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The KP Hierarchies in Matrix Models

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老子：

道可道，非常道；
名可名，非常名。

The principle that can be stated can not be the absolute principle.
The name that can be given can not be the permanent name.

LAO Tze (Ca. 500 B.C.)

公孙龙：

一尺之锤，
日取其半，
萬世不竭。

Take half from a foot–long stick every day;
you will not exhaust it in a billion years.

KUNGSUN Lung(Ca. 300 B.C.)

Chapter 1

Introduction

The aim of the subject is to construct the concepts out of which a comprehensive workable system of theoretical physics can be formulated. The system must be as simple as possible, and yet must lead by deductive reasoning to conclusions that correspond with empirical experience.

A. Einstein

1.1 Fundamental Forces

Since created by God, human being has been always puzzled by the tremendous mystery and the complexity of nature. A great deal of the outstanding thinkers in different countries and different times spent their talents and lifes in observing the environment and pursuing the scientific truth. However, they often face difficult choices. On the one hand, from time to time, there are always new domains of the knowledge and new ways of thinking arising. On the other hand, a formidable faith pushes scientists to simplify all of this knowledge, i.e. to search for some most fundamental assumptions which are as simple as possible, and from which we can deduce all known results. This leads thinkers to ask the eternal questions

What are the fundamental constituents of matter?

What are their fundamental laws?

The first answer to the questions could date back to the ancient Greek. The starting point of the Greeks in fact was gravity, they proposed a naive model which said that: *Everything* *is made up of the idealized point particles. If body falls down to the earth, the more the body weighs, the faster it is.*

This intuitive concept was deeply rooted in human's brain for a quite long time till the 17th century. Galilieo introduced the scientific method and together with Newton gave birth to *Physics*. Due to their contributions, as well as to their followers like, Laplace and Faraday, Maxwell and Boltzmann, etc. it was understood that

1. Matter is immersed in an absolute space and absolute time;
2. Matter has two distinct fundamental ingredients: massive point particles and waves;
3. There are 3 different forces in nature: gravitational force, electronic force and magnetic force;
4. The electric and magnetic forces travel at the speed of light, but travelling of the gravitational interactions takes place at infinite speed.

The first unified theory was due to Maxwell, his beatiful equation unifies the electronic and magnetic forces into a single interaction, that is, "*electromagnetic force*". Apart from some exceptions(like black body radiation), in the end of the last century, physicists could proundly claim that "give me the initial conditions, I can tell you all the story of the world in the future."

However such exciting time quickly ended with the birth of two fundamental theories of the physics, "*quantum theory*" and "*general relativity*". These are so profoundly revolutionary that in the history of physical science the twentieth century stands out as a tremendously particular period. The exploration of nature made the greatest progress covering from the widest expanses of cosmos to innermost recesses of matter. And physicists had to overthrow almost all our utterly unquestionable notions: the basic attitudes toward time and space, as well as the concepts about point particle and wave. Space and time are no longer "*empty box*", they united together as space–time, and could be identified with matter. Point particle and wave are not completely distinct any more, since

any matter is made up of the elementary particles which possess wave–particle duality.

However, not long later, physicists faced perplexity once again. The family of the discovered *elementary particles* acquired more and more members. Their interactions can be classified into four different types: strong, weak and electromagnetic as well as gravitational forces. It is unbelievable that all of these "elementary particles" are really *elementary*. A solution of this problem was with the help of the *gauge symmetry*. The local symmetry in Yang–Mills theories together with renormalization scheme enables us to banish the infinities of the quantum field theory and to unify the laws of the elementary particle physics into an elegant and comprehensive framework. In the late 1960s, electromagnetic force and weak forces united together as electroweak force in the Wenberg–Salam model. Later on, it was found that all the particle physics in fact is compatible with the minimal theory of $SU(3)\otimes SU(2)\otimes U(1)$. In the past decades, the theoretical physicists proposed many candidate models for the grand unified theories(GUTs), which are supposed to unify strong and electroweak interactions.

1.2 Why Strings?

The next step is to unify all the known forces. Our dream is to look for a "*final unified theory*". This theory must include gravity, give a unified description of all physics and predict a multitude of new fascinating phenomena. Since GUTs seem to be the unified theories for the strong and the electroweak interactions in the framework of quantum Yang–Mills theories, our task now is to combine quantum field theory and the general relativity together

$$\left.\begin{array}{l}\text{Quantum field theory}\\ \\ \text{General relativity}\end{array}\right\}\ \text{Unified field theory}$$

This is one of the greatest scientific challenges of the present century. Although much efforts have been made in the past six decades, such a final theory has not shown up yet. It seems that the quantum field theory and general relativity are totally incompatible because of the non–renormalizability of quantum gravity. Physicists are at a crossroad: we may accept that God has two minds, each working independently of the other in its own particular domain, otherwise we should abandon one or more of our cherished assumptions about our universe. Out of question, physicists prefer the latter way. Over the years, several proposals have been made. Among them, Superstring theories[1] are undoubtedly the most promising candidates, which only abandon the concept of the idealized point particles. With this limited amount of damage, supersting theories unify the various forces and particles in the same way as a violin string provides a unifying description of the musical tones. The string is the fundamental constituent, while the particles of nature are "musical tones". Exciting aspects of the superstring theory are

1. The theory is anomaly free, i.e. it is consistent.
2. There are no infinities in string theory.
3. There are very few parameters, i.e. it is very "simple".
4. The theory includes GUTs, super–Yang–Mills, supergravity, and Kaluza–Klein theories as "reductions".

1.3 Why 2–Dimensional Quantum Gravity?

Although superstring theory seems to be quite successful, as it provides a consistent renormalizable theory of quantum gravity and cancellations of anomalies and divergences, there are still some shortcomings in the model

1. There is as yet no experimental program for finding even indirect manifestations of the underlying string degrees of freedom in nature.

2. There is hardly a contact with conventional particle physics phenomenology. There are apparantly *thousands* of ways to break down the theory to low energies. This means that there are too many vacua, the string theory could not answer which one is correct.

3. The superstring theory describes the story in a 10–dimensional world, no one really knows how to break it down to 4–dimensions.

Among various objections to the model, the ones listed above are the most important. On the one hand, in order to choose the correct vacuum we must know the *nonperturbative properties* of the string theories. On the other hand, the last objection above simply states the inability to calculate dimensional breaking, since to every order in the perturbation theory, the dimension of space–time is stable. Thus, in order to have the theory spontaneously curl up into 4– and 6–dimensional universes, we must appeal to *nonperturbative effects*(or dynamical effects) too. There may be another way to solve this problem. We may consider the lower dimensional string theories from the beginning, this is called *non-critical string theories*. If we do so, we must leave off the critical dimensions. For the critical string theories, matter and gravity completely decouple. This is a very important property, since it means that we can deal with matter and gravity separately. However, this is no longer true for the non–critical string theories. Once we leave the critical dimensions, things are dramatically changed. Due to the anomaly cancellation, gravity and matter couple together. Therefore dealing with gravity is an inevitable step in non–critical string. These are the motivations for investigating 2–dimesional quantum gravity

$$\left.\begin{array}{l}\text{Nonperturbative properties}\\ \\ \text{Non}-\text{critical string theories}\end{array}\right\} 2-\text{Dimensional quantum gravity}$$

1.4 Why Matrix Models?

There are three different approaches to attack 2–dimesional quantum gravity . They are *Topological Field Theory* (TFT), *Liouville Field Theory Formalism* and *Matrix Model*. The main idea of TFT is to use the general invariance to calculate some topological invariants. We will briefly review this point in chapter 7. The Liouville field theory formalism in fact goes back to Polyakov's original paper[2], which said that a first quantized string propagating in R^d–spacetime can be most elegantly discribed as a theory of d–free bosons coupled to 2d–quantum gravity. The bosonic matter system has conformal invariance, therefore, the theory can be considered as a certain Conformal Field Theory (CFT) coupled to 2d–quantum gravity. We may use the perturbed CFT to calculate the string Susceptibility and the other critical exponents. But before the blossoming of the matrix model, we could not go further, for example, to calculate the correlation functions, due to the technique difficulty in continuous approach. In chapter 2, we will give a short review on this subject.

For pure 2–dimesional quantum gravity , the partition function can be defined

$$F = \sum_{\text{topologies}} \int \mathcal{D}g \, e^{-S} \, , \qquad (1.4.1)$$

where the action is the area of the Riemann surface. (If there is matter living in 2–dimensional Riemann surfaces, we should include the matter action see next chapter). In continuous approach, it is really difficult to perform the integration over the metric space. However, if we go to discrete case, i.e. discretizing Riemann surfaces, then the calculation is dramatically simplified. For example, if we replace a Riemann surface by a certain triangulation,

$$\text{Riemann surface } \Sigma \Longrightarrow \text{Triangulations}$$

then counting different triangulations of Riemann surface becomes an approximation to computing the integration over the metric space,

$$\sum_{\text{topologies}} \int \mathcal{D}g \quad \rightarrow \quad \sum_{\substack{\text{random} \\ \text{triangulations}}}$$

in other words, the summation over all such random triangulations is thus the discrete analog to the integral over all possible geometries. Fortunately enough, this triangulation of Riemann surface is dual to Feymann graph of the matrix field theory

$$Z = e^F = \int dM e^{-N(\frac{1}{2}\mathrm{Tr}M^2 + g\mathrm{Tr}M^3)} . \qquad (1.4.2)$$

where M is $N \times N$ hermitian matrix. Therefore the integral over geometries in continuous approach becomes the graphic enumeration of the matrix model. Of course, it is much more convenient to deal with matrix models than to investigate the continuous path integral (1.4.1).

1.5 Why Double Scaling Limit?

In matrix model (1.4.2), there are two distinct parameters, one is the size N of the matrix M, the other is the coupling constant g of matrix interaction*. Therefore the free energy of the matrix path integral can be expanded in two different ways

$$\text{Free energy} \begin{cases} \text{Topological expansion} \\ \text{Perturbation expansion} \end{cases}$$

The first expansion is the expansion in powers of N,

$$F(N, g) = \sum N^{2-2h} F_h(g) = N^2 F_0(g) + Z_1(g) + N^{-2} F_2(g) + \cdots \qquad (1.5.3)$$

* Generally, there could be infinite many coupling constants.

where h is the genus. This expansion in fact is a *topological expansion*, i.e. different powers of N correspond to contributions of graphs with different topologies. From eq.(1.5.3), we easily see that the free energy of fixed genus is suppressed by the powers of N^{2-2h}, for example, if we take the large N limit, we are only left with $F_0(g)$. Therefore, the large N limit causes loss of the contributions from the higher genus graphs. Another expansion is the perturbation expansion in powers of the coupling constant g,

$$F_h(g) \sim \sum_n n^{(\gamma_{str}-2)(1-h)-1}(g/g_c)^n \sim (g_c - g)^{(2-\gamma_{str})(1-h)} . \tag{1.5.4}$$

where γ_{str} is some constant. This expansion shows that, when the coupling constant g approaches the value g_c, $F_h(g)$ has a critical behavior. Now we have two different limiting procedures, the large N limit and going to the critical point of the coupling constant. If we properly unify these two limits, we probably could enhance the contributions from the graphs of the higher genus, so we were able to obtain nonperturbative properties. This procedure is known as *double scaling limit*[3]–[5].

$$\left.\begin{array}{l}\text{Large N limit}\\ \\ \text{Approaching critical coupling}\end{array}\right\}\text{Double Scaling Limit}$$

In fact, by means of this significant procedure, one is able to find that the partition function satisfies certain *differential equations.* If we denote

$$F_N(g) \Longrightarrow F(t)$$

where t is the scaling variable, then the specific heat

$$u = \partial^2 F(t)$$

satisfies Painleve I equation

$$\frac{1}{4}u''' + \frac{3}{2}uu' = 1. \tag{1.5.5}$$

This is of great importance. On the one hand, we can avoid calculating the partition function from the path–integral, which is an extremely difficult problem. On the other hand, if we can solve the equation, we can extract the *non–perturbative* properties of the string theory.

1.6 The Integrable Hierarchies

Till now, we only considered 1–matrix model with cubic polynomial interaction, which is known as the *second criticality*†. As we already pointed out, in matrix models there could appear infinite many polynomial interactions. How about the general *polynomial*

†The first critical point is trivial, the matrix potential only contains a Gaussian term.

interactions? In fact, we can perform the same "double scaling limit" analysis as we did before. If we restrict ourselves to the even potential case, we can properly choose polynomial interactions like

$$V_n(M,g) = -N\sum_{k=1}^{n} g_{2k}Tr(M^{2k}), \qquad n = 1,2,\cdots, \tag{1.6.6}$$

with

$$g_{2k} = (-1)^{k-1}\frac{n!(k-1)!}{(n-k)!(2k)!}, \qquad k = 1,2,\cdots,n; \tag{1.6.7}$$

which are called *critical potentials*[6]. In the double scaling limit, for each critical potential, the corresponding partition function of matrix path integral satisfies ONE differential equation. In this way we can obtain infinite many differential equations.

Now let us consider what will happen if the model leaves off critical point a little bit. Obviously it corresponds to a perturbation. The general perturbed potential can be written as

$$V_{\rm gen} = \sum_n t_n V_n, \tag{1.6.8}$$

with a series of infinite many perturbation parameters t. Then the specific heat satifies KdV hierarchical equations

$$\frac{\partial}{\partial t_r}u = [(\partial^2+u)_+^{r+\frac{1}{2}}, \partial^2+u]. \tag{1.6.9}$$

where the subindex "+" means keeping only the differential part of the pseudo-differential operator. Interesting enough, the partition function of 1-matrix model (with even potential) can be identified with the restricted τ-function of KdV hierarchy . The restrictions simply mean that the perturbations should obey the famous *Virasoro constraints*[7][8]. Furthermore, the correlation functions can be very easily calculated. They are nothing but the derivatives of the τ-function with respect to coupling constants(or flow parameters). These are quite essential properties of 1-matrix model , since they establish the intimate relations of matrix models with integrable hierarchy. As a consequence, the matrix model definition of 2-dimesional quantum gravity is completely manageable and solvable.

1.7 The Alternative Approach to Integrable Hierarchies in Matrix Models

Up to now, we have got good understanding on 1-matrix model with even potentials, but we still ignore at least two important points. On the one hand, the potentials considered above are not the most general ones, since they do not contain the odd powers of the polynomial interactions. As we discussed before, pure gravity corresponds to matrix model with cubic potential, so it is hard to be convinced that we can crudely exclude the odd power interactions. On the other hand, 1–matrix model is the simplest case of matrix models. How about multi–matrix models ? Do they correspond to certain integrable hierarchies too? In fact, it is quite difficult to derive their full hierarchical structures. Many people have conjectured that they should be the higher KdV hierarchy [9]–[14], but a systematic analysis is still lacking. Thus our present situation is

$$\begin{cases} 1-\text{matrix model with even potential} \rightarrow \text{KdV hierarchy} \\ 1-\text{matrix model with general potential} \rightarrow \quad ? \\ \text{multi}-\text{matrix models} \rightarrow \quad ? \end{cases}$$

Before running into the detailed explanations, it may be useful to get some intuitive expressions on the main differences between 1–matrix model and multi–matrix models (for more detail, see chapter **6**).

<u>1–matrix model</u> :

In the double scaling limit, one matrix model (with even potentials) is governed by the KdV hierarchy . From the eqs.(1.6.9), it is obvious that, when we do perturbations, the Schrödinger operator $(\partial^2 + u)$ is still a second order differential operator, while the higher flow equations get into the game. This property also shows up in the discrete level. The discrete version of the Schrödinger operator is a "*Jacobi matrix*", which has only three pseudo–diagonal lines(see eq.(3.3.8)), and this form is the same for any polynomial potential.

<u>multi–matrix models</u> :

However, in multi–matrix models , at discrete level, there are several Jacobi matrices(for q–matrix model, there are q Jacobi matrices, see eqs.(6.1.4)). Their forms depend on the choice of the potentials (see the eqs(6.1.7a–6.1.7c)). Thus, in double scaling limit, for different potentials, the corresponding differential operators of the Jacobi matrices probably have different orders. This intutive observation will tell us the following points. Firstly, it is very difficult to exhaust all the criticalities by taking double scaling limit, so nobody really knows how to get the full integrable structures in multi–matrix models . Secondly, it is hard to believe that we will only have a "*single*" higher KdV hierarchy , in other words, it seems possibly to get all higher KdV hierarchies. Futhermore, the double scaling limit in multi–matrix models is not as successful as in 1–matrix model .

In order to shed some light on these points, shall we appeal to some other approaches? In fact, we will try to analyse the models from another point of view. We simply avoid the double scaling limit. Instead, we propose the following procedure:

(i) At first we represent matrix models as certain discrete linear system(s), and extract their compatibility conditions(or *consistency conditions*). For 1–matrix model , these conditions are Toda chain lattice hierarchy and the discrete string equation. While they are 2–dimensional Toda lattice hierarchy(with certain additional flows) together with string equations in multi–matrix models .

(ii) Then we treat the first flow parameter(s) as the space coordinate(s). If so, surprisin gly, the lattice hierarchies can be completely reexpressed as *purely differential* hierarchies. In the 1—matrix model case, this is very easy to be done. The resulting continuous integrable hierarchy is the non-linear Schrödinger hierarchy , the partition function of 1-matrix model is nothing but the τ-function of non-linear Schrödinger hierarchy (subjected to $W_{1+\infty}$-constraints)[15]. When we make the even potential reduction, we will indeed obtain KdV hierarchy (see chapter 5 for more detail). In multi-matrix models , we found that the corresponding differential hierarchies are even much larger than KP hierarchy . We will call them the *generalized KP hierarchy* [16].

Apart from getting the continuous integrable hierarchies, our approach can tell us something more. It is wellknown that W_∞ algebras appear in many different subjects(see chapter 4). Whether or not all of them are the same, till now, this is open question. Fortunately, at least in the dispersionless limit, our approach provides a way to clas sify w_∞ algebras. The essential point is a new coordinization of KP hierarchy (4.5.1), the classical w_∞ algebras can be reduced to much smaller ones, which are formed by our new coodinates. To be fair, we should confess that we are not able to reduce these integrable hierarchies to the more familiar ones. We will explain this point in chapters 4,6.

1.8 Topological Meaning

The appearance of integrable hierarchies is extremely significant. Since these hierarchical structures enable us to do some calculations quite easily (even some non-perturbative quantities), without knowing the detailed information of 2-dimesional quantum gravity . So we may view integrable hierarchies (together with constraints) as the effective theories of 2-dimesional quantum gravity . This property probably reflects some underlying topological features of 2-dimesional quantum gravity . Roughly three years ago, Witten conjectured that the partition function of matrix path integral could be identified with the partition function of the topological gravity, and the correlation functions can be interpreted as the intersection numbers on Moduli space of Riemann surfaces [17][18]. This conjecture was proved by a young Russian mathematian—Kontsevich. He found a quite clever way to write down the generating function of the intersection numbers. His basic idea is to discretize the Moduli space in stead of triangulating Riemann surface in matrix models. Similar to matrix models, the lattice version of the path integral can also be represented in the form of matrix path integral, this is the famous Kontsevich model[19].

However, the equivalence of the matrix models and topological models is only valid for pure topological gravity, whose integrable structure is KdV hierarchy . Whether or not the non-linear Schrödinger hierarchy and the generalized KP hierarchies have some corresponding topological models, or they have counterparts in Kontsevich formulation are open questions. In the last chapter, we will discuss the topological model related to non-linear Schrödinger hierarchy [20]. It seems not to be coincided with all the known topological models. We still do not know what kind of generalized Kontsevich model it is equivalent to.

1.9 The organization

The thesis is organized as follows. At first, in the second chapter we will give a brief review on the Liouville continuous formalism of 2–dimesional quantum gravity . We will show how we can calculate the string Susceptibility and the other critical exponents.

Chapter **3** is devoted to the explanation of the discretization of 2–dimesional quantum gravity , and how we can represent it as matrix model path integral. The usual tricks in matrix model will be reviewed, i.e. the double scaling limit, KdV hierarchy and the Virasoro constraints. One of the most important properties of matrix models is their integrability, which opens a way to extract out the nonperturbative properties of 2–dimesional quantum gravity . This point is also discussed at discrete level in this chapter.

Since integrable systems play a very essential role in the study of the matrix models, we will discuss them in detail in chapter **4**. Our attention will only focus on the continuous integrable systems(We discuss the discrete integrable systems in Appendix A). At first we review some materials on the ordinary KP hierarchy like integrability, bi–hamiltonian structure, and the τ–function , etc. Then we propose one possible generalization of the ordinary KP hierarchy , that is the *generalized KP hierarchy* . Unlike the usual one, it contains several KP operators, each of them has a KP type of hierarchy, but all of them together should obey certain constraints[21]. This new hierarchy may exihibit the full properties of multi–matrix models .

After a systematic analysis of integrable systems, we come back to 1–matrix model in chapter **5**. We at first explain the procedure we claimed before— *passing from lattice to differential formalism*, then we show that the resulting differential hierarchy is nothing but the KP hierarchy . The important thing is that this KP hierarchy admits two bosonic field representation, and can be identified with the non–linear Schrödinger hierarchy . This non–linear Schrödinger hierarchy can even reduce to KdV hierarchy , which corresponds to even potential reduction in matrix model. Furthermore, we prove that the partition function of 1–matrix model is exactly the τ–function of non–linear Schrödinger hierarchy . In fact our procedure can tell us more. For each of the lattice integrable systems discussed in[22], if we treat their first time parameter as the space coordinate, then it possesses another *differential* integrable structure.

In chapter **6**, we try to perform the same procedure for multi–matrix models . We begin with representing the multi–matrix models as coupled discrete linear systems, and explain why multi–matrix models are quite distinct with 1–matrix model . This is in fact due to the *coupling conditions* among different discrete linear systems, which simply show that the forms of the Jacobi matrices in multi–matrix models depend on the choices of the matrix potentials, while the Jacobi matrix in 1–matrix model is always of three pseudo–diagonal lines for any potentials. This dependence is the origin of a great complexity. However, if we go to the differential language from lattice case, we will immediately get the generalized KP hierarchies, which are easier to be handled. Some of the coupling conditions can be rewritten as $W_{1+\infty}$–constraints.

In chapter **7** we give a brief review on topological field theory, and its coupling to gravity.

In chapter 8, we discuss the hidden topological models in 1–matrix model , which seems different from all the known topological models. The essential point is to treat the lattice size N as a flow parameter. So we will introduce a new operator coupled to it. The model is very similar to CP^1 topological Sigma model, but with a different hierarchical structure.

Finally we give some detailed calculations in Appendices. Appendix A is devoted to the constructions of the integrable structure of the general lattice models. In Appendix B, we give the detailed derivations of the discrete Virasoro constraints in 1–matrix model . The discrete $W_{1+\infty}$–constraints in two matrix model are derived in Appendix C. In our analysis, we often meet various W–infinity algebras. In order to avoid confusions about them, we give their definitions in Appendix D. Our notations will be given in Appendix E.

Chapter 2

Liouville Formalism of Gravity Coupled to CFT

In 1981, Polyakov pointed out in a famous paper[2], that a first quantized string propagating in R^d–spacetime can be most elegantly discribed as a theory of d–free bosons coupled to 2d–quantum gravity. The bosonic matter system has the conformal invariance, therefore, the theory can be considered as a certain CFT coupled to 2d–quantum gravity. This coupling is believed to be realized through anomaly cancellation, so, in critical dimension $d = 26$(or in superstring case, $d = 10$),the matter and the gravity essentially decouple, we can consider them separately. However, if the target space has non–critical dimension, the no–zero anomaly of matter sector should be compensated by the anomaly of gravity, so dealing 2d–quantum gravity is an unavoidable step.

Any theory including gravity must face the renormalization problem. In particular case of two dimesional quantum gravity, we may make use of some *large symmetries* to avoid the renormalization procedure. Based on this idea, considerable progress was also made by Polyakov[23], and later by KPZ[24]. They quantized the theory in light–cone gauge, in which they discovered a rich symmetry structure that is sl(2,R) Virasoro–Kac–Moody algebra. This symmetry survives the quantization of the theory and gives the exact solution of 2d–quantum gravity.

After the success in light–cone gauge formalism, Distler and Kawai and David[25] proposed a conformal gauge method. This is based on the fact that 2d–gravity can be represented as the Liouville action, whose free part is a conformal theory. 2d–gravity is certainly a certain CFT. Therefore we can treat the Liouville interaction term as a marginal deformation of the free action. In this way, we can derive the gravitational anomalous dimensions in a much easier manner. These two methods both are based on path integral formalism. Now, for the sake of the simplicity, we briefly review the second procedure [26]. We will use the conformal invariance to determine the form of the renormalization counter–terms, and use the translational invariance to calculate the critical exponents.

2.1 Path Integral Formalism in Conformal Gauge

Let Σ be a smooth two dimentional surface of genus h (no complex structure given), and let g be a metric on Σ. The space of metrics is an infinite dimensional Riemannian manifold, which will be denoted by MET_h. The inner product on its tangent vector space can be defined as

$$\| \delta g \|_g^2 = \int d^2\xi \sqrt{g}(Ag^{ab}g^{cd} + Bg^{ac}g^{bd})\delta g_{ab}\delta g_{cd}, \tag{2.1.1}$$

where A and B are non–negative constants. This determines a metric on MET_h, and thus formally, a Riemannian measure denoted as $\mathcal{D}g$, which is g–dependent. On the other hand, if there is matter fields living on Σ, in the same fashion, one can define the functional measure $\mathcal{D}_g X$ as

$$\int \mathcal{D}_g X \delta X \exp(-\| \delta X \|_g^2) = 1, \tag{2.1.2a}$$

$$\| \delta X \|_g^2 = \int d^2\xi \sqrt{g} \delta X \cdot \delta X. \tag{2.1.2b}$$

Consider a general action (which describes matter fields X^μ couple to 2d–gravity)

$$Z = \int_\Sigma \frac{\mathcal{D}g\mathcal{D}_g X}{\text{Vol.(Diff)}} \exp(-S_M(X;g) - S_c), \tag{2.1.3}$$

where S_M is the matter action, and S_c is its counter–term due to the renormalization scheme. Although we do not know how to perform the renormalization scheme, i.e. we do not know what are the counter–terms, but we are dealing an anomaly free theory, so conformal invariance will be preserved after renormalization. We will see that this conformal invariance restricts the form of the counter–term. The factor divided out is the volume of the symmetry group which are the diffeomorphisms of the Riemann surface. Once we make gauge fixing, as we usually do in quantum field theory, we should introduce Faddeev–Popov ghosts, then the partition function (2.1.3) becomes

$$Z = \int \frac{[d\tau]}{\text{Minimal Vol.}} \mathcal{D}_g X \mathcal{D}_g \Phi \mathcal{D}_g b \mathcal{D}_g c \exp(-S_M(X;g) - S_{gh}(b,c:g) - S_c), \tag{2.1.4}$$

where $S_{gh}(b,c:g)$ is the ghost action.

Both the matter action and the ghost action are invariant under reparametrization and Weyl scaling (or more precisely the conformal transformation),

$$\begin{cases} g \longrightarrow e^\sigma g, \\ S_M(X;g) \longrightarrow S_M(X;e^\sigma g) = S_M(X;g), \\ S_{gh}(b,c;g) \longrightarrow S_{gh}(b,c;e^\sigma g) = S_{gh}(b,c;g). \end{cases} \tag{2.1.5}$$

Furthermore, the measures are also totally reparametrization invariant. However, they are not invariant under the Weyl scaling (2.1.5). This is the crucial point of the theory which we should carefully analyze.

2.1.1 The translational invariant measures

Since all the measures are defined through the metric–dependent norms, and the metric is a dynamic variable in quantum gravity, the path integral is quite difficult to perform. In other words, quantizing the theory directly would require metric–dependent regulators, which is not known[27]. It is convenient to work with a translational invariant measure. This can be done by parametrizing g with a reference metric $\hat{g}$ and the Liouville field Φ. Due to the metric–dependence in the norms (2.1.1) and (2.1.2b), it turns out that the measures of matter and ghost fields will pick up an anomalous variations

$$\mathcal{D}_{\hat{g}e^{\Phi}} X = \mathcal{D}_{\hat{g}} X \exp(\frac{d}{48\pi} S_L(\hat{g};\Phi)), \tag{2.1.6a}$$

$$\mathcal{D}_{\hat{g}e^{\Phi}} b \mathcal{D}_{\hat{g}e^{\Phi}} c = \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \exp(-\frac{26}{48\pi} S_L(\hat{g};\Phi)), \tag{2.1.6b}$$

where

$$S_L(\hat{g};\Phi) = \int d^2\xi \sqrt{\hat{g}} \Big(\frac{1}{2}\hat{g}^{ab}\partial_a\Phi\partial_b\Phi + \hat{R}\Phi + \mu_0 e^{\Phi}\Big), \tag{2.1.7}$$

S_L is known as *Liouville action*, μ_0 is the bare cosmological constant, and $\hat{R}$ is the scalar curvature of the reference metric $\hat{g}$

$$\frac{1}{8\pi}\int d^2\xi \sqrt{\hat{g}}\hat{R} = 1 - h = \frac{1}{2}\chi(\Sigma). \tag{2.1.8}$$

The norm of the Liouville field Φ is induced by (2.1.1)

$$\| \delta\Phi \|_g^2 = \int d^2\xi \sqrt{\hat{g}} e^{\Phi} (\delta\Phi)^2. \tag{2.1.9}$$

So, it determines a functional measure for Φ (denoted by $\mathcal{D}_g\Phi$) , which is obviously Φ–dependent. Now what we want is the following translational invariant measure

$$\| \delta\Phi \|_{\hat{g}}^2 = \int d^2\xi \sqrt{\hat{g}} (\delta\Phi)^2. \tag{2.1.10}$$

In [25], it is simply *assumed* that, when we switch on this measure, the total measure picks up an overall Jacobian, which takes the form of an exponential of a *Local Liouville–like* action

$$\tilde{S}_L = \int d^2\xi \sqrt{\hat{g}} (\tilde{a}\hat{g}^{ab}\partial_a\Phi\partial_b\Phi + \tilde{b}\hat{R}\Phi + \mu e^{\tilde{c}\Phi}), \tag{2.1.11}$$

where $\tilde{a}, \tilde{b}$ and $\tilde{c}$ are constants that will be determined by requiring the overall conformal invariance. Several authors tried to justify this assumption, and to show that it can be obtained from (2.1.9) by the Weyl rescaling transformation (2.1.5) [27].

Under this assumption, partition function (2.1.4) becomes

$$Z = \int [d\tau] \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} \Phi \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \exp(-S_M(X;\hat{g}) - S_{gh}(b, c : \hat{g}) - \tilde{S}_L), \tag{2.1.12}$$

This path integral was defined to be reparametrization invariant, and should depend only on $e^{\Phi}\hat{g} = g$ (up tp diffeomorphism), not on the specific choice of $\hat{g}$. Due to diffeomorphism invariance, (2.1.12) should thus be invariant under the infinitesimal transformation

$$\delta\hat{g} = \varepsilon(\xi)\hat{g}, \qquad \delta\Phi = -\varepsilon(\xi), \tag{2.1.13}$$

which fixes the constants $\tilde{a}, \tilde{b}$

$$\tilde{a} = \frac{25-d}{96\pi}, \qquad \tilde{b} = \frac{25-d}{48\pi}. \tag{2.1.14}$$

Substituting them into (2.1.11) gives

$$\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\Big(\frac{25-d}{12}\hat{g}^{ab}\partial_a\Phi\partial_b\Phi + \frac{25-d}{6}\hat{R}\Phi\Big). \tag{2.1.15}$$

After rescaling $\Phi \longrightarrow \sqrt{\frac{12}{25-d}}\Phi$, we obtain the conventional Liouville action,

$$S_\Phi = \frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}\Big(\hat{g}^{ab}\partial_a\Phi\partial_b\Phi + Q\hat{R}\Phi\Big). \tag{2.1.16}$$

with

$$Q = \sqrt{\frac{25-d}{3}}. \tag{2.1.17}$$

The energy momentum tensor derived from (2.1.16) is

$$T_\Phi = -\frac{1}{2}\partial\Phi\partial\Phi + \frac{Q}{2}\partial^2\Phi. \tag{2.1.18}$$

The central charge can be read off immediately

$$c_L = 1 + 3Q^2. \tag{2.1.19}$$

It exactly cancels the anomalies of matter and ghost sectors.

2.1.2 The screening charge

In fact, the constant $\tilde{c}$ is related to the gravitational screening charge. In order to determine it, remembering that the rescaling of Φ changes $\tilde{c} \longrightarrow \tilde{c}\sqrt{\frac{12}{25-d}} \equiv \alpha$, since the last term in (2.1.11) represents the area of the surface, so $e^{\alpha\Phi}$ should have conformal dimension $(1,1)$, i.e.

$$-\frac{1}{2}\alpha(\alpha - Q) = 1. \tag{2.1.20}$$

Together with eq.(2.1.17), we have

$$\alpha = \frac{1}{\sqrt{12}}(\sqrt{25-d} - \sqrt{1-d}). \tag{2.1.21}$$

Finally, we can get the totally renormalized action and the stress tensor

$$S_{total} = S_M + S_{gh} + S_\Phi + \delta S_f, \tag{2.1.22a}$$
$$T_{total} = T_M + T_{gh} + T_\Phi + \delta T_f, \tag{2.1.22b}$$
$$Z(\beta_i) = \int \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \mathcal{D}_{\hat{g}} \Phi e^{-S_{total}}. \tag{2.1.22c}$$

The last term in (2.1.22a) is a finite counter–term. In order to determine it, we recall that the total action S_{total} is conformally invariant, meanwhile S_M and S_{gh} as well as S_Φ are all conformal invariant, so the counter–terms must be conformal invariant too. This suggests to us that it can be constructed from the primary fields (since they are conformal covariant) in the three sectors. We denote the matter primary field by $\Psi_i^M(\xi)$, the ghost one by $\Psi_i^{gh}(\xi)$ and the Liouville screening factor by $\exp(\alpha_i\Phi)$, which is called *gravitational dressing*. Then

$$\delta S_f = \sum_i \beta_i \int d^2\xi \sqrt{\hat{g}} \Psi_i^M(\xi) \Psi_i^{gh}(\xi) \exp(\alpha_i \Phi). \tag{2.1.23}$$

Since the dressed operator should have total conformal weight $(1,1)$, in order to perform the integration, so

$$\Delta_i^M + \Delta_i^{gh} + \Delta_i^\Phi = 1. \tag{2.1.24}$$

The conformal weight of the sceening factor is

$$\Delta_i^\Phi = -\frac{1}{2}{\alpha_i}^2 - \frac{1}{2}\alpha_i Q. \tag{2.1.25}$$

In particular, for Fadeev–Popov ghost independent operators, we have

$$\Psi_i^{gh}(\xi) = 1, \qquad \Delta_i^{gh} = 0.$$

Therefore, from the eq.(2.1.24) and eq.(2.1.25), we obtain the screening charge as follows

$$\alpha_i = \frac{1}{\sqrt{12}} \frac{[d - 25 - \sqrt{(25-d)(1-d+24\Delta_i^M)}]}{\sqrt{25-d}}. \tag{2.1.26}$$

A special case is when the matter primary field is the identity operator. It is also called *puncture operator*, denoted by $\mathcal{P}$. We see that this is nothing but the area term we discussed above.

For a unitary model, all the conformal weights are non–negative, therefore, among the primary fields, the identity operator has the minimal screening charge, which is α. However, in a non–unitary model, the conformal weight can be negative, so, the minimal screening charge corresponds to the primary field of the most negative conformal weight rather than the identity operator.

2.2 The String Susceptibility

Now, choose the following finite counterterm

$$\delta S_f = \mu \int d^2\xi \sqrt{\hat{g}} \exp(\alpha\Phi).$$

where, μ is the renormalized cosmological constant. The partition function

$$Z(\mu) = \frac{\int \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \mathcal{D}_{\hat{g}} \Phi e^{-(S_M + S_{gh} + S_\Phi + \mu \int d^2\xi \sqrt{\hat{g}} \exp(\alpha\Phi))}}{\int \mathcal{D}_{\hat{g}} X \mathcal{D}_{\hat{g}} b \mathcal{D}_{\hat{g}} c \mathcal{D}_{\hat{g}} \Phi e^{-(S_M + S_{gh} + S_\Phi)}}. \tag{2.2.27}$$

for small μ behaves like

$$Z(\mu) \approx \mu^{-\Gamma+2}. \tag{2.2.28}$$

Since the partition function (2.2.27) is invariant under the following translation

$$\begin{cases} \Phi \longrightarrow \Phi + \frac{\rho}{\alpha} \\ S_\Phi \longrightarrow S_\Phi - Q \int d^2 z \sqrt{\hat{g}} \hat{R} \frac{\rho}{\alpha} = S_\Phi - Q(1-h)\frac{\rho}{\alpha} \end{cases} \tag{2.2.29}$$

we have

$$Z(\mu) = Z(\mu e^\rho) \exp((1-h)\frac{Q}{\alpha}\rho). \tag{2.2.30}$$

If we set $e^\rho = \mu^{-1}$, then

$$Z(\mu) = \mu^{\frac{Q\chi}{2}} Z(1). \tag{2.2.31}$$

Now, we introduce the critical exponent in the following way

$$(2 - \Gamma) \equiv (2 - \gamma_{str.})(1-h). \tag{2.2.32}$$

Therefore, from eq.(2.2.28) and eq.(2.2.31), one can easily obtain

$$\gamma_{str} = \frac{1}{12}\left(d - 25 - \sqrt{(25-d)(1-d)}\right) + 2. \tag{2.2.33}$$

This is the so–called string susceptibility, which accounts for the contribution of the identity operator to the free energy, and indicates the singularity of the free energy in the infrared limit($\mu \longrightarrow 0$).

Now, suppose we choose another kind of finite counterterm

$$\delta S_f = \beta \int d^2\xi \sqrt{\hat{g}} \Psi_i^M(\xi) e^{(\alpha_i \Phi)} \tag{2.2.34}$$

then, making use of the translational invariance of the partition function, we get

$$Z(\beta) \sim \beta^{(h-1)\frac{Q}{\alpha_i}}, \tag{2.2.35}$$

therefore, from the eq.(2.2.28), one obtain

$$\gamma_{str} = \frac{Q}{\alpha_i} + 2 = \frac{1}{12(1-\Delta_i^M)}\left(d - 25 - \sqrt{(25-d)(1-d+24\Delta_i^M)}\right) + 2 \qquad (2.2.36)$$

Generally, the finite counterterm can contains all of the possible gravitational dressed physical primary fields. So, the string susceptibility is determined by the most sigular term, i.e. the term of the minimal screening charge, such term is also called the most relavent term.

For the unitary model, the most relevant operator is the identity operator(which minimizes $\gamma_{str.}$). So, for unitarity minimal BPZ series

$$d = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4, \ldots \qquad (2.2.37)$$

one gets the string susceptibility

$$\gamma_{str} = -\frac{1}{m}, \qquad m = 3, 4, \ldots \qquad (2.2.38)$$

For non–unitary BPZ series,

$$d = 1 - \frac{6(p-q)^2}{pq} \qquad (p, q) \text{ are relative prime} \qquad (2.2.39)$$

as we mentioned before, the most relevant operator is the primary field with the most negative conformal weight, which is

$$\Delta(p, q) = -\frac{(p-q)^2}{4pq} + \frac{1}{4pq}.$$

This leads to

$$\gamma_{str} = -\frac{2}{p+q-1}. \qquad (2.2.40)$$

Suppose

$$\begin{cases} p = 2, \\ q = 2m - 1 \end{cases}$$

We get the sigularity of Yang–Lee model

$$\gamma_{str} = -\frac{1}{m}.$$

We see that it has the same value as the unitary minimal model. This fact tells us that different theories can have the same critical exponents.

2.3 The gravitational Dressed Operators

Now we wish to determine the effective dimensions of fields after coupling to gravity. We only consider the sector of ghost number zero. As we already know from the first section, the physical operators have conformal weight $(1,1)$, so we could make integration over the Riemann surface. This integrated form has the symmetry as the original *total* action, thus can be added to the total action as a new kind of the finite counterterm(through a new coupling β, which plays the same role as μ.)

$$\delta S_f = \beta \int d^2\xi \sqrt{\hat{g}} \Psi_i^M(\xi) e^{\bar{\alpha}_i \Phi}, \tag{2.3.41}$$

then, the one point function takes the form

$$F_\Psi = < \Psi_i^M > = \frac{1}{Z(\mu)} \int [\text{measure}] e^{-S_{total}} \int d^2\xi \sqrt{\hat{g}} \Psi_i^M(\xi) e^{\alpha_i \Phi}. \tag{2.3.42}$$

Now we may associate a critical exponent to the large area behavior of this one point function, like

$$F_\Psi \sim \mu^{1-\nu}. \tag{2.3.43}$$

This definition coincides with the standard convention that $\nu < 1$ corresponds to a relevant operator, $\nu = 1$ to a marginal operator, and $\nu > 1$ to an irrelevant operator, in particular the relevant operators tend to dominate the infrared limit.

In order to determine ν, we employ the same scaling argument as the one used in the previous section. After making the translation, and set $e^{-\rho} = \mu$,

$$\Phi \longrightarrow \Phi + \frac{\rho}{\alpha},$$

we get

$$F_\Psi(\mu) = \frac{\mu^{Q\chi/2\alpha - \alpha_i/\alpha}}{\mu^{Q\chi/2\alpha}} F_\Psi(1) = \mu^{\alpha_i/\alpha} F_\Phi(1).$$

where the additional factor of $e^{\rho\alpha_i/\alpha} = \mu^{-\alpha_i/\alpha}$ comes from the $e^{\alpha_i\Phi}$ gravitational dressing of Ψ_i^M. The gravitational scaling dimension ν defined in (2.3.43) thus satisfies

$$\nu = 1 - \alpha_i/\alpha \tag{2.3.44}$$

Substituting the value of α_i in eq.(2.1.26) into the above equation,

$$\nu = \frac{\sqrt{1 - d + 24\Delta_i^M} - \sqrt{1-d}}{\sqrt{25-d} - \sqrt{1-d}} . \tag{2.3.45}$$

If $\Delta_i^M = 0$, i.e. we consider the puncture operator,

$$< \mathcal{P} > = A \sim \mu^{-1}. \tag{2.3.46}$$

We see that the cosmological constant μ is conjugate to the area of the surface A. This suggests us to transfor the formulation given above to the one expressed in terms of large area. The small cosmological behavior of partition function corresponds therefore to its large area behavior. We may do the path integral in two steps, by introducing the fixed area partition function

$$Z(A) = \int [\text{measure}] e^{-S_{total}} \delta\Big(\int d^2\xi \sqrt{\hat{g}} e^{\alpha\Phi} - A\Big), \tag{2.3.47}$$

and the total partition function is

$$Z(\mu) \sim \int dA Z(A). \tag{2.3.48}$$

then, for large A,

$$Z(A) \sim A^{\frac{(\gamma_{str}-2)\chi}{2}-1}. \tag{2.3.49}$$

We will see in next chapter that the matrix model approach really recovers this large area behavior.

Chapter 3

One Matrix Models

The matrix model was originally proposed to deal with nuclear spectra problem by Wigner, Dyson long time ago[28], later on it was applied on the lattice QCD by t'Hooft[29]. There were also some other applications, for examples, statistic mechanics[30], polymers[31], etc. However, only roughly three years ago, it became an amazing subject in theoretical physics. This is due to the discovery of the famous *double scaling limit.* By means of this significant procedure, three different groups of people[3]–[5]) were able to find that the partition function satisfies certain *differential equations.* This is of great importance. On the one hand, we can avoid calculating the partition function from the path–integral, which is an extremely difficult problem. On the other hand, if we can solve the equations, we can extract the *non–perturbative* properties of the string theory. In this chapter we will give a short review on this subject. The main references are listed in[32]. In the first section, we will explain why matrix models can be considered as the discrete analog of the 2–dimensional quantum gravity . In particular we will show that there exist two different types of expansion of the partition function of 1–matrix model , one is the *topological expansion*, the other is the perturbation in the coupling constants. The partition function of fixed genus has critical behavior when the coupling constants approach certain values. Correspondingly, there are two limiting procedures, the large N limit and going to the critical point. Therefore, if we properly unify these two limits, we would be able to obtain nonperturbative properties. In section **2**, we will give a particular example, 1–matrix model with quartic potential, to show how the proceeding idea works. One of the most important properties of matrix models is their integrability, which establishes the connection of the 2–dimensional quantum gravity with some well–known integrable systems, therefore we can completely solve the models. This point will be considered in section **3**, by means of the *discrete linear system.* The general 1–matrix model is defined on a parameter space of the infinite dimensions, all of these parameters can be regarded as perturbation parameters, which should be subjected to the Virasoro constraints even at discrete level, this will be derived in section 4. In order to get some hints about the full hierarchical structure in 1–matrix model , we proceed to the spheric limit in section **5**. Then we turn our attentions to the double scaling limit procedure for general even potential matrix models, and its KdV hierarchical structure in section **6**. Section **7** is devoted to the discussion of the continuum

Virasoro constraints. Finally we give some remarks in section 8.

3.1 Motivations and Basic Idea

The basic idea of [3]–[5] relied on a discretization of the string worldsheet to provide a method of taking the continuum limit which incorporated simultaneously the contribution of 2d surfaces with any number of handles. Thus it was possible not only to integrate over all possible deformation of a given genus surface (the analog of the integral over Feynman parameters for a given loop diagram), but also to sum over all genus. This would in principle free us from the mathematically fascinating but physically irrelevant problems of calculating conformal field theory correlation functions on surfaces of fixed genus with fixed moduli. Now let us see how to go to discrete language.

3.1.1 Discretized surfaces

We begin by considering a "$D = 0$ dimensional string theory", i.e. a pure theory of surfaces with no coupling to additional "matter" degrees of freedom on the string worldsheet. This is equivalent to the propagation of strings in a non-existent embedding space. For partition function we take

$$F = \sum_h \int \mathcal{D}g \; e^{-\beta A + \gamma \chi} \; , \tag{3.1.1}$$

where the sum over topologies is represented by the summation over h, the number of handles of the surface, and the action consists of couplings to the area $A = \int \sqrt{g}$, and to the Euler character $\chi = \frac{1}{4\pi}\int \sqrt{g}\, R = 2 - 2h$.

The integral $\int \mathcal{D}g$ over the metric on the surface in (3.1.1) is difficult to calculate in general. Let us denote by MET_h the metric space of genus h surface, and by $\mathrm{MET}_{A,h}$ the subspace of metrics corresponding to a fixed total area A of Σ surface. Then we can perform the integration in three steps, that is, at first integrating out the subspace $\mathrm{MET}_{A,h}$, secondly integrating over the area A, finally summing over the topologies[17]

$$F = \sum_h F(h), \qquad\qquad F(h) = \int d\,A F(h, A)$$

and

$$F(h, A) = \int_{\mathrm{MET}_{A,h}} \mathcal{D}g e^{(-\beta A + \gamma\chi(\Sigma))} = Vol(h, A) e^{(-\beta A + \gamma\chi(\Sigma))} \tag{3.1.2}$$

Obviously we see that the partition function with fixed genus and fixed area is only proportional to the volume $Vol(h, A)$ of subspace $MET_{A,h}$, therefore we are lead to calculate this volume $Vol(h, A)$, which is also a hard work. However, if we discretize the surface Σ, this turns out to be much easier, so as to calculate (3.1.1), even before removing the finite cutoff. We consider in particular a "random triangulation" of the surface with fixed area[33], in which the surface is constructed from triangles with the same area ϵ. The triangles are designated to be equilateral. Passing from continuum surface to its discrete analog, we see that

(i).*The geometric degrees of freedom* is encoded entirely into the coordination numbers N_i of the vertices, so that there is positive (negative) curvature at vertices i where the number N_i of incident triangles is more (less) than six, and zero curvature when $N_i = 6$.

(ii).*Every triangulation determines a metric on* Σ. Imagining triangulation embedded into some auxiliary Euclidian space of sufficiently high dimension, we say two triangulation are different if they have different configurations in the auxiliary space. Two different triangulation with the same topology can be related by a sequence of certain flips of links (edges of triangles). Denote the number of triangles by N, then the total area of the triangulation is $A = N\epsilon$. For any triangulation, fixing the area A and increasing the number of triangles, we can obviously approximate a given Riemann surface with any given accuracy. Thus, counting different triangulation of Σ becomes an approximation to computing the integration over the space MET_h, in other words, the summation over all such random triangulation is thus the discrete analog to the integral $\int \mathcal{D}g$ over all possible geometries,

$$\sum_{\text{genus } h} \int \mathcal{D}g \quad \to \quad \sum_{\substack{\text{random}\\ \text{triangulation}}}$$

(iii).*The discrete counterparts of continuum quantities.*

the infinitesimal volume element

$$d\xi^2\sqrt{g} \to \sigma_i = \epsilon N_i/3$$

The total area

$$\int \sqrt{g} \to A = \sum_i \sigma_i = N\epsilon$$

The factor of 1/3 in the definition of σ_i is because each triangle has three vertices and is counted three times.

Ricci scalar curvature

$$R(x) \to R_i = \pi(6 - N_i)/N_i$$

so that

$$\int \sqrt{g}\, R \to \sum_i 2\pi(1 - N_i/6) = 2\pi(V - \tfrac{1}{2}F) = 2\pi(V - E + F) = 2\pi\chi\ .$$

Here we have used the simplicial definition which gives the Euler character χ in terms of the total number of vertices, edges, and faces V, E, and F of the triangulation (and we have used the relation $3F = 2E$ obeyed by triangulation of surfaces, since each face has three edges each of which is shared by two faces).

In the above, triangles do not play an essential role and may be replaced by any set of polygons. General random polygonifications of surfaces with appropriate fine tuning of couplings may, as we shall see, have more general critical behavior , but can in particular always reproduce the pure gravity behavior of triangulations in the continuum limit.

3.1.2 The Graphic Enumeration and Matrix Models

We have seen that the integral over geometries in (3.1.1) may be performed in its discretized form as a sum over random triangulations. In fact

the triangulations of Riemann surfaces coicide with the daul diagrams of certain Feyman graphs of the matrix field theory with cubic interaction.

Therefore performing the integral over geometry in (3.1.1) becomes *the graphic enumeration* of a certain matrix field theory. This essential idea goes back to work [29] on the large N limit of QCD, followed by work on the saddle point approximation [34].

Without loss of generality, let us take the following 1–matrix model [35]

$$Z = e^F = \int dM e^{-\frac{1}{2}trM^2 + \frac{g}{N}trM^4}$$

where $M^i{}_j$ is an $N \times N$ hermitian matrix, and the Lebesgue measure

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{1 \le i < j \le N} d(ReM_{ij})d(ImM_{ij})$$

is invariant under $SU(N)$ transformation. In order to calculate correlation functions like

$$\langle (trM^4)^n \rangle = \int dM e^{-trM^2/2 + \frac{g}{N}trM^4} (\mathrm{tr}M^4)^n , \qquad (3.1.3)$$

We should at first work out the Feyman rules. Following the usual procedure in quantum field theory , we easily get

$$\text{propagator(fig. 1a)}: \qquad \langle M^i{}_j M^k{}_l \rangle = \delta^i_l \delta^k_j \qquad (3.1.4)$$

$$\text{vertex (fig. 1b)}: \qquad \langle trM^4 \rangle = \sum_{i,j,k,l=1}^{N} M_{ij}M_{jk}M_{kl}M_{li} \qquad (3.1.5)$$

The presence of upper and lower matrix indices is represented in fig. 1 by the double lines and it is understood that the sense of the arrows is to be preserved when linking together vertices. The resulting diagrams are similar to those of the scalar theory, except that each external line has an associated index i, and each internal closed line corresponds to a summation over an index $j = 1, \ldots, N$. The "thickened" structure is now sufficient to associate a Riemann surface to each diagram, because the closed internal loops uniquely specify locations and orientations of faces.

Fig.1: (a) the hermitian matrix propagator. (b) the hermitian matrix four-point vertex.

To make contact with the random triangulations discussed earlier, we consider the diagrammatic expansion of the matrix integral

$$Z = e^F = \int dM e^{-\frac{1}{2}trM^2+\frac{g}{\sqrt{N}}trM^3} \tag{3.1.6}$$

(with M an $N \times N$ hermitian matrix, and the integral again understood to be defined by analytic continuation in the coupling g.) Similarly we can write its Feyman rules and calculate the connected correlation functions

$$\langle (trM^3)^n \rangle = \int_M e^{-trM^2/2+\frac{g}{\sqrt{N}}trM^3} (\mathrm{tr}M^3)^n \ , \tag{3.1.7}$$

which counts the number of diagrams constructed with n vertices and is identically dual (i.e. in which each face, edge, and vertex is associated respectively to a dual vertex, edge, and face) to a random triangulation inscribed on some orientatible Riemann surface. We see that

- The matrix integral (3.1.6) automatically generates all such random triangulations.*
- The *free energy* of the matrix model is actually the *partition function* F of the 2d gravity (3.1.1). Since the matrix integral generates both connected and disconnected surfaces.
- The coupling constant g in matrix model is related to the cosmological constant in 2–dimensional quantum gravity .

  $$g = e^{-\beta\epsilon}$$

  Since in the g–expansion of Z defined in (3.1.6), the term corresponding to n vertices diagram contains a factor g^n, comparing with (3.1.2), we can immediately justify the above identification.

*Had we used real symmetric matrices rather than the hermitian matrices M, the two indices would be indistinguishable and there would be no arrows in the propagators and vertices of fig. 1. Such orientationless vertices and propagators generate an ensemble of both orientatible and non-orientatible surfaces.

- *Topological expansion*: If we change variables $M \rightarrow M\sqrt{N}$ in (3.1.6), the matrix action becomes $N\,tr(-\frac{1}{2}trM^2 + gtrM^3)$, with an overall factor of N.† This normalization makes it easy to count the power of N associated to any diagram. Each vertex contributes a factor of N, each propagator (edge) contributes a factor of N^{-1} (because the propagator is the inverse of the quadratic term), and each closed loop (face) contributes a factor of N due to the associated index summation. Thus each diagram has an overall factor

$$N^{V-E+F} = N^{\chi} = N^{2-2h}\,, \tag{3.1.8}$$

where χ is the Euler character of the surface associated to the diagram. Comparing with (3.1.2), we set $N = e^{\gamma}$ and we see that the large N–expansion of matrix model is in fact a topological expansion

$$F(g) = N^2 F_0(g) + Z_1(g) + N^{-2}F_2(g) + \ldots = \sum N^{2-2h}F_h(g) \tag{3.1.9}$$

Therefore, if we take $g = e^{-\beta\epsilon}$ and $N = e^{\gamma}$, we can identify the continuum limit of the free energy $\ln Z$ in (3.1.6) with the Z defined in (3.1.1). Furthermore, the matrix model with cubic action corresponds the "triangulation" of Riemann surfaces, the higher polynomial potential would result in more general "random polygonizations" of surfaces.

3.1.3 The spheric limit and critical property

In the conventional large N limit, we take $N \rightarrow \infty$ and only F_0 in eq.(3.1.9) survives, which is the contribution of the planar surface. On the other hand, we can also expand F_0 in a perturbation series in the coupling g, and for large order n behaves like (see [35] for a review)

$$F_0(g) \sim \sum_n n^{\gamma_{str}-3}(g/g_c)^n \sim (g_c - g)^{2-\gamma_{str}}\,. \tag{3.1.10}$$

which diverges as g approaches some critical coupling g_c. Conversely, we may also think that(although no "direct" proof exists) the divergence of the matrix integral when $g \rightarrow g_c$, is dominated by diagrams with infinite numbers of vertices. Since for a fixed area $A = n\epsilon$, $n \rightarrow \infty$ is equivalent to taking continuum limit $\epsilon \rightarrow 0$. Therefore We can extract the continuum limit of these discrete surfaces (or triangulations) by properly tuning $g \rightarrow g_c$.

Substituting $n = \frac{A}{\epsilon}$ into eq.(3.1.9), we see that for large fixed area

$$F_0 \sim A^{\gamma_{str}-3}$$

In general

$$F(h, A) \sim A^{(\gamma_{str}-2)\chi/2-1}\,. \tag{3.1.11}$$

So in this way we can calculate the *string susceptibility* γ_{str}.

†Note that N remains distinguished from the coupling g in the model, since it enters as well into the traces via the $N \times N$ size of the matrix.

3.1.4 The double scaling limit

Since γ_{str} is universal, i.e. independent of the genus, so we may expect that the successive coefficient functions $F_h(g)$ in (3.1.9) as well diverge at the same critical value of the coupling $g = g_c$ and takes the form (which can be proved in matrix model)

$$Z_h(g) \sim \sum_n n^{(\gamma_{str}-2)\chi/2-1}(g/g_c)^n \sim (g_c - g)^{(2-\gamma_{str})\chi/2} . \tag{3.1.12}$$

This shows that the contributions from higher genus are enhanced as $g \longrightarrow g_c$. Therefore, if we simultaneously take the limits $N \longrightarrow \infty$ and $g \longrightarrow g_c$, we may obtain a coherent contribution from all genus surfaces [3]–[5]. This is the so called *Double Scaling Limit.*

To see how this works explicitly, we write the leading singular piece of the $Z_h(g)$ as

$$Z_h(g) \sim f_h(g - g_c)^{(2-\gamma_{str})\chi/2} .$$

Then in terms of

$$\kappa^{-1} \equiv N(g - g_c)^{(2-\gamma_{str})/2} , \tag{3.1.13}$$

the expansion (3.1.9) can be rewritten ‡

$$Z = \kappa^{-2} f_0 + f_1 + \kappa^2 f_2 + \ldots = \sum_h \kappa^{2h-2} f_h . \tag{3.1.14}$$

The desired result is thus obtained by taking the limits $N \rightarrow \infty$, $g \rightarrow g_c$ while holding fixed the "renormalized" string coupling κ of (3.1.13). This is known as the "double scaling limit".

3.2 One–Matrix Model with Quartic Potential

In this section we will consider a particular example of 1–matrix model , which is of quartic potential, to show how the above idea works. We will perform the double scaling limit explicitly and derive the *string equation*, as well as determine the string susceptibility.

Let us begin with the following quartic interacted matrix model[5]

$$Z = \int dM e^{-\beta tr V(M)}, \qquad V(M) = M^2 + gM^4 \tag{3.2.1}$$

We can at first integrate out the angular variables and keep an integral over the N–eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ of the matrix M[35]

$$Z_N(\beta, g) = \frac{\Omega_N}{(2\pi)^N} \int_{-\infty}^{+\infty} \prod_{1\leq i\leq N} d\lambda_i \Delta_N^2(\lambda) \exp\left(-\beta \sum_{i=1}^N V(\lambda_i)\right) \tag{3.2.2}$$

‡Strictly speaking the first two terms here have additional non-universal pieces that need to be subtracted off.

where

$$V(\lambda) = \lambda^2 + g\lambda^4$$

and

$$\Delta_N(\lambda) = \prod_{1\le i\le N} (\lambda_i - \lambda_j) = det \parallel \lambda_i^{j-1} \parallel$$

which is the Vandermonde determinant. The quantity Ω_N is related to the volume of the unitary group. Disregarding the irrelevant constant, we define

$$d\,\mu(\lambda) \equiv e^{(-\beta V(\lambda))} d\,\lambda$$

$$\bar{Z}_N(\beta, c_i) \equiv Z_N(\beta, c_i)\frac{(2\pi)^N}{\Omega_N} = \int_{-\infty}^{+\infty} \prod_{1\le i\le N} d\,\mu(\lambda_i)\Delta_N^2(\lambda).$$

There are several ways to deal with this integral, for examples, *Dyson gas quantum mechanics*[5, 36], *Schwinger–Dyson equation* [7], *Discrete linear system*[37]. Among them, we explain the last one, which directly makes use of the powerful mathematical tool—the orthogonal polynomial technique[38], and directly leads to the integrability of the 1–matrix model .

3.2.1 The Orthogonal Polynomials

The set of orthogonal polynomials $P_n(\lambda)$ are defined as

$$P_n(\lambda) = \lambda^n + \text{lower powers of } \lambda$$

They satisfy the orthogonal relations

$$\int_{-\infty}^{+\infty} d\,\mu(\lambda)P_n(\lambda)P_m(\lambda) = h_n\delta_{nm} \tag{3.2.3}$$

and the recursion relations *

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + R_n P_{n-1}(\lambda) \tag{3.2.4}$$

---

*Suppose

$$\lambda P_n = P_{n+1} + \sum_{i=0}^{n} a_{ni}P_i$$

Then, from eq.(3.2.3), we should have

$$\delta_{n+1,m}h_m + a_{nm}h_m = \delta_{m+1,n}h_n + a_{mn}h_n$$

if $m < n$, the possible non-zero elements of (a_{nm}) are $a_{n,n-1} = R_n$ and a_{nn}, for even potential $a_{nn} = 0$(see footnote after eq.(3.3.7).

where we have introduced a new set of coefficients $h_n = R_n h_{n-1}$. After expanding the Vandermonde determinant in terms of the orthogonal polynomials ,

$$\Delta_N(\lambda) = det \parallel \lambda_i^{j-1} \parallel = det \parallel P_{j-1}(\lambda_i) \parallel$$

we can carry out the integration (3.2.2), and obtain

$$\bar{Z}_N(\beta, g) = N! h_0 h_1 \ldots h_{N-1} \tag{3.2.5}$$

$$F_N(\beta) = regular + \sum_{i=1}^{N-1} (N-k) \ln R_k \tag{3.2.6}$$

We see that the free energy is only a function of the coefficients R_n's. If we can calculate these coefficients, we completely solve the model. In order to do this, we at first show that these coefficients satisfy a certain set of equations.

Noting that

$$V' = 2\lambda + 4g\lambda^3$$

Using the recursion relation (3.2.4) repeatedly, we get

$$\begin{aligned} V' P_{n-1} = {} & 4gP_{n+2} + 2P_n + 4g(R_{n-1} + R_n + R_{n+1})P_n + 2R_{n-1}P_{n-2} \\ & +4g(R_n R_{n-1} + R_{n-1}^2 + R_{n-1}R_{n-2})P_{n-2} + 4gR_{n-1}R_{n-2}R_{n-3}P_{n-4} \end{aligned} \tag{3.2.7}$$

immediately the non-zero elements of the matrix V' are

$$\begin{aligned} V'_{n+3,n} &= 4gh_{n+3} \\ V'_{n-3,n} &= 4gR_n R_{n-1} R_{n-2} h_{n-3} \\ V'_{n+1,n} &= [2 + 4g(R_n + R_{n+1} + R_{n+2})]h_{n+1} \\ V'_{n-1,n} &= R_n[2 + 4g(R_n + R_{n-1} + R_{n+1})]h_n \end{aligned}$$

On the other hand,

$$\begin{aligned} nh_n &= \int_{-\infty}^{+\infty} d\mu(\lambda)\lambda P_n'(\lambda)P_n(\lambda) = \int_{-\infty}^{+\infty} d\mu(\lambda)P_n'(\lambda)(P_{n+1} + R_n P_{n-1}) \\ &= R_n \int_{-\infty}^{+\infty} d\mu P_n'(\lambda)P_{n-1} = R_n\beta \int_{-\infty}^{+\infty} d\mu P_n V' P_{n-1} \\ &= R_n \beta h_n[2 + 4g(R_{n-1} + R_n + R_{n+1})] \end{aligned}$$

The last equality shows that R_n's satisfy

$$R_n[2 + 4g(R_{n-1} + R_n + R_{n+1})] = \frac{n}{\beta} \tag{3.2.8}$$

This is the basic point of our further discussions. We will see that the string susceptibility and the string equation are both encoded in this relation.

3.2.2 The Spheric Limit

Now, let us introduce the continuum parameters

$$x \equiv \frac{n}{N}, \qquad \epsilon \equiv \frac{1}{N}, \qquad R(x) \equiv R_n, \qquad X \equiv \frac{N}{\beta}$$

then, eq.(3.2.8) can be written as

$$xX = 2R(x) + 16gR(x)[R(x-\epsilon) + R(x) + R(x+\epsilon)] \equiv W(R) \tag{3.2.9}$$

the spheric limit means that

$$\begin{cases} N \longrightarrow \infty \\ \beta \longrightarrow \infty \end{cases} \qquad \text{keeping} \qquad X \qquad \text{fixed} \tag{3.2.10}$$

In this limit, eq.(3.2.9) can be simplified

$$2R_0(x) + 12gR_0^2(x) = xX \tag{3.2.11}$$

where X can be viewed as an effective coupling constant. The reason is as following. Generally we say a function is regular at some point, if it is continuous and infinitely differentiable, otherwise, it is called singular at that point. In our case now, we want to tune the coupling constants such that R is singular, so that also F is. According to our argument in the first section, which corresponds to the continuum limit of discrete Riemann surface. Since there are two coupling parameters in quartic potential ($V(M) = g_2M^2 + g_4M^4$, $g_2 = \beta, g_4 = g\beta$), in order to have singular R at $X = 1$, we should set

$$W(R_c) = 1, \qquad W'(R_c) = 0.$$

This fixes one of the coupling constants $g = -\frac{1}{12}$(we have chosen $R_c = 1$), and results in

$$xX = 2R_0(x) - R_0^2(x) = 1 - [1 - R_0(x)]^2 \tag{3.2.12}$$

In other words

$$R_0(x) = 1 - (1 - xX)^{\frac{1}{2}} \tag{3.2.13}$$

Therefore, the free energy behaves like

$$\begin{aligned} N^{-2}F(N,\beta) &= N^{-2}\sum_{k=0}^{N-1}(N-k)\ln R_k \sim \int_0^1 dx(1-x)\ln R(x) \\ &\sim \int_0^1 dx(1-x)(1-xX)^{\frac{1}{2}} \sim (1-x)(1-xX)^{\frac{3}{2}}|_0^1 + \int_0^1 dx(1-xX)^{\frac{3}{2}} \\ &\sim (1-X)^{\frac{5}{2}} \sim \sum_n n^{-\frac{7}{2}}X^n \end{aligned} \tag{3.2.14}$$

Comparing with the eq.(3.1.10), we can immediately read off the string anomalous dimension

$$\gamma_{str.} = -\frac{1}{2} \tag{3.2.15}$$

Denoting the scaling variable $\tau = 1 - X$, we have

$$f(\tau) = \tau^{\frac{1}{2}} = F_0''(\tau) \tag{3.2.16a}$$

$$F(\tau) = \int_0^\tau dt(\tau - t)f(\tau) = \frac{4}{15}\tau^{\frac{5}{2}} \tag{3.2.16b}$$

Comparing with the eq.(3.1.13), we see that the relation between the scaling variable t and the renormalized string coupling constant is

$$\kappa^2 = \tau^{-\frac{5}{2}} \tag{3.2.17}$$

3.2.3 The Double Scaling Limit and the String Equation

In the previous subsection, we only considered the spheric limit, i.e. in order to get eq.(3.2.12) from eq.(3.2.9), we have ignored all the ϵ terms in the Taylor expansion of $R(x \pm \epsilon)$. In fact, a careful calculation will pick up ϵ^2 terms due to the singularity of $R(x)$. Let us change the notation a little bit†

$$x \equiv \frac{n}{\beta}, \qquad \epsilon \equiv \frac{1}{\beta}, \qquad R(x) \equiv R_n, \tag{3.2.18}$$

and introduce the scaling variables

$$t \equiv (1 - x)\beta^{\frac{4}{5}}, \qquad \tau \equiv (1 - X)\beta^{\frac{4}{5}}, \qquad f(t) \equiv (1 - R(x))\beta^{\frac{2}{5}}, \tag{3.2.19}$$

so eq.(3.2.12) is replaced by

$$x = 2R - R^2 - \frac{1}{3}R_{xx}\beta^{-2} \tag{3.2.20}$$

or equivalently

$$1 - x = (1 - R)^2 + \frac{1}{3}R_{xx}\beta^{-2} \tag{3.2.21}$$

where the subscripts mean the derivatives with respect to x, hereafter we will also use a similar notation for R_{tt}, $R_{\tau\tau}$. Using the scaling variables (3.2.19), and performing the double scaling limit, i.e.

$$\begin{cases} N \longrightarrow \infty \\ \beta \longrightarrow \infty \end{cases} \qquad \text{keeping} \quad \tau \quad \text{fixed} \tag{3.2.22}$$

we find

$$R_{xx} = R_{\tau\tau}\beta^{\frac{8}{5}}, \qquad f''(\tau) = -R_{\tau\tau}\beta^{\frac{2}{5}} \tag{3.2.23}$$

†We can do this because in the double scaling limit, $X = \frac{N}{\beta} \to 1$.

and (3.2.21) results in the Painlève–I equation

$$f^2 - \frac{1}{3} f'' = \tau \tag{3.2.24}$$

This is the so called *string equation.* In the limit $\tau \longrightarrow \infty$, f($\tau$) recovers the spheric approximation (3.2.16a). The solution to eq.(3.2.24) characterizes the behavior of the partition function of pure gravity to all orders in the genus expansion. If we know it, we would extract all the *perturbative* and *non–perturbative* information of pure gravity. However, mathematically this equation only can be solved perturbatively. The perturbative solution in the powers of $\tau^{-\frac{5}{2}} = \kappa^2$ takes the form

$$f(\tau) = \tau^{\frac{1}{2}}(1 - \sum_{k=1}^{\infty} f_k \tau^{-\frac{5k}{2}}) \tag{3.2.25}$$

where f_k's are all positive constants, and for large k it goes asymptotically as

$$k \to \infty \Longrightarrow f_k \sim (2k)! \tag{3.2.26}$$

so the perturbative solution is not Borel summable.

3.2.4 The Higher Critical Points

In fact, what we have analysed is the second non–trivial critical point of 1–matrix model , for the general even potential, we have n parameters

$$V(M, g_{2k}) = \sum_{k=1}^{n} g_{2k} Tr(M^{2k}) \tag{3.2.27}$$

In order to have a singularity, we should set

$$x\frac{N}{\beta} = V'(R) \equiv W(R), \tag{3.2.28}$$

$$W'(R_c) = \cdots = W^{(n-1)}(R_c) = 0 \tag{3.2.29}$$

which fix $(n-1)$ parameters,

$$g_{2k} = (-1)^{k-1} \frac{n!(k-1)!}{(n-k)!(2k)!} \tag{3.2.30}$$

the potential (3.2.27) with these parameters is called *critical potential*[6]. The overall constant X is an effective coupling constant(see §3.2.2). The matrix path–integral will diverge for large N, so as to define a continuum surface. The singularity of the function $R(x)$ at the point $x = 1$ is

$$R(x) = 1 - (1-x)^{\frac{1}{k}} \tag{3.2.31}$$

which suggests us to introduce the scaling variable for large $n \sim N$ as follows,

$$t \equiv (1 - \frac{n}{N}) N^{\frac{2k}{2k+1}}. \tag{3.2.32}$$

Since the first two relations of eq.(3.2.14) are universal, we can directly obtain

$$F(t) = \frac{k^2}{(k+1)(2k+1)} t^{(2+\frac{1}{k})} \tag{3.2.33}$$

and the specific heat is

$$f(t) = t^{\frac{1}{k}} \tag{3.2.34}$$

the string anomalous dimension is

$$\gamma_{str.} = -\frac{1}{k} \tag{3.2.35}$$

In the double scaling limit, the string equation will be a higher order differential equation.

3.3 The integrability

In the previous section we have discussed a particular example of 1–matrix model with quartic potential, which is in the double scaling limit described by Painlève–I equation. The question is what will happen if we include all the polynomial interactions consisting of both even and odd powers. In fact we will see that it is described by the Toda chain lattice hierarchy. This is remarkable, since it is a completely solvable system, and enables us to extract all the information of 2–dimensional quantum gravity . Now let us explain this point in detail.

3.3.1 The Discrete Linear System

We begin with the most general case of 1–matrix model , which includes both even and odd potentials[37]

$$Z_N(t) = \int dM e^{-TrV(M)} = const. \int \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) \exp(-\sum_{i=1}^{N} V(\lambda_i)) \tag{3.3.1}$$

where

$$V(M) = \sum_{r=1}^{\infty} t_r M^r$$

Doing the same thing as in last section, we introduce the orthogonal polynomials ,

$$P_n(\lambda) = \lambda^n + \dots$$

which satisfy the orthogonal and recursion relations

$$\int d\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = \delta_{n,m} h_n(t) \tag{3.3.2}$$

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda) \tag{3.3.3}$$

It is convenient to introduce another set of the polynomials

$$\xi_n(\lambda) \equiv \frac{1}{\sqrt{h_n}} e^{-\frac{1}{2}V(\lambda)} P_n(\lambda). \tag{3.3.4}$$

Then eqs.(3.3.2) and (3.3.3) become

$$\int d\lambda \xi_n(t,\lambda) \xi_m(t,\lambda) = \delta_{n,m} \tag{3.3.5a}$$

$$\lambda \xi_n = \sqrt{R_{n+1}} \xi_{n+1} + S_n \xi_n + \sqrt{R_n} \xi_{n-1} \tag{3.3.5b}$$

now the function $R_n(t)$'s and $S_n(t)$'s as well as $h_n(t)$'s are defined on the whole parameter space of $\{t_1, t_2, \cdots\}$. Unlike the even potential case, here $S_n(t) \neq 0$ for general potentials, and one can show that

$$S_n = -\frac{\partial}{\partial t_1} \ln h_n(t), \qquad n \geq 0 \tag{3.3.6}$$

or more generally*,

$$\frac{\partial}{\partial t_r} \ln h_n(t) = -(Q^r)_{nn} \qquad n \geq 0 \tag{3.3.7}$$

---

*Let us see what happen if we make the reduction $t_{2k+1} = 0, \forall k \geq 0$. Obviously

$$< Tr(M^{2k+1} >= \int dM Tr(M^{2k+1}) e^{-TrV(M)} = (-1)^{2k+1} < Tr(M^{2k+1}) >= 0$$

For the second equality, we have done $M \to -M$, which is always possible for dummy variables, and used the fact that $V(-M) = V(M)$ for even potentials. On the other hand(see below (3.3.10b),

$$< Tr(M^{2k+1} >= -\frac{\partial}{\partial t_{2k+1}} \ln Z_N(t)$$

for any positive integer N, therefore

$$\frac{\partial}{\partial t_{2k+1}} \ln \frac{Z_N(t)}{Z_{N-1}(t)} = \frac{\partial}{\partial t_{2k+1}} \ln h_N = 0$$

In particular, for $k = 0$, from eq.(3.3.6), we get

$$S_n = 0$$

That is to say, the even potentials correspond to $S_n = 0$'s.

where Q is the Jacobi matrix

$$\begin{aligned} Q_{nm} &\equiv \int d\lambda \xi_m(t,\lambda)\lambda\xi_n(t,\lambda) \\ &= \sqrt{R_{n+1}}\delta_{n,m-1} + S_n\delta_{n,m} + \sqrt{R_n}\delta_{n,m+1} \end{aligned} \tag{3.3.8}$$

In the same way, we introduce the matrix P

$$P_{nm} \equiv \int d\lambda \xi_m(\lambda)\frac{\partial}{\partial\lambda}\xi_n(\lambda) = \sum_{k=2}^{\infty} kt_k Q_a^{k-1} \tag{3.3.9}$$

Using the orthogonal polynomials, we can perform the integrations in (3.3.1), and obtain

$$Z_N(t) = const.N!h_0h_1h_2\ldots h_{N-1} \tag{3.3.10a}$$

$$\frac{\partial}{\partial t_r}\ln Z_N(t) = -\sum_{n=0}^{N-1} Q^r_{nn} \equiv -TrQ^r, \quad r \geq 1 \tag{3.3.10b}$$

Let us denote by ξ the column vector with components $\xi_0, \xi_1, \xi_2, \cdots$, use the recursion relations (3.3.5b) and differentiate the orthogonality relations (3.3.5a) with respect to t_r: we arrive at the following *discrete linear system* (DLS) of equations

$$\begin{cases} Q\xi = \lambda\xi, \\ \frac{\partial}{\partial t_r}\xi = Q^r_a\xi \\ \frac{\partial}{\partial\lambda}\xi = P\xi. \end{cases} \tag{3.3.11}$$

where the dependence on t and λ has been understood. Here and throughout this section we adopt the notation

$$(Q^r_a)_{nm} \equiv \begin{cases} \frac{1}{2}(Q^r)_{nm}, & m < n \\ 0, & m = n \\ -\frac{1}{2}(Q^r)_{nm}, & m > n \end{cases} \tag{3.3.12}$$

The product in the above equations is of course the matrix product.

The consistency conditions for this linear system give rise to the discrete KdV hierarchy[17]

$$\frac{\partial Q}{\partial t_r} = [Q^r_a, Q] \tag{3.3.13}$$

and to the so-called string equation†

$$[Q, P] = 1 \tag{3.3.15}$$

†Sometimes the string equation (3.3.15) can be represented in another form. Noting that the polyno-

All the integrability and criticality properties of the matrix model are encoded in the DLS[‡].

3.3.2 Gauge Symmetry and Integrability

We have seen that the 1–matrix model is characterized by the discrete linear system (3.3.11). Now we show that this system possesses a large gauge symmetry, which ensure

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mials $P_n = \lambda^n + \cdots$, we have

$$\int d\lambda e^{-V(\lambda)} P_{n-1} \frac{\partial}{\partial \lambda} P_n = n h_{n-1}.$$

From the definition of the polynomials ξ, we see that

$$\left(P + \frac{1}{2} V'(Q)\right)_{n,n-1} = n,$$

or equivalently

$$\sum_{k=2}^{\infty} k t_k Q^{k-1}_{n,n-1} = n. \tag{3.3.14}$$

we will use this expression in the section 5.

[‡]It is interesting to answer the question: to what extent is the correspondence between discrete linear systems and one–matrix models one to one? There certainly are linear systems that do not correspond to matrix models, however if we impose the matrix Q to have the Jacobi form (3.3.8) and choose the form of the polynomials ξ like (3.3.4), the correspondence is one to one. Indeed let us start from the infinite column vector ξ with orthonormalized components $\xi_0, \xi_1, \xi_2, \cdots$ as in eq.(3.3.5a), and write the system

$$Q\xi = \lambda\xi, \qquad \frac{\partial}{\partial t_r}\xi = Q^r_a \xi \tag{3.3.16}$$

Then we can reconstruct the partition function from

$$\frac{\partial}{\partial t_r} \ln Z_N(t) = -TrQ^r, \quad r \geq 1.$$

If we define now

$$\frac{\partial}{\partial \lambda}\xi = P\xi,$$

we have the consistency condition

$$\frac{\partial P}{\partial t_r} = [Q^r_a, P].$$

This admit the only solution

$$P = \sum_{k=2}^{\infty} k t_k Q_a^{k-1},$$

using simply eqs.(3.3.16) beside the orthonormality conditions. However, if choose other kind of polynomials, we will obtain different P matrix.

integrability.

Let us consider the following transformation (at fixed t_k's)

$$\xi \longrightarrow \hat{\xi} = G^{-1}\xi, \qquad Q \longrightarrow \hat{Q} = G^{-1}QG \tag{3.3.17}$$

where G is a unitary matrix. If the transformed Jacobi matrix $\hat{Q}$ has the same structure as Q, i.e. only the diagonal line and the first two off-diagonal lines are non-zero, and, moreover, if

$$\hat{Q}\hat{\xi} = \lambda\hat{\xi} \qquad \frac{\partial}{\partial t_r}\hat{\xi} = \hat{Q}^r_a\hat{\xi} \tag{3.3.18}$$

then, we say that our linear system is gauge invariant.

Let us examine these transformations more closely by considering the infinitesimal transformation

$$G = 1 + \varepsilon g.$$

Then, the invariance requires the matrix g to satisfy the equations

$$\hat{Q} = Q + \varepsilon[Q, g] \tag{3.3.19a}$$

$$\frac{\partial}{\partial t_r} g = [Q^r_a, g] - [Q^r, g]_a. \tag{3.3.19b}$$

A non-trivial solution is

$$g = \sum_k b_k Q^k_a, \tag{3.3.20}$$

where b_k's are time-independent constants. By abuse of language we will call this a "*time-independent gauge transformation*".

Let us consider the case when only b_k is nonzero. Then

$$\delta Q = \hat{Q} - Q = \varepsilon b_k[Q, Q^k_a] = -\varepsilon b_k \frac{\partial}{\partial t_k} Q. \tag{3.3.21}$$

This corresponds to the transformation $t_k \to t_k - \varepsilon b_k$, which can be rephrased by saying that the tuning of the time parameters is realized by means of the gauge transformation (3.3.20).

This transformation has remarkable properties. On the one hand it can be considered as the discrete version of the conformal transformations, on the other hand it leads to the integrability of the linear system (and consequently to that of the one–matrix model).

Let us consider in detail the latter claim. We can think of δQ given by eq.(3.3.21) as originating from a Poisson bracket in the following sense:

$$\delta Q \equiv \varepsilon\{A_r, Q\} \tag{3.3.22}$$

That is

$$\{A_r, Q\} \equiv [Q_a^r, Q]. \tag{3.3.23}$$

where A_r represents a Hamiltonian, which has been proven to be (see Appendix.A)

$$H_r \equiv \frac{1}{r}\sum_{n=0}^{\infty} Q_{nn}^r \qquad r = 1, 2, \ldots \tag{3.3.24}$$

and corresponding to the choice $A_r = H_r, H_{r-1}, H_{r-2}, \ldots$, we would obtain different Poisson brackets. More explicitly we write

$$\{H_{r-k+1}, Q\}_k = [Q_0^r, Q], \tag{3.3.25}$$

While explicitly working out the Poisson brackets one realizes that there are two distinct regimes according to whether $S_i = 0$ or $S_i \neq 0$.

(*i*) *First regime–Velterra Lattice*, i.e. $S_i = 0$. We find two meaningful Poisson brackets:

$$\{R_i, R_j\}_1 = R_i R_j(\delta_{j,i+1} - \delta_{i,j+1}) \tag{3.3.26}$$

and

$$\begin{aligned} \{R_i, R_j\}_3 &= R_i R_j(R_i + R_j)(\delta_{j,i+1} - \delta_{i,j+1}) \\ &+ R_j R_{j-1} R_{j-2}\delta_{j,i+2} - R_i R_{i-1} R_{i-2}\delta_{i,j+2} \end{aligned} \tag{3.3.27}$$

while

$$\{R_i, R_j\}_2 \equiv 0$$

(*ii*) *Second regime–Toda Lattice*, i.e. $S_i \neq 0$. We have two Poisson brackets:

$$\{R_i, R_j\}_1 = R_i R_j(\delta_{j,i+1} - \delta_{i,j+1}) \tag{3.3.28a}$$
$$\{R_i, S_j\}_1 = R_i S_j(\delta_{i,j} - \delta_{i,j+1}) \tag{3.3.28b}$$
$$\{S_i, S_j\}_1 = R_i\delta_{j,i+1} - R_j\delta_{i,j+1}. \tag{3.3.28c}$$

and

$$\{R_i, R_j\}_2 = 2R_i R_j(S_i\delta_{i,j-1} - S_j\delta_{i,j+1}), \tag{3.3.29a}$$
$$\{R_i, S_j\}_2 = R_i R_j(\delta_{i,j-1} + \delta_{i,j}) - R_i R_{j+1}(\delta_{i,j+1} + \delta_{i,j+2}) \tag{3.3.29b}$$
$$+R_i S_j^2(\delta_{i,j} - \delta_{i,j+1}), \tag{3.3.29c}$$
$$\{S_i, S_j\}_2 = (S_i + S_j)(R_j\delta_{i,j-1} - R_i\delta_{i,j+1}). \tag{3.3.29d}$$

For $k = 3$ eq.(3.3.25) does not define a consistent bracket[§]. In any case, there exist at least two compatible Poisson brackets, which guarantee the integrability of DLS, and of the 1–matrix model .

[§]In fact, there exists the simplest Poisson bracket, that corresponds to $k = 0$,

$$\{R_i, S_j\}_0 = R_i(\delta_{ij} - \delta_{i,j+1}),$$

all the others are vanished[39].

3.4 Virasoro constraints from DLS

An important piece of information for the matrix model is contained in the so-called Virasoro constraints[40][66]

$$L_n Z_N(t) = 0, \qquad n \geq -1 \tag{3.4.1}$$

where

$$L_n = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k-1}} - 2N\frac{\partial}{\partial t_n} + \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_k \partial t_{n-k}} + N^2 \delta_{n,0} \tag{3.4.2a}$$

$$[L_n, L_m] = (n-m)L_{n+m}. \tag{3.4.2b}$$

They completely determine the possible perturbations.

In this section we show that the Virasoro constraints result from the consistency conditions of the linear system (3.3.11), eqs.(3.3.13) and (3.3.15).

To this end we rewrite the string equation (3.3.15) in the following form

$$\sum_{k=2}^{\infty} kt_k \frac{\partial}{\partial t_{k-1}} Q = -1, \tag{3.4.3}$$

where we have used the KdV equations (3.3.13). Eq. (3.4.3) implies that

$$\hat{l} S_n = -1, \qquad n \geq 0; \qquad \hat{l} \equiv \sum_{k=2}^{\infty} kt_k \frac{\partial}{\partial t_{k-1}} \tag{3.4.4}$$

which, by (3.3.6), can be re–expressed as

$$\hat{l} \frac{\partial}{\partial t_1} \ln h_n = 1, \qquad \forall n \geq 0$$

since the operator $\hat{l}$ commutes with $\frac{\partial}{\partial t_1}$. After integrating over t_1, and using the formula (3.3.7), we get

$$\sum_{k=1}^{\infty} kt_k (Q^{k-1})_{nn} + \alpha = 0, \qquad \forall n \geq 0. \tag{3.4.5}$$

At first glance it seems that the integration constant α depends on $t_2, t_3, \ldots$, but using the discrete KP hierarchy and the string equation one can prove that α is actually a constant. Let us consider this point in detail. For convenience we introduce some more notations: for Jacobi matrix Q, Q_- means its pure lower triangular part, and $Q_{(0)}$ denotes the main diagonal line, while $Q_+ = Q - Q_-$. It is easy to see that

$$Q_a = \frac{1}{2}(Q_+ - Q_{(0)} - Q_-).$$

Therefore

$$[Q^r_a, Q^l]_{nn} = \frac{1}{2}[Q^r_+ - Q^r_{(0)} - Q_-, Q^l]_{nn}$$
$$= \frac{1}{2}\Big([Q^r_+, Q^l_-]_{nn} + [Q^l_+, Q^r_-]_{nn}\Big) = [Q^l_a, Q^r].$$

Using this result, we can immediately see that

$$\frac{\partial}{\partial t_r} Q^l_{nn} = [Q^r_a, Q^l]_{nn} = [Q^l_a, Q^r] = \frac{\partial}{\partial t_l} Q^r_{nn}.$$

Now let us take the derivative of the eq.(3.4.5) with respect to some time parameter $t_r (r \geq 2)$, obviously

$$\begin{aligned}
\frac{\partial}{\partial t_r}\alpha &= \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_r}(Q^{k-1})_{nn} + r Q^{r-1}_{nn} \\
&= \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}}(Q^{r-1})_{nn} + r Q^{r-1}_{nn} \\
&= [P, Q^r]_{nn} + r Q^{r-1}_{nn} = 0.
\end{aligned} \tag{3.4.6}$$

This means that α does not depend on any time parameter. Let us choose in particular the values $t_{2r} = 0, (r \geq 1)$, i.e. the even potential case. As we remarked before, this results in $S_n = 0, \forall n \geq 0$. So, from the eq.(3.4.5), we can immediately read off

$$\alpha = 0.$$

Therefore we finally derived

$$\sum_{k=1}^{\infty} k t_k (Q^{k-1})_{nn} = 0, \qquad \forall n \geq 0. \tag{3.4.7}$$

After taking the summation over n, and keeping the eq.(3.3.10b) in mind, we obtain

$$(\hat{l} - N t_1) Z_N(t) = 0,$$

or in another form

$$(\sum_{k=2}^{\infty} k t_k \frac{\partial}{\partial t_{k-1}} - N t_1) Z_N(t) = 0 \tag{3.4.8}$$

which is nothing but the L_{-1} constraint.

We remark that choosing even potentials would imply $S_n \equiv 0$; therefore eq.(3.3) would be meaningless and would forbid us to recover the L_{-1} Virasoro condition. We will see later on that in the continuum limit this obstruction is removed.

For later convenience, we shift the matrix P like

$$P \Longrightarrow M = P + \frac{1}{2}V'(Q).$$

Due to the eq.(3.4.7), we see obviously this new matrix M is a *purely lower triangular* matrix. On the other hand, we know that Q is symmetric, while the original P is anti–symmetric. So we immediately have the following identity

$$M + \bar{M} = V'(Q) = \sum_{k=2}^{\infty} kt_k Q^{k-1}. \tag{3.4.9}$$

where $\bar{M}$ means the transposition of the matrix.

In order to derive the other Virasoro constraints, we introduce the quantities

$$B_n^{(r)} \equiv \sqrt{R_n R_{n-1} \dots R_{n-r+1}} M_{n,n-r}; \qquad r \geq 0. \tag{3.4.10}$$

Due to the string equation (3.3.15), one finds for the above symbols the recursion relations

$$B_{n+1}^{(r+1)} - B_n^{(r+1)} = (S_{n-r} - S_n) B_n^{(r)} + R_{n-r} B_n^{(r-1)} - R_{n-1} B_{n-1}^{(r-1)} + \delta_{r,0} \tag{3.4.11}$$

The first few ones are as follows

$$B_n^{(0)} = 0, \qquad B_n^{(1)} = n, \tag{3.4.12a}$$

$$B_n^{(2)} = S_0 + S_1 + \dots + S_{n-2} - (n-1) S_{n-1}, \tag{3.4.12b}$$

$$\begin{aligned} B_n^{(3)} = & \sum_{i=0}^{n-3} S_i^2 + (n-2) S_{n-1} S_{n-2} - \sum_{i=0}^{n-3} S_i (S_{n-1} + S_{n-2}) \\ & + 2 \sum_{i=0}^{n-2} R_i - (n-2) R_{n-1} \\ & \dots\dots \end{aligned} \tag{3.4.12c}$$

On the other hand, from the KdV equations (3.3.13), it is easy to see that

$$\frac{\partial}{\partial t_1} TrQ = \frac{\partial}{\partial t_1} \sum_{i=0}^{N-1} S_i = -R_N. \tag{3.4.13}$$

With this proviso, we start to calculate the following objects. We multiply the eq.(3.4.9) by (Q^{n+1}), and take trace

$$Tr\Big(Q^{n+1}(M - V'(Q) + \bar{M})\Big) = 0, \quad n \geq -1. \tag{3.4.14}$$

Since the matrix M is purely lower triangular, so for each fixed integer n, we only need to calculate the finite terms of the traces $Tr\big(Q^{n+1} M (\text{or} \bar{M})\big)$. Meanwhile the second term in (3.4.14) can be represented as the derivatives of $\ln Z_N(t)$ with respect to t_r due to the eq.(3.3.10b). Therefore, making use of (3.3.10b), (3.4.12a–3.4.12c) and (3.4.13), we get the Virasoro constraints (or $W_{1+\infty}$–constraints,see Appendix.B),

$$L_n Z_N(t) = 0, \qquad n = 0, 1, 2 \tag{3.4.15}$$

$$L_n = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+n}} - 2N \frac{\partial}{\partial t_n} + \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_k \partial t_{n-k}} + N^2 \delta_{n,0}. \tag{3.4.16}$$

which is a subset of the constraints (3.4.1). The Virasoro algebraic structure (3.4.2b) ensures that the higher order constraints are also true.

3.5 The Spheric Limit of One–Matrix Model with General Potential

Now let us come to consider the spheric limit of 1–matrix model defined by (3.3.1). We want to deduce the classical limit of the discrete KP hierarchy , and derive the spheric limit of the Virasoro constraints.

In order to analyse the spheric limit, we at first rescale time parameters as we did in §3.2, like

$$t_r \Longrightarrow \beta t_r, \qquad \forall r \geq 1.$$

Then the discrete KP equations (3.3.13) become

$$\frac{\partial Q}{\partial t_r} = \beta [Q_a^r, Q], \tag{3.5.1}$$

and the M–matrix is rescaled by β

$$M = \beta \sum_{k=2} k t_k Q_-^{k-1}, \tag{3.5.2}$$

which still satisfies the eq.(3.4.9). Meanwhile the partition function satisfies

$$\frac{\partial}{\partial t_r} \ln Z_N(t) = -\beta \mathrm{tr} Q^r, \quad r \geq 1, \tag{3.5.3}$$

but the string equation (3.3.15) remains unchanged, however the other version (3.3.14) takes the following form

$$\sum_{k=2}^{\infty} k t_k Q_{n,n-1}^{k-1} = \frac{n}{\beta \sqrt{R_n}}. \tag{3.5.4}$$

Now we introduce the continuum parameters

$$x \equiv \frac{n}{N}, \qquad \epsilon \equiv \frac{1}{N}, \qquad R(x) \equiv R_n, \qquad S(x) \equiv S_n, \qquad X \equiv \frac{N}{\beta}.$$

The spheric limit means that

$$\begin{cases} N \longrightarrow \infty \\ \beta \longrightarrow \infty \end{cases} \qquad \text{keeping} \quad X \quad \text{fixed} \tag{3.5.5}$$

From the definition of polynomials ξ, in the spheric limit, we may set

$$\xi_n(t) = \exp\left(-\beta \eta(x,t)\right).$$

Then, for any integer j,

$$\xi_{n+j} \sim \exp\left(-\beta \eta(x + j\epsilon, t)\right) \sim \exp\left(-\beta \eta(x,t)\right) \exp\left(-X^{-1} \eta'(x,t)\right).$$

Throughout this section, the prime means the derivative with respect to x. Now we define[42]

$$z = \exp\left(-\frac{\eta'(x,t)}{X}\right). \tag{3.5.6}$$

In terms of this quantity, we see that

$$(Q\xi)_n = \sqrt{R_{n+1}}\xi_{n+1} + S_n\xi_n + \sqrt{R_n}\xi_{n-1} \sim \xi_n(z\sqrt{R(x)} + S(x) + \sqrt{R(x)}z^{-1}).$$

that is to say, in the spheric limit,

$$Q \Longrightarrow \mathcal{L} = z\sqrt{R(x)} + S(x) + \sqrt{R(x)}z^{-1}. \tag{3.5.7}$$

Similarly

$$Q^r_{nn} \Longrightarrow \mathcal{L}^r_{(0)}, \qquad Q^r_{n,n+l} \Longrightarrow \mathcal{L}^r_{(l)}, \qquad \forall l. \tag{3.5.8}$$

where the subscripts indicate the corresponding powers of z. Using these identification rules, we immediately see that the spheric limit of the eq.(3.5.4) is

$$\sum_{k=2}^{\infty} kt_k\bar{\mathcal{L}}^{k-1}_{(-1)} = xX. \tag{3.5.9}$$

The free energy behaves like

$$N^{-2}F(N,\beta) = N^{-2}\sum_{k=0}^{N-1}(N-k)\ln R_k \sim \int_0^1 dx(1-x)\ln R(x). \tag{3.5.10}$$

Therefore, we find that in the spheric limit

$$\ln Z_N(t) \sim N^2 \ln Z_0(x,t).$$

noting eq.(3.5.3), we get

$$\frac{\partial}{\partial t_r}\ln Z_0(x,t) = -X^{-1}\int_0^x (\mathcal{L}^r)_{(0)}, \quad r \geq 1, \tag{3.5.11}$$

or equivalently

$$\frac{\partial^2}{\partial x \partial t_r}\ln Z_0(x,t) = -X^{-1}(\mathcal{L}^r)_{(0)}. \tag{3.5.12}$$

In particular,

$$\frac{\partial^2}{\partial x\partial t_1}\ln Z_0(x,t) = -\frac{S(x)}{X}, \qquad \frac{\partial^2}{\partial x\partial t_2}\ln Z_0(x,t) = -\frac{2R+S^2}{X}. \tag{3.5.13}$$

3.5.1 The classical limit of the discrete KP hierarchy

Now let us see what happen to the discrete KP equations (3.5.1). We start with the first two flows

$$\frac{\partial}{\partial t_1} S_n = \beta(R_{n-1} - R_n) \sim -\frac{R'}{X}, \tag{3.5.14a}$$

$$\frac{\partial}{\partial t_1} R_n = \beta R_n(S_{n-1} - S_n) \sim -\frac{RS'}{X}. \tag{3.5.14b}$$

and

$$\frac{\partial}{\partial t_2} S_n = N\Big(R_n(S_n + S_{n-1}) - R_{n+1}(S_n + S_{n+1})\Big) \sim -\frac{2(RS)'}{X}, \tag{3.5.15a}$$

$$\frac{\partial}{\partial t_2} R_n = NR_n\Big(R_{n-1} + S_{n-1}^2 - R_{n+1} - S_n^2\Big) \sim -\frac{2R(R' + SS')}{X}. \tag{3.5.15b}$$

The remarkable effect is that we may rewrite these equations as Lax pair form, if we treat z as time independent parameter

$$\frac{\partial}{\partial t_r}\mathcal{L} = X^{-1}\Big(z\frac{\partial \mathcal{L}}{\partial z}\frac{\partial \mathcal{L}_+^r}{\partial x} - z\frac{\partial \mathcal{L}_+^r}{\partial z}\frac{\partial \mathcal{L}}{\partial x}\Big), \qquad r = 1, 2.$$

where the subindex "+" means keeping the non–negative powers of z. We will simply set $X = 1$, since it only means the rescaling of time parameters. These expressions remind us that z could be interpreted as the momentum conjugate to x, so we can define the basic Poisson bracket*

$$\{z, x\} = z. \tag{3.5.16}$$

Due to the appearance of z on the right hand side, we would refer z as "*twisted*" momentum conjugating to the space coordinate x. In fact this relation can be easily understood from another point of view. As usual we denote by E_{ij} the semi–infinite matrix $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$, and define

$$I_\pm \equiv \sum_{i=-\infty}^{\infty} E_{i,i\pm1}, \qquad \rho = \sum_{i=-\infty}^{\infty} iE_{ii}$$

Obviously, we have

$$[I_+, \rho] = I_+.$$

In the continuum limit, the diagonal matrix ρ approaches to space coordinate x, and the commutator becomes the Poisson bracket. If we denote the classical version of I_+ by z, then

*We may refer z to be *twisted momentum*, in order to its difference with the canonical one. The canonical momentum p can be introduced as following

$$p = \ln z, \qquad \{p, x\} = 1.$$

we obtain the basic Poisson bracket (3.5.16). Since the Jacobi matrix can be represented in terms of matrix I_+ in the way

$$Q = I_+\sqrt{R} + S + \sqrt{R}I_-.$$

where R and S mean the diagonal matrices $R_{ij} = R_i\delta_{ij}$ and $S_{ij} = S_i\delta_{ij}$ respectively, which become functions of x. Therefore, the classical limit of the Jacobi matrix is Q like eq.(3.5.7), this shows that the momentum variable z introduced in (3.5.6) is nothing but the classical limit of I_+. Now from the eq.(3.5.16), we can write down the spheric limit of the discrete KP equations (3.5.1) as follows

$$\frac{\partial}{\partial t_r}\mathcal{L} = \{\mathcal{L}^r_+, \mathcal{L}\}, \qquad \forall r \geq 1. \tag{3.5.17}$$

In terms of the coordinates R and S, we have

$$\frac{\partial}{\partial t_r}S = -(\mathcal{L}^r_{(-1)})', \tag{3.5.18a}$$

$$\frac{\partial}{\partial t_r}R = -\left[(\mathcal{L}^r_{(-2)})' + \mathcal{L}^r_{(-1)}S'\right]. \tag{3.5.18b}$$

The Toda chain equation corresponds to the zero–th flow

$$\frac{\partial}{\partial x}\mathcal{L} = \{(\ln\mathcal{L})_+, \mathcal{L}\}.$$

Till now the Poisson structure is defined on the auxilliary space (z, x). We may transfer it to the functional space of fields R and S. On the coordinates R and S, the Hamiltonians take the following form

$$\mathcal{H}_r = \frac{1}{r}\int(\mathcal{L}^r)_{(0)}, \qquad \forall r \geq 0. \tag{3.5.19}$$

The general Poisson structures read

$$\{R(x), R(y)\}_1 = 0, \qquad \{S(x), S(y)\}_1 = 0, \tag{3.5.20a}$$

$$\{R(x), S(y)\}_1 = R\partial\delta(x-y), \tag{3.5.20b}$$

and

$$\{R(x), R(y)\}_2 = 2(R^2\partial + RR')\delta(x-y), \tag{3.5.21a}$$

$$\{R(x), S(y)\}_2 = RS\partial\delta(x-y), \tag{3.5.21b}$$

$$\{S(x), S(y)\}_2 = (R\partial + R')\delta(x-y); \tag{3.5.21c}$$

as well as

$$\{R(x), R(y)\}_3 = (4R^2S\partial + 4RR'S + 2R^2S')\delta(x-y), \tag{3.5.22a}$$

$$\{R(x), S(y)\}_3 = (4R^2\partial + RS^2\partial + 2RR' + 2RSS')\delta(x-y), \tag{3.5.22b}$$

$$\{S(x), S(y)\}_3 = 2\Big(2RS\partial + (RS)'\Big)\delta(x-y). \tag{3.5.22c}$$

Based on these Poisson brackets we could find the W_∞ algebras.

3.5.2 The Virasoro constraints in the spheric limit

Let us start with L_{-1}-constraint. From eq.(3.5.9), using the eq.(3.5.18a) and eq.(3.5.12), we obtain

$$\sum_{k=2}^{\infty} kt_k \partial_1 \partial_{k-1} \ln Z_0(x,t) = x,$$

where we denote $\frac{\partial}{\partial t_r}$ by ∂_r. Now integrating over t_1 once, we get(discarding the integration constant)

$$\Big(\sum_{k=2}^{\infty} kt_k \partial_{k-1} - xt_1\Big) Z_0 = 0. \tag{3.5.23}$$

Essentially starting from this equation and the spheric limit version of the discrete KP hierarchy (3.5.17), we can derive all the other constraints.

In fact we can perform the same game as in the previous section. At first we notice that, in the spheric limit the quantities defined by (3.4.10) tend to

$$B_n^{(1)} \Longrightarrow x, \tag{3.5.24a}$$
$$B_n^{(2)} \Longrightarrow -xS(x) + \partial^{-1}S, \tag{3.5.24b}$$
$$B_n^{(1)} \Longrightarrow x(S^2 - R) - 2S\partial^{-1}S + \partial^{-1}(2R + S^2). \tag{3.5.24c}$$
$$\cdots\cdots\cdots\cdots$$

From eq.(3.4.10) again, we find

$$M \Longrightarrow \mathcal{M} \sim R^{-\frac{r}{2}} B_n^{(r)}. \tag{3.5.25}$$

On the other hand, the spheric limit of eq.(3.4.14) becomes

$$\Big(\mathcal{L}^{n+1}(\mathcal{M} - V'(\mathcal{L}) + \bar{\mathcal{M}})\Big)_{(0)} = 0, \quad n \geq -1. \tag{3.5.26}$$

where '*bar* " means changing z to z^{-1}. Using these results, we are able to derive the other constraints. Let us see some examples

(i). L_0-constraint:

$$\Big(\mathcal{L}(\mathcal{M} + \bar{\mathcal{M}})\Big)_{(0)} = 2(\mathcal{L}\mathcal{M})_{(0)} = 2x = -\sum_{k=1}^{\infty} kt_k \partial_k \partial^{-1} \ln Z_0(x,t),$$

integrating once over x, we get

$$\Big(\sum_{k=1}^{\infty} kt_k \partial_k + x^2\Big) Z_0(x,t) = 0. \tag{3.5.27}$$

(ii). L_1-constraint:

$$\sum_{k=1}^{\infty} kt_k\partial_{k+1}\partial^{-1}\ln Z_0(x,t), = -2\Big(\mathcal{L}^2\mathcal{M}\Big)_{(0)} = -2(xS+\partial^{-1}S),$$

Using eq.(3.5.12), we can write the above formula in another form (we simply discard the integration constant)

$$\Big(\sum_{k=1}^{\infty} kt_k\partial_{k+1} - 2x\partial_1\Big)Z_0(x,t) = 0. \tag{3.5.28}$$

(*iii*). L_2–constraint:

$$\Big(\mathcal{L}^3(\mathcal{M}+\bar{\mathcal{M}})\Big)_{(0)} = x(2R+S^2)+S\partial^{-1}S+\partial^{-1}(2R+S^2) = \partial\Big[\frac{1}{2}(\partial_1\ln Z_0)^2 - x\partial_2\ln Z_0\Big],$$

which leads to

$$\Big(\sum_{k=1}^{\infty} kt_k\partial_{k+2} - 2x\partial_2\Big)\ln Z_0(x,t) = (\partial_1\ln Z_0)^2. \tag{3.5.29}$$

Unfortunately, it can not be rewritten in a compact form. However, if we compare these formulas with the constraints (3.4.1), we find that here we lost the higher derivative terms like $(\frac{\partial}{\partial t_r}TrQ^{n-r})$, but we keep the terms like (TrQ^rTrQ^{n-r}), which could be considered as *contact terms*. Therefore, we see that the multi–point correlators are suppressed. This suggests a very simple way to derive the classical version of the Virasoro constraints (3.4.1). The trick is as follows: We start with eqs.(3.4.1), writing it as

$$\sum_k kt_k\partial_{k+n}\ln Z_N(t) - 2N\partial_n\ln Z_N(t) + \sum_{k=1}^{n-1}\Big(\partial_k\ln Z_N(t)\partial_{n-k}\ln Z_N(t) - \partial_k\partial_{n-k}\ln Z_N(t)\Big) = 0.$$

then, rescale $t \rightarrow \beta t$ as before

$$\ln Z_N(t) \Longrightarrow \beta^2\ln Z_N(t).$$

Now substituting Lattice size N by x, comparing the powers of β, we can get

$$\sum_k kt_k\partial_{k+n}\ln Z_0(t) - 2x\partial_n\ln Z_0(t) + \sum_{k=1}^{n-1}\partial_k\ln Z_0(t)\partial_{n-k}\ln Z_0(t) = 0. \tag{3.5.30}$$

Now we may summarize the spheric limit procedure as follows

1. Rescaling the time parameters by "N";
2. Introducing the continuum variables;

3. Mapping all the lattice quantities to their classical versions:

$$\begin{cases} I_+ \Longrightarrow z, \\ \text{diagonal matrices} \Longrightarrow \text{functions of } x, \\ \text{commutators} \Longrightarrow \text{Poisson braket}, \\ \text{trace operation} \Longrightarrow \text{integration over } x. \end{cases}$$

3.6 The Double Scaling Limit of One–Matrix Model with the General Even Potentials

In the proceeding sections we have shown that the gauge symmetry results in integrability. In fact there exists another kind of symmetry, which is time–dependent, and relates to the double scaling limit. This section is devoted to this problem.

3.6.1 Reparameterization and Time-Dependent Gauge Transformations

The DLS (3.3.11) is form invariant under reparameterization of the t_k couplings. That is

$$Q(\tilde{t})\xi(\tilde{t}) = \lambda\xi(\tilde{t}), \qquad \frac{\partial}{\partial \tilde{t}_r}\xi(\tilde{t}) = (Q_a^r)(\tilde{t})\xi(\tilde{t})$$

where $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, ...)$ and $\tilde{t}_k$ is a smooth functions of the t_k's.

This invariance is a formal one. However, by combining gauge and reparameterization transformations, we can obtain significant symmetries of the system. Let us consider the transformations

$$\begin{cases} \tilde{t}_k = t_k + \varepsilon(k-n)t_{k-n}, & \forall k > n, \quad n \geq -1 \\ \tilde{t}_n = t_n - 2N\varepsilon, & n \geq 1 \\ \hat{\xi}(\tilde{t}) = G^{-1}(\tilde{t})\xi(\tilde{t}), & \\ \hat{Q}(\tilde{t}) = G^{-1}(\tilde{t})Q(\tilde{t})G(\tilde{t}) & \end{cases} \tag{3.6.1}$$

In this kind of setup it is possible to find $G = 1 + \varepsilon g$ so that

$$\hat{Q}(\tilde{t}) = Q(t) \tag{3.6.2}$$

and the linear system becomes

$$Q(t)\hat{\xi}(\tilde{t}) = \lambda\hat{\xi}(\tilde{t}), \qquad \frac{\partial}{\partial t_r}\hat{\xi}(\tilde{t}) = Q_a^r(t)\hat{\xi}(\tilde{t}). \tag{3.6.3}$$

We refer to these as *time-dependent gauge transformations*.

In order to see the meaning of this kind of transformation, let us examine two particular examples

(*i*). $n = -1$ case, the transformation on parameter space is

$$\tilde{t}_k = t_k + \epsilon(k+1)t_{k+1}, \qquad \forall k \geq 1. \tag{3.6.4}$$

The invariance of eq.(3.6.2) requires the matrix g should be

$$g = P. \tag{3.6.5}$$

One can check that ξ is also invariant under this combined transformation, and can be viewed as $-\epsilon$ shift of the spectral parameter λ. Therefore, we see that the symmetry only states that the transformation (3.6.4) can be compensated by

$$\lambda \longrightarrow \lambda - \epsilon. \tag{3.6.6}$$

Usually this is called *nonisospectral symmetry*[43].

(*ii*). $n = 0$ case in (3.6.1) is nothing but the scaling transformation, which, as required by eq.(3.6.2), is compensated by

$$g = QP, \tag{3.6.7}$$

the corresponding variation of ξ is exactly the same as that generated by

$$\lambda \longrightarrow \lambda - \epsilon\lambda, \tag{3.6.8}$$

(i.e.$\delta_g \xi = \delta_{\delta\lambda}\xi$.)

From these two examples, we can learn the following things,

(*). This suggests us to view the time–dependent gauge transformation as an invariance of linear system under the transformation on *both* time parameter space *and* spectral parameter space. In fact we repeat a similar analysis for other values of n, however due to mathematical difficulty, it is not easy to write down compact form for the matrix g, so the corresponding λ transformation is complicated.

(**). This invariance also leads to the L_{-1} and L_0 Virasoro constraints. Similarly, we could as well obtain the other Virasoro constraints.

(* * *). If we consider a finite version of the transformation (3.6.1) above when $n = 0$:

$$t_r \longrightarrow \gamma^r t_r, \qquad \forall r \tag{3.6.9}$$

where γ is a finite constant, properly choosing matrix G, the Jacobi matrix Q would approach the continuous Schrödinger operator. In other words, the double scaling limit is connected with a singular case of a symmetry operation on our DLS. We will show this point in detail in next subsection.

3.6.2 The Double Scaling Limit

Let us consider a k–th order critical point and define, as usual, the continuum variables

$$x \equiv \frac{n}{\beta}, \qquad R(x) \equiv R_n, \qquad \xi(x) \equiv \xi_n.$$

Moreover we set

$$\epsilon \equiv (\frac{1}{\beta})^{\frac{1}{2k+1}}, \qquad \tilde{t}_0 = (1-\frac{n}{\beta})\beta^{\frac{2k}{2k+1}}, \qquad \partial \equiv \frac{\partial}{\partial \tilde{t}_0}, \tag{3.6.10}$$

The double scaling limit corresponds to

$$\beta \to \infty, \qquad N \to \infty, \qquad \tilde{t}_0 \text{ fixed.}$$

For large $n \sim N$ one has

$$x = 1 - \epsilon^{2k}\tilde{t}_0, \qquad R(x) = 1 + \epsilon^2 u(\tilde{t}_0) \tag{3.6.11}$$

where $u(\tilde{t}_0)$ is the specific heat.

The latter ansatz requires a comment. Let us consider the string equation (3.3.15), suitably rescaled

$$[Q,\bar{P}] = \frac{1}{\beta}, \qquad \bar{P} = \sum_{r=2}^{\infty} r\bar{t}_r Q_a^{r-1}, \qquad t_r = \beta\bar{t}_r \tag{3.6.12}$$

where $\bar{t}_r$ are renormalized coupling constants (see below). This equation establishes strong restrictions between the limiting expressions of Q and $\bar{P}$. With the simplifying assumption $t_{2r+1} = 0 \quad \forall r$, the above ansatz is correct, as is well known; if we switch on the odd interactions the analysis is more complicated, the above ansatz is still correct but we have not been able to exclude other solutions. In the following we will stick to the case of even potentials and to (3.6.11).

Let us now write down a few expansions which will be useful in the following. It is easy to see that

$$R_{n\mp 1} = R(x \mp \frac{1}{\beta}) = 1 + \epsilon^2 u(\tilde{t}_0 \pm \epsilon)$$

$$\xi_{n\mp 1} = \xi(x \mp \frac{1}{\beta}) = e^{\pm\epsilon\partial}\xi(\tilde{t}_0).$$

Then, using Taylor expansion, we have the following formulas

$$\begin{aligned} R_{n+b} &= 1 + \epsilon^2 u - b\epsilon^3 u' + \frac{b^2}{2}\epsilon^4 u'' - \frac{b^3}{6}\epsilon^5 u''' + \cdots, \\ \prod_{i=0}^{q} R_{n+b+i} &= 1 + (q+1)\epsilon^2 u - \frac{1}{2}(q+1)(2b+q)\epsilon^3 u' \end{aligned} \tag{3.6.13a}$$

$$
\begin{aligned}
&+ \epsilon^4\Big(\sum_{i=0}^{q}\frac{(b+i)^2}{2}u'' + \frac{(q+1)q}{2}u^2\Big)\\
&- \epsilon^5\Big(\sum_{i=0}^{q}\frac{(b+i)^3}{6}u''' + \frac{(2b+q)(q+1)q}{2}uu'\Big) + \cdots, \qquad (3.6.13b)
\end{aligned}
$$

$$
\begin{aligned}
\prod_{i=0}^{q}\sqrt{R_{n+b+i}} &= 1 + \frac{q+1}{2}\epsilon^2 u - \frac{(q+1)(2b+q)}{4}\epsilon^3 u'\\
&+ \epsilon^4\Big(\sum_{i=0}^{q}\frac{(b+i)^2}{4}u'' + \frac{(q+1)(q-1)}{8}u^2\Big)\\
&- \epsilon^5\Big(\sum_{i=0}^{q}\frac{(b+i)^3}{12}u''' + \frac{(2b+q)(q+1)(q-1)}{8}uu'\Big) + \cdots. \qquad (3.6.13c)
\end{aligned}
$$

Using these formulas one can derive

$$
\begin{aligned}
Q &= 2 + \epsilon^2(\partial^2 + u) + \mathcal{O}(\epsilon^3), \qquad (3.6.14a)\\
Q_a &= \epsilon\partial + \frac{1}{6}\epsilon^3(\partial^3 + 3u\partial + \frac{3}{2}u') + \frac{1}{8}\epsilon^4(2u'\partial + u'')\\
&+ \frac{1}{8}\epsilon^5\Big(\frac{1}{15}\partial^5 + \frac{2}{3}u\partial^3 + u'\partial^2 + u''\partial - u^2\partial - uu' + \frac{1}{3}u'''\Big) + \cdots, \qquad (3.6.14b)\\
Q_a^2 &= 2\epsilon\partial + \frac{4}{3}\epsilon^3(\partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u') + \frac{1}{2}\epsilon^4(2u'\partial + u'')\\
&+ \epsilon^5\Big(\frac{4}{15}\partial^5 + \frac{4}{3}u\partial^3 + 2u'\partial^2 + \frac{3}{2}u''\partial + \frac{5}{12}u'''\Big) + \cdots, \qquad (3.6.14c)\\
Q_a^3 &= 6\epsilon\partial + \epsilon^3(5\partial^3 + 9u\partial + \frac{9}{2}u') + \frac{9}{2}\epsilon^4(2u'\partial + u'') + \frac{\epsilon^5}{2}\Big(\frac{41}{10}\partial^5\\
&+ 15u\partial^3 + \frac{45}{2}u'\partial^2 + \frac{37}{2}u''\partial + \frac{9}{2}u^2\partial + \frac{9}{2}uu' + \frac{11}{2}u'''\Big) + \cdots, \qquad (3.6.14d)\\
Q_a^4 &= 12\epsilon\partial + 16\epsilon^3(\partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u') + 6\epsilon^4(2u'\partial + u'') + 2\epsilon^5\Big(\frac{24}{5}\partial^5\\
&+ 16u\partial^3 + 24u'\partial^2 + 19u''\partial + 6u^2\partial + 6uu' + \frac{11}{2}u'''\Big) + \cdots, \qquad (3.6.14e)\\
Q_a^5 &= 30\epsilon\partial + \epsilon^3(45\partial^3 + 75u\partial + \frac{75}{2}u') + 75\epsilon^4(2u'\partial + u'') + \frac{5}{4}\epsilon^5(29\partial^5\\
&+ 90u\partial^3 + 135u'\partial^2 + 111u''\partial + 45u^2\partial + 45uu' + 33u''') + \cdots. \qquad (3.6.14f)
\end{aligned}
$$

etc.

Let us see now some consequences of the above expansions. First of all let us notice that in the continuum limit the reduction to even potentials does not contradict the string equation as in the discrete case. We should remember that the contradiction is exposed in eq.(3.4.4). From the above expansions it is not difficult to see that in the continuum limit it does not make sense to single out an equation like (3.4.4), while the LHS of the string equation is replaced by a differential operator even if $t_{2r+1} = 0 \ \forall r$. So in the continuum limit on can safely choose an even potential.

Next we consider the continuum limit of (3.3.11). From eq.(3.6.14a) we see that in a neighbourhood of the critical point $\lambda \sim 2$. Therefore we introduce the renormalized quantities

$$\tilde{\lambda} \equiv \epsilon^{-2}(\lambda - 2), \qquad \tilde{Q} \equiv \partial^2 + u, \tag{3.6.15a}$$

$$\frac{\partial}{\partial \tilde{\lambda}}\xi = \tilde{P}\xi, \qquad \tilde{P} = \epsilon^2 P = \epsilon^{1-2k}\bar{P}. \tag{3.6.15b}$$

Then, the discrete Schrödinger equation goes over to its continuum version

$$(\partial^2 + u)\tilde{\xi}(\tilde{t}_0) = \tilde{\lambda}\tilde{\xi}(\tilde{t}_0). \tag{3.6.16}$$

Similarly the string equation becomes

$$[\tilde{Q}, \tilde{P}] = 1. \tag{3.6.17}$$

We are now in a condition to explicitly determine critical points. From eq.(3.3.11) and (3.6.15b) (recall that we are working with the simplifying assumption of even potential) we have

$$\begin{aligned} \tilde{P} &= \epsilon^2(2t_2 Q_a + 4t_4 Q_a^3 + 6t_6 Q_a^5 + \cdots) \\ &= \epsilon^{1-2k}(2\bar{t}_2 Q_a + 4\bar{t}_4 Q_a^3 + 6\bar{t}_6 Q_a^5 + \cdots) \end{aligned} \tag{3.6.18}$$

We remarked above that the string equation (3.6.17) puts severe restrictions not only in the discrete case but also in the double scaling limit. From eqs.(3.6.14a–) and (3.6.15a–), we see in particular that all the operators Q_a^r's are vanishing in the limit $\epsilon \to 0$, so that if one wants the string equation to be satisfied, one must let a certain subset of bare coupling constants in P go to infinity (DSL). The practical recipe is to look for combinations of $\bar{t}_{2r}$'s such that all the singular terms in the second expression of (3.6.18) vanish. Let us see a few significant examples.

<u>$(i) k = 2$.</u> In this case only $\bar{t}_2$ and $\bar{t}_4$ are nonzero, and $\beta = \epsilon^{-5}$. Then, from (3.6.14b,3.6.14d) we see that only if we set

$$t_2 = \frac{15}{8}\epsilon^{-5}\bar{t}_2 = \epsilon^{-5}\bar{t}_2, \qquad t_4 = -\frac{5}{32}\epsilon^{-5}\bar{t}_2 = \epsilon^{-5}\bar{t}_4,$$

are we able to eliminate the $\epsilon^{-1}\partial$ term in $\tilde{P}$, and we get the known operator

$$\tilde{P} = \frac{5}{2}\bar{t}_2(\partial^2 + u)_+^{\frac{3}{2}} + \mathcal{O}(\epsilon)$$

The string equation becomes

$$-\frac{5}{2}\bar{t}_2\mathcal{R}_2'[u] = 1$$

where we have introduced the Gelfand–Dickii polynomials

$$\mathcal{R}_k'[u] \equiv [(\partial^2 + u)_+^{k-\frac{1}{2}}, \partial^2 + u]. \tag{3.6.19}$$

As is well-known at the critical point $\tilde{t}_2^c = \frac{8}{15}$, the above string equation is the Painlevé equation of first kind

$$\tilde{t}_0 = \frac{1}{3}u'' + u^2.$$

(ii) $k = 3$. Only t_2, t_4 and t_6 are non-vanishing, $\beta = \epsilon^{-7}$. According to the above recipe we kill all the negative powers of ϵ in $\tilde{P}$ if we put

$$\begin{aligned} t_2 &= -\frac{105}{32}\epsilon^{-7}\tilde{t}_3 = \epsilon^{-7}\bar{t}_2, \\ t_4 &= \frac{35}{64}\epsilon^{-7}\tilde{t}_3 = \epsilon^{-7}\bar{t}_4, \\ t_6 &= -\frac{7}{192}\epsilon^{-7}\tilde{t}_3 = \epsilon^{-7}\bar{t}_6. \end{aligned}$$

It is straightforward to show that

$$\tilde{P} = \frac{7}{2}\tilde{t}_3(\partial^2 + u)_+^{\frac{5}{2}} + \mathcal{O}(\epsilon)$$

and the string equation becomes

$$-\frac{7}{2}\tilde{t}_3\mathcal{R}_3'[u] = 1.$$

The third critical point is at $\tilde{t}_3^c = -\frac{16}{35}$ and the differential equation is

$$\tilde{t}_0 = -(u^3 - \frac{1}{2}u'^2 - uu'' + \frac{1}{10}u^{(4)}).$$

One can proceed in this way and determine higher order critical points. As is well-known, on a very general ground the form of the operator $\tilde{P}$ must be as follows

$$\tilde{P} = \sum_{n=1}^{\infty}(n + \frac{1}{2})\tilde{t}_n\tilde{Q}_+^{n+\frac{1}{2}} + \mathcal{O}(\epsilon). \tag{3.6.20}$$

We conjecture that this form is induced by the following coupling redefinitions:

$$t_{2r} = -\sum_{n=r}^{\infty}(n + \frac{1}{2})\tilde{t}_n C_n a_r^{(n)}\epsilon^{-(2n+1)} \equiv \sum_{n=r}^{\infty}\Gamma_r^n\tilde{t}_n \tag{3.6.21}$$

where

$$a_r^{(n)} = (-1)^{r+1}\frac{n!(r-1)!}{(n-r)!(2r)!}$$

We checked eq.(6.13) for the first few cases and found

$$C_1 = 1, \quad C_2 = -\frac{3}{4}, \quad C_3 = \frac{5}{8}, \quad C_4 = -\frac{35}{64}, \cdots$$

In general one has

$$C_n = (-1)^{n+1}\frac{(2n-1)!!}{2^{n-1}n!}.$$

These factors are determined in such a way as to reproduce the standard KdV hierarchy.

It is worth noticing that (i) in eq.(3.6.20) and (3.6.21) we are considering all the critical points at a time, and (ii) the time transformation (3.6.21) is made of a reparameterization plus a scale transformation of the type (3.6.9).

What is left for us to do is to analyze the continuum limit of the KdV hierarchy. On the basis of eq. (3.6.21) one naively has

$$\frac{\partial}{\partial \bar{t}_n} = -\sum_{r=1}^{n}(n+\frac{1}{2})C_n\epsilon^{-(2n+1)}a_r^{(n)}\frac{\partial}{\partial t_{2r}}. \tag{3.6.22}$$

So, in particular, on the basis of (3.3.11) and (3.6.14a–) we must have

$$\begin{aligned}
\frac{\partial}{\partial \bar{t}_1}\xi &= \Big(\frac{3}{2}\epsilon^{-2}\partial + (\partial^2+u)_+^{\frac{3}{2}} + \mathcal{O}(\epsilon)\Big)\xi,\\
\frac{\partial}{\partial \bar{t}_2}\xi &= \Big(-\frac{15}{8}\epsilon^{-4}\partial + (\partial^2+u)_+^{\frac{5}{2}} + \mathcal{O}(\epsilon)\Big)\xi,\\
\frac{\partial}{\partial \bar{t}_3}\xi &= \Big(\frac{35}{32}\epsilon^{-6}\partial + (\partial^2+u)_+^{\frac{7}{2}} + \mathcal{O}(\epsilon)\Big)\xi.
\end{aligned}$$

etc. These are however naive formulae since $\{\bar{t}_n,\ n\geq 1\}$ is not a complete set of parameters after we take the continuum limit. To correct this we have to allow also for a ∂–dependent term in the RHS of (3.6.22). This additional term takes care exactly of the divergent terms (in the $\epsilon\to 0$ limit) in the RHS of the above equations.

Finally the evolution equations become

$$\frac{\partial}{\partial \bar{t}_n}\bar{\xi} = (\partial^2+u)_+^{n+\frac{1}{2}}\bar{\xi}, \qquad n\geq 0 \tag{3.6.23}$$

which result in the standard KdV flow

$$\frac{\partial}{\partial \bar{t}_n}u = [(\partial^2+u)_+^{n+\frac{1}{2}}, \partial^2+u], \qquad n\geq 0 \tag{3.6.24}$$

In eq.(3.6.23) $\bar{\xi}$ is the limit of ξ possibly multiplied by an ϵ–dependent factor which may be necessary in order to obtain a finite result.

3.7 The Virasoro Constraints in the Continuum Limit

In the previous section starting from the DLS we have obtained, near criticality, a continuous linear system

$$(\partial^2+u)\bar{\xi} = \bar{\lambda}\bar{\xi}, \tag{3.7.1a}$$

$$\frac{\partial}{\partial \bar{t}_n}\bar{\xi} = (\partial^2 + u)_+^{n+\frac{1}{2}}\bar{\xi}, \tag{3.7.1b}$$

$$\bar{P} = \sum_{n=1}^{\infty}(n + \frac{1}{2})\bar{t}_n \bar{Q}_+^{n+\frac{1}{2}}. \tag{3.7.1c}$$

whose consistency conditions are

$$[\bar{Q}, \bar{P}] = 1, \tag{3.7.2a}$$

$$\frac{\partial}{\partial \bar{t}_n}u = [(\partial^2 + u)_+^{n+\frac{1}{2}}, \partial^2 + u], \tag{3.7.2b}$$

$$\frac{\partial}{\partial \bar{t}_n}\bar{P} = [(\partial^2 + u)_+^{n+\frac{1}{2}}, \bar{P}]. \tag{3.7.2c}$$

We want now to recover the Virasoro constraints in this continuous system [8][7]. The strategy is the same as for the discrete case. We use essentially the string equation (3.7.2a). First of all, as is well known, in the continuum limit the partition function behaves like

$$\begin{aligned} \ln Z &= \sum_{k=1}^{N-1}(N-k)\ln R_k + \text{constant} \quad \text{terms} \\ &\longrightarrow \int_0^{\bar{t}_0} d\bar{t}_0'(\bar{t}_0 - \bar{t}_0')u(\bar{t}_0') + \mathcal{O}(\epsilon) + \text{regular} \quad \text{terms} \\ &\Rightarrow \partial^2 \ln \bar{Z} = u(\bar{t}_0) \end{aligned} \tag{3.7.3}$$

where, in taking the continuum limit, we passed to the normalized partition function

$$\bar{Z} \equiv Z(\bar{t})/Z(T),$$

the parameter T being a reference point connected with the extremum of integration $\bar{t}_0 = 0$.

Now, using eq.(3.7.1c), eq.(3.7.2a) can be written

$$-\sum_{k=1}^{\infty}(k + \frac{1}{2})\bar{t}_k \mathcal{R}_k'[u] = 1. \tag{3.7.4}$$

Integrating once with respect to t_0, we obtain

$$\sum_{k=1}^{\infty}(k + \frac{1}{2})\bar{t}_k\Big(\mathcal{R}_k[u] - \mathcal{R}_k[0]\Big) + \bar{t}_0 = 0 \tag{3.7.5}$$

where $\mathcal{R}_k[0]$ is $\mathcal{R}_k[u]$ computed at $\bar{t}_0 = 0^*$. In order to simplify the next formulas let us introduce the recursion operator

$$\hat{\phi} \equiv \frac{1}{4}\partial^2 + u + \frac{1}{2}u'\partial^{-1}$$

*If one directly considers the continuum limit of the discrete Virasoro constraints, it is not difficult to verify, at least for the first few critical points, that this is actually the case.

and define the recursion relation for the Gelfand–Dickii polynomials

$$\mathcal{R}'_{n+1} = \hat{\phi}\mathcal{R}'_n = \hat{\phi}^n \partial u. \tag{3.7.6}$$

Remembering that, on the basis of our conventions, we have

$$\partial^{-1}\mathcal{R}'_k[u] = \mathcal{R}_k[u] - \mathcal{R}_k[0]$$

eq.(3.7.5) can be rewritten

$$F \equiv \tilde{t}_0 + \sum_{k=1}^{\infty}(k+\frac{1}{2})\tilde{t}_k \partial^{-1}\hat{\phi}^{k-1}\partial u = 0. \tag{3.7.7}$$

On the same basis we can write

$$\partial^{-1}\hat{\phi}^{n+1}\partial F = 0, \qquad n \geq -1 \tag{3.7.8}$$

To obtain these equations we have used only the string equation. We can as well envisage eqs.(3.7.7) and (3.7.8) as a consequence of a symmetry of the system, precisely as a consequence of the fact that $u(\tilde{t}_0)$ and the KdV hierarchy are invariant under the transformations,which is called *non–isospectral symmetry*[43]

$$\begin{cases} \tilde{t}_k \longrightarrow \tilde{t}_k = \tilde{t}_k + \epsilon(k-n+\frac{1}{2})\tilde{t}_{k-n} \\ u(\tilde{t}) \longrightarrow \hat{\phi}^{n+1}\cdot 1 + u(\tilde{t}_k) \end{cases} \tag{3.7.9}$$

The generators associated with (3.7.9) are

$$\begin{aligned} L_{-1} &= \sum_{k=1}^{\infty}(k+\frac{1}{2})\tilde{t}_k\frac{\partial}{\partial \tilde{t}_{k-1}} + \frac{1}{4\rho}\tilde{t}_0^2, \\ L_0 &= \sum_{k=0}^{\infty}(k+\frac{1}{2})\tilde{t}_k\frac{\partial}{\partial \tilde{t}_k} + \frac{1}{16}, \\ L_n &= \sum_{k=0}^{\infty}(k+\frac{1}{2})\tilde{t}_k\frac{\partial}{\partial \tilde{t}_{k+n}} + \frac{\rho}{4}\sum_{k=1}^{n}\frac{\partial^2}{\partial \tilde{t}_{k-1}\partial \tilde{t}_{n-k}}, \qquad n \geq 1. \end{aligned} \tag{3.7.10}$$

Here, for later purposes, we have introduced a constant ρ (for example, by rescaling all the $\tilde{t}$'s).

Even though we will not use it in the following, it is worth mentioning that there is a larger symmetry of the system: the latter is also invariant under the transformations

$$\tilde{t}_k \longrightarrow \tilde{t}_k + \epsilon, \tag{3.7.11}$$

whose generators are given by

$$V_k = \frac{\partial}{\partial \tilde{t}_k}, \qquad k \geq 0. \tag{3.7.12}$$

The generators V_k's and L_n's characterize the master symmetry of the KdV hierarchy[44][45]. The corresponding algebra is

$$[V_k, V_l] = 0, \qquad k, l \geq 0, \tag{3.7.13a}$$

$$[V_k, L_n] = (k + \frac{1}{2})V_{k+n}, \qquad k \geq 0, k+n \geq 0, \tag{3.7.13b}$$

$$[V_0, L_{-1}] = \frac{1}{2\rho}\bar{t}_0, \tag{3.7.13c}$$

$$[L_n, L_m] = (n-m)L_{n+m}, \qquad n, m \geq -1. \tag{3.7.13d}$$

But let us return to the derivation of the Virasoro constraint. Representing now (3.7.8) in terms of the partition function, we get for $n = -1, 0$

$$\partial\Big(\sum_{n=1}^{\infty}(n+\frac{1}{2})\bar{t}_n\frac{\partial}{\partial\bar{t}_{n-1}}\ln\bar{Z} + \frac{1}{2\rho}\bar{t}_0^2\Big) = 0,$$

$$\partial\Big(\sum_{n=0}^{\infty}(n+\frac{1}{2})\bar{t}_n\frac{\partial}{\partial\bar{t}_n}\ln\bar{Z}\Big) = 0.$$

or, in general,

$$\partial\Big(\frac{l_n\sqrt{\bar{Z}}}{\sqrt{\bar{Z}}}\Big) = 0, \qquad n \geq -1. \tag{3.7.14}$$

where, by definition,

$$l_0 = L_0 - \frac{1}{16}, \qquad l_n = L_n, \qquad n \neq 0.$$

The most general solution of (3.7.14) has the form

$$l_n\sqrt{\bar{Z}} = b_n\sqrt{\bar{Z}} \tag{3.7.15}$$

where the b_n's are $\bar{t}_0$–independent but arbitrary functions of the other parameters. In order to determine them we remark that they must be compatible with the algebra (3.7.13a–). In particular they must satisfy

$$[l_n, b_m] - [l_m, b_n] = (n-m)b_{n+m} + \frac{1}{8}n\delta_{n+m,0}. \tag{3.7.16}$$

Moreover we remark that (3.7.13a–) is a graded algebra provided we define the degree as follows

$$\deg[\bar{t}_k] \equiv -2k-1, \qquad \deg[\frac{\partial}{\partial\bar{t}_l}] \equiv 2l+1, \qquad \deg[\rho] \equiv 0.$$

Therefore, $\deg[L_n] = 2n$ and, from (3.7.15),

$$\deg[b_n] = 2n, \qquad n \geq -1 \tag{3.7.17}$$

The general form of b_n will be a sum of monomials of the following type

$$\mathcal{M}_n(p, q_1, ..., q_a) = \text{const } \rho^p \, \bar{t}_{n_1}^{q_1} ... \bar{t}_{n_a}^{q_a}$$

where p is a real number and $q_1, q_2, ...$ are nonnegative real numbers (we exclude negative exponents as it is natural to require a smooth limit of b_n as any one of the couplings vanishes). Next we remember that the parameter ρ appeared on the scene because of a rescaling $\bar{t}_n \to \sqrt{\rho}\bar{t}_n$. Therefore if we perform the opposite rescaling the dependence of b_n on ρ must disappear. This implies the condition $p = \frac{1}{2}(q_1 + ... + q_a)$. So the degree of the above monomial would be

$$\deg[\mathcal{M}_n(p, q_1, ..., q_a)] = \sum_{i=1}^{a} (\frac{1}{2} - n_i) q_i \qquad (3.7.18)$$

Comparing eq.(3.7.17) with eq.(3.7.18), we see all the b_n's are zero except perhaps for b_0 which must be a constant, and b_{-1}, which could depend on $\bar{t}_1$ and ρ. We now use the consistency condition (3.7.16). For $n = 1$ and $m = -1$ it tells us that

$$\frac{5}{2}\bar{t}_2 \frac{\partial b_{-1}}{\partial \bar{t}_1} = 2b_0 + \frac{1}{8}.$$

This allows us to conclude that

$$b_0 = -\frac{1}{16}.$$

and b_{-1} does not depend on $\bar{t}_1$. Next we use again (3.7.16) for $n = 0$ and $m = -1$ and conclude that

$$b_{-1} = 0.$$

Collecting the above results we obtain the Virasoro constraints

$$L_n \sqrt{\bar{Z}} = 0, \qquad n \geq -1 \qquad (3.7.19)$$

with L_n given by (3.7.10).

Finally we recall that due to (3.7.2b) and (3.7.3), $\sqrt{\bar{Z}}$ is a τ-function of the KdV hierarchy.

3.8 Summary and Remarks

In the previous sections, we have considered the main objects in one hermitian matrix models. Till now what we can obtain is as follows

1. The partition function of 1–matrix model satisfies the differential equation, which opens a way to investigate the whole properties of gravity(possibly coupled to some matter system).

2. The even potential matrix models are described by KdV hierarchy, which could be a typic feature of 2–dimensional quantum gravity .

3. We can easily calculate the correlation functions , which is simply

$$< \sigma_r >= \frac{\partial}{\partial t_r} \ln Z \tag{3.8.1}$$

(at discrete level, $\sigma_r = Tr(M^r)$). The critical exponents can be obtained very easily.

4. For pure gravity case, we recover the usual perturbative expansions (see eq.(3.2.25) and (3.2.26)).

5. The perturbations are governed by the Virasoro constraints.

6. Finally, this approach stimulates the calculations in continuum formalism.

Although the matrix models are very successful, there is still a long way to go to understand gravity completely.

(i). Although the matrix model provided a way to extract the non–perturbative properties of gravity, we still can not get so much knowledge about this. The solution to Painléve I equation should be clarified further, in fact the real solutions to (3.2.24) do not satisfy the Schwinger–Dyson equations[46]. This suggests some people to investigate other kind of definition of pure gravity[47].

(ii). The matrix model corresponding to pure gravity is the cubic potential, which is not bounded below. Fortunately, this potential gives the same critical exponents as the quartic potential, we may use the quartic potential as the definition of pure gravity. However, eq.(3.2.27) shows quartic potential is not bounded below too. In order to get well–defined 2–dimensional quantum gravity , we way use KdV flows, i.e. move from the well–defined $k = 3$ critical point to $k = 2$ case(pure gravity). But due to the KdV structure, we can also move from the pure gravity back to $k = 3$ critical point, this indicates the non–perturbative instability of pure gravity. There are many works dealing with this problem[48].

(iii). In matrix model approach, there are infinite many operators, which seems to be much more than that of Liouville description. This identification of the operators is discussed in[49].

(iv). The double scaling limit of the general potentials (including odd powers) is analysed in[50]. However the gravity theory corresponding to non–Linear Schrödinger hierarchy is still unknown.

Chapter 4

KP Hierarchy, its Reductions and Generalizations

In the previous chapter we have seen that the 1–matrix model is intimately related to an integrable system, in particular the double scaling limit of 1–matrix model with even potential is described by the KdV hierarchy. In order to get a better understanding of matrix models, it is time to analyze the integrable system in more detail. This is our main task in this chapter. We will briefly review a large class of continuum integrable systems— KP hierarchy, the lattice integrable system is presented in Appendex A(see also [22]). Then we will consider two typical reductions of KP hierarchy , in one way it is reduced to the general KdV hierarchies, in the other case it is reduced to non–linear Schrödinger hierarchy, etc. Meanwhile we also present one possible kind of generalizations of KP hierarchy , whose roles in multi–matrix models will be explained in chapter 6.

4.1 KP hierarchy

In this section, we will consider the usual trick in KP system, that is, the pseudo–differential analysis, coadjoint orbit approach, as well as the associated linear systems. We refer the reader to nice books and notes[51][52] [53],etc. for more detailed explanation.

4.1.1 The pseudo-differential analysis

Let us begin with ordinary differential operators. Consider the following general form

$$A = \sum_{i=0}^{n} u_i(x)\partial^{n-i},$$

where $u_i(x)$'s are functions on a certain field, and we denote $\frac{\partial}{\partial x}$ by ∂, which acts on the functions according to the Leibnitz rule,

$$\partial u(x) = u(x)\partial + u'(x), \qquad [\partial, x] = 1. \tag{4.1.1}$$

This naturally defines an algebraic structure on the whole space of the ordinary differential operators. We could call it *pure differential algebra* $\wp_+$.

The essential point in the pseudo–differential anlysis is to include the integration operation, which as usual we denote by ∂^{-1}, i.e.

$$\partial^{-1}u(x) \equiv \int_{x_0}^{x} dx' u(x').$$

In the above definition, the subtle point is to choose the reference point x_0. The basic requirement for it is that we can always through away the boundary terms which appear when integrating by parts. This is equivalent to demanding the integrand to decrease rapidly(for $x_0 = -\infty$). Under this assumption, we could derive the important relations

$$\partial^{-1}\partial = \partial\partial^{-1} = 1, \tag{4.1.2}$$

$$\partial^{-j-1}u = \sum_{v=0}^{\infty}(-1)^v \binom{j+v}{v} u^{(v)}\partial^{-j-v-1},$$

where $u^{(v)}$ denotes $\frac{\partial^v u}{\partial x^v}$. The above formula can be considered as the generalized Leibnitz rules. Therefore we can extend $\wp_+$ to a larger algebra $\wp$, which contains the following *pseudo–differential operator*, denoted by PDO

$$A = \sum_{-\infty}^{n} u_i(x)\partial_i. \tag{4.1.3}$$

Obviously $\wp_+$ is the subalgebra of $\wp$, the integration operations form another subalgebra

$$\wp_- = \{A = \sum_{-\infty}^{-1} u_i(x)\partial_i\}.$$

The algebra $\wp$ has a direct sum decomposition as a vector space,

$$\wp = \wp_+ \oplus \wp_-.$$

For any given pseudo–differential operator A of type (4.1.3), we call $u_{-1}(x)$ its residue, denoted by*

$$\mathrm{res}_{\partial} A = u_{-1}(x) \qquad \text{or} \qquad A_{-1}.$$

The following functional integral will be very important in our discussion

$$< A >= \mathrm{Tr}(A) = \int u_{-1}(x)dx. \tag{4.1.4}$$

This functional integral naturally defines an inner scalar product on the algebra $\wp$, which is nondegenerated, symmetric and invariant †, therefore, it establishes a one–to–one correspondence between $\wp_+$ and $\wp_-$. This feature enables us to define a bi–hamiltonian structure on $\wp_+$(or, more generally on $\wp$). We will explain this point in the next subsection.

*The subindex ∂ means that the series are expanded in the powers of ∂. We will also consider the other expansions.

†The product is symmetric, i.e. $< AB >=< BA >$, the product also denoted as $< AB >= A(B) = Tr(AB)$. The invariance means $< A[B,C] >=< [A,B]C >$.

4.1.2 Hamiltonian structures

In this subsection we will show that the inner product (4.1.4) automatically leads to a bi–Hamiltonian structure. We start with a particular PDO of the following form

$$L = \partial + \sum_{i=0}^{\infty} u_i(x)\partial^{-i}. \tag{4.1.5}$$

Apart from the first term, L belongs to $\wp_-$. We call it KP *operator.* Similarly $u_i(x)$'s are refered to be KP *coordinates.* Any functional of KP coordinates can be expressed as

$$f_X(L) = L(X),$$

where X is an element of $\wp_+$

$$X = \sum_{i=0}^{\infty} \partial^i \chi_i(x),$$

with the testing functions $\chi_i(x)$'s. We denote this functional space by $\mathcal{F}(\wp_-)^{\ddagger}$.

The coadjoint action of $\wp_+$ on $\mathcal{F}(\wp_-)$ takes the following form

$$Ad_Y^* f_X(L) = L(Ad_{-Y}X) = L([X,Y]), \tag{4.1.6}$$

which naturally defines a Poisson bracket on $\mathcal{F}(\wp_-)$

$$\{f_X, f_Y\}_1(L) = L([X,Y]). \tag{4.1.7}$$

This is the so-called Kirillov–Konstant Poisson bracket, first given by Watanabe[55]. Hamiltonians are

$$H_r = \frac{1}{r}\mathrm{Tr}(L^r), \qquad \forall r \geq 1; \tag{4.1.8}$$

which generate the infinite many flows, which are called *the KP hierarchical equations*, or *KP hierarchy*

$$\frac{\partial}{\partial t_r} L = [L_+^r, L]. \tag{4.1.9}$$

Where the subindex "+" means choosing the non–negative powers of ∂. These equations can be considered as the equations of motion of the coordinates u_i's, or flow equations. $(t_1, t_2, t_3, \cdots)$ are time parameters. In particular, $t_1 = x$ is called *space coordinate.* A system has a bi–hamiltonian structure, if there exist two compatible Poisson brackets such that

$$\{H_{r+1}, F_Y\}_1 = \{H_r, F_Y\}_2. \tag{4.1.10}$$

‡For any given L, when running of X in $\wp_+$, $f_X(L)$ covers all of the functional space $\mathcal{F}(\wp_-)$.

This criterion enables us to derive the second Poisson bracket of the KP hierarchy (4.1.9)

$$\begin{aligned} \{f_X, f_Y\}_2(L) &= <(XL)_+YL> - <(LX)_+LY> \\ &+ \int [L,Y]_{-1}\Big(\partial^{-1}[L,X]_{-1}\Big), \end{aligned} \tag{4.1.11}$$

which is called Gelfand–Dickey bracket[56]. The derivation is quite similar to what we did in lattice case (see Appendix A), so we do not repeat it here. With respect to these two Poisson brackets, the KP coordinate u_i's form two different W_∞ algebras.

4.1.3 KP hierarchy and its integrability

We may understand the KP hierarchy from another point of view, for any given KP operator L of the form (4.1.5), it is obviously to see

$$[L, L^r_-] \in \wp_-.$$

Due to the trivial relation

$$[L^r, L^s] = 0, \qquad \forall r, s$$

we have

$$[L^r_+, L] = [L, L^r_-] \in \wp_-,$$

so we can always introduce the infinite number of the perturbations by eq.(4.1.9), and all these flows preserve the form of "L". But the essential point is that all these flows must commute among themselves, this is in fact easy to check. Furthermore, the commutativity of the flows implies "*zero curvature representation*"

$$\frac{\partial}{\partial t_m} L^n_+ - \frac{\partial}{\partial t_n} L^m_+ = [L^m_+, L^n_+], \qquad \forall n, m \tag{4.1.12}$$

or equivalently

$$\frac{\partial}{\partial t_m} L^n_- - \frac{\partial}{\partial t_n} L^m_- = [L^m_-, L^n_-], \qquad \forall n, m \tag{4.1.13}$$

Roughly speaking, two systems of equations (4.1.9) and (4.1.12) are equivalent to each other. However, rigorously they have some differences. At first eq.(4.1.9) contains infinite many quantities (or,*coordinates*) u_i's over the infinite dimensional parameter space. But, in the zero curvature representation (4.1.12), suppose $n > m$, it only forms a system of $(n-1)$ equations involving $(n-1)$ unknown quantities $u_1, \ldots, u_{n-1}$[§]. For example, in the case $n = 3, m = 2$, let

$$t_1 = x, \quad t_2 = y, \quad t_3 = t; \qquad u(x,y,t) = 2u_1(t_1, t_2, t_3)$$

[§]It is easy to see that u_0 is constant, could be set equal to zero.

eq.(4.1.12) are two equations of u and u_2, after eliminating u_2, we get

$$3u_{yy} = \left(4u_t - u''' - 6uu'\right)' \tag{4.1.14}$$

which is Kadomtsev–Petviashvili(KP) equation. Usually we use the terminology KP hierarchy for eq.(4.1.9) or eq.(4.1.12) with all positive integers n, m.

A system of finite degrees of freedom n is integrable if and only if there exist n independent quantities, which are in involution(commutative with respect to the Poisson brackets)[51]. As to the system of infinite many degrees of freedom, for example, the KP hierarchy , we may find various definitions for it. The essential point is that there must exist infinite many conserved quantities in involution. Therefore we can list some of the definitions below

1. *There exist two compatible Poisson brackets (or bi-hamiltonian structure).*

   Suppose that we have a set of equations of motion

$$\frac{\partial}{\partial t_1} u_i = f_i(u, u', u'', \cdots), \tag{4.1.15}$$

   Since there exist bi-hamiltonian structure, so we must be able to represent them in Hamiltonian form, i.e.

$$\frac{\partial}{\partial t_1} u_i = \{H_1, u_i\}_2 = \{H_2, u_i\}_1. \tag{4.1.16}$$

   Where H_1, H_2 are two different Hamiltonians. Evidently, $\{H_2, u_i\}_2$ will generate new flows like

$$\frac{\partial}{\partial t_2} u_i = \{H_2, u_i\}_2, \tag{4.1.17}$$

   where t_2 is a new flow parameter. Due to the existence of the bi-hamiltonian structure, there must be another Hamiltonian, say H_3, such that

$$\{H_2, u_i\}_2 = \{H_3, u_i\}_1. \tag{4.1.18}$$

   Now we can further use H_3 to generate other new flows. In this way, out of question, we can obtain an infinite series of Hamiltonians and flows ¶. The problem is to show their commutativity. We remember that in original system, we have

$$\{H_1, H_2\}_1 = \{H_1, H_2\}_2 = 0, \tag{4.1.19}$$

   since they are Hamiltonians. Furthermore, the compatibility of two Poisson brackets enables us to write down (see eq.(4.1.10))

$$\{H_n, H_m\}_1 = \{H_{n-1}, H_m\}_2 = \{H_{n-1}, H_{m+1}\}_1 = \cdots, \tag{4.1.20}$$

   Let $n = m = 2$, we get $\{H_1, H_3\}_1 = 0$. Let $n = 2, m = 3$, we have $\{H_1, H_4\}_1 = \{H_2, H_3\}_1 = 0$. Doing this procedure repeatedly, we are able to prove that all the Hamiltonians commute with one another.

¶For a system of finite degrees of freedom, all of these Hamiltonians are not totally independent.

2. *The flows are all commutative.*

   Suppose that we have some Poisson bracket. Since we have the infinite many flows, so we must have the infinite many 'Hamiltonians'(we can make this assumption even if we don't know what the Hamiltonians are), such that the equations of motion can be written as

$$\frac{\partial}{\partial t_n} f = \{H_n, f\}, \qquad \forall \mathrm{f}.$$

   Then the commutativity says

$$\frac{\partial}{\partial t_m}\Big(\frac{\partial}{\partial t_n} f\Big) = \frac{\partial}{\partial t_n}\Big(\frac{\partial}{\partial t_m} f\Big),$$

   or in other words

$$\{H_n, \{H_m, f\}\} + \{\{H_m, H_n\}, f\} = \{H_m, \{H_n, f\}\} + \{\{H_n, H_m\}, f\},$$

   using the Jacobi identity, we immediately get

$$\{\{H_m, H_n\}, f\} = 0, \qquad \forall \text{for any given function f.} \tag{4.1.21}$$

   which shows that the infinite many Hamiltonians are in involution, i.e.

$$\{H_m, H_n\}. \tag{4.1.22}$$

3. *There exists the zero curvature representation(4.1.12).*

   This is obviously equivalent to the second definition, since from the commutativity of the flows we immediately obtain the zero curvature representation, and vice versa.

We may find other definitions. But the above ones are the most important, since the bi-hamiltonian structure directly leads to the W_∞ algebraic symmetry, and using commutativity is the easiest way to prove the integrability ‖.

4.1.4 The Baker–Akhiezer function and the associated linear systems

The KP operator L can be expressed in terms of "dressing" operator

$$L = \hat{K}\partial\hat{K}^{-1}, \qquad \hat{K} = 1 + \sum_{i=1}^{\infty} w_i \partial^{-i}, \qquad K = 1 + \sum_{i=1}^{\infty} w_i \lambda^{-i}.$$

‖The commutativity of the flows reflects the bi-hamiltonian structure only in single Lax operator case. We will see later that in multi-Lax operator case, there are multi-series of hamiltonians which are not completely commutative but the flows do commute.

The KP hierarchy can be written as

$$\frac{\partial}{\partial t_r}\hat{K} = -L^r_-\hat{K}. \tag{4.1.23}$$

Define

$$\xi(t,\lambda) = \sum_{r=1}^{\infty} t_r\lambda^r, \tag{4.1.24}$$

then Baker–Akhiezer function and its adjoint take the following form

$$\Psi(t,\lambda) = \hat{K}e^{\xi(t,\lambda)}, \qquad \Psi^*(t,\lambda) = \hat{K}^{*-1}e^{-\xi(t,\lambda)}, \tag{4.1.25}$$

where the superscripts '*' means complex conjugation. In terms of these Baker–Akhiezer function, we may associate a linear system to KP hierarchy (4.1.9)

$$\begin{cases} L\Psi = \lambda\Psi, \\ \frac{\partial}{\partial t_r}\Psi = L^r_+\Psi. \end{cases} \tag{4.1.26}$$

and its dual form is

$$\begin{cases} L^*\Psi^* = \lambda\Psi^*, \\ \frac{\partial}{\partial t_r}\Psi^* = -(L^r_+)^*\Psi^*. \end{cases} \tag{4.1.27}$$

The consistency conditions give back the KP hierarchy. Since the flows preserve the forms of "L" and the Baker–Akhiezer function, they are nothing but the orbits of the symmetries generated by Hamiltonians.

Now we denote the residue of the usual Laurent expansion as

$$\mathrm{res}_\lambda \sum a_i\lambda^i = a_{-1}.$$

Let $P = \sum p_i\partial^i, Q = \sum q_i\partial^i$ are PDO's, then it is straightforward to see that

$$\mathrm{res}_\lambda\Big[\Big(Pe^{\xi(t,\lambda)}\Big)\cdot\Big(Qe^{-\xi(t,\lambda)}\Big)\Big] = \mathrm{res}_\partial PQ^*. \tag{4.1.28}$$

Choosing $P = \hat{K}, Q = \hat{K}^{*-1}$, we immediately obtain

$$\mathrm{res}_\lambda\Big(\Psi(t,\lambda)\cdot\Psi^*(t,\lambda)\Big) = res_\partial(\hat{K}\hat{K}^{-1}) = 0,$$

similarly

$$\mathrm{res}_\lambda\Big(\partial_r\Psi(t,\lambda)\cdot\Psi^*(t,\lambda)\Big) = \mathrm{res}_\partial(L^r_+\hat{K}\hat{K}^{-1}) = 0,$$

where we denote $\frac{\partial}{\partial t_r}$ by ∂_r, and we used the equations of motion of Ψ. Generally, we have

$$\mathrm{res}_\lambda\Big(\partial_r^n\Psi(t,\lambda)\cdot\Psi^*(t,\lambda)\Big) = 0,$$

These identities enable us to get the famous bilinear identities

$$\mathrm{res}_\lambda\Big(\Psi(t,\lambda)\cdot\Psi^*(t',\lambda)\Big) = 0. \tag{4.1.29}$$

This is a very useful formula both for calculating the residue of the operators and for deriving other identities.

4.2 The τ–function

The τ–function plays a central role in KP analysis. There are several ways to introduce the τ–function . In each of the definitions, the basic requirement on the τ–function is the Hirota's bilinear relation (4.1.29).

4.2.1 τ–function and Baker–Akhiezer function

Knowing Baker–Akhiezer function, we can use the following involved relation to extract τ–function, and vice versa

$$\Psi(t,\lambda) = \frac{\tau(t_1 - \frac{1}{\lambda}, t_2 - \frac{1}{2\lambda^2}, \ldots)}{\tau(t)} e^{\xi(t,\lambda)}. \tag{4.2.1}$$

Inverting this formula, we get

$$\partial_i \ln \tau = -\mathrm{res}_\lambda \Big(\lambda^i \big[\sum_{l\geq 1} \lambda^{-l-1} \partial_l - \frac{\partial}{\partial\lambda} \big] \ln K \Big). \tag{4.2.2}$$

For convenience, we introduce the vertex operators

$$V(t,\lambda) = e^{\xi(t,\lambda)} e^{-\xi(\tilde{D},\frac{1}{\lambda})} \qquad \tilde{V}(t,\lambda) = e^{-\xi(t,\lambda)} e^{\xi(\tilde{D},\frac{1}{\lambda})} \tag{4.2.3}$$

where

$$\tilde{D} = (\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots).$$

Then the eqs.(4.1.29) transforms into the Hirota's bilinear relation of the τ–function[57]

$$\mathrm{res}_\lambda \Big[\big(V(t,\lambda)\tau(t) \big) \cdot \big(\tilde{V}(t,\lambda)\tau(t) \big) \Big] = 0 \tag{4.2.4}$$

If we further introduce the Hirota's bilinear operator [57],

$$P(x) f(x) \cdot g(x) = P(\tilde{D}) f(x-y) g(x+y)|_{y=0}$$

for any functions $f(x), g(x)$ and any polynomial $P(x)$. Then the above relation becomes

$$\sum_{j=0}^{\infty} S_j(-2y) S_{j+1}(x) \exp(\sum_{i=1}^{\infty} y_i x_i) \tau(x) \cdot \tau(x) = 0, \tag{4.2.5}$$

where $(y = y_1, y_2, \cdots,)$ are arbitary parameters, and $S_j(t)$ are the elementary Schur polynomials

$$\sum_{j=0}^{\infty} S_j(t) \lambda^j = \exp(\sum_{i=1}^{\infty} t_i \lambda^i). \tag{4.2.6}$$

The elementary Schur polynomials are related to the complete symmetric functions h_k, which is the sum of all monomials of total degree k in the variables $z_1, z_2, \cdots, z_N$. The generating function for h_k is

$$\sum_{k \geq 0} h_k \lambda^k = \prod_{i=1}^{N} (1 - z_i \lambda)^{-1}, \tag{4.2.7}$$

for large N. Using the Miwa transformation[58]

$$t_j = \frac{z_1^j + z_2^j + \cdots + z_N^j}{j}, \tag{4.2.8}$$

we see that

$$S_k(t) = h_k(z_1, z_2, \cdots, z_N). \tag{4.2.9}$$

For a non–increasing finite sequence of positive integers $\nu = \{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k > 0\}$, we can associate the *Schur polynomial* defined by the $k \times k$ determinant

$$S_\nu = \det(S_{\nu_i + j - i}(t)) = \begin{vmatrix} S_{\nu_1} & S_{\nu_1+1} & S_{\nu_1+2} & \cdots \\ S_{\nu_2-1} & S_{\nu_2} & S_{\nu_2+1} & \cdots \\ S_{\nu_3-2} & S_{\nu_3-1} & S_{\nu_3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ S_{\nu_k-k+1} & S_{\nu_k-k+2} & S_{\nu_k-k+3} & \cdots \end{vmatrix}. \tag{4.2.10}$$

The Hirota's bilinear relation (4.2.5) is quite essential. It can be considered as the definition of KP hierarchy: if a τ function satisfies this bilinear equation, then it is a τ–function of KP hierarchy. This opens a way to find a lot of other expressions of the τ–function , and construct solutions of the KP hierarchy .

4.2.2 τ–function and conservation laws

One of the important properties of τ–function is its relation to the conservation laws. In order to see this point, we at first note that the eqs.(4.1.13) lead to

$$\frac{\partial}{\partial t_r}(\text{res}_{\,\partial} L^s) = \frac{\partial}{\partial t_s}(\text{res}_{\,\partial} L^r). \tag{4.2.11}$$

This implies the existence of the potential, which we call τ–function, which generates the conservation laws

$$\frac{\partial}{\partial t_r} \ln \tau = \partial^{-1} \text{res}_{\,\partial} L^r, \tag{4.2.12}$$

or in another form

$$\frac{\partial^2}{\partial t_1 \partial t_r} \ln \tau = \text{res}_{\,\partial} L^r. \tag{4.2.13}$$

The integrals of motion, or conservation laws have the form

$$\frac{\partial}{\partial t_i} J_l = 0, \qquad \forall i, l. \tag{4.2.14}$$

Now let us define

$$J_r = \int \text{res}_{\partial} L^r dx, \tag{4.2.15}$$

since for any two PDO, the residue of their commutator is a total derivative, i.e.

$$\text{res}_{\partial}[X, Y] = \partial h, \qquad \forall X, Y \in \wp, \tag{4.2.16}$$

we can immediately check that the integrals defined in the eqs.(4.2.15) are the integrals of motion[52]. Comparing the definition (4.2.15) and eqs.(4.2.13), we see that

$$J_r = \int \frac{\partial^2}{\partial t_1 \partial t_r} \ln \tau. \tag{4.2.17}$$

which is a *constant* only dependent on the boundary terms.

4.2.3 τ–function and Vertex operator: Bosonic Fock space I

Since $\{\frac{\partial}{\partial t_r}, t_s\}$ form Heisenberg algebras and can be considered as the oscillators of free bosonic field. The vacuum usually defined as

$$\frac{\partial}{\partial t_r}|0>= 0, \qquad \forall r \geq 1$$

and

$$V(t, \lambda) = e^{\xi(t,\lambda)} e^{-\xi(\tilde{D}, \frac{1}{\lambda})} =: e^{\phi(\lambda)} : \tag{4.2.18}$$
$$\tilde{V}(t, \lambda) = e^{-\xi(t,\lambda)} e^{\xi(\tilde{D}, \frac{1}{\lambda})} =: e^{-\phi(\lambda)} : \tag{4.2.19}$$

with the bosonic field

$$\phi(\lambda) = \sum_{r=1}^{\infty} t_r \lambda^r - \sum_{r=1}^{\infty} \frac{1}{r} \frac{\partial}{\partial t_r} \lambda^{-r}, \tag{4.2.20}$$

in terms of this field, we may introduce a current like

$$J(\lambda, \mu) =: e^{\phi(\lambda)} e^{-\phi(\mu)} : . \tag{4.2.21}$$

Generally a τ–function is a polynomial of the parameters $(t_1, t_2, \ldots)$. For example, the elementary Schur polynomials are solutions to (4.2.5). Thus we may identify τ–functions as a bosonic state. The most important point is that, under the action of the current $J(\lambda, \mu)$, the transformed τ–function still satisfy the Hirota's relation (4.2.5). This means that the space of τ-function possesses the symmetry generated by the current $J(\lambda, \mu)$, we will see, which is $GL(\infty)$.

In terms of this current (or vertex), we can construct the N-soliton solution

$$\begin{aligned} &\tau(t; \alpha_1, \lambda_1, \mu_1; \cdots; \alpha_N, \lambda_N, \mu_N) \\ &= e^{\alpha_1 J_1(\lambda_1, \mu_1)} e^{\alpha_2 J_2(\lambda_2, \mu_2)} \cdots e^{\alpha_N J_N(\lambda_N, \mu_N)} \cdot 1 \end{aligned} \tag{4.2.22}$$

Obviously $J(\lambda, \mu)$ mappes N-soliton solution to $(N+1)$-soliton solution. Furthermore, one can show that

$$J^2(\lambda, \mu)\tau(t) = 0.$$

Thus the eq.(4.2.22) can be written as

$$\begin{aligned} &\tau(t; \alpha_1, \lambda_1, \mu_1; \cdots; \alpha_N, \lambda_N, \mu_N) \\ &= \Big(1 + \alpha_1 J_1(\lambda_1, \mu_1)\Big)\Big(1 + \alpha_2 J_2(\lambda_2, \mu_2)\Big) \cdots \Big(1 + \alpha_N J_N(\lambda_N, \mu_N)\Big) \cdot 1. \end{aligned} \tag{4.2.23}$$

The proof that this τ-function really satifies the Hirota's bilinear equations can be found in [54][59]. Since the constant polynomial "1" is obviously a τ-function, the above formula (4.2.23) simply means that the space of τ-function is nothing but the orbit of vacuum under the action of $GL(\infty)$ group.

4.2.4 τ-function as VEV: Bosonic Fock space II

In the previous subsection we have given a construction of the τ-function . This τ-function is parametrized by the time parameters and the element specified by (α, λ, μ) of $GL(\infty)$ group. Obviously each point on the flow orbit belongs to the space of the τ-function . So setting $t = 0$, we get a particular parametrization of the τ-function , which only depends on the element A of $GL(\infty)$ group. This procedure is in fact equivalent to take the Vacuum Expectation Value (VEV), since the "*creation oscillators*" t_r's acting on the vacuum $< 0|$ equal to zero. Therefore we may identify τ-function with VEV

$$\tau(A) = < 0|R^B(A)|0 > . \tag{4.2.24}$$

where $R^B(A)$ means the bosonic representation of the $GL(\infty)$ element A.

4.2.5 τ-function as a fermionic state: Fermionic Fock space I

From the wellknown boson–fermion correspodence, we see that two vertex operators are nothing but fermion fields. Now we change our notations, denote them as

$$\Psi(\lambda) =: e^{\phi(\lambda)} := \sum_j \Psi_j \lambda^j \qquad \Psi^*(\lambda) =: e^{-\phi(\lambda)} := \sum_j \Psi_j^* \lambda^{-j}, \tag{4.2.25}$$

the anticommutators of the oscillators are

$$\{\Psi_i, \Psi_j\} = \{\Psi_i^*, \Psi_j^*\} = 0, \qquad \{\Psi_i, \Psi_j^*\} = \delta_{i,j}. \tag{4.2.26}$$

These anti-commutators imply the inner product, which can be denoted as

$$\Psi_i(\Psi_j^*) = \Psi_i^*(\Psi_j) = \delta_{ij}.$$

This inner product can be written in more transparant way, if we define the standard vacuum as

$$|0 >= \Psi_0 \wedge \Psi_{-1} \wedge \Psi_{-2} \wedge \cdots,$$

or more generally the $N-th$ vacuum

$$|N >= \Psi_N \wedge \Psi_{N-1} \wedge \Psi_{N-2} \wedge \cdots,$$

its dual form is

$$< N| = \cdots \wedge \Psi_{N+3} \wedge \Psi_{N+2} \wedge \Psi_{N+1}.$$

Due to the boson–fermion correspondence, any state (*for example, τ-function*), corresponds to a *fermionic state* reference to a certain vacuum(say $|0>$), in particular

$$S_{i_0, i_{-1}+1, i_{-2}+2, \cdots}(t) \Longrightarrow |W >= \Psi_{i_0} \wedge \Psi_{i_1} \wedge \Psi_{i_2} \wedge \cdots. \tag{4.2.27}$$

In terms of fermionic language, the current takes the form

$$J(\lambda, \mu) \Longrightarrow : \Psi(\lambda)\Psi^*(\mu) := \frac{1}{1-\frac{\mu}{\lambda}}\Psi(\lambda)\Psi^*(\mu) = \frac{1}{1-\frac{\mu}{\lambda}} \sum_{i,j} \lambda^i \mu^j \Psi_i \Psi_j^*. \tag{4.2.28}$$

This current is a bilinear operator, which has a natural representation on Fermionic Fock space (either the space formed by $\Psi(\lambda)$, or the one of $\Psi^*(\lambda)$), the representation is characterized by the infinite matrices, in particular $\Psi_i\Psi_j^*$ can be identified with E_{ij}. Therefore the current $J(\lambda, \mu)$ forms $gl(\infty)$ algebra.

Making use of the $GL(\infty)$ transformation, we may relate any state like (4.2.27) to the vacuum

$$\hat{R}^F(A)|0 >= |W >= \exp \sum_{i,j} A_{ij} \Psi_j \Psi_i^* |0 > . \tag{4.2.29}$$

After mapping back to the bosonic Fock space, the element of $GL(\infty)$ is represented by the current operator, so the above formula just says that the Schur polynomial lies in the orbit of vacuum under the $GL(\infty)$ action, therefore it is τ–function . More generally, if we have two states $|W_1>$ and $|W_2>$, corresponding to S_μ and S_ν, respectively, they are related by a $GL(\infty)$ transformation $\hat{R}(A)$, i.e.

$$|W_1 > \rightarrow S_\mu, \qquad |W_2 > \rightarrow S_\nu, \qquad |W_2 >= \hat{R}(A)|W_1 > .$$

then we can show that

$$S_\nu = \hat{R}(A) S_\mu = \sum_{\{\rho\}} \det(A^{\mu_1, \mu_2 - 1, \cdots}_{\rho_1, \rho_2 - 1, \cdots}) S_\rho. \tag{4.2.30}$$

If we set $\nu = 0$, we see that S_μ is a determinant of certain matrix A. So this confirms that the fermionic state, so as to τ–function , is completely characterized by the $GL(\infty)$ transformation.

4.2.6 τ–function and VEV: Fermionic Fock space II

As we saw in §4.2.4, we may consider the τ–function as VEV. In fermionic Fock space, this means that we may identify τ–function with

$$\tau(A) = <0|W> = \det(A). \tag{4.2.31}$$

In order to derive the Hirota's bilinear relation in fermionic language, we should take care of the *normalization* problem. Since the bosonic vertex operator acting on τ–function is equivalent to one fermion field insertion, i.e.

$$<0|W>, \qquad \text{insertion one fermion} \qquad \Longrightarrow <1|\Psi(\lambda)|W>.$$

Suppose $|W> = |0>$, we have

$$<1|\Psi(\lambda)|0> = \lambda <0|0> + \cdots.$$

The first term represents the standard vacuum to vacuum amplitude, but with a λ factor, so we may associate a λ dependent factor to non–standard vacua, i.e.

$$\Psi(\lambda) : |0> \Longrightarrow \lambda^{-1}|1> + \cdots.$$

Similarly, for anti–fermions

$$<-1|\Psi^*(\lambda)|0> = <0|0> + \cdots$$

which shows

$$\Psi^*(\lambda) : |0> \Longrightarrow |-1> + \cdots. \tag{4.2.32}$$

Remembering these identifications, we can reexpress the eq.(4.2.4) as follows

$$\mathrm{res}_\lambda \left[\frac{\lambda^{-1} <1|\Psi(\lambda)|\tau>}{\tau} \cdot \frac{<-1|\Psi^*(\lambda)|\tau>}{\tau} \right] = 0, \tag{4.2.33}$$

or in another form

$$\sum_j \Psi_j(\tau) \otimes \Psi_j^*(\tau) = 0. \tag{4.2.34}$$

This is the fermionic version of the Hirota's bilinear identity[54]. It is invariant under the $GL(\infty)$ transformation generated by current $J(\lambda, \mu)$, so all the orbit of vacuum under this transformation will correspond to τ–function.

4.2.7 The τ–function and determinant: Fermionic Fock Space III

Comparing the eq.(4.2.33) and the eq.(4.2.4), we see that $\lambda^{-1}\Psi(\lambda)$ insertion into the vacuum expectation value $<0|W>$ correponds to a vertex operator $V(t, \lambda)$ acting on τ–function . Therefore we may even view $<0|W>$ as the definition of τ–function .

Now let us consider the N–fermions insertion

$$< -N|\Psi^*(\lambda_1)\Psi^*(\lambda_2)\cdots\Psi^*(\lambda_N)|W> = <0|\Psi_0\cdots\Psi_{-N+1}\Psi^*(\lambda_1)\cdots\Psi^*(\lambda_N)|W>,$$

after pairing the fermions and the anti–fermions, which is just a particular combination of the elements of $GL(\infty)$ group. Therefore it indeed gives a τ–function. The normalization is choosen such that the vacuum to vacuum amplitude is constant "1", so

$$\tau(\lambda; w) = \frac{1}{\Delta(\lambda)} < -N|\Psi^*(\lambda_1)\Psi^*(\lambda_2)\cdots\Psi^*(\lambda_N)|W>, \tag{4.2.35}$$

where the normalization factor is

$$\Delta(\lambda) = < -N|\Psi^*(\lambda_1)\Psi^*(\lambda_2)\cdots\Psi^*(\lambda_N)|0> = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1} \end{vmatrix}$$

Substituting the eq.(4.2.29) into (4.2.35), and introducing a set of new polynomials

$$w_i(\lambda) = \lambda^i + \sum_{j=1}^{\infty} A_{ij}\lambda^{-j}, \tag{4.2.36}$$

we can derive the following determinant formula for τ–function

$$\tau(\lambda; w) = \frac{\Delta(\lambda; w)}{\Delta(\lambda)}, \tag{4.2.37}$$

where we denote the '*generalized Vandermonde determinant*' by

$$\Delta(\lambda; w) = \det(w_{i-1}(\lambda_j)) \begin{vmatrix} w_0(\lambda_1) & w_0(\lambda_2) & \cdots & w_0(\lambda_N) \\ w_1(\lambda_1) & w_1(\lambda_2) & \cdots & w_1(\lambda_N) \\ \vdots & \vdots & \cdots & \vdots \\ w_{N-1}(\lambda_1) & w_{N-1}(\lambda_2) & \cdots & w_{N-1}(\lambda_N) \end{vmatrix}$$

Right now the τ–function is parametrized by the fermionic state $|W>$ and the N–parameters $(\lambda_1, \lambda_2, \cdots, \lambda_N)$, which is related to the time parameter by the Miwa transformation (4.2.8)[58].

4.2.8 The τ–function and Grassmanian

The free fermions $\Psi(\lambda)$ and $\Psi^*(\lambda)$ are 2–dimensional chiral fields, any transformation of the space coordinates (λ) will result in the the transformation of the basis of the creation and annihillation oscillators, for example, on the *Hilbert space H* of the fermions Ψ_n's, any $GL(\infty)$ element $\hat{R}(A)$ will send this basis to a new one

$$|W> = \exp\Big(\sum_{ij} A_{ij}\Psi_j\Psi_i^*\Big)|0> = (\Psi_0 + \sum_{j=1}^{\infty} A_{o,j}\Psi_j)\wedge(\Psi_{-1} + \sum_{j=1}^{\infty} A_{-1,j}\Psi_j)\wedge\cdots,$$

we may consider

$$\tilde{\Psi}_{-i} = \Psi_{-i} + \sum_{j=1}^{\infty} A_{-i,j}\Psi_j, \qquad \forall i \geq 0,$$

as new basis*, then

$$|W >= \tilde{\Psi}_0 \wedge \tilde{\Psi}_{-1} \wedge \tilde{\Psi}_{-2} \wedge \cdots,$$

is a new vacuum. When A runs over $GL(\infty)$ group, all the different vacua form a *Grassmanian manifold.* As we see in the last subsection, this new vacuum can be identified with a τ–function, thus each point of the Grassmanian manifold gives a solution to KP hierarchy [53][60][61].

4.3 The conjugate operator and new flows

Now we come back to the KP hierarchy (4.1.9). Our purpose is to show that we may introduce other series of flows. In order to do so, we define a new operator[62],

$$M \equiv \hat{K} \sum_{i=1}^{\infty} r t_r \partial^{r-1} \hat{K}^{-1} = \sum_{i=-\infty}^{\infty} v_i \partial^i. \tag{4.3.1}$$

which is conjugate to KP operator L in the sense

$$[L, M] = \hat{K}[\partial, \sum_{i=1}^{\infty} r t_r \partial^{r-1}]\hat{K}^{-1} = 1, \tag{4.3.2}$$

Obviously, we can derive the equations of motion for M,

$$\begin{cases} \frac{\partial}{\partial t_r} M = [L_+^r, M], \\ \frac{\partial}{\partial \lambda}\Psi = M\Psi. \end{cases} \tag{4.3.3}$$

So we see that L and M are nothing but the operatorial expressions of $\lambda, \frac{\partial}{\partial \lambda}$ (acting on $\Psi(t, \lambda)$. With respect to KP operator L, we have a series of flows, we will see that with respect to the operator M, we may also introduce another series of flows.

4.3.1 Another series of flows

Our starting point is as follows, the t–series of perturbations is due to the fact $[L, L_-^r] \in \mathcal{P}_-, \forall r \geq 1$. Now we also have $[L, M_-^r] \in \mathcal{P}_-, \forall r \geq 1$, so we could introduce a new series of deformation parameters $y_1, y_2, y_3, \ldots$, such that

$$\begin{cases} \frac{\partial}{\partial y_r} L = [L, M_-^r], \\ \frac{\partial}{\partial y_r}\Psi = -M_-^r \Psi. \end{cases} \tag{4.3.4}$$

*We should properly choose $\tilde{\Psi}_i (i \geq 1)$ such that the new basis do satisfy the basic anticommutators (4.2.26).

All of these equations together result in the following enlarged KP system

$$\begin{cases} \frac{\partial}{\partial t_r} L = [L_+^r, L], \\ \frac{\partial}{\partial t_r} M = [L_+^r, M], \\ \frac{\partial}{\partial y_r} L = [L, M_-^r], \\ \frac{\partial}{\partial y_r} M = [M_+^r, M]. \end{cases} \tag{4.3.5}$$

Now we should prove that these new series of perturbations will not distroy consistency, that is to say, we should check the commutativity of all these flows. In the following we only consider an example, i.e.

$$\frac{\partial}{\partial t_r}(\frac{\partial}{\partial y_s}L) = \frac{\partial}{\partial y_s}(\frac{\partial}{\partial t_r}L). \tag{4.3.6}$$

Using the eqs.(4.3.5), we see that the left hand side is

$$\begin{aligned} \frac{\partial}{\partial t_r}(\frac{\partial}{\partial y_s}L) &= \frac{\partial}{\partial t_r}[L, M_-^s] \\ &= [[L_+^r, L], M_-^s] + [L, [L_+^r, M^s]_-] \\ &= [[L_+^r, L], M_-^s] + [L, [L_+^r, M_-^s]] - [L, [L_+^r, M_-^s]_+] \\ &= [L_+^r, [L, M_-^s]] + [[L^r, M_-^s]_+, L] \\ &= \text{r.h.s.} \end{aligned}$$

The other case can be checked in the similar way. So the perturbations we introduced before indeed give a enlarged KP hierarchy . Its associated linear system is

$$\begin{cases} L\Psi = \lambda\Psi, \\ \frac{\partial}{\partial t_r}\Psi = L_+^r\Psi, \\ \frac{\partial}{\partial y_r}\Psi = -M_-^r\Psi, \\ M\Psi = \frac{\partial}{\partial \lambda}\Psi. \end{cases} \tag{4.3.7}$$

One of the most important new features of this enlarged hierarchy (4.3.5) is the following point, the form of KP operator L (or M) does depend on the perturbations of y_r's (or t_r's). The usual KP hierarchy (4.1.9) is a particular case of eqs.(4.3.5) by fixing the y–series of the perturbations. We will show this point in more detail in the next subsection.

4.3.2 The new bi–hamiltonian structure

As we remarked a moment ago, when we disgard the y–series of flows, we recover the usual KP hierarchy discussed in §4.1, whose hamiltonians are

$$H_{r(L)} = \frac{1}{r} TrL^r,$$

here we use the subindex (L) to indicate that the Hamiltonians are constructed from the KP operator L. We may also use the same symbol to denote the Poisson brackets, $\{,\}_{(L)}$.

Now if we fix all the t–series of parameters, then we get another subset of the enlarged hierarchy (4.3.5), that is

$$\begin{cases} \frac{\partial}{\partial y_r} L = [L, M_-^r], \\ \frac{\partial}{\partial y_r} M = [M_+^r, M]. \end{cases} \tag{4.3.8}$$

The second equation is in fact a KP hierarchy with KP operator M of the the form (4.3.1). Since all these flow do commute, so it is an integrable system, and there should exist two compatible Poisson brackets written in terms of coordinates v_i's. However, these bi–hamiltonian structure is unknown since the positive powers of ∂ in M go to infinity. But we may guess that

$$\{H_{r(M)}, M\}_{(M)} = [M, M_-^r]. \tag{4.3.9}$$

where the Hamiltonians are

$$H_{r(M)} = \frac{1}{r}\mathrm{Tr}M^r, \qquad \forall r \geq 1. \tag{4.3.10}$$

Obviously we have

$$\frac{\partial}{\partial y_1}u_1 = [L, M_-]_{(-1)} = \{H_{1(M)}, u_1\}_{(M)} \neq 0.$$

(since u_1 is y–dependent). Using this fact, we get

$$\{H_{1(M)}, H_{1(L)}\}_{(M)} = \int dx \frac{\partial}{\partial y_1} u_1.$$

which is usually nonzero. Therefore we find that the two series of hamiltonians are not commuting.

4.3.3 The new basic derivatives

Till now what we have done is treating t_1 as the space coordinate. For later convenience, we denote $\frac{\partial}{\partial y_1}$ by $\bar{\partial}$. From the y_1–flows of Ψ, we may extract an operator identity

$$\bar{\partial} = -M_- = \sum_{i=1}^{\infty} \Gamma_i \partial^{-i}. \tag{4.3.11}$$

Since any positive powers of $\tilde{\partial}$ belongs to $\mathcal{P}_-(\partial)$, so $\{\tilde{\partial}^i; i \geq 1\}$ forms a basis of $\mathcal{P}_-(\partial)$, this enable us to express ∂ in terms of the new derivative* $\tilde{\partial}$

$$\partial = \sum_{i=1}^{\infty} \tilde{\Gamma}_i \tilde{\partial}^{-i}. \tag{4.3.12}$$

Using this fact, we get

$$M = -(\tilde{\partial} + \sum_{i=1}^{\infty} \tilde{v}_i \tilde{\partial}^{-i}) = -\tilde{K}\tilde{\partial}\tilde{K}^{-1}. \tag{4.3.13}$$

Obviously

$$M_-^r(\partial) = M_+^r(\tilde{\partial}), \qquad \forall r \geq 1, \tag{4.3.14}$$

where LHS is expanded in the powers of ∂, while RHS is expanded in the powers of $\tilde{\partial}$. Using eq.(4.3.12), we can reexpress all the formulas (4.3.5) in terms of this new derivative $\tilde{\partial}$, i.e.

$$\begin{cases} \frac{\partial}{\partial y_r}(-M(\tilde{\partial})) = (-1)^{r+1}[(-M)_+^r(\tilde{\partial}), (-M)(\tilde{\partial})], \\ \frac{\partial}{\partial y_r} L(\tilde{\partial}) = (-1)^{r+1}[(-M)_+^r(\tilde{\partial}), L(\tilde{\partial})], \\ \frac{\partial}{\partial t_r}(-M)(\tilde{\partial}) = -[(-M)(\tilde{\partial}), L_-^r(\tilde{\partial})], \\ \frac{\partial}{\partial t_r} L(\tilde{\partial}) = [L_+^r(\tilde{\partial}), L(\tilde{\partial})]. \end{cases} \tag{4.3.15}$$

apart from some additional signs, these equations are isomorphic to (4.3.5). This reminds us that we can even consider $(-M)$ as a KP operator, and alternatively interpret y_1 as coordinate, all the other parameters as time parameters. Therefore we can define two compatible Poisson brackets for KP operator $(-M)$ by simply replacing L in (4.1.7) and (4.1.11) by $(-M)$, which shows that on the space y_1, the fields v_i's form W_∞ algebra too.

4.4 Further perturbations and the full generalized KP hierarchy

In the previous section we have shown that the KP hierarchy can be perturbed by the conjugate operator M of the KP operator L. In fact, KP system allows further deformations. This is what we want to discuss in this section[16].

*Rigorously speaking, this is only true when it acts on the function Ψ. But we may think of it in the following way, starting from

$$\tilde{\partial}\Psi = \sum_{i=1}^{\infty} \Gamma_i \partial^{-i}\Psi.$$

properly choosing the combinations of $\tilde{\partial}$ such that we can reexpress the $\partial^{-1}\Psi$ in terms of new derivatives $\tilde{\partial}$, we replace all the derivatives ∂ in the linear system (4.3.7) by $\tilde{\partial}$. So we may interpret y_1 as another space coordinate.

4.4.1 The new series of the flows

In order to explain the further perturbations just mentioned, we change a little bit our notation. Denote t_r's and y_r's by t_{1r} and t_{2r} respectively. Furthermore we define

$$L(1) \equiv L, \qquad V'(1) \equiv \sum_{r=1}^{\infty} r t_{1r} L^{r-1}(1)$$

$$L(2) \equiv -\frac{1}{c_{12}} M \qquad V'(2) \equiv \sum_{r=1}^{\infty} r t_{2r} L^{r-1}(2)$$

Now let us introduce new operators in the following way

$$L(\alpha) \equiv -\frac{1}{c_{\alpha-1,\alpha}} \Big(c_{\alpha-2,\alpha-1} L(\alpha-2) + V'(\alpha-1) \Big) \tag{4.4.1a}$$

$$V'(\alpha) = \sum_{r=1}^{\infty} r t_{\alpha,r} L^{r-1}(\alpha), \qquad \alpha = 3, 4, \ldots, n \tag{4.4.1b}$$

where $c_{\alpha,\alpha+1}$'s are arbitary constants, which amount to rescaling the space coordinates, and n is arbitrary positive integer number. Then, in the same way, we can perturb the system further as follows

$$\frac{\partial}{\partial t_{\beta r}} L(\alpha) = [L_+^r(\beta), L(\alpha)], \qquad 1 \leq \beta < \alpha \tag{4.4.2a}$$

$$\frac{\partial}{\partial t_{\beta r}} L(\alpha) = [L(\alpha), L_-^r(\beta)], \qquad \alpha \leq \beta \leq n \tag{4.4.2b}$$

Now in order to justify the consistency of these perturbations, we once again should prove that all the flows commute among themselves. Let us check one example,

$$\frac{\partial}{\partial t_{\alpha l}} \Big(\frac{\partial}{\partial t_{\beta m}} L(\gamma) \Big) = \frac{\partial}{\partial t_{\beta m}} \Big(\frac{\partial}{\partial t_{\alpha l}} L(\gamma) \Big), \qquad \alpha < \beta < \gamma.$$

Using the above hierarchy and Jacobi identities, we see that

$$\text{l.h.s.} = \frac{\partial}{\partial t_{\alpha l}} [L_+^m(\beta), L(\gamma)] = [[L_+^l(\alpha), L^m(\beta)], L(\gamma)] + [L_+^m(\beta), [L_+^l(\alpha), L(\gamma)]].$$

and

$$\text{r.h.s.} = \frac{\partial}{\partial t_{\beta m}} [L_+^l(\alpha), L(\gamma)] = [[L^l(\alpha), L_-^m(\beta)]_+, L(\gamma)] + [L_+^l(\alpha), [L_+^m(\beta), L(\gamma)]].$$

The first term of "l.h.s" can be written as

$$\begin{aligned} \text{the 1st term} &= [[L_+^l(\alpha), L_+^m(\beta)], L(\gamma)] + [[L_+^l(\alpha), L_-^m(\beta)]_+, L(\gamma)] \\ &= [[L_+^l(\alpha), L_+^m(\beta)], L(\gamma)] + [[L^l(\alpha), L_-^m(\beta)]_+, L(\gamma)], \end{aligned}$$

therefore

$$\begin{aligned}
\text{l.h.s.} &= [[L^l(\alpha), L_-^m(\beta)]_+, L(\gamma)] + [[L_+^l(\alpha), L_+^m(\beta)], L(\gamma)] + [L_+^m(\beta), [L_+^l(\alpha), L(\gamma)]] \\
&= [[L^l(\alpha), L_-^m(\beta)]_+, L(\gamma)] + [L_+^l(\alpha), [L_+^m(\beta), L(\gamma)]] \\
&= \text{r.h.s.}
\end{aligned}$$

All the other cases can be done in the same way. Therefore (4.4.2a) and (4.4.2b) really define an integrable system. The associated linear system is

$$\begin{cases}
L(1)\Psi = \lambda\Psi, \\
\frac{\partial}{\partial t_{1,r}}\Psi = L_+^r(1)\Psi, \\
\frac{\partial}{\partial t_{\alpha,r}}\Psi = -L_-^r(\alpha)\Psi, \qquad \alpha = 2, 3, \ldots, n, \\
M\Psi = \frac{\partial}{\partial\lambda}\Psi.
\end{cases} \tag{4.4.3}$$

In fact we can rewrite this linear system in a better way by choosing a new function

$$\Psi(\lambda, t) \Longrightarrow \xi(\lambda, t) = exp(-\sum_{r=1}^{\infty} t_{1,r}\lambda_1^r)\Psi(\lambda, t),$$

then all the flows can be summarized by a single equation

$$\frac{\partial}{\partial t_{\alpha,r}}\xi = -L_-^r(\alpha)\xi. \tag{4.4.4}$$

The consistency conditions of the above linear system exactly give the hierarchy (4.4.2a) and (4.4.2b). We would like to remark that the hierarchy (4.4.2a) and (4.4.2b) have several important sub-hierarchies.

(*). $\alpha = \beta = 1$, the eqs.(4.4.2b) are nothing but the usual KP hierarchy (4.1.9).

(**). $2 \le \alpha = \beta \le n$, the eqs.(4.4.2b) give $(n-1)$ KP hierarchies whose KP operator possess the form (4.4.1a).

(* * *). Although all the flows commute, generally these n-series of hamiltonians are not commutative.

We may conclude that the hierarchy (4.4.2a) and (4.4.2b) possess n bi-hamiltonian structures, each of them generates a KP hierarchy , all of these hierarchies couple together. Although the hamiltonians in different series are not commutative, the commutativity of the flows really guarantees integrability.

4.4.2 New bi-hamiltonian structures

In the above analysis, all the operators are expanded in terms of ∂. However, if we use the same trick as the one in §4.3.3, it is not difficult to reexpress them in terms of any one of

$\frac{\partial}{\partial t_{\alpha,1}}$'s. Let us define

$$\partial_\alpha \equiv \frac{\partial}{\partial t_{\alpha,1}} \tag{4.4.5}$$

and expand $L_-(\alpha)$ in powers of ∂

$$L_-(\alpha) = -\sum_{i=1}^{\infty} \Gamma_i^{(\alpha)} \partial^{(-i)} \tag{4.4.6}$$

then the hierarchy and the linear system suggest

$$\partial_\alpha = \sum_{i=1}^{\infty} \Gamma_i^{(\alpha)} \partial^{(-i)} \tag{4.4.7}$$

similar to the argument in the previous subsection, we can invert these relations, such that

$$\partial = \sum_{i=1}^{\infty} \tilde{\Gamma}_i^{(\alpha)} \partial_\alpha^{(-i)} \tag{4.4.8}$$

Substituting them into the formulas (4.4.1a), we get the expansions of $L(\alpha)$ in $\frac{\partial}{\partial t_{\beta,1}}$ (*for any* α, β). In particular $L(\alpha)$ expanded in $\frac{\partial}{\partial t_{\alpha,1}}$ is also a KP operator,

$$L(\alpha) = -(\partial_\alpha + \sum_{i=1}^{\infty} v_i^{(\alpha)} \partial_\alpha^{-i}) \tag{4.4.9}$$

and its $\alpha - th$ series of flows is nothing but ordinary KP hierarchy

$$\frac{\partial}{\partial t_{\alpha,r}} L(\alpha) = (-1)^{r+1} [L_+^r(\alpha), L(\alpha)] \tag{4.4.10}$$

where the operators are expanded in powers of ∂_α, and the additional sign indicates rescaling of the parameters. Of course, for this subsystem, we can construct its integrable structure, by replacing L in (4.1.7) and (4.1.11) by $L(\alpha)$. Therefore, we have shown that KP system possesses multi bi-hamiltonian structures, and it contains n ordinary KP hierarchies, which, now, couple together. The coupling comes from the dynamical equations (4.4.2a) and (4.4.2b) with $\alpha \neq \beta$.

4.4.3 The τ–function of the generalized KP hierarchy

Using eqs.(4.4.2a) and eqs.(4.4.2b), we get

$$\frac{\partial}{\partial t_{\beta,s}} res_\partial L^r(\alpha) = \frac{\partial}{\partial t_{\alpha,r}} res_\partial L^s(\beta), \qquad \forall \alpha, \beta; \quad r, s. \tag{4.4.11}$$

These equalities implies the existence of τ–function

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{\alpha,r}} \ln \tau = res_\partial L^r(\alpha), \qquad \forall \alpha, r. \tag{4.4.12}$$

Using this τ–function, we can introduce a series of the Baker–Akhiezer functions,

$$\Psi_\alpha(t,\lambda_\alpha) = \frac{\tau(t_{\alpha,1} - \frac{1}{\lambda_\alpha}, t_{\alpha,2} - \frac{1}{2\lambda_\alpha^2}, \ldots)}{\tau(t)} e^{\xi(t,\lambda_\alpha)} \tag{4.4.13}$$

where $\alpha = 1, 2, \ldots, n$. For each of Ψ_α, we can associate one linear system. Among them, the $\alpha = 1$ case was discribed above. The other cases can be analysed in the similar way.

4.4.4 The dispersionless version of the generalized KP hierarchy

In this subsection we examine the *dispersionless* version of the generalized KP hierarchy discussed in the above. By dispersionless we simply means killing all the higher derivatives in the equations of motion. From the field theory point of view, this is equivalent to passing from quantum level to the classical level, i.e. taking the classical limit. Therefore the derivative ∂ will become to be the classical momentum p,

$$\partial \Longrightarrow p,$$

and the algebraic structure will be replaced by the Poisson bracket, i.e.

$$[\partial, x] = 1 \Longrightarrow \{p, x\} = 1. \tag{4.4.14}$$

This is our starting point. After doing this 'transformation', we find that the KP operator (4.1.5) and its conjugate (4.3.1) become

$$\bar{\mathcal{L}} = p + \sum_{i=1}^{\infty} u_i(x) p^{-i}, \tag{4.4.15}$$

and

$$\bar{\mathcal{M}} = \sum_{i=-\infty}^{\infty} v_i(x) p^i. \tag{4.4.16}$$

The other operators (4.4.1a) take the following form

$$\bar{\mathcal{L}}(1) \equiv \bar{\mathcal{L}}, \qquad \bar{\mathcal{L}}(2) \equiv -\frac{1}{c_12}\bar{\mathcal{M}} \tag{4.4.17a}$$

$$\bar{\mathcal{L}}(\alpha) \equiv -\frac{1}{c_{\alpha-1,\alpha}}\Big(c_{\alpha-2,\alpha-1}\bar{\mathcal{L}}(\alpha-2) + \mathcal{V}'(\alpha-1)\Big), \quad 2 < \alpha \le n; \tag{4.4.17b}$$

$$\mathcal{V}'(\alpha) = \sum_{r=1}^{\infty} r t_{\alpha,r} \bar{\mathcal{L}}^{r-1}(\alpha), \qquad \alpha = 1, 2, \ldots, n. \tag{4.4.17c}$$

In terms of the Poisson bracket (4.4.14), we can write down the dispersionless version of the hierarchy (4.4.2a–4.4.2b) as the following

$$\frac{\partial}{\partial t_{\beta,k}}\bar{\mathcal{L}}(\alpha) = \{\bar{\mathcal{L}}_+^k(\beta),\ \bar{\mathcal{L}}(\alpha)\}, \qquad 1 \le \beta \le \alpha \tag{4.4.18a}$$

$$\frac{\partial}{\partial t_{\beta,k}}\bar{\mathcal{L}}(\alpha) = \{\bar{\mathcal{L}}(\alpha),\ \bar{\mathcal{L}}_-^k(\beta)\}, \qquad \alpha \le \beta \le q \tag{4.4.18b}$$

Now let us consider a particular example $n = 2$ case. Our main purpose is to justify the definitions of the new derivatives introduced before. At first we observe that in this case, we only have two Lax operators $\bar{\mathcal{L}}$ and $\bar{\mathcal{M}}$, we may use them instead of $\bar{\mathcal{L}}(1,2)$, then we obtain the following set of equations of motion(the dispersionless version of eqs(4.3.5))

$$\frac{\partial}{\partial t_r}\bar{\mathcal{L}} = \{\bar{\mathcal{L}}^r_+, \bar{\mathcal{L}}\}, \tag{4.4.19a}$$

$$\frac{\partial}{\partial t_r}\bar{\mathcal{M}} = \{\bar{\mathcal{L}}^r_+, \bar{\mathcal{M}}\}, \tag{4.4.19b}$$

$$\frac{\partial}{\partial y_r}\bar{\mathcal{L}} = \{\bar{\mathcal{L}}, \bar{\mathcal{M}}^r_-\}, \tag{4.4.19c}$$

$$\frac{\partial}{\partial y_r}\bar{\mathcal{M}} = \{\bar{\mathcal{M}}^r_+, \bar{\mathcal{M}}\}. \tag{4.4.19d}$$

The Poisson relation should be understood in the following way

$$\{f, g\} \equiv \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial x}. \tag{4.4.20}$$

The new derivative $\tilde{\partial}$ defined by eq.(4.3.11) tends to a new canonical momemtum $\tilde{p}$ conjugate to y_1,

$$\tilde{p} = -\sum_{i=1}^{\infty} v_{-i} p^{-i}. \tag{4.4.21}$$

The new basic Poisson bracket is

$$\{\tilde{p}, y_1\} = 1. \tag{4.4.22}$$

Solving eq.(4.4.21), we can express p in terms of $\tilde{p}$,

$$p = \sum_{i=1}^{\infty} \tilde{v}_{-i}\tilde{p}^{-i}, \tag{4.4.23}$$

therefore, we can rewrite the hierarchy (4.4.19a–4.4.19d) based on the Poisson bracket (4.4.22). With respect to either (4.4.14) or (4.4.22), relation (4.3.2) becomes

$$\{\bar{\mathcal{L}}, \bar{\mathcal{M}}\} = 1. \tag{4.4.24}$$

Eq.(4.4.19a) and eq.(4.4.19d) are two dispersionless KP hierarchies. In order to see this point, we may take another point of view. Consider two KP dispersionless hierarchies

$$\frac{\partial}{\partial t_r}\bar{\mathcal{L}} = \frac{\partial \bar{\mathcal{L}}^r_+}{\partial p}\frac{\partial \bar{\mathcal{L}}}{\partial x} - \frac{\partial \bar{\mathcal{L}}}{\partial p}\frac{\partial \bar{\mathcal{L}}^r_+}{\partial x}. \tag{4.4.25}$$

and

$$\frac{\partial}{\partial y_r}\bar{\mathcal{M}} = \frac{\partial \bar{\mathcal{M}}^r_+}{\partial \tilde{p}}\frac{\partial \bar{\mathcal{M}}}{\partial y_1} - \frac{\partial \bar{\mathcal{M}}}{\partial \tilde{p}}\frac{\partial \bar{\mathcal{M}}^r_+}{\partial y_1}. \tag{4.4.26}$$

where

$$\bar{\mathcal{L}} = p + \sum_{i=1}^{\infty} a_i p^{-i}, \tag{4.4.27a}$$

$$\bar{\mathcal{M}} = \tilde{p} + \sum_{i=1}^{\infty} b_i \tilde{p}^{-i}. \tag{4.4.27b}$$

Now if we impose the restriction (4.4.24), use the Jacobi identities, we can immediately rederive the hierarchies (4.4.19a–4.4.19d). This confirms that $\tilde{p}$ given by (4.4.21) is indeed a new momentum.

4.5 Reductions of the ordinary KP hierarchy

It is wellknown that the KP hierarchy can be reduced to the generalized KdV hierarchies. This is obtained by imposing the following restriction on the KP opertor L.

$$L_-^N = 0, \qquad \frac{\partial}{\partial t_{mN}}\tau = 0, \qquad \forall m \geq 1.$$

This reduced system is still integrable, and usuallu it is refered to be *the N-th generalized KdV hierarchy*[52]. In this section we will not spend time on this subject, but focus our attention on anther kind of reduction. Which was first discovered in the study of matrix models[15].

4.5.1 The new coordinization of the KP hierarchy

The crucial point of the reduction we promised above is the new coordinization of the KP operator L, which could be parametrized in the following way

$$L = \partial + \sum_{i=1}^{\infty} a_i(x) \frac{1}{\partial - S_i} \frac{1}{\partial - S_{i-1}} \cdots \frac{1}{\partial - S_1}. \tag{4.5.1}$$

The full functional space $\mathcal{F}(\wp_-)$ is spanned by all the fields a_i's and fields S_i. Now the reductions come out if we set, for example,

$$a_i(x) = 0, \qquad \forall i \geq N,$$

the KP operator (4.5.1) becomes

$$L = \partial + \sum_{i=1}^{N} a_i(x) \frac{1}{\partial - S_i} \frac{1}{\partial - S_{i-1}} \cdots \frac{1}{\partial - S_1}. \tag{4.5.2}$$

In next chapter we will show that, passing from lattice to differential language, we indeed got such kind of the coordinization(see (5.3.26), or [15]), and this KP operator do satisfy the KP hierarchical equations (4.1.9). So after this reduction, we still have a KP–type of system.

4.5.2 The dispersionless hierarchies

In order to have better understanding of the new reductions of the KP hierarchy , it is instructive to consider their dispersionless versions. As From the argument in §4.4.4, we knew that going from the full hierarchy to its dispersionless version is simply completed by

$$\partial \Longrightarrow p,$$
$$[\partial, x] = 1 \Longrightarrow \{p, x\} = 1.$$

and the differences among fields S_i's can be ignored. Keeping this in mind, we find the dispersionless version of Lax operator (or KP operator) (4.5.2) has the following form

$$\bar{\mathcal{L}} = p + \sum_{i=1}^{N} \frac{a_i}{(p-S)^i} \equiv p + \sum_{i=1}^{\infty} u_i p^{-i}. \tag{4.5.3}$$

with the ordinary KP coordinates

$$u_i = \sum_{l=1}^{i} \begin{pmatrix} i-1 \\ l-1 \end{pmatrix} a_l S^{i-l}, \qquad \forall i \geq 1. \tag{4.5.4}$$

It is wellknown that they should satisfy w_∞ algebra.

$$\{u_i(x), u_j(y)\} = \Big[(i+j-2)u_{i+j-2}(x)\partial_x + (j-1)u'_{i+j-2}(x)\Big]\delta(x-y). \tag{4.5.5}$$

However, in terms of our new coordinates a_i's and S, the first Poisson bracket takes the following form

$$\{a_1(x), S(y)\} = \partial\delta(x-y), \tag{4.5.6a}$$
$$\{a_1(x), a_i(y)\} = 0, \qquad i \geq N; \tag{4.5.6b}$$
$$\{S(x), a_i(y)\} = 0, \qquad i \geq N; \tag{4.5.6c}$$
$$\{a_i(x), a_j(y)\} = \Big((i+j-2)a_{i+j-2}(x)\partial_x + (j-1)a'_{i+j-2}(x)\Big)\delta(x-y),$$
$$2 \leq i, j \leq N. \tag{4.5.6d}$$

We see that this algebra is the direct sum of two subalgebras. Surprisingly, in case of $N \longrightarrow \infty$, the subalgebra formed by $a_i (i \geq 2)$ is also a w_∞ algebra, which is isomorphic to (4.5.5). Since we can construct w_∞ algebra (4.5.5) from the above smaller algebra, we may think that the new coordinization gives a *classification* of w_∞ algebra. This result may remind us that the ordinary KP coordinates u_i's are not good one, which can not exihibit the rich structure of the KP hierarchy .

4.5.3 The non–linear Schrödinger Hierarchy

Now let us come back to the new reductions of KP hierarchy . We would like to examine the simplest example, $N = 2$ case. The KP operator is

$$L = \partial + \frac{1}{\partial - S} R. \tag{4.5.7}$$

We have

$$R = (\partial - S)(L - \partial) = (\partial - S)\sum_{i=1}^{\infty} u_i \partial^{-i},$$

which shows

$$u_1 = R, \qquad u_{i+1} = (\partial - S)\cdot u_i.$$

Therefore we find that all the KP coordinates can be represented as functions of two basic fields R and S

$$u_i = (\partial - S)^{i-1}\cdot R, \qquad \forall i \geq 1. \tag{4.5.8}$$

On the coordinates R and S the Poisson brackets (4.1.7) and (4.1.11) are as follows

$$\{R(x), R(y)\}_1 = 0 \qquad \{S(x), S(y)\}_1 = 0,$$
$$\{R(x), S(y)\}_1 = -\partial_x \delta(x-y),$$

and

$$\{R(x), R(y)\}_2 = (2R(x)\partial_x + R'(x))\delta(x-y), \tag{4.5.9a}$$
$$\{S(x), S(y)\}_2 = 2\partial_x \delta(x-y), \tag{4.5.9b}$$
$$\{R(x), S(y)\}_2 = (S(x)\partial_x - \partial_x^2)\delta(x-y). \tag{4.5.9c}$$

Such algebra also shows up in $SL(2)/U(1)$ WZW model[63] and the conformally affine Liouville theory[64]. Sometimes the KP hierarchy (4.1.9) with this R, S coordinates is referred to as *two bosonic representation* of KP hierarchy .

The flow equations can be expressed in compact form using the two polynomials

$$F_r(x) = \frac{\delta H_r}{\delta S(x)} \qquad G_r(x) = \frac{\delta H_r}{\delta R(x)} \qquad \forall r \geq 1$$

In particular, $F_1 = 0$ and $G_1 = 1$. They satisfy the recursion relations

$$F'_{r+1} = SF'_r - F''_r + 2RG'_r + R'G_r \tag{4.5.10}$$
$$G'_{r+1} = 2F'_r + G''_r + (SG_r)' \tag{4.5.11}$$

and in terms of them

$$\frac{\partial}{\partial t_r} S = G'_{r+1}, \qquad \frac{\partial}{\partial t_r} R = F'_{r+1} \tag{4.5.12}$$

We see immediately that this is the NLS hierarchy. In fact the t_2 flow equations are:

$$\frac{\partial S}{\partial t_2} = S'' + 2S'S + 2R', \qquad \frac{\partial R}{\partial t_2} = -R'' + 2(RS)' \tag{4.5.13}$$

If we change variables as follows

$$S = (\ln\psi)', \qquad R = -\psi\bar{\psi}$$

and denote by a dot the derivative with respect to t_2, eqs.(4.5.13) become

$$\dot{\psi} = \psi'' - 2\psi^2\bar{\psi}, \qquad \dot{\bar{\psi}} = -\bar{\psi}'' + 2\bar{\psi}^2\psi$$

which form the NLS equation. For this reason we will henceforth refer to this hierarchy as the NLS one.

4.5.4 The dispersionless non–linear Schrödinger hierarchy

In this subsection we will consider the *dispersionless* version of the non–linear Schrödinger hierarchy . In this case, Lax operator(or KP operator) (4.5.7) has a particular simple dispersionless form

$$\bar{\mathcal{L}} = p + \frac{1}{p-S}R. \tag{4.5.14}$$

and the KP coordinates are

$$u_i = RS^{i-1}, \qquad \forall i \geq 1. \tag{4.5.15}$$

Based on the basic Poisson bracket (4.4.14), we can define the general Poisson bracket on the functional space over the KP coordinates, which is

$$\{f,g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial x}. \tag{4.5.16}$$

In terms of this Poisson bracket, the dispersionless hierarchy takes the form

$$\frac{\partial}{\partial t_r}\bar{\mathcal{L}} = \{\bar{\mathcal{L}}^r_+, \bar{\mathcal{L}}\}, \tag{4.5.17}$$

where the subindex "+" means keeping the non–negative powers of p. On the coordinates S and R, we find

$$\{R(x), R(y)\}_1 = 0, \qquad \{S(x), S(y)\}_1 = 0, \tag{4.5.18a}$$
$$\{S(x), R(y)\}_1 = \partial\delta(x-y). \tag{4.5.18b}$$

and

$$\{R(x), R(y)\}_2 = (2R\partial + R')\delta(x-y), \tag{4.5.19a}$$
$$\{R(x), S(y)\}_2 = S\partial\delta(x-y), \tag{4.5.19b}$$
$$\{S(x), S(y)\}_2 = 2\partial\delta(x-y). \tag{4.5.19c}$$

as well as

$$\{R(x), R(y)\}_3 = \Big(4RS\partial + 2(RS)'\Big)\delta(x-y), \tag{4.5.20a}$$
$$\{R(x), S(y)\}_3 = (4R\partial + S^2\partial + 2R')\delta(x-y), \tag{4.5.20b}$$
$$\{S(x), S(y)\}_3 = (4S\partial + 2S')\delta(x-y). \tag{4.5.20c}$$

We see that the first Poisson algebra of R and S is nothing but one of the subalgebra in (4.5.6a–4.5.6d). Furthermore, the KP coordinate fields $u_l = RS^{l-1}(l \geq 1)$'s form reducible w_∞ algebras like

$$\{u_i(x), u_j(y)\}_1 = \Big[(i+j-2)u_{i+j-2}(x)\partial_x + (j-1)u'_{i+j-2}(x)\Big]\delta(x-y), \tag{4.5.21}$$

$$\begin{aligned}\{u_i(x), u_j(y)\}_2 \;&=\; \Big[(i+j)u_{i+j-1}(x)\partial_x + ju'_{i+j-1}(x) + 2(i-1)(j-1)\\ &\cdot\; \Big(u_{i-1}(x)u_{j-1}(x)\partial_x + u_{i-1}(x)u'_{j-1}(x)\Big)\Big]\delta(x-y),\end{aligned} \tag{4.5.22}$$

$$\begin{aligned}\{u_i(x), u_j(y)\}_3 \;&=\Big[(i+j+1)u_{i+j}\partial + (j+1)u'_{i+j}\\ &+2(i+1)(j-1)u_i(x)\partial u_{j-1}(x) + 2(i-1)(j+1)u_{i-1}(x)\partial u_j\\ &+2(j-i)u'_1u_{i+j-2}\Big]\delta(x-y).\end{aligned} \tag{4.5.23}$$

With respect to these three Poisson brackets, the Hamiltonians are as follows

$$\mathcal{H}_r = \frac{1}{r}\int (\bar{\mathcal{L}})^r_{(-1)} \forall r \geq 1.$$

Since

$$\bar{\mathcal{L}}^r_{(-1)} = \Big((p-S) + S + \frac{R}{p-S}\Big)^r_{(-1)} = \sum_{l=0}^{[\frac{r-1}{2}]} \binom{r}{2l+1}\binom{2l+1}{l} R^{l+1}S^{r-2l-1}.$$

we get

$$\mathcal{H}_r = \sum_{l=0}^{[\frac{r-1}{2}]} \frac{(r-1)!}{l!(l+1)!(r-2l-1)!}\int R^{l+1}S^{r-2l-1}. \tag{4.5.24}$$

Define two series of the polynomials like (4.5.10) and (4.5.11),

$$\mathcal{F}_r(x) = \frac{\delta\mathcal{H}_r}{\delta S(x)} = \sum_{l=0}^{[\frac{r-1}{2}]} \frac{(r-1)!}{l!(l+1)!(r-2l-2)!} R^{l+1}S^{r-2l-2}, \tag{4.5.25a}$$

$$\mathcal{G}_r(x) = \frac{\delta\mathcal{H}_r}{\delta R(x)} = \sum_{l=0}^{[\frac{r-1}{2}]} \frac{(r-1)!}{l!^2(r-2l-1)!} R^l S^{r-2l-1}. \tag{4.5.25b}$$

The flow equations take the follwing simple forms

$$\frac{\partial S}{\partial t_r} = \mathcal{G}'_{r+1} = \sum_{\substack{k \\ 0\le 2k\le r}} \binom{r}{2k}\binom{2k}{k}\left(R^k S^{r-2k}\right)' \tag{4.5.26}$$

$$\frac{\partial R}{\partial t_r} = \mathcal{F}'_{r+1} = \sum_{\substack{k \\ 2\le 2k\le r+1}} \binom{r}{2k-1}\binom{2k-1}{k}\left(R^k S^{r-2k+1}\right)' \tag{4.5.27}$$

In particular the second flow equations are

$$\frac{\partial S}{\partial t_2} = 2S'S + 2R', \qquad \frac{\partial R}{\partial t_2} = 2(RS)' \tag{4.5.28}$$

Now we remark that we can introduce another parameter, say, t_0, which gets into the game in the following way

$$\frac{\partial}{\partial t_1}S = \frac{\partial}{\partial t_0}R, \qquad \frac{\partial}{\partial t_1}R = R\frac{\partial}{\partial t_0}S. \tag{4.5.29}$$

With an abuse of notation, we denote $\frac{\partial}{\partial t_0}R(\frac{\partial}{\partial t_0}S)$ by $\dot{R}(\dot{S})$, then the eqs(4.5.28) can be recast into the form

$$\frac{\partial}{\partial t_2}S = 2\dot{R}S + 2R\dot{S}, \qquad \frac{\partial}{\partial t_2}R = 2R\dot{R} + 2Rs\dot{S}. \tag{4.5.30}$$

Let us introduce a new Lax operator

$$\mathcal{L} = z + S(t_0) + R(t_0)z^{-1}, \tag{4.5.31}$$

and new Poisson bracket

$$\{z, t_0\} = z. \tag{4.5.32}$$

Then the eq.(4.5.30) can be written in Lax pair form

$$\frac{\partial}{\partial t_2}\mathcal{L} = \{\mathcal{L}_+^2, \mathcal{L}\}, \tag{4.5.33}$$

where the subindex means keeping the non–negative powers of z. In fact, in terms of this new Lax operator (4.5.31) and the new basic Poisson bracket (4.5.32), we can express all the flows (4.5.26) and (4.5.27) into a universal form

$$\frac{\partial}{\partial t_0}\mathcal{L} = \{(\ln \mathcal{L})_+, \mathcal{L}\}, \tag{4.5.34a}$$

$$\frac{\partial}{\partial t_r}\mathcal{L} = \{\mathcal{L}_+^r, \mathcal{L}\}, \qquad \forall r \ge 1. \tag{4.5.34b}$$

On the coordinates R and S, the hamiltonians are

$$\mathcal{H}_r = \frac{1}{r}\int (\mathcal{L}^r)_{(0)}, \qquad \forall r \geq 0. \tag{4.5.35}$$

We recognize that this hierarchy is just the spheric limit of Toda Chain lattice hierarchy discussed in the previous chapter, and the general Poisson brackets can be found there.

Now let us start with the hierarchy (4.5.34b). When we set $S = 0$, from the eq.(4.5.35), we see that the odd Hamiltonians are vanishing, i.e.

$$\mathcal{H}_{2r+1} = 0, \qquad \forall r \geq 0$$

this means that the t_{2r+1}–flows dummy (i.e. the field R does not depend on the odd time parameters), so we are only left with the even flows, which takes the form

$$\frac{\partial}{\partial t_{2r}}\mathcal{L}_e = \{(\mathcal{L}_e)_+^{2r}, \mathcal{L}_e\} \tag{4.5.36}$$

where

$$\mathcal{L}_e = z + R(x)z^{-1} \tag{4.5.37}$$

the equations of motion for R are

$$\frac{\partial}{\partial t^w_{2(r-1)}}R = \begin{pmatrix} 2(r-1) \\ r+1 \end{pmatrix} (R^{r+1})' \tag{4.5.38}$$

4.5.5 Further reduction to KdV hierarchy

Now we come back to the full NLS hierarchy. we want to show that it admits further reduction to the well–known KdV hierarchy. This is achieved in the following way. Let us set $S = 0$ in (4.5.12) and discard the t_{2r} flows(for example set $t_{2r} = 0$). Then from eqs.(4.5.10,4.5.11) we get

$$G_{2r} = 0, \tag{4.5.39}$$
$$G'_{2r+1} = 2F'_{2r}, \tag{4.5.40}$$
$$F'_{2r+1} = -F''_{2r}, \tag{4.5.41}$$

and

$$\frac{\partial R}{\partial t_{2r+1}} = F'_{2r+2} = (\partial^3 + 4R\partial + 2R')F_{2r}, \tag{4.5.42}$$

where the initial condition $F_2 = R$ has been used. Eq.(4.5.42) is the recursion relation for the KdV hierarchy. In particular we have

$$\frac{\partial}{\partial t_3}R = R''' + 6RR'. \tag{4.5.43}$$

This KdV hierarchy can be written in Lax pair formulation

$$\frac{\partial}{\partial t_{2r+1}}R = 2^{2r}[(\partial^2 + R)_+^{r+\frac{1}{2}}, \partial^2 + R]. \tag{4.5.44}$$

Since we already discard the even time parameters, so we are only left with the odd time perturbations. Now let us remark that this reduction is really meaningful. Since eq.(4.5.39) simply says

$$\frac{\partial}{\partial t_{2r+1}}S = G'_{2r+2} = 0,$$

The constraints $S = 0$ is preserved by the odd time perturbations. However the non-vanishing $G_{2r+1} \neq 0, F_{2r+1} \neq 0$ means even time flows have no stable point, any small perturbation will move S to non-zero point. But we may simply think that the even time perturbations are completely fixed, therefore, the reduction condition $S = 0$ is locked.

4.5.6 The dispersionless KdV hierarchy

In this subsection we want to perform the same game as the one in the second subsection, i.e. consider the dispersionless version of the KdV hierarchy . What we do is to replace ∂ by p, and $[\partial, t_1] = 1$ by $\{p, t_1\} = 1$, respectively. Then the KdV hierarchy (4.5.44) becomes

$$\frac{\partial}{\partial t_r}R = 2^{2r}\{(p^2 + R)_+^{r+\frac{1}{2}}, p^2 + R\}. \tag{4.5.45}$$

Explicitly

$$\frac{\partial}{\partial t_r}R = \frac{2^r(2r+1)!!}{r!}R^r R'. \tag{4.5.46}$$

Of course, this hierarchy is isomorphic to the eqs.(4.5.38), since their difference is only the rescaling of time parameters. Therefore we found the dispersionless KdV hierarchy has two different representations, one is the equations (4.5.36) discussed in §4.5.3, the other is eqs.(4.5.45) given above. This may remind us that we could arrive to the KdV hierarchy in two different ways, one is doing the $S = 0$ reduction from the non-linear Schrödinger hierarchy , the other is to do the suitable continuum limit(i.e. double scaling limit) in the Volterra lattice system.

4.6 Discussion

We have shown that KP hierarchy can be extended to a much larger hierarchy by introducing additional KP operators. For each of the KP operators, we have one bi-hamiltonian structure. The commutativity of the flows generated by all these bi-hamiltonian structures gives strong evidence that these bi-hamiltonian structures should be compatible with each other.

As we know in the ordinary KP hierarchy case, the series of flows reflect the large symmetry of the system generated by its Hamiltonians, now in our case, the multi–series of flows imply that this new hierarchy (4.4.2a) and (4.4.2b) should possess much larger symmetry, among them, the largest one is in the case $n \longrightarrow \infty$, which may be related to the diffeomorphism of the following kind of vector space $\mathcal{V} = \{\sum_{r,s\geq 0} e_{r,s} L^r M^s\}$.

It is not clear if this new hierarchy relates to the multi–component KP hierarchy. Another interesting problem is to reduce this hierarchy to the more familiar cases. This is under investigation.

Chapter 5

The KP hierarchy in One–Matrix Model

In chapter **3**, we have shown that one–matrix models are characterized by a linear system whose integrability conditions form a discrete hierarchy (3.3.13) (i.e. a hierarchy of differential–difference equations). This discrete hierarchy is so–called Toda–chain lattice hierarchy. For even potential case, it reduces to Volterra lattice. Both of them turn out to be the reduced cases of the so–called Toda lattice hierarchy[65]. In the double scaling limit, the Volterra lattice hierarchy becomes the KdV hierarchy: it is formed by a hierarchy of purely differential equations, among which we find the celebrated KdV equation.

To avoid misunderstandings let us insist on the difference between the two above–mentioned types of hierarchies. They are typically represented by the two hierarchies (3.3.13) and (4.1.9)(see also (5.1.9) and (5.2.20) below). They have the same form, but in the first case Q is an (infinite) matrix and the equations can be interpreted as differential–difference equations; for this reason we call this hierarchy *discrete*. In the second case $L(\hat{Q}_n)$ is a pseudo–differential operator and the equations involved are purely differential; for this reason we call this hierarchy *differential*. Of the latter type is the KdV hierarchy met in the literature.

In this chapter we want to point out that taking a continuum limit is not necessary in order to get a differential hierarchy. There exists an alternative procedure by which we can extract a differential hierarchy from the discrete linear system associated to one–matrix models *without reference to any limiting procedure*. Said differently, the discrete linear system contains already a differential hierarchy, which can be reduced to the KdV hierarchy, without us being obliged to resort to a limiting procedure – which presumably causes some loss of information. In this hierarchy the first flow parameter t_1 plays the role of space coordinate.

From another point of view we may understand this alternative approach in the following way. As we know in the previous chapter, in order to solve the KP hierarchy , the best way is to construct its τ–function . There are many different representations of the τ–function , each of them has its own advantages. Using our proposal—passing from lattice to differential formulation, we can see that the path–integral of 1–matrix model gives a new expression of the τ–function . This means that even at the discrete level, matrix model gives solutions of the KP hierarchy . Since matrix model is the discretized version of 2–dimensional quantum gravity coupled to certain conformal field theory, or the non–critical string theory, this result may indicate some deep connections of the non–critical string with the integrable systems. Remarkably, for 1–matrix model , this KP hierarchy is nothing but non–linear Schrödinger hierarchy (NLS); with a further reduction, as we have shown in the preceeding chapter, it reduces to KdV hierarchy . So we may get differential hierarchies in a very simple way, without taking any continuum limit.

This chapter is organized as follows. We first introduce the Toda lattice in general (section 1), which involves $\infty \times \infty$ matrices, and we show in this general case how to pass from the matrix formulation to the (differential) operator formulation (section 2). Then we consider reductions of the above system to semi–infinite matrices (section 3). In section 4 we will consider the reduction to the linear system appropriate for one–matrix models (the Toda chain), and, in particular, *by a further reduction* we recover the KdV hierarchy. The section 5 is devoted to justify the identification of τ–function with the partition function of 1–matrix model . Finally (section 6) we discuss the $W_{1+\infty}$–constraints and Virasoro constraints. The main reference in this chapter is [15].

5.1 The Toda lattice hierarchy

In this section we introduce the so-called Toda lattice hierarchy. The main reference in this context is the paper of Ueno and Takasaki [65]. However we will present it in a form which is more suitable for our purposes, i.e. mainly by means of the associated linear system[22].

Let us first introduce some notations. Given a matrix M, we will denote by M_- the strictly lower triangular part and by M_+ the upper triangular part including the main diagonal. Unless otherwise specified, we will be dealing with $\infty \times \infty$ matrices. As usual E_{ij} will denote the matrix $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. We will also use

$$I_\pm \equiv \sum_{i=-\infty}^{\infty} E_{i,i\pm 1}, \qquad \rho = \sum_{i=-\infty}^{\infty} iE_{ii}$$

Throughout this chapter λ denotes the spectral parameter, and Λ represents an infinite dimensional column vector whose components Λ_n, $n \in Z$ are given by

$$\Lambda_n = \lambda^n$$

The vector Λ is our elementary starting point. From it, by means of matrix transformations, we can obtain other vectors. A useful one is η

$$\Lambda = \begin{pmatrix} \vdots \\ \lambda^{-1} \\ 1 \\ \lambda \\ \lambda^2 \\ \vdots \end{pmatrix}, \qquad \eta = exp(\sum_{r=1}^{\infty} t_r \lambda^r) \begin{pmatrix} \vdots \\ \lambda^{-1} \\ 1 \\ \lambda \\ \lambda^2 \\ \vdots \end{pmatrix}, \qquad \eta_n = exp(\sum_{r=1}^{\infty} t_r \lambda^r)\lambda^n. \tag{5.1.1}$$

Where t_r's are time or flow parameters. On η one can naturally define a (elementary) linear system

$$\begin{cases} \lambda\eta = \partial\eta = I_+\eta, \\ \lambda^r\eta = \frac{\partial}{\partial t_r}\eta = \partial^r\eta = I_+^r\eta, \\ \lambda^{-r}\eta = \partial^{-r}\eta = I_-^r\eta, \\ \frac{\partial}{\partial\lambda}\eta = P_0\eta, \qquad P_0 = \rho I_- + \sum_{r=1}^{\infty} r t_r I_+^{r-1}. \end{cases} \tag{5.1.2}$$

The same as in the former chapters, ∂ denotes the derivative $\frac{\partial}{\partial t_1}$ and ∂^{-1} denotes formal integration over t_1. Since for $(\infty \times \infty)$ matrices,

$$[I_+, \rho I_-] = 1,$$

we see that the spectral and flow equations in eqs.(5.1.2) are automatically compatible.

A crucial ingredient in the following construction is the (invertible) "wave matrix" W:

$$\begin{aligned} W &= 1 + \sum_{i=1}^{\infty} w_i I_-^i = 1 + \sum_{i=1}^{\infty}\sum_{n=-\infty}^{+\infty} w_i(n) E_{n,n-i} \\ &= \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 1 & 0 & 0 & 0 & \ddots \\ \ddots & w_1(1) & 1 & 0 & 0 & \ddots \\ \ddots & w_2(2) & w_1(2) & 1 & 0 & \ddots \\ \ddots & w_3(3) & w_2(3) & w_1(3) & 1 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \end{aligned} \tag{5.1.3}$$

So $w_i = \{w_i(n) | n \in Z\}$ are infinite diagonal matrices, and $w_i(n)$ are functions of the time parameters, we will call them as the w-coordinates of the system (5.1.2). Now let us impose them to be determined by the equations of motion

$$\frac{\partial}{\partial t_r} W = Q_+^r W - W I_+^r, \tag{5.1.4}$$

where Q is the infinite matrix

$$Q = WI_+W^{-1}. \tag{5.1.5}$$

Another important object is the vector Ψ

$$\Psi = W\eta. \tag{5.1.6}$$

In terms of all these objects the dynamical system we have defined can be written as

$$\begin{cases} Q\Psi = \lambda\Psi, \\ \frac{\partial}{\partial t_r}\Psi = Q_+^r\Psi, \\ \frac{\partial}{\partial\lambda}\Psi = P\Psi. \end{cases} \tag{5.1.7}$$

where

$$P = WP_0W^{-1}. \tag{5.1.8}$$

The compatibility conditions of this linear system form the so-called *discrete* KP–hierarchy*

$$\frac{\partial}{\partial t_r}Q = [Q_+^r, Q], \tag{5.1.9}$$

together with the *trivial* relation

$$[Q, P] = 1. \tag{5.1.10}$$

We should perhaps recall that what we have done so far is at a purely formal level and does not bear yet any relation to matrix models. In particular we insist that eq.(5.1.10) does not imply any constraint on the dynamical system.

To end this section let us make the above formulas more explicit and extract a few relations that will be useful in the following. From the equation of motion (5.1.4) we consider in particular

$$\frac{\partial}{\partial t_1}w_i(n) = w_{i+1}(n+1) - w_{i+1}(n) - w_i(n)(w_1(n+1) - w_1(n)). \tag{5.1.11}$$

Let us introduce a piece of terminology by saying that for any matrix its elements in the $n-th$ row belong to the "$n-th$ sector". Therefore, in regard to eq.(5.1.11) we can say that the flow in the $n-th$ sector depends only on the coordinates of the $n-th$ and $(n+1)-th$ sectors.

*Formally this discrete KP hierarchy (5.1.9) is the same as the one (3.3.13) derived from 1–matrix model , but rigorously they have some difference, since here we deal with $(\infty \otimes \infty)$ matrix, rather than the semi–infinite one, which is the case of 1–matrix model .

From (5.1.5) we see that

$$Q = I_+ + \sum_{i=0}^{\infty} a_i I_-^i. \tag{5.1.12}$$

The a_i's are new coordinates of the system, which can be uniquely expressed in terms of the w_i's. For example †

$$\begin{aligned}
a_0(n) &= w_1(n) - w_1(n+1), \\
a_1(n) &= w_1(n)(w_1(n+1) - w_1(n)) + w_2(n) - w_2(n+1), \\
a_2(n) &= w_2(n)(w_1(n+1) - w_1(n)) + w_1(n-1)(w_2(n+1) - w_2(n)), \\
&\quad + w_1(n)w_1(n-1)(w_1(n+1) - w_1(n)) + w_3(n) - w_3(n+1).
\end{aligned}$$

Another useful representation is obtained by inverting eq.(5.1.5)

$$I_+ = W^{-1}QW = Q + \sum_{i=0}^{\infty} q_i Q^{-i},$$

the q_i's are another set of diagonal matrices, which can be expressed in terms of a_i's or w_i's. It is worth noting that

$$I_- = (I_+)^{-1} = Q^{-1} + \ldots,$$

which results in the following equality

$$Q_+^r \equiv Q^r - Q_-^r = Q^r + \sum_{i=1}^{\infty} q_{r,i} Q^{-i} \qquad \forall r \geq 1. \tag{5.1.13}$$

Finally, from (5.1.8), we have

$$P = \sum_{r=1}^{\infty} r t_r Q^{r-1} + \sum_{i=0}^{\infty} v_i Q^{-i-1}. \tag{5.1.14}$$

Once again, v_i's are diagonal matrices, and $v_0 = \rho$.‡

We will see later on that the string equation of matrix models can be obtained by imposing a constraint on the coordinates v_i.

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†These equations show that the two sets of the variables a_i's and w_i's can be obtained from each other. However, strickly speaking, this one–to–one correspondence is only due to the fact that we have choosen a special form of W–matrix (5.1.3). Generally, for a given matrix Q, W is not uniquely determined.

‡Since $W\rho I_- W^{-1} = \rho Q^{-1} + [W,\rho] I_- W^{-1}$, the commutator is a strictly lower triangular matrix, so the second part at most contributes to the term Q^{-2}, which ensures that $v_o = \rho$.

5.2 From the discrete hierarchy to the differential hierarchy

In the previous section we introduced the usual Toda lattice. The discrete KP hierarchy we obtained is known as the Toda lattice hierarchy. It consists of an infinite set of differential-difference equations. In this section we show that passing from the matrix formalism of the previous section to a related (pseudo–differential) operator formalism, we can obtain a new hierarchy which consists merely of differential equations.

The operator formalism alluded to before is introduced as follows. We recall that equation (5.1.1) implies

$$\eta_n = \partial^{n-m}\eta_m, \qquad \forall n, m : \text{integers}.$$

This leads to

$$\Psi_n = (W\eta)_n = \sum_{i=-\infty}^{n} W_{ni}\eta_i = \sum_{i=-\infty}^{n} W_{ni}\partial^{i-n}\eta_n = \hat{W}_n\eta_n, \tag{5.2.15}$$

where we have defined

$$\hat{W}_n = 1 + \sum_{i<n} W_{ni}\partial^{i-n} = 1 + \sum_{i=1}^{\infty} w_i(n)\partial^{-i}. \tag{5.2.16}$$

This tells us that the "wave" matrix W can be considered as an infinite diagonal matrix, whose components are differential operators. The operator $\hat{W}_n$ can be inverted

$$\eta_n = \hat{W}_n^{-1}\Psi_n.$$

In this formalism the spectral equation in (5.1.7) becomes

$$\lambda\Psi_n = \lambda\hat{W}_n\eta_n = \hat{W}_n\partial\eta_n = \hat{W}_n\partial\hat{W}_n^{-1}\Psi_n = \hat{Q}_n\Psi_n.$$

Here we have introduced an infinite set of KP–type differential operators

$$\hat{Q}_n = \hat{W}_n\partial\hat{W}_n^{-1} = \partial + \sum_{i=1}^{\infty} u_i(n)\partial^{-i} \qquad \forall n \text{ integer}. \tag{5.2.17}$$

The variables u_i's are KP coordinates. Till now we have introduced several sets of the coordinates, they are $w_i(n)$'s in eq.(5.1.3), $a_i(n)$'s in eq.(5.1.12), and the others. All of them in fact are related together. The different choices may be suitable for different purposes. One should not make confusion about it. Now if we Invert the relation (5.2.17), we would obtain

$$\partial = \hat{W}_n^{-1}\hat{Q}_n\hat{W}_n = \hat{Q}_n + \sum_{i=0}^{\infty} q_i(n)\hat{Q}_n^{-i}, \qquad \forall n: \quad \text{integer}. \tag{5.2.18}$$

It is easy to see that this mapping from matrices to operators maps the upper triangular part of a given matrix into the differential part of the operator, and the lower triangular part of the matrix to the formal integration part of the operator. In particular we have

$$(Q^r_+\Psi)_n \longrightarrow (\hat{Q}_n)^r_+\Psi_n.$$

For an operator, the subscript "+" selects, as usual, the non-negative powers of the derivative ∂. Going on with the transcription of the Toda lattice linear system in the operator formalism, we can now rewrite the flow equations in eqs.(5.1.7)

$$\begin{aligned}\frac{\partial}{\partial t_r}\Psi_n &= (\frac{\partial}{\partial t_r}\hat{W}_n)\eta_n + \hat{W}_n\frac{\partial}{\partial t_r}\eta_n\\ &= \Big(\lambda^r + (\frac{\partial}{\partial t_r}\hat{W}_n)\hat{W}_n^{-1}\Big)\Psi_n\\ &= (\hat{Q}^r_n)_+\Psi_n \qquad \forall n \text{ integer.}\end{aligned}$$

Finally we can rewrite the linear system (5.1.7) as follows

$$\begin{cases}\hat{Q}_n\Psi_n = \lambda\Psi_n,\\ \frac{\partial}{\partial t_r}\Psi_n = (\hat{Q}^r_n)_+\Psi_n, \qquad \forall n \text{ integer,}\\ \frac{\partial}{\partial\lambda}\Psi_n = (\sum_{r=1}^{\infty} rt_r\hat{Q}_n^{r-1} + \sum_{i=0}^{\infty} v_i(n)\hat{Q}_n^{-i-1})\Psi_n.\end{cases} \tag{5.2.19}$$

Their compatibility conditions are

$$\frac{\partial}{\partial t_r}\hat{Q}_n = [(\hat{Q}_n)^r_+, \hat{Q}_n]. \tag{5.2.20}$$

or in other coordinates

$$\frac{\partial}{\partial t_r}\hat{W}_n = (\hat{Q}_n)^r_+\hat{W}_n - \hat{W}_n\partial^r. \tag{5.2.21}$$

The last two equations, like the previous ones, hold for any integer n. Eq.(5.2.20) or (5.2.21) specifies the *differential* hierarchy we promised in the introduction. It consists of an infinite set of differential equations: in the LHS we have the first order derivatives with respect to the flow parameters, in the RHS we have polynomials in the coordinates and derivatives of coordinates with respect to t_1.

It is a bit inappropriate to speak of one hierarchy: we have in fact an infinite number of hierarchies, one for each integer n. However these hierarchies are not independent as they are related by the t_1 flow; we will see later on that in the particular case of one-matrix model all these hierarchies are isomorphic.

5.3 Reduction: Semi–infinite matrices

Let us study now the problem of reducing the general system (5.2.19) defined in the two previous sections to a simpler one. In the next section we will consider a further reduction, i.e. to the Toda chain which is relevant for one-matrix models. We notice, first of all, that for the linear systems involved in matrix models Ψ does not contain negative powers of λ. Therefore the Jacobi matrix Q must be semi–infinite,

$$w_i(n) = 0, \qquad n < i.$$

This may be called as "Generalized Toda–Chain lattice system", which is of course integrable[22]. This is the reduction we will study in this section.

In this reduced system for any positive integer n we have an invertible operator with a finite number of terms

$$\hat{W}_n = 1 + \sum_{i=1}^{n} w_i(n)\partial^{-i}.$$

The KP–type operator is

$$\hat{Q}_n = \hat{W}_n \partial \hat{W}_n^{-1},$$

which still contains infinite many terms.

We remark here that we are still formally using $\infty \times \infty$ matrices in order to be able to fully exploit the formalism introduced in the previous sections. However three quadrants of these matrices become irrelevant.

So far we have been using mostly w_i coordinates, but henceforth it will be more convenient to shift to a_i coordinates. We recall that they are defined in the following way through the spectral equation

$$\lambda \Psi_n = \Psi_{n+1} + \sum_{i=0}^{n} a_i(n)\Psi_{n-i}. \tag{5.3.22}$$

Next we want to express the RHS of this equation in terms of Ψ_n only (this is an example of the procedure outlined in the last section). To this end we use the first flow equation

$$\partial \Psi_n = (Q_+\Psi)_n = \Psi_{n+1} + a_0(n)\Psi_n,$$

or equivalently

$$\Psi_{n+1} = (\partial - a_0(n))\Psi_n. \tag{5.3.23}$$

Inverting this relation, we get

$$\Psi_n = \hat{B}_n \Psi_{n+1}, \qquad \hat{B}_n = \partial^{-1}\sum_{l=0}^{\infty}(a_0(n)\partial^{-1})^l. \tag{5.3.24}$$

Using this relation repeatedly we can express any $\Psi_i (i < n)$ in terms of Ψ_n, i.e.

$$\Psi_{n-r} = \hat{B}_{n-r}\hat{B}_{n-r+1}\dots\hat{B}_{n-1}\Psi_n.$$

Therefore the spectral equation (5.3.22) can be rewritten as

$$\hat{Q}_n\Psi_n = \left(\partial + \sum_{i=1}^{n} a_i(n)\hat{B}_{n-i}\hat{B}_{n-i+1}\dots\hat{B}_{n-1}\right)\Psi_n, \tag{5.3.25}$$

and the n–th KP–type operator becomes

$$\begin{aligned}\hat{Q}_n &= \partial + \sum_{i=1}^{n} a_i(n)\hat{B}_{n-i}\hat{B}_{n-i+1}\dots\hat{B}_{n-1} \\ &= \partial + \sum_{i=1}^{n} a_i(n)\frac{1}{\partial - a_0(j-i)}\frac{1}{\partial - a_0(j-i+1)}\cdots\frac{1}{\partial - a_0(j-1)}.\end{aligned} \tag{5.3.26}$$

Expanding in the powers of ∂, we would have

$$\hat{Q}_n = \partial + \sum_{l=1} u_l(n)\partial^{-l},$$

where the KP coordinates $u_l(n)$'s only depend on the coordinates $a_i(n)(1 \le i \le n)$ in the n–th sector and n additional fields $a_0(j)(0 \le j \le n-1)$.

Comparing the above formula with the eq.(5.3.22), we immediately see that the terms of the negative powers of ∂ are represented the lower triangular part of Jacobi matrix Q. So we could formulate the rules passing from lattice to the purely differential language as follows:

(i). The Jacobi matrix goes to be KP operator, i.e.

$$(Q^r\Psi)_n \Longrightarrow (\hat{Q}_n)^r\Psi_n,$$

(ii). The lower triangular part of Jacobi matrix (or its powers) maps to the pure integration part of the KP operator (or its powers);

(iii). The upper triangular part together with the main diagonal line of the Jacobi matrix (or its powers) correspond to the purely differential part of the KP operator (or its powers).

(iv). Comparing the eq.(5.3.25) with the eq.(5.3.22), we find the residue of the KP operator (5.3.26) has a particular simple form,

$$\text{res}_{\partial}(\hat{Q}_n) = a_1(n) = Q_{n,n-1}. \tag{5.3.27}$$

On the right hand side, $Q_{n,n-1}$ means the element of the Jacobi matrix at n–th row and $(n-1)$–th column. More generally, we can obtain

$$\text{res}_{\partial}(\hat{Q}_n^r) = Q^r_{n,n-1}. \tag{5.3.28}$$

Keeping these arguments in mind, we can also rewrite the flow equations in terms of this KP operator. Therefore we finally obtain the linear system

$$\begin{cases} \hat{Q}_n\Psi_n = \lambda\Psi_n, \\ \frac{\partial}{\partial t_r}\Psi_n = (\hat{Q}_n^r)_+\Psi_n, \qquad \forall n \text{ integer}, \\ \frac{\partial}{\partial\lambda}\Psi_n = \left(\sum_{r=1}^{\infty} r t_r \hat{Q}_n^{r-1} + \sum_{i=0}^{\infty} v_i(n)\hat{Q}_n^{-i-1}\right)\Psi_n. \end{cases} \tag{5.3.29}$$

Which is of the same form as (5.2.19), but here the KP operator is (5.3.26). Their compatibility conditions are

$$\frac{\partial}{\partial t_r}\hat{Q}_n = [(\hat{Q}_n)_+^r, \hat{Q}_n]. \tag{5.3.30}$$

Furthermore, from the associated linear system (5.3.29), we can easily get the "zero curvature representation"

$$\frac{\partial}{\partial t_r}(\hat{Q}_n)_+^s - \frac{\partial}{\partial t_s}(\hat{Q}_n)_+^r = [(\hat{Q}_n)_+^r, (\hat{Q}_n)_+^s]. \tag{5.3.31}$$

Of course, which has the same form as eq.(4.1.12). This confirms the integrability of the system (5.3.29).

We may conclude that for any "Generalized Toda Chain lattice system", viewing the parameter t_1 as the space coordinate, we get a new hierarchy, which is purely differential hierarchy, and corresponds to the new reduction of KP hierarchy discussed in the end of the last chapter.

5.4 Toda chain and one-matrix models

The case relevant to one–matrix models is specified by the conditions

$$a_0(j) = S_j, \qquad a_1(j) = R_j, \qquad a_i(j) = 0, \qquad \forall i \geq 2.$$

The first (i.e. t_1) flow equation is

$$\frac{\partial}{\partial t_1}S_j = S_j' = R_{j+1} - R_j, \tag{5.4.32}$$

$$\frac{\partial}{\partial t_1}R_j = R_j' = R_j(S_j - S_{j-1}). \tag{5.4.33}$$

From the second equality we have

$$S_{j-1} = S_j - \frac{R_j'}{R_j}.$$

Therefore the "$j-th$ KP–type operator" is

$$L_j \equiv \hat{Q}_j = \partial + R_j\hat{B}_{j-1} = \partial + \frac{1}{\partial - S_j}R_j, \tag{5.4.34}$$

which is the same as the one given in the previous chapter (see (4.5.7)) as the KP operator of non–linear Schrödinger hierarchy . Thus we find that the Toda Chain Lattice hierarchy indeed contains a *differential hierarchy*—NLS.

If we expand the KP operator (5.4.34) in the powers of ∂, i.e.

$$L_j \equiv \partial + \sum_{l=1} u_l(j)\partial^{-l}. \tag{5.4.35}$$

Then from the eq.(5.4.34), we have

$$u_l = (\partial - S_j)^{i-1} \cdot R_j, \qquad \forall l \geq 1. \tag{5.4.36}$$

In particular the first few $u_l(j)$'s are as follows

$$\begin{aligned}
&u_1(j) = R_j, \qquad\qquad u_2(j) = -R_j' + R_jS_j \\
&u_3(j) = R_j'' - 2R_j'S_j - R_jS_j' + R_jS_j^2 \\
&u_4(j) = -R_j''' + 3R_j''S_j + 3R_j'S_j' - 3R_j'S_j^2 - 3R_jS_jS_j' + R_jS_j'' + R_jS_j^3 \\
&u_5(j) = R_j'''' - 4R_j'''S_j - 6R_j''S_j' + 6R_j''S_j^2 - 4R_j'S_j'' - 4R_j'S_j^3 \\
&\qquad +12R_j'S_jS_j' + 4R_jS_jS_j'' + 3R_jS_j'^2 \\
&\qquad -6R_jS_j^2S_j' - R_jS_j''' + R_jS_j^4
\end{aligned}$$

We find that all the KP coordinates $u_l(j)$'s are only the functions of R_j, S_j, in other words, they only depend on the coordinates in $j--th$ sector. The consequence of this property is that the infinite many KP–hierarchies (indexed by "j") are all isomorphic. We may conclude that the Toda chain and one-matrix models are characterized by NLS.

Since the following analysis is universal, i.e. is the same for all sectors, we will simply omit the subscript "j", and denote $R_j(t_1)$, $S_j(t_1)$ by $R(t_1)$ and $S(t_1)$, respectively.

As we have shown in the previous chapter, the equations of motion of NLS are

$$\frac{\partial}{\partial t_r} S = G'_{r+1}, \qquad \frac{\partial}{\partial t_r} R = F'_{r+1}. \tag{5.4.37}$$

Where the two sets of the polynomials F_r's and G_r's satisfy the recursion relations

$$F'_{r+1} = SF'_r - F''_r + 2RG'_r + R'G_r \tag{5.4.38}$$

$$G'_{r+1} = 2F'_r + G''_r + (SG_r)' \tag{5.4.39}$$

with $F_1 = 0$ and $G_1 = 1$. Since we have discussed all the properties of NLS in the chapter 4, we won't repeat it now.

5.5 The partition function of One–Matrix Model as the τ–function

In the first chapter we claimed that the partition function of 1–matrix model is the τ–function of non–linear Schrödinger hierarchy . Now let us give a proof.

For KP hierarchy , we already know that τ–function is related to the KP operator in the way

$$\frac{\partial^2}{\partial t_1 \partial t_r} \ln \tau = \text{res}_\partial(L^r). \tag{5.5.40}$$

In particular, for the KP hierarchy (5.3.30) whose $N-th$ KP operator has the form (5.4.34), its τ–function satisfy the following relations

$$\frac{\partial^2}{\partial t_1 \partial t_r} \ln \tau = u_1 = R_N,$$
$$\frac{\partial^2}{\partial t_1 \partial t_r} \ln \tau = u_1' + 2u_2 = 2R_N S_N - R_N'.$$
$$\cdots\cdots$$

Now let us turn our attention on the partition function of 1–matrix model . From the eq.(3.3.10b), the data of a one–matrix model can be encoded in a corresponding discrete linear system. Consider the partition function of 1–matrix model with the potential §

$$Z_N(t) = \int dM e^{\text{Tr}V(M)}, \qquad V(M) = \sum_{r=1}^{\infty} t_r M^r$$

which satisfies

$$\frac{\partial}{\partial t_r} \ln Z_N(t) = \text{Tr}Q^r = \sum_{i=0}^{N-1} Q^r_{ii}. \tag{5.5.41}$$

Taking its derivative with respect to t_1, we get

$$\frac{\partial^2}{\partial t_1 \partial t_r} \ln Z_N(t) = \frac{\partial}{\partial t_1} \text{Tr}Q^r,$$

using the first discrete flow equation

$$\frac{\partial}{\partial t_1} Q^r = [Q_+, Q^r],$$

we get

$$\frac{\partial^2}{\partial t_1 \partial t_r} \ln Z_N(t) = Q^r_{N,N-1}. \tag{5.5.42}$$

In particular,

$$\frac{\partial^2}{\partial t_1^2} \ln Z_N(t) = R_N. \tag{5.5.43}$$

Comparing eqs.(5.5.40) and (5.5.42), and keeping the eq.(5.3.28) in mind, we find the partition function of 1–matrix model is exactly τ–function of non–linear Schrödinger hierarchy .

§ We should remind you that we have changed a little bit our convention in this chapter, that is, we have replaced all the time parameters t_r's by $(-t_r)$'s.

5.6 The Virasoro constraints

At first let us quote some known results in the previous chapter. That is when we set $S=0$ in (5.4.37) and discard the t_{2r} flows(for example set $t_{2r}=0$), from eqs.(5.4.38) and (5.4.39) we get

$$G_{2r} = 0, \tag{5.6.44}$$
$$G'_{2r+1} = 2F'_{2r}, \tag{5.6.45}$$
$$F'_{2r+1} = -F''_{2r}, \tag{5.6.46}$$

and the recursion relation for the KdV hierarchy is

$$F'_{2r+2} = (\partial^3 + 4R\partial + 2R')F_{2r}. \tag{5.6.47}$$

The KdV hierarchy takes the form

$$\frac{\partial R}{\partial t_{2r+1}} = F'_{2r+2}. \tag{5.6.48}$$

In particular we have

$$\frac{\partial}{\partial t_3} R = R''' + 6RR'. \tag{5.6.49}$$

5.6.1 The discrete Virasoro constraints

Now let us come back to 1–matrix model . As we already discussed in the chapter **3**, Toda chain lattice describes 1–matrix model if and only if we imposing certain constraints on the model, this is represented by restricting the form of P

$$P_- = 0, \qquad \text{i.e.} \quad P = \sum_{r=1}^{\infty} r t_r Q_+^{r-1}. \tag{5.6.50}$$

When we pass from the lattice to the differential case, this becomes

$$\hat{P}_N = \sum_{r=1}^{\infty} r t_r (\hat{Q}_N)_+^{r-1}. \tag{5.6.51}$$

and the string equation (5.1.10) also goes to be differential equation

$$[\hat{Q}_N, \hat{P}_N] = 1. \tag{5.6.52}$$

By string equation we mean (5.6.52) together with the condition (5.6.51).

From the KP equations (5.2.20) and string equation, we can derive

$$d_{-1}R = 0, \qquad d_{-1}S + 1 = 0 \tag{5.6.53}$$

where

$$d_{-1} = \sum_{r=2}^{\infty} rt_r \frac{\partial}{\partial t_{r-1}}$$

Eqs.(5.6.53) can be written in the form

$$\sum_{r=2}^{\infty} rt_r F'_r = 0, \qquad \sum_{r=2}^{\infty} rt_r G'_r + 1 = 0.$$

If we integrate once the second equation, and discard the constant of integration, then we obtain

$$\sum_{r=2}^{\infty} rt_r G_r + t_1 = 0. \tag{5.6.54}$$

Using the equations of motion and the eq.(3.3.6), we get

$$\sum_{r=2}^{\infty} rt_r \frac{\partial}{\partial t_{r-1}} \ln h_N + t_1 = 0. \qquad \forall N \geq 1. \tag{5.6.55}$$

Now using the definition (5.5.41), and taking the summation over N in the eq.(5.6.55), we can recast the eq.(5.6.55) into the familiar form

$$\Big(\sum_{r=2}^{\infty} rt_r \frac{\partial}{\partial t_{r-1}} + Nt_1\Big) Z_N = 0.$$

Using the recursion relations (5.4.38,5.4.39) we can obtain other similar equations, for example

$$\sum_{r=1}^{\infty} rt_r F'_{r+1} + 2R = 0, \qquad \sum_{r=1}^{\infty} rt_r G'_{r+1} + S = 0.$$

which can be written as¶

$$\Big(\sum_{r=1}^{\infty} rt_r \frac{\partial}{\partial t_r} + N^2\Big) Z_N = 0.$$

Playing the same game, we can obtain all the other constraints. Finally we recover the discrete Virasoro constraints as discussed in §3.4

$$L_n Z_N = 0, \qquad n \geq -1 \tag{5.6.56}$$

where

$$\begin{aligned} L_{-1} &= \sum_{r=2}^{\infty} rt_r \frac{\partial}{\partial t_{r-1}} + Nt_1 \\ L_0 &= \sum_{r=1}^{\infty} rt_r \frac{\partial}{\partial t_r} + N^2 \\ L_n &= \sum_{r=1}^{\infty} rt_r \frac{\partial}{\partial t_{r+n}} + 2N \frac{\partial}{\partial t_n} + \sum_{r=1}^{n-1} \frac{\partial^2}{\partial t_r \partial t_{n-r}}, \qquad n > 0 \end{aligned} \tag{5.6.57}$$

¶We did not care about the integration constant.

5.6.2 The Virasoro constraints for reduced model

Now we want to show that through the reduction procedure we can derive the Virasoro constraints on the partition function , i.e.

$$\mathcal{L}_n Z = 0, \qquad n \geq -1 \tag{5.6.58}$$

where

$$\begin{aligned}
\mathcal{L}_{-1} &= \sum_{k=1}^{\infty}(k+\frac{1}{2})t_{2k+1}\frac{\partial}{\partial t_{2k-1}} + \frac{t_1^2}{8} \\
\mathcal{L}_0 &= \sum_{k=0}^{\infty}(k+\frac{1}{2})t_{2k+1}\frac{\partial}{\partial t_{2k+1}} + \frac{1}{16} \\
\mathcal{L}_n &= \sum_{k=0}^{\infty}(k+\frac{1}{2})t_{2k+1}\frac{\partial}{\partial t_{2k+2n+1}} + \frac{1}{2}\sum_{k=0}^{n-1}\frac{\partial^2}{\partial t_{2k+1}\partial t_{2n-2k-1}}, \qquad n > 0
\end{aligned} \tag{5.6.59}$$

Let us start with the first Virasoro constraint. We consider the eq.(5.6.54) as our starting relation and proceed to the reduction illustrated at the beginning of this section. Using the eqs.(5.6.44) and (5.6.45), we obtain

$$\sum_{k=1}^{\infty}(2k+1)t_{2k+1}\partial^{-1}\frac{\partial \ln Z}{\partial t_{2k-1}} + t_1 = 0 \tag{5.6.60}$$

We now integrate this equation with respect to t_1 and obtain

$$\mathcal{L}_{-1}Z = b_{-1}Z \tag{5.6.61}$$

which is almost the first Virasoro condition except for the integration constant b_{-1}, where $b_{-1} \equiv b_{-1}(t_3, t_5, \ldots)$ ‖.

Next we apply the operator

$$\mathcal{D} = \partial^3 + 4R\partial + 2R'$$

to eq.(5.6.60). Using (5.6.48) and integrating twice, we obtain

$$\mathcal{L}_0 Z = (a_0 t_1 + b_0)Z \tag{5.6.62}$$

where a_0 and b_0 are integration constants depending on $t_3, t_5, \cdots$. Notice that we have included a part of the integration constant $= \frac{1}{16}$ in the definition of $\mathcal{L}_0$. We could continue in this way by integrating once (5.6.61) and applying $\mathcal{D}$ and so on. We would obtain a series of integration constants a_n and b_n, $n > 0$ similar to the above ones. In order to understand how to calculate them we have to clarify preliminarily a few points.

When considering R and polynomials of R and its derivatives with respect to t_1 we are entitled to use the identities $\partial\partial^{-1} = 1$ and $\partial^{-1}\partial = 1$. In this sense we speak of the symbol

‖One might be tempted to integrate eq.(5.6.54) once more, but this does not make sense in the lattice.

∂^{-1} as a formal integration. The reason is that R and polynomials of R and its derivatives are supposed to belong to a family of objects to which pseudodifferential calculus applies (for example, to the family of functions rapidly decreasing at infinity in the variable t_1). However this is in general not guaranteed for an infinite sum such as, for example, $\sum_{k=2}^{\infty} kt_k G_k$. If we insist on applying ∂^{-1} to such objects as a formal integration, we are bound to run into inconsistencies. Therefore when applying ∂^{-1} to the infinite sums that appear in the string equations we have to interpret it as a true integration and take care of the corresponding integration constants.

On the other hand the KdV hierarchy and its solutions are characterized by homogeneity in t_n with degree assignement

$$deg(t_n) = n$$

We have consequently

$$deg(\mathcal{L}_n) = -2n = deg(a_n t_1 + b_n) \tag{5.6.63}$$

This is the first ingredient we will use. The second is the form of the constant a_n and b_n. A rather general form we can imagine for them is a sum of expressions like

$$\sim \prod_{i=1}^{s} t_{2k_i+1}^{a_i} \tag{5.6.64}$$

where a_i are real numbers. We can strongly restrict the enormous number of possibilities implied by equation (5.6.64) by using the path integral form of the model we are studying. We remember that in the path integral the terms involving $t_n, \;\; n \geq 3$ can be considered as perturbations. Therefore we do not expect anything dramatic to happen when only one of these couplings is set to zero. Thus we can conclude that in (5.6.64) all the exponents a_i are non–negative **. If we now make a degree analysis of the various possible constants, we conclude that only b_{-1} and b_0 can be non–vanishing, and the latter is a true constant, i.e. it does not depend on $t_3, t_5, \ldots$. What remains for us to do is to calculate these two constants. To this end we can repeat the analysis of the Appendix of [37].

Let us summarize the situation. So far we have found the relations

$$\mathcal{L}_n Z = b_n Z \tag{5.6.65}$$

where only b_{-1} and b_0 are non–vanishing. Since

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m}$$

the consistency conditions

$$[\mathcal{L}_n, b_m] - [\mathcal{L}_m, b_n] = (n-m) b_{n+m} \tag{5.6.66}$$

**This is certainly what happens in the lattice, see (5.6.56).

must be satisfied. Take these equations for $m = 1$ and $m = 0$. Studying the t_1 dependence we find

$$\frac{\partial b_{-1}}{\partial t_{2k+1}} = 0, \qquad \frac{\partial b_0}{\partial t_{2k+1}} = 0, \quad k \geq 0$$

So b_{-1} and b_0 are true constants (for b_0 this was already known independently to us). If we use (5.6.66) for $n = 0$ and $m = -1$ and for $n = 1$ and $m = -1$ we immediately conclude that

$$b_{-1} = 0, \qquad b_0 = 0$$

respectively. So finally we proved our claim (5.6.58) in the beginning of this subsection.

5.7 Discussion

Starting from the Toda lattice in the traditional form, we have extracted a continuous linear system (without taking a continuum limit). By reducing it we have been able to show that the discrete system corresponding to one–matrix models gives rise naturally to a differential hierarchy and in particular to the KdV hierarchy.

A few differences with the more common limiting treatment in one-matrix models should be stressed:

– the space parameter in our approach is t_1 (not t_0, which does not show up here);

– the Virasoro constraints we found in the previous section are the same as the Virasoro constraints of the discrete system [37], and can be reduced to the one obtained in the continuum limit;

– the complete differential hierarchy corresponding to one–matrix models does not seem to coincide exactly with any of the continuous hierarchies proposed so far [50]; its corresponding topological models will be considered in chapter 8.

Chapter 6

The Generalized KP Hierarchy in Multi–Matrix Models

Among the various matrix models, the 1–matrix model discussed till now is the simplest one, since at discrete level it is completely described by the associated linear system. The important ingredient—Jacobi matrix Q is formed by three pseudo–diagonal lines, and remains unchanged whatever the polynomial perturbations are. In other words, the form of the Jacobi matrix of the 1–matrix model is independent on the potential(or perturbations). However, once we turn our attention to the multi–matrix models , the situation dramatically changed. We have several Jacobi matrices, all of them completely determined by the potentials(or perturbations), i.e. different potentials will result in different Jacobi matrices. Therefore it is quite difficult to obtain the full KP hierarchy through the continuum limits, even for two–matrix models. Although it is believeed that the multi–matrix models should be characterized by generlized KdV-hierarchies(their partition functions satisfy W-type of constraints[10]), and related to the generalized Kontsevich models[9][53], however till now, it is still at the conjectural level, since we are not able to proceed the exact analysis except in some particular cases[12][13][14]. So the double scaling limit in multi–matrix models cases seems to be not as successful as in 1–matrix model . In order to get better understanding of 2–dimensional quantum gravity coupled to mimimal CFT, we are forced to investigate the deep relations between multi-matrix models and KP hierarchies systematically.

In the previous chapter we proposed a new approach to derive the KP hierarchy in 1–matrix model without taking any continuum limit. Its basic idea is passing from lattice to differential language by interpreting the first flow parameter t_1 as the space coordinate[15]. In this chapter we will follow the same line to deal with the multi–matrix models , try to shed some light on the inter–relations of the multi–matrix models and the generalized KP hierarchy discussed in chapter 4. In section 1,we will at first represent arbitary multi–matrix models as coupled discrete linear systems, and emphasize the importance of the coupling conditions, meanwhile we will show the existence of the hidden 2–dimensional Toda Lattice hierarchy. In order to get some knowledge about the full integrable structure in multi–matrix models , we proceed the spheric limit in the section 2. With this at hand, we explain the full KP hierarchy by passing from lattice to differential formalism in the section **3**, and prove that the multi–matrix model partition function in fact is τ–function of the generalized KP hierarchy discussed in the chapter **4**. We end this chapter by making some remarks in section 4.

6.1 The Multi–Matrix Models

The partition function of the q–matrix model is given by

$$Z_N(t,c)=\int dM_1dM_2\dots dM_q e^{TrU}$$

where $M_1,\dots,M_q$ are Hermitian $N\times N$ matrices, c's are coupling constants among adjacent matrices. And

$$U=\sum_{\alpha=1}^{q}V_\alpha+\sum_{\alpha=1}^{q-1}c_{\alpha,\alpha+1}M_\alpha M_{\alpha+1}$$

with the potentials

$$V_\alpha=\sum_{r=1}^{p_\alpha}t_{\alpha,r}M_\alpha^r \qquad \alpha=1,2\dots,q.$$

Right now we consider the potentials of finite terms, i.e. p_α's are finite positive integers, since in this way, we can see the meaning of the constraints more clearly.

The ordinary procedure could be formulated in the following three steps:

(i). Integrating out the angular parts such that only the integrations over the eigenvalues are left,

$$Z_N(t,c)=const.\int\prod_{\alpha=1}^{q}\prod_{i=1}^{N}d\lambda_{\alpha,i}\Delta(\lambda_1)e^{\tilde U}\Delta(\lambda_q),$$

where

$$\tilde U=\sum_{\alpha=1}^{q}\sum_{i=1}^{N}V(\lambda_{\alpha,i})+\sum_{\alpha=1}^{q-1}\sum_{i=1}^{N}c_{\alpha,\alpha+1}\lambda_{\alpha,i}\lambda_{\alpha+1,i},$$

and $\Delta(\lambda_1)$ and $\Delta(\lambda_q)$ are Vandermonde determinants

$$\Delta(\lambda_\alpha)=\begin{vmatrix}1 & 1 & \cdots & 1\\ \lambda_{\alpha,1} & \lambda_{\alpha,2} & \cdots & \lambda_{\alpha,N}\\ \vdots & \vdots & \cdots & \vdots\\ \lambda_{\alpha,1}^{N-1} & \lambda_{\alpha,2}^{N-1} & \cdots & \lambda_{\alpha,N}^{N-1}\end{vmatrix}, \qquad \alpha=1, \quad \text{or} \quad q.$$

Here the first subindex indicates α–th matrix, the second indices means the different eigenvalues of the matrix.

(ii). Introducing the orthogonal polynomials*

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers}, \qquad \eta_n(\lambda_q) = \lambda_q^n + \text{lower powers}$$

which satisfy the orthogonal relation

$$\int d\lambda_1 \dots d\lambda_q \xi_n(\lambda_1) e^{V_1(\lambda_1)+\mu+V_q(\lambda_q)} \eta_m(\lambda_q) = h_n(t,c)\delta_{nm} \tag{6.1.1}$$

where

$$\mu \equiv \sum_{\alpha=2}^{q-1} \sum_{r=1}^{\infty} t_{\alpha,r} \lambda_\alpha^r + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} \lambda_\alpha \lambda_{\alpha+1}.$$

(iii). Finally, expanding the Vandermonde determinants in terms of these orthogonal polynomials and using the orthogonality relation (6.1.1), we can easily calculate the partition function

$$Z_N(t,c) = const. N! \prod_{i=0}^{N-1} h_i \tag{6.1.2}$$

As in 1–matrix model , we see that the properties of the partition function are completely charcterized by those of the coefficients $h_n(t,c)$'s. The main purpose of this section is to show that the information about these coefficients can be obtained from certain discrete linear systems.

6.1.1 The Coupled Discrete Linear Systems in q–Matrix Model

For the later convenience, we introduce some more definitions. For any matrix M, we define

$$(\mathcal{M})_{ij} = M_{ij} \frac{h_j}{h_i}, \qquad \bar{M}_{ij} = M_{ji}, \qquad M_l(j) \equiv M_{j,j-l}.$$

Later on we will call the element $M_l(j)$ is belonging to the $j-th$ sector. As usual we also introduce a natural gradation

$$\deg[E_{ij}] = j - i.$$

For a given matrix M, if its all non–zero elements are at the pseudo–diagonal lines with degrees in the interval $[a,b]$, then we will simply denote by $M \in [a,b]$. We will see later that this notation is useful for discussing the dependences of Jacobi matrices on the perturbations.

Now we redefine the orthogonal polynomials in the following way

$$\Psi_n(\lambda_1) = e^{V_1(\lambda_1)} \xi_n(\lambda_1), \qquad \Phi_n(\lambda_q) = e^{V_q(\lambda_q)} \eta_n(\lambda_q).$$

*The choice of the polynomials is not unique, in fact for different purposes we may choose different ones such that the calculation is simpler.

As usual we denote the semi–infinite column vectors $\Psi_0, \Psi_1, \Psi_2, \ldots$, and $\Phi_0, \Phi_1, \Phi_2, \ldots$, by Ψ and Φ respectively. With these polynomials, the orthogonal relation (6.1.1) becomes

$$\int \prod_{\beta=1}^{q} d\lambda_\beta \Psi_n(\lambda_1) e^{\mu} \Phi_m(\lambda_q) = \delta_{nm} h_n(t,c). \tag{6.1.3}$$

This orthogonality relation plays a particularly crucial role in our analysis. In fact we will see that all our results come from this relation. In order to do so, we need to introduce the following q matrices

$$\int \prod_{\alpha=1}^{q} d\lambda_\alpha \Psi_n(\lambda_1) e^{\mu} \lambda_\alpha \Phi_m(\lambda_q) \equiv Q_{nm}(\alpha) h_m = \bar{Q}_{mn}(\alpha) h_n, \quad \alpha = 1, \ldots, q. \tag{6.1.4}$$

Among them, $Q(1), \bar{Q}(q)$ are Jacobi matrices, whose pure upper triangular parts are I_+. Besides, there are two more useful matrices worth mentioning, which are defined as

$$\int \prod_{\alpha=1}^{q} d\lambda_\alpha \Big(\frac{\partial}{\partial \lambda_1} \Psi_n(\lambda_1)\Big) e^{\mu} \Phi_m(\lambda_q) \equiv P_{nm} h_m \tag{6.1.5}$$

$$\int d\lambda_1 \ldots d\lambda_q \Psi_n(\lambda_1) e^{\mu} \Big(\frac{\partial}{\partial \lambda_q} \Phi_m(\lambda_q)\Big) \equiv \bar{P}_{mn}(q) h_n \tag{6.1.6}$$

Now all the materials are at hand, let us see what we can derive from the orthogonality eq.(6.1.3).

(i).The Coupling Conditions

At first we want to show that the matrices (6.1.4) we introduced above are not completely independent. Taking the derivatives of the integrand in eq.(6.1.3) with respect to $\lambda_\alpha, 1 \le \alpha \le q$, we can easily derive the constraints *or coupling conditions*

$$P + c_{12} Q(2) = 0, \tag{6.1.7a}$$

$$c_{\alpha-1,\alpha} Q(\alpha - 1) + V'_\alpha + c_{\alpha,\alpha+1} Q(\alpha + 1) = 0, \qquad 2 \le \alpha \le q-1, \tag{6.1.7b}$$

$$c_{q-1,q} Q(q-1) + \bar{P}(q) = 0. \tag{6.1.7c}$$

where we denote

$$V'_\alpha = \sum_{r=1}^{p_\alpha} r t_{\alpha,r} Q^{r-1}(\alpha), \qquad \alpha = 1, 2, \ldots, q$$

Although these conditions seem to be trivial, however they play extremely important roles in the study of multi–matrix models .

- Firstly, it is just these coupling conditions that lead to the famous $W_{1+\infty}$–constraints on the partition function at the discrete level [66](see also Appendex.C).

- Secondly, these conditions explicitly show that the Jacobi matrices depend on the choices of the potentials. We can immediately see that these coupling conditions completely determine the degrees of matrices $Q(\alpha)$. Since $(P - V_1')$ and $\left(P(q) - \bar{V}_q'\right)$ are *purely* lower triangular matrices, a simple calculation shows that

$$Q(\alpha) \in [-m_\alpha, n_\alpha], \qquad \alpha = 1, 2, \ldots, q \tag{6.1.8}$$

where

$$\begin{aligned} m_1 &= (p_q - 1)\ldots(p_3 - 1)(p_2 - 1), && n_1 = 1 \\ m_\alpha &= (p_q - 1)(p_{q-1} - 1)\ldots(p_{\alpha+1} - 1), && 2 \leq \alpha \leq q - 1 \\ n_\alpha &= (p_{\alpha-1} - 1)\ldots(p_2 - 1)(p_1 - 1), && 2 \leq \alpha \leq q - 1 \\ m_q &= 1; \qquad n_q = (p_{q-1} - 1)\ldots(p_2 - 1)(p_1 - 1) \end{aligned}$$

These directly show that, in order to preserve the forms of the Jacobi matrices, only a finite number of perturbations are allowed. Conversely, if we want to have the completely hierarchical structure—we should let $p_\alpha \longrightarrow \infty$, then all these matrices will have infinite many non–zero diagonal lines. So generally speaking, these coupling conditions reveal the major difference of the multi–matrix models with one–matrix model.

- Furthermore, these conditions may give some hint that the matrix models are really topological models. In order to see this point, we take a particular case—two matrix models. In this situation, the eqs.(6.1.7a–6.1.7c) reduce to

$$P + c_{12}Q(2) = 0, \qquad c_{12}Q(1) + \bar{\mathcal{P}}(2) = 0,$$

or eqivalently

$$c_{12}\bar{Q}_+(1) = -V_+'(Q(2)), \qquad c_{12}Q_+(2) = -V_+'(Q(1)). \tag{6.1.9}$$

we have $(p_1 + p_2 + 1)$–series of the coordinates(i.e. the elements of $Q(1,2)$ and R_i's) but we have $(p + q)$–series of the constraints. Finally, we are only left with one–series of independent coordinates, i.e. one can express all the elements of $Q(1,2)$ in terms of R_j's and some coupling constants. At first glance it seems to be strange, but that is the real story for multi–matrix models if we notice that all the information of the matrix model is included in the partition function which only involves R_i's. On the other hand, the string equation will provide one more series of the constraints, so there is no *local degree of freedom* left. This means matrix models are likely to be topological.

(*ii*). The coupled Discrete Linear Systems

Now it is time to derive the coupled linear systems we promised before. The calculation is very simple, that is, taking the derivatives of the eqs.(6.1.3) with respect to the time parameters $t_{\alpha,r}$, and using the eqs.(6.1.4), we can get the time evolutions of Ψ and Φ, which can be represented as two discrete linear systems

(*) *Discrete Linear System I*:

$$\left\{\begin{array}{ll} Q(1)\Psi(\lambda_1) = \lambda_1\Psi(\lambda_1), & \\ \frac{\partial}{\partial t_{1,k}}\Psi(\lambda_1) = Q_+^k(1)\Psi(\lambda_1), & 1 \le k \le p_1, \\ \frac{\partial}{\partial t_{\alpha,k}}\Psi(\lambda_1) = -Q_-^k(\alpha)\Psi(\lambda_1), & 1 \le k \le p_\alpha; \qquad 2 \le \alpha \le q, \\ \frac{\partial}{\partial \lambda}\Psi(\lambda_1) = P\Psi(\lambda_1). & \end{array}\right. \tag{6.1.10}$$

its consistent conditions are as follows

$$[Q(1), \quad P] = 1 \tag{6.1.11a}$$

$$\frac{\partial}{\partial t_{\alpha,k}}Q(1) = [Q(1), \quad Q_-^k(\alpha)] \tag{6.1.11b}$$

$$\frac{\partial}{\partial t_{\alpha,k}}P = [P, \quad Q_-^k(\alpha)] \tag{6.1.11c}$$

Among them, the first equation is nothing but the string equation, the others can be considered as discrete KP–Hierachies.

(**) *Discrete Linear System II*:

$$\left\{\begin{array}{ll} \bar{Q}(q)\Phi(\lambda_q) = \lambda_q\Phi(\lambda_q), & \\ \frac{\partial}{\partial t_{q,k}}\Phi(\lambda_q) = \bar{Q}_+^k(q)\Phi(\lambda_q), & \\ \frac{\partial}{\partial t_{\alpha,k}}\Phi(\lambda_q) = -\bar{Q}_-^k(\alpha)\Phi(\lambda_q), & 1 \le k \le p_\alpha; \qquad 1 \le \alpha \le q-1 \\ \frac{\partial}{\partial \lambda_q}\Phi(\lambda_q) = P(q)\Phi(\lambda_q). & \end{array}\right. \tag{6.1.12}$$

with consistent conditions

$$[\bar{Q}(q), \quad P(q)] = 1, \tag{6.1.13a}$$

$$\frac{\partial}{\partial t_{\alpha,k}}\bar{Q}(q) = [\bar{Q}(q), \quad \bar{Q}_-^k(\alpha)] \tag{6.1.13b}$$

$$\frac{\partial}{\partial t_{\alpha,k}}P(q) = [P(q), \quad \bar{Q}_-^k(\alpha)] \tag{6.1.13c}$$

The equations of motion of the matrices $Q(\alpha)$'s can be extracted from the eqs.(6.1.11a–6.1.11c)(or (6.1.13a–6.1.13c)) and eqs.(6.1.7a–6.1.7c), so finally we have the following equations of motion

$$\frac{\partial}{\partial t_{\beta,k}}Q(\alpha) = [Q_+^k(\beta),\ \ Q(\alpha)], \qquad 1 \leq \beta \leq \alpha \tag{6.1.14a}$$

$$\frac{\partial}{\partial t_{\beta,k}}Q(\alpha) = [Q(\alpha),\ \ Q_-^k(\beta)], \qquad \alpha \leq \beta \leq q \tag{6.1.14b}$$

where k runs from 1 to p_β. It is tedious, but straightforward to prove that all the flows commute with one another.

A few remarks are in order:

(*). The set of the eqs.(6.1.14a–6.1.14b) contain several subsystems, which are of particular interests. For examples, if we allow all the p_α's go to infinite, then

The case $\alpha = \beta = 1$: Which exihibits the full discrete hierarchical structure of the integrable lattice system discussed in [22],

$$\frac{\partial}{\partial t_{1,r}}Q(1) = [Q_+^r(1),\ \ Q(1)]. \tag{6.1.15}$$

One can obtain its two compatible Poisson brackets from the coadjoint orbit analysis. Meanwhile we also can simply write down its hamiltonians like

$$H_{r(1)} = \frac{1}{r}\tilde{\mathrm{Tr}}\Big(Q^r(1)\Big), \qquad \forall r \geq 1. \tag{6.1.16}$$

We will show in §6.3 that this system can be reformulated as KP hierarchy .

The case $2 \leq \alpha = \beta \leq q-1$: these subsystems can also be considered as discrete KP hierarchies, but the 'Jacobi matrices' have degrees $(-\infty, \infty)$, although their bi-hamiltonian structures are not clear, but the commutativity of the flows imply the integrability. The hamiltonians seem to be†

$$H_{r(\alpha)} = \frac{1}{r}\tilde{\mathrm{Tr}}\Big(Q^r(\alpha)\Big), \qquad \forall r \geq 1. \tag{6.1.17}$$

The case $\alpha = \beta = q$: this case is in fact quite similar to the first case. This is not surprising, since the first matrix and the last matrix in the matrix chain should play the same roles. The discrete hieraechy is

$$\frac{\partial}{\partial t_{q,r}}\bar{Q}(q) = [\bar{Q}_+^r(q),\ \ \bar{Q}(q)]. \tag{6.1.18}$$

†Since we may define the Poisson bracket in the following way

$$\{H_{r(\alpha)}, Q^s(\alpha)\} = [Q_+^r(\alpha), Q^s(\alpha)], \qquad \forall r, s.$$

when take trace operation, we would get vanishing Poisson bracket, this suggests the Hamiltonians to be the form (6.1.17).

Its hamiltonians are simply

$$H_{r(q)} = \frac{1}{r}\tilde{\mathrm{Tr}}\Big(Q^r(q)\Big), \qquad \forall r \geq 1. \tag{6.1.19}$$

This system can also be reformulated as KP hierarchy .

The non–commutativity of the hamiltonians: We see that there are q–series of the hamiltonians, i.e. the system (6.1.14a–6.1.14b) possesses multi–hamiltonian structures. Each of which generate a series of commuting flows described by the corresponding time parameters $t_{\alpha,r}$'s. It seems that hamiltonians in the different series do not commute, however we are not able to prove it.

(**). The second point we would like to mention is the following. For multi–matrix models , we have two Jacobi matrices, so we can introduce two discrete linear systems. In fact we can go further, we may even introduce the other $(2q-2)$ discrete linear systems. In order to do so, we start from the orthogonal relation (6.1.3) once again, and define two series of the functions like

$$\xi_n^{(\alpha)}(t,\lambda_\alpha) \equiv \int \prod_{\beta=1}^{\alpha-1} d\lambda_\beta \Psi_n(\lambda_1) e^{\mu_\alpha}. \tag{6.1.20}$$

and

$$\eta_n^{(\alpha)}(t,\lambda_\alpha) \equiv \int \prod_{\beta=\alpha+1}^{q} d\lambda_\beta e^{\nu_\alpha} \Phi_m(\lambda_q). \tag{6.1.21}$$

where we denote

$$\mu_\alpha \equiv \sum_{\beta=2}^{\alpha-1}\sum_{r=1}^{\infty} t_{\beta,r}\lambda_\beta^r + \sum_{\beta=1}^{\alpha-1} c_{\beta,\beta+1}\lambda_\beta\lambda_{\beta+1}.$$

$$\nu_\alpha \equiv \sum_{\beta=\alpha+1}^{q-1}\sum_{r=1}^{\infty} t_{\beta,r}\lambda_\beta^r + \sum_{\beta=\alpha}^{q-1} c_{\beta,\beta+1}\lambda_\beta\lambda_{\beta+1}.$$

Obviously we have

$$\xi_n^{(1)}(t,\lambda_1) = \Psi_n(\lambda_1), \qquad\qquad \eta_n^{(q)}(t,\lambda_q) = \Phi_m(\lambda_q).$$

Furthermore, we can recast the orthogonal relation (6.1.3) into the following form

$$\int d\lambda_\alpha \xi_n^{(\alpha)}(t,\lambda_\alpha) e^{V_\alpha(\lambda_\alpha)} \eta_n^{(\alpha)}(t,\lambda_\alpha) = \delta_{nm} h_n(t,c), \qquad \forall 1 \leq \alpha \leq q. \tag{6.1.22}$$

We may understand these relations in the following way. The q–series of the functions $\xi_n^{(\alpha)}(t,\lambda_\alpha)$ form *q–Hilbert spaces* $\mathcal{H}_\alpha$. The other q–series of the functions $\eta_n^{(\alpha)}(t,\lambda_\alpha)$ also form q Hilbert spaces $\mathcal{H}_\alpha^*$'s, which are dual to $\mathcal{H}_\alpha$. Then the orthogonal relations (6.1.22) just define the inner products between the Hilbert space $\mathcal{H}_\alpha$ and its dual $\mathcal{H}_\alpha^*$. This structure bears some similarity with topological field theory [53].

Now what we should do is to extract the spectral equations and the time evolutions of these new functions. From the eqs.(6.1.4), we immediately see that

$$\int d\lambda_\alpha \xi_n^{(\alpha)}(t,\lambda_\alpha)e^{V_\alpha(\lambda_\alpha)}\lambda_\alpha \eta_n^{(\alpha)}(t,\lambda_\alpha) = Q_{nm}(\alpha)h_m(t,c), \qquad 1 \le \alpha \le q. \qquad (6.1.23)$$

This reminds us that the spectral equations should be

$$\begin{aligned}
\lambda_\alpha \xi_n^{(\alpha)}(t,\lambda_\alpha) &= Q_{nm}(\alpha)\xi_m^{(\alpha)}(t,\lambda_\alpha),\\
\lambda_\alpha \eta_n^{(\alpha)}(t,\lambda_\alpha) &= \bar{Q}_{mn}(\alpha)\eta_m^{(\alpha)}(t,\lambda_\alpha),
\end{aligned} \qquad (6.1.24)$$

or in matrix form

$$\lambda_\alpha \xi^{(\alpha)} = Q(\alpha)\xi^{(\alpha)}, \qquad 1 \le \alpha \le q. \qquad (6.1.25)$$

$$\lambda_\alpha \eta^{(\alpha)} = \bar{Q}(\alpha)\eta^{(\alpha)}, \qquad 1 \le \alpha \le q. \qquad (6.1.26)$$

On the other hand, from the definitions (6.1.20) and (6.1.21), and making use of eqs.(6.1.10) and (6.1.12), we can derive the evolution equations of the functions $\xi^{(\alpha)}$ and $\eta^{(\alpha)}$, which take the following form

$$\frac{\partial}{\partial t_{\beta,r}}\xi^{(\alpha)} = Q_+^r(\beta)\xi^{(\alpha)}, \qquad 1 \le \beta \le \alpha - 1, \qquad (6.1.27)$$

$$\frac{\partial}{\partial t_{\beta,r}}\xi^{(\alpha)} = -Q_-^r(\beta)\xi^{(\alpha)}, \qquad \alpha \le \beta \le q. \qquad (6.1.28)$$

and

$$\frac{\partial}{\partial t_{\beta,r}}\eta^{(\alpha)} = \bar{Q}_+^r(\beta)\eta^{(\alpha)}, \qquad \alpha + 1 \le \beta \le q, \qquad (6.1.29)$$

$$\frac{\partial}{\partial t_{\beta,r}}\eta^{(\alpha)} = -\bar{Q}_-^r(\beta)\eta^{(\alpha)}, \qquad 1 \le \beta \le \alpha. \qquad (6.1.30)$$

Now combining the equations (6.1.25–6.1.30) together, we get $2q$ discrete linear systems, two of which are already presented (6.1.10) and (6.1.12), the other $(2q-2)$ are the additional discrete linear systems on the corresponding Hilbert space. All of them can be written in a unified form

$$\begin{cases}
Q(\alpha)\xi(\lambda_\alpha) = \lambda_\alpha \xi(\lambda_\alpha),\\
\frac{\partial}{\partial t_{\beta,r}}\xi(\lambda_\alpha) = Q_+^r(\beta)\xi(\lambda_\alpha), \qquad 1 \le \beta < \alpha,\\
\frac{\partial}{\partial t_{\beta,r}}\xi(\lambda_\alpha) = -Q_-^r(\alpha)\xi(\lambda_\alpha), \quad \alpha \le \beta \le q.
\end{cases} \qquad (6.1.31)$$

and

$$\begin{cases}
\bar{Q}(\alpha)\eta(\lambda_\alpha) = \lambda_\alpha \eta(\lambda_\alpha),\\
\frac{\partial}{\partial t_{\beta,r}}\eta(\lambda_\alpha) = \bar{Q}_+^r(\beta)\eta(\lambda_\alpha), \qquad \alpha + 1 \le \beta \le q,\\
\frac{\partial}{\partial t_{\beta,r}}\eta(\lambda_\alpha) = -\bar{Q}_-^r(\alpha)\eta(\lambda_\alpha), \quad 1 \le \beta \le \alpha.
\end{cases} \qquad (6.1.32)$$

From these additional $(2q-2)$ discrete linear systems, we can derive their consistency conditions, which are in fact nothing but the the string equations and the equations of motion (6.1.14a–6.1.14b) of the Jacobi matrices. This result suggests some fascinating interpretations. The introduction of the functions $\xi_n^{(\alpha)}(t,\lambda_\alpha)$'s and $\eta_n^{(\alpha)}(t,\lambda_\alpha)$ can be considered as formal reductions of the multi–matrix models to 1–matrix model , since we are left with the orthogonal relations (6.1.22), which are quite similar to the one in 1–matrix model . The major difference comes from the spectral (or *recursion*) relations (6.1.25), due to the fact that new Jacobi matrices $Q(\alpha)$'s have the degrees $(-\infty,\infty)$, $\xi_n^{(\alpha)}(t,\lambda_\alpha)$'s and $\eta_n^{(\alpha)}(t,\lambda_\alpha)$ are very complicated functions rather than the *Polynomials*. This probably means that the multi–matrix models can be realized as 1-matrix models if we choose suitable non–polynomial interactions. This is worth studying further.

(*iii*). <u>The Partition Function</u>

One of the byproducts in the above procedure is the equations of motion of the partition function.

$$\frac{\partial}{\partial t_{\alpha,r}} \ln Z_N(t,c) = \mathrm{Tr}\Big(Q^r(\alpha)\Big), \qquad 1 \le r \le p_\alpha; \qquad 1 \le \alpha \le q \tag{6.1.33}$$

Using the consistency conditions, we can rewrite these equations in the following form

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{\alpha,r}} \ln Z_N(t,c) = \Big(Q^r(\alpha)\Big)_{N,N-1}, \qquad 1 \le r \le p_\alpha; \qquad 1 \le \alpha \le q \tag{6.1.34}$$

or equivalently

$$\frac{\partial^2}{\partial t_{q,1}\partial t_{\alpha,r}} \ln Z_N(t,c) = \Big(\bar{Q}^r(\alpha)\Big)_{N,N-1}, \qquad 1 \le r \le p_\alpha; \qquad 1 \le \alpha \le q \tag{6.1.35}$$

We end this subsection with one more remark. All the information of the multi–matrix models is completely encoded in the discrete hierarchy (6.1.14a–6.1.14b). Once we solve these equations, we can reconstruct the partition function of the matrix models such that which satisfies (6.1.33).

6.1.2 The hidden 2-dimensional Toda lattice

In the previous subsection we have shown that the multi–matrix models are equivalent to certain discrete linear systems subjected to some constraints. Without reference to the coupling conditions, the discrete linear systems lead to discrete KP hierarchies. In fact we will show that these discrete hierarchies (or lattice hierarchies), in $q=2$ case, are nothing but the 2–dimensional Toda lattice[65], and in arbitrary q cases, they are 2–dimensional Toda lattice with additional flows.

At first, we give a coordinization of Jacobi matrices

$$Q(1) = I_+ + A, \qquad \bar{Q}(q) = I_+ + B \tag{6.1.36}$$

Explicitly

$$A = \sum_i \sum_{l=0}^{m_1} a_l(i) E_{i,i-l}, \qquad B = \sum_i \sum_{l=0}^{m_q} b_l(i) E_{i,i-l}$$

and for the supplementary matrices

$$Q(\alpha) = \sum_{l=-n_\alpha}^{m_\alpha} T_l^{(\alpha)}(i) E_{i,i-l}, \quad 2 \le \alpha \le q-1. \tag{6.1.37}$$

We can see immediately that

$$\Big(Q_+(1)\Big)_{ij} = \delta_{j,i+1} + a_0(i)\delta_{i,j}, \qquad \Big(Q_-(q)\Big)_{ij} = R_i \delta_{j,i-1}$$

so we can write down the $t_{1,1}$– and $t_{q,1}$–flows explicitly

$$\frac{\partial}{\partial t_{1,1}} a_l(j) = a_{l+1}(j+1) - a_{l+1}(j) + a_l(j)(a_0(j) - a_0(j-l)) \tag{6.1.38a}$$

$$\frac{\partial}{\partial t_{q,1}} a_l(j) = R_{j-l+1} a_{l-1}(j) - R_j a_{l-1}(j-1) \tag{6.1.38b}$$

$$\frac{\partial}{\partial t_{1,1}} T_l^{(\alpha)}(j) = T_{l+1}^{(\alpha)}(j+1) - T_{l+1}^{(\alpha)}(j) + T_l^{(\alpha)}(j)(a_0(j) - a_0(j-l)) \tag{6.1.38c}$$

$$\frac{\partial}{\partial t_{q,1}} T_l^{(\alpha)}(j) = R_{j-l+1} T_{l-1}^{(\alpha)}(j) - R_j T_{l-1}^{(\alpha)}(j-1) \tag{6.1.38d}$$

In particular

$$\frac{\partial}{\partial t_{q,1}} a_0(j) = R_{j+1} - R_j \tag{6.1.39}$$

on the other hand, from(6.1.33), one can show that

$$\frac{\partial}{\partial t_{1,1}} R_j = R_j(a_0(j) - a_0(j-1)) \tag{6.1.40}$$

Combining these two equations together we obtain the following 2-dimensional Toda lattice equation

$$\frac{\partial^2}{\partial t_{1,1} \partial t_{q,1}} \ln R_j = R_{j+1} - 2R_j + R_{j-1} \tag{6.1.41}$$

In terms of the coordinates $\phi_j = \ln h_j$'s, the above equation becomes

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{q,1}}\phi_j = e^{\phi_{j+1}-\phi_j} - e^{\phi_j-\phi_{j-1}} \tag{6.1.42}$$

These equations (6.1.41) and (6.1.42) show that the parameters $t_{1,1}$ and $t_{q,1}$ seem to be the space coordinates, and the other parameters are really time flow parameters. In fact, in $q = 2$ case (i.e. 2–matrix model), the other time flows generate the complete 2–dimensional Toda lattice hierarchy[65]. From the analysis in the previous subsection, we know that the system is restricted by the coupling conditions, therefore we can say that 2–matrix models are nothing but constrained 2–dimensional Toda lattice. When we consider the other multi–matrix models, we see that they contain not only this Toda lattice hierarchy, but also the additional series of flows.

6.2 The Spheric Limit of the Multi–Matrix Models

In the previous section we have already shown that the multi–matrix models is quite different from the 1–matrix model . The typical property is the dependence of its Jacobi matrices on the potentials. This causes a great difficulty in analysing the full hierarchical structure of the multi–matrix models by taking the double scaling limit. The usual conjecture is that the q–matrix model would described by $(q+1)$–th KdV hierarchy * in the double scaling limit, and the multi–criticality will represent the (q,p)–minimal model coupled to 2–dimensional quantum gravity . However, if we look the particular example—the two matrix model, we see that the 6–th even potentials could give us (4,5) and (3,8) models[12]. The second one is encoded in Bousinesque hierarchy, but the first one seems to belong to 4–th KdV hierarchy . As we already know from the 1–matrix model , criticality does not depend on the particular continuum limit, i.e. the 1–matrix model is characterized by non–linear Schrödinger hierarchy , its spheric limit version described by the dispersionless NLS. This suggests to us that we may perform the spheric limit as a useful exercise to obtain a certain dispersionless integrable hierarchy, whose non–dispersionless version would correspond to the full hierarchy in the multi–matrix models . This is what we want to do in this section.

Learning from the 1–matrix model in chapter **3**, going from lattice to the spheric limit is simply by rescaling the lattice "n" to the space coordinate x, and I_+ to the twisted momentum "z", such that

$$\{z, x\} = z. \tag{6.2.1}$$

The spheric version of the Jacobi matrices is

$$Q(1) \Longrightarrow \mathcal{L}(1) = z + \sum_{i=0}^{\infty} a_i(x) z^{-i}, \tag{6.2.2}$$

*For $q = 1$ case, it is usual KdV hierarchy , in the $q = 2$ case, it corresponds to Bousinesque hierarchy, etc.

$$Q(\alpha) \Longrightarrow \mathcal{L}(\alpha) = \sum_{i=-\infty}^{\infty} T_i^{(\alpha)}(x) z^{-i}, \qquad 2 \le \alpha \le q-1, \tag{6.2.3}$$

$$Q(q) \Longrightarrow \bar{\mathcal{L}}(1) = \frac{z}{R} + \sum_{i=0}^{\infty} b_i(x) R^i z^{-i}, \tag{6.2.4}$$

We should remind you that due to our different choices of the polynomials here the operation "bar" means the mapping from z to $\frac{R}{z}$.

In order to get the spheric limit of the discrete KP hierarchy (6.1.14a–6.1.14b), we only need to replace the matrix commutators by Poisson brackets (6.2.1),

$$\frac{\partial}{\partial t_{\beta,k}} \mathcal{L}(\alpha) = \{\mathcal{L}_+^k(\beta),\ \mathcal{L}(\alpha)\}, \qquad 1 \le \beta \le \alpha \tag{6.2.5a}$$

$$\frac{\partial}{\partial t_{\beta,k}} \mathcal{L}(\alpha) = \{\mathcal{L}(\alpha),\ \mathcal{L}_-^k(\beta)\}, \qquad \alpha \le \beta \le q \tag{6.2.5b}$$

without misunderstanding, these Poisson brackets in fact are rooted in the basic relation (6.2.1), and take the following form

$$\{f, g\} \equiv z\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial x}. \tag{6.2.6}$$

Besides the flows (6.2.5a–), there is an additional flow

$$\frac{\partial}{\partial x} \mathcal{L}(\alpha) = \{\left(\ln \mathcal{L}(1)\right)_+, \mathcal{L}(\alpha)\}. \tag{6.2.7}$$

In particular, for $q = 2$ case, we will have

$$\frac{\partial}{\partial t_{1,k}} \mathcal{L}(1) = \{\mathcal{L}_+^k(1),\ \mathcal{L}(1)\}, \tag{6.2.8a}$$

$$\frac{\partial}{\partial t_{2,k}} \mathcal{L}(1) = \{\mathcal{L}(1),\ \mathcal{L}_-^k(2)\}, \tag{6.2.8b}$$

$$\frac{\partial}{\partial t_{1,k}} \bar{\mathcal{L}}(2) = \{\mathcal{L}_+^k(1),\ \bar{\mathcal{L}}(2)\}, \tag{6.2.8c}$$

$$\frac{\partial}{\partial t_{2,k}} \bar{\mathcal{L}}(2) = \{\bar{\mathcal{L}}_+^k(2),\ \bar{\mathcal{L}}(2)\}, \tag{6.2.8d}$$

Now let us compare this hierarchy with the dispersionless generalized KP hierarchy introduced in chapter 4. At first we note that the Lax pairs are different, (4.4.15) is expressed in terms of the canonical mometum 'p'(due to the canonical Poisson bracket (4.4.14)), but $\mathcal{L}(\alpha)$ defined in the above are expanded in the powers of the twisted momentum 'z'(we will see that they are related by a shifting mapping). Due to this difference, the flows are written in terms of different Poisson brackets, i.e. eq.(6.2.6) and eq.(4.4.20). In the eq.(6.2.6), $x = t_0$ plays the role of the space coordinate and the t_1 is really a time parameter. However, in the eq.(4.4.20), the t_0 has been dummied, instead, t_1 plays the role of the space coordinate. Apart from these differences, these two hierarchies are almost the same as each other. This gives us some hint to extract the full KP hierarchy . We will be exploited this point in the next section.

6.3 The KP hierarchy in q–matrix model

Till now we have shown that multi–matrix models can be represented as coupled discrete linear systems, whose consistency conditions give discrete KP hierarchies and the string equations. The classical limit of the discrete hierarchy (6.1.14a–6.1.14b) is almost the same as the dispersionless version of the generalized KP hierarchy . In this section, we will use the appoach proposed in[15] to reexpress the lattice formulation as a purely differential hierarchy, that is exactly the generalized KP hierarchy .

6.3.1 The generalized KP hierarchy in q–matrix model

In our analysis, the first flows, i.e. $t_{\alpha,1}$–flows play an essential role. On the one hand, using these flows, we can express all the quantities involved in the theory as the functions of the coordinates in $j-th$ sector. On the other hand, it is just the first flows of Ψ and Φ that enable us to recast the discrete linear systems into pure differential systems. Now we analyze these points in detail.

(I). The first linear system associated to the generalized KP hierarchy:

In order to extract out the generalized KP hierarchy, our strategy is as follows. We first try to find the associated linear system, then consider its consistency conditions which should directly give us the generalized KP hierarchy. However, learning from the lattice formulation, there are $2q$ discrete linear systems encoded in the q-matrix model , so when we pass to the differential language, we would expect that there are also $2q$ linear systems associated to the generalized KP hierarchy. In the following we will begin with the DLS (6.1.10). By using the eqs.(6.1.40) and (6.1.39), we have

$$a_0(j-1) = a_0(j) - (\ln R_j)'$$
$$a_0(j-2) = a_0(j) - (\ln R_j)' - \Big(\ln\big[R_j - \dot{a}_0(j) + \frac{\partial^2 \ln R_j}{\partial t_{1,1}\partial t_{q,1}}\big]\Big)'$$

where for any function $f(t)$, we denote

$$f' \equiv \frac{\partial f}{\partial t_{1,1}}, \qquad \dot{f} \equiv \frac{\partial f}{\partial t_{q,1}} \tag{6.3.1}$$

generally we define

$$R_{j+r} \equiv F_r(j), \qquad a_0(j+r) \equiv G_r(j)$$
$$R_{j-r} \equiv \tilde{F}_r(j), \qquad a_0(j-r) \equiv \tilde{G}_r(j)$$

then they satisfy the recursion relation

$$F_{r+1}(j) = F_r(j) + \dot{G}_r(j) \tag{6.3.2a}$$
$$G_{r+1}(j) = G_r(j) + \big(\ln[F_r(j) + \dot{G}_r(j)]\big)' \tag{6.3.2b}$$

$$\tilde{F}_{r-1}(j) = \tilde{F}_r(j) - \dot{\tilde{G}}_r(j) + \frac{\partial^2 \ln \tilde{F}_r}{\partial t_{1,1} \partial t_{q,1}} \tag{6.3.2c}$$

$$\tilde{G}_{r-1}(j) = \tilde{G}_r(j) - (\ln \tilde{F}_r)' \tag{6.3.2d}$$

These results guarantee that all the $a_0(i)$'s and R_i's $(i \neq j)$ can be expressed in terms of $a_0(j)$ and R_j as well as their $t_{1,1}$ and $t_{q,1}$-derivatives. Furthermore, substituting these results into eqs.(6.1.38a–6.1.38d), we can recursively express all $a_l(i)$'s and $T_l^{(\alpha)}(i)$'s$(i \neq j)$ as the functions of the coordinates in the $j-th$ sector.

Now let us consider the $t_{1,1}-$,and $t_{q,1}$–flows of Ψ_j. From the eqs.(6.1.10), we can write down their explicit forms

$$\frac{\partial}{\partial t_{1,1}} \Psi_j = \Psi_{j+1} + a_0(j)\Psi_j \tag{6.3.3}$$

$$\frac{\partial}{\partial t_{q,1}} \Psi_j = -R_j \Psi_{j-1} \tag{6.3.4}$$

which lead to the following equalities

$$\Psi_{j+1} = \hat{A}_j \Psi_j, \qquad \Psi_{j-1} = \hat{B}_{j-1} \Psi_j \tag{6.3.5}$$

where

$$\hat{A}_j \equiv \partial - a_0(j) = -\bar{\partial}^{-1} R_{j+1}$$

$$\hat{B}_j \equiv \partial^{-1} \sum_{l=0}^{\infty} (a_0(j)\partial^{-1})^l = -\frac{1}{R_{j+1}} \bar{\partial}$$

we denote $\frac{\partial}{\partial t_{1,1}}$ and $\frac{\partial}{\partial t_{q,1}}$ by ∂_1, ∂_2 respectively. It is easy to see

$$\hat{A}_j \hat{B}_j = \hat{B}_j \hat{A}_j = 1 \qquad \forall j \geq 1$$

Using the eq.(6.3.5) we can rewrite the spectral equation in eq.(6.1.10) as a *purely* differential equation

$$L_j(1)\Psi_j = \lambda_1 \Psi_j \tag{6.3.6}$$

where

$$\begin{aligned} L_j(1) &= \partial + \sum_{l=1}^{j} a_l(j) \hat{B}_{j-l} \hat{B}_{j-l+1} \ldots \hat{B}_{j-1} \\ &= \partial + \sum_{l=1}^{j} a_l(j) \frac{1}{\partial - a_0(j-l)} \cdot \frac{1}{\partial - a_0(j-l+1)} \cdots \frac{1}{\partial - a_0(j-1)} \\ &\equiv \partial + \sum_{l=1}^{\infty} u_l(j) \partial^{-l} \end{aligned} \tag{6.3.7}$$

where all the functions $u_l(j)$ are only functions of the coordinates in $j-th$ sector.

$$u_1(j) = a_1(j),$$
$$u_2(j) = a_2(j) + a_1(j)\Big(a_0(j) - (\ln R_j)'\Big)$$
$$u_3(j) = a_3(j) + a_2(j)\Big[2a_0(j) - 2(\ln R_j)' - \Big(\ln\Big[R_j - \dot{a}_0(j) + \frac{\partial^2 \ln R_j}{\partial t_{1,1}\partial t_{q,1}}\Big]\Big)'\Big] \qquad (6.3.8)$$
$$+a_1(j)\Big([a_0(j) - (\ln R_j)']^2 - a_0'(j) + \frac{\partial^2 \ln R_j}{\partial t_{1,1}\partial t_{q,1}}\Big)$$

More generally, we could formulate the rules passing from lattice to the purely differential language, which is the same as the ones given in the previous chapter:

(i). The Jacobi matrices are mapped to KP operators, i.e.

$$\begin{cases} \Big(Q(\alpha)\Psi\Big)_j \Longrightarrow L_j(\alpha)\Psi_j, & \alpha = 1, 2, \ldots, q \\ \Big(P\Psi\Big)_j \Longrightarrow M_j\Psi_j. \end{cases} \qquad (6.3.9)$$

where

$$L(\alpha)_j = \sum_{l=1}^{\infty} T_{-l}^{(\alpha)}(j)\hat{A}_{j+l-1}\hat{A}_{j+l-2}\cdots\hat{A}_j + T_0^{(\alpha)}(j)$$
$$+\sum_{l=1}^{j} T_l^{(\alpha)}(j)\hat{B}_{j-l}\hat{B}_{j-l+1}\cdots\hat{B}_{j-1} \qquad (6.3.10a)$$
$$M_j = \sum_{l=1}^{\infty} P_{-l}(j)\hat{A}_{j+l-1}\hat{A}_{j+l-2}\cdots\hat{A}_j + P_0(j)$$
$$+\sum_{l=1}^{j} P_l(j)\hat{B}_{j-l}\hat{B}_{j-l+1}\cdots\hat{B}_{j-1} \qquad (6.3.10b)$$

(ii). The lower triangular part of Jacobi matrix (or its powers) maps to the pure integration part of the KP operator (or its powers);

(iii). The upper triangular part together with the main diagonal line of the Jacobi matrix (or its powers) correspond to the purely differential part of the KP operator (or its powers).

(iv). The residue of the KP operators have a particular simple form,

$$\text{res}_\partial(L_j(1)) = a_1(j) = Q_{j,j-1}(1). \qquad (6.3.11)$$

On the right hand side, $Q_{j,j-1}(1)$ means the element of the Jacobi matrix at n–th row and $(j-1)$–th column. More generally, we can obtain

$$\text{res}_\partial(L_j^r(1)) = Q_{j,j-1}^r(1). \qquad (6.3.12)$$

and

$$\mathrm{res}_{\partial}(L_j(\alpha)) = T_1^{(\alpha)}(j) = Q_{j,j-1}(\alpha).$$

as well as

$$\mathrm{res}_{\partial}(L_j^r(\alpha)) = \left(Q^r(\alpha)\right)^r_{j,j-1}(\alpha). \tag{6.3.13}$$

Expanding all the above operators in powers of ∂, the coefficients can be expressed as functions of the coordinates in $j-th$ sector. Collecting all the results together we get a linear system

$$\begin{cases} L_j(1)\Psi_j = \lambda_1\Psi_j \\ \frac{\partial}{\partial t_{1,r}}\Psi_j = \left(L_j^r(1)\right)_+\Psi_j \\ \frac{\partial}{\partial t_{\alpha,r}}\Psi_j = -\left(L_j^r(\alpha)\right)_-\Psi_j, \qquad \alpha = 2,3,\ldots,q \\ M_j\Psi_j = \frac{\partial}{\partial\lambda_1}\Psi_j \end{cases} \tag{6.3.14}$$

Since all the quatities are expressed in terms of the coordinates in $j-th$ sector, we can ommit the sub–index "j", we recoganize that this linear system is exactly the same as eqs.(4.4.3), whose consistency conditions give the generalized hierarchy (4.4.2a) and (4.4.2b).

$$\frac{\partial}{\partial t_{\beta r}}L(\alpha) = [L_+^r(\beta), L(\alpha)], \qquad 1 \le \beta \le \alpha \tag{6.3.15a}$$

$$\frac{\partial}{\partial t_{\beta r}}L(\alpha) = [L(\alpha), L_-^r(\beta)], \qquad \alpha \le \beta \le n \tag{6.3.15b}$$

We would like to remark here that if we impose the condition

$$a_l(j) = 0, \qquad \forall l \ge 2.$$

then the second expression of the eq.(6.3.7) gives the *so–called* two bosonic representation of the KP hierarchy . Therefore we may refer the full second expression in the eq.(6.3.7) to the multi bosonic field represention of KP hierarchy. We hope to explain this point in more detail somewhere else.

(II). The linear system related the polynomial Φ:

In the above analysis, we consider $t_{1,1}$ as the space coordinate, and ∂ as the basic derivative. In fact, if we replace them by $t_{q,1}$ and $\tilde{\partial}$ respectively, we get another linear system, in which Φ plays the role of Baker–Akhiezer function. Similar to the above discussion, we have

$$\Phi_{j+1} = \hat{\tilde{A}}_j\Phi_j, \qquad \Phi_{j-1} = \hat{\tilde{B}}_{j-1}\Phi_j \tag{6.3.16}$$

where

$$\hat{\tilde{A}}_j \equiv \tilde{\partial} - b_0(j) = -\partial^{-1} R_{j+1}$$
$$\hat{B}_j \equiv \tilde{\partial}^{-1} \sum_{l=0}^{\infty} (b_0(j)\tilde{\partial}^{-1})^l = -\frac{1}{R_{j+1}} \partial$$

the linear system is

$$\tilde{L}_j(q)\Phi_j = \lambda_q \Phi_j \tag{6.3.17a}$$
$$\frac{\partial}{\partial t_{q,r}} \Phi_j = \left(\tilde{L}_j^r(q)\right)_+ \Phi_j \tag{6.3.17b}$$
$$\frac{\partial}{\partial t_{\alpha,r}} \Phi_j = -\left(\tilde{L}_j^r(\alpha)\right)_- \Phi_j, \qquad \alpha = 1, 2, \ldots, q-1 \tag{6.3.17c}$$
$$\tilde{M}_j \Phi_j = \frac{\partial}{\partial \lambda_q} \Phi_j \tag{6.3.17d}$$

in which the KP operator takes the following form

$$\begin{aligned}
\tilde{L}_j(q) &= \tilde{\partial} + \sum_{l=1}^{j} b_l(j) \hat{\tilde{B}}_{j-l} \hat{\tilde{B}}_{j-l+1} \ldots \hat{\tilde{B}}_{j-1} \\
&= \tilde{\partial} + \sum_{l=1}^{j} b_l(j) \frac{1}{\tilde{\partial} - b_0(j-l)} \frac{1}{\tilde{\partial} - b_0(j-l+1)} \cdots \frac{1}{\tilde{\partial} - b_0(j-1)} \\
&\equiv \tilde{\partial} + \sum_{l=1}^{\infty} v_l(j) \tilde{\partial}^{-l}
\end{aligned} \tag{6.3.18}$$

all the functions $v_l(j)$ are only functions of the coordinates in $j-th$ sector.

$$v_1(j) = b_1(j),$$
$$v_2(j) = b_2(j) + b_1(j)\Big(b_0(j) - \tilde{\partial}(\ln R_j)\Big)$$
$$v_3(j) = b_3(j) + b_2(j)\Big[2b_0(j) - 2\tilde{\partial}(\ln R_j) - \tilde{\partial}\Big(\ln\Big[R_j - \partial b_0(j) + \frac{\partial^2 \ln R_j}{\partial t_{1,1}\partial t_{q,1}}\Big]\Big)\Big] \tag{6.3.19}$$
$$+ b_1(j)\Big([b_0(j) - \tilde{\partial}(\ln R_j)]^2 - \dot{b}_0(j) + \frac{\partial^2 \ln R_j}{\partial t_{1,1}\partial t_{q,1}}\Big)$$

etc. Using the rules going from lattice to differential laguange, we see

$$\left(\bar{Q}(\alpha)\Phi\right)_j \Longrightarrow \tilde{L}_j(\alpha)\Phi_j, \qquad \alpha = 1, 2, \ldots, q \tag{6.3.20a}$$
$$\left(P(q)\Phi\right)_j \Longrightarrow \tilde{M}_j \Phi_j. \tag{6.3.20b}$$

with the new operators like

$$\tilde{L}(\alpha)_j = \sum_{l=1}^{\infty} T_{-l}^{(\alpha)}(j) \hat{\tilde{A}}_{j+l-1} \hat{\tilde{A}}_{j+l-2} \cdots \hat{\tilde{A}}_j + T_0^{(\alpha)}(j)$$

$$+\sum_{l=1}^{j} T_l^{(\alpha)}(j)\hat{\bar{B}}_{j-l}\hat{\bar{B}}_{j-l+1}\cdots\hat{\bar{B}}_{j-1} \tag{6.3.21a}$$

$$\tilde{M}_j = \sum_{l=1}^{\infty} P_{-l}(q,j)\hat{\bar{A}}_{j+l-1}\hat{\bar{A}}_{j+l-2}\cdots\hat{\bar{A}}_j + P_0(q,j)$$

$$+\sum_{l=1}^{j} P_l(q,j)\hat{\bar{B}}_{j-l}\hat{\bar{B}}_{j-l+1}\cdots\hat{\bar{B}}_{j-1} \tag{6.3.21b}$$

Therefore we could write down another linear system

$$\begin{cases} \tilde{L}_j(q)\Phi_j = \lambda_q\Phi_j \\ \frac{\partial}{\partial t_{q,r}}\Phi_j = \left(\tilde{L}_j^r(q)\right)_+\Phi_j \\ \frac{\partial}{\partial t_{\alpha,r}}\Phi_j = -\left(\tilde{L}_j^r(\alpha)\right)_-\Phi_j, \qquad \alpha = 1,2,\ldots,q-1 \\ M_j\Phi_j = \frac{\partial}{\partial \lambda_1}\Phi_j \end{cases} \tag{6.3.22}$$

Whose consistency conditions also give the hierarchy (6.3.15a–6.3.15b).

(III). The coupling conditions:

Now our task is to reexpress the coupling conditions (6.1.7a–6.1.7b) in the differential laguange. This can be done very easily by replacing the matrices by their operatorial version, the only problem we should care about is the matrix $\bar{\mathcal{P}}(q)$, since its differential version is not $\tilde{M}$. We may simply denote its differential version by $M(q)$, then the coupling conditions are

$$M + c_{1,2}L(2) = 0, \tag{6.3.23a}$$
$$c_{\alpha-1,\alpha}L(\alpha-1) + V'(\alpha) + c_{\alpha,\alpha+1}L(\alpha+1) = 0, \qquad 2 \le \alpha \le q-1; \tag{6.3.23b}$$
$$c_{q-1,q}L(q-1) + M(q) = 0. \tag{6.3.23c}$$

where

$$V'(\alpha) = \sum_{r=1}^{\infty} r t_{\alpha,r}L^{r-1}(\alpha), \qquad \alpha = 1,2,\ldots,n.$$

We see that the eqs.(6.3.23b) are nothing but the eqs.(4.4.1a), which means that in the multi–matrix models, there indeed exist the generalized KP hierarchy. However the eq.(6.3.23a) and eq.(6.3.23c) impose some restrictions on the hierarchy, which, as we already knew before, are nothing but the $W_{1+\infty}$–constraints. Thus we may conclude that the multi–matrix models correspond to the generalized KP hierarchy subjected to certain constraints. On the other hand, we would like to point out that all the flows in (6.3.15a) and (6.3.15b) commute among themselves, since their lattice verions commute. This is very remarkable property. As we know that the usual additional KP flows are not compatible[67]. So we may say that the multi–matrix models give a natural realization of the commutative KP flows.

(IV). The other linear systems:

Quite similar to the analysis in the above, we could also start from the general form of the discrete linear systems (6.1.31) and (6.1.32), transfer them into the differential formalism by using the rules we listed before, the results are as follows

$$\begin{cases} L(\alpha)\xi(\lambda_\alpha) = \lambda_\alpha \xi(\lambda_\alpha), \\ \frac{\partial}{\partial t_{\beta,r}}\xi(\lambda_\alpha) = L_+^r(\beta)\xi(\lambda_\alpha), \quad 1 \leq \beta < \alpha, \\ \frac{\partial}{\partial t_{\beta,r}}\xi(\lambda_\alpha) = -L_-^r(\alpha)\xi(\lambda_\alpha), \quad \alpha \leq \beta \leq q. \end{cases} \tag{6.3.24}$$

and

$$\begin{cases} \bar{L}(\alpha)\eta(\lambda_\alpha) = \lambda_\alpha \eta(\lambda_\alpha), \\ \frac{\partial}{\partial t_{\beta,r}}\eta(\lambda_\alpha) = \bar{L}_+^r(\beta)\eta(\lambda_\alpha), \quad \alpha+1 \leq \beta \leq q, \\ \frac{\partial}{\partial t_{\beta,r}}\eta(\lambda_\alpha) = -\bar{L}_-^r(\alpha)\eta(\lambda_\alpha), \quad 1 \leq \beta \leq \alpha. \end{cases} \tag{6.3.25}$$

Once again they lead to the hierarchy (6.3.15a–6.3.15b). All the flows in (6.3.15a) and (6.3.15b) commute among themselves, since their lattice verions commute. This is very remarkable property. As we know that the usual additional KP flows do not compatible[67]. So we may say that the multi–matrix models give a natural realization of the commutative KP flows.

6.3.2 The partition function as τ–function

In order to justify the identification of the partition function and the τ–function of the generalized KP hierarchy, we only need prove that the eqs.(6.1.34) can be casted into the operatorial formalism, which has the same form as eqs.(4.4.12). This can be very easily to be done, if we note that the eq.(6.3.13) is valid for any $1 \leq \alpha \leq q$, and any positive integers r, j. So choosing $j = N$, we have

$$\left(Q^r(\alpha)\right)_{N,N-1} = res_\partial \left(L_N(\alpha)\right)^r \tag{6.3.26}$$

This equality together with eq.(6.1.34) give

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{\alpha,r}} \ln Z_N(t,c) = res_\partial \left(L_N(\alpha)\right)^r \tag{6.3.27}$$

This is nothing but the relation (4.4.12), which τ–function should satisfy. Therefore we may conclude that the partition function of multi–matrix models are exactly the τ–function of the generalized KP hierarchy. More precisely, the multi–matrix models are nothing but the generalized KP hierarchy subjected to $W_{1+\infty}$–constraints.

6.3.3 The dispersionless generalized KP hierarchy

Although we already discussed the dispersionless generalized KP hierarchy in chapter 4, but we prefer to say something more, since the differential hierarchy we derived before really gives a new representation of the KP hierarchy . This is quite transparant from the expression (6.3.7), in the dispersionless limit, which becomes

$$\bar{\mathcal{L}}(1) = p + \sum_{i=0}^{\infty} \frac{a_i}{p - a_0}, \tag{6.3.28}$$

In particular case

$$a_i = 0, \qquad \forall i \geq 2.$$

we recover the two bosonic representation of the dispersionless KP hierarchy .

For simplicity, we at first only consider the subsystem ($\alpha = \beta = 1$) case in the hierarchy (6.3.15a), which takes the following form

$$\frac{\partial}{\partial t_r} \bar{\mathcal{L}}(1) = \{\bar{\mathcal{L}}_+^r(1), \bar{\mathcal{L}}(1)\}, \tag{6.3.29}$$

where

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial p}\frac{\partial f}{\partial x}. \tag{6.3.30}$$

Now let us consider how we can go from this formalism to the one given in the second section. Starting with the dispersionless formalism. In the hierarchy (6.3.29), all the coordinate field u_l's are functions on the parameter space $(t_1, t_2, \ldots)$, among which t_1 can be interpreted as the space coodinate, and the others can be considered as time parameters. Now let us enlarge this parameter space by adding one more dimension, say, t_0 and require that the coordinate fields u_l's in this bigger space satisfy the constraints which are equivalent to

$$a_l' = \dot{a}_{l+1} - l\dot{a}_0 a_l \tag{6.3.31}$$

where $\dot{a} = \frac{\partial}{\partial t_0} a$. Substituting these formulas into the eqs.(6.3.29), we can express all the flows in terms the derivatives with respect to $t_0 = x$, fortunately enough, these new equations can also be written in the Lax formalism,

$$\frac{\partial}{\partial t_r} \mathcal{L} = z \frac{\partial \mathcal{L}_+^r}{\partial z}\frac{\partial \mathcal{L}}{\partial x} - z \frac{\partial \mathcal{L}}{\partial z}\frac{\partial \mathcal{L}_+^r}{\partial x} \tag{6.3.32}$$

where

$$\mathcal{L} = z + \sum_{i=0}^{\infty} a_i z^{-i} \tag{6.3.33}$$

which is the continuum limit of the first Jacobi matrix $Q(1)$. Comparing the two formulas (6.3.28) and (6.3.33), we find that if we set

$$z + a_0 = p, \tag{6.3.34}$$

then the two formalisms coincide with each other. This means that from the spheric limit of the lattice hierarchy to the dispersionless hierarchy is simply redefinition of the momentum. Similarly we can do the same thing with the other Jacobi matrices, finally we will recover the spheric limit discussed in section **2**.

6.4 Conclusion and Discussions

We have shown that any multi–matrix model can be reformulated as 2–dimensional Toda lattice hierarchy. Passing from lattice to purely differential hierarchy, it gives a natural realization of the generalized KP hierarchy. The coupling conditions together with the hierarchical structure lead to $W_{1+\infty}$–constraints. However, it is still far away from the complete understanding of the multi–matrix models.

1. At first, the topological phase of the general KP hierarchy is not clear. This is probably related to the reduction of the generalized KP hierarchy. For example, in 2–matrix model, the corresponding generlaized KP hierarchy is extremely large, such that it is hopefully to get any higher KdV hierarchies through some singular reductions. But this should be clarified further.

2. The second problem is to calculate the critical exponents baed on this scheme, once we do this, we could compare the results with ones obtained in the continuum approach, i.e. to see what is the matter content in multi–matrix models.

3. Finally, General speaking, we can do the same thing for infinite matrix chain, however, in that case, it is not easy to consider the coupling conditions.

   Probably we could conclude that multi–matrix models are nothing but KP hierarchies projected by $W_{1+\infty}$–constraints, here we should keep in mind that the different matrix models correspond to the different additional symmetries, and the infinite matrix chain corresponds to the full KP hierarchy with the full additional symmetries.

Chapter 7

The Introduction to Topological Field Theory

The topological field theories were originally studied by A.Schwarz[68] and E.Witten [17], and now they are called *Schwarz type* or *Witten type* topological field theory respectively. Their basic difference is due to their semi–classical limits. In such a limit, the Witten type topological field theory is still a topological theory, but the Schwarz type is not. In this chapter we will focus our attention on the Witten type of topological field theory . We will briefly discuss the topological matter theory, and its coupling to the topological gravity. The main references are[17, 18][53] [69]–[72].

7.1 The Definitions

Let us start with a quantum field theory defined on a compact manifold M of dimension D with some set of fundamental fields $\phi(x)$, its vacuum amplitude or partition function is given by the path-integral over all field configurations on M weighted with some action S

$$Z(M) = \int [d\phi]\, e^{-S[\phi]} \tag{7.1.1}$$

In general the partition function $Z(M)$ will depend on many geometrical data. For instance, in almost all quantum field theories we need a Riemannian structure on M, *i.e.*, a metric $g_{\mu\nu}(x)$. This metric enters both in the definition of the action S and in the definition of the measure $[d\phi]$. Other possible ingredients might be an orientation and, if fermions are involved, a choice of spin structure, or in the case of gauge fields a choice of fiber bundle. However the symmetries of the theory will dramatically simplify the analysis. The symmetry of quantum field theory simply indicates the independence of the partition function on some of goemetrical data. For example, string theory has world sheet reparametrization invariance which leads to the conservation law of the energy momentum tensor, a gauge theory possesses gauge invariance which shows the physical states should be BRST–invariant, and the conformal invariance of the theory garantees the traceless of the stress–energy tensor. Along this line, if we require the theory to be invariant under arbitrary smooth deformation of the metric:

$$\delta g_{\mu\nu} = \epsilon_{\mu\nu}.$$

then it is called *topological field theory* . In this case, the partition function $Z(M)$ and any physical correlation functions of local operators will be topological invariants(i.e. they will depend only on the topology of the manifold M). If we further consider the metric $g_{\mu\nu}$ as a dynamic variable, we get *topological gravity.*

The above definition of *topological field theory* through general covariance is very intuitive, but it is not tractable in practice. In fact, there are many other definitions, each of them has its own advantages. We now list a few below.

Definition 1: If the partition function and all the correlation functions of a quantum field theory are all metric independent, then the theory is topological.

Definition 2: If a quantum field theory satisfies the following two conditions

1. There exists a nipotent Q operator, the physical states $|\phi_i>$ are represented by Q–cohomology classes,

$$Q|\phi_i>= 0, \qquad |\phi_i>\sim |\phi_i> +Q|\lambda>$$

2. The energy–momentum tensor is Q–exact, i.e.

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\} \tag{7.1.2}$$

   for some functional $G_{\mu\nu}$ of the fields and the metric, then the theory is topological.

Definition 3: If the stress–energy tensor satisfies the Virasoro algebra with the central charge $c = 0$, it is also topological.

Definition 4: If the partition function and all correlation functions of a quantum field theory can be completely determined from a finite set of correlation functions , theh the theory is topological.

All of these definitions indeed are equivalent(although the regorous proof is unknown yet), they in fact describe a topological theory from different point of view. In the next section we will see what kind of general properties we can obtain only from these definitions.

7.2 The General Properties

7.2.1 Factorization

The most important property of the topological matter, is *factorization.* Roughly speaking, factorization means that we can reduce the multi–point correlation functions to a finite number of basic correlation functions . In order to understand it in detail, we need one more concept. Consider a quantum field theory on a general space–time M with a connected boundary B, its partition function depends on the configuration on the boundary,

$$Z(M,B) = \int_{\phi|_B=\phi'} [d\phi] e^{-S[\phi]} \tag{7.2.3}$$

if we regard B as a base manifold, and $\phi(x)$ as the fundamental field variables, then quantum mechanically the above partition function can be considered as the wave function. So we could associate a *Hilbert space* $\mathcal{H}_B$ to it *.

The definition of $\mathcal{H}_B$ in general will depend on the orientation of B. If $-B$ denotes the manifold with reversed orientation, we have

$$\mathcal{H}_{-B} = \mathcal{H}_B^*$$

If the boundaries of M have two connected pieces B_1 with the same orientation as M and B_2 with the reverse orientation, then

$$Z(M; B_1, B_2) = \int_{\phi|_{B_1}=\phi';\phi|_{B_2}=\phi''} e^{-S[\phi]} \tag{7.2.4}$$

defines an element in $\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}^*$, or equivalently a transition amplitude

$$Z(M; B_1, B_2) : \mathcal{H}_{B_2} \longrightarrow \mathcal{H}_{B_1} \tag{7.2.5}$$

Now suppose we cut the base manifold M into two parts M_1 and M_2 along a codimension one subspace B, then we can perform the path integral in two steps. At first we fix the configuration of the fields $\phi(x)$ on B and do the seperate integrals on M_1 and M_2 with these boundary conditions,

$$\Psi_{M_i}(\phi') = \int_{\phi|_B=\phi'} [d\phi] e^{-S[\phi]} \tag{7.2.6}$$

then the total path–integral over M is clearly given by

$$Z(M) = \int [d\phi'] \Psi_{M_1}(\phi') \Psi_{M_2}(\phi') \tag{7.2.7}$$

From the Hilbert space point of view, this is equivalent to say that we could insert $\mathcal{H}_B^* \otimes \mathcal{H}_B$ into the path–integral such that the original one reduces to two smaller ones. Repeating the procedure we are finally led to some basic path–integrals.

*At first glance it seems surpring, the Hilbert space is only related to the boundary not to the total manifold, however we should keep in mind that the wave function is dependent on the total manifold which means the dynamics is determined by total manifold.

7.2.2 Fröbenius algebra

Let us take 2–dimensioanl topological field theory as a example. In this case, our manifold is a surface of genus g. The only connected compact submanifold is the circle S^1, so we only have one Hilbert space

$$\mathcal{H} \equiv \mathcal{H}_{S^1}$$

By factorization we can cut M as a collection of spheres with one, two, and three holes. Each of them represents a particular element of $\mathcal{H}$.

(i) The disk gives rise to a particular state

$$\mathbf{1} \in \mathcal{H},$$

that we will denote as the identity.

(ii) The cylinder gives a bilinear map

$$\eta : \mathcal{H} \otimes \mathcal{H} \to \mathbf{C},$$

which we notate as

$$\eta(a, b) =< a, b > .$$

(iii) Finally the pair of pants,—or the sphere with three holes—corresponds, again with the appropriate choice of orientations, to a bilinear map

$$c : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, \tag{7.2.8}$$

If we introduce the notation

$$c(a, b) = a \times b, \tag{7.2.9}$$

this makes $\mathcal{H}$ into an algebra, the operator product algebra of the topological field theory. The other properties of this operator algebra follow from *duality*, which simply states that the inequivalent factorizations of a surface should give the same result. These additional structures are

$(*)$*normalization*

$$\phi_0 \phi_i = \phi_i, \qquad \forall i$$

which leads to

$$c_{ij0} = \eta_{ij} \tag{7.2.10}$$

$(**)$ *the compatibility of the metric η with the algebra $\mathcal{H}$*

$$< a \times b, c >=< a, b \times c > .$$

This is a simple consequence of the symmetry of the 3-punctured sphere.

$(***)$*the associativity*

$$(a \times b) \times c = a \times (b \times c). \tag{7.2.11}$$

LHS and RHS represent the two inequivalent ways of factorization of the sphere with four holes.

Therefore we finally get a commutative associative algebra of finite dimension N, which is called *Fröbenius algebra*[73]. Choosing its basis as $\{\phi_0, \ldots, \phi_{N-1}\}$, with $\phi_0 = 1$, the component notation is

$$\eta_{ij} =< \phi_i, \phi_j > \tag{7.2.12}$$

$$c_{ijk} =< \phi_i \phi_j \phi_k > \tag{7.2.13}$$

In this languange, it is easy to describe the factorization and duality.

$$\text{factorization :} \qquad < \phi_i \phi_j \phi_k \phi_l >= \sum_p c_{ij}{}^p c_{pkl} \tag{7.2.14}$$

$$\text{Duality or associativity :} \quad c_i{}^p{}_j c_p{}^k{}_l = c_i{}^k{}_p c_j{}^p{}_l \tag{7.2.15}$$

$$\text{commutativity :} \qquad c_{ijk} = c_{ikj} = \ldots \tag{7.2.16}$$

For higher genus partition and correlation functions, we can introduce the *handle operator H* in the following way[18]

$$< \phi_{i_1} \ldots \phi_{i_n} H^g >_0 =< \phi_{i_1} \ldots \phi_{i_n} >_g \tag{7.2.17}$$

In order to get the explicit form of H, we only need consider one point function on the torus case ($g = 1$). By factorization, we can cut this torus such that we obtain a cylinder. This is equivalent to inserting

$$1 = \sum_{ij} |\phi_i > \eta^{ij} < \phi_j| \tag{7.2.18}$$

so we have

$$< \phi_k >_1 = \sum_{ij} \eta^{ij} < \phi_i \phi_j \phi_k >_0 = \sum_{ij} \eta^{ij} c_{ijk} = \sum_i c_{ki}{}^i \tag{7.2.19}$$

From the definition we immediately get

$$H = \sum_{ij} c_i^{ij} \phi_j \tag{7.2.20}$$

So by factorization we can calculate partition function and correlation functions in any genus only from the genus zero two– and three–point functions.

7.2.3 Q–cohomolgy

Suppose the full Hilbert space of topological field theory is $\mathcal{H}'$('big' space), the physical sector (or 'small' space), from the second definition of topological field theory , is the cohomology ring of Q. Since Q is nipotent, it can be regarded as BRST charge. On the other hand, due to the fact that $T_{\mu\nu}$ is Q–exact, it can also be treated as supersymmetry charge. Let us see this in detail.

(i) *The supersymmetry*:

The conserved charges related to $T_{\mu\nu}$ and $G_{\mu\nu}$ are

$$P_\mu = \int T_{\mu 0}, \qquad G_\mu = \int G_{\mu 0},$$

where the integrals are taken over a $D-1$ dimensional spacelike surface. These charges form an anti-commuting extension of the translation group

$$P_\mu = \{Q, G_\mu\}, \qquad \{G_\mu, G_\nu\} = 0, \tag{7.2.21}$$

that can be viewed as a supersymmetry algebra with charges of spin zero and one, instead of the usual spin one-half.

(ii) *The descent equation*:

Since P_μ and G_μ form a supersymmetry algebra, we can go to a 'superspace' description by introducing D Grassmannian coordinates θ^μ. For any local operator $\phi^{(0)}(x)$, the superfield $\Phi(x,\theta)$ is given by

$$\begin{aligned}\Phi(x,\theta) &= \exp \theta^\mu G_\mu \cdot \phi(x) \\ &= \phi^{(0)}(x) + \phi^{(1)}_\mu(x)\theta^\mu + \ldots + \phi^{(D)}_{\mu_1\ldots\mu_D}(x)\theta^{\mu_1}\cdots\theta^{\mu_D}.\end{aligned}$$

Here the fields $\phi^{(k)}$ are generated from $\phi^{(0)}$ by repeated application of G_μ

$$\phi^{(k)}_{\mu_1\ldots\mu_k}(x) = \{G_{\mu_1}, \{G_{\mu_2}, \ldots, \{G_{\mu_k}, \phi^{(0)}(x)\}\ldots\}\}. \tag{7.2.22}$$

Since $\phi^{(k)}_{\mu_1\ldots\mu_k}$ is antisymmetric in all its indices, it represents a k-form, and we can write (7.2.22) symbolically as

$$\{G, \phi^{(k)}\} = \phi^{(k+1)}. \tag{7.2.23}$$

Since by (7.2.21) we have $\{Q, G\} = d$, these differential forms satisfy the important *descent equation*

$$d\phi^{(k)} = \{Q, \phi^{(k+1)}\}. \tag{7.2.24}$$

We can draw two conclusions from this equation. First, it suggests a new class of *non-local* physical observables. If C is a k-dimensional closed submanifold, the descent equation shows that

$$\phi(C) \equiv \int_C \phi^{(k)}(x) \tag{7.2.25}$$

is a physical observable, since

$$\{Q, \phi(C)\} = \int_C d\phi^{(k-1)} = \int_{\partial C} \phi^{(k-1)} = 0. \tag{7.2.26}$$

Secondly, the physical observable depends only on the homology class of C. That is, for each class in $H_k(M)$ and each element in $\mathcal{H}$ we can construct a non-local operator and the correlators of the local physical operators do not depend on the positions.

(*iii*) *The perturbations*

One very important non-local operator is the top-form

$$\phi(M) = \int_M \phi^{(D)}(x). \tag{7.2.27}$$

Since $\phi^{(D)}$ is a volume form, which is integrated over the space-time, it possesses the same symmetry as the action. So we can modify the action of the topological quantum field theory by adding (7.2.27) with some coupling coefficient t

$$S \to S - t \cdot \int_M \phi^{(D)}, \tag{7.2.28}$$

Therefore, the top-dimensional partner of any local observable defines a perturbation of the topological field theory. We can in this way define a multi-parameter family of topological field theories whose partition functions are given by

$$Z(M,t) = < \exp \sum_k t_k \cdot \int_M \phi_k^{(D)} > . \tag{7.2.29}$$

7.3 Topological Conformal Field Theory

A particular class of topological field theories is *topological conformal field theory*(TCFT), which means that we impose the conformal invariance on the general topological field theory . It is wellknown that any TCFT can be obtained by twisting suitable $N = 2$ superconformal field theory. So let us begin with the later object.

7.3.1 $N = 2$ Superconformal Field Theory

The $N = 2$ superconformal algebra contains the following four generators: a stress–energy tensor $T(z)$, two spin 3/2 supercurrents $Q^+(z), Q^-(z)$ and a $U(1)$ current $J(z)$. Their mode expansions satisfy the algebra[74]

$$\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\
&[L_n, J_m] = -mJ_{n+m}, \qquad [L_n, Q_r^{\pm}] = (\frac{n}{2} - r)Q^{\pm}_{n+r} \\
&[J_n, J_m] = \frac{c}{3}m\delta_{n+m,0}, \qquad [J_n, Q^{\pm}_{m\pm a}] = \pm Q^{\pm}_{n+m\pm a} \\
&\{Q_r^-, Q_s^+\} = 2L_{r+s} + (s-r)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{n+m,0}
\end{aligned}$$

where in NS–sector, $r, s \in \mathbf{Z} + \frac{1}{2}$; in R–sector, $r, s \in \mathbf{Z}$.

Unitarity gives the following restrictions on the conformal weight and $U(1)$ charge of the *chiral or anti–chiral* states

$$h \geq |q|/2, \qquad\qquad h \leq c/6 \tag{7.3.30}$$

the first equality only holds for *primary* states. This result immediately leads to a remarkable effect, that is, all the OPE of the primary fields are *non–singular* and still primary†. The second inequality shows the conformal weight is bounded above, so a non–degenerated theory only has *a finite number of the primary fields*, and they possess the *Ring* structure which is called *chiral primary ring.*

7.3.2 Twisting Procedure

The twisting procedure is first suggested by Witten[18], then proved by Eguchi and Yang[75]. Its basic idea is to improve the stress tensor such that the Virasoro algebra is centerless. Concretely

$$T(z) \longrightarrow \tilde{T}(z) = T(z) + \frac{1}{2}\partial J(z) \tag{7.3.31}$$

which gives in particular $L_0 \to L_0 + \frac{1}{2}J_0$ and accordingly adds the charge to the conformal dimensions

$$h \to \tilde{h} = h - \tfrac{1}{2}q. \tag{7.3.32}$$

So Q^+ becomes spin one current $Q(z)$, Q^- now is a spin two fermionic tensor $G(z)$. The algebra generated by the modes L_n, G_n, Q_n, J_n of the four currents is

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n}, & [J_m, J_n] &= d \cdot m\, \delta_{n+m,0}, \\
[L_m, G_n] &= (m-n)G_{m+n}, & [J_m, G_n] &= -G_{m+n}, \\
[L_m, Q_n] &= -n\, Q_{m+n}, & [J_m, Q_n] &= Q_{m+n},
\end{aligned}$$

$$\begin{aligned}
\{G_m, Q_n\} &= L_{m+n} + nJ_{m+n} + \tfrac{1}{2}d \cdot m(m+1)\delta_{n+m,0}, \\
[L_m, J_n] &= -nJ_{m+n} - \tfrac{1}{2}d \cdot m(m+1)\delta_{m+n,0}.
\end{aligned} \tag{7.3.33}$$

where

$$d = \frac{c}{3}$$

the above algebra explicitly shows that the Virasoro subalgebra has vanishing central charge. So we arrive at a topological field theory , in which Q_0 is BRST charge.

Since the physical states should be Q–closed, so they are ground states or *chiral primary fields.*

†This can be checked very easily, once we noting the fact that $U(1)$ charge is additive.

7.3.3 The Selection Rules

From the algebra (7.3.33), we see that the $U(1)$ current has a central charge d. This indicates that $J(z)$ does not transform as a proper current under coordinate transformations, and results in the background charge $d \cdot (g-1)$ on Reimann surface of genus g. Since the theory is general covariant, so this background charge has to be compensated. This requirement leads to the selection rule for the correlation functions .

$$< \phi_{i_1} \dots \phi_{i_s} >_g \tag{7.3.34}$$

is non–vanishing, only if

$$\sum_{i=1}^{s} q_i = d(1-g). \tag{7.3.35}$$

7.3.4 Generalized $SL(2,C)$ invariance and integrability

In the superspace, any local operator is associated with a superfield

$$\Phi(z,\bar{z};\theta,\bar{\theta}) = \phi^{(0)} + \phi^{(1,0)}\theta + \phi^{(0,1)}\bar{\theta} + \phi^{(2)}\theta\,\bar{\theta}. \tag{7.3.36}$$

Consider an arbitary correlation function on the sphere

$$\Big\langle \prod_{i=1}^{s} \int d^2z\, d^2\theta\ \Phi_i(z,\bar{z},\theta,\bar{\theta})\Big\rangle_0. \tag{7.3.37}$$

due to the generalized $SL(2,C)$ symmetry, generated by operators L_0, L_1, L_{-1} and G_0, G_1, G_{-1}, this correlator is ill–defined. In order to obtain a finite answer, we have to factorize out the infinite volume of this group, in other words, we should fix three of the z-coordinates, say z_1, z_2, z_3, at 0, 1, and ∞, and put three of the θ-coordinates to zero. Then we are left with a well–defined correlation function

$$\Big\langle \phi^{(0)}_{i_1}\phi^{(0)}_{i_2}\phi^{(0)}_{i_3} \int \phi^{(2)}_{i_4} \dots \int \phi^{(2)}_{i_s}\Big\rangle_0. \tag{7.3.38}$$

Since we started from an expression that was explicitly symmetric in *all* indices $i_1, \dots, i_s$, this correlator also has this permutational symmetry. That is, it does not matter which three operators we represent as zero-forms—the generalized $SL(2,C)$ invariance tells us that*we can interchange a zero and a two-form* [76, 77].

In terms of the coefficients $c_{ijk}(t)$ the permutation symmetry of (7.3.38) gives the important integrability condition

$$\frac{\partial c_{ijk}}{\partial t_l} = \frac{\partial c_{ijl}}{\partial t_k}. \tag{7.3.39}$$

which is followed from the conformal invariance, and immediately implies

1. All the structure constants can be derived from the τ-function of the hidden integrable hierarchy[‡]

$$c_{ijk}(t) = \frac{\partial^3 \log \tau}{\partial t_i \partial t_j \partial t_k}. \tag{7.3.40}$$

the free-energy $F = \log \tau$ can be symbolically defined as

$$F(t) = \left\langle \exp \sum_i t_i \int \Phi_i \right\rangle, \tag{7.3.41}$$

2. The metric $\eta_{ij}(t)$ on the Hilbert space of states is independent on the couplings t_k. Since $\phi_0^{(2)} = 0$[§].

3. Finally, the anomalous $U(1)$ symmetry will imply the following scaling relation for the free energy of the form(see, for example[76])

$$\sum_j (q_j - 1) t_j \frac{\partial}{\partial t_j} F(t) = (d-3) F(t). \tag{7.3.42}$$

Remarkably enough, these three ingredients will be sufficient to solve the theory in many special cases, in particular for all models with $d < 1$ [77].

7.4 Topological Landau-Ginzburg models

In the previous section we only gave a formal description of the TCFT. Now let us consider some of the examples, among them the Topological Landau–Ginzberg Model is of particular importance, since it maps the *operator ring* to a *polynomial ring*, so establishes the connection between the topological conformal matters and the Krichever's *dispersionless* KdV type of integrable hierarchies.

A general LG–theory is described by the following action

$$S = \int d^2z\, d^4\theta \cdot K(X^i, X^{\bar{i}}) + \int d^2z\, d^2\theta \cdot W(X^i) + c.c. \tag{7.4.43}$$

where X_i and $X_{\bar{i}}$ are a set of chiral $N=2$ superfields. $K(X_i, X_{\bar{i}})$ is the kinetic term,

$$K(X_i, X_{\bar{i}}) = X_i X_{\bar{i}}$$

for the free field realization of $N=2$ superconformal algebra. $W(X_i)$ is the superpotential, which completely determines the dynamics of theory. After the integration over $(\theta, \bar{\theta})$ and

[‡]In fact we can linearize eqs.(7.3.39) to get a linear integrable sysyem, see [73].

[§]This is typic property of the topological conformal matter, if it coupled to gravity, the gravitational screening allows non–vanishing 2–form of the puncture operator, which is area.

eliminating the auxiliary fields[78],

$$S = \int d^2z\, \Big(\partial_z x^{\bar{i}} \partial_{\bar{z}} x^i + \chi^{\bar{i}}_z \partial_{\bar{z}} \psi^i_L + \chi^{\bar{i}}_{\bar{z}} \partial_z \psi^i_R + \frac{\partial W}{\partial x^{\bar{i}}} \frac{\partial \overline{W}}{\partial x^i} + \psi^i_L \psi^j_R \frac{\partial^2 W}{\partial x^i \partial x^j} + \chi^{\bar{i}}_z \chi^{\bar{j}}_{\bar{z}} \frac{\partial^2 \overline{W}}{\partial x^{\bar{i}} \partial x^{\bar{j}}}\Big). \tag{7.4.44}$$

right now the superpotential $W(x_i)$ is only function of the bosonic fields, which should be *quasi-homogeneous*[79, 80]

$$W(\lambda^{q_i} x_i) = \lambda W(x_i). \tag{7.4.45}$$

such that Landau-Ginzburg models become conformal invariant at the renormalization group fixed point. The most important properties of Landau-Ginzburg models are the following

(i) Each chiral primary field ϕ corresponds to a polynomial $\phi(x_i)$ with weight q_i equal to its $U(1)$ charge[53]. So the chiral primary operator ring is isomorphic to a *polynomial ring* ¶.

$$\mathcal{H} = \frac{\mathbf{C}[x]}{\partial W(x)}. \tag{7.4.46}$$

which is equipped with the *ordinary algebraic multiplication*

$$\phi_i(x)\,\phi_j(x) = \sum_k c_{ij}{}^k \phi_k(x) \qquad (\text{mod } W'(x)). \tag{7.4.47}$$

(ii) The *inner product* should be compatible with this multiplication, and is thereby uniquely determined to be

$$< \phi_i \phi_j >= res_x\Big(\frac{\phi_i(x)\phi_j(x)}{W'(x)} dx\Big). \tag{7.4.48}$$

This enables us to calculate all the perturbed correlation functions .

(iii) The perturbed algebra is still of the form (7.4.46), but with a more general superpotential, and the dynamics is given by an *integrable hierarchy.*

Now let us take the A_{n-1} ring as an example to show what kind of integrable hierarchy we can get, and how the superpotential W completely determines all the correlation functions (before or after twisting).

¶Once we write down explicitly the BRST transformation, we find that in particular

$$\{Q, \Psi^i_L\} = \delta \Psi^i_L = \frac{\partial W}{\partial x_i}, \qquad \{Q, \Psi^i_R\} = \delta \Psi^i_R = -\frac{\partial W}{\partial x_{\bar{i}}}$$

so any function of the form $f = g^i \partial^i W$ is Q-exact. Therefore we should mode out the ideal generated by $\frac{\partial W}{\partial x_i} = 0$, in order to get the physical Hilbert space.

In this case, the non–perturbed superpotential is $W(x) = x^n$, and the physical states corresponds to $\phi_i = x^i$ with the charge $q_i = \frac{i}{n}$. The metric is

$$\eta_{ij} = \delta_{i+j,n-2} \tag{7.4.49}$$

After perturbation, it is expected to become

$$W = x^n + \sum_{i=0}^{n-2} u_i(t)x^i. \tag{7.4.50}$$

where $u_i(t)$'s are to be determined functions of $(n-1)$ coupling parameters $t_0, t_1, \cdots, t_{n-2}$.

Learning from the operatorial formulation, t_i is the coupling constant corresponding to the perturbation generated by the 2–form of ϕ_i, so ϕ_i can be considered as the response of W with respect to the change of t_i, i.e.

$$\phi_i(x,t) = \frac{\partial W}{\partial t_i} = \sum_{j=0}^{n-2} \frac{\partial g_j}{\partial t_{i+1}} x^j. \tag{7.4.51}$$

On the other hand, the constant metric indicates that all ϕ_i's should be *'orthogonal'*, therefore their explicit form is

$$\phi_i(x) = \frac{1}{i+1} dW_+^{(i+1)/n}. \tag{7.4.52}$$

Here $d = \partial/\partial x$, and the $+$ indicates a truncation to positive powers of x. Combining the equations (7.4.51) and (7.4.52), we obtain the important result

$$\frac{\partial W}{\partial t_i} = \frac{1}{i+1} \frac{\partial}{\partial x} W_+^{\frac{i+1}{n}}. \tag{7.4.53}$$

Krichever has proved that this is a solution of trucated KdV hierarchy subjected to Virasoro constraints[81].

Now let us consider the free energy of the model. Using the inner product (7.4.48), we get

$$c_{ijk}(t) = res_x\Big(\frac{\phi_i \phi_j \phi_k}{W'} dx\Big) \tag{7.4.54}$$

After integrating twice we have

$$\frac{\partial}{\partial t_i} F = \frac{res_x(W^{\frac{n+i+1}{n}})}{(i+1)(n+i+1)} \tag{7.4.55}$$

substituting it into eq.(7.3.42), we obtain the explicit expression of the free energy in genus zero

$$F = \sum_{i=0}^{n-2} \frac{(i-n)t_i res_x(W^{\frac{n+i+1}{n}})}{(i+1)n(n+i+1)(d-3)} \tag{7.4.56}$$

7.5 The coupling of topological matter to gravity

Let us consider the coupling of topological matter system to gravity. In the presence of gravity, the metric of the base manifold becomes a dynamical variable, its effect is that all the primary fields will be *gravitational dressed* and take the form

$$\mathcal{O}_{\alpha,i} = \mathcal{O}_\alpha \cdot \sigma_i$$

where $\mathcal{O}_\alpha$'s are primary fields of topological matter, and σ_i's are gravitational dressing operators with non–negative integer i. In particular

$$\mathcal{O}_{0,0} = \mathcal{O}_0 \cdot \sigma_0 = \sigma_0 = \mathcal{P}$$

is so–called *puncture* operater. For each $\mathcal{O}_{\alpha,i}$, we have one corresponding 2–form $\mathcal{O}^{(2)}_{\alpha,i}$, which can be added to the original action, therefore the general perturbed action is

$$S = S_0 + \sum_{\alpha,i} t_{\alpha,i} \int \mathcal{O}^{(2)}_{\alpha,i}$$

For each primary field we have one infinite series of perturbation parameters. The parameter space of all $t_{\alpha,i}$ is called *phase space*, while $t_{\alpha,0}$'s form the *small space*. Our task now is to calculate the correlation functions

$$< \mathcal{O}_{\alpha_1,i_1} \mathcal{O}_{\alpha_2,i_2} \cdots \mathcal{O}_{\alpha_n,i_n} >_g$$

As we already konw that in the small space TCFT is characterized by the first series of flows of the truncated integrable hierarchy together with the scaling laws. When we couple TCFT to gravity, due to the gravitational dressing procedure, we should replace the truncated hierarchy by the full integrable hierarchy, and the scaling law is replaced by the *puncture equation*. So the theory in the full phase space is described by the complete integrable hierarchy, and *puncture equation*, which reads

$$\frac{\partial}{\partial t_{0,0}} F = \frac{1}{2} \sum_{\alpha,\beta} t_{0,\alpha} t_{0,\beta} \eta^{\alpha,\beta} + \sum_{i,\alpha} (i+1) t_{i+1,\alpha} \frac{\partial}{\partial t_{i,\alpha}} F. \tag{7.5.57}$$

From this equation and the integrable hierarchy, we can easily derive that the partition function satisfies the W_N–constraints.

7.6 Summary

In the previous sections we have discussed topological field theories, and their coupling to gravity as well as their relations to integrable hierarchies. Now let us simply list the most important properties of the various theories.

<u>Topological Field Theory :</u>

1. The general covariance(or metric independence), which can be considered as both a BRST–symmetry and a supersymmetry, while the stress tensor is BRST–exact.

2. Factorization: all the partition function and correlation functions of a topological field theory are completely determined by a small (possibly finite) set of geometrical data.

3. Perturbation: the top–form operators preserve the general covariance, so as to generate the perturbed theories, whose algebraic structures are isomorphic to the original one.

<u>2–dimensional Topological Field Theory</u> :

Besides the above ones, they still have

4. Any 2–dimensional topological field theory are completely determined by two– and three–point correlation functions , so as to be characterized by a Fröbenius algebra.

<u>Topological Conformal Field Theory</u> :

They are 2–dimensional topological field theory with the following additional properties

1. The metric η_{ij} is independent on the perturbation parameters.

2. The structure constants are three times derivatives of the logarithm of the τ–function of the hidden integrable hierarchy.

3. There exists an anomous $U(1)$ symmetry, which leads to the selection rule at the conformal point and the scaling law for perturbed theory.

<u>Topological Landau–Ginzberg Model</u>:

This is a special class of TCFT, whose hidden integrable hierarchy is the truncated generalized dispersionless KdV type. This hierarchy together with the scaling law completely solve the model.

<u>Coupling to Gravity</u>:

Two ingradients are

(*) The full integrable hierarchies.

(**) The puncture equation.

Chapter 8

The Hidden Topological Models in Matrix Models

In chapter 3, we have shown that one–matrix models give rise to a lattice hierarchy of (differential–difference) equations, as well as to a discrete string equation. After taking the continuum limit, we obtain (in the even potential case) the KdV hierarchy and the continuum string equation. In chapter 5, we pointed out that both the KdV hierarchy and the corresponding string equation are intrinsic to the matrix model lattice – they are no mere result of the continuum limit, as oral tradition seems sometimes to imply. We do it by extracting the KdV hierarchy and string equations directly from the lattice counterparts without passing through a limiting procedure. In the last chapter we saw that both the KdV hierarchy and the string equation can be interpreted in terms of a topological field theory [17][18][53][69]. This may suggest to us that all the lattice quantities could have topological meaning. This at first seems to be surprising, however we may get self–confidence from the following wellknown fact: the Euler number of a Riemann surface Σ of genus h can be calculated with a continuum procedure by means of

$$2-2h=\frac{1}{4\pi}\int_{\Sigma}\sqrt{g}R \tag{8.0.1}$$

where R is the scalar curvature w.r.t. the metric g; but it can also be calculated by means of

$$2-2h=V-L+F \tag{8.0.2}$$

with reference to a simplicial approximation of Σ, in which V, L, F are the number of vertices, links and faces, respectively.

The KdV hierarchy and string equation obtained via a continuum limit are akin to the RHS of eq.(8.0.1). The KdV hierarchy and string equation we discuss here are parallel to the RHS of eq.(8.0.2). Since KdV hierarchy and corresponding string equation have a topological significance, the analogy just presented is more than a simple suggestion.

Actually in this chapter we will at first discuss the topological field theory coupled to gravity associated to the NLS hierarchy together with its perturbations. Our strategy is as follows: using the non–linear Schrödinger hierarchy and the string equation, we will try to show that the partition function and all the correlation functions can be derived from *only two* correlation functions . One a is *one-point function*, the other is a proper combination of the multi–point correlation functions . They may interpreted as the fundamental topological data of the model. This topological field theory looks different from the ones considered so far in the literature[20].

8.1 The KdV hierarchy and the corresponding TFT.

The KdV hierarchy in the context of matrix models has been repeatedly analyzed, and our results agree with the literature. However it is interesting to reconsider the problem from our point of view, both in itself and as a preparation for the next section.

The KdV hierarchy and the corresponding string equation are obtained by reducing non–linear Schrödinger hierarchy

$$\frac{\partial R}{\partial t_{2r+1}} = F'_{2r+2} = (\partial^3 + 4R\partial + 2R')F_{2r} \tag{8.1.3}$$

where the initial condition $F_2 = R$ has been used. Eq.(8.1.3) is the recursion relation for the KdV hierarchy. In particular we have

$$\frac{\partial}{\partial t_3}R = R''' + 6RR' \tag{8.1.4}$$

The connection with the partition function is the same as in §5.5, eq.(5.5.43), i.e.

$$\frac{\partial^2}{\partial t_1^2}\ln Z_N(t) = R_N. \tag{8.1.5}$$

This means in particular that the reduced partition function, Z, is a τ–function of the KdV hierarchy. As we already knew in §5.5, the τ–function of KdV hierarchy should satisfy Virasoro constraints

$$\mathcal{L}_n Z = 0, \qquad n \geq -1 \tag{8.1.6}$$

where

$$\begin{aligned}
\mathcal{L}_{-1} &= \sum_{k=1}^{\infty}(k+\frac{1}{2})t_{2k+1}\frac{\partial}{\partial t_{2k-1}} + \frac{t_1^2}{8} \\
\mathcal{L}_0 &= \sum_{k=0}^{\infty}(k+\frac{1}{2})t_{2k+1}\frac{\partial}{\partial t_{2k+1}} + \frac{1}{16} \\
\mathcal{L}_n &= \sum_{k=0}^{\infty}(k+\frac{1}{2})t_{2k+1}\frac{\partial}{\partial t_{2k+2n+1}} + \frac{1}{2}\sum_{k=0}^{n-1}\frac{\partial^2}{\partial t_{2k+1}\partial t_{2n-2k-1}}, \qquad n>0
\end{aligned} \tag{8.1.7}$$

Now all the tools we need are at hand.

It has been shown by Witten [17] and Kontsevitch [19] that the emergence of the KdV hierarchy and the string equation have a definite topological meaning. In particular, considered in the framework of a topological field theory coupled to 2D gravity they constitute a very efficient method to calculate correlation functions. Following [17] we define the action corresponding to the KdV system as

$$S = S_0 - \sum_{k=0}^{\infty} t_{2k+1} \int \sigma_{2k+1}^{(2)} \tag{8.1.8}$$

The corresponding free energy is

$$\mathcal{F} = \sum_{\{n_i\}} \frac{t_1^{n_1}!}{n_1!} \frac{t_2^{n_2}!}{n_2!} \cdots \frac{t_k^{n_k}!}{n_k!} < \sigma_1^{n_0} \sigma_3^{n_1} \ldots \sigma_{2k+1}^{n_k} >_\circ \tag{8.1.9}$$

where the sum extends over all the n-tuples of integers $\{n_i\} = \{n_1, n_2, ..., n_k\}$, and $< \cdot >_\circ$ stands for a correlation function calculated when all the couplings vanish. Let us also introduce the notation

$$\frac{\partial^{n_1}}{\partial t_1^{n_1}} \frac{\partial^{n_3}}{\partial t_3^{n_2}} \cdots \frac{\partial^{n_k}}{\partial t_{2k+1}^{n_k}} \mathcal{F} \equiv \ll \sigma_1^{n_1} \sigma_2^{n_2} \ldots \sigma_{2k+1}^{n_k} \gg$$

If we identify this free energy with the one introduced above, $\mathcal{F} = \ln Z$, then we have $R = \mathcal{F}'' = \ll PP \gg$, where $P \equiv \sigma_1$, we can use the KdV hierarchy and string equation to calculate the correlation functions for any genus. The calculation is particularly easy in genus 0 and we will limit ourselves to this case. Then the full KdV hierarchy can be replaced by the dispersionless KdV hierarchy

$$\frac{\partial}{\partial t_{2k+1}} R = 2^k \frac{(2k+1)!!}{(k+1)!} R^k R' \tag{8.1.10}$$

This can be easily obtained by rescaling t_k and R by suitable powers of N and keeping the dominant contribution in $\frac{1}{N}$, which simply amounts to discarding all the higher derivatives in t_1 in the flow equations. A straightforward result is then the following

$$\ll \sigma_{2k_1+1} \sigma_{2k_2+1} \ldots \sigma_{2k_n+1} \gg = \partial^{-2} \frac{\partial^n}{\partial t_1^n} \prod_{i=1}^{n} \frac{2^{k_i}(2k_i+1)!!}{k_i!} \frac{R^{k_1+k_2+\ldots+k_n+1}}{k_1+k_2+\ldots+k_n+1} \tag{8.1.11}$$

where, in the RHS, ∂^{-1} denotes here true integration with respect to t_1 (true integration as opposed to formal integration, since here we cannot exclude a priori non trivial integration constants). The expressions of the above correlation functions in terms of the coupling constants can be found by solving the Landau–Ginzburg type equation which is obtained by differentiating the $\mathcal{L}_{-1}$ Virasoro condition

$$\frac{\partial \mathcal{F}}{\partial t_1} = 2 \sum_{k=1}^{\infty} (2k+1) t_{2k+1} \frac{\partial \mathcal{F}}{\partial t_{2k-1}} + \frac{t_1^2}{2} \tag{8.1.12}$$

where we have shifted $t_3 \to t_3 - \frac{1}{6}$. For example the first critical point is met at $t_{2k+1} = 0$ for $k > 0$ (small phase space). In this case we have

$$< P >= \frac{t_1^2}{2}, \qquad < PP >= R = t_1, \qquad < PPP >= 1$$

$< \cdot >$ denotes a correlation function in the small phase space. Therefore if we replace R with t_1 in (8.1.11) we get immediately the correlation functions $< \sigma_{2k_1+1}\sigma_{2k_2+1}\dots\sigma_{2k_n+1} >$ in the small phase space, up to arbitrary constants coming from double integration with respect to t_1. These integration constants are determined by the string equation. For example, from the $\mathcal{L}_n$ Virasoro condition (8.1.7), after extracting the part relevant to genus 0 and shifting t_3 as above, we get the recursive relation

$$< \sigma_{n+3} >= 2t_1 < \sigma_{2n+1} > + \sum_{k=0}^{n-1} < \sigma_{2k+1} >< \sigma_{2n-2k-1} >$$

Similar recursive relations for multiple correlation functions can be obtained by first differentiating the $\mathcal{L}_n$ constraints with respect to the couplings and then repeating the above derivation. One can verify that all these recursive relations are satisfied by

$$< \sigma_{2k_1+1}\sigma_{2k_2+1}\dots\sigma_{2k_n+1} >= \frac{\partial^{n-2}}{\partial t_1^{n-2}} \prod_{i=1}^{n} \frac{2^{k_i}(2k_i+1)!!}{k_i!} \frac{t_1^{k_1+k_2+\dots+k_n+1}}{k_1+k_2+\dots+k_n+1} \quad (8.1.13)$$

In the case $n = 1$ the first symbol in the RHS has to be understood as formal integration w.r.t. t_1 (i.e. integration with vanishing integration constant – we will see in the next section that this is not always the case). The normalization we used for the σ_{2k+1} is completely natural in the context we presented, but leads to results which differ in normalization from the ones in the literature. However it is enough to define new couplings $\tilde{t}_r$

$$\tilde{t}_r = 2^r \frac{(2r+1)!!}{r!} t_{2r+1}, \quad r = 0, 1, 2, \dots \quad (8.1.14)$$

to recover the results of Gross and Migdal [5],.

Two comments are in order to end this section. What we have done so far tells us that the correlation functions of the fields σ_k can be calculated by using the path integral of one–matrix models as correlation functions of $Tr(M^{2k+1})$. In other words we have the correspondence

$$Tr(M^{2k+1}) \leftrightarrow \sigma_{2k+1}$$

The second remark is connected with the first. One should not confuse the fact that in one–matrix models the conditions $S_n = 0$ and $t_{2k+1} = 0$ are related with the fact that we put $S = 0$ at the beginning of this section. What we did is the following: the most general one–matrix model allowed us to write down an integrable system (with $S \neq 0$); next we defined a consistent reduction of the system specified by the condition $S = 0$; the reduced theory depends on the odd coupling constants. One should bear in mind that the constraint $S = 0$ is meant to be applied to the integrable system, not to the original one–matrix model path integral. Finally, we would like to point out that our analysis here is essentially based on the matrix model formalism, by the reduction procedure we obtained the same result as we derived from topological approach in the previous chapter. This confirms the eqivalence of the topological gravity and the 1–matrix model with even potential.

8.2 The NLS hierarchy and the corresponding TFT.

In this section we want to do for the larger NLS hierarchy what we have done in the previous section for the KdV hierarchy; i.e. we try to define a topological field theory coupled to gravity that corresponds to it.

One quickly realizes that it is not possible to represent this theory simply by means of the puncture operator P and its descendants as in the KdV case. There must be another field, which we call Q, coupled through some parameter in the theory. Since the only parameter in the theory not already associated to some field is the size N of the lattice, we assume that Q is coupled to the theory with coupling N. Of course we have to define the differentiation with respect to N. We do it as follows: for any function f_N on the lattice we define the derivative ∂_0 by means of

$$\left(e^{\partial_0} - 1\right) f_N \equiv f_{N+1} - f_N \tag{8.2.15}$$

Setting $D_0 = e^{\partial_0} - 1$ we have, conversely,

$$\partial_0 = \ln(D_0 + 1) \tag{8.2.16}$$

One can easily verify that ∂_0 can be identified with the derivative with respect to N *.

In particular, using the first flow equations, (5.4.32, 5.4.33), we obtain

$$\left(e^{\partial_0} - 1\right) R_N = R_{N+1} - R_N = S'_N \tag{8.2.17}$$

$$\left(e^{\partial_0} - 1\right) S_N = S_{N+1} - S_N = (\ln(R_N + S'_N))' \tag{8.2.18}$$

Next, R_N is connected to the partition function through the folowing equation (see (5.5.43))

$$\frac{\partial^2}{\partial t_1^2} \ln Z_N(t) = R_N. \tag{8.2.19}$$

The relation of S_N to the partition function is found by applying D_0 to both sides of the equation (see eq.(3.3.10b))

$$\frac{\partial}{\partial t_1} \ln Z_N(t) = \sum_{i=0}^{N-1} S_i. \tag{8.2.20}$$

and we obtain

$$\left(e^{\partial_0} - 1\right) \frac{\partial}{\partial t_1} \ln Z_N = S_N \tag{8.2.21}$$

Applying D_0 and $\frac{\partial}{\partial t_1}$ to eqs.(8.2.19, 8.2.21), we obtain identities, except when we apply D_0 to (8.2.21). In that case we obtain the following compatibility condition

$$\left(e^{\partial_0} - 1\right)^2 \ln Z_N = \ln(R_N + S'_N) + f(N) \tag{8.2.22}$$

*It is obvious that when differentiating with respect to N we understand a continous extension of the integer parameter N.

where we have integrated once with respect to t_1 and $f(N)$ is the corresponding arbitrary (model dependent) integration constant.

Eqs.(8.2.17–8.2.21) and the compatibility condition (8.2.22), together with the NLS flows (5.4.37) and the Virasoro constraints (5.6.56), are the basis of our subsequent discussion. Henceforth, for the sake of uniformity, we relabel the new coupling N as t_0. Therefore

$$t_0 = N, \qquad \partial_0 = \frac{\partial}{\partial t_0}, \qquad \text{etc.}$$

Moreover we drop the label N from R, S and $\ln Z$ and understand that it is included as t_0 in the collective label t which represent the coupling constants. In particular the first Virasoro constraint can be rewritten

$$\sum_{r=2}^{\infty} r t_r \frac{\partial \mathcal{F}}{\partial t_{r-1}} - \frac{\partial \mathcal{F}}{\partial t_1} + t_0 t_1 = 0 \tag{8.2.23}$$

where $\mathcal{F}(t) = \ln Z(t)$ and, for later purposes, we have shifted t_2 by $-\frac{1}{2}$ (which sets the first critical point at $t_2 = 0$).

The result of adding the new coupling t_0 can be regarded as an enlargement of the hierarchy (5.4.37). Beside the t_k flows with $k \geq 2$ we have now the t_0 flow as well. This is obtained by inverting eqs.(8.2.17) and (8.2.18)

$$\partial_0 R = S' - \frac{1}{2}\left(\ln(R + S')\right)'' + \dots \tag{8.2.24}$$

$$\partial_0 S = (\ln(R + S'))' - \frac{1}{2}\left\{\ln\left(1 + \frac{S'(\ln(R+S'))'}{R+S'}\right)\right\}' + \dots \tag{8.2.25}$$

where dots denote higher order derivatives in t_1.

8.3 The hidden TFT in 1–matrix model

Let us now pass to the field theory language. We want the free energy $\mathcal{F}$ to be generated by an action

$$S = S_0 - \sum_{k=0}^{\infty} t_k \int \sigma_k^{(2)} \tag{8.3.26}$$

The problem is now to express the (perturbed) correlation functions

$$\frac{\partial^{n_0}}{\partial t_0^{n_0}} \frac{\partial^{n_1}}{\partial t_1^{n_1}} \cdots \frac{\partial^{n_k}}{\partial t_k^{n_k}} \mathcal{F} \equiv \ll \sigma_0^{n_0} \sigma_1^{n_1} \dots \sigma_k^{n_k} \gg$$

in terms of R and S. Here

$$\sigma_0 = Q, \qquad \sigma_1 = P$$

So, in particular,

$$R = \ll PP \gg, \qquad S = \ll (e^Q - 1)P \gg = \sum_{n=1}^{\infty} \ll PQ^n \gg \tag{8.3.27}$$

and the compatibility condition (8.2.22) means

$$\ll \left(e^Q - 1\right)^2 \gg = \ln\left(\ll PPe^Q \gg\right) + f(t_0) \tag{8.3.28}$$

The tools to calculate the correlation functions are the NLS flows and the string equation, or, equivalently, the Virasoro constraints (5.6.56) which contain both. Let us draw first some conclusions concerning the small phase space, i.e. the space of couplings when all the t_k are set equal to zero except t_0 and t_1. The first Virasoro constraint (8.2.23) becomes

$$< P > = t_0 t_1$$

Therefore

$$< PQ > = t_1, \qquad < PP > = t_0 \tag{8.3.29}$$

and, as a consequence,

$$< PPQ > = 1$$

while all the other correlation functions containing at least one P insertion vanish. The correlation functions with only Q insertions depend on the arbitrary function $f(t_0)$ through (8.3.28).

So far the correlation functions look similar to those of the $\mathbf{CP}^1$ model studied by Witten, [17],[69]. But the hierarchy is different. Using the flows (5.4.37) we can write down equations for

$$\ll \sigma_{n_1} \dots \sigma_{n_k} Q \gg \qquad \text{or} \qquad \ll \sigma_{n_1} \dots \sigma_{n_k} P \gg$$

in terms of R, S and their derivatives. In order to obtain

$$\ll \sigma_{n_1} \dots \sigma_{n_k} \gg$$

we have simply to integrate either the first expression above with respect to t_0 or the second with respect to t_1. The integration constants have to be determined in such a way as to satisfy the string equation.

Let us consider, as an example, the one point correlation functions. The n–th Virasoro constraint can be rewritten, in the small phase space, as

$$< \sigma_{n+2} > = t_1 < \sigma_{n+1} > + 2t_0 < \sigma_n > + \sum_{k=1}^{n-1} \left(< \sigma_k \sigma_{n-k} > + < \sigma_k >< \sigma_{n-k} >\right) \tag{8.3.30}$$

The two–point functions can be obtained by differentiating the Virasoro constraints with respect to t_k, and so on. In this way we obtain a full set of recursive relations that allow us to calculate all the correlation functions.

8.4 The zero genus correlation functions

The calculation is particularly easy in genus 0 and from now on we limit ourselves to this case. In order to obtain the equations relevant to genus 0 we rescale all the quantities by suitable powers of N.

$$t_k \to N^{\frac{2-k}{2}} t_k, \qquad R \to NR, \qquad S \to N^{\frac{1}{2}} S$$

and keep the leading terms in $\frac{1}{N}$. For example

$$S = \ll PQ \gg + \sum_{l=1}^{\infty} \frac{1}{l! N^l} \ll PQ^l \gg$$

In conclusion in genus 0 we have

$$R = \ll PP \gg, \qquad S = \ll PQ \gg \tag{8.4.31}$$

while the compatibility condition (8.3.28) becomes

$$\ll QQ \gg = \ln \ll PP \gg + f_0(t_0) \tag{8.4.32}$$

where $f_0(t_0)$ is the appropriate genus zero term, derived from $f(t_0)$ (it is model dependent). The hierarchy (5.4.37) becomes the dispersionless hierarchy, i.e.

$$\frac{\partial S}{\partial t_r} = G'_{r+1} = \sum_{\substack{k \\ 0 \le 2k \le r}} \binom{r}{2k} \binom{2k}{k} \left(R^k S^{r-2k}\right)' \tag{8.4.33}$$

$$\frac{\partial R}{\partial t_r} = F'_{r+1} = \sum_{\substack{k \\ 2 \le 2k \le r+1}} \binom{r}{2k-1} \binom{2k-1}{k} \left(R^k S^{r-2k+1}\right)' \tag{8.4.34}$$

Notice that setting $S = 0$ and keeping only the odd flows and using the redefinitions of the previous section, we obtain (8.1.3).

From (8.4.33) we obtain

$$\ll \sigma_r Q \gg = \sum_{\substack{k \\ 0 \le 2k \le r}} \frac{r!}{(r-2k)!(k!)^2} \left(R^k S^{r-2k}\right) \tag{8.4.35}$$

while from (8.4.34) we get

$$\ll \sigma_r P \gg = \sum_{\substack{k \\ 2 \le 2k \le r+1}} \frac{r!}{(r-2k+2)!(k-1)!k!} R^k S^{r-2k+2} \tag{8.4.36}$$

What we said so far for genus 0 is valid in the large phase space. Now let us come to the the small phase space. To begin with we have

$$< QQ > = \ln t_0 + f_0(t_0)$$

and

$$R = t_0, \qquad S = t_1$$

If we insert this into (8.4.35) and integrate over t_0 (with a vanishing integration constant), we obtain, in the small phase space

$$< \sigma_r >= \sum_{\substack{k \\ 2\leq 2k \leq r+2}} \frac{r!}{(r-2k+2)!(k-1)!k!} t_0^k t_1^{r-2k+2} \tag{8.4.37}$$

We can obtain the same result by integrating (8.4.36) with respect to t_1, but in this case we have to add a suitable t_0–dependent integration constant.

Notice that, both here and in (8.4.35,8.4.36), we have made a choice for the integration constants. This choice can be justified on the basis of the string equation. For genus 0 eq.(8.3.30) takes the form

$$< \sigma_{n+2} >= t_1 < \sigma_{n+1} > +2t_0 < \sigma_n > + \sum_{k=1}^{n-1} < \sigma_k >< \sigma_{n-k} > \tag{8.4.38}$$

One can verify that eq.(8.4.37) does satisfy (8.4.38). In a similar way we can derive multi–point correlation functions.

A few final comments. First, it is not possible to carry out the reduction from the NLS TFT to the KdV TFT by simply killing particular degrees of freedom. For example, in the first case we have $< PPP >= 0$, in the second $< PPP >= 1$. Secondly, we do not address here the problem of interpreting the NLS TFT presented above in terms of algebraic geometry. The NLS TFT is rather different from the ones found in the literature. We intend to tackle this problem elsewhere. Thirdly, recently E. Brézin and J. Zinn-Justin also suggested to treat the size N of the matrix as a flow parameter. Their method seems to be similar to ours[82]. Finally we would like to add that a few subjects treated in this chapter may bear some relation with ref.[50].

Appendix A

The Integrability of the Generalized Lattice Systems

We generalize the Toda chain lattice system to the non–symmetric cases (see below eq.(A.1.2)), and show that these systems possess a bi–Hamiltonian structure[22].

At first we introduce some more notations.

$$M_\theta \equiv \frac{1}{2}M_{(0)} + M_-, \qquad M_\alpha \equiv M_+ - \frac{1}{2}M_{(0)}.$$

where as we used before, for any matrix M,

$$\begin{cases} M_- : & \text{the pure lower triangular part} \\ M_{(0)} : & \text{the diagonal line,} \\ M_+ : & \text{the diagonal line and pure upper triangular part} \end{cases}$$

We denote by $gl_0(\infty)$ the infinite dimensional algebra formed by the semi–infinite matrices under matrix commutation, which has the following decomposition

$$gl_0(\infty) = \mathcal{N}_+ \oplus \mathcal{H} \oplus \mathcal{N}_-,$$

the subalgebra $\mathcal{N}_+(\mathcal{N}_-)$ contains all the upper(lower) triangular matrices, while $\mathcal{H}$ represents the Abelian subalgebra formed by diagonal matrices. Furthermore, we introduce two subalgebras of $gl_0(\infty)$

$$\mathcal{P}_- \equiv \mathcal{H} \oplus \mathcal{N}_-, \qquad \mathcal{P}_+ \equiv \mathcal{H} \oplus \mathcal{N}_+.$$

A.1 The symmetries of the discrete linear systems

Let us begin with the following discrete linear system

$$\begin{cases} Q\xi = \lambda\xi, \\ \frac{\partial}{\partial t_r}\xi = (Q^r)_\theta\xi. \end{cases} \tag{A.1.1}$$

where the Jacobi matrix takes the following explicit form

$$Q_{ij} = \sqrt{R_j} E_{i,i+1}\delta_{j,i+1} + \sum_{l\geq 0}^{n} L_i^{(l)} E_{i,i-l}\delta_{j,i-l}, \qquad i,j \geq 0. \tag{A.1.2}$$

Evidently $Q \in [-n, 1]$ with arbitrary positive integer n, while R_i's and $L_i^{(l)}$'s are coordinate variables of the system, whose equations of motion are

$$\frac{\partial}{\partial t_r} Q = [(Q^r)_\theta, \;\; Q] \qquad \forall r \geq 1. \tag{A.1.3}$$

They are just the consistency conditions of the system (A.1.1), and we will call them discrete KP equations.

The discrete linear system (A.1.1) possesses two different kinds of the symmetries: a gauge symmetry and a Weyl symmetry. The gauge symmetry preserves completely the dynamics of the system, while the Weyl scaling can be used to reduce the system (A.1.1) to the standard form(see below (A.1.8)).

A.1.1 The gauge symmetry

By gauge symmetry we mean the invariance of the system under the following transformations

$$\begin{cases} Q \longrightarrow \bar{Q} = G^{-1} Q G, \\ \xi \longrightarrow \bar{\xi} = G^{-1}\xi. \end{cases} \tag{A.1.4}$$

where G is some invertible matrix such that the form of the system (A.1.1) remains unchanged, i.e.

$$\begin{cases} \bar{Q}\bar{\xi} = \lambda\bar{\xi}, \\ \frac{\partial}{\partial t_r}\bar{\xi} = (\bar{Q}^r)_\theta\bar{\xi}. \end{cases}$$

That is to say, the dynamics of the system does not change. In order to see what this symmetry is, we consider its infinitesimal form $G = 1 + \epsilon g$. The invariance requires that the matrix g should satisfy the equations

$$\begin{cases} \bar{Q} = Q + \epsilon[Q, g], \\ \frac{\partial}{\partial t_r} g = [Q_\theta^r, g] - [Q^r, g]_\theta. \end{cases} \tag{A.1.5}$$

Two solutions are

$$g_1 = \sum_k b_k Q^k, \qquad g_2 = \sum_k c_k Q_\alpha^k, \qquad \forall k \geq 1. \tag{A.1.6}$$

where b_k's and c_k's are time–independent constants. The effect of the transformations generated by g_1's is only to rescale the basis by a λ–dependent factor, while the transformations generated by the g_2's is in fact equivalent to tune the time parameters t_r's. They form an infinite dimensional gauge group. We will see that, it is just this symmetry that leads to the integrability of the discrete linear system.

A.1.2 Weyl Scaling Symmetry

Another kind of symmetry of the system (A.1.1) is Weyl scaling invariance. Suppose that we choose a diagonal matrix G. After the transformation (A.1.4), the time evolution equations of ξ will transform to another form, but the equations of motion of the coordinates R_i's and $L_i^{(l)}$'s keep unchanged. The only effect of this transformation is to rescale the basis of the vector ξ by a λ–independent factor. This symmetry has only one degree of freedom. Now let us choose the following special gauge $G_{ij} = \frac{1}{\sqrt{h_j}}\delta_{ij}$(where we have introduced the quantities $h_j/h_{j-1} = R_j$, h_0 is an undetermined constant which is not important in our analysis), then the Jacobi matrix becomes

$$Q = I_+ + \sum_{l=0}^{n} A^{(l)}, \qquad A^{(l)} \equiv \sum_{i=l}^{\infty} A_i^{(l)} E_{i,i-l}, \tag{A.1.7a}$$

$$A_j^{(0)} = L_j^{(0)}, \qquad A_j^{(l)} = L_j^{(l)} \sqrt{R_j R_{j-1} \ldots R_{j-l+1}}, \qquad \forall l \geq 1. \tag{A.1.7b}$$

In this gauge, it can be easily checked that the linear system (A.1.1) takes the following form

$$\begin{cases} Q\xi = \lambda\xi, \\ \frac{\partial}{\partial t_r}\xi = (Q^r)_-\xi. \end{cases} \tag{A.1.8}$$

The equations of motion (or consistency conditions of the system (A.1.8)) read

$$\frac{\partial}{\partial t_r} Q = [(Q^r)_-, \ Q] \qquad \forall r \geq 1. \tag{A.1.9}$$

We see that the equations of motion of the coordinates R_j's disappear, in fact it is implicitly involved in the equations of motion of the coordinates A_j's. Both system (A.1.1) and system (A.1.8) appear in multi–matrix models (see chapter **6**).

A.2 The integrability of the discrete linear system (A.1.8)

As we know a dynamical system with n degrees of freedom is integrable if and only if there are n–independent conserved quantities in involution[51][83]. For a system with infinite many degrees of freedom, there should be infinite many independent conserved quantities in involution*. One of the main approaches to show the integrability is the so-called bi–Hamitonian method, i.e. one should prove that there exist at least two compatible Poisson brackets. Compatibility of two Poisson brackets means

$$\{H_{k+2}, f_Y\}_1(Q) = \{H_{k+1}, f_Y\}_2(Q) \tag{A.2.10}$$

for any Hamiltonian H_k. Hereafter we will follow this line to construct two compatible Poisson brackets, which gives the discrete KP equations (A.1.9).

*Here we only consider a system with one Jacobi matrix.

A.2.1 The First Poisson Bracket

Our strategy is as follows. We try to define such a Poisson structure that gives the desired equations of motion, i.e. the discrete KP–equations (A.1.9). In order to do this, let us define the trace operation on the matrix space†

$$\tilde{\mathrm{Tr}}(M) \equiv \sum_{i=0}^{\infty} M_{ii}.$$

which gives a natural inner scalar product on $gl_0(\infty)$. It is easy to see that the product is symmetric and invariant under the action of $gl_0(\infty)$

$$\begin{cases} \tilde{\mathrm{Tr}}(AB) = \tilde{\mathrm{Tr}}(BA) \equiv A(B) = B(A), \\ \tilde{\mathrm{Tr}}(A[B,C]) = \tilde{\mathrm{Tr}}([A,B]C). \end{cases}$$

with respect to this product, $\mathcal{P}_+$ and $\mathcal{P}_-$ are dual algebras, their elements being in one–one correspondence. On the other hand, the functions of $A_j^{(l)}$'s, which span a functional space $\mathcal{F}$, can be defined as

$$f_X(Q) \equiv \tilde{\mathrm{Tr}}(QX), \qquad df_X(Q) = X \in \mathcal{P}_+.$$

This is a function on the algebra $\mathcal{P}_-$. Consequently the algebraic structure of $\mathcal{P}_-$ determines the Poisson structure on $\mathcal{F}$ through the Konstant–Kirillov bracket

$$\{f_X, f_Y\}_1(Q) = Q([X,Y]), \qquad X, Y \in \mathcal{P}_+ \tag{A.2.11}$$

Of course it is anti–symmetric and satisfies the Jacobi identity. Furthermore, one can show that with respect to this Poisson bracket, the conserved functions are

$$H_k = \frac{1}{k} Tr(Q^k) \qquad dH_k = Q_{[0,n]}^{(k-1)} \qquad \forall k \geq 1. \tag{A.2.12}$$

They are in involution, i.e.

$$\{H_{k+1}, H_{l+1}\}_1(Q) = 0.$$

So they can be chosen as Hamiltonians, and give the desired equations of motion, i.e. the discrete KP–Hierarchies. On the other hand, these Hamiltonians generate the symmetry transformations on the functional space $\mathcal{F}$. In order to see its relation to the gauge transformations (A.1.6), we recall that the co–adjoint action of $\mathcal{P}_+$ on the functional space $\mathcal{F}$ is defined through its adjoint action on the coordinates space $\mathcal{P}_-$, i.e.

$$ad_Y^* f_X(Q) \equiv f_X(ad_Y(Q)) = X([Y,Q]) \qquad X, Y \in \mathcal{P}_+.$$

Obviously the symmetry on $\mathcal{F}$ generated by the Hamiltonians is nothing but the gauge symmetry on the coordinate space. Therefore, for a system with infinite degrees of freedom, there must exist an infinite dimensional gauge symmetry, such that we could find infinite many independent involutive conserved quantities.

†We suppose that this summation is well–defined.

A.2.2 The Second Poisson Bracket

In order to get the second Poisson bracket, our starting point is compatibility of the two Poisson brackets. If we can represent the right hand side of eq.(A.2.10) in terms of dH_{k+1}, then we can extract the second Poisson bracket immediately. Consider the following matrix

$$\begin{aligned} F &= [Q, Q_+^{k+1}] \\ &= (Q(Q_+^k Q)_+ - (QQ_+^k)_+ Q) + (QQ_{[-1]}^k I_+ - I_+ Q_{[-1]}^k Q) \end{aligned}$$

From the equality

$$[Q, Q_+^k] = [Q_-^k, Q],$$

we obtain

$$Q_{[-1]}^k = \sum_{j=0}^{\infty} Q_{j+1,j}^k E_{j+1,j}, \quad Q_{j+1,j}^k = \sum_{l=0}^{j} [dH_{k+1}, Q]_{ll}.$$

Therefore we get

$$\begin{aligned} \{H_{k+2}, f_Y\}_1(Q) &= Q([dH_{k+2}, Y]) = Y(F) \\ &= <(dH_{k+1}Q)_+ YQ> - <(QdH_{k+1})_+ QY> \\ &+ \sum_{i,j} Q_{ij}(Q_{j,j-1}^k - Q_{i+1,i}^k) Y_{ji} \end{aligned}$$

substituting dH_{k+1} by X, and introduce a X–dependent matrix

$$\mathcal{D} \equiv \sum_{j=0}^{\infty} \mathcal{D}_j E_{j+1,j} \qquad \mathcal{D}_j \equiv \sum_{l=0}^{j} [X, Q]_{ll} \tag{A.2.13}$$

we finally obtain the second Poisson bracket

$$\begin{aligned} \{f_X, f_Y\}_2(Q) &= <(XQ)_+ YQ> - <(QX)_+ QY> \\ &+ <Q\mathcal{D}I_+ Y> - <QYI_+\mathcal{D}>. \end{aligned} \tag{A.2.14}$$

Writing explicitly in terms of the coordinates $A_j^{(l)}$'s, we have

$$\begin{aligned} \{A_i^{(a)}, A_j^{(b)}\}_2 &= A_j^{(a+b+1)} \delta_{i,j-b-1} - A_i^{(a+b+1)} \delta_{i,j+a+1} \\ &+ \sum_{l=0}^{a} (A_i^{(l)} A_j^{(a+b-l)} \delta_{i,j-b+l} - A_i^{(a+b-l)} A_j^{(l)} \delta_{i,j+a-l}) \\ &+ A_i^{(a)} A_j^{(b)} \sum_{l=j-b}^{j} (\delta_{il} - \delta_{i-a,l}). \end{aligned} \tag{A.2.15}$$

here we understand $A_i^{(r)} = 0$, if $r > n$. It is tiresome but straightforward to prove that the above Poisson bracket is antisymmetric and satisfies the Jacobi identity. The following is a simple example for $n = 2$ and

$$Q_{ij} = \delta_{j,i+1} + S_j \delta_{ij} + L_j \delta_{j,i-1} + A_j \delta_{j,i-2}.$$

Eq.(A.2.15) gives us the following Poisson brackets

$$\{S_i, S_j\}_2 = L_j\delta_{i,j-1} - L_i\delta_{i,j+1}, \tag{A.2.16a}$$
$$\{S_i, L_j\}_2 = S_iL_j(\delta_{i,j-1} - \delta_{ij}) + A_j\delta_{i,j-2} - A_i\delta_{i,j+1}, \tag{A.2.16b}$$
$$\{S_i, A_j\}_2 = S_iA_j(\delta_{i,j-2} - \delta_{ij}), \tag{A.2.16c}$$
$$\{L_i, S_j\}_2 = L_iS_j(\delta_{ij} - \delta_{i,j+1}) + A_j\delta_{i,j-1} - A_i\delta_{i,j+2}, \tag{A.2.16d}$$
$$\{L_i, L_j\}_2 = L_iL_j(\delta_{i,j-1} - \delta_{i,j+1}) + S_iA_j\delta_{i,j-1} - A_iS_j\delta_{i,j+1}, \tag{A.2.16e}$$
$$\{L_i, A_j\}_2 = L_iA_j(\delta_{i,j-2} + \delta_{i,j-1} - \delta_{ij} - \delta_{i,j+1}), \tag{A.2.16f}$$
$$\{A_i, S_j\}_2 = A_iS_j(\delta_{ij} - \delta_{i,j+2}), \tag{A.2.16g}$$
$$\{A_i, L_j\}_2 = A_iL_j(\delta_{i,j-1} + \delta_{ij} - \delta_{i,j+1} - \delta_{i,j+2}), \tag{A.2.16h}$$
$$\{A_i, A_j\}_2 = A_iA_j(\delta_{i,j-2} + \delta_{i,j-1} - \delta_{i,j+1} - \delta_{i,j-2}). \tag{A.2.16i}$$

From our construction of the Poisson brackets, at the first glance it seems to be possible to deduce higher order Poisson brackets. However it is very difficult to work it out, and, furthermore, the deduced higher order Poisson brackets probably do not satisfy Jacobi identity. The only exception is the ansatz $n = 1$, $r = 2$, in this case, we denote

$$Q = I_+ + S + R.$$

Then playing the same game as before, we can get the third Poisson bracket of the Toda-like lattice

$$\begin{aligned}\{f_X, f_Y\}_3(Q) &= \frac{1}{2}(< (XQ^2)_+YQ > - < (Q^2X)_+QY > \\ &+ < (QXQ)_+YQ > - < (QXQ)_+QY > \\ &+ < YQC_1 > - < QYC_2 >)\end{aligned} \tag{A.2.17}$$

where

$$\mathcal{B} \equiv \sum_{j=0}^{\infty} \mathcal{B}_jE_{j+2,j}, \qquad \mathcal{B}_j \equiv \sum_{l=0}^{j}[X, Q]_{l+1,l},$$
$$\mathcal{D} \equiv \sum_{j=0}^{\infty} \mathcal{D}_jE_{j+1,j}, \qquad \mathcal{D}_j \equiv \sum_{l=0}^{j}[X, Q]_{ll}.$$

and

$$\begin{aligned}C_1 &\equiv \mathcal{B}I_+^2 + I_+\mathcal{B}I_+ + \mathcal{D}Q_+^2 + 2I_+\mathcal{D}I_+ + I_+\mathcal{D}S + S\mathcal{D}I_+ \\ C_2 &\equiv I_+^2\mathcal{B} + I_+\mathcal{B}I_+ + Q_+^2\mathcal{D} + 2I_+\mathcal{D}I_+ + I_+\mathcal{D}S + S\mathcal{D}I_+.\end{aligned}$$

The Poisson brackets of the coordinates are as follows

$$\{R_i, R_j\}_1 = 0, \qquad \{S_i, S_j\}_1 = 0, \tag{A.2.18a}$$
$$\{R_i, S_j\}_1 = R_i(\delta_{i,j} - \delta_{i,j+1}). \tag{A.2.18b}$$

and

$$\{R_i, R_j\}_2 = R_i R_j(\delta_{j,i+1} - \delta_{i,j+1}), \qquad \text{(A.2.19a)}$$
$$\{R_i, S_j\}_2 = R_i S_j(\delta_{i,j} - \delta_{i,j+1}), \qquad \text{(A.2.19b)}$$
$$\{S_i, S_j\}_2 = R_j\delta_{j,i+1} - R_i\delta_{i,j+1}. \qquad \text{(A.2.19c)}$$

as well as

$$\{R_i, R_j\}_3 = 2R_iR_j(S_i\delta_{i,j-1} - S_j\delta_{i,j+1}) \qquad \text{(A.2.20a)}$$
$$\{R_i, S_j\}_3 = R_iR_j(\delta_{i,j-1} + \delta_{i,j}) - R_iR_{j+1}(\delta_{i,j+1} + \delta_{i,j+2}) + R_iS_j^2(\delta_{i,j} - \delta_{i,j+1}), \qquad \text{(A.2.20b)}$$
$$\{S_i, S_j\}_3 = (S_i + S_j)(R_j\delta_{i,j-1} - R_i\delta_{i,j+1}). \qquad \text{(A.2.20c)}$$

These Poisson brackets have been derived in[37] by a straightforward calculation, while now they have been obtained from the sytematic analysis. In the continuum limit, if we set the infinitesimal parameter $\epsilon = \frac{1}{N}$, and suppose that

$$\begin{cases} R_j \longrightarrow 1 + \frac{1}{2}\epsilon^2(u(x) + v(x)), \\ S_j \longrightarrow 2 + \frac{1}{2}\epsilon^2(u(x) - v(x)). \end{cases} \qquad \text{(A.2.21)}$$

after introducing a new Poisson bracket

$$\{,\} \equiv \frac{1}{4}\{,\}_3 - \{,\}_2,$$

we get two copies of Virasoro algebras

$$\{u(x), u(y)\} = \frac{1}{2}(\partial^3 + 4u(x)\partial + 2u'(x))\delta(x-y), \qquad \text{(A.2.22a)}$$
$$\{v(x), v(y)\} = -\frac{1}{2}(\partial^3 + 4v(x)\partial + 2v'(x))\delta(x-y), \qquad \text{(A.2.22b)}$$
$$\{u(x), v(y)\} = 0., \qquad \text{(A.2.22c)}$$

A.3 The Integrability of The System (A.1.1)

In the previous analysis, we have gauged away the Weyl symmetry killing in this way one degree of freedom. So the coordinates R_j's do not appear in the Poisson brackets, nor in the discrete KP equations. Now let us see what is the suitable Poisson brackets including them explicitly.

If we do not fix the Weyl symmetry, then we have the discrete linear system (A.1.1), whose integrability can be analysed in almost the same way as before. The only difference is that in this case $Q \in [-n, 1]$ but the tangent vector belongs to $[-1, n]$, so we should make use of R–matrix of $gl_0(\infty)$, which acts in the following way

$$\begin{cases} \mathcal{R}(E_{ij}) = E_{ij}, & i < j, \\ \mathcal{R}(E_{ii}) = 0, & \\ \mathcal{R}(E_{ij}) = -E_{ij}, & i > j. \end{cases} \tag{A.3.23}$$

which defines the following Lie–algebraic structure on the coordinate space

$$[X, Y]_{\mathcal{R}} = [\mathcal{R}(X), Y] + [X, \mathcal{R}(Y)]$$

Now let us consider the induced symmetry transformation on the functional space by gauge symmetry (A.1.6), which is in the coordinate space.

$$f_X(Q) \longrightarrow G^{-1} f_X(Q) G \equiv f_X(G^{-1}QG) \in \mathcal{F}.$$

In particular, set $g = Q^k_\alpha$, and introduce the quantities

$$H_k \equiv \frac{1}{k} Tr(Q^k), \qquad \forall k > 0,$$

then

$$\delta f_X(Q) = \epsilon X([Q, g]) = \epsilon \frac{1}{2} Q([dH_{k+1}, X]_{\mathcal{R}}),$$

which can be considered as the transformation generated by function H_{k+1} through some Poisson bracket, which is the first Poisson bracket

$$\{f_X, f_Y\}_1(Q) = \frac{1}{2} Q([X, Y]_{\mathcal{R}}) \tag{A.3.24}$$

With respect to this Poisson bracket, H_k's are the conserved quantities. Choosing one of them as Hamiltonian we recover the KP–Hierarchies. Obviously, this Poisson bracket is antisymmetric and satisfies the Jacobi identity.

In order to construct the second Poisson bracket, once again we use compatibility condition (A.2.10)(since it is valid for any integrable system). Finally the straightforward computation shows that the second Poisson bracket is

$$\{f_X, f_Y\}_2(Q) = \frac{1}{2} < QX\mathcal{R}(QY) - \mathcal{R}(YQ)XQ > \tag{A.3.25}$$

In particular, for the case $n = 2$, we have

$$Q_{ij} = \sqrt{R_j}\delta_{j,i+1} + S_j\delta_{ij} + l_j\delta_{j,i-1} + a_j\delta_{j,i-2},$$

where

$$\sqrt{R_j} l_j = L_j, \qquad \sqrt{R_j R_{j-1}} a_j = A_j,$$

the Poisson bracket (A.3.25) gives the Poisson algebra which contains (A.2.16a–A.2.16i) as a sub–algebra, besides, it also consists of the following ones

$$\{S_i, R_j\}_2 = S_i R_j(\delta_{i,j-1} - \delta_{ij}), \tag{A.3.26a}$$
$$\{L_i, R_j\}_2 = L_i R_j(\delta_{i,j-1} - \delta_{i,j+1}), \tag{A.3.26b}$$
$$\{A_i, R_j\}_2 = A_i R_j(\delta_{i,j-1} - \delta_{i,j+2}), \tag{A.3.26c}$$
$$\{R_i, R_j\}_2 = 0. \tag{A.3.26d}$$

Appendix B

The Derivation of the Virasoro Constraints

In this appendix we will derive the first few Virasoro constraints (3.4.15) explicitly.

Our starting point is eq.(3.4.14). Since we will use it repeatedly, so we simply quote it from chapter **3**

$$Tr\left(Q^{n+1}(M - V'(Q) + \bar{M})\right) = 0, \quad n \geq -1. \tag{B.0.1}$$

The important point is that matrix M above is *purely lower triangular*, so its transposition $\bar{M}$ is *purely upper triangular*. This property enables us to do very straightforward calculation, so as to prove that the eqs.(B.0.1) are true for $n = -1, 0, 1, 2$. Then, the algebraic structure (3.4.2b) guarantees the validity of the eqs.(B.0.1) for any $n \geq -1$. Concretely speaking, for any finite integer n, we at first express the "*trace*" in eq.(B.0.1) in terms of the quantities $B_n^{(r)}$ introduced in eq.(3.4.10); then use eq.(3.3.10b) to rewrite it as the derivatives of the partition function with respect to time parameters. Now let us see some simple examples.

<u>(*i*). L_{-1}–**constraint:**</u>

In this case $n = -1$, from eq.(B.0.1) we immediately have

$$\mathrm{Tr}(M) = \mathrm{Tr}(\bar{M}) = 0, \Longrightarrow \mathrm{Tr}\left(V'(Q)\right) = 0,$$

noting that

$$\mathrm{Tr}1 = N,$$

we get

$$Nt_1 - \sum_{r=2}^{\infty} rt_r \frac{\partial}{\partial t_{r-1}} \ln Z_N(t) = 0,$$

or equivalently

$$L_{-1}Z_N(t)=0, \qquad L_{-1}=\sum_{r=2}^{\infty} rt_r\frac{\partial}{\partial t_{r-1}}-Nt_1. \tag{B.0.2}$$

which is the string equation represented in terms of the partition function .

<u>*(ii)*. L_0–**constraint:**</u>

Now we consider the next case $n=0$, obviously we have

$$\mathrm{Tr}\Big(QM\Big)=\sum_{n=0}^{N-1}\Big(Q_{n,n+1}M_{n+1,n}\Big)=\sum_{n=0}^{N-1}B^{(1)}_{n+1}=\frac{1}{2}N(N+1),$$
$$\mathrm{Tr}\Big(Q\bar{M}\Big)=\sum_{n=0}^{N-1}\Big(Q_{n,n-1}(\bar{M})_{n-1,n}\Big)=\sum_{n=0}^{N-1}B^{(1)}_{n}=\frac{1}{2}N(N-1),$$

therefore eq.(B.0.1) shows that

$$\mathrm{Tr}\Big(QV'(Q)\Big)=\sum_{r=1}^{\infty}rt_r\mathrm{Tr}(Q^r)=N^2,$$

which leads to the L_0–constraint

$$L_0Z_N(t)=0, \qquad L_0=\sum_{r=1}^{\infty}rt_r\frac{\partial}{\partial t_r}+N^2. \tag{B.0.3}$$

<u>*(iii)*. L_1–**constraint:**</u>

This case corresponds to $n=1$ in eq.(B.0.1), at first we see that

$$\mathrm{Tr}(Q^2M)=\sum_{n=0}^{N-1}\Big(Q^2_{n,n+2}M_{n+2,n}+Q^2_{n,n+1}M_{n+1,n}\Big)=B^{(2)}_{n+2}+B^{(1)}_{n+1}(S_n+S_{n+1})$$
$$=\sum_{n=0}^{N-1}\Big(\sum_{i=0}^{n}S_i-(n+1)S_{n+1}+(n+1)(S_n+S_{n+1})\Big)$$
$$=\sum_{n=0}^{N-1}\Big(\sum_{i=0}^{n}S_i+(n+1)S_n\Big). \tag{B.0.4}$$

Similarly

$$\mathrm{Tr}(Q^2\bar{M})=\sum_{n=0}^{N-1}\Big(Q^2_{n,n-2}(\bar{M})_{n-2,n}+Q^2_{n,n-1}(\bar{M})_{n-1,n}\Big)=B^{(2)}_{n}+B^{(1)}_{n}(S_n+S_{n-1})$$
$$=\sum_{n=0}^{N-1}\Big(\sum_{i=0}^{n-2}S_i-(n-1)S_{n-1}+n(S_n+S_{n-1})\Big)=\sum_{n=0}^{N-1}\Big(\sum_{i=0}^{n}S_i+(n-1)S_n\Big). \tag{B.0.5}$$

Summing them together, and noting that

$$\sum_{n=0}^{N-1}\sum_{i=0}^{n} S_i = \sum_{i=0}^{N-1}\sum_{n=i}^{N-1} S_i = \sum_{i=0}^{N-1}(N-i)S_i,$$

we get

$$\mathrm{Tr}(Q^2M) + \mathrm{Tr}(Q^2\bar{M}) = 2\sum_{n=0}^{N-1}\Big(\sum_{i=0}^{n} S_i + nS_n\Big)$$
$$= 2\sum_{i=0}^{N-1}\Big((N-i)S_i + iS_i\Big) = 2N\sum_{i=0}^{N-1} S_i = 2N\mathrm{Tr}Q. \qquad \text{(B.0.6)}$$

Now using eq.(B.0.1) and (3.3.10b), we can get the L_1-constraint

$$\sum_{r=1}^{\infty} rt_r \mathrm{Tr}(Q^{r+1}) = 2N\mathrm{Tr}Q,$$

or

$$L_1 Z_N(t) = 0, \qquad L_1 = \sum_{r=1}^{\infty} rt_r \frac{\partial}{\partial t_{r+1}} - 2N\frac{\partial}{\partial t_1}. \qquad \text{(B.0.7)}$$

One remark should be mentioned. One can check that these three constraints L_{-1}, L_0, L_1 form a closed algebra, which is $sl(2)$.

(*iv*). L_2–**constraint:**

The most interesting case is the constraint L_2, which corresponds to set $n = 2$ in eq.(B.0.1). We also follow the same procedure as before

$$\mathrm{Tr}\Big(Q^3M\Big)$$
$$= \sum_{n=0}^{N-1}\Big(Q^3_{n,n+3}M_{n+3,n} + Q^3_{n,n+2}M_{n+2,n} + Q^3_{n,n+1}M_{n+1,n}\Big)$$
$$= \sum_{n=0}^{N-1}\Big(B^{(3)}_{n+3} + (S_n + S_{n+1} + S_{n+2})B^{(2)}_{n+2}$$
$$+(S_n^2 + S_nS_{n+1} + S_{n+1}^2 + R_n + R_{n+1} + R_{n+2})B^{(1)}_{n+1}\Big)$$

substituting the eqs.(3.4.12a–3.4.12c) into the above formula, we get a simpler expression

$$\mathrm{Tr}\Big(Q^3M\Big) = \sum_{n=0}^{N-1}\Big(2\sum_{i=0}^{n+1} R_i + (n+1)(R_n + R_{n+1})$$
$$+\sum_{i=0}^{n} S_i^2 + (n+1)S_n^2 + \sum_{i=0}^{n} S_iS_n\Big). \qquad \text{(B.0.8)}$$

On the other hand, we also have

$$\begin{aligned}
&\mathrm{Tr}\Big(Q^3(\bar{M})\Big)\\
&=\sum_{n=0}^{N-1}\Big(Q^3_{n,n-3}(\bar{M})_{n-3,n}+Q^3_{n,n-2}(\bar{M})_{n-2,n}+Q^3_{n,n-1}(\bar{M})_{n-1,n}\Big)\\
&=\sum_{n=0}^{N-1}\Big(B_n^{(3)}+(S_n+S_{n-1}+S_{n-2})B_n^{(2)}\\
&+(S_n^2+S_nS_{n-1}+S_{n-1}^2+R_n+R_{n+1}+R_{n-1})B_n^{(1)}\Big)\\
&=\sum_{n=0}^{N-1}\Big(2\sum_{i=0}^{n+1}R_i+(n-2)(R_n+R_{n+1})\\
&+\sum_{i=0}^{n}S_i^2+(n-2)S_n^2+\sum_{i=0}^{n}S_iS_n\Big).
\end{aligned} \tag{B.0.9}$$

Summing them together, we get

$$\begin{aligned}
\mathrm{Tr}\Big(Q^3(M+\bar{M})\Big)&=\sum_{n=0}^{N-1}\Big(4\sum_{i=0}^{n-1}R_i+(2n+3)(R_n+R_{n+1})\\
&+2\sum_{i=0}^{n}S_i^2+(2n+1)S_n^2+2\sum_{i=0}^{n-1}S_iS_n\Big).
\end{aligned} \tag{B.0.10}$$

Since

$$\begin{aligned}
\sum_{n=0}^{N-1}\sum_{i=0}^{n}R_i&=\sum_{i=0}^{N-1}\sum_{n=i}^{N-1}R_i=\sum_{i=0}^{N-1}(N-i)R_i,\\
\sum_{n=0}^{N-1}\sum_{i=0}^{n}S_i^2&=\sum_{i=0}^{N-1}\sum_{n=i}^{N-1}S_i^2=\sum_{i=0}^{N-1}(N-i)S_i^2,
\end{aligned}$$

and

$$\Big(\mathrm{Tr}Q\Big)^2=\sum_{n=0}^{N-1}\sum_{i=0}^{N-1}S_nS_i=\sum_{i=0}^{N-1}S_i^2+2\sum_{n=0}^{N-1}\sum_{i=0}^{n-1}S_iS_n.$$

We could write down the pure R–terms like

$$\begin{aligned}
&\sum_{n=0}^{N-1}\Big(4\sum_{i=0}^{n-1}R_i+(2n+3)(R_n+R_{n+1})\Big)\\
&=\sum_{i=0}^{N-1}\Big(4(N-i)R_i+(2i-1)R_r+(2i+3)R_{i+1}\Big)\\
&=4N\sum_{i=0}^{N-1}R_i+(2N+1)R_N\\
&=2N\sum_{i=0}^{N-1}(R_i+R_{i+1})+R_N.
\end{aligned} \tag{B.0.11}$$

Similarly, we can also rewrite the pure S–terms in (B.0.10) as

$$\sum_{n=0}^{N-1}\Big(2\sum_{i=0}^{n}S_i^2+(2n+1)S_n^2+2\sum_{i=0}^{n-1}S_iS_n\Big)=2N\sum_{i=0}^{N-1}S_i^2+\Big(\mathrm{Tr}Q\Big)^2. \qquad \text{(B.0.12)}$$

Combining the eqs.(B.0.11) and (B.0.12), we get finally

$$\mathrm{Tr}\Big(Q^3V'(Q)\Big)=2N\mathrm{Tr}(Q^2)+\Big(\mathrm{Tr}Q\Big)^2,$$

or in another form

$$L_2Z_N(t)=0, \qquad L_2=\sum_{r=1}^{\infty}rt_r\frac{\partial}{\partial t_{r+2}}-2N\frac{\partial}{\partial t_2}+\frac{\partial^2}{\partial t_1}. \qquad \text{(B.0.13)}$$

Now using the commutation relations (3.4.2b), we can derive all the other constraints. Thus the operator L_2 plays the role of "*creation*" operator.

Appendix C

$W_{1+\infty}$–Constraints in two–Matrix Model

In this appendix, starting from the string equations and the discrete KP–Heirachies, we try to prove that the partition function of the two–matrix model satisfies the $W_{1+\infty}$–constraints by using the algebraic structures.

C.1 The improved coupling conditions

In order to derive the $W_{1+\infty}$–constraints, we at first introduce the more general interaction terms into the partition function

$$Z_N(t,\tau) = \int dM_1 dM_2 e^U, \tag{C.1.1}$$

$$U(M;t;c) = \sum_{k=1}^{\infty} t_{1,k}\mathrm{Tr}(M_1^k) + \sum_{r=1}^{\infty} t_{2,r}\mathrm{Tr}(M_2^r) + \sum_{a,b\geq 1} C_{ab}\mathrm{Tr}(M_1^a M_2^b). \tag{C.1.2}$$

Then performing the same procedure as we did in the chapter 6, i.e. integrating out the angular part and introducing the orthogonal polynomials* like

$$\xi_n(\lambda_1,t) = \lambda_1^n + \cdots; \qquad \eta_n(\lambda_2,t) = \lambda_2^n + \cdots.$$

which satisfy the improved orthogonal relation

$$\int d\lambda_1 d\lambda_2 \xi_n(\lambda_1,t) e^{\mu} \eta_m(\lambda_2,t) = h_n \delta_{nm}. \tag{C.1.3}$$

where

$$\mu = V_1(\lambda_1) + V_2(\lambda_2) + \sum_{a,b\geq 1} C_{ab}\lambda_1^a \lambda_2^b, \tag{C.1.4}$$

$$V_1(\lambda_1) = \sum_{k=1}^{\infty} t_{1,k}\lambda_1^k, \qquad V_2(\lambda_2) = \sum_{r=1}^{\infty} t_{2,r}\lambda_2^r. \tag{C.1.5}$$

*For convenience, we choose polynomials different from those we used in the former chapters.

Furthermore, define the Jacobi matrices like (6.1.4)

$$\int d\lambda_1 d\lambda_2 \xi_n(\lambda_1,t)e^{\mu}\lambda_\alpha \eta_m(\lambda_2,t) = Q_{nm}(\alpha)h_m = \bar{Q}_{mn}h_n. \tag{C.1.6}$$

and their conjugations

$$P(1)\xi = \frac{\partial}{\partial\lambda_1}\xi, \qquad P(2)\eta = \frac{\partial}{\partial\lambda_2}\eta. \tag{C.1.7}$$

Then we can get the spectral equations

$$\lambda_1\xi = Q(1)\xi, \qquad \lambda_2\eta = \bar{Q}(2)\eta, \tag{C.1.8}$$

together with the equations of motion of the polynomials, which give two coupled linear systems(which have almost the same form as (6.1.10,6.1.12 with $q = 2$). The differences come from the coupling conditions and the equations of motion of the partition function . Noting the spectral equations, one can easily see that the coupling conditions become†

$$P(1) + V_1'(1) + \sum_{a,b\geq 1} aC_{ab}Q^{a-1}(1)Q^b(2) = 0, \tag{C.1.9a}$$

$$P(2) + V_2'(2) + \sum_{a,b\geq 1} bC_{ab}Q^{b-1}(2)Q^a(1) = 0. \tag{C.1.9b}$$

The time evolution of the partition function are given by eqs.(6.1.33), while the coupling dependences of the partition function are as follows

$$\frac{\partial}{\partial C_{ab}} \ln Z_N(t,c) = \mathrm{Tr}\Big(Q^a(1)Q^b(2)\Big), \qquad \forall a,b\geq 1. \tag{C.1.10}$$

These are all we need for the derivation of the constraints.

C.2 Virasoro constraints

The $W_{1+\infty}$–algebraic constraints possess a subset, which is Virasoro algebraic constraints. Now let us begin with this simpler case. At this moment the discussion is valid for both of the linear systems, so we temperially omit the system indices and consider general Jacobi matrix Q and its conjugation P

$$Q_{ij} = \delta_{j,i+1} + S_j\delta_{i,j} + L_j\delta_{j,i-1} + \dots, \qquad [Q,P] = 1. \tag{C.2.11}$$

†Hereafter we will use the notations

$$V_1'(1) = \sum_{k=1}^{\infty} kt_{1,k}Q^{k-1}(1),$$

$$V_1''(1) = \sum_{k=2}^{\infty} k(k-1)t_{1,k}Q^{k-2}(1),$$

etc.

which completely determines the quantities $B_n^{(r)}$'s defined by (3.4.10), which are given by eqs.(3.4.12a–3.4.12c). That is to say, the string equation does completely determine the matrix P. So we can get $P(1,2)$ in this way. Using these formulas and (3.4.13), doing the same thing as we did in the appendix B, one can easily re–express the following identities as the constraints on the partition function

$$\mathrm{Tr}\Big(Q^{n+1}(1)\big(P(1)+V'(1)+\big)+\sum_{a,b\geq 1} aC_{ab}Q^{a-1}(1)Q^{b}(2)\big)\Big)=0,\quad n\geq -1 \qquad \text{(C.2.12)}$$

which is equivalent to

$$(\mathcal{L}_n^{[1]}(1)+\mathcal{T}_n^{[1]}(1))Z_N(t;c)=0, \qquad n\geq -1. \qquad \text{(C.2.13)}$$

where

$$\begin{aligned}
\mathcal{L}_{-1}^{[1]}(1) &= \sum_{k=2}^{\infty} kt_{1,k}\frac{\partial}{\partial t_{1,k-1}}+Nt_{1,1},\\
\mathcal{L}_{0}^{[1]}(1) &= \sum_{k=1}^{\infty} kt_{1,k}\frac{\partial}{\partial t_{1,k}}+\frac{1}{2}N(N+1), \qquad \text{(C.2.14)}\\
\mathcal{L}_{n}^{[1]}(1) &\equiv \sum_{k=1}^{\infty} kt_{1,k}\frac{\partial}{\partial t_{1,k+n}}+(N+\frac{n+1}{2})\frac{\partial}{\partial t_{1,n}}+\frac{1}{2}\sum_{k=1}^{n-1}\frac{\partial^2}{\partial t_{1,k}\partial t_{1,n-k}},\quad n\geq 1.
\end{aligned}$$

and

$$\mathcal{T}_{-1}^{[1]}(1) \equiv \sum_{a\geq 2,b\geq 1} aC_{ab}\frac{\partial}{\partial C_{a-1,b}}+\sum_{b\geq 1}C_{1b}\frac{\partial}{\partial t_{2,b}}, \qquad \text{(C.2.15a)}$$

$$\mathcal{T}_{n}^{[1]}(1) \equiv \sum_{a,b\geq 1} aC_{ab}\frac{\partial}{\partial C_{a+n,b}},\qquad n\geq 0, \qquad \text{(C.2.15b)}$$

These operators satisfy the Virasoro algebras

$$[\mathcal{L}_n^{[1]}(1),\mathcal{L}_m^{[1]}(1)]=(n-m)\mathcal{L}_{n+m}^{[1]}(1),\qquad n,m\geq -1, \qquad \text{(C.2.16a)}$$

$$[\mathcal{L}_n^{[1]}(1),\mathcal{T}_m^{[1]}(1)]=0,\qquad n,m\geq -1, \qquad \text{(C.2.16b)}$$

$$[\mathcal{T}_n^{[1]}(1),\mathcal{T}_m^{[1]}(1)]=(n-m)\mathcal{T}_{n+m}^{[1]}(1),\qquad n,m\geq -1, \qquad \text{(C.2.16c)}$$

From these algebraic structures, we can learn the following thing: in order to prove the Virasoro constraints (C.2.13), we only need to check the first few cases, i.e. $n=-1,0,1,2$, the other higher constraints can be otained by these Virasoro algebraic structures. Using the same trick used in Appendix B, we can easily prove that eqs.(C.2.13) are true for $n=-1,0,1,2$, so they are correct for all $n\geq -1$.

C.3 The higher rank constraints

In the previous section we only obtained the Virasoro constraints, which may be referred to as rank 2 tensorial constraints. In order to get the spin–3 operators, we introduce the following notations

$$\begin{aligned}
\nu &\equiv \sum_{a,b\geq 1} C_{ab}Q^a(1)Q^b(2),\\
\nu' &\equiv \partial\nu = \sum_{a,b\geq 1} aC_{ab}Q^{a-1}(1)Q^b(2),\\
:\nu^2: &\equiv \sum_{a,b,a',b'\geq 1} C_{ab}C_{a'b'}Q^{a+a'}(1)Q^{b+b'}(2).
\end{aligned}$$

From the trivial relation

$$\int d\lambda_1 d\lambda_2 \frac{\partial^2}{\partial\lambda_1^2}\Big(\xi_n(\lambda_1,t)e^{\mu}\eta_m(\lambda_2,t)\Big)=0,$$

we get the following identity

$$-P^2(1)+V''(1)+V'^2(1)+2V'(1)\nu'+:\nu'^2:+\nu''=0, \tag{C.3.17}$$

then multiplying it by $Q^{n+2}(1)$ from the left and taking the trace operation, we obtain the rank–3 constraints

$$(\mathcal{L}_n^{[2]}(1)+T_n^{[2]}(1))Z_N(t;c)=0, \qquad n\geq -2. \tag{C.3.18}$$

with the definitions

$$\begin{aligned}
\mathcal{L}_{-2}^{[2]}(1) &\equiv \sum_{l_1+l_2=3}^{\infty} l_1l_2t_{1,l_1}t_{1,l_2}\frac{\partial}{\partial t_{1,l_1+l_2-2}}+\sum_{l=3}^{\infty} lt_{1,l}\sum_{k=1}^{l-3}\frac{\partial^2}{\partial t_{1,k}\partial t_{1,l-k-2}}\\
&\quad +2N(\sum_{l=3} lt_{1,l}\frac{\partial}{\partial t_{1,l-2}}+Nt_{1,2}+\frac{1}{2}t_{1,1}^2), \qquad \text{(C.3.19a)}\\
\mathcal{L}_{-1}^{[2]}(1) &\equiv \sum_{l_1,l_2=1}^{\infty} l_1l_2t_{1,l_1}t_{1,l_2}\frac{\partial}{\partial t_{1,l_1+l_2-1}}+\sum_{l=3}^{\infty} lt_{1,l}\sum_{k=1}^{l-2}\frac{\partial^2}{\partial t_{1,k}\partial t_{1,l-k-1}}\\
&\quad +2N\sum_{l=2} lt_{1,l}\frac{\partial}{\partial t_{1,l-1}}+\mathcal{L}_{-1}^{[1]}(1)+N^2t_{1,1}, \qquad \text{(C.3.19b)}\\
\mathcal{L}_{0}^{[2]}(1) &\equiv \sum_{l_1,l_2=1}^{\infty} l_1l_2t_{1,l_1}t_{1,l_2}\frac{\partial}{\partial t_{1,l_1+l_2}}+\sum_{l=2}^{\infty} lt_{1,l}\sum_{k=1}^{l-1}\frac{\partial^2}{\partial t_{1,k}\partial t_{1,l-k}}\\
&\quad +2(N+1)\sum_{l=1} lt_{1,l}\frac{\partial}{\partial t_{1,l}}+\frac{1}{3}N(N+1)(N+2), \qquad \text{(C.3.19c)}\\
\mathcal{L}_{n}^{[2]}(1) &\equiv \sum_{l_1,l_2=1}^{\infty} l_1l_2t_{1,l_1}t_{1,l_2}\frac{\partial}{\partial t_{1,l_1+l_2+n}}+\sum_{l=1}^{\infty} lt_{1,l}\sum_{k=1}^{l+n-1}\frac{\partial^2}{\partial t_{1,k}\partial t_{1,l+n-k}}\\
&+ \frac{1}{2(n+3)}\sum_{l=1}^{n-2}\sum_{k=1}^{n-l-1}(n-l+2)\frac{\partial^3}{\partial t_{1,l}\partial t_{1,k}\partial t_{1,n-l-k}}\\
&+ (N^2+(n+1)N+\frac{1}{6}(n+1)(n+2))\frac{\partial}{\partial t_{1,n}}\\
&+ (2N+n+2)\mathcal{L}_n^{[1]}(1) \qquad n\geq 1. \qquad \text{(C.3.19d)}
\end{aligned}$$

and

$$T^{[2]}_{-2}(1) \equiv \sum_{a+a'\geq 3,b,b'\geq 1} aa'C_{ab}C_{a'b'}\frac{\partial}{\partial C_{a+a'-2,b+b'}} + \sum_{a\geq 3,b\geq 1} a(a-1)C_{ab}\frac{\partial}{\partial C_{a-2,b}} + \sum_{b,b'\geq 1} C_{1b}C_{1b'}\frac{\partial}{\partial t_{2,b+b'}} + 2\sum_{b\geq 1} C_{2b}\frac{\partial}{\partial t_{2,b}}, \qquad \text{(C.3.20a)}$$

$$T^{[2]}_{n}(1) \equiv \sum_{a,b,a',b'\geq 1} aa'C_{ab}C_{a'b'}\frac{\partial}{\partial C_{a+a'+n,b+b'}} + \sum_{a,b\geq 1} a(a-1)C_{ab}\frac{\partial}{\partial C_{a+n,b}}, \qquad n\geq -1. \qquad \text{(C.3.20b)}$$

They satisfies the following algebra

$$[\mathcal{L}^{[2]}_n(1),\mathcal{L}^{[1]}_m(1)] = (n-2m)\mathcal{L}^{[2]}_{n+m}(1) + m(m+1)\mathcal{L}^{[1]}_{n+m}(1), \qquad \text{(C.3.21a)}$$
$$[\mathcal{L}^{[2]}_n(1),\mathcal{L}^{[2]}_m(1)] = 2(n-m)\mathcal{L}^{[3]}_{n+m}(1) + (n-m)(n+m+3)\mathcal{L}^{[2]}_{n+m}(1), \qquad \text{(C.3.21b)}$$
$$[T^{[2]}_n(1),T^{[1]}_m(1)] = (n-2m)T^{[2]}_{n+m}(1) + m(m+1)T^{[1]}_{n+m}(1), \qquad \text{(C.3.21c)}$$
$$[T^{[2]}_n(1),T^{[2]}_m(1)] = 2(n-m)T^{[3]}_{n+m}(1) + (n-m)(n+m+3)\mathcal{L}^{[2]}_{n+m}(1), \qquad \text{(C.3.21d)}$$

where we have introduced the spin–4 operator

$$T^{[3]}_n(1) \equiv \sum_{a_1,b_1,a_2,b_2,a_3,b_3\geq 1} a_1a_2a_3C_{a_1b_1}C_{a_2b_2}C_{a_3b_3}\frac{\partial}{\partial C_{a_1+a_2+a_3+n,b_1+b_2+b_3}} + \sum_{a_1,b_1,a_2,b_2\geq 1} \frac{3}{2}a_1a_2(a_1+a_2-2)C_{a_1b_1}C_{a_2b_2}\frac{\partial}{\partial C_{a_1+a_2+n,b_1+b_2}} + \sum_{a,b\geq 1} a(a-1)(a-2)C_{ab}\frac{\partial}{\partial C_{a+n,b}}, \qquad n\geq -3. \qquad \text{(C.3.22)}$$

Now let us prove our claim—the constraints (C.3.18). At first, noting that

$$\mathrm{Tr}\Big(P^2(1)\Big) = 0,$$

we can easily rewrite eq.(C.3.17) as follows

$$(\mathcal{L}^{[2]}_{-2}(1) + T^{[2]}_{-2}(1))Z_N(t;c) = 0, \qquad \text{(C.3.23)}$$

which is the particular case of eqs.(C.3.18) with $n=-2$. Then, the algebraic structures (C.3.21a) and (C.3.21b) guarantee that all the other constraints in eqs.(C.3.18) must be true. We see that the algebraic structures help us simplifying the calculations dramatically.

In fact, the algebra (C.3.21a–C.3.21d) can tell us more. From the eqs.(C.3.21b) and (C.3.21d), it is quite obviously to see that the algebra generated by $\{(\mathcal{L}^{[1]}_n(1)+T^{[1]}_n(1)), n\geq$

$-1; (\mathcal{L}_n^{[2]}(1) + T_n^{[2]}(1)), n \geq -2\}$ does not close. In order to get a closed algebra, we should include all the other higher rank operators, which are generated by the commuators of two or more rank–3 operators. For example, from eqs.(C.3.21b) and (C.3.21d), we obtain the rank–4 operators $\left(\mathcal{L}_n^{[3]}(1) + T_n^{[3]}(1)\right)$. Furthermore, these operators commute with $\left(\mathcal{L}_n^{[2]}(1) + T_n^{[2]}(1)\right)$, generate the rank–5 operators. In this way, we can obtain any higher rank operators, all of them together form a closed W_∞–algebra with the generators

$$T_n^{[r]}(1) \equiv \sum_{a_1,b_1,\ldots,a_r,b_r \geq 1} a_1 a_2 \ldots a_r C_{a_1 b_1} \ldots C_{a_r b_r} \frac{\partial}{\partial C_{a_1+\ldots+a_r+n, b_1+\ldots+b_r}}$$
$$+ \text{lower orders}$$

and the corrsponding $\mathcal{L}_n^{[r]}(1)$. The algebra is

$$[T_n^{[r]}(1), T_m^{[s]}(1)] = (sn - rm) T_{n+m}^{[r+s-1]}(1) + \ldots, \qquad \text{(C.3.24a)}$$
$$[\mathcal{L}_n^{[r]}(1), \mathcal{L}_m^{[s]}(1)] = (sn - rm) \mathcal{L}_{n+m}^{[r+s-1]}(1) + \ldots, \qquad \text{(C.3.24b)}$$
$$[T_n^{[r]}(1), \mathcal{L}_m^{[s]}(1)] = 0, \qquad \text{(C.3.24c)}$$

for $r, s \geq 1$; $n \geq -r, m \geq -s$.

This is very essential feature in our analysis. Since we have already shown that the rank–2 and rank–3 operators in the W_∞–algebra mentioned above *annihilate* the partition function , that is eqs.(C.2.13) and (C.3.18), due to the construction, all the other generators of the W_∞–algebra will vanish the partition function , that is to say

$$(\mathcal{L}_n^{[r]}(1) + T_n^{[r]}(1)) Z_N(t;c) = 0, \qquad r \geq 1; \qquad n \geq -r. \qquad \text{(C.3.25)}$$

For example, from eqs.(C.3.21b) and (C.3.21d), we obtain the rank–4 operatorial constraints,

$$\left(\mathcal{L}_n^{[3]}(1) + T_n^{[3]}(1)\right) Z_N(t;c) = 0, \qquad n \geq -3. \qquad \text{(C.3.26)}$$

Comparing with the analysis in the previous Appendix, we see that right now rank–3 constraints play the role of the "*creation*" operator, from which all the $W_{1+\infty}$–constraints are followed.

In fact, there is another way to get the higher rank operators, which is as follows: once again we start from the coupling condition (C.1.3), consider the trivial relation

$$\int d\lambda_1 d\lambda_2 \frac{\partial^r}{\partial \lambda_1^r} \left(\xi_n(\lambda_1, t) e^{\mu} \eta_m(\lambda_2, t)\right) = 0,$$

since

$$\frac{\partial^l}{\partial \lambda_1^l} e^{\mu(lm_1, \lambda_2)} = e^{\mu(lm_1, \lambda_2)} \left(\partial_{\lambda_1} + V'(\lambda_1) + \nu'(\lambda_1)\right) \cdot 1,$$

we get

$$\int d\lambda_1 d\lambda_2 \eta \frac{\partial^r}{\partial\lambda_1^r}\left(e^{\mu}\xi\right) = \sum_{l=0}^{r} \left(\begin{array}{c} r \\ l \end{array}\right) \left(\frac{\partial^l}{\partial\lambda_1^l} e^{\mu(lm_1,\lambda_2)}\right) \frac{\partial^{r-l}}{\partial\lambda_1^{r-l}}\xi$$

$$= \int d\lambda_1 d\lambda_2 \eta e^{\mu} \sum_{l=0}^{r} \left[\left(\partial_{\lambda_1} + V'(\lambda_1) + \nu'(\lambda_1)\right)^l \cdot 1\right] \frac{\partial^{r-l}}{\partial\lambda_1^{r-l}}\xi,$$

using the spectral relations (C.1.8) and the orthogonal relations (C.1.3), as well as the coupling conditions (C.1.9a–C.1.9b), we obtain

$$(-1)^{r+1}P^r(1) + :(\partial - V'(1) + \nu')^r \cdot 1 := 0, \tag{C.3.27}$$

multiplying it by $Q^{n+r}(1)$ from the left and taking the trace, we can get the r–th constraints (C.3.25).

For the same reason, we can derive another piece of W_∞–algebra, which is related to the second matrix.

$$(\mathcal{L}_n^{[r]}(2) + T_n^{[r]}(2)) Z_N(t;c) = 0, \qquad r \geq 1. \tag{C.3.28}$$

where $\mathcal{L}_n^{[r]}(2)$ can be obtained from $\mathcal{L}_n^{[r]}(1)$ once we replace $t_{1,k}$'s parameters by $t_{2,r}$'s parameters. However,

$$T_n^{[1]}(2) \equiv \sum_{a,b\geq 1} bC_{ab} \frac{\partial}{\partial C_{a,b+n}}, \qquad n \geq -1$$

$$T_n^{[2]}(2) \equiv \sum_{a,b,a',b'\geq 1} bb' C_{ab} C_{a'b'} \frac{\partial}{\partial C_{a+a',b+b'+n}} + \sum_{a,b\geq 1} b(b-1) C_{ab} \frac{\partial}{\partial C_{a,b+n}}, \qquad n \geq -2, \tag{C.3.29a}$$

$$T_n^{[3]}(2) \equiv \sum_{a_1,b_1,a_2,b_2,a_3,b_3\geq 1} b_1 b_2 b_3 C_{a_1 b_1} C_{a_2 b_2} C_{a_3 b_3} \frac{\partial}{\partial C_{a_1+a_2+a_3,b_1+b_2+b_3+n}} + \sum_{a_1,b_1,a_2,b_2\geq 1} \frac{3}{2} b_1 b_2 (b_1 + b_2 - 2) C_{a_1 b_1} C_{a_2 b_2} \frac{\partial}{\partial C_{a_1+a_2,b_1+b_2+n}} + \sum_{a,b\geq 1} b(b-1)(b-2) C_{ab} \frac{\partial}{\partial C_{a,b+n}}, \qquad n \geq -3, \tag{C.3.29b}$$

$$\cdots\cdots$$

$$T_n^{[r]}(2) \equiv \sum_{a_1,b_1,\ldots,a_r,b_r\geq 1} b_1 b_2 \ldots b_r C_{a_1 b_1} \ldots C_{a_r b_r} \frac{\partial}{\partial C_{a_1+\ldots+a_r,b_1+\ldots+b_r+n}} + \text{less c–terms}, \tag{C.3.29c}$$

with the following algebra which is isomorphic to (C.3.24a–C.3.24c)

$$[T_n^{[r]}(2), T_m^{[s]}(2)] = (sn - rm) T_{n+m}^{[r+s-1]}(2) + \ldots, \tag{C.3.30a}$$

$$[\mathcal{L}_n^{[r]}(2), \mathcal{L}_m^{[s]}(2)] = (sn - rm) \mathcal{L}_{n+m}^{[r+s-1]}(2) + \ldots, \tag{C.3.30b}$$

$$[T_n^{[r]}(2), \mathcal{L}_m^{[s]}(2)] = 0, \tag{C.3.30c}$$

for $r,s \geq 1$; $n \geq -r, m \geq -s$. We see that there are two isomorphic W_∞–algebraic constraints in case of the presence of general interactions. The closed algebra formed by these two pieces of W_∞ algebras gives the complete constraints(one should keep in mind that they are not direct product).

C.4 $W_{1+\infty}$–constraints

From eq.(C.3.27), it is easy to see that if we set all $C_{ab}, a, b \geq 1$ equal to zero, but only keep $C_{11} = c \neq 0$, we have

$$T_n^{[r]}(1)Z_N(t;c) = c^r \mathrm{Tr}\Big(Q^{r+n}(1)Q^r(2)\Big) \tag{C.4.31a}$$

$$T_n^{[r]}(2)Z_N(t;c) = c^r \mathrm{Tr}\Big(Q^{r+n}(2)Q^r(1)\Big). \tag{C.4.31b}$$

substituting these into eqs.(C.3.25) and (C.3.28), we get one piece of $W_{1+\infty}$–algebraic constraints

$$W_n^{[r]} Z_N(t,\tau,c) = 0, \quad r \geq 0; \ \ n \geq -r, \tag{C.4.32a}$$

$$W_n^{[r]} \equiv c^n \mathcal{L}_n^{[r]}(1) - \mathcal{L}_{-n}^{[r+n]}(2). \tag{C.4.32b}$$

One can explicitly check that this result coincides with the one in ref[66]. A further reduction would be possible. Suppose we set $t_{2,k} = 0, k > q$, then from the above equation, we have

$$\mathcal{L}_{-r}^{[r]}(1) + c^r \mathrm{Tr}(Q^r(2)) = 0 \tag{C.4.33}$$

substituting it into the other constraints, we can get another W_∞–algebraic constraints, which is only expressed in terms of t–parameters and $t_{2,k}, 1 \leq k \leq q$, and is a subalgebra of the $W_{1+\infty}$ in eq.(C.4.32a–C.4.32b).

Appendix D

W–infinity Algebras

In integrable systems, there appear several different W–infinity algebras. For those unfamiliar with this subject, it probably causes some confusion. We devote this Appendix to discuss them.

D.1 $W_{1+\infty}$ algebra

As we discussed in chapter 4, the KP hierarchy (4.1.9) possesses a bi–hamiltonian structure. The two compatible Poisson brackets are given by (4.1.7) and (4.1.11). Now let us begin with the first Poisson bracket

$$\{f_X, f_Y\}_1(L) = L([X,Y]). \tag{D.1.1}$$

where

$$L = \partial + \sum_{i=1}^{\infty} u_i(x)\partial^{-i}. \tag{D.1.2}$$

and

$$X = \sum_{i=0}^{\infty} \partial^i \chi_i(x) \in \wp_+,$$

with the testing functions $\chi_i(x)$'s(they are not functions of KP coordinates). Now set

$$X = \partial^{i-1}\chi(x), \qquad Y = \partial^{j-1}\mathcal{Y}(x);$$

Using the definition of the inner product on $\wp$ (4.1.4), it is straightforward to calculte that

$$f_X(L) = \int dx \chi(x) u_i(x), \qquad f_Y(L) = \int dx \mathcal{Y}(x) u_j(x),$$

and

$$
\begin{aligned}
< LXY > &= < \Big(\partial + \sum_{l=1}^{\infty} u_l(x)\partial^{-l}\Big)\partial^{i-1}\chi\partial^{j-1}\mathcal{Y} > \\
&= \sum_{l=0}^{j-1}(-1)^l\binom{j-1}{l}\int dx u_{i+j-l-1}\chi^{(l)}\mathcal{Y} \\
&= \sum_{l=0}^{j-1}\binom{j-1}{l}\int dx\chi\Big(u_{i+j-l-1}\mathcal{Y}\Big)^{(l)}
\end{aligned} \tag{D.1.3}
$$

Similarly

$$
< LYX > = \sum_{l=0}^{i-1}(-1)^l\binom{i-1}{l}\int dx\chi u_{i+j-l-1}\mathcal{Y}^{(l)} \tag{D.1.4}
$$

Substituting them into eq.(D.1.1), we get

$$
\begin{aligned}
&\int\int dxdy\chi(x)\{u_i(x),u_j(y)\}_1\mathcal{Y}(y) \\
&= \int dx\chi\Big[\sum_{l=0}^{j-1}\binom{j-1}{l}\partial^l u_{i+j-l-1} - \sum_{l=0}^{i-1}(-1)^l\binom{i-1}{l}u_{i+j-l-1}\partial^l\Big]\mathcal{Y}(x)
\end{aligned} \tag{D.1.5}
$$

Thus we can immediately extract out the Poisson algebra

$$
\{u_i(x),u_j(y)\}_1 = \Big[\sum_{l=0}^{j-1}\binom{j-1}{l}\partial^l u_{i+j-l-1} - \sum_{l=0}^{i-1}(-1)^l\binom{i-1}{l}u_{i+j-l-1}\partial^l\Big]\delta(x-y). \tag{D.1.6}
$$

In particular, we have

$$
\{u_2(x),u_2(y)\}_1 = \Big(2u_2\partial + u_2'\Big)\delta(x-y), \tag{D.1.7}
$$

and

$$
\{u_i(x),u_2(y)\}_1 = \Big(iu_i\partial + u_i'\Big)\delta(x-y) + \text{the higher derivative terms} \tag{D.1.8}
$$

These relations simply tell us that u_2 forms a (classical) Virasoro algebra. And if we ignore the higher derivative terms, u_i behaves like conformal tensor with conformal weight i(or spin–i tensor, $i \geq 1$). Since the algebra (D.1.6) includes spin–1 conformal tensor, we usually call it $W_{1+\infty}$ algebra.

D.2 W_∞ algebra

In the previous section we only considered the algebra based on the first Poisson bracket. In fact we can do the same thing for the second Poisson bracket. Then we are led to the following Poisson algebra

$$\{u_i(x),u_j(y)\}_2=\Big[\sum_{l=0}^{j}\binom{j}{l}\partial^l u_{i+j-l}-\sum_{l=0}^{i}(-1)^l\binom{i}{l}u_{i+j-l}\partial^l$$
$$-\sum_{l=1}^{i-1}\sum_{k=0}^{i-l-1}(-1)^k\binom{i-l-1}{k}u_{i+j-l-k-1}\partial^k u_l+\sum_{m=0}^{i-1}\sum_{l=1}^{i-m-1}$$
$$\sum_{n=0}^{i+j-l-m-2}(-1)^m\binom{i-1}{m}\binom{i+j-l-m-2}{n}u_l\partial^{m+n}\,u_{i+j-m-n-l-1}$$
$$-\sum_{l=1}^{i-1}\sum_{k=1}^{j-1}(-1)^l\binom{i-1}{l}\binom{j-1}{k}u_{i-l}\partial^{k+l-1}u_{j-k}\Big]\delta(x-y). \qquad \text{(D.2.9)}$$

Particularly

$$\{u_1(x),u_1(y)\}_2=\left(2u_1\partial+u_1'\right)\delta(x-y) \qquad \text{(D.2.10)}$$

It means that u_1 forms a (classical) Virasoro algebra. Generally, u_i is a conformal tensor with weight $(i+1)$(spin–$(i+1)$ tensor $i\geq 1$). We see that there is no spin–1 tensor in the above Poisson algebra. This algebra (D.2.9) is referred to as W_∞ algebra. One can check that when we take the particular reduction (4.5.8), we recover the algebra (4.5.9a–4.5.9c).

D.3 w_∞ Algebra

Now let us turn our attention to the dispersionless version of KP hierarchy . There is a very easy way to get the dispersionless versions of the Poisson algebras (D.1.6) and (D.2.9). The trick is the following: we simply disregard all the higher derivative terms. Then the algebra (D.2.9) becomes

$$\{u_i(x),u_j(y)\}_2^{\rm dis}=\Big[iu_{i+j-1}\partial+j\partial u_{i+j-1}$$
$$+\sum_{l=1}^{i-2}\Big((i-l-1)u_{i+j-l-2}\partial u_l+(j-l-1)u_l\partial u_{i+j-l-2}\Big)$$
$$+i(j-1)u_{i-1}\partial u_{j-1}\Big]\delta(x-y). \qquad \text{(D.3.11)}$$

This algebra is called w_∞ algebra. Setting

$$u_i=RS^{i-1}, \qquad \imath\geq 1.$$

We rederive the algebra (4.5.22).

D.4 $w_{1+\infty}$ Algebra

The dispersionless version of the algebra (D.1.6) is

$$\{u_i(x),u_j(y)\}_1^{\rm dis}=\Big[(j-1)\partial u_{i+j-2}+(i-1)u_{i+j-2}\partial\Big]\delta(x-y). \qquad \text{(D.4.12)}$$

This algebra is called $w_{1+\infty}$ algebra. Both $w_{1+\infty}$ and w_∞ algebras describe area preserving diffeomorphisms of Riemann surfaces[84].

D.5 The Quantized W–infinity Algebras

If we replace the Poisson brackets by operator product expansion (OPE), then $W_{1+\infty}$ and W_∞ will receive quantum corrections. The corrected algebras are referred to as $\hat{W}_{1+\infty}$ algebra and $\hat{W}_\infty$ respectively.

Appendix E

Notations

E.1 Semi–infinite matrix

In our analysis, we always meet the semi–infinite matrices. Now let us list below our notations for any matrix M.

The decompositions

$$\begin{cases} M_- : & \text{the purely lower triangular part} \\ M_{(0)} : & \text{the diagonal line,} \\ M_+ : & \text{the diagonal line and purely upper triangular part,} \\ M_{ij} : & \text{the element at i} - \text{th row and j} - \text{th column} \end{cases}$$

The truncations

$$M_\theta \equiv \frac{1}{2}M_{(0)} + M_-, \qquad M_\alpha \equiv M_+ - \frac{1}{2}M_{(0)}.$$

The operations

$$\begin{cases} \text{Transposition :} & \tilde{M}_{ij} = M_{ji}; \\ \text{Truncated trace :} & \mathrm{Tr}(M) = \sum_{i=0}^{N-1} M_{ii}; \\ \text{Non} - \text{truncated trace :} & \widetilde{\mathrm{Tr}}(M) = \sum_{i=0}^{\infty} M_{ii}; \\ & (\mathcal{M})_{ij} = M_{ij}\frac{h_j}{h_i}. \end{cases}$$

The sector

$$M_l(j) \equiv M_{j,j-l}.$$

We call the element $M_l(j)$ a coordinate belonging to the $j-th$ *sector.*

Some particular semi–infinite matrices

$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl}, \qquad I_{\pm} \equiv \sum_{i=0}^{\infty} E_{i,i\pm 1}, \qquad \rho = \sum_i iE_{ii}.$$

Gradation

$$\deg[E_{ij}] = j - i.$$

Using this notation, we intoduce one more definition

$$M_{[l]} = \sum_i M_{i,i+l}E_{i,i+l}$$

the subindex means selecting the particular part with the given degree. For a given matrix M, if its all non–zero elements are at the pseudo–diagonal lines with degrees in the interval $[a,b]$, then we will simply denote by $M \in [a,b]$.

E.2 Pseudo–differential operator

The integration operator

$$\partial^{-1}: \qquad \partial^{-1}u(x) \equiv \int_{x_0}^{x} dx'u(x').$$

The truncations and operations

For any pseudo–differential operator

$$A = \sum_{-\infty}^{n} u_i(x)\partial_i.$$

$$\begin{cases} A_+ = \sum_0^n u_i(x)\partial_i : & \text{the pure differential part;} \\ A_- = \sum_{-\infty}^{-1} u_i(x)\partial_i : & \text{the pure integration part;} \\ A_{(j)} = u_j(x) : & \text{selecting typic term;} \\ \text{res}_{\partial} A = u_{-1}(x) : & \text{residue;} \\ < A >= \text{Tr}(A) = \int u_{-1}(x)dx : & \text{inner scalar product.} \end{cases}$$

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