ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

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Dynamical Systems with Singular Potentials: Existence and Qualitative Properties of Periodic Motions

"DOCTOR PHILOSOPHIAE" THESIS

Supervisor: Prof. Antonio Ambrosetti.

Academic Year 1990–91.

TRIESTE

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Il presente lavoro costituisce la tesi presentata dal Dott. Enrico Serra, sotto la direzione del Prof. Antonio Ambrosetti, al fine di ottenere l'attestato di ricerca postuniversitaria "Doctor Philosophiae" presso la S.I.S.S.A., classe di Matematica, Settore di Analisi Funzionale e Applicazioni. Ai sensi del Decreto del Ministero della Pubblica Istruzione, n. 419 del 24 aprile 1987, tale diploma è equipollente al titolo di "Dottore di Ricerca in Matematica".

The present work is the thesis presented by Enrico Serra, under the supervision of Prof. Antonio Ambrosetti, in order to obtain the degree of "Doctor Philosophiae" at S.I.S.S.A., Mathematics Class, Sector of Functional Analysis and Applications. The Decreto del Ministero della Pubblica Istruzione n. 419, April 24th 1987, states that this degree is equivalent to the italian title of "Dottore di Ricerca in Matematica".

CONTENTS

INTRODUCTION

This thesis contains part of the author's researches undertaken during the last few years in collaboration with S. Terracini and V. Coti Zelati, under the supervision of Prof. A. Ambrosetti. It is devoted to the application of variational techniques to problems of existence of periodic solutions to singular Hamiltonian systems.

These are systems of ordinary differential equations which can be written in the form

$$-\ddot{x}(t) = \nabla_x F(t, x(t)), \tag{1}$$

where $F : \mathbb{R} \times \Omega \to \mathbb{R}$ is a regular function and $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, is an open set.

The potential F is said to be *singular* because we suppose that

$$\lim_{x \to \partial\Omega} F(t, x) = -\infty; \tag{2}$$

most of the times we shall assume that $\Omega = \mathbb{R}^N \setminus \{0\}$, so that the potential becomes infinite at the origin.

The *fixed period problem* associated to (1) consists in showing that for every positive number T (usually the period of the function F in the first variable) there exists a classical C^2 solution x of (1) such that $x(t+T) = x(t)$ for all t.

In the *fixed energy problem* one looks, given a time–independent potential F and a number h, for a periodic (the period is unknown) solution of (1) satisfying the energy conservation

$$\frac{1}{2}|\dot{x}(t)|^2 + F(x(t)) = h.$$

From the point of view of classical mechanics, system (1) describes the motion of a particle located in $x(t) \in \mathbb{R}^N$ at time t and moving in $\mathbb{R}^N$ under the force field given by $\nabla_x F(t, x)$; condition (2) expresses the fact that the potential presents a singularity at zero of attractive type.

CHAPTER 0

MOTIVATIONS AND OUTLINE OF CLASSICAL RESULTS

In the last few years there has been quite a large increase of interest towards the fixed period and the fixed energy problems associated to singular potentials, mainly due to the new approach derived from their variational structure. Indeed, as it is easy to see, classical T-periodic solutions of (1) are critical points of the action functional

$$I(x) = \frac{1}{2}\int_0^T |\dot{x}(t)|^2 dt - \int_0^T F(t, x(t))dt \tag{4}$$

in the set

$$\Lambda = \{x \in H^1(S^1; \mathbb{R}^N) \ / \ x(t) \neq 0 \ \ \forall t \in S^1\}.$$

Here we denote by $H^1(S^1; \mathbb{R}^N)$ the space of T-periodic functions such that $\int_0^T |\dot{x}(t)|^2 dt + \int_0^T |x(t)|^2 dt < +\infty$.

In order to understand what kind of difficulties one has to face in the study of critical points of the functional (4), it is enough to consider the model problem where $F(x) = -\frac{1}{|x|^\alpha}$, $\alpha > 0$; in this case the associated functional is $I_\alpha : \Lambda \to \mathbb{R}$ defined as

$$I_\alpha(x) = \frac{1}{2}\int_0^T |\dot{x}(t)|^2 dt + \int_0^T \frac{1}{|x(t)|^\alpha} dt. \tag{5}$$

One realizes at once that $\inf_{x\in\Lambda} I(x) = 0$ (as can be seen by taking a divergent sequence of constant functions) but the infimum is never attained, since the term $\int_0^T \frac{1}{|x|^\alpha} dt$ is always strictly positive. It is therefore hopeless to look for minima of I_α: critical points must come from other variational principles. The search of these principles has characterized the first papers concerning these problems; we recall here the pioneering work of Ambrosetti and Coti Zelati, [**2**], from which there followed an abstract result of Fadell and Husseini ([**25**]) on the category of the free loop space. Since then research in this field has received a powerful push and a deeper understanding of the variational properties of the solutions of

Probably the most familiar example of singular potential of attractive type is given by the Newtonian potential $F(x) = -\frac{a}{|x|}$, $a > 0$, which rules the classical theory of gravitation and is therefore present in all of the most important problems in celestial mechanics.

In this case the fixed period problem becomes

$$\text{(3)} \qquad \begin{cases} -\ddot{x}(t) = a\frac{x(t)}{|x(t)|^3} \\ x(t+T) = x(t) \quad \forall t. \end{cases}$$

This problem was solved by Newton himself by the introduction of the main concepts of calculus. He determined all the solutions of (3) and gave thereby a proof of the three Kepler's Laws. As is well–known all the T-periodic solutions of (3) form a family of ellipses having the origin as one of the foci. This family includes the circular solutions and the degenerate *collision* solutions, that is, segments emanating from zero and travelled back and forth.

In this thesis we are concerned with the existence of periodic solutions to (1) when the potential F behaves in some sense like $-\frac{a}{|x|^\alpha}$ near the origin, for some $\alpha > 0$.

The results gathered in this thesis are mainly taken from the papers **[22]**, **[23]**, **[39]**, **[40]**, to which we also refer for further references.

Acknowledgements. The author wishes to express his gratitude to the supervisor of this thesis, Prof. Antonio Ambrosetti, for his expert direction and kind availability.

Many thanks also to Susanna Terracini and Vittorio Coti Zelati, who shared with patience and enthusiasm the everyday work which led to the results presented herein.

A special mention to the International School for Advanced Studies in Trieste: the years spent there have been the most exciting and stimulating of my life.

(1) has given rise to very interesting results. Probably the most successful tool providing critical points of the functional I is the minimax method, in which critical levels appear as $\inf_{A\in\Gamma}\sup_{x\in A} I(x)$, for some suitable classes Γ of subsets of Λ. The application of variational methods has begun with the works of Bahri and Rabinowitz, [**10**], and of Greco, [**29**], who constructed natural, though not trivial generalizations to higher dimension of some earlier results by Gordon, [**27**], [**28**], valid only in the planar case.

The first difficulty encountered when working with a minimax method is the fact that functionals like (4) do not satisfy the Palais–Smale compactness condition at some levels (e.g. the level zero for the functional I_α), so that particular care must be taken in order to avoid the bad levels. The lack of compactness in the study of critical points of the functional I would justify by itself a technical interest in this kind of problems. However there is another main structural difficulty in the variational approach, and precisely the fact that one must work in the set Λ, which is open (and dense) in $H^1(S^1;\mathbb{R}^N)$. Since variational methods provide sequences of approximate solutions $(x_n)_{n\in\mathbb{N}}$ (which converge to a solution if the Palais–Smale condition is satisfied), it may happen that $x_n \to x \in \partial\Lambda = \{x \in H^1(S^1;\mathbb{R}^N)\ /\ x(t) = 0 \text{ for some } t \in S^1\}$. In other words, the limit orbit may cross the singularity of the potential. We speak in this case of a *collision* solution, since, thinking of Newton's Law of Gravitation, this orbit represents the moving body falling on the fixed one located in the origin and generating the field $\nabla_x F(t,x)$. Collision *periodic* solutions are physically meaningless, since one does not expect planets that intersecate each other or bounce. These solutions have nevertheless a relevant mathematical meaning and it is for this reason that Bahri and Rabinowitz have introduced in [**10**] the notion of *generalized* (collision) solution.

A consistent part of the papers appeared in the last few years are dedicated to the task of finding methods to avoid the collisions. Roughly speaking, the only general procedure applicable to some classes of problems consists in showing that no collisions are possible at certain levels, and therefore any minimax argument which can be shown to produce a safe level is liable to yield a classical noncollision solution. Various types of conditions on the potential have been introduced in order to obtain these estimates on the dangerous levels. The most popular and successful of these conditions is the so–called *Strong Force* assumption, appeared for the first time in the paper [28] of Gordon. A potential F is said to be a strong force if

$$(SF) \qquad \begin{aligned} &\exists \varepsilon > 0 \; \exists \; U \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R}) \text{ such that} \\ &-F(t,x) \geq |\nabla U(x)|^2 \quad \text{in} \quad S^1 \times \{0 < |x| \leq \varepsilon\}. \end{aligned}$$

The main consequence of the (SF) condition (as is shown in [28]) is that if F is a strong force then $\inf_{x \in \partial\Lambda} I(x) = +\infty$; therefore *any* minimax procedure for which some compactness can be checked provides a noncollision solution. Gordon's condition has been used repeatedly in the literature by many authors. In [2] for example, Ambrosetti and Coti Zelati obtained the following result.

Theorem 0.1. Suppose $F \in \mathcal{C}^2(\mathbb{R} \times \Omega; \mathbb{R})$ is T–periodic in the first variable and satisfies (SF) and

(F1) $\exists R > 0$ such that $F(t,x) < 0 \quad \forall |x| \geq R$,

(F2) $F(t,x)$ and $\nabla F(t,x)$ tend to zero as $|x| \to \infty$, uniformly in t.

Then (1) has infinitely many T–periodic solutions.

We also recall the existence and multiplicity results both for the fixed period and for the fixed energy problems with strong forces contained in [2], [10], [14], [20], [29], [33]. The major drawback in the use of (SF) is the fact that if F behaves like $-\frac{a}{|x|^\alpha}$ near the origin, then necessarily $\alpha \geq 2$: the strong force condition does not cover the case of the Newtonian potential ($\alpha = 1$) and its perturbations. This is the reason which has led some authors to seek alternative assumptions which guarantee the existence of noncollision solutions. The first step in this direction was moved by Degiovanni and Giannoni in [23]; they introduced a *pinching* condition on the potential, which consists in assuming that the inequality $\frac{a}{|x|^\alpha} \leq -F(t,x) \leq \frac{b}{|x|^\alpha}$ is verified for all t and $x \neq 0$, with a and b sufficiently close, depending on α. This condition allowed them to treat the case $\alpha \in]1,2[$. More precisely they proved

Theorem 0.2. Let $F \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfy

$$\nabla F(x) \to 0 \text{ as } |x| \to \infty.$$

Then there exists a nondecreasing lower semicontinuous function $\psi : [1,2[\to \mathbb{R}$ which verifies $\psi(1) = 1$ and $\lim_{\alpha\to 2}\psi(\alpha) = +\infty$ such that if

$$\frac{a}{|x|^\alpha} \le -F(x) \le \frac{b}{|x|^\alpha}$$

with $\frac{b}{a} \le \psi(\alpha)$, then (1) has at least one T–periodic classical solution.

Variants of the pinching condition were successively used by Terracini (see [44]–[47]) in the study of existence and multiplicity of noncollision solutions for the fixed period and the fixed energy problems. The use of the pinching inequality has produced many interesting results but, as one immediately sees, it is a *global* hypothesis on the potential: F must be close to a radial potential also at infinity, while problems should depend on the behavior of F in the neighborhood of the singularity. To complete this brief outline of techniques we also recall the paper [4], where mutiplicity of noncollision solutions without strong force or pinching assumptions was proved by Ambrosetti and Coti Zelati using a perturbation technique.

This was, very roughly speaking, the state of the art at the time when the author's researches began. We have recalled so far the principal approaches to problems with singular potentials in order to explain the motivations which have led us to the results contained in this thesis. We shall be rather sketchy here, referring the reader to the introductions of the single chapters for a more detailed exposition.

First of all, of the two main difficulties mentioned above, namely the lack of compactness and the problem of avoiding the collisions, we have directed our efforts towards the latter. The core of our results is an attempt to find new conditions leading to noncollision solutions. These conditions should cover the case of the Newtonian potential (actually the whole interval $\alpha \in]0,2[$) and should be weaker than the pinching inequality. The majority of the results contained in this thesis are intended to show that since the collision is a local phenomenon, there must be *local* hypotheses on the potential in the neighborhood of the singularity which yield classical solutions. This is actually the sense of Chapter 1, where these hypotheses are formulated, and corresponding existence results are proved. The analysis of the local behavior of the orbits near a collision, carried out in the first chapter, has allowed us to to increase the range of applicability of our assumptions to broader classes of problems, all of them discussed in the subsequent chapters. In order to succeed in our task, we had to bypass at first the problem of the lack of compactness mentioned above. In Chapter 1 this was done by assuming that the potential satisfies some symmetry conditions, so that working in a suitable subspace of symmetric functions the associated functional (4) turned out to be coercive. Solutions are then found as limits of minimizers of perturbed problems which satisfy the strong force condition. The main result of the local analysis of the collision orbits is the general principle that, in presence of rather weak local hypotheses, the property of minimizing the action functional can never be enjoyed by a collision function. This principle has been repeatedly applied in Chapters 3 and 4. In particular in the former it is used to prove the existence of a multiplicity of noncollision brake orbits derived from a minimization procedure, while in the latter it provides existence of a noncollision solution to the three–body problem in the tridimensional space. We point out that the symmetry condition imposed on the potential is of a purely technical nature in chapter 1 (where it could be replaced by an assumption on the existence of minimizers) while it is essential in the analysis of the geometry of the triple collisions carried out in Chapter 4. Lastly the local behaviour of collision orbits is studied in Chapter 2, where some conjectures formulated by Bahri and Rabinowitz in [**10**] are proved, and it is shown (with some ideas taken from [**19**]) that a generalized solution found in [**10**] is actually a noncollision solution.

Remark. In this Thesis theorems, propositions, remarks, etc. are labeled with two numbers, the first of which refers to the current chapter. Formulas are also labeled with two numbers, but the first one refers to the section in which the formula appears; in this way numeration of formulas restarts at each chapter.

CHAPTER 1

NONCOLLISION SOLUTIONS TO SINGULAR MINIMIZATION PROBLEMS

1.1. Statement of the results

In this first chapter we are going to prove the existence of noncollision periodic solutions for some classes of second order Hamiltonian systems with singular potentials. Precisely we shall study the existence of T-periodic solutions for the equation

$$-\ddot{x} = \nabla F(t, x) \tag{1.1}$$

where $F \in \mathcal{C}^2(\mathbb{R} \times \Omega; \mathbb{R})$ and $T > 0$ is fixed. Here $\Omega = \mathbb{R}^N \setminus \{0\}$, $N \geq 2$, F is T-periodic in t, and $\nabla F(t, x)$ denotes differentiation with respect to x.

The potential F is singular in the sense that

$$\lim_{x \to 0} F(t, x) = -\infty. \tag{1.2}$$

The existence of solutions to this kind of problems can be derived from the minimization of a suitable functional, the main difficulty being to avoid the "collisions", that is, the orbits which pass through the singularity of the potential.

In the last few years much attention has been devoted to these topics, especially in connection to the variational nature of the problems. The variational approach has been shown to be particularly successful in proving existence of one or multiple solutions both for the fixed period problem (1.1) and for the fixed energy problem. As a matter of fact, existence results have been given by Ambrosetti and Coti Zelati using a perturbation argument in [4] , where they were able to locate multiple periodic solutions near the circular orbits of a convenient unperturbed radially symmetric problem (we mention here only the results concerning the "weak force" case). On the other hand, Degiovanni, Giannoni and

Marino have introduced in [**23**] , [**24**] some *global* assumptions on the potential which allow them to rule out the collision solutions by means of some estimates on the critical level of the associated functional. Indeed the assumptions considered in their work are suitable in order to guarantee that the infimum of the functional on the set of collisions is larger or equal to the value of the functional evaluated on a particular noncollision orbit. However in that paper only the case $\alpha > 1$ was treated. The same kind of argument was successively used, when $\alpha \neq 1$, by Terracini (see e.g. [**44**] for a review of her results) to obtain multiple solutions for the fixed period and the fixed energy problem.

The main purpose of this chapter is to show that in a symmetric setting one can make use of *local* hypotheses on the potential (in some neighborhood of its singularity) in order to obtain the existence of at least one noncollision solution, for every $\alpha \in]0,2[$. The arguments used here will provide orbits which remain only for a short time in the neighborhood of zero where the assumptions are made, so that the method is not a plain perturbation technique.

We remark that concerning the existence of periodic solutions of (1.1) in a non symmetric setting there has recently been quite a large amount of papers. We address the reader to [**1**]–[**5**] , [**7**], [**10**], [**17**], [**23**], [**24**], [**29**], [**33**], [**35**], [**43**]–[**46**] and references therein.

We now begin by fixing some notation and by stating the main results of this chapter.

For a fixed $F \in C^2(\mathbb{R} \times \Omega; \mathbb{R})$, with $\Omega = \mathbb{R}^N \setminus \{0\}$, we write

$$F(t,x) = -\frac{a}{|x|^\alpha} + U(t,x) \tag{1.3}$$

for some $a > 0$ and $\alpha \in]0,2[$.

Concerning U we shall consider the following assumptions:

$$(H1) \qquad \limsup_{x \to 0} \left| \frac{\partial U}{\partial t}(t,x) \right| |x|^\alpha < +\infty \qquad \text{uniformly in } t;$$

$$(H2) \qquad \begin{aligned} &\exists C > 0 \ \exists \sigma > 0 \text{ such that} \\ &\limsup_{x \to 0} |\nabla^2 F(t,x)||x|^{\alpha+2-\sigma} \leq C \qquad \text{uniformly in } t; \end{aligned}$$

$$(H3) \qquad \exists \mu \geq 0 \text{ such that} \qquad \lim_{|x| \to \infty} \frac{|\nabla U(t,x)|}{|x|} \leq \mu \qquad \text{uniformly in } t.$$

Remark 1.1. It is easy to see that (H2) implies the following growth conditions at zero (provided C is chosen large enough):

$$\limsup_{x \to 0} |\nabla U(t,x)||x|^{\alpha+1-\sigma} \leq C \qquad \text{uniformly in } t; \tag{1.4}$$

$$\limsup_{x \to 0} |U(t,x)||x|^{\alpha-\sigma} \leq C \qquad \text{uniformly in } t. \tag{1.5}$$

Suitable conditions on the constant μ appearing in (H3) will be imposed in the statement of the results.

The main results are summarized in the following theorems. Although these are formulated for $N \geq 3$, we shall show in the next section that the case $N = 2$ can be treated as well, with similar but more restrictive hypotheses (see Theorem 2.1). As far as $N = 2$, we recall that other results of this type have been obtained by Coti Zelati in [**20**] in the case1 $< \alpha < 2$, and by Bessi and Coti Zelati in [**16**] for the N-body problem.

Theorem 1.2. Let F be defined as above, for some $N \geq 3$, $a > 0$ and $\alpha \in]0,2[$. Suppose that F is T-periodic in t and satisfies the symmetry condition

$$F(t + \frac{T}{2}, -x) = F(t,x) \qquad \forall t \in \mathbb{R} \quad \forall x \neq 0. \tag{S}$$

Assume moreover that (H1), (H2), (H3) hold with $\mu < (\frac{\pi}{T})^2$. Then the problem

$$\begin{cases} -\ddot{x} = \nabla F(t,x) & \forall t \in \mathbb{R} \\ x(t + \frac{T}{2}) = -x(t) & \forall t \in \mathbb{R} \\ x(t) \neq 0 & \forall t \in \mathbb{R} \end{cases} \tag{1.6}$$

has at least one solution.

Corollary 1.3. Under the hypotheses of Theorem 1.2, assume in addition that F does not depend on t. Then for every $T > 0$ problem (1.6) has at least one solution having T as minimal period.

Theorem 1.4. Let F be defined as above, for some $N \geq 3$, $a > 0$ and $\alpha \in]0,2[$. Let $T > 0$ and $x_0, x_1 \in \mathbb{R}^N \setminus \{0\}$ be fixed and let $\lambda(x_0, x_1)$ be the best Sobolev constant of the injection of $\{x \in H^1([0,1];\mathbb{R}^N) \,/\, x(0) = x_0,\ x(1) = x_1\}$ into $L^2([0,1])$. Assume (H1), (H2), (H3) hold with $\mu < \frac{\lambda(x_0,x_1)}{T^2}$. Then the problem

$$(1.7) \qquad \begin{cases} -\ddot{x} = \nabla F(t,x) & \forall t \in [0,T] \\ x(0) = x_0, \quad x(T) = x_1 & \\ x(t) \neq 0 & \forall t \in [0,T] \end{cases}$$

has at least one solution.

Remark 1.5. Hypotheses (H1) and (H2) are obviously fulfilled when U is regular (of class C^2) in the whole of $\mathbb{R}^N$ (and (H1) is trivially verified if U does not depend on t). Therefore problem (1.6) has at least one solution whenever $F(t,x) = -\frac{a}{|x|^\alpha} + U(t,x)$ is such that U is C^2 and verifies (S) and (U3) with $\mu < (\frac{\pi}{T})^2$.

In particular if we call U a *regular perturbation* when $U \in C^2(\mathbb{R} \times \mathbb{R}^N;\mathbb{R})$ and $\lim_{|x|\to\infty} |\nabla U(t,x)| = 0$, then the following result holds.

Corollary 1.6. Let $N \geq 3$, $a > 0$ and $\alpha \in]0,2[$. Let U be a regular perturbation T-periodic in t and such that (S) holds. Then the problem

$$\begin{cases} -\ddot{x} = -\frac{a\alpha}{|x|^{\alpha+2}} - \nabla U(t,x) & \forall t \in \mathbb{R} \\ x(t+\frac{T}{2}) = -x(t) & \forall t \in \mathbb{R} \\ x(t) \neq 0 & \forall t \in \mathbb{R} \end{cases}$$

has at least one solution.

Solving problems (1.6) and (1.7) is equivalent to finding critical points of the functional

$$(1.8) \qquad I(x) = \frac{1}{2}\int_0^T |\dot{x}(t)|^2 dt + \int_0^T \frac{a}{|x(t)|^\alpha} dt - \int_0^T U(t,x(t))dt$$

respectively on the sets

$$\Lambda_1 = \{x \in H^1(\mathbb{R};\mathbb{R}^N) \,/\, x(t+\frac{T}{2}) = -x(t),\ x(t) \neq 0\ \forall t \in \mathbb{R}\},$$

$$\Lambda_2 = \{x \in H^1([0,T];\mathbb{R}^N) \,/\, x(0) = x_0,\ x(T) = x_1,\ x(t) \neq 0\ \forall t \in [0,T]\}.$$

In the settings of Theorems (1.2) and (1.4) the functional I is weakly lower semicontinuous and coercive (due to the symmetry constraint and to (H3)) so that its infimum is always attained in $\overline{\Lambda}_1$ and $\overline{\Lambda}_2$ respectively. The main difficulty is thus to avoid the collisions, that is, to show that the infimum is attained *in* Λ_1 and Λ_2.

To this aim the main argument is the following: first we perturb I with a suitable "strong force" term in a neighborhood of zero, thereby obtaining a noncollision minimum for the corresponding functional; next we prove that any such minimum cannot interact with the perturbation, and therefore is a critical point of I in $\Lambda_i, i = 1,2$. The principal estimate consists in showing that if the minimum approaches too much the singularity, then a small variation making the functional decrease can be found.

As a by-product of this approach we shall show in Section 1.4 that in each case one actually has

$$\inf_{\Lambda_i} I < \inf_{\partial\Lambda_i} I, \qquad i = 1,2.$$

Remark 1.7. The two theorems can be proved with nearly the same arguments and the same technical lemmas. An additional difficulty arises in the proof of Theorem 1.2 because the variation mentioned above has to be symmetric. This is the reason why we shall prove in detail only Theorem 1.2.

Remark 1.8. The sets Λ_i, $i = 1,2$ defined above are open and dense in

$$H_1 = \{x \in H^1(\mathbb{R};\mathbb{R}^N) \;/\; x(t+\frac{T}{2}) = -x(t) \;\forall t \in \mathbb{R}\}$$

and

$$H_2 = \{x \in H^1([0,T];\mathbb{R}^N) \;/\; x(0) = x_0,\; x(T) = x_1\}$$

respectively. Therefore $\overline{\Lambda}_i$ and $\partial\Lambda_i$ are to be understood in H_i, $i = 1,2$.

1.2. Locally radial potentials

In this section we prove Theorem 1.2 under the additional assumption that the potential is radially symmetric in a neighborhood of the singularity. This result will be used in the proof of the general case in Section 1.3. However, in presence of local radial symmetries the hypotheses of Theorem 1.2 can be weakened (and the theorem holds for $N = 2$ too), so that next result is not just a particular case of Theorem 1.2.

According to the notations of Section 1.1 we suppose that $F \in \mathcal{C}^2(\mathbb{R} \times \Omega; \mathbb{R})$, with $\Omega = \mathbb{R}^N \setminus \{0\}$ and we write it as

$$F(t,x) = -\frac{a}{|x|^\alpha} + U(t,x) \qquad \forall t \in \mathbb{R} \quad \forall x \neq 0. \tag{2.1}$$

We consider the following assumptions on U:

$$\begin{aligned} &\exists \varepsilon > 0 \ \exists \phi :]0,\varepsilon] \to \mathbb{R} \text{ of class } \mathcal{C}^2 \text{ such that} \\ &U(t,x) = \phi(|x|) \qquad \forall t \in \mathbb{R} \quad \forall 0 < |x| \leq \varepsilon \end{aligned} \tag{$U1$}$$

$$\lim_{s \to 0} |\phi'(s)| s^{\alpha+1} = 0. \tag{$U2$}$$

Remark that by (U2) ϕ also verifies $\lim_{s\to 0} |\phi(s)| s^\alpha = 0$.

Theorem 2.1. Let F be as in (2.1) with $N \geq 2$, $a > 0$ and $\alpha \in]0,2[$. Suppose U is T-periodic in t and satisfies the symmetry condition (S). Assume that (U1), (U2) and (H3) hold with $\mu < (\frac{\pi}{T})^2$. Then problem (1.6) has al least one solution which minimizes I in Λ_1.

Proof. As we remarked in the previous section, we are going to find a solution of (1.6) as minimizer of the functional

$$I(x) = \frac{1}{2}\int_0^T |\dot{x}(t)|^2 dt + \int_0^T \frac{a}{|x(t)|^\alpha} dt - \int_0^T U(t,x(t)) dt. \tag{2.2}$$

Because of the symmetry properties of F and Λ_1 critical points of I in Λ_1 are solutions of (1.6). Indeed if x is such a critical point then $\nabla I(x) \cdot y = 0$ for all $y \in H_1$, so that $\nabla I(x)$ is orthogonal to H_1; however by symmetry, $\nabla I(x) \in H_1$, so that $\nabla I(x) = 0$.

Assumption (H3) with $\mu < (\frac{\pi}{T})^2$ implies that the functional I is coercive in H_1; indeed notice that by (H3) for every $R > 0$, there exists a constant $C_R > 0$ such that $|U(t,x)| \leq C_R + \frac{\mu}{2}|x|^2$ for all $|x| \geq R$ and all t. This, together with (U2) implies that for every $x \neq 0$ and every t, $\frac{1}{|x|^\alpha} - U(t,x) \geq -C_0 - \frac{\mu}{2}|x|^2$, for some constant $C_0 > 0$; therefore $I(x) \geq \frac{1}{2}\int_0^T |\dot{x}|^2 dt - \frac{\mu}{2}\int_0^T |x|^2 - TC_0$. Since by hypothesis the constant μ is strictly less than $\frac{\pi^2}{T^2}$, the best Sobolev constant of the injection of H_1 into L^2, there exist positive constants β_1 and β_2 such that

$$I(x) \geq \beta_1 ||\dot{x}||_2^2 - \beta_2, \qquad \forall x \in \Lambda_1. \tag{2.3}$$

We begin by perturbing I with a strong force term: for every $\delta > 0$ we take a function $f_\delta \in C^2(]0,+\infty[;\mathbb{R})$ such that

$$(i) \qquad f_\delta(s) = \begin{cases} 0 & \text{if } s \geq \delta \\ -s^{-2} & \text{if } 0 < s \leq \frac{\delta}{2} \end{cases}$$

$$(ii) \qquad f'_\delta(s) \geq 0 \qquad \forall s \in]0,\delta].$$

It is immediate to check that such a function exists for every $\delta > 0$. Now we define $F_\delta \in C^2(\mathbb{R}^N \setminus \{0\};\mathbb{R})$ as $F_\delta(x) = f_\delta(|x|)$ and we consider the functional

$$J_\delta(x) = I(x) - \int_0^T F_\delta(x)dt. \tag{2.4}$$

Since F_δ verifies the strong force assumption ([29]) we have that $\inf_{\partial\Lambda_1} J_\delta = +\infty$, and therefore no critical point of J_δ can cross the singularity. Moreover (2.3) holds for J_δ as well and hence we can conclude that there exists $x_\delta \in \Lambda_1$ such that

$$J_\delta(x_\delta) = \inf_{x \in \Lambda_1} J_\delta(x). \tag{2.5}$$

We shall now show that if δ is small enough, then $\min_{t\in[0,T]} |x_\delta(t)| \geq \delta$ so that x_δ is actually a solution of problem (1.5).

To this aim we suppose that for every $\delta > 0$ the minimum x_δ found above satisfies

$$\min_{t\in[0,T]} |x_\delta(t)| < \delta \tag{2.6}$$

and we show that this leads to a contradiction. In what follows we shall denote by C_j, $j = 1, 2, ...$ positive constants *independent* of δ.

Since x_δ is an absolute minimum for J_δ, in Λ_1, there exists C_1 such that

$$I(x_\delta) \leq J_\delta(x_\delta) \leq C_1; \tag{2.7}$$

then from (2.3) it follows that there exist C_2, C_3 such that

$$||\dot{x}_\delta||_2 \leq C_2 \tag{2.8}$$

and

$$\int_0^T \frac{a}{|x_\delta|^\alpha} dt \leq C_3. \tag{2.9}$$

This shows in particular that there exists $\varepsilon' \leq \varepsilon$ (independent of δ) such that if $A_{\varepsilon'} = \{t \in [0,T] \ / \ |x_\delta(t)| \leq \varepsilon'\}$, then $meas(A_{\varepsilon'}) < \frac{T}{2}$. At this point we can take an interval $[t_0, t_1] \subset [0,T]$ such that

i) $|x_\delta(t_0)| = |x_\delta(t_1)| = \varepsilon'$,

ii) $|x_\delta(t)| < \varepsilon' \quad \forall t \in]t_0, t_1[$,

iii) $\min_{t\in[t_0,t_1]} |x_\delta(t)| = \min_{t\in[0,T]} |x_\delta(t)| = |x_\delta(t_\delta)| < \delta$,

iv) $|t_1 - t_0| < \frac{T}{2}$;

we also remark that by (2.8)

$$|\varepsilon - \delta| \leq |x_\delta(t_0) - x_\delta(t_\delta)| \leq \int_{t_0}^{t_\delta} |\dot{x}_\delta| dt \leq C_2 |t_1 - t_0|^{\frac{1}{2}}$$

so that $|t_1 - t_0| \geq C_4$.

Repeating the above argument for $\frac{\varepsilon'}{2}$ we can find another interval $[s_0, s_1] \subset]t_0, t_1[$ such that $|x_\delta(s_0)| = |x_\delta(s_1)| = \frac{\varepsilon'}{2}$, $|x_\delta(t)| < \frac{\varepsilon'}{2}$ $\forall t \in]s_0, s_1[$ and $t_\delta \in]s_0, s_1[$. Exactly as above, then, there exist C_5, C_6 and C_7 such that $|s_1 - s_0| \geq C_5$ $|t_0 - s_0| \geq C_6$ $|t_1 - s_1| \geq C_7$.

Step 1: $N \geq 3$. Since the potential is radial in $\overline{B}_\varepsilon \setminus \{0\}$, x_δ is planar in the same set, and precisely it lies in the plane spanned by $x_\delta(t_0)$ and $\dot{x}_\delta(t_0)$. Assuming $N \geq 3$, there exists a vector $v_\delta \in S^{N-1}$ orthogonal to x_δ in $[t_0, t_1]$. We are going to use v_δ to show that there exists at least one direction along which the second derivative of J_δ is negative definite.

To this end let $\xi_\delta : [0,T] \to [0,1]$ be a continuous, piecewise linear function which satisfies

$$\xi_\delta = \begin{cases} 1 & \text{if } s \in [s_0, s_1] \\ 0 & \text{if } s \notin [t_0, t_1]. \end{cases} \tag{2.10}$$

We extend ξ_δ by periodicity to $\mathbb{R}$ and we set $w_\delta(t) = \big(\xi_\delta(t) - \xi_\delta(t + \frac{T}{2})\big)v_\delta$; then we have $x_\delta + w_\delta \in \Lambda_1$.

We want to show that $\nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta) < 0$, (for δ small) contradicting the fact that x_δ is a minimum point. To carry over this estimate we need to analyze the form of $\nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta)$. An easy computation shows that

$$\begin{aligned} \nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta) = & \int_0^T |\dot{w}_\delta|^2 dt + a\alpha(\alpha+2)\int_0^T \frac{(x_\delta \cdot w_\delta)^2}{|x_\delta|^{\alpha+4}} dt - a\alpha \int_0^T \frac{|w_\delta|^2}{|x_\delta|^{\alpha+2}} dt \\ & - \int_0^T [\nabla^2 U(t, x_\delta)(w_\delta, w_\delta) + \nabla^2 F_\delta(x_\delta)(w_\delta, w_\delta)] dt. \end{aligned} \tag{2.11}$$

Notice that whenever $w_\delta(t) \neq 0$ we have $|x_\delta(t)| \leq \varepsilon'$, so that we can substitute U with ϕ. Now, it is easy to see that in $\overline{B}_\varepsilon \setminus \{0\}$,

$$\nabla^2 \phi(|x|)(\zeta, \zeta) = K(x)(x \cdot \zeta)^2 + \frac{\phi'(|x|)}{|x|}|\zeta|^2, \tag{2.12}$$

for some suitable $K \in C^0(\overline{B}_\varepsilon \setminus \{0\}; \mathbb{R})$, and for all $\zeta \in \mathbb{R}^N$; moreover a similar expression holds for $\nabla^2 F_\delta$. Thus, since by definition we have $f'_\delta(|x|) \geq 0$ for all $x \neq 0$ and $x_\delta(t) \cdot w_\delta(t)$ for every t, we obtain, also taking into account the symmetry of x_δ and w_δ,

$$\begin{aligned} \nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta) & \leq \frac{2}{C_6} + \frac{2}{C_7} - a\alpha \int_0^T \frac{|w_\delta|^2}{|x_\delta|^{\alpha+2}} dt - \int_0^T \frac{\phi'(|x_\delta|)}{|x_\delta|}|w_\delta|^2 dt \\ & \leq \frac{2}{C_6} + \frac{2}{C_7} - 2a\alpha \int_{t_0}^{t_1} \frac{|w_\delta|^2}{|x_\delta|^{\alpha+2}} dt - \int_{t_0}^{t_1} \frac{\phi'(|x_\delta|)}{|x_\delta|}|w_\delta|^2 dt. \end{aligned} \tag{2.13}$$

Without loss of generality we can suppose ε' so small that by (U2)

$$-\frac{a\alpha}{|x|^{\alpha+2}} - \frac{\phi'(|x|)}{|x|} \leq -\frac{S}{|x|^{\alpha+2}},$$

for some $S > 0$ and for every $x \in \overline{B}_{\varepsilon'} \setminus \{0\}$, so that

$$\nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta) \leq \frac{2}{C_6} + \frac{2}{C_7} - 2S \int_{s_0}^{s_1} \frac{1}{|x_\delta|^{\alpha+2}} dt. \tag{2.14}$$

This completes the proof since, due to the strong force condition, if $\int_{s_0}^{s_1} |\dot{x}_\delta|^2 dt$ is bounded and $\min_{t\in[s_0,s_1]} |x_\delta(t)| \to 0$, then

$$\int_{s_0}^{s_1} \frac{1}{|x_\delta|^{\alpha+2}} dt \to +\infty.$$

This means that for δ small enough, $\nabla^2 J_\delta(x_\delta)(w_\delta, w_\delta) < 0$, and x_δ is not a minimum point for J_δ in Λ_1.

Step 2: $N = 2$. The case $N = 2$ can be proved in the following way. By the estimates until Step 1, one has that

$$\begin{aligned} &\int_{t_0}^{t_1} [\frac{1}{2}|\dot{x}_\delta|^2 + \frac{a}{|x_\delta|^\alpha} - \phi(|x_\delta|) - F_\delta(|x_\delta|)] dt \\ &= \min_{y\in\Sigma_2} \int_{t_0}^{t_1} [\frac{1}{2}|\dot{y}|^2 + \frac{a}{|y|^\alpha} - \phi(|y|) - F_\delta(|y|)] dt \end{aligned} \tag{2.15}$$

where

$$\Sigma_2 = \{y \in H^1([t_0,t_1];\mathbb{R}^2) \ / \ y(t_0) = x_\delta(t_0),\ y(t_1) = x_\delta(t_1),\ |y(t)| \leq \varepsilon'\ \forall t \in [t_0,t_1]\}.$$

Now, since the potential appearing in (2.15) is radial, we can immerge the minimization problem (2.15) in a higher dimension, that is, we consider a minimizer $\bar{x}$ of the integral in (2.15) in the set

$$\Sigma_N = \{y \in H^1([t_0,t_1];\mathbb{R}^N) \ / \ y(t_0) = x_\delta(t_0),\ y(t_1) = x_\delta(t_1),\ |y(t)| \leq \varepsilon'\ \forall t \in [t_0,t_1]\}$$

for some $N \geq 3$. Again, because of the radial symmetry of the potential and the constraint, we can assume that $\bar{x}$ is planar, and precisely that it lies in the same plane of x_δ, so that $\bar{x}$ minimizes the integral on the right–hand–side of (2.15) on Σ_2 as well as x_δ. Therefore we have $x_\delta = \bar{x}$ in $[t_0,t_1]$. At this point we know from Step 1 that (for δ small) $\bar{x}$ cannot interact with the perturbation. ∎

We remark that the argument used in the above proof can be repeated with some straightforward modifications (actually simplifications) to obtain the following result.

Theorem 2.2. Let F be as in (2.1), with $N \geq 2$, $a > 0$ and $\alpha \in]0, 2[$. Let $T > 0$ and $x_0, x_1 \in \mathbb{R}^N \setminus \{0\}$ be fixed and let $\lambda(x_0, x_1)$ be the best Sobolev constant of the injection of $\{x \in H^1([0,1]; \mathbb{R}^N) \ / \ x(0) = x_0,\ x(1) = x_1\}$ into L^2. Assume (U1), (U2), (H3) hold with $\mu < \frac{\lambda(x_0,x_1)}{T^2}$. Then problem (1.7) has at lest one solution which minimizes I in Λ_2.

1.3. The general case

This section is devoted to the proof of Theorem 1.2. To this end we shall make use of the results of Section 1.2, since we shall first work with a truncated potential which is radially symmetric in a neighborhood of zero. Then we shall obtain a more precise estimate of the behavior of the minimum found with the application of Theorem 2.1. With the aid of these estimates we shall show that any such minimum cannot interact with the truncation by the construction of a suitable variation along which the functonal decreases.

Throughout this section δ is the radius of the neighborhood where the truncation is located, and therefore δ can always be considered smaller than one.

We now introduce the family of truncations that we are going to use in the first part of the proof.

Let $\varphi : [0, +\infty[\to [0, 1]$ be a fixed cut-off function of class C^2 such that

(3.1)
$$\varphi(s) = \begin{cases} 0 & \text{if } s \leq \frac{1}{2} \\ 1 & \text{if } s \geq 1 \end{cases}$$

and define, for every $\delta > 0$,

(3.2)
$$\varphi_\delta(s) = \varphi\left(\frac{s}{\delta}\right).$$

With the aid of the function φ_δ we construct the truncated potential: we set

$$U_\delta(t, x) = \varphi_\delta(|x|)U(t, x), \qquad F_\delta(t, x) = -\frac{a}{|x|^\alpha} + U_\delta(t, x)$$

and finally

(3.3)
$$I_\delta(x) = \frac{1}{2}\int_0^T |\dot{x}|^2 dt - \int_0^T F_\delta(t, x)dt, \qquad \forall x \in \Lambda_1.$$

Proposition 3.1. Let U satisfy (H1) and (H2). Then U_δ satisfies (H1) and (H2) too, uniformly in δ.

Proof. It is clear that (H1) is fulfilled. Concerning (H2), remark that since φ is of class C^2 and its derivatives have compact support in $[0,1]$, then there exists a constant $d > 0$ such that $|\nabla\varphi_\delta(|x|)| \le \frac{d}{\delta}$ and $|\nabla^2\varphi_\delta(|x|)| \le \frac{d}{\delta^2}$. To complete the proof just consider that

$$|\nabla^2 U_\delta(t,x)| \le |U(t,x)||\nabla^2\varphi_\delta(|x|)| + 2|\nabla U(t,x)||\nabla\varphi_\delta(|x|)| + |\nabla^2 U(t,x)||\varphi_\delta(|x|)|. \quad \blacksquare$$

From now on the letter C will denote different positive constants *independent* of δ.

By the application of Theorem 2.1 (recall that $F_\delta(t,x) = -\frac{a}{|x|^\alpha}$ if $0 < |x| \le \frac{\delta}{2}$) and by Proposition 3.1 we can find an $x_\delta \in \Lambda_1$ such that

$$x_\delta \text{ is a local minimum for } I_\delta, \tag{3.4}$$

and

$$\exists C \text{ such that } I_\delta(x_\delta) \le C \quad \forall\delta > 0. \tag{3.5}$$

Of course if $\limsup_{\delta\to 0}\min_{t\in[0,T]}|x_\delta(t)| > 0$, then at least one x_δ solves (1.6), and there is nothing to prove. Thus we assume that

$$\limsup_{\delta\to 0}\min_{t\in[0,T]}|x_\delta(t)| = 0 \tag{3.6}$$

and we show that this leads to a contradiction. To this aim we begin by showing that each x_δ is "almost planar" in a neighborhood of zero *independent* of δ.

Keeping in mind the dependence on δ, we denote x_δ simply by x. Then x verifies the motion equation

$$-\ddot{x} = \nabla F_\delta(t,x), \qquad \forall t \in [0,T] \tag{3.7}$$

and the energy equality

$$\frac{1}{2}|\dot{x}(t)|^2 + F_\delta(t,x(t)) = E_\delta + \int_{t_0}^{t}\frac{\partial U_\delta}{\partial t}(s,x)ds \qquad \forall t \in [0,T] \tag{3.8}$$

for some constant $E_\delta \in \mathbb{R}$. We remark that there exists C such that

$$|E_\delta| \le C \qquad \forall\delta > 0. \tag{3.9}$$

Indeed, because of the growth assumptions on U_δ at the origin and at infinity and Proposition 3.1, the boundedness of $I_\delta(x)$ implies that each of the terms $\int_0^T \frac{1}{2}|\dot{x}(t)|^2 dt$, $\int_0^T \frac{a}{|x|^\alpha} dt$, $\int_0^T U_\delta(t,x)dt$ and $\int_0^T \frac{\partial U_\delta}{\partial t}(s,x)ds$ is bounded independently of δ. Then (3.9) follows from (3.8).

We now write $x = \rho\theta$, with $\rho = |x|$ and $\theta = \frac{x}{|x|}$. Then

$$\frac{1}{2}\ddot{\rho^2} = \frac{1}{2}\frac{d^2}{dt^2}|x|^2 = \frac{d}{dt}(x \cdot \dot{x}) = |\dot{x}|^2 + x \cdot \ddot{x}$$

so that by (3.7) and (3.8) we can write

$$\frac{1}{2}\ddot{\rho^2} = 2E_\delta + (2-\alpha)\frac{a}{\rho^\alpha} - 2U_\delta(t,x) - 2\nabla U_\delta(t,x)\cdot x + 2\int_{t_0}^t \frac{\partial U_\delta}{\partial t} dt. \tag{3.10}$$

Remark that by (H1), (H2) and Proposition 3.1, for every $r > 0$ there exists δ_0 (independent of δ) such that

$$|x|^\alpha \Big| \int_{t_0}^t \frac{\partial U_\delta}{\partial t} dt - U_\delta(t,x) - \nabla U_\delta(t,x)\cdot x \Big| \le r \qquad \forall 0 < |x| \le \delta_0. \tag{3.11}$$

Therefore there exist two constants $0 < a_1 < a < a_2$ independent of δ such that, from (3.10),

$$\frac{a_1}{\rho^\alpha} \le \frac{1}{2}\ddot{\rho^2} \le \frac{a_2}{\rho^\alpha} \tag{3.12}$$

for every instant t such that $\rho(t) \le \delta_0$.

We now consider an interval $[t_0, t_1]$ such that

i) $\rho(t_0) = \rho(t_1) = \delta_0$,

ii) $\rho(t) < \delta_0 \qquad \forall t \in]t_0, t_1[$,

iii) $\min_{t\in[0,T]} \rho(t) = \min_{t\in[t_0,t_1]} \rho(t) = \rho(t^*)$.

and we prove some technical lemmas. These hold whenever δ_0 is small enough (though fixed) independently of δ.

Proposition 3.2. Assume that (H1), (H2) and (H3) hold. Then there exists a constant $S > 0$ such that for every $\delta > 0$ and for exery x satisfying (3.4) and (3.5), there exists a vector $v \in S^{N-1}$ such that

$$\Big|v \cdot \frac{x}{|x|}\Big| \le S|x|^\sigma \qquad \forall t \in [t_0, t_1], \tag{3.13}$$

where σ is the exponent appearing in (H2).

Remark 3.3. If $F_\delta(t,x)$ were radially symmetric in $B_{\delta_0}\setminus\{0\}$ like it was the case in Section 1.2, then x_δ would be planar in the same set, and (since $N \geq 3$) (3.13) would trivially hold with $S=0$.

Proof. The proof will be carried out for $t \geq t^*$ (t^* is defined in iii) above). It will be clear that by reversing the time the same arguments work for $t \leq t^*$ as well.

We now write the motion equation (3.7) in the polar system of coordinates (ρ,θ) introduced above. To this aim we observe that $\ddot{x} = \ddot{\rho}\theta + 2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}$ and we recall that by definition we have $|\theta| = 1$, $\theta\cdot\dot{\theta} = 0$ and $\theta\cdot\ddot{\theta} = -|\dot{\theta}|^2$. Multiplying (3.7) by θ we have

$$-\ddot{\rho} + \rho|\dot{\theta}|^2 = \alpha\frac{a}{\rho^{\alpha+1}} + \nabla U_\delta(t,x)\cdot\theta \tag{3.14}$$

and subtracting (3.14) multiplied by θ from (3.7) we obtain

$$-2\dot{\rho}\dot{\theta} - \rho(\ddot{\rho} + |\dot{\theta}|^2\theta) = \nabla U_\delta(t,x) - (\nabla U_\delta(t,x)\cdot\theta)\theta. \tag{3.15}$$

If we denote by $\nabla_\rho U_\delta(t,x) = \nabla U_\delta(t,x)\cdot\theta$ and by $\nabla_\theta U_\delta(t,x) = \nabla U_\delta(t,x) - (\nabla U_\delta(t,x)\cdot\theta)\theta$, the radial and tangential parts of ∇U_δ respectively, then we see that (3.7) is equivalent to the system

$$\begin{cases} -\ddot{\rho} + \rho|\dot{\theta}|^2 = \nabla_\rho U_\delta(t,x) \\ -2\dot{\rho}\dot{\theta} - \rho(\ddot{\rho} + |\dot{\theta}|^2\theta) = \nabla_\theta U_\delta(t,x). \end{cases} \tag{3.16}$$

Notice that the second equation in (3.16) can be written

$$\frac{d}{dt}(\rho^2\dot{\theta}) = -\rho^2|\dot{\theta}|^2\theta - \rho\nabla_\theta U_\delta(t,x); \tag{3.17}$$

setting $\vec{B} = \rho^2\dot{\theta}$ and $B = |\vec{B}|$ we have from (3.17) that

$$\frac{d}{dt}B = \frac{\vec{B}}{B}\cdot\frac{d}{dt}\vec{B} = -\rho\nabla_\theta U(t,x)\frac{\dot{\theta}}{|\dot{\theta}|}, \tag{3.18}$$

so that by (H2),

$$|\frac{d}{dt}B| \leq \rho|\nabla_\theta U\delta(t,x)| \leq \frac{C}{\rho^{\alpha-\sigma}}, \tag{3.19}$$

and finally,

$$|B(t) - B(t^*)| \leq \int_{t^*}^{t} \frac{C}{\rho^{\alpha-\sigma}} dt. \tag{3.20}$$

We now want to estimate this integral. To this aim we multiply (3.12) by $2\rho^\sigma$ and we integrate, obtaining

$$\begin{aligned} \int_{t^*}^{t} \frac{2a_1}{\rho^{\alpha-\sigma}} dt &\leq \int_{t^*}^{t} \rho^\sigma \ddot{\rho^2} dt = \rho^\sigma(t)\dot{\rho^2}(t) - \rho^\sigma(t^*)\dot{\rho^2}(t^*) - \sigma \int_{t^*}^{t} \rho^{\sigma-1}\dot{\rho^2} dt \\ &= \rho^{\sigma+1}(t)\dot{\rho}(t) - \rho^{\sigma+1}(t^*)\dot{\rho}(t^*) - 2\sigma \int_{t^*}^{t} \rho^\sigma \dot{\rho} dt. \end{aligned} \tag{3.21}$$

Now the function ρ^2 is always positive and has a global minimum at $t = t^*$; therefore $\dot{\rho^2}(t^*) = 2\rho(t^*)\dot{\rho}(t^*) = 0$, and the second term in the right-hand-side of (3.21) vanishes. Moreover from (3.12) it follows $\ddot{\rho^2} > 0$ $\forall t \in]t^*, t_1]$, so that $\dot{\rho^2}$ is increasing in $]t^*, t_1]$. Since $\dot{\rho^2}(t^*) = 0$, we have $2\rho(t)\dot{\rho}(t) = \dot{\rho^2}(t) > 0$ for all $t \in]t^*, t_1]$. This proves that $\dot{\rho}(t) > 0$ in $]t^*, t_1]$. If we carry this information in (3.21) we obtain

$$\int_{t^*}^{t} \frac{2a_1}{\rho^{\alpha-\sigma}} dt \leq \rho^{\sigma+1}(t)\dot{\rho}(t) \qquad \forall t \in]t^*, t_1]. \tag{3.22}$$

Finally multiplying (3.8) by $|x|^2$, we obtain, (using (H2)),

$$|x||\dot{x}| \leq C|x|^{\frac{2-\alpha}{2}} = C\rho^{\frac{2-\alpha}{2}} \qquad \forall t \in]t^*, t_1] \tag{3.23}$$

and therefore

$$\frac{1}{2}\rho\dot{\rho} = \frac{1}{2}\frac{d}{dt}|x|^2 = x \cdot \dot{x} \leq |x||\dot{x}| \leq C_4\rho^{\frac{2-\alpha}{2}} \qquad \forall t \in]t^*, t_1]. \tag{3.24}$$

Using (3.22) and (3.24) we deduce the estimate

$$\int_{t^*}^{t} \frac{2a_1}{\rho^{\alpha-\sigma}} dt \leq \rho^{\sigma+1}(t)\dot{\rho}(t) \leq C\rho^{\frac{2-\alpha}{2}+\sigma}(t) \qquad \forall t \in]t^*, t_1]. \tag{3.25}$$

Recalling the definition of B, we see that (3.8) becomes

$$\frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\frac{B^2}{\rho^2} - \frac{a}{\rho^\alpha} + U_\delta(t, x) = E_\delta + \int_{t_0}^{t} \frac{\partial U_\delta}{\partial t} dt$$

and therefore, by (H2),

$$B^2 \leq C\rho^{2-\alpha} \qquad \forall t \in [t_0, t_1], \tag{3.26}$$

if δ_0 is small enough.

We derive therefore from (3.20), (3.25) and (3.26) that

$$|B^2(t) - B^2(t^*)| \leq |B(t^*)||B(t) - B(t^*)| + |B(t)||B(t) - B(t^*)| \leq C\rho^{2-\alpha+\sigma} \qquad \forall t \in]t^*, t_1]. \tag{3.27}$$

We now take a vector $v \in S^{N-1}$ such that $v \cdot \theta(t^*) = 0$ and $v \cdot \dot{\theta}(t^*) = 0$ and we set $\phi(t) = v \cdot \theta(t)$. If we multiply (3.17) by v we see that ϕ satisfies the Cauchy problem

$$\begin{cases} \rho^2 \frac{d}{dt} \rho^2 \dot{\phi} = -B^2 \phi + \rho^3 \nabla_\theta U_\delta \cdot v \\ \phi(t^*) = \dot{\phi}(t^*) = 0. \end{cases} \tag{3.28}$$

Let us denote $h(t) = \rho^3 \nabla_\theta U_\delta \cdot v$; as usual, by (H2), we have

$$|h(t)| \leq C\rho^{2-\alpha+\sigma}. \tag{3.29}$$

Moreover from (3.29) and (3.25) we deduce that

$$\int_{t^*}^t \frac{1}{\rho^2} \int_{t^*}^s \frac{|h(\tau)|}{\rho^2} d\tau ds \leq C \int_{t^*}^t \frac{1}{\rho^2} \int_{t^*}^s \frac{1}{\rho^{\alpha-\sigma}} d\tau ds \leq C \int_{t^*}^t \frac{\rho^{\sigma+1}\dot{\rho}}{\rho^2} ds \leq \tilde{C}\rho^\sigma \tag{3.30}$$

and likewise,

$$\int_{t^*}^t \frac{1}{\rho^2} \int_{t^*}^s \frac{|B^2(t) - B^2(t^*)|\rho^\sigma}{\rho^2} d\tau ds \leq C\rho^{2\sigma}. \tag{3.31}$$

With the aid of these two inequalities, we obtain the estimates on ϕ which we shall use in the proof of the following proposition, given in the appendix. We denote by $B_0^2 = B(t^*)^2$ and by $\beta(t) = B_0^2 - B(t)^2$; then the equation in (3.28) can be written

$$\rho^2 \frac{d}{dt} \rho^2 \dot{\phi} = -B_0^2 \phi + \beta(t)\phi + h(t) \tag{3.32}$$

Proposition 3.4. Let ϕ be the solution of

$$\begin{cases} \rho^2 \frac{d}{dt} \rho^2 \dot{\phi} = -B_0^2 \phi + \beta(t)\phi + h(t) \\ \phi(t^*) = \dot{\phi}(t^*) = 0 \end{cases} \tag{3.33}$$

and assume that there exists a function $f(t) > 0$ such that

$$\int_{t^*}^{t} \frac{1}{\rho^2} \int_{t^*}^{s} \frac{|h(\tau)|}{\rho^2} d\tau ds \leq f(t) \qquad \forall t \in [t^*, t_1] \tag{3.34}$$

and

$$\int_{t^*}^{t} \frac{1}{\rho^2} \int_{t^*}^{s} \frac{|\beta(\tau)| f(t)}{\rho^2} d\tau ds < \frac{1}{2} f(t) \qquad \forall t \in [t^*, t_1]. \tag{3.35}$$

Then

$$|\phi(t)| \leq 2f(t) \qquad \forall t \in [t^*, t_1]. \tag{3.36}$$

We can now easily complete the proof of Proposition 3.2, just by applying Proposition 3.4 with $f(t) = \tilde{C}\rho(t)^\sigma$ and by choosing $S = 2\tilde{C}$: we obtain

$$\left| v \cdot \frac{x}{|x|} \right| = |\phi| \leq 2f(t) = 2\tilde{C}\rho^\sigma. \quad \blacksquare$$

Next proposition provides an estimate of the time that a solution x takes in going from $|x(t_0)| = \delta_0$ to $|x(s_0)| = \frac{\delta_0}{2}$, when δ_0 is small.

Proposition 3.5. Let $[s_0, s_1] \subset]t_0, t_1[$ be an interval such that

$$\rho(s_0) = \rho(s_1) = \frac{\delta_0}{2}$$
$$\rho(t) \leq \frac{\delta_0}{2} \qquad \forall t \in [s_0, s_1].$$

Then there exist constants C', C'' such that

$$C'\delta_0^{\frac{2+\alpha}{2}} \geq \max\{t_1 - s_1, s_0 - t_0\} \geq \min\{t_1 - s_1, s_0 - t_0\} \geq C''\delta_0^{\frac{2+\alpha}{2}},$$

provided δ_0 is small enough, independently of δ.

Proof. Let us set $\omega = s_0 - t_0$; then by (3.23),

$$|x||\dot{x}| \leq C\delta_0^{\frac{2-\alpha}{2}}, \qquad \forall t \in [t_0, s_0].$$

Integrating (3.12) we obtain

$$(3.37) \qquad \frac{\omega}{\delta_0^\alpha} \le \int_{t_0}^{s_0} \frac{1}{|x|^\alpha} dt \le C|x||\dot{x}| \le C\delta_0^{\frac{2-\alpha}{2}};$$

therefore $\omega \le C'\delta_0^{\frac{2+\alpha}{2}}$.

Since integrating the energy equation one finds by (3.37)

$$\int_{t_0}^{s_0} |\dot{x}|^2 dt \le C \int_{t_0}^{s_0} \frac{1}{|x|^\alpha} dt \le C\delta_0^{\frac{2-\alpha}{2}},$$

then

$$\frac{\delta_0}{2} = |x(s_0)| - |x(t_0)| \le \int_{t_0}^{s_0} |\dot{x}| dt \le \omega^{\frac{1}{2}} \Big(\int_{t_0}^{s_0} |\dot{x}|^2 \Big)^{\frac{1}{2}} dt \le C\omega^{\frac{1}{2}} \delta_0^{\frac{2-\alpha}{2}},$$

that is, $\omega \ge C''\delta_0^{\frac{2+\alpha}{2}}$, the required estimate. Of course the same argument holds for $\omega = t_1 - s_1$ too. ∎

End of the proof of Theorem 1.2. In order to complete the proof of Theorem 1.2 we begin by making a truncation of the potential as in (3.1)–(3.3). By means of Theorem 2.1 the corresponding truncated functional I_δ possesses, for every $\delta > 0$, a local minimum x_δ satifying moreover (3.5). Therefore x_δ is a noncollision solution for the associated differential problem. As above, the problem is to show that x_δ does not interact with the truncation, or, in other words, that $\min_{t\in[0,T]} |x_\delta(t)| \ge \delta$ when δ is small. To do this we are going to apply the results of the previous analysis in order to show that, whenever (3.6) holds, one can find a small variation of x_δ making the functional decrease.

Let us fix δ_0 so small that all the results of the above propositions hold and consider a cut-off function $\psi \in C^1([0,T];\mathbb{R})$ such that

$$(3.38) \qquad \begin{cases} \psi(t) = 0 & \forall t \notin [t_0, t_1] \\ \psi(t) = 1 & \forall t \in [s_0, s_1] \\ \int_0^T |\dot{\psi}(t)|^2 dt \le 2\left(\frac{1}{s_0 - t_0} + \frac{1}{t_1 - s_1}\right). \end{cases}$$

Assuming (as is always possible) that $t_1 - t_0 < \frac{T}{2}$, let us set

$$(3.39) \qquad w_\delta(t) = \begin{cases} \psi(t) v_\delta & \forall t \in [t_0, t_1] \\ -\psi(t) v_\delta & \forall t \in [t_0 + \frac{T}{2}, t_1 + \frac{T}{2}], \end{cases}$$

and extend w_δ by periodicity to the whole of $\mathbb{R}$; we recall that $v \in S^{N-1}$ is the vector associated to x_δ by the application of Proposition 3.2. Remark that by Propositions 3.2 and 3.5 it follows that

$$|w_\delta \cdot \frac{x_\delta}{|x_\delta|}| \leq S|x_\delta|^\sigma \qquad \forall t \in [0,T] \tag{3.40}$$

and

$$\int_{t_0}^{t_1} |\dot{w}_\delta|^2 dt \leq C \frac{1}{\delta_0^{\frac{2+\alpha}{2}}}. \tag{3.41}$$

We are going to show that $\nabla^2 I_\delta(x_\delta)(w_\delta, w_\delta) < 0$ when $\min_t |x_\delta(t)| \to 0$ as $\delta \to 0$. This fact will end the proof since at every local minimum I_δ has a positive definite second derivative.

From assumption (H2) we can estimate $\nabla^2 I_\delta(x_\delta)(w_\delta, w_\delta)$ as

$$\nabla^2 I_\delta(x_\delta)(w_\delta, w_\delta) \leq \int_0^T |\dot{w}_\delta|^2 dt + C_1 \int_0^T \frac{(x_\delta \cdot w_\delta)^2}{|x_\delta|^{\alpha+4}} dt - C_2 \int_0^T \frac{|w_\delta|^2}{|x_\delta|^{\alpha+2}} dt =$$
$$2\int_{t_0}^{t_1} |\dot{w}_\delta|^2 dt + 2C_1 \int_{t_0}^{t_1} \frac{(x_\delta \cdot w_\delta)^2}{|x_\delta|^{\alpha+4}} dt - 2C_2 \int_{t_0}^{t_1} \frac{|w_\delta|^2}{|x_\delta|^{\alpha+2}} dt$$

provided δ_0 is small enough.

From (3.40), (3.41) and the last inequality, we deduce that

$$\nabla^2 I_\delta(x_\delta)(w_\delta, w_\delta) \leq C \frac{1}{\delta_0^{\frac{2+\alpha}{2}}} + C_3 \int_{t_0}^{t_1} \frac{1}{|x_\delta|^{\alpha+2-2\sigma}} dt - C_4 \int_{s_0}^{s_1} \frac{1}{|x_\delta|^{\alpha+2}} dt$$
$$= \frac{C}{\delta_0^{\frac{2+\alpha}{2}}} + \int_{t_0}^{s_0} \frac{C_3}{|x_\delta|^{\alpha+2-2\sigma}} dt + \int_{s_0}^{s_1} \frac{C_3}{|x_\delta|^{\alpha+2-2\sigma}} dt + \int_{s_1}^{t_1} \frac{C_3}{|x_\delta|^{\alpha+2-2\sigma}} dt - \int_{s_0}^{s_1} \frac{C_4}{|x_\delta|^{\alpha+2}} dt$$

Now from Proposition 3.5 we derive the estimate

$$\int_{t_0}^{s_0} \frac{1}{|x_\delta|^{\alpha+2-2\sigma}} dt + \int_{s_1}^{t_1} \frac{1}{|x_\delta|^{\alpha+2-2\sigma}} dt \leq C \frac{1}{\delta_0^{\frac{2+\alpha}{2}}},$$

while

$$\int_{s_0}^{s_1} \frac{C_3}{|x_\delta|^{\alpha+2-2\sigma}} dt - \int_{s_0}^{s_1} \frac{C_4}{|x_\delta|^{\alpha+2}} dt \to -\infty$$

as $\min_t |x_\delta(t)| \to 0$ (like in the Proof of Theorem 2.1). Since δ_0 is *fixed* independently of δ, we can conclude that (3.6) implies $\nabla^2 I_\delta(x_\delta)(w_\delta, w_\delta) < 0$ for δ small, in contradiction with the fact that x_δ is a local minimum for I_δ. Hence (3.6) cannot hold true, and the proof of Theorem 1.2 is complete. ∎

1.4. Further results

As it should be clear by now, the main idea used in the proof of Theorems 1.2 and 1.4 can be summarized in the following proposition.

Proposition 4.1. (A priori estimate). Assume that the hypotheses of Theorem 1.2 (or 1.4) are satisfied. Then for every $C \in \mathbb{R}$ there exists $\delta_0 > 0$ such that if $x \in \Lambda_1$ (resp. in Λ_2) is a local minimum of I with $I(x) \le C$, then $\min_{t\in[0,T]} |x(t)| \ge \delta_0$

The same results holds (with the same δ_0) for the perturbed functional I_δ, where

$$I_\delta(x) = \frac{1}{2}\int_0^T |\dot{x}|^2 dt + \int_0^T \frac{a}{|x|^\alpha} dt - \int_0^T U_\delta(t,x)dt - \int_0^T F_\delta(|x|)dt,$$

F_δ is the strong force term as in (2.4) and U_δ is as in (3.2).

A consequence of Proposition 4.1 is the following estimate.

Theorem 4.2. Under the hypotheses of Theorem 1.2 and 1.4 respectively,

$$\inf_{x\in\Lambda_i} I(x) < \inf_{x\in\partial\Lambda_i} I(x) \qquad i = 1,2.$$

Proof. Assume by contradiction that

$$c_i = \inf_{x\in\Lambda_i} I(x) = \inf_{x\in\partial\Lambda_i} I(x)$$

and denote by K_{c_i} the (compact) set of all the minima of I in $\Lambda_i \cup \partial\Lambda_i$. Then it follows from Proposition 4.1 that

$$dist(K_{c_i} \cap \Lambda_i, \partial\Lambda_i) = d_i > 0, \qquad i = 1,2.$$

We take an $\varepsilon > 0$ and we introduce a functional I^* such that

$$I^*(x) = \begin{cases} I(x) & \text{if } dist(x, K_{c_i} \cap \Lambda_i) \ge \frac{d_i}{2} \\ I(x) + \varepsilon & \text{if } dist(x, K_{c_i} \cap \Lambda_i) \le \frac{d_i}{4} \end{cases}$$

and $I^*(x) > I(x)$ if $dist(x, \mathcal{C}K_{c_i} \cap \Lambda_i) < \frac{d_i}{2}$. Then we have

$$c_i = \inf_{x \in \Lambda_i} I^*(x) = \inf_{x \in \partial\Lambda_i} I^*(x);$$

with a slight modification of the arguments used in the proof of Theorem 1.2 (resp. 1.4) one can easily prove that I^* admits a minimum $x^* \in \Lambda_i$, at level c_i. Therefore $dist(x, K_{c_i} \cap \Lambda_i) > \frac{d_i}{2}$, so that x^* minimizes I as well, and this is, by construction, a contradiction. ∎

1.5. Appendix

We prove Proposition 3.4. Remark that by the change of independent variable given by $s(t) = \int_{t^*}^{t} \frac{1}{\rho^2} dt$, the Cauchy problem (3.33) can be written

$$\begin{cases} \psi''(s) = -B_0^2\psi + \beta(s)\psi + h(s) \\ \psi(0) = \psi'(0) = 0. \end{cases} \tag{5.1}$$

Here $(\cdot)'$ denotes differentiation with respect to s and $\psi(s) = \phi(t(s))$. The assumptions of Proposition 3.4 become

$$\int_0^s \int_0^\tau |h(\sigma)| d\sigma d\tau \leq f(t(s)) \qquad \forall s \in [0, S] \tag{5.2}$$

$$\int_0^s \int_0^\tau |\beta(\sigma)| f(\sigma) d\sigma d\tau < \frac{1}{2} f(t(s)) \qquad \forall s \in [0, S], \tag{5.3}$$

where $S = \int_{t^*}^{t_1} \frac{1}{\rho^2} dt$.

Let us consider the solution of the Cauchy problem

$$\begin{cases} v''(s) = -B_0^2 v(s) + h(s) \\ v(0) = v'(0) = 0; \end{cases} \tag{5.4}$$

then $v(s) = \int_0^s \int_0^\tau \cos(B_0(s-\tau)) h(\sigma) d\sigma d\tau$, and therefore, from (5.2),

$$|v(s)| \leq \int_0^s \int_0^\tau |h(\sigma)| d\sigma d\tau \leq f(t(s)). \tag{5.5}$$

Now, setting $\xi(s) = \psi(s) - v(s)$, we have that ξ solves

$$\begin{cases} \xi''(s) = -B_0^2\xi(s) + \beta(s)\xi(s) + \beta(s)v(s) \\ \xi(0) = \xi'(0) = 0; \end{cases} \tag{5.6}$$

If we denote $g(s) = \beta(s)\xi(s) + \beta(s)v(s)$ we see that the Cauchy problem (5.6) is exactly of the type (5.4). Hence, if

$$\int_0^s \int_0^\tau |g(\sigma)| d\sigma d\tau \leq f(t(s)) \qquad \forall s \in [0, S] \tag{5.7}$$

then it follows (as above for v) that $|\xi(s)| \le f(t(s))$ and we are done since in this case $|\psi(s)| = |\xi(s) + v(s)| \le |\xi(s)| + |v(s)| \le 2f(t(s))$. Therefore we only have to show that (5.7) holds true. To this aim let

$$s^* = \sup\{\tau \in [0,S] \;/\; |\xi(s)| < f(t(s)) \quad \forall s \in [0,\tau]\};$$

we claim that $s^* = S$. Indeed if by contradiction $s^* < S$, then by (5.3)

$$\begin{aligned}\int_0^{s^*}\int_0^{\tau} |g(\sigma)| d\sigma d\tau &\le \int_0^{s^*}\int_0^{\tau} |\beta(\sigma)|(|\xi(\sigma)||v(\sigma)|) d\sigma d\tau \\ &\le 2\int_0^{s^*}\int_0^{\tau} |\beta(\sigma)| f(\sigma) d\sigma d\tau < f(t(s^*)).\end{aligned}$$

Therefore, like in (5.5)

$$|\xi(s^*)| \le \int_0^{s^*}\int_0^{\tau} |g(\sigma)| d\sigma d\tau < f(t(s^*)),$$

contradicting the definition of s^*. The proof is complete. ∎

CHAPTER 2

REGULARITY RESULTS FOR GENERALIZED SOLUTIONS

2.1. Motivations

In this chapter we continue the study of the existence of periodic solutions for second order Hamiltonian systems of the form

$$-q''(t) = \nabla V(t, q(t)) \tag{1.1}$$

where $q(t) \in \mathbb{R}^N$ and $V \in C^1(\mathbb{R} \times \mathbb{R}^N \setminus \{0\}; \mathbb{R})$ is T-periodic in t and $V(t, x) \to -\infty$ as $|x| \to 0$.

As we have seen in Chapter 1, in order to find classical solutions of (1.1) (i.e. solutions $q(t) \in C^2([0, T]; \mathbb{R}^N \setminus \{0\})$), an important role is played by the behavior of $V(t, x)$ near the singularity $x = 0$.

More precisely, defining as usual the functional $f : \Lambda \to \mathbb{R}$, where

$$\Lambda = \{\, u \in H^1(S^1; \mathbb{R}^N) \mid u(t) \neq 0 \quad \forall t \,\}$$

by

$$f(u) = \frac{1}{2} \int_0^T |u'(t)|^2 \, dt - \int_0^T V(t, u(t)) \, dt \tag{1.2}$$

we recall that one has $f(u) \to +\infty$ as $u \rightharpoonup \partial\Lambda$ weakly in H^1 if V is a "Strong Force", i.e. if

$$-V(t, x) \geq \frac{c}{|x|^2} \tag{1.3}$$

for some $c > 0$ when $|x|$ is close to 0, while, in case (1.3) is violated, one can have situations in which

$$f'(u_n) \to 0, \qquad f(u_n) \to c < +\infty, \qquad u_n \to \partial\Lambda.$$

As a consequence, one can use standard variational arguments to study (1.1) if (1.3) holds, while they generally fail if (1.3) is not satisfied.

To prove existence for (1.1) when (1.3) does not hold one possible approach, used the first time by A. Bahri and P.H. Rabinowitz in [**10**], is the following:

a) Perturb V by a strong force, for example setting $V_\varepsilon(t,x) = V(t,x) - \frac{\varepsilon}{|x|^2}$;

b) Prove existence of a solution q_ε for

$$(1.1)_\varepsilon. \qquad -q''(t) = \nabla V_\varepsilon(t, q(t))$$

This is possible using variational techniques since V_ε now satisfies the strong force condition (1.3).

c) Try to pass to the limit as $\varepsilon \to 0$ to find a solution $\bar{q}$ of (1.1).

This approach indeed works, but one cannot prove in general that such a solution $\bar{q}$ is a classical solution of (1.1). In Chapter 1 we were able to prove that the limit orbit was a classical (noncollision) solution because the approximating functions were found as *minima* of the action functional (actually this is the only role played by the symmetry constraint). Since for the nonsymmetric problem one deals with critical points which are not necessarily minima, Bahri and Rabinowitz have introduced the concept of *generalized solution* q of (1.1).

Definition 1.1. We say that $q \in C(S^1; \mathbb{R})$ is a generalized T-periodic solution of (1.1) if, setting

$$\mathcal{C}(q) = \{\, t \in S^1 \mid q(t) = 0 \,\}$$

(the "collision set") one has

a) $q \in H^1(S^1; \mathbb{R}^N)$ and $f(q) < +\infty$;

b) $\operatorname{meas} \mathcal{C}(q) = 0$;

c) $q \in C^2(S^1 \setminus \mathcal{C}(q); \mathbb{R}^N)$

d) q solves (1.1) pointwise in $S^1 \setminus \mathcal{C}(q)$.

In the case in which $V(t,x)$ does not depend on t one also asks that

e) $\exists E \in \mathbb{R}$ such that $\frac{1}{2}|q'(t)|^2 + V(t, q(t)) = E \ \forall t \in S^1 \setminus \mathcal{C}(q)$ (Energy conservation).

In the paper [**10**] existence of at least one generalized solution is proved. In particular, in such a paper it is asked if such a generalized solutions has additional regularity.

The main object of this paper it is to show that generalized solutions indeed have additional regularity under mild assumptions on the behaviour of V near the singularity. In particular we show, in Section 2.2, that every generalized solution of an autonomous system has only finitely many collisions; and we also show that the number of collisions can be bounded in terms of the value of the action functional (see Theorem 2.1 and Proposition 2.2).

In Section 2.3 such results are generalized to nonautonomous systems: we start by showing that for the generalized solutions which are obtained via the approximation scheme which we have described above, the mechanical energy (which is not conserved for nonautonomous systems) is continuous even through collisions. This fact permits us to prove results similar to those of Section 2.2.

In Section 2.4 we give a bound for the number of collisions of generalized solutions based on the Morse index. Actually, since collision solutions are not regular, one cannot speak of the Morse index of such solutions; and in fact we work with the Morse index of the sequence of approximated solutions.

While the first 3 sections are devoted to prove additional regularity of generalized solutions, in particular the finiteness of the collision set, in the last section we prove existence of a noncollision (i.e., of a classical) solution for a class of singular systems in the case V behaves near the singularity as $-\frac{1}{|x|^\alpha}$, $1 < \alpha < 2$. Such a problem has been studied by many authors; we recall here [4], [**23**], [**44**]. In all these paper global assumptions are made on the potential V which imply that it is not too far from a radial one; here we only make assumptions on the behaviour of V for x close to 0. Such a result is similar to that of [**18**], valid only for planar system, and to that of [**38**], valid only for even potentials.

As far as the case $\alpha = 1$ is concerned, we cannot prove the existence of a non-collision solution, but we prove, always in Section 2.5, that the solution found in [**10**] if it does collide, it is reflected back by the singularity.

During the writing of this thesis, we have received the paper [**43**], which contains results closely related to ours. In particular, in such a paper, a result very similar to our Theorem 5.1 is obtained using estimates on the Morse index which improves our results of Section 2.4.

2.2. Properties of generalized solutions: the autonomous case

In the previous section we have recalled the definition of generalized solution of (1.1). Here we shall establish some further properties of these solutions by making some assumptions on the potential V near its singularity. In the next section we shall show how to extend these results to the nonautonomous case.

We take $V \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ and we suppose that there exists $0 < \alpha < 2$ such that

(V1) $V(x) = -\frac{1}{|x|^\alpha} + U(x)$;

(V2) $|\nabla U(x)||x|^{\alpha+1} \to 0$ as $|x| \to 0$.

We remark that (V1), (V2) just say that V behaves like $-\frac{1}{|x|^\alpha}$ near the origin. Notice that U may be singular at zero.

We start by proving

Theorem 2.1. Suppose $V \in \mathcal{C}^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfies (V1),(V2) and let q be a generalized solution of (1.1). Then q has only finitely many collisions. Moreover, there exists $\delta > 0$ such that $\frac{d^2}{dt^2}|q(t)|^2 > 0$ for every $t \notin \mathcal{C}(q)$ for which $|q(t)| \leq \delta$.

Proof. Let $]t_1, t_2[$ be a connected component of $S^1 \setminus \mathcal{C}(q)$. Then

$$|q(t_1)| = |q(t_2)| = 0, \qquad |q(t)| \neq 0 \quad \forall t \in]t_1, t_2[.$$

Now $\forall t \in]t_1, t_2[$ we have

$$\frac{1}{2}\frac{d^2}{dt^2}|q(t)|^2 = |\dot{q}(t)|^2 - (q(t), \nabla V(q(t))).$$

From the conservation of the energy and the equation we obtain

$$\frac{1}{2}\frac{d^2}{dt^2}|q(t)|^2 = 2E + \frac{2}{|q(t)|^\alpha} - 2U(q(t)) - \frac{\alpha}{|q(t)|^\alpha} - (q(t), \nabla U(q(t))) =$$
$$= \frac{1}{|q(t)|^\alpha}\left[(2-\alpha) + |q(t)|^\alpha\big(2E - 2U(q(t)) - (q(t), \nabla U(q(t)))\big)\right].$$

By (V2) we se that if $|q(t)|$ is small enough, say $|q(t)| \leq \delta$, then

$$\frac{1}{2}\frac{d^2}{dt^2}|q(t)|^2 \geq \frac{2-\alpha}{2}\frac{1}{|q(t)|^\alpha} > 0,$$

so that $|q(t)|$ cannot have any (local) maximum if $|q(t)| \leq \delta$. This implies that between two collisions $\hat{t}$ and $\tilde{t}$ there is a point $\bar{s}$ such that $|q(\bar{s})| \geq \delta$. If $\mathcal{C}(q)$ is not finite there exists $\bar{t}$ accumulation point of $\mathcal{C}(q)$. Let $t_k \in \mathcal{C}(q)$ be a sequence we which converges to $\bar{t}$. Then for each k there exists s_k such that $t_k < s_k < t_{k+1}$ and $|q(s_k)| \geq \delta$. Then $s_k \to \bar{t}$ and from the continuity of q we deduce

$$0 = |q(\bar{t})| = \lim_{k\to\infty} |q(s_k)| \geq \delta,$$

a contradiction which proves the Theroem. ∎

More information can be given on the collision set under additional assumptions.

Proposition 2.2. Suppose $V \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfies

$$V(x) + \frac{1}{2}(x, \nabla V(x)) \leq -\frac{a}{|x|^\alpha} \qquad \forall x \in \mathbb{R}^N \setminus \{0\} \tag{2.1}$$

for some $a > 0$ and $0 < \alpha < 2$. Then, if $\mathcal{N}(q)$ is the cardinality of the set $\{t \in [0, T[$ such that $q(t) = 0\}$, we have

$$\mathcal{N}(q) \leq \frac{1}{a^{\frac{2}{\alpha}}} \frac{\alpha}{(2+\alpha)^{\frac{2+\alpha}{2}}} \frac{f(q)^{2+\alpha}}{T^{\frac{2-\alpha}{\alpha}}}$$

for every generalized solution q.

Proof. Let $]t_0, t_1[$ be a connected component of $[0, T] \setminus \mathcal{C}(q)$. As in Theorem 2.1 we have that

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}|q(t)|^2 &= 2\left[E - (V(q) + \frac{1}{2}(q, \nabla V(q))\right] \\ &\geq 2\left[E + \frac{a}{|q(t)|^\alpha}\right] \qquad \forall t \in]t_0, t_1[.\end{aligned}$$

Since $q(t_0) = q(t_1) = 0$, $\exists \bar{t} \in]t_0, t_1[$ such that

$$|q(\bar{t})| = \max_{t \in]t_0, t_1[} |q(t)|$$

Since $\bar{t}$ is a local maximum, we have

$$0 \geq \frac{1}{2}\frac{d^2}{dt^2}|q(\bar{t})|^2 \geq 2\left[E + \frac{a}{|q(\bar{t})|^\alpha}\right],$$

from which we deduce that $E < 0$ (we will set $h = -E > 0$) and

$$|q(\bar{t})| \geq \left(\frac{a}{h}\right)^{\frac{1}{\alpha}} \tag{2.2}$$

Suppose now that $\mathcal{C}(q) = \{t_1, ..., t_M\}$ so that $\mathcal{N}(q) = M$. Then we have

$$\int_0^T |\dot{q}|^2 = \sum_{i=1}^{M-1} \int_{t_i}^{t_{i+1}} |\dot{q}|^2 + \int_{t_M}^{T+t_1} |\dot{q}|^2 \tag{2.3}$$

From (2.2) we deduce that

$$\begin{aligned}\int_{t_i}^{t_{i+1}} |\dot{q}|^2 &\geq (t_{i+1} - t_i) 4 \left(\frac{a}{h}\right)^{\frac{2}{\alpha}} \frac{1}{(t_{i+1} - t_i)^2} \\ &= 4\left(\frac{a}{h}\right)^{\frac{2}{\alpha}} \frac{1}{(t_{i+1} - t_i)}\end{aligned}$$

and (2.3) becomes (setting $\Delta_i = t_{i+1} - t_i$)

$$\int_0^T |\dot{q}|^2 \geq \sum_{i=0}^{M} 4\left(\frac{a}{h}\right)^{\frac{2}{\alpha}} \frac{1}{\Delta_i}$$

where $\sum_{i=0}^{M} \Delta_i = T$, so that

$$\int_0^T |\dot{q}|^2 \geq 4\left(\frac{a}{h}\right)^{\frac{2}{\alpha}} \frac{M^2}{T} \tag{2.4}$$

On the other hand, from

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = E$$

we deduce

$$\begin{aligned}f(q) &= \frac{1}{2}\int_0^T |\dot{q}(t)|^2 - TE + \frac{1}{2}\int_0^T |\dot{q}(t)|^2 \\ &= \int_0^T |\dot{q}|^2 - TE = \int_0^T |\dot{q}|^2 + Th\end{aligned}$$

so that

$$\int_0^T |\dot{q}|^2 = f(q) - Th. \tag{2.5}$$

(2.4) and (2.5) give

$$4\frac{M^2}{T}\left(\frac{a}{h}\right)^{\frac{2}{\alpha}} \leq f(q) - Th$$

hence

$$\begin{aligned}
M^2 &\leq \frac{T}{4}\left(\frac{h}{a}\right)^{\frac{2}{\alpha}} f(q) - \frac{h}{4}T^2\left(\frac{h}{a}\right)^{\frac{2}{\alpha}} \\
&= \frac{T}{4a^{\frac{2}{\alpha}}}\left[h^{\frac{2}{\alpha}} f(q) - Th^{\frac{2}{\alpha}+1}\right] \\
&\leq \frac{T}{4a^{\frac{2}{\alpha}}}\frac{f(q)^{\frac{2}{\alpha}+1}}{T^{\frac{2}{\alpha}}}\frac{2^{\frac{2}{\alpha}}\alpha}{(2+\alpha)^{\frac{2}{\alpha}+1}} \\
&= \frac{1}{4^{\frac{\alpha-1}{\alpha}}a^{\frac{2}{\alpha}}}\frac{\alpha}{(2+\alpha)^{\frac{2+\alpha}{\alpha}}}\frac{f(q)^{\frac{2+\alpha}{\alpha}}}{T^{\frac{2-\alpha}{\alpha}}}
\end{aligned}$$

and the proposition follows. ∎

Remark 2.3. Proposition 2.2 can be used to estimate the number of collisions. For example, assuming

$$-V(q) \leq \frac{b}{|q|^\alpha}$$

it is possible to give an estimate of the critical level $f(q)$ corresponding to the generalized solution found in [4]. In such a case one finds

$$f(q) \leq \frac{T}{2}\left(\frac{2\pi}{T}\right)^2 R^2 + \frac{Tb}{R^\alpha}$$

where $\left(\frac{2\pi}{T}\right)^2 R = \frac{\alpha b}{R^{\alpha+1}}$; hence

$$\begin{aligned}
f(q) &= \left(\frac{1}{2} + \frac{1}{\alpha}\right)\frac{4\pi^2}{T}\left(\frac{\alpha b T^2}{4\pi^2}\right)^{\frac{2}{\alpha+2}} \\
&= \frac{\alpha+2}{2^{\frac{2-\alpha}{2+\alpha}}}\pi^{\frac{2\alpha}{\alpha+2}}\alpha^{\frac{\alpha}{\alpha+2}}T^{\frac{2-\alpha}{2+\alpha}}b^{\frac{2}{\alpha+2}}
\end{aligned}$$

which gives

$$\mathcal{N}(q) \leq \left(\frac{b}{a}\right)^{\frac{2}{\alpha}}\frac{\alpha^2(2+\alpha)^{\frac{4-\alpha^2}{2\alpha}}\pi^2}{2^{\frac{6\alpha-4-\alpha^2}{2\alpha}}}.$$

2.3. Properties of generalized solutions: the nonautonomous case

In Section 2 we have seen that generalized solutions of (1.1) enjoy some regularity properties under mild assumptions on the potential.

Here we shall extend some of the previous results in the case when the potential depends on time and the generalized solutions are obtained as limit of classical solutions of perturbed problems.

To this aim, let $\mathcal{G} = \{G \in C^\infty(\mathbb{R}^N \setminus \{0\}; \mathbb{R}) \;/\; G(x) \leq -\frac{a}{|x|^2} + b \text{ for some } a, b > 0\}$; then we give the following definition

Definition 3.1. We say that a generalized solution q of (1.1) is a variational solution of (1.1) if $\exists G \in \mathcal{G}$ such that $\forall \varepsilon > 0$ $\exists q_\varepsilon \in \Lambda$ satisfying

(i) q_ε is a classical solution of

$$-\ddot{q}_\varepsilon = \nabla U(t, q_\varepsilon) + \varepsilon \nabla G(q_\varepsilon); \tag{3.1$_\varepsilon$}$$

(ii) $f(q_\varepsilon) = \frac{1}{2}\int_0^T |\dot{q}_\varepsilon|^2 - \int_0^T V(t, q_\varepsilon) - \varepsilon \int_0^T G(q_\varepsilon) \leq C$, where C does not depend on ε;

(iii) $q_\varepsilon \to q$ in $H^1(S^1; \mathbb{R}^N)$.

Remark 3.2. (i) The generalized solution whose existence is proved in [**10**] is actually variational.

(ii) Since $q_\varepsilon \to q$ in H^1, q_ε is a classical solution in $[0, T]$ and q is a classical solution in $[0, T] \setminus \mathcal{C}(q)$, we easily deduce that $q_\varepsilon \to q$ in $C^2(B)$ $\forall B$ compact subset of $[0, T] \setminus \mathcal{C}(q)$.

We now make some assumptions on V near the singularity and show that these imply additional regularity for a variational solution.

Consider $V \in C^1(\mathbb{R} \times \mathbb{R}^N \setminus \{0\}; \mathbb{R})$ and assume that $\exists 0 < \alpha < 2$ such that

(V3) V is T-periodic in t;

(V4) $V(t, x) = -\frac{1}{|x|^\alpha} + U(t, x)$;

(V5) $|\nabla U(t, x)||x|^{\alpha+1} \to 0$ as $|x| \to 0$, uniformly in t;

(V6) $\exists \alpha' < \alpha$ such that $\left|\frac{\partial U}{\partial t}(t,x)\right| |x|^{\alpha'} \to 0$ as $|x| \to 0$, uniformly in t.

We point out that the only relevant difference between (V1-V2) and (V3-V6) is represented by (V6), which gives a control on the oscillation of $\frac{\partial U}{\partial t}$ near zero. We start by proving that for a variational solutions the energy varies as for classical ones.

Lemma 3.3. Suppose $V \in C^1(\mathbb{R} \times \mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfies (V3-V6). Let q be a variational solution of (1.1) with collision set $\mathcal{C}(q)$. Then $\exists E \in \mathbb{R}$ such that

$$\frac{1}{2}|\dot{q}(t)|^2 + V(t, q(t)) = E + \int_0^t \frac{\partial U}{\partial t}(s, q(s))\, ds \tag{3.2}$$

$\forall t \in [0, T] \setminus \mathcal{C}(q)$.

Proof. First of all, from (V5) one deduces that $\forall R > 0$, $\forall a > 0$ $\exists C_{a,R}$ such that

$$|U(t,x)| \le \frac{a}{|x|^\alpha} + C_{a,R} \qquad \forall |x| \le R$$

Hence, $\forall R > 0$, $\forall a > 0$ $\exists C_{a,R}$ such that

$$V(t,x) = -\frac{1}{|x|^\alpha} + U(t,x) \le -\frac{(1-a)}{|x|^\alpha} + C_{a,R} \qquad \forall |x| \le R \tag{3.3}$$

One also has, from (V6), that $\forall R$, $\forall a > 0$ $\exists C'_{a,R}$ such that

$$\left|\frac{\partial V}{\partial t}(t,x)\right| \le \frac{a}{|x|^{\alpha'}} + C'_{a,R} \qquad \forall |x| \le R \tag{3.4}$$

Since $f(q) < +\infty$, we have that

$$\frac{1}{2}\int_0^T |\dot{q}|^2 - \int_0^T V(t,q) \le C; \tag{3.5}$$

using (3.3) we find, taking $R = \|q\|_\infty$ and $a < 1$,

$$C \ge -\int_0^T V(t,q) \ge (1-a)\int_0^T \frac{1}{|q|^\alpha} - TC_{a,R} \tag{3.6}$$

which shows that $\frac{1}{|q|^\alpha} \in L^1(0,T)$. Using this fact and (3.4), we immediately have, $\forall t \in [0,T]$,

$$\int_0^t \left|\frac{\partial U}{\partial s}(s, q(s))\right| ds \le a\int_0^T \frac{1}{|q|^{\alpha'}} + C'_{a,R}T < +\infty \tag{3.7}$$

so that

$$\frac{\partial U}{\partial s}(s,q(s)) \in L^1(0,t) \qquad \forall t \in [0,T]$$

and (3.2) makes sense $\forall t \notin \mathcal{C}(q)$ (recall that $q \in C^2([0,T] \setminus \mathcal{C}(q))$).

We now consider

$$g_\varepsilon(t) = \frac{\partial V_\varepsilon}{\partial t}(t,q_\varepsilon(t)) = \frac{\partial U}{\partial t}(t,q_\varepsilon(t)).$$

We want to show that the g_ε are uniformly equi-integrable in $[0,t]$ $\forall t \in [0,T]$.

Since $q_\varepsilon \to q$ in H^1, $\exists R > 0$ such that $|q_\varepsilon(t)| \leq R$ $\forall t$ and $\forall \varepsilon$. Take any $A \subset [0,T]$ measurable, of measure $m(A) \leq \delta$. Then

$$\begin{aligned}\int_A g_\varepsilon(t)\,dt &\leq a\int_A \frac{1}{|q_\varepsilon|^{\alpha'}} + C'_{a,R}\delta \\ &\leq a(\delta)^{1-\frac{\alpha'}{\alpha}}\left(\int_0^T \frac{1}{|q_\varepsilon|^\alpha}\right)^{\frac{\alpha'}{\alpha}} + C'_{a,R}\delta\end{aligned}$$

which proves that the g_ε are uniformly equi-integrable since we claim that

$$\int_0^T \frac{1}{|q_\varepsilon|^\alpha} \leq C$$

with C independent from ε.

The claim follows from (ii) of definition 3.1 since

$$C \geq f_\varepsilon(q_\varepsilon) \geq f(q_\varepsilon) \geq -\int_0^T V(t,q_\varepsilon) \geq (1-a)\int_0^T \frac{1}{|q_\varepsilon|^\alpha} - TC_{a,R}.$$

Since $g_\varepsilon(t) = \frac{\partial V_\varepsilon}{\partial t}(t,q_\varepsilon(t)) \to \frac{\partial V}{\partial t}(t,q(t))$ almost everywhere and since $\frac{\partial V}{\partial t}(t,q(t)) \in L^1(0,t)$ the uniform equi-integrability implies

$$\int_0^t \frac{\partial V_\varepsilon}{\partial s}(s,q_\varepsilon(s))\,ds \to \int_0^t \frac{\partial V}{\partial s}(s,q(s))$$

Suppose now $0 \notin \mathcal{C}(q)$, $t \notin \mathcal{C}(q)$. Since q_ε is a classical solution of $(3.1)_\varepsilon$, we have

$$\frac{1}{2}|\dot{q}_\varepsilon(t)|^2 + V_\varepsilon(t,q_\varepsilon(t)) = E_\varepsilon + \int_0^t \frac{\partial V_\varepsilon}{\partial s}(s,q_\varepsilon(s))\,ds \tag{3.8}$$

where $E_\varepsilon = \frac{1}{2}|\dot{q}_\varepsilon(0)|^2 + V_\varepsilon(0,q_\varepsilon(0))$.

Since $0, t \notin \mathcal{C}(q)$, we have from Remark 3.2 (ii) that

$$\frac{1}{2}|\dot{q}_\varepsilon(t)|^2 + V_\varepsilon(t, q_\varepsilon(t)) \to \frac{1}{2}|\dot{q}(t)|^2 + V(t, q(t))$$

$$E_\varepsilon \to E = \frac{1}{2}|\dot{q}(0)|^2 + V(0, q(0))$$

Passing to the limit in (3.8) we then find (3.2). ∎

We can now prove a result similar to that of Theorem 1.1.

Theorem 2.4. Suppose V satisfies (V3-V6). Let q be a variational solution of (1.1). Then q has only finitely many collisions. Moreover $\exists \delta > 0$ such that $\frac{d^2}{dt^2}|q(t)|^2 > 0$ $\forall t$ such that $|q(t)| \leq \delta$.

Proof. The proof works almost exactly like that of Theorem 2.1. We suppose $0 \notin \mathcal{C}(q)$ and we take a connected component $]t_1, t_2[$ of $S^1 \setminus \mathcal{C}(q)$. Then $\forall t \in]t_1, t_2[$ we have

$$\frac{1}{2}\frac{d^2}{dt^2}|q(t)|^2 = |\dot{q}(t)|^2 - (q(t), \nabla V(t, q(t)))$$

and, using Lemma 3.3 we obtain

$$\frac{1}{2}\frac{d^2}{dt^2}|q(t)|^2 = 2\left[E + \int_0^t \frac{\partial U}{\partial s}(s, q(s))\,ds - \left(V(t,q) + \frac{1}{2}(q, \nabla V(t,q))\right)\right]$$
$$= \frac{2}{|q|^\alpha}\left[(1 - \frac{\alpha}{2}) + |q|^\alpha\left(E + \int_0^t \frac{\partial U}{\partial s}(s, q(s))\,ds - (U(t,q) + \frac{1}{2}(q, \nabla U(t,q))\right)\right]$$

Since

$$\left|\int_0^t \frac{\partial U}{\partial s}(s, q(s))\,ds\right| \leq \int_0^T \left|\frac{\partial U}{\partial s}(s, q(s))\right|\,ds < +\infty$$

$$\left|U(t,q) + \frac{1}{2}(q, \nabla U(t,q))\right| \leq \frac{a}{|q|^\alpha} + C_{a,R}$$

we have that, $\forall a > 0$

$$\frac{1}{2}\frac{d^2}{dt^2}|q(t)|^2 \geq \frac{2}{|q|^\alpha}\left[(1 - \frac{\alpha}{2}) - a + o(1)\right]$$

Which implies that $\exists \delta > 0$ such that

$$|q| \leq \delta \Rightarrow \frac{1}{2}\frac{d^2}{dt^2}|q(t)|^2 > 0$$

We then conclude like in the proof of Theorem 2.1 ∎

2.4. Morse index and collisions

In this section we will show that there is a relationship between the Morse index and the number of collisions of a generalized solution.

Actually, one cannot give a meaning to the Morse index $\underline{m}$ of a generalized solution (the functional is not C^2 in such a point), so we will take q to be a variational solution and bound the number of its collisions by the Morse index of the sequence of classical solutions of $(1.1)_\varepsilon$ which converge to q. We recall that the Morse index $\underline{m}(x)$ of a critical point x of a functional $J \in C^2(H;\mathbb{R})$ is the dimension of the maximum subspace of H where $d^2J(x)$ is negative definite.

Theorem 4.1. Suppose $V \in C^2(\mathbb{R}\times\mathbb{R}^N;\mathbb{R})$, $N \geq 3$, satisfies (V3-V6). Assume there exist $\sigma > 0$ and $C > 0$ such that

(V7) $|\nabla^2 U(t,y)||y|^{\alpha+2-\sigma} \leq C$ as $|y| \to 0$ uniformly in t

(V8) $\frac{|\nabla U(t,y)|}{|y|} \to 0$ as $|y| \to \infty$ uniformly in t.

Let q be a variational solution of (1.1). Then

$$\mathcal{N}(q)(N-2) \leq \liminf_{\varepsilon\to 0} \underline{m}(q_\varepsilon)$$

where q_ε are classical solutions of $(1.1)_\varepsilon$ such that $q_\varepsilon \to q$.

Proof. The proof relies on the results obtained in Chapter 1, to which we refer for the details. We recall that there it was shown how to construct functions ϕ_ε such that

$$(f''(q_\varepsilon)\phi_\varepsilon, \phi_\varepsilon) < 0$$

whenever q_ε tends to a collision solution q.

Moreover the functions $\phi_\varepsilon \in H_0^1([0,T];\mathbb{R})$ enjoy the following properties: suppose $q_\varepsilon \to q$ and let $\bar{t}_\varepsilon$ be a point of absolute minimum for $|q_\varepsilon|$; then for every ε small enough we can take ϕ_ε such that

(i) ϕ_ε is piecewise linear

(ii) $|\phi_\varepsilon(t)| = 1$ in a neighbourhood of $\bar{t}_\varepsilon$ independent from ε;

(iii) $\phi_\varepsilon = 0$ if $t \notin [t_\varepsilon - \delta, t_\varepsilon + \delta]$, for some δ independent of ε.

Then, taking any $w \in S^{N-1}$ such that

$$(w, q_\varepsilon(\bar{t}_\varepsilon)) = (w, \dot{q}_\varepsilon(\bar{t}_\varepsilon)) = 0$$

and setting

$$\psi_\varepsilon^w(t) = \phi_\varepsilon(t)w$$

one finds

$$(f''(q_\varepsilon)\psi_\varepsilon^w, \psi_\varepsilon^w) < 0.$$

Therefore q_ε has index at least $N-2$. Since this argument can be repeated for every t'_ε converging to a point of $C(q)$, it follows that $N(q)(N-2) \leq \underline{m}(q_\varepsilon)$, which proves the theorem. ∎

Remark 4.2. It is not difficult, using results contained in [9], [41], [43] and [48] to show that, in the setting of [**10**] (and also in the setting of Section 5), $\underline{m}(q_\varepsilon) \leq N-2$. This implies that the generalized solution found in [**10**] has at most one collision.

2.5. Existence of noncollision solutions in the case $1 < \alpha < 2$

In this section we will prove existence of a noncollision solution in the case V has the form

$$V(x) = -\frac{1}{|x|^\alpha} + U(t,x)$$

with $1 < \alpha < 2$.

Actually, we will show how the generalized solution found in [4] is actually a noncollision one in our situation.

Let us assume $V \in C^1(\mathbb{R} \times \mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfies (V1), (V2) and

(V7) $U(t,x) < 0\ \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$

(V8) $\lim_{|x|\to\infty} U(t,x) = \lim_{|x|\to\infty} |\nabla U(t,x)| = 0$

(V9) $\exists r > 0,\ \phi \in C^1(]0,r]; \mathbb{R})$ such that $U(t,x) = \phi(|x|)\ \forall 0 < |x| \le r,\ \forall t$

(V10) $\lim_{s\to 0} \phi'(s) s^{\alpha+1} = 0$

Our main result is the following

Theorem 5.1. Let $V \in C^1(\mathbb{R} \times \mathbb{R}^N \setminus \{0\}; \mathbb{R}^N)$ satisfy (V1), (V2), and (V7)-(V10), with $\alpha > 1$ and $N \ge 3$. Then there exists at least one noncollision solution of (1.1).

Proof. Since the case $\alpha \ge 2$ is well known (see for example [2]) we will restrict ourselves to the case $1 < \alpha < 2$.

The proof is divided in various steps.

Step 1. Existence of a variational solution ($0 < \alpha < 2$). Such a proof follows the one of [10], to which we refer for details.

Let $\Lambda = \{u \in H^1(S^1; \mathbb{R}^N) \mid u(t) \neq 0\}$, and, for $u \in \Lambda$,

$$f(u) = \frac{1}{2}\int_0^T |\dot{u}|^2\,dt - \int_0^T V(t,u)\,dt$$

$$f_\varepsilon(u) = \frac{1}{2}\int_0^T |\dot{u}|^2\,dt - \int_0^T V(t,u)\,dt + \varepsilon\int_0^T \frac{1}{|u|^2}\,dt$$

We define

$$\Gamma = \{\gamma : S^{N-2} \to \Lambda \mid \gamma \text{ is continuous}\}$$

To every $\gamma \in \Gamma$ we associate a continuous map $\tilde{\gamma} : S^{N-2} \times S^1 \to S^{N-1}$ defined by

$$\tilde{\gamma}(x,t) = \frac{\gamma(x)(t)}{|\gamma(x)(t)|} \tag{5.1}$$

and we set

$$\Gamma^* = \{\gamma \in \Gamma \mid \deg\tilde{\gamma} \neq 0\}$$

we then define a mini-max level by

$$c_\varepsilon = \inf_{\gamma\in\Gamma^*} \max_{x\in S^{N-2}} f_\varepsilon(\gamma(x)) \tag{5.2}$$

As in [**10**] one finds, $\forall\varepsilon > 0$, a critical point $q_\varepsilon \in \Lambda$ of f_ε which converges to a variational solution q of (1.1).

Step 2. Properties of q and q_ε $(1 < \alpha < 2)$.

Now we suppose that q has a collision and we will show that q_ε has a self-intersection for ε small. Tho proof is similar to that of [**18**].

By Theorem 3.4, we know that $\exists\delta > 0$ such that

$$\frac{d^2}{dt^2}|q(t)|^2 > 0 \qquad \forall t \notin C(q) \quad \text{such that} \quad |q(t)| \leq \delta. \tag{5.3}$$

Take any $\delta_0 < \min(\delta, r)$. Then, by (5.3), we have that $\exists t_0, t_1, t_2 \in [0,T]$ such that $t_0 < t_1 < t_2$ and

$$\begin{gathered} |q(t_0)| = |q(t_2)| = \frac{\delta_0}{2}, \\ |q(t_1)| = 0, \\ |q(t)| < \frac{\delta_0}{2} \qquad \forall t \in]t_0, t_2[, \\ \frac{d}{dt}|q(t)| > 0 \qquad \forall t \in]t_1, t_2[, \\ \frac{d}{dt}|q(t)| < 0 \qquad \forall t \in]t_0, t_1[. \end{gathered}$$

This implies that, $\forall\varepsilon$ small enough,

$$\frac{\delta_0}{4} < |q_\varepsilon(t_0)| < \delta_0, \qquad \frac{\delta_0}{4} < |q_\varepsilon(t_2)| < \delta_0,$$
$$|q_\varepsilon(t)| < \delta_0 \qquad \forall t \in [t_0, t_2]$$

and there exists $t_\varepsilon \in [t_0, t_2]$ such that

$$|q_\varepsilon(t_\varepsilon)| = \min_{t\in[t_0,t_2]} |q_\varepsilon(t)|,$$
$$\frac{d}{dt}|q_\varepsilon(t)| > 0 \qquad \forall t \in]t_\varepsilon, t_2],$$
$$\frac{d}{dt}|q_\varepsilon(t)| < 0 \qquad \forall t \in]t_0, t_\varepsilon].$$

Since in $\bar{B}_{\delta_0}(0) \setminus \{0\}$ V_ε is radially symmetric and q_ε is a classical solution of $(1.1)_\varepsilon$, q_ε is planar in $[t_0, t_2]$ and lies in the plane spanned by $q_\varepsilon(t_0)$ and $\dot{q}_\varepsilon(t_0)$. Passing to polar coordinates in this plane, we find that $q_\varepsilon = (\rho\cos\theta, \rho\sin\theta)$ satisfies

$$\frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\rho^2\dot{\theta}^2 - \frac{1}{\rho^\alpha} - \frac{\varepsilon}{\rho^2} + \phi(\rho) = E_\varepsilon \qquad \forall t \in [t_0, t_2] \tag{5.4}$$

$$\rho^2\dot{\theta} = J_\varepsilon \tag{5.5}$$

where E_ε is the energy of q_ε and J_ε its angular momentum (which we can take > 0 without loss of generality: remark that $J_\varepsilon \neq 0$ since q_ε is a non-collision solution).

From (5.5) we have

$$\Delta\theta = \int_{t_\varepsilon}^{t_2} \dot{\theta}(t)\,dt = J_\varepsilon \int_{t_\varepsilon}^{t_2} \frac{dt}{\rho^2}.$$

Since $\frac{d}{dt}\rho(t) > 0$ $\forall t \in]t_\varepsilon, t_2]$, we can invert $\rho(t)$ to obtain $t = t(\rho)$, so that $dt = \frac{dt}{d\rho}d\rho$. From energy conservation, we have

$$\left(\frac{d\rho}{dt}\right)^2 = 2\left(E_\varepsilon - \left(\frac{J_\varepsilon^2}{2} - \varepsilon\right)\frac{1}{\rho^2} + \frac{1}{\rho^\alpha} - \phi(\rho)\right) \qquad \forall t \in [t_0, t_2]$$

so that, setting $\mu_\varepsilon = \rho(t_\varepsilon)$,

$$\Delta\theta_\varepsilon = \int_{t_\varepsilon}^{t_2} \dot{\theta}(t)\,dt \geq J_\varepsilon \int_{\mu_\varepsilon}^{\frac{\delta_0}{4}} \frac{d\rho}{\rho^2\sqrt{2(E_\varepsilon - (\frac{J_\varepsilon^2}{2} - \varepsilon)\frac{1}{\rho^2} + \frac{1}{\rho^\alpha} - \phi(\rho))}}.$$

Let $K_\varepsilon = (J_\varepsilon^2 - 2\varepsilon)$. We find, with the change of variable $y = \frac{\mu_\varepsilon}{\rho}$,

$$\Delta\theta_\varepsilon \geq \frac{J_\varepsilon}{\sqrt{K_\varepsilon}} \int_{\frac{4\mu_\varepsilon}{\delta_0}}^{1} \frac{dy}{\sqrt{\frac{2\mu_\varepsilon^2 E_\varepsilon}{K_\varepsilon} - y^2 + \frac{2\mu_\varepsilon^{2-\alpha} y^\alpha}{K_\varepsilon} - \frac{2\mu_\varepsilon^2}{K_\varepsilon}\phi(\frac{\mu_\varepsilon}{y})}}$$

From $K_\varepsilon = J_\varepsilon^2 - 2\varepsilon$ we find

$$\frac{J_\varepsilon}{\sqrt{K_\varepsilon}} = \frac{J_\varepsilon}{\sqrt{J_\varepsilon^2 - 2\varepsilon}} \geq 1$$

so that

$$\Delta\theta_\varepsilon \geq \int_{\frac{4\mu_\varepsilon}{\delta_0}}^{1} \frac{dy}{\sqrt{g_\varepsilon(y)}}$$

with

$$g_\varepsilon(y) = \frac{2\mu_\varepsilon^2 E_\varepsilon}{K_\varepsilon} - y^2 + \frac{2\mu_\varepsilon^{2-\alpha}}{K_\varepsilon} y^\alpha - \frac{2\mu_\varepsilon^2}{K_\varepsilon}\phi\left(\frac{\mu_\varepsilon}{y}\right)$$

From energy conservation we deduce

$$\frac{2\mu_\varepsilon^2 E_\varepsilon}{K_\varepsilon} = 1 - \frac{2\mu_\varepsilon^{2-\alpha}}{K_\varepsilon} + \frac{2\mu_\varepsilon^2}{K_\varepsilon}\phi(\mu_\varepsilon)$$

so that

$$g_\varepsilon(y) = \left(1 - \frac{2\mu_\varepsilon^{2-\alpha}}{K_\varepsilon}\right)(1 - y^\alpha) + y^\alpha - y^2 + \frac{2\mu_\varepsilon^2}{K_\varepsilon}\left(\phi(\mu_\varepsilon) - \phi\left(\frac{\mu_\varepsilon}{y}\right)\right).$$

Always from energy conservation we deduce

$$\frac{1}{2}\frac{K_\varepsilon}{\mu_\varepsilon^{2-\alpha}} = E_\varepsilon \mu_\varepsilon^\alpha + 1 - \phi(\mu_\varepsilon)\mu_\varepsilon^\alpha$$

and since $E_\varepsilon \to E$, $\mu_\varepsilon \to 0$, we have, using (V10),

$$\frac{1}{2}\frac{K_\varepsilon}{\mu_\varepsilon^{2-\alpha}} \to 1$$

$$\frac{2\delta_\varepsilon^2}{K_\varepsilon}\left(\phi(\mu_\varepsilon) - \phi\left(\frac{\mu_\varepsilon}{y}\right)\right) \to 0$$

so that

$$g_\varepsilon(y) \to y^\alpha - y^2 \qquad \text{almost everywhere}$$

and from Fatou's lemma

$$\Delta\theta_0 = \liminf_{\varepsilon\to 0} \Delta\theta_\varepsilon \geq \liminf_{\varepsilon\to 0} \int_0^1 \chi_{\left[\frac{4\mu_\varepsilon}{\delta_0},1\right]} \frac{dy}{\sqrt{f_\varepsilon(y)}}$$
$$= \int_0^1 \frac{dy}{\sqrt{y^\alpha - y^2}} = \frac{\pi}{2-\alpha}$$

This implies that

$$\bar{\Delta}\theta_\varepsilon \equiv \int_{t_0}^{t_2} \dot{\theta}(t)\,dt = \int_{t_0}^{t_\varepsilon} \dot{\theta}(t)\,dt + \int_{t_\varepsilon}^{t_1} \dot{\theta}(t)\,dt$$
$$\geq \frac{2\pi}{2-\alpha}$$

In particular, if $\alpha > 1$, we have $\bar{\Delta}\theta_0 > 2\pi$, so that $\bar{\Delta}\theta_\varepsilon > 2\pi$ for ε small, which implies that $\exists s_0, s_1 \in [t_0, t_2]$, $s_0 < t_\varepsilon < s$, such that

$$q_\varepsilon(s_0) = q_\varepsilon(s_1)$$

So we have deduced that $\forall \delta_0 < \min(\delta, r)$ there exists an ε_0 such that q_ε has a self intersection $\forall 0 < \varepsilon < \varepsilon_0$.

We now use the fact that q_ε has a self–intersection to find a contradiction.

Step 3. The solutions q_ε cannot have self- intersections for ε small $(0 < \alpha < 2)$.

More precisely, we will show that $\exists\hat{\gamma} \in \Gamma^*$ such that

$$\max_{x\in S^{N-2}} f_\varepsilon(\hat{\gamma}(x)) = c_\varepsilon$$
$$\nabla f_\varepsilon(\hat{\gamma}(x)) \neq 0 \qquad \forall x \in S^{N-2}$$

contradicting the fact that c_ε is a mini-max value.

Indeed, let q_ε be our solution at level c_ε with self- intersection.

Let $q_\varepsilon(s_0) = q_\varepsilon(s_1)$ be the point of self–intersection of smallest norm, say $\bar{r} = |q_\varepsilon(s_0)|$.

We know that, for ε small, we can assume $\bar{r}$ as small as we please. Suppose $\bar{r} < \frac{\delta}{4}$ and let

$$u_\varepsilon(t) = \begin{cases} q_\varepsilon(t) & \text{if } t \notin [s_0, s_1] \\ q_\varepsilon(s_0 + s_1 - t) & \text{if } t \in [s_0, s_1] \end{cases}$$

which is nothing else but our solution q_ε with the loop $\hat{q}(t) = q_\varepsilon(t)|_{[s_0,s_1]}$ travelled backwards.

Remark that, from

$$\frac{d}{dt}|q_\varepsilon(t)| \neq 0 \qquad \text{for} \quad t = s_0, s_1$$

and

$$\theta'_\varepsilon = \frac{J_\varepsilon}{\bar{r}^2} \neq 0$$

we have that

$$q'_\varepsilon(s_0) \neq q'_\varepsilon(s_1)$$

so that u_ε is not of class C^2, which implies that u_ε cannot be a critical point of f.

We now introduce a new system of coordinates in $\mathbb{R}^N$ as follows. Take the first versor e_1 equal to $\frac{q_\varepsilon(s_0)}{|q_\varepsilon(s_0)|}$, the second e_2, such that $\{e_1, e_2\}$ span the plane which contains $\hat{q}$ and let $(e_3, ..., e_N)$ be such that $(e_1, e_2, ..., e_N)$ is a basis in $\mathbb{R}^N$.

We can write, $\forall t \in [s_0, s_1]$,

$$\begin{aligned}\hat{q}(t) &= a(t)e_1 + b(t)e_2 \\ &= (a(t) - \bar{r})e_1 + b(t)e_2 + \bar{r}e_1\end{aligned}$$

for some $a, b \in C^2([s_0, s_1]; \mathbb{R})$, $a(s_0) = a(s_1) = \bar{r}$, $b(s_0) = b(s_1) = 0$.

Let

$$S^{N-2} = S^{N-1} \bigcap \{e_2^\perp\}.$$

and, $\forall y \in S^{N-2}$ define

$$\psi(y,t) = \begin{cases} u_\varepsilon(t) & \text{if } t \notin [s_0, s_1] \\ (a(t) - \bar{r})y + b(t)e_2 + \bar{r}e_1 & \text{if } t \in [s_0, s_1] \end{cases}$$

It is easy to check that $|\psi(y,t)| \neq 0$ $\forall (y,t) \in S^{N-2} \times [0,T]$, so we can define $\gamma \in \Gamma$ as

$$\gamma(y)(\cdot) = \psi(y, \cdot).$$

From the fact that that V is rotationally symmetric in the ball of radius δ one also easily deduces that

$$c_\varepsilon = f_\varepsilon(q_\varepsilon) = f_\varepsilon(\gamma(e_1)) \leq \max_{x \in S^{N-2}} f_\varepsilon(\gamma(x)) \leq f_\varepsilon(q_\varepsilon) = c_\varepsilon.$$

To prove that $\gamma \in \Gamma^*$ we have to prove that

$$\tilde{\gamma}(y,t) = \frac{\psi(y,t)}{|\psi(y,t)|}$$

has a nonzero degree.

First of all, since $u_\varepsilon(t)$ for $t \in S^1 \setminus [s_0, s_1]$ is a closed loop in in Ω, which is simply connected, one can easily see that $\tilde{\gamma}$ is homotopic to

$$\bar{\gamma}(y,t) = \frac{\bar{\psi}(y,t)}{|\bar{\psi}(y,t)|}$$

where

$$\bar{\psi}(y,t) = \begin{cases} u_\varepsilon(s_0) & \forall t \in [0,T] \setminus [s_0, s_1] \quad \forall y \in S^{N-2} \\ \psi(y,t) & \forall t \in [s_0, s_1] \end{cases}$$

This easily implies that $\gamma \in \Gamma^*$ since all the point z in S^{N-1} such that $ze_1 \approx -1$ have a unique counterimage under $\bar{\psi}$, since from

$$\frac{(a(t) - \bar{r})y + b(t)e_2 + \bar{r}e_1}{|(a(t) - \bar{r})y + b(t)e_2 + \bar{r}e_1|} = z$$

we deduce (taking the scalar product with e_1)

$$(a(t) - \bar{r})ye_1 + \bar{r} \approx -|(a(t) - \bar{r})y + b(t)e_2 + \bar{r}e_1|$$

which implies

$$b(t) \approx 0.$$

We know that $b(t) = 0$ for $t = s_0, s_1, s_2$ (here s_2 denotes the only intersection of $\hat{q}$ with the e_1 axis apart from s_0, s_1. But if $t \approx s_0, s_1$ we immediately reach a contradiction, while for $t \approx s_2$ one gets that $z \approx -e_1$.

This proves step 3 and the theorem. ∎

In the case $\alpha = 1$ Theorem 5.1 cannot hold. However the method used to prove Theorem 5.1 shows that the solution found as a limit of solutions of approximated problems has still some additional properties. We can prove the following

Theorem 5.2. Suppose that the assumptions of Theorem 5.1 hold with $\alpha = 1$. Then there exists a generalized solution q of (1.1) such that for every $\bar{t} \in \mathcal{C}(q)$ one has

$$q(\bar{t} + t) = q(\bar{t} - t).$$

Moreover, such a solution has at most one collision.

Proof. In the case $\alpha = 1$ step 2 of the proof of Theorem 5.1 does not hold any more and we cannot say that q_ε has a self-intersection for ε small. However, notice that, if $q_\varepsilon(t)$ has no self-intersection for ε small, then

$$\bar{\Delta}\theta_\varepsilon < 2\pi$$

Since $\liminf_{\varepsilon \to 0} \bar{\Delta}\theta_\varepsilon \geq 2\pi$, we have

$$\lim_{\varepsilon \to 0} \bar{\Delta}\theta_\varepsilon = 2\pi$$

Taking, $\forall r$ small, t' and t'' such that

$$|q(t')| = |q(t'')| = r,$$

we have

$$\begin{aligned} q(t') &= \lim_{\varepsilon \to 0} q_\varepsilon(t') \\ &= \lim_{\varepsilon \to 0} (\rho_\varepsilon(t') \sin\theta_\varepsilon(t'), \rho_\varepsilon(t') \cos\theta_\varepsilon(t')) \\ &= \lim_{\varepsilon \to 0} (\rho_\varepsilon(t'') \sin[\theta_\varepsilon(t'') - 2\pi], \rho_\varepsilon(t'') \cos[\theta_\varepsilon(t'') - 2\pi]) \\ &= \lim_{\varepsilon \to 0} q_\varepsilon(t'') = q(t'') \end{aligned}$$

so that $q(t)$ has a self-intersection.

This, together with the fact that q is a classical solution outside the collision set implies that,

$$q(\bar{t} + t) = q(\bar{t} - t)$$

for all $\bar{t}$ in $\mathcal{C}(q)$.

To prove that q has at most one collision is enough to use Remark 4.2. ∎

CHAPTER 3

MULTIPLE BRAKE ORBITS AND SINGULAR POTENTIALS

3.1. Basic definitions

In this Chapter we study the existence of periodic solutions for the dynamical system

$$\ddot{q}(t) + V'(q(t)) = 0 \tag{1.1}$$

such that

$$\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h, \tag{1.2}$$

where h is a given constant and $V \in \mathcal{C}^2(E;\mathbb{R}^N)$, E open in $\mathbb{R}^N$.

More precisely, we shall look for brake orbits, i.e. solutions of

$$\begin{cases} \ddot{q}(t) + V'(q(t)) = 0 & \forall t \in [0,T] \\ \dot{q}(0) = \dot{q}(T) = 0 \\ \frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h & \forall t \in [0,T] \end{cases} \tag{1.3}$$

from which one can easily construct a periodic solution of period $2T$ by reflecting q around $t = 0, T$.

Our main purpose is to relate the number of solutions of (1.3) to some aspects of the topology of the sublevel set $\{x \in E \ / \ V(x) < h\}$.

If $\{x \in E \ / \ V(x) < h\}$ is homeomorphic to a ball then the existence of one brake orbit is a classical result of Seifert **[37]**. Generalizations of that result, as well as alternative proofs were given in **[12]**, **[26]**, **[30]**, **[31]**.

Here we are concerned with the case in which the topology of $\Omega = \{x \in E \ / \ V(x) < h\}$ is more complicated and we shall use this fact to prove the existence of multiple solutions. In particular we shall assume that Ω has a certain number k of "holes", in a sense that we shall make precise, and we shall show that (1.3) has at least k solutions.

we shall make precise, and we shall show that (1.3) has at least k solutions. This result has already been obtained by Bolotin and Kozlov in [**17**] in the case when the potential is assumed to be everywhere regular. Actually in Theorem 2.1 we prove the same result as in [**17**] with a slightly different argument.

The results obtained in Section 3.2 are then used in Sections 3.3 and 3.4, where we investigate the case in which V is singular in the sense that there exists x_0 such that $V(x) \to -\infty$ as $x \to x_0$. In such a case $x_0 \in \partial\Omega$ and the topology of Ω is richer; this fact has been indeed used in [**1**], [**2**], [**5**], papers which deal with existence of fixed period and fixed energy solutions of (1.1).

To deal with singular potentials, a notion of generalized solution has been introduced in [**10**] (see Definition 1.1 of Chapter 2). In the setting of brake orbits such a notion becomes

Definition 1.1. We say that a function $q \in H^1([0,T];\mathbb{R}^N)$ is a generalized brake orbit with collision set $\mathcal{C}(q) = \{t \in [0,T] \ / \ q(t) = 0\}$ if

i) $meas\mathcal{C}(q) = 0$

ii) $\int_0^T [h - V(u)]dt < +\infty$;

iii) $0, T \notin \mathcal{C}(q)$;

iv) (1.3) is verified $\forall t \notin \mathcal{C}(q)$;

We remark that there are situations where for every brake orbit q one has $\mathcal{C}(q) \neq \emptyset$. Indeed

Remark 1.2. If $V(x) = -\frac{1}{|x|^\alpha}$ for some $\alpha > 0$, so that $\overline{\Omega}$ is the ball of radius $(-\frac{1}{h})^{\frac{1}{\alpha}}$ centered at zero, then the only "brake orbits" are the rays u_θ ($\forall \theta \in S^{N-1}$) of the ball travelled back and forth with appropriate speed.

The solutions of (1.3) will be found as critical points of a suitable functional. Precisely, let

$$\Lambda = \{u \in H^1([0,1];\mathbb{R}^N) \ / \ u(t) \in E \ \forall t \in [0,1]\}$$

and define $f : \Lambda \to \mathbb{R}$ as

$$f(u) = \int_0^1 \frac{1}{2}|\dot{u}|^2 dt \cdot \int_0^1 [h - V(u)]dt. \tag{1.4}$$

Suppose now that $u \in \Lambda$ is a critical point of f such that $f(u) > 0$. Then (as is shown for example in [**12**]) setting

$$T^2 = \frac{\int_0^1 \frac{1}{2}|\dot{u}|^2 dt}{\int_0^1 [h - V(u)]dt} \tag{1.5}$$

one easily verifies that the function $q : [0, T] \to E$ defined by $q(t) = u(\frac{t}{T})$ verifies

$$\begin{cases} \ddot{q}(t) + V'(q(t)) = 0 & \forall t \in [0, T] \\ \frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h & \forall t \in [0, T] \\ \dot{q}(0) = \dot{q}(T) = 0 \end{cases} \tag{1.6}$$

namely, it is a (noncollision) brake orbit of energy h.

Remark 1.3. In this way to every critical point of f at a positive level there corresponds a brake orbit of energy h. From now on we shall implicitly make use of this fact by confining our attention to critical points of f.

In Section 3.2 we deal with regular potentials and we describe the main argument which will be used throughout this chapter. In Sections 3.3 and 3.4 we consider the problem of singular potentials, under different kinds of local hypotheses on V near $x = 0$. Existence results are given in each section.

3.2. Multiple brake orbits

In this section we are going to state sufficient conditions for the existence of multiple brake orbits in the case when the potential is assumed to be everywhere regular. We will assume that a function $V \in C^2(\mathbb{R}^N;\mathbb{R})$ and a real number h are given and we denote by Ω the set $\{x \in \mathbb{R}^N \; / \; V(x) < h\}$, so that $\partial\Omega = \{x \in \mathbb{R}^N \; / \; V(x) = h\}$. The main condition we shall need states that the topology of Ω is in some way "complicated". Precisely, we make the following assumptions. We suppose that Ω_i, $i = 0, ..., k$ are open bounded sets in $\mathbb{R}^N$ such that

(V1) $\Omega = \{x \in \mathbb{R}^N \; / \; V(x) < h\} = \Omega_0 \setminus \left(\cup_{i=1}^k \overline{\Omega}_i\right)$, with

i) $\overline{\Omega}_i \subset \Omega_0 \quad \forall i = 1, ..., k$,

ii) $\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset \quad \forall 1 \leq i \neq j \leq k$.

(V2) $V'(x) \neq 0 \quad \forall x \in \partial\overline{\Omega}$.

The main result of this section is the following

Theorem 2.1. Suppose $V \in C^2(\mathbb{R}^N;\mathbb{R})$ and h satisfy (V1) and (V2). Then there exist at least k distinct brake orbits of energy h.

The proof is divided in several steps and it is based on the minimization of the functional f defined in (1.4) over a set of functions connecting different components of the boundary of Ω. Since $x \in \partial\Omega$ implies $h - V(x) = 0$, the functional f cannot be coercive on

$$\Lambda = \{u \in H^1 \; / \; u(0) \in \partial\Omega_0, \; u(1) \in \partial\Omega \setminus \partial\Omega_0, \; u(t) \in \Omega \;\; \forall t \in]0,1[\,\}, \tag{2.1}$$

so that we shall start by studying an approximate problem.

Step 1. The approximate problem.

Let $\delta > 0$ and

$$\Omega_{0,\delta} = \{x \in \Omega \; / \; dist(x, \partial\Omega_0) > \delta\}, \tag{2.2}$$

$$(2.3) \qquad \Omega_{i,\delta} = \{x \in \Omega \ / \ dist(x, \partial\Omega_i) < \delta\} \qquad \forall i = 1, ..., k.$$

Next set $\Omega_\delta = \Omega_{0,\delta} \setminus \cup_{i=1}^{k} \Omega_{i,\delta}$; finally let $\mathcal{B}_1 = \cup_{i=1}^{k} \Omega_i$ and $\mathcal{B}_{1,\delta} = \cup_{i=1}^{k} \Omega_{i,\delta}$. We also introduce

$$(2.4) \qquad \Lambda_\delta = \{u \in H^1 \ / \ u(0) \in \partial\Omega_{0,\delta}, \ u(1) \in \partial\mathcal{B}_{1,\delta}, \ u(t) \in \mathcal{B}_{1,\delta} \ \forall t \in]0,1[\ \}$$

and we remark that the functional f is defined both in Λ and in Λ_δ.

Proposition 2.1. Let (V1)-(V2) hold. Then there exists $u_\delta \in \Lambda_\delta$ such that

$$(2.5) \qquad f(u_\delta) = \inf_{v \in \Lambda_\delta} f(v) > 0,$$

$$(2.6) \qquad \ddot{u}_\delta \int_0^1 [h - V(u_\delta)]dt + V'(u_\delta) \int_0^1 \frac{1}{2} |\dot{u}_\delta|^2 dt = 0 \qquad \forall t \in [0,1]$$

and

$$(2.7) \qquad \frac{1}{2} |\dot{u}_\delta|^2 \int_0^1 [h - V(u_\delta)]dt + (V(u_\delta) - h) \int_0^1 \frac{1}{2} |\dot{u}_\delta|^2 dt = 0 \qquad \forall t \in [0,1].$$

Proof. We start by showing that

$$\inf_{v \in \Lambda_\delta} f(v) > 0.$$

Indeed, notice that there exists $\sigma_\delta > 0$ such that for every $x \in \Omega_\delta$, $h - V(x) \geq \sigma_\delta$; next, take any $u \in \Lambda_\delta$ and remark that

$$d = dist(\partial\Omega_{0,\delta}, \partial\mathcal{B}_{1,\delta}) \leq |u(1) - u(0)| \leq \left(\int_0^1 |\dot{u}|^2 dt\right)^{\frac{1}{2}}.$$

Therefore $\forall u \in \Lambda_\delta$,

$$f(u) \geq \frac{1}{2} \sigma_\delta \int_0^1 |\dot{u}|^2 dt \geq \frac{1}{2} \sigma_\delta d^2 > 0.$$

Consider now a minimizing sequence u_k in Λ_δ; for k large enough one has

$$2 \inf_{\Lambda_\delta} f \geq f(u_k) \geq \frac{1}{2} \sigma_\delta \int_0^1 |\dot{u}_k|^2 dt,$$

and hence u_k is bounded in H^1. We can therefore assume that, up to a subsequence,

$$u_k \to u \quad \text{weakly in } H^1 \tag{2.8}$$

and

$$u_k \to u \quad \text{uniformly;} \tag{2.9}$$

notice that by (2.9), $u(0) \in \partial\Omega_{0,\delta}$, $u(1) \in \partial\mathcal{B}_{1,\delta}$ and $u(t) \in \overline{\Omega}_\delta$ $\forall t \in [0,1]$. Moreover since $\dot{u}_k \to \dot{u}$ weakly in L^2 and $u_k \to u$ uniformly, then

$$\begin{aligned} f(u) &\le \frac{1}{2}\big(\liminf_k ||\dot{u}_k||_2\big)^2 \lim_k \int_0^1 [h - V(u_k)]dt \le \\ &\le \frac{1}{2}\liminf_k\Big(\int_0^1 |\dot{u}_k|^2 \int_0^1 [h - V(u_k)]dt\Big) = \inf_{\Lambda_\delta} f. \end{aligned} \tag{2.10}$$

Since Λ_δ is dense in $\overline{\Lambda}_\delta$, equality holds in (2.10).

Next we prove that we can assume $u \in \Lambda_\delta$ without loss of generality. Indeed, since u is continuous, $[0,1] \setminus u^{-1}(\partial\Omega_\delta)$ is open. Necessarily it must contain an interval $]\bar{s},\bar{t}[$ such that

$$u(\bar{s}) \in \partial\Omega_{0,\delta}, \quad u(\bar{t}) \in \partial\mathcal{B}_{1,\delta}, \quad u(t) \in \Omega_\delta \quad \forall t \in]\bar{s},\bar{t}[.$$

Setting $w(t) = u(\bar{s} + t(\bar{t} - \bar{s}))$, it is easy to see that

$$f(w) \le f(u) = \inf_{\Lambda_\delta} f,$$

which proves (2.5), since $w \in \Lambda_\delta$.

From now on we denote, for each δ, by u_δ the minimizer $w \in \Lambda_\delta$ found above. Now, the fact that u_δ is a minimizer implies in a standard way that (2.6) and (2.7) hold, and the proof is complete. ∎

Step 2. The limiting procedure.

We now construct a suitable test function that we are going to use to pass to the limit for $\delta \to 0$.

To this aim we choose an $\eta > 0$ so small that $dist(x, \partial\overline{\Omega}) \leq 2\eta$ implies $|V'(x)| \geq \mu_\eta > 0$ (this is possible by (V2)). Next we take a cut-off function $\psi : \mathbb{R}^N \to [0,1]$ of class C^1 defined by

$$\psi(x) = \begin{cases} 0 & \text{if } dist(x, \partial\overline{\Omega}) \leq \eta \\ 1 & \text{if } dist(x, \partial\overline{\Omega}) \geq 2\eta \end{cases}$$

and we define $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ as

$$\varphi(x) = \psi(x)x + (1 - \psi(x))\frac{V'(x)}{|V'(x)|}. \tag{2.11}$$

Lemma 2.2. Assume (V1) and (V2) hold. Then, $\forall L > 0, \exists\beta > 0, \exists C > 0$ such that $\forall x \in \overline{\Omega}$,

$$\frac{1}{2L}V'(x)\varphi(x) \geq \beta(V(x) - h) + C.$$

Proof. Assume by contradiction that there exist $L > 0$ and three sequences $\beta_n \to +\infty$, $C_n \to 0$, $x_n \in \overline{\Omega}$ such that

$$\frac{1}{2L}V'(x_n)\varphi(x_n) < \beta_n(V(x_n) - h) + C_n. \tag{2.12}$$

Since Ω is bounded, $x_n \to x \in \overline{\Omega}$, up to a subsequence.

If $x_n \to x \in \Omega$, then the left-hand-side of (2.12) is bounded, while the right- hand-side tends to $-\infty$, and we reach a contradiction.

On the other hand, if $x \in \partial\Omega$; then, for n large, $\varphi(x_n) = \frac{V'(x_n)}{|V'(x_n)|}$ and from (2.12)

$$\frac{1}{2L}|V'(x)| = \lim_{n\to+\infty}\frac{1}{2L}|V'(x_n)| \leq \lim_{n\to+\infty} C_n = 0,$$

contradicting (V2). ∎

We shall use the function φ defined above to show that u_δ is bounded in H^1 independently of δ.

Let us define a function $v_\delta \in H^1$ by

$$v_\delta(t) = \varphi(u_\delta(t)),$$

where φ is defined in (2.11); differentiating v_δ yields $\dot{v}_\delta(t) = J\varphi(u_\delta(t))\dot{u}_\delta(t)$, where $J\varphi$ is the Jacobian of φ. Since φ is of class C^1, there exists a constant $L > 0$ such that $||J\varphi|| \leq L$.

In what follows we denote by $C_j, j = 1, 2, ...$ positive constants independent of δ. We now multiply (2.6) by v_δ and integrate: we obtain

$$
\begin{aligned}
&(v_\delta(1)\dot{u}_\delta(1) - v_\delta(0)\dot{u}_\delta(0))\int_0^1 [h - V(u_\delta)]dt - \\
&- \int_0^1 (J\varphi(u_\delta)\dot{u}_\delta, \dot{u}_\delta)dt \cdot \int_0^1 [h - V(u_\delta)]dt + \int_0^1 V'(u_\delta)\varphi(u_\delta)dt \cdot \int_0^1 \frac{1}{2}|\dot{u}_\delta|^2 dt = 0.
\end{aligned}
\tag{2.13}
$$

From (2.7) we have

$$
|\dot{u}_\delta(t)|^2 = \big(h - V(u_\delta(t))\big)\frac{\int_0^1 |\dot{u}_\delta|^2 dt}{\int_0^1 [h - V(u_\delta)]dt} \qquad \forall t \in [0,1]
\tag{2.14}
$$

so that

$$
|v_\delta(t)||\dot{u}_\delta(t)| = |v_\delta(t)|\big(h - V(u_\delta(t))\big)^{\frac{1}{2}} \frac{\left(\int_0^1 |\dot{u}_\delta|^2 dt\right)^{\frac{1}{2}}}{\left(\int_0^1 [h - V(u_\delta)]dt\right)^{\frac{1}{2}}}
$$

and

$$
\begin{aligned}
\Big|v_\delta(t)\dot{u}_\delta(t)\int_0^1 [h - V(u_\delta)]dt\Big| &\le |v_\delta(t)||\dot{u}_\delta(t)|\int_0^1 [h - V(u_\delta)]dt \\
&\le |v_\delta(t)|(h - V(u_\delta(t)))^{\frac{1}{2}} \frac{\left(\int_0^1 |\dot{u}_\delta|^2 dt\right)^{\frac{1}{2}}}{\left(\int_0^1 [h - V(u_\delta)]dt\right)^{\frac{1}{2}}} \int_0^1 [h - V(u_\delta)]dt \\
&\le \sqrt{2}|v_\delta(t))|(h - V(u_\delta(1)))^{\frac{1}{2}} f(u_\delta)^{\frac{1}{2}}.
\end{aligned}
$$

Now if $t = 0$ or $t = 1$ then $h - V(u_\delta(t)) \to 0$ as $\delta \to 0$. As a consequence we can deduce from (2.13) that

$$
\begin{aligned}
-C_2 &\le \int_0^1 (J\varphi(u_\delta)\dot{u}_\delta, \dot{u}_\delta)dt \cdot \int_0^1 [h - V(u_\delta)]dt - \int_0^1 V'(u_\delta)\varphi(u_\delta)dt \cdot \int_0^1 \frac{1}{2}|\dot{u}_\delta|^2 dt \\
&\le L\int_0^1 |\dot{u}_\delta|^2 dt \int_0^1 [h - V(u_\delta)]dt - \int_0^1 \frac{1}{2}|\dot{u}_\delta|^2 dt \cdot \int_0^1 V'(u_\delta)\varphi(u_\delta)dt \\
&= L\int_0^1 |\dot{u}_\delta|^2 dt \int_0^1 [h - V(u_\delta) - \frac{1}{2L}V'(u_\delta)\varphi(u_\delta)]dt.
\end{aligned}
\tag{2.15}
$$

By Lemma (2.2) we have

$$
\frac{1}{2L}V'(u_\delta)\varphi(u_\delta) \ge \beta\big(V(u_\delta) - h\big) + C,
$$

or, setting $\gamma = \beta + 1$,

$$
h - V(u_\delta) - \frac{1}{2L}V'(u_\delta)\varphi(u_\delta) \le \gamma\big(h - V(u_\delta)\big) - C
$$

which, carried into (2.15) yields

$$-C_2 \leq 2L\gamma f(u_\delta) - LC\int_0^1 |\dot{u}_\delta|^2 dt.$$

Therefore $\dot{u}_\delta$ must be bounded in L^2 independently of δ. At this point one easily proves like in Step 1 that u_δ converges, up to a subsequence, to a function u_1 that can be assumed to belong to Λ and such that $f(u_1) = \inf_\Lambda f > 0$ and

$$\ddot{u}_1 \int_0^1 [h - V(u_1)]dt + V'(u_1)\int_0^1 \frac{1}{2}|\dot{u}_1|^2 dt = 0 \qquad \forall t \in [0,1] \tag{2.16}$$

$$\frac{1}{2}|\dot{u}_1|^2 \int_0^1 [h - V(u_1)]dt + (V(u_1) - h)\int_0^1 \frac{1}{2}|\dot{u}_1|^2 dt = 0 \qquad \forall t \in [0,1]. \tag{2.17}$$

From u_1 it is straigtforward to construct a brake orbit q_1, as we pointed out in the Section 3.1. Remark that we can assume $u_1(0) \in \partial\Omega_0$ and $u_1(1) \in \partial\Omega_1$.

This proves the existence of a first brake orbit.

Step 3. Multiple solutions.

In the above proofs we have never used (nor even assumed) the fact that $\partial\Omega_0$ is connected. This allows us to repeat the above procedure simply by replacing Ω_0 by $\Omega_0 \setminus \Omega_1$ and $\mathcal{B}_1$ by $\mathcal{B}_2 = \cup_{i=2}^k \Omega_i$. Exactly the same argument proves then the existence of a second solution u_2 such that $u_2(0) \in (\partial\Omega_0 \cup \partial\Omega_1)$, $u_2(1) \in \partial\mathcal{B}_2$ and $u_2(t) \in \Omega$ $\forall t \in]0,1[$, which implies in particular that u_2 is different from u_1. Iterating this procedure the theorem follows.

3.3. Singular potentials: Strong Forces

In Section 3.2 we have seen that richness in the topology of the sublevel set $\{x \ / \ V(x) < h\}$ provides multiple brake orbits. In that case the potential was assumed to be everywhere regular, and in particular on the boundaries $\partial\Omega_j$. We now want to investigate what happens when the assumption of regularity is dropped, in the sense that there is a point x_0 where $\lim_{x\to x_0} V(x) = -\infty$. The potential presents therefore a singularity of attractive type in x_0.

In this section we assume that V satisfies the Strong Force condition of Gordon ([28]):

(SF) $\exists U \in C^1(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ with $\lim_{x\to 0} U(x) = \infty$ such that $-V(x) \geq |\nabla U(x)|^2$ for every $x \neq 0$ in a neighborhood of zero.

We recall that if V satisfies (SF) and $u \in H^1$ is such that $u(t_0) = 0$ for some t_0, then $f(u) = +\infty$.

Our goal is to show what kind of influence does the singularity have from the point of view of existence of brake orbits.

Remark 3.1. As we have seen in Remark 1.2 if $V(x) = -\frac{1}{|x|^\alpha}$ for some $\alpha \geq 0$, so that $\overline{\Omega}$ is the ball of radius $\left(-\frac{1}{h}\right)^{\frac{1}{\alpha}}$ centered at zero, then the only "brake orbits" are the rays u_θ ($\forall\theta \in S^{N-1}$).

If $\alpha \geq 2$, however, then (SF) is satisfied and $f(u_\theta) = +\infty \ \forall\theta$, due to the strong force condition and therefore such solutions do not fit into the variational framework. The presence of the singularity alone is thus not sufficient to ensure the existence of one (even generalized) solution. Compare this situation with the results of Section 3.4.

This is a very different situation from that of finding periodic solutions (not necessarily coming from brake orbits) of (1.1); indeed for such a problem it is the singularity that gives rise to multiple noncollision solutions.

On the other hand one can easily show that Theorem 2.1 holds true also in this case.

Theorem 3.2. Suppose $V \in C^2(\mathbb{R}^N \setminus \{0\}; \mathbb{R})$ satisfies (V1),(V2) and (SF) with $k \geq 1$. Then there exist at least k distinct noncollision brake orbit of energy h.

Proof. This proof follows step by step that of Theorem 2.1. The presence of the singularity does not interfere since, as is well-known, if V satisfies (SF) and $f(u) \leq C$ for some $C < +\infty$, then

$$\min_t |u(t)| \geq \lambda_C > 0.$$

This means that all the results used in proving Theorem 2.1 remain valid. ∎

3.4. Singular potentials: Weak Forces

In this section we shall deal with the same problem treated in Theorem 3.2 when the Strong Force condition is violated. Our goal will be twofold.

First we shall show that the presence of the singularity gives rise to a solution, in contrast with the strong force case. Of course we are concerned here with the existence of a generalized brake orbit.

Secondly we shall prove, under additional hypotheses in the neighborhood of the singularity, that one can still find multiple brake orbits which turn out to be noncollision solutions. This problem is particularly interesting since in the presence of weak forces not much is known concerning the existence of noncollision solutions (for the periodic problem see i.e. [**21**], [**23**], [**39**], [**44**]). The main point will be to show that the property of *minimizing* the functional f (on a suitable set) cannot be enjoyed by orbits which pass through the singularity of the potential, provided that in some arbitrarily small neighborhood of zero V differs from $-\frac{1}{|x|^\alpha}$ by a "small" radially symmetric function. This approach was used for the fixed period problem in [**39**] in the case of even potentials, and in [**21**] in the case $\alpha \in]1,2[$.

We begin by showing that in the case of weak forces a (generalized) brake orbit always exists.

Taken $V \in \mathcal{C}^2(\mathbb{R}^N \setminus \{0\};\mathbb{R})$ and $h \in \mathbb{R}$ we shall make the following assumptions:

(W1) the set $\Omega = \{x \in \mathbb{R}^N \ / \ V(x) < h\}$ is bounded and not empty;

(W2) $V'(x) \neq 0 \quad \forall x \in \partial\overline{\Omega}$;

(W3) $\exists \alpha \in]0,2[, \exists A, B > 0$ such that $-V(x) \leq A + \frac{B}{|x|^\alpha}$, $\forall x \neq 0$;

(W4) $\exists \lambda > 0$ such that $V'(x)x \geq \lambda V(x)$, $\forall x$ in a neighborhood of zero.

(W5) $\lim_{x \to 0} V(x) = -\infty$.

Remark that $\partial\Omega = \partial\overline{\Omega} \cup \{0\}$.

Then we can prove

Theorem 4.1. Assume V and h satisfy (W1)-(W5). Then there exists at least one generalized brake orbit of energy h.

We follow closely the proof of Theorem 2.1. In particular we are going to minimize a suitable functional over a set of orbits whose endpoints are constrained on two different components of $\partial\Omega$, namely $\partial\overline{\Omega}$ and $\{0\}$.

We begin by taking a small $\delta > 0$ and we set

$$\Omega_\delta = \{x \in \Omega \ / \ dist(x, \partial\overline{\Omega}) > \delta\},$$

and

$$\Lambda_\delta = \{u \in H^1([0,1];\mathbb{R}^N) \ / \ u(0) = 0,\ u(1) \in \partial\overline{\Omega}_\delta,\ u(t) \in \Omega_\delta\ \forall t \in]0,1[\}. \tag{4.1}$$

We define a functional $f : \Lambda_\delta \to \mathbb{R}$ by

$$f(u) = \begin{cases} \frac{1}{2}\int_0^1 |\dot{u}(t)|^2 dt \cdot \int_0^1 [h - V(u(t))]dt & \text{if } \int_0^1 [h - V(u(t))]dt < +\infty \\ +\infty & \text{otherwise} \end{cases} \tag{4.2}$$

Proposition 4.2. Let (W1)-(W5) hold. Then there exists $u_\delta \in \Lambda_\delta$ such that

$$0 < \inf_{v \in \Lambda_\delta} f(v) = f(u_\delta) < +\infty, \tag{4.3}$$

$$\ddot{u}_\delta \int_0^1 [h - V(u_\delta)]dt + V'(u_\delta) \int_0^1 \frac{1}{2}|\dot{u}_\delta|^2 dt = 0 \qquad \forall t \in]0,1], \tag{4.4}$$

and

$$\frac{1}{2}|\dot{u}_\delta|^2 \int_0^1 [h - V(u_\delta)]dt + (V(u_\delta) - h) \int_0^1 \frac{1}{2}|\dot{u}_\delta|^2 dt = 0 \qquad \forall t \in]0,1]. \tag{4.5}$$

Proof. Since we are testing f on collision functions we recall first of all that $\inf_{\Lambda_\delta} f < +\infty$ due to the weak force condition (take $v = \xi t^{\frac{2}{\alpha+2}}$, with $\xi \in \partial\Omega_\delta$: then $f(v) < +\infty$). The proof can then be performed exactly like that of Proposition 2.1. ∎

With the same notation as in Section 3.2 ($\varphi(x) = \psi(x)x + (1-\psi(x))\frac{V'(x)}{|V'(x)|}$) we now prove that the sequence u_δ of minimizers found by Proposition 4.2 is bounded in H^1.

Lemma 4.3 Assume (W1)-(W5) hold. Then, $\forall L > 0, \exists \beta > 0, \exists C > 0$ such that $\forall x \in \overline{\Omega} \setminus \{0\}$,

$$\frac{1}{2L} V'(x)\varphi(x) \geq \beta(V(x) - h) + C.$$

Proof. Assume by contradiction that there exist $L > 0$ and three sequences $\beta_n \to +\infty$, $C_n \to 0$, $x_n \in \overline{\Omega} \setminus \{0\}$ such that

$$\frac{1}{2L} V'(x_n)\varphi(x_n) < \beta_n(V(x_n) - h) + C_n. \tag{4.6}$$

Since Ω is bounded, $x_n \to x \in \overline{\Omega}$, up to a subsequence.

If $x \in \overline{\Omega} \setminus \{0\}$ we reach a contradiction arguing like in Lemma 2.1.

On the other hand, if $x_n \to 0$, then for n large $\varphi(x_n) = x_n$ and by (W4),

$$\frac{\lambda}{2L} V(x_n) \leq \frac{1}{2L} V'(x_n)x_n,$$

so that

$$0 \leq \beta_n \left(\frac{\beta_n - \frac{\lambda}{2L}}{\beta_n} V(x_n) - h \right) + C_n \to -\infty,$$

and again we reach a contradiction. ∎

Proof of Theorem 4.1. We set $v_\delta(t) = \varphi(u_\delta(t))$ and we multiply (4.4) by v_δ in $L^2(a, 0)$, with $a > 0$; we obtain

$$\begin{aligned} &(\varphi(u_\delta(1))\dot{u}_\delta(1) - \varphi(u_\delta(a))\dot{u}_\delta(a)) \int_0^1 [h - V(u_\delta)]dt \\ &- \int_a^1 (J\varphi(u_\delta)\dot{u}_\delta, \dot{u}_\delta)dt \cdot \int_0^1 [h - V(u_\delta)]dt + \int_a^1 V'(u_\delta)\varphi(u_\delta)dt \cdot \int_0^1 \frac{1}{2}|\dot{u}_\delta|^2 dt = 0. \end{aligned} \tag{4.7}$$

Now by (4.5),

$$|\dot{u}_\delta(a)|^2 = (h - V(u_\delta(a))) \frac{\int_0^1 |\dot{u}_\delta|^2 dt}{\int_0^1 [h - V(u_\delta)]dt}; \tag{4.8}$$

therefore, if a is so small that $\varphi(u_\delta(a)) = u_\delta(a)$, then by (W3) and (4.8) we have

$$\begin{aligned}|\varphi(u_\delta(a))\dot{u}_\delta(a)| &\leq |\varphi(u_\delta(a))||\dot{u}_\delta(a)| \\ &\leq |\varphi(u_\delta(a))|(h - V(u_\delta(a)))^{\frac{1}{2}} \frac{\left(\int_0^1 |\dot{u}_\delta|^2 dt\right)^{\frac{1}{2}}}{\left(\int_0^1 [h - V(u_\delta)]dt\right)^{\frac{1}{2}}} \\ &\leq C_1 |u_\delta(a)|^{\frac{2-\alpha}{2}} \frac{\left(\int_0^1 |\dot{u}_\delta|^2 dt\right)^{\frac{1}{2}}}{\left(\int_0^1 [h - V(u_\delta)]dt\right)^{\frac{1}{2}}};\end{aligned}$$

since $\alpha < 2$, the right-hand-side tends to 0 for $a \to 0$. Then we can pass to the limit in (4.7), and obtain

$$\begin{aligned}\varphi(u_\delta(1))\dot{u}_\delta(1)\int_0^1 [h - V(u_\delta)]dt = &\int_0^1 (J\varphi(u_\delta)\dot{u}_\delta, \dot{u}_\delta)dt \cdot \int_0^1 [h - V(u_\delta)]dt \\ &+ \int_0^1 V'(u_\delta)\varphi(u_\delta)dt \cdot \int_0^1 \frac{1}{2}|\dot{u}_\delta|^2 dt.\end{aligned} \tag{4.9}$$

At this point we can conclude exactly like in the first part of Theorem 2.1 to find a u such that $u(0) = 0$, $u(1) \in \partial\overline{\Omega}$, $u(t) \in \Omega$ $\forall t \in]0,1[$ and $f(u) = \inf_\Lambda f$. Such a function is a generalized solution of (1.1) which is not a generalized brake orbit; however we can construct a periodic generalized (brake) orbit just by reflecting u around $t = 0$. ∎

We now turn to the second result, namely the existence of k *noncollision* brake orbits when the topology of Ω satisfies (V1) and V is not too different from $-\frac{1}{|x|^\alpha}$ near zero. The precise assumption we shall use is the following

(W6) $\exists \alpha \in]0,2[$ $\exists r > 0$, $\exists \phi \in C^2(]0,+\infty[;\mathbb{R})$ such that

$$V(x) = -\frac{1}{|x|^\alpha} + \phi(|x|), \qquad \forall 0 < |x| \leq r, \tag{4.10}$$

$$\lim_{s \to 0} \phi'(s)s^{\alpha+1} = 0. \tag{4.11}$$

Then we can prove

Theorem 4.4. Assume (V1), (W2), and (W6) hold. Then there exist at least $k+1$ distinct brake orbits of energy h, and k of which are noncollision orbits.

Proof. We find a first solution connecting $\{0\}$ and $\partial\overline{\Omega}$ via Theorem 4.1. Then we show that each of the k orbits found with Theorem 2.1 is free of collisions. To this aim let K_0 and K_1 be two different connected components of $\partial\overline{\Omega}$ and set

$$\Lambda = \{u \in H^1 \;/\; u(0) \in K_0,\; u(1) \in K_1,\; u(t) \in \Omega \;\forall t \in]0,1[\}. \tag{4.12}$$

It is enough to show that if u satisfies

$$f(u) = \inf_{v\in\Lambda} f(v) \tag{4.13}$$

then $u \in \Lambda$, namely, it is free of collisions. In other words we shall prove that if

$$\Lambda_0 = \{u \in H^1 \;/\; u(0) \in K_0,\; u(1) \in K_1,\; \exists \bar{t} \in]0,1[\text{ suchthat } u(\bar{t}) = 0\} \tag{4.14}$$

then

$$\inf_{v\in\Lambda} f(v) < \inf_{v\in\Lambda_0} f(v). \tag{4.15}$$

Now the fact that $\inf_\Lambda f$ is achieved by some function in $\overline{\Lambda}$ follows as in the proof of by Theorem 4.1, because (W6) implies both (W3) and (W4). We shall therefore consider a minimizer $u \in \Lambda_0$ and we show that this leads to a contradiction.

Without loss of generality we assume that (after a rescaling) u verifies

$$u : [-1,1] \to \overline{\Omega}, \quad u(-1) \in K_0, \quad u(1) \in K_1, \quad u(0) = 0.$$

By (W6) there exists $\delta > 0$ such that $|u(t)| \le r \quad \forall t \in [-\delta, \delta]$. This implies that in $[-\delta,\delta]$ u is planar (actually $\frac{u(t)}{|u(t)|}$ is constant both in $[-\delta, 0[$ and in $]0,\delta]$).

Two cases may present:

i) After the collision u emeges from zero along a different direction (but always following a straight line). Then fix two instants $t_0 < t_1$ such that $|u(t_0)| = |u(t_1)| = r' < r$ and let $v : [t_0,t_1] \to \Omega$ be the projection of $u(t)$ on the segment through $u(t_0), u(t_1)$. Defining

$$w(t) = \begin{cases} v(t) & \text{if } t \in [t_0,t_1] \\ u(t) & \text{if } t \notin [t_0,t_1] \end{cases}$$

It is immediate to check that $w \in H^1$ and $f(w) < f(u)$, contradicting the minimizing property of u.

ii) the direction $u(t)$ is the same before and after the collision. In this case there exists $v \in S^{N-1}$ such that $v \cdot u(t) = 0 \quad \forall t \in [-\delta, \delta]$. We now define a piecewise linear function $w : [-1,1] \to \mathbb{R}^N$ as

$$w(t) = \begin{cases} 0 & \text{if } t \notin [-\delta, \delta] \\ \mu v & \text{if } t \in [-\frac{\delta}{2}, \frac{\delta}{2}] \end{cases}$$

where μ will be conveniently chosen later. It is clear that $w(t) \cdot u(t) = \dot{w}(t) \cdot \dot{u}(t) = 0 \quad \forall t \in [-1,1]$.

We are going to prove that if μ is small enough, then $f(u+w) < f(u)$, contradicting the fact that u is a minimum point. To this aim we evaluate

$$2f(u+w) - 2f(u) = \int_{-1}^{1} |\dot{u} + \dot{w}|^2 dt \cdot \int_{-1}^{1} [h - V(u+w)] dt - \int_{-1}^{1} |\dot{u}|^2 dt \cdot \int_{-1}^{1} [h - V(u)] dt =$$

$$\int_{-1}^{1} |\dot{u}|^2 dt \cdot \int_{-1}^{1} [h - V(u+w)] dt + \int_{-\delta}^{\delta} |\dot{w}|^2 dt \cdot \int_{-1}^{1} [h - V(u+w)] dt - \int_{-1}^{1} |\dot{u}|^2 dt \cdot \int_{-1}^{1} [h - V(u)] dt \leq$$

$$\int_{-1}^{1} |\dot{u}|^2 dt \cdot \int_{-1}^{1} [-V(u+w) + V(u)] dt + \int_{-\delta}^{\delta} |\dot{w}|^2 dt \cdot \int_{-1}^{1} [h - V(u)] dt =$$

$$\int_{-1}^{1} |\dot{u}|^2 dt \cdot \int_{-\delta}^{\delta} [-V(u+w) + V(u)] dt + \int_{-\delta}^{\delta} |\dot{w}|^2 dt \cdot \int_{-1}^{1} [h - V(u)] dt.$$

We now estimate separatly these two terms; in what follows the $C_j, j = 1, 2, \ldots$ will denote positive constants independent of μ.

To begin with, notice that $|\dot{w}(t)| = \frac{2\mu}{\delta} \quad \forall t \in [-\delta, -\frac{\delta}{2}] \cup [\frac{\delta}{2}, \delta]$ and $\dot{w}(t) = 0 \quad \forall t \notin [-\delta, \delta]$, so that

$$\int_{-\delta}^{\delta} |\dot{w}|^2 dt \cdot \int_{-1}^{1} [h - V(u)] dt \leq C_1 \mu^2.$$

Next we consider

$$\int_{-\delta}^{\delta} [-V(u+w) + V(u)] dt = \int_{-\delta}^{\delta} \int_0^1 \frac{d}{d\lambda} (-V(u + \lambda w)) d\lambda dt = \tag{4.16}$$

$$= \int_{-\delta}^{\delta} \int_0^1 (-V'(u + \lambda w) \cdot w) d\lambda dt.$$

Now, if δ and μ are small enough, then $|u(t) + \lambda w(t)| < r$, so that by (W6) and the fact that $u(t)$ and $w(t)$ are orthogonal,

$$\begin{aligned} V'(u+\lambda w)\cdot w =& \alpha\frac{u+\lambda w}{|u+\lambda w|^{\alpha+2}}\cdot w + \phi'(|u+\lambda w|)\frac{u+\lambda w}{|u+\lambda w|} = \\ & \frac{\lambda|w|^2}{|u+\lambda w|^{\alpha+2}}\left(\alpha + \phi'(|u+\lambda w|)|u+\lambda w|^{\alpha+1}\right) \geq \frac{\alpha}{2}\frac{\lambda|w|^2}{|u+\lambda w|^{\alpha+2}}. \end{aligned}$$

Inserting this inequality in (4.16) we find

$$\int_{-\delta}^{\delta}[-V(u+w)+V(u)]dt \leq -\frac{\alpha}{2}\int_{-\delta}^{\delta}\int_0^1 \frac{\lambda|w|^2}{|u+\lambda w|^{\alpha+2}}d\lambda dt \leq$$

$$-\frac{\alpha}{2}\mu^2\int_{\frac{-\delta}{2}}^{\frac{\delta}{2}}\int_0^1 \frac{\lambda}{|u+\lambda w|^{\alpha+2}}d\lambda dt \leq -\frac{\alpha}{4}\mu^2\int_{\frac{-\delta}{2}}^{\frac{\delta}{2}}\int_{\frac{1}{2}}^1 \frac{d\lambda dt}{|u+\lambda w|^{\alpha+2}}.$$

Since $u(0) = 0$, we have

$$|u(t)| = |u(t) - u(0)| \leq \int_0^t |\dot{u}|dt \leq t^{\frac{1}{2}}||\dot{u}||_2 = C_2 t^{\frac{1}{2}},$$

so that $|u(t)| \leq \mu$ whenever $|t| \leq \sigma_\mu = \frac{\mu^2}{C_2^2}$. Restricting the interval of integration to $[-\sigma_\mu, \sigma_\mu]$, and noticing that there $|u+\lambda w| \leq 2\mu$, we obtain

$$\int_{-\delta}^{\delta}[-V(u+w)+V(u)]dt \leq -\frac{\alpha}{4}\mu^2\int_{-\sigma_\mu}^{\sigma_\mu}\int_{\frac{1}{2}}^1 \frac{d\lambda dt}{2^{\alpha+2}\mu^{\alpha+2}} \leq -C_3\mu^{2-\alpha}.$$

Therefore

$$f(u+w) - f(u) \leq \frac{1}{2}(C_1\mu^2 - C_3\mu^{2-\alpha}) < 0$$

if μ is small enough, and a contradiction is reached. ∎

CHAPTER 4

THE THREE–BODY PROBLEM

4.1. Symmetric three–body problems

In this last chapter we present some results of existence of periodic solutions to the restricted three–body problem and to the full three–body problem in the tridimensional space.

Both these problems share such a long history that it is impossible to give here an extensive bibliography. The interested reader may consult upon this subject any classical text in Celestial Mechanics.

The results that we present here are derived from the application of the local analysis of the behavior of the orbits in a neighborhood of the collisions which was carried out in Chapter 1 and in Chapter 2. We shall concentrate here on *symmetric* problems, in the sense that the given force field verifies some symmetry condition like in Chapter 1.

From our point of view the three–body problem (and he restricted three–body problem) belong thus to the larger class of Hamiltonian systems with singular potentials of the type

$$(1.1) \qquad -\ddot{x} = \nabla F(x)$$

where $x \in \mathbb{R}^N$ and F presents a set of singularities of attractive type. For example, in the full three–body problem F is of the form $F(X) = F(x_1, x_2, x_3) = \sum_{i \neq j} \frac{-a_{ij}}{|x_i - x_j|}$. Here we have written $X = (x_1, x_2, x_3) \in (\mathbb{R}^3)^3$.

In the last few years quite a large amount of papers concerning periodic solutions to problems like (1.1) has appeared, concerning most of all the case in which $F(x) = -\frac{a}{|x|^\alpha}$ (see i.e. [1]–[7], [10], [11], [17], [23], [24], [29], [33], [35], [43]–[46]). In the papers treating the fixed period problem, periodic solutions are found as critical points of the associated action integral

$$I(x) = \frac{1}{2}\int_0^T |\dot{x}|^2 dt - \int_0^T F(x)dt$$

over a suitable function space. The application of these techniques to many–body problems is even newer; we recall the papers [6], [**11**], [**16**], [**19**], [**20**].

Also in the three–body problem the variational approach presents two main difficulties. First the lack of compactness characteristic of the two–body problem assumes here a more complicated form. Secondly, the problem of avoiding the collisions (which, talking about periodic solutions, are physically meaningless) becomes much more complicated since the geometry of the multiple collisions cannot be handled in a simple way. This is the reason why the results of this chapter cannot be generalized to more than three bodies, or even to $\mathbb{R}^N$, if $N > 3$.

Our approach follows the one introduced in [**20**]. A symmetry constraint on the function space will allow us to overcome the lack of compactness, so that existence of periodic solutions can be derived from the minimization of a suitable functional. Next we shall show, and this will be the technical part of the chapter, that solutions found in this way do not collide, that is, they are classical, $\mathcal{C}^2$ solutions of the problem. The main argument consists in proving that the property of *minimizing* the action integral cannot be enjoyed by a collision orbit, provided the potential satisfies some local assumptions in the neighborhood of the singularities.

We focus our attention on the existence of at least one noncollision solution. Multiplicity results have been given in [**16**], though in the larger class of (possibly) colliding functions; see also [**16**] for the N–body problem. Lastly we recall the paper [6] for the symmetric fixed energy problem for N bodies is treated. Existence results for the three–body problem in the strong force case without symmetries were provided in [**11**].

4.2. A restricted symmetric three–body problem

We begin with the application of the methods discussed in Chapter 1 to the restricted three–body problem in presence of symmetries. This means that we shall look for a periodic solution of the equations of motion which describe the behavior of one body moving in the force field generated by the other two. We suppose that the presence of the first body does not influence the motion of the others. This is the situation which occurs, for example, when the mass of the primary body can be neglected when compared to the total mass of the system. The classical model problem is the description of the motion of the Moon moving in the gravitational field generated by the Sun and the Earth: the effect of the Moon on the motion of the Sun and the Earth can be, in a first approximation, neglected.

We shall deal with a *symmetric* three–body problem in the sense that the two bodies creating the field are assumed to be moving on symmetric orbits, and we shall look for a symmetric solution as well.

We denote by $q_1(t), q_2(t) \in \mathbb{R}^3$ the coordinates of the "fixed" bodies at time t, and we assume that

$$q_1, q_2 \in H^1_T(\mathbb{R};\mathbb{R}^3) = \{x \in H^1_{loc}(\mathbb{R};\mathbb{R}^3) \;/\; x(t+T) = x(t)\ \forall t \in [0,T]\}. \tag{2.1}$$

The symmetry condition on the orbits is expressed by the relations

$$q_i(t + \frac{T}{2}) = -q_i(t) \qquad \forall t \in [0,T], \quad i = 1,2. \tag{2.2}$$

For any two orbits $x, y \in H^1_T(\mathbb{R};\mathbb{R}^3)$ we denote by $\mathcal{C}(x,y)$ their *collision set*, namely the (possibly empty) set

$$\mathcal{C}(x,y) = \{t \in [0,T] \;/\; x(t) = y(t)\}. \tag{2.3}$$

Since q_1, q_2 satisfy the equations of motion of the two–body problem, it is natural to require that

$$q_1, q_2 \in \mathcal{C}^2([0,T] \setminus \mathcal{C}(q_1,q_2);\mathbb{R}^3). \tag{2.4}$$

From now on we shall say that q_1, q_2 are *standard orbits* if the above hypotheses hold true, namely if (2.1)–(2.4) are satisfied.

The restricted symmetric three–body problem consists in determining, given two standard orbits and two functions $F_1, F_2 \in \mathcal{C}^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$, a $q \in \mathcal{C}^2([0,T] \setminus \mathcal{C}(q_1,q_2); \mathbb{R}^3)$ such that

$$(2.5) \qquad \begin{cases} -\ddot{q}(t) = \nabla F_1(q(t) - q_1(t)) + \nabla F_2(q(t) - q_2(t)) & \forall t \in [0,T] \setminus \mathcal{C}(q_1,q_2) \\ q(t + \frac{T}{2}) = -q(t) \qquad \forall t \in [0,T] \\ q(t) \neq q_1(t), \quad q(t) \neq q_2(t) \qquad \forall t \in [0,T] \setminus \mathcal{C}(q_1,q_2) \end{cases}$$

This problem can be tackled with the methods described in Chapter 1 because it has a variational structure. Indeed, let

$$H = \{q \in H^1_T(\mathbb{R};\mathbb{R}^3) \ / \ q(t + \frac{T}{2}) = -q(t) \ \forall t \in [0,T]\},$$

$$\Lambda = \{q \in H \ / \ q(t) \neq q_1(t), \ q(t) \neq q_2(t) \ \forall t \in [0,T] \setminus \mathcal{C}(q_1,q_2)\}$$

and assume that the potentials $F_i \in \mathcal{C}^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$ are even:

$$(S) \qquad F_i(-x) = F_i(x) \qquad \forall x \in \mathbb{R}^3 \setminus \{0\} \quad i = 1,2.$$

Then it is well–known that solutions of (2.5) are critical points of the action functional $I \in \mathcal{C}^2(\Lambda; \mathbb{R})$ defined as

$$(2.6) \qquad I(q) = \frac{1}{2}\int_0^T |\dot{q}(t)|dt - \int_0^T F_1(q(t) - q_1(t))dt - \int_0^T F_2(q(t) - q_2(t))dt.$$

We suppose that each F_i is of the form

$$(2.7) \qquad F_i(x) = -\frac{a_i}{|x|^\alpha} + \varphi_i(|x|) + U_i(x), \qquad i = 1,2$$

for some $a_i > 0$ and $\alpha \in]0,2[$. We remark that in the classical restricted three–body problem one has $\varphi_i = U_i = 0$, $i = 1,2$. Therefore the terms φ_i and U_i can be consisered perturbations of the classical potential.

Concerning the functions φ_i and U_i, we assume that $\varphi_i \in \mathcal{C}^2(]0,+\infty[; \mathbb{R})$ and that $U_i \in \mathcal{C}^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$, $i = 1,2$, and we make the following hypotheses:

$$(H1) \qquad \lim_{s \to 0} |\varphi_i'(s)| s^{\alpha+1} = 0 \qquad i = 1,2;$$

$$(H2)\qquad \begin{gathered}\exists C>0\ \exists\sigma>0 \text{ such that}\\ \limsup_{x\to 0}|\nabla^2 U_i(x)||x|^{\alpha+2-\sigma}\le C \qquad i=1,2;\end{gathered}$$

$$(H3)\qquad \lim_{|x|\to\infty}\frac{|\nabla F_i(x)|}{|x|}=0 \qquad i=1,2;$$

$$(H4)\qquad F_i(x)\le 0 \qquad \forall x\ne 0,\quad i=1,2.$$

We now state and prove the main result of this section.

Theorem 2.1. Let q_1, q_2 be standard orbits and suppose F_i verifies (S), (H1)–(H4). Then for every $T>0$ there exists at least one solution $q\in\Lambda$ of (2.5) having T as minimal period. Moreover

$$(2.8)\qquad I(q)=\inf_{x\in H} I(x).$$

Proof. We first remark that under our hypotheses the functional I admits a natural extension (which we still denote by I) to the whole of H, defined by

$$(2.9)\qquad I(q)=\begin{cases} I(q) & \text{if } \int_0^T F_i(q-q_i)dt<+\infty, \quad i=1,2\\ +\infty & \text{otherwise.}\end{cases}$$

It is straightforward to verify that such extension is weakly lower semicontinuous and coercive on H, due to the symmetry constraint and (H3). Therefore there exists $q\in H$ such that $I(q)=\inf_{x\in H} I(x)$. All we have to show is that actually $q\in\Lambda$.

To this aim we assume on the contrary that there exists a minimum q in $H\setminus\Lambda$ and we show that this leads to a contradiction; the arguments will follow closely those of Chapter 1 and will make use of the results obtained there.

By the definition of Λ, saying that $q\in H\setminus\Lambda$ is equivalent to saying

$$\exists t^*\in[0,T]\setminus\mathcal{C}(q_1,q_2),\ \exists i\in\{1,2\} \text{ such that } q(t^*)=q_i(t^*).$$

Of course we can assume without loss of generality that $i=1$, so that at time t^*, q collides with q_1. Since q_1 and q_2 are continuous functions, $[0,T]\setminus\mathcal{C}(q_1,q_2)$ is open in $[0,T]$; moreover, since the set $\mathcal{C}(q,q_1)$ is closed, we can find an interval $[a,b]\subset[0,T]$ such that

i) $a, b \notin \mathcal{C}(q, q_1)$,

ii) $a, b \notin \mathcal{C}(q, q_2)$,

iii) $t^* \in]a, b[$,

iv) $|q_1(t) - q_2(t)| \geq C > 0 \qquad \forall t \in [a, b]$.

Let us denote by $J : H \to \mathbb{R}$ the functional

$$J(x) = \frac{1}{2}\int_a^b |\dot{x}(t)| dt - \int_a^b F_1(x(t) - q_1(t))dt - \int_a^b F_2(x(t) - q_2(t))dt; \tag{2.10}$$

then the fact that q minimizes I implies that q also solves the minimization problem

$$\inf\{J(y) \;/\; y \in H^1([a, b]; \mathbb{R}^3),\ y(a) = q(a),\ y(b) = q(b)\}. \tag{2.11}$$

Remark that because of (H4), we have

$$\int_a^b |\dot{q}|^2 dt \leq 2J(q); \tag{2.12}$$

moreover, just by taking a smaller interval (if necessary), we can always assume that

$$(\sqrt{2J(q)} + ||\dot{q}_1||_2)(b - a)^{\frac{1}{2}} \leq \frac{C}{2}, \tag{2.13}$$

where C is the constant appearing in iv) above.

Now since

$$\begin{aligned} |q(t) - q_1(t)| &\leq |\int_{t^*}^t |\dot{q}(t) - \dot{q}_1(t)| dt| \leq \int_{t^*}^t |\dot{q}(t)| dt + \int_{t^*}^t |\dot{q}_1(t)| dt \\ &\leq \sqrt{|t^* - t|}(||\dot{q}||_2 + ||\dot{q}_1||_2) \leq \sqrt{b - a}(\sqrt{2J(q)} + ||\dot{q}_1||_2), \end{aligned}$$

we have that by (2.13) and iv),

$$|q(t) - q_2(t)| \geq |q_1(t) - q_2(t)| - |q(t) - q_1(t)| \geq \frac{C}{2} \qquad \forall t \in [a, b], \tag{2.14}$$

and the same estimate holds for every minimizer of (2.11).

By the change of variable $z = y - q_1$, we see that the minimization problem (2.11) is equivalent to

$$\inf_{z \in K} \left(\frac{1}{2}\int_a^b |\dot{z}| dt - \int_a^b F_1(z) dt - \int_a^b G(t, z) dt\right), \tag{2.15}$$

where $K = \{z \in H^1([a,b];\mathbb{R}^3),\ z(a) = q(a) - q_1(a),\ z(b) = q(b) - q_1(b)\}$ and

$$G(t,z) = \frac{1}{2}|\dot{q}_1(t)|^2 - z \cdot \ddot{q}_1(t) - F_2(z + q_1(t) - q_2(t)) + z(b) \cdot \dot{q}_1(b) - z(a) \cdot \dot{q}_1(a).$$

We now apply Theorems 1.4 and 2.2 of Chapter 1 to the minimization problem (2.15); remark that the functional to be minimized has the right form, since $F_1(z) = -\frac{a_1}{|z|^\alpha}$. Notice also that the only hypothesis which may fail to be satisfied is the regularity of F_2. However, thanks to (2.14), we can treat F_2 as regular, since every minimizer (or minimizing sequence) cannot interact with the set of singularities of $F_2(z + q_1 - q_2)$. Thus, from (H1)–(H3), the problem fits into Theorems 1.4 and 2.2 of Chapter 1. Now those results actually say that the infimum in (2.15) is attained in $K \cap \{z \ /\ z(t) \neq 0\ \forall t \in [a,b]\}$. Therefore, if z minimizes in (2.15), then $y = z + q_1$ minimizes in (2.11). Since q minimizes (2.11), we must conclude that $q - q_1 \in K \cap \{z \ /\ z(t) \neq 0\ \forall t \in [a,b]\}$, that is to say, $q(t) \neq q_1(t)\ \forall t \in [a,b]$, contradicting the fact that $q(t^*) = q_1(t^*)$. Of course the same argument applies for every other collision between q and q_1, or q and q_2, and the proof is complete. ∎

4.3. The symmetric three–body problem

The three–body problem is probably the most famous problem in Celestial Mechanics. It consists in describing the complete behavior of three particles attracting each other according to Newton's law of gravitation. Despite the efforts of generations of mathematicians and physicists, this problem is still unsolved, and it is generally believed that it will be so for a long time to come.

Our purpose here is to analize what can be done by means of the variational approach, which has been proved to be so useful in treating the easier two–body problem. The variational techniques have motivated in the very latest years a great rise of interest in this kind of problems, and especially concerning the search of periodic solutions (see i.e. [6], [**11**], [**19**], [**20**]). The analysis of the motion in the neighborhood of the singularity which was carried out in Chapter 1 and 2 has inspired the results contained in the present section. In particular, our main purpose is to investigate wether the arguments used in the first part of this work to avoid the collisions can be suitably modified and generalized in order to achieve the same kind of result for the double and triple collisions in the three–body problem. It turns out that this is indeed the case, at least if the hypotheses used in the first chapter are suitably strengthened. Unfortunately, the core of the arguments lies in the fact that we deal with three bodies in the tridimensional space, so that we do not think that any generalization of our techniques to higher dimensions, or, worst of all, to more than three bodies, is possible.

During this exposition we shall make use of known classical results, such as Sundman's Theorem; most of the time we shall need some slight generalizations of these results, which were originally proved only in the case of the Newtonian potentials. For the reader's convenience, we shall report here the original proofs (or their generalizations), in order to make the exposition self–contained.

We begin by stating the problem and the main results. The main theorem will be proved in Section 4.4.

We consider six functions $F_{ij} \in \mathcal{C}^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$, $1 \leq i \neq j \leq 3$, three positive numbers m_i (the masses of the bodies), and a period $T > 0$ and we look for a classical solution of the problem

$$
(3.1) \qquad \begin{cases} -m_i \ddot{q}_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^{3} \nabla F_{ij}(q_i(t) - q_j(t)) & i = 1,2,3 \quad \forall t \in [0,T] \\ q_i(t + \frac{T}{2}) = -q_i(t) & i = 1,2,3 \quad \forall t \in [0,T] \\ q_i(t) \neq q_j(t) & 1 \leq i \neq j \leq 3 \quad \forall t \in [0,T] \end{cases}
$$

The potentials $F_{ij} \in \mathcal{C}^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$ have a singularity at zero of actractive type.

In the classical three–body problem for instance the three particles located at q_i are subjcet to the mutual effect of the universal law of gravitation and must satisfy the equations

$$
-m_i \ddot{q}_i = \sum_{\substack{j=1 \\ j \neq i}}^{3} G \frac{m_i m_j}{|q_i - q_j|^3}(q_i - q_j) \qquad i = 1,2,3.
$$

Problem (3.1) has a variational structure provided that

$$
(S1) \qquad F_{ij}(x) = F_{ji}(-x), \qquad \forall x \in \mathbb{R}^3 \setminus \{0\},\ 1 \leq i,j \leq 3;
$$

as for the restricted three–body problem we shall deal with an even potential, that is we shall assume that

$$
(S2) \qquad F_{ij}(x) = F_{ij}(-x) \qquad \forall x \in \mathbb{R}^3 \setminus \{0\},\ 1 \leq i,j \leq 3.
$$

Remark that (S1) and (S2) actually reduce the number of the functions F_{ij} to three, since $F_{ij} = F_{ji}$. This identity permits to work in a space of symmetric orbits, such as in [**20**]; the conditions (S1) and (S2) are thus more restrictive than $F_{ij} = F_{ji}$, but yield a sharper result than that of [**20**].

If we denote by Q the vector $(q_1, q_2, q_3) \in (\mathbb{R}^3)^3$ and by

$$
(3.2) \qquad H = \{Q \in H^1_{loc}(\mathbb{R}, (\mathbb{R}^3)^3) \ / \ Q(t + \frac{T}{2}) = -Q(t),\ \forall t \in \mathbb{R}\},
$$

then the natural action functional associated to (3.1) is

$$
(3.3) \qquad I(Q) = I(q_1, q_2, q_3) = \frac{1}{2} \sum_{i=1}^{3} m_i \int_0^T |\dot{q}_i(t)|^2 - \frac{1}{2} \sum_{\substack{i,j=1 \\ j \neq i}}^{3} \int_0^T F_{ij}(q_i(t) - q_j(t)) dt.
$$

It is immediate to check that critical points of I in

$$\Lambda = \{Q \in H \ / \ q_i(t) \neq q_j(t) \ , \forall 1 \leq i,j \leq 3, \ \forall t \in [0,T]\} \tag{3.4}$$

are classical solutions of (3.1) Our main result is concerned with locally radially symmetric potentials. In order to state it we assume that each F_{ij} can be written in the form

$$F_{ij}(x) = -\frac{a_{ij}}{|x|^\alpha} + U_{ij}(x) \qquad \forall \, 1 \leq i,j \leq 3, \tag{$H1$}$$

for some $\alpha \in]0,2[$ and $a_{ij} > 0$. Concerning the functions U_{ij}, we consider the following assumptions:

$$\begin{aligned} &\exists \varepsilon > 0, \ \exists \phi_{ij} \in \mathcal{C}^2(]0,\varepsilon];\mathbb{R}) \ \text{ such that} \\ &U_{ij}(x) = \phi_{ij}(|x|) \qquad \forall \, 1 \leq i,j \leq 3, \quad \forall 0 < |x| \leq \varepsilon; \end{aligned} \tag{$H2$}$$

$$\lim_{s \to 0} |\phi'_{ij}(s)| s^{\alpha+1} = 0 \qquad \forall \, 1 \leq i,j \leq 3; \tag{$H3$}$$

$$\lim_{|x| \to \infty} \frac{|\nabla F_{ij}(x)|}{|x|} = 0 \qquad \forall \, 1 \leq i,j \leq 3; \tag{$H4$}$$

$$F_{ij}(x) \leq 0 \qquad \forall \, 1 \leq i,j \leq 3, \quad \forall x \neq 0. \tag{$H5$}$$

Remark 3.1. Hypothesis (H1) says that F_{ij} is a perturbation of the potential $-\frac{a_{ij}}{|x|^\alpha}$ (we obtain Newton's potential and the classical three–body problem when $\alpha = 1$ and $U_{ij} = 0$). Condition (H2) expresses the fact that the potential is locally radial at zero; this means that close to a collision the potential only depends on the distances between the bodies. Finally (H4) is a nonresonance condition at infinity which implies that the functional I is coercive on Λ.

We can now state the main result.

Theorem 3.2. Assume $F_{ij} \in \mathcal{C}^2(\mathbb{R}^3 \setminus \{0\};\mathbb{R})$ satisfy (S1), (S2) and (H1)–(H5). Then for every period T, problem (3.1) has at least one solution $Q = (q_1, q_2, q_3) \in \Lambda$ of minimal period T. Moreover

$$I(Q) = \inf_{X \in \Lambda} I(X) < \inf_{X \in \partial\Lambda} I(X). \tag{3.5}$$

Remark 3.3. We notice that when $U_{ij} = 0$, so that $F_{ij}(x) = -\frac{a_{ij}}{|x|^\alpha}$, then Theorem 3.2 applies, and in particular, (3.5) holds. This allows us to introduce pinching conditions, in the spirit of [**23**], in order to treat potentials without radial symmetry assumptions. The corresponding results are given in the following propositions.

Theorem 3.4. Let $F_{ij} \in C^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$ satisfy (S1), (S2) and

$$\begin{aligned}&\exists \alpha > 0,\ \exists a_{ij} > 0,\ \exists C > 1, \quad \text{such that} \\ &\frac{a_{ij}}{|x|^\alpha} \le -F_{ij}(x) \le C\frac{a_{ij}}{|x|^\alpha}, \qquad \forall\, 1 \le i,j \le 3 \quad \forall x \in \mathbb{R}^3 \setminus \{0\}.\end{aligned} \tag{3.6}$$

Then there exists a function $\Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{ij}}{a_{ik}})$ such that, when

$$C \le \Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{ij}}{a_{ik}}) \qquad \forall i,j,k = 1,2,3 \tag{3.7}$$

then for every period $T > 0$, (3.1) has at least one solution having T as minimal period. Moreover Ψ enjoys the following properties:

$$\begin{aligned}&\Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{ij}}{a_{ik}}) > 1 \qquad \forall \alpha > 0, \\ &\lim_{\alpha \to 2} \Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{ij}}{a_{ik}}) = +\infty, \\ &\lim_{\alpha \to 0} \Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{ij}}{a_{ik}}) = 1.\end{aligned}$$

Remark 3.5. Thinking of Newton's law of gravitation, a remarkable application of the above theorem holds when $F_{ij}(x) \simeq \frac{a m_i m_j}{|x|^\alpha}$. In that case, the pinching condition reduces to $C \le \Psi(\alpha, \frac{m_i}{m_j})$.

Remark 3.6. Here like in other results, the minimality of the period just follows from the well-known fact that each minimizer of a functional of the type of (3.3) over a space of T-periodic functions either is constant or has minimal period T.

As direct consequences of Theorem 3.4 we obtain the following Corollaries.

Corollary 3.7. Assume that $m_1 = m_2 = m_3 = m$, and let $F_{ij} \in C^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$ satisfy (S1), (S2) and

$$\begin{aligned} &\exists \alpha > 0 \ \exists a, b > 0 \ \text{ such that} \\ &= \frac{a}{|x|^\alpha} \leq -F_{ij}(x) \leq \frac{b}{|x|^\alpha}, \qquad 1 \leq i,j \leq 3 \quad \forall x \neq 0, \end{aligned}$$

then there exists a function $\Psi(\alpha)$ such that

$$\frac{b}{a} \leq \Psi(\alpha) \tag{3.8}$$

implies that for every period $T > 0$, (3.1) has at least one solution having T as minimal period. Moreover Ψ enjoys the following properties:

$$\begin{aligned} &\Psi(\alpha) > 1 \quad \forall \alpha > 0 \\ &\lim_{\alpha \to 2} \Psi(\alpha) = +\infty \\ &\lim_{\alpha \to 0} \Psi(\alpha) = 1. \end{aligned}$$

The following result is just an immediate consequence of Corollary 3.7:

Corollary 3.8. Let $F_{ij}, U_{ij} \in C^2(\mathbb{R}^3 \setminus \{0\}; \mathbb{R})$ satisfy (S1), (S2) and

$$\begin{aligned} &\exists \alpha > 0, \ \exists a_{ij} > 0 \text{ , such that} \\ &F_{ij}(x) = -\frac{a_{ij}}{|x|^\alpha} + U_{ij}(x) \qquad 1 \leq i,j \leq 3 \quad \forall x \neq 0 \\ &\lim_{x \to 0} |U_{ij}(x)||x|^\alpha = 0 \qquad 1 \leq i,j \leq 3. \end{aligned}$$

Then, for every $T > 0$, (3.1) has infinitely many solutions.

As we have anticipated above, the proof of Theorem 3.1 will be given in next section. Assuming that this result holds true we now turn to the proofs of the other propositions.

Proof of Theorem 3.4. Since the functional I is weakly lower semicontinuous and coercive in Λ, $\inf_\Lambda I$ is attained by some function Q in $\overline{\Lambda}$. However, if we show that

$$\inf_{X \in \Lambda} I(X) \leq \inf_{X \in \partial\Lambda} I(X) \tag{3.9}$$

then it follows that $Q \in \Lambda$, and therefore it is a classical noncollision solution of (3.1).

We introduce the following notations:

$$J(X;T,a_{ij},m_k) = \frac{1}{2}\sum_{i=1}^{3} m_i \int_0^T |\dot{x}_i(t)|^2 - \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{3} \int_0^T \frac{a_{ij}}{|x_i(t)-x_j(t)|^\alpha} dt \tag{3.10}$$

and

$$J_1(X;a_{ij},m_k) = \frac{1}{2}\sum_{i=1}^{3} m_i \int_0^1 |\dot{x}_i(t)|^2 - \frac{1}{2}\sum_{\substack{i,j=1\\ j\neq i}}^{3} \int_0^1 \frac{a_{ij}}{|x_i(t)-x_j(t)|^\alpha} dt. \tag{3.11}$$

Obviously J_1 is defined on sets of 1–periodic functions; we denote by Λ_1 the corresponding set of symmetric noncollision 1–periodic orbits. If we make the change of variable $X(t) = (T^2 \frac{a_{pq}}{m_r})^{\frac{1}{\alpha+2}} Y(\frac{t}{T})$, where $p,q,r \in \{1,2,3\}$, an easy computation shows that

$$J(X;T,a_{ij},m_k) = (T^{2-\alpha} a_{pq}^2 m_r^\alpha)^{\frac{1}{\alpha+2}} J_1(Y;\frac{a_{ij}}{a_{pq}},\frac{m_k}{m_r});$$

therefore

$$J(X;T,Ca_{ij},m_k) = C^{\frac{2}{\alpha+2}} (T^{2-\alpha})^{\frac{1}{\alpha+2}} J_1(Y;a_{ij},m_k) = C^{\frac{2}{\alpha+2}} J(X;T,a_{ij},m_k). \tag{3.12}$$

This also means that

$$\inf_{X\in\Lambda} J(X;T,Ca_{ij},m_k) = C^{\frac{2}{\alpha+2}} \inf_{X\in\Lambda} J(X;T,a_{ij},m_k) \tag{3.13}$$

and

$$\inf_{X\in\partial\Lambda} J(X;T,Ca_{ij},m_k) = C^{\frac{2}{\alpha+2}} \inf_{X\in\partial\Lambda} J(X;T,a_{ij},m_k). \tag{3.14}$$

We now define Ψ as

$$\Psi(\alpha,\frac{m_i}{m_j},\frac{a_{ij}}{a_{ik}}) = \max_{p,q,r}\left(\frac{\inf_{Y\in\partial\Lambda_1} J_1(Y;\frac{m_i}{m_r},\frac{a_{ij}}{a_{pq}})}{\inf_{Y\in\Lambda_1} J_1(Y;\frac{m_i}{m_r},\frac{a_{ij}}{a_{pq}})}\right) \tag{3.15}$$

and we remark that by Theorem 3.2 we have $\Psi(\alpha,\frac{m_i}{m_j},\frac{a_{ij}}{a_{ik}}) \geq 1$. The fact that Ψ enjoys the properties in the statement of the theorem is an easy consequence of the definition of Ψ; we refer the reader to [**23**], where the proof is carried out in detail for the two–body problem.

Suppose now that the constant C is chosen in such a way that

$$C^{\frac{2}{\alpha+2}} \leq \Psi(\alpha, \frac{m_i}{m_j}, \frac{a_{ij}}{a_{ik}}). \tag{3.16}$$

Then we have

$$\begin{aligned}\inf_{X\in\Lambda} I(X) \leq \inf_{X\in\Lambda} J(X;T,Ca_{ij},m_k) \leq C^{\frac{2}{\alpha+2}} \inf_{X\in\Lambda} J(X;T,a_{ij},m_k) \\ \leq \inf_{X\in\partial\Lambda} J(X;T,a_{ij},m_k) \leq \inf_{X\in\partial\Lambda} I(X)\end{aligned}$$

and the proof is complete. ∎

Proof of Corollary 3.8. From the above discussion one easily sees that, under the assumptions of Theorem 3.4, there exists a constant C_1 (depending only on the masses and the values of the a_{ij}'s) such that the minimizer X found by the application of Theorem 3.4 satisfies $||\dot{X}||_2 \leq C_1 T^{\frac{2-\alpha}{2+\alpha}}$. One then concludes just by taking a small T, since condition (3.7) holds with a constant C as close to the value one as we wish, provided the motion is constrained in a sufficiently small neighborhood of the origin. In order to obtain an infinity of solutions, just apply the above reasoning to the sequence $\frac{T}{k}$, for each k large enough. ∎

4.4. Proof of the main result

This section is entirely devoted to the proof of Theorem 3.2. We are going to show that a minimizer of the functional I is free of both double and triple collisions, thereby proving that $\inf_\Lambda I < \inf_{\partial\Lambda} I$.

We are going to accomplish our goal by means of a main estimate based on a series of preliminary lemmas. Some of these lemmas are almost immediate extensions of classical results, such as Sundman's Theorem. For the sake of completeness we shall carry out all the proofs, including those which can easily be deduced from known classical results.

The section is structured as follows: first we show, with the aid of the results established in Section 2, that minimizers are free of double collisions. Next we obtain a geometric description of the triple collisions through a local analysis of the equations of motion; this description will enable us to rule out the triple collisions arguing by contradiction.

We start by remarking that the functional I admits a natural extension to the whole of H defined as usual by

$$I(X) = \begin{cases} I(X) & \text{if } \int_0^T F_{ij}(x_i - x_j)dt < +\infty \quad \forall 1 \le i \ne j \le 3 \\ +\infty. & \text{otherwise} \end{cases}$$

Similarly to the functional associated to the two–body problem, this extension is lower semicontinuous and coercive on H, due to the symmetry costraint and (H4). Therefore there exist $Q \in H$ such that

$$I(Q) = \inf_{X \in \Lambda} I(X). \tag{4.1}$$

We want to show that actually $Q \in \Lambda$. To this aim we suppose by contradiction that

$$I(Q) = \inf_{X \in \Lambda} I(X) = \inf_{X \in \partial\Lambda} I(X) = \inf_{X \in H} I(X) \tag{4.2}$$

so that the point of minimum Q verifies $Q \in \partial\Lambda$. By definition, $Q \in \partial\Lambda$ just means that

$$\exists \bar{t} \in [0, T] \,, \exists i, j \in \{1, 2, 3\}, \; i \ne j \;, q_i(\bar{t}) = q_j(\bar{t}) \tag{4.3}$$

We say that $\bar{t}$ is a time of *double collision* if $q_k(\bar{t}) \neq q_i(\bar{t}) = q_j(\bar{t})$, for $k \neq i$, $k \neq j$, while $\bar{t}$ is a time of *triple collision* if $q_1(\bar{t}) = q_2(\bar{t}) = q_3(\bar{t})$.

We first prove the following result.

Proposition 4.1. Assume that (H1), (H2) and (H3) hold, and let Q be such that $I(Q) = \inf_{X \in H} I(X)$. Then Q can not have any double collisions.

Proof. We assume by contradiction that there exists a time $\bar{t}$ of double collision, say $q_2(\bar{t}) = q_3(\bar{t})$ and $q_1(\bar{t}) \neq q_2(\bar{t})$.

We observe that q_3 is a minimizer of the functional

$$\begin{aligned} I_{q_1,q_2}(q_3) =& \frac{1}{2}\int_0^T |\dot{q}_3|^2 dt - \int_0^T [F_{13}(q_1 - q_3) - F_{23}(q_2 - q_3)]dt \\ &+ \frac{1}{2}\int_0^T \left(|\dot{q}_1|^2 + |\dot{q}_2|^2\right) dt - \int_0^T F_{12}(q_1 - q_2)dt \end{aligned} \tag{4.4}$$

over

$$\tilde{H} = \{x_3 \in H^1_{loc}(\mathbb{R};\mathbb{R}^3) \ / \ x_3(t + \frac{T}{2}) = -x_3(t), \forall t \ \in \mathbb{R}\}.$$

We can then apply the results of Theorem 2.1 to prove that $\min_{\tilde{H}} I_{x_1,x_2}(x_3)$ is attained by a function q_3 which does not cross neither q_1 nor q_2 for all $\bar{t}$ such that $q_1(\bar{t}) \neq q_2(\bar{t})$. Since this argument works for every instant of double collision, we can conclude that the minimizer Q can at most have triple collisions. ∎

The argument to be used in proving that Q is free of triple collisions will take the remaining part of this section. We begin by introducing some notations.

For a generic orbit $X = (x_1, x_2, x_3) \in H$, we set

$$T = T(\dot{X}) = \frac{1}{2}\sum_{i=1}^{3} m_i|\dot{x}_i|^2, \qquad G = G(X) = \sum_{i=1}^{3} m_i|x_i|^2$$

and

$$F(X) = \sum_{\substack{i,j=1\\ i\neq j}}^{3} F_{ij}(x_i - x_j).$$

The following Proposition makes precise the kind of orbits we are dealing with.

Proposition 4.2. Suppose $Q \in H$ is such that $I(Q) = \inf_{X \in H} I(X)$; then

i) Q is free of double collisions.

ii) Q has at most two triple collisions, both at zero.

iii) $\exists\, E \in \mathbb{R}$ such that $T(\dot{Q}) + F(Q) = E$ in every instant which is not of triple collision (conservation of energy).

iv) $\sum_{i=1}^{3} m_i q_i = 0$ for every $t \in [0,T]$ and $\sum_{i=1}^{3} m_i \dot{q}_i = 0, \quad \forall t \in [0,T]$ not of triple collision (conservation of center of mass and momentum).

v) Suppose that there are no collisions for all t in some interval $]a,b[$ and that

$$|q_i(t) - q_j(t)| \leq \varepsilon \qquad \forall t \in]a,b[\quad \forall\, 1 \leq i \neq j \leq 3,$$

where ε is the constant introduced in (H2). Then there exists $B_0 \in \mathbb{R}^3$ such that

$$B(t) = \sum_{i=1}^{3} m_i \dot{q}_i(t) \times q_i(t) = B_0 \qquad \forall t \in]a,b[.$$

(Conservation of angular momentum for small distances).

Proof. We set $I(Q) = \int_0^T L(Q,\dot{Q})dt$, and $I_{[a,b]}(Q) = \int_a^b L(Q,\dot{Q})dt$. We recall that Q satisfies the equations of motion associated to I:

$$-m_i \ddot{q}_i = \sum_{\substack{j=1 \\ j \neq i}}^{3} \nabla F_{ij}(q_i - q_j)$$

in every instant which is not of double or triple collision.

Proof of i). See Proposition 4.1.

Proof of ii). Suppose that Q has more than two triple collisions and that $t = 0$ and $t = T_1$ are instants of triple collision; it is not restrictive to assume moreover that

$$I_{[0,T_1]}(Q) \leq \frac{2T_1}{T} I(Q).$$

Setting

$$V(t) = Q(t) - \frac{\sum_{i=1}^{3} m_i q_i(t)}{\sum_{i=1}^{3} m_i},$$

we see that V has only triple collisions at zero and $I(V) \leq I(Q)$ (strictly if the center of mass of Q is not zero). Let $U(t) = V(\frac{2T_1}{T}t)$ for $t \in [0, \frac{T}{2}]$ (and symmetrically in $[\frac{T}{2}, T]$); then $U \in H$ and, because of (H5),

$$I(U) = \frac{T}{2T_1} I_{[0,T_1]}(V) + (\frac{2T_1}{T} - \frac{T}{2T_1}) \sum_{i=1}^{3} m_i \int_0^{T_1} |\dot{v}_i|^2 dt \leq \frac{T}{2T_1} I_{[0,T_1]}(V) \leq I(Q),$$

and the strict inequality holds when Q has more than two triple collisions. This is a contradiction since Q minimizes I.

Proof of iii). Since F is a function of the q_i's only, it follows immediately that if Q is a solution of the equations of motion, then there $T + F$ is constant of each connected component of $[0, T] \setminus \mathcal{C}(q_1, q_2, q_3)$. By ii) this set has only two components and therefore, by the symmetry of Q the constants are the same:

$$T + F = E. \tag{4.5}$$

Proof of iv). By summing the three equations of motion one obtains

$$-\sum_{i=1}^{3} m_i \ddot{q}_i = \sum_{\substack{i,j=1 \\ i \neq j}}^{3} \nabla F_{ij}(q_i - q_j);$$

but since, by (S1),

$$\nabla F_{ij}(q_i - q_j) = -\nabla F_{ji}(q_j - q_i),$$

we have $\sum_{i=1}^{3} m_i \ddot{q}_i = 0$, which implies $\sum_{i=1}^{3} m_i q_i = ct + d$, for some constants $c, d \in \mathbb{R}^3$. By periodicity, $c = 0$, and by the symmetry condition, $d = 0$, which means that both the center of mass $\sum_{i=1}^{3} m_i q_i$ and the linear momentum $\sum_{i=1}^{3} m_i \dot{q}_i$ are identically zero throughout the motion.

Proof of v). Indeed, by virtue of (H2), we have

$$-m_i \ddot{q}_i \times q_i = \sum_{\substack{j=1 \\ j \neq i}}^{3} \alpha m_i m_j \frac{q_j \times q_i}{|q_i - q_j|^{\alpha+2}} + \phi'_{ij}(|q_i - q_j|) \frac{q_i \times q_j}{|q_i - q_j|} \qquad \forall t \in]a, b[.$$

Summing over i one sees that the j–th term of the i–th equation cancels the i–th term of the j–th equation. Therefore

$$\dot{B}(t) = \frac{d}{dt}\sum_{i=1}^{3} m_i\dot{q}_i \times q_i = \sum_{i=1}^{3} m_i\ddot{q}_i \times q_i = 0 \qquad \forall t \in]a,b[,$$

and B is constant in $]a,b[$. ∎

Now we prove some preliminary lemmas; here and below we denote by $\frac{\partial F}{\partial q_i}$ the gradient of F with respect to the vector q_i.

Lemma 4.3. For every $\gamma > 0$, there exists $\sigma_\gamma > 0$ such that, if $|q_i| \leq \sigma_\gamma$, $i = 1,2,3$, then

$$Q \cdot \nabla F(Q) = \sum_{i=1}^{3} q_i \cdot \frac{\partial F}{\partial q_i}(Q) \leq -(\alpha+\gamma)F(Q). \tag{4.6}$$

Proof. We remark that equality holds in (4.6), with $\gamma = 0$, if F is homogeneous of degree $-\alpha$; this is the case, for example, of the classical three body problem ($\alpha = 1$).

In our case it is enough to compute

$$\sum_{i=1}^{3} q_i \cdot \frac{\partial F}{\partial q_i}(Q) = \sum_{\substack{i,j=1\\ i\neq j}}^{3} -\alpha\frac{m_i m_j}{|q_i - q_j|^\alpha} + \phi'_{ij}(|q_i - q_j|)|q_i - q_j|,$$

and to see that, for σ_γ small one has, by (H3),

$$\begin{aligned}&\sum_{i=1}^{3} q_i \frac{\partial F}{\partial q_i}(Q) + (\alpha+\gamma)F(Q) = \\ &= \sum_{\substack{i,j=1\\ i\neq j}}^{3} \frac{-1}{|q_i - q_j|^\alpha}\{\gamma m_i m_j + \phi'_{ij}(|q_i - q_j|)|q_i - q_j|^{\alpha+1} + \\ &+ (\alpha+\gamma)\phi_{ij}(|q_i - q_j|)|q_i - q_j|^\alpha\} \leq 0.\end{aligned}$$ ∎

Remark 4.4. The following result is crucial for the understanding of the geometrical behavior of the solution near a point of triple collision. The result, in the case of the plain Newton potential, was already known to Weierstrass, and it was first proved by Sundman in 1913. The argument of the proof here is derived from [40].

Lemma 4.5. (Sundman). Let $t=0$ be an isolated triple collision instant for Q. Then there exists $\bar{t}>0$ such that

$$B(t)=0 \qquad \forall t\in]-\bar{t},\bar{t}[\setminus\{0\}. \tag{4.7}$$

Proof. By the conservation of center of mass, the triple collision can occur only at zero. Moreover we can assume that Q is regular of class C^2 in $(-\delta,\delta)\setminus\{0\}$, for some $\delta>0$.

We start by the algebraic estimate

$$\Big|\sum_{i=1}^{3} a_k\times b_k\Big|^2=\sum_{i=1}^{3}|a_k\times b_k|^2+2\sum_{i<j}(a_i\times b_i)\cdot(a_j\times b_j)\le 3\sum_{i=1}^{3}|a_k\times b_k|^2$$
$$\le 3\sum_{i=1}^{3}|a_k|^2|b_k|^2\le 3\Big(\sum_{i=1}^{3}|a_k|^2\Big)\Big(\sum_{i=1}^{3}|b_k|^2\Big),$$

which holds for every $a_k,b_k\in\mathbb{R}^3$, $\quad k=1,2,3$. Setting $a_k=\sqrt{m_k}q_k$ and $b_k=\sqrt{m_k}\dot{q}_k$, the above inequality shows that

$$|B|^2=\Big|\sum_{i=1}^{3} m_kq_k\times\dot{q}_k\Big|^2\le 3\Big(\sum_{i=1}^{3}m_k|q_k|^2\Big)\Big(\sum_{i=1}^{3}m_k|\dot{q}_k|^2\Big)=3G(Q)\cdot 2T(\dot{Q}). \tag{4.8}$$

Let $\eta=\frac{|B|^2}{3}$; then from (4.8) we have

$$G(Q)\cdot 2T(\dot{Q})\ge\eta. \tag{4.9}$$

We want to show that $\eta=0$ in $]-\bar{t},\bar{t}[\setminus\{0\}$, if $\bar{t}$ is small enough. Therefore we suppose from now on that $\bar{t}$ has been chosen so small that all the previous local results hold.

Differentiating twice G yields

$$\frac{1}{2}G''=\sum_{i=1}^{3}m_i|\dot{q}_i|^2+\sum_{i=1}^{3}m_iq_i\cdot\ddot{q}_i=2T(\dot{Q})-\sum_{i=1}^{3}q_i\frac{\partial F}{\partial q_i};$$

hence, by Lemma (4.3) and by the conservation of the energy,

$$\frac{1}{2}G''\ge 2T(\dot{Q})+(\alpha+\gamma)F(Q)=(2-\alpha-\gamma)T(\dot{Q})+(\alpha+\gamma)E. \tag{4.10}$$

Multiplying this inequality by G and using (4.9) we obtain

$$\frac{1}{2}GG'' \geq (2-\alpha-\gamma)TG + (\alpha+\gamma)EG \geq \frac{2-\alpha-\gamma}{2}\eta + (\alpha+\gamma)EG. \tag{4.11}$$

Let us set $\alpha^* = \alpha + \gamma < 2$.

We claim that for $t < 0$ small enough we have $G' < 0$, and the opposite inequality if $t > 0$. Assuming for the moment that this is true, we work from now on in $]\bar{t}, 0[$. It will be clear that the same argument also works in $]0, \bar{t}[$. Then we can multiply (4.11) by $-4\frac{G'}{G} > 0$ to obtain

$$-2G'G'' \geq -2(2-\alpha^*)\eta\frac{G'}{G} - 4\alpha^* EG',$$

that is,

$$-\frac{d}{dt}|G'|^2 \geq -2(2-\alpha^*)\eta\frac{d}{dt}\log G - 4\alpha^* EG'. \tag{4.12}$$

We now integrate this relation over an interval $[s, t]$, with $s < t < 0$:

$$-G'(t)^2 + G'(s)^2 \geq -2(2-\alpha^*)\eta\log\frac{G(t)}{G(s)} - 4\alpha^* E(G(t) - G(s)), \tag{4.13}$$

and we finally obtain

$$G'(s)^2 \geq 2(2-\alpha^*)\eta\log\frac{G(s)}{G(t)} - 4\alpha^*|E|G(s). \tag{4.14}$$

Now, if we let t tend to zero with s fixed, we see that the first term in the right–hand–side of (4.14) tends to $+\infty$. Clearly this is possible only if $\eta = 0$.

We only have to show that the claim holds true: $G'(t) < 0$ if $t < 0$ is small. To this aim let us consider again (4.10): using the conservation of the energy we can write it as

$$\frac{1}{2}G'' \geq 2E - (2-\alpha-\gamma)F; \tag{4.15}$$

recalling that in $t = 0$ a triple collision occurs, we see from (4.15) that $G''(t) \to +\infty$ as $t \to 0$. Therefore G'' is strictly positive around $t = 0$ and hence G' can change sign at most once in any neighborhood of zero. Remark also that by the equations of motion it follows $G'' \in L^1$, and therefore $G \in C^1$. Since G has a local minimum at $t = 0$ we conclude that $G'(0) = 0$, which together with the above remarks yields $G'(t) < 0$ if $t < 0$ is small enough. The claim is thus proved and the proof is complete. ∎

In the next proposition the crucial hypothesis is that we are working with three bodies in $\mathbb{R}^3$.

Lemma 4.6. Suppose that no collisions take place in $]a,b[$, and also that $B(t) = 0 \quad \forall t \in]a,b[$. Then the components of Q,(q_1, q_2, q_3) lie on the same fixed plane of $\mathbb{R}^3 \quad \forall t \in]a,b[$.

Proof. ([40]). Let (x,y,z) be an orthonormal reference frame in $\mathbb{R}^3$ and denote by (x_k, y_k, z_k), $k = 1,2,3$, the coordinates of the k-th body q_k. Since $|B|^2$ and the equations of motions are invariant under rotations and since the center of mass is assumed to be at zero, we may also assume that at some time $t = \tau$, the three particles lie in the plane $z = 0$. By hypothesis we have that at time $t = \tau$,

$$\sum_{k=1}^{3} m_k y_k \dot{z}_k = 0, \qquad \sum_{k=1}^{3} m_k x_k \dot{z}_k = 0, \tag{4.16}$$

and since the center of mass is at rest, we also have

$$\sum_{k=1}^{3} m_k \dot{z}_k = 0. \tag{4.17}$$

If we look at (4.16), (4.17) as a linear homogeneous systems in the three unknowns $m_k \dot{z}_k$ at $t = \tau$, we see that either $\dot{z}_k = 0$, $k = 1,2,3$, or

$$\det \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3, \\ 1 & 1 & 1 \end{pmatrix} = 0.$$

In the first case, the initial directions of motion for the three bodies lie in the plane $z = 0$, and by uniqueness, the bodies remain in this plane for all $t \in]a,b[$. In the second case the three bodies lie initially on a line, and the coordinate axes can still be rotated so that $\dot{z}_3$ vanishes at $t = \tau$. If we then exclude the case already treated, the above equations give

$$x_1 = x_2 \qquad y_1 = y_2 \qquad \text{at } t = \tau,$$

implying that two of the bodies (q_1 and q_2) collide ad $t = \tau$, contrary to our assumption. ∎

From the previous Lemma it follows thus that before and after a triple collision the motion takes place on fixed planes. One can ask if these planes coincide, that is if the motion is planar in a neighborhood of a collision. The answer is in general negative. However we show in next proposition that when one deals with minimizers of the action, then it can be assumed without loss of generality that the planes do coincide. This is one of the cases in which the variational approach is determinant.

Proposition 4.7. Suppose the hypotheses of Theorem 3.2 hold and let $Q \in H^1$ be such that $I(Q) = \inf_{X \in \Lambda} I(X)$. Assume that an isolated triple collision takes place at zero at time $t = 0$. Then there exists $\tilde{Q} \in H^1$ such that $I(\tilde{Q}) = I(Q)$ and $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ lie on the same fixed plane of $\mathbb{R}^3$ for all $|t| < \delta$, with $\delta > 0$ small enough.

Proof. By Lemma 4.5 we can assume $B(t) = 0$ in $]-\delta, 0[\cup]0, \delta[$ for some $\delta > 0$. By Lemma 4.6 Q is planar both in $]-\delta, 0[$ and in $]0, \delta[$, but the two planes need not coincide.

Set $I(Q) = \int_0^T L(Q, \dot{Q})dt$; it is not restrictive to assume that

$$\int_0^{\frac{T}{4}} L(Q, \dot{Q})dt \leq \int_{\frac{T}{4}}^{\frac{T}{2}} L(Q, \dot{Q})dt,$$

so that by the symmetry of Q it follows

$$\int_{\frac{T}{2}}^{\frac{3T}{4}} L(Q, \dot{Q})dt \leq \int_{\frac{3T}{4}}^{T} L(Q, \dot{Q})dt.$$

Now consider the orbit defined by

$$\tilde{Q}(t) = \begin{cases} Q(t) & \text{if } t \in [0, \frac{T}{4}] \cup [\frac{T}{2}, \frac{3T}{4}] \\ -Q(-t) & \text{otherwise in } [0, T] \end{cases}$$

it is immediate to check that $\tilde{Q} \in H^1, I(\tilde{Q}) \leq I(Q)$, and $\tilde{Q}(-s) = -\tilde{Q}(s)$ for all s small enough, which means that $\tilde{Q}$ minimizes I and enters and leaves the collision on the same plane. ∎

Having gathered the estimates we need, we can now prove the main result of this section. We recall that if Q minimizes I, then it is free of double collisions. We now are in a position to show that Q can not have any triple collision either, thereby completing the proof of Theorem 3.2.

Proposition 4.8. Assume the hypotheses of Theorem 3.2 hold and let Q be such that $I(X) = \inf_{X\in\Lambda} I(X)$. Then Q can not have any triple collision.

Proof. Let us suppose by contradiction that Q has an isolated triple collision at time $t = 0$. By Lemma 4.8 we can replace Q by another minimizer (still denoted by Q) with the further property that Q is planar in $[-\delta, \delta]$, for some $\delta > 0$. Therefore, there exists $w \in S^2$ such that

$$q_j(t) \cdot w = \dot{q}_j(t) \cdot w = 0 \qquad \forall j = 1,2,3 \quad \forall t \in [-\delta, \delta].$$

Let $\overline{w}$ be the piecewise linear function defined by

$$\overline{w}(t) = \begin{cases} 0 & \text{if } t \notin [-\delta, \delta] \\ \mu w & \text{if } t \in [-\frac{\delta}{2}, \frac{\delta}{2}] \end{cases}$$

where $\mu \in \mathbb{R}$ will be conveniently chosen later; extend $\overline{w}$ to the whole of $\mathbb{R}$ by periodicity and define

$$\overline{\overline{w}}(t) = \overline{w}(t) - \overline{w}(t + \frac{T}{2}).$$

Finally let $V \in H$ be the function

$$V(t) = (v_1(t), v_2(t), v_3(t)) = (\overline{\overline{w}}(t), 0, -\overline{\overline{w}}(t));$$

it is clear that $v_j(t)+q_j(t) \neq v_k(t)+q_k(t)$, $\forall j \neq k$, $\forall t \in \mathbb{R}$ and moreover that $v_k(t)\cdot q_j(t) = 0$, $\forall j, k$, $\forall t \in \mathbb{R}$.

We are going to show that $I(Q + V) < I(Q)$, for μ small enough, contradicting the fact that $I(Q)$ is the infimum of I over Λ.

We start by remarking that by symmetry and by the choice of V,

$$I(Q+V) - I(Q) = 2\{\sum_{i=1}^{3} \frac{m_i}{2} \int_{-\delta}^{\delta} [|\dot{q}_i + \dot{v}_i|^2 - |\dot{q}_i|^2] dt -$$

$$-\frac{1}{2} \sum_{\substack{i,j=1 \\ j\neq i}}^{3} \int_{-\delta}^{\delta} [F_{ij}(q_i + v_i - q_j - v_j) - F_{ij}(q_i - q_j)] dt\}.$$

The estimate of the kinetic part leads to

$$\sum_{i=1}^{3} m_i \int_{-\delta}^{\delta} [|\dot{q}_i + \dot{v}_i|^2 - |\dot{q}_i|^2] dt = \sum_{i=1}^{3} m_i \int_{-\delta}^{\delta} |\dot{v}_i|^2 dt \le C_1 \mu^2, \tag{4.18}$$

where $C_1 > 0$ is independent of μ.

Therefore

$$\begin{aligned} &I(Q+V) - I(Q) \le \\ &\le - \sum_{\substack{i,j=1\\ j\neq i}}^{3} \int_{-\delta}^{\delta} [F_{ij}(q_i + v_i - q_j - v_j) - F_{ij}(q_i - q_j)] dt + C_1 \mu^2. \end{aligned} \tag{4.19}$$

We want to show that if μ is taken sufficiently small, then this quantity is strictly negative. To do this, we can certainly assume that δ and μ are so small that the potential takes the form given by (H2) throughout the interval $[-\delta, \delta]$. In other words, if we set

$$\begin{aligned} T_{ij} = \int_{-\delta}^{\delta} [&\frac{m_i m_j}{|q_i + v_i - q_j - v_j|^\alpha} - \phi_{ij}(|q_i + v_i - q_j - v_j|) - \\ &- \frac{m_i m_j}{|q_i - q - j|^\alpha} + \phi_{ij}(|q_i - q_j|)] \, dt, \end{aligned}$$

then (4.19) becomes

$$I(Q+V) - I(Q) \le \sum_{\substack{i,j=1\\ j\neq i}}^{3} T_{ij} + C_1 \mu^2. \tag{4.20}$$

Now we estimate the generic term T_{ij} of (4.20):

$$\begin{aligned} T_{ij} &= \int_{-\delta}^{\delta} \int_0^1 \frac{d}{d\lambda} \Big(\frac{m_i m_j}{|q_i - q_j + \lambda v_i - \lambda v_j|^\alpha} - \phi_{ij}(|q_i - q_j + \lambda v_i - \lambda v_j|) \Big) d\lambda dt \\ &= \int_{-\delta}^{\delta} \int_0^1 [- \frac{\lambda |v_i - v_j|^2}{|q_i - q_j + \lambda v_i - \lambda v_j|^{\alpha+2}} \cdot \\ &\cdot \big(\alpha m_i m_j + \phi'_{ij}(|q_i - q_j + \lambda v_i - \lambda v_j|) |q_i - q_j + \lambda v_i - \lambda v_j|^{\alpha+1} \big)] d\lambda dt. \end{aligned}$$

Now for μ and δ small enough, the quantity in the round brackets is larger or equal than $\frac{1}{2} \alpha m_i m_j$, because of (H3). Hence

$$\begin{aligned} T_{ij} &\le - \frac{\alpha m_i m_j}{2} \int_{-\delta}^{\delta} \int_0^1 \frac{\lambda |v_i - v_j|^2}{|q_i - q_j + \lambda v_i - \lambda v_j|^{\alpha+2}} d\lambda dt \\ &\le - \frac{\alpha m_i m_j}{2} \int_{\frac{-\delta}{2}}^{\frac{\delta}{2}} \int_0^1 \frac{\lambda \mu^2}{|q_i - q_j + \lambda v_i - \lambda v_j|^{\alpha+2}} d\lambda dt \\ &\le - \frac{\alpha m_i m_j}{4} \mu^2 \int_{\frac{-\delta}{2}}^{\frac{\delta}{2}} \int_{\frac{1}{2}}^1 \frac{1}{|q_i - q_j + \lambda v_i - \lambda v_j|^{\alpha+2}} d\lambda dt. \end{aligned}$$

Now since

$$|q_i(t)| \le \int_0^t |\dot{q}_i(t)| dt \le C_2 t^{\frac{1}{2}},$$

for some constant C_2 independent of μ, we have that $|q_i - q_j| \le \mu$, for all t such that $2C_2 t^{\frac{1}{2}} \le \mu$.

Restricting the interval of integration to $[-\sigma, \sigma] := [-\left(\frac{\mu}{2C_2}\right)^2, \left(\frac{\mu}{2C_2}\right)^2]$, we obtain

$$T_{ij} \le -\frac{\alpha m_i m_j}{4}\mu^2 \int_{-\sigma}^{\sigma} \int_{\frac{1}{2}}^{1} \frac{1}{3^{\alpha+2}\mu^{\alpha+2}} d\lambda dt \le -C_3 \mu^{2-\alpha},$$

where C_3 is a positive constant independent of μ.

Summing the T_{ij}'s and recalling (4.18) we obtain

$$I(Q+V) - I(Q) \le C_1 \mu^2 - C_4 \mu^{2-\alpha}$$

and therefore $I(Q+V) - I(Q) < 0$, when μ is small enough, contradicting the fact that $I(Q) = \inf_{X \in \Lambda} I(X)$. Theorem 3.2 is proved. ∎

REFERENCES

1. A. Ambrosetti and U. Bessi, *Bifurcation of periodic solutions for Keplerian problems*, Rend. Mat. Acc. Lincei S.9 **2** (1991), 11–15.
2. A. Ambrosetti and V. Coti Zelati, *Critical points with lack of compactness and singular dynamical systems*, Ann. Mat. Pura Appl. (4) **149** (1987), 237–259.
3. A. Ambrosetti and V. Coti Zelati, *Periodic solutions of singular dynamical systems*, in "Periodic solutions of Hamiltonian systems and related topics," Nato ASI series, Reidel, 1987, pp. 1–10.
4. A. Ambrosetti and V. Coti Zelati, *Perturbations of Hamiltonian systems with Keplerian potentials*, Math. Zeit. **201** (1989), 227–242.
5. A. Ambrosetti and V. Coti Zelati, *Closed orbits of fixed energy for singular Hamiltonian systems*, Arch. Rational Mech. Anal. **112** (1990), 339–362.
6. A. Ambrosetti and V. Coti Zelati, *Closed orbits of fixed energy for a class of N-body problems*, Ann. IHP Anal. non Lin. (to appear).
7. A. Ambrosetti and V. Coti Zelati, *Noncollision orbits for a class of Keplerian-like potentials*, Ann. IHP, Anal. non lin. **5** (1988), 287–295.
8. V. I. Arnol'd, "Mathematical methods of classical mechanics," Springer–Verlag, 1978.
9. A. Bahri and P.L. Lions, *Morse index of some min-max critical points. I. Application to multiplicity results*, Comm. Pure Appl. Math. **41** (1988), 1027–1037.
10. A. Bahri and P.H. Rabinowitz, *A minimax method for a class of Hamiltonian systems with singular potential*, Jour. Funct. Anal. **82** (1989), 412–428.
11. A. Bahri and P.H. Rabinowitz, *Solutions of the three-body problem via critical points at infinity*, Preprint.
12. V. Benci and F. Giannoni, *A new proof of the existence of a brake orbit*, in "Advanced topics in the theory of dynamical systems (Trento 1987)," Notes Rep. Math. Sci. Energ. 6, Academic Press, 1990.
13. V. Benci and F. Giannoni, *Periodic solutions of prescribed energy for a class of Hamil-*

tonian systems with singular potentials, J. Diff. Eq. **82** (1989), 60–70.

14. U. Bessi, *Multiple closed orbits for singular conservative systems via geodesics theory*, Preprint SNS, 1990.
15. U. Bessi, *Multiple closed orbits of fixed energy for gravitational potentials*, J. Diff. Eq. (to appear).
16. U. Bessi and V. Coti Zelati, *Symmetries and noncollision closed orbits for planar N-body type problems*, Nonlin Anal. TMA **16** (1991), 587–598.
17. S.V. Bolotin and V.V. Kozlov, *Librations in systems with many degrees of freedom*, J. Appl. Math. Mech. **42** (1978), 256–261.
18. A. Capozzi, S. Solimini and S. Terracini, *On a class of dynamical systems with singular potentials*, Nonlin. Anal. TMA **16** (1991), 805–815.
19. V. Coti Zelati, *Periodic solutions for a class of planar, singular dynamical systems*, J. Math. Pures et Appl. **68** (1989), 109–119.
20. V. Coti Zelati, *A class of periodic solutions of the N-body problem*, Cel. Mech. and Dyn. Astr. **46** (1989), 177–186.
21. V. Coti Zelati, *Periodic solutions for N-body type problems*, Ann. IHP, Anal. Non Lin. **7** (1990), 477–492.
22. V. Coti Zelati and E. Serra, *Collision and non-collision solutions for a class of Keplerian-like dynamical systems*, preprint SISSA, 1990.
23. V. Coti Zelati and E. Serra, *Multiple brake orbits for some classes of Hamiltonian systems*, Preprint SISSA, 1991.
24. M. Degiovanni and F. Giannoni, *Dynamical systems with Newtonian type potentials*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. **4** (1989), 467–494.
25. M. Degiovanni, F. Giannoni and A. Marino, *Periodic solutions of dynamical systems with Newtonian type potentials*, Atti Accad. Naz. Lincei, Rend. Cl. Sc. MFN **LXXXI** (1987), 271–278.
26. E. Fadell and S. Husseini, *A note on the category of the free loop space*, Proc. AMS **107** (1989), 527–536.
27. H. Gluck and W. Ziller, *Existence of periodic motions of conservative systems*, Seminar on minimal submanifolds, Princeton University Press, 1986, pp. 155–161.

28. W. Gordon, *A minimizing property of Keplerian orbits*, Amer. J. Math. **99** (1975), 961–971.

29. W. Gordon, *Conservative dynamical systems involving Strong Forces*, Trans. AMS **204** (1975), 113–135.

30. C. Greco, *Periodic solutions of a class of singular Hamiltonian systems*, Nonlin. Anal. TMA **12** (1988), 259–269.

31. E. W. C. van Groesen, *Analytical mini-max methods for Hamiltonian brake orbits of prescribed energy*, J. Math. Anal. Appl. **132** (1988), 1–12.

32. K. Hayashi, *Periodic solutions of classical Hamiltonian systems: a survey*, SIAM Jour. Math. Anal. **13** (1982), 343–352.

33. H. Hofer and E. Zehnder, *Periodic solutions on hypersurfaces and a result by C. Viterbo*, Inv. Math. **90** (1987), 1–9.

34. P. Majer, *Ljusternik-Schnirelman theory without Palais-Smale condition and singular dynamical systems*, Ann. IHP Anal. non Lin. (to appear).

35. K. R. Meyer, *Periodic solutions of the N-body problem*, J. Diff. Eq. **39** (1981), 2–38.

36. J. Moser, *Regularization of Kepler's problem and the averaging method on a manifold*, Comm. Pure Appl. Math. **23** (1970), 609–636.

37. P. H. Rabinowitz, *On a theorem of Hofer and Zehnder*, Periodic solutions of Hamiltonian systems and related topics, Nato ASI series, Reidel, 1987, pp. 245–253.

38. H. Seifert, *Periodischer Bewegungen Mechanischen Systeme*, Math. Zeit. **51** (1948), 197–216.

39. E. Serra and S. Terracini, *Noncollision solutions to some singular minimization problems with Keplerian-like potentials*, Nonlin. Anal. TMA (to appear).

40. E. Serra and S. Terracini, *Noncollision periodic solutions to some three-body like problems*, Preprint SISSA, 1990.

41. C. L. Siegel and J. K. Moser, *Lectures on celestial mechanics*, GMW 187, Springer Verlag, 1971.

42. S. Solimini, *Morse index estimates in min-max theorems*, Manuscripta Math. **63** (1989), 421–453.

43. K. Tanaka, *Morse index at critical points related to the symmetric Mountain-Pass*

theorem and applications, Comm. PDE **14** (1989), 119–128.

44. K. Tanaka, *Non–collision solutions for a second order singular Hamiltonian system with weak forces*, Preprint, 1991.
45. S. Terracini, *Periodic solutions to dynamical systems with Keplerian type potentials*, Ph.D. Thesis - SISSA, Trieste (1990).
46. S. Terracini, *An homotopical index and multiplicity of periodic solutions to dynamical systems with singular potentials*, J. Diff. Eq. (to appear).
47. S. Terracini, *Second order conservative systems with singular potentials: noncollision periodic solutions to the fixed energy problem*, Preprint, 1991.
48. S. Terracini, *Multiplicity of periodic solutions having a prescribed energy for a class of dynamical systems with singular potentials*, Preprint, 1991.
49. C. Viterbo, *Indice de Morse des points critiques obtenus par minimax*, Ann. IHP, Anal. non lin. **3** (1988), 221–225.
50. A. Weinstein, *Periodic orbits for convex Hamiltonian systems*, Ann. of Math. **108** (1978), 507–518.