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Moduli spaces and geometrical aspects of two–dimensional conformal field theories.

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"Ma gli elettròlogi non ne vollero sapere d'una simile ipotesi, e sfoderarono delle equazioni differenziali: che pervennero anche ad integrare, con quale gioia del cav. Bertoloni si può presumere."

C. E. Gadda, La cognizione del dolore, 1963

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Introduction.

In the last few years, a remarkable attention has grown around geometrical and topological methods in quantum field theory. This was due to the great interest of string and superstring theory as the ultimate unifying theory of matter and forces [GSW], and because of some break–through in the study of two dimensional critical phenomena that was achieved by means of field–theoretical methods [BPZ].

The mathematical backgrounds for such improvements can be readily identified in complex geometry and in the theory of infinite dimensional Lie algebras and super–algebras.

In this thesis we will apply algebro–geometrical tools to the analysis of some open problems in this field. In particular, we will be mainly interested in the study of moduli spaces of algebraic objects that are naturally attached to 2–dimensional field theories. This follows in some sense the modular geometry programme [FS] which, roughly speaking, proposes to get all the relevant informations about 2–dimensional conformal field theory (i.e. central charges for the energy–momentum tensor and anomalous dimensions for primary fields) from suitable vector bundles over the moduli space of curves of generic genus.

Such a programme has been fully motivated at the physical level. Still, its mathematical foundations need further efforts. Our main goal is indeed to work out as far as possible the mathematical methods which may be relevant in two–dimensional conformal and superconformal quantum field theory. Although the examples we will dwell on might seem quite unrelated, they have a common root, namely the investigation of the properties of non–abelian sheaves on algebraic curves. In fact, the real aim of this thesis is to extend to the non–abelian realm results which are well known for classical domains such as curves and line bundles. Non–abelian structures are plugged into the game in two different aspects. The first is to consider sheaves of

$\mathbb{Z}_2$–graded algebras, i.e. to make contact with the geometry of superconformal field theories. The second is to study ordinary locally free sheaves of rank greater than 1 on curves and their variations. Ours is to be considered quite an attempt in view of the non–linearities involved. Nevertheless, some nice results can be quickly obtained.

To make the thesis as self–consistent as possible, we devoted the first two chapters to a collection of results from the theory of algebraic curves and their moduli and from the theory of graded–commutative varieties and their deformations.

In Chapter 3 we investigate some questions about the global structure of N=1 supermoduli space. The main string–theoretical motivation for such a study is the fact that it is the ultimate domain of integration for the computation of string correlation functions in the Polyakov approach. The analysis of its geometrical properties, besides being interested on its own is prompted by the following problem which arises in string theory. While the bosonic piece of the Polyakov path integral is well understood as an integral over moduli spaces of algebraic curves, the fermionic part is more delicate as one has to Berezin–integrate along odd directions, and this may give rise to ambiguities in the resulting ordinary integral (see [DP] and references quoted therein.)

We analyze the graded–holomorphic structure of supermoduli spaces in the framework of stack theory by constructing explicit "coordinate charts" and studiyng their transition functions. Such an analysis boils down to the result that the simplest choices one can make in defining universal deformations of N=1susy–curves are actually plagued by the ambiguities we mentioned above, except for the genus 2 case.

The last section of chapter 3 deals with the moduli space of N=2 superconformal supersymmetry in two dimensions. The nicest mathematical aspect of such a theory is that when dealing with $N > 1$–supersymmetry, one is lead to consider locally free sheaves of rank greater than one on algebraic curves. From the physical point of view, $N = 2$ $d = 2$ field theoretical models are interesting because the richness of their symmetry algebra allows a thorough study of some of their global aspects. On the other hand N=2 superconformal models have also a string–theoretical interest. In fact, the most popular compactification scheme of superstring theory consists in compactifying the six extra dimensions in a Calabi–Yau space, i.e. a complex threefold with $SU(3)$ holonomy, which leaves an unbroken supersymmetry generator in the four dimensional Minkowski space. As discovered in [Ge], the propagation of a string in such a vacuum is consistently achieved by considering representations of N=2 supersymmetry algebra, or, in other words, N=1 space–time supersymmetry requires N=2 world-sheet supersymmetry.

We approach the problem from a deformation theory standpoint, getting a description of the reduced space of N=2 supermoduli space as a suitable quotient of the Picard variety over the moduli of curves and of its first infinitesimal neighbourhood as the first direct image of some natural sheaf on it. The properties of the system one expects on physical grounds (such as the existence of modular parameters for the $U(1)$ current mixing supersymmetries) emerge very neatly.

In the last chapter we give a geometric description of some representations of the semidirect sum of the Virasoro and Kac-Moody algebras in terms of line bundles on the moduli stacks of stable vector bundles over smooth Riemann surfaces. In particular, we dwell on a problem originating from works by Arbarello, De Concini Kac and Procesi, who managed to settle up a bridge between the representation theory of some infinite dimensional Lie algebras and the Picard groups of the moduli spaces of curves. More specifically, they considered the (classical) Virasoro algebra $diff^{\mathbb{C}}\, S^1$ and its semidirect product with the (classical) Heisenberg algebra $Lu(1)$ and proved that their central extensions can be put in a bijective correspondence with the Picard group respectively of the moduli variety M_g'' parameterizing curves, points and a non–zero tangent vector at that point and the moduli variety F_d'' which is fibered over M_g'' with fiber $Pic_d(C)$ parameterizing the data of M_g'' plus the datum of a degree d line bundle on C. In particular, they were able to deduce in a algebraic setting the central charges for spin j $b-c$–systems on a Riemann surface of arbitrary genus. We want to extend to the non–abelian case the link between the topology of moduli spaces of bundles on curves and the cohomology of the relevant algebras of infinitesimal symmetries of rank $r > 1$ bundles. Namely, we want to deal with higher rank vector bundles so that the algebra $\mathcal{F} = diff^{\mathbb{C}}\, S^1 \ltimes Lu(1)$ is replaced by

$\mathcal{D} = diff^{\mathbb{C}}\, S^1 \ltimes L\mathfrak{gl}(n, \mathbb{C})$. The analysis will show that the abovementioned methods can be applied with minor modifications also to our case. Also (even though we can give proofs in a weaker form) there is an isomorphism between the second cohomology of $\mathcal{D}$ and the Picard group of the relevant moduli space. The main physical motivation for such an investigation is that the moduli space of stable bundles is the classical phase space for 2-d Yang–Mills and 3-d Chern–Simons theories (at least in suitable topologies of the three–manifold on which the Chern–Simons action is originally defined). These latter are, in turn, connected to rational conformal field theories since it has been shown that every model of the discrete series (i.e. with central charge less that one) can be obtained by a Chern–Simons theory via an appropriate choice of the gauge group. Then, the moduli space of vector bundles over variable curves appears as the natural arena in which the modular geometry of such theories can be investigated.

In the spirit of classifying quantum field theories by classifying the representations of their symmetry groups, this setting seems to suggest that there should be a way of getting Sugawara formula for the stress energy tensor of Wess–Zumino–Novikov–Witten [GO] σ–models from purely geometrical data, as some results contained in [Hit] confirm. Actually, we can get the correct relation between the Kač–Moody central extension and the central charge of the Virasoro generators whenever the WZNW model can be fermionized, but, unfortunately, we do not have yet a complete control of the generic case. Achieving such a control, and more generically, achieving a systematic way to construct out of our geometrical data sheaves of modules for the Virasoro algebra along the lines of [BS] is still not completely clear to us and deserves further investigation.

This thesis is essentially based on the papers [FR1], [FR2], [FR3], [FR4].

Chapter 1.

Curves and their Moduli.

This chapter is devoted to a collection of some properties of algebraic curves, in order to outline the set up in which we will work and recall some results to be used in the sequel. After listing some primary facts about the analytic geometry of curves and their Picard groups, we will introduce in some details the moduli problem. We will recall some details of the Kodaira-Spencer theory of deformations since we will use it extensively in the subsequent chapters.

Section 1.3 deals with a presentation of Mumford's theorem on divisor theory on moduli space of curves, while Section 1.4 is devoted to a collection of perhaps less known results about the theory of moduli for vector bundles of rank greater than one.

Throughout this chapter, we will be mainly interested in the case of genus $g \geq 2$ curves and will pay a special attention to θ-characteristics .

1.1 Algebraic curves and their Picard group

Definition. A (smooth) Riemann surface (or a smooth algebraic curve) is a complex 1-dimensional manifold, or equivalently, a 2-dimensional real oriented manifold together with a conformal class of metrics.

From the differentiable point of view, any compact Riemann surface C is isomorphic to a sphere with a number of handles. This number is called the (topological) genus of the curve. The first homotopy group of C is the free group on $2g$ generators $\{a_i, b_i\}_{i=1,\dots,g}$ subjected to the relation

$$\prod_{i=1}^{g}[a_i, b_i] = 1$$

and its first homology group $H_1(C, \mathbb{Z})$ is freely generated by $\{a_i, b_i\}_{i=1,\dots,g}$.
A canonical choice of such generators is the one in which their intersection matrix is the $2g\times 2g$ symplectic matrix $\begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$ The Picard group Pic(C) of isomorphism classes of line bundles on C can be read off the exact sequence

$$0 \rightarrow \mathcal{Z} \longrightarrow \mathcal{O}_C \stackrel{\exp\, 2\pi i}{\longrightarrow} \mathcal{O}_C^* \rightarrow 1$$

where $\mathcal{O}_C^*$ is the sheaf of nowhere vanishing holomorphic functions and $\mathcal{Z}$ is the sheaf of $\mathbb{Z}$-valued continuous functions. The associated long exact cohomology sequence reads

$$\rightarrow \check{H}^1(C, \mathcal{Z}) \rightarrow \check{H}^1(C, \mathcal{O}_C) \rightarrow \check{H}^1(C, \mathcal{O}_C^*) \rightarrow \check{H}^2(C, \mathcal{Z}) \rightarrow \check{H}^2(C, \mathcal{O}_C) \rightarrow$$

which can be shrunk to

$$\begin{array}{ccccccccc} 0 & \rightarrow & \check{H}^1(C, \mathcal{O}_C)/H_1(C, \mathbb{Z}) & \rightarrow & \check{H}^1(C, \mathcal{O}_C^*) & \stackrel{\delta^*}{\longrightarrow} & H_2(C, \mathbb{Z}) & \rightarrow 0 \\ & & & & & & \| & \\ & & & & & & \mathbb{Z} & \end{array}$$

as the Čech cohomology groups of constant sheaves are isomorphic to the singular homology groups of the topological space and $\check{H}^2(C, \mathcal{O}_C)$ vanishes as $\mathcal{O}_C$ is coherent analytic on a 1-dimensional space. Then Pic(C) $\equiv \check{H}^1(C, \mathcal{O}_C^*)$ is the semidirect product $\mathbb{Z} \ltimes \mathrm{Pic}_0(C)$ and $\mathrm{Pic}_0(C)$ is a complex g-dimensional torus isomorphic to the variety built in the following way.
Pick a basis $\omega_1, \dots, \omega_g$ of the space $H_{\bar\partial}^{(0,1)}(C, K_C)$ of abelian differentials normalized according to $\oint_{a_i} \omega_j = \delta_{ij}$. Then

$$\oint_{b_i} \omega_j = \Omega_{ij}$$

is a symmetric matrix with positive definite imaginary part, called the period matrix of the Riemann surface C. If Λ denotes the lattice in $\mathbb{C}^g$ generated by the columns of the $g \times 2g$ matrix $(\delta_{ij}|\Omega_{ij})$, then $\mathrm{Pic}_0(C) \simeq \mathbb{C}^g/\Lambda$. The latter is usually called the *Jacobian* variety of C and denoted $J(C)$. As for the other connected components of the Picard group notice that $\mathrm{Pic}_0(C)$ acts freely and transitively on $\mathrm{Pic}_d(C)$ turning it into a principal homogeneous space.

Definition. A (Cartier) divisor on a smooth curve C is the datum of an open cover $\{U_\alpha\}_{\alpha \in I}$ and, $\forall \alpha \in I$ a holomorphic function f_α such that, if $U_\alpha \cap U_\beta \neq \emptyset$, $\frac{f_\beta}{f_\alpha}$ is holomorphic and nowhere vanishing.

By the compactness of C, the collection of f_α defines a finite set of points counted with multiplicities, and so one gets the notion of a (Weyl) divisor as an element of the free abelian group generated by the points of C, i.e. a finite formal sum $\sum_{i=1}^{n} k_i p_i \; n_i \in \mathbb{Z}$. It is straightforward to check that a divisor defines a line bundle. Also the converse is true, i.e.

Proposition. On a smooth algebraic curve every line bundle is the line bundle associated to a divisor.

In particular every line bundle admits a meromorphic section and hence its first Chern class can be computed as

$$c_1(L) = \int_C \partial\bar{\partial} \log |s|^2$$

where s is any meromorphic section of L and $|\cdot|^2$ is any hermitean metric along the fibres.

Two divisors D and D' are called linearly equivalent whenever $D - D'$ is the divisor of a global meromorphic function. The set $\mathrm{Div}C/\sim$ of divisors on C modulo linear equivalence is actually isomorphic to the Picard group of C, and, furthermore, if $L = [D]$ is a line bundle in the equivalence class of $D = \sum n_i p_i$ its first Chern class is given by

$$c_1(L) \equiv \deg L = \deg D = \sum n_i$$

The cornerstones of the theory of algebraic curves are the Riemann–Roch theorem and the Serre duality theorem. The Riemann–Roch theorem computes the Euler characteristics of any coherent analytic sheaf on a (smooth) algebraic genus g curve C. If E is a locally free sheaf of rank r then

$$\chi(E) = dim H^0(C,E) - dim H^1(C,E) = \deg E + r(1-g)$$

where deg$E$ is the degree of the invertible sheaf det E.

Serre duality assumes, in one complex dimension, the following simple form. Let K_C denote the sheaf of Abelian differentials on C; then

$$H^0(C, E^* \otimes K_C) \simeq H^1(C,E)^\vee$$

where $'\vee'$ means the dual vector space and the duality is given (in Dolbeault cohomology), by integrating the evaluation of a holomorphic (1,0)-form with values in E^* on a non–$\bar\partial$ exact(0,1)– form with values in E.

In the case of invertible sheaves, Riemann–Roch and Serre duality give a complete solution to the Riemann–Roch problem, i.e. to the computation of the dimension $h^0(C,L)$ of $H^0(C,L)$ for deg $L \geq 2g-2$ and deg $L \leq 0$. In fact the following results hold:

$$\deg L < 0 \Rightarrow h^0(C,L) = 0$$

$$\deg L = 0 \Rightarrow h^0(C,L) = \begin{cases} 1 & \text{and} \quad L = \mathcal{O}_C \\ 0 & \quad \text{otherwise;} \end{cases}$$

$$\deg L = 2g-2 \Rightarrow h^0(C,L) = \begin{cases} g & \text{and} \quad L = K_C \\ g-1 & \quad \text{otherwise;} \end{cases}$$

$$\deg L > 2g-2 \Rightarrow h^0(C,L) = \deg L - g + 1$$

In the range $0 < \deg L < 2g-2$ Riemann-Roch gives only the lower bound $h^0(C,L) \geq \deg L - g + 1$. An upper bound is given by Clifford's theorem, stating that

$$h^0(C,L) \leq \frac{1}{2}(\deg L + 1)$$

with equality reached only if $L = \mathcal{O}_C$, $L = K_C$ or C is hyperelliptic, i.e. a double covering of the rational curve $\mathbb{P}^1$. Nonetheless, if L is a generic line bundle with $0 \leq \deg L \leq 2g-2$ then $h^0(C,L) = \min(0, \deg L - g + 1)$.

By means of line bundles, one can define maps of C into projective spaces. Given a divisor D, a the projective space associated to a vector subspace of the set $|D|$ of effective divisors linearly equivalent to D is called a linear series. A base point for $|D|$ is a point common to all divisors in $|D|$ Consider now $L = \mathcal{O}(D)$ and suppose that the linear series associated to the whole $H^0(C,L)$ is base point free. Then we can define

$$\phi_L : C \to \mathbb{P}(H^0(C,L))^\vee \equiv \mathbb{P}H^1(C, KL^{-1})$$

by means of

$$\phi_L(p) = \{s \in H^0(C,L) | s(p) = 0\}$$

In homogeneous coordinates, picking up a basis $s_0, \ldots, s_h$ of $H^0(C,L)$ one has the explicit representation

$$\phi_L(p) = [s_0(p), \ldots, s_h(p)]$$

A very simple fact we will use in the sequel is the following

Proposition. The canonical bundle K is base point free.

Proof. Suppose $q \in C$ is a base point for K. Then $h^0(C, K(-q)) = g$ and so $h^0(C, \mathcal{O}(q)) = 2$, which is impossible if C is not $\mathbb{P}^1$. ■

The last topic we will cover is the notion of θ-characteristics .

Definition. A θ-characteristics on a smooth curve C is a line bundle $\mathcal{L}$ s.t. $\mathcal{L} \otimes \mathcal{L} \simeq K_C$.

The defining equation makes sense as $\deg K_C = 2g-2$ is always even. In particular the degree of a θ-characteristics is $g-1$. In terms of divisors the defining relation becomes $2[D] = [K_C]$, so that the quickest way of computing the number of θ-characteristics is the following. Let us fix a θ-characteristics $\mathcal{L}_0$. Then for any

θ-characteristics $\mathcal{L}$ it holds

$$\left(\mathcal{L}\otimes\mathcal{L}_0^{-1}\right)^2=(\mathcal{L})^2\otimes\left(\mathcal{L}_0^{-1}\right)^2\simeq K_C\otimes K_C^{-1}=\mathcal{O}_C$$

so that θ-characteristics are in a bijective (non canonical) correspondence with points of order two in the Jacobian $J(C)$. Then, as $J(C)$ is a g-dimensional torus, the number of such points, i.e. the number of different θ-characteristics is 2^{2g}. As $\deg\mathcal{L}=g-1$, $\mathcal{L}$ lies in the unstable range, so that one should not expect $h^0(C,\mathcal{L})$ to be independent of $\mathcal{L}$. In fact the best one can do is to classify θ-characteristics according to their parity, i.e. to $h^0(\mathcal{L})\text{mod } 2$. Actually there is some merit in doing so, because the parity of a θ-characteristics is invariant under deformations of the curve, and the number of even (resp. odd)θ-characteristics is known to be [Mul] $2^{g-1}(2^g+1)$ (resp. $2^{g-1}(2^g-1)$).

Being $\mathcal{L}$ a "square root" of the canonical bundle, one naturally thinks of its sections as of spinor fields on the curve C. In a more classical setting, recall that, on an m-dimensional real riemannian manifold M a spin structure is defined to be a principal fiber bundle $\tilde{P}\xrightarrow{\tilde{\pi}}M$ with structure group $\mathrm{Spin}(n)$ such that, if $P\xrightarrow{\pi}M$ is the bundle of orthonormal frames on M, and α is the non trivial double covering $\mathrm{Spin}(n)\xrightarrow{\alpha}\mathrm{SO}(n)$ there exists a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Spin}(n) & \times & \tilde{P} & \longrightarrow & \tilde{P} & & \\
 & & & & & \tilde{\pi}\searrow & \\
\downarrow\alpha & & \downarrow & & & & M \\
 & & & & & \pi\nearrow & \\
\mathrm{SO}(n) & \times & P & \longrightarrow & P & &
\end{array}$$

It is known that such a commutative diagram exists iff the obstruction class (the 2^{nd} Stiefel-Whitney class) $w_2\in H^2(M,\mathbb{Z}_2)$ vanishes, and the number of non–equivalent diagrams is the order of $H^1(M,\mathbb{Z}_2)$. Actually, due to the evenness of the first Chern class of the tangent bundle, every Riemann surface is a spin manifold, and the following theorem holds, relating θ-characteristics to spin-structures [A].

Theorem. The spin structures on a compact complex spin manifold correspond bijectively to the isomorphism classes of holomorphic line bundles $\mathcal{L}$ with $\mathcal{L}^2\simeq K$, where K is the canonical bundle, i.e. the top exterior power of the cotangent bundle.

1.2 The moduli problem

The moduli problem is a central one in algebraic geometry, as it consists in the non–topological part of the classification problem for algebraic varieties. In fact, roughly speaking, algebraic varieties are classified by some discrete invariant (such as the genus for curves) and some "continuous" invariants which, for historical reasons are called *moduli*. For the sake of concreteness, in this section we will stick to the case of moduli of algebraic curves.

Definition. A family of smooth algebraic curves is a proper surjective holomorphic map $\pi : X \longrightarrow S$ between two complex analytic manifolds such that $\forall s \in S$ the fiber $\pi^{-1}(s)$ is a smooth algebraic curve.

Theorem. [K] A family of algebraic curves is differentiably locally trivial, i.e. locally isomorphic, in the C^∞ category, to the product $C_* \times S$, C_* being any of the fibers of π.

For a family of Riemann surfaces over a connected base S, the genus of the fibers is constant and, more generally, all discrete topological invariants of the fibers will not vary over S.

Given a family $\pi : X \longrightarrow S$ and a map $f : S' \longrightarrow S$ one can define the pull back family as the fibered product $f^*(X) = X \times_f S'$. It is a family over S' and comes equipped with a commutative diagram

$$\begin{array}{ccc} f^*(X) & \xrightarrow{\phi} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

This notion leads to define in a natural way a contravariant functor (the Riemann functor) from the category of complex manifolds and holomorphic maps to the category of sets and maps by sending:

$$S \rightsquigarrow \mathcal{M}_g(S)$$

where $\mathcal{M}_g(S)$ is the set of isomorphism classes of genus g curves parameterized by S, and

$$f \in Hom(S, S') \rightsquigarrow \mathcal{M}_g(f) = [\phi] \in Hom(\mathcal{M}_g(S'), \mathcal{M}_g(S))$$

where ϕ is the map defined in the diagram above up to isomorphisms.

If $\mathcal{M}_g$ were a representable functor, i.e. there would exist a complex manifold M_g such that $\mathcal{M}_g(S)$ is isomorphic to the functor $S \rightsquigarrow Hom(S, M_g)$, then M_g would be called a *fine* moduli space for $\mathcal{M}_g$.
Actually it is known [see, e.g. Hu] that a fine moduli space for algebraic curves does not exist. A way to envisage the failure of representability of the Riemann functor is the following. If the Riemann functor were representable, its moduli space would come equipped with a "universal" family $\mathcal{C}_g \xrightarrow{\pi} M_g$ from which any other family $X \xrightarrow{f} B$ could be obtained by means of a unique diagram like

$$\begin{array}{ccc} X & \xrightarrow{F} & \mathcal{C}_g \\ \downarrow f & & \downarrow \pi \\ B & \xrightarrow{\varphi} & M_g \end{array}$$

Now, suppose $C \longrightarrow p$ is a curve with a non trivial automorphism γ over a point p. Then one can get a different diagram from the one above simply by twisting F by γ, thus loosing uniqueness.

The non existence of a fine moduli space for curves of genus g can be circumvented in at least two ways. The first is to relax the assumption of representability of the functor $\mathcal{M}_g$ thus getting the notion of *coarse* moduli space as follows

Definition. M_g is called a coarse moduli space for genus g curves if there is a morphism of functors

$$\Phi : \mathcal{M}_g \longrightarrow Hom(\cdot, M_g)$$

satisfying

i) if B is a point, $\Phi(B)$ is an isomorphism

ii) for any other morphism of functors $\xi : \mathcal{M}_g \longrightarrow Hom(\cdot, X_g)$ there is a unique map $\varrho : M_g \longrightarrow X_g$ for which the corresponding morphism of functors $[\varrho] : M_g \longrightarrow X_g$

satisfies

$$\xi = [\varrho]\circ\Phi.$$

In practice, when considering a coarse moduli space one gives up uniqueness of the classifying map and simply requires its non–uniqueness to be under control. This leads to the theory of moduli stacks, a very brief sketch of which is contained in Appendix A.2. A coarse moduli space does exist for genus g curves. The other way to face the non–existence of fine moduli spaces is to consider the moduli space of curves together with some additional structures, so that non- -trivial automorphisms are ruled out. The best known example of such a way out is Teichmüller theory, which can be described as originating from the following analysis of the automorphisms group of a smooth algebraic curve [ACGH].

The cases of genus 0 and 1 are easily dealt with. For the Riemann sphere $\mathbb{P}^1$, $Aut(\mathbb{P}^1)$ is $PGL(2,\mathbb{C})$. For $g = 1$, $Aut(C)$ is the extension by C itself of a group F of order $2, 4,$ or 6 and, actually, there are unique tori for which $ord\ F = 4$ or 6. In the case $g \geq 2$ things are much sharpened as the following proposition holds

Proposition Let C be a smooth genus g curve. Then $Aut(C)$ is a finite group of order at most $84(g-1)$.

Furthermore Hurwitz's theorem holds:

Theorem. In the above hypothesis, if $\varphi \in Aut(C)$ and φ is homotopic to the identity, then φ is the identity.

The above considerations lead naturally to the following

Definition. Let Σ be a closed oriented 2-dimensional manifold of genus g. A *marked Riemann surface* is a pair $(C, [f])$ where C is a Riemann surface , $f : C \longrightarrow \Sigma$ is a homeomorphism and $[f]$ denotes the homotopy class of f.

Two marked Riemann surface $(C, [f])$and $(C', [f'])$ are equivalent iff there is a conformal map $C \xrightarrow{h} C'$ such that $[f' \circ h] = [f]$.

The family version of this construction can be defined as follows [Hu]. Let $\pi : X \longrightarrow S$ be a family of curves. A Teichmüller structure of type Σ over X is the datum of an equivalence class of diffeomorphisms $[\Psi] : S \times \Sigma \longrightarrow X$ commuting with π where two diffeomorphisms are said to be equivalent if they are homotopic via a fiber map over S. Then, by a fibrewise argument, if $\Psi : S \times \Sigma \longrightarrow X$ is a representative of a Teichmüller structure, then any automorphism of X over S preserving $[\Psi]$ is the identity. In simpler words two curves related by a conformal automorphism not homotopic to the identity are to be considered as *different* in Teichmüller theory. Let $Diff^+(\Sigma)/Diff_0^+(\Sigma) \equiv \Gamma_\Sigma$ the mapping class group of the manifold Σ, i.e. the group of connected components of the orientation preserving diffeomorphism group $Diff^+(\Sigma)$. Given any family of curves $\pi : X \longrightarrow S$ one can consider the following principal fiber bundle

$$\begin{array}{ccc} \Gamma_\Sigma & \longrightarrow & \Gamma_X \\ & & \downarrow \\ & & S \end{array}$$

Definition. The Teichmüller functor of type Σ, $\mathcal{T}_g$ associates to an analytic space S the set of isomorphism classes of family of curves $\pi : X \longrightarrow S$ equipped with a section of Γ_X

By the previous discussion one expects good properties for the Teichmüller functor. In fact it holds [Hu]

Theorem. The Teichmüller functor $\mathcal{T}_g$ is representable by means of a Stein variety of dimension $3g-3$ isomorphic to an open ball in $\mathbb{C}^{3g-3}$, which is called the Teichmüller space T_g.

A real coordinatization of T_g can be given by means of the Fenchel-Nielsen coordinates, defined as follows. Consider an hexagon in the hyperbolic plane, which is determined, up to isometries, by the lengths l_1, l_2, l_3 of alternating sizes. Considering its double across the remaining sizes we get a pair of pants, which are building blocks for a Riemann surface in the sense that a genus g Riemann surface can be obtained by glueing $2g-2$ pair of pants. This works as follows: fix a collection $\{\gamma_1 \dots \gamma_{3g-3}\}$ of disjoint simple closed curves such that $\Sigma \setminus \{\gamma_i\}$ is the disjoint union of pair of pants . Then Σ can be completely reconstructed by attaching these pair of pants along the $\{\gamma_i\}$'s. The Fenchel Nielsen coordinates are the free parameters in this construction:

they are the geodesic lengths l_i of the γ_i and the hyperbolic distances τ_i between the feet of perpendiculars to γ_i dropped from fixed boundary points [Ha].

The study of the geometry of the Teichmüller space is a major step forward in understanding the geometry of the (coarse) moduli space of curves. In fact, T_g can be realized as the quotient $Conf(\Sigma)/Diff_0(\Sigma)$ of the conformal group by the group of diffeomorphisms homotopic to the identity, while M_g is given by $Conf(\Sigma)/Diff^+(\Sigma)$ this latter being the full orientation preserving diffeomorphisms. Thus, one can use the topological triviality of T_g to get some insight into the topology of M_g, since $M_g = T_g/\Gamma_g$, where Γ_g is the mapping class group of the Riemann surface . In fact one has the following results [Ha]

Proposition. If $g \geq 3$ the action of Γ_g over T_g is properly discontinuous but not free. Its fixed points correspond to algebraic curves with non trivial automorphisms group. Correspondingly, the moduli space M_g has the structure of a complex space and a complex V-manifold (or *orbifold*). The lower cohomology groups of M_g are computed as follows

$$H^0(M_g, \mathbb{Z}) = \mathbb{Z}$$

$$H^1(M_g, \mathbb{Z}) = 0$$

$$H^2(M_g, \mathbb{Z}) = \mathbb{Z}$$

A deeper understanding of moduli space and of its complex structure is achieved by means of deformation theory [K]. Let X be a smooth algebraic curve.

Definition. A deformation of X, parameterized by a pointed analytic space (Y, y_0) is a proper holomorphic map

$$\varphi : \mathcal{X} \longrightarrow Y$$

plus a given isomorphism $\psi : X \to \varphi^{-1}(y_0)$ between X and the central fiber $\varphi^{-1}(y_0)$.

The notion of deformation thus differs from the one of family by the prescribed identification of the central fiber with the object to be deformed. A it first order deformation of X is a deformation of X parameterized by $S = Spec\ \mathbb{C}[\epsilon]$, the spectrum of the dual numbers. In the sequel we will pursue the Kodaira-Spencer approach to deformation theory, in which on thinks of the curve C as being qualified by patching data $\{U_\alpha, z_\alpha, f_{\alpha\beta}\}$, and thinks of deforming it by deforming the patching data. Here

$\{U_\alpha\}_{\alpha\in I}$ is a finite covering of C

z_α is a holomorphic coordinate in U_α

$z_\alpha = f_{\alpha\beta}(z_\beta)$ in $U_\alpha \cap U_\beta$

In any triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ the cocycle rule holds

$$f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma)) = f_{\alpha\gamma}(z_\gamma)$$

A first order deformation, can then be thought of as being given by glueing products $U_\alpha \times S$ by means of

$$z_\alpha = \tilde{f}_{\alpha\beta}(z_\beta, \epsilon) \equiv f_{\alpha\beta}(z_\beta) + \epsilon \cdot b_{\alpha\beta}(z_\beta)$$

For any fixed ϵ the $\tilde{f}_{\alpha\beta}$'s must be transition functions for a curve, so that they must satisfy the cocycle rule. This latter translates into the cocycle condition for the $f_{\alpha\beta}$'s and the following condition for the $b_{\alpha\beta}$'s

$$b_{\alpha\beta} + \frac{\partial f_{\alpha\beta}}{\partial z_\beta} \cdot b_{\alpha\beta} = b_{\beta\gamma}$$

But, by the chain rule $\frac{\partial f_{\alpha\beta}}{\partial z_\beta}\frac{\partial}{\partial z_\alpha} = \frac{\partial}{\partial z_\beta}$ so that, putting $X_{\alpha\beta} := b_{\alpha\beta}\frac{\partial}{\partial z_\alpha}$ yields

$$X_{\alpha\beta} + X_{\beta\gamma} - X_{\alpha\gamma} = 0$$

i.e. $X_{\alpha\beta}$ defines a class

$$[X_{\alpha\beta}] \in \check{H}^1(C, T_C)$$

which is called the Kodaira-Spencer class of the first order deformation φ.

The sheaf–theoretical version of this construction is given as follows. For any first order deformation $\varphi : \mathcal{X} \to S$ one gets the following exact sequence of $\mathcal{O}_{\mathcal{X}}$-modules

$$0 \to T_C \longrightarrow T_{\mathcal{X}} \xrightarrow{\varphi_*} \varphi^* T_S \to 0$$

which induces the exact cohomology sequence

$$\cdots \to \check{H}^0(C, T_{\mathcal{X}}) \xrightarrow{\varphi_*} \check{H}^0(C, \varphi^*(T_S) \xrightarrow{\delta^*} \check{H}^1(C, T_C) \to \cdots$$

Then $X_{\alpha\beta}$ is just $\delta^*\left([\varphi^* \frac{\partial}{\partial\epsilon}]\right)$. Given two isomorphic deformations $\varphi : \mathcal{X} \longrightarrow S$ and $\varphi' : \mathcal{X}' \longrightarrow S$ the commutative diagram

$$\begin{array}{cccccccc}
\cdots & \check{H}^0(C, T_{\mathcal{X}}) & \xrightarrow{\varphi_*} & \check{H}^0(C, \varphi^*(T_S)) & \xrightarrow{\delta^*} & \check{H}^1(C, T_C) & \to \cdots \\
 & & & \| & & \downarrow & \\
\cdots & \check{H}^0(C, T'_{\mathcal{X}}) & \xrightarrow{\varphi_*} & \check{H}^0(C, \varphi'^*(T_S)) & \xrightarrow{\delta^*} & \check{H}^1(C, T_C) & \to \cdots
\end{array}$$

insures that $[X_{\alpha\beta}]$and $[X'_{\alpha\beta}]$ are the same class in $\check{H}^1(C, T_C)$.
Conversely, taking two cocycles in the same class, the difference between the two infinitesimal deformations they induce is simply a holomorphic coordinate change, so that the two are really indistinguishable.

Given an arbitrary deformation $\phi : \mathcal{X} \longrightarrow (Y, y_0)$ and a map $(S, s_0) \xrightarrow{f} (Y, y_0)$ the pull–back $f^*(\mathcal{X})$ is the first order approximation of $\phi : \mathcal{X} \longrightarrow (Y, y_0)$ in the direction of the tangent vector corresponding to f. Notice that the spectrum of dual numbers embodies the notion of tangent vector to an algebraic space, in the sense that $T_{y_0}Y \simeq Hom((S, s_0), (Y, y_0))$. Thus we get a homomorphism

$$\rho_\phi : T_{y_0}(Y) \longrightarrow H^1(C, T_C)$$

called the Kodaira-Spencer homomorphism associated to $\phi : \mathcal{X} \longrightarrow (Y, y_0)$.
Definition. Let $\pi : \mathcal{X} \longrightarrow (Y, y_0)$ be a deformation of C. $\mathcal{X}$ is said to be complete at y_0 (or *versal*) at y_0 if for any other deformation of C,

$$\pi' : \mathcal{Y} \longrightarrow (Y', y'_0)$$

there exists a neighbourhood $V' \ni y'_0$ and a holomorphic map $g : V' \longrightarrow Y$ sending y'_0 to y_0 such that the restricted family $\mathcal{Y} \restriction V'$ is isomorphic to $f^*(\mathcal{X})$ over (V', y'_0).

Definition. Let $\pi : \mathcal{X} \longrightarrow (Y, y_0)$ be a versal deformation of C. We say it is *universal at* y_0 if the germ of the classifying map above is unique.

Remark (Uni)versality is a local property, in the sense that if $\pi : \mathcal{X} \longrightarrow (Y, y_0)$ is (uni)versal at y_0, then it is (uni)versal in a whole neighbourhood of y_0.

The rôle of the Kodaira-Spencer map is clarified by the following

Theorem. Let $\pi : \mathcal{X} \longrightarrow (Y, y_0)$ be a deformation of C such that the Kodaira-Spencer map ρ_π is an isomorphism. Then $\pi : \mathcal{X} \longrightarrow (Y, y_0)$ is universal.

A universal deformation will also be called a *Kuranishi* deformation. The theorem above allows one to compute the dimension of moduli space of genus g curves. In fact, if $g \geq 2$ the tangent sheaf T_C has negative degree, so the dimension of $H^1(C, T_C)$ is read off the Riemann-Roch theorem as

$$dim H^1(C, T_C) \equiv dim M_g = 3g - 3.$$

A satisfactory deformation theory can be settled up for algebraic curves by using a particular class of Kuranishi families, namely the so–called *Schiffer variations.* Let p be a generic point of C. and consider the following exact sequence

$$0 \to T_C \longrightarrow T_C(p) \longrightarrow \mathcal{S}_p^{\mathbb{C}} \to 0$$

The coboundary map δ_p sends $H^0(C, \mathcal{S}_p^{\mathbb{C}}) = \mathbb{C} \longrightarrow H^1(C, T_C)$. In terms of Čech cocycles, considering the acyclic cover $\{U, V\}$, where $V = C \setminus p$ and U is a small disk centered at p and parameterized by z then a representative of $\delta_p(1)$ is

$$X_{UV} = \frac{1}{z}\frac{\partial}{\partial z}$$

By Serre duality the map

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{P}H^1(C, T_C) \\ p & \rightsquigarrow & [\delta_p] \end{array}$$

is the bicanonical map, so that Schiffer variations generate $H^1(C, T_C)$. Schiffer variations have the advantage of being easily integrated. Namely one can find a deformation

$$\Psi : \mathcal{D} \longrightarrow \Delta_\epsilon = \{t \in \mathbb{C} \,|\, |t| < \epsilon\}$$

whose Kodaira-Spencer map is δ_p. In fact, such a $\mathcal{D}$ can be obtained, if Δ' is another small disk of radius ϵ', by glueing

$$C \setminus \{p \in C s.t. |z(p)| < \frac{\epsilon}{2}\} \times \Delta_\epsilon \text{ with } \Delta' \times \Delta_\epsilon$$

via the glueing map

$$\begin{cases} t = \tilde{t} \\ w = z + \frac{t}{z} \end{cases}$$

In general, by choosing $3g-3$ distinct general points p_i and removing $3g-3$ small disks around them one gets a family parameterized by a $3g-3$-dimensional polydisk, defined by the glueing law

$$\begin{cases} t_i = \tilde{t}_i \\ w = z_i + \dfrac{t_i}{z_i} \end{cases}$$

The Kodaira-Spencer map of this $3g-3$-dimensional Schiffer variation is the coboundary map δ in the exact sequence

$$H^0(C, T_C(\textstyle\sum_1^{3g-3} p_i)) \xrightarrow{\oplus_i res_{p_i}} \oplus_{j=1}^{3g-3} S_{p_j}^{\mathbb{C}} \xrightarrow{\delta} \cdots$$
$$\cdots \xrightarrow{\delta} H^1(C, T_C) \longrightarrow H^1(C, T_C(\textstyle\sum_1^{3g-3} p_i)) \to \cdots$$

Now, denoting as usual $h^i(C, L) := dim H^i(C, L)$, one has

$$h^0(C, T_C(\textstyle\sum_1^{3g-3} p_i)) = h^1(C, T_C(\textstyle\sum_1^{3g-3} p_i)) = h^0(C, K_C^2(-\textstyle\sum_1^{3g-3} p_i))$$

where the second equality is just Serre duality, while the first comes from Riemann-Roch noticing that $deg(T_C(\sum_1^{3g-3} p_i)) = g-1$. Then δ is an isomorphism, and hence the Schiffer variation above is a Kuranishi family if and only if $h^0(C, K^2(\sum_1^{3g-3} p_i)) = 0$. This can be achieved by choosing the points p_i according to the following strategy. Let s_1 be a non-zero holomorphic section of K^2 and let p_1 be such that $s_1(p_1) \neq 0$ so that $h^0(C, K^2(-p_1)) = h^0(C, K^2) - 1$. Then let s_2 be a non-zero section of $K^2(-p_1)$ and choose p_2 where s_2 is not vanishing. Again

$$h^0(C, K^2(-p_1 - p_2)) = h^0(C, K^2) - 2$$

Inductively, one can find $p_1, \dots, p_j$ such that

$$h^0(C, K^2(-\sum_{i=1}^{j} p_i)) = h^0(C, K^2) - j$$

Obviously this process ends after $3g-3$ steps as $h^0(C, K^2) = 3g-3$. Notice that there is no flaw in this argument, as, at each step we are dealing with line bundles of degree $\geq g$ and again Riemann-Roch insures that they have at least one non-zero holomorphic section.

To complete the discussion, let us sketch how a natural complex structure can be given to the set M_g. Natural here means that it must be induced by the notion of family, in a sense that we are going to describe.

First of all, notice that every genus g curve C can be holomorphically embedded in $\mathbb{P}^{5g-6}$ by the means of the tricanonical map. Every automorphisms γ of C will induce an automorphisms of $H^0(C, K^3)$ but, as C is embedded in $\mathbb{P}H^0(C, K^3)$, γ is the restriction of an automorphisms of $\mathbb{P}H^0(C, K^3)$ so that $Aut(C)$ is a discrete algebraic subgroup of $PGL(5g-6, \mathbb{C})$ and hence is finite.

Now let C be a genus g smooth curve and suppose

$$\begin{array}{ccc} C & \xrightarrow{\Psi} & \mathcal{C} \\ & & \downarrow \pi \\ & & (S, s_0) \end{array}$$

is its universal deformation; as was noticed when dealing with deformation theory, one can assume that it is a universal deformation for every $s \in S$. Taking $\gamma \in Aut(C)$ and replacing Ψ with $\Psi \circ \gamma$ one gets another universal deformation of C, say $\mathcal{C}'$. But, thanks to the defining property of universal deformations, there exist unique automorphisms a_γ and b_γ respectively of S and $\mathcal{C}$ such that

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{b_\gamma} & \mathcal{C}' \\ \downarrow \pi & & \downarrow \pi' \\ (S, s_0) & \xrightarrow{a_\gamma} & (S, s_0) \end{array}$$

commutes, $a_\gamma(s_0) = s_0$ and $\Psi \circ \gamma = b_\gamma \circ \Psi$

If some other element $\gamma' \in Aut(C)$ carries s to s', then $\pi^{-1}(s) \simeq \pi^{-1}(s)$ so that the map from S to M_g factors through $S/Aut(C)$, i.e.

$$\begin{array}{ccc} S & & \\ & {}^{\delta}\searrow & \\ \downarrow\chi & & M_g \\ & {}^{\eta}\nearrow & \\ S/G & & \end{array}$$

Then, by means of geometric invariant theory, one can show that [Mu2,DM] S/G has a natural complex structure under which χ is holomorphic that can be transported via η to an open subset of M_g. Namely, a local continuous function on M_g will be called holomorphic iff its composition with δ is holomorphic , or, in other words, we can give M_g the unique complex structure obtained by glueing together bases of universal families of isomorphic curves.
We end this section recalling the main features of Mumford's compactification $\overline{M}_g$ of M_g. This issue is of great relevance as, for instance, most of the known algebro-geometrical techniques one could hope to use in studying M_g work only in the case of complete varieties. Mumford's compactification scheme solves this trouble by choosing a specific set of singular curves to be plugged into $\bar{M}_g$ as its boundary, the so called *stable* ones.

A node curve is a curve whose only singularities are described locally by the equation $xy = 0$. By compactness, the number of nodes is finite; the normalization N_C of a node curve C is the smooth curve obtained by C by pulling apart meeting branches.

Definition. A stable curve (resp. a semistable one) is a connected node curve of genus $g > 1$ such that any of its rational components meets the rest of the curve in at least three (resp. two) points.
Stable curves can be deformed in a way much alike to the one we described before [ACGH] and the number of moduli of a genus g stable curve is the same as that of a genus g smooth curve. In fact a stable curve C can be thought of as being qualified

by its normalization N_C and the identification of the preimages of the nodes $\tilde{p}_i$ as $p_1 \sim q_1, \ldots, p_r \sim q_r$. Then we start from a universal deformation of N_C thus getting $\sum_{i=1}^{\nu} m(N_i)$-parameters, where the N_i's are the connected components of N_C and

$$m(N_i) = \begin{cases} 0 & \text{if} N_i \simeq \mathbb{P}^1 \\ 1 & \text{if} g(N_i) = 1 \\ 3g(N_i) - 3 & \text{otherwise.} \end{cases}$$

Taking into account that $\mathbb{P}^1$ has $PGL(2, \mathbb{C})$ and an elliptic curve a complex torus as automorphisms groups, we see that the total contribution of any component N_i is $3g(N_i)-3$. Then we can deform by identifying a point near p_i to a point near q_i getting $2r$ more parameters, and r additional parameters occur as a result of smoothing the nodes (these are the transversal parameters to the boundary). Summing up we have that the dimension of the base space T of a universal deformation of C is

$$dim\ T = \sum_{i=1}^{\nu}(3g(N_i) - 3) + 3r = 3\left(\sum_{i=1}^{\nu} g(N_i) \ \ - \nu + r + 1\right) - 3 = 3g(C) - 3$$

Final step is to show that the procedure of adjoining to M_g stable curves is "exhaustive" and gives rise to a compact space. This is achieved by means of the stable reduction theorem which, roughly speaking, asserts that every family of algebraic curves admits a stable limit. Without entering the details – the hard part of the proof is covered by a semi–stable reduction theorem and uniqueness of the limit requires stability – we can state it in the following form.

Theorem. Let X be a complex space and $X \xrightarrow{\pi} \Delta$ a proper map such that $\pi^{-1}(t)$ is a smooth algebraic curve $\forall t \neq 0$.

Then there are an integer n, a family $\Xi \xrightarrow{\sigma} \Delta$ of stable curves and a commutative diagram

$$\begin{array}{ccc} \Xi \setminus \sigma^{-1}(0) & \xrightarrow{\beta} & X \setminus \pi^{-1}(0) \\ \downarrow \sigma & & \downarrow \pi \\ \Delta \setminus \{0\} & \xrightarrow{\alpha} & \Delta \setminus \{0\} \end{array}$$

where $\alpha(t) = t^n$ and$\beta\restriction_{\sigma^{-1}(t)}$ is an isomorphism $\forall t \neq 0$. In other words, when a family of curves degenerates to a generic singular one, one can safely construct a family which is isomorphic to the previous one, except for the fact that the central fiber is replaced with a stable curve.

1.3 Mumford's theorem

Let us now describe divisor theory on M_g, following works by Mumford, Harris, Arbarello, Cornalba and others. What one wants to end up with is not only a formal description of $Pic(M_g)$ but a classification of line bundles on M_g built by means of geometrically significant objects. Let X be a smooth projective variety, and let K_X its canonical sheaf, i.e. the maximum wedge product of the cotangent sheaf. K_X is an object that "allows" to do duality in the sense that

i) $H^n(X, K_X) \simeq H^{n,n}_{\bar\partial}(X) \simeq H^{2n}(X, \mathbb{Z}) \otimes \mathbb{C} \simeq \mathbb{C}$

and the isomorphisms above are all natural

ii) Given any vector bundle $E \xrightarrow{\pi} X$ a canonical isomorphism is given

$$H^{i,0}_{\bar\partial}(X, E) \xrightarrow{\sim} H^{n-i,i}_{\bar\partial}(X, K_X \otimes E^*)^{\vee}$$

Then, given a possibly singular projective variety X one is naturally lead to the following

Definition . A dualizing sheaf ω_X for X is a coherent sheaf together with an explicit homomorphism $H^n(X, \omega_X) \to \mathbb{C}$, called the *trace* homomorphism, such that for all coherent sheaves $\mathcal{F}$ on X the pairing

$$Hom(\mathcal{F}, \omega_X) \times H^n(X, \omega_X) \longrightarrow H^n(X, \omega_X) \longrightarrow \mathbb{C}$$

is non-degenerate.

The following facts are known

i) such an ω_X, if existing, is unique.

ii) ω_X exists and is locally free if X is a projective variety, subjected to the technical condition of being locally a complete intersection.

iii) if $\mathcal{F}$ is locally free and coherent

$$H^{n-i}(X,\mathcal{F}) \xrightarrow{\sim} \left(H^i(X,\omega_X \otimes \mathcal{F}^*)\right)^*$$

iv) if X is smooth, then ω_X coincides with the canonical sheaf.

Now consider a proper smooth morphism of algebraic varieties $\pi : \mathcal{X} \longrightarrow B$ and the associated sheaf exact sequence

$$0 \to \pi^*\Omega^1_B \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/B} \to 0$$

where $\Omega^1_{\mathcal{X}/B} \simeq \Omega^1_{\mathcal{X}}/\pi^*\Omega^1_B$ is the sheaf of one-forms along the fibers. Taking the maximum wedge product in the exact sequence above one gets

$$\Omega^n_{\mathcal{X}/B} = \omega_{\mathcal{X}} \otimes \left(\pi^*\omega_B\right)^*$$

so that

Definition. The relative dualizing sheaf of the family $\pi : \mathcal{X} \longrightarrow B$ is the invertible sheaf

$$\omega_{\mathcal{X}/B} = \omega_{\mathcal{X}} \otimes \left(\pi^*\omega_B\right)^*$$

The ultimate reason for doing so relies in the following

Proposition . The restriction of $\omega_{\mathcal{X}/B}$ to any fiber F of π is the dualizing sheaf ω_F.

We are now in a position to discuss Mumford's theorem and the Picard group of the moduli space of algebraic curves. Let us restrict our discussion to the subvariety M^0_g of automorphisms –free curves, because when dealing with M_g the nonexistence of the universal curve

$$\mathcal{C} \xrightarrow{\pi} M_g$$

generates subtleties to be treated with more sophisticated techniques. Also, for genus $g \geq 4$, the set of curves with automorphisms is of codimension $d = g - 2$ in M_g so that removing it does not affect divisor theory. Let us consider the universal curve $\mathcal{C}\longrightarrow M_g^0$ the relative dualizing sheaf $\omega_{\mathcal{C}/M_g^0}$ and its powers $\omega^n_{\mathcal{C}/M_g^0}$. The Grothendieck-Riemann-Roch theorem for n-canonical relative forms gives

$$Ch(\pi_!\omega^n_{\mathcal{C}/M_g^0}) = \pi_*\left(Ch\omega_{\mathcal{C}/M_g^0}\cdot Td\Omega^1_{\mathcal{C}}\cdot\pi^*(Td\Omega^1_{M_g^0})^{-1}\right)$$

which can be simplified, by means of the following exact sequence

$$0\to\Omega^1_{\mathcal{C}/M_g^0}\longrightarrow\Omega^1_{\mathcal{C}}\xrightarrow{\pi_*}\Omega^1_{M_g^0}\to 0$$

yielding

$$Td\Omega^1_{\mathcal{C}}\cdot(\pi^*Td\Omega^1_{M_g^0})^{-1} = Td\Omega^1_{\mathcal{C}/M_g^0}$$

to

$$Ch(\pi_!\omega_{\mathcal{C}/M_g^0}) = \pi_*\left[Ch\omega^n_{\mathcal{C}/M_g^0}Td\Omega^1_{\mathcal{C}/M_g^0}\right]$$

Now one has that $\pi_!\omega^n_{\mathcal{C}/M_g^0} = \pi_*\omega^n_{\mathcal{C}/M_g^0}\ominus R^1\pi_*\omega^n_{\mathcal{C}/M_g^0}$ as $\omega^n_{\mathcal{C}/M_g^0}$ is invertible and so its higher direct image sheaves vanish. Also, ${\omega_{\mathcal{C}/M_g^0}}^n$ restricts to each fiber C_t of the family to $\omega^n_{C_t}$ so that, for $n>1$ $\;deg_{rel}\omega^n_{\mathcal{C}/M_g^0}<0$ and hence $R^1\pi_*\omega^n_{\mathcal{C}/M_g^0}=0$ and for $n=1$ $\;R^1\pi_*\omega_{\mathcal{C}/M_g^0}\simeq\mathcal{O}_{M_g^0}$ and hence does not affect $\pi_!$ so that one can rewrite the formula above as

$$Ch(\pi_*\omega_{\mathcal{C}/M_g^0}) = \pi_*\left[Ch\omega^n_{\mathcal{C}/M_g^0}Td\Omega^1_{\mathcal{C}/M_g^0}\right]$$

Expanding both sides one gets

$$Ch(\pi_*\omega^n_{\mathcal{C}/M_g^0}) = \pi_*\Bigg[\left(1+c_1(\omega^n_{\mathcal{C}/M_g^0})+\frac{c_1(\omega^n_{\mathcal{C}/M_g^0})^2}{2}+\cdots\right)\cdot$$
$$\cdot\left(1-\frac{c_1(\Omega^1_{\mathcal{C}/M_g^0})}{2}+\frac{c_1(\Omega^1_{\mathcal{C}/M_g^0})^2+c_2(\Omega^1_{\mathcal{C}/M_g^0})}{12}+\cdots\right)\Bigg]$$

By extracting the right codimension piece one has

$$c_1(\pi_*\omega^n_{\mathcal{C}/M_g^0}) = c_1(\det\,\pi_*\omega^n_{\mathcal{C}/M_g^0}) = \pi_*\left[\frac{c_1(\Omega^1_{\mathcal{C}/M_g^0})^2+c_2(\Omega^1_{\mathcal{C}/M_g^0})}{12}+\right.$$

$$-\frac{c_1(\omega^n_{C/M^0_g})c_1(\Omega^1_{C/M^0_g})}{2}+\frac{c_1(\omega^n_{C/M^0_g})^2}{2}\Bigg]$$

On M_g the dualizing sheaf and the sheaf of Kähler differentials coincide, so that, denoting $\lambda_n := \pi_*(\text{ det } \omega^n_{C/M_g})$ one has

$$c_1(\lambda_n) = \frac{1}{12}(6n^2 - 6n + 1)\pi_*[c_1(\omega^n_{C/M_g})^2]$$

and, in particular,

$$c_1(\lambda) = \frac{1}{12}\pi_*[c_1(\omega_{C/M_g})^2]$$

yielding Mumford's formula

$$c_1(\lambda_n) = (6n^2 - 6n + 1)c_1(\lambda)$$

1.4 Moduli space of θ-characteristics .

In this section we want to describe the moduli space of θ-characteristics because of their relevance in the supersymmetric case.

Given a family of curves $\mathcal{C} \stackrel{\pi}{\longrightarrow} S$, a (relative) θ-characteristics is an invertible sheaf $\mathcal{L}$ over $\mathcal{C}$ which is a "square root" of the relative canonical sheaf $\omega_{\mathcal{C}/S}$ i.e. such that, $\forall s \in S \quad L^2 \restriction_{\pi^{-1}(s)} \simeq \omega_{\pi^{-1}(s)}$. A θ-characteristics is called even or odd according to the parity of $dim\ H^0(C, L)$. There are $2^{g-1}(2^g + 1)$ even and $2^{g-1}(2^g - 1)$ odd θ-characteristics , adding up to a total of 2^{2g}.

Mimicking deformation theory of stable curves, one can define a deformation of a (smooth) θ-characteristics (C, L) to be a relative θ-characteristics [H2], i.e. a diagram

$$\begin{array}{ccccc} L & & & \mathcal{L}_\pi & \\ & \searrow & & & \searrow \\ & & C & \stackrel{i}{\longrightarrow} & \mathcal{X} \\ & & \downarrow & & \downarrow \pi \\ & & p & \longrightarrow & S \end{array}$$

such that the isomorphism $i : C \to \pi^{-1}(s_0)$ induces an isomorphism of L and $i^*\mathcal{L}_\pi$. More generally, one can deform all θ-characteristics $(C, L_1, ...L_m)$, $(m = 2^{2g})$ on C, by giving 2^{2g} sheaves $\mathcal{L}_{\pi_1}, ..., \mathcal{L}_{\pi_m}$ on $\mathcal{X}$ satisfying the above property. In this way we get a 2^{2g}-fold covering of the base space S of $\mathcal{X}$. A remark is due at this point. When considering families of line bundles over moving curves $\mathcal{L}_\pi \to \mathcal{X} \to S$, the requirement that the line bundle $\mathcal{L}_\pi$ restricts to each fiber to a prescribed line bundle, does not fix it completely, since it can be tensored with a line bundle coming from the base S of the deformation without affecting such a requirement. The case of θ-characteristics is special for the fact that its square is fixed to be the relative dualizing sheaf of the family , which is uniquely defined. Hence the only ambiguity in defining the relative θ-characteristics will be a square root of the structure sheaf, an ambiguity that can be easily dealt with.

Since θ-characteristics on an algebraic smooth curve are in a (non-natural) one-to-one correspondence to points of order 2 on the Jacobian $J(C)$, a more concrete way of describing such a covering is to consider the family $\tau : \mathcal{J} \to S$ of Jacobians associated to the deformation $\pi : \mathcal{X} \to S$ of C, whose fibre $\tau^{-1}(s)$ is precisely the Jacobian of $\pi^{-1}(s)$. The choice of (C, L_1) gives us its deformation $\mathcal{L}_{\pi_1}$ over X and 2^{2g} sections $\sigma_1, ..., \sigma_m$ of $\mathcal{J}$ over S gotten by setting $\sigma_i = \mathcal{L}_{\pi_i} \otimes \mathcal{L}_{\pi_1}^{-1}$. Their image is the desired covering of S. When dealing with smooth curves , this local covering extends by isomorphisms to the whole M_g, thus realizing the moduli space of θ-characteristics over smooth curves S_g as a 2^{2g}-fold of M_g. The problem is that, when considering singular curves besides the smooth ones, trouble can arise, as the following example shows.

Let us consider a family of elliptic curves parameterized by a small disk $\Delta \in \mathbb{C}$ as follows. Set $\tau = ln(b)/2\pi i$, $b \in \Delta$, and consider the lattice $\Lambda_\tau \subset \mathbb{C}$ generated by 1 and τ. This acts holomorphycally on $\Delta \times \mathbb{C}$ by translations on the second factor. The quotient $X = \Delta \times_{\Lambda_\tau} \mathbb{C}$ is a family of tori degenerating to a single-node curve for $b = 0$. At genus one all θ-characteristics have degree 0 and one of them is isomorphic to the structure sheaf $\mathcal{O}_{c_\tau}$. So the other three naturally corresponds to points of order two on the Jacobian, which in turn coincides with the torus itself. So, on $\Delta \setminus \{0\}$ we get the following sections of $J = X \to \Delta$

$$\sigma_1 = 0; \qquad \sigma_2 = 1/2$$
$$\sigma_3 = \tau/2; \qquad \sigma_4 = \tau/2 + 1/2 \qquad (\text{mod}\, \Lambda_\tau)$$

where $\tau = \tau(b)$ as above.

We can now clearly see three phenomena. First of all, we have monodromy in the covering, because a rotation around $b = 0$ exchanges the two sections σ_3 and σ_4. Second, these two sections are 'asymptotic' for $b \to 0$ ($|\tau| \to \infty$), meaning that there is branching in the covering (recall that the Jacobian of a torus with one node can be compactified getting again the same torus; being asymptotic here means that the two section above go to the node in the limit.) Finally, this limit point cannot be interpreted any more as an invertible sheaf, but corresponds to a more general coherent sheaf.

If we abstract from the peculiarities of genus 1, the picture we get from this example is general. In particular, the three phenomena mentioned above, i.e. monodromy, branching and the appearance of more general sheaves than sheaves of sections of line bundles enter the game at all genera. For instance, such sheaves occur in the compactification of the moduli of θ–characteristics recently constructed by Deligne [D]. A different way for getting a compactified moduli space of θ-characteristics has been given by Cornalba [C]. This involves the addition to the moduli space of smooth curves of a wider class of singular curves, (namely a certain subclass of *semistable* ones), but have the desirable feature of yielding invertible sheaves as "limits of θ-characteristics ".

Without entering too much the details of this compactification scheme, we simply quote the following results [C];

1) $\overline{S}_g$ has a natural structure of a normal projective variety, $\partial\overline{S}_g = \overline{S}_g \setminus S_g$ is a closed proper analytic subvariety of $\overline{S}_g$, and therefore S_g is an open subvariety.

2) The natural map $\chi : \overline{S}_g \longrightarrow \overline{M}_g$ given by forgetting spin structures and reverting to stable models is finite.

3) Since the parity of a θ-characteristics is invariant under deformations, $\overline{S}_g$ is the disjoint union

$$\overline{S}_g = \overline{S}_g^+ \sqcup \overline{S}_g^-$$

of the two closed irreducible subvarieties of even and odd spin curves of genus g.

1.5 Stable bundles on Riemann surfaces

A holomorphic rank r vector bundle E over a Riemann surface C is determined by the datum of a local trivializing system, i.e. a covering $\{U_\alpha\}_{\alpha \in I}$ and for each non–void intersection the transition function

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(r, \mathbb{C})$$

Two bundles E and E' are equivalent when there is a collection of $GL(r, \mathbb{C})$–valued holomorphic functions λ_α, defined on the whole U_α such that

$$g'_{\alpha\beta} = \lambda_\alpha \cdot g_{\alpha\beta} \lambda_\beta^{-1}$$

Since on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ the transition functions must satisfy the co-cycle condition $g_{\alpha\beta} g_{\beta\gamma} g_{\alpha\gamma}^{-1} = 1$, equivalence classes of holomorphic rank r vector bundles on a Riemann surface C are parameterized by the cohomology *set* $H^1(C, \mathcal{GL}(r))$. Only in the case $r = 1$ this space has simple geometrical properties (actually its connected components $\mathrm{Pic}_d(C)$ are principally polarized abelian varieties). When $r > 1$ $H^1(C, \mathcal{GL}(r))$ is a non-Hausdorff space, so that one has to restrict the allowable bundle in order to get sensible moduli spaces. This problem was solved by Mumford [Mu4] by introducing the notion of stable (and semi-stable) bundle. This was done in the framework of geometric invariant theory, but just like in the case of the corresponding notion for singular curves, the condition of stability of a vector bundle translates into a simple geometric requirement.

Let $E \to C$ be a rank r vector bundle over C; define its slope to be the rational number $\mu(E) := \deg(E)/\mathrm{rank}(E)$.

Definition. $E \to C$ is called a stable bundle if for every subbundle $0 \to L \to E \to E/L \to 0$, $\mu(L) < \mu(E)$, or equivalently, $\mu(E/L) > \mu(E)$. E is called semistable if inequalities hold in weak sense.

Notice that this condition strenghtens of the fundamental principle that curvature decreases in holomorphic subbundles and increases in quotients. The stability condition can be rephrased stating that E is stable if for every proper subbundle $W \subset E$ $\deg(E^* \otimes W) < 0$.

In the following we will collect some of the most relevant features of stability, referring to [AB] and [FSCA] for proofs and a more complete account of the subject.

A stable bundle is simple, i.e., every holomorphic automorphism

$$\begin{array}{ccc} E & \stackrel{T}{\longrightarrow} & E \\ & {}^{\pi}\searrow \quad \swarrow^{\pi} & \\ & C & \end{array}$$

is a constant multiple of the identity automorphisms . This can be proven as follows [NS]. If E and F are vector bundles on C,a homomorphism $E \stackrel{f}{\longrightarrow} F$ is said to be of maximal rank if the induced map $\det E \stackrel{\det f}{\longrightarrow} \det F$ is a non–zero homomorphism. Notice that, if f is of maximal rank, then $\deg E \leq \deg F$ and, if equality holds, then f is an isomorphism. Moreover, any non–zero homomorphism $E \stackrel{f}{\longrightarrow} F$ has the following canonical factorization

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \longrightarrow & E & \stackrel{\pi}{\longrightarrow} & E_2 & \longrightarrow & 0 \\ & & & & \downarrow f & & \downarrow g & & \\ 0 & \longleftarrow & F_2 & \longleftarrow & F & \stackrel{i}{\longleftarrow} & F_1 & \longleftarrow & 0 \end{array}$$

where g is of maximal rank.

Proposition. If E and F are of the same rank and degree, and at least one of them is stable, any non–zero homomorphism $E \stackrel{f}{\longrightarrow} F$ is an isomorphism.

Proof. If f is of maximal rank there is nothing to prove. So let us suppose that f is not of maximal rank, and let us factorize it as $f = i \circ g \circ \pi$, where now g is of maximal rank. We have $0 < \mathrm{rk}(E_2) = \mathrm{rk}(F_1) < \mathrm{rk}(F)$. Obviously

$$0 = \deg(E^* \otimes E) = \deg(E^* \otimes E_1) + \deg(E^* \otimes E_2)$$

$$0 = \deg(F^* \otimes F) = \deg(F^* \otimes F_1) + \deg(F^* \otimes F_2)$$

Let us suppose that F is stable and E semistable. Then $\deg(E^* \otimes E_1) \leq 0 \Rightarrow \deg(E^* \otimes E_2) \geq 0$. Since g is of maximal rank, $\deg F_1 \geq \deg E_2$ so that

$$\deg(F^* \otimes F_1) = \deg F_1 \mathrm{rk} F - \deg F \mathrm{rk} F_1 = \deg F_1 \mathrm{rk} E - \deg E \mathrm{rk} F_1 \geq$$
$$\geq \deg E_2 \mathrm{rk} E - \deg E \mathrm{rk} E_2 = deg(E^* \otimes E_2) \geq 0$$

which contradicts the assumption of stability of F.

On the other hand, if we suppose E stable, we have $\deg(F^* \otimes F_1) \leq 0 \Rightarrow \deg(F^* \otimes F_2) \geq 0$ so that

$$\deg(E^* \otimes E_1) = \deg E_1 \mathrm{rk} E - \deg E \mathrm{rk} E_1 = \deg E_1 \mathrm{rk} F - \deg F \mathrm{rk} E_1 \geq$$
$$\geq \deg F_2 \mathrm{rk} F - \deg F \mathrm{rk} F_2 = deg(F^* \otimes F_2) \geq 0$$

and we have another contradiction. ∎

Then considering the case $E \equiv F$, the $\mathbb{C}$–algebra $H^0(C, EndE)$ satisfies the hypotheses of the Gel'fand–Mazur theorem and hence is $\mathbb{C}$.

Furthermore, for semi–stable bundles a Kodaira vanishing theorem holds, i.e. if E is semi–stable and $\deg E < 0$ then E does not have any holomorphic section. Tensoring with line bundles does not alter (semi)–stability. This is a very useful property, because, as tensoring with a degree d line bundle L changes the Chern class of E by the quantity $d \cdot \mathrm{rank}\ E$, and its determinant is changed into $\det\ E \cdot L^{\mathrm{rank} E}$, equivalence classes of rank r vector bundles are classified according to the residue $mod\, r$ of the degree. Also, the dual E^* of a (semi)–stable bundle is (semi)–stable, and, given two semistable bundles E and F, their tensor product $E \otimes F$ is still semistable. Notice that this is not true for stable bundles, since, for instance, $E^* \otimes E \equiv EndE$, is strictly semi–stable.

Vector bundles can also be analyzed from the point of view of extension theory. Actually [Gu] every vector bundle admits a meromorphic section and so it admits a rank 1 sub–line bundle $\mathcal{L}$. Then the exact sheaf sequence

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{E}/\mathcal{L} \equiv \mathcal{F} \rightarrow 0$$

exhibits E as an extension of F by L. As it is well known, such extension are parameterized by their extension class, i.e. the image class of the identity map $\mathcal{F} \xrightarrow{id} \mathcal{F}$ in the first cohomology group $H^1(C, \mathcal{L} \otimes \mathcal{F}^*)$ under the coboundary map.

Proposition. Stable bundles are indecomposable, i.e. the extension class is never trivial.

Proof. A splitting of an exact sequence

$$0 \rightarrow W \xrightarrow{i} E \xrightarrow{\pi} V \rightarrow 0$$

is given by a homomorphism $V \xrightarrow{a} E$ such that $a \circ \pi = id_V$ which exhibits V as a subbundle of E. But this contradicts the stability of E since $\mu(V) > \mu(E)$ ■

A complementary way of looking at vector bundles over a curve is the following. Recall that the first homotopy group of a genus $g(\geq 2)$ algebraic curve C is the free group on $2g$ symbols $\{a_i; b_i\}$ modulo the relation $\prod_{i=1}^{g}[a_i, b_i] = 1$. Introducing one more generator J and writing the above relation as

$$\prod_{i=1}^{g}[a_i, b_i] = J,$$

one gets the universal central extension

$$0 \rightarrow \mathbb{Z} \rightarrow \Gamma \rightarrow \pi_1(C) \rightarrow 1.$$

Then, E is associated to a representation of Γ i.e. to a projective representation of $\pi_1(C)$. A fundamental result of Narasihman and Seshadri states that stable bundles are those arising from irreducible unitary representations (and the semistable ones are those ones where irreducibility fails to hold). In other words one has the following

Theorem. An indecomposable holomorphic vector bundle $E \xrightarrow{\pi} C$ is stable iff it admits a unitary connection having constant central curvature $\star\Omega = -2\pi i\mu(E)$. Such a connection is unique.

We will sketch a gauge–theoretical proof of this theorem following Donaldson [Do]. The link between gauge theory (i.e. the differential geometrical approach to vector bundles) and the algebraic geometry of their sheaves of local holomorphic sections relies on the following remarkable fact. Given a C^∞ hermitean vector bundle E over a complex manifold M with a connection A, the covariant derivative $d_A : \Omega^0(E) \to \Omega^1(E)$ splits into

$$d'_A : \Omega^0(E) \to \Omega^{1,0}(E)$$
$$d''_A : \Omega^0(E) \to \Omega^{0,1}(E)$$

Thus, by promoting the local sections in the kernel of d''_A to be the holomorphic ones, one gets a holomorphic structure on E. Conversely, if E is a holomorphic vector bundle , there is a unique unitary connection A such that $d''_A = \bar{\partial}$.

Now the group $\mathcal{G}$ of unitary automorphisms of E acts on the space of unitary connections $\mathcal{A}$ as $A \to A^u = A - d_A u \cdot u^{-1}$ and the action extends to its complexification $\mathcal{G}^{\mathbb{C}}$, which is the group of linear automorphisms of E as $A \to A^g = A - \bar{\partial}_A g \cdot g^{-1} + \left(\bar{\partial}_A g \cdot g^{-1}\right)^*$.

Obviously, A and A' define isomorphic holomorphic structures whenever they lie in the same $\mathcal{G}^{\mathbb{C}}$–orbit, and hence $\mathcal{A}/\mathcal{G}^{\mathbb{C}}$ parameterizes all non–equivalent holomorphic bundles of the same degree and rank. Given the bundle E, in the following $[E]$ will denote the $\mathcal{G}^{\mathbb{C}}$–orbit of connections on the underlying C^∞ bundle. The theorem above is true for $r = 1$, since the curvature of a connection can be always expressed as $\Omega = \partial\bar{\partial} \log |s|^2$ for s any section of the line bundle L. Then $\Omega \in H^2(C, \mathbb{R}) \Rightarrow \star\Omega \in H^0(C, \mathbb{R})$ and so is a constant by Hodge theory. In higher rank, one considers the functional (defined on the space of self adjoint sections of $End(E)$) $N(s) = \left(\int_C \star Tr(s^* \cdot s)\right)^{\frac{1}{2}}$ and define

$$J : \mathcal{A} \longrightarrow \mathbb{C}$$
$$A \rightsquigarrow N\left(\frac{\star\Omega}{2\pi i} + \mu \cdot \mathbb{I}\right)$$

This functional vanishes iff A satisfies the theorem; it is not smooth, but has the following semicontinuity property. Given a sequence $\{A_i\}$ weakly convergent to A

in the Sobolev space L_1^2 (so that Ω_{A_i} weakly converges to Ω_A in L^2, then $J(A) \leq \liminf J(A_i)$. Then consider a weakly convergent minimizing sequence of connections $\{A_i\}$ for $J(\cdot)$ with $\{A_i\} \in [E]$. By Uhlenbeck's weak compactness theorem this sequence will admit a limit $\tilde{A}$. Then, it can be proven [Do] that either $\tilde{A}$ is in $[E]$, and $J(\tilde{A}) = 0$ or E is not stable.

According to this, one can compute the dimension of the "moduli space" of rank r and degree d stable bundles, hereinafter denoted by $U(r,d)$, computing the cohomology of the complex of Lie algebra associated to the representation of $\pi_1(C)$ which defines the vector bundle E, but we prefer to do such a computation by means of a Kodaira-Spencer like approach.

Definition. A deformation of the vector bundle $E \to C$ parameterized by the pointed analytic space (T, t_0) is a vector bundle $W \to C \times T$ such that, for any $t \in T$ the restriction to the fibre over t of the projection onto the second factor W_t is a vector bundle over C, plus a fixed isomorphism between W_{t_0} and E.

In terms of transition functions, consider a local trivializing system $\{U_\alpha, g_{\alpha\beta}\}$, and assume, for the sake of simplicity, that the open sets U_α are also coordinate patches, so that, actually, one can think of the transition functions as holomorphic maps from domains of $\mathbb{C}$ to $GL(n,\mathbb{C})$.Then a deformation will be given by introducing extra parameters $t_1, \ldots, t_n$ and considering $GL(n,\mathbb{C})$–valued functions $\tilde{g}_{\alpha\beta}(z_\beta; t)$ which satisfy

a) $\tilde{g}_{\alpha\beta}(f_{\beta\gamma}(z_\gamma); t)\tilde{g}_{\beta\gamma}(z_\gamma; t) = \tilde{g}_{\alpha\gamma}(z_\gamma; t)$ for every fixed $t \in T$, $f_{\alpha\beta}$ being the clutching functions on M;

b)$\tilde{g}_{\alpha\beta}(z_\beta; t_0) = g_{\alpha\beta}(z_\beta)$.

To get the dimension of the base space of a universal deformation, one has to consider infinitesimal deformations, i.e., deformations parameterized by the ring of the dual numbers. This amounts to considering T a ball in $\mathbb{C}$, taking $t_0 = 0$ and writing $\tilde{g}_{\alpha\beta}(z_\beta; t) = g_{\alpha\beta}(z_\beta) + t b_{\alpha\beta}(z_\beta)$. A simple computation (which we will describe in more details when considering vector bundles over variable curves) shows that the Maurer–Cartan local forms $g_{\alpha\beta}^{-1} \cdot b_{\alpha\beta}$ give rise to a one–cocycle in $C^1(C, End E)$. Then by the Riemann–Roch theorem one has that if $W \to C \times T$ is a universal deformation,

$$\dim T = \dim H^1(C, End(E)) = r^2(g-1)+1$$

Of course, the same result could have been gotten by noticing that the sheaf of infinitesimal automorphisms of E is actually the sheaf of sections of $End(E)$.
Given a family of vector bundles $W \to C \times T$, the set of points $t \in T$ such that W_t is stable is open in T and so the set $\mathcal{A}_s$ of connections such that the associated holomorphic vector bundle E_A is stable is open in the affine Banach space $\mathcal{A}$ and is invariant under the action of $\mathcal{G}^{\mathbb{C}}$. Also, thanks to the fact that $H^0(C, End(E)) = \mathbb{C}$ if E is stable, the quotient $\overline{\mathcal{G}} = \mathcal{G}^{\mathbb{C}}/\mathbb{C}^*$ acts properly and freely on $\mathcal{A}_s$. Hence $U(r,d) = \mathcal{A}_s/\overline{\mathcal{G}}$ inherits the structure of a $r^2(g-1)+1$–dimensional smooth complex manifold. This is also a (coarse) moduli space for stable rank r and degree d vector bundles over C. It has sound geometrical properties: it is a quasi–projective algebraic variety and admits (for $g > 2$) a natural compactification of codimension greater than 2 whose points are (isomorphism classes) of semistable bundles. When r and d are coprime the situation is greatly simplified (essentially because the notion of stability and semi–stability coincide) and the following results hold:
a) $U(r,d)$ is projective
b) $U(r,d)$ represent the functor $T \longrightarrow U(r,d)(T)/Pic(T)$ where two vector bundles W_1 and W_2 over $T \times C$ are identified whenever there is a line bundle $L \to T$ such that $W_1 \simeq W_2 \otimes \mathrm{pr}_1^*(L)$.

Taking determinants gives rise to a map $\det : U(r,d) \to \mathrm{Pic}_d$ whose fibre over L is clearly the moduli space of stable bundles with fixed determinant $L \in \mathrm{Pic}_d$, Pic_d being the degree d component of the Picard group of C. This space will be denoted $U_0(r,L)$. Its dimension is readily computed as $\dim U_0(r,L) = (r^2-1)(g-1)$, since the sheaf of infinitesimal automorphisms of a bundle with fixed determinant is the sheaf of trace–free endomorphisms of E, $End_0(E)$. The fact that the dimensions of $U(r,d)$ and $U_0(r,L)$ differ by the genus of the algebraic curve is apparent from the following observation. The Jacobian $J(C) \simeq \mathrm{Pic}_d$ acts both on $U(r,d)$ and on Pic_d by tensor product; if we let $J(C)$ act on Pic_d by the rth power map, then the diagram

$$\begin{array}{ccc} U(r,d) & \xrightarrow{\det} & \text{Pic}_d \\ \downarrow \otimes L & & \downarrow \otimes L^r \\ U(r,d) & \xrightarrow{\det} & \text{Pic}_d \end{array}$$

commutes and shows that [AB]

$$U(r,d) = (U_0(r,L) \times \text{Pic}_d) / H^1(C, \mathbb{Z}_r)$$

As proven by Harder and Narasihman [HS], the action of $H^1(C, \mathbb{Z}_r)$ on the rational cohomology of $U_0(r,L)$ is trivial, so that

$$H^*(U(r,d), \mathbb{Q}) = H^*(U_0(r,L), \mathbb{Q}) \otimes H^*(J_C, \mathbb{Q})$$

An analogous result holds for the first homotopy group: $U_0(r,L)$ is simply connected and

$$\pi_1(U(r,d)) = H^1(C, \mathbb{Z}).$$

Some results about the analytic cohomology of $U_0(r,L)$, still in the coprime case, are the following.

a) The Picard group $\text{Pic}(U_0(r,L))$ is free cyclic and it is generated by the inverse of the square root of the canonical bundle;

b) $H^i(U_0(r,L), \mathcal{T}_{U_0(r,L)}) = 0$ if $i \neq 1$

and $H^1(U_0(r,L), \mathcal{T}_{U_0(r,L)}) \simeq H^1(C, \mathcal{T}_C)$

c) the structure sheaf $\mathcal{O}_{U_0(r,L)}$ has no cohomology except for the zeroth group which is obviously $\mathbb{C}$.

We end this section discussing about the Poincaré vector bundle over $U(r,d)$. Let us consider a family vector bundles parameterized by the moduli space $U(r,d)$. A vector bundle $V \to U(r,d) \times C$ is called a Poincaré vector bundle if

$$V \restriction_{[E] \times C} \equiv E \; \forall E \in U(r,d)$$

The Poincaré vector bundle has the following universal property; given any other family of vector bundles over C parameterized by an analytic space S,

$$\tilde{V} \to S \times C$$

there is a unique map $S \xrightarrow{f} U(r,d)$ such that

$$(id_C \times f)^* V \equiv \tilde{V} \otimes p_1^*(L)$$

where $p_1 : S \times C \to S$ is the projection onto the first factor and L is a line bundle over S.

It is apparent from the definition that a Poincaré vector bundle is defined up to tensorization with a line bundle on $U(r,d)$. Hence one has a normalization problem, which can be solved in the following way [AB]. Let V be any Poincaré vector bundle, and consider the line bundles $E \equiv \det\, p_{1!}V$ and $F \equiv \det\, V\restriction_{U(r,d)\times\{p\}}$. Since the Poincaré vector bundle is defined only up to tensorization with pull–backs of line bundles on the base, it is important to control the behaviour of $c_1(E)$ and $c_1(F)$ under such tensorizations. By the Grothendieck-Riemann-Roch theorem,

$$c_1(p_{1!}(V \otimes p_1^*(L))) = p_{1*}(ChV \otimes p_1^*(L) \cdot Td_{T/})$$

so that

$$c_1(p_{1!}(V \otimes p_1^*(L))) = c_1(V) + [r(1-g)+d]c_1(L) = c_1(V) + \chi(V)c_1(L)$$

On the other hand,

$$c_1(\det\, V\restriction_{U(r,d)\times\{p\}} \otimes p_1^*(L)) = c_1(\det\, V\restriction U(r,d)\times\{p\}) + rc_1(L)$$

If we define $\tilde{V} = v \otimes p_1^*(E_V^p F_V^q)$ then $\tilde{V} \simeq \tilde{W} \equiv \widetilde{V \otimes p_1^*}(L)$ if and only if the diophantine equation

$$1 + \chi(V)p + rq = 0$$

admits a solution. Taking into account the expression of the Euler characteristics $\chi(V)$, one arrives at

$$p = q(g-1) - \frac{(qd+1)}{r}$$

which can be solved thanks to the fact that $(r,d)=1$. Then if q_0 is the unique integer $-r < q_0 < 0$ satisfying $q_0 \equiv -1 \mod r$, $\tilde{V}$ is uniquely defined up to isomorphism. This procedure gives us a relation between $c_1(E_{\tilde{V}})$ and $c_1(F_{\tilde{V}})$ since if (p_0,q_0) are determined by the request above,

$$\tilde{V} \simeq \tilde{V} \otimes p_1^*(E_{\tilde{V}}^{p_0} \otimes F_{\tilde{V}}^{q_0})$$

and hence

$$p_0 c_1(E_{\tilde{V}}) + q_0 c_1(F_{\tilde{V}}) = 0$$

Restricting V to a fiber of $U(r,d) \xrightarrow{det} \mathrm{Pic}_d(C)$ gives rise to line bundles $E_{\tilde{V}}^0$ and $F_{\tilde{V}}^0$ which satisfy the same relation. Notice that, in the general case, neither $E_{\tilde{V}}^0$ nor $F_{\tilde{V}}^0$ can be taken as the free generator of $Pic(U_0(r,L))$.

A representative for the generator can be obtained by means of the fact that [NR2] the tangent bundle to $U_0(r,L)$ is the first direct image sheaf $R^1_{p_{1!}} End_0 \tilde{V}$ and the already recalled fact that its first Chern class is minus the square of the ample generator of $Pic(U_0(r,L))$. In fact the Grothendieck-Riemann-Roch theorem, when applied to the sheaf $End_0(V)$ gives

$$c_1(p_{1!}(End_0(\tilde{V})) = -c_1(R^1_{p_{1!}} End_0 \tilde{V}) = -p_{1*}(Ch End_0(\tilde{V}) Td_{T/})$$

Now, $Ch(End_0\tilde{V}) = Ch(End\tilde{V}) - 1 = Ch(\tilde{V}^* \otimes \tilde{V}) - 1 = -(1 - Ch(\tilde{V}) \cdot Ch(\tilde{V}^*)$. Then one has

$$c_1(R^1_{p_{1!}} End_0 \tilde{V}) = p_{1*}\{1 - \{(n + Ch_1(\tilde{V}) + Ch_2(\tilde{V}) + \cdots) \cdot \\ (n + Ch_1(\tilde{V}^*) + Ch_2(\tilde{V}^*) + \cdots) \cdot (1 - \frac{1}{2} c_1(\omega_/))\}\}$$

where $\omega/$ is the relative dualizing sheaf. Considering only the appropriate pieces,

$$c_1(R^1_{p_{1!}} End_0 \tilde{V}) = p_{1*}\{1 + \{n + Ch_1(\tilde{V}) + Ch_2(\tilde{V}) +) \cdot \\ \cdot (n - Ch_1(\tilde{V}) + Ch_2(\tilde{V}) +)(1 - \frac{1}{2} c_1(\omega_/) +)\}\} = -2 \cdot p_{1*}(Ch_2(\tilde{V}))$$

which exhibits $p_{1*}(Ch_2(\tilde{V}))$ as the generator for $Pic(U_0(r,L))$.

Chapter 2.

Super–Algebraic Geometry.

Supersymmetry, first invented in the early days of string theory, has received in the last two decades a growing attention in theoretical quantum field theory especially in view of the hope of getting finite theories of elementary interactions. Apart from that, in two–dimensions it acquires a remarkable significance, since some well known statistical models such as the Ising model with vacancies exhibit such a symmetry at their critical points.

Its main mathematical interest lies in the fact that it is the first example of (mildly) non–commutative geometry, and its study is an ideal training in testing new dimension in geometry in the spirit as advocated by Manin [M1]. There are several geometric structures which are generically called supermanifolds both in the physical and in the mathematical literature. In this thesis work we will mainly stick to the Berezin-Kostant-Leites approach to "supergeometry"[Be] [L], and this chapter is devoted to a collection of some basic definitions and results concerning this theory in the holomorphic framework. The main motivations for this choice are the following. First, in this picture the "anticommuting coordinates" will emerge as local generators of the "minimal" extension of the structure sheaf of an ordinary manifold. This keeps us as close as possible to the framework suggested by works on supersymmetry in physics. Second, from a mathematical standpoint, as will be apparent in the sequel, this category is very close to the one of ordinary complex spaces, thus allowing the use of powerful techniques of sheaf theory and complex geometry without the need of considering sheaves with infinitely many generators. In any case, the spirit of the

most correct (i.e. closest to physics) approach to supergeometry is [S2] to regard a supermanifold as a functor from the category of Grassmann algebras to the category of sets. This will be respected throughout our analysis.
Section 2.1 deals with some structure theory of supervarieties, (which, in the complex case happens to be far richer than in the C^∞ case), with special emphasis on the so–called splitting problem. Section 2.2 will be devoted to an outline of the set up of their deformation theory.

2.1. Complex Superanalytic Spaces.

Whatever mathematical environment for a supersymmetrical field theory one has in mind, complex superspaces are topological spaces X together with a structure sheaf which is a $\mathbb{Z}_2$-graded extension of the ordinary structure sheaf $\mathcal{O}_X$. Let us recall their definition, deferring to appendix A.3 for a listing of some of the main properties of $\mathbb{Z}_2$-graded rings and algebras.

Definition . (i) A ringed space $(X, \mathcal{A}_X)$ is a topological space X together with a sheaf of rings $\mathcal{A}_X$ over it. X is commonly called the underlying space and $\mathcal{A}_X$ the structure sheaf.

(ii) a map between two ringed spaces $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$ is a continuous map $f : X \to Y$ and a sheaf homomorphism $f_\sharp : \mathcal{A}_Y \to f_* \mathcal{A}_X$ (or equivalently $f^\sharp : f^* \mathcal{A}_Y \to \mathcal{A}_X$)

Definition. (i) Let F be a field. An F-ringed space is a ringed space $(X, \mathcal{A}_X)$ such that the restriction $\mathcal{A}_X{\restriction_U}$ of the structure sheaf to any open set $U \subset X$ has the structure of an F-algebra with unity. One also assumes that for all stalks $\mathcal{A}_p$ of $\mathcal{A}_X$ a morphism of F-algebras $c_p : \mathcal{A}_p \to F$ is given. Its kernel is then a maximal two-sided ideal I_p.

(ii) a map of F-ringed spaces is a map of ringed spaces such that $f_\sharp$ is a morphism of sheaves of F-algebras making the diagram

$$\begin{array}{ccc} & F & \\ {}^{c_{f(p)}}\nearrow & & {}^{c_p}\nwarrow \\ \mathcal{A}_{Y_{f(p)}} & \xrightarrow{f_\sharp} & f_*\mathcal{A}_{X_p} \end{array}$$

commutative.

Definition. A complex supervariety is a C–ringed space $(X, \mathcal{A}_X)$ such that

(i) X is Hausdorff with countable basis.

(ii) $A_X = \mathcal{A}_X^0 \oplus A_X^1$ is a sheaf of graded commutative C-algebras.

(iii) $(X, \mathcal{A}_X^0)$ is an analytic space.

(iv) $\mathcal{A}_X^1$ is a coherent $\mathcal{A}_X^0$ - module.

(v) if $\mathcal{N} \hookrightarrow A_X$ is the ideal of nilpotents and $\mathcal{A}_{red} = \mathcal{A}_X/\mathcal{N}$, then $(X, \mathcal{A}_{red})$ is an ordinary complex space.

(vi) the $\mathcal{A}_{red}$-module $\mathcal{E} = \mathcal{N}/\mathcal{N}^2$ is locally free and $\mathcal{A}_X$ is locally isomorphic to its Grassmann algebra $\wedge^*(\mathcal{E})$.

Moreover, $(X, \mathcal{A}_X)$ is said to have dimension $m|n$ if $m = dim(X, \mathcal{A}_{red})$ and $n = rk_{\mathcal{A}_{red}}\ \mathcal{E}$.

Remark. What actually characterizes the category of Berezin – Kostant – Leites supermanifolds among all possible geometric realizations of supersymmetry is condition (v).

The underlying complex space $(X, \mathcal{A}_{red})$ is often denoted X_{red} for the sake of brevity. As in the case of ordinary manifolds, one can define a supermanifold as a space locally isomorphic to a prototypical one, a "model space". Such models are called superdomains. As in this thesis we are primarily interested in holomorphic graded manifolds from now on we will stick to that case. Quite obviously, all what we are recalling now is true also for C^∞ real manifolds, provided one substitutes C with R and "'holomorphic" with "infinitely differentiable".

Definition . A superdomain $\overline{U}$ of dimension $m|n$ is a ringed space of the form

$$\overline{U} := (U, \mathcal{O}_U \otimes \wedge^*(\mathbb{C}^n))$$

where U is a domain in $\mathbb{C}^m$, $\mathcal{O}_U$ is the ring of holomorphic functions on U and $\wedge^*(\mathbb{C}^n)$ is the Grassmann algebra of $\mathbb{C}^n$. Notice that the ring $\mathcal{A} := \mathcal{O}_U \otimes \wedge^*(\mathbb{C}^n)$, has a natural $\mathbb{Z}$ - grading, and hence inherits a rougher $\mathbb{Z}_2$ grading. This apparently trivial observation will be of primary interest in the discussion of splitness and projectedness of supermanifolds.

Remark. An ordinary domain $(U, \mathcal{O}_U)$ is thus naturally a superdomain of dimension $m|0$. Moreover, considering the subalgebra $\mathcal{N}$ generated by nilpotent elements in A and taking the quotient $\mathcal{A}/\mathcal{N}$ one gets the natural map

$$\mathcal{A}/\mathcal{N} \longrightarrow \mathcal{O}_U$$

i.e., (recalling the definition of maps between ringed spaces) an embedding

$$(U, \mathcal{O}_U) \longrightarrow (U, \mathcal{O}_U \otimes \wedge^*(\mathbb{C}^n))$$

Morphisms of superdomains can be characterized in the following way. Let $\overline{U} = (U, \mathcal{O}_U \otimes \wedge^*(\mathbb{C}^n))$ and $\overline{V} = (V, \mathcal{O}_V \otimes \wedge^*(\mathbb{C}^p))$ and let x^i, ξ^j be coordinates in $\overline{U}$ (this means that the x's are coordinates in U and the ξ's are a free set of odd generators for $\wedge^*(\mathbb{C}^n)$). Then, given m ($m = dimV$) even sections y^i and p odd sections η^k of the sheaf $\mathcal{O}_U \otimes \wedge^*(\mathbb{C}^n)$ such that $y^i(x, 0)$ lies in V, one defines a morphism of superdomains by means of

$$\mathcal{A}_{\overline{V}} \ni b(y, \eta) \rightsquigarrow b(y(x, \xi), \eta(x, \xi)) \in \mathcal{A}_{\overline{U}}$$

Conversely, any morphism of superdomains has this form.

One can give the constructive definition of a supermanifold as an object built up by glueing superdomains.

Definition . A complex supermanifold of dimension $m|n$ is a $\mathbb{C}$-ringed space $(X, \mathcal{A}_X)$ satisfying the following conditions.

(i) Every point $p \in X$ has a neighbourhood U_p s.t. there exists an isomorphism of $\mathbb{C}$-ringed spaces

$$(f, f_\sharp) : (U, \mathcal{A}\restriction_U) \rightarrow (W, \mathcal{O}_W \otimes \wedge^*(\mathbb{C}^n))$$

where $(W, \mathcal{O}_W \otimes \wedge^*(\mathbb{C}^n))$ is an $(m|n)$ dimensional superdomain. Such an $\overline{f} = (f, f_\sharp)$ will be called a chart.

(ii) if $U \cap V \neq \emptyset$ and $(g, g_\sharp) : (V, \mathcal{A}\restriction_V) \to (Y, \mathcal{O}_Y \otimes \wedge^*(\mathbb{C}^n))$ is another chart, then the composite map $(f, f_\sharp) \circ (g, g_\sharp)^{-1}$ is an isomorphism of superdomains wherever defined. Such maps will be called transition functions.

Proposition .[Be] Suppose we are given a collection of superdomains

$$\overline{U}_\alpha = (U_\alpha, \mathcal{O}_{U_\alpha} \otimes \wedge^*(\mathbb{C}^n))$$

and, $\forall$ ordered pair α, β of indices an open subspace $U_{\alpha\beta}$ of U_α together with morphisms

$$\overline{\varphi}_{\alpha\beta} : \overline{U}_{\alpha\beta} \longrightarrow \overline{U}_{\beta\alpha}$$

such that they satisfy the cocycle condition

$$\overline{\varphi}_{\alpha\beta} \circ \overline{\varphi}_{\beta\gamma} \circ \overline{\varphi}_{\gamma\alpha} = Id_{\overline{U}_{\alpha\beta\gamma}}$$

Then there exists a unique (up to isomorphisms) complex supermanifold $(X, \mathcal{A}_X)$ having the $\overline{\varphi}_{\alpha\beta}$'s as transition functions.

Via the identification of analytic locally free sheaves of constant rank on X and (sheaves of local sections of) vector bundles on X, roughly speaking a supermanifold is a ringed space whose structure sheaf is *locally* isomorphic to the sheaf of sections of a Grassmann algebra of a vector bundle. This interpretation raises a natural question, i.e. how far this local isomorphism can be "globalized". To be concrete, from the very definition of a supermanifold, one gets the following two sheaf exact sequences [R1]

$$0 \to \mathcal{N} \hookrightarrow \mathcal{A} \to \mathcal{A}/\mathcal{N} = \mathcal{O} \to 0 \qquad (a)$$

$$0 \to \mathcal{N}^2 \hookrightarrow \mathcal{A} \to \mathcal{A}/\mathcal{N}^2 = \mathcal{O} \oplus \mathcal{E} \to 0 \qquad (b)$$

If there is a splitting $0 \to \mathcal{O} \overset{i}{\hookrightarrow} \mathcal{A}$ of (a) the supermanifold is said to be *projected* and if (b) splits as $0 \to \mathcal{O} \oplus \mathcal{E} \overset{\mu}{\hookrightarrow} \mathcal{A}$ it is said to be *split*. Notice that μ must satisfy

$$\mu(f\xi) \;=\; \mu(f) \cdot \mu(\xi) \quad \forall f \in O \text{ and } \xi \in \mathcal{O} \oplus \mathcal{E}.$$

The terminology deserves a bit of explanation. For what projectedness is concerned, notice that, having an injective map $\mathcal{O} \xrightarrow{i} \mathcal{A}$ gives a map (id, i) between the ringed spaces

$$(id, i) : (X, \mathcal{A}_X) \longrightarrow (X, \mathcal{O}_X)$$

so that the supermanifold "projects" down to the underlying manifold.

As for the splitness, suppose (b) splits. Then we can consistently extend $\mu : \mathcal{O} \oplus \mathcal{E} \to \mathcal{A}$ to $\hat{\mu} : \wedge^*(\mathcal{E}) \longrightarrow \mathcal{A}$ by means of

$$\hat{\mu}(f \cdot \xi_1 \wedge \cdots \wedge \xi_n) = \mu(f) \cdot \mu(\xi_1) \wedge \cdots \wedge \mu(\xi_n)$$

which is clearly a $\mathbb{Z}_2$–ring isomorphism. Then splitness of a supermanifold means that the structure sheaf $\mathcal{A}$ is globally isomorphic to the sheaf of section of a Grassmann algebra of a vector bundle $\mathcal{E} \xrightarrow{\pi} X$.

Remark. Every $m|1$-dimensional supermanifold is trivially split. In fact in this case, $\mathcal{N}^2 = 0$ and hence the sequence (b) collapses to

$$0 \to \mathcal{A} \to \mathcal{O} \oplus \mathcal{E}(= \wedge^*(\mathcal{E})) \to 0.$$

We next want to enter in more details the issues of splitness and projectedness of supermanifolds. A keen starting point, due to Rothstein [R1], is to try to to characterize how far a given supermanifold is different from its "split counterpart". Namely, given $(X, \mathcal{A})$ and $(X, \wedge^*(\mathcal{E}))$ with $\mathcal{E} \simeq \mathcal{A}/\mathcal{N}$ we have two supermanifolds which agree, by construction, up the so-called first infinitesimal neighbourhood, and one wants to set up a machinery telling how far this isomorphism can be pushed on.

Let $Aut \wedge^* (\mathcal{E})$ denote the sheaf of parity preserving $\mathbb{C}$-linear automorphisms of $\wedge^*(\mathcal{E})$ and $\wedge_{(k)}(\mathcal{E}) := \sum_{j \geq k} \wedge^j(\mathcal{E})$. If $g : \wedge^*(\mathcal{E}) \longrightarrow \wedge^*(\mathcal{E})$ is an automorphism, then it induces naturally an automorphism $\tilde{g}$ of $\mathcal{E}$. Let then $Aut^+ \wedge^*(\mathcal{E})$ denote the subsheaf of $Aut \wedge^*(\mathcal{E})$ s.t. $\tilde{g} = id_{\mathcal{E}}$. $Aut^+ \wedge^* (\mathcal{E})$ can be identified with a more tractable object. In fact, let k be an even integer and let $Der_k \wedge^* (\mathcal{E})$ be the sheaf of derivations in $\wedge^*(\mathcal{E})$ which increase the degree by k. As above, let $Der^{(k)} := \sum_{j \leq 2k \leq n} Der_{2k} \wedge^* (\mathcal{E})$. The following holds [R1]

Proposition

$$\exp : Der^{(2)} \wedge^* (\mathcal{E}) \rightarrow Aut^+ \wedge^* (\mathcal{E})$$

is a bijection.

Now, let us consider an open cover $\{U_\alpha\}_{\alpha \in A}$ such that the sheaf $\mathcal{A}\restriction_{U_\alpha}$ is isomorphic to $O_{U_\alpha} \otimes \wedge^*(\mathbb{C}^n)$ and also U_α trivializes the vector bundle E. Let us denote

$$Gr(\mathcal{A}) := \mathcal{A}/\mathcal{N} \oplus \mathcal{N}/\mathcal{N}^2 \oplus \cdots \oplus \mathcal{N}^{(n-1)}/\mathcal{N}^n$$

and suppose that the supercoordinatization

$$\varphi^\sharp_\alpha : \mathcal{A}\restriction_{U_\alpha} \longrightarrow \wedge^*(\mathcal{E})\restriction_{U_\alpha}$$

is the identity when thought as being defined on $Gr(\mathcal{A})\restriction_{U_\alpha}$. Then the cocycle $\varphi^\sharp_\alpha \circ {\varphi^\sharp_\beta}^{-1}$ defines $\mathcal{A}$ up to isomorphisms and hence one gets the following

Proposition [R1,Gr] The isomorphism classes of supermanifolds $(X, \mathcal{A}_X)$ with underlying $\mathcal{O}$-module $\mathcal{E}$ are in a natural $1-1$ correspondence with the cohomology set

$$H^1(X, Aut^+ \wedge^* (\mathcal{E})) \simeq H^1(X, Der^{(2)} \wedge^* (\mathcal{E})).$$

Then a (smooth) supermanifold is characterized by the datum of

(i) a complex manifold X

(ii) a holomorphic vector bundle $E \xrightarrow{\pi} X$

(iii) a cohomology class $\tau \in H^1(X, Aut^+ \wedge^* (\mathcal{E}))$ or $\log \tau \in H^1(X, Der^{(2)} \wedge^* (\mathcal{E}))$.

Notice that, being $Aut^+ \wedge^* (\mathcal{E})$ non-abelian, $H^1(X, Aut^+ \wedge^* (\mathcal{E}))$ is a pointed set rather than a group, and its distinguished point labels the isomorphism class of the split supermanifold.

To sharpen the analysis above, one can express the obstruction to splitness and projectedness by means of "a chain of obstructions" [Gr].

Proposition . For any holomorphic vector bundle $E \xrightarrow{\pi} X$, $Aut^+ \wedge^* (\mathcal{E})$ admits a decreasing filtration $Aut^+_k \wedge^* (\mathcal{E})$ satisfying

(i) $Aut^+_2 \wedge^* (\mathcal{E}) = Aut^+_1 \wedge^* (\mathcal{E})$

(ii) if k is even $Aut_k^+ \wedge^* (\mathcal{E})/Aut_{k+1}^+ \wedge^* (\mathcal{E}) \simeq Der_{\mathcal{O}_X}(\mathcal{O}_X, \wedge^k(\mathcal{E})) \simeq TX \otimes \wedge^k(\mathcal{E})$

(iii) if k is odd $Aut_k^+ \wedge^* (\mathcal{E})/Aut_{k+1}^+ \wedge^* (\mathcal{E}) \simeq Hom_{\mathcal{O}_X}(\wedge^1(\mathcal{E}), \wedge^k(\mathcal{E})) \simeq \mathcal{E}^* \otimes \wedge^k(\mathcal{E})$

Then it is clear that the obstruction to splitness is given by cohomology classes $\tau_k \in H^1(X, Aut_k^+ \wedge^*(\mathcal{E})/Aut_{k+1}^+ \wedge^*(\mathcal{E}))$ and, in particular, the obstruction to projectedness is given by the even classes τ_{2k}.

The proposition above gives immediately a nice proof of Batchelor's theorem which states that, in the category of C^∞ supermanifold, every object is the wedge product of a vector bundle. This follows at once by noticing that in this case $Der^{(2)} \wedge^* (\mathcal{E})$ is a sheaf of $C^\infty(X)$–modules and hence, being $C^\infty(X)$ fine, its first cohomology vanishes.

The above analysis can be easily in terms of transition functions. Let $\{U_\alpha\}$ be a (locally finite) open covering of X and suppose we can express the cocycle $\varphi_\alpha^\sharp \circ {\varphi_\beta^\sharp}^{-1}$ by means of

$$\begin{cases} x_\alpha = f_{\alpha\beta}(x_\beta, \xi_\beta) \\ \xi_\alpha = g_{\alpha\beta}(x_\beta, \xi_\beta) \end{cases}$$

Expanding in power series in the odd generators ξ_β^i's one has

$$\begin{cases} x_\alpha^\mu = f_{\alpha\beta}^\mu(x_\beta) + f_{\alpha\beta\, ij}^\mu(x_\beta)\xi_\beta^i\xi_\beta^j + \cdots \\ \xi_\alpha^i = g_{\alpha\beta\, j}^i(x_\beta)\xi_\beta^j + g_{\alpha\beta\, jkl}^i\xi_\beta^j\xi_\beta^k\xi_\beta^l + \cdots \end{cases}$$

Whenever the supermanifold is (isomorphic to) $(X, \wedge^*(\mathcal{E}))$, one can find a refinement $\{V_\alpha\}$ of $\{U_\alpha\}$ such that the transition functions for the generators are those for a holomorphic vector bundle, i.e.

$$\begin{cases} x_\alpha^\mu = f_{\alpha\beta}^\mu(x_\beta) \\ \xi_\alpha^i = g_{\alpha\beta\, j}^i(x_\beta)\xi_\beta^j \end{cases}$$

which means that the "higher order terms" appearing in the above most general transformation law are, roughly speaking, cohomologous to zero.

To give a more concrete meaning to the preceeding discussion we show here that the set of non-split supermanifolds is non-void by constructing a (very elementary) example.

Let us consider two copies of $(\mathbb{C}^*, \mathcal{O}_{\mathbb{C}^*} \otimes \wedge(\mathbb{C}^2))$ parameterized by (z, ξ_1, ξ_2) and (w, η_1, η_2) and glue them by means of the following cocycle:

$$\begin{cases} z = \frac{1}{w} + \frac{\eta_1 \cdot \eta_2}{w^3} \\ \xi_i = -\frac{1}{w^2}\eta_i \end{cases}$$

Thus the underlying manifold is the Riemann sphere $\mathbb{P}^1$ and $\mathcal{E} = K \oplus K$ is the direct sum of two copies of the canonical bundle K. The obstruction class reduces to τ_2 represented by the cocycle

$$\frac{1}{w^3} \cdot \frac{\partial}{\partial z} \otimes \eta_1 \wedge \eta_2 \text{ in } H^1(\mathbb{P}^1, K^{-1} \otimes \wedge^2(K \oplus K))$$

Now, under the identification $H^1(\mathbb{P}^1, K^{-1} \otimes \wedge^2(K \oplus K)) \simeq H^1(\mathbb{P}^1, K)$ the obstruction cocycle becomes dw/w which is the generator of $H^1(\mathbb{P}^1, K) = \mathbb{C}$.

Finally, as it should be expected, most of the constructions of ordinary differential geometry carries over the graded-commutative case. For instance, one has a sound notion of what are to be considered the correct generalizations of the notion of vector bundle. Obviously, as Berezin–Kostant–Leites supermanifolds are introduced as ringed spaces, vector bundles are to be generically defined in sheaf–theoretical terms as follows.

Definition . Let $(X, \mathcal{A}_X)$ be a supermanifold. We define a rank $r|s$ super vector bundle over to be an $\mathcal{A}_X$ – locally free sheaf $\mathcal{F}$ over X of rank $r|s$. In particular, a line bundle $\hat{\mathcal{L}}$ is a rank $1|0$ vector bundle over $(X, \mathcal{A}_X)$.

Definition . The tangent sheaf $\hat{\mathcal{T}}X$ to $(X, \mathcal{A}_{\mathcal{X}})$ is the sheaf defined by the presheaf

$$U \longrightarrow \hat{D}er(\mathcal{A}_{\mathcal{X}})\restriction_U$$

where $\hat{D}er(\mathcal{A}_{\mathcal{X}})\restriction_U$ is the sheaf of graded derivations of the ring $\mathcal{A}_{\mathcal{X}})\restriction_U$.

To be definite, we will choose left derivations, and consequently $\hat{\mathcal{T}}X\restriction_U$ admits a natural structure of free left $\mathcal{A}_X$-module, of rank equal (by definition!) to the dimension of $(X, \mathcal{A}_X)$.

2.2. Deformation theory of complex superspaces.

The aim of this section is to give some definitions of the deformation theory of complex superspaces, and to show that, the usual technology of the ordinary case (discussed, for the case of curves, in chapter 1) can be carried over to the graded commutative category.

By a complex analytic superspace we mean, in analogy with ordinary reduced complex spaces, a ringed space $(X, \mathcal{A}_X)$, where $\mathcal{A}_X$ is a sheaf of graded commutative $\mathbb{C}$-algebras, which is locally isomorphic to a complex analytic superspace patch, where the latter is defined as follows.
Let us consider a superdomain $\hat{U} := (U, \mathcal{A}_U) = (U, \mathcal{O}_U \otimes \wedge^*(\mathbb{C}^q))$, a set $\{f_1, \cdots, f_k\}$ of sections of $\mathcal{A}_U$ and the ideal $\mathcal{J}$ they define in $\mathcal{A}_U$. The reduction modulo nilpotents defines a complex analytic space patch V so that one defines the complex analytic superspace patch (defined by the f_i's) as the ringed space

$$\hat{V} = (V, \mathcal{A}_U/\mathcal{J})$$

Let $\hat{V}$ and $\hat{W}$ two complex analytic space patches, both subsuperspaces of $\mathbb{C}^{p|q}$. They are called equivalent at $x \in \mathbb{C}^p$ iff there is a neighbourhood U of x such that $\mathcal{A}_V \restriction_{U\cap V} \tilde{\rightarrow} \mathcal{A}_W \restriction_{U\cap W}$ are isomorphic.

Definition . A *germ* of complex superspace at x is an equivalence class of complex superspaces.

Morphisms between such objects are defined by taking representatives and morphisms between them and requiring the correspondent equivalence condition.

Definition . Let $(\hat{S}, s)$ be a germ of complex superspace at s. A deformation $(\hat{\mathcal{X}}, \hat{S})$ of a complex superspace $(X, \mathcal{A}_X)$ over $(\hat{S}, s)$ is a commutative diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{i} & \hat{\mathcal{X}} \\ \downarrow & & \downarrow \pi \\ \{s\} & \longrightarrow & \hat{S} \end{array}$$

where $\pi : \hat{\mathcal{X}} \to \hat{S}$ is a flat complex superspace morphism and i is a fixed isomorphism between $\hat{X}$ and the central fiber $\pi^{-1}(s)$.

A morphism

$$(\hat{\mathcal{X}}, \hat{S}) \longrightarrow (\hat{\mathcal{Y}}, \hat{T}')$$

is a pair of complex superspace and germ morphisms $\mathcal{X} \xrightarrow{\Phi} \mathcal{Y}$ and $S \xrightarrow{f} T$ such that the following diagram

$$\begin{array}{ccccc}
\mathcal{X} & & \xrightarrow{\Phi} & & \mathcal{Y} \\
 & {}^{i}\nwarrow & & {}_{j}\nearrow & \\
\downarrow\pi & & X & & \downarrow\sigma \\
S & & \overset{f}{\longrightarrow}\!\!\!\!\downarrow & & T \\
 & \nwarrow & & \nearrow & \\
 & & \{*\} & &
\end{array}$$

is commutative.

Given a deformation of a complex superspace $(\mathcal{X}, (S, s))$ and a germ morphism

$$(T, t) \xrightarrow{f} (S, s)$$

one defines the pullback deformation $f^*(\mathcal{X})$ over (T, t) as the fibered product $\mathcal{X} \times_f S$, meaning that, as topological spaces,

$$F^*(\mathcal{X}) = \{(x, t) \in \mathcal{X} \times T \big| f(t) = \pi(x)\}$$

and, as for the structure sheaf,

$$\mathcal{A}_{f^*\mathcal{X}} = \mathcal{A}_T \hat{\otimes} \mathcal{A}_{\mathcal{X}} / \left((id_T \times \pi)^* \mathcal{J}\right)$$

where $\mathcal{J}$ is the ideal defining the graph of f.

We denote by $Def(X, S)$ the set of isomorphism classes of deformations of X over S.

The pull-back deformation comes equipped with two morphisms p_1 and p_2, mapping onto the first and second component of each pair (x,t), which make the diagram

$$\begin{array}{ccc} f^*\mathcal{X} & \xrightarrow{p_1} & \mathcal{X} \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow{f} & S \end{array}$$

commutative.

Recall that the pull-back deformation has the following property: for any deformation $\pi' : \mathcal{X}' \to B'$ and any morphisms of deformations $(\Gamma, f) : \mathcal{X}' \to \mathcal{X}$ there exists a unique morphism $(\Psi, h) : \mathcal{X}' \to f^*\mathcal{X}$ such that the diagram

$$\begin{array}{ccccc} \mathcal{X}' & & \xrightarrow{\Psi} & & f^*(\mathcal{X}) \\ & {\scriptstyle\Gamma}\searrow & & \swarrow & \\ \downarrow{\scriptstyle \pi'} & & \mathcal{X} & & \downarrow \\ S' & & \downarrow \xrightarrow{id} & & S' \\ & {\scriptstyle f}\searrow & & {\scriptstyle f}\swarrow & \\ & & S & & \end{array}$$

commutes.

Given a complex superspace $(X, \mathcal{A}_X)$, the pull-back makes $Def(X, \cdot) : \mathcal{CS} \to Ens$ into a contravariant functor from the category $\mathcal{CS}$ of (germs of pointed) complex superspaces to the category Ens of sets, which assigns to each $B \in \mathcal{CS}$ the set $Def(X,B)$ of isomorphism classes of deformations of X over B.

As usual in deformation theory, we say that a deformation $\mathcal{X} \in Def(X,B)$ is i) complete, if for any other deformation $\mathcal{X}' \in Def(X,B')$ there exists a morphism

$$f : B' \to B$$

such that $\mathcal{X}'$ is isomorphic to the pull-back deformation $f^*\mathcal{X}$,

ii) universal if such an f is unique or versal if all the morphisms f satisfying condition i) have the same differential.

The same definitions can be repeated when B is a purely even superspace. In this case one speaks of even completeness, even versality etc.. Notice that a purely even superspace is the same thing as a (non reduced) complex space.

Having introduced the moduli functor for complex superspaces, one can argue about moduli (super)spaces, defined as (see §1.2) spaces M realizing an isomorphism of functors $Def_X(\cdot, M) \simeq Hom(\cdot, M)$. The search for a fine moduli space in this category is almost hopeless. Namely, $\mathbb{Z}_2$-graded commutative algebras always admit the canonical involution

$$\alpha(x) = (-1)^{\tilde{x}} x$$

which play the rôle of non trivial automorphisms in the case of moduli spaces of Riemann surfaces . Nonetheless, what one is really interested in is the existence of a coarse moduli space for superspaces of a certain "topological" class. Solving this problem is tantamount to solving the problem of existence of versal deformations for $(X, \mathcal{A}_X)$. That they exist has been recently proven by Vaintrob [V] considering first evenversal deformations and then considering their extensions to "odd" directions.

What is outstandingly relevant is the extension to the supersymmetric case of the Kodaira-Spencer formalism, and, namely the definition of the super Kodaira-Spencer map, $\widehat{KS}$. Mimicking what happens in standard deformation theory, one first studies infinitesimal deformations. To this purpose, one introduces the super–commutative ring of super–dual numbers $O_S = \mathbb{C}[t, \zeta]/(t^2, t\zeta)$, where $(t, \zeta) \in \mathbb{C}^{1,1}$, $\mathbb{C}[t, \zeta]$ is the polynomial ring and $(t^2, t\zeta)$ is the ideal generated by t^2 and $t\zeta$. Associated to this ring there is a superspace $S = (\{*\}, O_S)$, which embodies the idea of a super-tangent vector.

Definition . Let $(X, \mathcal{A}_X)$ be a complex superspace. A deformation of $(X, \mathcal{A}_X)$ over S will be called an infinitesimal deformation.

Given a complex superspace (B, b_0), the tangent space $T_{b_0}B$ at b_0 is isomorphic to the linear superspace $Mor(S, B) = \{f : S \to B \mid f(*) = b_0\}$ of superspace morphisms.

Now, given a deformation $\mathcal{X} \to B$ of X, we can think of a tangent vector in $T_{b0}B$ as a map $f \in Mor(S, B)$ and the pull-back deformation $f^*\mathcal{X} \to S$ is a first order deformation of C. The Kodaira–Spencer class of is obtained by considering the exact sheaf sequence

$$0 \to f^*\mathcal{TX}_/ \longrightarrow f^*\mathcal{TX} \longrightarrow f^*(Der\hat{B}) \to 0$$

where $\mathcal{TX}_/$ is the relative tangent sheaf. Taking the coboundary map one has

$$\widehat{KS}_f : H^0(f^*(Der\hat{B})) \equiv A \in T_{b_0}B \to H^1(f^*\mathcal{TX}_/) \equiv T\hat{X}$$

and then, letting f vary one gets the $\widehat{KS}$ homomorphism

$$\widehat{KS} : T_{b_0}B \to H^1(X, \hat{T}X)$$

The fundamental theorems of the Kodaira-Spencer theory extend to the graded commutative case [V]. In particular, when X is of dimension $1|p$ it holds:

Theorem . A deformation of $(X, \mathcal{A}_X)$ such that $\widehat{KS}$ is surjective (an isomorphism) is complete (resp. versal).

Chapter 3.

Super–Algebraic Curves

The issue of checking the consistency of superstring theory as the "theory of everything" is plagued with difficulties both of technical and substantial nature. Among the problems which are not yet completely solved, stems the question of the global structure of supermoduli space, which plays in superstring theory the rôle of moduli space of algebraic curves in the bosonic model. For instance, when computing amplitudes in superstring theory via a path integral approach, one faces the problem of dealing with odd variables. While the bosonic piece of the Polyakov path integral is well understood as an integral over moduli spaces of algebraic curves, the fermionic part is more embarrassing as the discovery of ambiguities in performing the integration over odd variables has pointed out (for a review see e.g. [DP]). To cut a long story short, the basic trouble comes from the fact that in a given supersymmetric gauge the measure for superstrings reduces to a Berezin form, which unluckily is gauge dependent. This is because a supersymmetry transformation induces a small variation of one's gauge choice in a way which mixes up even and odd variables. In other words, the modular parameters change by a nilpotent contribution inducing a change of the string measure by a "total derivative" which deprives any computation of sensible physical meaning.

Although the local problem may be handled within the correct treatment of Berezin forms under non-split coordinate transformations [R2], this approach is not completely satisfactory for the following reasons. First, due to the non–naturality of the splitting of the Berezinian sequence even in the smooth case, possible divergences in the amplitudes (before GSO–projection) may give rise to boundary terms. Second, one would like to set up a formalism which manifestly keeps into account super–holomorphic structures step by step.

In this chapter we consider such problems from the mathematical point of view, by studying the global structure of supermoduli "spaces". Our strategy is to describe the supersymmetric analogue of the moduli stacks [Mu5], but we do not attempt to formalize such a structure here. We choose instead a direct coordinate approach, giving explicit representatives for the odd moduli which, as we shall see, can be identified with gravitino zero–modes.

Our analysis boils down to give results which are essentially of a negative type (except for genus $g = 2$) and namely that the simplest choices one can make in defining universal deformations of susy–curves give non–projected "atlases" on supermoduli spaces, a fact that has been suspected for a long time in the physical literature.

This does not mean, of course, that there are no subtler choices of gravitino zero–modes which may eventually lead to projected atlases. The nontrivial part is however to study whether the obstruction cocycle can be made trivial, and, in any case, this may possibly remove only the first obstruction to projectedness leaving open the question of higher order obstructions.

Needless to say, a full study of the problem requires a more careful handling of sheaves on the moduli stack of spin curves and their cohomology, including families with singular curves. Anyhow, the fact that the most natural choices one can make do not work has severe consequences for physical applications, where one needs a detailed computational control of the matter. Accordingly, we feel more promising than looking for abstruse fine tunings of choices, the strategy of setting up a formalism which is insensitive to splitness obstructions. For instance, a satisfactory measure for supermoduli can be indeed obtained in the operator formalism [AGMNV]. On the other hand, when relying on the Polyakov's path–integral method, one may try to work on the split model of supermoduli, and extract from it the relevant informations. The results on super–Mumford forms of Manin and others [BMFS,M2,Vo] seem to support this possible way–out.

Finally, although studying "two supersymmetries" may seem the most obvious step beyond N=1, it is already a non trivial matter as noticed in some works [Co,Me] recently appeared in the literature. In fact, for $N > 1$ one is lead to consider locally free sheaves of rank greater than one on algebraic curves. Actually, for N=2 the superconformal structure to be imposed on such objects will bypass most of the subtleties related with moduli of vector bundles over variable curves, leaving us with a supervariety whose reduced space is a suitable quotient of the Picard variety over the moduli space of genus g curves.

From the physical point of view, there is some string–theoretical interest in the study of N=2 superconformal models, as it has been pointed out [Ge] that in the Calabi–Yau compactification scheme of the extra dimensions, space-time N=1 supersymmetry requires N=2 world-sheet supersymmetry. It is also widely believed that viewing N=1 supermoduli spaces as embedded in N=2 supermoduli spaces could be a keen standpoint for investigating the peculiarities of the first (provided one has a good control of the second). Furthermore, as it will be proven later on, the moduli space of $susy_2$-curves is locally isomorphic to the moduli space of $1|1$–dimensional supermanifolds. This is quite interesting, since it is by now clear that the supersymmetric generalization of the Krichever construction, in the field theoretic interpretation, involves *this* moduli space as a building block [Ra].

In the last section of this chapter we describe the reduced space of N=2 supermoduli space and its first infinitesimal neighbourhood. The properties of the system one expects on physical grounds (such as the existence of modular parameters for the $U(1)$ current mixing supersymmetries) emerge most neatly.

3.1 N=1 susy–curves and their deformations.

The analysis of two-dimensional superconformal supergravity [Ho] lead to the remarkable observation that real 2-dimensional supermanifolds are locally superconformally flat, i.e. one can always find supersymmetry gauges in which the superderivative $\mathcal{D}$ looks locally like

$$\mathcal{D} = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$$

By analogy with ordinary complex manifolds theory, one then is naturally lead to the definition of a Super Riemann surface (alias a susy–curve) as a supermanifold locally built with coordinate patches that preserve in some sense such a structure.

Namely, considering a generic 1|1-dimensional complex manifold, we see that, considering holomorphic transition functions

$$\begin{cases} \tilde{z} = \tilde{z}(z, \theta) \\ \tilde{\theta} = \tilde{\theta}(z, \theta) \end{cases}$$

the superderivative transforms as [Fr]

$$\mathcal{D} = \left[\mathcal{D}\tilde{\theta}\right] \tilde{\mathcal{D}} + \left[\mathcal{D}\tilde{z} - \tilde{\theta}\mathcal{D}\tilde{\theta}\right] \tilde{\mathcal{D}}^2$$

A holomorphic map of the above form will be called a *superconformal* transformation iff the transformation law for the superderivative $\mathcal{D}$ is homogeneous, i.e. if

$$\mathcal{D} = \left[\mathcal{D}\tilde{\theta}\right] \tilde{\mathcal{D}} \quad \Leftrightarrow \left[\mathcal{D}\tilde{z} - \tilde{\theta}\mathcal{D}\tilde{\theta}\right] \tilde{\mathcal{D}}^2 = 0$$

Notice that $\mathcal{D}^2 = \frac{1}{2}\lceil \mathcal{D}, \mathcal{D} \rfloor = \frac{\partial}{\partial z}$ and $(\mathcal{D}, \frac{\partial}{\partial z})$ span the tangent space $\hat{T}_{\mathbb{C}}\Sigma$. Accordingly, one is lead to the following

Definition. A super–curve is the datum of

a) an algebraic curve C_{red}, with structure sheaf $\mathcal{O}$,

b) a sheaf $\mathcal{A}$ over C_{red} of super-commutative $\mathbb{C}$-algebras (with nilpotent ideal $\mathcal{N}$) such that

i) $\mathcal{A}/\mathcal{N} = \mathcal{O}$,

ii) the $\mathcal{O}$–module $\mathcal{N}$ is locally free of rank 1.

Definition. A susy–curve $\hat{C}$ is a supercurve, together with

c) a locally free rank 0|1 subsheaf $\mathcal{D}$ of the tangent sheaf $\widehat{TC}$

$$0 \to \mathcal{D} \to \widehat{TC} \to \widehat{TC}/\mathcal{D} \to 0$$

such that

iii) the commutator (mod $\mathcal{D}$)

$$\lceil\, , \rfloor_{\mathcal{D}} : \mathcal{D} \otimes \mathcal{D} \longrightarrow \hat{T}C/\mathcal{D}$$

is an isomorphism, so that $\mathcal{D}$ and $\lceil \mathcal{D}, \mathcal{D} \rfloor_{\mathcal{D}}$ generate $\hat{T}C$.

To make contact with the coordinate approach to superconformal field theory one can argue in the following way. Considering obviously the smooth case, one can identify a local generator for the distribution $D\restriction_{U_\alpha}$ with the coordinate expression $\partial/\partial\theta_\alpha + \theta_\alpha \otimes \partial/\partial z_\alpha$. The equivalence between the two definitions will be given below; anyhow as a consequence on each intersection $U_\alpha \cap U_\beta$ we have that both $\partial/\partial\theta_\alpha + \theta_\alpha \otimes \partial/\partial z_\alpha$ and $\partial/\partial\theta_\beta + \theta_\beta \otimes \partial/\partial z_\beta$ generate $\mathcal{D}$ and therefore should be proportional. An easy computation yields the following clutching functions

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) \\ \theta_\alpha = g_{\alpha\beta}(z_\beta)\, \theta_\beta \end{cases}$$

with $g^2_{\alpha\beta} = f'_{\alpha\beta}$, $(f'_{\alpha\beta} = df_{\alpha\beta}/dz_\beta)$, showing that the $g_{\alpha\beta}$ are transition functions for a θ- characteristics $\mathcal{L}$ on C_{red}. Conversely, given a pair $(C_{red}, \mathcal{L})$ we can construct a susy-curve $\hat{C}$ just setting $\mathcal{A} = \mathcal{O} \oplus \Pi\mathcal{L}$, where Π is the parity changing functor, whose effect is to make sections of $\mathcal{L}$ anticommute. Summing up we have the

Proposition. There are as many susy-curves C on a fixed smooth algebraic curve C_{red} as reduced space as θ- characteristics on C_{red}.

The notion of a $(N = 1)$ susy–curve depending on a parameter space $(B, \mathcal{O}_B)$ is given as follows.

Definition. A family of susy-curves X parameterized by a complex superspace B, or, for the sake of brevity, a susy-curve X over B, is a proper surjective map $\pi\colon X \longrightarrow B$ of complex superspaces having $1|1$-dimensional fibres, together with a $0|1$-dimensional distribution $\mathcal{D}_\pi$ in the relative tangent sheaf $\mathcal{T}_\pi X$ such that the supercommutator *mod* $\mathcal{D}_\pi$, $[\, ,]_{\mathcal{D}} : \mathcal{D}_\pi^{\otimes^2} \longrightarrow \mathcal{T}_\pi X/\mathcal{D}_\pi$ is an isomorphism.

When B reduces to a point $\{*\}$, one will speak of 'isolated' or 'single' susy-curves, thus recovering the definition we have just given. A relative coordinate system on the susy-curve $X \stackrel{\pi}{\longrightarrow} B$ given by coordinate charts $\{(U_\alpha, z_\alpha, \theta_\alpha, b)\}$ in which the local generator for $\mathcal{D}_\pi$ is expressed as $D_\alpha = \frac{\partial}{\partial \theta_\alpha} + \theta_\alpha \frac{\partial}{\partial z_\alpha}$ is called canonical. Then $\mathcal{D}^{\otimes 2}$ is locally generated by $\frac{\partial}{\partial z_\alpha}$.

Proposition. Any Super Riemann surface admits a canonical atlas [LR].

Proof. Let (w, ϕ) be a coordinate system. Then, as $\mathcal{D}$ is locally free of rank $0|1$, it has a generator of the form $\frac{\partial}{\partial \phi} + h\frac{\partial}{\partial w}$, with h odd. Then $[\mathcal{D}, \mathcal{D}]_{\mathcal{D}}$ looks locally like $\frac{\partial h}{\partial \phi} \cdot \frac{\partial}{\partial z}$ so that for the commutator to be an isomorphism, $\frac{\partial h}{\partial \phi}$ must be invertible. Let us introduce coordinates (z, η) with $\eta = \phi$. Now the local generator for $\mathcal{D}$ looks like

$$\frac{\partial}{\partial \phi} + h\frac{\partial}{\partial w} = \frac{\partial}{\partial \eta} + (h\frac{\partial z}{\partial w} + \frac{\partial z}{\partial \phi})\frac{\partial}{\partial z}$$

so that one has to solve the equation $h\frac{\partial z}{\partial w} + \frac{\partial z}{\partial \phi} = \eta$. Expanding both z and h in powers of ϕ as

$$z = z_0 + \phi z_1 \qquad h = h_0 \phi h_1$$

and equating terms of the same degree in ϕ one obtains the equations

$$\begin{cases} z_1 + h_1 \frac{\partial z_0}{\partial w} \\ h_0 \frac{\partial z_0}{\partial w} + h_1 \frac{\partial z_1}{\partial w} = 1 \end{cases}$$

As $h_1 \equiv \frac{\partial h}{\partial \phi}$ is invertible this system has solutions.

Besides being mathematically natural, the need for families of susy-curves in physical applications follows from the fact that world-sheet supersymmetry requires the presence of a gravitino field on a given single susy-curve C. One can fix local superconformal gauges, which amount to choosing local complex coordinates, a local holomorphic trivialization of $\mathcal{L}$ and to identifying a chiral piece of the gravitino field with a section χ of $A^{0,1}(C_{red}, \mathcal{L}^{-1})$, i.e. with a smooth antiholomorphic one form with values in $\mathcal{L}^{-1}$ or, passing to Čech cohomology, with a Čech 1–cocycle $\epsilon_{\alpha\beta}$ with values in $\mathcal{L}^{-1}$. Notice that the action of supersymmetry has no effect on the $\epsilon_{\alpha\beta}$'s, while we have a local symmetry generated by holomorphic sections η_α of $\mathcal{L}^{-1}$ acting via Čech coboundaries, i.e. as $\epsilon_{\alpha\beta} \to \epsilon_{\alpha\beta} + \eta_\alpha - \eta_\beta$. In other words, we can benefit of the isomorphism $H^{0,1}_{\overline{\partial}}(C_{red}, \mathcal{L}^{-1}) = H^1(C_{red}, \mathcal{L}^{-1})$ to represent chiral gravitino fields χ (up to supersymmetries) via Čech cocycles $\epsilon_{\alpha\beta}$ (up to coboundaries).

The datum of $[\epsilon]$ can be encoded in an extension of the structure sheaf of C as follows. Forgetting about parity, consider $H^1(C_{red}, \mathcal{L}^{-1})$ as a constant sheaf with group $\mathbb{C}^{2g-2}$ on C_{red}. If $\epsilon_{\alpha\beta}$ is a representative of $[\epsilon]$ and $\epsilon^i_{\alpha\beta}$ represent a basis $[\epsilon^i]$ for $H^1(C_{red}, \mathcal{L}^{-1})$, we set $\epsilon_{\alpha\beta}(\zeta_i) = \epsilon^i_{\alpha\beta}\zeta_i$, (sum over $i = 1, ..., 2g-2$) and construct an extension $\mathcal{F}$ of $\mathcal{L}$ by $\mathbb{C}^{2g-2}$ by stating that $\mathcal{F}$ is the sheaf of sections of a rank $2g-1$ vector bundle locally generated by θ_α, ζ_i with transition functions *

$$\begin{pmatrix} \theta_\alpha \\ \zeta_i \end{pmatrix} = \begin{pmatrix} \pm\sqrt{f'_{\alpha\beta}} & \epsilon^i_{\alpha\beta} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \theta_\beta \\ \zeta_i \end{pmatrix}$$

Notice that $\mathcal{F}$ is independent (up to isomorphisms) both of the basis $[\epsilon^i]$ and of its representatives. The supermanifold $(C_{red}, \wedge\mathcal{F})$ is not yet a susy-curve, but we can cook out of the same data a deformation $\mathcal{A}$ of $\wedge\mathcal{F}$ making $(C_{red}, \mathcal{A})$ a susy-curve. It is enough to find a superconformal coordinate patching $z_\alpha = z_\alpha(z_\beta, \theta_\beta, \zeta_i)$, $\theta_\alpha = \theta_\alpha(z_\beta, \theta_\beta, \zeta_i)$ which reproduces the transition functions above for $\mathcal{F}$ and $z_\alpha = f_{\alpha\beta}(z_\beta)$ mod $\mathcal{N}^2$. (here $\mathcal{N}$ is the nilpotent ideal locally generated by θ_α, ζ_i). The "minimal" answer is

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) + \theta_\beta\sqrt{f'_{\alpha\beta}(z_\beta)}\epsilon_{\alpha\beta}(z_\beta, \zeta_i) \\ \theta_\alpha = \sqrt{f'_{\alpha\beta}(z_\beta) + \epsilon_{\alpha\beta}(z_\beta)\epsilon'_{\alpha\beta}(z_\beta)} \cdot \theta_\beta + \epsilon_{\alpha\beta}(z_\beta, \zeta_i) \end{cases}$$

By minimal here we mean that it depends only on the data already encoded in $\mathcal{F}$ at the lowest order compatible with superconformal structures. Unfortunately, we see that, in spite the local model $\wedge\mathcal{F}$ was independent of choices, $\mathcal{A}$ is not. In particular it is not independent of the choice of the representatives $\epsilon^i_{\alpha\beta}$ because of the non-linear therm $\epsilon\epsilon'$ entering the transition functions. In any case, $(C_{red}, \mathcal{A})$ gives us an example of a non trivial susy-curve encoding informations about gravitino fields.

A first step in the construction of supermoduli spaces is to study some deformation theory of susy-curves. As the notion of susy–curves encodes more data than a generic $1|1$-dimensional supermanifold, their deformations must be defined as follows;

* Hereinafter $f'_{\alpha\beta}$ means $\partial f_{\alpha\beta}/\partial z_\beta$ The sign ambiguity refers to the choice of a θ-characteristics on C and will be left implicit in the following.

Definition . A deformation of a susy-curve C over (the germ of a pointed) complex superspace (B, b_0) at $b_0 \in B$ is a family $\pi: X \longrightarrow B$ of susy-curves over B together with a fixed isomorphism $i : C \to \pi^{-1}(b_0)$ between C and the special fibre over b_0.

This makes sense because each fibre $\pi^{-1}(b), b \in B$, is itself a single susy curve with the subsheaf $\mathcal{D}$ induced by $\mathcal{D}_\pi$. Notice that an isomorphism of single susy-curves may be thought of as induced by an isomorphism of the underlying θ-characteristics. We remark also that fixing the isomorphism $i : C \to \pi^{-1}(b_0)$ is vital as in the ordinary case, since it allows the study of the action of the automorphism group of C on the base space B of its deformations.

Let us recall that, given a deformation $\pi : X \to B$ of a susy-curve, we have two natural subsheaves of the tangent sheaf $\mathcal{T}X$. Along with the relative tangent sheaf $\mathcal{T}_\pi X = \ker \pi_*$ there is the sheaf $\mathcal{T}^{\mathcal{D}}X$ of derivations which commute with sections of $\mathcal{D}_\pi$. A basic role is played by the sheaf $\mathcal{T}_\pi^{\mathcal{D}} =: \mathcal{T}_\pi X \cap \mathcal{T}^{\mathcal{D}}X$ of infinitesimal automorphisms of X.

Lemma. [LR] There is an isomorphism $\mathcal{T}_\pi^{\mathcal{D}} \simeq \mathcal{D}_\pi^{\otimes 2}$.

Proof. Let (z, θ) be relative canonical coordinates. Then, any element $V \in \mathcal{T}_\pi^{\mathcal{D}}$ can be written as

$$V = a\frac{\partial}{\partial z} + b\mathcal{D}$$

Then supposing V homogeneous of degree l

$$[\mathcal{D}, V] = (\mathcal{D}a)\frac{\partial}{\partial z} - (-1)^l b\frac{\partial}{\partial z} + (\mathcal{D}b)\mathcal{D}$$

so that

$$V = a\frac{\partial}{\partial z} + (-1)^l(\mathcal{D}a)\mathcal{D}$$

and the isomorphism is proven. ∎

Clearly enough, the sequence

$$0 \to \mathcal{T}_\pi^{\mathcal{D}}X \to \mathcal{T}^{\mathcal{D}}X \to \pi^*\mathcal{T}B \to 0$$

is exact.

Let us first examine infinitesimal deformations. If S is the ring of the super dual numbers, and $f : S \to B$ represents a tangent vector to B, its Kodaira–Spencer class is obtained by considering the exact sheaf sequence

$$0 \to f^*\mathcal{T}^{\mathcal{D}}_{\pi}X \longrightarrow f^*\mathcal{T}^{\mathcal{D}}X \longrightarrow f^*\mathcal{T}B \to 0$$

Taking the coboundary map one has

$$\widehat{KS}_f : H^0(f^*\mathcal{T}B) \equiv \{[f] \in T_{b_0}B\} \to H^1(f^*\mathcal{T}^{\mathcal{D}}_{\pi}X) \equiv H^1(C, \mathcal{D}^{\otimes 2}).$$

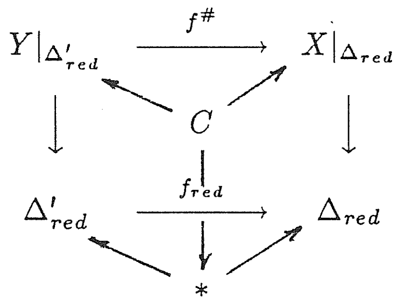
Letting f vary we get the Kodaira-Spencer homomorphism

$$\widehat{KS} : T_{b_0}B \to H^1(C, \mathcal{D}^{\otimes 2}).$$

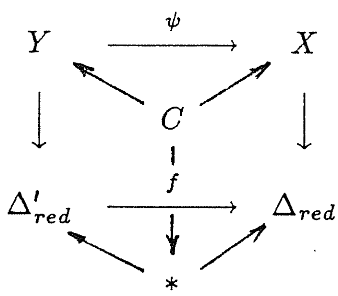
Exactly as in the case of deformations of bare supermanifolds, a family of susy-curves for which KS is an isomorphism is 'modular'[LR]. On the other hand there is merit in considering deformations, because as in the ordinary case, one can prove the following

Theorem. A deformation of a susy-curve $C \overset{i}{\hookrightarrow} X \overset{\pi}{\longrightarrow} \Delta$ for which $\widehat{KS}$ is an isomorphism is universal.

Proof. Promoting a modular family to be a deformation by adding the datum of the isomorphism $i : C \to \pi^{-1}(0)$, helps in killing the possible $\mathbb{Z}_2$ ambiguities. In fact a deformation of $(C_{red}, \wedge\mathcal{L})$ over a reduced base is the same as a deformation of a θ-characteristics . So if $C \overset{j}{\hookrightarrow} Y \overset{\pi'}{\longrightarrow} \Delta'$ is any deformation of $(C, \wedge\mathcal{L})$ there exists a unique $f_{red} : \Delta'_{red} \longrightarrow \Delta_{red}$ such that the diagram



commutes. Notice that $f^\#$ is uniquely fixed by i and j. Hence the proposition above tells us that there is a unique extension $f : \Delta' \longrightarrow \Delta$ of f_{red} and an isomorphism ψ such that the diagram



commutes as well. The only possible ambiguity concerns now the uniqueness of the isomorphism ψ. If ψ_1, ψ_2 were two such isomorphisms, then $\psi_1 \circ \psi_2^{-1}$ is either the identity or the canonical automorphism of Y [LR]. But the commutativity of the latter diagram fixes it as $\psi_1 \circ \psi_2^{-1} = \mathrm{id}_Y$.

To classify infinitesimal deformations of susy-curves in the spirit of the "original" Kodaira-Spencer approach one can proceed as follows, regarding a susy-curve as built by patching together $1|1$-dimensional superdomains by means of superconformal transformations, and singling out 'infinitesimal' moduli as non-trivial parameters in the transition functions. Namely, consider a canonical atlas $\{U_\alpha, z_\alpha, \theta_\alpha,\}$ for C with clutching functions

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) \\ \theta_\alpha = \sqrt{f'_{\alpha\beta}}\,\theta_\beta \end{cases}$$

They obviously satisfy the cocycle condition $f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma)) = f_{\alpha\gamma}(z_\gamma)$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

We can cover a first order deformation $\pi : X \to S$ of C glueing the $U_\alpha \times S$ via the identification

$$z_\alpha = f_{\alpha\beta}(z_\beta) + t b_{\alpha\beta}(z_\beta) + \theta_\beta \zeta g_{\alpha\beta}(z_\beta) F_{\alpha\beta}(z_\beta)$$

$$\theta_\alpha = F_{\alpha\beta}\theta_\beta + \zeta g_{\alpha\beta}$$

where $F_{\alpha\beta} = \sqrt{f'_{\alpha\beta} + t b'_{\alpha\beta}}$, so that the clutching functions are superconformal for any t, ζ. The cocycle condition for these transformation rules reduce to the cocycle condition for the $f_{\alpha\beta}$'s as before, plus

$$b_{\alpha\beta} + f'_{\alpha\beta}\, b_{\beta\gamma} = b_{\alpha\gamma}$$

$$g_{\alpha\beta}\theta_\alpha + f'_{\alpha\beta}\, g_{\beta\gamma}\theta_\beta = g_{\alpha\gamma}\theta_\gamma$$

Taking the tensor product by $\partial/\partial z_\alpha$, one sees that the one cochains

$$v^0_{\alpha\beta} = \{b_{\alpha\beta}\partial/\partial z_\alpha\}$$

$$v^1_{\alpha\beta} = \{g_{\alpha\beta}\theta_\alpha \otimes \partial/\partial z_\alpha\}$$

are actually cocycles. They define a class in

$$H^1(C_{red},\omega^{-1}) \oplus \Pi H^1(C_{red},\mathcal{L}^{-1}) = H^1(C,\mathcal{D}^{\otimes 2}) \subset H^1(C,\mathcal{T}C)$$

called the Kodaira-Spencer class of the first order deformation* $X \xrightarrow{\pi} S$.
A similar computation, considering local superconformal reparametrizations with local odd parameters λ_α, shows that they leave the cocycle v_0 invariant and send v_1 into $\tilde{v}^1_{\alpha\beta} = v^1_{\alpha\beta} + (\lambda_\alpha - \sqrt{f'_{\alpha\beta}} \cdot \lambda_\beta)\theta_\alpha \otimes \frac{\partial}{\partial z_\alpha}$ which leads to the

Theorem. The set of equivalence classes of first order deformations of a susy-curve C is a linear complex superspace with dimension $3g-3|2g-2$.

Proof. The dimensions of $H^1(C_{red},\omega^{-1})$ and $H^1(C_{red},\mathcal{L}^{-1})$ are computed to be $3g-3$ and $2g-2$ by means of Riemann-Roch theorem. ■

Since C is split, $\mathcal{D}^{\otimes 2} \simeq \mathcal{T}C_{red} \oplus \Pi\mathcal{L}^{-1}$, and $H^1(C,\mathcal{D}^{\otimes 2})$ naturally splits into even and odd subspaces and we can speak about even and odd Kodaira-Spencer homomorphisms KS_0 and KS_1, by composing $\widehat{KS}$ with the projections of $H^1(C,\mathcal{D}^{\otimes 2}) = H^1(C_{red},\omega^{-1}) \oplus \Pi H^1(C_{red},\mathcal{L}^{-1})$ onto the first and second summand. It follows that, if B is a purely even superspace (i.e. an ordinary complex space), $KS_1 = 0$ and $KS_0; T_{b_0}B \to H^1(C_{red},\omega^{-1})$ is the ordinary Kodaira-Spencer map. Using the natural map $i : B_{red} \to B$, we get that a deformation $X \to B$ is versal on a purely even B, if and only if the induced deformation $i^*X \to B_{red}$ is. As we need the datum of a θ-characteristics on C_{red}, the deformation $X \to B_{red}$ has to be be considered as a deformation of a θ-characteristics. We have therefore the following

Proposition. Even-versal deformations of a susy-curve exist and are in 1-1 correspondence with pull-backs under maps $f : B \to B_{red}$ of versal deformations over B_{red} of the underlying θ-characteristic.

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* Here we obviously assume that C is smooth. Deformation theory of SUSY-curves with nodes requires the handling of θ -characteristics in the singular case.

This proposition tells us that the reduced space of moduli of susy-curves can be given in terms of isomorphism classes of pairs $(C_{red}, \mathcal{L})$. This space S_g comes equipped with the "universal curve" $\pi : \mathcal{C} \to S_g$ together with the "universal (dual) θ-characteristics" $\mathcal{L}_\pi^{-1} \to \mathcal{C}$ on $\mathcal{C}$. From the construction above it is then clear that the first infinitesimal neighbourhood of supermoduli space is the sheaf $R^1\pi_*(\mathcal{L}_\pi^{-1})$ on S_g. Accordingly the local model for supermoduli spaces is given by the supermanifold $(S_g, \wedge R^1\pi_*\mathcal{L}_\pi^{-1})$. We will discuss in the next section how much actual supermoduli spaces may differ from being split.

3.2 Supermoduli space building.

From the discussion outlined in the previous sections, it should be clear that, when dealing with $N = 1$ supermoduli space, one has a good control of two of the three ingredients needed to define a supermanifold. This section is devoted to the study of what happens when we try to glue together local patches of supermoduli space. It is clear that restricting oneself to work with purely even objects (a procedure which, in the physical literature is sometimes referred to as "considering split super Riemann surface ") is not correct since a deformation depending trivially on odd parameters has identically vanishing odd super Kodaira-Spencer map, and thus its basis cannot be taken as building block for supermoduli space.

In the following we will restrict ourselves to describe quite informally some of the ingredients entering the construction of the graded analogue of the moduli stack. In practice, we will forget about the existence of automorphism, and pretend that universal deformations $X \to \Delta$ of susy-curves give "coordinate charts" on "supermoduli spaces". Our strategy to get some insight to the geometry of these "spaces" is first to select some very special classes of versal deformations, and then trying to glue their bases requiring that a superconformal isomorphism exists between the families.

First we give concrete examples of versal deformations of a susy-curve. To this purpose we need the following two lemmas.

Lemma . Let $p \in C_{red}$ be a generic point and L a θ-characteristics on C; then for $n \geq 1$ the connecting homomorphism $\rho_p^n : \mathbb{C}^{(n+1)(g-1)} \to H^1(C_{red}, \mathcal{L}^{-n})$ associated to the exact sequence

$$0 \to \mathcal{L}^{-n} \to \mathcal{L}^{-n}((n+1)(g-1)p) \to \mathcal{L}^{-n}((n+1)(g-1)p)/\mathcal{L}^{-n} \to 0$$

is an isomorphism.

Proof. A segment of the long cohomology sequence reads

$$\to H^0(C_{red}, \mathcal{L}^{-n}(Np)) \to \mathbb{C}^N \to H^1(C_{red}, \mathcal{L}^{-n}) \to H^1(C_{red}, \mathcal{L}^{-n}(Np)) \to$$

$(N = (n+1)(g-1))$ and since the first and the last space have the same dimension, we need only to prove that one of them vanishes. By Serre duality, this is the same thing as showing that $H^0(C_{red}, \mathcal{L}^{n+2}(-Np)) = 0$, that is that there are no sections of $\mathcal{L}^{n+2}$ vanishing of order $\geq N$ at p. Let σ_0 be a local trivializing section of $\mathcal{L}$ around p, and $\sigma_i = f_i(z)\sigma_0$ $(i = 1, ..., N)$ be the local expression for a basis of $H^0(C_{red}, \mathcal{L}^{n+2})$. The matrix

$$\begin{pmatrix} f_1(z) & f_1'(z) & \dots & f_1^{(N-1)} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ f_N(z) & f_N'(z) & \dots & f_N^{(N-1)} \end{pmatrix}$$

has vanishing determinant whenever one of the f_i's vanishes of order $\geq N$ at p. This cannot be the case for almost all $p \in C_{red}$ because in this case a line of the matrix above would be linear combination of the others, i.e. we would get a differential equation of order $N-1$ with N linearly independent solutions.

Lemma. For a generic point $p \in C_{red}$ and $n \geq 1$, the connecting homomorphism $\delta_p^n : \mathbb{C} \to H^1(C_{red}, \mathcal{L}^{-n})$ associated to the exact sequence

$$0 \to \mathcal{L}^{-n} \to \mathcal{L}^{-n}(p) \to \mathcal{L}^{-n}(p)/\mathcal{L}^{-n} \to 0$$

is injective. The map $\delta^n : C_{red} \to H^1(C_{red}, \mathcal{L}^{-n})$ given by $p \to \delta_p^n(1)$ is full, i.e. there are $(n+1)(g-1)$ points p_i such that the classes $\delta_{pi}^n(1)$ form a basis of $H^1(C_{red}, \mathcal{L}^{-n})$.

Proof. The relevant cohomology sequence reads

$$\dots \to H^0(C_{red}, \mathcal{L}^{-n}(p)) \to \mathbb{C} \to H^1(C_{red}\mathcal{L}^{-n}) \to \dots$$

Since $deg\mathcal{L}^{-n}(p) = n+1-ng$ is negative for $g \geq 1+1/n$, injectivity follows at any p for $g \geq 2$ and $n > 1$.

The same is true in the case $n = 1$ at $g = 2$ because, if $\mathcal{L}$ is even its divisor is not effective and $\mathcal{L}^{-1}(p)$ cannot be trivial. In the odd sector $\mathcal{L}$ has as divisor one of the six Weierstrass points and again $H^0(C_{red}, \mathcal{L}^{-1}(p)) = 0$, provided p is not a Weierstrass point. To show that δ^n is full, it is enough to notice that if $Im\delta^n$ was contained in a hyperplane in $H^1(C_{red}, \mathcal{L}^{-n})$, then there would be an element ϕ of the dual space $H^1(C_{red}, \mathcal{L}^{-n})^\vee = H^0(C_{red}, \mathcal{L}^{n+2})$ such that $<\phi, \delta^n_p(1)>= 0$ for all $p \in C_{red}$. Here $< .,. >$ is Serre duality i.e. $<\phi, \delta^n_p(1)>= res_p\phi.\sigma$ where σ is a representative of $\delta^n_p(1)$ i.e. a section of $\mathcal{L}^{-n}$ with a first order pole at p. Then this would imply that ϕ itself vanishes, an absurdity. ∎

Example. A very simple example of a versal deformation of a susy-curve $C = (C_{red}, \mathcal{L})$ can be constructed by concentrating the deformation at a generic point $p \in C_{red}$. Let $\{U_\alpha, z_\alpha, \theta_\alpha\}$ be an atlas for C and assume $p \in U_0$ with $z_0(p) = 0$. We glue a superdisk with coordinates x_0, ϕ_0 with $C_{red} - \{p\}$ by means of the map

$$\begin{cases} x_0 = z_0 + \sum_{i=1}^{3g-3} \frac{t_i}{z_0^i} + \theta_0 \sqrt{1 - \sum_{i=1}^{3g-3} \frac{it_i}{z_0^{i+1}}} \quad \sum_{k=1}^{2g-2} \frac{\epsilon_k}{z_0^k} \\ \phi_0 = \sqrt{1 - \sum_{i=1}^{3g-3} \frac{it_i}{z_0^{i+1}} - \sum_{k=1}^{2g-2} \frac{\epsilon_k}{z_0^k} \sum_{k=1}^{2g-2} \frac{k\epsilon_k}{z_0^{k+1}}} \quad \theta_0 + \sum_{k=1}^{2g-2} \frac{\epsilon_k}{z_0^k} \end{cases}$$

with (t_i, ϵ_k) in a small superpolydisk Δ. Now

$$KS_0(\frac{\partial}{\partial t_i}) = [\frac{1}{z_0^i}\frac{\partial}{\partial z_0}] = \rho_p^2(e_i)$$

$$KS_1(\frac{\partial}{\partial \epsilon_k}) = [\frac{1}{z_0^k}\theta_0 \otimes \frac{\partial}{\partial z_0}] = \rho_p^1(e_k)$$

where $\{e_i\}$ and $\{e_k\}$ are standard basis in $\mathbb{C}^{3g-3}$ and $\mathbb{C}^{2g-2}$ respectively. The first lemma above then tells us that KS is an isomorphism and our family is versal.

Example. Another class of versal deformations of C can be associated to $5g-5$ generic points p_i. This is closer to what is done in the physical literature (see, e.g. [B]), as it corresponds to considering gravitino zero modes as δ-functions on $2g-2$ distinct points. We glue superdisks with coordinates x_i, ϕ_i with $C - \{p_i\}$ by means of the maps

$$\begin{cases} x_i = z_i + \frac{t_i}{z_i} \\ \phi_i = \sqrt{1 - \frac{t_i}{z_i^2}}\theta_i \end{cases}$$

for $i = 1, ..., 3g-3$ and by

$$\begin{cases} x_i = z_i + \theta_i \frac{\epsilon_{i-3g+3}}{z_i} \\ \phi_i = \theta_i + \frac{\epsilon_{i-3g+3}}{z_i} \end{cases}$$

for $i = 3g-2, ..., 5g-5$. Then

$$KS_0(\frac{\partial}{\partial t_i}) = [\frac{1}{z_i}\frac{\partial}{\partial z_i}] = \delta^2_{p_i}(1), \qquad i = 1, ..., 3g-3$$

$$KS_1(\frac{\partial}{\partial \epsilon_{i-3g+3}}) = [\frac{1}{z_i}\theta_0 \otimes \frac{\partial}{\partial z_i}] = \delta^1_{p_i}(1), \qquad i = 3g-2, ..., 5g-5.$$

Again, by the second lemma above, this family is modular.
By analogy with the ordinary case, we will call such deformations *Schiffer* deformations.

Both these examples yield "local coordinates" (up to automorphisms of the central fibre) on supermoduli "space" by

$$\Delta \xrightarrow{\Psi} \hat{S}_g$$

where $\Psi(t, \epsilon)$ =[isomorphism class of $\pi^{-1}(t, \epsilon)$] and we consider on $\Psi(\Delta)$ the sheaf $\Psi_* \mathcal{O}_\Delta$. Whenever two such "charts" overlap, i.e. $X_k \to \Delta_k$ $(k = 1, 2)$ are deformations of C_k such that $\Psi_1(\Delta_1) \cap \Psi(\Delta_2) = V \neq \emptyset$, then the restrictions X'_k of X_k to $\Psi_k^{-1}(V)$ are isomorphic as families of susy-curves, that is there are maps g, h making the diagram

$$\begin{array}{ccc} X'_1 & \xrightarrow{g} & X'_2 \\ \downarrow & & \downarrow \\ \Psi_1^{-1}(V) & \xrightarrow{h} & \Psi_2^{-1}(V) \end{array}$$

commute. The map h is then the "clutching" function for these two charts on supermoduli space.

Obviously enough, the global structure of supermoduli spaces is quite subtle because of the presence of automorphisms of susy–curves , and in particular of the canonical

$\mathbb{Z}_2$ automorphism. Here we will limit ourselves to considering a somewhat simplyfied problem in the framework of the theory of moduli stacks. To make things intuitive, a 'covering' on the moduli stack is a collection of versal families $\{X^j \xrightarrow{\pi^j} B^j\}_{j\in I}$ of genus g susy–curves such that

1) the reduced families $\{X^j{}_{red} \xrightarrow{\pi^j{}_{red}} B^j{}_{red}\}_{j\in I}$ give a covering of the stack Σ_g of genus g spin curves [C];

2) whenever $X^j{}_{red}$ and $X^i{}_{red}$ contain isomorphic spin curves, the isomorphism of reduced families

$$\begin{array}{ccc} X^j{}_{red} & \xrightarrow{L_{ji}} & X^i_{red} \\ \downarrow \pi^j_{red} & & \downarrow \pi^i_{red} \\ B^j_{red} & \xrightarrow{h_{ji}} & B^i_{red} \end{array}$$

over suitably restricted base spaces, comes together with maps $L^{\#}_{ji} : L^{-1}_{ji}\mathcal{A}_{X_i} \longrightarrow \mathcal{A}_{X_j}$ and $h^{\#}_{ji} : h^{-1}_{ji}\mathcal{A}_{B_i} \longrightarrow \mathcal{A}_{B_j}$ which make the two families superconformally isomorphic. We shall say for short that, in this latter case, X_j and X_i *partially overlap*. Notice that being the two families modular, h_{ji} and $h^{\#}_{ji}$ are essentially unique.

An obvious way to construct an 'atlas' on the stack is to choose the B_j's to be superpolydisks and the X_j's to be universal deformations of susy–curves C_j. So each family X_j comes with the open covering $\{\mathcal{U}_{\alpha_j}\}$ with coordinates $\{x_{\alpha_j}, \varphi_{\alpha_j}, t_j, \eta_j\}$. We can as well give the coordinate description of the maps $L^{\#}_{\alpha_j\alpha_i}$ as

$$\begin{cases} x_{\alpha_j} = f_{\alpha_j\beta_j}(x_{\beta_j};t_j,\eta_j) - \sqrt{f'_{\alpha_j\beta_j}(x_{\beta_j};t_j,\eta_j)}\,\mu_{\alpha_j\beta_j}(x_{\beta_j};t_j,\eta_j)\phi_{\beta_j} \\ \phi_{\alpha_j} = \sqrt{f'_{\alpha_j\beta_j}(x_{\beta_j};t_j,\eta_j) + \mu_{\alpha_j\beta_j}\mu'_{\alpha_j\beta_j}(x_{\beta_j};t_j,\eta_j)} + \mu_{\alpha_j\beta_j}(x_{\beta_j};t_j,\eta_j) \\ t_j = t_j \\ \eta_j = \eta_j \end{cases} \qquad (*)$$

As a morphism of complex supermanifolds is completely specified by expressing its effect on the 'coordinate functions', when X_j and X_k overlap one can locally describe

$h^{\#}_{kj}$ and $L^{\#}_{kj}$ in terms of the following maps

$$\begin{cases} x_{\beta_k} = g_{\beta_k\beta_j}(x_{\beta_j};t_j,\eta_j) - \sqrt{g'_{\beta_k\beta_j}(x_{\beta_j};t_j,\eta_j)}\sigma_{\beta_k\beta_j}(x_{\beta_j};t_j,\eta_j)\phi_{\beta_j} \\ \phi_{\beta_k} = \sqrt{g'_{\beta_k\beta_j}(x_{\beta_j};t_j,\eta_j) + \sigma_{\beta_k\beta_j}\sigma'_{\beta_k\beta_j}(x_{\beta_j};t_j,\eta_j)} + \sigma_{\beta_k\beta_j}(x_{\beta_j};t_j,\eta_j) \\ t_k = h_{kj}(t_j,\eta_j) \\ \eta_k = \Xi_{kj}(t_j\eta_j) \end{cases} \tag{**}$$

Notice that the last two lines give the coordinate description of $h^{\#}_{kj}$. These give rise to a superconformal isomorphism provided that

$$L^{\#}{}_{\alpha_j\beta_j} \circ L^{\#}{}_{\beta_j\beta_k} = L^{\#}{}_{\alpha_j\alpha_k} \circ L^{\#}{}_{\alpha_k\beta_k} \tag{***}$$

More explicitly, considering a common open covering $\mathcal{U}_\alpha$ for the reduced families X_i and $h^*(X_j)$, redefining $L^{\#}{}_{\alpha_j\beta_j} \equiv L^{j\#}_{\alpha\beta}$, $L^{\#}{}_{\beta_j\beta_k} \equiv L^{\#}_{\beta}$, $t_j \equiv t$, $\eta_j \equiv \eta$, $t_k \equiv s$, $\eta_k \equiv \xi$, these equations take the form

$$L^j_{\alpha\beta}(L_\beta(x_\beta,\phi_\beta;t,\eta);t,\eta) = L_\alpha(L^k_{\alpha\beta}(x_\beta,\phi_\beta;h(t,\eta),\Xi(t,\eta));t,\eta) \tag{3}$$

An easy but tedious computation shows that this is equivalent to imposing the following patching conditions on the building blocks of the superconformal maps (*) and (**)

$$\begin{aligned} &f_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k} - \sqrt{f'_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k}} \cdot (\mu_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k}) \cdot \sigma_{\beta_j\beta_k} = \\ &= g_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k} - \sqrt{g'_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k}} \cdot (\sigma_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k}) \cdot \mu_{\alpha_k\beta_k} \end{aligned}$$

where the meaning of $\circ$ is as given in eq. (***) and

$$\begin{aligned} &\sqrt{f'_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k} + \mu_{\alpha_j\beta_j}\mu'_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k}} \cdot \sigma_{\beta_j\beta_k} + \mu_{\alpha_j\beta_j} \circ g_{\beta_j\beta_k} = \\ &= \sigma_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k} + \sqrt{g'_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k} + \sigma_{\alpha_j\alpha_k}\sigma'_{\alpha_j\alpha_k} \circ f_{\alpha_k\beta_k}}\mu_{\alpha_k\beta_k} \end{aligned}$$

Notice that, as in the ordinary case (see e.g. [K]), when dealing with universal deformations these relations actually determine the maps $t_k = h_{kj}(t_j,\eta_j)$ $\eta_k = \Xi_{kj}(t_j\eta_j)$.

It is convenient to expand these equations in powers of odd generators. One possibility is to quotient the patching conditions by $\pi^{-1}\mathcal{N}^n_{B_j}$ $(n = 1, 2, \ldots)$, $\mathcal{N}_{B_j}$ being the ideal of nilpotents in the base $(B_j, \mathcal{A}_{B_j})$, which leads to the notion of susy–curves over a thickened basis as in [LR]. For our purposes, however, it is sufficient to quotient the equations above by the full 3^{rd} power $\mathcal{N}^3_{X_j}$ of the nilpotent ideal in $\mathcal{A}_{X_j}$. In fact, as one easily checks, up to $\mathcal{N}^3_{B_j}$ the maps $h^{\#}_{jk}$ are the same in both procedures. On the other hand, we gain a nice intuitive description of what is going on in terms of the sheaf cohomology on the moduli stack, entering the details of which is outside the aims of this thesis.

Eq.s (3) modulo $\mathcal{N}_{X_j}$ give us the reduced structure of the moduli stack of spin curves. Up to $\mathcal{N}^2_{B_j}$ we get

$$\sigma^1{}_{\alpha_j\alpha_k} \circ f^0{}_{\alpha_k\beta_k} + \sqrt{g'^0{}_{\alpha_j\alpha_k} \circ f^0{}_{\alpha_k\beta_k}} \cdot \mu^1{}_{\alpha_k\beta_k}) =$$
$$= \sqrt{f'^0{}_{\alpha_j\beta_j} \circ g^0{}_{\beta_j\beta_k}} \cdot \sigma^1{}_{\beta_j\beta_k} + \mu^1{}_{\alpha_j\beta_j} \circ g^0{}_{\beta_j\beta_k}$$

(here superscripts denote the order of the η–expansion) which, upon tensorization with $\frac{\partial}{\partial\phi_{\alpha_k}}$ tells us that $\mu_{\alpha_i\beta_i}\frac{\partial}{\partial\phi_{\alpha_i}}$ and $L^*_{ij}\mu_{\alpha_j\beta_j}\frac{\partial}{\partial\phi_{\alpha_j}}$ are cohomologous in $H^1(C_t, \mathcal{L}^{-1})$ via the coboundary $\sigma_{\alpha_i\alpha_j}\frac{\partial}{\partial\phi_{\alpha_i}}$. Then we get the already remarked fact that, up to order 1, we can safely take cohomology classes getting the first infinitesimal neighbourhood of Σ_g as $R^1\pi_*\mathcal{L}^{-1}$.

The first obstruction to projectedness of the supermoduli stack can be seen at the next order. One gets that h^2_{ij} must satisfy the condition

$$\frac{\partial g_{\alpha_j\alpha_j}}{\partial t_j} h^2_{ij} = f^2{}_{\alpha_i\beta_i} - f^2{}_{\alpha_j\beta_j} + \sigma_{\alpha_i\alpha_j}\mu_{\alpha_i\beta_i} - \mu_{\alpha_j\beta_j}\sigma_{\beta_i\beta_j}$$

In spite this description deserves a detailed formalization, we will use it in its present rough form as an intuitive clue to construct an explicit representative of the first obstruction to projectedness of supermoduli stack.

3.3. Schiffer variations and obstructions to projectedness.

This section is devoted to study what happens when we try to glue two universal deformations of susy–curves of the special kind we described above (Schiffer deformations).

An atlas for $X \longrightarrow \Delta^{n|m}$ can be constructed along the lines discussed in §3.2 as follows. We can cover C_0 with an atlas $(U_\alpha; z_\alpha, \theta_\alpha)$ such that for $1 \le \alpha, \beta \le n+m, U_\alpha \cap U_\beta = \emptyset$, and with transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$; $\theta_\alpha = \sqrt{f'_{\alpha\beta}(z_\beta)}$. Then an atlas for X is given by $\mathcal{U}_\alpha = U_\alpha \times \Delta_{red}{}^{m|n}$ with coordinates $(x_\alpha, \varphi_\alpha; t_i, \eta_i)$ and with transition functions

$$\begin{cases} x_\alpha = F_{\alpha\beta}(x_\beta) + \varphi_\alpha \sqrt{F'_{\alpha\beta}(x_\beta)}\mu_{\alpha\beta} \\ \varphi_\alpha = \sqrt{F'_{\alpha\beta}(x_\beta)}\varphi_\beta + \mu_{\alpha\beta}(x_\beta) \end{cases}$$

where:

for $1 \le \alpha \le n$ and any $\beta \ge m+n$

$$\begin{cases} F_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(x_\beta) + \frac{t_\alpha}{f_{\alpha\beta}(x_\beta)} \\ \mu_{\alpha\beta}(x_\beta) = 0 \end{cases}$$

for $n+1 \le \alpha \le n+m$ and any $\beta \ge n+m$

$$\begin{cases} F_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(x_\beta) \\ \mu_{\alpha\beta}(x_\beta) = \dfrac{\eta_{\alpha-n}}{f_{\alpha\beta}(x_\beta)} \end{cases}$$

for $\alpha, \beta \ge n+m$

$$\begin{cases} F_{\alpha\beta}(x_\beta) = f_{\alpha\beta}(x_\beta) \\ \mu_{\alpha\beta}(x_\beta) = 0 \end{cases}$$

The crucial property of such a deformation we want to capture can be summarized in the following

Proposition. There exist universal deformations $X \longrightarrow \Delta^{n|m}$, $(n|m = 3g-3|2g-2)$ of a susy–curve C whose transition functions

i) depend linearly on the odd deformations parameters;

ii) are split but for a finite number of intersections

iii)the relative one–cocycle with values in $\mathcal{L}_\pi^{-1}$ $\mu_{\alpha\beta}$ has $m = 2g-2$ simple poles on each fibre of X.

More generally, we have complete deformations with the same properties when $n|m \geq 3g-3|2g-2$. Needles to say, given any other universal deformation $X' \rightarrow \Delta'^{\,n|m}$ of C (e.g. one given by a generic choice of non–trivial $\mu_{\alpha\beta}$'s) there is a unique base change $h: \Delta^{n|m} \rightarrow \Delta'^{\,n|m}$ such that h^*X' is isomorphic to X (possibly after a suitable shrinking of the bases). This makes us free to choose Schiffer 'atlases' on the supermoduli stack of the form $\{C_j; X_j \rightarrow \Delta_j^{n|m}\}$, the $X_j \rightarrow \Delta_j^{n|m}$ being universal Schiffer deformations of C_j. To handle such deformations, and in particular the choices of the $\mu_{\alpha\beta}$, we need some more technicalities. Given two different choices of $m = 2g-2$ points $\{p_i\}$, $\{p_{m+i}\}$ $i = 1, \ldots, 2g-2$ on each fibre* C_t of a deformation X, we have a Stein covering of C_{red} made of the disjoint union of $U_0 \equiv C_{red} \setminus \{p_k\}_{k=1,\ldots,4g-4}$, and small disks $\{U_k\}_{k=1,\ldots,4g-4}$ around each p_k such that the only non–empty intersections are punctured disks given by $U_{0k} = U_0 \cap U_k$. Now two choices of one–cocycles $\mu_{\alpha\beta}$ can be represented on this covering by a collection of *meromorphic* sections of $\mathcal{L}_\pi^{-1}\restriction_{U_k}$ with simple poles at p_i, $1 \leq i \leq 2g-2$ and at p_j, $2g-1 \leq j \leq 4g-4$, i.e.

$$\begin{cases} \mu_{i0} = \sum_{j=1}^{2g-2} \delta_i^j \dfrac{\eta^i}{x_i} \varphi_i \otimes \dfrac{\partial}{\partial x_i} \\ \nu_{i0} = \sum_{j=2g-1}^{4g-4} \delta_i^j \dfrac{\epsilon^{i-n}}{x_i} \varphi_i \otimes \dfrac{\partial}{\partial x_i} \end{cases}$$

We shall need the following

Lemma. There exists a unique (up to a sign) linear map $\epsilon^k = A^k{}_i \eta^i$ such that μ_{0i} and ν_{0i} are cohomologous.

Proof. The difference $\lambda_{i0} \equiv \mu_{0i} - \nu_{0i}$ has simple poles at $4g-4$ points. From the exact sequence

$$0 \rightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{L}^{-1}\left(\sum_{i=1}^{4g-4} p_i\right) \longrightarrow \mathcal{L}^{-1}\left(\sum_{i=1}^{4g-4} p_i\right)/\mathcal{L}^{-1} \rightarrow 0$$

we have

$$\begin{array}{ccccccccc} 0 & \rightarrow & H^0(\mathcal{L}^{-1}(\sum_{i=1}^{4g-4} p_i)) & \longrightarrow & \mathbb{C}^{4g-4} & \xrightarrow{\delta} & H^1(\mathcal{L}^{-1}) & \rightarrow & 0 \\ & & \| & & & & & & \\ & & \mathbb{C}^{2g-2} & & & & & & \end{array}$$

* These actually can be made to define a divisor on X, but we prefer to work fibrewise.

so there $\lambda \in ker\delta$ iff there are $2g-2$ sections $s_0^i \in H^0(\mathcal{L}^{-1}(\sum_{i=1}^{4g-4} p_i))$ such that *

$$\lambda_{i0} = \sum_k \eta_k (s_0^k - s_i^k)$$

where s_i^k denotes the holomorphic tail of s_0^k on U_i., i.e. $s_0^k \restriction_{U_i} = \frac{\mathsf{B}_i^k}{x_i} + s_i^k$, with $\mathsf{B} = (\mathsf{I}, \mathsf{A})$. ∎

We can now prove the following

Proposition. For $g \geq 3$ Schiffer atlases are not projected.

Proof. Let X_j and X_k be two universal Schiffer deformations which partially overlap, and let $\mu_{\alpha_j\beta_j}$, $\mu_{\alpha_k\beta_k}$ the corresponding $\mathcal{L}^{-1}$–valued one–cocycles. Then there exists an $\mathcal{L}^{-1}$–valued coboundary which make them cohomologous. Then the obstruction to projectedness $mod\ \mathcal{N}^3$ now reads

$$\tau^2 = \frac{\partial g_{\alpha_i\alpha_j}}{\partial t_j} h_{ij}^{(2)} = \sigma_{\alpha_i\alpha_j}\mu_{\alpha_i\beta_i} - \mu_{\alpha_j\beta_j}\sigma_{\beta_i\beta_j} \stackrel{def}{=}$$

$$\stackrel{def}{=} \sigma_\alpha \mu_{\alpha\beta} - \nu_{\alpha\beta}\sigma_\beta$$

We can take $\mu_{\alpha\beta}$ ($\nu_{\alpha\beta}$) with support on the punctured disks $U_{0\alpha}$ for $\alpha = 1, \ldots, 2g-2$ and for $\alpha = 2g-1, \ldots, 4g-4$ respectively. Then, after the suitable identification of the odd parameters, we have $\mu_{\alpha\beta} - \nu_{\alpha\beta} = \sum_k \eta^k (s_\beta^k - s_\alpha^k)$ and $\sigma_\alpha = \sum_k \eta^k s_\alpha^k$ i.e.

$$\mu_{\alpha\beta} - \nu_{\alpha\beta} = \mu_{\alpha 0} - \nu_{\alpha 0}\restriction_{U_\beta} = \sum_k \eta^k (s_0^k - s_\alpha^k)$$

Now τ^2 reads

$$\tau^2 = \begin{cases} \sum_l \eta^l s_\alpha^l (\sum_k \eta^k (s_o^k - s_\alpha^k))\restriction_{U_\beta} & 1 \leq \alpha \leq 2g-2 \\ \sum_k \eta^k s_0^k (\sum_l \eta^l (s_o^l - s_\alpha^l))\restriction_{U_\beta} & 2g-1 \leq \alpha \leq 4g-4 \end{cases}$$

i.e.

$$\tau^2 = \begin{cases} \sum_{kl} \eta^l \eta^k (s_\alpha^l s_0^k - s_\alpha^l s_\alpha^k) & 1 \leq \alpha \leq 2g-2 \\ \sum_{kl} \eta^k \eta^l (s_0^k s_0^l - s_k^l s_\alpha^l) & 2g-1 \leq \alpha \leq 4g-4 \end{cases}$$

---

* We take advantage here from the fact that for a generic choice of the points p_i, $H^0(\mathcal{L}^{-1}(p_j + \sum_{i=1}^{2g-2} p_{m+i}))$ is one–dimensional for $j = 1, \ldots, 2g-2$.

As the last row can be written as $\sum_{kl}\eta^l\eta^k(s^l_k s^l_\alpha - s^k_0 s^l_0)$, then $\tau^2 = \xi_{0\alpha} - \xi_0 + \xi_\alpha$ where

$$\xi_{0\alpha} = \sum_{kl}\eta^l\eta^k s^k_0 s^l_\alpha \qquad 1\le\alpha\le 4g-4$$
$$\xi_0 = \sum_{kl}\eta^l\eta^k s^k_0 s^l_0 \qquad 1\le\alpha\le 4g-4$$
$$\xi_\alpha = \begin{cases} -\sum_{kl}\eta^l\eta^k s^l_\alpha s^k_\alpha & 1\le\alpha\le 2g-2 \\ 0 & 2g-1\le\alpha\le 4g-4\end{cases}$$

This shows that τ^2 is a one–cocycle with values in $\mathcal{L}^{-2}$ cohomologous to $\xi_{0\alpha}$, which, in turn can be identified with a family of local meromorphic sections of $\mathcal{L}^{-2}$ with simple poles at the points p_α. This is cohomologous to zero if and only if the $\frac{1}{2}(2g-2)(2g-3) = (g-1)(2g-3)$ sections $s^k_0 s^l_\alpha - s^l_0 s^k_\alpha$ are cohomologous to zero. Now the exact sequence

$$0\to\mathcal{L}^{-2}\longrightarrow\mathcal{L}^{-2}(\textstyle\sum_\alpha p_\alpha)\longrightarrow\mathcal{L}^{-2}(\sum_\alpha p_\alpha)/\mathcal{L}^{-2}\to 0$$

gives us on each fibre

$$0\to H^0(\mathcal{L}^{-2}(\textstyle\sum_\alpha p_\alpha))\longrightarrow\mathbb{C}^{4g-4}\longrightarrow H^1(\mathcal{L}^{-2})$$

Now for any choice of p_α's, $\dim H^0(\mathcal{L}^{-2}(\sum_\alpha p_\alpha))\le g$ and so for $g\ge 3$, τ^2 cannot be cohomologous to 0. Since the representative for the splitting cocycle is $h^{(2)}_{ij}\frac{\partial}{\partial t_j} = (KS^{ev}_t)^{-1}(\tau^2)$, we see that $h^{(2)}_{ij}$ cannot vanish. ■

For $g = 2$ this dimensional argument clearly breaks down. In fact, special choices can be made (see e.g. [GIS]) which allow us a constructive proof of Deligne's result about the splitness of genus 2 supermoduli space. Deligne's proof is contained in an unpublished letter to D. Kazhdan, whose content is actually unknown to us. The splitness of genus 2 supermoduli spaces can be understood also in terms of the holomorphic geometry of moduli of spin curves as follows. It is known that the moduli space M_2 of genus $g=2$ *smooth* curves is an affine variety [Mu3] and hence it is Stein. But the moduli space of θ-characteristics is a finite covering of M_g and hence it is a Stein variety as well, so that the higher cohomology of coherent analytic sheaves vanishes.

The constructive proof makes use of the properties of double coverings of $\mathbb{P}^1$ as follows.

Proposition. There exist split Schiffer 'atlases' on the even supermoduli stack at $g = 2$.

Proof. Let $p_\pm$ be two conjugate points under the hyperelliptic involution h. The exact sequence

$$0 \to \mathcal{L}^{-1} \longrightarrow \mathcal{L}^{-1}(p_- + p_+)) \longrightarrow \mathcal{L}^{-1}(p_- + p_+))/\mathcal{L}^{-1} \to 0$$

yields

$$\cdots \to 0 \longrightarrow H^0(\mathcal{L}^{-1}(p_- + p_+)) \longrightarrow \mathbb{C}^2 \xrightarrow{\delta} H^1(\mathcal{L}^{-1}) \longrightarrow H^1(\mathcal{L}^{-1}(p_- + p_+)) \to \cdots$$

Now $\deg \mathcal{L}^{-1}(p_- + p_+) = 1 = g - 1$, and therefore $H^i(\mathcal{L}^{-1}(p_- + p_+))(i = 0, 1)$ have the same dimension. So to prove that δ is an isomorphism we notice that $p_- + p_+$ is a canonical divisor, so that $H^1(\mathcal{L}^{-1}(p_- + p_+)) \equiv H^1(\mathcal{L}^{-1} \otimes \mathcal{K}) \simeq H^0(\mathcal{L}) = \{0\}$ because genus two even θ-characteristics are non-singular.

Accordingly we can choose the $\mu_{\alpha_i\beta_i}$ defining the Schiffer deformations with simple poles at $p^i_\pm$. * Now glueing two such families X_j and X_k we see that the obstruction cocycle ${\tau^2}_{0\alpha} = s^1_o s^2_\alpha - s^2_0 s^1_\alpha$ to splitness (which is the same thing as projectedness at $g = 2$) is an $\mathcal{L}^{-2}$–valued one–cocycle with simple poles at $p^j_\pm$ and $p^k_\pm$ depending linearly on two parameters.

Now τ^2 is cohomologous to a global section in $H^0(\mathcal{L}^{-2}(p^j_- + p^j_+ + p^k_- + p^k_+))$ if and only if the family $\omega_\alpha \equiv z_\alpha \cdot {\tau^2}_{0\alpha}$ is cohomologous to a global section of $\mathcal{K}$ (indeed $\mathcal{L}^{-2}(p^j_- + p^j_+ + p^k_- + p^k_+) \equiv \mathcal{K}$). Since abelian differentials are anti–invariant under the hyperelliptic involution h we need only to show that the leading terms of ω_α satisfy $\omega_1 = -h^*\omega_2$ and $\omega_3 = -h^*\omega_4$. This turns out to be true for the following reason. The section s^1_0 (resp. s^2_0) is the *unique* (up to multiplication by

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* We stress that genus $g = 2$ is the only case for which this choice is significant. In fact, for $g \geq 3$ the variety of curves admitting singular θ-characteristics is a *divisor* in Σ^g_0.

a constant) meromorphic section in $\mathcal{L}^{-1}$ having simple poles at p_-^j, p_-^k, p_+^k (resp. at p_+^j, p_-^k, p_+^k.) Since $h^*(s_0^1)$ has the same behaviour as s_0^2 and conversely, there must hold $h^*(s_0^1) = \lambda s_0^2$ and $h^*(s_0^2) = \mu s_0^1$ with $\lambda\mu = 1$. Now a glance at the expression of $\tau_{0\alpha}^2$ shows that $h^*(\tau_{0\alpha}^2) = -\lambda\mu\tau_{0\alpha}^2$ and this concludes the proof. ∎

3.4 N=2 Susy–Curves.

Recall that N=1 susy–curves are 1|1 dimensional supercurves equipped with a distinguished distribution $\mathcal{D}$ in the tangent sheaf, spanned by the supersymmetry generator. In the same way the structure sheaf $\mathcal{A}_C$ of a N=2 susy–curve (hereinafter susy$_2$–curves) is somehow special as it should embody the notion of the superconformal structure. In the physical literature this is realized in terms of coordinate transformations [Co][Me]. Here we give a definition which naturally extends that of N=1 susy–curves.

Definition. A family of susy$_2$-curves $(C, \mathcal{A}_C)$ parameterized by the complex superspace $(S, \mathcal{A}_S)$ i.e. a susy$_2$-curve over S is the datum of

i) a sheaf homomorphism $\pi^\# : \pi^{-1}\mathcal{A}_S \to \mathcal{A}_C$ of relative dimension 1|2 over a proper surjective flat map $C \xrightarrow{\pi} S$

ii) a 0|2-dimensional locally free distribution $\mathcal{D}_\pi$ in the relative tangent sheaf $\mathcal{T}_\pi$ such that the commutator $mod \mathcal{D}_\pi$

$$\{\ ,\ \}_\mathcal{D} : \mathcal{D}_\pi \otimes \mathcal{D}_\pi \to \mathcal{T}_\pi / \mathcal{D}_\pi$$

is a symmetric non degenerate bilinear map of sheaves of $\mathcal{A}_C$-modules.

As usual, a susy$_2$-curve over the trivial superspace $\{*\}$ will be called a single susy$_2$-curve . The connection between the above definition and the usual coordinate approach, as given e.g. in [Fr], is a simple generalization of the $N = 1$ case. Indeed, one can easily prove that there exist generators D^i for $\mathcal{D}_\pi$ and $\frac{\partial}{\partial z}$ for $\mathcal{T}_\pi/\mathcal{D}_\pi$ such that

$$\{D^i, D^j\}_\mathcal{D} = \delta^{ij}\frac{\partial}{\partial z}$$

A simple computation shows then that $D^i = \frac{\partial}{\partial \theta^i} + \theta^i \frac{\partial}{\partial z}$. Besides matching with physical applications, our definition allows an immediate characterization of single susy$_2$-curves .

Proposition. Let $(C, \mathcal{A}_C)$ be single susy$_2$-curve with reduced canonical sheaf ω. Then there exists a rank 2 locally free sheaf $\mathcal{E}$ such that

i) $\mathcal{A}_C \simeq \wedge\mathcal{E}$, i.e. $\mathcal{A}_C$ is split

ii) $\mathcal{E} \simeq \mathcal{E}^* \otimes \omega$, i.e. $\mathcal{E}$ is Serre self-dual.

Proof. Let $(U_\alpha, z_\alpha, {\theta^i}_\alpha)$ be a canonical atlas with transition functions

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) + g_{\alpha\beta}\epsilon_{ij}\theta^i\theta^j \\ {\theta_\alpha}^i = [m_{\alpha\beta}]^i_j {\theta_\beta}^j \end{cases}$$

The existence of the distribution $\mathcal{D}_\pi$ is then equivalent to the superconformal condition

$$D^i{}_\beta z_\alpha = \theta^k{}_\alpha D^i{}_\beta \theta^k{}_\alpha$$

(sum over repeated latin indices) which gives

$$\epsilon_{ij}g_{\alpha\beta} + \delta_{ij}f'_{\alpha\beta} = [{}^t m_{\alpha\beta} m_{\alpha\beta}]_{ij},$$

where $f'_{\alpha\beta} = \frac{\partial f_{\alpha\beta}}{\partial z_\beta}$. Looking at the symmetric and the antisymmetric part of this equation we have:

i) $g_{\alpha\beta} = 0$, so $\mathcal{A}_C$ splits to $\wedge\mathcal{E}$, where $\mathcal{E}$ is locally generated by the ${\theta^i}_\alpha$'s,

ii) ${}^t m_{\alpha\beta} m_{\alpha\beta} = \mathbf{1}\cdot f'_{\alpha\beta}$, where $m_{\alpha\beta}$ are the transition functions of $\mathcal{E}$.

So $m_{\alpha\beta} ={}^t m^{-1}_{\alpha\beta} f'_{\alpha\beta}$ i.e. $\mathcal{E} \simeq \mathcal{E}^* \otimes \omega$. ∎

We want to remark at this point the power of superconformal structures. Indeed, a generic supercurve of dimension $1|2$ is by no means split, as opposite to the trivial $1|1$ case, and as the example given in §2.1 shows. Nevertheless (single) susy$_2$-curves are split.

According to the physical literature, a susy$_2$-curve is called *twisted*, whenever the $O(2)$ symmetry of the (anti)commutation relations for the local supersymmetry generators D^i_α cannot be reduced to an $SO(2)$ symmetry [Co]. This is related to the vanishing of a class in $H^2(C,\mathbb{Z}_2)$ gotten by taking the determinant of the transition functions for the locally free sheaf $\mathcal{E}$. Namely, as any rank two locally free sheaf can be represented as the extension of a invertible sheaf $\mathcal{L}_1$ by another $\mathcal{L}_2$ fitting the exact sequence $0 \to \mathcal{L}_2 \to \mathcal{E} \to \mathcal{L}_1 \to 0$, we have det $\mathcal{E} = \mathcal{L}_1 \otimes \mathcal{L}_2$. But Serre self–duality implies det $\mathcal{E} = \omega \otimes \mathcal{N}_2$, $\mathcal{N}_2$ being a point of order 2 on the Jacobian of C. Then a susy$_2$-curve is untwisted whenever $\mathcal{N}_2$ is trivial.

From the holomorphic point of view, Serre self-dual rank 2 locally free sheaves are quite simple objects.

Proposition The characteristic sheaf $\mathcal{E}$ of untwisted susy$_2$-curves decomposes as the direct sum $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$, with $\mathcal{L}_1 \otimes \mathcal{L}_2 \simeq \omega$.

Proof. In a superconformal gauge the transition functions $\mu_{\alpha\beta}(z_\beta)^i{}_j$ of $\mathcal{E}$ satisfy ${}^t m_{\alpha\beta} \cdot m_{\alpha\beta} = f'_{\alpha\beta} \cdot \mathbf{1}$ and hence they can be given the form $m_{\alpha\beta} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ -b_{\alpha\beta} & a_{\alpha\beta} \end{pmatrix}$, with $a^2_{\alpha\beta} + b^2_{\alpha\beta} = f'_{\alpha\beta}$. A simple computation shows that there is a one–cochain λ_α with values in the sheaf of $GL(2,\mathbb{C})$–valued holomorphic functions which diagonalizes $m_{\alpha\beta}$ showing that, actually, $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$. Imposing the Serre self–duality condition in this gauge gives

$$\mathcal{L}_1 \oplus \mathcal{L}_2 \simeq \mathcal{L}_1^{-1} \otimes \omega \oplus \mathcal{L}_2^{-1} \otimes \omega$$

This ends the proof since $\mathcal{L}_1 \oplus \mathcal{L}_2 \simeq= \mathcal{L}'_1 \oplus \mathcal{L}'_2$ if and only if either $\mathcal{L}_1 \simeq \mathcal{L}'_1$ or $\mathcal{L}_1 \simeq \mathcal{L}'_2$ ∎

Proposition. Any twisted Serre self-dual locally free sheaf $\mathcal{E}$ of rank 2 on C is holomorphically isomorphic to the direct sum of two different θ-characteristics, i.e. $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$, with $\mathcal{L}_i^2 = \omega$.

Proof. The first thing we prove is that a twisted semistable Serre self–dual locally free sheaf is strictly semistable, i.e. it admits only degree $g-1$ invertible subsheaves. Notice that if $\mathcal{E}$ is untwisted it is not stable. If $\mathcal{E}$ is twisted, there is a point $\mathcal{M}$

of order 4 on the Jacobian of C such that $\mathcal{E} \otimes \mathcal{M}$ is untwisted in the sense that det $(\mathcal{E} \otimes \mathcal{M}) = \omega$. As $\mathcal{E} \otimes \mathcal{M}$ is stable if and only if $\mathcal{E}$ is stable, we are again in the above situation.

Next, we prove that an unstable Serre self dual locally free sheaf is strictly semistable as well. To this purpose, we have to use a lemma about uniqueness of maximal sub–line bundles of an unstable rank 2 vector bundle [Gu]. Given a vector bundle $\mathcal{E} \to C$, its divisor is defined as

$$\mathrm{div}\mathcal{E} = \max_{L \subset \mathcal{E}} c_1(L)$$

Then, if $\mu(\mathcal{E})$ is the slope of $\mathcal{E}$, it holds that if $\mathrm{div}\mathcal{E} > \mu(\mathcal{E})$ there is a unique line bundle $L \hookrightarrow \mathcal{E}$ with $c(L) = \mathrm{div}\mathcal{E}$; such uniqueness holds also if $\mathrm{div}\mathcal{E} = \mu(\mathcal{E})$ and Eis indecomposable. Now suppose that $\mathcal{E}$ is given as $0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0$ with $c_1(\mathcal{L}_1) \geq g-1$. Serre–dualizing we get $0 \to \mathcal{L}_2^\vee \to \mathcal{E} \to \mathcal{L}_1^\vee \to 0$. Then, supposing $c_1(\mathcal{L}_1) > g-1$, then $\mathcal{L}_1 \simeq \mathcal{L}_2^\vee$ and hence as det $\mathcal{E} = \mathcal{L}_1 \otimes \mathcal{L}_2 = \omega$ we get a contradiction with the assumption of twisting of $\mathcal{E}$.

We have only to discuss the case $0 \to \mathcal{L}_1 \to \mathcal{E} \to \mathcal{L}_2 \to 0$, with $c_1(\mathcal{L}_i) = g-1$, $\mathcal{L}_1 \otimes \mathcal{L}_2 \neq \omega$. If $\mathcal{E}$ were indecomposable, the lemma we recalled above tells us that there would be a unique invertible subsheaf $\mathcal{L} \subset \mathcal{E}$ of degree $g-1$ contradicting the assumption that $\mathcal{L}_1 \not\simeq \mathcal{L}_2^\vee$. Finally, given $\mathcal{E} \simeq \mathcal{L}_1 \oplus \mathcal{L}_2$ the Serre–self duality condition implies that $\mathcal{L}_i^2 \simeq \omega$. ∎

Summing up we have that superconformal structures force the characteristic sheaf $\mathcal{E}$ to be, in the twisted case, the direct sum of two non–isomorphic square roots of the canonical bundle. The untwisted case has a richer structure, since here $\mathcal{E}$ decomposes as $\mathcal{L} \oplus \omega \otimes \mathcal{L}^{-1}$, $\mathcal{L} \in \mathrm{Pic}C$. Notice that the reduced moduli space of untwisted susy$_2$-curves is then infinitely connected, with zeroth homotopy group parameterized by $\mathrm{deg}\mathcal{L}$. For the sake of concreteness, we will restrict the general discussion to the case $\mathrm{deg}\mathcal{L} = g-1$, which is the only case that gives rise to a semi–stable $\mathcal{E}$, pointing out the necessary modifications in the general case.

We next want to deform (at least at the infinitesimal level) the structure of a susy$_2$-curve .

Definition. A deformation of a susy$_2$-curve C over a pointed superspace $(B,\{*\})$ is a family $\mathcal{C}\xrightarrow{\pi}B$ of susy$_2$-curves together with an isomorphism ψ of C with the 'central fibre' $\pi^{-1}(\{*\})$ fitting the commutative diagram

$$\begin{array}{ccc} C & \overset{\psi}{\hookrightarrow} & \mathcal{C} \\ \downarrow & & \downarrow \pi \\ \{*\} & \hookrightarrow & B \end{array}$$

As usual, the starting point to set up deformation theory is to identify the sheaf of infinitesimal automorphisms of the object to be deformed. In our case this is the subsheaf $T_\pi^{\mathcal{D}}$ of the relative tangent sheaf, whose elements are germs of vector fields along the fibres which preserve $\mathcal{D}$,

$$T_\pi^{\mathcal{D}} := \{X \in T_\pi \mid [D,X] \in \mathcal{D}\, \forall D \in \mathcal{D}\}$$

In perfect analogy with the case of $N=1$ Susy–curves we find

Proposition. There is an isomorphism $T_\pi^{\mathcal{D}} \simeq (T_\pi)_{red} \otimes \mathcal{A}_C$ as sheaves of $\pi^{-1}(\mathcal{A}_B)$–modules.

Proof. The condition for X to belong to $T_\pi^{\mathcal{D}}$ reads $[D^i, X] \in \mathcal{D}$, where D^i are generators of $\mathcal{D}$. Introducing canonical coordinates (z,θ^i), so that $D^i = \dfrac{\partial}{\partial\theta^i} + \theta^i\dfrac{\partial}{\partial z}$ and setting $X = a\cdot\dfrac{\partial}{\partial z} + b_i \cdot D^i.$, one has

$$[D^i, X] = D^i a \frac{\partial}{\partial z} - (-1)^{|X|} b_k \delta^{ki} \frac{\partial}{\partial z} + D^i b_k \cdot D^k$$

Therefore $X \in T_\pi^{\mathcal{D}}$ if and only if $b_i = (-1)^{|a|} D^i a$ and the isomorphism is given by $a\cdot\dfrac{\partial}{\partial z} \rightsquigarrow a\dfrac{\partial}{\partial z} + (-1)^{|a|} D^i a \cdot D^i$ ■

Thanks to this lemma we have, for $\mathcal{E}$ semistable, the following

Proposition. Versal deformations of susy$_2$-curves exist. The dimension of the base of such deformations is $3g-3+g-a|4g-4$, with $a=0,1$ in the untwisted and twisted case respectively.

Proof. From the Kodaira-Spencer deformation theory, we know that possible obstructions lie in the second cohomology group of the sheaf of infinitesimal automorphisms $T_\pi^{\mathcal{D}}$. By the analysis of the structure sheaf $\mathcal{A}_C$ of a susy$_2$-curve one gets

$$T_\pi^{\mathcal{D}} = \omega^{-1} \oplus \Pi(\omega^{-1} \otimes \mathcal{E}) \oplus \det\,\mathcal{E} \otimes \omega^{-1}$$

the above sum being direct sum of sheaves of $\mathcal{O}_C$–modules*. Then $H^2(T_\pi^{\mathcal{D}}) = \{0\}$, showing the existence of versal deformations. The second part of the proposition follows from Serre self–duality of $\mathcal{E}$ and from proposition 1.4. Indeed $dim H^1(\omega^{-1} \oplus \det \mathcal{E} \otimes \omega^{-1}) = dim H^1(\omega^{-1} \oplus \mathcal{N})$, where $\mathcal{N} = \det \mathcal{E} \otimes \omega^{-1} = \mathcal{O}$ for untwisted susy$_2$-curves , while it is a point of order two in the Jacobian of C in the twisted case. As for the odd dimension, notice that $dim H^1(\omega^{-1} \otimes (\mathcal{L}_1 \oplus \mathcal{L}_2)) = dim H^1(\mathcal{L}_1^{-1}) + dim H^1(\mathcal{L}_2^{-1})$ ∎

Remark. As for the computation of the odd dimension q of the "would be" moduli space of susy$_2$-curves in the general untwisted case, one can argue as follows. As $\mathcal{E} \simeq \mathcal{L} \oplus \omega \otimes \mathcal{L}^{-1}$, $H^1(C, \mathcal{E})$ is invariant under the Kummer map $\mathcal{L} \rightsquigarrow \omega \otimes \mathcal{L}^{-1}$ Hence one can restrict himself to discuss the case $\deg \mathcal{L} \equiv d > g-1$ only. By the Riemann–Roch theorem one has

a) if $g-1 \leq d \leq 2g-2$ then $q = 4g-4$;

b) if $2g-2 \leq d \leq 3g-3$ *and* $\mathcal{L}$ *is generic* then $q = 4g-4$;

c) if $3g-3 \leq d \leq 4g-4$ *and* $\mathcal{L}$ *is generic* then $q = d+g-1$;

d) if $4g-4 < d$ then $q = d+g-1$.

Notice that, in the cases b) and c), the odd dimension of "moduli space" jumps on analytic submanifolds of the reduced space, a fact that renders quite subtle its structure in the framework of Kostant–Leites supermanifold theory.

From a more computative point of view, one can consider infinitesimal deformations, as being given by deforming the clutching functions of the central fibre. As we proved above, these are of the form

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) \\ \theta_\beta^i = [m_{\alpha\beta}(z_\beta)]^i_j \theta_\beta^j \end{cases}$$

the matrix $\mu_{\alpha\beta}(z_\beta)^i{}_j$ being of the form $\mu_{\alpha\beta}(z_\beta)^i{}_j = \begin{pmatrix} g_{1_{\alpha\beta}} & 0 \\ 0 & g_{2_{\alpha\beta}} \end{pmatrix}$ with either $g_{i_{\alpha\beta}}^2 = f'_{\alpha\beta}$ or $g_{1_{\alpha\beta}} \cdot g_{2_{\alpha\beta}} = f'_{\alpha\beta}$

The most general deformation of such clutching functions, i.e. one generated by a

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* The parity change operator Π has, strictly speaking, no effective meaning in this context. We just use it as a parity bookkeeper.

vector field in the whole $\mathcal{T}_\pi$, over $\hat{S}$ is given by

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) + t(b_{\alpha\beta}(z_\beta) + \frac{1}{2}g_{\alpha\beta}(z_\beta)\epsilon_{ij}\theta^i_\beta\theta^j_\beta) + \zeta\eta_{i_{\alpha\beta}}(z_\beta)\theta^i_\beta \\ \theta^i_\beta = [m_{\alpha\beta}(z_\beta)]^i_j\theta^j_\beta + t([l_{\alpha\beta}(z_\beta)]^i_j\theta^j_\beta + \zeta(\psi^i_{\alpha\beta}(z_\beta) + \frac{1}{2}\nu^i_{\alpha\beta}\epsilon_{jk}\theta^j_\beta\theta^k_\beta) \end{cases}$$

Imposing the superconformal condition shows that * $g_{\alpha\beta}(z_\beta) = 0$, and that the only independent data are $b_{\alpha\beta}(z_\beta)$, $\psi^i_{\alpha\beta}(z_\beta)$ and $[l_{\alpha\beta}(z_\beta)]^i_j$. The cocycle condition leads easily to the identification of $\{b_{\alpha\beta}(z_\beta)\cdot\frac{\partial}{\partial z_\alpha}\}$ as a one–cocycle with values in the relative tangent sheaf ω_π^{-1}, and $\{\psi^i_{\alpha\beta}(z_\beta)\}$ as a one–cocycle with values in $\mathcal{E}^*$.

As for the rôle of the matrix $[l_{\alpha\beta}(z_\beta)]^i_j$, one can argue as follows. Since the even and odd infinitesimal deformations give decoupled equations, one can limit himself to discussing a deformation of the form

$$\begin{cases} z_\alpha = f_{\alpha\beta}(z_\beta) + tb_{\alpha\beta}(z_\beta) \\ \theta^i_\beta = [m_{\alpha\beta}(z_\beta)]^i_j\cdot\{\delta^j_k + t([m^{-1}\cdot l_{\alpha\beta}(z_\beta)]^j_k\}\cdot\theta^k_\beta \end{cases}$$

The superconformal condition translates into the equation

$$O_{\alpha\beta} + {}^tO_{\alpha\beta} = \frac{1}{f'_{\alpha\beta}}\frac{\partial b_{\alpha\beta}}{\partial z_\beta}\cdot\mathbf{1}$$

for the matrix $O_{\alpha\beta} \equiv m^{-1}_{\alpha\beta}\cdot l_{\alpha\beta}$. Hence, $O_{\alpha\beta} = \begin{pmatrix} \tau_{\alpha\beta} & \alpha_{\alpha\beta} \\ -\alpha_{\alpha\beta} & \tau_{\alpha\beta} \end{pmatrix}$ and its only free part is the off diagonal $\tilde{O}_{\alpha\beta} = \begin{pmatrix} 0 & \alpha_{\alpha\beta} \\ -\alpha_{\alpha\beta} & 0 \end{pmatrix}$. This decomposition is obviously due to the fact that when deforming the underlying curve C according to $b_{\alpha\beta}$, line bundles on C are deformed as well. The cocycle condition for $O_{\alpha\beta}$ gives

$$m_{\alpha\beta}O_{\alpha\beta}m_{\beta\gamma} + m_{\alpha\beta}m_{\beta\gamma}O_{\beta\gamma} = m_{\alpha\gamma}O_{\alpha\gamma} - b_{\alpha\beta}m'_{\alpha\beta}m_{\beta\gamma}$$

Looking once again at the off–diagonal part one has (multiplying on the left by $m^{-1}_{\alpha\gamma}$)

$$m^{-1}_{\beta\gamma}\tilde{O}_{\alpha\beta}m_{\beta\gamma} + \tilde{O}_{\beta\gamma} = \tilde{O}_{\alpha\gamma}$$

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* this fact can be also grasped by writing explicitly the superconformal vector fields that generate the deformations

A simple algebra shows that $\tilde{O}_{\alpha\beta} m_{\beta\gamma} = \frac{f'_{\beta\gamma}}{det\ m_{\beta\gamma}} \cdot m_{\beta\gamma} \tilde{O}_{\alpha\beta}$ yielding

$$\frac{f'_{\beta\gamma}}{det\ m_{\beta\gamma}} \tilde{O}_{\alpha\beta} + \tilde{O}_{\beta\gamma} = \tilde{O}_{\alpha\gamma}.$$

Then, considering local generators $\{\varphi_\alpha\}$ of $\omega^{-1} \otimes \det \mathcal{E}$ one readily identifies the collection $\{\alpha_{\alpha\beta} \cdot \varphi_\beta\}$ as a one–cocycle with values in $\omega^{-1} \otimes \det \mathcal{E}$.

Summing up, a complete infinitesimal deformation of a susy$_2$-curves consists of a deformation of the underlying algebraic curve and the couple of line bundles which define the 'single' object plus the deformation specified by $l_{\alpha\beta}$. Since this latter is completely qualified by an element in $H^1(C, \omega^{-1} \otimes \det \mathcal{E})$ we find a complete agreement with the results of the more formal analysis we discussed before. As a final remark we notice that $H^1(C, \omega^{-1} \otimes \det \mathcal{E})$, which can also be thought of as the space of superconformally non–equivalent susy$_2$–structures on a fixed curve, coincides with the space of holomorphically non–equivalent extensions of a θ-characteristics $\mathcal{L}_1$ by another one $\mathcal{L}_2$.

As a last topic, we can give now an alternative description of the reduced moduli spaces of susy$_2$-curves in a group–theoretical setting. Holomorphic isomorphism classes of twisted susy$_2$-curves are in one-to-one correspondence with isomorphisms classes of couples $(C, \mathcal{L}_{12})$, where $\mathcal{L}_{12}$ is an unordered couple of non–equivalent θ-characteristics. This sits inside the second symmetric power $\Sigma^{(2)}$ of the spin covering $\Sigma \to \mathcal{M}$ of the moduli space (at some fixed genus). Also, the reduced moduli space of untwisted susy$_2$-curves with semistable characteristic sheaf $\mathcal{E}$ can be identified with the universal degree $g-1$ Picard variety $Pic_{g-1} \to \mathcal{M}_g$ over the moduli variety of genus g algebraic curves, modulo the Kummer map $\mathcal{L} \rightsquigarrow \omega \otimes \mathcal{L}^{-1}$.

From the group–theoretical point of view, the sheaf $\mathcal{E}$ should be regarded not merely as a holomorphic sheaf, because the presence of the superconformal structure amounts to saying that it is the sheaf of sections of a vector bundle E with structure group the conformal group

$$G = \{m \in GL(2, \mathbb{C}) \mid {}^t m \cdot m = \lambda \mathbb{I}\} \equiv G_0 \sqcup \eta \cdot G_0$$

where G_0 is the identity component and $\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$). The map $\varphi: G \to \mathbb{C}^*$ given by $\varphi(m) = {}^t m \cdot m$ gives rise to the exact diagram of complex groups

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & SO(2) & \longrightarrow & G_0 & \xrightarrow{\varphi} & \mathbb{C}^* & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & O(2) & \longrightarrow & G & \xrightarrow{\varphi} & \mathbb{C}^* & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array}$$

Notice that the first row is an exact sequence of central subgroups of the groups in the second row. This is vital at the level of exact sequences of sheaves of germs of group–valued functions given by sheafifying the above diagram. Indeed, pushing the induced cohomology sequences as far as possible, we get an exact diagram

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & H^1(C, \mathcal{SO}_2) & \longrightarrow & H^1(C, \mathcal{G}_0) & \longrightarrow & H^1(C, \mathcal{O}^*) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & H^1(C, \mathcal{O}_2) & \longrightarrow & H^1(C, \mathcal{G}) & \longrightarrow & H^1(C, \mathcal{O}^*) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & H^1(C, \mathbb{Z}_2) & \longrightarrow & H^1(C, \mathbb{Z}_2) & \longrightarrow & 1 & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \qquad (*)$$

Here we used some results of non–abelian sheaf cohomology, and the following

Lemma. The cohomology groups $H^*(C, \mathcal{O}^*)$ and $H^*(C, \mathcal{SO}_2)$ coincide.

Proof. The exact sheaf sequence

$$0 \to \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{m} \mathcal{SO}_2 \to 1$$

where the map m is defined by $m(f) = \begin{pmatrix} \cos(2\pi i f) & \sin(2\pi i f) \\ -\sin(2\pi i f) & cos(2\pi i f) \end{pmatrix} \quad \forall f \in \Gamma(U, \mathcal{O})$ fits together with the standard exponential sequence into the commutative diagram of sheaves (of abelian groups)

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{exp} & \mathcal{O}^* & \to & 1 \\ & & \downarrow{id} & & \downarrow{id} & & \downarrow{m^*} & & \\ 0 & \to & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{m} & \mathcal{SO}_2 & \to & 1 \end{array}$$

where

$$m^*(\psi) = \begin{pmatrix} (\psi + \psi^{-1})/2 & (\psi - \psi^{-1})/2i \\ -(\psi - \psi^{-1})/2i & (\psi + \psi^{-1})/2 \end{pmatrix} \quad \forall \psi \in \Gamma(U, \mathcal{O}^*)$$

This gives rise to a long commutative sequence of cohomology groups proving the lemma. ■

Remark. Notice that the sequence above shows that the cohomology group $H^1(C, \mathcal{SO}_2)$ is isomorphic to the group $Pic\ C$ of invertible sheaves on C.

The basic fact for our concern is the following

Lemma. The action of $H^1(C, \mathcal{SO}_2)$ is free and transitive on the fibre of $H^1(C, \mathcal{G}_0) \rightarrow H^1(C, \mathcal{O}^*)$ over each class $\tau \in H^1(C, \mathcal{O}^*)$. The same is true for the action of $H^1(C, \mathcal{G}_0)$ on the fibre of $H^1(C, \mathcal{G}) \rightarrow H^1(C, \mathbb{Z}_2)$ over $\tau' \in H^1(C, \mathbb{Z}_2)$.

Proof. Since both $\mathcal{SO}_2 \hookrightarrow \mathcal{G}_0$ and $\mathcal{G}_0 \hookrightarrow \mathcal{G}$ are central and abelian, we can apply a (simplyfied) argument of non–abelian sheaf cohomology (see, *e.g.* lemma 2.4 of [LR]) to get the proof. This runs as follows. Given an exact sequence of sheaves of groups $0 \rightarrow \mathcal{P} \rightarrow \mathcal{Q} \rightarrow \mathcal{R} \rightarrow 0$ in which $\mathcal{P}$ is central and abelian and $\mathcal{R}$ is abelian, one has the following results:

i) there is a connecting map $H^1(\mathcal{R}) \xrightarrow{\delta_1} H^2(\mathcal{P})$ so that the sequence

$$H^1(\mathcal{Q}) \longrightarrow H^1(\mathcal{R}) \xrightarrow{\delta_1} H^2(\mathcal{P})$$

is exact.

ii) whenever $\tau \in \operatorname{Ker}\,\delta_1$, $H^1(\mathcal{P})$ acts transitively on the fiber of $H^1(\mathcal{Q})$ over τ, with kernel given by the image of $H^0(\mathcal{R}) \xrightarrow{\delta_0} H^1(\mathcal{Q})$. In our case the lemma follows from the fact that $H^2(C, \mathcal{SO}_2) \simeq H^2(C, \mathcal{O}^*) = \{0\}$ and the observation that elements in $H^0(C, \mathbb{Z}_2)$ (($H^0(C, \mathcal{O}^*)$)) are mapped into locally constant matrices by the connecting homomorphisms δ_0 and so are trivial cocycles. ■

Using the above lemma, we can give the following description of the (reduced) moduli space of untwisted susy$_2$–structures over a fixed curve C.

Proposition. Non–equivalent untwisted susy$_2$–structures on a fixed (smooth) algebraic curve C are parameterized by the fibre of $H^1(C, \mathcal{SO}_2)$ in $H^1(C, \mathcal{G}_0)$ over $[\omega] \in H^1(C, \mathcal{O}^*)$.

Proof. This follows at once by noticing that the map $H^1(C, \mathcal{G}_0) \longrightarrow H^1(C, \mathcal{O}^*)$ in the diagram $(*)$ is surjective and the map $H^1(C, \mathcal{SO}_2) \longrightarrow H^1(C, \mathcal{G}_0)$ is injective. The last assertion follows from the Serre self–duality condition. ∎

Chapter 4.

Vector bundles and Grassmannians.

In the last few years, great attention have been paid to works by Mumford, Sato and others which relate solutions of certain non–linear partial differential equations (the so–called Kodomchev–Petviashvilii hierarchy of equations) to line bundles over algebraic curves. One of the most striking results of such a theory is that it leads to a solution of the long–standing Schottky problem for algebraic curves, which can be formulated as follows.

Given a (smooth) algebraic genus g curve C, a normalized basis for the space of abelian differentials $\{\omega_1, ..., \omega_g\}$, one gets a natural structure of principally polarized abelian variety over the Jacobian variety $J(C)$. Conversely, given $J(C)$ together with its Θ–divisor, Torelli's theorem enables one to reconstruct the curve C. Actually, a principally polarized abelian variety is characterized by the assignment of a symmetric $g \times g$ matrix with nondegenerate imaginary part, so that a rough count of parameters gives the dimension of the moduli space of dimension g principally polarized abelian varieties as $\frac{(g+1)\cdot g}{2}$.

Since the dimension of the moduli space of curves is strictly less than this for $g > 3$, it is clear that not all deformations of principally polarized abelian varieties come from deformations of curves, i.e., there is plenty of principally polarized abelian varieties which are not Jacobians of a curve C. Then the natural problem of characterizing Jacobians among principally polarized abelian varieties , commonly referred to as the Schottky problem, arises.

It was given recently a solution by considering maps of a suitably enlarged ("dressed") moduli space of curves into the so called infinite or universal grassmann manifold. Such a space has been the subject of a wide investigation in the last years both in the physical and in the mathematical literature (see, e.g. [PS] and [AGR], and, for its supersymmetrical extension, [S]). Using such techniques, as a very interesting byproduct, Arbarello, De Concini Kac and Procesi [ADKP] managed to settle up a bridge between the representation theory of some infinite dimensional Lie algebras and the Picard groups of the moduli spaces of "dressed" curves. More specifically, they considered the (classical) Virasoro algebra $diff^{\mathbb{C}}\, S^1$ and its semidirect product with the (classical) Heisenberg algebra $lu(1)$ and proved that their central extensions can be put in a bijective correspondence with the Picard group respectively of the moduli variety M_g'' parameterizing curves, points and a non–zero tangent vector at that point and the moduli variety F_d'' which is fibered over M_g'' with fiber $Pic_d(C)$ and parameterizes the data of M_g'' plus the datum of a degree d line bundle on C. In particular, they were able to give a "geometrical" explanation of the following numerical coincidence. The algebra $diff^{\mathbb{C}}\, S^1$ acts naturally via Lie derivative on the space of weight j differential on the punctured plane $\mathbb{C} \setminus 0$ i.e. the vector space of expressions of the form $f(z)dz^j$ with $f(z)$ possibly singular at the origin. In this way one gets a representation ρ_j of $diff^{\mathbb{C}}\, S^1$ in a (suitably defined) algebra of infinite size matrices a_∞. The pull–backs of the standard non–trivial two–cocycle $\psi \in H^2(a_\infty, \mathbb{C})$ satisfy

$$\rho_n^*(\psi) = (6n^2 - 6n + 1)\rho_1^*(\psi).$$

On the other hand, Mumford's formula for the Chern classes of n–th powers of the relative canonical sheaf $\omega_/$ over a family of curves $C \xrightarrow{\pi} M_g$ gives (see §1.3)

$$c_1(\pi_!(\omega_/^n)) = (6n^2 - 6n + 1)c_1(\pi_!(\omega_/)).$$

The issue of generalizing the abovementioned constructions has been already tackled by various authors; for instance, the extension of Mumford's work to vector bundles has been analyzed in [Mul] [PW], while the remarkable paper [BS] was mainly concentrated on the representation theory of the Virasoro algebra in terms of $\mathcal{D}$–

modules on moduli spaces. In this last chapter, we want to extend to the non–abelian case the link between the topology of moduli spaces of bundles on curves and the cohomology of the algebras of infinitesimal symmetries of higher rank vector bundles. Namely, we want to deal with rank r vector bundles so that the semidirect product of $diff^{\mathbb{C}}\,S^1$ times the Heisenberg algebra is replaced by $\mathcal{D} = diff^{\mathbb{C}}\,S^1 \ltimes L\mathfrak{gl}(n, \mathbb{C})$. The analysis will show that the work of [ADKP] can be repeated almost *verbatim* in such a case, and also that (even though we can prove it in a weaker form) there is an isomorphism between the second cohomology of $\mathcal{D}$ and the Picard group of the relevant moduli space.

This setting seems to suggest that there should be a way of getting Sugawara formula for the stress energy tensor of Wess–Zumino–Novikov–Witten [GO] models from purely geometrical data, as some results contained in [Hit] confirm, but at the moment, we are not able to achieve such a result.

4.1 The infinite Grassmannian.

Let V be a complex locally convex topological vector space. A polarization in V is an explicit isomorphism $V \xrightarrow{\varphi} V_+ \oplus V_-$, $V_\pm$ being two closed subspaces in V. A continuous operator $T : V \to V$ is Fredholm if both its kernel and cokernel are finite dimensional. The index ind_T is the difference dim ker T − dim coker T.

Definition. The Grassmannian manifold of V, $Gr(V)$ is the set of closed subspaces $0 \to W \to V$ such that the projection p_W on V_- is Fredholm.

$Gr(V)$ can be given the structure of a complex manifold modelled on the vector space $Hom(V_-, V_+)$. It is infinitely connected, and its connected components are labelled by the index of p_W.

Let us specialize the construction to the case in which the vector space V is the space H^r of germs at 0 of $\mathbb{C}^r$–valued holomorphic functions possibly singular at 0. For the sake of simplycity, let us stick to the case $r = 1$. Here one can take H_- to be the space of holomorphic functions on $\mathbb{P} \setminus \{\infty\}$ which vanish at ∞ and H_+ to be space of germs of holomorphic functions at 0. H_- can be given a Fréchet topology considering the family of norms $\| f \|_n = \max_{|z|=n} |f(z)|$ and H_+ is the direct limit of the Banach spaces H_ϵ of holomorphic functions in the disk D_ϵ of radius ϵ. One can prove the following

Theorem. The map $H_+ \times H_- \to \mathbb{C}$ given by

$$\mathrm{Res}_{z=0} < f(z), g(z) >$$

is a perfect pairing. Moreover, the space of polynomials in z (respectively, in z^{-1}) is dense in H_+ (H_-).

A standard result on the infinite Grassmannian is the following [PS]: let S_+, S_- be a Dirac partition of the integers $\mathbb{Z}$, i.e. a pair of disjoint infinite ordered sets of integers such that $s^i_- = -i$ for $i \gg 0$ and $s^i_+ = i$ for $i \gg 0$. Then if H_S is the closure of the linear span of the monomials $z^{s^i_-}$ for $s^i_- \in S_-$, then H_S is in $Gr(H)$ and $Gr^S(H) = \{W \in Gr(H) | p_S : W \to H_- \text{ is an isomorphism } \}$ is a coordinate chart. The determinant line bundle $\mathrm{Det} \to Gr(H)$ is defined by

$$\bigcup_{W \in Gr(H)} \wedge^{max} \mathrm{Ker}\, p\restriction_W \wedge^{max} (\mathrm{Coker}\, p\restriction_W)^*$$

$\mathrm{Det} \to Gr(H)$ is canonically trivialized on the open sets $Gr^S(H)$. Given a polarized topological vector space H, any continuous endomorphism $A : H \to H$ admits a canonical "matrix" decomposition

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$$

$GL_{\mathrm{res}}(H)$ is the group of invertible operators such that both A_{--} and A_{++} are Fredholm of index zero. It acts transitively on $Gr(H)$ which acquires the structure of a homogeneous space. It does not act on the determinant line bundle, which is acted upon by its universal central extension $\tilde{G}L_{\text{res}}(H)$ which is defined as the quotient of the group

$$\mathcal{E} = \{(A, q) \in GL_{\text{res}}(H) \times GL(H_-) | A_{--} - q \quad \text{is trace class}\},$$

modulo those with unit determinant.

We remark that the notion of infinite Grassmannian depends on which kind of topological vector space one starts from. Actually, one could choose the Hilbert space $V = L^2(S^1, \mathbb{C})$ and define $Gr(V)$ to be the set of all closed subspaces W whose projection to V_- is Fredholm and whose projection to V_+ is in his favourite Schatten ideal of the space of bounded operators in $L^2(S^1, \mathbb{C})$. For instance, if one assumes p_W^+ to be in the Hilbert-Schmidt class, then $Gr(V)$ acquires the structure of a infinite dimensional Hilbert manifold. We will rest on the choice done in [ADKP] since it is best suited for algebraic and algebro–geometrical applications.

4.2 Vector bundles on variable curves.

In §1.5 we have given some results about deformation theory of a vector bundle over a fixed curve C. Now we want to allow the curve to vary, and deform the whole structure. Let E be a stable rank r vector bundle of degree d over a curve C.

The first step for settling up infinitesimal deformations is to identify the sheaf of the infinitesimal automorphisms of the structure to be deformed. An automorphism of a vector bundle $E \to C$ is a fibrewise linear biholomorphic map $\tilde{\mu}$ and a biholomorphic map μ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\mu}} & E \\ \downarrow & & \downarrow \\ C & \xrightarrow{\mu} & C \end{array}$$

is commutative. An automorphism of the form $(\tilde{\mu}, id)$ is called vertical. The sheaf $Aut_v E$ of germs of vertical automorphisms of E is a subsheaf of $AutE$, i.e. we have an exact sequence

$$1 \to Aut_v E \to AutE \to AutE/Aut_v E \to 1$$

where the last sheaf is isomorphic to the sheaf of germs of biholomorphic maps $\mu : C \to C$. The infinitesimal version of the sequence above reads

$$0 \to EndE \to \Sigma_E \to K^{-1} \to 0$$

where Σ_E is the sheaf of germs of differential operators of order less or equal to one on E of the form $a(z)\frac{\partial}{\partial z} + \tau(z)$, with $a(z)\frac{\partial}{\partial z}$ is a local holomorphic vector field and $\tau(z)$ is a local endomorphism of E. Being an extension of the tangent sheaf K^{-1} by $EndE$, the sheaf Σ_E corresponds to a class in $H^1(C, K \otimes EndE) \simeq H^0(C, (EndE)^*)$. Now stability implies that $H^0(C, (EndE)^*) \simeq H^0(C, E^* \otimes E) \simeq \mathbb{C}$, i. e. global homomorphisms are of the form $\zeta\ id_E$, with $\zeta \in \mathbb{C}$. We shall be also interested in the case of traceless endomorphisms $End_0 E$ fitting in the sequence

$$0 \to End_0 E \to EndE \xrightarrow{tr} \mathcal{O} \to 0$$

for which the extension

$$0 \to End_0 E \to \Sigma_E^0 \to K^{-1} \to 0$$

is trivial for E stable because $H^0(C, End_0 E) = 0$. This corresponds to deforming infinitesimally the projective bundle $P(E)$ associated to E.

A simple coordinate computation sheds light on the discussion above. Let E be a rank r vector bundle over C. It will be qualified by the assignment of a local trivializing system $\{U_\alpha, g_{\alpha\beta}\}$ for E and patching functions $\{U_\alpha, f_{\alpha\beta}\}$ for the curve C. With no loss of generality we have assumed that the open sets U_α are also coordinate patches. A deformation of $E \to C$ will be given by introducing extra parameters $t_1, \ldots, t_n$ and considering $GL(r, \mathbb{C})$-valued functions $\tilde{g}_{\alpha\beta}(z_\beta; t)$ which satisfy

a) $\tilde{g}_{\alpha\beta}(f_{\beta\gamma}(z_\gamma);t)\tilde{g}_{\beta\gamma}(z_\gamma;t) = \tilde{g}_{\alpha\gamma}(z_\gamma;t)$ for every fixed $t \in T$, $f_{\alpha\beta}$ being the clutching functions on C;

b)$\tilde{g}_{\alpha\beta}(z_\beta;t_0) = g_{\alpha\beta}(z_\beta)$,

together with a variation of the clutching functions;

c) $\tilde{f}_{\alpha\beta} = \tilde{f}_{\alpha\beta}(z_\beta, t)$

Infinitesimally one can

$$\tilde{g}_{\alpha\beta}(z_\beta;t) = g_{\alpha\beta}(z_\beta) + tb_{\alpha\beta}(z_\beta)$$
$$\tilde{f}_{\alpha\beta} = f_{\alpha\beta}(z_\beta) + tc_{\alpha\beta}(z_\beta)$$

Imposing the cocycle condition shows that $c_{\alpha\beta}\cdot\frac{\partial}{\partial z_\beta}$ defines a class $[c_{\alpha\beta}] \in H^1(C, K^{-1})$ while $g_{\alpha\beta}^{-1}\cdot b_{\alpha\beta}$ give rise to a one–cochain in $C^1(C, EndE)$ whose coboundary is (minus) the cup product of $g_{\alpha\beta}^{-1}dg_{\alpha\beta}$ times $[c_{\alpha\beta}]$.

Quite obviously, the failure of the cochain to be closed is remnant of the fact that a deformation of the underlying curve is reflected in a deformation of the vector bundle. Whenever the deformation of the curve is trivial, we recover the results of §1.5. Also, in such a case, one can also give concrete meaning to deformations of vector bundles with a fixed determinant. As long as infinitesimal deformations are concerned, one can mimick the construction above, with the difference that $g_{\alpha\beta}^{-1} \cdot b_{\alpha\beta}$ are now 1–cochains with values in the sheaf $End_0(E)$ of traceless automorphisms of E. At the finite level, the best trick off the hook to the fact that the notion of fixed determinant is meaningless when deforming the underlying curve is to consider deformations of the projective bundle $P(E)$ associated to E. It is clear that the projectivization $P(E)$ of a vector bundle determines E up to its determinant and an r–th root of the structure sheaf $\mathcal{O}_C$. We are not going to dwell on these subtleties any longer, because in the setting we will work in, we will also have at our disposal a different way to "fix the determinant", as it will be apparent from the sequel.

Definition. A family of pointed curves

$$C \underset{\sigma}{\overset{\pi}{\rightleftarrows}} S$$

parameterized by S is a family of curves together with a section σ of π.

It is clear that the moduli space of pointed curves is represented (in the weak sense as discussed in §1.2) by the universal curve $M_g^{(1)}$ over the moduli space of genus g curves. A way to get rid of automorphisms and to obtain a fine moduli space is considering the moduli space M_g'' of pointed curves and non–zero tangent vector at that point. This is a fine moduli space since there is no non–trivial automorphisms of an algebraic curve fixing a point and a non–zero tangent vector at that point [ADKP]. We obviously have projections

$$M_g'' \to M_g^{(1)} \to M_g$$

We will denote respectively with $Pic_d'' \to Pic_d^{(1)} \to Pic_d$ the relative Picard varieties of degree d line bundles, and with $\mathcal{U}(r,d)'' \to \mathcal{U}(r,d)^{(1)} \to \mathcal{U}(r,d)$ the relative varieties of moduli of vector bundles of rank n and degree d. Let us concentrate for one moment on the relative Picard variety. It is known [ACGH] that a Poincaré line bundle $\mathcal{L}_d$ exists on $Pic_d(C) \times C$ and such a line bundle can be normalized by the request that, given a point $q \in C$,

$$\mathcal{L}_d \restriction \{q\} \times Pic_d = \mathcal{O}_{Pic_d}$$

Let us consider a family of pointed curves; it comes equipped with the following diagram

$$\begin{array}{ccccc} \mathcal{L}_d & & & & \\ & \overset{q}{\searrow} & & & \\ & & \tilde{C} & \underset{\tilde{\sigma}}{\overset{\tilde{\pi}}{\rightleftarrows}} & Pic_d(S) \\ & & \downarrow & & \downarrow p \\ & & \tilde{C} & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & S \end{array}$$

Then, the section $\tilde{\sigma}$ will be used to normalize $\mathcal{L}_d$ by means of the request

$$\tilde{\sigma}^*(\tilde{\mathcal{L}}_d) = \mathcal{O}_{Pic_d(S)}$$

Furthermore, in the diagram above also the projection $Pic_d(S) \xrightarrow{p} S$ admits a canonical section ξ, namely $\xi(s) = (s, \mathcal{O}_{C_s}(d \cdot \sigma(s)))$.

Dealing with vector bundles, recall that in §1.5 we described how (on a fixed curve), a distinguished Poincaré vector bundle was defined by the request that it was independent of the operation of tensoring it with line bundles coming from the base of the deformation. To give meaning to such construction, however, we needed to fix a point on the curve. In the family version, let us consider a Poincaré vector bundle

$$\begin{array}{ccc} \mathcal{E} & & \\ & \searrow & \\ & & \tilde{C} \underset{\tilde{\sigma}}{\overset{\tilde{\pi}}{\rightleftarrows}} \mathcal{U}(r,d)(S) \end{array}$$

Then the vector bundle

$$\tilde{\mathcal{E}} = \mathcal{E} \otimes \tilde{\pi}^*((\det\, \tilde{\pi}_! \mathcal{E})^r \cdot (\det\, \tilde{\sigma}^* \mathcal{E})^q) \to \tilde{C}$$

is independent of tensoring with bundles coming from the base, and hence well defined on overlapping of modular families when p and q satisfy $1 + (d + r(1-g))p + rq = 0$ and q is the unique integer $-r < q < 0$ satisfying $qd \equiv -1 \mod r$,

Also, we have a canonical commutative diagram of the form

$$\begin{array}{ccc} \mathcal{U}(r,d)(S) & \xrightarrow{det} & Pic_d(S) \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & S & \end{array}$$

and, as discussed above, $Pic_d(S) \xrightarrow{p_1} S$ admits the canonical section ξ. Hence, we will define the moduli space of rank n with fixed determinant $U_L(r,d)$ to be the fiber of det over the image of ξ. The rationale for this is obviously the fact that this space maps to some moduli space of "dressed" curves, and the fiber of this map over a point $(C, \cdots)$ is clearly the moduli space of vector bundles with fixed determinant over C.

4.3 Atiyah algebras and their cohomology.

This section is devoted to the study of some aspects of the infinite dimensional Lie

algebra which naturally appears when dealing with infinitesimal deformations of vector bundles over variable curves. In a general setting, given a vector bundle W over a manifold M, one defines the Atiyah algebra $\mathcal{A}_W$of W to be the algebra of infinitesimal automorphisms of W. When M is a smooth algebraic curve, this algebra can be quite easily handled. In fact, an affine curve Σ is a Stein variety so that every vector bundle $E \to \Sigma$ is analytically trivial. Given $p \in C$ and the preimage U_0 of a small coordinate disk D the covering $\{C \setminus p, U_0\}$ is a local trivializing system for E Then the algebra of its vertical automorphisms can be identified with the algebra of 1–cochains relative to the covering $\{D, C \setminus p\}$ with values in the Lie algebra $\mathfrak{end}E$.
The infinitesimal sequence of automorphisms of a vector bundle E induces the sequence

$$0 \to C^1(C, \mathrm{End}_V E) \to C^1(C, \mathcal{A}_E) \to C^1(C, T_C) \to 0$$

Hence, given a local trivialization of E and a local parameter z on D, the fact that polynomials are a dense set in all the algebras listed below yields identifications* between $diff^{\mathbb{C}}\, S^1$ and $C^1(C, T_C)$ and between $L\mathfrak{u}(n)^{\mathbb{C}}$ and $C^1(C, \mathrm{End}_V E)$. and hence between $\mathcal{A}_E$ and the semidirect product $diff^{\mathbb{C}}\, S^1 \ltimes L\mathfrak{u}(n)^{\mathbb{C}}$.
Even if we are essentially interested in the loop algebras of $\mathfrak{su}(n)$ or $\mathfrak{u}(n)$ it takes no extra effort to describe the low cohomology of slightly more general Lie algebras, i.e. the semidirect product $\mathcal{D} = diff^{\mathbb{C}}\, S^1 \ltimes L\mathfrak{g}^{\mathbb{C}}$ where $\mathfrak{g}$ is a Lie algebra with at most a one–dimensional centre and $L\, \mathfrak{g}^{\mathbb{C}}$ denotes the loop algebra of its complexification.
We have a "natural" basis in $\mathcal{D}$, namely the one given by the three sets of elements $\{l_n\}, \{d_n\}, \{T^a_n\}$ where
$l_n = z^{n+1}\frac{\partial}{\partial z}$ is a basis for $diff^{\mathbb{C}}\, S^1$
$T^a_n = T^a \otimes z^n$ is a basis for $L\, \tilde{\mathfrak{g}}^{\mathbb{C}}$ (here T^a is a basis for $\tilde{\mathfrak{g}} \equiv \mathfrak{g}/Z_{\mathfrak{g}}$)

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* We are obviously disregarding all subtleties of functional analytic nature involved in the definition of loop spaces as manifolds

$d_n = H \otimes z^n$ is a basis for $L\ Z_{\mathfrak{g}}$. The following commutation relations hold;

$$[l_n, l_m] = (m-n)l_{n+m}$$

$$[l_n, d_m] = m d_{m+n}$$

$$[l_n, T^a_m] = m T^a_{m+n}$$

$$[d_n, d_m] = [d_n, T^a_m] = 0$$

$$[T^a_n, T^b_m] = f^{ab}_c T^c_{m+n}$$

where f^{ab}_c are the structure constants of the simple lie algebra $\mathfrak{g}/Z_{\mathfrak{g}}$. Obviously, when , f.i. $\mathfrak{g} = su(n)$, all the d_n's vanish.

Proposition. $H^1(\mathcal{D}, \mathbb{C}) = \{0\}; \qquad H^2(\mathcal{D}, \mathbb{C}) \equiv \mathbb{C}^4$

Proof. The first part is a consequence of the fact that finite combinations of the basis elements (i.e. polynomials "loops") are dense in $\mathcal{D}$ and hence any continuous n–cocycle is determined by its values on the (tensor product of) the basis vectors, and of the fact that, as it is apparent from the structure of the commutation relations, $[\mathcal{D}, \mathcal{D}] \supset \mathcal{D}$.

As for the computation of the second cohomology one can argue as follows. Consider the basis set $\{l_n, d_n, T^a_n\}$. Any 2–cocycle ω on $\mathcal{D}$ with values in $\mathbb{C}$ will be completely specified by the datum of

$$\omega(l_n, l_m) := A_{n,m}, \quad \omega(l_n, d_m) := B_{n,m}, \quad \omega(d_n, d_m) := C_{n,m},$$

$$\omega(l_n, T^a_m) := a^a_{n,m}, \quad \omega(T^a_n, T^b_m) := b^{ab}_{n,m}, \quad \omega(d_n, T^a_m) := c^a_{n,m}$$

The Jacobi identity, applied to the specified sets of basis vector gives

$$(l,l,l) \qquad (n-h)A_{h+n,m} + (m-n)A_{n+m,h} + (h-m)A_{h+m,n} = 0$$

$$(l,l,d) \qquad (n-h)B_{h+n,m} + m(B_{h,n+m} - B_{n,h+m}) = 0$$

$$(l,d,d) \qquad n \cdot C_{h+n,m} - m \cdot C_{h+m,n} = 0$$

$$(l,l,T) \qquad m \cdot a^a_{h,n+m} - h \cdot a^a_{m,n+h} - (m-n) \cdot a^a_{m+h,n} = 0$$

$$(l,T,T) \qquad f_{abc} \cdot a^c_{h,m+n} - h \cdot b^{ab}_{m,n+h} + h \cdot b^{ba}_{n,m+h} = 0$$

$$(T,T,T) \qquad f_{bcd} b^{ad}_{h,n+m} + f_{cad} b^{bd}_{m,n+h} + f_{abd} b^{cd}_{n,m+h} = 0$$

$$(l,d,T) \qquad m \cdot c^a_{n,m+h} + n \cdot c^a_{m,n+h} = 0$$

$$(d,T,T) \qquad f_{abc} c^c_{m,n+h} = 0$$

Being the f_{abc} the structure constants of a simple Lie algebra, the last equation gives $c^c_{m,n+h} = 0$. Furthermore equation (T,T,T) is the classical cocycle condition for the loop algebra of a simple Lie algebra, which has, as a unique solution, the Kac–Moody cocycle

$$b^{ab}_{m,n} = \lambda\delta^{ab}_{m,-n}$$

so that, substituting this into equation (l,T,T) yields $a^a_{m,n} = 0$, a solution which is consistent with equation (l,l,T). One ends up, then, with the following set of equations

$$\begin{array}{ll}(l,l,l) & (n-h)A_{h+n,m} + (m-n)A_{n+m,h} + (h-m)A_{h+m,n} = 0 \\ (l,l,d) & (n-h)B_{h+n,m} + m(B_{h,n+m} - B_{n,h+m}) = 0 \\ (l,d,d) & n \cdot C_{h+n,m} - m \cdot C_{h+m,n} = 0 \\ (T,T,T) & f_{bcd}b^{ad}_{h,n+m} + f_{cad}b^{bd}_{m,n+h} + f_{abd}b^{cd}_{n,m+h} = 0\end{array}$$

which have a 4–dimensional space of solutions, as the first three equations describe the central extensions of the semidirect product of $diff^{\mathbb{C}}\, S^1$ times an Heisenberg algebra, whose second cohomology is $\mathbb{C}^3$ [ADKP] and the last one contributes with one more independent element. ∎

Remark. It is apparent from the computation above that, if $\mathfrak{g}$ is a simple Lie algebra, $H^2(\mathcal{D},\mathbb{C}) \equiv \mathbb{C}^2$.

For reasons that will become apparent later on, we want now collect some results about the algebra of infinite size matrices and its cohomology. Let a_∞ denote the algebra of infinite size matrices $(a_{ij})\ i,j \in \mathbb{Z}$ such that $a_{ij} = 0\ |i-j| \gg 0$ i.e. the algebra of matrices that are "concentrated" along a strip of arbitrary width along the diagonal.

Its second cohomology is generated by the 2- -cocycle ψ [KP] given by

$$\begin{cases} \psi(E_{ij},E_{ji}) = -\psi(E_{ji},E_{ij}) = 1 & \text{if } i \le 0, j > 0 \\ \psi(E_{ij},E_{kl}) = 0 & \text{otherwise.} \end{cases}$$

the E_{ij} being the standard generators for a_∞.

More generally, if we represent an element $A \in a_\infty$ as

$$\begin{pmatrix} a_{++} & a_{+-} \\ a_{-+} & a_{--} \end{pmatrix}$$

the general form for the cocycle is

$$\psi(A,B) = Tr(a_{+-}b_{-+} - a_{-+}b_{+-})$$

Let us digress for a moment on the algebra $\mathcal{F}$ of regular differential operators of order less or equal than one on $\mathbb{C}^* \equiv \mathbb{C} \setminus \{0\}$. A standard set of generators is then given by $l_n = z^{n+1}\frac{\partial}{\partial z}$, $d_n = z^n \cdot$. It is represented on the space T_j of regular j–differentials on $\mathbb{C}^*$ as

$$\begin{aligned} l_n f(dz)^j &= (z^{n+1}\frac{\partial}{\partial z} + j(n+1)z^n)f)(dz)^j \\ d_n f(dz)^j &= (z^n f)(dz)^j \end{aligned}$$

Let us call $\rho_j^{(1)}$ such a representation. A basis for $H^2(\mathcal{F},\mathbb{C})$ is given by [ADKP]

$$\begin{cases} \alpha_1(g_1\frac{\partial}{\partial z} + h_1, g_2\frac{\partial}{\partial z} + h_2) = \mathrm{Res}_{z=0}\; g_1 dg'' \\ \alpha_2(g_1\frac{\partial}{\partial z} + h_1, g_2\frac{\partial}{\partial z} + h_2) = \mathrm{Res}_{z=0}\; g_1 dh_2' - g_2 dh_1' \\ \alpha_3(g_1\frac{\partial}{\partial z} + h_1, g_2\frac{\partial}{\partial z} + h_2) = \mathrm{Res}_{z=0}\; h_1 dh_2 \end{cases}$$

Considering the basis $\varphi_k^j = z^k(dz)^j$ of T_j gives rise to a matrix representation of $\mathcal{F}$under which one can pull back the standard generator of a_∞. In particular, for $j = 0$ it holds

$$(\rho_0^{(1)})^*(-\psi) = -\alpha_1/6 + \alpha_2/2 + \alpha_1$$

A different representation $\rho_j^{(r)}$ can be obtained considering $\mathcal{F}$ as acting on the space $T_j^{(r)}$ of $\mathbb{C}^r$–valued j–differentials. The relations between the two representations is easily gotten as

$$\rho_j^{(r)}(g\frac{\partial}{\partial z_n} + h) = (\rho_j^{(1)}(g\frac{\partial}{\partial z_n} + h)) \otimes \mathbf{1}_r$$

so that, as long as central extensions are concerned,

$$(\rho_j^{(r)})^*(-\psi) = r \cdot (\rho_0^{(1)})^*(-\psi)$$

The relevance of such representations is apparent in our context; in fact, the space $T_j^{(r)}$ is acted upon by the Atiyah algebra $\mathcal{D}$, and when viewing $\mathcal{F} \hookrightarrow \mathcal{D}$ the appropriate representation is exactly $\rho_j^{(r)}$. We have proven that $H^2(\mathcal{D},\mathbb{C}) = \mathbb{C}^4$. A set of generators can be obtained by adjoining to any set of generators for $H^2(\mathcal{F}, \mathbb{C})$ the generator α_4 which is defined as follows. Any element of $\mathcal{D}$ can be represented uniquely as $X = g\frac{\partial}{\partial z} \otimes \mathbf{1}_r + h \otimes \mathbf{1}_r + A$ with A an element of the loop group of the simple algebra $\mathfrak{g}/Z_{\mathfrak{g}}$ Then

$$\alpha_4(X, Y) = \mathrm{Res}_{z=0} < A_X, dA_Y >$$

where $< \cdot, \cdot >$ is the Killing form on $\mathfrak{g}/Z_{\mathfrak{g}}$. Then it holds that, if $\tilde{\rho}_0^{(r)}$ is the natural representation of $\mathcal{D}$ in $T_0^(r)$ that

$$(\tilde{\rho}_0^{(r)})^*(-\psi) = r \cdot (-\alpha_1/6 + \alpha_2/2 + \alpha_1) + \alpha_4$$

This means, in particular, that given a set $\{\omega_i\}_{i=1,\dots,3}$ of generators of the second cohomology of $\mathcal{F}$, the sets $\{\{\omega_i\}_{i=1,\dots,3}, \alpha_4\}$ and $\left\{\{\omega_i\}_{i=1,\dots,3}, \tilde{\rho}_0^{(1)*}(-\psi)\right\}$ are a set of generators for $H^2(\mathcal{D}, \mathbb{C})$

As a final remark, we notice that the relations between the representations $\rho_j^{(r)}$ and $\rho_j^{(1)}$ is easily visualized in the grassmannian set–up as follows. There is a natural isomorphism between the spaces H and H^r as introduced in §4.1 given by the so–called lexicographic transcription [PS]. In terms of standard bases $e_i z^k$ of H^r where e_i is the standard basis of $\mathbb{C}^n$ and ζ^j of H, this correspondence sends $e_i z^k$ to ζ^{nk+i-1}. Under the inverse map, a simple computation shows that the standard generator of the Virasoro algebra l_k can be represented as $l_k \otimes \mathbf{1}_r$, while d_n becomes d_{nr} whose matrix representation is the same as that of $d_n \otimes \mathbf{1}_r$.

4.4 The geometry of the Krichever map

We will now describe the construction of some infinite dimensional complex varieties which will be embedded by the so–called generalized Krichever map into a suitable infinite Grassmannian. Consider a pointed curve

$$C \underset{\sigma}{\overset{\pi}{\rightleftarrows}} S$$

parameterized by S; and choose a small neighbourhood V of $s \in S$ over which the family is C^∞ trivial. A tubular neighbourhood $\mathcal{V}$ of $\sigma(V) \subset \mathcal{C}\restriction V$ if a family of disks parameterized by V and so it is holomorphically trivial. This gives the pointed curve a local coordinate centered at $\sigma(v)$,$v \in V$. Given another open set V' in S, the new local parameter will be related to the old one by a holomorphic map with non–vanishing derivative at $z = 0$. In this way one can construct a natural infinite dimensional variety $\hat{S}$, whose points can be considered as pairs (s, z_s), where $s \in S$ and z_s is a local coordinate on C_s near $\sigma(s)$. This comes equipped with a universal curve $\hat{\mathcal{C}}$ obtained by pulling back the curve

$$\mathcal{C} \quad \underset{\sigma}{\overset{\pi}{\rightleftarrows}} \quad S$$

under the natural projection. Obviously, $\hat{S}$ can be considered as a deformations of the data (C, p, z).

Pasting together Kuranishi families of such objects one can construct the smooth infinite dimensional manifold $\hat{M}_g$ modelled on $H_+ \times \mathbb{C}^{3g-3}$ whose points parameterize triples made of (curve C, points $p \in C$, local parameters z near p). Observe that, if M''_g parameterizes pointed curves and a non–zero tangent vector at that point, we have a natural projection

$$\hat{M}_g \xrightarrow{p''} M''_g$$

sending (C, p, z) into $(C, p, \frac{\partial}{\partial z})$ with fibres isomorphic to the vector space H_+ so that p'' induces an isomorphism in cohomology

$$H^*(\hat{M}_g, \mathbb{Z}) \equiv H^*(M''_g \mathbb{Z})$$

Let us now plug vector bundles into the game. Let (C, p) be a pointed curve, let $\mathcal{E}$ be a rank r stable vector bundle over C. A local trivialization ϕ of $\mathcal{E}$ at p is an isomorphism of $\mathcal{E}_p$ with $\oplus_{i=1,\dots r}\mathcal{O}_p$. Two locally trivialized vector bundles $(\mathcal{E}_1, \phi_1)$ and $(\mathcal{E}_2, \phi_2)$ will be called equivalent if there is an isomorphism $\mathcal{E}_1 \xrightarrow{T} \mathcal{E}_2$ such that

$$\oplus_{i=1,\dots r}\mathcal{O}_p \overset{\phi_2 T \phi_1^{-1}}{\longrightarrow} \oplus_{i=1,\dots r}\,\mathcal{O}_p.$$

is a the identity. It is clear that the set of equivalence classes of such pairs maps onto the moduli space of stable rank r vector bundles on C. The fiber of this map over $(C,\mathcal{E})$ is the group $\mathcal{GL}(\mathcal{E})_p$ (i.e. the stalk at p of the sheaf of automorphisms of $\mathcal{E}$) modulo the group of global invertible elements in $H^0(C, End(E))$. Hence, after choosing a local parameter z at p, this in turn can be identified with the space of elements of the form $g_p\cdot(\mathbf{1}_r+z\cdot A(z))$, with $A(z)$ a matrix valued holomorphic function of z and g_p is in $Gl(r,\mathbb{C})/H^0(C,End(E))^*$. Thanks to stability $H^0(C,End(E))^*$ is reduced to constant multiples of the identity, and so the fiber is $PGL(r,\mathbb{C}) \times End(E)_+$, this latter being the space of local holomorphic matrix valued functions. Again, by pasting together Kuranishi deformations of quintuples $(C,p,z,[\mathcal{E},\phi])$ one defines the infinite dimensional moduli space of such objects, $\hat{\mathcal{U}}(r,d)$.

For later use we observe that $\hat{\mathcal{U}}(r,d)$ maps onto $\hat{\mathcal{U}}(r,d)''$ which is the pull–back of $\mathcal{U}(r,d) \to \mathcal{M}_g$ to $\mathcal{M}_g''$ and this map induces an isomorphism up to second rational cohomology as the fibers have the same homotopy type as $SU(r)/\mathbb{Z}_r$.

Let us consider $x = (C,p,z,[\mathcal{E},\phi]) \in \hat{\mathcal{U}}(n,d)$ and the space of sections of $\mathcal{E}$ over $C \setminus p$.

Proposition. There exists a natural map of $\hat{\mathcal{U}}(r,d)$ into the infinite Grassmannian $Gr(H^r)$.

Proof. Consider the open covering of C made of $C \setminus p$ and the preimage D of a small coordinate disk of radius ϵ. Since every affine curve is a Stein manifold, this covering computes the cohomology of every coherent analytic sheaf on C. Namely, the Mayer–Vietoris sequence associated to such a covering reads:

$$0 \to H^0(C,\mathcal{E}) \to \Gamma(D,\mathcal{E}) \oplus \Gamma(C \setminus p,\mathcal{E}) \to \Gamma(D',\mathcal{E}) \to H^1(C,\mathcal{E}) \to 0$$

Factoring out $\Gamma(D,\mathcal{E})$ one gets the exact sequence

$$0 \to H^0(C,\mathcal{E}) \to \Gamma(C \setminus p,\mathcal{E}) \to \Gamma(D',\mathcal{E})/\Gamma(D,\mathcal{E}) \to H^1(C,\mathcal{E}) \to 0$$

Now, the local trivialization ϕ and the local parameter z enables one to identify $\Gamma(D',\mathcal{E}) \equiv H^r$ and $\Gamma(D,\mathcal{E}) \equiv H^r_+$ so that the sequence can be rewritten as

$$0 \to H^0(C,\mathcal{E}) \to \Gamma(C \setminus p,\mathcal{E}) \to H^r/H^r_+ \equiv H^r_- \to H^1(C,\mathcal{E}) \to 0$$

which, by the completeness of the curve C exhibits $\Gamma(C\setminus p,\mathcal{E})$ as a Fredholm subspace of H^r. As a side remark, we notice that the image of $\hat{\mathcal{U}}(r,d)$ is contained in the component of index $\chi(\mathcal{E})$ of $Gr(H^r)$, $\chi(\mathcal{E})$ being the Euler characteristics of $\mathcal{E}$. ■

The map

$$\begin{array}{ccc} \hat{\mathcal{U}}(r,d) & \xrightarrow{K} & Gr(H^r) \\ x & \rightarrow & \Gamma(C\setminus p,\mathcal{E}) \end{array}$$

is called "generalized" or "non–abelian" Krichever map [Mul][PW].

Remark. Notice that the moduli space $\hat{\mathcal{M}}_g$ of dressed curves and the moduli space of dressed degree d line bundles $\hat{\mathcal{M}}_d$ are naturally embedded into $Gr(H)$ [ADKP]. However we can get a commutative diagrams of embedding into $Gr(H^r)$, where r = rank $\mathcal{E}$ sending $Gr(H)\rightarrow Gr(H^r)$ by means of the lexicographic transcription described in §4.3.

We are now going to study the Krichever map $\hat{\mathcal{U}}(r,d)\rightarrow Gr(H^r)$ and the diagram

$$\begin{array}{ccc} \hat{\mathcal{U}}(r,d) & \longrightarrow & Gr(H^r) \\ & \searrow \quad \swarrow & \\ & \hat{M}_g & \end{array}$$

Consider a local universal family of curves $\mathcal{C}\rightarrow\mathcal{M}$ and the relative variety of moduli of vector bundles $\mathcal{U}(r,d)\xrightarrow{\pi}\mathcal{M}$. As we already discussed, the tangent space exact sequence reads

$$0\rightarrow H^1(C,End(E))\rightarrow H^1(C,\Sigma_E)\rightarrow H^1(C,K^{-1})\rightarrow 0$$

where Σ_E is the sheaf of differential operators of order less or equal to 1 with scalar symbol acting on sections of E. This exact sequence describes, as pointed out in §4.2, the infinitesimal deformation of a vector bundle over a curve. Turning to the infinite dimensional moduli space, the sequence above will be "enlarged", since some coboundaries will act in a non– trivial way on the data we want to parameterize.

Given points

$$x=(C,p,z,[E,\phi])\in\hat{\mathcal{U}}(r,d)$$

$$y=\hat{\pi}(x)=(C,p,z)\in\hat{\mathcal{M}}_g$$

we have natural identifications of

$$\Sigma_E\restriction p = H_-^{r^2} \oplus H_- \cdot \partial \subset \mathcal{D}$$

$$K^{-1}\restriction p = H_- \cdot \partial \subset diff^{\mathbb{C}}\, S^1$$

and Lie algebras inclusions

$$\mathcal{D}_x \equiv \Gamma(C \setminus p, \Sigma_E) \hookrightarrow \mathcal{D}, \quad diff^{\mathbb{C}}\, S^1{}_y \equiv \Gamma(C \setminus p, K^{-1}) \hookrightarrow diff^{\mathbb{C}}\, S^1$$

In analogy with the abelian case one has the following

Proposition. For every $x \in \hat{\mathcal{U}}(r,d)$ one has the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{D}_x & \to & \mathcal{D} & \to & T_x(\hat{\mathcal{U}}(r,d)) & \to & 0 \\ & & \sigma \downarrow\uparrow i & & \sigma \downarrow\uparrow i & & \downarrow dp & & 0 \\ 0 & \to & diff^{\mathbb{C}}\, S^1{}_x & \to & diff^{\mathbb{C}}\, S^1 & \to & T_y(\hat{\mathcal{M}}_g) & \to & 0 \end{array}$$

where σ is the symbol map and the horizontal sequences are exact.

Proof . The tangent space $T_x(\hat{\mathcal{U}}(r,d))$ is the space of isomorphism classes of deformations of the quintuple $(C, p, z, E, [\phi])$ parameterized by the ring of dual numbers $S \equiv \mathbb{C}[\epsilon]/\epsilon^2$. Take $A(z) + b(z)\frac{\partial}{\partial z} \in \mathcal{D}$ and consider, once more, a Stein covering of C made of a small disk D around p and $C \setminus p$. By the very definition of the Lie algebra $\mathcal{D}$, one can shrink D so that, for $z \in D$, $A(z)$ is a holomorphic matrix–valued function, and $b(z)$ is a holomorphic function. Then one can define a family of pointed curves with local parameter and a vector bundle $\mathcal{E} \to \mathcal{C} \underset{\sigma}{\overset{\pi}{\rightleftarrows}} S$ considering

$$\mathcal{C} = (D \times S)\coprod(C \setminus p \times S)/\sim$$

$$(z, \epsilon) \sim (z + \epsilon b(z), \epsilon)$$

and

$$\mathcal{E} = (E\restriction_D \times S)\coprod(E\restriction_{C\setminus p} \times S)/\sim$$

$$(\phi, \epsilon) \sim (\phi \cdot (\mathbf{1} + \epsilon A(z)), \epsilon)$$

The map which associates to $A(z)+b(z)\frac{\partial}{\partial z}$ the pair $(\mathcal{C},\mathcal{E})$ defines a surjective homomorphism $\mathcal{D}\to T_x(\hat{\mathcal{U}}(r,d))$. What is left to prove is that the kernel of this homomorphism is $\mathcal{D}_x$. If $A(z)+b(z)\frac{\partial}{\partial z}$ is in the kernel, then $(\mathcal{C},\mathcal{E})$ is a trivial deformation of (C,E) and so, by Kodaira-Spencer theory, must lie in the image of the difference map

$$\mathcal{D}_x\oplus\mathcal{O}_p(\Sigma_E)\xrightarrow{\delta}\mathcal{D}$$

where $\mathcal{O}_p(\Sigma_E)$ is the stalk at p of the sheaf Σ_E. Looking at the local parameter and trivialization, it is clear that the isomorphism between $(\mathcal{C},\mathcal{E})$ and the trivial deformation $E\times S$ induces the identity on $D\times E$ and hence is in $\mathcal{D}_x$. ∎

Let us now consider the Krichever map

$$K:\hat{\mathcal{U}}(r,d)\to Gr(H^r)$$

and let $x=(C,p,z,E,[\phi])\in\hat{\mathcal{U}}(r,d)$. The tangent map

$$T_x(\hat{\mathcal{U}}(r,d))\equiv\mathcal{D}/\mathcal{D}_x\xrightarrow{dK}T_{K(x)}(Gr(H^r))\equiv End(K(x),H^r/K(x))$$

can be represented, if $O(z)=A(z)+b(z)\frac{\partial}{\partial z}$ is a matrix valued differential operator which maps to $v\in T_x(\hat{\mathcal{U}}(r,d))$ and $f(z)\in K(x)$ as

$$dK(v)(f)=O\cdot f\ \mathrm{mod}K(x)$$

Let us consider the algebra of infinite size matrices a_∞ and the natural representation $\rho_0^{(r)}$ of $\mathcal{D}$ in a_∞. The analysis above can be summarized in the following commutative diagram;

$$\begin{array}{ccc}\mathcal{D}\times\hat{\mathcal{U}}(r,d) & \xrightarrow{\rho_0^{(r)}\times K} & a_\infty\times Gr(H^r)\\ \downarrow & & \downarrow\\ T(\hat{\mathcal{U}}(r,d)) & \xrightarrow{dK} & T(Gr(H^r))\end{array}$$

Now we can show how central extensions of the Lie algebra $\mathcal{D}$ are related to the Picard group of $\hat{\mathcal{U}}(r,d)$.

Lemma. Let X be a complex manifold, and let $\mathfrak{g}$ be an algebra acting on X such that

i) $\forall x \in X$ the evaluation map $\mathfrak{g} \xrightarrow{\phi_x} T_x(X)$ is surjective;

ii) $\mathfrak{g}_x \equiv \ker \phi_x$ is such that $[\mathfrak{g}_x, \mathfrak{g}_x] = \mathfrak{g}_x$

Then for any Lie algebra continuous extension

$$0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

which is trivial on $\mathfrak{g}_x$ there is associated a continuous extension

$$0 \to X \times \mathbb{C} \to F \to T_x(X) \to 0$$

This defines a homomorphism

$$\cap_{x \in X} \mathrm{ker} r_x \to Ext^1(\mathcal{T}_X, \mathcal{O}_X)$$

where $r_x : H^2(\mathfrak{g}, \mathbb{C}) \to H^2(\mathfrak{g}_x, \mathbb{C})$ is defined by restriction.

Proof. See [ADKP].

We can apply this lemma to our situation thanks to the following two propositions.

Proposition. Let C be an affine curve, and let $\mathcal{E}$ be a holomorphic vector bundle on C. If $\mathcal{D}_x$ is the algebra of global differential operators of order ≤ 1 with scalar symbol acting on sections of $\mathcal{E}$, then $[\mathcal{D}_x, \mathcal{D}_x] = \mathcal{D}_x$.

Proof. Every vector bundle over an affine curve is analytically trivial, so that we we can suppose that $\mathcal{E} \simeq C \times \mathbb{C}^r$, and $\mathcal{D}_x$ is the algebra of global differential operators of order ≤ 1 and scalar symbol acting on vector–valued functions. Then, if ∂ is a nowhere vanishing vector field on C, and T^a is a basis for the algebra of $n \times n$ matrices $Mat_n(\mathbb{C})$ any element of $\mathcal{D}_x$ can be uniquely written as

$$f\partial + \sum_a g^a T^a$$

with f and $g^a \in \mathcal{O}_C$. One notices that $\mathcal{D}_x$ is an $\mathcal{O}_C$–module as

$$s[a\partial + A, b\partial + B] = [sa\partial, \frac{1}{2}b\partial + B] + [\frac{1}{2}a\partial + A, sb\partial] + [A, B]$$

Hence one is left to prove that the elements ∂, and T^a are in the commutator subalgebra. If C is a Zariski open set in $\mathbb{C}$ this is immediate since

$$\partial = [\partial, z\partial] \quad T^a = [\partial, zT^a]$$

In the other cases, one can consider projections of C on the affine line $\mathbb{C}$ with disjoint ramification divisors and pull-back the standard generators. ■

To prove the second proposition we need to recall the following facts. The algebra $\mathcal{D}$ is embedded via $\rho_0^{(r)}$ into the algebra a_∞ of infinite size matrices. The second cohomology of the latter is generated by the 2-cocycle ψ given by

$$\begin{cases} \psi(E_{ij}, E_{ji}) = -\psi(E_{ji}, E_{ij}) = 1 & \text{if } i \leq 0, j > 0 \\ \psi(E_{ij}, E_{kl}) = 0 & \text{otherwise.} \end{cases}$$

the E_{ij} being the standard generators for a_∞. A basis for the second cohomology of $\mathcal{D}$ is given by the four generators listed in section 4.3, and all these are pull-backs under suitable representations of the cocycle $\psi \in H^2(a_\infty, \mathbb{C})$.

Proposition. Let us consider the tangent to the Krichever map

$$0 \to \mathcal{D}_x \to \mathcal{D} \to T_x(\hat{\mathcal{U}}(r,d)) \to 0$$

Then every Lie algebra extension of $\mathcal{D}$ is trivial on $\mathcal{D}_x$.

Proof. In view of what we have discussed above, we can work on the image of $\hat{\mathcal{U}}(r,d)$ in the Grassmannian $Gr(H^r)$. The image of $\mathcal{D}_x$ is contained in the Lie algebra $a_{K(x)}$ of the stabilizer of the image of $x \in \hat{\mathcal{U}}(r,d)$ in $Gr(H^r)$. But, thanks to the fact that the group A_∞ of invertible infinite size matrices acts transitively on $Gr(H^r)$ and $Adg(a_{H^r_+}) = a_{gH^r_+}$ it suffices to show that any extension of a_∞ is trivial when restricted to $a_{H^r_+}$. But this is true in view of the facts that in the canonical decomposition of operators induced by the polarization of H^r, the stabilizer of H^r_+ appears as the group of "diagonal matrices" i.e.

$$A \in a_{H^r_+} \Rightarrow A_{+-} = A_{-+} = 0$$

and the form of the standard cocycle is given as

$$\psi(A,B) = tr(A_{+-}B_{-+} - B_{+-}A_{-+})$$

∎

Summing up, by using the lemma above, we have proven the

Theorem. There exists a homomorphism

$$H^2(\mathcal{D},\mathbb{C}) \xrightarrow{\rho} Ext^1(\mathcal{T}_{\hat{\mathcal{U}}(r,d)}, \mathcal{O}_{\hat{\mathcal{U}}(r,d)}) \equiv H^1(\Omega^1_{\hat{\mathcal{U}}(r,d)}).$$

The same argument yields a homomorphism

$$H^2(a_\infty) \to Ext^1(\mathcal{T}_{Gr(H^r)}, \mathcal{O}_{Gr(H^r)})$$

This latter is spelled out by the following

Proposition. [ADKP] The extension Σ induced by the standard extension of a_∞ is the sheaf Σ_{det} of differential operators of order ≤ 1 acting on sections of the determinant bundle.

We next want to show that the extensions induced by the action of $\mathcal{D}$ on $T_{\hat{\mathcal{U}}(r,d)}$ actually come from line bundles. There is a canonical homomorphism

$$c : H^1(\mathcal{O}^*_{\hat{\mathcal{U}}(r,d)}) \to H^1(\Omega^1_{\hat{\mathcal{U}}(r,d)})$$

which associates to a line bundle L the isomorphism class of extensions represented by the sheaf of differential operators of order ≤ 1 acting on sections of L.

Proposition. The image of the homomorphism

$$H^2(\mathcal{D}) \xrightarrow{\rho} Ext^1(\mathcal{T}_{\hat{\mathcal{U}}(r,d)}, \mathcal{O}_{\hat{\mathcal{U}}(r,d)})$$

is contained in the image of the homomorphism

$$H^1(\mathcal{O}^*_{\hat{\mathcal{U}}(r,d)}) \xrightarrow{c} H^1(\Omega^1_{\hat{\mathcal{U}}(r,d)})$$

Proof. We noticed in §4.3 that $H^2(\mathcal{D},\mathbb{C})$ is generated by a basis of the second cohomology of $\mathcal{F} \equiv diff^{\mathbb{C}} S^1 \ltimes Lu(1)$ plus the pull–back $\alpha_0 = (\tilde{\rho}_0^{(r)})^*(-\psi)$ of the generator of $H^2(a_\infty, \mathbb{C})$under the standard representation of $\mathcal{D}$ in a_∞. We know, still from [ADKP], that the first three cocycles actually define line bundles over the moduli space $\hat{P}ic_d$ parameterizing curves, points, local parameters, and equivalence classes of line bundles and trivializations. Since we have an obvious projection

$$\hat{\mathcal{U}}(r,d) \xrightarrow{det} \hat{P}ic_d$$

under which we can pull- back such line bundles, what we have to prove is that there is a further line–bundle associated to α_0. As we have discussed in §4.2, we can define a normalized universal vector bundle $\tilde{\mathcal{E}}$ over the universal curve $\tilde{\mathcal{C}} \xrightarrow{\tilde{\pi}} \mathcal{U}(r,d)$ which we can pull back to the universal curve over $\hat{\mathcal{U}}(r,d)$ For the sake of simplicity, let us call this bundle still $\tilde{\mathcal{E}}$, and Ω the line bundle $det\ \hat{\tilde{\pi}}_! \tilde{\mathcal{E}} \to \hat{\mathcal{U}}(r,d)$.
Let us consider the diagram

$$\begin{array}{ccc} \mathcal{D} \times \hat{\mathcal{U}}(r,d) & \longrightarrow & a_\infty \times Gr(H^r) \\ \downarrow & & \downarrow \\ T(\hat{\mathcal{U}}(r,d)) & \longrightarrow & T(Gr(H^r)) \end{array}$$

By construction, if K is the non–abelian Krichever map, $K^*(det^{-1}) = \Omega$ so that, by the proposition above and the functoriality of the Krichever construction, the pull–back of $\Sigma_{det^{-1}}$ is exactly Σ_Ω ∎

The paper of Arbarello, De Concini, Kac and Procesi ended with the remarkable proof of the existence of an isomorphism between the group of line–bundles on the moduli space of pointed curves with a non zero cotangent vector and a degree $g-1$ line bundle and the second cohomology of the semidirect product of the Virasoro and Heisenberg algebra. What we can prove is a somewhat weaker result, and namely that there is an isomorphism between $H^2(\mathcal{D}, \mathbb{C})$ and the second rational cohomology of $\mathcal{U}(r,d)''$ for the case of rank 2 and 3 bundles. Actually, the considerations based on the non–abelian Krichever construction show that there is a homomorphism between central extensions of the Lie algebra $\mathcal{D}$ and line bundles over $\hat{\mathcal{U}}(r,d)$. To prove the isomorphism (although in the weaker form concerning rational cohomology) in low rank, we can proceed as follows.

Recall that $\hat{\mathcal{U}}(r,d)$ maps onto $\mathcal{U}(r,d)'$ and the map induces an isomorphism in second rational cohomology. Let

$$\mathcal{C} \quad \overset{\pi}{\underset{\sigma}{\rightleftarrows}} \quad S$$

be a pointed curve parameterized by S and consider the diagram :

$$\begin{array}{ccc} \mathcal{U}(2,d)(S) & \stackrel{\det}{\longrightarrow} & Pic_d(S) \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & S & \end{array}$$

Proposition. The bundles $\mathcal{U}(2,d)(\pi)\xrightarrow{p_2}S$ and $Pic^d(\pi)\xrightarrow{p_1}S$ come equipped with canonical sections ξ_2 and ξ_1 which make the associated diagram commutative.

Proof. The case of the relative Picard variety can be dealt with as in [ADKP] by

$$\begin{array}{rccc} \xi_1 : & S & \rightarrow & Pic^d(\pi) \\ & s & \rightsquigarrow & (s, \mathcal{O}_{C_s}(d\sigma)) \end{array}$$

For the case of vector bundles, the construction is a bit more complicated, but can be achieved by means of the following considerations. As discussed in §1.5, we can restrict the discussion to the cases $d=0$ and $d=1$. Let us consider first $d=0$. Given $p \in C$, isomorphism classes of non–trivial extensions $0 \to \mathcal{O}(-p) \to \mathcal{E} \to \mathcal{O}(p) \to 0$ are parameterized by the projective space $\mathbb{P}H^1(C, \mathcal{O}(-2p))$. Let us choose the distinguished class $[\eta_0]$ which is dual to the class ω_0 in $H^0(C, K_C(2p))$ represented by the unique abelian differential which has a true singularity in p. We want to prove then that bundle $\mathcal{E}_{[\eta_0]}$ is stable, i.e. it does not admit any subline bundle of non–negative degree.

Suppose we have $0 \to L \xrightarrow{\varphi} \mathcal{E}_{[\eta_0]}$; Then, composing with the projection π one would get a non–zero homomorphism $s : L \to \mathcal{O}(p)$. It is clearly impossible if $\deg L > 1$. Moreover, if $\deg L = 1$, and $s \neq 0$ then $L = \mathcal{O}(p)$ and s is nowhere zero, which is impossible since $\mathcal{E}_{[\eta_0]}$ is a non–trivial extension. This shows that $\mathcal{E}_{[\eta_0]}$ is semi–stable.

As for stability, let us consider a degree 0 line bundle L and a map $0 \to L \xrightarrow{\varphi} \mathcal{E}_{[\eta_0]}$; One then would have an element in $H^0(C, L^{-1}(p))$, which implies that $L = \mathcal{O}_C$. In such a case, then, one ends up with the following diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}(-p) & \to & \mathcal{E}_{[\eta_0]} & \to & \mathcal{O}(p) & \to & 0 \\ & & & & & & \uparrow \rho & & \\ & & & & & & \mathcal{O} & & \end{array}$$

and the existence of the homomorphism $\mathcal{O} \xrightarrow{\varphi} \mathcal{E}_{[\eta_0]}$ is equivalent to the possibility of lifting the map ρ in the diagram above. This, in turn, happens if and only if the characteristic class of the extension, $[\eta_0]$ lies in the kernel of the natural map $\delta : H^1(C, \mathcal{O}(-2p)) \to H^1(C, \mathcal{O}(p))$ induced by ρ [NS]. The dual statement is that ω_0 is in the image of the map of C into the projective space induced by the linear system $|K_C(2p)|$. But it cannot be so, because the canonical system is base–point free.

Then, given any family of pointed curves,

$$\mathcal{C} \underset{\sigma}{\overset{\pi}{\rightleftarrows}} S$$

one can define the section

$$\begin{array}{rcl} \xi_2 \ : S & \longrightarrow & \mathcal{U}(2, 2k)(\pi) \\ s & \rightsquigarrow & (s, \mathcal{E}_{[\eta_0(\sigma(s))]}) \otimes \mathcal{O}(k \cdot (\xi_1(s))) \end{array}$$

Notice that $\det \mathcal{E}_{[\eta_0(\sigma(s))]} = \mathcal{O}(2k\sigma(s))$.

For the coprime case, let us consider the case $d = 1$ We borrow from Ramanan [R] the following argument.

Given two coprime numbers r and d, let l be the unique number $0 < l < r$ such that $ld \equiv 1 \mod r$. Then there is a unique number $0 \leq k < r$ such that $ld - kr = 1$. If V and W are stable vector bundles respectively of rank l and degree k and of rank $r - l$ and degree $d - k$, any non–trivial extension

$$0 \to V \to E \to W \to 0$$

is stable.

For $r = 2$ and $d = 1$, we have $l = 1$ and $k = 0$ so that we will consider an extension

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(p) \to 0$$

The problem is to choose canonically a non–trivial extension. Such extensions are qualified as usual by $H^1(C, \mathcal{O}(-p))$ which is isomorphic to $H^1(C, \mathcal{O})$. This latter has a very nice description, which is also natural in our setting. In fact, given the Stein covering $\{D, C \setminus p\}$ elements in $H^1(C, \mathcal{O})$ are identified by the Weierstrass gap sequence of the curve C in the following sense. If z is a local coordinate at p, a one–cochain relative to the covering $\{D, C \setminus p\}$ with values in $\mathcal{O}$ is a holomorphic function defined in $D \setminus p$. Since there are no triple intersections, every one–cochain is a cocycle, and this will be non–trivial as long as it cannot be written as the difference of a holomorphic function on D and a holomorphic function in $C \setminus p$. If $f \in \mathcal{O}(C, D \setminus p)$, and $\tilde{f}$ denotes its Laurent tail, then $[\tilde{f}]$ is a non–trivial element in $H^1(\mathcal{O})$ iff there is no global meromorphic function on C having $\tilde{f}$ as its only polar part. If we consider the functions $f_k = 1/z^k$ $k = 1, \ldots, 2g - 2$ we see that $[f_k] \in H^1(C, \mathcal{O})$ is non–trivial precisely whenever k is not a Weierstrass gap. Since $k = 1$ is never a Weierstrass gap (as long as C is not $\mathbb{P}^1$) the extension

$$0 \to \mathcal{O} \to \mathcal{E}_{[1/z]} \to \mathcal{O}(p) \to 0$$

is stable. We remark that the class $[1/z]$ is only optically dependent on the local parameter z at p. The definition of the section $\xi_2 : S \to \mathcal{U}(2, 2k + 1)$ follows in complete analogy with the coprime case ∎

Remark. The difference between the choices in the coprime and in the non–coprime case are due to the fact that by the residue theorem, there is no generator in $H^0(C, K(p))/H^0(C, K)$. Actually, the Serre dual picture of the choice we do in the coprime case is to choose the dual to the generator of $H^0(C, K)/H^0(C, K(-p))$.

Summing up we have shown that, once the datum of a smooth algebraic curve C and a point $p \in C$, is given we can choose in a canonical way a stable vector bundle of rank 2 over C and arbitrary degree d by tensoring $\mathcal{E}_{[\eta_0]}$ or, according to the parity of d, $\mathcal{E}_{[1/z]}$ by $\mathcal{O}(j \cdot p)$ getting $\mathcal{E}^j_{[\eta_0]}$ and $\mathcal{E}^j_{[1/z]}$ which are of degree respectively $2j$ and $2j+1$. Notice that $(\mathcal{E}^j_{[\eta_0]})^* = \mathcal{E}^{(-1-j)}_{[\eta_0]}$ and the same relation holds for $(\mathcal{E}^j_{[1/z]})^*$ and $\mathcal{E}^j_{[1/z]}$.

We are now in the position to prove the isomorphism between the second cohomology of $\mathcal{U}(2,d)$ and the second cohomology of $\mathcal{D}$. For the sake of simplicity, let us deal with the coprime case. Consider, for $d = 2k+1$ the universal vector bundle $\tilde{\mathcal{E}}$ over the universal curve $\tilde{C}'' \xrightarrow{\pi''} \mathcal{U}(r,d)''$. In the notations of §4.2, $\sigma^*(det\tilde{\mathcal{E}})$ and $det\pi''_! \tilde{\mathcal{E}}$ generate (with a relation) the second cohomology of the fibre to the fibration

$$\mathcal{U}(2,d)'' \xrightarrow{det} Pic''_d$$

This means that $\dim H^2(\mathcal{U}(2,d)'', \mathbb{Q})/det^*(H^2(Pic''_d, \mathbb{Q})) \geq 1$. But the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{M}(2,d) & \xrightarrow{\det} & Pic_d \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & M_g & \end{array}$$

and the existence of the section ξ_2 give a section of

$$\mathcal{U}(2,d)'' \xrightarrow{det} Pic''_d$$

so that

$$H^2(Pic''_d, \mathbb{Q}) \xrightarrow{det^*} H^2(\mathcal{U}(2,d)'', \mathbb{Q})$$

is an injection. Since [ADKP] $H^2(Pic''_d, \mathbb{Q})$ has dimension 3, we have then that the dimension of $H^2(\mathcal{U}(2,d)'', \mathbb{Q})$ is at least four. But by the Kunneth formula $\dim H^2(Pic''_d, \mathbb{Q}) \leq 4$ so that actually we have proven

Proposition. $\dim H^2(Pic''_d, \mathbb{C}) = \dim H^2(\mathcal{D}, \mathbb{C}) = 4$ ∎

The same constructions can be done with the Atiyah algebra $\mathcal{D}_0 = diff^{\mathbb{C}}\, S^1 \ltimes L\mathfrak{sl}(2,\mathbb{C})$ which is associated to the algebra of 1–cocycles with values in $End_0 E$. It obviously is associated to the moduli space of vector bundles with fixed determinant $\mathcal{U}(2,d)''_L$ in the sense as explained in §4.2. This space fibers directly over the "dressed" moduli space M_g''. This fibration admits a section too, since we have chosen sections of

$$\hat{\mathcal{U}}(r,d) \rightarrow \hat{Pic}_d$$

whose determinants is precisely $\mathcal{O}(d \cdot p)$, and since [AC] $\dim H^2(M_g'', \mathbb{Z}) = 1$, one has the analogous

Corollary. $\dim\, H^2(\mathcal{U}(2,d)_L'', \mathbb{C}) = \dim\, H^2(\mathcal{D}_0, \mathbb{C}) = 2$ ■

For the case of rank $r = 3$ one can argue as follows. A sufficient condition for the existence of the isomorphism between the second cohomology of the Lie algebra $\mathcal{D}$ and second cohomology of $\hat{\mathcal{U}}(r,d)$, is the existence of a section ξ to the fibration $\mathcal{U}(r,d)'' \rightarrow M_g''$. We can still use Ramanan's argument, using as building blocks the rank 2 bundles $\mathcal{E}^j_{[1/z]}$ and $\mathcal{E}^j_{[\eta_0]}$. Sticking to the coprime case, we have to discuss the cases $d \equiv 1 \bmod r$ and $d \equiv 2 \bmod r$; actually, solving the problem for one of the two cases is enough since if $E$is such that $\deg E \equiv 1 \bmod r$ then $\deg E^* \equiv 2 \bmod r$. Some arithmetics shows that we have to seek for a non–trivial extension

$$0 \rightarrow \mathcal{O} \rightarrow F_3 \rightarrow \mathcal{E}^j_{[1/z]} \rightarrow 0$$

Such extensions are classified by $\mathbb{P}H^1(C, (\mathcal{E}^j_{[1/z]})^*)$. The cohomology sequence for $(\mathcal{E}^j_{[1/z]})^*$ reads

$$0 \rightarrow H^0(C,\mathcal{O}) \xrightarrow{\delta} H^1(C, \mathcal{O}(-p)) \rightarrow H^1(C, (\mathcal{E}^j_{[1/z]})^*) \rightarrow H^1(C,\mathcal{O}) \rightarrow 0$$

Then, to keep things as simple as possible, we can try to assign an element in $H^1(C, (\mathcal{E}^j_{[1/z]})^*)$ giving an element in $H^1(C, \mathcal{O}(-p)$ as long as it is not in the image of $H^0(C,\mathcal{O})$ under the connecting homomorphism δ. Here our favourite cocycle $[1/z]$ unfortunately does not make the job, since a simple computation based on the fact that an explicit representative for the transition function of $\mathcal{E}^j_{[1/z]}$ is (still using the covering $\{D, C \setminus p\}$)

$$G = \begin{bmatrix} 1 & 1/z^2 \\ 0 & 1/z \end{bmatrix}$$

shows that $\delta(1) = [1/z]$. The minimal choice we can then do is to consider the class $[1/z^2]$. This means that actually we are defining a section of $\mathcal{U}(3,1)^{0\prime\prime} \to M_g^0$ where M_g^0 is the moduli space of automorphism–free curves (actually we need only to delete the hyperelliptic locus in M_g). Since for $g \geq 5$ the set of curves with automorphisms is of sufficiently high codimension, we can safely apply the argument above to prove the isomorphism

$$H^2(\mathcal{D}, \mathbb{C}) \simeq H^2(\mathcal{U}(3,d)'', \mathbb{C}) \quad d = 1, 2$$

Appendix 1.

The Grothendieck-Riemann-Roch Theorem

The Grothendieck-Riemann-Roch theorem computes the Chern character of formal sums of direct image of sheaves under analytic maps between complex varieties. Before stating it, let us collect some terminology (see, e.g. [Hart]). Let X be a complex manifold of dimension n with structure sheaf $\mathcal{O}$.

Definition. A sheaf of abelian groups on X is called analytic if

i) the stalks $\mathcal{S}_x$ are $\mathcal{O}_x$-modules

ii) the map $\bigcup_{x\in X} \mathcal{S}_x \times \mathcal{O}_x \longrightarrow \mathcal{S}_x$ defined by the module operation is continuous.

Definition. An analytic sheaf S is called coherent if $\forall x \in X$ there is a neighbourhood U_x of x and a short right exact sheaf sequence

$$\mathcal{O}^p \restriction_{U_x} \to \mathcal{O}^q \restriction_{U_x} \to \mathcal{S} \restriction_{U_x} \to 0$$

Here $\mathcal{O}^p = \mathcal{O} \oplus \cdots \oplus \mathcal{O}$ p-times

What matters for us is the following

Fact. if E is a complex analytic vector bundle on X, then the sheaf $\mathcal{E}$ of local holomorphic sections of E is coherent analytic [Hirz].

Direct image sheaves are defined as follows.

Let $X \xrightarrow{f} Y$ be a holomorphic map and $\mathcal{S}$ an analytic sheaf over X. One defines the q^{th} – direct image sheaf $R^q f_*(\mathcal{S})$ by means of a suitable presheaf in the following way. Let V be open in Y. Then the cohomology space $H^q(f^{-1}(V), \mathcal{S})$ is an $\mathcal{O}_X \restriction_{f^{-1}(V)}$-module. By composition of maps it is also an $\mathcal{O}_Y \restriction_V$-module. The map $V \rightsquigarrow H^q(f^{-1}(V), \mathcal{S})$ defines a presheaf on Y whose associated sheaf is, by definition, $R^q f_*(\mathcal{S})$. Naively, the stalk at $y \in Y$ of $R^q f_*(\mathcal{S})$ can be identified with $H^q(f^{-1}(y), \mathcal{S})$.

Coherent analytic sheaves have 'simple' cohomological properties. Namely

Proposition 1. if $\mathcal{S}$ is a coherent analytic sheaf over an n-dimensional manifold X, then

$$H^q(X, \mathcal{S}) = 0 \quad \text{for } q > n$$

Proposition 2. If $X \xrightarrow{f} Y$ is a proper holomorphic map,

$$R^q f_*(\mathcal{S}) = 0 \quad \text{for } q > \dim X$$

and $R^q f_*(\mathcal{S})$ is coherent for $q \geq 0$.

Let $Coh(X)$ denote the set of isomorphism classes of coherent analytic sheaves over a complex manifold X, and let $F(X)$ denote the free abelian group generated by $Coh(X)$. If $R(X)$ is the subgroup generated by all elements of the form $\mathcal{S} - \mathcal{S}' - \mathcal{S}"$ where

$$0 \to \mathcal{S}' \to \mathcal{S} \to \mathcal{S}" \to 0$$

is short exact, one defines the Grothendieck group of coherent analytic sheaves over X as

$$K_\omega(X) := F(X)/R(X)$$

Given a proper holomorphic map $X \xrightarrow{f} Y$ one gets an homomorphism

$$\begin{array}{ccccc} F(X) & \xrightarrow{\tilde{f}_!} & F(Y) & & \\ \mathcal{S} & \rightsquigarrow & \tilde{f}_!(\mathcal{S}) & := & \sum_{q=0}^{n} (-)^q R^q f_*(\mathcal{S}) \end{array}$$

As $\tilde{f}_!$ maps $R(X)$to $R(Y)$ it induces an homomorphism

$$f_! : K_\omega(X) \longrightarrow K_\omega(Y)$$

The Grothendieck-Riemann-Roch theorem for analytic sheaves is an equality between Chern characters of coherent analytic sheaves.

Lemma. Let $\mathcal{S}$ be a coherent analytic sheaf aver an n-dimensional algebraic manifold X. Then there are complex vector bundles $W_0 \cdots W_n$ over X and an exact sequence (called resolution by vector bundles)

$$0 \to \mathcal{W}_0 \to \cdots \to \mathcal{W}_n \to \mathcal{S} \to 0$$

of analytic sheaves over X.(where $\mathcal{W}_i$ denotes the sheaf of local holomorphic sections of W_i).

Then the Chern character $Ch(\mathcal{S})$ is defined as

$$Ch(\mathcal{S}) = \sum_{i=0}^{n} (-)^i \; Ch(W_i)$$

The Chern character is a homomorphism

$$Ch : K_\omega(X) \longrightarrow H^*(X, \mathbb{Q})$$

Theorem (Grothendieck-Riemann-Roch). Let $b \in K_\omega(X)$ and $f\colon X \to Y$ a proper holomorphic map between algebraic varieties. Then

$$Ch(f_!(b)) = f_* \left(Ch(b) \cdot Td(X) \cdot (f^*(Td(Y)))^{-1} \right)$$

Here $Td(\cdot)$ is the total Todd class of the tangent sheaf to $\cdot$ and f_* is the so-called Gysin homomorphism (represented, in the smooth case, by integration along the fibers in De Rham cohomology).

The formula above is clarified to a great extent when working with smooth objects by means of the Chern-Weil construction, which gives explicit expressions for characteristic classes in terms of polynomial invariants built out of the curvature of the relevant bundles. Recall that [K] given a holomorphic vector bundle $E \xrightarrow{\pi} X$ with a hermitean structure $< \cdot, \cdot >_E$, there is a unique unitary connection ∇_E which is compatible with the holomorphic structure, in the sense that the (0,1)-component of ∇_E coincides with $\bar{\partial}_E$.

The Chern-Weil construction associates to $(E \quad, < \cdot, \cdot >_E, \nabla_E)$ a set of distinguished differential forms which represent in De Rham cohomology the Chern classes of E (and hence any characteristic class) and are built out of the curvature R_E of ∇_E. In particular, the relevant polynomials entering the $\bar{\partial}_E$-complex are the Todd genus of the tangent bundle TX and the Chern character of E, given by

$$Ch(E) = \mathrm{tr}\; e^{\frac{i}{2\pi} R_E}$$

$$Td(X) = \sqrt{\det \frac{R_{TX}/4\pi}{\sinh R_{TX}/4\pi}} \quad \mathrm{tr}\ e^{\frac{i}{2\pi} R_{TX}}$$

By the splitting principle one can deduce the following formulas:

$$Ch(E) = \sum_{j=1}^{rk\ E} e_j^x = \mathrm{rk}\ E + c_1(E) + \frac{1}{2}\left(c_1(E)^2 - 2c_2(E)\right) + \cdots$$

$$Td(X) = \prod_{j=1}^{dimX} \frac{y_j}{1 - e^{-y_j}} = 1 + \frac{1}{2} c_1(TX) + \frac{1}{12}\left(c_1(TX)^2 + c_2(TX)\right) + \cdots$$

Let us specialize this construction to the case in which X is a holomorphic family of and $\mathcal{E}$ is the sheaf of sections of a holomorphic vector bundle on X. In this case $f_!(\mathcal{E})$ is the formal difference $H^0\left(f^{-1}(s), \mathcal{E}\right) \ominus H^1\left(f^{-1}(s), \mathcal{E}\right)$ so that

$$c_1\left(f_!(\mathcal{E})\right) = c_1\left(\det f_!(\mathcal{E})\right) = \int_{fibers} \left(Ch(E) \cdot Td(X) \cdot \left(f^*(Td(Y))\right)^{-1}\right)_{(4)}$$

Appendix 2.

Moduli stacks.

If the moduli spaces M_g of genus g curves existed as manifolds, one would have been able to use the bases B of universal deformations $C \overset{i}{\hookrightarrow} X \to B$ to give local coordinates on M_g by setting [isomorphism class of $\pi^{-1}(b)$] $\rightsquigarrow b \in B$. Unfortunately this is not the case because of the presence of automorphism. Indeed, for any automorphisms $\alpha \in Aut(C)$, we get another deformation $C \overset{i\circ\alpha}{\hookrightarrow} X \to B$ of C and by the universal property there is a base change $\phi(\alpha) : B \to B$ making the two deformations isomorphic. So, $Aut(C)$ acts on B; in other words, B overparameterizes the curves "near" C, while the correct local model turns out to be $B/Aut(C)$. This way of thinking leads to the construction of the "coarse moduli spaces". Incidentally, these turn out to be complex spaces but generically not smooth manifolds.

Another possibility of dealing with the moduli problem is to enlarge the very concept of "manifolds" by first enlarging that of the underlying topological space. This generalization is actually a *stack* an we want to describe here its basic features, referring to the literature for the complete set up [Mu5][DM][Po].

Let us first work at the topological level. The basic idea of Grothendieck is to forget about points (i.e. isomorphism classes of curves in our case) and to construct a generalized topology by allowing more open sets and 'inclusions' than usual. Recall that a topology on a space can be considered as a category, whose objects are open sets and morphisms are inclusions. Intersections and unions correspond to products and sums in the category and of course one finite products and any sum exist in the category itself. The basic property one is going to generalize is that, in ordinary topologies, the morphisms between two objects U, V are either empty or consist of a single morphism; namely the inclusion of U in V. One gets in this way a category $\mathcal{M}_g$, which in our case (for $g \geq 2$) can be described as follows;

1) objects ('open sets') are versal families of smooth curves of genus g $\pi_j : X_j \to B_j$ over smooth bases B_j, with final object X,

2) morphisms ('inclusions') are morphisms of families of curves and projections on the final object X,

3) the category is closed under finite product ('intersections') and generic sums ('unions')

4) a collection of morphisms

$$\begin{array}{ccc} X_\alpha & \stackrel{G_\alpha}{\to} & X \\ \downarrow & & \downarrow \\ S_\alpha & \stackrel{g_\alpha}{\to} & S \end{array}$$

is a covering of S if $S = \cup_\alpha g_\alpha(S_\alpha)$. A collection of projections of families onto the final object X is a covering if every curve occurs in at least one of the families.

Loosely speaking, one forgets about automorphisms by using versal families (instead of their bases up to automorphisms) as open sets. Notice that this is by no means an ordinary topology, that is Σ is not an ordinary topological space, because morphisms of two objects are not required to be unique when they are defined.

This topology for moduli problems comes together other properties, which are the distinctive features of stalks. Among these, we want to recall that;

5) for any morphism $\phi : B' \to B$" and any family $X" \to B"$, there is a unique pull-back family $\phi^* X" \to B'$ over B'.

6) for any covering $\phi_j : B_j \to B'$ of B', denote by

$$B_{ij} = B_i \times_{B'} B_j =: \{(b_i, b_j) \in B_i \times B_j | \phi_i(b_i) = \phi_j(b_j)\}$$

$$B_{ijk} = B_i \times_{B'} B_j \times_{B'} B_k.$$

Then, there exist some family $X' \to B'$ and isomorphisms $\Phi_{ij} : \pi_i^* X' \to \phi_j^* X'$ over B_{ij}, satisfying an obvious cocycle condition over B_{ijk}.

The notion of a sheaf on the moduli stack is given as follows. For every family of algebraic curves $\mathcal{C} \stackrel{\pi}{\longrightarrow} B$ one gives a sheaf $\mathcal{S}_B$ over the base S, plus isomorphisms Φ_{ij} between $\mathcal{S}_{B_i}$ and $f^*(\mathcal{S}_{B_j}$ satisfying the cocycle condition whenever a cartesian diagram

$$\begin{array}{ccc} \mathcal{C}_i & \longrightarrow & \mathcal{C}_j \\ \downarrow & & \downarrow \\ B_i & \xrightarrow{f} & B_j \end{array}$$

is given.

Appendix 3.

Superalgebra.

Definition. Let $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded ring . We will denote by $\tilde{a}$ the degree of any of its homogeneous elements. Given a pair (a, b) of homogeneous elements of A, their *supercommutator* is defined to be

$$\lceil a, b\rfloor = a \cdot b - (-1)^{\tilde{a}\tilde{b}} b \cdot a$$

The definition of supercommutator is then extended to arbitrary elements $x, y \in A$ by linearity. A $\mathbb{Z}_2$-graded ring A is called supercommutative (or graded commutative) iff

$$\forall a, b \in A \quad \lceil a, b\rfloor = 0.$$

In a complete analogy one can introduce the notion of $\mathbb{Z}_2$-graded algebra and of supercommutative $\mathbb{Z}_2$-graded algebra . Notice that in both cases the supercommutator satisfies the following fundamental identities:

$$(1)\lceil a, b\rfloor = -(-1)^{\tilde{a}\tilde{b}}\lceil b, a\rfloor$$

$$\lceil a, \lceil b, c\rfloor\rfloor + (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}\lceil b, \lceil c, a\rfloor\rfloor + (-1)^{\tilde{c}(\tilde{a}+\tilde{b})}\lceil c, \lceil a, b\rfloor\rfloor = 0$$

Given a $\mathbb{Z}_2$-graded ring A, one can define right and left A-modules, which, consistently, will be $\mathbb{Z}_2$-graded abelian groups. What actually happens is that every (say right) A-module is a bimodule, whenever left multiplication by $a \in A$ is defined taking into account a "sign rule", i.e. if M is a right A-module we will define the left action as

$$m \cdot a \overset{\text{def}}{=} (-1)^{\tilde{a}\tilde{m}} a \cdot m$$

Tensor products and the usual operations of linear algebra carry over to the graded case with the only precaution of taking correctly into account the sign rule.

Definition. An additive map $f : S \rightarrow T$ between two A-modules is called a homomorphism whenever it is A-linear and *preserves* grading.

Definition. Let S be an A-module. We define the A-module ΠS by means of the following prescriptions:

i) $\Pi S_0 = S_1 \qquad \Pi S_1 = S_0$

ii) $\Pi S \simeq S$ *qua* abelian groups

iii) right multiplication differs by a sign factor:

$$a \cdot \Pi s = (-1)^{\tilde{a}} \Pi(a \cdot s)$$

Example . The prototypical $\mathbb{Z}_2$-graded rings A we will deal with are the following:

(1) The Grassmann algebra $\wedge^* V$ of a n-dimensional vector space V

(2) The ring of "regular" functions on a domain in $\mathbb{C}^m$ with values in $\wedge^* V$. This second ring can be thought of as generated by considering it as the quotient of the polynomial ring in $m+n$ indeterminates $x_1, \cdots, x_m; \xi_1, \cdots, \xi_n$ by the ideal generated by the following relations:

$$\begin{cases} x_i x_j = x_j x_i \\ x_i \xi_\alpha = \xi_\alpha x_i \\ \xi_\alpha \xi_\beta = -\xi_\beta \xi_\alpha \end{cases}$$

An A-module S is said to be free of rank $p|q$ iff it is isomorphic to the A-module $A^{p|q} := A^p \oplus (\Pi A)^q$. Notice that $A_0^{p|q} = A_0^p \oplus (\Pi A_1)^q$ and conversely. The rank of a free A-module shares (thanks to graded-commutativity) with the dimension of vector spaces the property of being uniquely defined, in the sense that, two free A-modules S and S' will be isomorphic iff they have the same rank. This property enables one to discuss of matrices as representative of (even) homomorphisms between free A-modules. An $(m|n \times p|q)$ matrix with entries in A will be said to be in standard form if it is in block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $a_{ir}, d_{\alpha\beta} \in A_0$ and $c_{i\alpha}, b_{\beta r} \in A_1$. The set of matrices in standard form with entries in A is commonly denoted by

$$M(m|n, p|q; A)$$

It is a $\mathbb{Z}_2$-graded algebra naturally isomorphic to $Hom(A^{p|q}, A^{m|n})$, via the usual isomorphism given by considering the natural bases in $A^{p|q}$ and $A^{m|n}$. Given an element $X \in M(m|n, m|n; A)$ one defines its *Supertrace* to be

$$StrM = TrA_M - TrD_M.$$

Definition. A derivation in A is an additive map $X : A \longrightarrow A$ satisfying the graded Leibnitz rule:

$$X(ab) = (Xa)b + (-1)^{\tilde{a}\tilde{X}} a(Xb)$$

where $\tilde{X}$ is the parity of X *qua* additive map.

If A is an algebra over a field F, we will say that X is a derivation over F if

$$Xf = 0 \quad \forall f \in F.$$

The set of F-derivations in A are made into a Lie $\mathbb{Z}_2$-graded algebra by defining

$$[X, Y] = X \circ Y - (-1)^{\tilde{X}\tilde{Y}} y \circ X.$$

which naturally has the structure of A-module.

The last definition we want to recall here is the one of *Berezinian* or *Superdeterminant.* Let $B \in GL(p|q; A)$ an even automorphisms of $A^{p|q}$. Writing B in standard form

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

one defines

$$BerB = det\ (B_1 - B_2 B_4^{-1} B_3)/det\ B_4$$

The meaning of this definition and the reason why it is the right generalization of the notion of determinant is clarified by the following

Proposition . $Ber : GL(p|q;A) \longrightarrow Gl(1|0;A_0)$ is the unique group homomorphism satisfying

$$Ber(\exp M) = \exp(StrM)$$

Berezin-Kostant-Leites supermanifolds are substantially complex spaces together with a sheaf of $\mathbb{Z}_2$-graded rings , i.e. they are built by pasting together collections of the objects we have described above.

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