi_{\mathcal{H}}$ is an automorphism of $\mathbb{P}^{n-3}$ fixing $n-1$ points in general position, this forces $\varphi_{\mathcal{H}} = \mathrm{Id}$. Then χ is injective and $\mathrm{Aut}(\overline{M}_{0,A_{r,s}[n]}) \cong S_n$. □

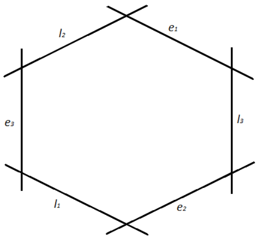
Remark 3.2.5. The Hassett's space $\overline{M}_{0,A_{1,2}[5]}$ is the blow-up of $\mathbb{P}^2$ in three points in general position, that is a Del Pezzo surface $\mathcal{S}_6$ of degree 6. By Theorem 3.2.4 we recover the classical result on its automorphism group $\mathrm{Aut}(\mathcal{S}_6) \cong (\mathbb{C}^*)^2 \times S_3 \times S_2$. For a proof not using the theory of moduli of curves see [DI, Section 6].

Furthermore, note that we are allowed to permute the points labeled by $1, 2, 3$ and to exchange the marked points $4, 5$. However any permutation mapping $1, 2$ or 3 to 4 or 5 contracts a boundary divisor isomorphic to $\mathbb{P}^1$ to the point $\rho_1(E_{5,4})$, so it does not induce an automorphism. Furthermore the Cremona transformation lift to the automorphism of $\overline{M}_{0,A_{1,2}[5]}$ corresponding to the transposition $4 \leftrightarrow 5$.

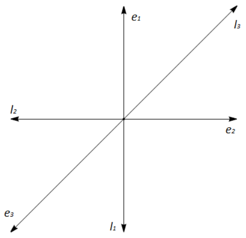
Remark 3.2.6. In Remark 3.0.12 we identified the step $r = 1, s = n - 3$ of Construction 3.0.11 with the Losev-Manin's space $\overline{L}_{n-2}$. This space is a toric variety of dimension $n - 3$. By Theorem 3.2.4 we recover $(\mathbb{C}^*)^{n-3} \subset \mathrm{Aut}(\overline{L}_{n-2})$. The automorphisms in $S_{n-2} \times S_2$ reflect on the toric setting as automorphisms of the fan of $\overline{L}_{n-2}$.
For example consider the Del Pezzo surface of degree six $\overline{M}_{0,A_{1,2}[5]} \cong \overline{L}_3 \cong \mathcal{S}_6$. Let us say that $\mathcal{S}_6$ is the blow-up of $\mathbb{P}^2$ at the coordinate points p_1, p_2, p_3 with exceptional divisors e_1, e_2, e_3 and let us denote by $l_i = \langle p_j, p_k \rangle$, $i \neq j, k$, $i = 1, 2, 3$, the three lines generated by p_1, p_2, p_3.
Such a surface can be realized as the complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$ cut out by the equations $x_0y_0 = x_1y_1 = x_2y_2$. The six lines are given by $e_i = \{x_j = x_k = 0\}$, $l_i = \{y_j = y_k = 0\}$ for $i \neq j, k$, $i = 1, 2, 3$. The torus $T = (\mathbb{C}^*)^3/\mathbb{C}^*$ acts on $\mathbb{P}^2 \times \mathbb{P}^2$ by

$$(\lambda_0, \lambda_1, \lambda_2) \cdot ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) = ([\lambda_0 x_0 : \lambda_1 x_1 : \lambda_2 x_2], [\lambda_0^{-1} y_0 : \lambda_1^{-1} y_1 : \lambda_2^{-1} y_2]).$$

This torus action stabilizes $\mathcal{S}_6$. Furthermore S_2 acts on $\mathcal{S}_6$ by the transpositions $x_i \leftrightarrow y_i$, and S_3 acts on $\mathcal{S}_6$ by permuting the two sets of homogeneous coordinates separately. The action of S_3 corresponds to the permutations of the three points of $\mathbb{P}^2$ we are blowing-up, while the S_2-action is the switch of roles of exceptional divisors between the sets of lines $\{e_1, e_2, e_3\}$ and $\{l_1, l_2, l_3\}$. These six lines are arranged in a hexagon inside $\mathcal{S}_6$



which is stabilized by the action of $S_3 \times S_2$. The fan of $\mathcal{S}_6$ is the following



where the six 1-dimensional cones correspond to the toric divisors e_1, l_3, e_2, l_1, e_3 and l_2. It is clear from the picture that the fan has many symmetries given by permuting $\{e_1, e_2, e_3\}$, $\{l_1, l_2, l_3\}$ and switching e_i with l_i for $i = 1, 2, 3$.

Remark 3.2.7. From the description of $\overline{L}_{n-2}$ given in Remark 3.0.12 it is clear that S_{n-2} gives the permutations of $x_1, ..., x_{n-2}$ while S_2 corresponds to the transposition $x_0 \leftrightarrow x_\infty$.

The Hassett's spaces of Construction 3.0.14 are more symmetric and simpler from the automorphisms viewpoint.

Theorem 3.2.8. *The automorphism groups of the Hassett's spaces appearing in Construction 3.0.14 are given by*

$$\mathrm{Aut}(X_k[n]) \cong S_n$$

for any $k = 1, ..., n-4$.

Proof. We use the same notations of Theorem 3.2.4. Since step $k = 1$ we have blown-up $n-1$ points, so we have n forgetful morphisms and a surjective morphism of groups

$$\chi : \mathrm{Aut}(X_k[n]) \to S_n.$$

As in Theorem 3.2.4 any automorphism fixing all the forgetful morphisms preserves the lines L_i through p_i and the rational normal curves C through $p_1, ..., p_{n-1}$. By the equalities 3.2.4 we get $d = 1$ and $\varphi_{\mathcal{H}} = \mathrm{Id}$. ☐

Higher genera

Now, we switch to curves of positive genus. First observe that $\overline{M}_{1,A[1]} \cong \overline{M}_{1,1} \cong \mathbb{P}^1$ for any weight data. Therefore we can restrict to the cases $g = 1, n \geqslant 2$ and $g \geqslant 2, n \geqslant 1$.

Lemma 3.2.9. *If* $g = 1, n \geqslant 2$ *or* $g \geqslant 2, n \geqslant 1$ *then all the forgetful morphisms* $\overline{M}_{g,A[n]} \to \overline{M}_{g,A[n-1]}$ *are well defined morphisms.*

Proof. If $g = 1$ then $2g-2+a_1+...+a_{n-1} = a_1+...+a_{n-1} > 0$ being $n \geqslant 2$. If $g = 2$ we have $2g-2+a_1+...+a_{n-1} \geqslant 2+a_1+...+a_{n-1} > 0$ for any $n \geqslant 1$. To conclude it is enough to apply [Has, Theorem 4.3]. ☐

Since by Lemma 3.2.9 all the forgetful morphism are well defined we get a morphism of groups

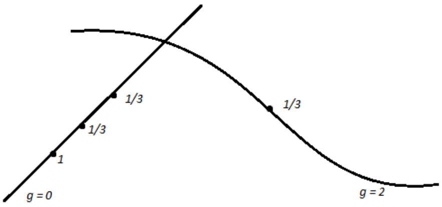
$$\chi : \mathrm{Aut}(\overline{M}_{g,A[n]}) \to S_n,\ \varphi \mapsto \sigma_\varphi$$

where

$$\sigma_\varphi : \{1, ..., n\} \to \{1, ..., n\},\ i \mapsto j_i.$$

In the case of $\overline{M}_{g,n}$ this morphism is clearly surjective and turns out to be injective as soon as $2g-2+n \geqslant 3$, see Theorem 2.2.9 of Chapter 2. However in the more general setting of Hassett's spaces the image of χ depends on the weight data. We are wondering which permutations actually induce automorphisms of $\overline{M}_{g,A[n]}$. To better understand this issue let us consider the following example.

Example 3.2.10. In $\overline{M}_{2,A[4]}$ with weights $(1, 1/3, 1/3, 1/3)$ consider the divisor parametrizing reducible curves $C_1 \cup C_2$, where C_1 has genus zero and markings $(1, 1/3, 1/3)$, and C_2 has genus two and marking $1/3$.



After the transposition $1 \leftrightarrow 4$ the genus zero component has markings $(1/3, 1/3, 1/3)$, so it is contracted. This means that the transposition induces a birational map

$$\begin{tikzcd} \overline{M}_{2,A[4]} \arrow[r, dashed, "1\leftrightarrow 4"] & \overline{M}_{2,A[4]} \end{tikzcd}$$

contracting a divisor on a codimension two subscheme of $\overline{M}_{2,A[4]}$. Consider the locus of curves $C_1 \cup C_2$ with $C_1 \cong \mathbb{P}^1$, $x_2 = x_3 = x_4 \in C_1$ and $x_1 \in C_2$. Since $a_1 + a_2 + a_3 > 1$ the birational map induced by $1 \leftrightarrow 4$ is not defined on such locus.

This example suggests us that troubles come from rational tails with at least three marked points and leads us to the following definition.

Definition 3.2.11. A transposition $i \leftrightarrow j$ of two marked points is *admissible* if and only if for any $h_1, ..., h_r \in \{1, ..., n\}$, with $r \geqslant 2$,

$$a_i + \sum_{k=1}^{r} a_{h_k} \leqslant 1 \iff a_j + \sum_{k=1}^{r} a_{h_k} \leqslant 1.$$

We need the following lemma which, in the complex setting, in nothing but an immediate consequence of Hartog's extension theorem.

Lemma 3.2.12. *Let $\varphi : X \to Y$ be a continuous map of separated schemes defining a morphism in codimension at least two. If X is S_2 then φ is a morphism.*

Proof. Let $\mathcal{U} \subset X$ be an open set, whose complementary have codimension at least two, where φ is a morphism. Let f be a regular function on Y, then $f \circ \varphi_{|\mathcal{U}} \in \mathcal{O}_X(\mathcal{U})$ is a regular function on $\mathcal{U}$. Since X is S_2 $f \circ \varphi_{|\mathcal{U}}$ extends to a regular function on X. So we get a morphism of sheaves $\mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ and $\varphi : X \to Y$ is a morphism of schemes. $\square$

Any transposition $i \leftrightarrow j$ in S_n defines a birational map $\tilde{\varphi}_{i,j} : \overline{M}_{g,A[n]} \dashrightarrow \overline{M}_{g,A[n]}$. We aim to understand when this map is an automorphism, our main tool is the following proposition.

Proposition 3.2.13. *The following are equivalent:*

(a) $i \leftrightarrow j$ *is admissible,*

(b) $\tilde{\varphi}_{i,j}$ *is an automorphism,*

(c) $\overline{M}_{g,A_i[n-1]} \cong \overline{M}_{g,A_j[n-1]}$, *where* $A_i = \{a_1, ..., \hat{a}_i, ..., a_n\}$ *and* $A_j = \{a_1, ..., \hat{a}_j, ..., a_n\}$.

Proof. (a) $\Rightarrow$ (b) By [Has, Theorem 4.1] we have a birational reduction morphism

$$\rho : \overline{M}_{g,n} \to \overline{M}_{g,A[n]}.$$

Let $\varphi_{i,j} \in \mathrm{Aut}(\overline{M}_{g,n})$ be the automorphism induced by the transposition $i \leftrightarrow j$. Then we have a commutative diagram

$$\begin{tikzcd} \overline{M}_{g,n} \arrow[r, "\varphi_{i,j}"] \arrow[d, "\rho"'] & \overline{M}_{g,n} \arrow[d, "\rho"] \\ \overline{M}_{g,A[n]} \arrow[r, dashed, "\tilde{\varphi}_{i,j}"] & \overline{M}_{g,A[n]} \end{tikzcd}$$

where a priori $\tilde{\varphi}_{i,j}$ is just a birational map. By [Has, Proposition 4.5] ρ contracts the divisors $\Delta_{I,J}$ whose general points correspond to curves with two irreducible components, a genus

zero smooth curve with $I=\{i_1,\dots,i_r\}$ as marking set and a genus g curve with marking set $J=\{j_1,\dots,j_{n-r}\}$, such that $a_{i_1}+\dots+a_{i_r}\leqslant 1$ and $2<r\leqslant n$. A priori $\tilde{\varphi}_{i,j}$ is defined just on the open subset of $\overline{M}_{g,A[n]}$ parametrizing curves where x_i,x_j coincide at most with another marked point. Let $\mathcal{U}\subset\overline{M}_{g,A[n]}$ be the open subset parametrizing such curves.
Let us consider a curve $[C,x_1,\dots,x_i,\dots,x_j,\dots,x_n]$ with $x_i=x_{i_2}=\dots=x_{i_r}$, $2<r\leqslant n-1$. By Definition 3.0.8 we have $a_i+a_{i_2}+\dots+a_{i_r}\leqslant 1$. Then $\rho^{-1}([C,x_1,\dots,x_i,\dots,x_j,\dots,x_n])$ lies on a divisor of type $\Delta_{I,J}$. By Definition 3.2.11 we have $a_j+a_{i_2}+\dots+a_{i_r}\leqslant 1$. So $(\rho\circ\varphi_{i,j}\circ\rho^{-1})([C,x_1,\dots,x_i,\dots,x_j,\dots,x_n])=[C,x_1,\dots,x_j,\dots,x_i,\dots,x_n]$ with $x_j=x_{i_2}=\dots=x_{i_r}$. We consider the same construction for curves $[C,x_1,\dots,x_i,\dots,x_j,\dots,x_n]$ with $x_j=x_{i_2}=\dots=x_{i_r}$, $2<r\leqslant n-1$ and extend $\tilde{\varphi}_{i,j}$ as a continuous map by

$$\tilde{\varphi}_{i,j}([C,x_1,\dots,x_i,\dots,x_j,\dots,x_n]):=[C,x_1,\dots,x_j,\dots,x_i,\dots,x_n].$$

The continuous map $\tilde{\varphi}_{i,j}:\overline{M}_{g,A[n]}\to\overline{M}_{g,A[n]}$ is an isomorphism between two open subsets $\mathcal{U},\mathcal{V}$ whose complementary have codimension at least two. This is enough to conclude, by Remark 3.0.9 and Lemma 3.2.12, that $\tilde{\varphi}_{i,j}$ is an isomorphism.
$(b)\Rightarrow(c)$ By Proposition 3.1.7 and Lemma 3.1.8 in the cases $g\geqslant 2$ and $g=1$ respectively we produce a commutative diagram

$$\begin{tikzcd}
\overline{M}_{g,A[n]} \arrow[r, "\tilde{\varphi}_{i,j}^{-1}"] \arrow[d, "\pi_j"'] & \overline{M}_{g,A[n]} \arrow[d, "\pi_i"] \\
\overline{M}_{g,A_i[n-1]} \arrow[r, "\overline{\varphi}_{i,j}"] & \overline{M}_{g,A_j[n-1]}
\end{tikzcd}$$

where $\overline{\varphi}_{i,j}$ is invertible and hence an isomorphism.
$(c)\Rightarrow(a)$ We may assume that $a_i\geqslant a_j$. Then, by [Has, Proposition 4.5], the reduction morphism $\rho_{A_i[n-1],A_j[n-1]}:\overline{M}_{g,A_j[n-1]}\to\overline{M}_{g,A_i[n-1]}$ is an isomorphism. Therefore, again by [Has, Proposition 4.5], $a_j+\sum_{k=1}^{r}a_{h_k}\leqslant 1$ and $a_i+\sum_{k=1}^{r}a_{h_k}>1$ is possible only if $r\leqslant 1$. This shows that $i\leftrightarrow j$ is admissible. □

Let us consider the subgroup $\mathcal{A}_{A[n]}\subseteq S_n$ generated by admissible transpositions and the morphism

$$\chi:\operatorname{Aut}(\overline{M}_{g,A[n]})\to S_n.$$

Clearly $\mathcal{A}_{A[n]}\subseteq\operatorname{Im}(\chi)$. In what follows we aim to study the image and the kernel of χ.

Lemma 3.2.14. *For any $g\geqslant 1$ and n such that $2g-2+n\geqslant 3$ we have $\operatorname{Im}(\chi)=\mathcal{A}_{A[n]}$.*

Proof. Let $\sigma_\varphi=\chi(\varphi)$ be the permutation induced by $\varphi\in\operatorname{Aut}(\overline{M}_{g,A[n]})$. Up to taking its decomposition as a product of disjoint cycles we can assume σ_φ to be a cycle $(i_1\dots i_r)$. Let us consider its decomposition

$$(i_1\dots i_r)=(i_1i_r)(i_1i_{r-1})\dots(i_1i_3)(i_1i_2)$$

as product of transpositions. We want to prove that (i_1i_h) is admissible for any $h=2,\dots,r$. We proceed by induction on the length r of the cycle. If $r=2$ then (i_1i_2) is admissible by Proposition 3.2.13.
Now, note that the cycle $(i_1\dots i_r)$ maps i_r to i_1. This means that $\pi_{i_r}\circ\varphi^{-1}$ factors through π_{i_1} and the following commutative diagram

$$\begin{tikzcd}
\overline{M}_{g,A[n]} \arrow[r, "\varphi^{-1}"] \arrow[d, "\pi_{i_1}"'] & \overline{M}_{g,A[n]} \arrow[d, "\pi_{i_r}"] \\
\overline{M}_{g,A_{i_1}[n-1]} \arrow[r, "\overline{\varphi}"] & \overline{M}_{g,A_{i_r}[n-1]}
\end{tikzcd}$$

guaranties that $\overline{M}_{g,A_{i_r}[n-1]} \cong \overline{M}_{g,A_{i_1}[n-1]}$ Then, by Proposition 3.2.13, the transposition $(i_1 i_r)$ is admissible and $(i_1 i_r) = \chi(\tilde{\varphi}_{i_1,i_r})$ with $\tilde{\varphi}_{i_1,i_r} \in \mathrm{Aut}(\overline{M}_{g,A[n]})$. We have $\chi(\varphi) = \chi(\tilde{\varphi}_{i_1,i_r})(i_1,i_{r-1})...(i_1,i_2)$ and

$$\chi(\varphi \circ \tilde{\varphi}_{i_1,i_r}^{-1}) = (i_1 i_{r-1})...(i_1 i_2) = (i_1 ... i_{r-1}).$$

Since $\varphi \circ \tilde{\varphi}_{i_1,i_r}^{-1} \in \mathrm{Aut}(\overline{M}_{g,A[n]})$, by induction hypothesis, we have that $(i_1 i_h)$ is admissible for any $h = 2, ..., r-1$. We conclude that $(i_1 i_h)$ is admissible for any $h = 2, ..., r$, and $\sigma_\varphi \in \mathcal{A}_{A[n]}$. □

Proposition 3.2.15. *For any* $g \geqslant 2$ *the only automorphism of* $\overline{M}_{g,A[1]}$ *is the identity. Furthermore* $\mathrm{Aut}(\overline{M}_{1,A[1]}) \cong PGL(2)$, $\mathrm{Aut}(\overline{M}_{1,A[2]}) \cong (\mathbb{C}^*)^2$ *and* $\mathrm{Aut}(\overline{M}_{1,A[3]}) \cong \mathcal{A}_{A[3]} \cong S_3$.

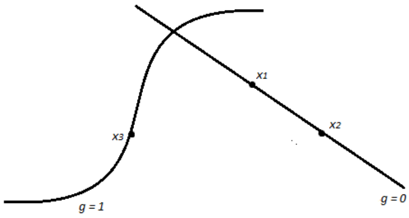
Proof. If $n \leqslant 2$, by [Has, Corollary 4.7], the reduction morphism $\rho : \overline{M}_{g,n} \to \overline{M}_{g,A[n]}$ is an isomorphism and we conclude by Propositions 2.2.5 and 2.2.7 of Chapter 2.
Consider now the case $g = 1, n = 3$. By Lemma 3.2.14 we have a surjective morphism

$$\chi : \mathrm{Aut}(\overline{M}_{1,A[3]}) \to \mathcal{A}_{A[3]}.$$

Let φ be an automorphism of $\overline{M}_{1,A[3]}$ inducing the trivial permutation. Then φ^{-1} induces the trivial permutation as well and we have three commutative diagrams

$$\begin{tikzcd}
\overline{M}_{1,A[3]} \arrow[r, "\varphi"] \arrow[d, "\pi_i"'] & \overline{M}_{1,A[3]} \arrow[d, "\pi_i"] \\
\overline{M}_{1,A[2]} \arrow[r, "\overline{\varphi}"] & \overline{M}_{1,A[2]}
\end{tikzcd}$$

Let $[C, x_1, x_2] \in \overline{M}_{1,A[2]}$ be a general point. The fiber $\pi_i^{-1}([C, x_1, x_2])$ intersects the boundary divisors $\Delta_{0,2} \subset \overline{M}_{1,A[3]}$ in two points corresponding to curves of the following type



The two points in $\pi_i^{-1}([C, x_1, x_2]) \cap \Delta_{0,2}$ can be identified with x_1, x_2. Now let $[C', x_1', x_2']$ be the image of $[C, x_1, x_2]$ via $\overline{\varphi}$. Similarly $\pi_i^{-1}([C', x_1', x_2']) \cap \Delta_{0,2} = \{x_1', x_2'\}$. We have $\varphi(\pi_i^{-1}([C, x_1, x_2]) \cap \Delta_{0,2}) = \pi_i^{-1}([C', x_1', x_2']) \cap \Delta_{0,2}$, $[C', x_1', x_2'] = [C, x_1, x_2]$ and $\overline{\varphi}$ has to be the identity.
So φ restricts to an automorphism of the elliptic curve $\pi_1^{-1}([C, x_1, x_2]) \cong C$ mapping the set $\{x_1, x_2\}$ into itself. On the other hand φ restricts to an automorphism of the elliptic curve $\pi_2^{-1}([C, x_1, x_2]) \cong C$ with the same property. Note that $\pi_2^{-1}([C, x_1, x_2]) \cap \pi_1^{-1}([C, x_1, x_2]) = \{x_1\}$. Combining these two facts we have that φ restricts to an automorphism of $\pi_1^{-1}([C, x_1, x_2]) \cong C$ fixing x_1 and x_2. Since C is a general elliptic curve we have that $\varphi_{|\pi_1^{-1}([C,x_1,x_2])}$ is the identity, and since $[C, x_1, x_2] \in \overline{M}_{1,A[2]}$ is general we conclude that $\varphi = \mathrm{Id}_{\overline{M}_{1,A[3]}}$. The isomorphism $\mathcal{A}_{A[3]} \cong S_3$ is immediate from Definition 3.2.11. □

Theorem 3.2.16. *The automorphism group of* $\overline{M}_{g,A[n]}$ *is isomorphic to the group of admissible permutations*

$$\mathrm{Aut}(\overline{M}_{g,A[n]}) \cong \mathcal{A}_{A[n]}$$

for any $g \geqslant 1, n$ *such that* $2g-2+n \geqslant 3$.

Proof. We proceed by induction on n. Proposition 3.2.15 gives the cases $g \geqslant 2, n = 1$ and $g = 1, n = 3$. By Lemma 3.2.14 we know that the morphism χ is surjective on $\mathcal{A}_{A[n]} \subseteq S_n$. Let us compute its kernel.
Let $\varphi \in \mathrm{Aut}(\overline{M}_{g,A[n]})$ be an automorphism such that $\chi(\varphi)$ is the identity, that is for any $i = 1, ..., n$ the fibration $\pi_i \circ \varphi^{-1}$, and the fibration $\pi_i \circ \varphi$ as well, factor through π_i and we have n commutative diagrams

$$\begin{tikzcd}
\overline{M}_{g,A[n]} \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & \overline{M}_{g,A[n]} \arrow[d, "\pi_1"] \\
\overline{M}_{g,A[n-1]} \arrow[r, "\overline{\varphi}_1"] & \overline{M}_{g,A[n-1]}
\end{tikzcd} \quad \cdots \quad \begin{tikzcd}
\overline{M}_{g,A[n]} \arrow[r, "\varphi"] \arrow[d, "\pi_n"'] & \overline{M}_{g,A[n]} \arrow[d, "\pi_n"] \\
\overline{M}_{g,A[n-1]} \arrow[r, "\overline{\varphi}_n"] & \overline{M}_{g,A[n-1]}
\end{tikzcd}$$

The morphisms $\overline{\varphi}_i$ are automorphisms of $\overline{M}_{g,A[n-1]}$ and by induction hypothesis $\overline{\varphi}_1, ..., \overline{\varphi}_n$ act on $\overline{M}_{g,A[n-1]}$ as permutations.
The action of $\overline{\varphi}_i$ on the marked points $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n$ has to lift to the same automorphism φ for any $i = 1, ..., n$. So the actions of $\overline{\varphi}_1, ..., \overline{\varphi}_n$ have to be compatible and this implies $\overline{\varphi}_i = \mathrm{Id}_{\overline{M}_{g,A[n-1]}}$ for any $i = 1, ..., n$. We distinguish two cases.

- Assume $g \geqslant 3$. It is enough to observe that φ restricts to an automorphism on the fibers of π_1. Then φ restricts to the identity on the general fiber of π_1, so $\varphi = \mathrm{Id}_{\overline{M}_{g,A[n]}}$.

- Assume $g = 1, 2$. Note that φ restricts to an automorphism on the fibers of π_1 and π_2. So φ defines an automorphism of the fiber of π_1 with at least two fixed points in the case $g = 1, n \geqslant 3$ and at least one fixed point in the case $g = 2, n \geqslant 2$. Since the general 2-pointed genus 1 curve and the general 1-pointed genus 2 curves do not have non trivial automorphisms we conclude as before that φ restricts to the identity on the general fiber of π_1, so $\varphi = \mathrm{Id}_{\overline{M}_{g,A[n]}}$.

This proves that χ is injective and defines an isomorphism between $\mathrm{Aut}(\overline{M}_{g,n})$ and $\mathcal{A}_{A[n]}$. ☐

Example 3.2.17. Consider $\overline{M}_{g,A[4]}$ with $g \geqslant 1$ and weight data $(1, 1/3, 1/3, 1/3)$. The transpositions $1 \leftrightarrow 2$, $1 \leftrightarrow 3$ and $1 \leftrightarrow 4$ induce just birational maps. The group $\mathcal{A}_{A[4]}$ is generated by the admissible transpositions $2 \leftrightarrow 3$, $2 \leftrightarrow 4$ and $3 \leftrightarrow 4$.
For $\overline{M}_{g,A[4]}$ with $g \geqslant 1$ and weight data $(1/12, 2/3, 1/4, 1/3)$ the automorphism group $\mathcal{A}_{A[4]}$ is generated by the two admissible transpositions $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$.

Automorphisms of $\overline{\mathcal{M}}_{g,A[n]}$

Let us consider the Hassett's moduli stack $\overline{\mathcal{M}}_{g,A[n]}$ and the natural morphism $\pi : \overline{\mathcal{M}}_{g,A[n]} \to \overline{M}_{g,A[n]}$ on its coarse moduli space. Since π is universal for morphism to schemes for any $\varphi \in \mathrm{Aut}(\overline{\mathcal{M}}_{g,A[n]})$ there exists an unique $\tilde{\varphi} \in \mathrm{Aut}(\overline{M}_{g,A[n]})$ such that $\pi \circ \varphi = \tilde{\varphi} \circ \pi$. So we get a morphism of groups

$$\tilde{\chi} : \mathrm{Aut}(\overline{\mathcal{M}}_{g,A[n]}) \to \mathrm{Aut}(\overline{M}_{g,A[n]}).$$

Proposition 3.2.18. *If* $2g-2+n \geqslant 3$ *then the morphism* $\tilde{\chi}$ *is injective.*

Proof. For the values of g and n we are considering $\overline{\mathcal{M}}_{g,A[n]}$ is a normal Deligne-Mumford stack with trivial generic stabilizer. To conclude it is enough to apply Proposition 2.3.4 of Chapter 2. ☐

By Proposition 3.2.18 for any $g \geqslant 1, n$ such that $2g-2+n \geqslant 3$ the group $\mathrm{Aut}(\overline{\mathcal{M}}_{g,A[n]})$ is a subgroup of $\mathcal{A}_{A[n]}$. Note that an admissible transposition $i \leftrightarrow j$ defines an automorphism of $\overline{\mathcal{M}}_{g,A[n]}$. Indeed the contraction of a rational tail with three special points does not affect neither the coarse moduli space nor the stack because it is a bijection on points and preserves the automorphism groups of the objects. However, it may induce a non trivial transformation on the universal curve.

Theorem 3.2.19. *The automorphism group of the stack* $\overline{\mathcal{M}}_{g,A[n]}$ *is isomorphic to the group of admissible permutations*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{g,A[n]}) \cong \mathcal{A}_{A[n]}$$

for any $g \geqslant 1, n$ *such that* $2g-2+n \geqslant 3$. *Furthermore* $\mathrm{Aut}(\overline{\mathcal{M}}_{1,A[1]}) \cong \mathbb{C}^*$ *while* $\mathrm{Aut}(\overline{\mathcal{M}}_{1,A[2]})$ *is trivial.*

Proof. By Proposition 3.2.18 the surjective morphism

$$\tilde{\chi} : \mathrm{Aut}(\overline{\mathcal{M}}_{g,A[n]}) \rightarrow \mathcal{A}_{A[n]}$$

is an isomorphism. The isomorphism $\mathrm{Aut}(\overline{\mathcal{M}}_{1,A[1]}) \cong \mathbb{C}^*$ derives from $\overline{\mathcal{M}}_{1,A[1]} \cong \overline{\mathcal{M}}_{1,1} \cong \mathbb{P}(4,6)$. Since a rational tail with three special points in automorphisms-free the reduction morphism

$$\rho : \overline{\mathcal{M}}_{1,2} \rightarrow \overline{\mathcal{M}}_{1,A[2]}$$

is a bijection on points and preserves the automorphism groups of the objects. The stacks $\overline{\mathcal{M}}_{1,2}$ and $\overline{\mathcal{M}}_{1,A[2]}$ are isomorphic. We conclude by Proposition 2.3.7 of Chapter 2. ☐

4

KONTSEVICH'S MODULI SPACES: SOME CONJECTURES

Let X be a projective variety, $\beta \in H_2(X, \mathbb{Z})$ be a homology class, and $Z_1, ..., Z_n \subset X$ cycles in general position. We want to study the following set of curves

$$\{C \subset X \text{ of genus } g, \text{ homology } \beta, \text{ and } C \cap Z_i \neq \emptyset \text{ for any } i\}. \tag{4.0.1}$$

In [Kh] *M. Kontsevich* observed that the curve $C \subset X$ should be replaced by a pointed curve $(C, (x_1, ..., x_n))$ and a holomorphic map $f : C \to X$ such that $f(x_i) \in Z_i$ for any $i = 1, ..., n$. The key idea, in order to give an algebraic definition of *Gromov-Witten classes* and *invariants*, is to introduce a suitable compactification done by *stable maps* of the space of curves 4.0.1.

Definition 4.0.20. An n-pointed, genus g, *quasi-stable* curve $[C, (x_1, ..., x_n)]$ is a projective, connected, reduced, at most nodal curve of arithmetic genus g, with n distinct, and smooth marked points.
A family of n-pointed genus g quasi-stable curves parametrized by a scheme S over $\mathbb{C}$ is a flat, projective morphism $\pi : \mathcal{C} \to S$, with n-sections $x_1, ..., x_n : S \to \mathcal{C}$, such that the fiber $[C_s, (x_1(s), ..., x_n(s))]$ is a n-pointed, genus g, quasi-stable curve, for any geometric point $s \in S$.

Let X be a scheme over $\mathbb{C}$. A family of maps over S to X is a collection

$$(\pi : \mathcal{C} \to S, (x_1, ..., x_n), \alpha : \mathcal{C} \to X)$$

such that

- $(\pi : \mathcal{C} \to S, (x_1, ..., x_n))$, is a family of n-pointed genus g quasi-stable curves parametrized by S.
- $\alpha : \mathcal{C} \to X$ is a morphism.

The families $(\pi : \mathcal{C} \to S, (x_1, ..., x_n), \alpha)$ and $(\pi^{'} : \mathcal{C}^{'} \to S, (x_1^{'}, ..., x_n^{'}), \alpha^{'})$ are isomorphic if there is an isomorphism of schemes $\varphi : \mathcal{C} \to \mathcal{C}^{'}$ such that $\pi = \pi^{'} \circ \varphi$, $x_i^{'} = \varphi \circ x_i$ for any $i = 1, ..., n$, and $\alpha = \alpha^{'} \circ \varphi$.
Let $(C, (x_1, ..., x_n), \alpha)$ be a map from an n-pointed genus g curve to X, the *special points* of an irreducible component $E \subseteq C$ are the marked points of C on E and the points in $E \cap \overline{C \setminus E}$.

Definition 4.0.21. A map $(C, (x_1, ..., x_n), \alpha)$ from an n-pointed genus g quasi-stable curve to X is *stable* if:

- any component $E \cong \mathbb{P}^1$ of C contracted by α contains at least three special points,
- any component $E \subseteq C$ of arithmetic genus 1 contracted by α contains at least one special point.

A family $(\pi : \mathcal{C} \to S, (x_1, ..., x_n), \alpha)$ is stable if each geometric fiber is stable.

Remark 4.0.22. In the case $X = \mathbb{P}^N$ the map $(\pi : \mathcal{C} \to S, (x_1, ..., x_n), \alpha)$ is stable if and only if $\omega_{\mathcal{C}/S}(x_1 + ... + x_n) \otimes \alpha^*(\mathcal{O}_{\mathbb{P}^N}(3))$ is π-ample.

Let X be a scheme over $\mathbb{C}$, and let $\beta \in A_1X$. To any scheme S over $\mathbb{C}$ we associate the set of isomorphism classes of stable families $(\pi : \mathcal{C} \to S, (x_1, ..., x_n), \alpha)$ parametrized by S of n-pointed genus g curves to X such that $\alpha_*(C_s) = [\beta]$, where $[\beta]$ denotes the fundamental class of β. In this way we get a controvariant functor

$$\overline{\mathcal{M}}_{g,n}(X, \beta) : \mathfrak{Schemes} \to \mathfrak{Sets}.$$

If X is a projective scheme over $\mathbb{C}$ then there exists a projective scheme $\overline{M}_{g,n}(X, \beta)$ coarsely representing the functor $\overline{\mathcal{M}}_{g,n}(X, \beta)$, [FP, Theorem 1]. The spaces $\overline{M}_{g,n}(X, \beta)$ are called *moduli spaces of stable maps*, or *Kontsevich's moduli spaces*.
Recall that a smooth variety X is said to be *convex* if $H^1(\mathbb{P}^1, \alpha^*T_X) = 0$ for any morphism $\alpha : \mathbb{P}^1 \to X$.

Remark 4.0.23. The tangent bundle of an homogeneous variety is generated by global section, so it is convex. On the other hand to be convex for an uniruled variety is a strong condition, for instance the blow-up of a convex variety is not convex.

Let X be a projective, nonsingular, convex variety, then $\overline{M}_{0,n}(X, \beta)$ is a normal, projective variety of pure dimension

$$\dim(X) + \int_\beta c_1(T_X) + n - 3.$$

Furthermore $\overline{M}_{0,n}(X, \beta)$ is locally a quotient of a nonsingular variety by a finite group, that is $\overline{M}_{0,n}(X, \beta)$ has at most finite quotient singularities, [FP, Theorem 2].
In the special case $X = \mathbb{P}^N$ we have $\beta \sim d[\text{line}]$ for some integer d and the scheme $\overline{M}_{0,n}(\mathbb{P}^N, d)$ is irreducible.

Examples

In the following we give a list of examples in which moduli of stable maps have a clear geometric description.

- The moduli space of stable maps to a point is isomorphic to the moduli space of curves

  $$\overline{M}_{g,n}(\mathbb{P}^0, 0) \cong \overline{M}_{g,n}.$$

  For the space of degree zero stable maps we have

  $$\overline{M}_{g,n}(X, 0) \cong \overline{M}_{g,n} \times X.$$

- The moduli space of degree one maps to $\mathbb{P}^N$ is the Grassmannian

  $$\overline{M}_{0,0}(\mathbb{P}^N, 1) \cong \mathbb{G}(1, N),$$

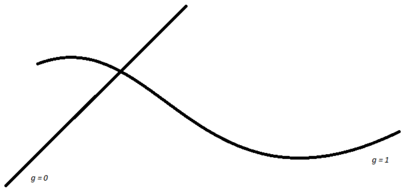
  and similarly the moduli space of degree one maps to a smooth quadric hypersurface $Q \subset \mathbb{P}^N$, with $N \geqslant 3$, is the orthogonal Grassmannian

  $$\overline{M}_{0,0}(Q, 1) \cong \mathbb{OG}(1, N).$$

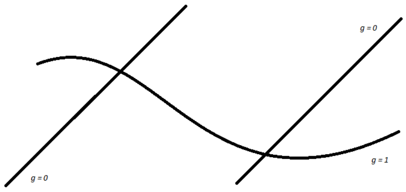
- The Kontsevich moduli space $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the space of complete conics that is to the blow up of the $\mathbb{P}^5$ parametrizing conics in $\mathbb{P}^2$ along the Veronese surface V of double lines

  $$\overline{M}_{0,0}(\mathbb{P}^2, 2) \cong Bl_V\mathbb{P}^5.$$

- Consider now $\overline{M}_{1,0}(\mathbb{P}^2,3)$. Smooth plane cubic are parametrized by an open subset of $\mathbb{P}^9 = \mathbb{P}(k[x_0,x_1,x_2]_3)$. On the other hand we have maps from a reducible curve with a component of genus zero and a component of genus one, contracting the genus one component and of degree three on the genus zero component.

  For any curve of genus one we have a 1-dimensional choice for the genus zero component, namely the connecting node. So we get a component of dimension 10 of $\overline{M}_{1,0}(\mathbb{P}^2,3)$. Finally we have a curve with three components: an elliptic curve and two rational tails. The map contracts the elliptic curve and maps the rational tails to a line and a conic.

  Here we have a 2-dimensional choice for the two nodes on the elliptic curve, a 2-dimensional choice for the line, and a 5-dimensional choice for the conic. We conclude that $\overline{M}_{1,0}(\mathbb{P}^2,3)$ has three irreducible components: two of dimension 9 and one of dimension 10.

- Let $X \subset \mathbb{P}^7$ be a smooth degree seven hypersurface containing a $\mathbb{P}^3$. Writing down an explicit equation for X one can see that $\overline{M}_{0,0}(X,2)$ has two irreducible components: one component is 5-dimensional and covers X, the second component parametrizes conics in the $\mathbb{P}^3$ and so has dimension $5+3=8$.
  Generalizing this construction one can show that $\overline{M}_{0,0}(X,2)$ can have a component of dimension arbitrary larger than the dimension of the main component even if X is a Fano hypersurface in $\mathbb{P}^N$.

Natural maps

Kontsevich's moduli spaces, as moduli spaces of curves, admit natural morphisms.

- Forgetful morphisms
$$\pi_I : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n-j}(X,\beta),$$
  forgetting the the points marked by $i_1,...,i_j$ for $j \leqslant n$.

- Evaluation morphisms
$$ev_i : \overline{M}_{g,n}(X,\beta) \to X,$$
  mapping $(C,\{x_1,...,x_n,\alpha\})$ to $\alpha(x_i)$.

- If $2g+n-3 \geqslant 0$ we have morphisms forgetting the map α,
$$\rho : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}.$$

4.1 THE STACK $\overline{\mathcal{M}}_{g,n}(x, \beta)$

In this section we follow the clear and detailed discussion worked out by *F. Poma* in [Po]. The construction of the moduli of stable maps can be transposed into the realm of algebraic stacks. Let k be a field. Consider the functor

$$\mathcal{F} : \mathfrak{Schemes}_{/k} \to \mathfrak{Groupoids},$$

associating to a scheme S the groupoids $\mathcal{F}(S)$ of flat projective families $\pi : C \to S$ of nodal curves of genus g,

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\pi"] & X \\ S \arrow[u, bend left, "s_i"] & \end{tikzcd}$$

where s_i are disjoint smooth sections of π, $\alpha_*[C_s] = \beta$ for any fiber $C_s = \pi^{-1}(s)$, and $\mathrm{Aut}(C, \alpha, \pi, s_i)$ is finite over S.

Theorem 4.1.1. *(Abramovich-Oort '01) There exists a proper algebraic stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of finite type over k which represents $\mathcal{F}$.*

Theorem 4.1.2. *(Kontsevich '95, Behrend-Fantechi '97) If $\mathrm{ch}\, k = 0$, then $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is of Deligne-Mumford type.*

Recall that a *Dedekind domain* D is an integral domain which is not a field, satisfying one of the following equivalent conditions:

- D is noetherian, and the localization at each maximal ideal is a Discrete Valuation Ring.
- D is an integrally closed, noetherian domain with Krull dimension one.
- Every nonzero proper ideal of D factors into primes ideals.
- Every fractional ideal of D is invertible.

Example 4.1.3. Let C be an affine smooth curve over a field k. The coordinate ring $A(C)$ of C is a finitely generated k-algebra, and so noetherian, it has dimension one since C is a curve. Furthermore, since C is smooth and so normal $A(C)$ is integrally closed. So $A(C)$ is a Dedekind domain.

Consider now the functor

$$\mathcal{F}_D : \mathfrak{Schemes}_{/D} \to \mathfrak{Groupoids},$$

exactly defined as $\mathcal{F}$ but from the category of schemes over a Dedekind domain D.

Theorem 4.1.4. *(Abramovich-Oort '01) There exists a proper algebraic stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of finite type over D which represents $\mathcal{F}_D$.*

In the case $\mathrm{ch}\, k = p$, in general $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a proper *Artin stack*. As instance consider the element $(\mathbb{P}^1, \alpha) \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, p)$ given by

$$\alpha : \mathbb{P}^1 \to \mathbb{P}^1,\ [x_0, x_1] \mapsto [x_0^p, x_1^p].$$

Then $\mathrm{Aut}(\mathbb{P}^1, \alpha) = \mu_p = \operatorname{Spec} k[\xi]/(\xi^p - 1) = \operatorname{Spec} k[\xi]/(\xi - 1)^p$, which is not reduced over $\operatorname{Spec} k$. However even in the characteristic p case the stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is a global quotient stack and the functor

$$\theta : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \mathfrak{M}_{g,n}$$

is representable. This led *A. Kresch* to define an intersection theory for Artin stacks over a field [Kr].

Recall that a *ring of mixed characteristic* is a commutative ring R having characteristic zero, having an ideal I such that R/I has positive characteristic. For instance the ring of integers $\mathbb{Z}$ has characteristic zero, and for any prime number p, $\mathbb{Z}/(p)$ is a finite field of characteristic p. Recently *F. Poma* in [Po] extended the construction of the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ in [BF] to schemes in positive and mixed characteristic. This leads to a rigorous definition of Gromov-Witten invariants for these classes of schemes.

4.2 VIRTUAL DIMENSION

If X is a homogeneous variety then it is smooth and its tangent bundle is generated by global sections, in particular X is convex. In this case $\overline{M}_{0,n}(X, \beta)$ is a normal, projective variety of pure dimension. Furthermore if $X = \mathbb{P}^N$ then $\overline{M}_{0,n}(\mathbb{P}^N, d)$ is irreducible. On the other hand when $g \geqslant 1$, and even when $g = 0$ for most schemes $X \neq \mathbb{P}^N$ the space $\overline{M}_{g,n}(X, \beta)$ may have many components of dimension greater than the expected dimension. To overcome this gap and to give a rigorous definition of Gromov-Witten invariants we have to introduce the notions of *virtual fundamental class* and *virtual dimension*.

The normal cone

In this section we follow [BF]. Let E be a rank r vector bundle on a smooth variety Y, $s \in H^0(E)$ a section, and $Z = Z(s) \subset Y$ the zero scheme of s. As s varies Z can become reducible or even of non pure dimension. Let $\mathcal{I}$ be the ideal sheaf of Z in Y, the *normal cone* of Z in Y is the affine cone over Z defined by

$$C_Z Y = \operatorname{Spec}(\bigoplus_{k=0}^{\infty} \mathcal{I}^k/\mathcal{I}^{k+1}).$$

Note that the $C_Z Y$ has pure dimension $n = \dim Y$. Multiplication by s induces a surjective map

$$\bigoplus_k \mathrm{Sym}^k(\mathcal{O}(E^*/\mathcal{I}\mathcal{O}(E^*))) \to \bigoplus_k \mathcal{I}^k/\mathcal{I}^{k+1},$$

and applying Spec we get an embedding

$$C_Z Y \to E_{|Z}.$$

The normal cone gives a class $[C_Z Y] \in A_n(E_{|Z})$, so we have $s^*[C_Z Y] \in A_{n-r}(Z)$.

Let $\mathcal{M}$ be a Deligne-Mumford stack. Since $\mathcal{M}$ admits an étale open cover by schemes we can consider a scheme U and take an embedding $U \hookrightarrow W$, where W is a smooth scheme. Now, consider the ideal sheaf $\mathcal{I}$ of U in W, and form the normal cone $C_U W$. The differentiation map

$$\bigoplus_k \mathcal{I}^k \to \Omega^1_W, \ f \mapsto df$$

induces a map

$$\bigoplus_k \mathcal{I}^k/\mathcal{I}^{k+1} \to \bigoplus_k \mathrm{Sym}^k(\Omega^1_W/\mathcal{I}\Omega^1_W),$$

finally applying Spec we get a map

$$T_{W|U} = \mathrm{Spec}(\bigoplus_k \mathrm{Sym}^k(\Omega^1_W/\mathcal{I}\Omega^1_W)) \to C_U W.$$

The intrinsic normal cone $\mathcal{C}_U$ is defined as the stack quotient $[C_U W/T_{W|U}]$. Now, given an étale open cover $\{U_i\}$ of $\mathcal{M}$ the intrinsic normal cones C_{U_i} glue to give the intrinsic normal cone $\mathcal{C}_\mathcal{M}$ of $\mathcal{M}$.

If $L^\bullet_\mathcal{M}$ is the *cotangent complex* of $\mathcal{M}$, an *obstruction theory* for $\mathcal{M}$ is a complex of sheaves $\mathcal{E}^\bullet$ on $\mathcal{M}$ with a morphism $\mathcal{E}^\bullet \to L^\bullet_\mathcal{M}$, which is an isomorphism on h^0 and a surjection on h^{-1}.

Given an arbitrary complex $\mathcal{E}^\bullet$ we define $h^1/h^0(\mathcal{E}^\bullet)$ to be the quotient stack of the kernel of $\mathcal{E}^1 \to \mathcal{E}^2$ by the cokernel of $\mathcal{E}^{-1} \to \mathcal{E}^0$.

By the definition of perfect obstruction theory the intrinsic normal cone $\mathcal{C}_\mathcal{M}$ embeds in $h^1/h^0((\mathcal{E}^\bullet)^*)$.

Let C be the fiber product of $(E^{-1})^*$ with $\mathcal{C}_\mathcal{M}$ over $h^1/h^0((\mathcal{E}^\bullet)^*)$, where $\mathcal{O}(E^{-1}) = \mathcal{E}^{-1}$. This is a cone contained in the vector bundle $(E^{-1})^*$. The *virtual fundamental class* is defined to be the intersection of C with the zero section of $(E^{-1})^*$.

In this part we mainly follow [De] and [Po]. Let X be a smooth connected projective scheme, $\mathfrak{M}_{g,n}$ the Artin stack parametrizing pre-stable n-pointed genus g connected nodal curves, and C its universal curve. We define an algebraic stack $\mathrm{Mor}(C, X)$ as follows:

- for any scheme S objects in $\mathrm{Mor}(C, X)(S)$ are pre-stable curves $(C_S \to S, s_i)$ over S with a morphism $f_S : C_S \to X$,
- for any scheme S a morphism from $(C_S \to S, s_i)$ to $(C'_S \to S, s'_i)$ is an isomorphism α of pre-stable curves such that $f'_S \circ \alpha = f_S$.

There is a natural functor $\theta : \mathrm{Mor}(C, X) \to \mathfrak{M}_{g,n}$ forgetting the map to X, furthermore $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is an open substack of $\mathrm{Mor}(C, X)$. The fiber product $\overline{\mathcal{C}} \times_{\mathfrak{M}_{g,n}} \mathrm{Mor}(C, X)$ is a universal family for $\mathrm{Mor}(C, X)$ and we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[rr, bend left, "\psi"] \arrow[d, "\pi"] & \overline{\mathcal{C}} \arrow[r, "\overline{\psi}"'] \arrow[d, "\pi"] & X \\
\overline{\mathcal{M}}_{g,n}(X, \beta) \arrow[r] \arrow[u, bend left, "s_i"] & \mathrm{Mor}(C, X) \arrow[u, bend left, "\overline{s}_i"] &
\end{tikzcd}$$

where $\mathcal{C} = \overline{\mathcal{C}} \times_{\mathrm{Mor}(C,X)} \overline{\mathcal{M}}_{g,n}(X, \beta)$ is the universal stable map.

It turns out that considering the complex $F^\bullet = (R\pi_* \overline{\psi}^* T_X)^*$ we get a vector bundle stack $h^1/h^0(F^\bullet)$. Similarly $E^\bullet = (R\pi_* \psi^* T_X)^*$ gives a perfect obstruction theory for θ, and so a virtual fundamental class for $\overline{\mathcal{M}}_{g,n}(X, \beta)$.

In what follows we try to understand more concretely the tangent and the obstruction spaces to $\mathrm{Mor}(Y, X)$, where X, Y are projective varieties over a field. The scheme $\mathrm{Mor}(Y, X)$, parametrizing morphisms $Y \to X$, is a locally noetherian scheme having countably many components. However fixing an ample divisor H on X we can consider the scheme $\mathrm{Mor}(P)(Y, X)$ parametrizing morphisms $Y \to X$ with fixed Hilbert polynomial $P(m) = \chi(Y, mf^*H)$. This is a quasi-projective scheme.

The tangent space $T_{[f]}Mor(Y,X)$ in a point $[f] \in Mor(Y,X)$ parametrizes morphisms Spec $k[\epsilon]/(\epsilon^2) \to Mor(Y,X)$, and hence $k[\epsilon]/(\epsilon^2)$-morphisms

$$f_\epsilon : Y \times \operatorname{Spec} k[\epsilon]/(\epsilon^2) \to X \times \operatorname{Spec} k[\epsilon]/(\epsilon^2),$$

which should be interpreted as first order deformations of f.

Proposition 4.2.1. *Let X, Y be projective varieties. The tangent space to $Mor(Y,X)$ in a point $[f]$ is given by*

$$T_{[f]}Mor(Y,X) = H^0(Y, \mathcal{H}om(f^*\Omega_X, \mathcal{O}_Y)).$$

Proof. Assume $X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B)$ to be affine, where A, B are finitely generated k-algebras. Let $f^\sharp : A \to B$ be the morphism induced by f. We are looking for $k[\epsilon]/(\epsilon^2)$-algebras homomorphisms $f^\sharp_\epsilon : A[\epsilon] \to B[\epsilon]$ of the type $f^\sharp_\epsilon(a) = f^\sharp(a) + \epsilon g(a)$. Notice that the since $f^\sharp_\epsilon(aa') = f^\sharp_\epsilon(a)f^\sharp_\epsilon(a')$ we get $\epsilon g(aa') = (f^\sharp(a)+\epsilon g(a))(f^\sharp(a')+\epsilon g(a')) - f^\sharp(a)f^\sharp(a') = \epsilon(f^\sharp(a)g(a') + f^\sharp(a')g(a))$. Then $f^\sharp_\epsilon(aa') = f^\sharp_\epsilon(a)f^\sharp_\epsilon(a')$ is equivalent to

$$g(aa') = f^\sharp(a)g(a') + f^\sharp(a')g(a),$$

that is $g : A \to B$ is a k-derivation of the A-module B and then it has to factorize as $g : A \to \Omega_A \to B$. Such extensions are therefore parametrized by $Hom_A(\Omega_A, B) = Hom_B(\Omega_A \otimes_A B, B)$.
Now, let us cover X by open affine $U_i = \operatorname{Spec}(A_i)$ and Y by open affine $V_i = \operatorname{Spec}(B_i)$ such that $f(V_i) \subseteq U_i$. By the previous part of the proof first order deformations of $f_{|V_i}$ are parametrized by $h_i \in Hom_{B_i}(\Omega_{A_i} \otimes_{A_i} B_i, B_i) = H^0(V_i, \mathcal{H}om(f^*\Omega_X, \mathcal{O}_Y))$. To glue these together we need the compatibility condition $h_{i|V_{ij}} = h_{j|V_{ij}}$ which means that the collection $\{h_i\}$ defines a global section on Y. ☐

Notice that when X is smooth along the image of f we have

$$T_{[f]}Mor(Y,X) = H^0(Y, f^*T_X).$$

Furthermore when Y is smooth $H^0(Y, T_Y)$ is the tangent space to the automorphism group of Y at the identity, its elements are called infinitesimal automorphisms. The image of the morphism $H^0(Y, T_Y) \to H^0(Y, f^*T_X)$ parametrizes deformation of f by reparametrizations.

Let $0 \mapsto I \to R \to R/I \mapsto 0$ be a semi-small extension in the category of local Artinian k-algebras. That is $I \subseteq \mathfrak{M}$ and $I\mathfrak{M} = 0$, where $\mathfrak{M}$ is the maximal ideal of R. Let $f : Y \to X$ be a morphism. Assume as before X, Y affine. Since X is smooth along the image of f and $I^2 = 0$ by the infinitesimal lifting property [Ha, Exercise 8.6 - Chap 2], there exists a lifting of $f^\sharp_{R/I} : A \otimes_k R/I \to B \otimes_k R/I$ to a morphism $f^\sharp_R : A \otimes_k R \to B \otimes_k R$, and two different liftings differ by an R-derivation $A \otimes_k R \to B \otimes_k I$, that is by an element of $H^0(Y, f^*T_X) \otimes_k I$.
In the general case we need to glue two extensions h_i, h_j on each $V_i \cap V_j$. These two extensions differ by an element $\nu_{ij} \in H^0(V_i \cap V_j, f^*T_X) \otimes_k I$. We have $\nu_{ij}h_{i|V_{ij}} = h_{j|V_{ij}}$. On the triple intersection $V_i \cap V_j \cap V_k$ we have $\nu_{jk}\nu_{ij}h_{i|V_{ijk}} = \nu_{jk}h_{j|V_{ijk}} = h_{k|V_{ijk}} = \nu_{ik}h_{i|V_{ijk}}$. So $\nu_{ik} = \nu_{jk}\nu_{ij}$ and the collection $\{\nu_{ij}\} \in C^1(\{V_i\}, f^*T_X \otimes_k I)$ is a cocycle. We have a global lifting if and only if $\nu_{ij} = 0$, and the *obstruction space* is $H^1(Y, f^*T_X) \otimes I$.

Locally around a point $[f] \in Mor(Y,X)$ the space $Mor(Y,X)$ can be defined by a set of polynomial $\{P_i\}$ in some affine space $\mathbb{A}^N$. The rank r of the Jacobian $J(P_i)$ is the codimension of the Zariski tangent space $T_{[f]}Mor(Y,X) \subseteq k^N$. Let V be a variety defined by r equations

among the P_i for which the corresponding rows in the Jacobian have rank r, then V is smooth at $[f]$ and has the same Zariski tangent space of $Mor(Y, X)$. By 6.3.1 the variety V has dimension $h^0(Y, f^*T_X)$ in $[f]$. We want to show that in the regular local ring $R = \mathcal{O}_{V,[f]}$ the ideal I of regular functions vanishing on $Mor(Y, X)$ can be generated by $h^1(Y, f^*T_X)$ elements. Since the Zariski tangent spaces are the same the ideal I is contained in the square of the maximal ideal $\mathfrak{M}$ of R. Furthermore by Nakayama's lemma it is enough to show that the k-vector space $I/\mathfrak{M}I$ has dimension at most h^1.
The morphism $Spec(R/I) \to Mor(Y, X)$ corresponds to an extension $f_{R/I} : Y \times Spec(R/I) \to X \times Spec(R/I)$ of f. We know that the obstruction to lift this extension to an extension $f_{R/\mathfrak{M}I} : Y \times Spec(R/\mathfrak{M}I) \to X \times Spec(R/\mathfrak{M}I)$ lies in

$$H^1(Y, f^*T_X) \otimes_k I/\mathfrak{M}I.$$

Let $\sum_{i=1}^{h^1} a_i \otimes \overline{b}_i$ be the obstruction, where $b_i \in I$. Since the obstruction vanishes modulo the ideal $(b_1, ..., b_{h^1})$ the morphism $Spec(R/I) \to Mor(Y, X)$ lifts to a morphism $Spec(R/\mathfrak{M}I + (b_1, ..., b_{h^1})) \to Mor(Y, X)$. In other words the identity $R/I \to R/I$ factors through the projection as $R/I \to R/\mathfrak{M}I + (b_1, ..., b_{h^1}) \to R/I$. Then $I = \mathfrak{M}I + (b_1, ..., b_{h^1})$, which means that $I/\mathfrak{M}I$ is generated by the classes of $b_1, ..., b_{h^1}$.

Remark 4.2.2. Locally around $[f]$ the space $Mor(Y, X)$ can be defined by at most $h^1(Y, f^*T_X)$ equations in a smooth variety of dimension $h^0(Y, f^*T_X)$. In particular any irreducible component of $Mor(Y, X)$ through $[f]$ has dimension at least

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X).$$

The equations defining $Mor(Y, X)$ locally around $[f]$ can intersect badly so that the actual dimension is not the expected one. My naive way of understanding the deformation to the normal cone and the virtual fundamental class is to imagine a deformation of these equations that make the intersection transverse. If there is such a deformation, which formally means that there exists a perfect obstruction theory, then the object we obtain would be a virtual fundamental class.

Theorem 4.2.3. *Let X be a smooth projective variety. The virtual dimension of the moduli space $\overline{M}_{g,n}(X, \beta)$ is given by*

$$virdim(\overline{M}_{g,n}(X, \beta)) = (1 - g)(\dim(X) - 3) - \int_\beta \omega_X + n.$$

Proof. Consider the stable map $(C, \{x_1, ..., x_n\}, \alpha\}) \in \overline{M}_{g,n}(X, \beta)$. Let $Def(C, \{x_1, ..., x_n\}, \alpha\})$ be the space of first order deformations of $(C, \{x_1, ..., x_n\}, \alpha\})$, and let $Def_\alpha(C, \{x_1, ..., x_n\}, \alpha\})$ be the space of first order deformations with C held rigid. There is an exact sequence

$$0 \mapsto Def(C, \{x_1, ..., x_n\}) \to Def(C, \{x_1, ..., x_n\}, \alpha\}) \to Def_\alpha(C, \{x_1, ..., x_n\}, \alpha\}) \mapsto 0.$$

Note that since $(C, \{x_1, ..., x_n\}, \alpha\})$ is stable it does not have infinitesimal automorphisms, and this gives the injectivity of the map on the left.

- First we compute the dimension of $Def(C, \{x_1, ..., x_n\})$. The curve C is a stable nodal curve. By the spectral sequence of Ext functors we have

$$0 \mapsto H^1(C, \mathcal{H}om(\Omega_C, \mathcal{O}_C)) \to Ext^1(\Omega_C, \mathcal{O}_C) \to H^0(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) \mapsto 0,$$

there being no H^2 on a curve. We denote by δ the number of nodes in C. Since the sheaf Ω_C is locally free on the smooth locus of C, the sheaf $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))$ is just k at each node, then $\dim(H^0(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))) = \delta$. The curve C is l.c.i, then the dualizing sheaf ω_C is an invertible sheaf, and since $\omega_C \cong \Omega_C$ on the open set of regular points, we have an injective morphism $\check{\omega}_C \rightarrow \mathcal{H}om(\Omega_C, \mathcal{O}_C)$, and an exact sequence

$$0 \mapsto \check{\omega}_C \rightarrow \mathcal{H}om(\Omega_C, \mathcal{O}_C) \rightarrow \mathcal{O}_Z \mapsto 0,$$

where $Z = Sing(C)$. Since C is stable $h^0(\mathcal{H}om(\Omega_C, \mathcal{O}_C)) = 0$, by the cohomology exact sequence we get $h^0(\check{\omega}_C) = 0$, and

$$0 \mapsto H^0(C, \mathcal{O}_Z) \rightarrow H^1(C, \check{\omega}_C) \rightarrow H^1(\mathcal{H}om(\Omega_C, \mathcal{O}_C)) \mapsto 0.$$

By Riemann-Roch for singular curves we get $h^1(\check{\omega}_C) = 3g-3$, and since $h^0(\mathcal{O}_Z) = \delta$ we get $h^1(\mathcal{H}om(\Omega_C, \mathcal{O}_C)) = 3g-3-\delta$. Finally

$$\dim(Ext^1(\Omega_C, \mathcal{O}_C)) = h^1(T_C) + h^0(\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) = 3g-3-\delta+\delta = 3g-3.$$

and

$$\dim Def(C, \{x_1, ..., x_n\}) = 3g-3+n.$$

- By Remark 4.2.2 the expected dimension of $Def_\alpha(C, \{x_1, ..., x_n\}, \alpha\})$ is $h^0(\alpha^* T_X) - h^1(\alpha^* T_C)$. By Riemann-Roch theorem we get

$$\text{expdim}\, Def_\alpha(C, \{x_1, ..., x_n\}, \alpha\}) = \chi(\alpha^* T_C) = -K_X \cdot \alpha_* C + (1-g)\dim(X).$$

We conclude that

$$\text{expdim}\, Def(C, \{x_1, ..., x_n\}, \alpha\}) \geqslant -K_X \cdot \alpha_* C + (1-g)\dim(X) + 3g - 3 + n,$$

and the virtual dimension of $\overline{M}_{g,n}(X, \beta)$ is given by

$$-K_X \cdot \alpha_* C + (1-g)\dim(X) + 3g-3+n = (1-g)(\dim(X)-3) - \int_\beta \omega_X + n.$$

☐

4.3 CONJECTURES

Let us consider the space $\overline{M}_{0,n}(\mathbb{P}^N, d)$. This is an irreducible projective variety with at most finite quotient singularities and of dimension

$$\dim(\overline{M}_{0,n}(X, \beta)) = N(d+1) + d + n - 3.$$

The symmetric group S_n, and the automorphism groups $Aut(\mathbb{P}^N)$ act on $\overline{M}_{0,n}(\mathbb{P}^N, d)$.

- The action of S_n is given by

$$S_n \times \overline{M}_{0,n}(\mathbb{P}^N, d) \rightarrow \overline{M}_{0,n}(\mathbb{P}^N, d),\ (\sigma, [C, (x_1, ..., x_n), \alpha]) \mapsto [C, (x_{\sigma(1)}, ..., x_{\sigma(n)}), \alpha].$$

- The action of $Aut(\mathbb{P}^N)$ is given by

$$Aut(\mathbb{P}^N) \times \overline{M}_{0,n}(\mathbb{P}^N, d) \rightarrow \overline{M}_{0,n}(\mathbb{P}^N, d),\ (f, [C, (x_1, ..., x_n), \alpha]) \mapsto [C, (x_1, ..., x_n), f \circ \alpha].$$

Clearly the two actions commute.

The groups S_n and $\mathrm{Aut}(\mathbb{P}^N)$ induce automorphisms of $\overline{M}_{0,n}(\mathbb{P}^N, d)$.

Proposition 4.3.1. *The automorphisms of $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ are exactly the ones induced by automorphisms of $\mathbb{P}^2$, that is*

$$\mathrm{Aut}(\overline{M}_{0,0}(\mathbb{P}^2, 2)) \cong PGL(3).$$

Proof. It is well known that the space $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the space of complete conics, that is the blow up of $\mathbb{P}^5$ along the Veronese surface $V \subset \mathbb{P}^5$ parametrizing double lines:

$$\overline{M}_{0,0}(\mathbb{P}^2, 2) \cong Bl_V\mathbb{P}^5.$$

Then the automorphisms of $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ are induced by automorphisms of $\mathbb{P}^5$ stabilizing $V \cong \mathbb{P}^2$. On the other hand these are exactly the automorphisms of $\mathbb{P}^5$ induced by automorphisms of $\mathbb{P}^2$. $\square$

Let $\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2)$ be the Kontsevich moduli space parametrizing stable maps of degree $n-2$ from n-pointed genus zero curves to $\mathbb{P}^{n-2}$. In [Ka, Theorem 0.1] *M. Kapranov* considers the subscheme $V_0(p_1, ..., p_n)$ of the Hilbert scheme $\mathcal{H}$ of $\mathbb{P}^{n-2}$, parametrizing rational normal curves in $\mathbb{P}^{n-2}$ through n points $p_1, ..., p_n$ in linear general position. *Kapranov* proves that the closure $V(p_1, ..., p_n)$ in $\mathcal{H}$ of $V_0(p_1, ..., p_n)$ is indeed isomorphic to $\overline{M}_{0,n}$.
Let $\rho : \overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2) \rightarrow \overline{M}_{0,n}$ be the natural morphism forgetting the map $C \rightarrow \mathbb{P}^{n-2}$, and let $ev_i : \overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2) \rightarrow \mathbb{P}^{n-2}$ be the evaluation on the i-th marked point. [Ka, Theorem 0.1] implies that the morphism

$$\rho \times ev_1 \times ... \times ev_n : \overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2) \rightarrow \overline{M}_{0,n} \times \mathbb{P}^{n-2} \times ... \times \mathbb{P}^{n-2}$$

is an isomorphism on the open subset of $\mathbb{P}^{n-2} \times ... \times \mathbb{P}^{n-2}$ parametrizing points in general position. The projection on $\mathbb{P}^{n-2} \times ... \times \mathbb{P}^{n-2}$

$$\begin{tikzcd}
\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2) \arrow[r, "\rho\times ev_1\times\ldots\times ev_n"] \arrow[dr, "\pi"'] & \overline{M}_{0,n} \times \mathbb{P}^{n-2} \times ... \times \mathbb{P}^{n-2} \arrow[d] \\
 & \mathbb{P}^{n-2} \times ... \times \mathbb{P}^{n-2}
\end{tikzcd}$$

gives a fibration π of $\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2)$ whose general fiber is isomorphic to $\overline{M}_{0,n}$.

Conjecture 4.3.2. *Let $\varphi \in \mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2))$ be an automorphism. If $n \geqslant 5$ there exists an automorphism σ of $\mathbb{P}^{n-2} \times ... \times \mathbb{P}^{n-2}$ such that the diagram*

$$\begin{tikzcd}
\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2) \arrow[r, "\varphi"] \arrow[d, "\pi"] & \overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2) \arrow[d, "\pi"] \\
\mathbb{P}^{n-2} \times ... \times \mathbb{P}^{n-2} \arrow[r, "\sigma"] & \mathbb{P}^{n-2} \times ... \times \mathbb{P}^{n-2}
\end{tikzcd}$$

is commutative.

The Conjecture 4.3.2 implies the following theorem.

Theorem 4.3.3. *The automorphisms of $\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2)$ are the ones induced by automorphisms of $\mathbb{P}^{n-2}$ and permutations for any $n \geqslant 5$. More precisely*

$$\mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2)) \cong PGL(n-1) \times S_n,$$

for any $n \geqslant 5$.

Proof. Let $\varphi \in \mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2))$ be an automorphism. Consider a general point $(p_1, ..., p_n) \in \mathbb{P}^{n-2} \times ... \times \mathbb{P}^{n-2}$ and the fiber $\pi^{-1}(p_1, ..., p_n) \cong \overline{M}_{0,n}$. By Conjecture 4.3.2 the automorphism φ maps $\pi^{-1}(p_1, ..., p_n)$ onto another fiber, say $\pi^{-1}((q_1, ..., q_n))$.
Since the points $\{p_1, ..., p_n\}$ and $\{q_1, ..., q_n\}$ are in general position in $\mathbb{P}^{n-2}$ there exists an unique automorphism $\sigma \in \mathrm{Aut}(\mathbb{P}^{n-2})$ such that $\sigma(p_i) = q_i$ for any $i = 1, ..., n$.
So, up to an automorphism of $\mathbb{P}^{n-2}$, we can assume

$$\varphi_{|\pi^{-1}(p_1,...,p_n)} : \pi^{-1}(p_1, ..., p_n) \to \pi^{-1}(p_1, ..., p_n),$$

and consider $\varphi_{|\pi^{-1}(p_1,...,p_n)}$ as an automorphism of $\overline{M}_{0,n}$.
Since $n \geqslant 5$, by [BM2, Theorem 4.3] $\varphi_{|\pi^{-1}(p_1,...,p_n)}$ is a permutation of the marked points. Summing up, the automorphism $\varphi \in \mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2))$, up to a unique automorphism of $\mathbb{P}^{n-2}$, induces a permutation of the markings on the general fiber of π. This permutation necessarily comes from the automorphism of $\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2)$ acting as the permutation itself. In other words we have the following exact sequence of groups:

$$0 \mapsto \mathrm{Aut}(\mathbb{P}^{n-2}) \to \mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2)) \to S_n \mapsto 0.$$

Clearly there is a section $S_n \to \mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2))$ and

$$\mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2)) \cong \mathrm{Aut}(\mathbb{P}^{n-2}) \rtimes S_n$$

is a semi-direct product. Furthermore, since

$$\mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2)) / \mathrm{Aut}(\mathbb{P}^{n-2}) \cong S_n$$

is a group, $\mathrm{Aut}(\mathbb{P}^{n-2}) \lhd \mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2))$ is a normal subgroup. It is enough to observe that $\mathrm{Aut}(\mathbb{P}^{n-2}) \cap S_n = \{Id\}$, and that the actions of the two subgroups commute, to conclude that $\mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2))$ is the direct product of $\mathrm{Aut}(\mathbb{P}^{n-2})$ and S_n. $\square$

Now, let $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2)$ be the Deligne-Mumford moduli stack parametrizing n-pointed, genus zero, stable maps; and let

$$\chi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2) \to \overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2),$$

be the natural map on the coarse moduli space.

Proposition 4.3.4. *The automorphism group of* $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2)$ *is given by*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2)) \cong PGL(n-1) \times S_n,$$

for any $n \geqslant 5$.

Proof. The map χ induces a surjective morphism of groups

$$\overline{\chi} : \mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^{n-2}, n-2)) \to \mathrm{Aut}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2)).$$

For any $n \geqslant 5$ the general stable map in $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2)$ is automorphisms-free. Since $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^{n-2}, n-2)$ is a normal stack, by Proposition 2.3.4 of Chapter 2 the morphism $\overline{\chi}$ is injective. We conclude by Theorem 4.3.3. $\square$

These arguments give enough evidence to believe in the following conjecture.

Conjecture 4.3.5. *For any* $n \geqslant 5$ *we have*

$$\mathrm{Aut}(\overline{\mathcal{M}}_{0,n}(\mathbb{P}^N, d)) \cong \mathrm{Aut}(\overline{M}_{0,n}(\mathbb{P}^N, d)) \cong S_n \times PGL(N+1).$$

Part II

VSP - VARIETIES OF SUMS OF POWERS

5

BIRATIONAL ASPECTS OF THE GEOMETRY OF VARIETIES OF SUMS OF POWERS

We work over the complex field. We mainly follow notation and definitions of [Do]. The set of all decomposition $\{L_1, ..., L_h\}$ of a general polynomial $F \in k[x_0, ..., x_n]_d$ is denoted by $VSP(F, h)^o$. Via this construction it is easy to embed $VSP(F, h)^o$ into $Hilb_h((\mathbb{P}^n)^*)$.

Definition 5.0.6. The closure

$$VSP(F, h) := \overline{VSP(F, h)^o} \subseteq Hilb_h((\mathbb{P}^n)^*)$$

is the *Variety of Sums of Powers* of F.

Using the smoothness of $Hilb_h((\mathbb{P}^n)^*)$, when $n = 1, 2$, one gets the following classical result, see for instance [Do].

Proposition 5.0.7. *In the cases* $n = 1, 2$ *for a general polynomial* $F \in k[x_0, ..., x_n]_d$ *the variety* $VSP(F, h)$ *is either empty or a smooth variety of dimension*

$$\dim(VSP(F, h)) = h(n + 1) - \binom{n+d}{d}.$$

It is important to notice that an additive decomposition of F induces an additive decomposition of its partial derivatives.

Remark 5.0.8 (Partial Derivatives). Let $\{[L_1], ..., [L_h]\}$ be a decomposition of a homogeneous polynomial $F \in k[x_0, ..., x_n]_d$. We write

$$F = L_1^d + ... + L_h^d.$$

The partial derivatives of F are homogeneous polynomials of degree $d - 1$ decomposed in h linear factors

$$\frac{\partial F}{\partial x_i} = \alpha_{i_1} dL_1^{d-1} + ... + \alpha_{i_h} dL_h^{d-1}, \text{ for any } i = 0, ..., n.$$

Hence, as long as $h < \binom{d-1+n}{n}$, $VSP(F, h)^o \subseteq VSP(\frac{\partial F}{\partial x_i}, h)^o$, and taking closures we have

$$VSP(F, h) \subseteq VSP(\frac{\partial F}{\partial x_i}, h).$$

The polynomial F has $\binom{n+l}{l}$ partial derivatives of order l. Clearly these derivatives are homogeneous polynomials of degree $d - l$ decomposed in h-linear factors. Then, when $h < \binom{d-l+n}{n}$, we have $VSP(F, h) \subseteq VSP(\frac{\partial^l F}{\partial x_0^{l_0}, ..., \partial x_n^{l_n}}, h)$, where $l_0 + ... + l_n = l$.

As remarked in the introduction we are interested in a different compactification of additive decompositions. Consider the span of an additive decomposition in the Veronese embedding. We can associate to a decomposition of F an $(h - 1)$-plane h-secant to the Veronese variety $V_{d,n} \subset \mathbb{P}^N$. Note that by the generalized trisecant lemma, [CC, Proposition 2.6], when $h < N - n + 1$ the general h-secant linear space intersects transversely the Veronese variety in exactly h points. Hence we may embed a non empty open set $U \subset VSP(F, h)$ into $\mathbb{G}(h - 1, N)$, where $\mathbb{G}(k, n)$ is the Grassmannian variety of k-linear spaces of $\mathbb{P}^n$. To make this observation more useful we start recalling definitions and results concerning secant varieties.

Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non degenerate variety,

$$\Gamma_h(X) \subset X \times ... \times X \times \mathbb{G}(h-1, N),$$

the reduced closure of the graph of

$$\alpha : X \times ... \times X \dashrightarrow \mathbb{G}(h-1, N),$$

taking h general points to their linear span $\langle x_1, ..., x_h \rangle$. Observe that $\Gamma_h(X)$ is irreducible and reduced of dimension hn. Let $\pi_2 : \Gamma_h(X) \to \mathbb{G}(h-1, N)$ be the natural projection. Denote by

$$\mathcal{S}_h(X) := \pi_2(\Gamma_h(X)) \subset \mathbb{G}(h-1, N).$$

Again $\mathcal{S}_h(X)$ is irreducible and reduced of dimension hn. Finally let

$$\mathcal{I}_h = \{(x, \Lambda) \mid x \in \Lambda\} \subset \mathbb{P}^N \times \mathbb{G}(h-1, N),$$

with natural projections π_h and ψ_h onto the factors. Furthermore observe that $\psi_h : \mathcal{I}_h \to \mathbb{G}(h-1, N)$ is a $\mathbb{P}^{h-1}$-bundle on $\mathbb{G}(h-1, N)$.

Definition 5.0.9. Let $X \subset \mathbb{P}^N$ be an irreducible and reduced, non degenerate variety. The *abstract* h*-Secant variety* is the irreducible and reduced variety

$$\mathrm{Sec}_h(X) := (\psi_h)^{-1}(\mathcal{S}_h(X)) \subset \mathcal{I}_h.$$

While the h*-Secant variety* is

$$\mathbb{S}ec_h(X) := \pi_h(Sec_h(X)) \subset \mathbb{P}^N.$$

It is immediate that $\mathrm{Sec}_h(X)$ is a $(hn + h - 1)$-dimensional variety with a $\mathbb{P}^{h-1}$-bundle structure on $\mathcal{S}_h(X)$. One says that X is h*-defective* if

$$\dim \mathbb{S}ec_h(X) < \min\{\dim \mathrm{Sec}_h(X), N\}$$

In what follows we need to extend this classical notion to a relative set-up. Let S be a noetherian scheme, and let $X \to S$ be a scheme over S such that there exists a coherent sheaf E on S with a closed embedding of X into $\mathbb{P}(E) := \mathbb{P}Sym_{\mathcal{O}_S}(E)$ over S. Equivalently we may assume that there exists a relatively ample line bundle L on X over S.
There exists a scheme $\mathrm{Grass}(h, E)$ finely parametrizing locally free sub-sheaves of rank h of E. Furthermore $\mathrm{Grass}(h, E)$ is projective over S.
Now suppose E to be a rank $N + 1$ vector bundle, the fiber of the morphism $\mathrm{Grass}(h, E) \to S$ over a closed point $s \in S$ is the Grassmannian $\mathrm{Grass}(h, E_s) \cong \mathbb{G}(h, N)$, where E_s is the fiber of E over $s \in S$. There is a well defined rational map over S

$$\begin{tikzcd}
X \times_S ... \times_S X \arrow[rr, dashed, "\alpha"] \arrow[dr] & & \mathrm{Grass}(h, E) \arrow[dl] \\
& S &
\end{tikzcd}$$

mapping $(x_1, ..., x_h)$ to the linear span $\langle x_1, ..., x_h \rangle$. Note that being α a map over S we are taking $x_i \in X_s \subset \mathbb{P}(E_s) \cong \mathbb{P}^N$ for some $s \in S$. Take $\Gamma_h^S(X)$ to be the reduced closure of the graph of α in $X \times_S ... \times_S X \times_S \mathrm{Grass}(h, E)$, then $\Gamma_h^S(X)$ is irreducible and reduced of dimension hn over S.
Let $\pi : \Gamma_h^S(X) \to \mathrm{Grass}(h, E)$ be the projection, denote by

$$\mathcal{S}_h^S(X) := \pi(\Gamma_h^S(X)) \subseteq \mathrm{Grass}(h, E).$$

Again $\mathcal{S}^S_h(X)$ is irreducible and reduced of dimension hn over S, where $n = \dim_S(X)$. Now, consider the incidence correspondence

$$\begin{tikzcd}
& \mathcal{I}^S_h := \{(z,F) \mid z \in F\} \subseteq \mathbb{P}(E) \times_S \mathrm{Grass}(h,E) \arrow[dl, "\pi_h"'] \arrow[dr, "\psi_h"] & \\
\mathbb{P}(E) \arrow[dr] & & \mathrm{Grass}(h,E) \arrow[dl] \\
& S &
\end{tikzcd}$$

Definition 5.0.10. Let $X \to S$ be an irreducible and reduced scheme over S, together with a closed embedding into $\mathbb{P}(E)$. The *abstract relative* h*-secant variety* of X over S is

$$\mathrm{Sec}^S_h(X) := \psi_h^{-1}(\mathcal{S}^S_h(X)) \subseteq \mathcal{I}^S_h,$$

while the *relative* h*-secant variety* of X over S is

$$\mathcal{S}ec^S_h(X) := \pi_h(\mathrm{Sec}^S_h(X)) \subseteq \mathbb{P}(E).$$

Remark 5.0.11. The scheme $\mathcal{S}ec^S_h(X)$ naturally comes with a morphism $\mathcal{S}ec^S_h(X) \to S$ whose fiber over a closed point $s \in S$ is the h-secant variety $\mathcal{S}ec_h(X_s) \subseteq \mathbb{P}(E_s) \cong \mathbb{P}^N$ of the fiber X_s of $X \to S$ over $s \in S$.

The scheme $\mathrm{Sec}^S_h(X)$ has dimension $hn + h - 1$ over S. Next we introduce the new compactification we want to study.

Definition 5.0.12. Let $X \subset \mathbb{P}^N$ be an irreducible non degenerate variety of dimension n, and $p \in \mathbb{P}^N$ a general point. For $h + n < N + 1$ consider the h-secant map $\pi_h : \mathcal{S}ec_h(X) \to \mathbb{P}^N$ and define

$$\mathrm{VSP}^X_G(h)_p := \pi_h^{-1}(p).$$

We may omit X or p or both and set

$$\mathrm{VSP}_G(h) := \mathrm{VSP}^X_G(h) := \mathrm{VSP}^X_G(h)_p,$$

if no confusion is likely to arise. For the Veronese variety we also use the notation

$$\mathrm{VSP}_G(F,h) := \mathrm{VSP}^{V_{d,n}}_G(h)_{[F]}.$$

Remark 5.0.13. We already observed that $\mathrm{VSP}_G(F,h)$ is birational to $\mathrm{VSP}(F,h)$. On the other hand the variety $\mathrm{VSP}_G(F,h)$ contains limits of h-secant planes. We expect, in general, that there are no morphisms between $\mathrm{VSP}_G(F,h)$ and $\mathrm{VSP}(F,h)$. Indeed not all degree h zero dimensional subschemes of the Veronese variety span a linear space of dimension $h-1$ and not all limits of h-secant planes cut a zero dimensional scheme. Both directions are clearly true when $n = 1$ and in this case we have $\mathrm{VSP}(F,h) \cong \mathrm{VSP}_G(F,h)$.

The bound on h in the definition is harmless. Our usual approach is to study a special value of h satisfying this bound and then derive conclusions on bigger h via the chain construction in Section 5.1.

As a closing remark note the following improvement of the partial derivative Remark 5.0.8.

Remark 5.0.14 (Partial Derivatives II)**.** The partial derivatives Remark 5.0.8 can be strengthened as follows. Let $[F] \in \mathbb{P}^N$ be a general point. The partial derivatives of F span a linear space, say H_∂, in the corresponding projective space $\mathbb{P}^{N'}$. Remark 5.0.8 tell us that linear spaces associated to a general decomposition have to contain H_∂.

We recall the definitions and properties we need about rational connected varieties. The main reference is Kollár's book [Ko].

Definition 5.0.15. [Ko, Definition IV.3.2] Let X be a variety. We say that X is rationally chain connected if there is a family of proper and connected algebraic curves $g : U \to Y$ whose geometric fibers have only rational components with cycle morphism $u : U \to X$ such that

$$u^{(2)} : U \times_Y U \to X \times X \text{ is dominant,}$$

where the image of $u^{(2)}$ consist of pairs $(x_1, x_2) \in X$ such that $x_1, x_2 \in u(U_y)$ for some $y \in Y$. We say that X is rationally connected if there is a family of proper and connected algebraic curves $g : U \to Y$ whose geometric fibers are irreducible rational curves with cycle morphism $u : U \to X$ such that $u^{(2)}$ is dominant.

It is clear that the cone over a variety Z is rationally chain connected, but it is not rationally connected, unless Z is. For smooth proper varieties in characteristic zero, this does not happen.

Theorem 5.0.16. *[Ko, Theorem IV.3.10] Let X be a smooth proper variety over an algebraically closed field of zero characteristic. Then X is rationally chain connected if and only if it is rationally connected.*

We conclude recalling the following result of Graber-Harris-Starr.

Theorem 5.0.17. *[GHS, Corollary 1.3] Let $f : X \to Y$ be any dominant morphism of complex varieties. If Y and the general fiber of f are rationally connected, then X is rationally connected.*

5.1 CHAINS IN $\mathrm{VSP}(f, h)$

Let $F \in k[x_0, \dots, x_n]_d$ be a general homogeneous polynomial of degree d. Consider a general additive decomposition

$$F = \sum_1^h L_i^d$$

Let $p \in \mathrm{VSP}(F, h)$ the corresponding point. In this set up also the polynomial

$$F - L_1^d$$

is general and we can identify $\mathrm{VSP}(F - L_1^d, h-1)$ as a subvariety of $\mathrm{VSP}(F, h)$ passing through p. More generally we can identify a flag of subvarieties

$$\mathrm{VSP}(F, h) \supset \mathrm{VSP}(F - L_1^d, h-1) \supset \dots \supset \mathrm{VSP}(F - \sum_1^r L_i^d, h-r) \ni p,$$

that is we can cover any variety of sums of powers via VSP with less addends. Under suitable numerical assumption we may also connect two very general points of $\mathrm{VSP}(F, h)$ with chains of $\mathrm{VSP}(\bullet, h-1)$. Before stating it explicitly we adopt a convention.

Convention 1. When working with a general decomposition, say $\sum_1^h L_i^d$,we will always tacitly consider the irreducible component of $\mathrm{VSP}(F, h)^o$ containing this general decomposition and keep denoting its compactifications $\mathrm{VSP}(F, h)$, and $\mathrm{VSP}_G(F, h)$.

Theorem 5.1.1. *Let* $F \in k[x_0, \dots, x_n]_d$ *be a general polynomial of degree* d. *Assume that* $h \geqslant \frac{\binom{n+d}{d}}{n+1} + 2$, *or equivalently that* $\dim \mathrm{VSP}(F, h-1) \geqslant n+1$. *Then two very general points* p_1, p_2 *of an irreducible component of* $\mathrm{VSP}(F, h)$ *are joined by a chain (of length at most three) of* $\mathrm{VSP}(\bullet, (h-1))$. *Let* $W_i^{p_1,p_2}$ *be the elements of this chain, then* $W_i^{p_1,p_2} \cap W_j^{p_1,p_2}$ *intersects the smooth locus of* $\mathrm{VSP}(F, h)$. *Assume moreover that any irreducible component of* $\mathrm{VSP}(\bullet, h-1)$ *is rationally connected and* $\dim \mathrm{VSP}(\bullet, h-1) \geqslant n$ *then any irreducible component of* $\mathrm{VSP}(F, h)$ *is rationally connected.*

Proof. We have

$$\dim \mathrm{VSP}(F, h-1) = n(h-1) + h - 2 - \binom{n+d}{d} + 1 = (h-1)(n+1) - \binom{n+d}{d}.$$

Hence the numerical assumption yields

$$\dim \mathrm{VSP}(F, h-1) - (n+1) = (n+1)(h-2) - \binom{n+d}{d} \geqslant 0. \qquad (5.1.1)$$

Let p_1 and p_2 be two points in $\mathrm{VSP}(F, h)$ with associated decompositions, respectively,

$$\sum_1^h L_i^d \text{ and } \sum_1^h G_i^d.$$

Along the proof we will always consider $\mathrm{VSP}(\bullet, h-1)$ as irreducible subvarieties of $\mathrm{VSP}(F, h)$, keep in mind Convention 1. Let $q \in \mathrm{VSP}(F - L_1^d, h-1) \subset \mathrm{VSP}(F, h)$ be a general point with associated decomposition

$$L_1^d + \sum_2^h B_i^d.$$

Let $\nu : Z \to \mathrm{VSP}(F, h)$ be a resolution of singularities. Assume that

($\star$) $\nu^{-1}(\mathrm{VSP}(F - L_1^d, h-1))$ and $\nu^{-1}(\mathrm{VSP}(F - G_1^d, h-1))$ belong to the same irreducible component of $\mathrm{Hilb}(Z)$, and ν is an isomorphism in a neighborhood of q.

The Hilbert scheme of Z has countably many irreducible components hence the points satisfying assumption ($\star$) are very general.

The construction yields

$$q \in \mathrm{VSP}(F - L_1^d, h-1) \cap \mathrm{VSP}(F - B_2^d, h-1).$$

As soon as $\dim \mathrm{VSP}(\bullet, h-1) \geqslant 0$ we have

$$\mathrm{codim}_{\mathrm{VSP}(F,h)} \mathrm{VSP}(F - L_1^d, h-1) = n+1.$$

Hence by equations (5.1.1), and assumption ($\star$) we conclude that

$$\mathrm{VSP}(F - G_1^d, h-1) \cap \mathrm{VSP}(F - B_2^d, h-1) \neq \emptyset.$$

To conclude observe that q, a point in the intersection of two elements of the chain, is a general point in $\mathrm{VSP}(F - L_1^d, h-1)$, hence

$$W_i^{p_1,p_2} \cap W_j^{p_1,p_2} \not\subset \mathrm{Sing}(\mathrm{VSP}(F, h)).$$

To have the better bound in the rational connected case, we want to produce a higher dimensional rational connected variety starting from $\mathrm{VSP}(F, h-1)$. Let $p \in \mathrm{VSP}(F, h)$ be a point associated to a decomposition

$$A_1^d + \ldots + A_h^d$$

and consider

$$V_p := \overline{\bigcup_{\lambda} VSP(F - \lambda A_1^d, h-1)} \subset VSP(F,h).$$

Then V_p has a natural map onto $\mathbb{P}^1$ with rationally connected fibers. Hence, via Theorem 5.0.17, we conclude that V_p is a rationally connected variety of dimension $n+1$. Now substitute $VSP(F - L_1^d, h-1)$ with V_p in the above argument. Then for a pair of points, p_1 and p_2, satisfying the $(\star)$ condition, the general $q \in V_{p_1}$ is such that $V_q \cap V_{p_i} \neq \emptyset$ for $i = 1,2$. In particular $VSP(F,h)$ is rationally chain connected by irreducible rational curves intersecting in smooth points. This is enough, by Theorem 5.0.16, to conclude that $VSP(F,h)$ is rationally connected. $\square$

Theorem 5.1.1 allows us to describe birational properties of $VSP(F,h)$ starting from those of $VSP(\bullet, h-1)$. The following is our best tool to study rational connectedness of $VSP_G(F,h)$.

Proposition 5.1.2. *For any triple of integers (a,b,c), with $0 < c < n$, there is an irreducible and reduced rationally connected variety $W^n_{a,b,c} \subset \mathrm{Hilb}(\mathbb{P}^n)$ with the following properties:*

- *a general point in $W^n_{a,b,c}$ represents a rational subvariety of $\mathbb{P}^n$ of codimension c;*
- *for any $Z \subset \mathbb{P}^n \setminus \{(x_0 = \ldots = x_{n-c} = 0)\}$ reduced zero dimensional scheme of length $\leqslant b$, there is a rationally connected subvariety $W_{Z,c} \subset W^n_{a,b,c}$, of dimension at least a, whose general element $[Y] \in W_{Z,c}$ represents a rational subvariety of $\mathbb{P}^n$ of codimension c containing Z.*

Proof. We prove the statement by induction on c. Assume $c = 1$, and consider an equation of the form

$$Y = (x_n A(x_0, \ldots, x_{n-1})_{d-1} + B(x_0, \ldots, x_{n-1})_d = 0),$$

then, for A and B generic, Y is a rational hypersurface of degree d with a unique singular point of multiplicity $d-1$ at the point $[0, \ldots, 0, 1]$.

Fix $d > ab$ and let $W^n_{a,b,1} \subset \mathbb{P}(k[x_0, \ldots, x_n]_d)$ be the linear span of these hypersurfaces. For any triple $(a,b,1)$ and a subset $Z \subset \mathbb{P}^n \setminus \{[0, \ldots, 0, 1]\}$ consider $W_{Z,1} \subset W^n_{a,b,1}$ as the sublinear system of hypersurfaces containing Z.

Assume, by induction, that $W^n_{a,b,i-1} \subset \mathrm{Hilb}(\mathbb{P}^{n-1})$ exists for any n and b. Define, for $i \geqslant 2$,

$$\tilde{W}^n_{a,b,i} := W^n_{a,b,1} \times W^{n-1}_{a,b,i-1} \subset \mathrm{Hilb}(\mathbb{P}^n) \times \mathrm{Hilb}(\mathbb{P}^{n-1}).$$

Let $[X]$ be a general point in $W^n_{a,b,1}$. By construction X has a point of multiplicity $d-1$ at the point $[0, \ldots, 0, 1] \in \mathbb{P}^n$. Then the projection $\pi_{[0,\ldots,0,1]} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ restricts to a birational map $\varphi_X : X \dashrightarrow \mathbb{P}^{n-1}$. Hence we may associate the general element $([X],[Y]) \in \{[X]\} \times W^{n-1}_{a,b,i-1}$ to the codimension i subvariety $\varphi_X^{-1}(Y) \subset \mathbb{P}^n$. This, see for instance [Ko, Proposition I.6.6.1], yields a rational map

$$\chi : \tilde{W}^n_{a,b,i} \dashrightarrow \mathrm{Hilb}(\mathbb{P}^n).$$

Let $W^n_{a,b,i} := \overline{\chi(\tilde{W}^n_{a,b,i})} \subset \mathrm{Hilb}(\mathbb{P}^n)$. For any Z we may then define

$$\tilde{W}_{Z,i} := W_{Z,1} \times W_{\pi_{[1,0,\ldots,0]}(Z), i-1},$$

and as above $W_{Z,i} = \overline{\chi(\tilde{W}_{Z,i})}$. $\square$

In this section we prove some rationality result for VSP's. The first interesting case is that of $\mathbb{P}^1$, namely polynomials in two variables. This is probably known but we where not able to find an appropriate reference.

Theorem 5.2.1. *Let* $h > 1$ *be a fixed integer. For any integer* d *such that*

$$h \leqslant d \leqslant 2h - 1,$$

we have $VSP(F, h) \cong \mathbb{P}^{2h-d-1}$.

Proof. We already noticed, see Remark 5.0.13, that in this case

$$VSP(F, h) \cong VSP_G(F, h).$$

Let F be a homogeneous polynomial of degree d and let $\{[L_1], ..., [L_h]\}$ be a decomposition of F, then

$$F = L_1^d + ... + L_h^d.$$

We consider the partial derivatives of order $d - h > 0$ of F. This partial derivatives are

$$\binom{d-h+1}{d-h} = d - h + 1 \leqslant h$$

homogeneous polynomials of degree h.
Let X be the rational normal curve of degree h in $\mathbb{P}^h$. The partial derivatives span a $(d - h)$-plane $H_\partial \subset \mathbb{P}^h$. The general choice of F ensures that $H_\partial \cap X = \emptyset$. By Remark 5.0.14 the points $[L_1^h], ..., [L_h^h] \in X$ span a hyperplane containing H_∂.
The hyperplanes of $\mathbb{P}^h$ containing H_∂ are parametrized by $\mathbb{P}^{2h-d-1}$ and any hyperplane containing H_∂ intersects X in a zero dimensional scheme of length h. This gives rise to an injective morphism

$$\varphi : \mathbb{P}^{2h-d-1} \to VSP(F, h),\ \Pi \mapsto \Pi \cap X.$$

The varieties $VSP(F, h)$ and $\mathbb{P}^{2h-d-1}$ are both smooth by Proposition 5.0.7 and

$$\dim(VSP(F, h)) = 2h - \binom{d+1}{d} = 2h - d - 1.$$

Hence the injective morphism φ is an isomorphism. □

The next rationality result is for quadratic polynomials, this is known to experts but we could not find a reference. Our proof is based on the simultaneous diagonalization of two general quadrics.

Theorem 5.2.2. *Let* $F \in k[x_0, \ldots, x_n]_2$ *be a general homogeneous polynomial of degree two. Then* $VSP(F, n + 1)$ *is rational.*

Proof. Up to an automorphism of $\mathbb{P}^n$ we may assume that F is given by

$$F = x_0^2 + ... + x_n^2.$$

Let Π be a general $(N - n)$-plane in $\mathbb{P}^N = \mathbb{P}(k[x_0, \ldots, x_n]_2)$, and $[G] \in \Pi$ a general point.
The quadrics F and G are general. Then we may assume that the pencil they generate contains exactly $n + 1$ distinct singular quadric cones, say $C_0, ..., C_n$. Let $v_i \in \mathbb{P}^n$ the vertex of the cone C_i for $i = 0, ..., n$. Via the Veronese embedding $\nu_2 : \mathbb{P}^n \to \mathbb{P}^N$ we find $n + 1$ points $\nu_2(v_i)$ on the Veronese variety $V_{2,n} \subset \mathbb{P}^N$.
Let A be the matrix of G. Then the cones in the pencil $\lambda F - G$ are determined by the values of λ such that $\det(\lambda I - A) = 0$. In other words the cones C_i correspond to the eigenvalues of

A and the singular points v_i are given by the eigenvectors of A. In particular v_i's are linearly independent and in the basis $\{v_0, ..., v_n\}$ the matrix A is diagonal

$$\begin{pmatrix} \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

We may further assume that $\{v_0, ..., v_n\}$ is an orthonormal base. Therefore after the automorphism induced by this change of variables we have that F is still represented by the identity and G is diagonal.

Any automorphism of $\mathbb{P}^n$ induces an automorphism on $\mathbb{P}^N$ that stabilizes $V \subset \mathbb{P}^N$. Hence after the needed automorphisms we have

$$\nu_2(v_i) = \nu_2([0, \ldots, 0, 1, 0, \ldots, 0]) = [x_i^2].$$

Therefore the linear space $\langle [x_0^2], ..., [x_n^2] \rangle$ contains both $[F]$ and $[G]$. This construction gives a map

$$\psi : \Pi \dashrightarrow VSP(F, n+1), [G] \mapsto \{v_0, ..., v_n\}.$$

The birationality of ψ is immediate once remembered that Π is a codimension n linear space, and $\dim(VSP(F, n+1)) = N - n$. □

For conics a bit improvement is at hand.

Theorem 5.2.3. *Let* $F \in k[x_0, x_1, x_2]_2$ *be a general homogeneous polynomial of degree two. Then* $VSP(F, 4)$ *is birational to the Grassmannian* $\mathbb{G}(1, 4)$*, and hence rational.*

Proof. The map is quite simple. The 3-planes passing through $[F] \in \mathbb{P}^5$ are parametrized by $\mathbb{G}(1, 4)$ and a general linear space cuts exactly 4 points on the Veronese surface $V_{2,2} \subset \mathbb{P}^5$. To conclude it is enough to check that $\dim VSP(F, 4) = \dim \mathbb{G}(1, 4) = 6$. □

We are not able to prove rationality for arbitrary n and h. Nonetheless the proof of Theorem 5.2.2 allows us to prove the following unirationality statement.

Theorem 5.2.4. *Let* $F \in k[x_0, \ldots, x_n]_2$ *be a general homogeneous polynomial of degree two. Then* $VSP(F, h)$ *is unirational.*

Proof. We have to prove the statement for $h > n + 1$. Let $\Pi \subset \mathbb{P}^N$ be a codimension n linear space and $q \in \Pi$ a point. The proof of Theorem 5.2.2 shows that for a general $[F] \in \mathbb{P}^N$ there is a well defined decomposition associated to q. This can be seen as a rational section

$$\sigma_q : \mathbb{P}^N \dashrightarrow Sec_n(V_{2,n}).$$

We proved that the general fiber of the map $\pi_n : Sec_n(V_{2,n}) \to \mathbb{P}^N$ is rational. Hence we have a well defined birational map

$$\chi : \mathbb{P}^N \times \mathbb{P}^{N-n} \dashrightarrow Sec_n(V_{2,n}).$$

This means that given a general quadratic polynomial, say q, and a point in $\mathbb{P}^{N-n}$ it is well defined an additive decomposition of q into h factors. This allows us to define the following map, for $h > n + 1$

$$\psi_h : \mathbb{P}^{N-n} \times (V_{2,n} \times \mathbb{P}^1)^{h-(n+1)} \dashrightarrow VSP_G(F, h)$$

given by

$$(p,[L_1^2],\lambda_1,\dots,[L_{h-(n+1)}^2],\lambda_{h-(n+1)})\mapsto(\lambda_1L_1^2+\dots+\lambda_{h-(n+1)}L_{h-(n+1)}^2+$$
$$+\chi([F-\sum_{i=1}^{h-(n+1)}\lambda_1L_i^2],p)).$$

The map ψ_h is clearly generically finite, of degree $\binom{h}{n+1}$, and dominant. This is enough to show that $VSP_G(F,h)$ is unirational for $h>n+1$. ☐

5.3 RATIONAL CONNECTEDNESS

In this section we prove the result on rational connectedness taking advantage of the preparatory work of the previous sections.
In higher degrees one cannot expect a result like the one of quadratic polynomials . It is enough to think of either *Mukai* Theorem [Mu1], where is proven that $VSP(F,10)$ is a $K3$ surface for $F\in k[x_0,x_1,x_2]_6$ general, or *Iliev* and *Ranestad* example of a symplectic VSP, [IR1]. On the other hand we found a nice behavior for infinitely many degrees and number of variables. Keep in mind that $VSP(F,h)$ are not empty only for $h\geqslant\frac{\binom{n+d}{n}}{n+1}$.

Theorem 5.3.1. *Assume that for some positive integer* $0<k<n$ *the number* $\frac{\binom{d+n}{n}-1}{k+1}$ *is an integer. Then the irreducible components of* $VSP(F,h)$ *are rationally connected for* $F\in k[x_0,\dots,x_n]_d$ *general and* $h\geqslant\frac{\binom{n+d}{n}-1}{k+1}$.

To prove the Theorem we use [Me2, Remark 4.6].

Proposition 5.3.2. *Let* $V_{\delta,n}\subset\mathbb{P}^N$ *be a Veronese embedding, for* $\delta\geqslant 4$. *Assume that* $\operatorname{codim}\operatorname{Sec}_h(V)\geqslant n+1$. *Then through a general point of* $\operatorname{Sec}_h(V)$ *there is a unique* $(h-1)$*-linear space* h*-secant to* V.

Proof. Let $z\in\operatorname{Sec}_h(V)$ be a general point. Assume that $\langle p_1,\dots,p_h\rangle\ni z$ and $z\in\langle q_1,\dots,q_h\rangle$ for h-tuple of points in V. Then Terracini Lemma, [CC, Theorem 1.1], yields

$$\mathfrak{T}_z\operatorname{Sec}_h(V)=\langle\mathfrak{T}_{q_1}V,\dots,\mathfrak{T}_{q_h}V\rangle=\langle\mathfrak{T}_{p_1}V,\dots,\mathfrak{T}_{p_h}V\rangle.$$

Therefore the general hyperplane section $H\cap V$ singular at $\{p_1,\dots,p_h\}$ is singular at $\{q_1,\dots,q_h\}$ as well. On the other hand, by [Me2, Corollary 4.5], V is not h-weakly defective. Then by [CC, Theorem 1.4] the general hyperplane section $H\cap V$ tangent at h-general points $\{p_1,\dots,p_h\}$, of V is singular only at those points. This gives $\{p_1,\dots,p_h\}=\{q_1,\dots,q_h\}$ and proves the proposition. ☐

Proof of Theorem 5.3.1. Without loss of generality, to simplify notation, we may assume that $VSP_G(F,h)$ is irreducible. Fix $h=\frac{\binom{n+d}{n}-1}{k+1}=\frac{N}{k+1}$, and assume that $[\Lambda_x],[\Lambda_y]\in VSP_G(F,h)$ are two general points, with $\Lambda_x=\langle x_1,\dots x_h\rangle$ and $\Lambda_y=\langle y_1,\dots y_h\rangle$.

In the notation of Proposition 5.1.2, let $W_1:=W^n_{a,2h,n-k}$, for $a\gg 0$. Let $[X]\in W_1$ be a general element.

Claim 2. We may assume the following properties of $\operatorname{Sec}_h(X)$:

i) $\mathbb{S}\mathrm{ec}_h(X)\subset\mathbb{P}^N$ is a hypersurface of degree, say α,

ii) through the general point of $\mathbb{S}\mathrm{ec}_h(X)\subset\mathbb{P}^N$ there is a unique h-secant linear space and $\mathbb{S}\mathrm{ec}_h(X)$,

iii) $\mathrm{Sec}_h(X)$ is singular in codimension 1.

Proof. Let $d' \gg h$ and $V_{d',n} \subset \mathbb{P}^M$ the associated Veronese variety. For any element $D \subset |\mathcal{O}_{\mathbb{P}^n}(d'-d)|$ we have a birational projection $\pi_D : \mathbb{P}^M \dashrightarrow \mathbb{P}^N$ such that $\pi_{D|V_{d',n}}$ is an isomorphism onto $V_{d,n} \subset \mathbb{P}^N$. Let $Y := \nu_{d'}(X) \subset V_{d',n}$ be the embedding of X in this Veronese variety. We may assume that $\langle Y\rangle = \mathbb{P}^M$. The bound $d' \gg h$ yields $\mathrm{Sec}_h(Y) \cap V_{d',n} = Y$ and $\mathrm{Sec}_h(V_{d',n}) \subsetneq \mathbb{P}^M$. In particular by Proposition 5.3.2 there is a unique h-secant linear space through the general point of $\mathrm{Sec}_h(V_{d',n})$. Hence the latter is true for $\mathrm{Sec}_h(Y)$ and

$$\dim \mathrm{Sec}_h(Y) = h(k+1) - 1 = N - 1.$$

To prove (i) and (ii) in the claim it is enough to show that $\mathrm{Sec}_h(X)$ is a birational projection of $\mathrm{Sec}_h(Y)$. Assume that the projection of $\mathrm{Sec}_h(Y)$ is not birational. The variety X is a birational projection of Y hence, as already noticed in the proof of Proposition 5.3.2, by Terracini's Lemma and [CC, Theorem 1.4], our assumption forces X to be h-weakly defective. In other words a hyperplane of $\mathbb{P}^M$ containing $\langle D\rangle$ and tangent to Y at the points $\{x_1,\dots,x_h\}$ is tangent along a positive dimensional subvariety $Z \subset Y$ containing the points x_i. On the other hand for $a \gg 0$ the proof of Proposition 5.1.2 shows that, in a neighborhood of $\{x_1,\dots x_h\}$, the elements in $\mathcal{W}$ tangent to Y at the points $\{x_1,\dots,x_h\}$ intersect only at the points x_i. This contradiction proves i) and ii).

To conclude iii) note that, for a general D we have

$$\langle D\rangle \supsetneq \langle \mathrm{Sec}_h(Y) \cap \langle D\rangle\rangle.$$

This shows that π_D can be factored via a linear projection $\pi_1 : \mathbb{P}^M \dashrightarrow \mathbb{P}^{N+1}$ followed by a projection $\pi_2 : \mathbb{P}^{N+1} \dashrightarrow \mathbb{P}^N$ from a point $p \notin \pi_1(\mathrm{Sec}_h(Y))$. We already know that $\mathrm{Sec}_2(\pi_1(\mathrm{Sec}_h(Y))) = \mathbb{P}^{N+1}$ hence the singular locus of $\pi_D(\mathrm{Sec}_h(Y))$ has dimension $2(N-1)+1-(N+1) = N-2$. □

Then Remark 5.0.11 allows us to define a rational map as follows

$$\varphi : \mathcal{W}_1 \dashrightarrow \mathbb{P}(k[x_0,\dots,x_N]_\alpha)$$

defined sending X to its h-secant.

Claim 3. The map φ is generically injective.

Proof. Let $[X] \in \mathcal{W}_1$ be a general point and $[Z] \in \varphi^{-1}(\varphi([X])) \setminus [X]$. Let $V := V_{\delta,k} \subset \mathbb{P}^M$ be the Veronese variety and $\Lambda_X, \Lambda_Z \subset \mathbb{P}^M$ two linear spaces that project V onto X and Z, respectively. This yields two projection maps $p_X : \mathrm{Sec}_h(V) \dashrightarrow S$, $p_Z : \mathrm{Sec}_h(V) \dashrightarrow S$ onto $\mathrm{Sec}_h(Z) = \mathrm{Sec}_h(X) := S$. The composition $\chi := p_X \circ p_Z^{-1}$ induces a birational self map on S. Let $\Omega \subset S$ be the locus of singularities, then, by Claim 2, Ω is codimension 1. Hence χ is defined on the general point of Ω. If $w \in \Omega$ is a general point and $x, y \in p_Z^{-1}(w)$ is a pair points then $p_X(x) = p_X(y) = w' \in W$. In particular the line $r_{x,y} := \langle x, y\rangle$ intersects both Λ_X and Λ_Z. Then there is at least a codimension 1 set $V \subset \Omega$ such that for $p_X(x) = p_X(y) \in V$ we have $\Lambda_X \cap r_{x,y} = \Lambda_Z \cap r_{x,y}$. This is enough to conclude recursively that $\Lambda_X = \Lambda_Z$. □

Let $\mathcal{SW}_1 := \overline{\varphi(\mathcal{W}_1)}$ and $H_{[F]} \subset \mathbb{P}(k[x_0,\dots,x_N]_\alpha)$ be the hyperplane parametrizing the hypersurfaces passing through [F]. We are interested in the intersection $\mathcal{SW}_1 \cap H_{[F]}$ that parametrizes secant varieties through the point [F]. Let $\mathcal{SW}_{1[F]}$ be an irreducible component of maximal dimension of $\mathcal{SW}_1 \cap H_{[F]}$.

By Claim 2 there is a unique h-secant linear space to X through a general point of $\mathrm{Sec}_h(X)$.

We may then define a rational map

$$\psi : SW_{1\,[F]} \dashrightarrow VSP_G(F,h) \subset \mathbb{G}(h-1,N) \tag{5.3.1}$$

sending a general secant in $SW_{1\,[F]}$ to the unique h-secant linear space passing through $[F] \in \mathbb{P}^N$.

Claim 4. The map ψ is dominant.

Proof. The variety W_1, see Proposition 5.1.2, is such that for any zero dimensional scheme $Z \subset V_{d,n}$ of length at most $2h$ there is a rationally connected subvariety in W_1 parametrizing rational varieties through Z. In particular a h-secant linear space to $V_{d,n}$ is h-secant to some $X' \subset V_{d,n}$ with $[X'] \in W_1$. □

In the notation of Proposition 5.1.2 we have

$$\overline{\psi^{-1}([\Lambda_x])} \supseteq \varphi(W_{\{x_1,\dots x_h\},n-k}),$$

$$\overline{\psi^{-1}([\Lambda_y])} \supseteq \varphi(W_{\{y_1,\dots,y_h\},n-k}),$$

and

$$\overline{\psi^{-1}([\Lambda_x])} \cap \overline{\psi^{-1}([\Lambda_y])} \supseteq \varphi(W_{\{x_1,\dots,x_h,y_1,\dots,y_h\},n-k}).$$

The subvarieties $W_{\{x_1,\dots x_h\},n-k}$ and $W_{\{y_1,\dots,y_h\},n-k}$ are rationally connected. Therefore $SW_{1\,[F]}$ is rationally chain connected by two rational curves intersecting in a general point of $\varphi(W_{\{x_1,\dots,x_h,y_1,\dots,y_h\},n-k})$.

We aim to prove that the variety $SW_{1\,[F]}$ is rationally connected. The variety $SW_1 \subset \mathbb{P}(k[x_0,\dots,x_N]_\alpha)$ parametrizes divisors in $\mathbb{P}^N$. By Claim 2 a general point $[T] \in SW_1$ represents a hypersurface singular in codimension 1, with $T = Sec_h(X)$. Assume that a general point of $\mathrm{Sing}(T)$ is of multiplicity m. That is, by Proposition 5.3.2, for $t \in \mathrm{Sing}(T)$ general point there are m linear spaces h-secant to X passing through t, with $m \geqslant 2$. In particular $[T] \in \varphi(W_{\{z_1,\dots,z_h,w_1,\dots,w_h\},n-k})$ for some $\{z_1,\dots,z_h,\},\dots,\{w_1,\dots,w_h\}$.

Let $\Sigma_{[F]} \subset SW_{1\,[F]}$ be the subvariety parametrizing secant varieties with more than one $(h-1)$-linear space h-secant passing through $[F]$.

Claim 5. $\mathrm{codim}_{SW_{1\,[F]}} \Sigma_{[F]} = 1$.

Proof. We already observed that for $[T] \in SW_1$ the hypersurface T is singular along a codimension 1 set. Therefore the set of hypersurfaces singular at a general point $[F] \in \mathbb{P}^N$ is in codimension 2 in SW_1,

$$\mathrm{codim}_{SW_1} \Sigma_{[F]} = 2.$$

All these hypersurfaces are clearly contained in $SW_{1\,[F]}$, therefore we conclude that

$$\mathrm{codim}_{SW_{1\,[F]}} \Sigma_{[F]} = 1.$$

□

Our construction shows that $SW_{1\,[F]}$ is rationally chain connected by chains of rational curves passing through general points of $\Sigma_{[F]}$.

Let $\nu : Z \to SW_{1\,[F]}$ be the normalization.

Claim 6. The variety Z is rationally chain connected by chains of rational curves passing through general points of the strict transform of $\Sigma_{[F]}$.

Proof. Fix two general decompositions and let

$$S_{\{x_i\}\{y_j\}} := \varphi(W_{\{x_1,\dots,x_h\},n-k}) \cap \varphi(W_{\{y_1,\dots,y_h\},n-k})$$

be the intersection. By construction $\dim S_{\{x_i\}\{y_j\}} \geqslant a$. Let us consider $\Sigma_{[F]}$ with its complex topology. Let $Z_\Sigma := \nu^{-1}\Sigma_{[F]}$ be the preimage of the locus we are interested in and $\nu_\Sigma := \nu_{|Z_\Sigma}$ the restricted morphism. Then the morphism ν_Σ is a finite étale covering outside a codimension 1 set, say K. For any point $s \in \Sigma_{[F]} \setminus K$ there is an open neighborhood (in the complex topology), say B_s, such that $\nu_{\Sigma|\nu^{-1}(B_s)}$ is finite and étale. The set K is closed and of measure zero. That is for any $\epsilon > 0$ there is an open $V \subset \Sigma_{[F]}$ such that $V \supset K$ and V has measure bounded by ϵ. The set V^c is compact and we may cover it with finitely many open sets $\{B_{s_i}\}_{i=1,\dots,m}$ as above.
The map $\nu_{\Sigma|\nu^{-1}(B_s)}$ is étale hence the general choice of the decompositions, the irreducibility of SW_1 and the finite number of the $\{B_{s_i}\}$ allow us to conclude that

$$\dim \nu_\Sigma^{-1}(\varphi(W_{\{x_1,\dots,x_h\},n-k})) \cap \nu_\Sigma^{-1}(\varphi(W_{\{y_1,\dots,y_h\},n-k})) > 0,$$

and prove the claim. □

The variety Z is rationally chain connected by chains of curves intersecting in smooth points. Hence, by Theorem 5.0.16, it is rationally connected. Then $SW_{1[F]}$ and $VSP_G(F,h)$, via the map ψ of equation (5.3.1), are rationally connected. To conclude the proof for $h > \frac{\binom{n+d}{n}}{k+1}$ it is then enough to apply Theorem 5.1.1. □

For special values a more precise statement con be obtained.

Theorem 5.3.3. *The variety* $VSP(F,h)$ *is rationally connected in the following cases:*

- *a)* $F \in k[x_0,x_1,x_2]_4$ *and* $h \geqslant 6$,
- *c)* $F \in k[x_0,\dots,x_4]_3$ *and* $h \geqslant 8$,
- *b)* $F \in k[x_0,\dots,x_3]_3$ *and* $h \geqslant 6$,
- *d)* $F \in k[x_0,x_1,x_2]_3$ *and* $h \geqslant 4$,

The variety $VSP(F,h)$ *is uniruled for* $F \in k[x_0,\dots,x_4]_3$ *and* $h \geqslant 7$.

Proof. In cases a) and b) we know that $VSP(F,6)$, [Mu1], and $VSP(F,8)$, [RS], respectively are rational of dimension $n+1$. Then to conclude it is enough to apply Theorem 5.1.1.

In case c) observe that there is a twisted cubic in $\mathbb{P}^3$ through 6 points. Then Theorem 5.2.1 produces a chain of $\mathbb{P}^2$ through very general points of $VSP(F,6)$. Then we apply Theorem 5.1.1 to conclude for arbitrary $h \geqslant 7$. In case d) we have $\mathbb{P}^2 \cong VSP(F,4)$ and we conclude again by Theorem 5.1.1.

Finally observe that there is a rational quartic in $\mathbb{P}^4$ through 7 points. Then Theorem 5.2.1 produce a $\mathbb{P}^1$ through a general point of $VSP(F,h)$, for $h \geqslant 7$. □

Remark 5.3.4. Theorem 5.3.1 is sharp. In [IR1] *A. Iliev* and *K. Ranestad* proves that $VSP(F,10)$ with $d=3$ and $n=5$ is a Hyperkähler manifold deformation equivalent to the Hilbert square of a K3 surface of genus 8. In particular $VSP(F,10)$ can not be rationally connected. In this case we have $\binom{n+d}{n} - 1 = 55$, so $k+1=5$, and Theorem 5.3.1 holds for $h \geqslant 11$.

Finally we show how the existence of a canonical decomposition yields the unirationality of $VSP(F,h)$.

Proposition 5.3.5. *Let* $F \in k[x_0, x_1, x_2, x_3]_3$ *be a general homogeneous polynomial. For any* $h \geqslant 5$ *the variety* $VSP(F, h)$ *is unirational.*

Proof. If $h = 5$ then $VSP(F, 5)$ is a single point. If $h \geqslant 6$ consider the incidence variety

$$\begin{tikzcd} & \mathcal{I} = \{(l_1, ..., l_{h-5}, G) \mid G \in \langle F, l_1^3, ..., l_{h-5}^3\rangle\} \subseteq (\mathbb{P}^3)^{h-5} \times \mathbb{P}^{19} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ (\mathbb{P}^3)^{h-5} & & \mathbb{P}^{19} \end{tikzcd}$$

The map φ is dominant and its general fiber is a linear subspace of dimension $h-5$ in $\mathbb{P}^{19}$. Then $\mathcal{I}$ is a rational variety of dimension $3(h-5) + h - 5 = 4h - 20$.
Let $(l_1, ..., l_{h-5}, G) \in \mathcal{I}$ be a general point. By Sylvester pentahedral theorem the polynomial G admits a unique decomposition $G = L_1^3 + ... + L_5^3$ as sum of five cubes of linear forms. Since $G \in \langle F, l_1^3, ..., l_{h-5}^3\rangle$ we have $L_1^3 + ... + L_5^3 = \alpha F + \sum_{i=1}^{h-5} \lambda_i l_i^3$, and

$$F = \frac{1}{\alpha}L_1^3 + ... + \frac{1}{\alpha}L_5^3 - \sum_{i=1}^{h-5} \frac{\lambda_i}{\alpha} l_i^3.$$

We get a generically finite rational map

$$\chi : \mathcal{I} \dashrightarrow VSP(F, h),\ (l_1, ..., l_{h-5}, G) \mapsto \{L_1, ..., L_5, l_1, ..., l_{h-5}\}.$$

Since $\dim(VSP(F, h)) = 4h - 20 = \dim(\mathcal{I})$ the map χ is dominant and $VSP(F, h)$ is unirational.
□

Remark 5.3.6. Consider a general homogeneous polynomial $F \in k[x_0, x_1, x_2]_5$. By Hilbert theorem F admits a unique decomposition as sum of seven 5-powers of linear forms. The argument used in Proposition 5.3.5 in this case shows that $VSP(F, h)$ is unirational for any $h \geqslant 7$.

6

GENERALIZED VARIETIES OF SUMS OF POWERS

In Definition 5.0.12 we used the map $\pi_h : \mathrm{Sec}_h(X) \to \mathbb{P}^N$ to define *varieties of sums of powers* for an irreducible variety $X \subset \mathbb{P}^N$. Now, let us consider the following more general definition.

Definition 6.0.7. Let $X \subset \mathbb{P}^N$ be an irreducible variety, and let $p_1, ..., p_k \in \mathbb{P}^N$ be $k \leqslant h$ general points. We define

$$\mathrm{VSP}_G^X(h,k) := (\pi_h)^{-1}(\langle p_1, ..., p_k \rangle) \subseteq \mathrm{Sec}_h(X).$$

Using the Hilbert scheme $\mathrm{Hilb}_h(X)$ parametrizing length h zero-dimensional subschemes of X we can define

$$\mathrm{VSP}_H^X(h,k)^o := \{\{x_1, ..., x_h\} \in \mathrm{Hilb}_h(X) \mid p_1, ..., p_k \in \langle x_1, ..., x_h \rangle\} \subseteq \mathrm{Hilb}_h(X),$$

then we can consider a compactification taking its closure in $\mathrm{Hilb}_h(X)$,

$$\mathrm{VSP}_H^X(h,k) := \overline{\mathrm{VSP}_H^X(h,k)^o}.$$

We will write $\mathrm{VSP}_G^X(h) := \mathrm{VSP}_G^X(h,1)$ and $\mathrm{VSP}_H^X(h) := \mathrm{VSP}_H^X(h,1)$.

Remark 6.0.8. The variety $\mathrm{VSP}_G^X(h,k)$ parametrizes $(h-1)$-linear spaces h-secant to X and containing $\langle p_1, ..., p_k \rangle$. Clearly there is a dominant rational map

$$\tau : \mathrm{VSP}_H^X(h,k) \dashrightarrow \mathrm{VSP}_G^X(h,k),\ \{x_1, ..., x_h\} \mapsto \langle x_1, ..., x_h \rangle.$$

Furthermore if $n+h-1 < N$ the general $(h-1)$-linear space parametrized by $\mathrm{VSP}_G^X(h,k)$ intersects X in subscheme consisting of h distinct points, so $\tau : \mathrm{VSP}_H^X(h,k) \dashrightarrow \mathrm{VSP}_G^X(h,k)$ is birational.

Proposition 6.0.9. *Assume the general $(k-1)$-linear space $\Lambda \subseteq \mathbb{P}^N$ to be contained in a $(h-1)$-linear space h-secant to X. Then the variety $\mathrm{VSP}_H^X(h,k)$ has dimension*

$$\dim(\mathrm{VSP}_H^X(h,k)) = h(n+k) - kN - k.$$

Furthermore if $n = 2$ and X is a smooth surface then for Λ varying in an open Zariski subset of $\mathbb{G}(k-1,N)$ the varieties $\mathrm{VSP}_H^X(h,k)$ are smooth and irreducible.

Proof. Consider the incidence variety

$$\begin{tikzcd}
 & \mathcal{I} = \{(Z, \langle p_1, ..., p_k \rangle) \in \mathrm{Hilb}_h(X) \times \mathbb{G}(k-1,N) \mid Z \in \mathrm{VSP}_H^X(h,k)\} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
\mathrm{Hilb}_h(X) & & \mathbb{G}(k-1,N)
\end{tikzcd}$$

The morphism φ is surjective and there exists and open subset $U \subseteq \mathrm{Hilb}_h(X)$ such that for any $Z \in U$ the fiber $\varphi^{-1}(Z)$ is isomorphic to the Grassmannian $\mathbb{G}(k-1,h-1)$, so $\dim(\varphi^{-1}(Z)) = k(h-k)$. The fibers of ψ are the varieties $\mathrm{VSP}_H^X(h,k)$. Under our hypothesis the morphism ψ is dominant and

$$\dim(\mathrm{VSP}_H^X(h,k)) = \dim(\mathcal{I}) - k(N-k+1) = h(n+k) - kN - k.$$

If $n = 2$ and X is a smooth surface then $\mathrm{Hilb}_h(X)$ is smooth. The fibers of φ over U are open Zariski subset of Grassmannians. So $\mathcal{I}$ is smooth and irreducible. Since the varieties $VSP_H^X(h,k)$ are the fibers of ψ we conclude that for the linear space $\langle p_1,...,p_k\rangle$ varying in an open Zariski subset of $\mathbb{G}(k-1,N)$ the varieties $VSP_H^X(h,k)$ are smooth and irreducible. $\square$

Remark 6.0.10. In the case $k = 1$ our assumption on the morphism ψ means $\mathcal{S}ec_h(X) = \mathbb{P}^N$.

6.1 VARIETIES OF MINIMAL DEGREE

Let k be an algebraically closed field of any characteristic, and $X \subset \mathbb{P}_k^N$ be an irreducible and reduced variety over k. There is a lower bound on the degree of X.

Proposition 6.1.1. *If* $X \subset \mathbb{P}_k^N$ *is a nondegenerate variety, then* $\deg(X) \geqslant \mathrm{codim}(X)+1$.

Proof. If $\mathrm{codim}(X) = 1$, being X nondegenerate we have $\deg(X) \geqslant 2 = \mathrm{codim}(X)+1$. We proceed by induction on $\mathrm{codim}(X)$. Let $x \in X$ be a general point, and

$$\pi_x : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$$

be the projection from x. The variety $Y = \overline{\pi_x(X)} \subset \mathbb{P}^{N-1}$ has degree $\deg(Y) = \deg(X)-1$, and codimension $\mathrm{codim}(Y) = \mathrm{codim}(X)-1$. By induction hypothesis we have $\deg(Y) \geqslant \mathrm{codim}(Y)+1$, which implies $\deg(X) \geqslant \mathrm{codim}(X)+1$. $\square$

Definition 6.1.2. We say that a nondegenerate variety $X \subset \mathbb{P}^N$ is a *variety of minimal degree* if $\deg(X) = \mathrm{codim}(X)+1$.

If $\mathrm{codim}(X) = 1$ then X is a quadric hypersurface, and then classified by its dimension and its singular locus. In higher codimension the following result holds.

Theorem 6.1.3. *If* $X \subset \mathbb{P}^N$ *is a variety of minimal degree, then* X *is a cone over a smooth such variety. If* X *is smooth and* $\mathrm{codim}(X) \geqslant 2$*, then* X *is either a rational normal scroll or the Veronese surface* $V_4^2 \subset \mathbb{P}^5$.

For a very nice survey on varieties of minimal degree see [EH].

Proposition 6.1.4. *Let* $X \subset \mathbb{P}^N$ *be a variety of minimal degree* d *and dimension* $\dim(X) = n$*. Then* $VSP_H^X(h)$ *is rational if* $h = d$*, and rationally connected for any* $h \geqslant d$.

Proof. Let $p \in \mathbb{P}^N$ be a general point. Since $\dim(X)+(d-1) = N-\mathrm{codim}(X)+d-1 = N$ a general $(d-1)$-plane Λ through p intersects X in d distinct points $\Lambda \cap X = \{x_1,...,x_d\}$. Clearly $p \in \Lambda = \langle x_1,...,x_d\rangle$, and $\mathcal{S}ec_d(X) = \mathbb{P}^N$. The $(d-1)$-plane in $\mathbb{P}^N$ passing through p are parametrized by the Grassmannian $\mathbb{G}(N-d,N-1)$. We have a generically injective rational map

$$\chi : \mathbb{G}(N-d,N-1) \dashrightarrow VSP_H^X(d),\ \Lambda \mapsto \Lambda \cap X.$$

Now, it is enough to observe that $\dim(\mathbb{G}(N-d,N-1)) = (N-d+1)(d-1) = n(d-1) = d(n+1)-N-1 = \dim(VSP_H^X(d))$ to conclude that $VSP_H^X(d)$ is rational.
Now, let $p \in \mathbb{P}^N$ be a general point. For $h > d$ consider the incidence variety

$$\begin{tikzcd}[column sep=small]
 & Y := \{((x_1,\lambda_1),...,(x_{h-d},\lambda_{h-d}),\Lambda) \mid p - \sum_{i=1}^{h-d}\lambda_i x_i \in \Lambda\} \subseteq (X\times\mathbb{P}^1)^{h-d}\times\mathbb{G}(\deg(X)-1,N) \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\
(X\times\mathbb{P}^1)^{h-d} & & \mathbb{G}(d-1,N)
\end{tikzcd}$$

The morphism $\varphi : Y \to (X \times \mathbb{P}^1)^{h-d}$ is surjective and its fibers are isomorphic to the Grassmannian $\mathbb{G}(N-d, N-1)$, that is Y is a $\mathbb{G}(N-d, N-1)$-bundle over $(X \times \mathbb{P}^1)^{h-d}$. Note that $(X \times \mathbb{P}^1)^{h-d}$ is rational being X of minimal degree and hence rational. By Theorem 5.0.17 the variety Y is rationally connected. Since χ is birational, for $((x_1, \lambda_1), ..., (x_{h-d}, \lambda_{h-d}), \Lambda) \in Y$ general the intersection $\Lambda \cap X = \{\hat{x}_1, ..., \hat{x}_d\}$ determines a decomposition $p - \sum_{i=1}^{h-d} = \sum_{j=1}^{d} \hat{\lambda}_j \hat{x}_j$. The map

$$\alpha : Y \dashrightarrow VSP_H^X(h), \ ((x_1, \lambda_1), ..., (x_{h-d}, \lambda_{h-d}), \Lambda) \mapsto \{x_1, ..., x_{h-d}, \hat{x}_1, ..., \hat{x}_d\}$$

is a generically finite, rational map, of degree $\binom{h}{h-d}$. Now, it is enough to observe that

$$\dim(Y) = (n+1)(h-d) + (N-d+1)(d-1) = h(n+1) - N - 1 = \dim(VSP_H^X(h))$$

to conclude that α is dominant. The variety $VSP_H^X(h)$ is dominated by a rationally connected variety, then it is rationally connected as well. □

Example 6.1.5. Let $Q \subset \mathbb{P}^3$ be a smooth quadric. Since any line through a general point $p \in \mathbb{P}^3$ cuts on Q a length two zero-dimensional subscheme, in this case the morphism

$$\chi : \mathbb{P}^2 \to VSP_H^Q(2)$$

is an injective regular morphism. Moreover $VSP_H^Q(2)$ is a smooth surface, so χ is an isomorphism and $VSP_H^Q(2) \cong \mathbb{P}^2$.

6.2 STRATIFICATION OF $VSP_H^X(h, k)$

Assume $VSP_H^X(h, k) \neq \emptyset$, and let $\{x_1, ..., x_h\} \in VSP_H^X(h, k)$ be a general point. Then there exist $p_1, ..., p_k \in \mathbb{P}^N$ general points such that

$$p_1 = \sum_{i=1}^{h} \lambda_i^1 x_i, ..., p_k = \sum_{i=1}^{h} \lambda_i^k x_i.$$

The points $p_i - \lambda_1^i x_1$ are general for any $i = 1, ..., k$, and we get a generically injective rational map

$$VSP_H^X(h-1, k) \dashrightarrow VSP_H^X(h, k).$$

This construction yield a stratification

$$VSP_H^X(h-r, k) \subset VSP_H^X(h-r+1, k) \subset ... \subset VSP_H^X(h-1, k) \subset VSP_H^X(h, k).$$

Convention 2. When we refer to a general decomposition we always consider the irreducible component of $VSP_H^X(h, k)^o$ containing this general decomposition, and we still denote by $VSP_H^X(h, k)$ its compactification.

Proposition 6.2.1. *Let $X \subset \mathbb{P}^N$ be a non-degenerate variety such that the general $(k-1)$-linear space $\Lambda \subseteq \mathbb{P}^N$ to be contained in a $(h-1)$-linear space h-secant to X. If*

$$h \geqslant \frac{k(N+1)}{n+k} + 2$$

then two very general points of $VSP_H^X(h, k)$ are joined by a chain, of at most length three, of $VSP_H^X(h-1, k)$. If V_i are the elements of this chain and $q \in V_i \cap V_j$ is a general points, then we can assume q to be a smooth point in V_i, V_j and $VSP_H^X(h, k)$.

Proof. Let $x = \{x_i\}, y = \{y_i\} \in VSP_H^X(h,k)$ be two very general points, and write

$$p_j = \sum_{i=1}^{h} \lambda_i^j x_i = \sum_{i=1}^{h} \gamma_i^j y_i$$

Let $z \in VSP_H^X(p_j - \lambda_1^j x_1, h-1, k)$ be a general point associated to the decomposition

$$p_j - \lambda_1^j x_1 = \sum_{i=2}^{h} \alpha_i z_i.$$

Let $\nu : Z \to VSP_H^X(h,k)$ be a resolution of singularities. Since x and y are two very general point we can assume that

(i) $\nu^{-1}(VSP_H^X(p_j - \lambda_1^j x_1, h-1, k))$ and $\nu^{-1}(VSP_H^X(p_j - \gamma_1^j y_1, h-1, k))$ belong to the same irreducible component of $\mathrm{Hilb}(Z)$.

(ii) ν is an isomorphism in a neighborhood of q.

Since $z \in VSP_H^X(h,k)$ is associated to $p_j = \lambda_1 x_1 + \sum_{i=2}^{h} \alpha_i z_i$ we have

$$z \in VSP_H^X(p_j - \lambda_1^j x_1, h-1, k) \cap VSP_H^X(p_j - \alpha_2^j z_2, h-1, k).$$

Under our numerical hypothesis we have

$$\dim(VSP_H^X(p_j - \alpha_2^j z_2, h-1, k)) \geqslant \mathrm{codim}_{VSP_H^X(h,k)}(VSP_H^X(p_j - \alpha_2^j z_2, h-1, k)),$$

and by (i) and (ii) we conclude that

$$VSP_H^X(p_j - \alpha_2^j z_2, h-1, k) \cap VSP_H^X(p_j - \gamma_1^j y_1, h-1, k) \neq \emptyset,$$

moreover the general point of this intersection is a smooth point of $VSP_H^X(p_j - \alpha_2^j z_2, h-1, k)$, $VSP_H^X(p_j - \gamma_1^j y_1, h-1, k)$ and $VSP_H^X(h,k)$. □

In particular Theorem 6.2.1 tells us that we can join two general points of $VSP_H^X(h)$ by a chain of length at most three of $VSP_H^X(h-1)$.

6.3 RATIONAL CONNECTEDNESS RESULTS

In this section we generalize Theorem 5.3.1 substituting the Veronese varieties with arbitrary unirational varieties. Then first step is the following generalization of Proposition 5.1.2.

Proposition 6.3.1. *Let X be an irreducible, unirational variety. For any triple of integers (a,b,c), with $0 < c < n$, there is a rationally connected variety $V_{a,b,c}^n \subset \mathrm{Hilb}(X)$ with the following properties:*

- *a general point in $V_{a,b,c}^n$ represents a rational subvariety of X of codimension c;*
- *for a general $Z \subset X$ reduced zero dimensional scheme of length $l \leqslant b$, there is a rationally connected subvariety $V_{Z,c} \subset V_{a,b,c}^n$, of dimension at least a, whose general element $[Y] \in V_{Z,c}$ represents a rational subvariety of X of codimension c containing Z.*

Proof. Since X is unirational there is a generically finite, dominant map $\varphi : \mathbb{P}^n \dashrightarrow X$. For any Hilbert polynomial $P \in \mathbb{Q}[z]$ the map φ induces a generically finite rational map

$$\chi : \mathrm{Hilb}_P(\mathbb{P}^n) \dashrightarrow \mathrm{Hilb}_Q(X),\ Z \mapsto \varphi(Z).$$

We prove the statement by induction on c. Assume $c = 1$, and consider an equation of the form

$$Y = (x_n A(x_0, \dots, x_{n-1})_{d-1} + B(x_0, \dots, x_{n-1})_d = 0),$$

then, for A and B general, $Y \subset \mathbb{P}^n$ is a rational hypersurface of degree d with a unique singular point of multiplicity $d-1$ at the point $[0, \dots, 0, 1]$. Take A and B general. Let $\overline{Y} := \overline{\varphi(Y)}$ be the closure of the image of Y in X. If $\overline{y} \in \overline{Y}$ is a general point the fiber $\varphi^{-1}(\overline{y})$ intersects Y in a point, that is $\varphi_{|Y} : Y \to \overline{Y}$ is birational.
Fix $d > ab$ and let $W^n_{a,b,1} \subset \mathbb{P}(\mathbb{C}[x_0, \dots, x_n]_d)$ be the linear span of these hypersurfaces. We take $V^n_{a,b,1} := \chi(W^n_{a,b,1})$. Let $Z = \{x_1, ..., x_l\} \subset X$ be a zero dimensional subscheme of length $l \leqslant b$, and take $p_i \in \varphi^{-1}(x_i)$ for $i = 1, ..., l$.
For any triple $(a, b, 1)$ consider $W_{Z,1} \subset W^n_{a,b,1}$ as the sublinear system of hypersurfaces containing $\{p_1, ..., p_l\}$. Now take $V_{Z,1} := \chi(W_{Z,1})$. Then on a general point $[Y] \in W_{Z,1}$ the map φ restricts to a birational map and a general point of $V_{Z,1}$ parametrizes a rational subvariety of codimension 1 in X containing Z.
Assume, by induction, that $W^n_{a,b,i-1} \subset \mathrm{Hilb}(\mathbb{P}^{n-1})$ exist for any n and b. Define, for $i \geqslant 2$,

$$\tilde{W}^n_{a,b,i} := W^n_{a,b,1} \times W^{n-1}_{a,b,i-1} \subset \mathrm{Hilb}(\mathbb{P}^n) \times \mathrm{Hilb}(\mathbb{P}^{n-1}).$$

Let $[Y]$ be a general point in $W^n_{a,b,1}$. By construction Y has a point of multiplicity $d-1$ at the point $[0, \dots, 0, 1] \in \mathbb{P}^n$. Then the projection $\pi_{[0,\dots,0,1]} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ restricts to a birational map $\varphi_Y : Y \dashrightarrow \mathbb{P}^{n-1}$. Hence we may associate the general element $([Y], [S]) \in \{[Y]\} \times W^{n-1}_{a,b,i-1}$ to the codimension i subvariety $\varphi_Y^{-1}(S) \subset \mathbb{P}^n$. This, see for instance [Ko, Proposition I.6.6.1], yields a rational map

$$\alpha : \tilde{W}^n_{a,b,i} \dashrightarrow \mathrm{Hilb}(\mathbb{P}^n),\ ([Y], [S]) \mapsto [\varphi_Y^{-1}(S)].$$

Let $W^n_{a,b,i} := \overline{\alpha(\tilde{W}^n_{a,b,i})} \subset \mathrm{Hilb}(\mathbb{P}^n)$. For any Z we may then define

$$\tilde{W}_{Z,i} := W_{Z,1} \times W_{\pi_{[1,0,\dots,0]}(Z),i-1},$$

and as above $W_{Z,i} = \overline{\alpha(\tilde{W}_{Z,i})}$.
By construction a general point of $W^n_{a,b,c}$ is the inverse image of a rational subvariety of codimension $c-1$ in $\mathbb{P}^{n-1}$ via the projection from the singular point of a general rational hypersurface in $W^n_{a,b,1}$. Then on the general subvariety parametrized by $W^n_{a,b,c}$ and $\tilde{W}_{Z,c}$ the map φ restricts to a birational map. We take $V^n_{a,b,c} := \chi(W^n_{a,b,c})$ and $V_{Z,c} := \chi(\tilde{W}_{Z,c})$. The varieties $V^n_{a,b,c}$ and $V_{Z,c}$ are dominated by rationally connected varieties, so they are rationally connected as well. $\square$

Remark 6.3.2. Let $X \subset \mathbb{P}^N$ be a rational, nondegenerate variety of dimension n, and let $\varphi : \mathbb{P}^n \dashrightarrow X$ be a birational map. Let $B \subset \mathbb{P}^n$ be the indeterminacy locus of φ, then B has codimension at least two in $\mathbb{P}^n$. The linear system $\mathcal{H} = \varphi^*\mathcal{O}_{\mathbb{P}^N}(1)$ is a sub-system of $\mathcal{O}_{\mathbb{P}^n}(d)$ for some integer d. We can embed $\mathbb{P}^n$ via the Veronese embedding $\nu_{d,n}$ in $\mathbb{P}^{N_{d,n}}$. The variety X is a birational projection

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "\nu_{d,n}"] \arrow[dr, dashed] & V_{\delta,n} \subset \mathbb{P}^{N_{d,n}} \arrow[d, dashed, "\pi"] \\
 & X \subset \mathbb{P}^N
\end{tikzcd}$$

of $V_{d,n}$. This means that a rational variety can be seen as a birational projection of a suitable Veronese variety.

Thanks to Remark 6.3.2, with minor changes in the proof of Theorem 5.3.1 we get the following Theorem.

Theorem 6.3.3. *Let* $X \subset \mathbb{P}^N$ *be a unirational variety. Assume that for some positive integer* $k < n$ *the number* $\frac{N}{k+1}$ *is an integer. Then the irreducible components of* $VSP_H^X(h)$ *are rationally connected for* $h \geqslant \frac{N}{k+1}$.

6.4 RATIONAL HOMOGENEOUS VARIETIES

The most interesting varieties from the viewpoint of the decomposition of symmetric, antisymmetric and mixed tensors are *Veronese* varieties, *Grassmannians*, and *Segre-Veronese* varieties. We recall some basic facts about homogeneous varieties.

Definition 6.4.1. An *algebraic group* is an abstract group G with a structure of algebraic variety such that the map $G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2^{-1}$ is a morphism of algebraic varieties. An algebraic subgroup is a subgroup H of G which is a closed subset of G. A projective irreducible algebraic group is called an *abelian variety*.

The group G acts transitively on itself. By considering this action it is immediate that an algebraic group is smooth as variety. As a generalization of this fact we introduce the notion of homogeneous variety.

Definition 6.4.2. An algebraic variety X endowed with the action of an algebraic group G is called a G-variety. When G acts transitively X is said to be *homogeneous*. Finally, X is said to be *quasi-homogeneous* if it is the closure of the orbit of some $x \in X$.

Clearly, as for algebraic groups, any homogeneous variety is smooth. The basic results on the topic are the following:

- (*C. Chevalley*) A projective algebraic group is an abelian variety.
- (*A. Borel, R. Remmert*) A homogeneous projective variety is isomorphic to a product $A \times X$, where A is an abelian variety and X is a rational homogeneous variety. More generally a homogeneous compact Kähler manifold is isomorphic to a product $T \times X$, where $T \cong \mathbb{C}^n/\Lambda$ is a complex torus and X is rational homogeneous.
- (*A. Borel, R. Remmert*) A rational homogeneous variety is isomorphic to a product $G_1/P_1 \times ... \times G_k/P_k$, where thr G_i are simple groups and the P_i are parabolic subgroups.

In what follows we work out some numbers which make Theorem 6.3.3 working.

Grassmannians

It is well known that the Grassmannian $\mathbb{G}(r,n)$ parametrizing r-linear subspaces of $\mathbb{P}^n$ is a rational homogeneous variety of dimension $(r+1)(n-r)$, and has a natural embedding

$$\mathbb{G}(r,n) \hookrightarrow \mathbb{P}^N,$$

with $N = \binom{n+1}{r+1} - 1$, called the Plücker embedding. Furthermore the Grassmannian of lines $\mathbb{G}(1,n)$ is 1-defective of defect 4.

r	n	$\dim(\mathbb{G}(r,n))$	N	k	h
1	4	6	9	2	$\geqslant 3$
1	5	8	14	6	$\geqslant 3$
2	6	12	34	1	$\geqslant 17$
2	7	15	55	10	$\geqslant 5$
3	8	20	125	4	$\geqslant 25$

Segre-Veronese Varieties

Combining the Segre and the Veronese embeddings we can define the Segre-Veronese embedding

$$\psi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N,$$

with $N = \binom{a+n}{n}\binom{b+m}{m} - 1$, using the sheaf $\mathcal{O}_{\mathbb{P}^n}(a)$ on $\mathbb{P}^n$ and the sheaf $\mathcal{O}_{\mathbb{P}^m}(b)$ on $\mathbb{P}^m$. Let $SV^{n,m}_{a,b} = \psi(\mathbb{P}^n \times \mathbb{P}^m)$ be the Segre-Veronese variety.
A homogeneous polynomial of degree r on $SV^{n,m}_{a,b}$ corresponds to a bihomogeneous polynomial of bidegree (ar, br) on $\mathbb{P}^n \times \mathbb{P}^m$. Then the Hilbert polynomial of $SV^{n,m}_{a,b}$ is given by

$$h_{SV^{n,m}_{a,b}}(r) = \binom{ar+n}{n}\binom{br+m}{m} = \frac{a^n b^m}{n!m!} r^{n+m} + \ldots$$

We have that $\dim(SV^{n,m}_{a,b}) = n + m$ and $\deg(SV^{n,m}_{a,b}) = \frac{(n+m)!}{n!m!} a^n b^m = \binom{n+m}{n} a^n b^m$.

n	m	a	b	$\dim(SV^{n,m}_{a,b})$	N	k	h
2	3	1	3	5	39	2	$\geqslant 13$
4	4	2	3	8	524	3	$\geqslant 131$
4	4	3	3	8	1224	3	$\geqslant 153$
5	5	3	3	10	3135	4	$\geqslant 627$
5	5	3	4	10	7055	4	$\geqslant 1411$

7

POLYNOMIALS DECOMPOSITION AS SUMS OF POWERS

We work over an algebraically closed field of characteristic zero. We mainly follow notations and definitions of [Do]. Let V be a vector space of dimension $n+1$ and let $\mathbb{P}(V) = \mathbb{P}^n$ be the corresponding projective space. For any finite set of points $\{p_1, ..., p_h\} \subseteq \mathbb{P}^n$ we consider the linear space of homogeneous forms F of degree d on $\mathbb{P}^n$ such that $Z(F)$ contains the points $p_1, ..., p_h$, and we denote it by

$$L_d(p_1, ..., p_h) = \{F \in k[x_0, ..., x_n]_d \mid p_i \in Z(F) \ \forall\ 1 \leqslant i \leqslant h\}.$$

Definition 7.0.3. An unordered set of points $\{[L_1], ..., [L_h]\} \subseteq \mathbb{P}V^*$ is a polar h-polyhedron of $F \in k[x_0, ..., x_n]_d$ if

$$F = \lambda_1 L_1^d + ... + \lambda_h L_h^d,$$

for some nonzero scalars $\lambda_1, ..., \lambda_h \in k$ and moreover the L_i^d are linearly independent in $k[x_0, ..., x_n]_d$.

Apolarity

We briefly introduce the concept of Apolar form to a given homogeneous form to state the connection between the set of h-polyhedra of F and the space of apolar forms of F. This correspondence will be very important to reconstruct the h-polyhedra of F.
We fix a system of coordinates $\{x_0, ..., x_n\}$ on V and the dual coordinates $\{\xi_0, ..., \xi_n\}$ on V^*. Let $\varphi = \varphi(\xi_0, ..., \xi_n)$ be a homogeneous polynomial of degree t on V^*. We consider the differential operator

$$D_\varphi = \varphi(\partial_0, ..., \partial_n), \text{ with } \partial_i = \frac{\partial}{\partial x_i}.$$

This operator acts on φ substituting the variable ξ_i with the partial derivative $\partial_i = \frac{\partial}{\partial x_i}$. For any $F \in k[x_0, ..., x_n]_d$ we write

$$< \varphi, F >= D_\varphi(F).$$

We call this pairing the apolarity pairing.
In general φ is of the form $\varphi(\xi_0, ..., \xi_n) = \sum_{i_0+...+i_n=t} \alpha_{i_0,...,i_n} \xi_0^{i_0} ... \xi_n^{i_n}$ and F is of the form $F(x_0, ..., x_n) = \sum_{j_0+...+j_n=d} f_{i_0,...,i_n} x_0^{j_0} ... x_n^{j_n}$. Then

$$D_\varphi(F) = (\textstyle\sum_{i_0+...+i_n=t} \alpha_{i_0,...,i_n} \partial_0^{i_0} ... \partial_n^{i_n})(F).$$

We see that F is derived $i_0 + ... + i_n = t$ times. So we obtain a homogeneous polynomial of degree $d - t$ on V.
Once fixed $F \in k[x_0, ..., x_n]_d$ we have the map

$$ap_F^t : k[\xi_0, ..., \xi_n]_t \to k[x_0, ..., x_n]_{d-t}, \ \varphi \mapsto D_\varphi(F).$$

The map ap_F^t is linear and we can consider the subspace $Ker(ap_F^t)$ of $k[\xi_0, ..., \xi_n]_t$.

Definition 7.0.4. A homogeneous form $\varphi \in k[\xi_0, ..., \xi_n]_t$ is called apolar to a homogeneous form $F \in k[x_0, ..., x_n]_d$ if $D_\varphi(F) = 0$, in other words if $\varphi \in Ker(ap_F^t)$. The vector subspace of $k[\xi_0, ..., \xi_n]_t$ of apolar forms of degree t to F is denoted by $AP_t(F)$.

Lemma 7.0.5. *[Do, Lemma 3.1] The set* $\mathcal{P} = \{[L_1], ..., [L_h]\}$ *is a polar* h*-polyhedron of* F *if and only if*

$$L_d([L_1], ..., [L_h]) \subseteq AP_d(F),$$

and the inclusion is not true if we delete any $[L_i]$ *from* $\mathcal{P}$.

Proof. Let $\varphi \in S^dV$ be a homogeneous polynomial of degree d and let $L_i \in V*$ be a linear form on V. We have $< \varphi, L_i^d >= 0$ if and only if $(\sum_{i_0+...+i_n=k} \varphi_{i_0,...,i_n} \partial_0^{i_0}...\partial_n^{i_n})(L_i^d) = 0$ if and only if $(\sum_{i_0+...+i_n=k} \alpha_{i_0,...,i_n} L_0^{i_0}...L_n^{i_n}) = 0$ if and only if $\varphi([L_i]) = 0$. Therefore

$$\left\langle L_1^d, ..., L_h^d \right\rangle^{\perp} = \{\varphi \in S^dV \mid < \varphi, L_i^d >= 0\} = \{\varphi \in S^dV \mid \varphi([L_i]) = 0\} = L_d(\mathbb{P}V, [L_1], ..., [L_h]).$$

If the conditions of the lemma are satisfied we have

$$F \in AP_d(F)^{\perp} \subseteq L_d(\mathbb{P}V, [L_1], ..., [L_h])^{\perp} = \left\langle L_1^d, ..., L_h^d \right\rangle$$

and F is a linear combination of the L_i^d. If the $L_1^d, ..., L_h^d$ are linearly dependent there exists a proper subset $\mathcal{Q}$ of $\mathcal{P}$ such that $\langle \mathcal{Q} \rangle = \langle \mathcal{P} \rangle$, we can suppose $\mathcal{Q} = \{[L_1], ..., [L_{h-1}]\}$. Then

$$AP_d(F)^{\perp} \subseteq L_d(\mathbb{P}V, p_1, ..., p_h)^{\perp} = \langle \mathcal{Q} \rangle .$$

We have $\langle \mathcal{Q} \rangle^{\perp} = L_d(\mathbb{P}V, [L_1], ..., [L_h]) \subseteq AP_d(F)$ contradicting the hypothesis. This proves that $\mathcal{P}$ is a polar polyhedron of F.
Now suppose that $\mathcal{P}$ is a polar polyhedron of F. Then $F \in \langle \mathcal{P} \rangle$ and $L_d(\mathbb{P}V, [L_1], ..., [L_h]) = \langle \mathcal{P} \rangle^{\perp} \subseteq \langle F \rangle^{\perp} = AP_d(F)$.
Suppose that $L_d(\mathbb{P}V, [L_1], ..., [L_h]) \subseteq AP_d(F)$. Then $F \in AP_d(F)^{\perp} \subseteq L_d(\mathbb{P}V, [L_1], ..., [L_h])^{\perp} = \left\langle L_1^d, ..., L_{h-1}^d \right\rangle$. So we can write

$$F = \lambda_1 L_1^d + ... + \lambda_h L_h^d = \alpha_1 L_1^d + ... + \alpha_{h-1} L_{h-1}^d.$$

This implies

$$\lambda_1 - \alpha_1 L_1^d + ... + (\lambda_{h-1} - \alpha_{h-1}) L_{h-1}^d + \lambda_h L_h^d = 0$$

in contradiction with the linear independence of $L_1^d, ..., L_h^d$. □

7.1 THE CASE $\mathrm{Sec}_h(V_d^n) = \mathbb{P}^N$

In this section we consider cases in which the secant varieties of the Veronese varieties fill $\mathbb{P}^N$. We present a way to rebuild decomposition under some special hypothesis.

Construction 7.1.1. Let $F \in k[x_0, ..., x_n]_d$ be an homogeneous polynomial and let $F_1^l, ..., F_{D_l}^l \in k[x_0, ..., x_n]_{d-l}$ be the partial derivatives of order l, with $D_l = \binom{n+l}{l}$. We denote by $\mathbb{P}^{N_l}$ the projective space parametrizing the homogeneous polynomials of degree $d-l$ and consider the hyperplanes $AP^{d-l}(F_1^l), ..., AP^{d-l}(F_{D_l}^l) \subseteq \mathbb{P}^{N_l}$.
Let $h \in \mathbb{Z}$ be a positive integer such that $h - 1 < N_l$ and let $\{[l_1], ..., [l_h]\}$ be an h-polar polyhedron of F. Then by remark 5.0.8 and lemma 7.0.5 we know that

$$L_{d-l}(l_1, ..., l_h) \subseteq \textstyle\bigcap_{i=1}^{D_l} AP^{d-l}(F_i^l) = H^{d-l} \cong \mathbb{P}^{N_l - D_l}.$$

Since for a general h-polar polyhedron $\{[l_1], ..., [l_h]\}$ we have $\dim(L_{d-l}(l_1, ..., l_h)) = N_l - h$, we get the rational map

$$\varphi : VSP(F, h) \dashrightarrow \mathbb{G}(N_l - h, N_l - D_l), \ \{[l_1], ..., [l_h]\} \mapsto L_{d-l}(l_1, ..., l_h).$$

Suppose that the general $(h-1)$-plane containing $(AP^{d-l})^*$ intersects the corresponding Veronese variety in at least h points, so that the map φ is dominant.
In this case a general $(N_l - h)$-plane contained in H^{d-l} represents a linear system of the type $L_{d-l}(l_1, ..., l_h)$. If the intersection of n elements of this linear system consists of $(d-l)^n = t$ points $p_1, ..., p_t$, if $h \leqslant t$ then choosing h points from the p_i we get an h-polar polyhedron of F.
If $L_{d-l}(l_1, ..., l_h)$ has a base locus $\mathcal{B}$ of positive dimension we can construct an h-polar polyhedron of F simply by choosing h points on $\mathcal{B}$.
This construction gives a method to find the h-polyhedra of F under the required hypothesis.

For instance in the case $d = 3, n = 2, h = 4$ *I. V. Dolgachev* and *V. Kanev* proved that $VSP(F, 4) \cong \mathbb{P}^2$ [DK]. We give a simple proof of this result based on classical constructions of projective geometry.

Theorem 7.1.2. *Let* $F \in k[x, y, z]_3$ *be a general homogeneous polynomial. Then* $VSP(F, 4) \cong \mathbb{P}^2$.

Proof. The partial derivatives of F are three general homogeneous polynomials $F_x, F_y, F_z \in k[x, y, z]_2$. Let $H_\partial := \langle F_x, F_y, F_z \rangle$ be the plane in $\mathbb{P}(k[x, y, z]_2) \cong \mathbb{P}^5$ spanned by the partial derivatives. Any decomposition $\{L_1, ..., L_4\}$ of F induces a decomposition of the partial derivatives, and the 3-plane $\langle L_1^2, ..., L_4^2 \rangle$ contains H_∂. Since the 3-planes containing H_∂ are parametrized by $\mathbb{P}^2$ we get a morphism

$$\varphi : VSP(F, 4) \to \mathbb{P}^2,\ \{L_1, ..., L_4\} \mapsto \left\langle L_1^2, ..., L_4^2 \right\rangle.$$

Now, since $\deg(V_2^2) = 4$ any 3-plane containing H_∂ intersects V_2^2 in a subscheme of dimension zero and length four. We conclude that φ is an injective morphism between two smooth varieties of the same dimension. So it is an isomorphism. □

In the following example we explicitly reconstruct a decomposition for a cubic polynomial.

Example 7.1.3. Consider the cubic polynomial

$$F = x^3 + x^2y + x^2z + xy^2 + xyz + xz^2 + y^3 + y^2z + yz^2 + z^3.$$

The operator D_φ is given by

$$D_\varphi = \alpha_0 \frac{\partial^2}{\partial x^2} + \alpha_1 \frac{\partial^2}{\partial y^2} + \alpha_2 \frac{\partial^2}{\partial z^2} + \alpha_3 \frac{\partial^2}{\partial x \partial y} + \alpha_4 \frac{\partial^2}{\partial x \partial z} + \alpha_5 \frac{\partial^2}{\partial y \partial z}.$$

We are in the situation of construction 7.1.1, an the spaces of apolar forms are the following

$$AP_2(\tfrac{\partial F}{\partial x}) = Z(6\alpha_0 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5);$$

$$AP_2(\tfrac{\partial F}{\partial y}) = Z(2\alpha_0 + 6\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5);$$

$$AP_2(\tfrac{\partial F}{\partial z}) = Z(2\alpha_0 + 2\alpha_1 + 6\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5).$$

Now we choose a line on the plane determined by these three equations, for instance intersecting with the hyperplane $H_0 = Z(\alpha_0)$. Choosing two conics in this pencil and computing the base locus we get the following decomposition for F.

$$L_1 = (-0.005006 - i0.278616)x + (-0.008344 - i0.464361)y + (-0.012516 - i0.696541)z,$$
$$L_2 = (0.438881 - i0.986000)x,$$
$$L_3 = (-0.579402 - i0.878415)y,$$
$$L_4 = (-0.027303 - i0.199112)x + (-0.081910 - i0.597338)y + (-0.081910 - i0.597338)z.$$

7.1.1 *Uniqueness of the decomposition*

When the secant varieties of the Veronese embedding fills the projective space there are few cases in which we have the uniqueness of the decomposition. The cases examined here are two of these. In this context we recall the following theorem.

Theorem 7.1.4. *[Me2, Theorem 1] Fix integers $d > n > 1$ and $h \geqslant 1$ such that $(h+1)(n+1) = \binom{n+d}{n}$. Then the generic homogeneous polynomial of degree d in $n+1$ variables can be expressed as a sum of $h+1$ d-th powers of linear forms in a unique way if and only if $d = 5$ and $n = 2$.*

Polynomials on $\mathbb{P}^1$

We consider the decomposition of a polynomial $F \in k[x,y]_{2h-1}$ as sum of h linear forms. More generally if $F \in k[x,y]_d$ then $VSP(F,h) \cong \mathbb{P}^{2h-d-1}$. When $h > \frac{d+1}{2}$ we have infinitely many decompositions which can be reconstructed by construction 7.1.1.

Theorem 7.1.5. *(Sylvester) Let F be a generic homogeneous polynomial of degree $2h-1$ in two variables. There exists a unique decomposition of F as sum of h linear forms.*

Proof. : Let X be the rational normal curve of degree $2h-1$ in $\mathbb{P}^{2h-1}$. Since $\dim(Sec_h(X)) = h+(h-1) = 2h-1$ there exists a decomposition of F.
Suppose that $\{l_1,...,l_h\}$ and $\{L_1,...,L_h\}$ are two distinct decompositions of F. Let Λ_l and Λ_L be the two $(h-1)$-planes generated by the decompositions. The point F_{2h-1} belongs to $\Lambda_l \cap \Lambda_L$ so the linear space $\Gamma = \langle \Lambda_l, \Lambda_L \rangle$ has dimension

$$\dim(\Gamma) \leqslant (h-1)+(h-1) = 2h-2.$$

If $\Lambda_l \cap \Lambda_L = \{F\}$, then $\dim(\Gamma) = (h-1)+(h-1) = 2h-2$. So Γ is a hyperplane in $\mathbb{P}^{2h-1}$ and $\Gamma \cdot X \geqslant 2h$. A contradiction because $\deg(X) = 2h-1$.
If Λ_l and Λ_L have k common points, then Λ_l and Λ_L intersect in $k+1$ points $Q_1,...,Q_k,F$. In this case $\Lambda_l \cap \Lambda_L$ is a $\mathbb{P}^k$ and $\dim(\Gamma) = 2h-2-k$. We choose k points $P_1,...,P_k$ on X in general position so $H = \langle \Gamma, P_1,...,P_k \rangle$ is a hyperplane such that $H \cdot X \geqslant 2h-k+k = 2h$, a contradiction. We conclude that the decomposition of F in h linear factors is unique. □

In order to reconstruct the decomposition we consider the following construction.

Construction 7.1.6. The partial derivatives of order $h-2$ of F are $\binom{h-2+1}{1} = h-1$ homogeneous polynomials of degree $h+1$. Let $\nu_{h+1} : \mathbb{P}^1 \to \mathbb{P}^{h+1}$ be the $(h+1)$-Veronese embedding and let $X = \nu_{h+1}(\mathbb{P}^1)$ be the corresponding rational normal curve. Consider the projection

$$\pi : \mathbb{P}^{h+1} \setminus H_\partial \to \mathbb{P}^2$$

from the $(h-2)$-plane H_∂ spanned by the partial derivatives. Since the decomposition $\{L_1,...,L_h\}$ of F is unique, the projection $\overline{X} = \pi(X)$ will have a unique singular point $p_L = \pi(\langle L_1^{h+1},...,L_h^{h+1}\rangle)$ of multiplicity h. Now to find the decomposition we have to compute the intersection $H \cdot X = \{L_1^{h+1},...,L_h^{h+1}\}$, where $H = \langle H_\partial, p_L \rangle$.

Example 7.1.7. We consider the polynomial

$$F = x^3 + x^2y - xy^2 + y^3 \in k[x,y]_3.$$

i.e. the point $[F] = [1:1:1:1] \in \mathbb{P}^3$. The projection from $[F]$ to the plane $(X = 0) \cong \mathbb{P}^2$ is given by

$$\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2,\ [X:Y:Z:W] \mapsto [Y-X : X+Z : W-X].$$

Using Script 1 we compute the projection $C = \pi(X)$ of the twisted cubic curve X, and by Script 2 we compute the singular point of C,

$$p = \mathrm{Sing}(C) = [4 : 10 : 9].$$

The line $L = \langle p, [F] \rangle$ is given by the following equations

$$\begin{cases} 3X - 5Y - 2Z = 0, \\ 5 - 9Y + 4W = 0. \end{cases}$$

We compute the intersection $X \cdot L$, where X is the twisted cubic curve, using Script 3 we find $L_1^3 = [0.0515957 : 0.4157801 : 1.1168439 : 1]$ and $L_2^3 = [155.0515957 : 86.5842198 : 16.1168439 : 1]$. These points correspond to the linear forms

$$L_1 = -0.3722812x + y \text{ and } L_2 = 5.3722813x + y.$$

Indeed we have

$$F = 0.99322 \cdot (-0.3722812x + y)^3 + 0.00678 \cdot (5.3722813x + y)^3.$$

Hilbert and Sylvester Theorems

We consider the cases $d = 5$, $n = 2$, $h = 7$ (*Hilbert*), and $d = 3$, $n = 3$, $h = 5$ (*Sylvester*). Our aim is to provide a method by which explicitly reconstructing the decompositions in these two cases. We begin with the case $d = 5$, $n = 2$, $h = 7$.

Theorem 7.1.8. *(Hilbert) Let $F \in k[x, y, z]_5$ be a general homogeneous polynomial of degree five in three variables. Then F can be decomposed as sum of seven linear forms*

$$F = L_1^5 + \dots + L_7^5.$$

Furthermore the decomposition is unique.

Proof. A computation, together with [AH] main result, shows that $\dim \mathrm{VSP}(F, 7) = 0$. Assume that F admits two different decompositions, say $\{[L_1], \dots, [L_7]\}$ and $\{[l_1], \dots, [l_7]\}$. Consider the second partial derivatives of F. Those are six general homogeneous polynomials of degree three. Let $H_\partial \subseteq \mathbb{P}^9$ be the linear space they generate. Then, by Remark 5.0.8, we have

$$H_L := \langle [L_1^3], \dots, [L_7^3] \rangle \supset H_\partial \subset \langle [l_1^3], \dots, [l_7^3] \rangle =: H_l$$

The general choice of F ensures that both H_L and H_l intersect the Veronese surface $V_3^2 \subseteq \mathbb{P}^9$ at 7 distinct points.
Let

$$\pi : \mathbb{P}^9 \dashrightarrow \mathbb{P}^3$$

be the projection from H_∂, and $\overline{V} = \pi(V)$. Then $\overline{V}$ is a surface of degree $\deg(\overline{V}) = 9$ with seven points corresponding to $\pi(H_L)$ and $\pi(H_l)$. This shows that the 7-dimensional linear space $H := \langle H_L, H_l \rangle$ intersect V along a curve, say Γ. The construction of Γ yields

$$\deg \Gamma \leqslant \#(H_L \cap V) = 7.$$

On the other hand $\deg \Gamma = 3j$ therefore we end up with the following possibilities.

Case 1 (deg $\Gamma = 3$)**.** Then Γ is a twisted cubic curve contained in H and

$$H_l \cdot \Gamma = H_L \cdot \Gamma = 3$$

We may assume that $H_l \cap \Gamma = \{[l_1^3], [l_2^3], [l_3^3]\}$ and $H_L \cap \Gamma = \{[L_1^3], [L_2^3], [L_3^3]\}$. Let Λ be the pencil of hyperplanes containing H, and $\nu_3 : \mathbb{P}^2 \to V$ the Veronese embedding. The linear system $\nu_3^*(\Lambda_{|V})$ is a pencil of conics and therefore $\#(\mathrm{Bl}\,\Lambda_{|V}) \leqslant 4$.

To conclude observe that $\mathrm{Bl}\,\Lambda_{|V} \supset H \cap V$. This forces

$$\{[L_4^3], [L_5^3], [L_6^3], [L_7^3]\} = \{[l_4^3], [l_5^3], [l_6^3], [l_7^3]\},$$

and consequently the impossible $H_L = H_l$.

Case 2 (deg $\Gamma = 6$)**.** Then

$$H_l \cdot \overline{\Gamma} = H_L \cdot \overline{\Gamma} = 6$$

We may assume that $\Gamma \supset \{[L_1^3], \dots, [L_6^3]\} \cup \{[l_1^3], \dots, [l_6^3]\}$. Let Λ be the pencil of hyperplanes containing H. Let $\nu_3 : \mathbb{P}^2 \to V$ be the Veronese embedding. The linear system $\nu_3^*(\Lambda_{|V})$ is a pencil of lines and therefore $\#(\mathrm{Bl}\,\Lambda_{|V}) \leqslant 1$. This forces

$$[L_7^3] = [l_7^3],$$

and consequently the impossible $H_L = H_l$.

$\square$

The following construction is inspired by the proof of Theorem 7.1.8, and provides a method to reconstruct the decomposition starting from the polynomial.

Construction 7.1.9. If $\{[L_1], ..., [L_7]\}$ is a decomposition of F, then it is also a decomposition for its partial derivatives of any order. In particular F has six partial derivatives of order 2 that are homogeneous polynomials of degree three in x, y, z. We consider these derivatives as points in the projective space $\mathbb{P}^9 = \mathbb{P}(k[x, y, z]_3)$, parametrizing the homogeneous polynomials of degree three in three variables. We denote by $H_\partial \subseteq \mathbb{P}^9$ the 5-plane spanned by the derivatives, and with V the Veronese variety $V = \nu(\mathbb{P}^2)$, where $\nu : \mathbb{P}^2 \to \mathbb{P}^9$ is the Veronese embedding of degree 3.
Since all the derivatives can be decomposed as sum of $L_1^3, ..., L_7^3$ the 5-plane H_∂ is contained in the 6-plane 7-secant to the the Veronese variety $V \subseteq \mathbb{P}^9$, given by $H_L = \langle L_1^3, ..., L_7^3 \rangle$. Consider now the projection

$$\pi : \mathbb{P}^9 \dashrightarrow \mathbb{P}^3$$

from the linear space H_∂. The image of the Veronese variety $\pi(V) = \overline{V}$ is a surface of degree 9 in $\mathbb{P}^3$, furthermore it has a point p_L of multiplicity 7, which comes from the contraction of H_L. This is the unique point of multiplicity 7 on $\overline{V}$ by the uniqueness of the decomposition. From this discussion we derive an algorithm to find the decomposition divided into the following steps.

1. Compute the partial derivative of order 2 of F.
2. Compute the equation of the 5-plane H_∂ spanned by the derivatives.
3. Project the Veronese variety V in $\mathbb{P}^3$ from H_∂.
4. Compute the point p_L of multiplicity 7 on $\overline{V}$.
5. Compute the 6-plane $H = \langle H_\partial, p_L \rangle$ spanned by H_∂ and the point p_L.

6. Compute the intersection $V \cdot H = \{L_1^3, ..., L_7^3\}$.

Example 7.1.10. Consider the polynomial $F \in k[x, y, z]_5$ given by
$F = x^5 + x^4y^2 - x^2y^3 - y^5 + z^5 + x^3z^2 + x^2z^3 - x^4y + x^4z - 4x^3yz + 6x^2y^2z - 6x^2yz^2 + xy^4 - 4xy^3z + 6xy^2z^2 - 4xyz^3 + xz^4 + y^4z - 2y^3z^2 + 2y^2z^3 - yz^4$.
On $\mathbb{P}^9 = \mathbb{P}(k[x, y, z]_3)$ we fix homogeneous coordinates $[X_0 : ... : X_9]$ corresponding respectively to the monomials $\{x^3, x^2y, x^2z, xyz, xy^2, xz^2, y^3, y^2z, yz^2, z^2\}$. In these coordinates the linear space H_∂ spanned by the second partial derivatives is given by the following equations.

$$\begin{cases} -1701X_0 - 4455X_1 + 567X_2 - 4455X_3 - 567X_5 - 1458X_6 + 81X_7 = 0, \\ -4536X_0 - 13392X_1 - 13392X_3 - 4455X_6 + 216X_7 - 567X_9, \\ 216X_1 + 216X_2 + 216X_3 - 216X_5 + 81X_6 + 81X_9 = 0, \\ 13392X_4 - 26784X_8 = 0. \end{cases}$$

We project on the linear space $\{X_0 = X_1 = X_2 = X_3 = X_4 = X_5 = 0\} \cong \mathbb{P}^3$. The projection $\pi : \mathbb{P}^9 \setminus H_\partial \to \mathbb{P}^3$ has equations
$\pi(X_0, ..., X_9) = [-(42X_0 + 110X_1 - 14X_2 + 110X_3 + X_4 + 14X_5 + 36X_6) : -18(X_4 + 2X_7) : 18(X_4 - 2X_8) : (42X_0 + 14X_1 - 110X_2 + 14X_3 + X_4 + 110X_5 - 36X_9)]$.
We compute the projection of the Veronese variety V by Script 3. In this way we obtain the equation of $\overline{V} = Z(F)$ where $F = F(X, Y, Z, W)$ is a homogeneous polynomial of degree $9 = \deg(V)$. Now we use Script 4 to compute the point of multiplicity 7 on $\overline{V}$. The singular point is $p_L = [-5.0632364198314 : 0 : 0 : 35.442654938835]$. By Script 4 we compute the intersection $V \cdot H = \{L_1^3, ..., L_7^3\}$ and we obtain the linear forms

$$\begin{aligned}
L_1 &= 0.98274177184x - 0.12482457140y, \\
L_2 &= -0.65071281231x + 0.65071281231y, \\
L_3 &= 0.12482457140x - 0.98274177184y, \\
L_4 &= (0.18975376061 - i0.33683479696)x + (0.83442021400 - i0.082003524422)z, \\
L_5 &= (0.04447250903 - i0.38403953709)x - (0.62685967129 + i0.556802140865)z, \\
L_6 &= (-0.12154672768 + i0.37408236279)x + (0.18089826609 - i0.55674761546)z, \\
L_7 &= 0.72477966367x - 0.72477966495y + 0.72477965837z.
\end{aligned}$$

These forms give the unique decomposition of our polynomial.

Now we consider the case $d = 3$, $n = 3$, $h = 5$. Sylvester pentahedral Theorem can be proved following the proof of Theorem 7.1.8 with a slightly more convoluted argument. *G. Ottaviani* informed me of a very nice and neat proof using apolarity.

Theorem 7.1.11. *(Sylvester) Let $F \in k[x, y, z, w]_3$ be a generic homogeneous polynomial of degree three in four variables. Then F can be decomposed as sum of seven linear forms*

$$F = L_1^3 + ... + L_5^3.$$

Furthermore the decomposition is unique.

Proof. Let $F = F_3 \in \mathbb{P}^9$ be a homogeneous form of degree three. We know that a 5-polar polyhedron of F exists. The polar form of F in a point $\xi = [\xi_0 : \xi_1 : \xi_2 : \xi_3] \in \mathbb{P}^3$ is the quadric

$$P_\xi F = \xi_0 \frac{\partial F}{\partial x_0} + \xi_1 \frac{\partial F}{\partial x_1} + \xi_2 \frac{\partial F}{\partial x_2} + \xi_3 \frac{\partial F}{\partial x_3}.$$

Let $\{L_1, ..., L_5\}$ be a 5-polar polyhedron of F, then $F = L_1^3 + ... + L_5^3$. The polar form is of the type

$$P_\xi F = \sum_{i=1}^{5} \xi_i \lambda_i L_i^2$$

and it has rank 2 on the points $\xi \in \mathbb{P}^3$ on which three of the linear form L^i vanish simultaneously. These points are $\binom{5}{3} = 10$.
Now we consider the subvariety X_2 of $\mathbb{P}^9$ parametrizing the quadrics of rank 2. A quadric Q of rank 2 is the union of two planes, then $\dim(X_2) = 6$. To find the degree of X_2 we have to intersect with a 3-plane, that is intersection of 6 hyperplanes. So the degree of X_2 is equal to the number of quadrics of rank 2 passing through 6 general points of $\mathbb{P}^3$. If we choose three points then the plane through these points is determined, and the quadric is also determined. Then these quadrics are $\frac{1}{2}\binom{6}{3} = 10$. We have seen that $\dim(X_2) = 6$ and $\deg(X_2) = 10$.
Now the linear space

$$\Gamma = \{P_\xi F \mid \xi \in \mathbb{P}^3\} \subseteq \mathbb{P}^9$$

is clearly a 3-plane in $\mathbb{P}^9$.
Then $\Gamma \cap X_2 = \{P_\xi F \mid \mathrm{rk}(P_\xi F) = 2\}$ is a set of 10 points. These points have to be the 10 points we have found in the first part of the proof. Then the decomposition of F in five linear factors is unique. ☐

The argument used in the proof suggests us an algorithm to reconstruct the decomposition.

Construction 7.1.12. Consider F and its first partial derivatives.

1. Compute the 3-plane Γ spanned by the partial derivatives of F.

2. Compute the intersection $\Gamma \cdot X_2$, where X_2 is the variety parametrizing the rank 2 quadrics in $\mathbb{P}^3$.

3. Consider the 10 points in the intersection. By construction on each plane we are looking for there are 6 of these points, furthermore on each plane there are 4 triples of collinear points. Then with these 10 points we can construct exactly $\frac{\binom{10}{3}}{\binom{6}{3}+4} = 5$ planes. These planes gives the decomposition of F. Note that a priori we have $\binom{10}{6} = 210$ choices, but we are interested in combinations of six points $\{P_{j_1}, ..., P_{j_6}\}$ which lie on the same plane. We know that there are exactly five of these. To find the five combinations we use Script 5 which constructs a matrix A whose lines are the ten points and then computes the 6×4 submatrices of rank 3 of A.

Example 7.1.13. Consider the polynomial
$F = x^3 + x^2y + x^2z + x^2w + xy^2 + xyz + xyw + xz^2 + xzw + xw^2 + y^3 + y^2z + y^2w + yz^2 + yzw + yw^2 + z^3 + z^2w + zw^2 + w^3$.
We compute the equations of the linear space Γ, the equations of the variety X_2, and verify that their intersection is a subscheme of dimension zero and length 10. In the $\mathbb{P}^9$ parametrizing the quadrics on $\mathbb{P}^3$ we fix homogeneous coordinates $[X_0 : ... : X_9]$, corresponding to the

monomials $\{x^2, xy, xz, xw, y^2, yz, yw, z^2, zw, w^2\}$. Check what we have said using the Script 6. In these coordinates the 3-plane spanned by the partial derivatives has equations

$$\begin{cases} X_7 - 2X_8 + X_9 = 0, \\ X_5 - X_6 - X_8 + X_9 = 0, \\ X_4 - 2X_6 + X_9 = 0, \\ X_2 - X_3 - X_8 + X_9 = 0, \\ X_1 - X_3 - X_6 + X_9 = 0, \\ X_0 - 2X_3 + X_9 = 0. \end{cases}$$

Script 7 allows us to calculate the intersection of H_∂ with the variety X_2 parametrizing the quadrics of rank 2.
We find $10 = deg(X_2)$ points on H_∂ that corresponds to the following points in $\mathbb{P}^3$.

$$\begin{aligned}
P_1 &= [-0.0538 - 0.0089i : -0.0538 - 0.0089i : -0.0538 - 0.0089i : 0.2692 + 0.0447i], \\
P_2 &= [0.9291 + 0.1127i : 0 - 0.9291 - 0.1127i : 0], \\
P_3 &= [0 : 0 : -0.3198 - 0.0488i : 0.3198 + 0.0488i], \\
P_4 &= [0 : 0.4297 + 0.7502i : -0.4297 - 0.7502i : 0], \\
P_5 &= [0 : -0.3850 + 0.0834i : 0 : 0.3850 - 0.0834i], \\
P_6 &= [0.4850 - 0.8736i : -0.4850 + 0.8736i : 0 : 0], \\
P_7 &= [-0.4873 - 0.0825i : 0 : 0 : 0.4873 + 0.0825i], \\
P_8 &= [0.7990 + 0.1275i : -0.1598 - 0.0255i : -0.1598 - 0.0255i : -0.1598 - 0.0255i], \\
P_9 &= [2.3960 - 1.8505i : 2.3960 - 1.8505i : -11.9800 + 9.2523i : 2.3960 - 1.8505i], \\
P_{10} &= [-0.0652 - 0.1273i : 0.3260 + 0.6364i : -0.0652 - 0.1273i : -0.0652 - 0.1273i].
\end{aligned}$$

Thanks to Script 5 we can compute the five combinations of six coplanar points, and then the linear forms.

$$\begin{aligned}
L_1 &= (0.0149652 + 0.0069738i)x + (0.0449377 + 0.020996i)y \\
&\quad +(0.0149652 + 0.0069738i)z + (0.0149652 + 0.0069738i)w, \\
L_2 &= (0.00927286 + 0.0448705i)x + (0.00310162 + 0.0149327i)y \\
&\quad +(0.00310162 + 0.0149327i)z + (0.00310162 + .0149327i)w, \\
L_3 &= (0.0278039 - 0.0573066i)x + (0.0278039 - 0.0573066i)y \\
&\quad +(0.0834118 - 0.17192i)z + (0.02780390.0573066i)w, \\
L_4 &= (-0.0642594 - 0.253748i)x + (-0.0642594 - 0.253748i)y \\
&\quad +(-0.0642594 - 0.253748i)z + (-0.06425940.253748i)w, \\
L_5 &= (-0.0312783 - 0.127146i)x + (-0.0312783 - 0.127146i)y \\
&\quad +(-0.0312783 - 0.127146i)z + (-0.0938348 - 0.381437i)w.
\end{aligned}$$

7.2 THE CASE $\mathrm{Sec}_h(V_d^n) \neq \mathbb{P}^N$

Let $\nu : \mathbb{P}^n \to \mathbb{P}^{N_d}$ be the d-Veronese embedding, and let $V_d^n = \nu(\mathbb{P}^n)$ be its image. Let $[F] \in \mathbb{P}^N = \mathbb{P}(k[x_0, ..., x_n]_d)$ be a degree d homogeneous polynomial. Fixed a positive integer h such that $\mathrm{Sec}_h(V_d^n) \neq \mathbb{P}^N$ we want to determine whether $[F] \in \mathrm{Sec}_h(V_d^n)$. We begin with the following simple observation:

Remark 7.2.1. If $F = \sum_{i=1}^{h} \lambda_i L_i^d$ then its partial derivatives of order l lie in the linear space $\langle L_1^{d-l}, ..., L_h^{d-l} \rangle$ for any $l = 1, ..., d-1$.

The partial derivatives of order l are $\binom{n+l}{l}$ homogeneous polynomials of degree $d-l$, so the previous observation is meaningful when $h < \binom{n+l}{l}$ and $h < \binom{d-l+n}{n}$. The latter condition ensures that $\langle L_1^{d-l}, ..., L_h^{d-l} \rangle$ is a proper subspace of the projective space $\mathbb{P}^{N_{d-l}}$ parametrizing homogeneous polynomials of degree $d-l$.
Consider the partial derivatives $F^l_{l_0,...,l_n} := \frac{\partial^l F}{\partial x_0^{l_0},...,\partial x_n^{l_n}}$ and the incidence variety

$$\begin{tikzcd}
 & \mathcal{J}_{l,h} = \{(F,H) \mid \in F^l_{l_0,...,l_n} \in H,\ \forall\ l_0 + ... + l_n = l\} \subset \mathbb{P}^N \times \mathbb{G}(h-1, N_{d-l}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^N & & \mathbb{G}(h-1, N_{d-l})
\end{tikzcd}$$

where $\mathcal{S}_h V^n_{d-l} \subseteq \mathbb{G}(h-1, N_{d-l})$ is the abstract h-secant variety of V^n_{d-l}. Note that when $h < \binom{n+l}{l}$ the map π_1 is generically injective. Let $X_{l,h} = \pi_1(\mathcal{J}_{l,h}) \subseteq \mathbb{P}^N$ be its image, note that $X_{l,h}$ is irreducible. By remark 7.2.1 we get $\mathrm{Sec}_h(V^n_d) \subseteq X_{l,h}$. By construction $X_{l,h}$ is not too difficult to describe, so we want to find cases when the equality holds in order to get a simple criterion to establish whether $[F] \in \mathrm{Sec}_h(V^n_d)$.

Remark 7.2.2. The equality holds trivially when $d = 2$. Let $F \in k[x_0, ..., x_n]_2$ be a polynomial and let $\mathcal{M}_F$ the matrix of the quadratic symmetric form associated to F. Then $F \in \mathrm{Sec}_h(V^n_2)$ if and only if $\mathrm{rk}(\mathcal{M}_F) \leqslant h$. But the rows of $\mathcal{M}_F$ are exactly the partial derivatives of F.

Consider the partial derivatives $F_1, ..., F_m \in k[x_0, ..., x_n]_{d-l}$ of order l of F. Let $\varphi : \mathbb{P}^n \times \mathbb{P}^{N_{d-l}} \to \mathbb{P}^M$ be the Segre-Veronese embedding induced by $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^{N_{d-l}}}(d-l, 1)$, and let $\Sigma_{d-l,1}$ be its image.

Proposition 7.2.3. *If the partial derivatives* $F_1, ..., F_m$ *lie in a* $(h-1)$*-plane* $H \subset \mathbb{P}^{N_{d-l}}$ *which is* h*-secant to the Veronese variety* $V^n_{d-l} \subset \mathbb{P}^{N_{d-l}}$ *, with* $h - 1 < N_{d-l}$*, then* $[F] \in \mathrm{Sec}_h(\Sigma_{d-l,1})$.

Proof. By assumption $F^l_{l_0,...,l_n} = \sum_{i=1}^{h} \lambda_i^{l_0,...,l_n} L_i^{d-l}$. Recursively applying Euler formula we get $F = P_1 L_1^{d-l} + ... + P_h L_h^{d-l}$ where $P_i \in k[x_0, ..., x_n]_l$, and this means that $[F] \in \mathrm{Sec}_h(\Sigma_{d-l,1})$. □

Remark 7.2.4. Suppose that $F_{x_0}, ..., F_{x_n} \in k[x_0, ..., x_n]_{d-1}$ are the partial derivatives of a homogeneous polynomial $F \in k[x_0, ..., x_n]_d$. Furthermore suppose that $F_{x_i} \in \langle L_1^{d-1}, ..., L_h^{d-1} \rangle$ for any i. By Euler formula we get

$$F = P_1 L_1^{d-1} + ... + P_h L_h^{d-1},$$

where the P_i's are linear forms, i.e. $F \in \mathrm{Sec}_h(\Sigma_{d-1,1})$. Since $F \in \mathbb{P}^N$ by hypothesis we have $F \in \mathrm{Sec}_h(\Sigma_{d-1,1}) \cap \mathbb{P}^N$. Consider the following two statements

(i) $\mathrm{Sec}_h(\Sigma_{d-1,1}) \cap \mathbb{P}^N = \mathrm{Sec}_h(V^n_d)$;

(ii) $F_{x_i} \in \langle L_1^{d-1}, ..., L_h^{d-1} \rangle$ for any $i = 0, ..., n$, implies $[F] \in \mathrm{Sec}_h(V^n_d)$.

From the above discussion we deduce that (i) implies (ii).

The Case $n = 1$

We begin with the simplest case $n = 1$. We denote by $C_d \subset \mathbb{P}^d$ the degree d rational normal curve, in this case $\mathcal{S}ec_h(C_d) \neq \mathbb{P}^d$ if and only if $h \leqslant \frac{d}{2}$.

Lemma 7.2.5. *Let* $F = \sum_{i+j=d} \alpha_{i,j} x_0^i x_1^j \in k[x_0, x_1]_d$ *be a homogeneous polynomial, and let* $c = c(\alpha_{i,j})$ *be the coefficient of* x_0^h *in the partial derivative* $\frac{\partial^{d-h}F}{\partial x_0^m \partial x_1^s}$*, with* $h \geqslant 1$*. Then* $c = C \cdot \alpha_{d-s,s}$*, where* C *is a constant.*

Proof. Since the only monomial of F producing c is $x_0^{d-s} x_1^s$ the assertion follows. ☐

Theorem 7.2.6. *For any* $h \leqslant \frac{d}{2}$ *we have* $\mathcal{S}ec_h(C_d) = X_{d-h,h}$*. Consequently if the partial derivatives of order* $d-h$ *of a homogeneous polynomial* $F \in k[x_0, x_1]_d$ *lie in a hyperplane of* $\mathbb{P}^h$ *then* $[F]$ *lies in* $\mathcal{S}ec_h(C_d)$*.*

Proof. The partial derivatives of order $d-h$ of F are $d-h+1$ homogeneous polynomials of degree h. If $F = \sum_{i=1}^{h} \lambda_i L_i^d$ the partial derivatives lie in $\langle L_1^h, ..., L_h^h \rangle$ which is a hyperplane h-secant to C_h, but $\deg(C_h) = h$ and the latter condition is irrelevant. Let H be a general hyperplane in $\mathbb{P}^h$, forcing the partial derivatives of a degree d polynomial $G = \sum_{i+j=d} \alpha_{i,j} x_0^i x_1^j \in k[x_0, x_1]_d$ to lie in H gives $d-h+1$ linear equations in the coefficients of G. Without loss of generality we can suppose H to be the defined by the vanishing of the first homogeneous coordinate on $\mathbb{P}^h$, then by 7.2.5 the fiber of π_2 is the linear subspace of $\mathbb{P}^N$ defined by

$$\pi_2^{-1}(H) = \{\alpha_{d-s,s} = 0, \ \forall \ s = 0, ..., d-h\}.$$

The equations of $\pi_2^{-1}(H)$ are independent so

$$\dim(\pi_2^{-1}(H)) = d - (d-h+1) = h-1,$$

and the dimension of $X_{d-h,h}$ is

$$\dim(X_{d-h,h}) = \dim(\mathcal{I}_{d-h,h}) = h-1+h = 2h-1.$$

Finally $\dim(\mathcal{S}ec_h(C_d)) = h+h-1 = 2h-1$ yields $\mathcal{S}ec_h(C_d) = X_{d-h,h}$. ☐

Remark 7.2.7. The partial derivatives of order $d-h$ of a homogeneous polynomial $F \in k[x_0, x_1]_d$ depend on $d+1$ parameters. We consider the matrix $\mathcal{M}_{d,h}$ whose lines are the partial derivatives. From 7.2.6 we get equations for $\mathcal{S}ec_h(C_d)$ imposing $\mathrm{rk}(\mathcal{M}_{d,h}) \leqslant h$, that is the classical determinantal description of $\mathcal{S}ec_h(C_d)$.

Proposition 7.2.8. *If* $[F] \in \mathcal{S}ec_h(C_d)$ *is general then its decomposition in powers of linear forms is unique.*

Proof. Let $H_\partial \subset \mathbb{P}^h$ be the hyperplane spanned by the partial derivatives of order $d-h$ of F. Since $\deg(C_h) = h$ and F is general we have $H_\partial \cdot C_h = \{L_1^h, ..., L_h^h\}$. Then $\{L_1, ..., L_h\}$ is the unique h-polyhedron of F. ☐

Theorem 7.2.6 and proposition 7.2.8 immediately suggest an algorithm.

Construction 7.2.9. Given $F \in k[x_0, x_1]_d$ to establish if F admits a decomposition in $h \leqslant \frac{d}{2}$ linear forms, and eventually to find it we proceed as explained in the following diagram.

$$\begin{tikzcd}
\{\text{Compute} \dim(H_\partial)\} \arrow[d, "\dim(H_\partial)=h"] \arrow[dr, "\dim(H_\partial)=h-1"] & & \\
 & \{F \text{ admits a } h-\text{polyhedron}\} \arrow[dr] & \\
\{F \text{ does not admit a } h-\text{polyhedron}\} & & \{\text{Compute } H_\partial \cdot C_h\}
\end{tikzcd}$$

Then $H_\partial \cdot C_h = \{L_1^h, ..., L_h^h\}$ and $F = \sum_{i=1}^h \lambda_i L_i^d$.

Example 7.2.10. Consider the case $d = 4, h = 2$ and write $F = \sum_{i_0+i_1=4} \alpha_{i,j} x_0^i x_1^j$. Forcing $\frac{\partial^2 F}{\partial x_0 \partial x_1} \in \langle \frac{\partial^2 F}{\partial x_0^2}, \frac{\partial^2 F}{\partial x_1^2} \rangle$ we get

$$Sec_2(C_4) = \{54\alpha_{3,1}^2\alpha_{0,4} - 18\alpha_{3,1}\alpha_{2,2}\alpha_{1,3} - 144\alpha_{4,0}\alpha_{2,2}\alpha_{0,4} + 4\alpha_{2,2}^3 + 54\alpha_{4,0}\alpha_{1,3}^2 = 0\}.$$

Now consider the polynomial

$$F = 9(x_0^4 + x_0^3x_1 + x_0^2x_1 + x_0x_1^3) + 4x_1^4.$$

The second partial derivatives of F lie in the line

$$H_\partial = \{X_0 - 3X_1 + 3X_2 = 0\} \subset \mathbb{P}(k[x_0, x_1]_2).$$

Now we have to compute the intersection $H_\partial \cdot C_2$, where $C_2 = \{X_1^2 - 4X_0X_2 = 0\}$ is the conic parametrizing squares of linear forms, we have

$$H_\partial \cdot C_2 = \{[15 + 6\sqrt{6} : 6 + 2\sqrt{6} : 1], [15 - 6\sqrt{6} : 6 - 2\sqrt{6} : 1]\}.$$

Finally we compute the linear forms giving the decomposition

$$L_1 = 5.44948x_0 + x_1 \text{ and } L_2 = 0.55051x_0 + x_1.$$

The Case $h \leqslant n$

Now we consider the variety $X_{d-1,h}$. The partial derivatives of order $d-1$ of F are linear forms i.e. points in $(\mathbb{P}^n)^*$, so we restrict our attention on the case $h \leqslant n$ to have significant constraints. First we compute the dimension of the general fiber of $\pi_2 : \mathcal{I}_{d-1,h} \rightarrow \mathbb{G}(h-1, n)$.

Theorem 7.2.11. *The fiber of* $\pi_2 : \mathcal{I}_{d-1,h} \rightarrow \mathbb{G}(h-1, n)$ *on a general* $(h-1)$*-plane* $H \in \mathbb{G}(h-1, n)$ *is a linear subspace of* $\mathbb{P}^N$ *of dimension*

$$\dim(\pi_2^{-1}(H)) = \binom{d+h-1}{d} - 1.$$

Furthermore the dimension of X_{d-1} *is given by*

$$\dim(X_{d-1,h}) = h(n-h+1) + \binom{d+h-1}{d} - 1.$$

Proof. We can suppose $H = \{X_0 = ... = X_{n-h} = 0\}$, where $\{X_0, ..., X_n\}$ are homogeneous coordinates on $\mathbb{P}^n$. We write a general polynomial $[F] \in \mathbb{P}^N$ in the form

$$F = \sum_{i_0+...+i_n=d} \alpha_{i_0,...,i_n} x_0^{i_0} ... x_n^{i_n}.$$

The fiber $\pi_2^{-1}(H)$ is the linear subspace of $\mathbb{P}^N$ defined by the vanishing of the coefficients of $x_0, ..., x_{n-h}$ in the derivatives of F. Many of these equations are redundant, the difficulty is in counting the exact number of independent equations. We prove that this number is $\binom{d+n-1}{d-1} + \binom{d+n-1}{d} - \binom{d+h-1}{d}$ by induction on $n-h$. If $n-h=0$ then H is an hyperplane and the condition on the derivatives are all independent, so the number of conditions is exactly the number of derivatives $\binom{d-1+n}{d-1}$. Furthermore our formula for $n-h=0$ gives $\binom{d+n-1}{d-1} + \binom{d+n-1}{d} - \binom{d+n-1}{d} = \binom{d+n-1}{d-1}$, and the case $n-h=0$ is verified. Consider now the general case, let $\overline{H} = \{X_0 = ... = X_{n-h-1} = 0\}$, let C_{n-h-1} the number of independent conditions obtained forcing the partial derivatives to lie in $\overline{H}$. Adding the condition $\{X_{n-h} = 0\}$ gives new equations coming from the coefficients of the form $\alpha_{0,...,0,i_{n-h},i_{n-h+1},...,i_n}$, with $i_{n-h} \neq 0$. These correspond to monomials of degree d in the variables $x_{n-h}, ..., x_n$ that contain the variable x_{n-h}. Now the monomials of degree d not containing x_{n-h} are the monomials of degree d in $x_{n-h+1}, ..., x_n$. So in the final step we are adding

$$\binom{d+h}{d} - \binom{d+h-1}{d}$$

conditions. Then the number if independent equations is $C_{n-h} = C_{n-h-1} + \binom{d+h}{d} - \binom{d+h-1}{d}$, by induction hypothesis

$$C_{n-h-1} = \binom{d+n-1}{d-1} + \binom{d+n-1}{d} - \binom{d+n-(n-h-1)-1}{d}.$$

So $C_{n-h} = \binom{d+n-1}{d-1} + \binom{d+n-1}{d} - \binom{d+n-(n-h-1)-1}{d} + \binom{d+h}{d} - \binom{d+h-1}{d} = \binom{d+n-1}{d-1} + \binom{d+n-1}{d} - \binom{d+h-1}{d}$. Finally we have $\dim(X_{d-1,h}) = \dim(\mathbb{G}(h-1,n)) + \dim(\pi_2^{-1}(H)) = h(n-h+1) + \binom{d+h-1}{d} - 1$. ☐

Remark 7.2.12. Consider the case $d = 2$. By Alexander-Hirshowitz theorem [AH], $\mathcal{S}ec_h(V_2^n) \neq \mathbb{P}^N$ if and only if $h \leqslant n$. By theorem 7.2.11 and remark 7.2.2 we recover the effective dimension of $\mathcal{S}ec_h(V_2^n)$,

$$\dim(\mathcal{S}ec_h(V_2^n)) = \frac{2nh - h^2 + 3h - 2}{2},$$

and consequently the formula for the h-secant defect of V_2^n,

$$\delta_h(V_2^n) = \frac{h(h-1)}{2}.$$

At this point we have a complete description for polynomials of arbitrary degree in two variables and for polynomials of degree two in any number of variables. So we concentrate on the case $n \geqslant 2$ and $d \geqslant 3$.

Theorem 7.2.13. *Let $n \geqslant 2, d \geqslant 3, h \leqslant n$ be positive integers. Then $\mathcal{S}ec_h(V_d^n)$ is a subvariety of $X_{d-1,h}$ of codimension*

$$\operatorname{codim}_{\mathcal{S}ec_h(V_d^n)}(X_{d-1,h}) = \binom{d+h-1}{d} - h^2.$$

Proof. Since $n \geqslant 2$, $d \geqslant 3$, and $h \leqslant n$, by Alexander-Hirshowitz theorem the effective dimension of $\mathcal{S}ec_h(V_d^n)$ is the expected one

$$\dim(\mathcal{S}ec_h(V_d^n)) = \min\{hn + (h-1), N_d\}.$$

Furthermore $n \geqslant 2$, $d \geqslant 3$, $h \leqslant n$ implies $hn + (h-1) < N_d$. So

$$\dim(\mathcal{S}ec_h(V_d^n)) = hn + (h-1).$$

Finally $\operatorname{codim}_{\mathcal{S}ec_h(V_d^n)}(X_{d-1,h}) = h(n-h+1) + \binom{d+h-1}{d} - 1 - hn - (h-1) = \binom{d+h-1}{d} - h^2$. ☐

Corollary 7.2.14. *If $d = 3$ then $\mathcal{S}ec_2(V_3^n) = X_{2,2}$ for any $n \geqslant 2$. Consequently if the second partial derivatives of a homogeneous polynomial $F \in k[x_0, ..., x_n]_3$ lie in a line of $\mathbb{P}^n$ then $[F]$ lies in $\mathcal{S}ec_2(V_3^n)$.*

Proof. For $h = 2, d = 3$ we have $\binom{d+h-1}{d} - h^2 = 0$. We conclude by theorem 7.2.13. ☐

7.2.1 *The variety* $X_{l,h}$

Let's look closer at the variety $X_{l,h}$. This variety parametrizes polynomials $F \in k[x_0, ..., x_n]_d$ whose partial derivatives of order l span a $(h-1)$-plane. Let $\mathcal{M}_{l,h}$ be the $\binom{n+l}{l} \times \binom{n+d-l}{d-l}$ matrix whose lines are the l-th derivatives of $F = \sum_{i_0+...+i_n=d} \alpha_{i_0,...,i_n} x_0^{i_0}...x_n^{i_n}$. Then $X_{l,h}$ is the determinantal variety defined in $\mathbb{P}^N$ by $\mathrm{rk}(\mathcal{M}_{l,h}) \leqslant h$, where the $\alpha_{i_0,...,i_n}$ are the homogeneous coordinates on $\mathbb{P}^N$. Let $\mathbb{P}^M$ be the projective space parametrizing $\binom{n+l}{l} \times \binom{n+d-l}{d-l}$ matrices, and let $M_h \subset \mathbb{P}^M$ be the variety of matrices of rank less or equal than h. Then M_h is an irreducible variety of dimension $M - \left(\binom{n+l}{l} - h\right) \cdot \left(\binom{n+d-l}{d-l} - h\right)$. Clearly the variety $X_{l,h}$ is a special linear section of M_h.

Lemma 7.2.15. *The varieties* $X_{l,h}$ *and* $X_{d-l,h}$ *are isomorphic.*

Proof. The matrix $\mathcal{M}_{d-l,h}$ whose lines are the $(d-l)$-th partial derivatives of F is the $\binom{n+d-l}{d-l} \times \binom{n+l}{l}$ matrix given by

$$\mathcal{M}_{d-l,h} = \mathcal{M}_{l,h}^t,$$

where $\mathcal{M}_{l,h}^t$ is the transposed matrix of $\mathcal{M}_{d-l,h}$. Then the assertion follows. □

Proposition 7.2.16. *Consider the case* $h \leqslant n$*. The variety* $X_{1,h}$ *is irreducible.*

Proof. By Lemma 7.2.15 it is equivalent to prove that $X_{d-1,h}$ is irreducible. Consider the map $\pi_2 : \mathcal{I}_{d-1,h} \to G(h-1, n)$. By Theorem 7.2.11 the general fiber of π_2 is a linear subspace of $\mathbb{P}^N$ of dimension $\dim(\pi_2^{-1}(H)) = \binom{d+h-1}{d} - 1$ and π_2 is surjective on $G(h-1, n)$, so $X_{d-1,h}$ is irreducible. □

In the cases $d = 2$ and $d = 3, h = 2$ we have that $\dim(X_{1,h}) = \dim(\mathcal{S}ec_h(V_d^n))$, since $X_{1,h}$ is irreducible we get $\mathcal{S}ec_h(V_d^n) = X_{1,h}$. So if the first partial derivatives of a polynomial F span a linear space of dimension $h-1$ then F can be decomposed into a sum of h powers of linear forms.

Example 7.2.17. Consider a polynomial of degree three in three variables

$$F = a_0x^3 + a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xz^2 + a_6y^3 + a_7y^2z + a_8yz^2 + a_9z^3.$$

The variety $X_{1,2}$ is defined by

$$\mathrm{rk}\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \mathrm{rk}\begin{pmatrix} 3a_0 & 2a_1 & 2a_2 & a_3 & a_4 & a_5 \\ a_1 & 2a_3 & a_4 & 3a_6 & 2a_7 & a_8 \\ a_2 & a_4 & 2a_5 & a_7 & 2a_8 & 3a_9 \end{pmatrix} \leqslant 2.$$

Consider the projective space $\mathbb{P}^{17}$ of 3×6 matrix with homogeneous coordinates

$$X_{0,0}, ..., X_{0,5}, X_{1,0}, ..., X_{1,5}, X_{2,0}, ..., X_{2,5}.$$

The determinantal variety M_2 defined by

$$\mathrm{rk}\begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} & X_{0,3} & X_{0,4} & X_{0,5} \\ X_{1,0} & X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} & X_{1,5} \\ X_{2,0} & X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} & X_{2,5} \end{pmatrix} \leqslant 2$$

is irreducible of dimension $17-4=13$. The linear space

$$H := \begin{cases} 2X_{1,0}-X_{0,1}=0, \\ 2X_{2,0}-X_{0,2}=0, \\ 2X_{0,3}-X_{1,1}=0, \\ X_{0,4}-X_{1,2}=0, \\ 2X_{0,5}-X_{2,2}=0, \\ 2X_{2,3}-X_{1,4}=0, \\ 2X_{2,4}-X_{1,5}=0, \\ X_{0,4}-X_{2,1}=0. \end{cases}$$

cuts out on M_2 the variety $X_{1,2}$, which is irreducible of dimension $5 = \dim(\mathcal{S}ec(V_3^2))$.

Remark 7.2.18. Considering a polynomial $F \in k[x,y,z]_4$ and proceeding as in example 7.2.17 one gets $\dim(X_{1,2}) = 6$, so

$$\mathcal{S}ec_2(V_4^2) \subsetneqq X_{1,2}.$$

Proposition 7.2.19. *Let* $d = 2k$ *be an even integer such that* $\binom{n+k}{k} \geqslant N_{d-k}$*, where* $N_{d-k} = \binom{d-k+n}{n} - 1$*. The variety* $X_{k,N_{d-k}}$ *is an irreducible hypersurface of degree* $\binom{n+k}{k}$ *in* $\mathbb{P}^N$*.*

Proof. The map $\pi_2 : \mathcal{I}_{k,N_{d-k}} \to \mathbb{G}(N_{d-k}-1, N_{d-k}) \cong \mathbb{P}^{N_{d-k}}$ is dominant, so $\mathcal{I}_{k,N_{d-k}}$ and $X_{k,N_{d-k}}$ are irreducible. The assertion follows observing that $X_{k,N_{d-k}}$ is defined by the vanishing of the determinant of a $\binom{n+k}{k} \times \binom{n+k}{k}$ matrix. ☐

Let us look at some consequences of the previous proposition.

Example 7.2.20. Consider a polynomial

$$F = a_0x^4 + a_1x^3y + a_2x^3z + a_3x^2y^2 + a_4x^2yz + a_5x^2z^2 + a_6xy^3 + a_7xy^2z + a_8xyz^2 \\ + a_9xz^3 + a_{10}y^4 + a_{11}y^3z + a_{12}y^2z^2 + a_{13}yz^3 + a_{14}z^4.$$

The map $\pi_2 : \mathcal{I}_{2,4} \to \mathbb{G}(3,5)$ is dominant, so $X_{2,4}$ is irreducible. Let $Z_0, Z_1, Z_2, Z_3, Z_4, Z_5$ be homogeneous coordinates on $\mathbb{P}^5$ corresponding to $x^2, xy, xz, y^2, yz, z^2$ respectively. To compute the dimension of the general fiber of π_2 we can take the $3-plane$ $H = \{Z_0 = Z_3 = 0\}$ which intersect V_2^2 in a subscheme of dimension zero. Computing the second partial derivatives of F it turns out that

$$\pi_2^{-1}(H) = \{a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_{10} = a_{11} = a_{12} = 0\}.$$

So $\dim(\pi_2^{-1}(H)) = 14-11 = 3$ and $\dim(X_{2,4}) = 3+8 = 11$. Since $\dim(\mathcal{S}ec_4V_4^2) = 11$ we get

$$\mathcal{S}ec_4V_4^2 = X_{2,4}.$$

Consider now $\pi_2 : \mathcal{I}_{2,5} \to \mathbb{P}^5$. This map is dominant, so $X_{2,5}$ is irreducible. We have $\dim(\pi_2^{-1}(H)) = 14-6 = 8$, where $H = \{Z_0 = 0\}$. So $\dim(X_{2,5}) = 13$ and

$$\mathcal{S}ec_5V_4^2 = X_{2,5}$$

is an hypersurface of degree 6 in $\mathbb{P}^{14}$.
Consider now the case $d = 4, n = 3, h = 9$ and the second partial derivatives. The map

$\pi_2 : \mathcal{J}_{2,9} \to \mathbb{P}^9$ is dominant and $X_{2,9}$ is irreducible. The general fiber of π_2 has dimension 24. Then $\dim(X_{2,9}) = 24 + 9 = 33$ and

$$\mathcal{S}ec_9 V_4^3 = X_{2,9}$$

is an hypersurface of degree 10 in $\mathbb{P}^{34}$.
Finally in the case $d = 4, n = 4, h = 14$ as before one can verify that $X_{2,14}$ is irreducible of dimension 68, so

$$\mathcal{S}ec_{14} V_4^4 = X_{2,14}$$

is an hypersurface of degree 15 in $\mathbb{P}^{69}$.

Example 7.2.21. Consider now a polynomial $F \in k[x,y,z]_6$ and the partial derivative of order 3. For $h = 8,9$ the map π_2 is dominant, so $X_{3,8}$ and $X_{3,9}$ are irreducible. First let us take $h = 8$. Proceeding as before we get $\dim(\pi_2^{-1}(H)) = 27 - 19 = 8$ and $\dim(X_{3,8}) = 24$. So $\mathcal{S}ec_8 V_6^2 \subset X_{3,8}$ is a divisor.
In the case $h = 9$ we have $\dim(\pi_2^{-1}(H)) = 27 - 10 = 17$ and $\dim(X_{3,9}) = 17 + 9 = 26$. So

$$\mathcal{S}ec_9 V_6^2 = X_{3,9}$$

is an hypersurface of degree 10 in $\mathbb{P}^{27}$.

7.2.2 *The first secant variety of* V_d^n

We focus on the case $h = 2$. Without any assumptions on d and n we obtain set-theoretical equations for the first secant variety of V_d^n. In the proof we use all the time the equality

$$\sum_{k=0}^{n} \binom{d-1+k}{d-1} = \binom{d+n}{d},$$

which can be easily proved by induction on n. In [Kan] *V. Kanev*, adopting a different approach, proved that the same equations cut out the ideal of $\mathcal{S}ec_2(V_d^n)$.

Theorem 7.2.22. *If* $h = 2$ *for the first secant variety of* V_d^n *we have*

$$\mathcal{S}ec_2(V_d^n) = X_{2,d-2}$$

for any n *and* $d \geqslant 3$.

Proof. Consider the diagram

$$\begin{tikzcd}
 & \mathcal{J}_{2,d-2} = \{(F,H) \mid \in F^{l}_{l_0,\dots,l_n} \in H,\ \forall\, l_0 + \dots + l_n = d-2\} \subset \mathbb{P}^N \times \mathbb{G}(1,N_2) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
\mathbb{P}^N & & \mathbb{G}(1,N_2)
\end{tikzcd}$$

clearly $\mathcal{S}_2 V_2^n \subseteq \mathrm{Im}(\pi_2)$. Let $F \in k[x_0, \dots, x_n]_d$ be a polynomial whose partial derivatives of order $d-2$ lie on a line $H \subset \mathbb{P}^{N_2}$. The derivatives of order $d-3$ of F are cubic polynomials whose first partial derivatives are collinear. By 7.2.14 $X_{2,1} = X_{2,2} = \mathcal{S}ec_2 V_3^n$, so if we denote by G a partial derivative of order $d-3$ of F we get a decomposition $G = L_1^3 + L_2^3$. Then $G_{x_0}, \dots, G_{x_n}$ (which are partial derivatives of order $d-2$ of F) lie on the line $\langle L_1^2, L_2^2 \rangle$, and so the line containing the partial derivative of order $d-2$ of F is exactly the secant line to V_2^n given by $\langle L_1^2, L_2^2 \rangle$. This means that

$$\mathcal{S}_2 V_2^n = \mathrm{Im}(\pi_2).$$

Since the fibers of π_2 are linear spaces we conclude that $\mathcal{I}_{2,d-2}$ and $X_{2,d-2}$ are irreducible. We compute now the dimension of the fiber of π_2. We fix on $\mathbb{P}^{N_2}$ homogeneous coordinates $Z_0, ..., Z_{N_2}$ corresponding to the monomials in lexicographic order $x_0^2, x_0x_1, ..., x_n^2$, and consider the line $H = \{Z_0 = Z_1 = ... = Z_{N_2-2} = 0\}$.
First consider monomials containing x_0. Forcing the derivatives to lie in $\{Z_0 = 0\}$ we get $\binom{d-2+n}{n}$ conditions (the monomials containing x_0^2, whose number is equal to the number of degree $d-2$ monomials in $x_0, ..., x_n$). Imposing $\{Z_1 = 0\}$ we get $\binom{d-2+n-1}{n-1}$ conditions (the monomials containing x_0x_1, whose number is equal to the number of degree $d-2$ monomials in $x_1, ..., x_n$). Proceeding in this way when we force $\{Z_n = 0\}$ we get $\binom{d-2+n-n}{n-n} = 1$ condition (the monomials containing x_0x_n, whose number is equal to the number of degree $d-2$ monomials in x_n). Up to now we have

$$\sum_{k=0}^{n} \binom{d-2+k}{k} = \binom{d-1+n}{d-1}$$

conditions.
Consider now the monomials containing x_1. Forcing $\{Z_{n+1} = 0\}$ we get $\binom{d-2+n-1}{n-1}$ conditions (the monomials containing x_1^2, whose number is equal to the number of degree $d-2$ monomials in $x_1, ..., x_n$). Imposing $\{Z_{n+2} = 0\}$ we get $\binom{d-2+n-2}{n-2}$ conditions (the monomials containing x_1x_2, whose number is equal to the number of degree $d-2$ monomials in $x_2, ..., x_n$). Proceeding in this way we get

$$\sum_{k=0}^{n-1} \binom{d-2+k}{k} = \binom{d-1+n-1}{d-1}$$

conditions.
Proceeding in this way at the step x_{n-2} we have

$$\sum_{k=0}^{2} \binom{d-2+k}{k} = \binom{d-1+2}{d-1}$$

more conditions. At the step x_{n-1} we have only to force $\{Z_{N_2-2} = 0\}$, and we get $\binom{d-1}{1} = d-1$ conditions.
Summing up the fiber $\pi_2^{-1}(H)$ is a linear subspace of $\mathbb{P}^N$ defined by

$$\sum_{k=2}^{n} \binom{d-1+k}{d-1} + d - 1 = \sum_{k=0}^{n} \binom{d-1+k}{d-1} - 1 - d + d - 1 = \binom{d+n}{d} - 2.$$

So the fiber has dimension

$$\dim(\pi_2^{-1}(H)) = N - \binom{d+n}{d} + 2 = 1,$$

recalling that $N = \binom{d+n}{d} - 1$. Finally we look at the map $\pi_2 : \mathcal{I}_{2,d-2} \to \mathcal{S}_2 V_2^n$, since π_2 is dominant we have

$$\dim(X_{2,d-2}) = \dim(\mathcal{I}_{2,d-2}) = 2n + 1.$$

Since $\dim(\mathcal{S}ec_2 V_d^n) = 2n + 1$ the assertion follows. □

7.2.3 *The case* $n = 2, h = 4$

In the same spirit of Theorem 7.2.22 we obtain the following result.

Theorem 7.2.23. *If* $n = 2, h = 4$ *for the variety of 4-secant 3-planes of* V_d^2 *we have*

$$\mathcal{S}ec_4(V_d^2) = X_{4,\lfloor \frac{d}{2} \rfloor}$$

for any d *positive integer.*

Proof. The case $d = 4$ is the Example 7.2.20. Consider now the case $d = 5$. The map $\pi_2 : \mathcal{I}_{4,3} \to \mathbb{G}(3,5)$ is dominant, so $X_{4,3}$ and hence $X_{4,2}$ are irreducible. Let $F \in k[x,y,z]_5$ be a polynomial, looking at the proof of theorem 7.2.22 we get that forcing the partial derivatives of order 3 of F to lie in $\{Z_0 = Z_3 = 0\}$ gives

$$\binom{5-2+2}{2} + \binom{5-2+2}{2} - \sharp\{\text{monomials containing } x^2y^2\} = 20 - 3 = 17$$

conditions. Since $\dim(X_{4,2}) = \dim(X_{4,3}) = 20 - 17 + \dim(\mathbb{G}(3,5)) = 11$ we conclude

$$\mathcal{S}ec_4(V_5^2) = X_{4,2}.$$

Consider the case $d = 6$ and the partial derivative of order 3. If the 3-th derivatives of F lie in a 3-plane then the first partial derivative of F are degree 5 polynomials whose second partial derivatives lie in a 3-plane. By the same trick of Theorem 7.2.22 we prove that the 3-plane containing the 3-th partial derivative has to be 4-secant to V_3^2. So $X_{4,3}$ is irreducible, and as usual by counting dimension we get the equality

$$\mathcal{S}ec_4(V_6^2) = X_{4,3}.$$

Now we treat the general case by induction on d. Let $F \in k[x,y,z]_d$ be a polynomial whose $\lfloor \frac{d}{2} \rfloor$-th derivative lies in a 3-plane. Then the first partial derivative of F are polynomials of degree $d-1$ whose $\lfloor \frac{d-1}{2} \rfloor$-th derivatives lie in a 3-plane. So F_x, F_y, F_z can be decomposed as sums of four powers of linear forms. As before we conclude that the map $\pi_2 : \mathcal{I}_{4,\lfloor \frac{d}{2} \rfloor} \to \mathbb{G}(3, N_{d-\lfloor \frac{d}{2} \rfloor})$ is dominant, so $X_{4,\lfloor \frac{d}{2} \rfloor}$ is irreducible. We conclude, by combinatorial computations similar to the previous one, computing $\dim(X_{4,\lfloor \frac{d}{2} \rfloor}) = \dim(\mathcal{S}ec_4(V_d^2))$. ☐

Remark 7.2.24. In a completely analogous way one can show that $\mathcal{S}ec_5(V_d^2)$ is defined by size 6 minors of the matrix of partial derivatives of order $\lfloor \frac{d}{2} \rfloor$ for $d = 4$ and $d \geqslant 6$.

7.2.4 *Reconstructing decompositions*

First, we report part of a table in [LO] summarizing the known cases in which a secant of a Veronese variety coincides at least set theoretically with a catalecticant variety. Indeed in these cases the equations of catalecticants cut scheme theoretically the secant variety and in some cases even the ideal. We denote by $\mathcal{M}_l$ the matrix whose lines are the partial derivatives of order l of a homogeneous polynomial $F \in k[x_0, ..., x_n]_d$.

Secant	Catalecticant	Reference
$Sec_h V_2^n$	$h+1$ minors of $\mathcal{M}_1$	Classical
$Sec_h V_d^1$	$h+1$ minors of $\mathcal{M}_{d-h}$	Iarrobino – Kanev and Th 7.2.6
$Sec_2 V_d^n$	3 minors of $\mathcal{M}_{d-2}$	Kanev and Th 7.2.22
$Sec_4 V_d^2$	5 minors of $\mathcal{M}_{\lfloor \frac{d}{2} \rfloor}$	Schreier and Th 7.2.23
$Sec_5 V_d^2,\ d=4,\ d \geqslant 6$	6 minors of $\mathcal{M}_{\lfloor \frac{d}{2} \rfloor}$	Th 3.2.1 [BCS]
$Sec_6 V_d^2,\ d \geqslant 6$	7 minors of $\mathcal{M}_{\lfloor \frac{d}{2} \rfloor}$	Th 3.2.1 [CG]
$Sec_9 V_6^2$	determinant of $\mathcal{M}_3$	Ex 7.2.21

The following proposition gives conditions under which a simultaneous decomposition of the derivatives lifts to a decomposition of the polynomial and is very useful in reconstructing decompositions.

Proposition 7.2.25. *Let $F \in k[x_0, ..., x_n]_d$ be a homogeneous polynomial. Suppose that its partial derivatives admit a decomposition*

$$F_{x_0} = \sum_{i=1}^{h} \alpha_i^0 L_i^{d-1}, ..., F_{x_n} = \sum_{i=1}^{h} \alpha_i^n L_i^{d-1},$$

in h linear forms $L_i = A_i^0 x_0 + ... + A_i^n x_n$ such that $L_1^{d-2}, ..., L_h^{d-2}$ are independent in $k[x_0, ..., x_n]_{d-2}$. Then there are the following relations between the coefficients

$$\alpha_i^t A_i^s = \alpha_i^s A_i^t, \quad t, s = 0, ..., n; \quad i = 1, ..., h.$$

These relations force the decomposition of the partial derivatives to be of the following form

$$F_{x_0} = \sum_{i=1}^{h} \alpha_i^0 \lambda_i^{d-1} (\alpha_i^0 x_0 + ... + \alpha_i^n x_n)^{d-1}, ..., F_{x_n} = \sum_{i=1}^{h} \alpha_i^n \lambda_i^{d-1} (\alpha_i^0 x_0 + ... + \alpha_i^n x_n)^{d-1},$$

where $\lambda_i = \frac{A_i^0}{\alpha_i^0} = ... = \frac{A_i^n}{\alpha_i^n}$. Furthermore the decomposition lifts to a decomposition of the polynomial

$$F = \sum_{i=1}^{h} \frac{1}{\lambda_i} L_i^d.$$

Proof. The 1-form $F_{x_0} dx_0 + ... + F_{x_n} dx_n$ is exact on $\mathbb{P}^n$ so it is closed, then $F_{x_t x_s} = F_{x_s x_t}$ for any $t, s = 0, .., n$. Since $L_1^{d-2}, ..., L_h^{d-2}$ are independent these equalities forces $\alpha_i^t A_i^s = \alpha_i^s A_i^t$, $t, s = 0, ..., n$; $i = 1, ..., h$.

Then $A_i^1 = \alpha_i^1 \frac{A_i^0}{\alpha_i^0}, ..., A_i^n = \alpha_i^n \frac{A_i^n}{\alpha_i^n}$. Define $\lambda_i = \frac{A_i^0}{\alpha_i^0} = ... = \frac{A_i^n}{\alpha_i^n}$ for any $i = 1, ..., h$. Substituting in $L_i^{d-2} = (A_i^0 x_0 + ... + A_i^n x_n)^{d-2}$ we get

$$L_i = \lambda_i^{d-2} (\alpha_i^0 x_0 + ... + \alpha_i^n x_n)^{d-2}, \quad i = 1, ..., h.$$

Then the expressions for the partial derivatives become

$$F_{x_0} = \sum_{i=1}^{h} \alpha_i^0 \lambda_i^{d-1} (\alpha_i^0 x_0 + ... + \alpha_i^n x_n)^{d-1}, ..., F_{x_n} = \sum_{i=1}^{h} \alpha_i^n \lambda_i^{d-1} (\alpha_i^0 x_0 + ... + \alpha_i^n x_n)^{d-1}.$$

To lift the decomposition on F consider the Euler formula $F = \sum_{i=1}^{n} x_i F_{x_i}$. Substituting the above expressions for the partial derivatives and by straightforward computations we get $F = \sum_{i=1}^{h} \frac{1}{\lambda_i} L_i^d$. $\square$

Remark 7.2.26. Clearly Proposition 7.2.25 can be easily generalized replacing the first partial derivatives with derivatives of any order.

In the following we consider the case $h \leqslant n+1$ in order to make meaningful the constraints on the derivatives. To check whether a polynomial F admits a decomposition into a given number of factors and, if it is so, to compute the linear form, we implement the following algorithm:

Construction 7.2.27. The starting data is a homogeneous polynomial $F \in k[x_0, ..., x_n]_d$ and we look for a decomposition in h linear forms. We proceed with the following steps:

1. Compute the partial derivatives of F and let H_∂ be their linear span. Now we have three possibilities:
   - 1A The derivatives generated a linear span of dimension bigger than $h-1$. In this case the decomposition does not exist.
   - 1B $\dim(H_\partial) = h-1$ but $H_\partial \cap V_{d-1}^n$ contains less than h points. So the decomposition does not exist.
   - 1C $\dim(H_\partial) = h-1$ and $H_\partial \cap V_{d-1}^n$ contains more than h points. In this case we proceed.
2. Compute the intersection $X = H_\partial \cdot V_{d-1}^n$.
   - 2A If X does not span H_∂ the decomposition does not exist.
   - 2B If X span H_∂ choose h-independent points $L_1^{d-1}, ..., L_h^{d-1} \in X$. By Proposition 7.2.25 the linear forms $L_1, ..., L_h$ give a decomposition of F.

Example 7.2.28. The partial derivatives of the polynomial $F = x^3 + x^2z + xz^2 + z^3$ lie on the line $H = \{Z_1 = Z_3 = Z_4 = Z_0 - 2Z_2 + Z_5 = 0\}$. By Theorem 7.2.22 we know that F admits a decomposition as sum of two linear forms. To compute the intersection $H \cdot V_2^2$ we have to solve the following system

$$\begin{cases} Z_4^2 - 4Z_3Z_5 = 0, \\ Z_2Z_4 - 2Z_1Z_5 = 0, \\ 2Z_2Z_3 - Z_1Z_4 = 0, \\ Z_2^2 - 4Z_0Z_5 = 0, \\ Z_1Z_2 - 2Z_0Z_4 = 0, \\ Z_1^2 - 4Z_0Z_3 = 0, \\ Z_1 = Z_3 = Z_4 = 0, \\ Z_0 - 2Z_2 + Z_5 = 0. \end{cases}$$

We found that the decomposition of F is given by the linear forms $L_1 = (2+\sqrt{3})x + z$ and $L_2 = (2-\sqrt{3})x + z$.

Example 7.2.29. The partial derivatives of the polynomial

$$F = \frac{2}{3}x^3 + x^2z + xz^2 + \frac{2}{3}z^3 + x^2y + xy^2 + \frac{2}{3}y^3 + y^2z + yz^2$$

span a plane 3-secant to the Veronese surface V_2^2 at the points $(x+z)^2, (x+y)^2, (y+z)^2$. A priori this is not a meaningful condition. However proposition 7.2.25 ensures that the decomposition lifts and we have $F = \lambda_1(x+z)^3 + \lambda_2(x+y)^3 + \lambda_3(y+z)^3$.

SCRIPTS

In this appendix we report the scripts used in the work. Scripts 1, 3, 6 are realized with MacAulay2 [Mc2], Scripts 2, 4, 7 with Bertini [Be], finally Script 5 with MatLab.

Script 1. `Macaulay2, version1.3.1`

```
i1 :  P3 = QQ[X,Y,Z,W]
o1 = P3
o1 :  PolynomialRing
i2 :  P1 = QQ[s,t]
o2 = P1
o2 :  PolynomialRing
i3 :  TC = map(P1, P3, s^3, 3s^2t, 3st^2, t^3)
o3 = map(P1, P3, s^3, 3s^2t, 3st^2, t^3)
o3 :  RingMap P1 < P3
i4 :  ITC = kernelTC
o4 = ideal(Z^2-3YW, YZ-9XW, Y^2-3XZ)
o4 :  Idealof P3
i5 :  RTC = P3/ITC
o5 = RTC
o5 :  QuotientRing
i6 :  P2 = QQ[A,B,C]
o6 = P2
o6 :  PolynomialRing
i7 :  projmap = map(RTC, P2, Y-X, X+Z, W-X)
o7 = map(RTC, P2, -X+Y, X+Z, -X+W)
o7 :  RingMap RTC < P2
i8 :  I = kernelprojmap
o8 = ideal(14A^3+15A^2B+15AB^2-13B^3-18A^2C+45ABC-18B^2C+54AC^2)
o8 :  Ideal of P2
```

Script 2. `CONFIG`

```
END;
INPUT
homvariablegroup A,B,C;
function f1, f2, f3, f4;
f1 = 14A^3+15A^2B+15AB^2-13B^3-18A^2C+45ABC-18B^2C+54AC^2);
f2 = (42(A^2))+(30AB)+(45CB)-(36CA)+(15(B^2))+(54(C^2));
f3 = (15(A^2))+(30AB)+(45AC)-(39(B^2))-(36*B*C);
f4 = (45AB)+(108AC)-(18(A^2))-(18(B^2));
END;
```

Script 3. `Macaulay2, version 1.3.1`

```
i1 :  P2 = QQ[x,y,z]
o1 = P2
o1 :  PolynomialRing
i2 :  P9 = QQ[X_0,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,X_9]
o2 = P9
o2 :  PolynomialRing
i3 :  VerMap = map(P2,P9,x^3,3x^2y,3x^2z,6xyz,3xy^2,3xz^2,y^3,3y^2z,3yz^2,z^3)
o3 = map(P2,P9,x^3,3x^2y,3x^2z,6xyz,3xy^2,3xz^2,y^3,3y^2z,3yz^2,z^3)
```

```
o3 :  RingMap P2 <-- P9
i4 :  IVer = kernel VerMap
o4 :  Ideal of P9
i5 :  RVer = P9/IVer
o5 = RVer
o5 :  QuotientRing
i6 :  P3 = QQ[X,Y,Z,W]
o6 = P3
o6 :  PolynomialRing
i7 :  Projection = map(RVer,P3,"Equations of the Projection")
o7 = map(RVer,P3,"Equations of the Projection")
o7 :  RingMap RVer <-- P3
i8 :  IProjVer = kernel Projection
o8 :  Ideal of P3
```

Script 4. `CONFIG`

```
TRACKTOLBEFOREEG: 1e-8;
TRACKTOLDURINGEG: 1e-11;
FINALTOL: 1e-14;
MPTYPE: 1;
PRECISION: 128;
END;
INPUT
homvariablegroup X,Y,Z,W;
function f1, f2, f3, f4, f5;
f1 = F;
```

`f2 =` $\frac{\partial^6 F}{\partial X^6}$`;`

`f3 =` $\frac{\partial^6 F}{\partial Y^6}$`;`

`f4 =` $\frac{\partial^6 F}{\partial Z^6}$`;`

`f5 =` $\frac{\partial^6 F}{\partial W^6}$`;`

```
END;
```

Script 5. `P1 = input('Point 1:');`

```
⋮
P10 = input('Point 10:');
q = input('Precision:');
A = [P1;P2;P3;P4;P5;P6;P7;P8;P9;P10];
t = 1;
B = [];
for a=1:5,
for b=a+1:6,
for c=b+1:7,
for d=c+1:8,
for f=d+1:9,
for g=f+1:10,
M = [A(a,:);A(b,:);A(c,:);A(d,:);A(f,:);A(g,:)];
disp(t);
t = t+1;
v = [];
for a1 = 1:3,
```

```
for a2 = a1+1:4,
for a3 = a2+1:5,
for a4 = a3+1:6,
v = [v,det([M(a1,:);M(a2,:);M(a3,:);M(a4,:)])];
end; end; end; end;
if abs(v(1))<q,abs(v(2))<q,abs(v(3))<q,abs(v(4))<q,abs(v(5))<q,
abs(v(6))<q,abs(v(7))<q,abs(v(8))<q,abs(v(9))<q,abs(v(10))<q,
abs(v(11))<q,abs(v(12))<q,abs(v(13))<q,abs(v(14))<q,abs(v(15))<q,
B = [B M];
end; end; end; end; end; end; end;
[n,m] = size(B);
s = 1;
for r=1:4:m-3,
disp('Matrix'), disp(s),
s = s+1;
B(:,r:r+3),
end;
```

Script 6. `Macaulay2, version 1.3.1`

```
i1 :  P9 = QQ[X_0,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,X_9]
o1 = P9
o1 :  PolynomialRing
i2 :  MDer = matrix {{X_0,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,X_9},{3,2,2,2,1,1,1,1,1,1},
{1,2,1,1,3,2,2,1,1,1},{1,1,2,1,1,2,1,3,2,1},{1,1,1,2,1,1,2,1,2,3}}
o2 :  Matrix P9 <-- P9
i3 :  IDer = minors(5,MDer)
o3 :  Ideal of P9
i4 :  MQuad = matrix {{X_0,X_1/2,X_2/2,X_3/2},{X_1/2,X_4,X_5/2,X_6/2},{X_2/2,X_5/2,X_7,X_8/2},
{X_3/2,X_6/2,X_8/2,X_9}}
o4 :  Matrix P9 <-- P9
i5 :  IRTQuad = minors(3,MQuad)
o5 :  Ideal of P9
i6 :  X2 = variety IRTQuad
o6 = X2
o6 :  ProjectiveVariety
i7 :  DerSpace = variety IDer
o7 = DerSpace
o7 :  ProjectiveVariety
i8 :  IdInt = IDer+IRTQuad
o8 :  Ideal of P9
i9 :  Int = variety IdInt
o9 = Int
o9 :  ProjectiveVariety
i10 :  dim Int
o10 = 0
i11 :  degree Int
o11 = 10
```

Script 7. `CONFIG`

```
END;
INPUT
```

```
homvariablegroup X_0,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,X_9;
function f1,f2,f3,f4,f5,f6,f7,...,f22;
f1 = X_7-2X_8+X_9;
f2 = X_5-X_6-X_8+X_9;
f3 = X_4-2X_6+X_9;
f4 = X_2-X_3-X_8+X_9;
f5 = X_1-X_3-X_6+X_9;
f6 = X_0-2X_3+X_9;
f7 = ....;
⋮
f22 = ...;
END;
```

8

RANK OF MATRIX MULTIPLICATION

Let V, W be two complex vector spaces of dimension n and m. The contraction morphism

$$\begin{array}{ccc} V^* \otimes W & \to & \mathrm{Hom}(V, W) \\ T = \sum_{i,j} f_i \otimes w_j & \mapsto & L_T \end{array},$$

where $L_T(v) = \sum_{i,j} f_i(v) w_j$, defines an isomorphism between $V^* \otimes W$ and the space of linear maps from V to W.
Then, given three vector spaces A, B, C of dimension a, b and c, we can identify $A^* \otimes B$ with the space of linear maps $A \to B$, and $A^* \otimes B^* \otimes C$ with the space of bilinear maps $A \times B \to C$. Let $T : A^* \times B^* \to C$ be a bilinear map. Then T induces a linear map $A^* \otimes B^* \to C$ and may also be interpreted as:

- an element of $(A^* \otimes B^*)^* \otimes C = A \otimes B \otimes C$,
- a linear map $A^* \to B \otimes C$.

Segre varieties and their secant varieties

Let A, B and C be complex vector spaces. The three factor Segre map is defined as

$$\begin{array}{rcl} \sigma_{1,1,1} : \mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C) & \to & \mathbb{P}(A \otimes B \otimes C) \\ ([a], [b], [c]) & \mapsto & [a \otimes b \otimes c], \end{array}$$

where $[a]$ denotes the class in $\mathbb{P}(A)$ of the vector $a \in A$. The notation $\sigma_{1,1,1}$ is justified by the fact that the Segre map is induced by the line bundle $\mathcal{O}(1,1,1)$ on $\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$. The two factor Segre map

$$\sigma_{1,1} : \mathbb{P}(B) \times \mathbb{P}(C) \to \mathbb{P}(B \otimes C)$$

is defined in a similar way. The Segre varieties are defined as the images of the Segre maps: $\Sigma_{1,1,1} = \sigma_{1,1,1}(\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C))$, $\Sigma_{1,1} = \sigma_{1,1}(\mathbb{P}(B) \times \mathbb{P}(C))$. For each integer $r \geqslant 0$ we define the open secant variety and the secant variety of $\Sigma_{1,1,1}$ respectively as

$$\mathbb{S}ec_r(\Sigma_{1,1,1})^o = \bigcup_{x_1, \ldots, x_{r+1} \in \Sigma_{1,1,1}} \langle x_1, \ldots, x_{r+1} \rangle, \quad \mathbb{S}ec_r(\Sigma_{1,1,1}) = \overline{\mathbb{S}ec_r(\Sigma_{1,1,1})^o}.$$

In the above formulas $\langle x_1, \ldots, x_{r+1} \rangle$ denotes the linear space generated by the points x_i and $\mathbb{S}ec_r(\Sigma_{1,1,1})$ is the closure of $\mathbb{S}ec_r(\Sigma_{1,1,1})^o$ with respect to the Zariski topology. Let us notice that with the above definition $\mathbb{S}ec_0(\Sigma_{1,1,1}) = \Sigma_{1,1,1}$.

Rank and border rank of a bilinear map

The *rank* of a bilinear map $T : A^* \times B^* \to C$ is the smallest natural number $r := \mathrm{rk}(T) \in \mathbb{N}$ such that there exist $a_1, ..., a_r \in A$, $b_1, ..., b_r \in B$ and $c_1, ..., c_r \in C$ decomposing $T(\alpha, \beta)$ as

$$T(\alpha, \beta) = \sum_{i=1}^{r} a_i(\alpha) b_i(\beta) c_i$$

for any $\alpha \in A^*$ and $\beta \in B^*$. The number $\mathrm{rk}(T)$ has also two additional interpretations.

- Considering T as an element of $A \otimes B \otimes C$ the rank r is the smallest number of rank one tensors in $A \otimes B \otimes C$ needed to span a linear space containing the point T. Equivalently, $\mathrm{rk}(T)$ is the smallest number of points $t_1, ..., t_r \in \Sigma_{1,1,1}$ such that $[T] \in \langle t_1, ..., t_r \rangle$. In the language of secant varieties this means that $[T] \in \mathcal{S}ec_{r-1}(\Sigma_{1,1,1})^o$ but $[T] \notin \mathcal{S}ec_{r-2}(\Sigma_{1,1,1})^o$.
- Similarly, if we consider T as a linear map $A^* \to B \otimes C$ then $\mathrm{rk}(T)$ is the smallest number of rank one tensors in $B \otimes C$ need to span a linear space containing the linear space $T(A^*)$. As before we have a geometric counterpart. In fact $\mathrm{rk}(T)$ is the smallest number of points $t_1, ..., t_r \in \Sigma_{1,1}$ such that $\mathbb{P}(T(A^*)) \subseteq \langle t_1, ..., t_r \rangle$.

The *border rank* of a bilinear map $T : A^* \times B^* \to C$ is the smallest natural number $r := \underline{\mathrm{rk}}(T)$ such that T is the limit of bilinear maps of rank r but is not a limit of tensors of rank s for any $s < r$. There is a geometric interpretation also for this notion: T has border rank r if $[T] \in \mathcal{S}ec_{r-1}(\Sigma_{1,1,1})$ but $[T] \notin \mathcal{S}ec_{r-2}(\Sigma_{1,1,1})$. Clearly $\mathrm{rk}(T) \geqslant \underline{\mathrm{rk}}(T)$.

Matrix Multiplication

Now, let us consider a special tensor. Given three vector spaces $L = \mathbb{C}^l, M = \mathbb{C}^m$ and $N = \mathbb{C}^n$ we define $A = N \otimes L^*$, $B = L \otimes M^*$ and $C = N^* \otimes M$. We have a matrix multiplication map

$$M_{n,l,m} : A^* \times B^* \to C$$

As a tensor $M_{n,l,m} = \mathrm{Id}_N \otimes \mathrm{Id}_M \otimes \mathrm{Id}_L \in (N^* \otimes L) \otimes (L \otimes M^*) \otimes (N^* \otimes M) = A \otimes B \otimes C$, where $\mathrm{Id}_N \in N^* \otimes N$ is the identity map. If $n = l$ the choice of a linear map $\alpha^0 : N \to L$ of maximal rank allows us to identify $N \cong L$. Then the multiplication map $M_{n,n,m} \in (N \otimes N^*) \otimes (N \otimes M^*) \otimes (N^* \otimes M)$ induces a linear map $N^* \otimes N \to (N^* \otimes M) \otimes (N^* \otimes M)^*$ which is an inclusion of Lie algebras

$$M_A : \mathfrak{gl}(N) \to \mathfrak{gl}(B),$$

where $\mathfrak{gl}(N) \cong N^* \otimes N$ is the algebra of linear endomorphisms of N. In particular, the rank of the commutator $[M_A(\alpha^1), M_A(\alpha^2)]$ of $nm \times nm$ matrices is equal to m times the rank of the commutator $[\alpha^1, \alpha^2]$ of $n \times n$ matrices. This equality reflects a general philosophy, that is to translate expressions in commutators of $\mathfrak{gl}_{n^2}$ into expressions in commutators in $\mathfrak{gl}_n$.

Matrix Equalities

The following lemmas are classical in linear algebra. However, for completeness, we give a proof.

Lemma 8.0.30. *The determinant of a* 2×2 *block matrix is given by*

$$\det \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \det(X)\det(W - ZX^{-1}Y),$$

where X *is an invertible* $n \times n$ *matrix,* Y *is a* $n \times m$ *matrix,* Z *is a* $m \times n$ *matrix, and* W *is a* $m \times m$ *matrix.*

Proof. The statement follows from the equality

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} -X^{-1}Y & \mathrm{Id}_n \\ \mathrm{Id}_m & 0 \end{pmatrix} = \begin{pmatrix} 0 & X \\ W - ZX^{-1}Y & Z \end{pmatrix}.$$

□

Lemma 8.0.31. *Let* A *be an* $n \times n$ *invertible matrix and* U, V *any* $n \times m$ *matrices. Then*

$$\det_{n\times n}(A + UV^t) = \det_{n\times n}(A) \det_{m\times m}(\mathrm{Id} + V^t A^{-1} U),$$

where V^t *is the transpose of* V.

Proof. It follows from the equality

$$\begin{pmatrix} A & 0 \\ V^t & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \mathrm{Id} & -A^{-1}U \\ 0 & \mathrm{Id} + V^t A^{-1} U \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -V^t & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} A + UV^t & -U \\ 0 & \mathrm{Id} \end{pmatrix}.$$

□

8.1 LANDSBERG - OTTAVIANI EQUATIONS

In [LO] *J.M. Landsberg* and *G. Ottaviani* generalized Strassen's equations as introduced by *V. Strassen* in [S1]. We follow the exposition of [La1, Section 2].
Let $T \in A \otimes B \otimes C$ be a tensor, and assume $b = c$. Let us consider T as a linear map $A^* \to B \otimes C$, and assume that there exists $\alpha \in A^*$ such that $T(\alpha) : B^* \to C$ is of maximal rank b. Via $T(\alpha)$ we can identify $B \cong C$, and consider $T(A^*) \subseteq B^* \otimes B$ as a subspace of the space of linear endomorphisms of B.
In [S1] *Strassen* considered the case $a = 3$. Let $\alpha^0, \alpha^1, \alpha^2$ be a basis of A^*. Assume that $T(\alpha^0)$ has maximal rank and that $T(\alpha^1), T(\alpha^2)$ are diagonalizable, commuting endomorphisms. Then $T(\alpha^1), T(\alpha^2)$ are simultaneously diagonalizable and it is not difficult to prove that in this case $\mathrm{rk}(T) = b$. In general, $T(\alpha^1), T(\alpha^2)$ are not commuting. The idea of Strassen was to consider their commutator $[T(\alpha^1), T(\alpha^2)]$ to obtain results on the border rank of T. In fact, Strassen proved that, if $T(\alpha^0)$ is of maximal rank, then $\underline{\mathrm{rk}}(T) \geqslant b + \mathrm{rank}[T(\alpha^1), T(\alpha^2)]/2$ and $\underline{\mathrm{rk}}(T) = b$ if and only if $[T(\alpha^1), T(\alpha^2)] = 0$.
Now let us consider the case $a = 3, b = c$. Fix a basis a_0, a_1, a_2 of a A, and let a^0, a^1, a^2 be the dual basis of A^*. Choose bases of B and C, so that elements of $B \otimes C$ can be written as matrices. Then we can write $T = a_0 \otimes X_0 - a_1 \otimes X_1 + a_2 \otimes X_2$, where the X_i are $b \times b$ matrices. Consider $T \otimes \mathrm{Id}_A \in A \otimes B \otimes C \otimes A^* \otimes A = A^* \otimes B \otimes A \otimes A \otimes C$,

$$T \otimes \mathrm{Id}_A = (a_0 \otimes X_0 - a_1 \otimes X_1 + a_2 \otimes X_2) \otimes (a^0 \otimes a_0 + a^1 \otimes a_1 + a^2 \otimes a_2)$$

and its skew-symmetrization in the A factor $T_A^1 \in A^* \otimes B \otimes \bigwedge^2 A \otimes C$, given by

$$\begin{aligned} T_A^1 = \; & a^1 X_0 (a_0 \wedge a_1) + a^2 X_0 (a_0 \wedge a_2) - a^0 X_1 (a_1 \wedge a_0) - a^2 X_1 (a_1 \wedge a_2) + a^0 X_2 (a_2 \wedge a_0) + \\ & a^1 X_2 (a_2 \wedge a_1) \end{aligned}$$

where $a^i X_j (a_j \wedge a_i) := a^i \otimes X_j \otimes (a_j \wedge a_i)$. It can also be considered as a linear map

$$T_A^1 : A \otimes B^* \to \bigwedge^2 A \otimes C.$$

In the basis a_0, a_1, a_2 of A and $a_0 \wedge a_1, a_0 \wedge a_2, a_1 \wedge a_2$ of $\bigwedge^2 A$ the matrix of T_A^1 is the following

$$\mathrm{Mat}(T_A^1) = \begin{pmatrix} X_1 & -X_2 & 0 \\ X_0 & 0 & -X_2 \\ 0 & X_0 & -X_1 \end{pmatrix}$$

Assume X_0 is invertible and change bases such that it is the identity matrix. By Lemma 8.0.30, on the matrix obtained by reversing the order of the rows of $\mathrm{Mat}(T^1_A)$, with

$$X = \begin{pmatrix} 0 & X_0 \\ X_0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} -X_1 \\ -X_2 \end{pmatrix}, \; Z = \begin{pmatrix} X_1 & -X_2 \end{pmatrix}, \; W = 0$$

we get

$$\det(\mathrm{Mat}(T^1_A)) = \det(X_1X_2 - X_2X_1) = \det([X_1, X_2]).$$

Now we want to generalize this construction as done in [LO]. We consider the case $a = 2p+1$, $T \otimes \mathrm{Id}_{\bigwedge^p A} \in A \otimes B \otimes C \otimes \bigwedge^p A^* \otimes \bigwedge^p A = (\bigwedge^p A^* \otimes B) \otimes (\bigwedge^{p+1} A \otimes C)$, and its skew-symmetrization

$$T^p_A : \bigwedge^p A \otimes B^* \to \bigwedge^{p+1} A \otimes C.$$

Note that $\dim(\bigwedge^p A \otimes B^*) = \dim(\bigwedge^{p+1} A \otimes C) = \binom{2p+1}{p} b$. After choosing a basis $a_0, \ldots, a_{2p}$ of A we can write $T = \sum_{i=0}^{2p} (-1)^i a_i \otimes X_i$. The matrix of T^p_A with respect the basis $a_0 \wedge \ldots \wedge a_{p-1}, \ldots, a_{p+1} \wedge \ldots \wedge a_{2p}$ of $\bigwedge^p A$, and $a_0 \wedge \ldots \wedge a_p, \ldots, a_p \wedge \ldots \wedge a_{2p}$ of $\bigwedge^{p+1} A$ is of the form

$$\mathrm{Mat}(T^p_A) = \begin{pmatrix} Q & 0 \\ R & \overline{Q} \end{pmatrix} \tag{8.1.1}$$

where the matrix is blocked $(\binom{2p}{p+1} b, \binom{2p}{p} b) \times (\binom{2p}{p+1} b, \binom{2p}{p} b)$, the lower left block is given by

$$R = \begin{pmatrix} X_0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & X_0 \end{pmatrix}$$

and Q is a matrix having blocks $X_1, \ldots, X_{2p}$ and zero, while $\overline{Q}$ is the block transpose of Q except that if an index is even, the block is multiplied by -1. We derive below the expression (8.1.1) in the case $p = 2$; the general case can be developed similarly, see [La1, Section 3].

Example 8.1.1. Consider the case $p = 2$. The matrix of T^2_A is

$$\begin{aligned} T^2_A = \; & (a^1 \wedge a^2)X_0(a_0 \wedge a_1 \wedge a_2) + (a^1 \wedge a^3)X_0(a_0 \wedge a_1 \wedge a_3) + (a^1 \wedge a^4)X_0(a_0 \wedge a_1 \wedge a_4) + \\ & (a^2 \wedge a^3)X_0(a_0 \wedge a_2 \wedge a_3) + (a^2 \wedge a^4)X_0(a_0 \wedge a_2 \wedge a_4) + (a^3 \wedge a^4)X_0(a_0 \wedge a_3 \wedge a_4) - \\ & (a^0 \wedge a^2)X_1(a_1 \wedge a_0 \wedge a_2) - (a^0 \wedge a^3)X_1(a_1 \wedge a_0 \wedge a_3) - (a^0 \wedge a^4)X_1(a_1 \wedge a_0 \wedge a_4) - \\ & (a^2 \wedge a^3)X_1(a_1 \wedge a_2 \wedge a_3) - (a^2 \wedge a^4)X_1(a_1 \wedge a_2 \wedge a_4) - (a^3 \wedge a^4)X_1(a_1 \wedge a_3 \wedge a_4) + \\ & (a^0 \wedge a^1)X_2(a_2 \wedge a_0 \wedge a_1) + (a^0 \wedge a^3)X_2(a_2 \wedge a_0 \wedge a_3) + (a^0 \wedge a^4)X_2(a_2 \wedge a_0 \wedge a_4) + \\ & (a^1 \wedge a^3)X_2(a_2 \wedge a_1 \wedge a_3) + (a^1 \wedge a^4)X_2(a_2 \wedge a_1 \wedge a_4) + (a^3 \wedge a^4)X_2(a_2 \wedge a_3 \wedge a_4) - \\ & (a^0 \wedge a^1)X_3(a_3 \wedge a_0 \wedge a_1) - (a^0 \wedge a^2)X_3(a_3 \wedge a_0 \wedge a_2) - (a^0 \wedge a^4)X_3(a_3 \wedge a_0 \wedge a_4) - \\ & (a^1 \wedge a^2)X_3(a_3 \wedge a_1 \wedge a_2) - (a^1 \wedge a^4)X_3(a_3 \wedge a_1 \wedge a_4) - (a^2 \wedge a^4)X_3(a_3 \wedge a_2 \wedge a_4) + \\ & (a^0 \wedge a^1)X_4(a_4 \wedge a_0 \wedge a_1) + (a^0 \wedge a^2)X_4(a_4 \wedge a_0 \wedge a_2) + (a^0 \wedge a^3)X_4(a_4 \wedge a_0 \wedge a_3) + \\ & (a^1 \wedge a^4)X_4(a_4 \wedge a_1 \wedge a_2) + (a^1 \wedge a^3)X_4(a_4 \wedge a_1 \wedge a_3) + (a^2 \wedge a^3)X_4(a_4 \wedge a_2 \wedge a_3) \end{aligned}$$

The matrix of T_A^2 is

$$\mathrm{Mat}(T_A^2) = \begin{pmatrix} X_2 & -X_3 & X_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_1 & 0 & 0 & -X_3 & X_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_1 & 0 & -X_2 & 0 & X_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_1 & 0 & -X_2 & X_3 & 0 & 0 & 0 & 0 \\ X_0 & 0 & 0 & 0 & 0 & 0 & -X_3 & X_4 & 0 & 0 \\ 0 & X_0 & 0 & 0 & 0 & 0 & -X_2 & 0 & X_4 & 0 \\ 0 & 0 & X_0 & 0 & 0 & 0 & 0 & -X_2 & X_3 & 0 \\ 0 & 0 & 0 & X_0 & 0 & 0 & -X_1 & 0 & 0 & X_4 \\ 0 & 0 & 0 & 0 & X_0 & 0 & 0 & -X_1 & 0 & X_3 \\ 0 & 0 & 0 & 0 & 0 & X_0 & 0 & 0 & -X_1 & X_2 \end{pmatrix}$$

If X_0 is the identity by Lemma 8.0.30 on $R = \mathrm{Id}, Q$ and $\overline{Q}$ the determinant of $\mathrm{Mat}(T_A^p)$ is equal to the determinant of

$$\begin{pmatrix} 0 & [X_1, X_2] & [X_1, X_3] & [X_1, X_4] \\ -[X_1, X_2] & 0 & [X_2, X_3] & [X_2, X_4] \\ -[X_1, X_3] & -[X_2, X_3] & 0 & [X_3, X_4] \\ -[X_1, X_4] & -[X_2, X_4] & -[X_3, X_4] & 0 \end{pmatrix}$$

In general the determinant of $\mathrm{Mat}(T_A^p)$ is equal to the determinant of the $2pb \times 2pb$ matrix of commutators

$$\begin{pmatrix} 0 & X_{1,2} & X_{1,3} & X_{1,4} & \cdots & X_{1,2p-1} & X_{1,2p} \\ -X_{1,2} & 0 & X_{2,3} & X_{2,4} & \cdots & X_{2,2p-1} & X_{2,2p} \\ -X_{1,3} & -X_{2,3} & 0 & X_{3,4} & \cdots & X_{3,2p-1} & X_{3,2p} \\ -X_{1,4} & -X_{2,4} & -X_{3,4} & 0 & \cdots & X_{4,2p-1} & X_{4,2p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -X_{1,2p-1} & -X_{2,2p-1} & -X_{3,2p-1} & -X_{4,2p-1} & \cdots & 0 & X_{2p-1,2p} \\ -X_{1,2p} & -X_{2,2p} & -X_{3,2p} & -X_{4,2p} & \cdots & -X_{2p-1,2p} & 0 \end{pmatrix}$$

where $X_{i,j}$ denotes the commutator matrix $[X_i, X_j] = X_i X_j - X_j X_i$.

8.2 KEY LEMMA

We use the same notation of [La1] throughout the text.

Lemma 8.2.1. *[La2, Lemma 11.5.0.2] Let* V *be a* n*-dimensional vector space and let* $P \in S^d V^* \setminus \{0\}$ *be a polynomial of degree* $d \leqslant n-1$ *on* V*. For any basis* $\{v_1, ..., v_n\}$ *of* V *there exists a subset* $\{v_{i_1}, ..., v_{i_s}\}$ *of cardinality* $s \leqslant d$ *such that* $P_{|\langle v_{i_1}, ..., v_{i_s} \rangle}$ *is not identically zero.*

Proof. Let $x = \sum_{i=1}^n x_i v_i$ be an element of U and consider $P(x)$ as a polynomial in $x_1, ..., x_n$. For instance take the first non-zero monomial appearing in $P(x)$. Since it can involve at most d of the x_i's the polynomial P restricted to the span of the corresponding v_i's is not identically zero. ☐

Lemma 8.2.1 says, for instance, that a quadric surface in $\mathbb{P}^3$ can not contain six lines whose pairwise intersections span $\mathbb{P}^3$. Note that as stated Lemma 8.2.1 is sharp in the sense that under the same hypothesis the bound $s \leqslant d$ can not be improved. For example the polynomial $P(x, y, z, w) = xy$ vanishes on the four points $[1 : 0 : 0 : 0], ..., [0 : 0 : 0 : 1] \in \mathbb{P}^3$.

Lemma 8.2.2. *Let $A = N^* \otimes L$, where $l = n$. Given any basis of A, there exists a subset of at least $n^2 - (2p+3)n$ basis vectors, and elements $\alpha^0, \alpha^1, \dots, \alpha^{2p}$ of A^*, such that*

- *α^0 is of maximal rank, and thus may be used to identify $L \simeq N$ and A as a space of endomorphisms. (I.e. in bases α^0 is the identity matrix.)*
- *Choosing a basis of L, so the α^j become $n \times n$ matrices, the size $2pn$ block matrix whose (i, j)-th block is $[\alpha^i, \alpha^j]$ has non-zero determinant, and*
- *The subset of $n^2 - (2p+3)n$ basis vectors annihilate $\alpha^0, \alpha^1, \dots, \alpha^{2p}$.*

Proof. Let $\mathcal{B}$ be a basis of A, and consider the polynomial $P_0 = \det_n$. By Lemma 8.2.1 we get a subset S_0 of at most n elements of $\mathcal{B}$ and $\alpha^0 \in S_0$ with $\det_n(\alpha^0) \neq 0$. Now, via the isomorphism $\alpha^0 : L \to N$ we are allowed to identify $A = \mathfrak{gl}(L)$ as an algebra with identity element α^0. So, from now on, we work with $\mathfrak{sl}(L) = \mathfrak{gl}(L)/\langle \alpha^0 \rangle$ instead of $\mathfrak{gl}(L)$.
Following the proof of [La1, Lemma 4.3], let $v_{1,0}, ..., v_{2p,0} \in \mathfrak{sl}(L)$ be linearly independent and not equal to any of the given basis vectors, and let us work locally on an affine open neighborhood $\mathbb{V} \subset G(2p, \mathfrak{sl}(L))$ of $E_0 = \langle v_{1,0}, ..., v_{2p,0} \rangle$. We extend $v_{1,0}, ..., v_{2p,0}$ to a basis $v_{1,0}, ..., v_{2p,0}, w_1, ..., w_{n^2-2p-1}$ of $\mathfrak{sl}(L)$, and take local coordinates (f^μ_s) with $1 \leqslant s \leqslant 2p$, $1 \leqslant \mu \leqslant n^2 - 2p - 1$, on V, so that $v_s = v_{s,0} + \sum_{\mu=1}^{n^2-2p-1} f^\mu_s w_\mu$.
We denote $v_{i,j} = [v_i, v_j]$ and let us define

$$A_{i,i+1} = \begin{pmatrix} 0 & v_{i,i+1} \\ -v_{i,i+1} & 0 \end{pmatrix}$$

for $i = 1, \dots, 2p$ and let A be the following diagonal block matrix

$$A = \operatorname{diag}(A_{1,2}, A_{3,4}, \dots, A_{2p-3,2p-2}, \mathrm{Id}_{2n \times 2n})$$

which is a squared matrix of order $4pn$. Consider the $4pn \times 4pn$ matrix

$$M = \begin{pmatrix} 0 & v_{1,2} & v_{1,3} & v_{1,4} & \cdots & v_{1,2p-1} & v_{1,2p} \\ -v_{1,2} & 0 & v_{2,3} & v_{2,4} & \cdots & v_{2,2p-1} & v_{2,2p} \\ -v_{1,3} & -v_{2,3} & 0 & v_{3,4} & \cdots & v_{3,2p-1} & v_{3,2p} \\ -v_{1,4} & -v_{2,4} & -v_{3,4} & 0 & \cdots & v_{4,2p-1} & v_{4,2p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -v_{1,2p-1} & -v_{2,2p-1} & -v_{3,2p-1} & -v_{4,2p-1} & \cdots & 0 & v_{2p-1,2p} \\ -v_{1,2p} & -v_{2,2p} & -v_{3,2p} & -v_{4,2p} & \cdots & -v_{2p-1,2p} & 0 \end{pmatrix}$$

The polynomial $\det_{4pn\times 4pn}(M)$ is not identically zero on $G(2p,\mathfrak{sl}(L))$, so it is not identically zero on $\mathbb{V}$. Furthermore we can write $M = A + U\mathrm{Id}_{4pn\times 4pn}$, where

$$U = \begin{pmatrix} 0 & 0 & v_{1,3} & v_{1,4} & \dots & v_{1,2p-1} & v_{1,2p} \\ 0 & 0 & v_{2,3} & v_{2,4} & \dots & v_{2,2p-1} & v_{2,2p} \\ -v_{1,3} & -v_{2,3} & 0 & 0 & \dots & v_{3,2p-1} & v_{3,2p} \\ -v_{1,4} & -v_{2,4} & 0 & 0 & \dots & v_{4,2p-1} & v_{4,2p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -v_{1,2p-1} & -v_{2,2p-1} & -v_{3,2p-1} & -v_{4,2p-1} & \dots & -\mathrm{Id}_{n\times n} & v_{2p-1,2p} \\ -v_{1,2p} & -v_{2,2p} & -v_{3,2p} & -v_{4,2p} & \dots & -v_{2p-1,2p} & -\mathrm{Id}_{n\times n} \end{pmatrix}$$

By Lemma 8.0.31 we have

$$\det(M) = \det(A)\det(\mathrm{Id} + A^{-1}U) = \det([v_1,v_2])^2 \dots \det([v_{2p-3},v_{2p-2}])^2 \det(\mathrm{Id} + A^{-1}U).$$

The entries of the $n \times n$ matrices $[v_k, v_{k+1}]$ are quadratic in the f_s^{μ}'s, so the polynomials $\det([v_k, v_{k+1}])$ have degree $2n$, and

$$P_1 = \det([v_1,v_2])^2 \dots \det([v_{2p-3},v_{2p-2}])^2 = (\det([v_1,v_2]) \dots \det([v_{2p-3},v_{2p-2}]))^2$$

is a polynomial of degree $4n(p-1)$. Since P_1 is a square, we can consider the polynomial $\widetilde{P}_1 = \det([v_1,v_2]) \dots \det([v_{2p-3},v_{2p-2}])$ which has degree $2n(p-1)$. Applying Lemma 8.2.1 to $\widetilde{P}_1$ we find a subset S_1 of at most $2n(p-1)$ elements of our basis such that $\widetilde{P}_1$, and hence P_1, is not identically zero on $\langle S_1 \rangle$.

Now, let us fix some particular value of the coordinates f_s^{μ} such that on the corresponding matrices $\overline{v}_1, \dots, \overline{v}_{2p-2}$ the matrix A is invertible. For these values the expression $\det(\mathrm{Id} + A^{-1}U)$ makes sense. Let us consider the matrix

$$\mathrm{Id} + A^{-1}U = \begin{pmatrix} \mathrm{Id} & 0 & -v_{1,2}^{-1}v_{2,3} & -v_{1,2}^{-1}v_{2,4} & \dots & -v_{1,2}^{-1}v_{2,2p-1} & -v_{1,2}^{-1}v_{2,2p} \\ 0 & \mathrm{Id} & v_{1,2}^{-1}v_{1,3} & v_{1,2}^{-1}v_{1,4} & \dots & v_{1,2}^{-1}v_{1,2p-1} & v_{1,2}^{-1}v_{1,2p} \\ v_{3,4}^{-1}v_{1,4} & v_{3,4}^{-1}v_{2,4} & \mathrm{Id} & 0 & \dots & -v_{3,4}^{-1}v_{4,2p-1} & -v_{3,4}^{-1}v_{4,2p} \\ -v_{3,4}^{-1}v_{1,3} & -v_{3,4}^{-1}v_{2,3} & 0 & \mathrm{Id} & \dots & v_{3,4}^{-1}v_{3,2p-1} & v_{3,4}^{-1}v_{3,2p} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -v_{1,2p-1} & -v_{2,2p-1} & -v_{3,2p-1} & -v_{4,2p-1} & \dots & 0 & v_{2p-1,2p} \\ -v_{1,2p} & -v_{2,2p} & -v_{3,2p} & -v_{4,2p} & \dots & -v_{2p-1,2p} & 0 \end{pmatrix}$$

By Lemma 8.0.30 on $\mathrm{Id} + A^{-1}U$ with

$$X = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix},\ Y = \begin{pmatrix} -v_{1,2}^{-1}v_{2,3} & -v_{1,2}^{-1}v_{2,4} & \dots & -v_{1,2}^{-1}v_{2,2p-1} & -v_{1,2}^{-1}v_{2,2p} \\ v_{1,2}^{-1}v_{1,3} & v_{1,2}^{-1}v_{1,4} & \dots & v_{1,2}^{-1}v_{1,2p-1} & v_{1,2}^{-1}v_{1,2p} \end{pmatrix},$$

$$Z = \begin{pmatrix} v_{3,4}^{-1}v_{1,4} & v_{3,4}^{-1}v_{2,4} \\ -v_{3,4}^{-1}v_{1,3} & -v_{3,4}^{-1}v_{2,3} \\ \vdots & \vdots \\ -v_{1,2p-1} & -v_{2,2p-1} \\ -v_{1,2p} & -v_{2,2p} \end{pmatrix},\ W = \begin{pmatrix} \mathrm{Id} & 0 & \dots & -v_{3,4}^{-1}v_{4,2p-1} & -v_{3,4}^{-1}v_{4,2p} \\ 0 & \mathrm{Id} & \dots & v_{3,4}^{-1}v_{3,2p-1} & v_{3,4}^{-1}v_{3,2p} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -v_{3,2p-1} & -v_{4,2p-1} & \dots & 0 & v_{2p-1,2p} \\ -v_{3,2p} & -v_{4,2p} & \dots & -v_{2p-1,2p} & 0 \end{pmatrix}$$

we get $\det(\mathrm{Id} + A^{-1}U) = \det(W - ZY)$. Note that the coordinates f^{μ}_{s} appear in the terms indexed by $2p-1$ and $2p$, while all the other terms are constant once we fixed $\bar{v}_1, ..., \bar{v}_{2p-2}$. Then $P_2 = \det(W - ZY)$ is a polynomial of degree $4n$. By Lemma 8.2.1 we find a subset S_2 of at most $4n$ elements of the basis $\mathcal{B}$ such that P_2 is not identically zero on $\langle S_2 \rangle$.
Summing up we found a subset S of at most $n + 2n(p-1) + 4n = (2p+3)n$ elements of $\mathcal{B}$ such that $\det(M)$ is not identically zero on $\langle S \rangle$. ☐

Remark 8.2.3. In [La1, Lemma 4.3] the author proved the analogous statement for $n^2 - (4p + 1)n$.

We are ready to prove our main Theorem following the proof of [La1, Theorem 1.2].

Theorem 8.2.4. *Let* $p \leqslant \frac{n}{2}$ *be a natural number. Then*

$$\mathrm{rk}(M_{n,n,m}) \geqslant (1 + \frac{p}{p+1})nm + n^2 - (2p+3)n. \tag{8.2.1}$$

For example, when $\sqrt{\frac{n}{2}} \in \mathbb{Z}$*, taking* $p = \sqrt{\frac{n}{2}} - 1$*, we get*

$$\mathrm{rk}(M_{n,n,m}) \geqslant 2nm + n^2 - 2\sqrt{2}nm^{\frac{1}{2}} - n.$$

When $n = m$ *we obtain*

$$\mathrm{rk}(M_{n,n,n}) \geqslant (3 - \frac{1}{p+1})n^2 - (2p+3)n. \tag{8.2.2}$$

This bound is maximized when $p = \lceil \sqrt{\frac{n}{2}} - 1 \rceil$ *or* $p = \lfloor \sqrt{\frac{n}{2}} - 1 \rfloor$*, hence when* $\sqrt{\frac{n}{2}} \in \mathbb{Z}$ *we have*

$$\mathrm{rk}(M_{n,n,n}) \geqslant 3n^2 - 2\sqrt{2}n^{\frac{3}{2}} - n.$$

In general we have the following bound

$$\mathrm{rk}(M_{n,n,n}) \geqslant 3n^2 - 2\sqrt{2}n^{\frac{3}{2}} - 3n. \tag{8.2.3}$$

Proof. Let φ be a decomposition of the matrix multiplication tensor $M_{n,n,m}$ as sum of $r = \mathrm{rk}(M_{n,n,m})$ rank one tensors. Recall that the left kernel of a bilinear map $f : V \times U \to W$ is defined as $\mathrm{Lker}(f) = \{v \in V \mid f(v, u) = 0 \, \forall \, u \in U\}$. Since $\mathrm{Lker}(M_{n,n,m}) = 0$, that is for any $\alpha \in A^* \setminus \{0\}$, there exists $\beta \in B^*$ such that $M_{n,n,m}(\alpha, \beta) \neq 0$ we can write $\varphi = \varphi_1 + \varphi_2$ with $\mathrm{rk}(\varphi_1) = n^2$, $\mathrm{rk}(\varphi_2) = r - n^2$ and $\mathrm{Lker}(\varphi_1) = 0$.
The n^2 elements of A^* appearing in φ_1 form a basis of A^*. By Lemma 8.2.2 there exists a subset of $n^2 - (2p+3)n$ of them annihilating a maximal rank element α^0 and some $\alpha^1, ..., \alpha^{2p}$ such that, choosing bases, the determinant of the matrix $([\alpha^i, \alpha^j])$ is non-zero.
Let ψ_1 be the sum of all monomials in φ_1 whose terms in A^* annihilate $\alpha^0, ..., \alpha^{2p}$. By Lemma 8.2.2 there are at least $n^2 - (2p+3)n$ of them. Then $\mathrm{rk}(\psi_1) \geqslant n^2 - (2p+3)n$. Furthermore consider $\psi_2 = \varphi_1 - \psi_1 + \varphi_2$ so that $\varphi = \psi_1 + \psi_2$ and the terms appearing in ψ_2 does not annihilate $\alpha^0, ..., \alpha^{2p}$.
Let $A' = \langle \alpha^0, ..., \alpha^{2p} \rangle \subseteq A^*$. Again by Lemma 8.2.2 the determinant of the linear map $M_{n,n,m|A' \otimes B^* \otimes C^*} : \bigwedge^p A' \otimes B^* \to \bigwedge^{p+1} A' \otimes C$ is non-zero. Then $\underline{\mathrm{rk}}(\varphi_2) \geqslant nm\frac{2p+1}{p+1} = \dim(\bigwedge^p A' \otimes B^*)$. We conclude that

$$\mathrm{rk}(\varphi) = \mathrm{rk}(\varphi_1) + \mathrm{rk}(\varphi_2) \geqslant n^2 - (2p+3)n + nm\frac{2p+1}{p+1} = (1 + \frac{p}{p+1})nm + n^2 - (2p+3)n.$$

This concludes the proof of (8.2.1).
To prove the other assertions, let us consider the function $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}$ defined by $f(p) =$

$(3 - \frac{1}{p+1})n^2 - (2p+3)n$. The first derivative is $f'(p) = \frac{1}{(p+1)^2}n^2 - 2n$, which vanishes in $p = \sqrt{\frac{n}{2}} - 1$. Moreover $f''(p) = -\frac{2}{(p+1)^3}n^2 < 0$, hence $p = \sqrt{\frac{n}{2}} - 1$ is the maximum of f.
Then the bound (8.2.2) is maximized for $p = \lceil \sqrt{\frac{n}{2}} - 1 \rceil$ or $p = \lfloor \sqrt{\frac{n}{2}} - 1 \rfloor$, depending on the value of n.
If $(\sqrt{\frac{n}{2}} - 1) - \lfloor \sqrt{\frac{n}{2}} - 1 \rfloor \geqslant \frac{1}{2}$ we may consider $p = \lceil \sqrt{\frac{n}{2}} - 1 \rceil$. In this case $\sqrt{\frac{n}{2}} - 1 \leqslant p \leqslant \sqrt{\frac{n}{2}} - \frac{1}{2}$, and we get $f\big(\lceil \sqrt{\frac{n}{2}} - 1 \rceil\big) \geqslant \lceil f \rceil(n) := 3n^2 - 2\sqrt{2}n^{\frac{3}{2}} - 2n$.
If $(\sqrt{\frac{n}{2}} - 1) - \lfloor \sqrt{\frac{n}{2}} - 1 \rfloor < \frac{1}{2}$ we consider $p = \lfloor \sqrt{\frac{n}{2}} - 1 \rfloor$. Then $\sqrt{\frac{n}{2}} - \frac{3}{2} \leqslant p \leqslant \sqrt{\frac{n}{2}} - 1$, and we have $f\big(\lfloor \sqrt{\frac{n}{2}} - 1 \rfloor\big) \geqslant \lfloor f \rfloor(n) := (3 - \frac{2\sqrt{2}}{2n - \sqrt{2}})n^2 - \sqrt{2}n^{\frac{3}{2}} - n$.
Finally to prove (8.2.3) it is enough to observe that both $\lceil f \rceil(n)$ and $\lfloor f \rfloor(n)$ are greater than $3n^2 - 2\sqrt{2}n^{\frac{3}{2}} - 3n$. ☐

The bound (8.2.3) improves Bläser's one, $\frac{5}{2}n^2 - 3n$, for $n \geqslant 32$. Nevertheless, when $p = 2$, the bound in (8.2.2) becomes $\frac{8}{3}n^2 - 7n$, which improves Bläser's one for every $n \geqslant 24$. Compared with Landsberg's bound $3n^2 - 4n^{\frac{3}{2}} - n$, our bound (8.2.3) is better for $n \geqslant 3$.

ACKNOWLEDGMENTS

First of all I thank my parents *Aldo* and *Marina*. Then *Valentina Pastorello*, *Giulio Battaglia*, *Davide Mangolini*, *Alessio Lo Giudice*, *Paolo Bailo* and *Serena Dipierro*. A special thank to my advisor *Massimiliano Mella*, who taught me maths and what a mathematician should be. And above all, who in the last years has been able to point me in the right direction whenever I felt lost. Finally I thank all the mathematicians that in the last years taught me a lot of mathematics and shared with me a lot of frustration and some success. I will try to list them: *Michele Bolognesi*, *Ada Boralevi*, *Ugo Bruzzo*, *Alessandro Chiodo*, *Philippe Ellia*, *Barbara Fantechi*, *Noah Giansiracusa*, *Paltin Ionescu*, *Andrew Kresch*, *Joseph Landsberg*, *Stefano Maggiolo*, *Simone Marchesi*, *Nicola Pagani*, *Rahul Pandharipande*, *Emanuele Raviolo* and *José Carlos Sierra*.

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