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G-CONVERGENCE AND HOMOGENIZATION OF MONOTONE OPERATORS

Thesis submitted for the degree of Doctor Philosophiae
Academic Year 1989/90

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Il presente lavoro costituisce la tesi presentata dalla Dott.ssa Valeria Chiadò Piat, sotto la direzione del Prof. Gianni Dal Maso, al fine di ottenere l'attestato di ricerca postuniversitaria "Doctor Philosohiae" presso la S.I.S.S.A., classe di Matematica, settore di Analisi Funzionale e Applicazioni. Ai sensi del Decreto del Ministro della Pubblica Istruzione 24.4.1987, n. 419, tale diploma è equipollente al titolo di "Dottore di ricerca in Matematica".

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INTRODUCTION

This thesis deals with the asymptotic analysis of sequences of nonlinear boundary value problems of elliptic type.

To fix the ideas, let us consider as a model case a sequence of Dirichlet problems of the form

$$(0.1)\qquad \begin{cases} -\mathrm{div}(a_h(x,Du_h)) = f & \text{on } \Omega\,, \\ u_h \in H^1_0(\Omega)\,, \end{cases}$$

where Ω is a bounded subset of $\mathbf{R}^n$ and $f \in L^2(\Omega)$. The functions $a_h : \Omega\times\mathbf{R}^n \to \mathbf{R}^n$ are measurable and satisfy regularity and monotonicity conditions of the type

$$(0.2)\qquad |a_h(x,\xi_1) - a_h(x,\xi_2)| \le \Lambda\, |\xi_1 - \xi_2|$$

$$(0.3)\qquad (a_h(x,\xi_1) - a_h(x,\xi_2), \xi_1 - \xi_2) \ge \lambda\, |\xi_1 - \xi_2|^2$$

for a.e. $x \in \Omega$, for every $\xi_1, \xi_2 \in \mathbf{R}^n$, and for two given constants $0 < \lambda \le \Lambda < +\infty$, independent of h. In this case problem (0.1) has a unique solution $u_h \in H^1_0(\Omega)$, and the sequence (u_h) is bounded in the H^1-norm. Hence, by Rellich's theorem, it has a subsequence, still denoted by (u_h), that converges to some function $u \in H^1_0(\Omega)$, weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$, as h tends to $+\infty$.

At this point it is natural to ask whether u solves a boundary value problem of the type (0.1), i.e.

$$(0.4)\qquad \begin{cases} -\mathrm{div}(a(x,Du)) = f & \text{on } \Omega\,, \\ u \in H^1_0(\Omega) & . \end{cases}$$

In that case, it is interesting to know which are the properties of the function a defining the limit equation in (0.4), and how it can be computed from the given sequence (a_h).

In this thesis we try to answer to these and other related questions for a more general class of problems including the model case (0.1). The approach we follow uses the theory of G-convergence.

A first notion of G-convergence was introduced by S. Spagnolo in [67] as the convergence, in a suitable topology, of the Green's operators associated to the Dirichlet boundary value problems, in the case where $a_h(x,\xi) = a_h(x)\xi$ is linear in ξ, and $a_h(x)$ is a positive definite symmetric matrix (see also [66], [68]). He proves that the class of such matrices is sequentially compact with respect to G-convergence, i.e., given an arbitrary sequence of matrices (a_h) there exist a positive definite symmetric matrix a (called the G-limit), and an increasing sequence of integers $(\sigma(h))$, such that for every $f \in L^2(\Omega)$ the sequence $(u_{\sigma(h)})$ of the solutions to (0.1) corresponding to $a_{\sigma(h)}$ converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ to the solution u to (0.4), with $a(x,\xi) = a(x)\xi$. A localization property also holds, in the following sense: if two sequences (a_h) and (b_h) coincide a.e. in an open subset of Ω , then the corresponding G-limits do the same.

For what concerns the dependence of a on the sequence (a_h) , E. De Giorgi and S. Spagnolo prove in [36] that it can be expressed through the associated energy functionals, whose convergence follows from the G-convergence. A simpler formula for the G-limit is obtained when (a_h) is a sequence of periodic matrices with decreasing periods, of the type $a_h(x) = \alpha(hx)$, and α is a given matrix of periodic functions.

This case, that was previously studied by E. Sanchez-Palencia ([62], [63], [64]) and others, is an example of a homogenization problem, and has a relevant physical meaning, that justifies the terminology and emphasizes some interesting applicative aspects of this theory. If we suppose that the function u_h represents a quantity of physical interest, like the temperature, or the electric potential, or, in the vector valued case, the elastic displacement, then problem (0.1) provides a good model for the study of the physical behaviour of a heterogeneous body with a fine periodic structure. In this connection, the coefficients $a_h(x)$ describe the properties of the different materials constituting the body, and stand for the thermic or electrostatic conductivity, or the elastic coefficients. When the period of the structure is very small, i.e. for large h, a direct numerical computation of the solution to (0.1) may be very heavy, or even impossible. Then homogenization (or G-convergence) provides an alternative way of approximating such solution by means of the solution u to (0.4). In this case the G-limit, that is usually called the homogenized operator, turns out to be a constant matrix b , that may be interpreted as the physical parameters of a homogeneous body, whose behaviour is equivalent, from a macroscopic point of view, to the behaviour of the material with the given periodic microstructure.

These kinds of problems were subsequently studied in a number of different situations, like more general assumptions on the functions a_h , different boundary conditions, higher order operators, different kind of equations (parabolic, hyperbolic, stochastic...), variational inequalities, optimization problems, and so on, giving rise to a very rich literature. Here we confine ourselves to the study of 2nd order elliptic equations in divergence form.

In this connection the notion of G-convergence was first extended to linear problems of the type (0.1) defined by non symmetric matrices by F. Murat and L. Tartar, under the name of H-convergence (see [70], [71], [53]). For the related problem of homogenization we refer to the books [10], [61], [6], that contain also an extensive bibliography on these topics.

The properties of the G-convergence for quasi-linear elliptic operators were studied by L. Boccardo, Th. Gallouet, and F. Murat in [13], [14], and [11].

The first results in the nonlinear case are due to L. Tartar, who studied (in [73]) the properties of the G-convergence for monotone problems of the type (0.1), assuming that the maps a_h are uniformly Lipschitz-continuous, and uniformly strictly monotone on $\mathbf{R}^n$, i.e. satisfy (0.2), (0.3). The corresponding homogenization results can be found in [4], while the vector valued case is considered in [69]. More general classes of uniformly equicontinuous strictly monotone operators defined on the Sobolev space $H^{1,p}(\Omega)$, for $p \geq 2$, are considered by Raitum in [59], while the corresponding homogenization case is studied by N. Fusco and G. Moscariello in [38], [39] for $p > 1$.

One of the main purposes of our work is to weaken the strict monotonicity and the equicontinuity of the functions a_h, which are constant assumptions in the quoted literature, assuming just the monotonicity condition

$$(0.5) \qquad (a_h(x,\xi_1) - a_h(x,\xi_2),\xi_1 - \xi_2) \geq 0$$

for a.e. $x\in\Omega$, for every $\xi_1, \xi_2\in\mathbf{R}^n$, and the estimates

$$(0.6) \qquad |a_h(x,\xi)| \leq c_1(1 + |\xi|^2)$$

$$(0.7) \qquad |\xi|^2 \leq c_2(1 + (a_h(x,\xi),\xi))$$

for a.e. $x\in\Omega$, for every $\xi\in\mathbf{R}^n$, where $c_1 > 0$, $c_2 > 0$ are fixed. This permits to include in the framework of G-convergence the Euler equations of minimum problems of the form

$$(0.8) \qquad \min_{u\in H_0^1(\Omega)} \left(\int_\Omega \psi_h(x,Du)dx - \int_\Omega fu\, dx \right)$$

where $\psi_h = \psi_h(x,\xi) : \Omega\times\mathbf{R}^n \to \mathbf{R}$ is a non negative function that is measurable in x, convex in ξ, and satisfies inequalities of the type

$$(0.9) \qquad c_3|\xi|^2 \leq \psi_h(x,\xi) \leq c_4(1 + |\xi|^2)$$

for a.e. $x\in\Omega$, for every $\xi\in\mathbf{R}^n$, and for two given constants $0 < c_3 \leq c_4$, independent of h. The limit behaviour of minimum problems like (0.8) has been studied, for instance, in [49], [21], [16], using the techniques of Γ-convergence introduced by E. De Giorgi and T. Franzoni (see [34], [35]).

We remark that the lack of strict monotonicity and equicontinuity of a_h produces a deep change in the limit behaviour of the sequence (0.1): the limit map a may be multivalued, and the limit problem (0.4) becomes a differential inclusion

$$(0.10)\qquad \begin{cases} -\mathrm{div}(a(x,Du)) \ni f & \text{on } \Omega \ , \\ u \in H^1_0(\Omega) & . \end{cases}$$

An example of this situation is shown in Chapter 2 (see Example 4.7). This fact suggests to study G-convergence in a multivalued setting from the beginning, i.e. to make an asymptotic analysis of sequences boundary value problems of the form (0.10). This approach includes also the case of minimum problems (0.8) whose differential counterpart is no longer an Euler equation, but consists in a differential inclusion of the type (0.10), where $a(x,\xi) = \partial_\xi \psi_h(x,\xi)$ is the subdifferential of ψ_h with respect to ξ.

To study these problems, in the first two chapters of this thesis we determine an appropriate class of maximal monotone multivalued functions a_h, including as a model case the subdifferential of convex functions ψ_h satisfying (0.9), and we give a corresponding notion of G-convergence. The main purpose is to prove a compactness result for this general class of multifunctions, and to obtain a localization property for the G-convergence. Moreover, we characterize subclasses which are closed with respect to G-convergence, recovering, in particular, the compactness results in the existing literature, and we determine a precise formula for the G-limit in the case of homogenization.

This multivalued setting we deal with requires appropriate tools. Some measure theory for multifunctions and properties of maximal monotone operators between Banach spaces are needed for the existence of solutions to the differential inclusions we consider. Moreover the notion of Kuratowski set convergence is required to express the G-convergence of the maps a_h in terms of the convergence of their graphs. A deep change occurs in the proof of the main compactness result (see Chapter 1, Theorem 4.1). In fact, while the proofs in the existing literature are based essentially on a density argument, which is made possible by the continuity of the resolvent operators associated to the boundary value problems (0.1), our proof relies on a theorem by F. Hiai and H. Umegaki concerning the representation of all measurable selections of a suitable multivalued map (see [40]).

The subsequent part of the thesis concerns single valued operators. Here we use more standard techniques of G-convergence theory to study problems that present some difficulties due to their particular structure. Chapter 3 contains a compactness result for the G-convergence of a sequence of boundary value problems with mixed boundary conditions on perforated domains, while Chapter 4 is devoted to the homogenization of elliptic differential equations containing nonlinear lower order terms.

The perforated domains Ω_h we consider in Chapter 3 are obtained by removing a sequence (B_h) of compact sets from a given bounded open set $\Omega \subseteq \mathbf{R}^n$. Given $f \in L^2(\Omega)$, and a sequence of strictly monotone, continuous functions $a_h : \Omega_h \times R^n \to R^n$, we study the limit behaviour of the solutions $u_h \in H^1(\Omega_h)$ to

$$(0.11) \qquad \begin{cases} -\mathrm{div}(a_h(x,Du_h)) = f & \text{on } \Omega_h , \\ u_h = 0 & \text{on } \partial\Omega , \\ (a_h(x,Du_h),\nu_h) = 0 & \text{on } \partial B_h , \end{cases}$$

where ν_h denotes the outer unit normal to the boundary of the holes B_h. The assumptions we make on the sets Ω_h guarantee that the holes do not concentrate in any part of Ω and that the solutions $u_h \in H^1(\Omega_h)$ can be extended to the whole of Ω, with suitable uniform estimates of their H^1-norms. We prove that the extensions of u_h converge to the solution u of a Dirichlet problem of the form

$$(0.12) \qquad \begin{cases} -\mathrm{div}(a(x,Du)) = bf & \text{on } \Omega , \\ u \in H^1_0(\Omega) , \end{cases}$$

where the function $b \in L^\infty(\Omega)$ is the weak limit of the characteristic functions of the sets Ω_h, and the map a has the same qualitative properties of a_h. The interesting feature is that the limit problem is independent of the particular way one extends the u_h's. A similar problem for the case of variational integrals is studied in [23], [24].

The results presented in Chapter 4 concern the homogenization of a quasi-linear equation containing a nonlinear lower order term with natural growth. Let us consider, as a model case, the problem

$$(0.13) \qquad \begin{cases} -\mathrm{div}(a_h(x,Du_h)) + \gamma u_h = |Du_h|^2 + f(x) & \text{on } \Omega , \\ u_h \in H^1_0(\Omega) \cap L^\infty(\Omega) , \end{cases}$$

where Ω is a bounded open subset of R^n, γ is a positive constant, $f \in L^\infty(\Omega)$, $a_h(x,\xi) = a(hx,\xi)$, and the map $a : R^n \times R^n \to R^n$ is periodic in x, strictly monotone and continuous in ξ. While the limit behaviour of the left hand side of (0.13) follows from the known homogenization theory, the asymptotic analysis of the quadratic term presents some difficulties. We prove that the limit problem has the form

$$(0.14) \qquad \begin{cases} -\mathrm{div}(b(Du)) + \gamma u = H(Du) + f(x) & \text{on } \Omega , \\ u \in H^1_0(\Omega) \cap L^\infty(\Omega) , \end{cases}$$

where the monotone map $b : \mathbf{R}^n \to \mathbf{R}^n$ is the usual homogenized operator, while $H : \mathbf{R}^n \to \mathbf{R}^n$ is a function with quadratic growth, depending on a. The proof of this result follows the method introduced by A. Bensoussan, L. Boccardo, and F. Murat in [9] for the case where $a(x,\xi)$ is linear in ξ, and is based on a comparison argument. Moreover it requires the knowledge of a corrector result for the homogenization of quasi-linear equations defined by monotone maps, for which we refer to [33].

The content of this thesis, which is published in the papers [25], [26], [27], [28], is the result of a research activity carried on by the Author during her graduate studies at the International School for Advanced Studies in Triest, under the guide of Prof. Gianni Dal Maso, and in collaboration with Dr. Anneliese Defranceschi.

CHAPTER 1

G-CONVERGENCE OF MONOTONE OPERATORS

In this chapter we introduce a general notion of G-convergence for sequences of maximal monotone operators of the form $\mathcal{A}_h u = -\mathrm{div}(a_h(x,Du))$, in terms of the asymptotic behaviour, as $h \to +\infty$, of the solutions u_h to the equations $\mathcal{A}_h u_h = f_h$ and of their momenta $a_h(x,Du_h)$. The main results we prove are the local character of the G-convergence and the G-compactness of some classes of nonlinear monotone operators. The content of this chapter is published in [25].

INTRODUCTION

The aim of this chapter is to study a general notion of G-convergence for nonlinear monotone operators $\mathcal{A} : H_0^{1,p}(\Omega) \to H^{-1,q}(\Omega)$ of the form

$$\mathcal{A}u = -\mathrm{div}(a(x,Du)), \tag{0.1}$$

where Ω is a bounded open subset of $\mathbf{R}^n$, $1 < p < +\infty$, and $1/p + 1/q = 1$. We assume that the (possibly multivalued) map $a : \Omega\times\mathbf{R}^n \to \mathbf{R}^n$ which occurs in (0.1) is measurable on $\Omega\times\mathbf{R}^n$, is maximal monotone on $\mathbf{R}^n$ for almost every $x\in\Omega$, and satisfies suitable coerciveness and boundedness conditions (see Section 2). The class of all these maps will be denoted by $M_\Omega(\mathbf{R}^n)$.

The main examples of maps of the class $M_\Omega(\mathbf{R}^n)$ have the form

$$a(x,\xi) = \partial_\xi \psi(x,\xi), \tag{0.2}$$

where ∂_ξ denotes the subdifferential with respect to ξ and $\psi : \Omega\times\mathbf{R}^n \to [0,+\infty[$ is measurable in (x,ξ), convex in ξ, and satisfies the inequalities

$$c_1|\xi|^p \le \psi(x,\xi) \le c_2(1 + |\xi|^p)$$

for suitable constants $0 < c_1 \leq c_2$. In this case the operator (0.1) is the subdifferential of the functional

(0.3) $$\Psi(u) = \int_\Omega \psi(x,Du)dx$$

and the notion of G-convergence of the operators (0.1) can be studied in connection with the notion of Γ-convergence of the corresponding functionals (0.3) (see [2], [50], [5]).

Let us return to the general case of maps of the class $M_\Omega(\mathbf{R}^n)$ for which the representation (0.2) is not always possible. Let (a_h) be a sequence in $M_\Omega(\mathbf{R}^n)$ and let $a \in M_\Omega(\mathbf{R}^n)$. To introduce the notion of G-convergence in $M_\Omega(\mathbf{R}^n)$ we begin with the simpler case where a_h and a are single-valued and strictly monotone on $\mathbf{R}^n$. We then say that (a_h) G-converges to a if, for every $f \in H^{-1,q}(\Omega)$ and for every sequence (f_h) converging to f strongly in $H^{-1,q}(\Omega)$, the solutions u_h of the equations

(0.4) $$\begin{cases} -\mathrm{div}(a_h(x,Du_h)) = f_h & \text{on } \Omega , \\ u_h \in H_0^{1,p}(\Omega) , & \end{cases}$$

satisfy the following conditions:

$$u_h \to u \qquad \text{weakly in } H^{1,p}(\Omega) ,$$

$$a_h(x,Du_h) \to a(x,Du) \qquad \text{weakly in } (L^q(\Omega))^n ,$$

where u is the solution of the equation

(0.5) $$\begin{cases} -\mathrm{div}(a(x,Du)) = f & \text{on } \Omega , \\ u \in H_0^{1,p}(\Omega) . & \end{cases}$$

If we drop the hypothesis that a_h and a are single-valued and strictly monotone, then the definition of G-convergence is more delicate, due to the non-uniqueness of the solutions of the equations (0.4) and (0.5).

In the general case we say that (a_h) G-converges to a if for every increasing sequence of integers $\tau(h)$, for every $f \in H^{-1,q}(\Omega)$, for every sequence (f_h) converging to f strongly in $H^{-1,q}(\Omega)$, for every sequence (u_h) of solutions of the equations

$$(0.6)\qquad \begin{cases} -\mathrm{div}(a_{\tau(h)}(x,Du_h)) \ni f_h & \text{on } \Omega\,, \\ u_h \in H_0^{1,p}(\Omega)\,, \end{cases}$$

and for every sequence (g_h) in $(L^q(\Omega))^n$ with

$$g_h(x) \in a_{\tau(h)}(x,Du_h(x)) \text{ a.e. in } \Omega \quad \text{and} \quad -\mathrm{div} g_h = f_h \quad \text{in } \Omega\,,$$

there exists an increasing sequence of integers $\sigma(h)$ such that

$$u_{\sigma(h)} \to u \qquad \text{weakly in } H^{1,p}(\Omega)$$

and

$$g_{\sigma(h)} \to g \qquad \text{weakly in } (L^q(\Omega))^n\,,$$

where u is a solution of the equation

$$(0.7)\qquad \begin{cases} -\mathrm{div}(a(x,Du)) \ni f & \text{on } \Omega\,, \\ u \in H_0^{1,p}(\Omega)\,, \end{cases}$$

and

$$g(x) \in a(x,Du(x)) \text{ a.e. in } \Omega\,.$$

Let us emphasize that the notion of G-convergence in $M_\Omega(R^n)$ is independent of the particular boundary condition chosen in the definition, in the sense that, given $\varphi \in H^{1,p}(\Omega)$, we can replace $H_0^{1,p}(\Omega)$ by $\varphi + H_0^{1,p}(\Omega)$ in (0.4), (0.5), (0.6), (0.7) without changing the G-convergent sequences and their limits.

The main result of this chapter is the compactness of the class $M_\Omega(R^n)$ with respect to G-convergence. Moreover we prove the following localization property: if (a_h) G-converges to a, (b_h) G-converges to b, and $a_h(x,\cdot) = b_h(x,\cdot)$ for almost every x in an open subset Ω' of Ω, then $a(x,\cdot) = b(x,\cdot)$ for almost every $x \in \Omega'$.

Finally we determine some subsets of $M_\Omega(R^n)$ which are closed under G-convergence. This allows us to prove in a unified way the compactness, with respect to G-convergence, of all general classes of linear or nonlinear operators of the form (0.1) which have been considered in the literature.

The notion of G-convergence for second order linear elliptic operators was studied by E. De Giorgi and S. Spagnolo in the symmetric case (see [66], [67], [68], [36]), and then extended to the non-symmetric case by F. Murat and L. Tartar under the name of H-convergence (see [70], [71], and [53]). We refer to [10] and [61] for the related problem of the homogenization of elliptic equations and to [76] for the extension of the notion of G-convergence to higher order linear elliptic operators.

The properties of the G-convergence for quasilinear elliptic operators were studied by L. Boccardo, Th. Gallouet , and F. Murat in [13], [14], and [11].

The first results in the nonlinear case (0.1), with $p = 2$, are due to L. Tartar, who studied (in [73]) the properties of the G-convergence in a suitable class of monotone operators of the form (0.1), assuming that the maps a are uniformly Lipschitz continuous and uniformly strictly monotone on $\mathbf{R}^n$. The corresponding homogenization results are contained in [4].

A similar theory of G-convergence for more general classes of uniformly equicontinuous strictly monotone operators was developed by U. E. Raitum in the case $2 \leq p < +\infty$ (see [59]). For the corresponding homogenization results we refer to [38] and [39].

We remark that, in order to include the case (0.2), we do not assume the maps of our class $M_\Omega(\mathbf{R}^n)$ to be continuous or strictly monotone on $\mathbf{R}^n$, and this requires a deep change in the proof of the compactness of $M_\Omega(\mathbf{R}^n)$ under G-convergence. While all proofs in the quoted papers are based essentially on a density argument, which is made possible by the continuity of the operators $\mathcal{A}$ or of the inverse operators $\mathcal{A}^{-1}$, our proof relies on a theorem by F. Hiai and H. Umegaki concerning the representation of every closed decomposable subset of L^p as the set of all measurable selections of a suitable multivalued map (see [40]).

1. MULTIVALUED FUNCTIONS

In this section we fix the notation and recall some results concerning multivalued functions and their measurability. Furthermore, we summarize the main theorems for multivalued monotone operators on Banach spaces which will be applied in this chapter.

If x, y are elements of a set X , by [x,y] we denote the ordered pair formed by x and y , whereas (x,y) denotes the scalar product of x and y , provided X is a Hilbert space.

Multivalued functions. Let X and Y be two sets. A *multivalued function* F from X to Y is a map that associates with any $x \in X$ a subset Fx of Y . The subsets Fx are called the *images* or *values* of F . The sets

$$D(F) = \{x \in X : Fx \neq \emptyset\} \qquad \text{and} \qquad G(F) = \{[x,y] \in X \times Y : y \in Fx\}$$

are called the *domain* of F and the *graph* of F, respectively. The *range* of F is, by definition, the set

$$R(F) = \bigcup_{x\in X} Fx .$$

If for every $x\in X$ the set Fx contains exactly one element of Y, we say that F is single-valued.

In general, we shall identify every multivalued function F with its graph in $X\times Y$. The *inverse* F^{-1} of the multivalued map F from X to Y is the multivalued function from Y to X defined by $x\in F^{-1}y$ if and only if $y\in Fx$; in other words, F^{-1} is the multivalued function, whose graph is symmetric to the graph of F.

Measurable multivalued functions. Let $(X,\mathcal{T})$ be a measurable space, and let $F : X \to \mathbf{R}^n$ be a multivalued function from the space X to the family of non-empty subsets of the space $\mathbf{R}^n$. For every $B \subseteq \mathbf{R}^n$ the inverse image of B under F is denoted by

$$F^{-1}(B) = \{x\in X : B\cap Fx \neq \emptyset\} .$$

We shall consider the following measurability conditions:

(1.1) for each Borel set $B \subset \mathbf{R}^n$, $F^{-1}(B)\in \mathcal{T}$;

(1.2) for each closed set $C \subset \mathbf{R}^n$, $F^{-1}(C)\in \mathcal{T}$;

(1.3) for each open set $U \subset \mathbf{R}^n$, $F^{-1}(U)\in \mathcal{T}$;

(1.4) there exists a sequence (σ_h) of measurable selections such that $Fx = \mathrm{cl}\{\sigma_h(x) : h\in N\}$ for each x (a selection of F is a map $\sigma : X \to \mathbf{R}^n$ such that $\sigma(x)\in Fx$ for every x);

(1.5) $G(F)\in \mathcal{T}\otimes\mathcal{B}(\mathbf{R}^n)$, where $\mathcal{B}(\mathbf{R}^n)$ is the σ-field of all Borel subsets of $\mathbf{R}^n$.

We say that a multivalued function $F : X \to \mathbf{R}^n$ is *measurable* (with respect to $\mathcal{T}$ and $\mathcal{B}(\mathbf{R}^n)$) if (1.2) is verified. Let us state a theorem which links this definition of measurability of a multivalued function F to the other conditions on F listed above.

Theorem 1.1. *Let $(X,\mathcal{T})$ be a measurable space. Let $F : X \to \mathbf{R}^n$ be a multivalued function with non-empty closed values. Then the following conditions hold:*

(i) (1.1) ⇒ (1.2) ⇔ (1.3) ⇔ (1.4) ⇒ (1.5) ;

(ii) If there exists a complete σ-finite measure μ defined on $\mathcal{T}$, then all conditions (1.1)–(1.5) *are equivalent.*

The proof of the above theorem can be found in [22], Chapter III, Section 2. A useful tool for problems of this type is given by the projection theorem below (see [22], Theorem III.23).

Theorem 1.2. *Let* $(X,\mathcal{T},\mu)$ *be a measurable space, where* μ *is a complete* σ*–finite measure defined on* $\mathcal{T}$*. If* G *belongs to* $\mathcal{T}\otimes\mathcal{B}(\mathbf{R}^n)$ *, then the projection* $\mathrm{pr}_X G$ *belongs to* $\mathcal{T}$*.*

The next theorem states the equivalence between conditions (1.2) and (1.5) for certain multivalued functions even if the measure space is not complete.

Theorem 1.3. *Let* $(X,\mathcal{T},\mu)$ *be a measurable space, where* μ *is a complete* σ*–finite measure defined on* $\mathcal{T}$*. Let* $F : X \to \mathbf{R}^n\times\mathbf{R}^m$ *be a multivalued function with non-empty closed values. Let* $H : X\times\mathbf{R}^n \to \mathbf{R}^m$ *be the multivalued function defined by*

$$H(x,\xi) = \{\eta\in\mathbf{R}^m : [\xi,\eta]\in Fx\} \ . \tag{1.6}$$

Then the following conditions are equivalent:

(i) F *is measurable with respect to* $\mathcal{T}$ *and* $\mathcal{B}(\mathbf{R}^n)\otimes\mathcal{B}(\mathbf{R}^m)$ *;*

(ii) $G(F)\in\mathcal{T}\otimes\mathcal{B}(\mathbf{R}^n)\otimes\mathcal{B}(\mathbf{R}^m)$ *;*

(iii) H *is measurable with respect to* $\mathcal{T}\otimes\mathcal{B}(\mathbf{R}^n)$ *and* $\mathcal{B}(\mathbf{R}^m)$ *;*

(iv) $G(H)\in\mathcal{T}\otimes\mathcal{B}(\mathbf{R}^n)\otimes\mathcal{B}(\mathbf{R}^m)$ *.*

Proof. By Theorem 1.1(ii) we have that (i) $\Leftrightarrow$ (ii). Moreover, Theorem 1.1(i) guarantees that (iii) $\Rightarrow$ (iv). Since $G(F) = G(H)$, we obtain easily that (ii) $\Leftrightarrow$ (iv). To conclude the proof of the theorem we shall show that (ii) $\Rightarrow$ (iii). To this aim it is enough to prove that (ii) yields $H^{-1}(C)\in\mathcal{T}\otimes\mathcal{B}(\mathbf{R}^n)$ for every compact subset C of $\mathbf{R}^m$. Let us fix a compact set $C\subseteq\mathbf{R}^m$. By taking (1.6) into account we have that

$$H^{-1}(C) = \{[x,\xi]\in X\times\mathbf{R}^n : \exists\ \eta\in\mathbf{R}^m \ : [\xi,\eta]\in Fx\cap(\mathbf{R}^n\times C)\} \ . \tag{1.7}$$

Let B denote the set of all $x\in X$ such that $Fx\cap(\mathbf{R}^n\times C)$ is non-empty. By (ii) and the projection Theorem 1.2 it follows that $B\in\mathcal{T}$. If Φ is the multivalued function from X to $\mathbf{R}^n\times\mathbf{R}^m$ defined by $\Phi x = Fx\cap(\mathbf{R}^n\times C)$, then $D(\Phi) = B$ and (1.7) becomes

$$H^{-1}(C) = \{[x,\xi]\in X\times\mathbf{R}^n : \exists\ \eta\in\mathbf{R}^m \ : [\xi,\eta]\in \Phi x\} \ . \tag{1.8}$$

Since $G(\Phi) = G(F)\cap(X\times\mathbf{R}^n\times C)\in\mathcal{T}\otimes\mathcal{B}(\mathbf{R}^n)\otimes\mathcal{B}(\mathbf{R}^m)$, by Theorem 1.1 there exists a sequence $[\varphi_h,g_h]$ of measurable functions from B to $\mathbf{R}^n\times\mathbf{R}^m$ such that

$$\Phi x = \mathrm{cl}\{[\varphi_h(x),g_h(x)] : h\in\mathbf{N}\} \tag{1.9}$$

for every $x\in B$. By taking (1.9) into account let us define the set

(1.10) $$M = \{[x,\xi] \in X \times R^n : x \in B,\ \xi \in \mathrm{cl}\{\varphi_h(x) : h \in N\}\}\ .$$

We shall prove that $M = H^{-1}(C)$. The inclusion $H^{-1}(C) \subseteq M$ follows easily from (1.8), (1.9), and (1.10). To prove that $M \subseteq H^{-1}(C)$, let us fix $[x,\xi] \in M$. By definition there exists a subsequence $(\varphi_{\sigma(h)})$ of (φ_h) such that $(\varphi_{\sigma(h)}(x))$ converges to ξ. Moreover, the corresponding sequence $(g_{\sigma(h)}(x))$ belongs to the compact set C. Hence, by passing, if necessary, to a subsequence we may assume that $(g_{\sigma(h)}(x))$ converges to some $\eta \in R^m$. By (1.9) we have $[\xi,\eta] \in \Phi x$, hence $[x,\xi] \in H^{-1}(C)$, which concludes the proof of the equality $M = H^{-1}(C)$. Since $M = \{[x,\xi] \in X \times R^n : x \in B,\ \inf_{h \in N} |\xi - \varphi_h(x)| = 0\}$, we have that $M \in \mathcal{T} \otimes \mathcal{B}(R^n)$ and the proof of the theorem is accomplished. ♦

Finally, let us give a more general theorem for the existence of a measurable selection of a multivalued function due to Aumann and von Neumann (see [22], Theorem III.22).

Theorem 1.4. *Let* $(X,\mathcal{T})$ *be a measurable space and let* F *be a multivalued function from* X *to* R^n *with non-empty values. If the graph* $G(F)$ *belongs to* $\mathcal{T} \otimes \mathcal{B}(R^n)$ *and there exists a complete* σ*-finite measure defined on* $\mathcal{T}$*, then* F *has a measurable selection.*

Maximal monotone operators. Our present aim is to remind the definition and some basic properties of multivalued maximal monotone operators in Banach spaces.

Let X be a Banach space and let X^* be its topological dual. By $<\ ,\ >$ we denote the duality pairing between X^* and X.

Definition 1.5. A subset $A \subseteq X \times X^*$ is called *monotone* (resp. *strictly monotone*)if

$$<y_1 - y_2\,,\, x_1 - x_2> \geq 0 \qquad (\text{resp.} > 0)$$

for any $[x_1,y_1] \in A$, $[x_2,y_2] \in A$.

Definition 1.6. A monotone subset $A \subseteq X \times X^*$ is called *maximal monotone* if it is not properly contained in any other monotone subset of $X \times X^*$, i.e. for every $[x,y] \in X \times X^*$ such that

$$<y - \eta\,,\, x - \xi> \geq 0 \qquad \forall\ [\xi,\eta] \in A$$

it follows that $[x,y] \in A$.

We say that a multivalued operator $F: X \to X^*$ is *monotone* (resp. *maximal monotone*) if its graph is a monotone (resp. maximal monotone) subset of $X \times X^*$.

Remark 1.7. Since the monotonicity is invariant under transposition of the domain and the range of a map, F is (maximal) monotone if and only if F^{-1} has this property.

Let us note that if F is a (multivalued) maximal monotone operator on X, then for any $x \in D(F)$ the image Fx is a closed convex subset of X^* (see, for example, [58], Chapter III.2).

Before giving the statement of the next theorem, which will be heavily applied in Sections 2 and 5, we recall the definition of the concept of upper-semicontinuous multivalued operator.

Definition 1.8. Let S_1 and S_2 be two topological spaces, and let F be a multivalued function of S_1 into S_2. Then F is said to be *upper-semicontinuous* if for every $s_0 \in S_1$ and for every open neighborhood V of Fs_0 in S_2 there exists a neighborhood U of s_0 in S_1 such that $Fs \subseteq V$ for every $s \in U$.

The following result provides a useful criterion for maximal monotonicity (see [20], Theorem (3.18)).

Theorem 1.9. *Let X be a Banach space and let X^* be its dual. Let F be a multivalued monotone operator of X into X^*. Suppose that for each x in X, Fx is a non empty weak* closed convex subset of X^* and that for each line segment in X, F is an upper-semicontinuous multivalued operator from the line segment to X^*, with X^* given its weak* topology. Then F is maximal monotone.*

Finally, we state a surjectivity result for a class of multivalued monotone operators which is of crucial importance in the proof of our theorems in Sections 2 and 4.

Theorem 1.10. *Let X be a reflexive Banach space and let X^* be its dual. Let F be a multivalued maximal monotone operator from X to X^*. If F is coercive, then $R(F) = X^*$.*

We remind that the (multivalued) operator $F: X \to X^*$ is called *coercive* if

$$\lim_{\|x\| \to \infty} \frac{\langle Fx, x\rangle}{\|x\|} = +\infty .$$

The proof of Theorem 1.10 can be found in [58], Chapter III, Theorem 2.10.

2. MULTIVALUED MONOTONE OPERATORS IN SOBOLEV SPACES

In this section we study a class of multivalued monotone operators on Sobolev spaces of the type $-\mathrm{div}(a(x,Du))$.

Throughout this chapter we denote by p a fixed real number, $1 < p < +\infty$, and by q its dual exponent, $1/p + 1/q = 1$. Moreover we fix a bounded open subset Ω of $\mathbf{R}^n$, two non-negative functions $m_1, m_2 \in L^1(\Omega)$, and two constants $c_1 > 0$, $c_2 > 0$. By $\mathcal{L}(\Omega)$ we denote the σ-field of all Lebesgue measurable subsets of Ω, and by $\mathcal{B}(\mathbf{R}^n)$ the σ-field of all Borel subsets of $\mathbf{R}^n$. The Euclidean norm and the scalar product in $\mathbf{R}^n$ are denoted by $|\cdot|$ and $(\cdot,\cdot)$, respectively.

Definition 2.1. By $M_\Omega(\mathbf{R}^n)$ we denote the class of all multivalued functions $a : \Omega\times\mathbf{R}^n \to \mathbf{R}^n$ with closed values which satisfy the following conditions:

(i) for a.e. $x\in\Omega$ the multivalued function $a(x,\cdot) : \mathbf{R}^n \to \mathbf{R}^n$ is maximal monotone;

(ii) a is measurable with respect to $\mathcal{L}(\Omega)\otimes\mathcal{B}(\mathbf{R}^n)$ and $\mathcal{B}(\mathbf{R}^n)$, i.e.

$$a^{-1}(C) = \{[x,\xi]\in\Omega\times\mathbf{R}^n : a(x,\xi)\cap C \neq \emptyset\}\in\mathcal{L}(\Omega)\otimes\mathcal{B}(\mathbf{R}^n)$$

for every closed set $C \subseteq \mathbf{R}^n$;

(iii) the estimates

$$|\eta|^q \leq m_1(x) + c_1(\eta,\xi)\,, \tag{2.1}$$

$$|\xi|^p \leq m_2(x) + c_2(\eta,\xi) \tag{2.2}$$

hold for a.e. $x\in\Omega$, for every $\xi\in\mathbf{R}^n$, and $\eta\in a(x,\xi)$.

Remark 2.2. Conditions (2.1) and (2.2) imply that there exist two functions $m_3\in L^q(\Omega)$, $m_4\in L^1(\Omega)$ and two constants $c_3 > 0$, $c_4 > 0$ such that

$$|\eta| \leq m_3(x) + c_3|\xi|^{p-1}\,, \tag{2.3}$$

$$(\eta,\xi) \geq m_4(x) + c_4\,|\xi|^p \tag{2.4}$$

for a.e. $x\in\Omega$, for every $\xi\in\mathbf{R}^n$, and $\eta\in a(x,\xi)$. Conversely, if a satisfies (2.3) and (2.4), then (2.1) and (2.2) hold for suitable m_1, m_2, c_1, c_2.

Remark 2.3. For a.e. $x\in\Omega$ and for every $\xi\in\mathbf{R}^n$ the set $a(x,\xi)$ is closed and convex in $\mathbf{R}^n$ by (i) (see, for instance, [58], Section III.2.3). Moreover, (ii) and Theorem 1.1(i) imply that the graph of a belongs to $\mathcal{L}(\Omega)\otimes\mathcal{B}(\mathbf{R}^n)\otimes\mathcal{B}(\mathbf{R}^n)$. By (2.3), for a.e. $x\in\Omega$ the maximal monotone operator $a(x,\cdot)$ is locally bounded, hence $a^{-1}(x,\cdot)$ is surjective (see [58], III.4.2). This implies that $a(x,\xi) \neq \emptyset$ for a.e. $x\in\Omega$ and for every $\xi\in\mathbf{R}^n$.

Given $a \in M_\Omega(\mathbf{R}^n)$, $f \in H^{-1,q}(\Omega)$, and $\varphi \in H^{1,p}(\Omega)$ we consider the Dirichlet boundary value problem

$$(2.5) \qquad \begin{cases} -\mathrm{div}(a(x,Du)) \ni f & \text{on } \Omega, \\ u \in H^{1,p}_\varphi(\Omega), \end{cases}$$

where $H^{1,p}_\varphi(\Omega) = \{u \in H^{1,p}(\Omega) : u - \varphi \in H^{1,p}_0(\Omega)\}$.

To study the solutions of (2.5), and in particular their dependence on f and a, we shall give some equivalent formulations of this problem which are used in the sequel.

Definition 2.4. Let $\varphi \in H^{1,p}(\Omega)$. By $M(H^{1,p}_\varphi)$ (resp. $M(H^{1,p})$) we denote the class of all multivalued operators $A : H^{1,p}_\varphi(\Omega) \to (L^q(\Omega))^n$ (resp. $A : H^{1,p}(\Omega) \to (L^q(\Omega))^n$) satisfying the following conditions:

(i) if $u_i \in H^{1,p}_\varphi(\Omega)$ (resp. $H^{1,p}(\Omega)$) and $g_i \in Au_i$, $i = 1,2$, then

$$(Du_1 - Du_2, g_1 - g_2) \geq 0 \qquad \text{a.e. on } \Omega;$$

(ii) the estimates

$$(2.6) \qquad |g|^q \leq m_1 + c_1(Du,g) \qquad \text{a.e. on } \Omega,$$

$$(2.7) \qquad |Du|^p \leq m_2 + c_2(Du,g) \qquad \text{a.e. on } \Omega,$$

hold for every $u \in H^{1,p}_\varphi(\Omega)$ (resp. $u \in H^{1,p}(\Omega)$) and $g \in Au$.

By $\mathcal{M}(H^{1,p}_\varphi)$ (resp. $\mathcal{M}(H^{1,p})$) we denote the class of all multivalued operators $\mathcal{A} : H^{1,p}_\varphi(\Omega) \to H^{-1,q}(\Omega)$ (resp. $\mathcal{A} : H^{1,p}(\Omega) \to H^{-1,q}(\Omega)$) of the form

$$(2.8) \qquad \mathcal{A}u = \{-\mathrm{div} g : g \in Au\},$$

with $A \in M(H^{1,p}_\varphi)$ (resp. $A \in M(H^{1,p})$).

Remark 2.5. In the case $\varphi = 0$ the operators of the class $\mathcal{M}(H^{1,p}_0)$ are monotone according to Definition 1.5 in consequence of (i). If $\mathcal{A} \in \mathcal{M}(H^{1,p}_0)$ is maximal monotone, then $D(\mathcal{A}) = H^{1,p}_0(\Omega)$. Indeed $\mathcal{A}$ is locally bounded by (2.6), hence $\mathcal{A}^{-1}$ is surjective (see [58], III.4.2).

Definition 2.6. Let $\varphi \in H^{1,p}(\Omega)$. To every $a \in M_\Omega(\mathbf{R}^n)$ we associate the operators $A \in M(H^{1,p})$ and $\mathcal{A} \in \mathcal{M}(H^{1,p})$ defined by

$$Au = \{g \in (L^q(\Omega))^n : g(x) \in a(x,Du(x)) \text{ for a.e. } x \in \Omega\},$$

$$\mathcal{A}u = \{-\mathrm{div}\, g : g \in Au\} .$$

Their restrictions to $H^{1,p}_{\varphi}(\Omega)$ belong to $M(H^{1,p}_{\varphi})$ and $\mathcal{M}(H^{1,p}_{\varphi})$ and will be denoted by A^{φ} and $\mathcal{A}^{\varphi}$, respectively.

By taking these definitions into account, problem (2.5) becomes then equivalent to the following one: given $f \in H^{-1,q}(\Omega)$, find $u \in H^{1,p}_{\varphi}(\Omega)$ such that

$$(2.9) \qquad \begin{cases} f \in \mathcal{A}^{\varphi} u , \\ u \in H^{1,p}_{\varphi}(\Omega) , \end{cases}$$

or equivalently, find $u \in H^{1,p}_{\varphi}(\Omega)$ and $g \in (L^q(\Omega))^n$ such that

$$(2.10) \qquad \begin{cases} g \in A^{\varphi} u , \\ -\mathrm{div}\, g = f , \\ u \in H^{1,p}_{\varphi}(\Omega) . \end{cases}$$

Let us denote by I the (single-valued) monotone operator from $L^p(\Omega)$ to $L^q(\Omega)$ defined by $Iu = |u|^{p-2}u$. The next theorem is more than needed for solving problem (2.9) in the case $\varphi = 0$, but it is used in its generality in Section 6.

Theorem 2.7. *Let $\mathcal{A}^0$ be the operator in $\mathcal{M}(H^{1,p}_0)$ associated to a function $a \in M_{\Omega}(\mathbb{R}^n)$ in the case $\varphi = 0$ (Definition 2.6). Then*

(i) $\mathcal{A}^0$ is maximal monotone ;

(ii) $R(\mathcal{A}^0 + \lambda I) = H^{-1,q}(\Omega)$ for every $\lambda \geq 0$.

Proof. Let us start with the proof of (i). To this aim we show that the operator $\mathcal{A}^0$ satisfies the assumptions of Theorem 1.9.

(a) For every $u \in H^{1,p}_0(\Omega)$, we have $\mathcal{A}^0 u \neq \emptyset$. To prove this assertion let us fix $u \in H^{1,p}_0(\Omega)$. By Remark 2.3 the set $a(x,Du(x))$ is non-empty, closed, and convex in $\mathbb{R}^n$ for a.e. $x \in \Omega$. Therefore, by taking Theorem 1.1 into account we conclude that there exists a measurable function $g : \Omega \to \mathbb{R}^n$ such that $g(x) \in a(x,Du(x))$ for a.e. $x \in \Omega$. Finally, the estimate (2.3) yields $g \in (L^q(\Omega))^n$, which concludes the proof of (a).

(b) For every $u \in H_0^{1,p}(\Omega)$, $\mathcal{A}^0 u$ is a convex subset of $H^{-1,q}(\Omega)$. This follows easily from the fact that $a(x,Du(x))$ is a convex subset of $\mathbf{R}^n$ for a.e. $x \in \Omega$ (Remark 2.3).

(c) For every $u \in H_0^{1,p}(\Omega)$, $\mathcal{A}^0 u$ is a weakly closed subset of $H^{-1,q}(\Omega)$ and the multivalued operator $\mathcal{A}^0$ is upper-semicontinuous from the strong topology of $H_0^{1,p}(\Omega)$ to the weak topology of $H^{-1,q}(\Omega)$. By the boundedness condition (2.3), to prove this assertion it is enough to show that, if (u_h) converges to u strongly in $H_0^{1,p}(\Omega)$, (f_h) converges to f weakly in $H^{-1,q}(\Omega)$, and $f_h \in \mathcal{A}^0 u_h$ for every $h \in \mathbf{N}$, then $f \in \mathcal{A}^0 u$. Under these assumptions on f_h, f, u_h, u, the boundedness condition (2.3) guarantees the existence of a sequence of functions $g_h \in (L^q(\Omega))^n$ and of a function $g \in (L^q(\Omega))^n$ such that (up to a subsequence) (g_h) converges to g weakly in $(L^q(\Omega))^n$, $g_h(x) \in a(x,Du_h(x))$ for a.e. $x \in \Omega$, $-\mathrm{div}\, g_h = f_h$, and $-\mathrm{div}\, g = f$. Therefore, it remains to verify that $g(x) \in a(x,Du(x))$ for a.e. $x \in \Omega$. If we show that the set

$$M = \{x \in \Omega : \exists\, \xi \in \mathbf{R}^n , \exists\, \eta \in a(x,\xi) : (g(x) - \eta, Du(x) - \xi) < 0\}$$

has Lebesgue measure zero, then the maximal monotonicity of a yields $g(x) \in a(x,Du(x))$ a.e. on Ω, which concludes the proof of (c). To prove that $|M| = 0$, let us write $M = \{x \in \Omega : Gx \neq \emptyset\}$, where

$$Gx = \{[\xi,\eta] \in \mathbf{R}^n \times \mathbf{R}^n : \eta \in a(x,\xi) , \ (g(x) - \eta, Du(x) - \xi) < 0\} .$$

By Remark 2.3 the graph of G belongs to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^n)$, thus $M \in \mathcal{L}(\Omega)$ by the projection Theorem 1.2. By the Aumann-von Neumann Theorem 1.4 there exists a measurable selection $[\xi,\eta]$ of G defined on M . Therefore $\eta(x) \in a(x,\xi(x))$ and

$$(g(x) - \eta(x), Du(x) - \xi(x)) < 0 \tag{2.11}$$

for every $x \in M$. On the other hand, the monotonicity assumption on a implies that

$$(g_h(x) - \eta(x), Du_h(x) - \xi(x)) \geq 0 \qquad \text{a.e. on } M \tag{2.12}$$

for every $h \in \mathbf{N}$. If $|M| > 0$, there exists a measurable subset M' of M with $|M'| > 0$ such that $[\xi(x),\eta(x)]$ is bounded on M' . By integrating (2.12) on M' and by passing to the limit as $h \to +\infty$, we obtain

$$\int_{M'} (g(x) - \eta(x), Du(x) - \xi(x))\, dx \geq 0 ,$$

which contradicts (2.11) being $|M'| > 0$. Therefore we have to conclude that $|M| = 0$. This proves (c) and completes the proof of (i).

Proof of (ii). By (i) we have that $\mathcal{A}^0$ is maximal monotone. Since $D(\mathcal{A}^0) = D(I) = H_0^{1,p}(\Omega)$, and I is maximal monotone on $H_0^{1,p}(\Omega)$, the operator $\mathcal{A}^0 + \lambda I$ is maximal monotone for every $\lambda \geq 0$ (see [58], III.3.6). By (2.2) it is also coercive and therefore $R(\mathcal{A}^0 + \lambda I) = H^{-1,q}(\Omega)$ by Theorem 1.10. ♦

Remark 2.8. Problem (2.9) has a solution for every $\varphi \in H^{1,p}(\Omega)$. Indeed, let us define the multivalued function $a_\varphi(x,\xi) = a(x,\xi + D\varphi(x))$ which still belongs to the class $M_\Omega(R^n)$. If $\mathcal{A}_\varphi^0$ denotes the operator in $\mathcal{M}(H_0^{1,p})$ associated to the function a_φ by Definition 2.6, it follows easily that $\mathcal{A}^\varphi(u+\varphi) = \mathcal{A}_\varphi^0 u$ for every $u \in H_0^{1,p}(\Omega)$. Since by Theorem 2.7(ii) we have that $R(\mathcal{A}_\varphi^0) = H^{-1,q}(\Omega)$, our assertion follows immediately.

Finally, the following result is a useful tool to check the maximality of certain monotone operators on $H_0^{1,p}(\Omega)$.

Lemma 2.9. *Let $\mathcal{A}$ be a (multivalued) monotone operator from $H_0^{1,p}(\Omega)$ into $H^{-1,q}(\Omega)$, let $\lambda > 0$, and let I be the (single-valued) function from $L^p(\Omega)$ to $L^q(\Omega)$ defined by $Iu = |u|^{p-2}u$. If $R(\mathcal{A} + \lambda I) = H^{-1,q}(\Omega)$, then $\mathcal{A}$ is maximal monotone.*

Proof. Let $\mathcal{B} : H_0^{1,p}(\Omega) \to H^{-1,q}(\Omega)$ be a (multivalued) monotone operator such that $\mathcal{A} \subseteq \mathcal{B}$. The proof will be accomplished if we show that $\mathcal{B} \subseteq \mathcal{A}$. Let $f \in \mathcal{B}u$. It is clear that

$$f + \lambda Iu \in \mathcal{B}u + \lambda Iu . \tag{2.13}$$

On the other hand, since $R(\mathcal{A} + \lambda I) = H^{-1,q}(\Omega)$ there exists $v \in H_0^{1,p}(\Omega)$ such that $f + \lambda Iu \in \mathcal{A}v + \lambda Iv$. Then the assumption $\mathcal{A} \subseteq \mathcal{B}$ implies

$$f + \lambda Iu \in \mathcal{B}v + \lambda Iv . \tag{2.14}$$

By taking (2.13) and (2.14) into account, the strict monotonicity of the operator $\mathcal{B} + \lambda I$ yields $v = u$ a.e. on Ω. Thus, $f + \lambda Iu \in \mathcal{A}u + \lambda Iu$, or equivalently, $f \in \mathcal{A}u$, which concludes the proof of the lemma. ♦

3. G-CONVERGENCE OF MONOTONE OPERATORS

In this section we introduce a notion of convergence in the class of multivalued functions $M_\Omega(R^n)$ which permits a satisfactory analysis of the perturbations of Dirichlet problems of the form (2.5).

The convergence considered here is defined in terms of a general concept of set-convergence named Kuratowski convergence (see [45], Section 29) which can be formulated in abstract terms in an arbitrary topological space (X,τ) as follows.

Definition 3.1. Let (E_h) be a sequence of subsets of X. We define the *sequential lower limit* and the *sequential upper limit of* (E_h) by

(3.1) $$K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_h = \{u\in X : \exists\, u_h \xrightarrow{\tau} u\,,\ \exists\, k\in N\,,\ \forall\, h\geq k : u_h\in E_h\}\,,$$

and

(3.2) $$K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h = \{u\in X : \exists\, \sigma(h)\to+\infty\,,\ \exists\, u_h \xrightarrow{\tau} u\,,\ \forall\, h\in N : u_h\in E_{\sigma(h)}\}\,.$$

Then, we say that the sequence (E_h) $K_{seq}(\tau)$-*converges to a set* E *in* X if

(3.3) $$K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_h = K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h = E$$

and in this case we write $K_{seq}(\tau)\text{-}\lim\limits_{h\to\infty} E_h = E$.

Remark 3.2. From the definitions above it follows immediately that

$$K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_h \subseteq K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h\,.$$

Therefore (E_h) $K_{seq}(\tau)$-converges to E if and only if

$$E \subseteq K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_h \quad\text{and}\quad K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h \subseteq E\,.$$

Remark 3.3. It is easy to prove that

$$K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h \subseteq E$$

if and only if every subsequence $(E_{\sigma(h)})$ of (E_h) has a further subsequence $(E_{\sigma(\tau(h))})$ such that

$$K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_{\sigma(\tau(h))} \subseteq E\,.$$

This notion of set-convergence has been particularized to obtain the graph-convergence of sequences of maximal monotone operators on reflexive Banach spaces (see [5], Definition 3.58), which is useful for handling convergence problems for the stationary and evolution equations associated to such operators.

To study perturbations of Dirichlet problems of the form (2.5) we introduce here a stronger notion of convergence.

We denote by w the weak topology on $H^{1,p}(\Omega)$. If σ_1 denotes the weak topology of $(L^q(\Omega))^n$ and σ_2 the topology on $(L^q(\Omega))^n$ induced by the pseudo-metric $d(g_1,g_2) = \|\mathrm{div} g_1 - \mathrm{div} g_2\|_{H^{-1,q}}$, we denote by σ the weakest topology on $(L^q(\Omega))^n$ which is stronger than σ_1 and σ_2. In other words, (g_h) converges to g in σ if and only if (g_h) converges to g weakly in $(L^q(\Omega))^n$ and $(-\mathrm{div} g_h)$ converges to $-\mathrm{div} g$ strongly in $H^{-1,q}(\Omega)$.

The connection between w and σ is explained by the following lemma, which will be frequently used in the sequel.

Lemma 3.4. *Let* (u_h) *be a sequence converging to* u *weakly in* $H^{1,p}(\Omega)$, *and let* (g_h) *be a sequence in* $(L^q(\Omega))^n$ *converging to* g *in the topology* σ. *Then*

$$\int_\Omega g_h D u_h \varphi\, dx \quad \to \quad \int_\Omega g D u \varphi\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

Proof. The lemma is a simple case of compensated compactness (see [54], [72]). It can be proved by observing that

$$\int_\Omega g_h D u_h \varphi\, dx = \langle -\mathrm{div} g_h, u_h \varphi \rangle - \int_\Omega g_h u_h D\varphi\, dx$$

for every $\varphi \in C_0^\infty(\Omega)$. ♦

Having in mind the usual identification of a multivalued map with its graph we give the following definition.

Definition 3.5. We say that a sequence (a_h) in $M_\Omega(R^n)$ *G-converges to* $a \in M_\Omega(R^n)$ if

$$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A_h^0 \subseteq A^0 ,$$

where A_h^0 and A^0 are the operators in $M(H_0^{1,p})$ associated to a_h and a by Definition 2.6 in the case $\varphi = 0$.

Remark 3.6. The condition $K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A_h^0 \subseteq A^0$ in the above definition determines uniquely the G-limit a, as we shall prove in Corollary 5.9.

Remark 3.7. Using Remarks 3.2 and 3.3 it is easy to prove that the G-convergence satisfies the following axioms:

(i) axiom of the constant sequences: if $a_h = a$ for every $h \in N$, then (a_h) G-converges to a;

(ii) axiom of the subsequences: if (a_h) G-converges to a, and $(a_{\sigma(h)})$ is a subsequence of (a_h), then $(a_{\sigma(h)})$ G-converges to a;

(iii) Urysohn axiom: (a_h) G-converges to a if every subsequence of (a_h) contains a further subsequence which G-converges to a.

In the sequel we enunciate some results regarding the G-convergence on the class $M_\Omega(R^n)$ and make some comments connecting these results to our investigation on convergence of solutions to sequences of Dirichlet problems of type (2.5). We shall prove the following Theorem in Section 6.

Theorem 3.8. *Let* $\varphi \in H^{1,p}(\Omega)$, *let* (a_h) *be a sequence in* $M_\Omega(R^n)$ *and let* $a \in M_\Omega(R^n)$. *Then the following conditions are equivalent:*

(i) (a_h) *G-converges to* a,

(ii) $K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A_h \subseteq A$,

(iii) $K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A_h^\varphi \subseteq A^\varphi$,

where A_h, A *are the operators in* $M(H^{1,p})$ *associated to* a_h *and* a *by Definition* 2.6 *and* A_h^φ, A^φ *are the corresponding operators in* $M(H_\varphi^{1,p})$.

Remark 3.9. It follows immediately from the boundedness hypothesis (2.6) that the inclusion

(3.4)
$$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to+\infty} A_h \subseteq A$$

is equivalent to the following condition: if $\sigma(h)$ is a sequence of integers, (f_h) is a sequence in $H^{-1,q}(\Omega)$, and (u_h) is a sequence of local solutions in $H^{1,p}(\Omega)$ of the equations

$$-\mathrm{div}(a_{\sigma(h)}(x,Du_h)) \ni f_h \qquad \text{on } \Omega$$

with

$$\sigma(h) \to +\infty ,$$

$$u_h \to u \qquad \text{weakly in } H^{1,p}(\Omega) ,$$

$$f_h \to f \qquad \text{strongly in } H^{-1,q}(\Omega) ,$$

then u is a solution to the equation

$$-\mathrm{div}a(x,Du) \ni f \qquad \text{on } \Omega ,$$

and for every sequence (g_h) in $(L^q(\Omega))^n$, with

$$g_h(x)\in a_{\sigma(h)}(x,Du_h(x)) \text{ a.e. in } \Omega \quad \text{and} \quad -\mathrm{div}g_h = f_h \quad \text{in } \Omega ,$$

there exists a subsequence $(g_{\tau(h)})$ such that

$$g_{\tau(h)} \to g \qquad \text{weakly in } (L^q(\Omega))^n$$

and

$$g(x)\in a(x,Du(x)) \quad \text{a.e. in } \Omega .$$

Remark 3.10. The inclusion

(3.5)
$$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to+\infty} A^{\varphi}_h \subseteq A^{\varphi}$$

is equivalent to the following condition: for every increasing sequence of integers $\tau(h)$,for every $f\in H^{-1,q}(\Omega)$, for every sequence (f_h) converging to f strongly in $H^{-1,q}(\Omega)$, for every sequence (u_h) of solutions of the equations

$$\begin{cases} -\mathrm{div}(a_{\tau(h)}(x,Du_h)) \ni f_h & \text{on } \Omega , \\ u_h\in H^{1,p}_{\varphi}(\Omega) , \end{cases}$$

and for every sequence (g_h) in $(L^q(\Omega))^n$ with

$$g_h(x) \in a_{\tau(h)}(x, Du_h(x)) \text{ a.e. in } \Omega \quad \text{and} \quad -\operatorname{div} g_h = f_h \quad \text{in } \Omega ,$$

there exists an increasing sequence of integers $\sigma(h) \to +\infty$ such that

$$u_{\sigma(h)} \to u \qquad \text{weakly in } H^{1,p}(\Omega)$$

and

$$g_{\sigma(h)} \to g \qquad \text{weakly in } (L^q(\Omega))^n ,$$

where u is a solution of the equation

(3.6)
$$\begin{cases} -\operatorname{div}(a(x,Du)) \ni f & \text{on } \Omega , \\ u \in H^{1,p}_{\varphi}(\Omega) , \end{cases}$$

and

(3.7)
$$g(x) \in a(x,Du) \text{ a.e. in } \Omega .$$

In fact, assume (3.5) and suppose that $\tau(h)$, f, f_h, u_h, g_h satisfy the above assumptions. By the coerciveness condition (2.7) the sequence (u_h) is bounded in $H^{1,p}(\Omega)$ and therefore (g_h) is bounded in $(L^q(\Omega))^n$ by the growth condition (2.6). Thus, there exists a subsequence $[u_{\sigma(h)}, g_{\sigma(h)}]$ of $[u_h, g_h]$ which converges to $[u,g]$ weakly in $H^{1,p}(\Omega) \times (L^q(\Omega))^n$. This implies that $(-\operatorname{div} g_h)$ converges to $-\operatorname{div} g$ weakly in $H^{-1,q}(\Omega)$, hence $f = -\operatorname{div} g$. Therefore $[u_{\sigma(h)}, g_{\sigma(h)}]$ converges to $[u,g]$ in the topology $w \times \sigma$ and the assumption (3.5) implies $g \in A^{\varphi} u$, hence (3.7). This yields that u is a solution of (3.6), being $f = -\operatorname{div} g$.

The converse implication is trivial.

The following result, which will be proved in Section 6, shows the relationship between our definition of G-convergence and that one considered by Ambrosetti and Sbordone in [2].

Let us denote by ρ the strong topology in $H^{-1,q}(\Omega)$.

Theorem 3.11. *Let* $\varphi \in H^{1,p}(\Omega)$. Let (a_h) *be a sequence in* $M_\Omega(R^n)$ *which G-converges to* $a \in M_\Omega(R^n)$. *Then*

(i) $K_{seq}(w \times \rho)\text{-}\lim_{h\to\infty} \mathcal{A}_h = \mathcal{A}$,

(ii) $K_{seq}(w \times \rho)\text{-}\lim_{h\to\infty} \mathcal{A}^{\varphi}_h = \mathcal{A}^{\varphi}$,

where $\mathcal{A}_h$, $\mathcal{A}$ *are the operators in* $\mathcal{M}(H^{1,p})$ *associated to* a_h *and* a *by Definition* 2.6, *and* $\mathcal{A}_h^\varphi$, $\mathcal{A}^\varphi$ *are the corresponding operators in* $\mathcal{M}(H_\varphi^{1,p})$.

Remark 3.12. The condition

$$K_{seq}(w\times\rho)\text{-}\lim_{h\to\infty} \mathcal{A}_h^\varphi = \mathcal{A}^\varphi \tag{3.8}$$

can be expressed in terms of convergence of solutions of differential equations. More precisely, (3.8) holds if and only if both the following conditions (a) and (b) are satisfied:

(a) if (f_h) converges to f strongly in $H^{-1,q}(\Omega)$, (u_h) converges to u weakly in $H^{1,p}(\Omega)$, and u_h satisfies the equation

$$\begin{cases} -\mathrm{div}(a_h(x,Du_h)) \ni f_h & \text{on } \Omega , \\ u_h \in H_\varphi^{1,p}(\Omega) , \end{cases} \tag{3.9}$$

for infinitely many $h\in \mathbf{N}$, then u is a solution to

$$\begin{cases} -\mathrm{div}(a(x,Du)) \ni f & \text{on } \Omega , \\ u \in H_\varphi^{1,p}(\Omega) ; \end{cases} \tag{3.10}$$

(b) if $f\in H^{-1,q}(\Omega)$ and u is a solution to (3.10), then there exist (f_h) converging to f strongly in $H^{-1,q}(\Omega)$ and (u_h) converging to u weakly in $H^{1,p}(\Omega)$ such that u_h satisfies the equation (3.9) for every $h\in \mathbf{N}$.

Remark 3.13. Conditions (i) and (ii) in Theorem 3.11 do not imply that (a_h) G-converges to a . The reason lies in the fact that a is not uniquely determined by the associated operator $\mathcal{A}$ as the following example shows.

Assume n = 3, and let $\varphi\in C_0^\infty(\Omega)$. Let us define

$$a(x,\xi) = \xi$$

and

$$b(x,\xi) = \xi + D\varphi(x)\times\xi ,$$

where $\times$ denotes the external product in $\mathbf{R}^3$. It is easy to see that a and b belongs to the class $M_\Omega(\mathbf{R}^3)$ with $p = 2$, $m_1 = m_2 = 0$, $c_1 = (1 + \max_\Omega|D\varphi|^2)$, and $c_2 = 1$.

Since

$$\int_\Omega (D\varphi \times Du)Dv\,dx = 0 \qquad \text{for every } u, v \in H^{1,2}(\Omega) ,$$

it follows that

(3.11) $$\int_\Omega a(x,Du)Dv\,dx = \int_\Omega b(x,Du)Dv\,dx \qquad \text{for every } u, v \in H^{1,2}(\Omega) .$$

This implies that the operators in $\mathcal{M}(H^{1,2})$ associated to a and b according to Definition 2.6 coincide.

♦

4. A COMPACTNESS THEOREM

The main purpose of this section is to prove the following compactness result for the G-convergence on the class of multivalued functions $M_\Omega(R^n)$.

Theorem 4.1. *Let* (a_h) *be a sequence in* $M_\Omega(R^n)$. *Then there exists a subsequence* $(a_{\sigma(h)})$ *of* (a_h) *which* G-*converges to a function* a *of the class* $M_\Omega(R^n)$.

Without any difficulty Theorem 4.1 comes out from the next two theorems and from the definition of G-convergence.

Theorem 4.2. *Let* (a_h) *be a sequence in* $M_\Omega(R^n)$, *and let* (A^0_h) *be the sequence of operators in* $M(H_0^{1,p})$ *associated to* (a_h) *by Definition* 2.6 . *Then there exist a subsequence* $(A^0_{\sigma(h)})$ *of* (A^0_h) *and an operator* $B\in M(H_0^{1,p})$ *such that*

$$K_{seq}(w\times\sigma)\text{-}\lim_{h\to\infty} A^0_{\sigma(h)} = B .$$

Moreover $D(B) = H_0^{1,p}(\Omega)$.

Theorem 4.3. *Let* $B\in M(H_0^{1,p})$ *with* $D(B) \supseteq C_0^\infty(\Omega)$. *Then there exists a unique function* $a\in M_\Omega(R^n)$ *such that* $B \subseteq A^0$, *where* A^0 *denotes the operator in* $M(H_0^{1,p})$ *associated to* a *by Definition* 2.6.

The proofs of these theorems are quite technical and will be divided in several steps. We devote this section to the proof of Theorem 4.2, whereas Theorem 4.3 will be proved in the next section.

The following proposition is the first step of the proof of Theorem 4.2.

Proposition 4.4. *Let* (B_h) *be a sequence of operators of the class* $M(H_0^{1,p})$. *Then there exists a subsequence* $(B_{\sigma(h)})$ *which* $K_{seq}(w\times\sigma)$*-converges to an operator* $B\in M(H_0^{1,p})$.

Proof. On every separable reflexive Banach space X there exists a metric d such that for every sequence (x_h) in X the following conditions are equivalent:

(4.1) $$x_h \to x \text{ weakly in } X;$$

(4.2) $$(x_h) \text{ is norm-bounded in } X \text{ and } d(x_h,x)\to 0.$$

By τ_1 we denote the topology induced by a metric on $H_0^{1,p}(\Omega)$ which satisfies (4.1) and (4.2). By τ_2 we denote the topology on $(L^q(\Omega))^n$ induced by the metric

$$d_2(g_1,g_2) = d(g_1,g_2) + \|\mathrm{div}g_1 - \mathrm{div}g_2\|_{H^{-1,q}(\Omega)},$$

where d is a metric on $(L^q(\Omega))^n$ which satisfies (4.1) and (4.2).

Since $\tau_1\times\tau_2$ has a countable base, by the Kuratowski compactness theorem (see [45], Section 29, Theorem VIII) there exists a subsequence of (B_h), still denoted by (B_h), which $K_{seq}(\tau_1\times\tau_2)$-converges to a set $B\subseteq H_0^{1,p}(\Omega)\times(L^q(\Omega))^n$.

By Remark 3.2, to prove that (B_h) $K_{seq}(w\times\sigma)$-converges to B, it is enough to show that

(4.3) $$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} B_h \subseteq B,$$

and

(4.4) $$B \subseteq K_{seq}(w\times\sigma)\text{-}\liminf_{h\to\infty} B_h.$$

Let us verify (4.3). Let $[u,g]\in K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} B_h$. Then, there exist $\sigma(h)\to+\infty$ and $[u_h,g_h]$ converging to $[u,g]$ in the topology $w\times\sigma$ such that $[u_h,g_h]\in B_{\sigma(h)}$ for every $h\in N$. By (4.1) and (4.2) we get immediately that $[u_h,g_h]$ converges to $[u,g]$ in $\tau_1\times\tau_2$ and we conclude that $[u,g]\in B$.

Let us prove (4.4). Let $[u,g]\in B$. Then there exists a sequence $[u_h,g_h]$ which converges to $[u,g]$ in $\tau_1\times\tau_2$ such that $[u_h,g_h]\in B_h$ for h large enough. Since $(\mathrm{div}g_h)$ is bounded in $H^{-1,q}(\Omega)$, condition (2.7) implies that (u_h) is bounded in $H_0^{1,p}(\Omega)$, hence (g_h) is bounded in $(L^q(\Omega))^n$ by (2.6). Then the equivalence between (4.1) and (4.2) yields that (u_h)

converges to u weakly in $H_0^{1,p}(\Omega)$ and (g_h) converges to g weakly in $(L^q(\Omega))^n$. Since $(-\mathrm{div} g_h)$ converges to $-\mathrm{div} g$ strongly in $H^{-1,q}(\Omega)$, we conclude that $[u_h, g_h]$ converges to $[u,g]$ in the topology $w\times\sigma$, which implies (4.4).

Finally, let us prove that the operator B belongs to the class $M(H_0^{1,p})$. We verify here only condition (i) of Definition 2.4. The boundedness and coerciveness conditions (2.6) and (2.7) can be proved in the same way. Let us fix $u^i \in H_0^{1,p}(\Omega)$ and $g^i \in Bu^i$, $i = 1,2$. By (4.4) there exists a sequence $[u_h^i, g_h^i]$ converging to $[u^i, g^i]$ in the topology $w\times\sigma$ such that $[u_h^i, g_h^i] \in B_h$ for h large enough. Since $B_h \in M(H_0^{1,p})$, we have

$$\int_\Omega (Du_h^1 - Du_h^2, g_h^1 - g_h^2)\varphi\, dx \geq 0$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ on Ω. By Lemma 3.4 it follows that

$$\int_\Omega (Du^1 - Du^2, g^1 - g^2)\varphi\, dx \geq 0$$

for every $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$ on Ω. This implies that

$$(Du^1 - Du^2, g^1 - g^2) \geq 0 \quad \text{a.e. on } \Omega,$$

hence B satisfies condition (i) of Definition 2.4. ♦

The second step to achieve the proof of Theorem 4.2 is based on the next proposition.

Proposition 4.5. *Let* (B_h) *be a sequence of operators in* $M(H_0^{1,p})$ *and* $(\mathcal{B}_h)$ *be the corresponding sequence in* $\mathcal{M}(H_0^{1,p})$ *according to* (2.8). *Assume that*

$$B = K_{seq}(w\times\sigma)\text{-}\lim_{h\to\infty} B_h .$$

Then

$$\mathcal{B} = K_{seq}(w\times\rho)\text{-}\lim_{h\to\infty} \mathcal{B}_h ,$$

where $\mathcal{B}$ *is the operator of the class* $\mathcal{M}(H_0^{1,p})$ *associated to* $B \in M(H_0^{1,p})$ *according to* (2.8) *and* ρ *denotes the strong topology of* $H^{-1,q}(\Omega)$.

Proof. The inclusion $\mathcal{B} \subseteq K_{seq}(w\times\rho)\text{-}\liminf_{h\to\infty} \mathcal{B}_h$ is trivial. To prove the inclusion $K_{seq}(w\times\rho)\text{-}\limsup_{h\to\infty} \mathcal{B}_h \subseteq \mathcal{B}$, let us fix $[u,f] \in K_{seq}(w\times\rho)\text{-}\limsup_{h\to\infty} \mathcal{B}_h$. By (3.2) there exist

$\sigma(h) \to +\infty$, and a sequence $[u_h,f_h]$ converging to $[u,f]$ in $w\times\rho$ such that $[u_h,f_h]\in \mathcal{B}_{\sigma(h)}$ for every $h\in N$. By Definition 2.6 this implies that there exists $g_h\in B_{\sigma(h)}u_h$ such that $-\mathrm{div}g_h = f_h$. By (2.6) we have

$$\int_\Omega |g_h|^q dx \le c[1 + \int_\Omega |Du_h|^p dx]$$

for a suitable constant c, which implies that the sequence (g_h) is bounded in $(L^q(\Omega))^n$. Thus there exists a subsequence $(g_{\tau(h)})$ converging weakly in $(L^q(\Omega))^n$ to a function g , which yields that $(-\mathrm{div}g_{\tau(h)})$ converges to $-\mathrm{div}g$ weakly in $H^{-1,q}(\Omega)$. Since, by assumption, (f_h) converges to f strongly in $H^{-1,q}(\Omega)$, we conclude that $f = -\mathrm{div}g$. Therefore, $[u_{\tau(h)},g_{\tau(h)}]$ converges to $[u,g]$ in the topology $w\times\sigma$ and $[u_{\tau(h)},g_{\tau(h)}]\in B_{\sigma(\tau(h))}$. Thus $[u,g]\in B$ and $[u,f]\in \mathcal{B}$. ♦

We are now able to prove Theorem 4.2.

Proof of Theorem 4.2. By Proposition 4.4 there exist a subsequence $(A^0_{\sigma(h)})$ of (A^0_h) and an operator B belonging to $M(H_0^{1,p})$ such that

(4.5)
$$K_{seq}(w\times\sigma)\text{-}\lim_{h\to\infty} A^0_{\sigma(h)} = B .$$

Let us prove that $D(B) = H_0^{1,p}(\Omega)$. Since the K-convergence is stable with respect to continuous perturbations, Proposition 4.5 together with (4.5) implies that for every $\lambda \ge 0$, we have

(4.6)
$$K_{seq}(w\times\rho)\text{-}\lim_{h\to\infty} (\mathcal{A}^0_{\sigma(h)} + \lambda I) = \mathcal{B} + \lambda I ,$$

where $\mathcal{B}$ is the operator in $\mathcal{M}(H_0^{1,p})$ associated to B according to (2.8). Let us prove that $R(\mathcal{B} + \lambda I) = H^{-1,q}(\Omega)$. Let $f\in H^{-1,q}(\Omega)$. By Theorem 2.7(ii), for every $h\in N$ there exists $u_h\in H_0^{1,p}(\Omega)$ such that

$$\mathcal{A}^0_{\sigma(h)}u_h + \lambda I u_h \ni f .$$

By (2.7) the sequence (u_h) is bounded in $H_0^{1,p}(\Omega)$, thus it contains a subsequence which converges to a function u weakly in $H_0^{1,p}(\Omega)$. By (4.6) we have

$$\mathcal{B}u + \lambda I u \ni f ,$$

which gives $R(\mathcal{B} + \lambda I) = H^{-1,q}(\Omega)$.

By Lemma 2.9 the operator $\mathcal{B}$, hence $\mathcal{B}^{-1}$, is maximal monotone. By (2.6) the operator $\mathcal{B}^{-1}$ is coercive on $H^{-1,q}(\Omega)$. Therefore Theorem 1.10 implies that $R(\mathcal{B}^{-1}) = H_0^{1,p}(\Omega)$, which is equivalent to $D(\mathcal{B}) = H_0^{1,p}(\Omega)$. This yields $D(B) = H_0^{1,p}(\Omega)$ and concludes the proof of the theorem. ♦

5. A REPRESENTATION THEOREM

The main goal of this section is the proof of the following theorem, which contains Theorem 4.3 of Section 4.

Theorem 5.1. *Let* $B \in M(H^{1,p})$ *with* $D(B) \supseteq C_0^\infty(\Omega)$. *Then there exists a unique multivalued function* $a \in M_\Omega(R^n)$ *such that* $B \subseteq A$, *where* A *denotes the operator in* $M(H^{1,p})$ *associated to* a *by Definition* 2.6.

The following representation theorem for maximal monotone operators in the class $\mathcal{M}(H_0^{1,p})$ is an easy consequence of Theorem 5.1 and Remark 2.5.

Theorem 5.2. *Any maximal monotone operator in* $\mathcal{M}(H_0^{1,p})$ *is associated to a function* $a \in M_\Omega(R^n)$ *according to Definition* 2.6.

Before starting with the proof of Theorem 5.1 we shall introduce some notions and results related to measurable multivalued functions.

By $\mathcal{F}$ we denote the family of all measurable multivalued functions $F : \Omega \to R^n \times R^n$ with non-empty closed values, and for every $F \in \mathcal{F}$ we indicate by $\mathcal{S}_F^{p,q}$ the set of all $(L^p(\Omega))^n \times (L^q(\Omega))^n$-selections of F, i.e.

$$\mathcal{S}_F^{p,q} = \{f \in (L^p(\Omega))^n \times (L^q(\Omega))^n : f(x) \in Fx \text{ a.e. on } \Omega\}.$$

Then the following results hold (see, for instance, [40], Lemma 1.1 and Corollary 1.2).

Lemma 5.3 (Castaign representation). *Let* $F \in \mathcal{F}$. *If* $\mathcal{S}_F^{p,q}$ *is non-empty, then there exists a sequence of functions* (f_n) *belonging to* $\mathcal{S}_F^{p,q}$ *such that* $Fx = \mathrm{cl}\{f_h(x) : h \in N\}$ *for all* $x \in \Omega$.

Lemma 5.4. *Let* $F_1, F_2 \in \mathcal{F}$. *If* $\mathcal{S}_{F_1}^{p,q} = \mathcal{S}_{F_2}^{p,q} \neq \emptyset$, *then* $F_1x = F_2x$ *a.e. on* Ω.

Let M be a set of single-valued measurable functions $f : \Omega \to R^n \times R^n$. We call M *decomposable* (with respect to $\mathcal{L}(\Omega)$), if $f_1, f_2 \in M$ and $U \in \mathcal{L}(\Omega)$ imply $1_U f_1 + 1_{\Omega\setminus U} f_2 \in M$, where 1_U and $1_{\Omega\setminus U}$ indicate the characteristic functions of U and of $\Omega\setminus U$, respectively. The following theorem gives a characterization of the closed decomposable subsets of $(L^p(\Omega))^n \times (L^q(\Omega))^n$ (for the proof see [40], Theorem 3.1).

Theorem 5.5. *Let* M *be a non-empty closed subset of* $(L^p(\Omega))^n \times (L^q(\Omega))^n$. *Then* M *is decomposable if and only if there exists* $F \in \mathcal{F}$ *such that* $M = \mathcal{S}_F^{p,q}$.

Proof of Theorem 5.1. Let E be the subset of $(L^p(\Omega))^n \times (L^q(\Omega))^n$ defined by

$$E = \{[Du,g] \in (L^p(\Omega))^n \times (L^q(\Omega))^n : u \in H^{1,p}(\Omega) , g \in Bu\} . \tag{5.1}$$

Then, E is non-empty and satisfies the following monotonicity condition:

$$\text{if } [\varphi_1,g_1], [\varphi_2,g_2] \in E , \text{ then } (\varphi_1 - \varphi_2, g_1 - g_2) \geq 0 \quad \text{a.e. on } \Omega . \tag{5.2}$$

Moreover for every $[\varphi,g] \in E$ we have

$$|g|^q \leq m_1 + c_1(\varphi,g) \quad \text{a.e. on } \Omega , \tag{5.3}$$

$$|\varphi|^p \leq m_2 + c_2(\varphi,g) \quad \text{a.e. on } \Omega . \tag{5.4}$$

Let decE be the smallest decomposable set containing E. It is easy to prove that $[\varphi,g] \in$ decE if and only if there exists a finite Borel partition $(\Omega_i)_{i\in I}$ of Ω and a finite family $([\varphi_i,g_i])_{i\in I}$ of elements of E such that $[\varphi,g] = [\varphi_i,g_i]$ a.e. on Ω_i. Therefore, decE is non-empty and (5.2), (5.3), (5.4) hold with E replaced by decE .

Besides decE , let us consider also the set

$$\tilde{E} = \mathrm{cl}_{s\times w}(\mathrm{dec}E) , \tag{5.5}$$

defined as the closure of decE in $(L^p(\Omega))^n \times (L^q(\Omega))^n$, with $(L^p(\Omega))^n$ endowed with its strong topology and $(L^q(\Omega))^n$ endowed with its weak topology. The next proposition, whose proof will be given later, summarizes the main properties of $\tilde{E}$.

Proposition 5.6. *Let* $\tilde{E}$ *be defined by* (5.5). *Then the following properties hold:*

(a) *for every* $[\varphi,g] \in \tilde{E}$ *there exists a sequence* $[\varphi_h,g_h] \in$ decE *such that* (φ_h) *converges to* φ *strongly in* $(L^p(\Omega))^n$ *and* (g_h) *converges to* g *weakly in* $(L^q(\Omega))^n$;

(b) $\tilde{E}$ *is decomposable and* (5.2), (5.3), (5.4) *hold with* E *replaced by* $\tilde{E}$;

(c) $\tilde{E}$ *is maximal monotone.*

Proof of Theorem 5.1 (Continuation). Since $\tilde{E}$ is a non-empty, closed, and decomposable subset of $(L^p(\Omega))^n \times (L^q(\Omega))^n$, by Theorem 5.5 there exists a measurable multivalued function $F : \Omega \to \mathbf{R}^n \times \mathbf{R}^n$ with non-empty closed values such that

$$\tilde{E} = \{[\varphi,g] \in (L^p(\Omega))^n \times (L^q(\Omega))^n : [\varphi(x),g(x)] \in Fx \text{ for a.e. } x \in \Omega\} . \tag{5.6}$$

Let us define the multivalued function $a : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ by

$$a(x,\xi) = \{\eta \in \mathbf{R}^n : [\xi,\eta] \in Fx\} . \tag{5.7}$$

We shall prove in Lemma 5.7 that a belongs to the class $M_\Omega(\mathbf{R}^n)$. By (5.1), (5.6), and (5.7) we have $B \subseteq A$, where A denotes the operator in $M(H^{1,p})$ associated to a by Definition 2.6. The uniqueness of a will be proved in Proposition 5.8. ♦

Proof of Proposition 5.6. Let us start with (a). Let $[\varphi_0,g_0] \in \tilde{E}$, and let $\mathcal{U}_1$ be the ball in $(L^p(\Omega))^n$ with center φ_0 and radius 1. Since (5.3) holds for $\mathrm{dec}E$, there exists a constant $R = R(c_1,m_1,\varphi_0)$ such that, if $[\varphi,g] \in \mathrm{dec}E$ and $\varphi \in \mathcal{U}_1$, then $g \in \mathcal{B}_R$, where $\mathcal{B}_R$ denotes the ball in $(L^q(\Omega))^n$ with center 0 and radius R. We may also assume that $g_0 \in \mathcal{B}_R$. Therefore,

$$\mathrm{dec}E \cap (\mathcal{U} \times (\mathcal{V} \cap \mathcal{B}_R)) = \mathrm{dec}E \cap (\mathcal{U} \times \mathcal{V}) \neq \emptyset \tag{5.8}$$

for every neighborhood $\mathcal{V}$ of g_0 in the weak topology of $(L^q(\Omega))^n$ and for every neighborhood $\mathcal{U}$ of φ_0 in the strong topology of $(L^p(\Omega))^n$ such that $\mathcal{U} \subseteq \mathcal{U}_1$.

Since the weak topology is metrizable on $\mathcal{B}_R$, there exists a countable base $(\mathcal{V}_h)$ for the neighborhood system of g_0 in $\mathcal{B}_R$ endowed with the weak topology of $(L^q(\Omega))^n$. We may also assume that $\mathcal{V}_{h+1} \subseteq \mathcal{V}_h$ for every $h \in \mathbf{N}$. Let us denote by $\mathcal{U}_h$ the ball in $(L^p(\Omega))^n$ with center φ_0 and radius $1/h$. By (5.8) the sets $\mathrm{dec}E \cap (\mathcal{U}_h \times \mathcal{V}_h)$ are non-empty, thus for every $h \in \mathbf{N}$ we may pick up $[\varphi_h,g_h] \in \mathrm{dec}E$ such that $\varphi_h \in \mathcal{U}_h$ and $g_h \in \mathcal{V}_h$. This yields that (φ_h) converges to φ_0 strongly in $(L^p(\Omega))^n$ and (g_h) converges to g_0 weakly in $(L^q(\Omega))^n$, concluding the proof of (a).

By applying (a), we obtain easily property (b) of $\tilde{E}$ from the analogous property of $\mathrm{dec}E$.

Finally, let us prove (c). To this aim we apply Theorem 1.9 to $\tilde{E}$. We prove first that for every $\varphi \in (L^p(\Omega))^n$, the set $\tilde{E}(\varphi)$ is non-empty. In the case $\varphi \in (L^p(\Omega))^n$, φ piecewise constant and with compact support on Ω, the proof follows easily from the assumption $D(B) \supseteq C_0^\infty(\Omega)$ and the definition of $\mathrm{dec}E$. The general case can be obtained by approximation of $\varphi \in (L^p(\Omega))^n$ in the strong topology of $(L^p(\Omega))^n$ with functions (φ_h) of the previous type. In fact, from above it follows that there exists $g_h \in (L^q(\Omega))^n$ such that $g_h \in \tilde{E}(\varphi_h)$. Then, the estimate (5.3) for $\tilde{E}$ (proved in (b)) implies that (g_h) is bounded in $(L^q(\Omega))^n$. By passing, if necessary, to a subsequence , (g_h) converges to a function g in the weak topology of $(L^q(\Omega))^n$ and g lies in $\tilde{E}(\varphi)$; the first assumption of Theorem 1.9 is so guaranteed. It is clear that for every $\varphi \in (L^p(\Omega))^n$ the set $\tilde{E}(\varphi)$ is decomposable and weakly closed in $(L^q(\Omega))^n$. Let us prove that $\tilde{E}(\varphi)$ is convex. Fix $g_1 , g_2 \in \tilde{E}(\varphi)$ and $t \in (0,1)$. There exists a sequence (U_h) of subsets of Ω such that $1|_{U_h} \to t$ and $1|_{\Omega \setminus U_h} \to (1-t)$ in the weak* topology of $L^\infty(\Omega)$. Since $\tilde{E}(\varphi)$ is decomposable we have $1|_{U_h} g_1 + 1|_{\Omega \setminus U_h} g_2 \in \tilde{E}(\varphi)$. Since $\tilde{E}(\varphi)$ is weakly closed in $(L^q(\Omega))^n$, taking the limit as $h \to +\infty$, we obtain $tg_1 + (1-t)g_2 \in \tilde{E}(\varphi)$, which proves that $\tilde{E}(\varphi)$ is convex. Finally, let us prove that $\tilde{E}$ is upper semi-continuous from $(L^p(\Omega))^n$, with the strong topology, into $(L^q(\Omega))^n$, with the weak topology. Fix $\varphi \in (L^p(\Omega))^n$, and let $\mathcal{V}$ be an open neighborhood of $\tilde{E}(\varphi)$ in the weak topology of $(L^q(\Omega))^n$. We claim that for every sequence (φ_h) converging to φ strongly in $(L^p(\Omega))^n$ there exists $k \in N$ such that $\tilde{E}(\varphi_h) \subseteq \mathcal{V}$ for every $h \geq k$. Assume the contrary. Then there exists a subsequence $(\varphi_{\sigma(h)})$ of (φ_h) and a sequence (g_h) such that $g_h \in \tilde{E}(\varphi_{\sigma(h)})$ and $g_h \notin \mathcal{V}$ for every $h \in N$. By the estimate (5.3) for $\tilde{E}$ (proved in (b)) the sequence (g_h) is bounded in $(L^q(\Omega))^n$, thus there exists a subsequence, $(g_{\tau(h)})$ of (g_h) which converges weakly in $(L^q(\Omega))^n$ to a function g . Since $[\varphi_{\sigma(\tau(h))}, g_{\tau(h)}] \in \tilde{E}$ for every $h \in N$ we have $g \in \tilde{E}(\varphi)$, hence $g \in \mathcal{V}$. But the last fact requires that $g_h \in \mathcal{V}$ for h large enough, which contradicts our assumption. This implies that $\tilde{E}$ is upper-semicontinuous and concludes the proof of (c). ♦

Lemma 5.7. *The function* a *defined by* (5.7) *belongs to* $M_\Omega(R^n)$.

Proof of Lemma 5.7. The measurability of a follows immediately from the measurability of F and from Theorem 1.3. Moreover, the property (5.2) for $\tilde{E}$ and the Castaign representation of F (Lemma 5.3) imply that Fx is monotone for a.e. $x \in \Omega$. We come now to the maximal monotonicity of a . By (5.7) it is enough to show that the set M defined by

$$M = \{x \in \Omega : \exists\, [\xi,\eta] \in R^n \times R^n : [\xi,\eta] \notin Fx \text{ and } (\xi-\xi',\eta-\eta') \geq 0 \;\; \forall\, [\xi',\eta'] \in Fx\}$$

has Lebesgue measure zero. To this aim let us write $M = \{x \in \Omega : \Phi x \neq \varnothing\}$, where

$$\Phi x = \{[\xi,\eta] \in R^n \times R^n : [\xi,\eta] \notin Fx \text{ and } (\xi-\xi',\eta-\eta') \geq 0 \;\; \forall\, [\xi',\eta'] \in Fx\} \; .$$

Since $F \in \mathcal{F}$ and $\tilde{E} = \mathcal{S}_F^{p,q} \neq \varnothing$, by Lemma 5.3 there exists a sequence $[\varphi_h, g_h] \in (L^p(\Omega))^n \times (L^q(\Omega))^n$ such that

$$\Phi x = \{[\xi,\eta] \in \mathbf{R}^n \times \mathbf{R}^n : [\xi,\eta] \notin Fx \text{ and } (\xi - \varphi_h(x), \eta - g_h(x)) \geq 0 \ \ \forall\, h \in \mathbf{N}\} =$$

$$= \bigcap_{h \in \mathbf{N}} \{[\xi,\eta] \in \mathbf{R}^n \times \mathbf{R}^n : [\xi,\eta] \notin Fx \text{ and } (\xi - \varphi_h(x), \eta - g_h(x)) \geq 0\} .$$

Since φ_h, g_h are measurable and F is measurable, it follows easily that the graph of Φ belongs to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^n)$, thus $M \in \mathcal{L}(\Omega)$ by the projection Theorem 1.2. By the Aumann-von Neumann Theorem 1.4 there exists a measurable selection $[\varphi_0, g_0]$ of Φ defined on M . Therefore, for every $x \in M$ we have

(5.9) $$[\varphi_0(x), g_0(x)] \notin Fx ,$$

and

(5.10) $$(\varphi_0(x) - \xi, g_0(x) - \eta) \geq 0 \qquad \text{for every } [\xi,\eta] \in Fx .$$

If $|M| > 0$, there exists a measurable subset M' of M with $|M'| > 0$ such that $[\varphi_0(x), g_0(x)]$ is bounded on M' . Given $[\varphi_*, g_*] \in \tilde{E}$ we define the functions

$$\bar{\varphi}(x) = \begin{cases} \varphi_0(x) & \text{if } x \in M' , \\ \varphi_*(x) & \text{if } x \in \Omega \setminus M' , \end{cases}$$

and

$$\bar{g}(x) = \begin{cases} g_0(x) & \text{if } x \in M' , \\ g_*(x) & \text{if } x \in \Omega \setminus M' . \end{cases}$$

Then $[\bar{\varphi}, \bar{g}] \in (L^p(\Omega))^n \times (L^q(\Omega))^n$. By (5.10) and by property (5.2) of $\tilde{E}$ we have that

$$\int_\Omega (\bar{\varphi} - \varphi, \bar{g} - g)dx = \int_{M'} (\varphi_0 - \varphi, g_0 - g)dx + \int_{\Omega \setminus M'} (\varphi_* - \varphi, g_* - g)dx \geq 0$$

for every $[\varphi, g] \in \tilde{E}$. Since $\tilde{E}$ is maximal monotone (Proposition 5.6(c)), the above inequality yields $[\bar{\varphi}, \bar{g}] \in \tilde{E}$, or equivalently, $[\bar{\varphi}(x), \bar{g}(x)] \in Fx$ a.e. on Ω . But this implies that $[\varphi_0(x), g_0(x)] \in Fx$ for a.e. $x \in M'$, which contradicts (5.9) being $|M'| > 0$. Therefore, we have to conclude that the set M has Lebesgue measure zero, which guarantees that $a(x,\cdot)$ is maximal monotone for a.e. $x \in \Omega$. To conclude that $a \in M_\Omega(\mathbf{R}^n)$ it remains to verify that a satisfies (2.1) and (2.2), but this is an easy consequence of Lemma 5.3 and of properties (5.3) and (5.4) for $\tilde{E}$ (Proposition 5.6(b)). ♦

The following proposition will be crucial in the proof of the localization property considered in the next section.

Proposition 5.8. *Let* $C \in M(H^{1,p})$ *with* $D(C) \supseteq \psi + C_0^\infty(\Omega)$ *for a given* $\psi \in H^{1,p}(\Omega)$. *Let* a *and* b *be two functions of the class* $M_\Omega(R^n)$ *and let* A *and* B *be the corresponding operators of the class* $M(H^{1,p})$. *If* $C \subseteq A$ *and* $C \subseteq B$, *then* $a(x,\xi) = b(x,\xi)$ *for a.e.* $x \in \Omega$ *and for every* $\xi \in R^n$.

Proof. It is enough to prove the proposition when $\psi = 0$, since the general case can be obtained easily by translation (Remark 2.8).

Let E be the subset of $(L^p(\Omega))^n \times (L^q(\Omega))^n$ defined as in (5.1) with B replaced by C and let

$$E_a = \{[\varphi,g] \in (L^p(\Omega))^n \times (L^q(\Omega))^n : g(x) \in a(x,\varphi(x)) \text{ a.e. on } \Omega\} .$$

It is clear that $C \subseteq A$ implies $E \subseteq E_a$. Since E_a is decomposable we have $\mathrm{dec}E \subseteq E_a$. Since E_a is maximal monotone (see [19], Example 2.3.3), it is sequentially closed in $(L^p(\Omega))^n \times (L^q(\Omega))^n$ with $(L^p(\Omega))^n$ endowed with its strong topology and $(L^q(\Omega))^n$ endowed with its weak topology. Therefore, $\tilde{E} \subseteq E_a$, hence $\tilde{E} = E_a$ by the maximal monotonicity of $\tilde{E}$ (Proposition 5.6(c)).

Analogously, we obtain $\tilde{E} = E_b$, therefore Lemma 5.4 implies that $a(x,\xi) = b(x,\xi)$ for a.e. $x \in \Omega$ and for every $\xi \in R^n$. ♦

The following corollary proves the uniqueness of the G-limit.

Corollary 5.9. *Let* $\varphi \in H^{1,p}(\Omega)$, *let* (a_h) *be a sequence of functions of the class* $M_\Omega(R^n)$, *and let*

$$C = K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} A_h^\varphi ,$$

where A_h^φ *are the operators in* $M(H_\varphi^{1,p})$ *associated to* a_h *by Definition* 2.6. *Let* a *and* b *be two functions of the class* $M_\Omega(R^n)$ *and let* A^φ *and* B^φ *be the corresponding operators of the class* $M(H_\varphi^{1,p})$. *If* $C \subseteq A^\varphi$ *and* $C \subseteq B^\varphi$, *then* $a(x,\xi) = b(x,\xi)$ *for a.e.* $x \in \Omega$ *and for every* $\xi \in R^n$.

Proof. It is enough to prove the corollary when $\varphi = 0$, since the general case can be obtained easily by translation (Remark 2.8).

Assume that $C \subseteq A^0$ and $C \subseteq B^0$. Since $A^0 \in M(H_0^{1,p})$ we have immediately $C \in M(H_0^{1,p})$, and by Theorem 4.2 we get $D(C) = H_0^{1,p}(\Omega)$. The conclusion follows now from Proposition 5.8. ♦

6. LOCALIZATION AND BOUNDARY CONDITIONS

In the first part of this section we prove the local character of the G-convergence in the class $M_\Omega(\mathbf{R}^n)$. In the second part we study the convergence of solutions to non-homogeneous Dirichlet problems of the form (2.5).

Let Ω' be an open subset of Ω. Besides the topologies w and σ on $H^{1,p}(\Omega)$ and $(L^q(\Omega))^n$ introduced in Section 3, we consider the topologies w' and σ' defined analogously on $H^{1,p}(\Omega')$ and $(L^q(\Omega'))^n$. For every $a \in M_\Omega(\mathbf{R}^n)$ we denote by a' the function of $M_{\Omega'}(\mathbf{R}^n)$ defined by

(6.1) $$a' = a|_{\Omega' \times \mathbf{R}^n} .$$

Then the following localization property holds.

Theorem 6.1. *Let* (a_h) *be a sequence in* $M_\Omega(\mathbf{R}^n)$ *which G-converges to* a *in* $M_\Omega(\mathbf{R}^n)$. *Then* (a_h') *G-converges to* a' *in* $M_{\Omega'}(\mathbf{R}^n)$.

This theorem is an easy consequence of the next result.

Theorem 6.2. *Let* (a_h) *be a sequence in* $M_\Omega(\mathbf{R}^n)$ *which G-converges to* $a \in M_\Omega(\mathbf{R}^n)$. *Then*

(6.2) $$K_{seq}(w' \times \sigma')\text{-}\limsup_{h\to\infty} A'_h \subseteq A' ,$$

where A'_h *and* A' *are the operators in* $M(H^{1,p}(\Omega'))$ *associated to* a'_h *and* a' *by Definition* 2.6.

Proof. By Remark 3.3 it is enough to show that for every subsequence $(a_{\sigma(h)})$ of (a_h) there exists a further subsequence $(a_{\sigma(\tau(h))})$ such that

(6.3) $$K_{seq}(w' \times \sigma')\text{-}\liminf_{h\to\infty} A'_{\sigma(\tau(h))} \subseteq A' .$$

By the definition of G-convergence and by Theorem 4.2 for every subsequence $(a_{\sigma(h)})$ of (a_h) there exists a further subsequence $(a_{\sigma(\tau(h))})$ such that

(6.4) $$C = K_{seq}(w\times\sigma)\text{-}\lim_{h\to\infty} A^0_{\sigma(\tau(h))} \subseteq A^0 ,$$

where $A^0_{\sigma(\tau(h))}$ and A^0 are the operators of $M(H^{1,p}_0(\Omega))$ associated to $a_{\sigma(\tau(h))}$ and a by Definition 2.6. This implies that

(6.5) $$D(K_{seq}(w'\times\sigma')\text{-}\liminf_{h\to\infty} A'_{\sigma(\tau(h))}) \supseteq C^\infty_0(\Omega') .$$

Indeed, let $u'\in C^\infty_0(\Omega')$ and let $u\in C^\infty_0(\Omega)$ such that $u|_{\Omega'} = u'$. Since $D(C) = H^{1,p}_0(\Omega)$ (Theorem 4.2), there exists $g\in (L^q(\Omega))^n$ such that $[u,g]\in C$. Thus, there exists a sequence $[u_h,g_h]$ converging to $[u,g]$ in the topology $w\times\sigma$ such that $[u_h,g_h]\in A^0_{\sigma(\tau(h))}$ for every $h\in N$. It is clear that $[u_h|_{\Omega'},g_h|_{\Omega'}]$ converges to $[u|_{\Omega'},g|_{\Omega'}]$ in the topology $w'\times\sigma'$; therefore $[u',g|_{\Omega'}]\in K_{seq}(w'\times\sigma')\text{-}\liminf_{h\to\infty} A'_{\sigma(\tau(h))}$, which proves (6.5).

Proceeding as in proof of Proposition 4.4 we can also show that $(K_{seq}(w'\times\sigma')\text{-}\liminf_{h\to\infty} A'_{\sigma(\tau(h))})\in M(H^{1,p}(\Omega'))$. Therefore, by Theorem 5.1 there exists $b'\in M_{\Omega'}(R^n)$ such that

(6.6) $$K_{seq}(w'\times\sigma')\text{-}\liminf_{h\to\infty} A'_{\sigma(\tau(h))} \subseteq B' ,$$

where B' denotes the operator of $M(H^{1,p}(\Omega'))$ associated to b' by Definition 2.6. We define $C' = \{[u|_{\Omega'},g|_{\Omega'}] : [u,g]\in C\}$. It is clear that $C'\in M(H^{1,p}(\Omega'))$ and $D(C')\supseteq C^\infty_0(\Omega')$, being $D(C) = H^{1,p}_0(\Omega)$. By (6.4) we have

(6.7) $$C' \subseteq A' .$$

Moreover, without any difficulty it can be shown that

(6.8) $$C' \subseteq K_{seq}(w'\times\sigma')\text{-}\liminf_{h\to\infty} A'_{\sigma(\tau(h))} \subseteq B' .$$

By taking (6.7) and (6.8) into account, Proposition 5.8 guarantees that $a' = b'$. Therefore $A' = B'$ and (6.6) implies (6.3). ♦

The following corollary is an easy consequence of Theorem 6.2.

Corollary 6.3. *Let* (a_h) *and* (b_h) *be sequences in* $M_\Omega(R^n)$ *which G-converge to* a *and* b *, respectively. If* $a'_h = b'_h$ *, then* $a' = b'$.

Let $(\Omega^i)_{i\in I}$ be a family of open subsets of Ω such that $|\Omega\setminus\bigcup_{i\in I}\Omega^i| = 0$. For every $a\in M_\Omega(R^n)$ we denote by a^i the restriction of a to $\Omega^i\times R^n$. The next corollary follows immediately from the compactness Theorem 4.1 and Corollary 6.3.

Corollary 6.4. *A sequence* (a_h) *in* $M_\Omega(R^n)$ *G-converges to* $a\in M_\Omega(R^n)$ *if and only if* (a^i_h) *G-converges to* a^i *in* $M_{\Omega^i}(R^n)$ *for every* $i\in I$.

We now prove the results stated in Section 3 regarding the convergence of solutions to non-homogeneous Dirichlet problems.

Proof of Theorem 3.8.

(i) $\Rightarrow$ (ii). It follows from Theorem 6.2 with $\Omega' = \Omega$.

(ii) $\Rightarrow$ (iii). Let $[u,g]\in K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A^\varphi_h$. By (3.2) there exist a sequence of integers $\sigma(h)\to+\infty$, and a sequence $[u_h,g_h]$ converging to $[u,g]$ in the topology $w\times\sigma$ such that $[u_h,g_h]\in A^\varphi_{\sigma(h)} \subseteq A_{\sigma(h)}$ for every $h\in N$, hence $[u,g]\in A$ by (ii). Since clearly $u-\varphi\in H^{1,p}_0(\Omega)$, we have $[u,g]\in A^\varphi$, which gives (iii).

(iii) $\Rightarrow$ (i). The compactness Theorem 4.1 implies that for every subsequence $(a_{\sigma(h)})$ of (a_h) there exist a further subsequence $(a_{\sigma(\tau(h))})$ of $(a_{\sigma(h)})$ and a function $b\in M_\Omega(R^n)$ such that $(a_{\sigma(\tau(h))})$ G-converges to b . Since (i) implies (iii), we get

(6.9)
$$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A^\varphi_{\sigma(\tau(h))} \subseteq B^\varphi ,$$

where B^φ is the operator of $M(H^{1,p}_\varphi)$ associated to b .
On the other hand, by assumption we have

(6.10)
$$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A^\varphi_{\sigma(\tau(h))} \subseteq A^\varphi .$$

By Corollary 5.9 we deduce that $a(x,\xi) = b(x,\xi)$ for a.e. $x\in\Omega$ and for every $\xi\in R^n$. The proof can now be completed by using the Urysohn axiom (Remark 3.7). ♦

We conclude this section by giving the proof of Theorem 3.11.

Proof of Theorem 3.11. Let us prove (ii). To this aim we show first that the G-convergence of the sequence (a_h) to the function a in $M_\Omega(R^n)$ implies that

(6.11) $$K_{seq}(w\times\rho)\text{-}\lim_{h\to\infty} \mathcal{A}_h^0 = \mathcal{A}^0 .$$

By the definition of G-convergence and by Theorem 4.2 for every subsequence $(a_{\sigma(h)})$ of (a_h) there exists a further subsequence $(a_{\sigma(\tau(h))})$ such that

$$B = K_{seq}(w\times\sigma)\text{-}\lim_{h\to\infty} A^0_{\sigma(\tau(h))} \subseteq A^0 .$$

By Proposition 4.5 this implies that

(6.12) $$\mathcal{B} = K_{seq}(w\times\rho)\text{-}\lim_{h\to\infty} \mathcal{A}^0_{\sigma(\tau(h))} \subseteq \mathcal{A}^0 .$$

Since $R(\mathcal{A}^0_{\sigma(\tau(h))}+\lambda I) = H^{-1,q}(\Omega)$ for every $\lambda\geq 0$ (Theorem 2.7(ii)), it follows that $R(\mathcal{B}+\lambda I) = H^{-1,q}(\Omega)$ (see the proof of Theorem 4.2), hence $\mathcal{B}$ is maximal monotone (Lemma 2.9). Therefore, by the monotonicity of $\mathcal{A}^0$, the inclusion (6.12) implies that

$$K_{seq}(w\times\rho)\text{-}\lim_{h\to\infty} \mathcal{A}^0_{\sigma(\tau(h))} = \mathcal{A}^0$$

and (6.11) follows from the Urysohn property of the K-convergence.

To prove (ii) in the general case $\varphi\in H^{1,p}(\Omega)$, for every $A\in M(H^{1,p})$ we consider the operator A^0_φ of the class $M(H^{1,p}_0)$ defined by

$$A^0_\varphi v = A^\varphi(v+\varphi) \qquad \text{for every } v\in H^{1,p}_0(\Omega) ,$$

and the operator $\mathcal{A}^0_\varphi$ of $\mathcal{M}(H^{1,p}_0)$ associated to A^0_φ by (2.8). By Theorem 3.8 the G-convergence of the sequence (a_h) to the function a in $M_\Omega(R^n)$ can be expressed by

$$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A^\varphi_h \subseteq A^\varphi ,$$

which implies that

(6.13) $$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} (A_h)^0_\varphi \subseteq A^0_\varphi .$$

Since $(A_h)^0_\varphi, A^0_\varphi$ are operators of $M(H^{1,p}_0)$, the inclusion (6.13) implies, as already seen, that

$$K_{seq}(w\times\rho)\text{-}\lim_{h\to\infty} (\mathcal{A}_h)^{\varphi}_{\varphi} = \mathcal{A}^{\varphi}_{\varphi} ,$$

which gives immediately (ii).

Proof of (i). By Theorem 3.8 the G-convergence of (a_h) to a in $M_\Omega(\mathbf{R}^n)$ implies that

$$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A_h \subseteq A .$$

Arguing as in the proof of Proposition 4.5 we obtain

$$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} \mathcal{A}_h \subseteq \mathcal{A} .$$

By (ii) it follows that

$$\mathcal{A}^{\varphi} = K_{seq}(w\times\rho)\text{-}\lim_{h\to\infty} \mathcal{A}^{\varphi}_h \subseteq K_{seq}(w\times\rho)\text{-}\liminf_{h\to\infty} \mathcal{A}_h \subseteq K_{seq}(w\times\rho)\text{-}\limsup_{h\to\infty} \mathcal{A}_h \subseteq \mathcal{A}$$

for every $\varphi\in H^{1,p}(\Omega)$, which yields (i). ♦

7. SOME G-CLOSED CLASSES OF OPERATORS

In this section we consider some subsets of $M_\Omega(\mathbf{R}^n)$, which are closed under G-convergence. These classes are obtained by imposing to the operator a some additional conditions of uniform equicontinuity or strict monotonicity.

Definition 7.1. Given a non-negative function $m\in L^1(\Omega)$ and two constants α and c, with $0\le\alpha\le(p/2)\wedge(p-1)$ and $c>0$, we denote by $U=U(\alpha,c,m)$ the class of all operators $a\in M_\Omega(\mathbf{R}^n)$ such that

$$m(x)+(\eta_1,\xi_1)+(\eta_2,\xi_2)\ \ge 0 \tag{7.1}$$

and

$$|\eta_1-\eta_2| \le c\Phi^{(p-1-\alpha)/p}(\eta_1-\eta_2,\xi_1-\xi_2)^{\alpha/p} \tag{7.2}$$

for a.e. $x\in\Omega$, for every $\xi_1, \xi_2\in \mathbf{R}^n$ and $\eta_1\in a(x,\xi_1)$, $\eta_2\in a(x,\xi_2)$, where $\Phi = \Phi(x,\xi_1,\xi_2,\eta_1,\eta_2)$ denotes the left hand side of (7.1).

Definition 7.2. Given a non-negative function $m \in L^1(\Omega)$ and two constants β and c, with $p \vee 2 \le \beta < +\infty$ and $c > 0$, we denote by $S = S(\beta,c,m)$ the class of all operators $a \in M_\Omega(\mathbf{R}^n)$ such that

$$m(x) + (\eta_1,\xi_1) + (\eta_2,\xi_2) \ge 0$$

and

$$(7.3) \qquad (\eta_1 - \eta_2, \xi_1 - \xi_2) \ge c\Phi^{(p-\beta)/p}|\xi_1 - \xi_2|^\beta$$

for a.e. $x \in \Omega$, for every $\xi_1, \xi_2 \in \mathbf{R}^n$ and $\eta_1 \in a(x,\xi_1)$, $\eta_2 \in a(x,\xi_2)$, where $\Phi = \Phi(x,\xi_1,\xi_2,\eta_1,\eta_2)$ denotes the left hand side of (7.1).

Remark 7.3. Conditions (2.1) and (2.2) imply that there exists a non-negative function $m \in L^1(\Omega)$ such that (7.1) holds for every $a \in M_\Omega(\mathbf{R}^n)$.

Moreover, by (7.2) every function a of the class U is single-valued.

Remark 7.4. By using the estimates (2.1) and (2.2) it is easy to see that, if $0 \le \alpha' \le \alpha \le (p/2) \wedge (p-1)$, then $U(\alpha,c,m) \subseteq U(\alpha',c',m')$ for suitable c' and m'. In the same way it can be proved that, if $p \vee 2 \le \beta \le \beta' < +\infty$, then $S(\beta,c,m) \subseteq S(\beta',c',m')$ for suitable c' and m'.

The model example of operator of the classes U and S is given by

$$a(x,\xi) = b(x)|\xi|^{p-2}\xi \ .$$

Indeed, if $0 < b_1 \le b(x) \le b_2 < +\infty$ for every $x \in \Omega$, then

$$a \in U(\tfrac{p}{2} \wedge (p-1), c', m') \cap S(p \vee 2, c'', m'')$$

for suitable c', c", m', and m".

Before proving that the classes U and S are closed under G-convergence we compare them with some other classes of monotone operators which are not closed, but are defined in a simpler way.

Definition 7.5. Given a non-negative function $m \in L^p(\Omega)$ and two constants α and c, with $0 \le \alpha \le 1 \wedge (p-1)$ and $c > 0$, we denote by $U^* = U^*(\alpha,c,m)$ the class of all single-valued operators $a \in M_\Omega(\mathbf{R}^n)$ such that

(7.4) $$|a(x,\xi_1) - a(x,\xi_2)| \le c(m(x) + |\xi_1| + |\xi_2|)^{p-1-\alpha}|\xi_1 - \xi_2|^{\alpha}$$

for a.e. $x \in \Omega$ and for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

Definition 7.6. Given a non-negative function $m \in L^p(\Omega)$ and two constants β and c, with $p \vee 2 \le \beta < +\infty$ and $c > 0$, we denote by $S^* = S^*(\beta,c,m)$ the class of all operators $a \in M_\Omega(\mathbf{R}^n)$ such that

(7.5) $$(\eta_1 - \eta_2, \xi_1 - \xi_2) \ge c(m(x) + |\xi_1| + |\xi_2|)^{p-\beta}|\xi_1 - \xi_2|^{\beta}$$

for a.e. $x \in \Omega$, for every $\xi_1, \xi_2 \in \mathbf{R}^n$ and $\eta_1 \in a(x,\xi_1)$, $\eta_2 \in a(x,\xi_2)$.

Remark 7.7. From (2.3) we obtain that

(7.6) $$U(\alpha,c,m) \subseteq U^*(\frac{\alpha}{p-\alpha}, c', m')$$

for suitable c' and m'. Conversely, given c', c", m', and m", from (2.4) it follows that

(7.7) $$U^*(\alpha,c',m') \cap S^*(\beta,c'',m'') \subseteq U(\frac{\alpha p}{\beta}, c, m)$$

for suitable c and m. Moreover, given c and m, we have

(7.8) $$S(\beta,c,m) \subseteq S^*(\beta,c',m')$$

(7.9) $$S^*(\beta,c,m) \subseteq S(\beta,c'',m'')$$

for suitable c', c", m', and m".

In particular, if $2 \le p < +\infty$, (7.6) and (7.7) imply

(7.10) $$U(\frac{p}{2}, c, m) \subseteq U^*(1,c',m')$$

(7.11) $$U^*(1,c',m') \cap S^*(p,c'',m'') \subseteq U(1,c,m) \ .$$

Finally, if $1 < p \le 2$, (7.6) and (7.7) yield

(7.12) $$U(p-1,c,m) \subseteq U^*(p-1,c',m')$$

(7.13) $$U^*(p-1,c',m') \cap S^*(2,c'',m'') \subseteq U(\frac{(p-1)p}{2}, c, m) \ .$$

The following lemma is crucial in the proof of the closedness of the classes U and S .

Lemma 7.8. *Let* γ *and* δ *be two non-negative constants with* $\gamma+\delta\le 1$. *Let* ψ , ζ , θ *be functions in* $L^1(\Omega)$ *and let* (ψ_h) , (ζ_h) , (θ_h) *be sequences in* $L^1(\Omega)$ *converging to* ψ , ζ , θ *in the weak sense of distributions. Suppose that* $\zeta_h\ge 0$, $\theta_h\ge 0$, *and*

(7.14) $$|\psi_h|\le(\zeta_h)^{\gamma}(\theta_h)^{\delta}\quad a.e.\ in\ \Omega\ .$$

Then

(7.15) $$|\psi|\le\zeta^{\gamma}\theta^{\delta}\quad a.e.\ in\ \Omega\ .$$

Proof. Let $\varepsilon=1-\gamma-\delta$. By (7.14), for every $\varphi\in C_0^{\infty}(\Omega)$, $\varphi\ge 0$ we have

(7.16) $$\int_\Omega|\psi_h|\varphi dx\le\left(\int_\Omega\zeta_h\varphi dx\right)^{\gamma}\left(\int_\Omega\theta_h\varphi dx\right)^{\delta}\left(\int_\Omega\varphi dx\right)^{\varepsilon}\ .$$

Since $(\psi_h\varphi)$ converges to $\psi\varphi$ in the weak sense of distributions we obtain

$$\int_\Omega|\psi|\varphi dx\le\liminf_{h\to+\infty}\int_\Omega|\psi_h|\varphi dx\ .$$

Therefore, by taking the limit in (7.16) as $h\to+\infty$ we get

(7.17) $$\int_\Omega|\psi|\varphi dx\le\left(\int_\Omega\zeta\varphi dx\right)^{\gamma}\left(\int_\Omega\theta\varphi dx\right)^{\delta}\left(\int_\Omega\varphi dx\right)^{\varepsilon}$$

for every $\varphi\in C_0^{\infty}(\Omega)$, $\varphi\ge 0$. By standard approximation argument we obtain (7.17) for every for every $\varphi\in L^{\infty}(\Omega)$, $\varphi\ge 0$. In particular, we have

$$\int_{B_\rho(x)}|\psi|\varphi dx\le\left(\int_{B_\rho(x)}\zeta\varphi dx\right)^{\gamma}\left(\int_{B_\rho(x)}\theta\varphi dx\right)^{\delta}\left(\int_{B_\rho(x)}\varphi dx\right)^{\varepsilon}$$

for every $x\in\Omega$ and $\rho\ge 0$ small enough and this implies (7.15) by the Lebesgue derivation theorem. ♦

Theorem 7.9. *The classes* U *and* S *are closed under* G*-convergence.*

Proof. Let us fix α, c, and m as in Definition 7.1. Let (a_h) be a sequence in $U(\alpha,c,m)$ which G-converges to a function $a \in M_\Omega(\mathbf{R}^n)$. We have to prove that $a \in U(\alpha,c,m)$. By hypothesis we have

$$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A_h^0 \subseteq A^0 ,$$

where A_h^0 and A^0 are the operators of $M(H_0^{1,p})$ associated to a_h, a by Definition 2.6. By Theorem 4.2 there exists a subsequence of (a_h), still denoted by (a_h), such that

$$B = K_{seq}(w\times\sigma)\text{-}\lim_{h\to\infty} A_h^0 \subseteq A^0 . \tag{7.18}$$

As in the proof of Theorem 5.1 we introduce the set E defined by

$$E = \{[Du,g] \in (L^p(\Omega))^n \times (L^q(\Omega))^n : g \in Bu\}$$

and we denote by decE the smallest decomposable subset of $(L^p(\Omega))^n \times (L^q(\Omega))^n$ containing E. Moreover, we consider the set

$$\tilde{E} = cl_{s\times w}(decE) ,$$

defined as the closure of decE in $(L^p(\Omega))^n \times (L^q(\Omega))^n$, with $(L^p(\Omega))^n$ endowed with its strong topology and $(L^q(\Omega))^n$ endowed with its weak topology. As in the proof of Proposition 5.8 it follows that

$$\tilde{E} = \{[\varphi,g] \in (L^p(\Omega))^n \times (L^q(\Omega))^n : g(x) \in a(x,\varphi(x)) \text{ a.e. in } \Omega\} . \tag{7.19}$$

We are now in a position to prove (7.1) and (7.2). This will be done in three steps.

Step 1. If $[u^1,g^1], [u^2,g^2] \in B$, then

$$|g^1 - g^2| \le c\zeta^{(p-1-\alpha)/p}(g^1 - g^2, Du^1 - Du^2)^{\alpha/p} \tag{7.20}$$

a.e. on Ω, where

$$\zeta = m + (g^1,Du^1) + (g^2,Du^2) \ge 0 .$$

Step 2. If $[\varphi^1,g^1], [\varphi^2,g^2] \in \tilde{E}$, then

$$|g^1 - g^2| \le c\omega^{(p-1-\alpha)/p}(g^1 - g^2, \varphi^1 - \varphi^2)^{\alpha/p} \tag{7.21}$$

a.e. on Ω, where

$$\omega = m + (g^1,\varphi^1) + (g^2,\varphi^2) \geq 0 .$$

Step 3. The inequalities (7.1) and (7.2) holds for a.e. $x\in\Omega$, for every $\xi^1, \xi^2 \in R^n$ and $\eta^1\in a(x,\xi^1)$, $\eta^2\in a(x,\xi^2)$.

Proof of Step 1. Let $[u^i,g^i]\in B$, $i = 1,2$. By (7.18) there exists a sequence $[u^i_h,g^i_h]$ converging to $[u^i,g^i]$ in the topology $w\times\sigma$, such that $[u^i_h,g^i_h]\in A^0_h$ for every $h\in N$. Since $a_h\in U(\alpha,c,m)$ we have

$$|g^1_h - g^2_h| \leq c\zeta_h^{(p-1-\alpha)/p}(g^1_h - g^2_h,Du^1_h - Du^2_h)^{\alpha/p} ,$$

where

$$\zeta_h = m + (g^1_h,Du^1_h) + (g^2_h,Du^2_h) \geq 0 .$$

Let us define

$$\psi_h = g^1_h - g^2_h \ , \qquad \psi = g^1 - g^2 \ ,$$

$$\theta_h = (g^1_h - g^2_h,Du^1_h - Du^2_h) , \qquad \theta = (g^1 - g^2,Du^1 - Du^2) .$$

By Lemma 3.4 (ζ_h) converges to ζ and (θ_h) converges to θ weakly in the sense of distributions. Therefore $\zeta \geq 0$ a.e. in Ω and Lemma 7.8 yields

$$|\psi| \leq \zeta^{(p-1-\alpha)/p}\theta^{\alpha/p} \qquad \text{a.e. in } \Omega ,$$

proving (7.20). •

Proof of Step 2. The result of Step 1 can be reformulated by saying that (7.21) holds for $[\varphi^i,g^i]\in E$. The characterization of decE mentioned in the proof of Theorem 5.1 implies (7.21) for $[\varphi^i,g^i]\in \text{decE}$. Let us prove the same property for $\tilde{E}$. Let $[\varphi^i,g^i]\in \tilde{E}$, $i = 1,2$. By Proposition 5.6(a) there exists a sequence $[\varphi^i_h,g^i_h]\in \text{decE}$ such that (φ^i_h) converges to φ^i strongly in $(L^p(\Omega))^n$ and (g^i_h) converges to g^i weakly in $(L^q(\Omega))^n$. Since (7.21) holds on decE , we have

$$|g^1_h - g^2_h| \leq c\omega_h^{(p-1-\alpha)/p}(g^1_h - g^2_h,\varphi^1_h - \varphi^2_h)^{\alpha/p} ,$$

where

$$\omega_h = m + (g^1_h,\varphi^1_h) + (g^2_h,\varphi^2_h) \geq 0 .$$

By applying Lemma 7.8 to

$$\psi_h = g_h^1 - g_h^2 \ , \qquad \text{and} \qquad \theta_h = (g_h^1 - g_h^2, \varphi_h^1 - \varphi_h^2)$$

we obtain (7.21) for $[\varphi^1, g^1]$ and $[\varphi^2, g^2]$. Moreover $\omega \geq 0$ a.e. in Ω, being $\omega_h \geq 0$ a.e. in Ω for $h \in \mathbf{N}$.

•

Proof of Step 3. Let us denote by M the set of all $x \in \Omega$ such that (7.2) is not satisfied for some $\xi^1, \xi^2, \eta^1, \eta^2$ with $\eta^1 \in a(x,\xi^1)$, $\eta^2 \in a(x,\xi^2)$. We have to prove that $|M| = 0$. To this aim we write $M = \{x \in \Omega: Gx \neq \emptyset\}$, where $G : \Omega \to (\mathbf{R}^n)^4$ is the multivalued function defined by

$$Gx = \{[\xi^1, \xi^2, \eta^1, \eta^2] : |\eta_1 - \eta_2| > c\Phi^{(p-1-\alpha)/p}(\eta_1 - \eta_2, \xi_1 - \xi_2)^{\alpha/p}, \eta^i \in a(x,\xi^i), i=1,2\} ,$$

where $\Phi = m + (\eta^1,\xi^1) + (\eta^2,\xi^2)$. By Remark 2.3 the graph of G belongs to $\mathcal{L}(\Omega) \otimes \mathcal{B}(\mathbf{R}^n)^4$, thus $M \in \mathcal{L}(\Omega)$ by the projection Theorem 1.2. By the Aumann-von Neumann Theorem 1.4 there exists a measurable selection $[\varphi_0^1, \varphi_0^2, g_0^1, g_0^2]$ of G defined on M. Therefeore, for every $x \in M$ we have

(7.22) $$|g_0^1 - g_0^2| > c[m + (g_0^1, \varphi_0^1) + (g_0^2, \varphi_0^2)]^{(p-1-\alpha)/p}(g_0^1 - g_0^2, \varphi_0^1 - \varphi_0^2)^{\alpha/p}$$

and

(7.23) $$g_0^i(x) \in a(x, \varphi_0^i(x)) \quad , \quad i = 1,2 \ .$$

If $|M| > 0$, there exists a measurable subset M' of M with positive measure such that $[\varphi_0^1, \varphi_0^2, g_0^1, g_0^2]$ is bounded on M'. By Step (a) in the proof of Theorem 2.7 there exists $g_* \in (L^q(\Omega))^n$ such that

(7.24) $$g_*(x) \in a(x,0) \quad \text{a.e. in } \Omega \ .$$

For $i = 1,2$, we define

(7.25) $$\varphi^i(x) = \begin{cases} \varphi_0^i(x) & \text{if } x \in M' , \\ 0 & \text{if } x \in \Omega \setminus M' , \end{cases}$$

(7.26) $$g^i(x) = \begin{cases} g_0^i(x) & \text{if } x \in M' , \\ g_*(x) & \text{if } x \in \Omega \setminus M' . \end{cases}$$

Then $[\varphi^i, g^i] \in (L^p(\Omega))^n \times (L^q(\Omega))^n$ and $g^i(x) \in a(x,\varphi^i(x))$ a.e. in Ω by (7.23) and (7.24). Therefore $[\varphi^i, g^i] \in \tilde{E}$ by (7.19), hence

$$|g_0^1 - g_0^2| \leq c[m + (g_0^1,\varphi_0^1) + (g_0^2,\varphi_0^2)]^{(p-1-\alpha)/p}(g_0^1 - g_0^2,\varphi_0^1 - \varphi_0^2)^{\alpha/p} \qquad \text{a.e. in } M'$$

by Step 2. This contradicts (7.22) being $|M'| > 0$. Therefore, we have to conclude that M has Lebesgue measure 0, which proves that (7.2) holds for a.e. $x \in \Omega$.

The proof of (7.1) is analogous, therefore the class $U(\alpha,c,m)$ is closed with respect to G-convergence. •

To prove that the class $S(\beta,c,m)$, $p \vee 2 \leq \beta < +\infty$, is closed, we note that (7.3) is equivalent to

$$|\xi_1 - \xi_2| \leq c\Phi^{(\beta-p)/\beta p}(\eta_1 - \eta_2,\xi_1 - \xi_2)^{1/\beta}$$

and the proof can be concluded as in the case $U(\alpha,c,m)$. ♦

Theorem 7.9 and Remark 7.7 allows us to obtain some compactness results concerning the classes U^* and S^*.

Corollary 7.10. *Assume* $p = 2$. *Given two non-negative functions* m', $m'' \in L^2(\Omega)$ *and two constants* $c' > 0$, $c'' > 0$, *there exist two non-negative functions* $\tilde{m}'$, $\tilde{m}'' \in L^2(\Omega)$ *and two constants* $\tilde{c}' > 0$, $\tilde{c}'' > 0$ *with the following property: if*

$$a_h \in U^*(1,c',m') \cap S^*(2,c'',m'')$$

for every $h \in N$, *then there exists a subsequence* $(a_{\sigma(h)})$ *of* (a_h) *which* G*-converges to a function*

$$a \in U^*(1,\tilde{c}',\tilde{m}') \cap S^*(2,\tilde{c}'',\tilde{m}'') .$$

The same result was obtained by different methods by L. Tartar in [73].

Corollary 7.11. *Assume* $1 < p \leq 2$. *Given two non-negative functions* m', $m'' \in L^p(\Omega)$ *and two constants* $c' > 0$, $c'' > 0$, *there exist two non-negative functions* $\tilde{m}'$, $\tilde{m}'' \in L^p(\Omega)$ *and two constants* $\tilde{c}' > 0$, $\tilde{c}'' > 0$ *with the following property: if*

$$a_h \in U^*(p-1,c',m') \cap S^*(2,c'',m'')$$

for every $h\in N$, *then there exists a subsequence* $(a_{\sigma(h)})$ *of* (a_h) *which G-converges to a function*

$$a\in U^*(\tfrac{p-1}{3-p},\tilde{c}',\tilde{m}')\cap S^*(2,\tilde{c}'',\tilde{m}'') \ .$$

A similar result was obtained by N. Fusco and G. Moscariello in the case of the homogeneization (see [38], [39]).

Corollary 7.12. *Assume* $2\le p<+\infty$ *and* $0\le\alpha\le 1$. *Given two non-negative functions* $m', m''\in L^p(\Omega)$ *and two constants* $c'>0$, $c''>0$, *there exist two non-negative functions* $\tilde{m}'$, $\tilde{m}''\in L^p(\Omega)$ *and two constants* $\tilde{c}'>0$, $\tilde{c}''>0$ *with the following property: if*

$$a_h\in U^*(\alpha,c',m')\cap S^*(p,c'',m'')$$

for every $h\in N$, *then there exists a subsequence* $(a_{\sigma(h)})$ *of* (a_h) *which G-converges to a function*

$$a\in U^*(\frac{\alpha}{p-\alpha},\tilde{c}',\tilde{m}')\cap S^*(p,\tilde{c}'',\tilde{m}'') \ .$$

Compare this result with those obtained by U. E. Raitum in [59]. We refer also to [38], [39] for the case $\alpha = 1$.

Definition 7.13. By $L(c_1,c_2)$ we denote the class of all operators $a:\Omega\times\mathbf{R}^n\to\mathbf{R}^n$ of the form

$$a(x,\xi) = a(x)\xi \qquad \text{for a.e. } x\in\Omega\text{ , for every } \xi\in\mathbf{R}^n \ ,$$

where $a(x) = (a_{ij}(x))$ is a n×n-matrix of bounded measurable functions such that

(7.27) $$|a(x)\xi|^2 \le c_1(a(x)\xi,\xi) \ ,$$

(7.28) $$|\xi|^2 \le c_2(a(x)\xi,\xi)$$

for a.e. $x\in\Omega$ and for every $\xi\in\mathbf{R}^n$.

By $L_{sym}(c_1,c_2)$ we denote the class of all operators of $L(c_1,c_2)$ corresponding to a symmetric matrix $(a_{ij}(x))$.

Remark 7.14. It is easy to see that $L(c_1,c_2)$ is the set of all operators $a \in M_\Omega(\mathbf{R}^n)$, with $p=2$, such that for a.e. $x \in \Omega$ the multivalued function $a(x,\cdot)$ is linear, i.e. its graph is a linear subspace of $\mathbf{R}^n \times \mathbf{R}^n$.

Theorem 7.15. *The classes* $L(c_1,c_2)$ *and* $L_{sym}(c_1,c_2)$ *are closed under* G-*convergence.*

Proof. We give a sketch of the proof only for $L(c_1,c_2)$, the case of $L_{sym}(c_1,c_2)$ being analogous. By arguing as in the proof of Theorem 7.9, for which we refer for the notation, the result will be achieved in three steps.

Step 1. B is a linear subspace of $H_0^{1,2}(\Omega) \times (L^2(\Omega))^n$.

Step 2. $\tilde{E}$ is a linear subspace of $(L^2(\Omega))^n \times (L^2(\Omega))^n$.

Step 3. For a.e. $x \in \Omega$, the multivalued function $a(x,\cdot)$ is linear.

The proof of each step is completely analogous to the proof of the corresponding step in Theorem 7.9, and is therefore omitted. ♦

The compactness under G-convergence of the class $L(c_1,c_2)$ was proved by different methods by F. Murat and L. Tartar in [53] and [70]. The symmetric case was studied earlier by S. Spagnolo and E. De Giorgi (see [66], [67], [68], [36]).

CHAPTER 2

HOMOGENIZATION OF MONOTONE OPERATORS

In this chapter we deal with the homogenization of a sequence of nonlinear monotone operators $\mathcal{A}_h : H_0^{1,p}(\Omega) \rightarrow H^{-1,q}(\Omega)$ of the form $\mathcal{A}_h u = -\mathrm{div}(a(\frac{x}{\varepsilon_h}, Du))$. We prove a representation formula for the homogenized operator under the assumptions that $a = a(y,\xi)$ is periodic in y, maximal monotone in ξ for almost every $y \in \mathbf{R}^n$, and satisfies suitable coerciveness and boundedness conditions. The results of this chapter are published in [26].

INTRODUCTION

In this chapter we deal with the homogenization of a sequence of nonlinear monotone operators $\mathcal{A}_h : H_0^{1,p}(\Omega) \rightarrow H^{-1,q}(\Omega)$ of the form

(0.1) $$\mathcal{A}_h u = -\mathrm{div}(a(\frac{x}{\varepsilon_h}, Du)) ,$$

where Ω is a bounded open subset of $\mathbf{R}^n$, $1 < p < +\infty$, $1/p + 1/q = 1$, and (ε_h) is a sequence of positive numbers converging to 0. The (possibly multivalued) map a, which appears in (0.1), is defined on $\mathbf{R}^n \times \mathbf{R}^n$ and takes its values $a(y,\xi)$ in $\mathbf{R}^n$. We assume that a is periodic in y, measurable in $[y,\xi]$, maximal monotone in ξ for almost every $y \in \mathbf{R}^n$, and satisfies suitable coerciveness and boundedness conditions (see Section 1).

To present our main results, we begin with the simpler case where a is single-valued and strictly monotone, i.e.

$$(a(y,\xi_1) - a(y,\xi_2), \xi_1 - \xi_2) > 0$$

for a.e. $y \in \mathbf{R}^n$, for every $\xi_i \in \mathbf{R}^n$, $\xi_1 \neq \xi_2$. By using the general notion of G-convergence for sequences of maximal monotone operators of type (0.1), which has been introduced in Chapter 1, we prove the convergence, as ε_h tends to 0, of the solutions u_h and of the momenta $a(\frac{x}{\varepsilon_h}, Du_h)$ of the Dirichlet boundary value problem

$$(0.2)\qquad \begin{cases} -\mathrm{div}(a(\frac{x}{\varepsilon_h}, Du_h)) = f & \text{on } \Omega\ , \\ u_h \in H_0^{1,p}(\Omega) \end{cases}$$

to the solution u and to the momentum b(Du) of the homogenized problem

$$(0.3)\qquad \begin{cases} -\mathrm{div}(b(Du)) = f \quad \text{on } \Omega, \\ u \in H_0^{1,p}(\Omega)\ . \end{cases}$$

The operator $b : \mathbf{R}^n \to \mathbf{R}^n$ in (0.3) turns out to be defined by

$$b(\xi) = \int_Y a(y, Dv(y)+\xi)dy \quad ,$$

where v is the solution to the following local problem on the unit cube Y of $\mathbf{R}^n$:

$$(0.4)\qquad \begin{cases} -\mathrm{div}\, a(y, Dv(y)+\xi) = 0 & \text{on } Y\ , \\ v \in H_\#^{1,p}(Y)\ . \end{cases}$$

By $H_\#^{1,p}(Y)$ we denote the subset of $H^{1,p}(Y)$ of all the functions u with mean value zero which have the same trace on the opposite faces of Y .

The same results have been obtained in [70], [69] for $p = 2$ under the additional assumptions of uniform Lipschitz-continuity and uniform strict monotonicity for a , i.e.

$$(0.5)\qquad |a(y,\xi_1) - a(y,\xi_2)| \le \Lambda|\xi_1 - \xi_2| \quad ,$$

$$(0.6)\qquad (a(y,\xi_1) - a(y,\xi_2), \xi_1 - \xi_2) \ge \lambda|\xi_1 - \xi_2|^2$$

for a.e. $y \in \mathbf{R}^n$, for every ξ_1, $\xi_2 \in \mathbf{R}^n$, and $0 < \lambda \le \Lambda < +\infty$.

The case $1 < p < +\infty$ has been studied under analogous hypotheses of equicontinuity and uniform strict monotonicity for a by N. Fusco and G. Moscariello in [38] and [39].

In the present chapter, besides the generalization of the results just mentioned, which permits to weaken the assumptions (0.5)-(0.6), we analyse the general case where a is multivalued. More precisely, we study the asymptotic behaviour, as (ε_h) tends to 0 , of the solutions u_h and the momenta g_h to the Dirichlet boundary value problem

$$(0.7)\qquad \begin{cases} g_h(x) \in a(\frac{x}{\varepsilon_h}, Du_h(x)) & \text{for a.e. } x \in \Omega \ , \\ -\operatorname{div} g_h = f \ , \\ u_h \in H_0^{1,p}(\Omega) \ . \end{cases}$$

We prove that, up to a subsequence, (u_h) and (g_h) converge to a solution u and to a momentum g of the homogenized problem

$$(0.8)\qquad \begin{cases} g(x) \in b(Du(x)) & \text{for a.e. } x \in \Omega \ , \\ -\operatorname{div} g = f \ , \\ u \in H_0^{1,p}(\Omega) \ . \end{cases}$$

The operator $b : \mathbf{R}^n \to \mathbf{R}^n$ is now multivalued and for every $\xi \in \mathbf{R}^n$ the set $b(\xi)$ is defined by means of the solutions to the following local problem on the unit cube Y of $\mathbf{R}^n$:

$$(0.9)\qquad \begin{cases} v \in H_{\#}^{1,p}(Y) \ , \ k \in (L^q(Y))^n \ , \\ k(y) \in a(y, Dv(y)+\xi) & \text{for a.e. } y \in Y \ , \\ \int_Y (k(y), Dw(y))dy = 0 & \text{for every } w \in H_{\#}^{1,p}(Y) \ . \end{cases}$$

More precisely,

$$b(\xi) = \{\upsilon \in \mathbf{R}^n : \upsilon = \int_Y k(y)dy \ , v \text{ and } k \text{ satisfying (0.9)}\}.$$

This "multivalued approach" finds its motivation in the fact that, under our general assumptions, the additional hypothesis on a to be single-valued is not enough to ensure the same property for b. An example of this phenomenon is illustrated in Section 4 together with some special cases in which b turns out to be strictly monotone and single-valued.

The homogenization problems have been investigated in recent years by several authors. For a wide bibliography on this topic we address the reader to the books [10], [61], and [6], where one can find also the physical motivation to this research.

For the homogenization of linear elliptic operators of the form $-\operatorname{div}(a(\frac{x}{\varepsilon_h})Du)$ we refer to [36]. The case where a is just almost-periodic has been treated in [44]. Analogous results for the quasi-linear case have been stated in [11], [13], and [14].

On the other hand, the homogenization of a class of variational integrals of the form

(0.10)
$$\Psi_h(u) = \int_\Omega \psi(\tfrac{x}{\varepsilon_h}, Du)\,dx \ ,$$

which is related to the homogenization of the operators $-\mathrm{div}(\partial_\xi\psi(\frac{x}{\varepsilon_h},\xi))$, has been studied in [49] and in [21] using the techniques of Γ-convergence introduced by E. De Giorgi. Homogenization results for functionals of type (0.10) under almost periodicity assumptions have been proven in [17] and in [18].

1. NOTATION AND PRELIMINARY RESULTS

Let p be a real constant, $1 < p < +\infty$, and let q be its dual exponent, $1/p + 1/q = 1$. For every open subset U of $\mathbf{R}^n$ we denote by $\mathcal{L}(U)$ the σ-field of all Lebesgue measurable subsets of U , and by $\mathcal{B}(\mathbf{R}^n)$ the σ-field of all Borel subsets of $\mathbf{R}^n$. The Euclidean norm and the scalar product in $\mathbf{R}^n$ are denoted by $|\cdot|$ and $(\cdot,\cdot)$, respectively. If x , y are elements of a set X , by $[x,y]$ we indicate the ordered pair formed by x and y . Finally, let us fix two constants $m_1 \geq 0$, $m_2 \geq 0$, and two constants $c_1 > 0$, $c_2 > 0$.

In order to study the homogenization of a class of nonlinear multivalued monotone operators of the type $-\mathrm{div}(a(x,Du))$ we recall the definition of the class $M_U(\mathbf{R}^n)$ introduced in Chapter 1, together with some related results which will be used in the sequel.

Definition 1.1. By $M(\mathbf{R}^n)$ we denote the class of all multivalued functions $a : \mathbf{R}^n \to \mathbf{R}^n$ which satisfy the following conditions:

(i) a is maximal monotone;

(ii) the estimates

(1.1) $$|\eta|^q \leq m_1 + c_1(\eta,\xi) \ ,$$

(1.2) $$|\xi|^p \leq m_2 + c_2(\eta,\xi)$$

hold for every $\xi \in \mathbf{R}^n$, and $\eta \in a(\xi)$.

For every open subset U of $\mathbf{R}^n$, by $M_U(\mathbf{R}^n)$ we denote the class of all multivalued functions $a : U \times \mathbf{R}^n \to \mathbf{R}^n$ with closed values which satisfy the following conditions:

(iii) for a.e. $y \in U$, $a(y,\cdot) \in M(\mathbf{R}^n)$;

(iv) a is measurable with respect to $\mathcal{L}(U) \otimes \mathcal{B}(\mathbf{R}^n)$ and $\mathcal{B}(\mathbf{R}^n)$, i.e.
$$a^{-1}(C) = \{[y,\xi] \in U \times \mathbf{R}^n : a(y,\xi) \cap C \neq \emptyset\} \in \mathcal{L}(U) \otimes \mathcal{B}(\mathbf{R}^n)$$
for every closed set $C \subseteq \mathbf{R}^n$.

Remark 1.2. Let us note that the conditions above ensure that for a.e. $y \in U$ and for every $\xi \in \mathbf{R}^n$ the set $a(y,\xi)$ is non-empty, closed and convex in $\mathbf{R}^n$ (see Chapter 1, Section 2). Note also that, if $b \in M(\mathbf{R}^n)$ and $a(y,\xi)=b(\xi)$ for every $y \in U$ and $\xi \in \mathbf{R}^n$, then $a \in M_U(\mathbf{R}^n)$. In fact, the measurability condition (iv) is satisfied since, by the maximal monotonicity of b, the graph of b is closed. Therefore, for every closed subset C of $\mathbf{R}^n$, we have

$$b^{-1}(C) = \mathrm{pr}(\mathrm{graph}\, b \cap (\mathbf{R}^n \times C)) \in \mathcal{B}(\mathbf{R}^n) ,$$

where $\mathrm{pr} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is the projection defined by $\mathrm{pr}([x,y]) = x$.

Moreover, for every $a \in M(\mathbf{R}^n)$ the inverse (possibly multivalued) operator a^{-1} still belongs to $M(\mathbf{R}^n)$ (see, for instance, [58], Section III.2.3). Finally, if $a \in M_U(\mathbf{R}^n)$ the inverse operator a^{-1}, defined by $\xi \in a^{-1}(y,\eta)$ if and only if $\eta \in a(y,\xi)$, also belongs to $M_U(\mathbf{R}^n)$. In fact, the measurability property for a^{-1} can be obtained by using Theorem 1.3 in Chapter 1.

Let us fix from now on a bounded open subset Ω of $\mathbf{R}^n$. To every $a \in M_\Omega(\mathbf{R}^n)$ we associate the multivalued operator $A : H_0^{1,p}(\Omega) \to (L^q(\Omega))^n$ given by

$$Au = \{g \in (L^q(\Omega))^n : g(x) \in a(x,Du(x)) \text{ for a.e. } x \in \Omega\} . \tag{1.3}$$

It can be shown that for every $f \in H^{-1,q}(\Omega)$ there exist $u \in H_0^{1,p}(\Omega)$ and $g \in (L^q(\Omega))^n$ solving the following Dirichlet boundary value problem

$$\begin{cases} g \in Au , \\ -\mathrm{div}\, g = f , \\ u \in H_0^{1,p}(\Omega) . \end{cases} \tag{1.4}$$

For a proof we refer to Chapter 1, Section 2.

In order to study the behaviour of problem (1.4) under perturbations of the operator a the notion of G-convergence in $M_\Omega(\mathbf{R}^n)$ has been introduced in Chapter 1. In this chapter such convergence, which is based on the set convergence in the sense of Kuratowski (see [45], Section 29), will be applied to the case of homogenization.

Let us first recall the Kuratowski convergence defined in abstract terms in an arbitrary topological space (X,τ) as follows.

Definition 1.3. Let (E_h) be a sequence of subsets of X. We define the *sequential lower limit* and the *sequential upper limit of* (E_h) by

(1.5) $$K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_h = \{u\in X : \exists\, u_h \overset{\tau}{\to} u\,,\ \exists\, k\in N\,,\ \forall\, h\ge k : u_h\in E_h\}\,,$$

and

(1.6) $$K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h = \{u\in X : \exists\, \sigma(h)\to+\infty\,,\ \exists\, u_h \overset{\tau}{\to} u\,,\ \forall\, h\in N : u_h\in E_{\sigma(h)}\}\,.$$

Then, we say that the sequence (E_h) $K_{seq}(\tau)$*-converges to a set* E *in* X if

(1.7) $$K_{seq}(\tau)\text{-}\liminf_{h\to\infty} E_h = K_{seq}(\tau)\text{-}\limsup_{h\to\infty} E_h = E\,,$$

and in this case we write $K_{seq}(\tau)\text{-}\lim_{h\to\infty} E_h = E$.

By w we denote the weak topology on $H^{1,p}(\Omega)$. Moreover, if σ_1 denotes the weak topology of $(L^q(\Omega))^n$ and σ_2 the topology on $(L^q(\Omega))^n$ induced by the pseudo-metric $d(g_1,g_2) = \|\mathrm{div} g_1 - \mathrm{div} g_2\|_{H^{-1,q}}$, we denote by σ the weakest topology on $(L^q(\Omega))^n$ which is stronger than σ_1 and σ_2. In other words, (g_h) converges to g in σ if and only if (g_h) converges to g weakly in $(L^q(\Omega))^n$ and $(-\mathrm{div} g_h)$ converges to $-\mathrm{div} g$ strongly in $H^{-1,q}(\Omega)$.

Having in mind the usual identification of a multivalued map with its graph we recall the definition of G-convergence.

Definition 1.4. We say that a sequence (a_h) in $M_\Omega(R^n)$ *G-converges to* $a\in M_\Omega(R^n)$ if

$$K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A_h \subseteq A\,,$$

where A_h and A are the operators from $H_0^{1,p}(\Omega)$ into $(L^q(\Omega))^n$ associated to a_h and a by (1.3).

Although the above definition requires only an inclusion, it determines the G-limit uniquely, as stated in the following theorem.

Theorem 1.5. *Let* (a_h) *be a sequence of functions of the class* $M_\Omega(R^n)$, *and let*

$$C = K_{seq}(w\times\sigma)\text{-}\limsup_{h\to\infty} A_h\,,$$

where A_h *are the operators associated to* a_h *by* (1.3). *Let* a *and* b *be two functions of the class* $M_\Omega(\mathbf{R}^n)$ *and let* A *and* B *be the corresponding operators. If* $C \subseteq A$ *and* $C \subseteq B$ *, then* $a(x,\xi) = b(x,\xi)$ *for a.e.* $x \in \Omega$ *and for every* $\xi \in \mathbf{R}^n$.

For the proof of this theorem we refer to Chapter 1, Corollary 5.9. Finally, we recall a lemma of compensated compactness type (see [54], [72]) which will be used in the sequel. For its proof see Lemma 3.4 in Chapter 1.

Lemma 1.6. *Let* (u_h) *be a sequence converging to* u *weakly in* $H^{1,p}(\Omega)$, *and let* (g_h) *be a sequence in* $(L^q(\Omega))^n$ *converging to* g *in the topology* σ . *Then*

$$\int_\Omega (g_h, Du_h)\varphi dx \quad \to \quad \int_\Omega (g, Du)\varphi dx$$

for every $\varphi \in C_0^\infty(\Omega)$.

2. A BOUNDARY VALUE PROBLEM ON THE UNIT CUBE

Let $Y =]0,1[^n$ be the unit cube in $\mathbf{R}^n$. We say that a function $u : \mathbf{R}^n \to \mathbf{R}$ is Y-periodic if $u(x+e_i) = u(x)$ for every $x \in \mathbf{R}^n$, and for every $i = 1,..., n$, where (e_i) is the canonical base of $\mathbf{R}^n$. By $H^{1,p}_\#(Y)$ we denote the subset of $H^{1,p}(Y)$ of all the functions u with mean value zero which have the same trace on the opposite faces of Y . Every function u of $H^{1,p}_\#(Y)$ can be extended by periodicity to a function of $H^{1,p}_{loc}(\mathbf{R}^n)$.

Let a be a function in $M_Y(\mathbf{R}^n)$, and let us fix $\xi \in \mathbf{R}^n$. The goal of this section is the study of the existence of functions $v \in H^{1,p}_\#(Y)$ and $k \in (L^q(Y))^n$ satisfying the following local problem:

$$(2.1) \qquad \begin{cases} v \in H^{1,p}_\#(Y) , \\ k(y) \in a(y, Dv(y)+\xi) & \text{for a.e. } y \in Y , \\ \int_Y (k(y), Dw(y))dy = 0 & \text{for every } w \in H^{1,p}_\#(Y) . \end{cases}$$

If we still denote by k its Y-periodic extension to $\mathbf{R}^n$, the last equation in (2.1) is equivalent to $-\text{div}\, k = 0$ in $\mathcal{D}'(\mathbf{R}^n)$.

In order to give an equivalent formulation of problem (2.1), which plays a crucial role in the next section, let us set

$$V = \{k\in (L^q(Y))^n : \int_Y (k(y),Dw(y))dy = 0 \text{ for every } w\in H^{1,p}_{\#}(Y)\}$$

endowed with the strong topology of $(L^q(Y))^n$.
Then, problem (2.1) is equivalent to the following one: find $k\in V$ and $\varphi\in (L^p(Y))^n$ such that

$$(2.2)\qquad \begin{cases} k\in V , \\ \varphi(y)+\xi \in a^{-1}(y,k(y)) \quad \text{for a.e. } y\in Y , \\ \int_Y (\varphi(y),h(y))dy = 0 \quad \text{for every } h\in V , \end{cases}$$

where $a^{-1}(y,\cdot)$ is the inverse of the maximal monotone map $a(y,\cdot)$, i.e. $\xi\in a^{-1}(y,\eta)$ if and only if $\eta\in a(y,\xi)$, and $a^{-1}\in M_Y(R^n)$ by Remark 1.2. In fact, it can be proved that $[v,k]\in H^{1,p}_{\#}(Y)\times(L^q(Y))^n$ is a solution to problem (2.1) if and only if $[k,\varphi]\in V\times(L^p(Y))^n$, with $\varphi = Dv$, is a solution to problem (2.2). This result follows easily by the definition of the space V and by noticing that $\varphi\in (L^p(Y))^n$ satisfies

$$\int_Y (\varphi(y),h(y))dy = 0 \qquad \text{for every } h\in V$$

if and only if there exists $v\in H^{1,p}_{\#}(Y)$ such that $\varphi = Dv$.

In order to prove the existence of solutions to problem (2.2) (and equivalently to (2.1)), we introduce the multivalued operator $\mathcal{B} : V \to V'$ defined for every $k\in V$ as follows: $\Phi\in \mathcal{B}k$ if and only if there exists $\varphi\in (L^p(Y))^n$ with $\varphi(y)+\xi\in a^{-1}(y,k(y))$ for a.e. $y\in Y$ such that

$$(2.3)\qquad <\Phi,h> = \int_Y (\varphi(y),h(y))dy \qquad \text{for every } h\in V ,$$

where $<\cdot,\cdot>$ denotes here the duality pairing between V and its dual space V'.

Remark 2.1. With this notation problem (2.2), and equivalently problem (2.1), has a solution if and only if 0 belongs to the range of $\mathcal{B}$, i.e. $0\in R(\mathcal{B})$.

In the remaining part of this section we shall prove more than needed for solving problem (2.2) since we shall obtain the following result.

Theorem 2.2. *Let $\mathcal{B}$ be the operator from V into V' defined by (2.3). Then $\mathcal{B}$ is maximal monotone and $R(\mathcal{B}) = V'$.*

In the proof of this theorem we make use of the following definition and results.

Definition 2.3. Let S_1 and S_2 be two topological spaces, and let F be a multivalued function of S_1 into S_2. Then F is said to be *upper-semicontinuous* if for every $s_0 \in S_1$ and for every open neighbourhood V of Fs_0 in S_2 there exists a neighbourhood U of s_0 in S_1 such that $Fs \subseteq V$ for every $s \in U$.

Theorem 2.4. *Let X be a reflexive Banach space and let X' be its dual. Let F be a multivalued monotone operator of X into X'. Suppose that for each x in X, Fx is a non empty weakly closed convex subset of X' and that for each line segment in X, F is an upper-semicontinuous multivalued operator from the line segment to X', with X' given its weak topology. Then F is maximal monotone.*

Theorem 2.5. *Let X be a reflexive Banach space and let X' be its dual. Let F be a multivalued maximal monotone operator from X to X'. If F is coercive, then the range of F is X'.*

The proof of Theorem 2.4 can be found in [20], Theorem 3.18, while that of Theorem 2.5 is contained in [58], Chapter III, Theorem 2.10.

Proof of Theorem 2.2. By using Theorem 2.4 we prove first the maximal monotonicity of $\mathcal{B}$.

(a) $\mathcal{B}$ is monotone. To prove this assertion let us fix $k_i \in V$ and $\Phi_i \in \mathcal{B}k_i$, $i = 1, 2$. By definition of $\mathcal{B}$ there exist $\varphi_i \in (L^p(Y))^n$, $i = 1, 2$, with $\varphi_i(y)+\xi \in a^{-1}(y,k_i(y))$ for a.e. $y \in Y$ such that

$$<\Phi_1-\Phi_2,k_1-k_2> = \int_Y (\varphi_1-\varphi_2,k_1-k_2)dy .$$

By the monotonicity of $a^{-1}(y,\cdot)$ we have

$$\int_Y ((\varphi_1+\xi)-(\varphi_2+\xi),k_1-k_2)dy \geq 0 ,$$

that proves the monotonicity of $\mathcal{B}$.

(b) For every $k \in V$ we have $\mathcal{B}k \neq \emptyset$. To prove this fact let us fix $k \in V$. By Remark 1.2 the set $a^{-1}(y,k(y))$ is non-empty, closed, and convex in R^n for a.e. $y \in Y$. Therefore, by taking the measurability of a^{-1} into account, we conclude that there exists a measurable function $\varphi : Y \to \mathbf{R}^n$ such that $\varphi(y)+\xi \in a^{-1}(y,k(y))$ for a.e. $y \in Y$. Finally, the estimate (1.2) yields $\varphi \in (L^p(Y))^n$, which concludes the proof of (b).

(c) For every $k \in V$, $\mathcal{B}k$ is a convex subset of V'. This follows easily from the fact that $a^{-1}(y,k(y))$ is a convex subset of $\mathbf{R}^n$ for a.e. $y \in Y$.

(d) For every $k \in V$, $\mathcal{B}k$ is weakly closed in V' and the operator $\mathcal{B}$ is upper-semicontinuous from V, with the strong topology, into V', with the weak topology. Let us fix $k \in V$ and let W be an open neighbourhood of $\mathcal{B}k$ in the weak topology on V'. We claim that for every sequence (k_h) in V converging to k strongly in $(L^q(Y))^n$, there exists $j \in N$ such that $\mathcal{B}k_h \subseteq W$ for every $h \geq j$. Assume the contrary. Then there exist a subsequence $(k_{\sigma(h)})$ of (k_h) and a sequence (Φ_h) such that $\Phi_h \in \mathcal{B}k_{\sigma(h)}$ and $\Phi_h \notin W$ for every $h \in N$. Since $\Phi_h \in \mathcal{B}k_{\sigma(h)}$ there exists $\varphi_h \in (L^p(Y))^n$ such that $\varphi_h(y)+\xi \in a^{-1}(y,k_{\sigma(h)}(y))$ for a.e. $y \in Y$ and

$$<\Phi_h,w> \;=\; \int_Y (\varphi_h(y),w(y))dy \qquad \text{for every } w \in V .$$

By the estimates in $M_Y(\mathbf{R}^n)$, and the strong convergence of $(k_{\sigma(h)})$ to k in $(L^q(Y))^n$, the sequence (φ_h) is bounded in $(L^p(Y))^n$; thus there exists a subsequence $(\varphi_{\tau(h)})$ of (φ_h) which converges weakly in $(L^p(Y))^n$ to a function φ. If $\varphi(y)+\xi \in a^{-1}(y,k(y))$ for a.e. $y \in Y$, then $\Phi \in \mathcal{B}k$, hence $\Phi \in W$. But the last fact requires that $\Phi_h \in W$ for h large enough, which contradicts our assumption. Therefore, to get a contradiction, it remains to show that $\varphi(y)+\xi \in a^{-1}(y,k(y))$ for a.e. $y \in Y$. To this aim let us introduce the measurable multivalued map $F : Y \to \mathbf{R}^n \times \mathbf{R}^n$ defined by $Fy = \{[\eta,\zeta] \in \mathbf{R}^n \times \mathbf{R}^n : \zeta \in a^{-1}(y,\eta)\}$. By the Castaign representation for F (see [22], Chapter III) there exists a sequence $([\eta_m,\zeta_m])$ of measurable selections of F such that for every $y \in Y$ the sequence $([\eta_m(y),\zeta_m(y)])$ is dense in Fy. The monotonicity of a^{-1} implies then

$$(2.4) \qquad (k_{\sigma(\tau(h))}(y)-\eta_m(y),\varphi_{\tau(h)}(y)+\xi-\zeta_m(y)) \geq 0 \qquad \text{for a.e. } y \in Y .$$

By passing to the limit as h tends to $+\infty$ we get

$$(k(y)-\eta_m(y),\varphi(y)+\xi-\zeta_m(y)) \geq 0$$

for a.e. $y \in Y$ and for every $m \in N$, which implies easily

$$(k(y)-\eta,\varphi(y)+\xi-\zeta) \geq 0$$

for a.e. $y \in Y$, for every $\eta \in \mathbf{R}^n$ and $\zeta \in a^{-1}(y,\eta)$. By the maximal monotonicity of a^{-1} we get $\varphi(y)+\xi \in a^{-1}(y,k(y))$ for a.e. $y \in Y$, which concludes the proof of (d). •

We now prove that $\mathcal{B}$ is coercive, i.e.

$$\lim_{\|k\| \to \infty} \frac{\langle \Phi,k \rangle}{\|k\|} = +\infty \tag{2.5}$$

for each $\Phi \in \mathcal{B}k$, where $\|\cdot\|$ denotes the norm on V. Given $\Phi \in \mathcal{B}k$, there exists $\varphi \in (L^p(Y))^n$ with $\varphi(y)+\xi \in a^{-1}(y,k(y))$ for a.e. $y \in Y$ such that

$$\langle \Phi,k \rangle = \int_Y (\varphi(y),k(y))dy \ .$$

By the boundedness condition (1.1) it follows easily that

$$\lim_{\|k\| \to \infty} \frac{\langle \Phi,k \rangle}{\|k\|} \geq c \lim_{\|k\| \to \infty} \|k\|^{q-1}$$

for some suitable constant $c > 0$, which proves (2.5). •

Finally, since $\mathcal{B}$ is maximal monotone and coercive, by applying Theorem 2.5 we conclude that $R(\mathcal{B}) = V'$. ♦

3. HOMOGENIZATION

By $M_\#(\mathbf{R}^n)$ we denote the set of all operators $a \in M_{\mathbf{R}^n}(\mathbf{R}^n)$ such that $a(\cdot,\xi)$ is Y-periodic for every $\xi \in \mathbf{R}^n$. Given $a \in M_\#(\mathbf{R}^n)$, we consider the following Dirichlet boundary value problem on the bounded open subset Ω of $\mathbf{R}^n$:

$$\text{(3.1)} \qquad \begin{cases} g_h(x) \in a(\frac{x}{\varepsilon_h}, Du_h(x)) & \text{for a.e. } x \in \Omega \ , \\ -\operatorname{div} g_h = f \ , & \\ u_h \in H_0^{1,p}(\Omega) \ , & \end{cases}$$

where $f \in H^{-1,q}(\Omega)$ and (ε_h) is a sequence of positive real numbers converging to 0.

In this section we shall prove the convergence, as (ε_h) tends to 0, of the solutions u_h and the momenta g_h of (3.1) to the solutions u and the momenta g of the following homogenized problem

$$(3.2)\qquad \begin{cases} g(x)\in b(Du(x)) & \text{for a.e. } x\in\Omega\ , \\ -\mathrm{div}\, g \;=\; f\ , & \\ u\in H_0^{1,p}(\Omega)\ , & \end{cases}$$

where, for every $\xi\in\mathbf{R}^n$, the set $b(\xi)$ is defined by

$$(3.3)\qquad b(\xi) = \{\upsilon\in\mathbf{R}^n\colon \exists v\in H_{\#}^{1,p}(Y), \exists k\in (L^q(Y))^n \text{ satisfying (2.1), and } \upsilon = \int_Y k(y)dy\}\ .$$

Remark 3.1. Let us note that the equivalence between problem (2.1) and problem (2.2) implies that $b(\xi) = \{\upsilon\in\mathbf{R}^n\colon \exists k\in V, \exists\varphi\in (L^p(Y))^n \text{ satisfying (2.2), and } \upsilon = \int_Y k(y)dy\}$.

By taking Theorem 3.11 and Remark 3.12 in Chapter 1 into account one obtains the results mentioned above directly from the next theorem.

Theorem 3.2. *Let* $a\in M_{\#}(\mathbf{R}^n)$ *and let* (ε_h) *be a sequence of positive real numbers converging to* 0 *. Let us define* $a_h(x,\xi) = a(\frac{x}{\varepsilon_h},\xi)$ *for every* $x\in\Omega$, $\xi\in\mathbf{R}^n$ *. Then* (a_h) *G-converges to the operator* b *defined by* (3.3).

To prove this theorem we need the following result.

Proposition 3.3. *The operator* $b:\mathbf{R}^n\to\mathbf{R}^n$ *defined by* (3.3) *belongs to the class* $M(\mathbf{R}^n)$.

Proof. First of all we show that b satisfies the estimates (1.1) and (1.2). Given $\xi\in\mathbf{R}^n$ and $\upsilon\in b(\xi)$, by the definition of b there exist $v\in H_{\#}^{1,p}(Y)$ and $k\in(L^q(Y))^n$ satisfying problem (2.1) with

$$\upsilon = \int_Y k(y)dy\ .$$

By the boundedness condition (1.1) for a we have

$$|\upsilon|^q \le m_1 + c_1\int_Y (k(y),Dv(y))dy + c_1\int_Y (k(y),\xi)dy\ .$$

By taking the Y-periodicity of v into account we immediately obtain (1.1) for b. Analogously, by the coerciveness assumption (1.2) for a and the Y-periodicity of v we get

$$|\xi|^p \le m_2 + c_2 \int_Y (k(y), Dv(y)+\xi)dy = m_2 + c_2(\upsilon,\xi),$$

which concludes the proof of (1.2) for b . •

Now, to prove the maximal monotonicity of b we use Theorem 2.4.

(a) b is monotone. Let us fix $\xi_i \in \mathbf{R}^n$, $\upsilon_i \in b(\xi_i)$, $i = 1, 2$. By the definition of b there exist $v_i \in H^{1,p}_{\#}(Y)$, $k_i \in (L^q(Y))^n$, $i = 1, 2$, satisfying problem (2.1) with ξ replaced by ξ_i, and

$$\upsilon_i = \int_Y k_i(y)dy \qquad i = 1, 2.$$

We have

$$(\upsilon_1-\upsilon_2,\xi_1-\xi_2) = \int_Y (k_1(y)-k_2(y),\xi_1-\xi_2)dy =$$

$$= \int_Y (k_1(y)-k_2(y),(Dv_1(y)+\xi_1)-(Dv_2(y)+\xi_2))dy ,$$

where the last equality is justified by the Y-periodicity of the function v_1-v_2. The monotonicity of b follows now easily from that of a .

(b) For every $\xi \in \mathbf{R}^n$, we have $b(\xi) \ne \varnothing$. This fact follows easily from Remark 2.1 and Theorem 2.2.

(c) For every $\xi \in \mathbf{R}^n$, $b(\xi)$ is convex. Since Remark 3.1 and Remark 2.1 yield

$$b(\xi) = \{\upsilon \in \mathbf{R}^n : \upsilon = \int_Y k(y)dy , \ k \in \mathcal{B}^{-1}0\} ,$$

our assertion is proved if we ensure that $\mathcal{B}^{-1}0$ is convex. This follows easily from the maximal monotonicity of $\mathcal{B}$ obtained in Theorem 2.2.

(d) For every $\xi \in \mathbf{R}^n$, the set $b(\xi)$ is closed in $\mathbf{R}^n$ and the operator b is upper-semicontinuous. Since b satisfies the boundedness condition (1.1), to prove this assertion it is enough to show that, if (ξ_h) converges to ξ in $\mathbf{R}^n$, $\upsilon_h \in b(\xi_h)$, and (υ_h) converges to υ in $\mathbf{R}^n$, then $\upsilon \in b(\xi)$. By the definition of b there exist $v_h \in H^{1,p}_{\#}(Y)$, $k_h \in (L^q(Y))^n$ satisfying problem (2.1) with ξ replaced by ξ_h such that

$$\upsilon_h = \int_Y k_h(y)dy .$$

Since (ξ_h) converges to ξ, the estimate (1.1) for a implies that the sequence (k_h) is bounded in $(L^q(Y))^n$, and therefore (v_h) is bounded in $H^{1,p}_{\#}(Y)$ by the coerciveness condition (1.2) for a . Hence, up to a subsequence, we have that $[v_h,k_h]$ converges to $[v,k]$ weakly in $H^{1,p}_{\#}(Y) \times (L^q(Y))^n$. Since it is clear that

$$\int_Y (k(y),Dw(y))dy = 0 \qquad \text{for every } w \in H^{1,p}_{\#}(Y)$$

and

$$\upsilon = \int_Y k(y)dy \quad ,$$

it remains to prove that $k(y) \in a(y,Dv(y)+\xi)$ for a.e. $y \in Y$. To this aim let us introduce the measurable multivalued map $F : Y \to \mathbf{R}^n \times \mathbf{R}^n$ defined by $Fy = \{[\zeta,\eta] \in \mathbf{R}^n \times \mathbf{R}^n : \eta \in a(y,\zeta)\}$. By the Castaign representation for F (see [22], Chapter III) there exists a sequence $([\zeta_m,\eta_m])$ of measurable selections of F such that for every $y \in Y$ the sequence $([\zeta_m(y),\eta_m(y)])$ is dense in Fy. The monotonicity of a implies then

(3.4) $$(k_h(y) - \eta_m(y), Dv_h(y) + \xi_h - \zeta_m(y)) \geq 0 \qquad \text{for a.e. } y \in Y .$$

Since $[v_h,k_h]$ converges to $[v,k]$ in the topology $w \times \sigma$, by taking Lemma 1.6 into account and by passing to the limit in $\mathcal{D}(\Omega)$ as h tends to $+\infty$ we get

$$(k(y) - \eta_m(y), Dv(y) + \xi - \zeta_m(y)) \geq 0$$

for a.e. $y \in Y$ and for every $m \in \mathbf{N}$, which implies easily

$$(k(y) - \eta, Dv(y) + \xi - \zeta) \geq 0$$

for a.e. $y \in Y$, for every $\zeta \in \mathbf{R}^n$ and $\eta \in a(y,\zeta)$. By the maximal monotonicity of a we get $k(y) \in a(y,Dv(y)+\xi)$ for a.e. $y \in Y$, which concludes the proof of (d).

Hence, b satisfies all the assumptions of Theorem 2.4, which ensures that b is maximal monotone. ♦

Proof of Theorem 3.2. We have to prove that

$$K_{seq}(w \times \sigma)\text{-}\limsup_{h \to \infty} A_h \subseteq B ,$$

where A_h and B are the operators associated to a_h and b by (1.3), respectively. To this aim, let us take $[u,g] \in K_{seq}(w \times \sigma)\text{-}\limsup_{h\to\infty} A_h$. By definition there exist a sequence $\tau(h)$ of integers, with $\tau(h) \to +\infty$, and a sequence $[u_h,g_h] \in A_{\tau(h)}$ such that $[u_h,g_h]$ converges to $[u,g]$ in the topology $w \times \sigma$ on $H^{1,p}(\Omega) \times (L^q(\Omega))^n$. In order to prove that $[u,g] \in B$ we define suitable functions $v_h \in H^{1,p}(\Omega)$, $k_h \in (L^q(\Omega))^n$, both $\varepsilon_h Y$-periodic, in the following way.

Given $\xi \in \mathbf{R}^n$, let us consider a solution $v \in H^{1,p}_{\#}(Y)$, $k \in (L^q(Y))^n$ to problem (2.1). Let us still denote by v and k their Y-periodic extensions to $\mathbf{R}^n$. It can be proved that $v \in H^{1,p}_{loc}(\mathbf{R}^n)$, $k \in (L^q_{loc}(\mathbf{R}^n))^n$ with

(3.5) $$\int_{\mathbf{R}^n} (k(y), Dw(y))dy = 0$$

for every $w \in C^\infty_0(\mathbf{R}^n)$ (for a proof see, for instance, [69]). Now we are in a position to define

$$v_h(x) = (\xi, x) + \varepsilon_h v(\tfrac{x}{\varepsilon_h}) \qquad \text{for a.e. } x \in \Omega,$$

$$k_h(x) = k(\tfrac{x}{\varepsilon_h}) \qquad \text{for a.e. } x \in \Omega.$$

The periodicity properties of these functions yield easily that

(3.6) $$v_h \to (\xi, \cdot) \qquad \text{weakly in } H^{1,p}(\Omega),$$

(3.7) $$Dv_h \to \xi \qquad \text{weakly in } (L^p(\Omega))^n,$$

(3.8) $$k_h \to \upsilon = \int_Y k(y)dy \qquad \text{weakly in } (L^q(\Omega))^n,$$

(3.9) $$k_h(x) \in a(\tfrac{x}{\varepsilon_h}, Dv_h(x)) \qquad \text{for a.e. } x \in \Omega.$$

Note that (3.5)–(3.8) guarantee that $[v_h, k_h]$ converges to $[(\xi,\cdot),\upsilon]$ in the topology $w \times \sigma$ on $H^{1,p}(\Omega) \times (L^q(\Omega))^n$. By the monotonicity of a we have

$$\int_\Omega (g_h(x) - k_h(x), Du_h(x) - Dv_h(x))\varphi(x)dx \geq 0$$

for every $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$. By passing to the limit as h tends to $+\infty$, Lemma 1.6 implies that

(3.10) $$\int_\Omega (g(x) - \upsilon, Du(x) - \xi)\varphi(x)dx \geq 0$$

for every $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$; therefore, for every $\xi \in \mathbf{R}^n$ and every $\upsilon \in b(\xi)$ we have

(3.11) $$(g(x) - \upsilon, Du(x) - \xi) \geq 0 \qquad \text{for a.e. } x \in \Omega.$$

In particular, if we denote by $\{[\xi_m, \upsilon_m] : m \in \mathbf{N}\}$ a dense subset of the graph of b, (3.11) yields that

(3.12) $$(g(x) - \upsilon_m, Du(x) - \xi_m) \geq 0 \qquad \text{for a.e. } x \in \Omega, \text{ for every } m \in \mathbf{N}.$$

This implies easily that

$$(g(x) - \upsilon, Du(x) - \xi) \geq 0$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbf{R}^n$ and $\upsilon \in b(\xi)$. By taking the maximal monotonicity of b into account this guarantees that $g(x) \in b(Du(x))$ for a.e. $x \in \Omega$, i.e. $[u,g] \in B$, which was our goal. ♦

4. PROPERTIES OF THE HOMOGENIZED OPERATOR

In this section we illustrate some different situations in which the homogenized operator b defined in Section 3 turns out to be strictly monotone or single-valued.

To this aim let us recall that, given $a \in M(\mathbf{R}^n)$, the function a is said to be strictly monotone if

(4.1) $$(\eta_1 - \eta_2, \xi_1 - \xi_2) > 0$$

for any $\xi_i \in \mathbf{R}^n$, $\eta_i \in a(\xi_i)$, $i = 1, 2$, and $\xi_1 \neq \xi_2$.

Proposition 4.1. *Let* $a \in M_\#(\mathbf{R}^n)$. *Assume that* $a(y,\cdot)$ *is strictly monotone for a.e.* $y \in \mathbf{R}^n$. *Then the operator* b *defined by* (3.3) *is strictly monotone.*

Proof. Let $\xi_i \in \mathbf{R}^n$, $\upsilon_i \in b(\xi_i)$, $i = 1, 2$, with $\xi_1 \neq \xi_2$. By the definition of b there exist $[v_i,k_i] \in H^{1,p}_\#(Y) \times (L^q(Y))^n$, $i = 1, 2$, such that

(4.2) $$k_i(y) \in a(y, Dv_i(y) + \xi_i) \quad \text{for a.e. } y \in Y,$$

(4.3) $$\int_Y (k_i(y), Dw(y))dy = 0 \quad \text{for every } w \in H^{1,p}_\#(Y),$$

and

$$\upsilon_i = \int_Y k_i(y)dy .$$

By taking the strict monotonicity of a and (4.2) into account we have

(4.4) $$(k_1(y) - k_2(y), (Dv_1(y) + \xi_1) - (Dv_2(y) + \xi_2)) > 0$$

on the set $M = \{y \in Y: Dv_1(y)+\xi_1 \neq Dv_2(y)+\xi_2\}$. If M has positive Lebesgue measure, by integrating (4.4) on Y and taking (4.3) into account, we get

$$(\upsilon_1-\upsilon_2,\xi_1-\xi_2) > 0 .$$

Therefore, the proof of the strict monotonicity of b is accomplished if we show that M has positive Lebesgue measure. Assume the contrary, i.e.

$$Dv_1(y)+\xi_1 = Dv_2(y)+\xi_2 \tag{4.5}$$

for a.e. $y \in Y$. Since $v_i \in H^{1,p}_{\#}(Y)$, the integration of (4.5) on Y yields $\xi_1 = \xi_2$, that clearly contradicts our assumption. ♦

Proposition 4.2. *Let* $a \in M_{\#}(\mathbf{R}^n)$. *Assume that* $a^{-1}(y,\cdot)$ *is strictly monotone for a.e.* $y \in \mathbf{R}^n$. *Then the operator* b *defined by* (3.3) *is single-valued.*

Proof. Let us argue by contradiction. Let $\xi \in \mathbf{R}^n$, and assume that there exist υ_1 , $\upsilon_2 \in b(\xi)$ with $\upsilon_1 \neq \upsilon_2$. By the definition of b there exist $[v_i,k_i] \in H^{1,p}_{\#}(Y) \times (L^q(Y))^n$, $i = 1, 2$ such that

$$k_i(y) \in a(y,Dv_i(y)+\xi) \quad \text{for a.e. } y \in Y , \tag{4.6}$$

$$\int_Y (k_i(y),Dw(y))dy = 0 \quad \text{for every } w \in H^{1,p}_{\#}(Y) \tag{4.7}$$

and

$$\upsilon_i = \int_Y k_i(y)dy .$$

By (4.6) we have $Dv_i(y)+\xi \in a^{-1}(y,k_i(y))$ for a.e. $y \in Y$, which by the strict monotonicity of a^{-1} gives

$$(k_1(y)-k_2(y),Dv_1(y)-Dv_2(y)) > 0 \tag{4.8}$$

on the set $M = \{y \in Y: k_1(y) \neq k_2(y)\}$. Since $\upsilon_1 \neq \upsilon_2$, the set M has positive Lebesgue measure. Then, by integrating (4.8) on Y we obtain

$$\int_Y (k_1(y)-k_2(y),Dv_1(y)-Dv_2(y))dy > 0 ,$$

which contradicts (4.7). Therefore, we have to conclude that $\upsilon_1 = \upsilon_2$, i.e. b is single-valued. ♦

Corollary 4.3. *Let* $a \in M_\#(\mathbf{R}^n)$. *Assume that* a *is single-valued and* $a(y,\cdot)$ *is strictly monotone for a.e.* $y \in \mathbf{R}^n$. *Then the operator* b *defined by* (3.3) *is single-valued and strictly monotone.*

Proof. By Proposition 4.1 we get immediately that b is strictly monotone. Furthermore, by Proposition 4.2 the proof of the corollary is accomplished if we show that a^{-1} is strictly monotone. Since a belongs to $M_\#(\mathbf{R}^n)$ and a is strictly monotone, a^{-1} is everywhere defined on $\mathbf{R}^n \times \mathbf{R}^n$ and single-valued. Then, noticing that

$$(\eta_1 - \eta_2, a^{-1}(y,\eta_1) - a^{-1}(y,\eta_2)) = (a(y,\xi_1) - a(y,\xi_2), \xi_1 - \xi_2) ,$$

for $\xi_i = a^{-1}(y,\eta_i)$, the strict monotonicity of a^{-1} comes out from the strict monotonicity of a and from the fact that $\eta_1 \neq \eta_2$ implies $\xi_1 \neq \xi_2$. ♦

Let us consider $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ satisfying the following conditions:

(4.9) $f(y,\xi)$ is Lebesgue measurable and Y-periodic in y , convex in ξ ;

(4.10) $c_3|\xi|^p - c_4 \leq f(y,\xi) \leq c_5(1 + |\xi|^p)$ for every $[y,\xi] \in \mathbf{R}^n \times \mathbf{R}^n$,

where c_3 , c_4 , and $c_5 \in \mathbf{R}$, with $0 < c_3 \leq c_5 < +\infty$, $0 \leq c_4 < +\infty$.

Remark 4.4. It can be proved without any difficulty that for every f satisfying (4.9) and (4.10) the subdifferential of f with respect to ξ , denoted by $\partial_\xi f$, belongs to the class $M_\#(\mathbf{R}^n)$ for some suitable positive constants m_1 , m_2 , c_1 , and c_2 . In fact, the condition (iii) of Definition 1.1 follows easily from the maximal monotonicity of $\partial_\xi f$ and conditions (4.9) and (4.10) , respectively, whereas the measurability condition (iv) comes out from Theorem 2.3 in [3] and the measurability condition for f in (4.9).

Proposition 4.5. *Let* $f : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ *be a function satisfying* (4.9) *and* (4.10). *Assume that* $f(y,\cdot)$ *is differentiable in* $\mathbf{R}^n$ *for a.e.* $y \in \mathbf{R}^n$ *and let* $a = \partial_\xi f$. *Then the operator* b *defined by* (3.3) *is single-valued.*

Proof. By Proposition 4.2 it is enough to prove that $a^{-1}(y,\cdot)$ is strictly monotone for a.e. $y \in \mathbf{R}^n$. Since f is differentiable, its conjugate function with respect to ξ , denoted by f^* , turns out to be strictly convex (this follows by Theorem 26.3 in [60] taking into account that, under our assumptions on f , we have $f^{**} = f$). Directly from the definition of the subdifferential it follows that $\partial_\xi f^*$ is strictly monotone. Now, the proof of the proposition is complete by noticing that $\partial_\xi f^* = (\partial_\xi f)^{-1}$ by Corollary 23.5.1 in [60]. ♦

Remark 4.6. Let $n = 1$. Let a be a single-valued operator in $M_{\#}(\mathbf{R})$. Then a satisfies automatically the assumptions of Proposition 4.5. In fact, since $n = 1$ we can take

$$f(y,\xi) = \int_0^\xi a(y,t)dt \quad ,$$

which fulfills the conditions (4.9) and (4.10) for some suitable constants c_3, c_4 and c_5.

In the last part of this section we give an example of a single-valued operator $a \in M_{\#}(\mathbf{R}^n)$ whose associated homogenized operator b defined by (3.3) turns out to be multivalued.

Example 4.7. Let $n = 2$. Let us consider the Y-periodic function $r : \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$ defined on $Y \times \mathbf{R}^2$ as

$$r(y,\xi) = \begin{cases} (\xi_2, -\xi_1) & \text{if } y_1 \in \,]0,1/2[\quad , \\ (-\xi_2, \xi_1) & \text{if } y_1 \in [1/2,1[\quad , \end{cases}$$

for every $y = [y_1,y_2] \in Y$, $\xi = [\xi_1,\xi_2] \in \mathbf{R}^2$, and let us introduce the function $f : \mathbf{R}^2 \to \mathbf{R}$ such that

$$f(\xi) = \tfrac{1}{2}[(|\xi|-1) \vee 0]^2$$

for every $\xi \in \mathbf{R}^2$. We set

$$a(y,\xi) = \partial f(\xi) + r(y,\xi) = [(|\xi|-1) \vee 0] \frac{\xi}{|\xi|} + r(y,\xi) \ . \tag{4.11}$$

It is an easy matter to prove that the single-valued map a belongs to $M_{\#}(\mathbf{R}^2)$. In fact, the operator a is clearly monotone and continuous in ξ. Therefore, $a(y,\cdot)$ is maximal monotone on $\mathbf{R}^2$ (see, for instance, Proposition 2.4 in [19], Chapter II.3). However, a is not a subdifferential, since $a(x,\cdot)$ is not cyclically monotone (see, for instance, [19], Example 2.8.1).

We are now in a position to prove that the homogenized operator b, associated to a in Section 3, is actually multivalued; more precisely, we shall show that $b(0)$ has at least two elements. Since by our construction $0 \in b(0)$ it is enough to prove that there exists $\upsilon \in b(0)$, with $\upsilon \neq 0$. To this aim let us consider any function $w : [0,1] \to \mathbf{R}$, $w \in H_0^{1,p}(]0,1[)$, $|w'| \leq 1$, with mean value zero and $w(1/2) \neq 0$. We set $v : Y \to \mathbf{R}$, $v(y_1,y_2) = w(y_1)$ and $k(y) = r(y,Dv(y))$ for every $y \in Y$. Now, it turns out that $[v,k]$ is a solution to the local problem (2.1) corresponding to the operator a defined by (4.11) and $\xi = 0$. Furthermore, an easy calculation gives

$$\upsilon = \int_Y k(y)dy = [0,-2w(\tfrac{1}{2})] \in \mathbf{R}^2$$

and proves our assertion that b is multivalued. ♦

CHAPTER 3

ASYMPTOTIC BEHAVIOUR OF QUASI-LINEAR PROBLEMS WITH NEUMANN BOUNDARY CONDITIONS ON PERFORATED DOMAINS

In this chapter we deal with the limit behaviour of the solutions u_h of quasi-linear equations $-\mathrm{div}(a_h(x,Du_h)) = f$ on perforated domains $\Omega_h \subseteq \Omega$ with homogeneous Neumann boundary conditions on the holes. Under suitable assumptions on a_h and Ω_h we prove that certain extensions of u_h converge weakly in $H^{1,p}(\Omega)$ to the solution u of a quasi-linear equation of the form $-\mathrm{div}(a(x,Du)) = bf$, where the function $b \in L^\infty(\Omega)$ is the weak limit of the characteristic functions of the sets Ω_h, and the map a has the same qualitative properties of a_h. The results of this chapter are published in [27].

INTRODUCTION

In this chapter we deal with the asymptotic behaviour of the solutions of quasi-linear equations associated to monotone operators of the form

$$-\mathrm{div}(a_h(x,Du_h)) \tag{0.1}$$

on perforated domains of $\mathbf{R}^n$ with homogeneous Neumann boundary conditions on the holes. In (0.1) we indicate by $a_h = a_h(x,\xi)$ the functions $a_h : \Omega_h \times \mathbf{R}^n \to \mathbf{R}^n$ which are monotone and continuous in ξ, and satisfy suitable coerciveness and growth conditions (see Section 1). Let Ω be a bounded open subset of $\mathbf{R}^n$, let (B_h) be a sequence of compact subsets of Ω with regular boundary, and let $\Omega_h = \Omega \setminus B_h$. The main hypothesis we set on the perforated domains is that (Ω_h) has the uniform local extension property (see Definition 1.1). Moreover, we assume that the sequence of the characteristic functions (1_{Ω_h}) converges in $L^\infty(\Omega)$-weak* to a limit function $b \in L^\infty(\Omega)$ such that

$$0 < \beta \leq b(x) \qquad \text{for a.e. } x \in \Omega ,$$

where β is a fixed real constant.

Under these assumptions, we study the limit behaviour, as $h \to +\infty$, of the sequence of solutions $u_h \in H^{1,p}(\Omega_h)$ to the problem

$$(0.2) \qquad \begin{cases} -\mathrm{div}(a_h(x,Du_h)) = f & \text{on } \Omega_h , \\ u_h = 0 & \text{on } \partial\Omega , \\ (a_h(x,Du_h),\nu_h) = 0 & \text{on } \partial B_h , \end{cases}$$

where p is a real constant with $1 < p < +\infty$, $f \in L^q(\Omega)$ $(1/p+1/q = 1)$, ν_h is the unit outer normal to ∂B_h, and $(\cdot,\cdot)$ denotes the scalar product in $\mathbf{R}^n$.

We shall prove the following compactness result: there exist a subsequence $\sigma(h) \to +\infty$ and a suitable function $a : \Omega\times\mathbf{R}^n \to \mathbf{R}^n$ such that for every $f \in L^q(\Omega)$ we have

$$\tilde{u}_{\sigma(h)} \to u \quad \text{weakly in } H^{1,p}(\Omega) ,$$

$$g_{\sigma(h)} \to a(x,Du) \quad \text{weakly in } (L^q(\Omega))^n ,$$

where $\tilde{u}_{\sigma(h)}$ is a suitable extension to Ω of the solution $u_{\sigma(h)} \in H^{1,p}(\Omega_{\sigma(h)})$ to problem (0.2),

$$g_{\sigma(h)} = \begin{cases} a_{\sigma(h)}(x,Du_{\sigma(h)}) & \text{on } \Omega_{\sigma(h)} , \\ 0 & \text{on } \Omega\setminus\Omega_{\sigma(h)} , \end{cases}$$

and $u \in H_0^{1,p}(\Omega)$ is the unique solution to the Dirichlet boundary value problem

$$\begin{cases} -\mathrm{div}(a(x,Du)) = bf & \text{on } \Omega , \\ u \in H_0^{1,p}(\Omega) . \end{cases}$$

The proofs follow the method introduced in [73]. For simplicity, they are given in detail only for the case $p = 2$, while in the last section we just point out the modifications needed for the case $p \neq 2$.

When a_h is linear, this problem has been treated in [42]. The limit behaviour of the solutions to the equations $-\Delta u_h = f$ (or, of more general elliptic equations) on perforated domains with a periodic distribution of the holes and homogeneous Neumann boundary conditions has been studied extensively in [30], [31] by the energy method and in [48] by the method of multi-scales. The corresponding non-homogeneous case has been analyzed in [29], [32]. Analogous problems for a system of elliptic equations has been investigated by [55], [56], [57]. Furthermore, some problems related to the homogenization of eigenvalue problems in perforated domains has been studied in [65], [41], [74], [75].

In the periodic case a homogenization result for a class of nonlinear elliptic problems on perforated domains with non-homogeneous Neumann boundary conditions on the holes has been obtained in [37].

Finally, in the particular case in which

$$a_h(x,\xi) = \partial_\xi \psi_h(x,\xi) ,$$

where ∂_ξ denotes the subdifferential with respect to ξ and $\psi_h : \Omega_h \times \mathbf{R}^n \to [0,\infty[$ are measurable in (x,ξ), convex in ξ and satisfy suitable coerciveness and growth conditions, problem (0.2) has been studied by using Γ-convergence techniques applied to the non-equicoercive integral functionals associated to ψ_h extended to 0 outside Ω_h. We refer to [52] for the linear case and to [23] for the nonlinear one. Homogenization results for non-coercive functionals when u is a vector valued function can be found in [1].

1. NOTATION AND STATEMENT OF THE PROBLEM

Let Ω be a bounded open subset of $\mathbf{R}^n$. The Euclidean norm and the scalar product in $\mathbf{R}^n$ are denoted by $|\cdot|$ and $(\cdot,\cdot)$, respectively. For every $h \in \mathbf{N}$ let B_h be a compact subset of Ω that lies locally on one side of its boundary and has Lipschitz-continuous boundary ∂B_h. The complement of B_h in Ω will be denoted by Ω_h. Moreover, we assume that the sequence of the characteristic functions (1_{Ω_h}) converges in $L^\infty(\Omega)$-weak* to a limit function $b \in L^\infty(\Omega)$ such that

(1.1) $$0 < \beta \le b(x) \qquad \text{for a.e. } x \in \Omega ,$$

where β is a fixed real constant. Finally, we suppose that the sequence (Ω_h) has the uniform local extension property in the sense of the following definition (see [23]).

Definition 1.1. The sequence of sets (Ω_h) satisfies the *uniform local extension property* with constant $c_0 > 0$, if for every pair U', U of open subsets of $\mathbf{R}^n$ such that $U' \subset\subset U$ and for every $h \in \mathbf{N}$ there exists a linear and continuous extension operator $E_h : H^1(U \cap \Omega_h) \to H^1(U' \cap \Omega)$ such that

(1.2) $E_h u = u$ a.e. in $U' \cap \Omega_h$,

(1.3) $$\int_{U' \cap \Omega} |D(E_h u)|^2 dx \le c_h \int_{U \cap \Omega_h} |Du|^2 dx,$$

for every $u \in H^1(U \cap \Omega_h)$, with $\limsup_{h\to\infty} c_h \le c_0$.

Remark 1.2. Let $B \subseteq \mathbf{R}^n$ be a given closed set with non-empty interior and regular boundary (say, Lipschitz continuous). Moreover, assume $0 \in B$, $\mathrm{diam} B \le 1$ and $B \subset\subset D$, where D is a fixed open subset of $\mathbf{R}^n$. Let B_h be the union of a finite family $\{B_h^i : i \in I_h\}$ of closed sets of the form $B_h^i = x_h^i + r_h^i B$, where we assume that $x_h^i \in \mathbf{R}^n$, $r_h^i > 0$ and $x_h^i + r_h^i D \subseteq \Omega$. Then, the sequence of sets defined by $\Omega_h = \Omega \backslash B_h$ satisfies the uniform local extension property as shown in [31], Lemma 3.

It is clear that the uniform local extension property for the sequence (Ω_h) implies the weaker extension assumption known as the strong connectivity condition (see [42]) which is given by the following definition.

Definition 1.3. The sequence of sets (Ω_h) satisfies the *strong connectivity condition* with constant $c_0 > 0$ if for every $h \in \mathbf{N}$ there exists a linear and continuous extension operator $E_h : H^1(\Omega_h) \to H^1(\Omega)$ such that

(1.4) $\quad E_h u = u$ a.e. in Ω_h,

(1.5) $\quad \displaystyle\int_\Omega |D(E_h u)|^2 dx \le c_0 \int_{\Omega_h} |Du|^2 dx,$

for every $u \in H^1(\Omega_h)$.

Let us note that most of the results of this chapter are obtained by using only the strong connectivity condition for (Ω_h). However, the proof of the main theorem in Section 2 requires for (Ω_h) to satisfy the uniform local extension property.

Throughout this chapter, c_1 and c_2 are two real constants such that $0 < c_1 \le c_2$. Let us fix a sequence of functions $a_h : \Omega_h \times \mathbf{R}^n \to \mathbf{R}^n$ satisfying the following conditions:

(1.6) $\quad a_h(\cdot,\xi)$ is measurable for every $\xi \in \mathbf{R}^n$,

(1.7) $\quad a_h(x,0)$ is bounded in $(L^2(\Omega_h))^n$, uniformly with respect to h.

Moreover, we make the following monotonicity and continuity assumptions:

(1.8) $\quad (a_h(x,\xi_1) - a_h(x,\xi_2), \xi_1 - \xi_2) \ge c_1 |\xi_1 - \xi_2|^2$,

(1.9) $\quad |a_h(x,\xi_1) - a_h(x,\xi_2)| \le c_2 |\xi_1 - \xi_2|$

for a.e. $x \in \Omega_h$, for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

Given $f \in L^2(\Omega)$, we consider the following Neumann boundary value problem: find $u_h \in H^1(\Omega_h)$ such that

$$(1.10) \qquad \begin{cases} -\mathrm{div}(a_h(x,Du_h)) = f & \text{on } \Omega_h , \\ u_h = 0 & \text{on } \partial\Omega , \\ (a_h(x,Du_h),\nu_h) = 0 & \text{on } \partial B_h , \end{cases}$$

where ν_h is the unit outer normal to ∂B_h . Problem (1.10) is intended in the usual weak sense, i.e. u_h is a solution of (1.10) if and only if

$$(1.11) \qquad \begin{cases} u_h \in H^1(\Omega_h) \, , \; u_h=0 \text{ on } \partial\Omega \, , \\ \displaystyle\int_{\Omega_h} (a_h(x,Du_h),Dw)dx = \int_{\Omega_h} fw dx \quad \forall \; w \in H^1(\Omega_h) \, , \; w=0 \text{ on } \partial\Omega \, . \end{cases}$$

Remark 1.4. It is well known that the Neumann boundary value problem (1.10) has a unique solution $u_h \in H^1(\Omega_h)$. Moreover, the uniform strict monotonicity assumption (1.8) and the boundedness condition (1.7) imply the following a priori estimate

$$(1.12) \qquad \int_{\Omega_h} |Du_h|^2 dx \le c$$

for the solution u_h to problem (1.11), where $c \in \mathbf{R}$ is independent of h . From (1.12), (1.7) and the equicontinuity assumption (1.9) it follows also that

$$(1.13) \qquad \int_{\Omega_h} |a_h(x,Du_h)|^2 dx \le c' ,$$

where $c' \in \mathbf{R}$ is still independent of h .

The purpose of this chapter is to study the asymptotic behaviour, as h tends to $+\infty$, of the solutions $u_h \in H^1(\Omega_h)$ and the momenta $a_h(x,Du_h) \in (L^2(\Omega_h))^n$ corresponding to problem (1.10). To this aim for every $h \in \mathbf{N}$ we consider the extension operator E_h given by Definition 1.3 and we set

$$\tilde{u}_h = E_h u_h \; ,$$

$$g_h = \begin{cases} a_h(x,Du_h) & \text{on } \Omega_h , \\ 0 & \text{on } B_h , \end{cases}$$

so $\tilde{u}_h \in H_0^1(\Omega)$ and $g_h \in (L^2(\Omega))^n$. By (1.12) and (1.5) the sequence $(\tilde{u}_h)$ turns out to be uniformly bounded in $H_0^1(\Omega)$. Moreover, by (1.13) the sequence (g_h) is also uniformly bounded in $(L^2(\Omega))^n$. We shall prove in the remaining sections that there exist a sequence $\sigma(h) \to +\infty$ and a suitable function $a(x,\xi) : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ such that for every $f \in L^2(\Omega)$ we have

$$\tilde{u}_{\sigma(h)} \to u \quad \text{weakly in } H^1(\Omega) \,,$$
$$g_{\sigma(h)} \to a(x,Du) \quad \text{weakly in } (L^2(\Omega))^n \,,$$

where u is the unique solution to the following Dirichlet boundary value problem

$$\text{(1.14)} \qquad \begin{cases} -\operatorname{div}(a(x,Du)) = bf & \text{on } \Omega \,, \\ u \in H_0^1(\Omega) \,, \end{cases}$$

and b is the $L^\infty(\Omega)$-weak* limit of (1_{Ω_h}).

Remark 1.5. Let us note that the limit function u and the map a which appear in (1.14) do not depend on the particular extension operator E_h we have considered. This is an easy consequence of the following lemma.

Lemma 1.6. *Let* U *be an open subset of* $\mathbf{R}^n$ *and let* (u_h) *and* (v_h) *be two sequences in* $H^1(U)$ *converging weakly in* $H^1(U)$ *to* u *and* v, *respectively. Assume that* $u_h = v_h$ *a.e. on* $U \cap \Omega_h$, *where* (1_{Ω_h}) *converges in* $L^\infty(\Omega)$*-weak* to a function* $b \in L^\infty(\Omega)$ *satisfying (1.1). Then* $u = v$ *a.e. on* U.

Proof. For every open set $U' \subset\subset U$, U' with a smooth boundary, by the Rellich theorem the sequences (u_h) and (v_h) converge in the strong topology of $L^2(U')$ to u and v, respectively. Since

$$\int_{U'} (u_h - v_h) 1_{\Omega_h} dx = 0 \,,$$

by passing to the limit as h tends to $+\infty$, we get

$$\int_{U'} (u - v) b dx = 0 \,.$$

Hence, by the strict positivity of b assumed in (1.1) we may conclude that $u = v$ a.e. on U.

♦

2. THE MAIN RESULT

In this section we prove the main theorem of this chapter by using some results whose proofs are given in the next section.

Theorem 2.1. *Let* (a_h) *be the sequence of functions given in Section 1. Then there exist* $\sigma(h) \to +\infty$ *and a Carathéodory function* $a : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ *such that for every* $f \in L^2(\Omega)$ *we have*

$$\tilde{u}_{\sigma(h)} \to u \quad \textit{weakly in } H^1(\Omega)\,, \tag{2.1}$$

$$g_{\sigma(h)} \to a(x,Du) \quad \textit{weakly in } (L^2(\Omega))^n\,, \tag{2.2}$$

where $\tilde{u}_{\sigma(h)} = E_{\sigma(h)} u_{\sigma(h)}$ *with* $E_{\sigma(h)}$ *given by Definition 1.3,* $u_{\sigma(h)} \in H^1(\Omega_{\sigma(h)})$ *is the solution to problem (1.10) in the sense of (1.11),*

$$g_{\sigma(h)} = \begin{cases} a_{\sigma(h)}(x, Du_{\sigma(h)}) & \text{on } \Omega_{\sigma(h)}\,, \\ 0 & \text{on } \Omega \setminus \Omega_{\sigma(h)}\,, \end{cases}$$

and $u \in H^1_0(\Omega)$ *is the unique solution to problem (1.14). Moreover, the function* a *satisfies the following monotonicity and continuity conditions*

$$(a(x,\xi_1) - a(x,\xi_2)\,, \xi_1 - \xi_2) \ \geq \ \frac{c_1}{c_0}\, |\xi_1 - \xi_2|^2 \tag{2.3}$$

$$|a(x,\xi_1) - a(x,\xi_2)| \ \leq \ \frac{c_0}{c_1} c_2^2\, |\xi_1 - \xi_2| \tag{2.4}$$

for a.e. $x \in \Omega$ *and for every* $\xi_1, \xi_2 \in \mathbf{R}^n$.

For the proof of this theorem we introduce several operators whose properties are only stated therein, while their proofs are given in Section 3.

Proof of Theorem 2.1. Let V be a countable dense subset of $L^2(\Omega)$. Since for every $f \in V$ the solution u_h and the momentum $a_h(x,Du_h)$ of problem (1.10) corresponding to f satisfy the a priori bounds (1.12), (1.13), by means of a diagonalization argument we can find a sequence $\sigma(h) \to +\infty$ such that for every $f \in V$

$$\tilde{u}_{\sigma(h)} \to u \quad \text{weakly in } H^1(\Omega)\,, \tag{2.5}$$

$$g_{\sigma(h)} \to g \quad \text{weakly in } (L^2(\Omega))^n\,, \tag{2.6}$$

where $\tilde{u}_{\sigma(h)}$ and $g_{\sigma(h)}$ are defined as in the statement of the theorem.

Now, in the following lemmas, whose proofs are given in Section 3, we introduce some suitable operators satisfying nice properties which allow us to complete the proof of Theorem 2.1.

Lemma 2.2. *Let* $\mathcal{B} : V \to H_0^1(\Omega)$ *be the operator defined by* $\mathcal{B}f = u$ *with* u *given by (2.5). Then,* $\mathcal{B}$ *has a unique extension to* $L^2(\Omega)$, *still denoted by* $\mathcal{B}$, *such that*

$$(2.7)\qquad \|\mathcal{B}f_1 - \mathcal{B}f_2\|_{H_0^1(\Omega)} \le \frac{c_0}{c_1}\|bf_1 - bf_2\|_{H^{-1}(\Omega)}$$

for every $f_1, f_2 \in L^2(\Omega)$.

Lemma 2.3. *Let* $\mathcal{J} : L^2(\Omega) \to L^2(\Omega)$ *be the isomorphism of* $L^2(\Omega)$ *given by* $\mathcal{J}f = bf$ *for every* $f \in L^2(\Omega)$. *Let* $\mathcal{T} : L^2(\Omega) \subseteq H^{-1}(\Omega) \to H_0^1(\Omega)$ *be the operator defined by*

$$(2.8)\qquad \mathcal{T}f = (\mathcal{B}\mathcal{J}^{-1})f$$

for every $f \in L^2(\Omega)$, *where* $\mathcal{B} : L^2(\Omega) \to H_0^1(\Omega)$ *is the operator given by Lemma 2.2. Then,* $\mathcal{T}$ *has a unique extension to* $H^{-1}(\Omega)$, *still denoted by* $\mathcal{T}$, *that satisfies*

$$(2.9)\qquad \langle f_1 - f_2, \mathcal{T}f_1 - \mathcal{T}f_2\rangle \ge \frac{c_1}{c_0}\frac{1}{c_2^2}\|f_1 - f_2\|^2_{H^{-1}(\Omega)},$$

$$(2.10)\qquad \|\mathcal{T}f_1 - \mathcal{T}f_2\|_{H_0^1(\Omega)} \le \frac{c_0}{c_1}\|f_1 - f_2\|_{H^{-1}(\Omega)}$$

for every $f_1, f_2 \in H^{-1}(\Omega)$.

Lemma 2.4. *Let* $\mathcal{T} : H^{-1}(\Omega) \to H_0^1(\Omega)$ *be the operator given by Lemma 2.3. Then,* $\mathcal{T}$ *is invertible and the operator* $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ *defined by*

$$(2.11)\qquad \mathcal{A}u = \mathcal{T}^{-1}u$$

for every $u \in H_0^1(\Omega)$ *satisfies*

$$(2.12)\qquad \langle \mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2\rangle \ge \left(\frac{c_1}{c_0}\right)^3\frac{1}{c_2^2}\|u_1 - u_2\|^2_{H_0^1(\Omega)},$$

(2.13) $$\|\mathcal{A}u_1 - \mathcal{A}u_2\|_{H^{-1}(\Omega)} \le \frac{c_0}{c_1} c_2^2 \|u_1 - u_2\|_{H_0^1(\Omega)}$$

for every $u_1, u_2 \in H_0^1(\Omega)$.

Lemma 2.5. *Let* $B : V \to (L^2(\Omega))^n$ *be the operator defined by* $Bf = g$ *with* g *given by (2.6). Then, it has a unique extension to* $L^2(\Omega)$ *, still denoted by* B *, such that*

(2.14) $$\|Bf_1 - Bf_2\|_{(L^2(\Omega))^n} \le \frac{c_2}{c_1} \|f_1 - f_2\|_{L^2(\Omega)}$$

for every $f_1, f_2 \in L^2(\Omega)$.

Lemma 2.6. *Let* $\mathcal{T}$ *and* $\mathcal{J}$ *be the operators given in Lemma 2.3. Let* Y *be the countable dense subset of* $H_0^1(\Omega)$ *given by* $Y = (\mathcal{T}\mathcal{J})(V)$ *and let* $A : Y \to (L^2(\Omega))^n$ *be*

the operator defined by

(2.15) $$Au = (B\mathcal{J}^{-1}\mathcal{A})u$$

for every $u \in Y$, *where* $B : L^2(\Omega) \to (L^2(\Omega))^n$ *and* $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ *are the operators given in Lemmas 2.5 and 2.4, respectively. Then,* A *has a unique extension to* $H_0^1(\Omega)$ *, still denoted by* A *, satisfying*

(2.16) $$\|Au_1 - Au_2\|_{(L^2(\Omega))^n} \le \left(\frac{c_0}{c_1}\right)^2 c_2^3 \|u_1 - u_2\|_{H_0^1(\Omega)}$$

for every $u_1, u_2 \in H_0^1(\Omega)$. *Moreover,*

(2.17) $$(Au_1(x) - Au_2(x), Du_1(x) - Du_2(x)) \ge \frac{c_1}{c_0} |Du_1(x) - Du_2(x)|^2 ,$$

(2.18) $$|Au_1(x) - Au_2(x)| \le \frac{c_0}{c_1} c_2^2 |Du_1(x) - Du_2(x)|$$

for a.e. $x \in \Omega$ *, for every* $u_1, u_2 \in H_0^1(\Omega)$.

Proof of Theorem 2.1. Continuation. Since V is dense in $L^2(\Omega)$, by the continuity of the operators $\mathcal{B}$ and B and by the properties (1.8) and (1.9) of a_h we easily get that for every $f \in L^2(\Omega)$ the convergences (2.5) and (2.6) hold when $\tilde{u}_{\sigma(h)}$ is the extended solution to problem (1.10) for f and $g_{\sigma(h)}$ is the corresponding extended momentum.

Now, let us define the function $a : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ as follows: given $\xi \in \mathbf{R}^n$, an open set $\omega \subset\subset \Omega$ and a function $\psi \in C_0^\infty(\Omega)$ with $\psi \equiv 1$ on ω , we set

(2.19) $$a(x,\xi) = (Av)(x) \quad \text{for a.e. } x\in\omega ,$$

where $A : H_0^1(\Omega) \to (L^2(\Omega))^n$ is the operator given in Lemma 2.6 and

(2.20) $$v(x) = \psi(x)(\xi,x) \quad \text{for every } x\in\Omega .$$

In order to show that the function a given in (2.19) does not depend on ω and ψ, let us consider $\xi\in\mathbf{R}^n$, two different open sets $\omega_1 \subset\subset \Omega$, $\omega_2 \subset\subset \Omega$ and the corresponding functions ψ_1 and ψ_2. By (2.18) it follows

$$|Av_1(x) - Av_1(x)| \leq \frac{c_0}{c_1} c_2^2 |Dv_1(x) - Dv_2(x)| = 0 \quad \text{for a.e. } x\in\omega_1\cap\omega_2 ,$$

where $v_i(x) = \psi_i(x)(\xi,x)$ for $i = 1, 2$. Thus, using an invading sequence of sets $\omega_n \subset\subset \Omega$, we obtain that the function $a(x,\xi)$ is well defined for a.e. $x\in\Omega$ and for every $\xi\in\mathbf{R}^n$. Moreover, the estimates (2.3) and (2.4) are an easy consequence of the definition of a and the inequalities (2.17) and (2.18).

Now, in order to prove (2.2) let us fix $f\in L^2(\Omega)$ and let us show that

(2.21) $$g(x) = a(x,Du(x)) \quad \text{for a.e. } x\in\Omega ,$$

where $u = \mathcal{B}f$ and $g = Bf$ are given by (2.5) and (2.6), respectively. Let us note first that (2.15) holds for every $u\in \mathcal{T}(L^2(\Omega))$ which is the set of all functions $u\in H_0^1(\Omega)$ with $\mathcal{A}u\in L^2(\Omega)$. This is due to the density of Y in $\mathcal{T}(L^2(\Omega))$ and the continuity of the operators B, $\mathcal{J}^{-1}$ and $\mathcal{A}$ which define A. Now, since u in (2.21) can be expressed as $u = (\mathcal{T}\mathcal{J})f$, we have $u\in \mathcal{T}(L^2(\Omega))$ which by (2.15) implies that $Au = (B\mathcal{J}^{-1}\mathcal{A})\mathcal{B}f = Bf = g$. Therefore, (2.21) will be proved if we show that $(Au)(x) = a(x,Du(x))$ for a.e. $x\in\Omega$. To this aim it is enough to prove that

(2.22) $$((Au)(x) - a(x,\xi),Du(x) - \xi) \geq 0$$

holds for a.e. $x\in\Omega$ and for every $\xi\in\mathbf{R}^n$. In fact, by taking $\xi = Du(x) + t\eta$ with $\eta\in\mathbf{R}^n$, $t\in\mathbf{R}$, $t > 0$, it follows that

$$((Au)(x) - a(x,Du(x) + t\eta),t\eta) \geq 0 .$$

Dividing by t and passing to the limit as t goes to 0, the continuity property (2.4) yields

$$((Au)(x) - a(x,Du(x)),\eta) \geq 0$$

for a.e. $x \in \Omega$ and for every $\eta \in \mathbf{R}^n$. Therefore, $(Au)(x) = a(x,Du(x))$ for a.e. $x \in \Omega$ which proves (2.21).

To prove (2.22) let us fix $\xi \in \mathbf{R}^n$ and let us consider the function $v \in H^1_0(\Omega)$ given by (2.20). By Proposition 2.6 there exists a sequence $v_n \in Y$ such that (v_n) converges to v strongly in $H^1(\Omega)$ and $\mathcal{A}v_n \in L^2(\Omega)$ for every $n \in \mathbf{N}$. We may assume that (Dv_n) converges to Dv a.e. in Ω, and therefore (Dv_n) converges to ξ a.e. in ω. Let us fix now $n \in \mathbf{N}$ and denote by $\tilde{v}_{n,h}$ the extended solution to problem (1.10) corresponding to $f_n = (\mathcal{J}^{-1}\mathcal{A})v_n$, and by $g_{n,h}$ the corresponding extended momentum. By the definition of $\mathcal{B}$ and B we get, by passing to the limit as $h \to +\infty$, that

$$\tilde{v}_{n,\sigma(h)} \to \mathcal{B}(\mathcal{J}^{-1}\mathcal{A})v_n = v_n \qquad \text{weakly in } H^1(\Omega) ,$$

$$g_{n,\sigma(h)} \to B(\mathcal{J}^{-1}\mathcal{A})v_n = Av_n \qquad \text{weakly in } (L^2(\Omega))^n .$$

By the monotonicity assumption on $a_{\sigma(h)}(x,\cdot)$ it follows that

$$\int_\Omega (g_{\sigma(h)} - g_{n,\sigma(h)}, D\tilde{u}_{\sigma(h)} - D\tilde{v}_{n,\sigma(h)})\varphi dx \ \geq\ 0$$

for every $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$ on Ω. Since

$$-\mathrm{div}(g_{\sigma(h)} - g_{n,\sigma(h)}) = (f - f_n)1_{\Omega_{\sigma(h)}}$$

and $((f-f_n)1_{\Omega_{\sigma(h)}})$ converges to $(f-f_n)b$ weakly in $L^2(\Omega)$ (hence, strongly in $H^{-1}(\Omega)$), by a compensated compactness result (see, for instance, Lemma 3.4 in Chapter 1) the last inequality implies, as h tends to $+\infty$, that

$$\int_\Omega (Au - Av_n, Du - Dv_n)\varphi dx \ \geq\ 0$$

for every $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$ on Ω. Now, by taking the limit as n tends to $+\infty$, by the continuity of A (see (2.16)) we get

$$\int_\Omega (Au - Av, Du - Dv)\varphi dx \ \geq\ 0$$

for every $\varphi \in C^\infty_0(\Omega)$, $\varphi \geq 0$ on Ω. Since φ is arbitrary, we may conclude that

$$((Au)(x) - a(x,\xi), Du(x) - \xi) \ \geq\ 0$$

for a.e. $x \in \omega$. Thus, using an increasing sequence of sets $\omega_n \subset\subset \Omega$, and taking the continuity of $a(x,\cdot)$ into account, we obtain (2.22) and therefore (2.21).

Finally, in order to prove that $u \in H_0^1(\Omega)$ is the unique solution to problem (1.14) it is enough to pass to the limit in

$$\int_\Omega (g_{\sigma(h)}, Dw)dx = \int_\Omega 1_{\Omega_{\sigma(h)}} fw dx$$

for every $w \in H_0^1(\Omega)$, by means of (2.2) and the convergence of the sequence (1_{Ω_h}) to the function b. ♦

3. PROOFS OF THE TECHNICAL LEMMAS

This section is completely devoted to the proofs of the lemmas stated in the previous section.

Proof of Lemma 2.2. Let us prove (2.7). Let $f_1, f_2 \in V$, and let $u_{1,\sigma(h)}, u_{2,\sigma(h)}$ be the solutions to

$$\begin{cases} u_{i,\sigma(h)} \in H^1(\Omega_{\sigma(h)}), \ u_{i,\sigma(h)}=0 \ \text{on } \partial\Omega ; \\ \displaystyle\int_{\Omega_{\sigma(h)}} (a_{\sigma(h)}(x, Du_{i,\sigma(h)}), Dw)dx = \int_{\Omega_{\sigma(h)}} f_i w dx \\ \forall \ w \in H^1(\Omega_{\sigma(h)}), \ w=0 \ \text{on } \partial\Omega . \end{cases}$$

By using $w = u_{1,\sigma(h)} - u_{2,\sigma(h)}$ as test function in the above equations and by taking their difference we get

$$\int_{\Omega_{\sigma(h)}} (a_{\sigma(h)}(x, Du_{1,\sigma(h)}) - a_{\sigma(h)}(x, Du_{2,\sigma(h)}), Du_{1,\sigma(h)} - Du_{2,\sigma(h)})dx = \int_{\Omega_{\sigma(h)}} (f_1 - f_2, u_{1,\sigma(h)} - u_{2,\sigma(h)})dx .$$

Then, the strict monotonicity assumption (1.8) and the extension estimate (1.5) imply

$$\frac{c_1}{c_0} \int_\Omega |D\tilde{u}_{1,\sigma(h)} - D\tilde{u}_{2,\sigma(h)}|^2 dx \le \int_\Omega 1_{\Omega_{\sigma(h)}} (f_1 - f_2, u_{1,\sigma(h)} - u_{2,\sigma(h)})dx .$$

Since, by the definition of $\mathcal{B}$ the sequences $(\tilde{u}_{i,\sigma(h)})$ converge to $\mathcal{B}f_i$ weakly in $H^1(\Omega)$, by passing to the limit in the last inequality, the weak lower semicontinuity of the L^2-norm ensures that

$$\frac{c_1}{c_0}\|\mathcal{B}f_1 - \mathcal{B}f_2\|_{H_0^1(\Omega)} \leq \|bf_1 - bf_2\|_{H^{-1}(\Omega)} \tag{3.1}$$

which proves (2.7) on V. Now, it is clear that $\mathcal{B}$ can be extended by continuity in a unique way to $L^2(\Omega)$, still preserving (3.1). ♦

Proof of Lemma 2.3. The proof of (2.10) on $L^2(\Omega)$ follows directly from the definition of $\mathcal{T}$ and (2.7). Since $L^2(\Omega)$ is dense in $H^{-1}(\Omega)$, $\mathcal{T}$ has a unique extension to an operator on $H^{-1}(\Omega)$, still denoted by $\mathcal{T}$, satisfying (2.10) on $H^{-1}(\Omega)$.

Let us show (2.9). Let $f_1, f_2 \in L^2(\Omega)$ and $u_1 = \mathcal{T}f_1$, $u_2 = \mathcal{T}f_2$. By the definition of $\mathcal{T}$, the sequences $(\tilde{u}_{i,\sigma(h)})$ of the extended solutions to problems

$$\begin{cases} u_{i,\sigma(h)} \in H^1(\Omega_{\sigma(h)}),\ u_{i,\sigma(h)}=0 \ \text{ on } \partial\Omega , \\ \displaystyle\int_{\Omega_{\sigma(h)}} (a_{\sigma(h)}(x,Du_{i,\sigma(h)}),Dw)dx = \int_{\Omega_{\sigma(h)}} (\mathcal{J}^{-1}f_i)w\,dx \\ \forall\ w \in H^1(\Omega_{\sigma(h)}),\ w=0 \ \text{ on } \partial\Omega \end{cases}$$

converge to $\mathcal{T}f_i$ weakly in $H^1(\Omega)$, for $i = 1, 2$. Now, by the continuity assumption (1.9) and the monotonicity condition (1.8), for every $w \in H_0^1(\Omega)$ with $\|w\|_{H_0^1(\Omega)} \leq 1$ we get

$$|<1_{\Omega_{\sigma(h)}}(\mathcal{J}^{-1}f_1) - 1_{\Omega_{\sigma(h)}}(\mathcal{J}^{-1}f_2),w>| = |\int_{\Omega_{\sigma(h)}} (a_{\sigma(h)}(x,Du_{1,\sigma(h)}) - a_{\sigma(h)}(x,Du_{2,\sigma(h)}),Dw)dx|$$

$$\leq c_2\,\|Du_{1,\sigma(h)} - Du_{2,\sigma(h)}\|_{(L^2(\Omega_{\sigma(h)}))^n}$$

$$\leq \sqrt{\frac{c_0}{c_1}}\,c_2\Big(\int_{\Omega_{\sigma(h)}} (a_{\sigma(h)}(x,Du_{1,\sigma(h)}) - a_{\sigma(h)}(x,Du_{2,\sigma(h)}),Du_{1,\sigma(h)} - Du_{2,\sigma(h)})dx\Big)^{1/2}$$

$$= \sqrt{\frac{c_0}{c_1}}\,c_2\,(<1_{\Omega_{\sigma(h)}}(\mathcal{J}^{-1}f_1) - 1_{\Omega_{\sigma(h)}}(\mathcal{J}^{-1}f_2),\tilde{u}_{1,\sigma(h)} - \tilde{u}_{2,\sigma(h)}>)^{1/2}.$$

Since $(1_{\Omega_{\sigma(h)}}(\mathcal{J}^{-1}f_i))$ converges to f_i strongly in $H^{-1}(\Omega)$, $i = 1, 2$, by passing to the limit in the above inequalities, as h tends to $+\infty$, we have

$$|<f_1 - f_2,w>| \leq \sqrt{\frac{c_0}{c_1}}\,c_2\,(<f_1 - f_2,\mathcal{T}f_1 - \mathcal{T}f_2>)^{1/2}$$

for every $w \in H_0^1(\Omega)$, $\|w\|_{H_0^1(\Omega)} \le 1$, which proves (2.9) on $L^2(\Omega)$. By the continuity of $\mathcal{T}$ given by (2.10) we may conclude immediately that (2.9) holds on $H^{-1}(\Omega)$. ♦

Proof of Lemma 2.4. Since $\mathcal{T}$ is strictly monotone, coercive, and continuous, it is a one to one operator from $H^{-1}(\Omega)$ into $H_0^1(\Omega)$. Moreover, it follows immediately by the definition of $\mathcal{A}$ and the estimates (2.9) and (2.10) that (2.12) and (2.13) hold. ♦

Proof of Lemma 2.5. Let us prove (2.14). Let $f_1, f_2 \in V$ and let $u_{1,\sigma(h)}$, $u_{2,\sigma(h)}$ be the solutions to

$$\begin{cases} u_{i,\sigma(h)} \in H^1(\Omega_{\sigma(h)}) \,,\; u_{i,\sigma(h)}=0 \;\text{ on } \partial\Omega \,, \\[2mm] \displaystyle\int_{\Omega_{\sigma(h)}} (a_{\sigma(h)}(x, Du_{i,\sigma(h)}), Dw)dx = \int_{\Omega_{\sigma(h)}} f_i w dx \\[2mm] \forall\; w \in H^1(\Omega_{\sigma(h)}) \,,\; w=0 \;\text{ on } \partial\Omega \,. \end{cases}$$

By the definition of B the extended momenta $(g_{i,\sigma(h)})$ converge to Bf_i weakly in $(L^2(\Omega))^n$, $i = 1, 2$. Now, by using $w = u_{1,\sigma(h)} - u_{2,\sigma(h)}$ as test function in the above equations, by taking their difference, and by applying the monotonicity assumption (1.8) we get

$$c_1 \int_{\Omega_{\sigma(h)}} |Du_{1,\sigma(h)} - Du_{2,\sigma(h)}|^2 dx \;\le\; \int_{\Omega_{\sigma(h)}} (f_1 - f_2, u_{1,\sigma(h)} - u_{2,\sigma(h)})dx \;,$$

which implies easily

$$c_1 \|Du_{1,\sigma(h)} - Du_{2,\sigma(h)}\|_{(L^2(\Omega_{\sigma(h)}))^n} \le \|f_1 - f_2\|_{L^2(\Omega)} \;.$$

Then, by the Lipschitz-continuity (1.9) it follows that

$$\|a_{\sigma(h)}(x, Du_{1,\sigma(h)}) - a_{\sigma(h)}(x, Du_{2,\sigma(h)})\|_{(L^2(\Omega_{\sigma(h)}))^n} \le \frac{c_2}{c_1} \|f_1 - f_2\|_{L^2(\Omega)} \;.$$

Finally, the definition of $g_{i,\sigma(h)}$ and the weak lower semicontinuity of the L^2-norm imply (2.14) on V.

Now, it is clear that B can be extended by continuity in a unique way to $L^2(\Omega)$, still preserving (2.14). ♦

Let us state the following well-known Poincaré inequality which will be useful in the proof of Lemma 2.6.

Proposition 3.1. *Let* $1 < p < +\infty$ *, and let* Ω *be a bounded open connected subset of* $\mathbb{R}^n$ *with Lipschitz continuous boundary* $\partial\Omega$ *. Then, for every* $\varepsilon > 0$ *there exists a constant* $c = c(\varepsilon,p,\Omega) > 0$ *such that*

$$\int_\Omega |u|^p dx \le c \int_\Omega |Du|^p dx$$

for every $u \in H^{1,p}(\Omega)$ *with* $|\{x\in\Omega : u(x) = 0\}| \ge \varepsilon$ *, where* $|\cdot|$ *denotes the n-dimensional Lebesgue measure.*

Proof of Lemma 2.6. Let us start by proving (2.16) on Y . Let $u_1, u_2 \in Y$ and let $u_{1,\sigma(h)}$, $u_{2,\sigma(h)}$ be the solutions to

$$(3.2)\qquad \begin{cases} u_{i,\sigma(h)} \in H^1(\Omega_{\sigma(h)}) \ , \ u_{i,\sigma(h)}=0 \ \text{ on } \partial\Omega \ , \\ \displaystyle\int_{\Omega_{\sigma(h)}} (a_{\sigma(h)}(x,Du_{i,\sigma(h)}),Dw)dx = \int_{\Omega_{\sigma(h)}} (\mathcal{J}^{-1}\mathcal{A}u_i)w\,dx \\ \forall\ w \in H^1(\Omega_{\sigma(h)}) \ , \ w=0 \ \text{ on } \partial\Omega \ . \end{cases}$$

Now, by taking $w = u_{1,\sigma(h)} - u_{2,\sigma(h)}$ as test function in the difference of the two equations, the monotonicity assumption (1.8) and the extension estimate (1.5) guarantee that

$$\frac{c_1}{c_0} \|D\tilde{u}_{1,\sigma(h)} - D\tilde{u}_{2,\sigma(h)}\|_{(L^2(\Omega))^n} \le \|1_{\Omega_{\sigma(h)}}[(\mathcal{J}^{-1}\mathcal{A})u_1 - (\mathcal{J}^{-1}\mathcal{A})u_2]\|_{H^{-1}(\Omega)} \ .$$

This inequality together with the continuity condition (1.9) gives

$$\|g_{1,\sigma(h)} - g_{2,\sigma(h)}\|_{(L^2(\Omega))^n} \le \frac{c_0}{c_1} c_2 \|1_{\Omega_{\sigma(h)}}[(\mathcal{J}^{-1}\mathcal{A})u_1 - (\mathcal{J}^{-1}\mathcal{A})u_2]\|_{H^{-1}(\Omega)} \ .$$

Since $(g_{i,\sigma(h)})$ converges to Au_i weakly in $(L^2(\Omega))^n$, $i = 1, 2$, by the weak lower semicontinuity of the L^2-norm we get

$$\|Au_1 - Au_2\|_{(L^2(\Omega))^n} \le \frac{c_0}{c_1} c_2 \|\mathcal{A}u_1 - \mathcal{A}u_2\|_{H^{-1}(\Omega)} \ ,$$

which by (2.13) yields (2.16) on Y. Therefore, the operator A has a unique continuous extension to $H^1_0(\Omega)$, still denoted by A, which satisfies (2.16).

By the continuity of A it is enough to prove (2.17) and (2.18) for every $u_1, u_2 \in Y$. Let us start by proving (2.17) on Y. To this aim, given $u_1, u_2 \in Y$ and an open connected set $U'' \subseteq \Omega$, we show that

$$\int_{U''} (Au_1 - Au_2, Du_1 - Du_2)dx \geq \frac{c_1}{c_0} \int_{U''} |Du_1 - Du_2|^2 dx \ , \tag{3.3}$$

from which (2.17) follows by the arbitrariness of U'' and the Lebesgue derivation theorem. Let $u_{i,\sigma(h)}$ be the solution to problem (3.2). Given two open connected sets U' and U such that $U' \subset\subset U \subset\subset U''$, U' connected with a regular boundary, and $\varphi \in C^\infty_0(U'')$ with $\varphi \equiv 1$ on U and $0 \leq \varphi \leq 1$ on U'', by the local extension property (1.3) we have

$$\int_{U'} |D\hat{u}_{1,\sigma(h)} - D\hat{u}_{2,\sigma(h)}|^2 \varphi dx \leq c_{\sigma(h)} \int_{U \cap \Omega_{\sigma(h)}} |Du_{1,\sigma(h)} - Du_{2,\sigma(h)}|^2 dx \ ,$$

where $\hat{u}_{i,\sigma(h)} \in H^1(U')$ denotes the extension of $u_{i,\sigma(h)} \in H^1(U \cap \Omega_{\sigma(h)})$ given by Definition 1.1. By the monotonicity assumption (1.8) it follows that

$$\int_{U'} |D\hat{u}_{1,\sigma(h)} - D\hat{u}_{2,\sigma(h)}|^2 dx \leq \tag{3.4}$$

$$\leq \frac{c_{\sigma(h)}}{c_1} \int_{U \cap \Omega_{\sigma(h)}} (a_{\sigma(h)}(x, Du_{1,\sigma(h)}) - a_{\sigma(h)}(x, Du_{2,\sigma(h)}), Du_{1,\sigma(h)} - Du_{2,\sigma(h)}) \varphi dx$$

$$\leq \frac{c_{\sigma(h)}}{c_1} \int_{U''} (g_{1,\sigma(h)} - g_{2,\sigma(h)}, D\tilde{u}_{1,\sigma(h)} - D\tilde{u}_{2,\sigma(h)}) \varphi dx \ ,$$

where $\tilde{u}_{i,\sigma(h)} = E_{\sigma(h)} u_{i,\sigma(h)} \in H^1_0(\Omega)$ and $g_{i,\sigma(h)} \in (L^2(\Omega))^n$ are the extended solution and momentum corresponding to problem (3.2), with $i = 1, 2$, and $E_{\sigma(h)}$ is the global extension operator given by Definition 1.3. Since the sequences $(\tilde{u}_{i,\sigma(h)})$ converges to u_i weakly in $H^1(\Omega)$, the sequences $(g_{i,\sigma(h)})$ converges to Au_i weakly in $(L^2(\Omega))^n$, and

$$-\text{div}(g_{1,\sigma(h)} - g_{2,\sigma(h)}) = 1_{\Omega_{\sigma(h)}}((\mathcal{J}^{-1}\mathcal{A})u_1 - (\mathcal{J}^{-1}\mathcal{A})u_2) \rightarrow (\mathcal{A}u_1 - \mathcal{A}u_2)$$

weakly in $L^2(\Omega)$ (hence, strongly in $H^{-1}(\Omega)$), by a compensated compactness result (see, for example, Lemma 3.4 in Chapter 1) the last inequality implies

(3.5) $$\limsup_{h\to+\infty} \int_{U'} |D\hat{u}_{1,\sigma(h)} - D\hat{u}_{2,\sigma(h)}|^2 dx \le \frac{c_0}{c_1} \int_{U''} (Au_1 - Au_2, Du_1 - Du_2) dx \quad .$$

By the uniform boundedness of $(\tilde{u}_{1,\sigma(h)} - \tilde{u}_{2,\sigma(h)})$ in $H^1(U')$ and the estimate (3.5) it follows that the derivatives of the functions $v_h = (\hat{u}_{1,\sigma(h)} - \hat{u}_{2,\sigma(h)}) - (\tilde{u}_{1,\sigma(h)} - \tilde{u}_{2,\sigma(h)})$ are bounded in $L^2(U')$, uniformly with respect to h. By the Poincaré inequality (Proposition 3.1) this implies that the sequence (v_h) is bounded in $H^1(U')$ uniformly with respect to h. Therefore, up to a subsequence, $(\hat{u}_{1,\sigma(h)} - \hat{u}_{2,\sigma(h)})$ converges to a function w weakly in $H^1(U')$. Since $(\tilde{u}_{1,\sigma(h)} - \tilde{u}_{2,\sigma(h)})$ converges to $u_1 - u_2$ weakly in $H^1(U')$ and $(\hat{u}_{1,\sigma(h)} - \hat{u}_{2,\sigma(h)}) = (\tilde{u}_{1,\sigma(h)} - \tilde{u}_{2,\sigma(h)})$ on $U' \cap \Omega_{\sigma(h)}$, by Lemma (1.6) we may conclude that $w = u_1 - u_2$ a.e. on U'. Hence, by the weak lower semicontinuity of the L^2-norm the inequality (3.5) implies

$$\int_{U'} |Du_1 - Du_2|^2 dx \le \frac{c_0}{c_1} \int_{U''} (Au_1 - Au_2, Du_1 - Du_2) dx \quad .$$

By taking the supremum over $U' \subset\subset U''$ we achieve (3.3) and conclude the proof of (2.17).

Let us prove (2.18) on Y. Let $u_1, u_2 \in Y$ and let $u_{i,\sigma(h)}$ be the solution to problem (3.2) for $i = 1, 2$. Moreover let U', U, U'', and φ be as in the proof of (2.17). By the continuity assumption (1.9) we have

$$\int_{U' \cap \Omega_{\sigma(h)}} |a_{\sigma(h)}(x, Du_{1,\sigma(h)}) - a_{\sigma(h)}(x, Du_{2,\sigma(h)})|^2 dx \le c_2^2 \int_{U'} |D\hat{u}_{1,\sigma(h)} - D\hat{u}_{2,\sigma(h)}|^2 dx \ ,$$

where $\hat{u}_{i,\sigma(h)} \in H^1(U')$ denotes the local extension of $u_{i,\sigma(h)} \in H^1(U \cap \Omega_{\sigma(h)})$ given by Definition 1.1. This implies by (3.4) that

(3.6) $$\int_{U'} |g_{1,\sigma(h)} - g_{2,\sigma(h)}|^2 dx \le$$

$$\le \frac{c_{\sigma(h)}}{c_1} c_2^2 \int_{U''} (g_{1,\sigma(h)} - g_{2,\sigma(h)}, D\tilde{u}_{1,\sigma(h)} - D\tilde{u}_{2,\sigma(h)}) \varphi dx \ ,$$

where $\tilde{u}_{i,\sigma(h)} = E_{\sigma(h)} u_{i,\sigma(h)} \in H_0^1(\Omega)$ and $g_{i,\sigma(h)} \in (L^2(\Omega))^n$ are the extended solution and momentum corresponding to problem (3.2), with $i = 1, 2$, and $E_{\sigma(h)}$ is the global extension operator given by Definition 1.3. Since $(\tilde{u}_{i,\sigma(h)})$ converges to u_i weakly in $H^1(\Omega)$ and $(g_{i,\sigma(h)})$ converge to Au_i weakly in $(L^2(\Omega))^n$, $i = 1, 2$, by applying the weak lower semicontinuity of the L^2-norm to the left hand side and a compensated compactness result to the right hand side of (3.6) we get

$$\int_{U'} |Au_1 - Au_2|^2 dx \;\leq\; \frac{c_0}{c_1} c_2^2 \int_{U''} (Au_1 \;\; Au_2, Du_1 - Du_2) dx \quad .$$

By taking the supremum over $U' \subset\subset U''$ and by noticing that U'' is an arbitrary open subset of Ω, we obtain finally (2.18) on Y as an application of the Lebesgue derivation theorem. ♦

4. THE CASE $p \neq 2$

The aim of this section is to state the analogue of Theorem 2.1 in the case $H^{1,p}(\Omega)$, $1 < p < +\infty$. We assume that the sequence of sets (Ω_h) considered in Section 1 satisfies the extension properties (1.3) and (1.5) with the exponent 2 replaced by p and that the sequence of functions (a_h) is as follows.

Given two constants α and β, with $0 \leq \alpha \leq 1 \wedge (p-1)$ and $p \vee 2 \leq \beta < +\infty$, let $a_h : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ be satisfying the following conditions:

(4.1) $\quad a_h(\cdot,\xi)$ is measurable for every $\xi \in \mathbf{R}^n$,

(4.2) $\quad a_h(\cdot,0)$ is bounded in $(L^q(\Omega_h))^n$ by a constant $c_3 > 0$, and $1/p + 1/q = 1$.

Moreover, the following monotonicity and continuity assumptions hold:

(4.3) $\quad (a_h(x,\xi_1) - a_h(x,\xi_2), \xi_1 - \xi_2) \;\geq\; c_1(1 + |\xi_1| + |\xi_2|)^{p-\beta} |\xi_1 - \xi_2|^\beta$

(4.4) $\quad |a_h(x,\xi_1) - a_h(x,\xi_2)| \;\leq\; c_2(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha} |\xi_1 - \xi_2|^\alpha$,

for a.e. $x \in \Omega_h$, for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

It is well known that for every $f \in L^q(\Omega)$ the Neumann boundary value problem (1.10) corresponding to a_h has a unique solution $u_h \in H^{1,p}(\Omega_h)$, i.e. there exists u_h such that

$$(4.5) \qquad \begin{cases} u_h \in H^{1,p}(\Omega_h)\ ,\ u_h = 0 \ \text{ on } \partial\Omega\ , \\[2ex] \displaystyle\int_{\Omega_h} (a_h(x, Du_h), Dw) dx = \int_{\Omega_h} fw dx \qquad \forall\ w \in H^{1,p}(\Omega_h)\ ,\ w = 0 \ \text{ on } \partial\Omega\ . \end{cases}$$

We are now in a position to state the result announced at the beginning of this section.

Theorem 4.1. *Let* (a_h) *be the sequence of functions satisfying (4.1)-(4.4). Then there exist* $\sigma(h) \to +\infty$ *and a Carathéodory function* $a : \Omega\times\mathbf{R}^n \to \mathbf{R}^n$ *such that for every* $f\in L^q(\Omega)$ *we have*

$$\tilde{u}_{\sigma(h)} \to u \quad \textit{weakly in } H^{1,p}(\Omega)\,,$$
$$g_{\sigma(h)} \to a(x,Du) \quad \textit{weakly in } (L^q(\Omega))^n\,,$$

where $\tilde{u}_{\sigma(h)} = E_{\sigma(h)}u_{\sigma(h)}$ *with* $E_{\sigma(h)}$ *the global extension operator given by Definition 1.3,* $u_{\sigma(h)}\in H^{1,p}(\Omega_{\sigma(h)})$ *is the solution to problem (1.10) in the sense of (4.5),*

$$g_{\sigma(h)} = \begin{cases} a_{\sigma(h)}(x,Du_{\sigma(h)}) & \text{on } \Omega_{\sigma(h)}\,, \\ 0 & \text{on } \Omega\setminus\Omega_{\sigma(h)}\,, \end{cases}$$

and $u\in H_0^{1,p}(\Omega)$ *is the unique solution to the Dirichlet boundary value problem*

$$\begin{cases} -\mathrm{div}(a(x,Du)) = bf & \text{on } \Omega\,, \\ u\in H_0^{1,p}(\Omega)\,. \end{cases}$$

Moreover, the function a *satisfies the following monotonicity and continuity conditions*

$$(a(x,\xi_1) - a(x,\xi_2)\,,\,\xi_1-\xi_2) \ \geq\ c(1+|\xi_1|+|\xi_2|)^{p-\beta}|\,\xi_1-\xi_2|^\beta$$

$$|a(x,\xi_1) - a(x,\xi_2)| \ \leq\ c(1+|\xi_1|+|\xi_2|)^{p-1-\gamma}|\xi_1-\xi_2|^\gamma\,,$$

for a.e. $x\in\Omega$ *and for every* $\xi_1\,,\,\xi_2\in\mathbf{R}^n$*, where* $\gamma=\alpha/(\beta-\alpha)$ and $c>0$ *depends only on* n, p, α, β, c_0, c_1, c_2, *and* c_3.

Since the proof of Theorem 4.1 can be obtained by following the same scheme of the proof of Theorem 2.1, we state here only the modifications of the lemmas, given in Section 2, needed for the case $p\neq 2$. The main tool in the proof of these lemmas is the following estimate which is an easy consequence of the Hölder inequality: for every φ_1, $\varphi_2\in(L^p(\Omega))^n$

$$(4.6)\qquad \|\varphi_1-\varphi_2\|_{(L^p(\Omega))^n} \leq$$

$$\leq c\Big(\int_\Omega|\varphi_1-\varphi_2|^\beta(1+|\varphi_1|+|\varphi_2|)^{p-\beta}dx\Big)^{1/\beta}\big(|\Omega|^{1/p}+\|\varphi_1\|_{(L^p(\Omega))^n}+\|\varphi_2\|_{(L^p(\Omega))^n}\big)^{(\beta-p)/\beta}$$

with $c > 0$ depending only on n and p. The proofs of the following lemmas can be obtained proceeding as in the corresponding lemmas for the case $p = 2$ and are based essentially on the Hölder and Young inequalites and on (4.6), and are therefore omitted.

The letter c will denote various positive constants, whose value depends only on $n, p, \alpha, \beta, c_0, c_1, c_2$, and c_3.

Lemma 4.2. *Let* $\mathcal{B} : V \to H_0^{1,p}(\Omega)$ *be the operator defined by* $\mathcal{B}f = u$ *with* u *given by (2.5) corresponding to* $p \neq 2$. *Then,* $\mathcal{B}$ *has a unique extension to* $L^q(\Omega)$, *still denoted by* $\mathcal{B}$, *such that*

$$\|\mathcal{B}f_1 - \mathcal{B}f_2\|_{H_0^{1,p}(\Omega)} \leq c(1+\|bf_1\|_{H^{-1,q}(\Omega)}+\|bf_2\|_{H^{-1,q}(\Omega)})^{q-1-1/(\beta-1)}\|bf_1 - bf_2\|_{H^{-1,q}(\Omega)}^{1/(\beta-1)}$$

for every $f_1, f_2 \in L^q(\Omega)$.

Lemma 4.3. *Let* $\mathcal{J} : L^q(\Omega) \to L^q(\Omega)$ *be the isomorphism of* $L^q(\Omega)$ *given by* $\mathcal{J}f = bf$ *for every* $f \in L^q(\Omega)$. *Let* $\mathcal{T}: L^q(\Omega) \subseteq H^{-1,q}(\Omega) \to H_0^{1,p}(\Omega)$ *be the operator defined by*

$$\mathcal{T}f = (\mathcal{B}\mathcal{J}^{-1})f$$

for every $f \in L^q(\Omega)$, *where* $\mathcal{B} : L^q(\Omega) \to H_0^{1,p}(\Omega)$ *is the operator given by Lemma 4.2. Then,* $\mathcal{T}$ *has a unique extension to* $H^{-1,q}(\Omega)$, *still denoted by* $\mathcal{T}$, *that satisfies*

$$<f_1 - f_2, \mathcal{T}f_1 - \mathcal{T}f_2> \;\geq\; c(1+\|f_1\|_{H^{-1,q}(\Omega)} +\|f_2\|_{H^{-1,q}(\Omega)})^{q-\beta/\alpha}\|f_1 - f_2\|_{H^{-1,q}(\Omega)}^{\beta/\alpha}$$

$$\|\mathcal{T}f_1 - \mathcal{T}f_2\|_{H_0^{1,p}(\Omega)} \leq c(1+\|f_1\|_{H^{-1,q}(\Omega)} +\|f_2\|_{H^{-1,q}(\Omega)})^{q-1-1/(\beta-1)}\|f_1 - f_2\|_{H^{-1,q}(\Omega)}^{1/(\beta-1)}$$

for every $f_1, f_2 \in H^{-1,q}(\Omega)$.

Lemma 4.4. *Let* $\mathcal{T}: H^{-1,q}(\Omega) \to H_0^{1,p}(\Omega)$ *be the operator given by Lemma 4.3. Then,* $\mathcal{T}$ *is invertible and the operator* $\mathcal{A}: H_0^{1,p}(\Omega) \to H^{-1,q}(\Omega)$ *defined by*

$$\mathcal{A}u = \mathcal{T}^{-1} u$$

for every $u \in H_0^{1,p}(\Omega)$ *satisfies*

$$<\mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2> \;\geq\; c(1+\|u_1\|_{H_0^{1,p}(\Omega)} +\|u_2\|_{H_0^{1,p}(\Omega)})^{p-\beta(\beta-1)/\alpha}\|u_1 - u_2\|_{H_0^{1,p}(\Omega)}^{\beta(\beta-1)/\alpha}$$

$$\|\mathcal{A}u_1 - \mathcal{A}u_2\|_{H^{-1,q}(\Omega)} \leq c(1+\|u_1\|_{H_0^{1,p}(\Omega)} + \|u_2\|_{H_0^{1,p}(\Omega)})^{p-1-\alpha/(\beta-\alpha)} \|u_1 - u_2\|_{H_0^{1,p}(\Omega)}^{\alpha/(\beta-\alpha)}$$

for every $u_1, u_2 \in H_0^{1,p}(\Omega)$.

Lemma 4.5. *Let* $B : V \rightarrow (L^q(\Omega))^n$ *be the operator defined by* $Bf = g$ *with* g *given by (2.6) corresponding to* $p \neq 2$ *. Then, it has a unique extension to* $L^q(\Omega)$ *, still denoted by* B *, such that*

$$\|Bf_1 - Bf_2\|_{(L^q(\Omega))^n} \leq c(1+ \|f_1\|_{L^q(\Omega)} + \|f_2\|_{L^q(\Omega)})^{1-\alpha/(\beta-1)]} \|f_1 - f_2\|_{L^q(\Omega)}^{\alpha/(\beta-1)}$$

for every $f_1, f_2 \in L^q(\Omega)$.

Lemma 4.6. *Let* $\mathcal{T}$ *and* $\mathcal{J}$ *be the operators given in Lemma 4.3. Let* Y *be the countable dense subset of* $H_0^{1,p}(\Omega)$ *given by* $Y = (\mathcal{T}\mathcal{J})(V)$ *and let* $A : Y \rightarrow (L^q(\Omega))^n$ *be the operator defined by*

$$Au = (B\mathcal{J}^{-1}\mathcal{A})u$$

for every $u \in Y$ *, where* $B : L^q(\Omega) \rightarrow (L^q(\Omega))^n$ *and* $\mathcal{A} : H_0^{1,p}(\Omega) \rightarrow H^{-1,q}(\Omega)$ *are the operators given in Lemmas 4.5 and 4.4, respectively. Then,* A *has a unique extension to* $H_0^{1,p}(\Omega)$ *, still denoted by* A *, satisfying*

$$\|Au_1 - Au_2\|_{(L^q(\Omega))^n} \leq c(1+\|u_1\|_{H_0^{1,p}(\Omega)} + \|u_2\|_{H_0^{1,p}(\Omega)})^{p-1-\gamma} \|u_1 - u_2\|_{H_0^{1,p}(\Omega)}^{\gamma}$$

for every $u_1, u_2 \in H_0^1(\Omega)$, *where* $\gamma = \alpha/(\beta-\alpha)$. *Moreover,*

$$(Au_1(x) - Au_2(x), Du_1(x) - Du_2(x)) \geq c(1+|Du_1(x)| + |Du_2(x)|)^{p-\beta} |Du_1(x) - Du_2(x)|^{\beta} ,$$
$$|Au_1(x) - Au_2(x)| \leq c(1+|Du_1(x)| + |Du_2(x)|)^{p-1-\gamma} |Du_1(x) - Du_2(x)|^{\gamma}$$

for a.e. $x \in \Omega$ *, for every* $u_1, u_2 \in H_0^{1,p}(\Omega)$.

CHAPTER 4

HOMOGENIZATION OF QUASI-LINEAR EQUATIONS WITH NATURAL GROWTH TERMS

In this chapter we deal with the limit behaviour of the bounded solutions u_ε of quasi-linear equations of the form $-\mathrm{div}(a(\frac{x}{\varepsilon},Du_\varepsilon)) + \gamma|u_\varepsilon|^{p-2}u_\varepsilon = H(\frac{x}{\varepsilon},u_\varepsilon,Du_\varepsilon) + h(x)$ on Ω with Dirichlet boundary conditions on $\partial\Omega$. The map $a = a(x,\xi)$ is periodic in x, monotone in ξ, and satisfies suitable coerciveness and growth conditions. The function $H = H(x,s,\xi)$ is assumed to be periodic in x, continuous in $[s,\xi]$ and to grow at most like $|\xi|^p$. Under these assumptions on a and H we prove that there exists a function $H^0 = H^0(s,\xi)$ with the same behaviour of H, such that, up to a subsequence, (u_ε) converges to a solution u of the homogenized problem $-\mathrm{div}(b(Du)) + \gamma|u|^{p-2}u = H^0(u,Du) + h(x)$ on Ω, where b depends only on a and has analogous qualitative properties. The results of this chapter are contained in [28].

INTRODUCTION

This chapter is concerned with a homogenization result for quasi-linear boundary value problems of the type

$$(0.1)\qquad \begin{cases} -\mathrm{div}(a(\frac{x}{\varepsilon},Du_\varepsilon)) + \gamma|u_\varepsilon|^{p-2}u_\varepsilon = H(\frac{x}{\varepsilon},u_\varepsilon,Du_\varepsilon) + h(x) \quad \text{on } \Omega \,, \\ u_\varepsilon \in H_0^{1,p}(\Omega)\cap L^\infty(\Omega) \,, \end{cases}$$

where Ω is a bounded open subset of $\mathbf{R}^n$, γ is a positive constant, p satisfies $1 < p < +\infty$, and $h \in L^\infty(\Omega)$. We assume that the map $a : \mathbf{R}^n\times\mathbf{R}^n \to \mathbf{R}^n$, $a = a(x,\xi)$, is periodic in x, monotone and continuous in ξ, and satisfies suitable coerciveness and growth conditions. The main feature of the function $H : \mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$, $H = H(x,s,\xi)$, which is also periodic in x, and continuous with respect to s and ξ, is its growth of order p in ξ.

Under these assumptions an existence result for problems of the type (0.1) has been proved by Boccardo, Murat and Puel in [15].

In the present chapter we construct a function $H^0 : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$, $H^0 = H^0(s,\xi)$, with the same qualitative behaviour of H, and prove that every sequence (u_ε) of solutions to problems (0.1) converges, up to a subsequence, in the weak topology of $H_0^{1,p}(\Omega)$, to a solution u of the homogenized boundary value problem

$$(0.2)\qquad \begin{cases} -\mathrm{div}(b(Du)) + \gamma|u|^{p-2}u = H^0(u,Du) + h(x) \quad \text{on } \Omega\,, \\ u\in H_0^{1,p}(\Omega)\cap L^\infty(\Omega)\,. \end{cases}$$

The monotone map $b : \mathbf{R}^n \to \mathbf{R}^n$ in (0.2) depends only on the map a which appears in the principal part $-\mathrm{div}(a(\frac{x}{\varepsilon},Du_\varepsilon))$ and can be expressed by a homogenization formula, whose proof is given in Chapter 2. On the contrary, H^0 depends both on a and H, and is independent of h.

As a main tool for our proof we first obtain a corrector result for the solutions (u_ε) to problems (0.1). More precisely, we show that the family of correctors $p_\varepsilon(x,\xi)$, introduced in [33] and depending only on the function a, allows us to prove that

$$Du_\varepsilon = p_\varepsilon(\cdot,M_\varepsilon Du) + r_\varepsilon\,,$$

where the rest r_ε, depending on u and u_ε, converges to 0 strongly in $(L^p(\Omega))^n$. The sequence (M_ε) is a family of approximations of the identity map on $(L^p(\Omega))^n$ such that $M_\varepsilon\varphi$ is a step function for every $\varphi\in(L^p(\Omega))^n$. The family (p_ε) has the form

$$p_\varepsilon(x,\xi) = p(\tfrac{x}{\varepsilon},\xi)$$

for a suitable periodic function $p(\cdot,\xi)$ which is obtained by solving the same auxiliary problem used in the construction of the homogenized operator b.

The corrector result permits us to pass to the limit on the right hand side of (0.1), overcoming the difficulties due to the "natural" growth of $H(x,s,\xi)$ in ξ.

The proofs follow the method introduced in [9] for the case where $a(x,\cdot)$ is linear.

The homogenization problems have been investigated in recent years by several authors. For the general framework of homogenization theory, the methods, several examples, and references we address the reader to the books [10], [61], [6].

The homogenization of linear elliptic operators of the form $-\mathrm{div}(a(\frac{x}{\varepsilon},Du))$ has been treated, among others, in [36], [67]. Results for the quasi-linear case and further references on this topic are given in Chapter 2. The asymptotic behaviour of problems of the form (0.1) where $a(x,\cdot) = a(x)\xi$ and $H(x,s,\xi)$ has quadratic growth with respect to ξ have been investigated in [14], while the homogenization of elliptic equations with principal part not in divergence form has been studied in [8].

1. NOTATION AND PRELIMINARY RESULTS

Given $\lambda > 0$, we say that a function $u : \mathbf{R}^n \to \mathbf{R}$ is λ-periodic if $u(x+\lambda e_i) = u(x)$ for every $x \in \mathbf{R}^n$ and for every $i = 1,\dots,n$, where (e_i) is the canonical base of $\mathbf{R}^n$. Let $Y =]0,1[^n$ be the unit cube in $\mathbf{R}^n$. Let p be a real constant, $1 < p < +\infty$, and let q be its dual exponent, $1/p + 1/q = 1$. By $H^{1,p}_{\#}(Y)$ we denote the set of all functions $u \in H^{1,p}(Y)$ with mean value zero which have the same trace on the opposite faces of Y. Every function u of $H^{1,p}_{\#}(Y)$ can be extended by periodicity to a function of $H^{1,p}_{loc}(\mathbf{R}^n)$.

The Euclidean norm and the scalar product in $\mathbf{R}^n$ are denoted by $|\cdot|$ and $(\cdot,\cdot)$, respectively.

Given two constants $c_1, c_2 > 0$, and two constants α and β, with $0 \le \alpha \le 1 \wedge (p-1)$ and $p \vee 2 \le \beta < +\infty$, let us fix a function $a : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ which fulfills the following conditions:

(1.1) for every $\xi \in \mathbf{R}^n$, $a(\cdot,\xi)$ is 1-periodic and Lebesgue measurable,

(1.2) $a(x,0) = 0$ for a.e. $x \in \mathbf{R}^n$.

Moreover, we make the following continuity and monotonicity assumptions:

(1.3) $|a(x,\xi_1) - a(x,\xi_2)| \le c_1(1 + |\xi_1| + |\xi_2|)^{p-1-\alpha}|\xi_1-\xi_2|^{\alpha}$,

(1.4) $(a(x,\xi_1) - a(x,\xi_2),\xi_1 - \xi_2) \ge c_2(|\xi_1| + |\xi_2|)^{p-\beta}|\xi_1-\xi_2|^{\beta}$

for a.e. $x \in \mathbf{R}^n$, for every $\xi_1, \xi_2 \in \mathbf{R}^n$.

Let us fix from now on a bounded open subset Ω of $\mathbf{R}^n$ with boundary $\partial\Omega$ regular enough to guarantee the Meyers estimate in Theorem 1.3. For every $\varepsilon > 0$ we define $a_\varepsilon(x,\xi) = a(\frac{x}{\varepsilon},\xi)$ for every $x \in \mathbf{R}^n$ and for every $\xi \in \mathbf{R}^n$. By a classical result in existence theory for boundary value problems defined by monotone operators (see, for instance, [47], Chapter 2, Theorem 2.1, or [43], Chapter 3, Corollary 1.8) it follows that for every $f \in H^{-1,q}(\Omega)$ and for every $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in H^{1,p}_0(\Omega)$ to the following Dirichlet boundary value problem

$$(1.5)\qquad \begin{cases} -\mathrm{div}(a_\varepsilon(x,Du_\varepsilon)) + \gamma|u_\varepsilon|^{p-2}u_\varepsilon = f \quad \text{on } \Omega , \\ u_\varepsilon \in H^{1,p}_0(\Omega) \end{cases}$$

with $\gamma > 0$.

Now, we state a homogenization theorem which is a particular case of a more general result proved in Theorem 3.2 of Chapter 2.

Theorem 1.1. *Let* $f \in H^{-1,q}(\Omega)$ *and let* u_ε *be the solution to problem (1.5). Then, as* $\varepsilon \to 0$ *, we have*

$$u_\varepsilon \to u \quad \textit{weakly in } H^{1,p}(\Omega) \ ,$$
$$a_\varepsilon(x,Du_\varepsilon) \to b(Du) \quad \textit{weakly in } (L^q(\Omega))^n \ ,$$

where u *is the solution to the homogenized problem*

$$(1.6) \qquad \begin{cases} -\mathrm{div}(b(Du)) + \gamma|u|^{p-2}u = f & \textit{on } \Omega \ , \\ u \in H_0^{1,p}(\Omega) \ . \end{cases}$$

The function $b : \mathbf{R}^n \to \mathbf{R}^n$ *is independent of* f *and is defined, for every* $\xi \in \mathbf{R}^n$ *, by*

$$b(\xi) = \int_Y a(x,\xi+Dv_\xi(x))dx \ ,$$

where v_ξ *is the solution to the following local problem on* Y :

$$(1.7) \qquad \begin{cases} \displaystyle\int_Y (a(x,\xi+Dv_\xi),Dw)dx = 0 & \textit{for every } w \in H_\#^{1,p}(Y) \ , \\ v_\xi \in H_\#^{1,p}(Y) \ . \end{cases}$$

Remark 1.2. By Remark 7.7 and by Theorem 7.9 in Chapter 1 it follows that the function $b : \mathbf{R}^n \to \mathbf{R}^n$ satisfies the inequalities

$$(1.8) \qquad |b(\xi_1) - b(\xi_2)| \le \tilde{c}_1(1 + |\xi_1| + |\xi_2|)^{p-1-\mu}|\xi_1-\xi_2|^\mu \ ,$$
$$(1.9) \qquad (b(\xi_1) - b(\xi_2),\xi_1 - \xi_2) \ge \tilde{c}_2(1 + |\xi_1| + |\xi_2|)^{p-\beta}|\xi_1-\xi_2|^\beta$$

for every $\xi_1, \xi_2 \in \mathbf{R}^n$, where $\mu = \alpha/(p-\alpha)$, and $\tilde{c}_1, \tilde{c}_2$ are strictly positive constants depending on $n, p, \alpha, \beta, c_1, c_2$.

A useful tool for our purposes will be the following regularity result known as the Meyers estimate (for a proof, see for instance, [51]).

Theorem 1.3. *Let* $f \in H^{-1,\sigma}(\Omega)$, $\sigma > q$. *Let* $w \in H_0^{1,p}(\Omega)$ *be a weak solution to the equation*

$$\begin{cases} -\mathrm{div}(a(x,Dw)) + \gamma |w|^{p-2} w = f \quad on \ \Omega , \\ w \in H_0^{1,p}(\Omega) , \end{cases}$$

where γ *is a non-negative constant. Then there exists* $\eta > 0$ *such that we have* $w \in H_0^{1,p}(\Omega) \cap H^{1,p+\eta}(\Omega)$ *and*

$$\|w\|_{H^{1,p+\eta}(\Omega)} \leq c \|w\|_{H^{1,p}(\Omega)} . \tag{1.10}$$

The constant η *depends only on* n, p, σ, c_1, c_2, *while* $c > 0$ *depends in addition on* Ω *and on* $\|f_i\|_{L^q(\Omega)}$, *when* f *is represented by* $f = \sum_{i=1}^{n} (-D_i f_i) + f_0$.

If nothing else is specified, in this chapter the letter c will denote various positive constants, whose value can change from one line to the other.

Now, in order to state a corrector result for problem (1.5) we begin by defining the family (M_ε) of approximations of the identity map on $(L^p(\Omega))^n$. To this aim let us consider for $i \in \mathbf{Z}^n$ and $\varepsilon > 0$ the cube $Y_\varepsilon^i = i + \varepsilon Y$, i.e. the translated image of $Y_\varepsilon = \varepsilon Y$ by the integer vector $i \in \mathbf{Z}^n$. Given $\varphi \in (L^p(\Omega))^n$ let us define the function $M_\varepsilon \varphi : \mathbf{R}^n \to \mathbf{R}^n$ by

$$(M_\varepsilon \varphi)(x) = \sum_{i \in I_\varepsilon} 1_{Y_\varepsilon^i}(x) \frac{1}{|Y_\varepsilon^i|} \int_{Y_\varepsilon^i} \varphi(y) dy , \tag{1.11}$$

where $I_\varepsilon = \{i \in \mathbf{Z}^n : Y_\varepsilon^i \subseteq \Omega\}$, 1_A is the characteristic function of a set $A \subseteq \mathbf{R}^n$, and $|A|$ denotes its Lebesgue measure. For some properties satisfied by M_ε we refer to [33], Section 2.

Finally, let us define the function $p : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ by

$$p(x,\xi) = \xi + Dv_\xi(x) , \tag{1.12}$$

where v_ξ is the unique solution to the local problem (1.7). Let us remark that v_ξ can be extended by periodicity to a function of $H_{loc}^{1,p}(\mathbf{R}^n)$ and that problem (1.7) is equivalent to $-\mathrm{div}(a(x,\xi+Dv_\xi)) = 0$ in $\mathcal{D}'(\mathbf{R}^n)$. It follows that $p(\cdot,\xi)$ is 1-periodic and that the function $p_\varepsilon : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$p_\varepsilon(x,\xi) = p(\tfrac{x}{\varepsilon},\xi) = \xi + Dv_\xi(\tfrac{x}{\varepsilon}) \tag{1.13}$$

is ε-periodic in x. Among the properties of p_ε, that can be found in [33], Section 2, we recall here just the ones we need in the sequel.

Lemma 1.4. *For every* $\xi, \xi_1, \xi_2 \in \mathbf{R}^n$ *we have*

$$(1.14)\qquad \|p_\varepsilon(\cdot,\xi)\|^p_{(L^p(Y_\varepsilon))^n} \le m_1(1+|\xi|^p)|Y_\varepsilon| ,$$

$$(1.15)\qquad \|p_\varepsilon(\cdot,\xi_1) - p_\varepsilon(\cdot,\xi_2)\|^p_{(L^p(Y_\varepsilon))^n} \le m_2(1+|\xi_1|^p + |\xi_2|^p)^{(\beta-\alpha-1)/(\beta-\alpha)}|\xi_1-\xi_2|^{p/(\beta-\alpha)}|Y_\varepsilon| ,$$

where $m_1 > 0$ *depends only on* n, p, β, c_1, c_2, *while* m_2 *depends in addition on* α.

Lemma 1.5. *Let* $\varphi \in (L^p(\Omega))^n$ *and let* ψ *be a simple function of the form*

$$\psi = \sum_{j=1}^{m} \eta_j 1_{\Omega_j}$$

with $\eta_j \in \mathbf{R}^n \setminus \{0\}$, $\Omega_j \subset\subset \Omega$, $|\partial\Omega_j| = 0$, $\Omega_j \cap \Omega_k = \emptyset$ *for* $j \ne k$. *Then*

$$(1.16)\qquad \limsup_{\varepsilon\to 0} \|p_\varepsilon(\cdot,M_\varepsilon\varphi) - p_\varepsilon(\cdot,\psi)\|_{(L^p(\Omega))^n} \le$$

$$\le c(|\Omega| + \|\varphi\|_{(L^p(\Omega))^n} + \|\psi\|_{(L^p(\Omega))^n})^{(\beta-\alpha-1)/(\beta-\alpha)} \|\varphi-\psi\|^{1/(\beta-\alpha)}_{(L^p(\Omega))^n} ,$$

where the constant c *depends only on* $n, p, \alpha, \beta, c_1, c_2$.

Finally, we conclude this section by stating the following corrector result which can be obtained modifying slightly the proof of Theorem 2.1 in [33].

Theorem 1.6. *Let* $f \in H^{-1,q}(\Omega)$, *let* u_ε *be the solutions to problem (1.5) and let* u *be the solution to problem (1.6). Then*

$$(1.17)\qquad Du_\varepsilon = p_\varepsilon(\cdot,M_\varepsilon Du) + r_\varepsilon ,$$

where p_ε *is defined by (1.13) and* (r_ε) *converges to* 0 *strongly in* $(L^p(\Omega))^n$ *as* ε *tends to* 0.

2. HOMOGENIZATION OF QUASI-LINEAR EQUATIONS WITH BOUNDED SOLUTIONS

In this section we study the asymptotic behaviour, as ε tends to 0, of solutions u_ε to the Dirichlet boundary value problems

$$(2.1)\qquad \begin{cases} -\operatorname{div}(a_\varepsilon(x,Du_\varepsilon)) + \gamma|u_\varepsilon|^{p-2}u_\varepsilon = H_\varepsilon(x,u_\varepsilon,Du_\varepsilon) + h_\varepsilon(x) \quad \text{on } \Omega\ , \\ u_\varepsilon \in H_0^{1,p}(\Omega)\cap L^\infty(\Omega)\ , \end{cases}$$

where a_ε are the functions introduced in Section 1 and γ is a positive constant. The functions H_ε and h_ε satisfy suitable assumptions that are given below. Let us recall that a solution u_ε to problem (2.1) is a function in $H_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that

$$\int_\Omega (a_\varepsilon(x,Du_\varepsilon),Dv)dx + \gamma\int_\Omega |u_\varepsilon|^{p-2}u_\varepsilon v dx = \int_\Omega H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)v dx + \int_\Omega h_\varepsilon(x)v dx$$

for every test function $v\in H_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

The corresponding linear case, where $a_\varepsilon(x,\xi) = a_\varepsilon(x)\xi$, has been considered by [9].

Given the constants $k_1, k_2 > 0$ and the continuous and increasing functions $\sigma_1, \sigma_2, \sigma_3, \omega : \mathbf{R}^+ \to \mathbf{R}^+$ with $\omega(0) = 0$, let us fix a function $H : \mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ satisfying the following conditions:

(2.2) for every $s\in\mathbf{R}$, for every $\xi\in\mathbf{R}^n$, $H(\cdot,s,\xi)$ is 1-periodic and Lebesgue measurable,

(2.3) $|H(x,s,\xi)| \le k_1 + \sigma_1(|s|)\,|\xi|^p$ for a.e. $x\in\mathbf{R}^n$, and for every $s\in\mathbf{R}$, $\xi\in\mathbf{R}^n$.

Moreover, we require the following continuity properties:

(2.4) $|H(x,s_1,\xi) - H(x,s_2,\xi)| \le \omega(|s_1-s_2|)\sigma_2(|s_1|+|s_2|)|\xi|^p$ for a.e. $x\in\mathbf{R}^n$, and for every $s_1, s_2\in\mathbf{R}$, $\xi\in\mathbf{R}^n$,

(2.5) $|H(x,s,\xi_1) - H(x,s,\xi_2)| \le \sigma_3(|s|)(1 + |\xi_1| + |\xi_2|)^{p-\alpha}|\xi_1-\xi_2|^\alpha$ for a.e. $x\in\mathbf{R}^n$, and for every $s\in\mathbf{R}$, for every $\xi_1 , \xi_2\in\mathbf{R}^n$.

Furthermore, assume that $h_\varepsilon\in L^\infty(\Omega)$ satisfies

(2.6) (h_ε) converges to h in $L^\infty(\Omega)$-weak* and $\|h_\varepsilon\|_{L^\infty(\Omega)} \le k_2$.

For every $\varepsilon > 0$ we define $H_\varepsilon(x,s,\xi) = H(\frac{x}{\varepsilon},s,\xi)$ for every $x\in\mathbf{R}^n$, $s\in\mathbf{R}$, and for every $\xi\in\mathbf{R}^n$. Our assumptions on the functions a , H and h_ε are more then needed for the existence of at least one solution u_ε to problem (2.1). For a proof see [15]. In order to study the limit behaviour of a sequence (u_ε) of solutions to problem (2.1) we begin by stating some a priori estimates on u_ε which come out from the existence theory established in [15].

Lemma 2.1. *Let* (u_ε) *be a sequence of solutions to problems (2.1). Then*

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \leq c_3 \ , \qquad \|u_\varepsilon\|_{H_0^{1,p}(\Omega)} \leq c_3 \ ,$$

where c_3 *is a positive constant depending only on* p, γ, k_1, k_2, c_2, *and is independent of* ε.

In order to state the main result of this chapter we need to introduce a function $H^0 : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ related to the function H as follows

$$H^0(s,\xi) = \int_Y H(y,s,p(y,\xi))dy \tag{2.7}$$

for every $s\in\mathbf{R}$ and for every $\xi\in\mathbf{R}^n$. Its properties are summarized by Proposition 2.3.

Theorem 2.2. *Let* (u_ε) *be a sequence of solutions to problems (2.1), where* a, H *and* h_ε *satisfy (1.1)-(1.4) and (2.2)-(2.6). Let* b *be the homogenized operator given by Theorem 1.1. Then, up to a subsequence, we have*

$$u_\varepsilon \to u \quad \textit{weakly in } H^{1,p}(\Omega) \ , \tag{2.8}$$

$$u_\varepsilon \to u \quad L^\infty(\Omega)\textit{-weak*} \ , \tag{2.9}$$

$$Du_\varepsilon - p_\varepsilon(\cdot,M_\varepsilon Du) \to 0 \quad \textit{strongly in } (L^p(\Omega))^n \ , \tag{2.10}$$

where M_ε *and* $p_\varepsilon(\cdot,M_\varepsilon Du)$ *are given by (1.11) and (1.13). We have also*

$$a_\varepsilon(x,Du_\varepsilon) \to b(Du) \quad \textit{weakly in } (L^q(\Omega))^n \ , \tag{2.11}$$

$$-\mathrm{div}(a_\varepsilon(x,Du_\varepsilon)) \to -\mathrm{div}(b(Du)) \quad \textit{strongly in } H^{-1,q}(\Omega) \ , \tag{2.12}$$

where u *is a solution to the problem*

$$\begin{cases} -\mathrm{div}(b(Du)) + \gamma|u|^{p-2}u = H^0(u,Du) + h & on \ \Omega \ , \\ u\in H_0^{1,p}(\Omega)\cap L^\infty(\Omega) \ . \end{cases} \tag{2.13}$$

Finally,

$$H_\varepsilon(x,u_\varepsilon,Du_\varepsilon) \to H^0(u,Du) \quad \textit{weakly in } L^1(\Omega) \ , \tag{2.14}$$

$$\int_\Omega H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)u_\varepsilon dx \to \int_\Omega H^0(u,Du)u dx \ . \tag{2.15}$$

The equation (2.13) is intended to be satisfied in the weak sense with test functions in $H_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, as for problem (2.1).

We first prove the qualitative properties of the limit function H^0 introduced by (2.7). Sequently, in Lemma 2.5 we construct suitable approximations v_ε of the given functions u_ε by means of which we shall prove Theorem 2.2 at the end of this section.

Proposition 2.3. *Let* $H^0 : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ *be the function defined in (2.7). Then*

(2.16) $|H^0(s,\xi)| \leq \tilde{k}_1 + \tilde{\sigma}_1(|s|)(1 + |\xi|^\beta)$ *for every* $s\in\mathbf{R}$ *, and* $\xi\in\mathbf{R}^n$,

(2.17) $|H^0(s_1,\xi) - H^0(s_2,\xi)| \leq \tilde{\omega}(|s_1-s_2|)\tilde{\sigma}_2(|s_1|+|s_2|)(1 + |\xi|^\beta)$ *for every* $s_1, s_2\in\mathbf{R}$ *, and* $\xi\in\mathbf{R}^n$,

(2.18) $|H^0(s,\xi_1) - H^0(s,\xi_2)| \leq \tilde{\sigma}_3(|s|)(1 + |\xi_1| + |\xi_2|)^{p-\nu}|\xi_1-\xi_2|^\nu$ *for every* $s\in\mathbf{R}$ *, and for every* $\xi_1 , \xi_2\in\mathbf{R}^n$,

where $\tilde{k}_1$ *is a positive constant,* $\tilde{\sigma}_1 , \tilde{\sigma}_2 , \tilde{\sigma}_3$ *and* $\tilde{\omega} : \mathbf{R}^+ \to \mathbf{R}^+$ *are continuous increasing functions with* $\tilde{\omega}(0) = 0$ *, and* $\nu = \alpha/(\beta-\alpha)$ *. Moreover, we have*

(2.19) $$H_\varepsilon(\cdot,\varphi_\varepsilon,p_\varepsilon(\cdot,M_\varepsilon\psi)) \to H^0(\varphi,\psi) \quad \textit{weakly in } L^1(\Omega)$$

for every $\psi\in(L^p(\Omega))^n$ *and for every sequence* (φ_ε) *which is bounded in* $L^\infty(\Omega)$ *and converges pointwise to* φ .

Proof. Since $H_\varepsilon(\cdot,s,p_\varepsilon(\cdot,\xi))$ is ε-periodic it follows immediately that

(2.20) $$H_\varepsilon(\cdot,s,p_\varepsilon(\cdot,\xi)) \to H^0(s,\xi) \quad \text{weakly in } L^p(\Omega)$$

for every $s\in\mathbf{R}$ and for every $\xi\in\mathbf{R}^n$. By the definition of H^0 and the condition (2.3) we get

$$|H^0(s,\xi)| \leq k_1 + \sigma_1(|s|)\int_Y |p(y,\xi)|^p dy .$$

By applying the estimate (1.14), condition (2.16) follows immediately. Let us prove (2.17). By the definition of H^0 and (2.4) we obtain

$$|H^0(s_1,\xi) - H^0(s_2,\xi)| \leq \omega(|s_1-s_2|)\sigma_2(|s_1|+|s_2|)\int_Y |p(y,\xi)|^p dy .$$

The estimate (1.14) provides then (2.17). Moreover, by (2.5) and Hölder's inequality it follows that

$$|H^0(s,\xi_1) - H^0(s,\xi_2)| \le$$

$$\le c\sigma_3(|s|)(1+ \|p(\cdot,\xi_1)\|_{(L^p(Y))^n} + \|p(\cdot,\xi_2)\|_{(L^p(Y))^n})^{p-\alpha}\|p(\cdot,\xi_1) - p(\cdot,\xi_2)\|^{\alpha}_{(L^p(Y))^n} .$$

By applying both (1.14) and (1.15) the estimate (2.18) follows immediately.
Finally, let us prove (2.19). This will be done in four steps.

Step 1. By (2.20) it easily follows that

$$H_\varepsilon(\cdot,\varphi,p_\varepsilon(\cdot,\psi)) \to H^0(\varphi,\psi) \quad \text{weakly in } L^p(\Omega) \tag{2.21}$$

for every step function $\varphi \in L^\infty(\Omega)$ and for every step function $\psi \in (L^p(\Omega))^n$.

Step 2. We have

$$H_\varepsilon(\cdot,\varphi,p_\varepsilon(\cdot,M_\varepsilon\psi)) \to H^0(\varphi,\psi) \quad \text{weakly in } L^p(\Omega) \tag{2.22}$$

for every step function $\varphi \in L^\infty(\Omega)$ and for every function $\psi \in (L^p(\Omega))^n$.

In fact, given φ and ψ as above, for every $\delta > 0$ there exists a step function

$$\eta(x) = \sum_{j=1}^{m} \eta_j 1_{\Omega_j}(x)$$

with $\eta_j \in R^n \setminus \{0\}$, $\Omega_j \subset\subset \Omega$, $|\partial\Omega_j| = 0$, $\Omega_j \cap \Omega_k = \varnothing$ for $j \ne k$, such that

$$\|\psi - \eta\|_{(L^p(\Omega))^n} < \delta . \tag{2.23}$$

Since

$$H_\varepsilon(\cdot,\varphi,p_\varepsilon(\cdot,M_\varepsilon\psi)) - H^0(\varphi,\psi) = \tag{2.24}$$

$$= [H_\varepsilon(\cdot,\varphi,p_\varepsilon(\cdot,M_\varepsilon\psi)) - H_\varepsilon(\cdot,\varphi,p_\varepsilon(\cdot,\eta))] + [H_\varepsilon(\cdot,\varphi,p_\varepsilon(\cdot,\eta)) - H^0(\varphi,\eta)] +$$

$$+ [H^0(\varphi,\eta) - H^0(\varphi,\psi)]$$

and by (2.21)

$$H_\varepsilon(\cdot,\varphi,p_\varepsilon(\cdot,\eta)) \to H^0(\varphi,\eta) \quad \text{weakly in } L^p(\Omega) , \tag{2.25}$$

to get (2.22) it is enough to evaluate the first and the third bracket on the right hand side of (2.24). By (2.5) and Hölder's inequality we have

$$\int_\Omega |H_\varepsilon(x,\varphi,p_\varepsilon(x,M_\varepsilon\psi)) - H_\varepsilon(x,\varphi,p_\varepsilon(x,\eta))|dx \le$$

$$\leq c\sigma_3(\|\varphi\|_{(L^\infty(\Omega))^n})(1+\|p(\cdot,M_\varepsilon\psi)\|_{(L^p(\Omega))^n}+\|p(\cdot,\eta)\|_{(L^p(\Omega))^n})^{p-\alpha}\cdot$$

$$\cdot\|p(\cdot,M_\varepsilon\psi)-p(\cdot,\eta)\|^{\alpha}_{(L^p(\Omega))^n}\,.$$

Arguing as in (4.2) of [33] we can show that $p_\varepsilon(\cdot,M_\varepsilon\psi)$ and $p_\varepsilon(\cdot,\eta)$ are bounded in $(L^p(\Omega))^n$ uniformly with respect to ε. Hence,

$$\limsup_{\varepsilon\to0}\int_\Omega|H_\varepsilon(x,\varphi,p_\varepsilon(x,M_\varepsilon\psi))-H_\varepsilon(x,\varphi,p_\varepsilon(x,\eta))|dx\leq$$

$$\leq c\limsup_{\varepsilon\to0}\|p(\cdot,M_\varepsilon\psi)-p(\cdot,\eta)\|^{\alpha}_{(L^p(\Omega))^n}\,,$$

where c is a positive constant independent of ε. By Lemma 1.5 and by the preceeding inequality it turns out that

$$(2.26)\quad \limsup_{\varepsilon\to0}\int_\Omega|H_\varepsilon(x,\varphi,p_\varepsilon(x,M_\varepsilon\psi))-H_\varepsilon(x,\varphi,p_\varepsilon(x,\eta))|dx\leq$$

$$\leq c(|\Omega|+\|\psi\|_{(L^p(\Omega))^n}+\|\eta\|_{(L^p(\Omega))^n})^{\alpha-\nu}\|\psi-\eta\|^{\nu}_{(L^p(\Omega))^n}$$

$$< c(|\Omega|+\|\psi\|_{(L^p(\Omega))^n}+\|\eta\|_{(L^p(\Omega))^n})^{\alpha-\nu}\delta^{\nu}$$

with $\nu=\alpha/(\beta-\alpha)$, where in the last inequality we have used (2.23).
Finally, by (2.18) and Hölder's inequality we conclude that

$$(2.27)\quad \int_\Omega|H^0(\varphi,\eta)-H^0(\varphi,\psi)|dx\leq$$

$$\leq c\tilde{\sigma}_3(\|\varphi\|_{(L^\infty(\Omega))^n})(|\Omega|+\|\psi\|_{(L^p(\Omega))^n}+\|\eta\|_{(L^p(\Omega))^n})^{p-\nu}\|\psi-\eta\|^{\nu}_{(L^p(\Omega))^n}$$

$$\leq c(|\Omega|+\|\psi\|_{(L^p(\Omega))^n}+\|\eta\|_{(L^p(\Omega))^n})^{p-\nu}\delta^{\nu}\,,$$

still taking (2.23) into account. Now, by (2.24)-(2.27) and the arbitrariness of δ we obtain (2.22). •

Step 3. Let $\varphi\in L^\infty(\Omega)$ and $\psi\in(L^p(\Omega))^n$. By approximating φ uniformly with a sequence of step functions in $L^\infty(\Omega)$ and by taking (2.22), (2.4) and (2.17) into account we get

$$(2.28)\qquad H_\varepsilon(\cdot,\varphi,p_\varepsilon(\cdot,M_\varepsilon\psi))\to H^0(\varphi,\psi)\quad\text{weakly in } L^1(\Omega)\,.$$ •

Step 4. Finally, (2.19) follows from (2.28) by using (2.4) and (2.17). ♦

Lemma 2.4. *Let* u_ε *and* u *be the solutions to problem (1.5) and (1.6), respectively, with* $f \in H^{-1,\sigma}(\Omega)$, $\sigma > q$. *Then, for every bounded sequence* (φ_ε) *in* $L^\infty(\Omega)$ *such that* (φ_ε) *converges to* φ *a.e. on* Ω *we have*

$$(2.29) \qquad \int_\Omega (a_\varepsilon(x,Du_\varepsilon),Du_\varepsilon)\varphi_\varepsilon dx \;\to\; \int_\Omega (b(Du),Du)\varphi dx \;.$$

Proof. By Theorem 1.1 the sequence (u_ε) is uniformly bounded in $H_0^{1,p}(\Omega)$, and hence by Meyers estimate (1.10) also in $H^{1,p+\eta}(\Omega)$, $\eta > 0$. By means of (1.2) and (1.3) this implies that the functions $a_\varepsilon(\cdot,Du_\varepsilon)$ are uniformly bounded in $(L^s(\Omega))^n$ for some $s > q$. Therefore, there exists $\tau > 1$ such that

$$\|(a_\varepsilon(\cdot, Du_\varepsilon),Du_\varepsilon)\|_{L^\tau(\Omega)} \le c ,$$

uniformly with respect to ε. Hence, up to a subsequence, $(a_\varepsilon(\cdot, Du_\varepsilon),Du_\varepsilon)$ converges weakly to a function $g \in L^\tau(\Omega)$ as ε tends to 0. Since u_ε solves (1.5), a compensated compactness result (see, for example, Lemma 3.4 in Chapter 1) implies that $(a_\varepsilon(\cdot, Du_\varepsilon),Du_\varepsilon)$ converges to $(b(Du),Du)$ weakly in $\mathcal{D}'(\Omega)$ as ε tends to 0. Therefore, we may conclude that $g = (b(Du),Du)$. Now, since by assumption the sequence (φ_ε) converges to φ strongly in $L^{\tau'}(\Omega)$, $1/\tau + 1/\tau' = 1$, (2.29) follows immediately. ♦

Lemma 2.5. *Let* (u_ε) *be a sequence of solutions to problem (2.1), where* a, H *and* h_ε *satisfy (1.1)-(1.4) and (2.3), (2.6). Let* b *be the homogenized operator given by Theorem 1.1 and assume that* (u_ε) *converges to* $u \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *weakly in* $H^{1,p}(\Omega)$ *and a.e. on* Ω. *Let* v_ε *be the solution to problem*

$$(2.30) \qquad \begin{cases} -\mathrm{div}(a_\varepsilon(x,Dv_\varepsilon)) + \gamma|v_\varepsilon|^{p-2}v_\varepsilon = -\mathrm{div}(b(Du)) + \gamma|u|^{p-2}u \quad on\ \Omega , \\ v_\varepsilon \in H_0^{1,p}(\Omega) \;. \end{cases}$$

Then,

$$(2.31) \qquad u_\varepsilon - v_\varepsilon \;\to\; 0 \quad \textit{strongly in } H^{1,p}(\Omega) .$$

Remark 2.6. Let us note that Theorem 1.1 guarantees that (v_ε) converges to u weakly in $H^{1,p}(\Omega)$. Hence, by the assumption of Lemma 2.5 the sequence $(u_\varepsilon - v_\varepsilon)$ converges to 0 weakly in $H^{1,p}(\Omega)$. Actually, we shall prove the strong convergence.

Proof. Since $u \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ there exists a sequence of smooth functions (u_λ) such that

$$u_\lambda \to u \quad \text{strongly in } H^{1,p}(\Omega) . \tag{2.32}$$

Let $v_{\varepsilon,\lambda}$ be the solution to the problem

$$\begin{cases} -\operatorname{div}(a_\varepsilon(x, Dv_{\varepsilon,\lambda})) + \gamma |v_{\varepsilon,\lambda}|^{p-2} v_{\varepsilon,\lambda} = -\operatorname{div}(b(Du_\lambda)) + \gamma |u_\lambda|^{p-2} u_\lambda \text{ on } \Omega , \\ v_{\varepsilon,\lambda} \in H_0^{1,p}(\Omega) . \end{cases} \tag{2.33}$$

By the regularity theory for quasi-linear elliptic equations (see, for instance, [46], Chapter 4, Theorem 7.1, or [12]) the smoothness of u_λ implies that $v_{\varepsilon,\lambda} \in L^\infty(\Omega)$. Moreover, by (2.30) and (2.33) it follows immediately that

$$\int_\Omega (a_\varepsilon(x, Dv_{\varepsilon,\lambda}) - a_\varepsilon(x, Dv_\varepsilon), Dv_{\varepsilon,\lambda} - Dv_\varepsilon) dx + \gamma \int_\Omega (|v_{\varepsilon,\lambda}|^{p-2} v_{\varepsilon,\lambda} - |v_\varepsilon|^{p-2} v_\varepsilon)(v_{\varepsilon,\lambda} - v_\varepsilon) dx =$$

$$= \int_\Omega (b(Du_\lambda) - b(Du), Dv_{\varepsilon,\lambda} - Dv_\varepsilon) dx + \gamma \int_\Omega (|u_\lambda|^{p-2} u_\lambda - |u|^{p-2} u)(v_{\varepsilon,\lambda} - v_\varepsilon) dx .$$

By using (1.4), an easy consequence of Hölder's inequality (see, for example, Lemma 3.1 in [33]) and the strict monotonicity of the function $I : L^p(\Omega) \to L^q(\Omega)$ defined by $Iw = |w|^{p-2} w$, we get

$$c \|Dv_{\varepsilon,\lambda} - Dv_\varepsilon\|^\beta_{(L^p(\Omega))^n} (|\Omega|^{1/p} + \|Dv_{\varepsilon,\lambda}\|_{(L^p(\Omega))^n} + \|Dv_\varepsilon\|_{(L^p(\Omega))^n})^{(p-\beta)} \le$$

$$\le \int_\Omega (b(Du_\lambda) - b(Du), Dv_{\varepsilon,\lambda} - Dv_\varepsilon) dx + \gamma \int_\Omega (Iu_\lambda - Iu)(v_{\varepsilon,\lambda} - v_\varepsilon) dx .$$

By estimating the right hand side of the last inequality using (1.8) and Hölder's inequality we obtain

$$c \|Dv_{\varepsilon,\lambda} - Dv_\varepsilon\|^\beta_{(L^p(\Omega))^n} (|\Omega|^{1/p} + \|Dv_{\varepsilon,\lambda}\|_{(L^p(\Omega))^n} + \|Dv_\varepsilon\|_{(L^p(\Omega))^n})^{(p-\beta)} \le$$

$$\le c(1 + \|Du_\lambda\|_{(L^p(\Omega))^n} + \|Du\|_{(L^p(\Omega))^n})^{p-1-\mu}_{(L^p(\Omega))^n} \|Du_\lambda - Du\|^\mu_{(L^p(\Omega))^n} \|Dv_{\varepsilon,\lambda} - Dv_\varepsilon\|_{(L^p(\Omega))^n} +$$

$$+ \gamma \|Iu_\lambda - Iu\|_{L^q(\Omega)} \|v_{\varepsilon,\lambda} - v_\varepsilon\|_{L^p(\Omega)} .$$

Finally, taking into account that $v_{\varepsilon,\lambda}$, v_ε and u_λ are bounded in $H^{1,p}(\Omega)$ uniformly with respect to ε and λ, and using properly Young's inequality one gets

$$(2.34)\qquad \|v_{\varepsilon,\lambda}-v_\varepsilon\|_{H_0^{1,p}(\Omega)} \le c(\|Du_\lambda-Du\|^{\mu/\beta}_{(L^p(\Omega))^n} + \|Iu_\lambda-Iu\|^{1/\beta}_{L^q(\Omega)}) \ .$$

Setting

$$(2.35)\qquad g_\lambda = -\mathrm{div}(b(Du_\lambda)) + \gamma|u_\lambda|^{p-2}u_\lambda$$

from (2.33) and (2.1) it follows that

$$(2.36)\quad -\mathrm{div}(a_\varepsilon(x,Du_\varepsilon) - a_\varepsilon(x,Dv_{\varepsilon,\lambda})) + \gamma(|u_\varepsilon|^{p-2}u_\varepsilon - |v_{\varepsilon,\lambda}|^{p-2}v_{\varepsilon,\lambda}) = H_\varepsilon(x,u_\varepsilon,Du_\varepsilon) + h_\varepsilon - g_\lambda \, .$$

As in [15] and [9], we shall define the function

$$(2.37)\qquad \varphi_\mu(s) = s\, e^{\vartheta s^2}$$

with ϑ a positive parameter. Since $u_\varepsilon-v_{\varepsilon,\lambda}$ belongs to $H_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, we can use $\varphi_\vartheta(u_\varepsilon-v_{\varepsilon,\lambda})$ as a test function in (2.36) to obtain by means of (2.3) and Lemma 2.1

$$(2.38)\quad \int_\Omega (a_\varepsilon(x,Du_\varepsilon)-a_\varepsilon(x,Dv_{\varepsilon,\lambda}),Du_\varepsilon-Dv_{\varepsilon,\lambda})\varphi_\vartheta'(u_\varepsilon-v_{\varepsilon,\lambda})dx +$$

$$+\ \gamma \int_\Omega (|u_\varepsilon|^{p-2}u_\varepsilon - |v_{\varepsilon,\lambda}|^{p-2}v_{\varepsilon,\lambda})\varphi_\vartheta(u_\varepsilon-v_{\varepsilon,\lambda})dx \le$$

$$\le \int_\Omega (k_1 + \sigma_1(c_3)|Du_\varepsilon|^p)|\varphi_\vartheta(u_\varepsilon-v_{\varepsilon,\lambda})|dx + \int_\Omega h_\varepsilon\varphi_\vartheta(u_\varepsilon-v_{\varepsilon,\lambda})dx - \langle g_\lambda, \varphi_\vartheta(u_\varepsilon-v_{\varepsilon,\lambda})\rangle$$

$$\le \int_\Omega (k_1 + 2^{p-1}\sigma_1(c_3)|Du_\varepsilon-Dv_{\varepsilon,\lambda}|^p + 2^{p-1}\sigma_1(c_3)|Dv_{\varepsilon,\lambda}|^p)|\varphi_\vartheta(u_\varepsilon-v_{\varepsilon,\lambda})|dx +$$

$$+ \int_\Omega h_\varepsilon\varphi_\vartheta(u_\varepsilon-v_{\varepsilon,\lambda})dx - \langle g_\lambda, \varphi_\vartheta(u_\varepsilon-v_{\varepsilon,\lambda})\rangle \ .$$

Noticing that $\varphi_\vartheta' > 0$, by (1.4) and the strict monotonicity of the function $I : L^p(\Omega) \to L^q(\Omega)$ defined by $Iw = |w|^{p-2}w$, the inequality (2.38) becomes

$$c_2 \int_\Omega |Du_\varepsilon - Dv_{\varepsilon,\lambda}|^\beta (1 + |Du_\varepsilon| + |Dv_{\varepsilon,\lambda}|)^{p-\beta} \varphi_\vartheta'(u_\varepsilon - v_{\varepsilon,\lambda})dx \le$$

$$\le \int_\Omega (k_1 + 2^{p-1}\sigma_1(c_3)|Du_\varepsilon - Dv_{\varepsilon,\lambda}|^p + 2^{p-1}\sigma_1(c_3)|Dv_{\varepsilon,\lambda}|^p)|\varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})|dx +$$

$$+ \int_\Omega h_\varepsilon \varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})dx - \langle g_\lambda, \varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})\rangle .$$

Since by Young's inequality we have

$$|Du_\varepsilon - Dv_{\varepsilon,\lambda}|^p \le c_4|Du_\varepsilon - Dv_{\varepsilon,\lambda}|^\beta (1 + |Du_\varepsilon| + |Dv_{\varepsilon,\lambda}|)^{p-\beta} + c(1 + |Dv_{\varepsilon,\lambda}|^p) ,$$

then

$$(2.39) \quad \int_\Omega |Du_\varepsilon - Dv_{\varepsilon,\lambda}|^\beta (1 + |Du_\varepsilon| + |Dv_{\varepsilon,\lambda}|)^{p-\beta} (c_2\varphi_\vartheta'(u_\varepsilon - v_{\varepsilon,\lambda}) - 2^{p-1}\sigma_1(c_3)c_4|\varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})|)dx \le$$

$$\le c \int_\Omega (1 + |Dv_{\varepsilon,\lambda}|^p)|\varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})|dx + \int_\Omega h_\varepsilon \varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})dx - \langle g_\lambda, \varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})\rangle .$$

Now, for $\vartheta \ge [2^{p-2}\sigma_1(c_3)c_4/c_2]^2$, it follows that

$$c_2\, \varphi_\vartheta'(s) - 2^{p-1}\sigma_1(c_3)c_4|\varphi_\vartheta(s)| \ge \frac{c_2}{2}$$

for every $s \in \mathbf{R}$. By making such a choice for ϑ , we deduce from (2.39) that

$$(2.40) \quad \frac{c_2}{2} \int_\Omega |Du_\varepsilon - Dv_{\varepsilon,\lambda}|^\beta (1 + |Du_\varepsilon| + |Dv_{\varepsilon,\lambda}|)^{p-\beta} dx \le$$

$$\le c \int_\Omega (1 + c_2^{-1}(a_\varepsilon(x, Dv_{\varepsilon,\lambda}), Dv_{\varepsilon,\lambda}))|\varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})|dx +$$

$$+ \int_\Omega h_\varepsilon \varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})dx - \langle g_\lambda, \varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})\rangle .$$

By an easy consequence of Hölder's inequality (see, for example, Lemma 3.1 in [33]) the left hand side of (2.40) can be estimated as follows

$$c\|Du_\varepsilon - Dv_{\varepsilon,\lambda}\|^\beta_{(L^p(\Omega))^n}(|\Omega|^{1/p} + \|Du_\varepsilon\|_{(L^p(\Omega))^n} + \|Dv_{\varepsilon,\lambda}\|_{(L^p(\Omega))^n})^{p-\beta} \le$$

$$\le \frac{c_2}{2}\int_\Omega |Du_\varepsilon - Dv_{\varepsilon,\lambda}|^\beta(1 + |Du_\varepsilon| + |Dv_{\varepsilon,\lambda}|)^{p-\beta}\,dx\ .$$

Then, by taking into account that u_ε and $v_{\varepsilon,\lambda}$ are bounded in $H^{1,p}(\Omega)$ uniformly with respect to ε and λ, we obtain

$$\|Du_\varepsilon - Dv_{\varepsilon,\lambda}\|^\beta_{(L^p(\Omega))^n} \le c\int_\Omega (1 + c_2^{-1}(a_\varepsilon(x,Dv_{\varepsilon,\lambda}),Dv_{\varepsilon,\lambda}))|\varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})|dx +$$

$$+ c\int_\Omega h_\varepsilon\varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})dx - c<g_\lambda, \varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda})>.$$

Let us note now that $(\varphi_\vartheta(u_\varepsilon - v_{\varepsilon,\lambda}))_\varepsilon$ converges to $\varphi_\vartheta(u-u_\lambda)$ strongly in $L^\tau(\Omega)$ for every $1 \le \tau < +\infty$ and weakly in $H_0^{1,p}(\Omega)$, which together with Lemma 2.4 ensures that

$$\limsup_{\varepsilon\to 0} \|Du_\varepsilon - Dv_{\varepsilon,\lambda}\|^\beta_{(L^p(\Omega))^n} \le c\int_\Omega (1 + c_2^{-1}(b(Du_\lambda),Du_\lambda))|\varphi_\vartheta(u-u_\lambda)|dx + \tag{2.41}$$

$$+ c\int_\Omega h\varphi_\vartheta(u-u_\lambda)dx - c<g_\lambda, \varphi_\vartheta(u-u_\lambda)>.$$

Since

$$\|u_\varepsilon - v_\varepsilon\|_{H_0^{1,p}(\Omega)} \le \|u_\varepsilon - v_{\varepsilon,\lambda}\|_{H_0^{1,p}(\Omega)} + \|v_{\varepsilon,\lambda} - v_\varepsilon\|_{H_0^{1,p}(\Omega)}\ ,$$

by (2.41) and (2.34) we get

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon - v_\varepsilon\|_{H_0^{1,p}(\Omega)} \le c\int_\Omega (1 + c_2^{-1}(b(Du_\lambda),Du_\lambda))|\varphi_\vartheta(u-u_\lambda)|dx +$$

$$+ c\int_\Omega h\varphi_\vartheta(u-u_\lambda)dx - c<g_\lambda, \varphi_\vartheta(u-u_\lambda)> + c(\|Du_\lambda - Du\|^{\mu/\beta}_{(L^p(\Omega))^n} + \|Iu_\lambda - Iu\|^{1/\beta}_{L^q(\Omega)})\ .$$

By means of (2.32), the condition (2.31) follows now immediately. ♦

Proof of Theorem 2.2. Let (u_ε) be a sequence of solutions to problems (2.1). By Lemma 2.1 we can assume that

$$u_\varepsilon \to u \quad \text{weakly in } H^{1,p}(\Omega) \text{ and a.e. in } \Omega\,,$$
$$u_\varepsilon \rightharpoonup u \quad L^\infty(\Omega)\text{-weak}^*\,.$$

By applying the corrector Theorem 1.6 to the solution v_ε to problem (2.30) we have that

$$Dv_\varepsilon - p_\varepsilon(\cdot, M_\varepsilon Du) \to 0 \quad \text{strongly in } (L^p(\Omega))^n\,.$$

Then, by (2.31) we get

$$Du_\varepsilon - p_\varepsilon(\cdot, M_\varepsilon Du) \to 0 \quad \text{strongly in } (L^p(\Omega))^n \tag{2.42}$$

which proves the corrector result (2.10) for u_ε. Let $H^0 : \mathbf{R}\times\mathbf{R}^n \to \mathbf{R}$ be the function defined by (2.7). By (2.19) we have

$$H_\varepsilon(\cdot, u_\varepsilon, p_\varepsilon(\cdot, M_\varepsilon Du)) \to H^0(u, Du) \quad \text{weakly in } L^1(\Omega)\,. \tag{2.43}$$

Moreover, by using Hölder's inequality we obtain by means of (2.42) and (2.5) that

$$H_\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon) - H_\varepsilon(\cdot, u_\varepsilon, p_\varepsilon(\cdot, M_\varepsilon Du)) \to 0 \quad \text{strongly in } L^1(\Omega)\,. \tag{2.44}$$

Then, (2.43) and (2.44) guarantee

$$H_\varepsilon(\cdot, u_\varepsilon, Du_\varepsilon) \to H^0(u, Du) \quad \text{weakly in } L^1(\Omega) \tag{2.45}$$

proving (2.14). By using (1.3) and (2.31) it follows that

$$a_\varepsilon(x, Du_\varepsilon) - a_\varepsilon(x, Dv_\varepsilon) \to 0 \quad \text{strongly in } (L^q(\Omega))^n\,.$$

On the other hand, by the weak convergence of the momenta in Theorem 1.1 applied to problem (2.30) we have

$$a_\varepsilon(x, Dv_\varepsilon) \to b(Du) \quad \text{weakly in } (L^q(\Omega))^n\,.$$

Hence, (2.11) and (2.12) follow easily. Moreover, (2.13) is satisfied. Finally, let us show (2.15). First of all

$$\left| \int_\Omega H_\varepsilon(x, u_\varepsilon, Du_\varepsilon)(u_\varepsilon - u)\,dx \right| \le \tag{2.46}$$

$$\leq \int_\Omega |H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)-H_\varepsilon(x,u_\varepsilon,p_\varepsilon(x,M_\varepsilon Du))||u_\varepsilon-u|dx + \left|\int_\Omega H_\varepsilon(x,u_\varepsilon,p_\varepsilon(x,M_\varepsilon Du))(u_\varepsilon-u)dx\right| .$$

By (2.44) and the fact that $(u_\varepsilon-u)$ is uniformly bounded in $L^\infty(\Omega)$ the first term on the right hand side of (2.46) vanishes as ε tends to 0. By taking (2.43) into account and by noticing that the sequence $(u_\varepsilon-u)$ converges a.e. to 0 and is uniformly bounded in $L^\infty(\Omega)$ we have

$$\int_\Omega H_\varepsilon(x,u_\varepsilon,p_\varepsilon(x,M_\varepsilon Du))(u_\varepsilon-u)dx \ \rightarrow\ 0 .$$

Finally, from (2.45) we get

$$\int_\Omega H_\varepsilon(x,u_\varepsilon,Du_\varepsilon)udx \ \rightarrow\ \int_\Omega H^0(u,Du)udx \ ,$$

which together with (2.46) yields (2.15). ♦

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