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Inverse Problem for Semisimple Frobenius Manifolds Monodromy Data and the Painlevé VI Equation

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Introduction

The WDVV equations of associativity were introduced by Witten [54], Dijkgraaf, Verlinde E., Verlinde H. [13]. They are differential equations satisfied by the *primary free energy* $F(t)$ in two-dimensional topological field theory. $F(t)$ is a function of the coupling constants $t := (t^1, t^2, ..., t^n)$ $t^i \in \mathbf{C}$. Let $\partial_\alpha := \frac{\partial}{\partial t_\alpha}$. Given a non-degenerate symmetric matrix $\eta_{\alpha\beta}$, $\alpha, \beta = 1, ..., n$, and numbers $q_1 = 0, q_2, ..., q_n, d$, the WDVV equations are

$$\partial_\alpha \partial_\beta \partial_\lambda F \; \eta^{\lambda\mu} \; \partial_\mu \partial_\gamma \partial_\delta F \; = \; \text{the same with } \alpha, \, \delta \text{ exchanged}, \tag{1}$$

$$\partial_1 \partial_\alpha \partial_\beta F = \eta_{\alpha\beta}, \qquad F(\lambda^{1-q_1} t^1, ..., \lambda^{1-q_n} t^n) = \lambda^{3-d} F(t^1, ...t^n), \quad \lambda \in \mathbf{C} \backslash \{0\}$$

The theory of Frobenius manifolds was introduced by B. Dubrovin [14] to formulate in geometrical terms the WDVV equations. It has links to many branches of mathematics like singularity theory and reflection groups [48] [49] [19] [16], algebraic and enumerative geometry [33] [36], isomonodromic deformations theory, boundary value problems and Painlevé equations [17].

If we define $c_{\alpha\beta\gamma}(t) := \partial_\alpha \partial_\beta \partial_\gamma F(t)$, $c_{\alpha\beta}^\gamma(t) := \eta^{\gamma\mu} c_{\alpha\beta\mu}(t)$ (sum omitted), and we consider a vector space A=span$(e_1, ..., e_n)$, then we obtain a family of algebras A_t with the multiplication $e_\alpha \cdot e_\beta := c_{\alpha\beta}^\gamma(t) e_\gamma$. Equation (1) is equivalent to associativity.

A *Frobenius manifold* is a smooth/analytic manifold M over $\mathbf{C}$ whose tangent space $T_t M$ at any $t \in M$ is an *associative, commutative algebra.* Moreover, there exists a non-degenerate bilinear form $< \;\; , \;\; >$ defining a *flat metric.* The variables $t^1, .., t^n$ are the flat coordinates for a point $t \in M$ and $\eta_{\alpha\beta} = < \partial_\alpha, \partial_\beta >$, $\alpha, \beta = 1, 2, ..., n$. The structure constants in $T_t M$ with respect to the basis $\partial_1, ..., \partial_n$ are $c_{\alpha\beta\gamma}(t) = \partial_\alpha \partial_\beta \partial_\gamma F(t)$.

The manifold is characterized by a family of flat connections $\tilde{\nabla}(z)$, parametrized by a complex number z, such that for $z = 0$ the connection is associated to $< \;\; , \;\; >$. To find a flat coordinate $\tilde{t}(t, z)$ we impose $\tilde{\nabla}(z) d\tilde{t} = 0$, which becomes the linear system

$$\partial_\alpha \xi = z C_\alpha(t) \xi, \qquad \partial_z \xi = \left[\mathcal{U}(t) + \frac{\hat{\mu}}{z} \right] \xi,$$

where $\hat{\mu} := \text{diag}(\mu_1, ..., \mu_n)$, $\mu_\alpha := q_\alpha - \frac{d}{2}$, ξ is a column vector of components $\xi^\alpha = \eta^{\alpha\mu} \partial \tilde{t} / \partial t^\mu$, $\alpha = 1, ..., n$ and $C_\alpha(t) := \left(c_{\alpha\gamma}^\beta(t) \right)$, $\mathcal{U} := \left((1 - q_\mu) t^\mu c_{\mu\gamma}^\beta(t) \right)$ are $n \times n$ matrices (sum over repeated indices is omitted).

We restrict to *semisimple* Frobenius manifolds, namely analytic Frobenius manifolds such that the matrix $\mathcal{U}$ can be diagonalized with distinct eigenvalues on an open dense subset $\mathcal{M} \subset M$. Then, there exists an invertible matrix $\Psi(t)$ such that $\Psi \mathcal{U} \Psi^{-1} = \text{diag}(u_1, ..., u_n) =: U$, $u_i \neq u_j$ for $i \neq j$ on $\mathcal{M}$. The second equation of the above system becomes:

$$\frac{dY}{dz} = \left[U + \frac{V(u)}{z} \right] Y, \qquad u := (u_1, ..., u_n), \quad u_i \in \mathbf{C}, \qquad V = \Psi^{-1} \hat{\mu} \Psi. \tag{2}$$

As it is proved in [16] [17], u_1,...,u_n are local coordinates on $\mathcal{M}$. Locally we obtain a change of coordinates, $t^\alpha = t^\alpha(u)$, then $\Psi = \Psi(u)$, $V = V(u)$. A local chart of $\mathcal{M}$ is reconstructed by parametric formulae:

$$t^\alpha = t^\alpha(u), \qquad F = F(u) \tag{3}$$

where $t^\alpha(u)$, $F(u)$ are certain meromorphic functions of $(u_1, ..., u_n)$, $u_i \neq u_j$, which can be obtained from the coefficients of the system (2).

The dependence of the system on u is *isomonodromic* [29]. This means that the monodromy data of the system, to be introduced below, do not change for a small deformation of u. Therefore, the coefficients of the system in every local chart of $\mathcal{M}$ are naturally labeled by the monodromy data. To calculate the functions (3) in every local chart one has to reconstruct the system (2) from its *monodromy data*. This is the *inverse problem.*

The inverse problem can be formulated as a *Riemann-Hilbert boundary value problem.* It can be proved [40] [35] [17] that if the boundary value problem has solution at $u = u^0$ (such that $u_i^0 \neq u_j^0$) for given monodromy data, then the solution is unique and it defines $V(u)$, $\Psi(u)$ and 3 as analytic functions in a neighborhood of u^0.

Moreover, $V(u)$, $\Psi(u)$ and (3) can be continued analytically as meromorphic functions on the universal covering of $\mathbf{C}^n\backslash$diagonals, where "diagonals" stands for the union of all the sets $\{u \in \mathbf{C}^n \mid u_i = u_j, \;\; i \neq j\}$. Since $(u_1, ..., u_n)$ are local coordinates on $\mathcal{M}$ they are defined up to permutation. Thus, the analytic continuation of the local structure of $\mathcal{M}$ is described by the *braid group*, namely the fundamental group of $(\mathbf{C}^n\backslash\text{diagonals})/\mathcal{S}_n$ ($\mathcal{S}_n$ is the symmetric group of n elements). Since every local chart of the atlas covering the manifold is labelled by monodromy data, then there exists an action of the braid group itself on the monodromy data corresponding to the change of coordinate chart. This action is described in [17] and chapter 1.

In order to understand the global structure of the manifold M we have to study the solution of the inverse problem and (3) when two or more distinct coordinates u_i, u_j, etc, merge. $\Psi(u)$, $V(u)$ and (3) are multi-valued meromorphic functions of $u = (u_1, ..., u_n)$ and the branching occurs when u goes around a loop around the set of diagonals $\bigcup_{ij}\{u \in \mathbf{C}^n \mid u_i = u_j, \;\; i \neq j\}$. $\Psi(u)$, $V(u)$ and (3) have singular behaviour if $u_i \to u_j$ $(i \neq j)$. We call such behavior *critical behaviour.* Although it is impossible to solve the boundary value problem exactly, except for special cases occurring for 2×2 systems, we may hopefully compute the asymptotic/critical behaviour of the solution, using the isomonodromic deformation method [29] [26]. We will face the problem in the first non-trivial case, namely for three dimensional Frobenius manifolds.

Instead of analyzing the boundary value problem directly, we exploit the isomonodromic dependence of the system (2) on u, which implies that the solution of the inverse problem must satisfy the nonlinear equations

$$\frac{\partial V}{\partial u_k} = [V_k, V] \tag{4}$$

$$\frac{\partial \Psi,}{\partial u_k} = V_k \Psi, \qquad k = 1, ..., n \tag{5}$$

where V_k is a $n \times n$ matrix whose entries are $(V_k)_{ij} = \frac{\delta_{ki} - \delta_{kj}}{u_i - u_j} V_{ij}$. The WDVV equations are equivalent to (4) (5). For 3-dimensional Frobenius manifolds, (4) (5) are reduced to a special case of the Painlevé 6 equation [17]:

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right]\frac{dy}{dx}$$

$$+\frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu-1)^2 + \frac{x(x-1)}{(y-x)^2}\right], \qquad \mu \in \mathbf{C}, \qquad x = \frac{u_3 - u_1}{u_2 - u_1} \tag{6}$$

The parameter μ appears in the matrix $\hat{\mu}$ =diag$(\mu, 0, -\mu)$ of (2). In chapter 1 and 6 we show that the entries of $V(u)$ and $\Psi(u)$ are rational functions of x, $y(x)$, $\frac{dy}{dx}$. The critical behaviour of $V(u)$, $\Psi(u)$ and (3) is a consequence of the critical behaviour of the transcendent $y(x)$ close to the *critical points* $x = 0, 1, \infty$.

Let $F_0(t) := \frac{1}{2}\left[(t^1)^2 t^3 + t^1 (t^2)^2\right]$. We prove in chapter 6 that for generic μ the parametric representation (3) becomes

$$t^2(u) = \tau_2(x, \mu)\ (u_2 - u_1)^{1+\mu}, \qquad t^3(u) = \tau_3(x, \mu)\ (u_2 - u_1)^{1+2\mu} \tag{7}$$

$$F(u) = F_0(t) + \mathcal{F}(x,\mu)\ (u_2-u_1)^{3+2\mu}, \qquad x = \frac{u_3-u_1}{u_2-u_1} \tag{8}$$

where $\tau_2(x,\mu), \tau_3(x,\mu), \mathcal{F}(x,\mu)$ are certain rational functions of x, $y(x)$, $\frac{dy}{dx}$ and μ, which we computed explicitly. More details will be given below.

The two integration constants in $y(x)$ – and thus in the corresponding solution of (4) (5) – and the parameter μ are contained in the three entries (x_0, x_1, x_∞) of the *Stokes's matrix*

$$S = \begin{pmatrix} 1 & x_\infty & x_0 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{such that } x_0^2 + x_1^2 + x_\infty^2 - x_0 x_1 x_\infty = 4\sin^2(\pi\mu)$$

of the system (2). The Stokes matrix is part of the *monodromy data* of the system (2). They will be discussed in chapter 1. Here we briefly introduce the Stokes matrix. At $z=\infty$ there is a formal solution of (2) $Y_F = \left[I + \frac{F_1}{z} + \frac{F_2}{z^2} + ...\right]\ e^{z\ U}$ where F_j's are $n\times n$ matrices. It is a well known result that fundamental matrix solutions exist which have Y_F as asymptotic expansion for $z\to\infty$ [2] . Let l be a generic oriented line passing through the origin. Let l_+ be the positive half-line and l_- the negative one. Let $\mathcal{S}_L$ and $\mathcal{S}_R$ be two sectors in the complex plane to the left and to the right of l respectively. There exist unique fundamental matrix solutions Y_L and Y_R having the asymptotic expansion Y_F for $x\to\infty$ in $\mathcal{S}_L$ and $\mathcal{S}_R$ respectively [2]. They are related by a connection matrix S, called *Stokes matrix*, such that $Y_L(z) = Y_R(z)S$ for $z\in l_+$.

A further step is to invert (3) in order to obtain a closed form $F = F(t^1,...,t^n)$. The final purpose of the inversion is to understand the analytic properties of the solution $F(t)$ of the WDVV equations.

The entire procedure described above is an application of the isomonodromic deformation theory to the WDVV equations and is the object of the thesis.

The first step is to properly choose the monodromy data in order to arrive at physically or geometrically interesting Frobenius manifolds. We consider the quantum cohomology of projective spaces as an important example of semisimple Frobenius manifold. For this example we compute the monodromy data.

The *quantum cohomology* of $\mathbf{CP}^d$, denoted by $QH^*(\mathbf{CP}^d)$, is a $(d+1)$-dimensional semisimple Frobenius manifold ([33] [37] and chapter 2 below). It has relations to enumerative geometry. A well known example is the quantum cohomology of $\mathbf{CP}^2$. It corresponds to the solution of the WDVV equations for $n=3$ which generates the numbers N_k of rational curves $\mathbf{CP}^1 \longrightarrow \mathbf{CP}^2$ of degree k passing through $3k-1$ generic points. Namely

$$F(t^1,t^2,t^3) = \frac{1}{2}\left[(t^1)^2t^3 + t^1(t^2)^2\right] + \sum_{k=1}^{\infty} \frac{N_k}{(3k-1)!}(t^3)^{3k-1}e^{kt^2}, \quad \text{for } (t^3)^3e^{t^2}\to 0 \tag{9}$$

The global analytic properties of this function are unknown, except for $(t^3)^3e^{t^2}\to 0$, and the inverse reconstruction of the corresponding Frobenius manifold starting from its monodromy data may shed some light on these properties.

- **Results of the Thesis**

i) The monodromy data of $QH^*(\mathbf{CP}^d)$. In chapter 4 we prove the following

Theorem [chapter 4]: *There exists a chart of $QH^*(\mathbf{CP}^d)$ where the Stokes's matrix $S=(s_{ij})$, $i,j=1,2,...,d+1$, has the canonical form:*

$$s_{ii}=1, \qquad s_{ij} = (-1)^{j-i}\binom{d+1}{j-i}, \qquad s_{ji}=0, \qquad i<j$$

For any other local chart of $QH^*(\mathbf{CP^d})$ the Stokes matrix is obtained from the canonical form by the action of the braid group.

The canonical form is very simple, but it was never computed before, except for $d=2$ [17]. In general, the computation of the Stokes' matrices of a linear system of differential equations is hard, and we can rarely obtain an explicit form. In our case, we need to reduce the system (2) to a linear differential equation of order n and study the Stokes phenomenon for such equation, which is a special case of a generalized confluent hypergeometric equation whose Stokes factors (which will be introduced in chapter 4) were studied in [20].

The main difficulty concerns the reduction of the Stokes matrix to the above canonical form. We start from a system (2) corresponding to a special point $t_0 \in QH^*(\mathbf{CP}^d)$ such that the matrix U has both distinct eigenvalues and a very simple form which makes the computation of S feasible. But the matrix S turns out to be very complicated (see section 4.10). It corresponds to the local chart containing t_0. Hence, we have to move to other charts by the action of the braid group, which is non linear and requires a hard analysis. We could devise the right braid and we obtained the canonical form.

It is to be remarked that the above computation proves, for projective spaces, a long-lasting conjecture about the connections between quantum cohomology and the theory of derived categories of coherent sheaves.

It was conjectured [18] that the Stokes matrix for the quantum cohomology of a good Fano variety X is equal to the Gram matrix of the bilinear form $\chi(E,F) := \sum_k (-1)^k \dim Ext^k(E,F)$ computed on a full collection of exceptional objects in the derived category $Der^b(Coh(X))$ of coherent sheaves on X. More precisely, let $Der^b(Coh(X))$ be the derived category of coherent sheaves on a smooth projective variety X of dimension d. An object E of $Der^b(Coh(X))$ is called *exceptional* if $Ext^i(E,E) = 0$ for $0 < i < d$, $Ext^0(E,E) = \mathbf{C}$ and $Ext^d(E,E)$ is of the smallest dimension (if X is a projective space, then $Ext^d(E,E) = 0$). A collection $\{E_1, ..., E_s\}$ of exceptional objects is an *exceptional collection* if for any $1 \le m < n \le s$ we have $Ext^i(E_n, E_m) = 0$ for any $i \ge 0$, $Ext^i(E_m, E_n) = 0$ for any $i \ge 0$ except possibly for one value of i. A *full exceptional collection* is an exceptional collection which generates $Der^b(Coh(X))$ as a triangulated category. This theory is developed in [46] [47] [8]. We say that a Fano variety is *good* if it has a full exceptional collection.

It is known that $X = \mathbf{CP}^d$ is good, the collection of sheaves on $\mathbf{CP}^d$ $\{\mathcal{O}(n)\}_{n\in\mathbf{Z}}$ is exceptional, and $\{E_1, E_2, ..., E_{d+1}\} := \{\mathcal{O}, \mathcal{O}(1), \ ..., \mathcal{O}(d)\}$ is a full exceptional collection [4], [24]. In this case, $g_{ij} := \chi(\mathcal{O}(i-1), \mathcal{O}(j-1))$, $i,j = 1,2,...,d+1$ has the form:

$$g_{ij} = \binom{d+j-i}{j-i} \qquad i<j$$

$$g_{ii} = 1, \qquad g_{ij} = 0 \quad i>j$$

The inverse to this matrix has entries a_{ij}

$$a_{ij} = (-1)^{j-i}\binom{d+1}{j-i} \qquad i<j$$

$$a_{ii} = 1, \qquad a_{ij} = 0 \quad i>j$$

This matrix is equivalent to the one above with respect to the action of the braid group.

The mentioned conjecture claims that the Stokes matrix of the quantum cohomology of $\mathbf{CP}^d$ is equal to the above Gram matrix (modulo the action of the braid group: remarkably, this action on the Stokes matrix for the Frobenius manifold coincides with the natural action of the braid group on the collections of exceptional objects [55] [46]). The conjecture is proved in chapter 4 by our theorem above.

This conjecture has its origin in the paper by Cecotti and Vafa [9], where another Stokes matrix introduced in [15] for the tt^* equations was found in the case of the $\mathbf{CP}^2$ topological σ model. It was suggested, on physical arguments, that the entries of the Stokes' matrix $S = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$ are integers. They must satisfy a Diophantine equation $x^2 + y^2 + z^2 - xyz = 0$ whose integer solutions (x,y,z) are all equivalent to (3,3,3) modulo the action of the braid group. The authors of [9] also suggested that their matrix must coincide with the Stokes matrix defined in the theory of WDVV equations. We remark that it has not yet been proved that the Stokes' matrix for tt^* equations and the Stokes' matrix for the corresponding Frobenius manifold coincide. This point deserves further investigation.

Later, in [55], the links between $N = 2$ super-symmetric field theories and the theory of derived categories were further investigated and the coincidence of $\chi(E_i, E_j)$ with the Stokes matrix of tt^* for $\mathbf{CP}^d$ was conjectured.

The conjecture may probably be derived from more general conjectures by Kontsevich in the framework of categorical mirror symmetry. To my knowledge, the subject was discussed in [32] (I thank B. Dubrovin for this reference).

In chapter 4 we also study the structure of the monodromy group of $QH^*(\mathbf{CP}^d)$ and we prove that it is related to the hyperbolic triangular groups (the case $d = 2$ was already studied in [17]). The concept of *monodromy group of a Frobenius manifold* is explained in chapter 1.

ii) Once the monodromy data, in particular the Stokes' matrix, are known, we have to solve the inverse problem for the system (2) in order to obtain $V(u)$, $\Psi(u)$ and the parametric representation (3). We face the problem for the first non-trivial case $n = 3$, which is reduced to the Painlevé 6 equation. Therefore, we devote chapter 5 to the investigation of the behaviour of $y(x)$ close to the *critical points* $x = 0, 1, \infty$ and to the connection problem between the parameter governing that behaviour at different critical points. Moreover we give the explicit dependence of the parameters on the entries (x_0, x_1, x_∞) of the Stokes' matrix.

The classical Painlevé equation was discovered by Painlevé [43] and Gambier [23], who classified all the second order ordinary differential equations of the type

$$\frac{d^2y}{dx^2} = \mathcal{R}\left(x, y, \frac{dy}{dx}\right)$$

where $\mathcal{R}$ is rational in $\frac{dy}{dx}$, meromorphic in x and y. The Painlevé 6 equation satisfies the *Painlevé property* of absence of movable critical singularities. The general solution can be analytically continued to a meromorphic function on the universal covering of $\mathbf{CP}^1 \backslash \{0, 1, \infty\}$. For generic values of the integration constants and of the parameters in the equation, the solution can not be expressed via elementary or classical transcendental functions. For this reason, the solution is called a *Painlevé transcendent.*

The connection problem for a class of solutions to the Painlevé 6 equation was solved by Jimbo [28] for the general Painlevé equation with generic values of its coefficients α, β, γ δ (in standard notation of [26]), using the isomonodromic deformation theory developed in [29] [30]. Later, Dubrovin-Mazzocco [21] applied Jimbo's procedure to (6), with the restriction $2\mu \not\in \mathbf{Z}$. The connection problem was solved by the authors above for the class of transcendents having the following local behaviour at the critical points $x = 0, 1, \infty$:

$$y(x) = a^{(0)} x^{1-\sigma^{(0)}} (1 + O(|x|^\delta)), \qquad x \to 0, \tag{10}$$

$$y(x) = 1 - a^{(1)} (1-x)^{1-\sigma^{(1)}} (1 + O(|1-x|^\delta)), \qquad x \to 1, \tag{11}$$

$$y(x) = a^{(\infty)} x^{-\sigma^{(\infty)}} (1 + O(|x|^{-\delta})), \qquad x \to \infty, \tag{12}$$

where δ is a small positive number, $a^{(i)}$ and $\sigma^{(i)}$ are complex numbers such that $a^{(i)} \neq 0$ and

$$0 \leq \Re\sigma^{(i)} < 1.$$

This behaviour is true if x converges to the critical points inside a sector with vertex on the corresponding critical point, along a radial direction in the x-plane. The *connection problem*, i.e. the problem of finding the relation among the three pairs $(\sigma^{(i)}, a^{(i)})$, $i = 0, 1, \infty$, was solved thanks to the link between the Painleve' equation and a Fuchsian system of differential equations

$$\frac{dY}{dz} = \left[\frac{A_0(x)}{z} + \frac{A_x(x)}{z-x} + \frac{A_1(x)}{z-1}\right] Y$$

where the 2×2 matrices $A_i(x)$ ($i = 0, x, 1$ are labels) have isomonodromic dependence on x and satisfy Schlesinger equations. The local behaviours (10), (11), (12) were proved using a result on the asymptotic behaviour of a class of solutions of Schlesinger equations proved by Sato, Miwa, Jimbo in [50]. The connection problem was solved because the parameters $\sigma^{(i)}$, $a^{(i)}$ were expressed as functions of the monodromy data of the fuchsian system associated to (6). For studies on the asymptotic behaviour of the coefficients of Fuchsian systems and Schlesinger equations see also [7].

The monodromy data of the Fuchsian system turn out to be expressed in terms of the triple (x_0, x_1, x_∞) of entries of the Stokes matrix [21]. There exists a one-to-one correspondence between triples and branches of the Painlevé transcendents.[1] In other words, any branch $y(x)$ is parametrized by a triple, namely $y(x) = y(x; x_0, x_1, x_\infty)$. As it is proved in [21], the transcendents (10), (11), (12) are parametrized by a triple according to the formulae

$$x_i^2 = 4\sin^2\left(\frac{\pi}{2}\sigma^{(i)}\right), \quad i = 0, 1, \infty, \qquad 0 \leq \Re\sigma^{(i)} < 1.$$

A more complicated expression gives $a^{(i)} = a^{(i)}(x_0, x_1, x_\infty)$.

Due to the restriction $0 \leq \Re\sigma^{(i)} < 1$, the formulae of Dubrovin-Mazzocco do not work if at least one x_i $(i = 0, 1, \infty)$ is real and $|x_i| \geq 2$. This is the case of $QH^*(\mathbf{CP}^2)$, because $(x_0, x_1, x_\infty) = (3, 3, 3)$. To overcome this limitation, in chapter 5 we find the critical behaviour and we solve the connection problem for all the triples satisfying

$$x_i \neq \pm 2 \implies \sigma^{(i)} \neq 1, \quad i = 0, 1, \infty$$

The method used is an extension of Jimbo and Dubrovin-Mazzocco's method and relies on the isomonodromy deformation theory. We prove that

Theorem 1 [chapter 5]: *Let $\mu \neq 0$. For any $\sigma^{(0)} \notin (-\infty, 0) \cup [1, +\infty)$, for any $a^{(0)} \in \mathbf{C}$, $a \neq 0$, for any $\theta_1, \theta_2 \in \mathbf{R}$ and for any $0 < \tilde{\sigma} < 1$, there exists a sufficiently small positive $\epsilon^{(0)}$ such that the equation (6) has a solution $y(x; \sigma^{(0)}, a^{(0)})$ with the behaviour*

$$y(x; \sigma^{(0)}, a^{(0)}) = a^{(0)} x^{1-\sigma^{(0)}} \left(1 + O(|x|^\delta)\right) \quad , 0 < \delta < 1,$$

as $x \to 0$ in the domain $D(\sigma^{(0)}) = D(\epsilon^{(0)}; \sigma^{(0)}; \theta_1, \theta_2, \tilde{\sigma})$ defined by

$$|x| < \epsilon^{(0)}, \qquad \Re\sigma^{(0)} \log|x| + \theta_2 \Im\sigma^{(0)} \leq \Im\sigma^{(0)} \arg(x) \leq (\Re\sigma^{(0)} - \tilde{\sigma}) \log|x| + \theta_1 \Im\sigma^{(0)}$$

For $\sigma^{(0)} = 0$ the domain is simply $|x| < \epsilon^{(0)}$.

We note that $\epsilon^{(0)}$ depends on the choice of θ_1 and a. The critical behaviour in theorem 1 coincides with (10) for $0 \leq \Re\sigma^{(0)} < 1$, but for $\Re\sigma^{(0)} < 0$ and $\Re\sigma^{(0)} \geq 1$ it holds true if $x \to 0$ along a spiral, according to the shape of $D(\epsilon^{(0)}; \sigma^{(0)}; \theta_1, \theta_2, \tilde{\sigma})$. Instead, the behaviour when $x \to 0$ along a radial path may be more complicated and no indication is given by the theorem.

By symmetries of (6), we also prove the existence of solutions with local behaviour at $x = 1$

$$y(x, \sigma^{(1)}, a^{(1)}) = 1 - a^{(1)}(1-x)^{1-\sigma^{(1)}}(1 + O(|1-x|^\delta)) \quad x \to 1$$

$$a^{(1)} \neq 0, \quad \sigma^{(1)} \notin (-\infty, 0) \cup [1, +\infty)$$

and

$$y(x; \sigma^{(\infty)}, a^{(\infty)}) = a^{(\infty)} x^{\sigma^{(\infty)}} \left(1 + O(\frac{1}{|x|^\delta})\right) \quad \tilde{x} \to \infty$$

$$a^{(\infty)} \neq 0, \quad \sigma^{(\infty)} \notin (-\infty, 0) \cup [1, +\infty)$$

in suitable domains $D(\sigma^{(1)})$, $D(\sigma^{(\infty)})$ which will be described in chapter (5).

We also prove that the analytic continuation of a branch $y(x; x_0, x_1, x_\infty)$ to the domains of theorem 1 is governed by parameters $\sigma^{(i)}$, $a^{(i)}$ given by the following

Theorem 2 [chapter 5]: *For any set of monodromy data (x_0, x_1, x_∞) such that $x_0^2 + x_1^2 + x_\infty^2 - x_0 x_1 x_\infty = 4\sin^2(\pi\mu)$ and $x_i \neq \pm 2$ there exist a unique solution $y(x; \sigma^{(i)}, a^{(i)})$ in $D(\sigma^{(i)})$ with parameters $\sigma^{(i)}$ and $a^{(i)}$ obtained as follows:*

$$x_i^2 = 4\sin^2\left(\frac{\pi}{2}\sigma^{(i)}\right), \qquad \sigma^{(i)} \in \mathbf{C} \backslash \{(-\infty, 0) \cup [1, +\infty)\}$$

$$a^{(0)} = \frac{iG(\sigma^{(0)}, \mu)^2}{2\sin(\pi\sigma^{(0)})} \left[2(1 + e^{-i\pi\sigma^{(0)}}) - f(x_0, x_1, x_\infty)(x_\infty^2 + e^{-i\pi\sigma^{(0)}} x_1^2)\right] f(x_0, x_1, x_\infty)$$

[1]There are only some exceptions to the one-to-one correspondence above, which are already treated in [37]. In order to rule them out we require that at most one of the entries x_i of the triple may be zero and that $(x_0, x_1, x_\infty) \notin \{(2, 2, 2)$ $(-2, -2, 2)$, $(2, -2, -2)$, $(-2, 2, -2)\}$. See [37].

where

$$f(x_0,x_1,x_\infty) := \frac{4-x_0^2}{2-x_0^2-2\cos(2\pi\mu)}, \qquad G(\sigma^{(0)},\mu) = \frac{1}{2}\frac{4^{\sigma^{(0)}}\Gamma(\frac{\sigma^{(0)}+1}{2})^2}{\Gamma(1-\mu+\frac{\sigma^{(0)}}{2})\Gamma(\mu+\frac{\sigma^{(0)}}{2})}$$

The parameters $a^{(1)}$, $a^{(\infty)}$ are obtained like $a^{(0)}$, provided that we do the substitutions $(x_0,x_1,x_\infty) \mapsto (x_1,x_0,x_0x_1-x_\infty)$ and $(x_0,x_1,x_\infty) \mapsto (x_\infty,-x_1,x_0-x_1x_\infty)$ respectively (and $\sigma^{(0)} \mapsto \sigma^{(1)}$ and $\sigma^{(0)} \mapsto \sigma^{(\infty)}$ respectively) in the above formulae.

We remark that the theorem is actually a bit more complicated, we need to distinguish some subcases and to be careful about the definition of the branch cuts for $y(x;x_0,x_1,x_\infty)$; we refer to chapter 5 for details.

The *connection problem* for the transcendents $y(x;\sigma^{(i)},a^{(i)})$ is now solved, because we are able to compute $(\sigma^{(i)},a^{(i)})$ for $i=0,1,\infty$ in terms of a fixed triple (x_0,x_1,x_∞).

We also discuss the problem of the analytic continuation of the branch $y(x;x_0,x_1,x_\infty)$, namely we discuss how (x_0,x_1,x_∞) change when x describes loops around $x=0,1,\infty$.

The above theorem implies that we can always restrict to the case $0 \le \Re\sigma^{(i)} \le 1$, $\sigma^{(i)} \ne 1$, so the critical behaviours $y(x;\sigma^{(i)},a^{(i)})$ coincide with (10), (11), (12), except for the case $\Re\sigma^{(i)}=1$, where the critical behaviour holds true only if x converges to a critical point along a spiral.

We use the elliptic representation of the transcendents in order to investigate this last case and its critical behaviour along radial paths. We can restrict here to $x=0$ because the symmetries of (6) yield the behaviour close to the other critical points. The elliptic representation was introduced by R.Fuchs in [22]:

$$y(x) = \wp\left(\frac{u(x)}{2};\omega_1(x),\omega_2(x)\right)$$

Here $u(x)$ solves a non-linear second order differential equation and $\omega_1(x)$, $\omega_2(x)$ are two elliptic integrals, expanded for $|x|<1$ in terms of hypergeometric functions:

$$\omega_1(x) = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left[(\frac{1}{2})_n\right]^2}{(n!)^2}x^n$$

$$\omega_2(x) = -\frac{i}{2}\left\{\sum_{n=0}^{\infty}\frac{\left[(\frac{1}{2})_n\right]^2}{(n!)^2}x^n\ \ln(x) + \sum_{n=0}^{\infty}\frac{\left[(\frac{1}{2})_n\right]^2}{(n!)^2}2\left[\psi(n+\frac{1}{2})-\psi(n+1)\right]x^n\right\}$$

where $\psi(z) := \frac{d}{dz}\ln\Gamma(z)$.

We study the critical behaviour implied by this representation. We show that the representation also provides the critical behaviour along radial paths for $\Re\sigma^{(i)}=1$. More precisely we prove the following

Theorem 3 [chapter 5]: *For any complex ν_1, ν_2 such that*

$$\nu_2 \notin (-\infty,0]\cup[2,+\infty)$$

there exists a sufficiently small r such that

$$y(x) = \wp(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x);\omega_1(x),\omega_2(x))$$

in the domain $\mathcal{D}(r;\nu_1,\nu_2)$ defined as

$$|x|<r, \qquad \Re\nu_2\ln|x| + C_1 - \ln r < \Im\nu_2\arg x < (\Re\nu_2-2)\ln|x| + C_2 + \ln r,$$

$$C_1 := -\left[4\ln 2\Re\nu_2 + \pi\Im\nu_1\right], \qquad C_2 := C_1 + 8\ln 2.$$

The function $v(x)$ is holomorphic in $\mathcal{D}(r;\nu_1,\nu_2)$ and has convergent expansion

$$v(x) = \sum_{n\ge1}a_nx^n + \sum_{n\ge0,\ m\ge1}b_{nm}x^n\left(\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right)^m + \sum_{n\ge0,\ m\ge1}c_{nm}x^n\left(\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right)^m$$

where a_n, b_{nm}, c_{nm} are certain rational functions of ν_2. Moreover, there exists a constant $M(\nu_2)$ depending on ν_2 such that $v(x) \le M(\nu_2)\left(|x| + \left|\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right| + \left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right|\right)$ in $\mathcal{D}(r;\nu_1,\nu_2)$.

We note that for $\mu = \frac{1}{2}$ the function $v(x)$ vanishes, and we obtain *Piccard solutions* [44] whose critical behaviour is studied in [37].

The transcendent of theorem 3 coincides with $y(x;\sigma^{(0)},a^{(0)})$ of theorem 1 on the domain $D(\epsilon^{(0)},\sigma^{(0)})\cap \mathcal{D}(r;\nu_1,\nu_2)$ with critical behaviour specified by $\sigma^{(0)} = 1-\nu_2$ and $a^{(0)} = -\frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]$. The identification of $a^{(0)}$ and $\sigma^{(0)}$ makes it possible to connect ν_1 and ν_2 to the monodromy data (x_0,x_1,x_∞) according to theorem 2.

On the other hand, we will prove that the behaviour implied by the elliptic representation is oscillatory along paths contained in $\mathcal{D}(r;\nu_1,\nu_2)$ which are parallel to the boundaries of the domain in the $(\ln|x|,\arg(x))$ plane, namely $\Im\nu_2 \arg x = (\Re\nu_2-2)\ln|x|+C_2+\ln r$ and $\Im\nu_2\arg x = \Re\nu_2\ln|x|+C_1-\ln r$. This follows from the Fourier expansion of the Weierstrass elliptic function which will be discussed in section 5.4.

In particular the case $\Re\nu_2 = 0$ ($\nu_2 \neq 0$) coincides with $\Re\sigma^{(0)} = 1$ ($\sigma^{(0)} \neq 1$) and the paths parallel to the boundary $\Im\nu_2 \arg x = C_1 - \ln r$ are radial paths. As a consequence of theorem 3, the critical behaviour along a radial path (equivalently, inside a sector) is

$$y(x) = O(x) + \frac{1+O(x)}{\sin^2\left(\frac{\nu}{2}\ln x - \nu\ln 16 + \frac{\pi\nu_1}{2} + \sum_{m=1}^{\infty} c_{om}(\nu)\left[\left(\frac{e^{i\pi\nu_1}}{e^{i\nu}}\right)x^{i\nu}\right]^m\right)}, \quad x\to 0. \tag{13}$$

The number ν is real, $\nu\neq 0$ and $\sigma^{(0)} = 1-i\nu$. The series $\sum_{m=1}^{\infty} c_{om}(\nu)\left[\left(\frac{e^{i\pi\nu_1}}{e^{i\nu}}\right)x^{i\nu}\right]^m$ converges and defines a holomorphic and bounded function in the domain $\mathcal{D}(r;\nu_1,i\nu)$

$$|x|<r, \qquad C_1 - \ln r < \nu\arg x < -2\ln|x| + C_2 + \ln r$$

Note that not all the values of $\arg x$ are allowed, namely $C_1 - \ln r < \nu\arg(x)$. Our belief is that if we extend the range of $\arg x$, then $y(x)$ may have (movable) poles. We are not able to prove it in general, but we will produce an example in section 5.4.

We finally remark that the critical behaviour of Painlevé transcendents can also be investigated using a representation due to S.Shimomura [52] [27]

$$y(x) = \frac{1}{\cosh^2\left(\frac{u_s(x)}{2}\right)}$$

where $u_s(x)$ solves a non linear differential equation of the second order. We will discuss it in chapter 5. However, the connection problem in this representation was not solved.

In the thesis we give an extended and unified picture of both elliptic and Shimomura's representations and Dubrovin-Mazzocco's works, and we solve the connection problem for elliptic and Shimomura's representations.

iii) In Chapter 6 we apply the results on the critical behaviour of Painlevé transcendents to obtain the behaviour of the parametric representation (3) for $n=3$.

Let $F_0(t) := \frac{1}{2}\left[(t^1)^2t^3 + t^1(t^2)^2\right]$. We prove that for generic μ the parametric representation is

$$t^2(u) = \tau_2(x,\mu)\ (u_2-u_1)^{1+\mu}, \qquad t^3(u) = \tau_3(x,\mu)\ (u_2-u_1)^{1+2\mu} \tag{14}$$

$$F(u) = F_0(t) + \mathcal{F}(x,\mu)\ (u_2-u_1)^{3+2\mu}, \qquad x = \frac{u_3-u_1}{u_2-u_1} \tag{15}$$

where $\tau_2(x,\mu),\tau_3(x,\mu),\mathcal{F}(x,\mu)$ will be computed explicitly as rational functions of x, $y(x)$, $\frac{dy}{dx}$ and μ. The ratio $\frac{t^2}{(t^3)^{\frac{1+\mu}{1+2\mu}}}$ is independent of (u_2-u_1). This is actually the crucial point, because now the closed form $F = F(t)$ must be:

$$F(t) = F_0(t) + (t^3)^{\frac{3+2\mu}{1+2\mu}}\varphi\left(\frac{t^2}{(t^3)^{\frac{1+\mu}{1+2\mu}}}\right)$$

where the function φ has to be determined by the inversion of (14) (15).

In the case of $QH^*(\mathbf{CP}^2)$ we prove that we just need to take the limit of the above t^3 and F for $\mu\to -1$, but such a limit does not exist for t^2 and the correct form is

$$t^2(u) = 3\ln(u_2-u_1) + 3\int^x d\zeta \frac{1}{\zeta + f(\zeta)} \tag{16}$$

where $f(x)$ is again computed explicitly as a rational functions of x, $y(x)$, $\frac{dy}{dx}$. This time $e^{t^2}(t^3)^3$ is independent of $(u_2 - u_1)$ and so

$$F(t) = F_0(t) + \frac{1}{t^3}\,\varphi\left(e^{t^2}(t^3)^3\right)$$

To our knowledge, this is the first time the *explicit* parameterization (14) (15) (16) is given; although its proof is mainly a computational problem (the theoretical problem being already solved by the reduction to the Painlevé 6 eq. [16]), it is very hard. Moreover, the knowledge of this explicit form is necessary to proceed to the inversion of the parametric formulae close to the diagonals.

When the transcendent behaves like (10), or (11), or (12) with rational exponents, then t and F in (14) (15) are expanded in Puiseux series in x, $1-x$ of $\frac{1}{x}$. The expansion can be inverted, in order to obtain $F = F(t)$ in closed form as an expansion in t. We apply the procedure staring from the algebraic solutions [21] of (6) and we obtain the polynomial solutions of the WDVV equations.

We also apply the procedure for $QH^*(\mathbf{CP}^2)$. This time, $\Re\sigma^{(i)} = 1$, the transcendent has oscillatory behaviour and therefore the reduction of (16) (14) (15) to closed form is hard. Hence, we expand the transcendent in Taylor series close to a regular point x_{reg}, we plug it into (16) (14) (15) and we obtain t and F as a Taylor series in $(x - x_{\text{reg}})$. We invert the series and we get a closed form $F = F(t)$. We prove that it is precisely the solution (9). Thus, our procedure is an alternative way to compute the numbers N_k as an application of the isomonodromic deformations theory.

We have just started to investigate the possibilities offered by the formulae (14) (15) (16). We believe they will be a good tool to understand some analyticity properties of $F(t)$ in future investigations. Particularly, we hope to better understand the connection between the monodromy data of the quantum cohomology and the number of rational curves. This problem will be the object of further investigations.

The entire procedure at points i), ii), iii) is a significant application of the theory of isomonodromic deformations to a problem of mathematical physics: solve the WDVV equations or, at least, investigate the analytic properties of $F(t)$.

The thesis is organized as follows. Chapter 1 is a review on Frobenius manifolds where we discuss in detail the parameterization of the manifold through monodromy data and the reduction to a Painlevé 6 equation. This chapter is mainly a synthesis of [16] [17].

In chapter 2 we introduce the quantum cohomology of projective spaces and its connections to enumerative geometry. We also propose a numerical computation which we did in order to investigate the nature of the singular point which determines the radius of convergence of the solution (9).

Chapter 3 is a didactic exposition of the reconstruction of a 2-dimensional Frobenius manifold starting from the isomonodromic deformation of the associated linear system. The 2-dim case is exactly solvable, but it is a good model for the general procedure in any dimension.

Chapters 4, 5, 6 contain the main results i), ii), iii) of the thesis.

We would like to spend a word to explain the nature of this thesis. Although many objects studied here come from enumerative and algebraic geometry, we never make use of tools from those fields. We only require some knowledge of differential geometry and elementary topology. All the work is mainly analytical. We use concepts and tools and we face problems of complex analysis, asymptotic expansions, theory of linear systems of differential equations, isomonodromic deformations theory, Riemann-Hilbert boundary value problems, Painlevé equations.

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Contents

Chapter 1

Introduction to Frobenius Manifolds

This chapter is a review on the theory of Frobenius manifolds. The connection between Frobenius manifolds and the theory of isomonodromic deformations will be studied, in view of the inverse reconstruction of a Frobenius structure starting from a set of monodromy data of a system of linear differential equations.

1.1 The WDVV Equations of Associativity

The theory of Frobenius manifolds was introduced by B. Dubrovin in [14] for the Witten-Dijkgraaf-Verlinde-Verlinde equations of associativity (WDVV) [54] [13]. The WDVV equations are differential equations satisfied by the *primary free energy* $F(t)$ of a family of two-dimensional topological field theories. $F(t)$ is a function of the coupling constants $t := (t^1, t^2, ..., t^n)$ $t^i \in \mathbf{C}$. Given a non-degenerate symmetric matrix $\eta^{\alpha\beta}$, $\alpha, \beta = 1, ..., n$, $F(t)$ satisfies:

$$\partial_\alpha\partial_\beta\partial_\lambda F \; \eta^{\lambda\mu} \; \partial_\mu\partial_\gamma\partial_\delta F \;=\; \text{the same with } \alpha,\, \delta \text{ exchanged}, \tag{1.1}$$

where $\partial_\alpha := \frac{\partial}{\partial t^\alpha}$. Sum over repeated indices is omitted. The relation between $\eta^{\alpha\beta}$ and $F(t)$ is given by

$$\partial_1\partial_\alpha\partial_\beta F = \eta_{\alpha\beta}, \tag{1.2}$$

where the matrix $(\eta_{\alpha\beta})$ is the inverse of the matrix $(\eta^{\alpha\beta})$. Finally, $F(t)$ must satisfy the *quasi-homogeneity condition*: given numbers $q_1, q_2, ..., q_n, d, r_1, ..., r_n$ ($r_\alpha = 0$ if $q_\alpha \neq 1$) we require:

$$E(F(t)) = (3-d)F(t) + \text{ (at most) quadratic terms}, \tag{1.3}$$

where $E(F(t))$ means a differential operator E applied to $F(t)$ and defined as follows:

$$E = \sum_{\alpha=1}^{n} E^\alpha \partial_\alpha, \qquad E^\alpha = (1-q_\alpha)t^\alpha + r_\alpha, \qquad \alpha = 1, ..., n,$$

It is called *Euler vector field.* The equation (1.3) is the differential form of $F(\lambda^{1-q_1}t^1, ..., \lambda^{1-q_n}t^n) = \lambda^{3-d}F(t^1, ...t^n)$, where $\lambda \neq 0$ is any complex number. If $q_\alpha = 1$ we must read $r_\alpha \ln\lambda + t^\alpha$ instead of $\lambda^{1-q_\alpha}t^\alpha$ in the argument of F.

The equations (1.1), (1.2), (1.3) are the *WDVV equations.*

If we define $c_{\alpha\beta\gamma}(t) := \partial_\alpha\partial_\beta\partial_\gamma F(t)$, $c^\gamma_{\alpha\beta}(t) := \eta^{\gamma\mu}c_{\alpha\beta\mu}(t)$ (sum omitted), and we consider a vector space A=span$(e_1, ..., e_n)$, then we obtain a family of algebras A_t with the multiplication $e_\alpha \cdot e_\beta := c^\gamma_{\alpha\beta}(t)e_\gamma$. The algebra is *commutative* by definition of $c^\gamma_{\alpha\beta}$. Equation (1.1) is equivalent to *associativity.* The vector e_1 is the *unit element* because (1.2) implies $c^\gamma_{1\beta} = \delta^\gamma_\beta$. The bilinear form $< .,. >$ defined by

$$< e_\alpha, e_\beta > = \eta_{\alpha\beta}$$

is symmetric, non degenerate and *invariant*, namely $< e_\alpha \cdot e_\beta, e_\gamma > = < e_\alpha, e_\beta \cdot e_\gamma >$.

The equation (1.2) is integrated and yields:

$$F(t)=\frac{1}{6}\eta_{11}(t^1)^3+\frac{1}{2}[\sum_{\alpha\neq 1}\eta_{1\alpha}t^\alpha](t^1)^2+\frac{1}{2}[\sum_{\alpha\neq 1,\beta\neq 1}\eta_{\alpha\beta}t^\alpha t^\beta]\ t^1+f(t^2,...,t^n)$$

The function $f(t^2,...,t^n)$ is so far arbitrary and it is defined *up to quadratic and linear terms* which do not affect the third derivatives of $F(t)$.

By a linear change of coordinates $t'^\alpha = A^\alpha_\beta t^\beta$ which does not modify (1.1), (1.2) and (1.3) we can reduce $\eta := (\eta_{\alpha\beta})$ to the form:

$$\eta=\begin{pmatrix}\eta_{11} & 0 & 0 & & 0 & 0\\ 0 & & & & & 1\\ 0 & & & & 1 & \\ \vdots & & & \cdot^{\cdot^{\cdot}} & & \\ 0 & & 1 & & & \\ 0 & 1 & & & & \end{pmatrix},\qquad \text{if } \eta_{11}\neq 0$$

$$\eta=\begin{pmatrix} & & & 1\\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}\qquad \text{if } \eta_{11}=0$$

All the other entries are zero. If $n\geq 2$, we are going to deal with Frobemius manifolds such that

$$q_1=0,\qquad ,q_\alpha+q_{n-\alpha+1}=d,\quad \eta=\begin{pmatrix} & & & 1\\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}$$

The condition $q_1=0$ is not very restrictive, because if $q_1\neq 1$ we can always reduce to $q_1=0$ by rescaling all the q_α's and d in the quasi-homogeneity condition. The free energy in this case is

$$F(t)=\frac{1}{2}(t^1)^2t^n+\frac{1}{2}t^1\sum_{\alpha=2}^{n-1}t^\alpha t^{n-\alpha+1}\ +f(t^2,...,t^n)$$

1.1.1 Examples

• $n=1$

$$F(t)=\frac{1}{6}\eta\ (t)^3+A\ (t)^2+B\ t+C$$

• $n=2$, $\eta_{11}=0$

$$F(t^1,t^2)=\frac{1}{2}(t^1)^2t^2+f(t^2)$$

(1.1) is automatically satisfied. (1.3) is:

- If $d\neq 1$

$$(1-d)t^2\frac{df}{dt^2}=(3-d)f\ \Rightarrow\ f(t^2)=C\ (t^2)^{\frac{3-d}{1-d}}$$

where C is a constant.

- If $d=1$

$$r\frac{df}{dt^2}=2f\ \Rightarrow\ f(t^2)=C\ \left[e^{t^2}\right]^{\frac{2}{r}}$$

- If $d=1$, $r=0$, then $f=0$.

- For $d = -1$, there is also the solution

$$f(t^2) = C(t^2)^2 \ln(t^2)$$

- For $d = 3$,there is also the solution

$$f(t^2) = C \ln(t^2)$$

• $n = 3$, $\eta_{11} = 0$

$$F(t^1, t^2, t^3) = F_0(t^1, t^2, t^3) + f(t^2, t^3), \quad \text{where } F_0(t^1, t^2, t^3) = \frac{1}{2}\left[(t^1)^2 t^3 + t^1 (t^2)^2\right]$$

(1.1) becomes

$$f_{222} f_{233} + f_{333} = (f_{223})^2 \tag{1.4}$$

where the subscripts mean derivatives w.r.t. t^2 and t^3. The charges are:

$$q_1 = 0, \quad q_2 = \frac{d}{2}, \quad q_3 = d$$

and (1.3) is translated into the following quasi-homogeneity conditions:

- If $d \neq 1, 2$

$$f(\lambda^{1-\frac{d}{2}} t^2, \lambda^{1-d} t^3) = \lambda^{3-d} f(t^2, t^3)$$

thus

$$f(t^2, t^3) = (t^2)^{\frac{3-d}{1-d/2}} \varphi(t^2 (t^3)^q), \quad q = \frac{1 - d/2}{d - 1}$$

where φ is an function of $t^2(t^3)^q$ such that (1.4) is satisfied .

- If $d = 2$

$$f(t^2 + r_2 \ln\lambda, \frac{1}{\lambda} t^3) = \lambda f(t^2, t^3)$$

which implies

$$f(t^2, t^3) = \frac{1}{t^3} \varphi((t^3)^{r_2} e^{t^2})$$

For $r = 3$ this is the case of the *Quantum Cohomology* of the projective space CP^2 (see chapter 2).

- If $d = 1$

$$f(\lambda^{\frac{1}{2}} t^2, t^3 + r_3 \ln\lambda) = \lambda^2 f(t^2, t^3)$$

thus

$$f(t^2, t^3) = (t^2)^4 \varphi((t^2)^{-2r_3} e^{t^3})$$

In all the above cases, the equation (1.4) becomes an O.D.E. for φ. In particular, there are four polynomial solutions. Let $a \in \mathbf{C}\backslash\{0\}$:

$$F(t) = F_0(t) + a\,(t^2)^2 (t^3)^2 + \frac{4}{15} a^2\,(t^3)^5, \quad d = \frac{1}{2}, \tag{1.5}$$

$$F(t) = F_0(t) + a\,(t^2)^3 t^3 + 6a^2\,(t^2)^2 (t^3)^3 + \frac{216}{35} a^4\,(t^3)^7, \quad d = \frac{2}{3}, \tag{1.6}$$

$$F(t) = F_0(t) + a\,(t^2)^3 (t^3)^2 + \frac{9}{5} a^2\,(t^2)^2 (t^3)^5 + \frac{18}{55} a^4\,(t^3)^{11}, \quad d = \frac{4}{5}, \tag{1.7}$$

$$F(t) = F_0(t) + a\,(t^2)^4, \quad d = 1.$$

Note that we can fix a (it is an integration constant). This means that each of the above solutions is considered as *one* solution and not as a one-parameter family.

For any positive integer m there are analytic solutions at $t = 0$ consisting in a one-parameter family if $d = 2\frac{m+1}{m+2}$, one solution if $d = 2\frac{m+2}{m+4}$, one solution if $d = 2\frac{m+3}{m+6}$. Note that $d \neq 1, 2$.

For $d = 1$ there are three analytic solutions at $t = 0$ (plus the polynomial solution above). We give them for a fixed value of the integration constant a:

$$F(t) = F_0(t) - \frac{1}{24}(t^2)^4 + t^2 e^{t^3}, \quad r_3 = \frac{3}{2},$$

$$F(t) = F_0(t) + \frac{1}{2}(t^2)^4 + (t^2)^2 e^{t^3} - \frac{1}{48}e^{3t^3}, \quad r_3 = 1,$$

$$F(t) = F_0(t) - \frac{1}{72}(t^2)^4 + \frac{2}{3}(t^2)^3 e^{t^3} + \frac{2}{3}(t^2)^2 e^{2t^3} + \frac{9}{16}e^{4t^3}, \quad r_3 = \frac{1}{2}$$

For $r_3 = 0$ we also have

$$F(t) = F_0(t) - \frac{(t^2)^4}{16}\gamma(t^3)$$

where $\gamma(\zeta)$ satisfies

$$\gamma''' = 6\gamma\gamma'' - 9(\gamma')^2, \quad (\text{here}\gamma' := d\gamma(\zeta)/d\zeta).$$

This is the *Chazy equation* and it has a solution analytic at $\zeta = i\infty$:

$$\gamma(\zeta) = \sum_{n \geq 0} a_n e^{2\pi i n \zeta}$$

The Chazy equation determines the a_n's.

For $d = 2$, $r_2 = 3$ there is a solution analytic at $t^2 = -\infty$, $t^3 = 0$, discovered by Kontsevich [33]:

$$F(t) = F_0(t) + \frac{1}{t^3}\sum_{k \geq 0} A_k \; (t^3)^{3k} e^{kt^2}$$

The A_k are uniquely determined (once the integration constant A_1 is chosen) and the series converges around $(t^3)^3 e^{t^2} = 0$.

We will return later to some of the above solutions.

1.2 Frobenius Manifolds

Let's consider a smooth/analytic manifold M of dimension n over $\mathbf{C}$, whose tangent space T_tM at any $t \in M$ is an *associative, commutative algebra* with *unit element* e, equipped with a non-degenerate bilinear form $< .,. >$. Let denote by $\cdot$ the product of two vectors; the bilinear form is *invariant* w.r.t. the product, namely $< u \cdot v, w > = < u, v \cdot w >$ for any $u, v, w \in T_tM$. T_tM is called a *Frobenius algebra* (the name comes from a similar structure studied by Frobenius in group theory).

We further suppose that

1) $< .,. >$ is *flat*. Therefore, there exist flat coordinates $t^1,...,t^n$ such that

$$< \partial_\alpha, \partial_\beta > =: \eta_{\alpha\beta} \quad \text{constant}$$

where $\partial_\alpha = \frac{\partial}{\partial t^\alpha}$ is a basis. We denote by ∇ the Levi-Civita connection. In particular $\nabla_\alpha = \partial_\alpha$.

2) $\nabla e = 0$. So we can choose $e = \partial_1$.

3) the tensors $c(u, v, w) := < u \cdot v, w >$ and $\nabla_y c(u, v, w)$, $u, v, w, y \in T_tM$, are symmetric.

We define $c_{\alpha\beta\gamma}(t) := < \partial_\alpha \cdot \partial_\beta, \partial_\gamma >$; the symmetry becomes the complete symmetry of $\partial_\delta c_{\alpha\beta\gamma}(t)$ in the indices. This implies the existence of a function $F(t)$ such that $\partial_\alpha\partial_\beta\partial_\gamma F(t) = c_{\alpha\beta\gamma}(t)$. F satisfies the equation (1.1) because of the associativity of the algebra T_tM.

The equation (1.2) follows from the axiom $\nabla e = 0$ and the choice $e = \partial_1$.

4) There exist an *Euler vector field* E such that

i) $\nabla\nabla E = 0$

ii) $\text{Lie}_E \; c = c$

$$\text{iii)}\quad \mathrm{Lie}_E\ e = -e$$

$$\text{iv)}\quad \mathrm{Lie}_E\ <.,.>= (2-d) <.,.>$$

i) implies that $E^\gamma = \sum_\beta a^\gamma_\beta t^\beta + r_\beta$ in flat coordinates. Assuming *diagonalizability* of ∇E we reduce to

$$E = \sum_{\alpha=1}^{n} \left((1-q_\alpha)t^\alpha + r_\alpha\right)\partial_\alpha$$

iii) implies $q_1 = 0$; iv) implies $(q_\alpha + q_\beta - d)\eta_{\alpha\beta} = 0$ and ii) implies $E(c^\gamma_{\alpha\beta}) = (q_\alpha + q_\beta + q_\gamma - d)c^\gamma_{\alpha\beta}$ (no sums), and then $E(F) = (3-d)F+$ quadratic terms. The condition $(q_\alpha + q_\beta - d)\eta_{\alpha\beta} = 0$ can be put in the form $q_\alpha + q_{n-\alpha+1} = d$ if $\eta_{11} = 0$.

Definition: The manifold M equipped with such a structure is called a *Frobenius manifold.*

In this way, the WDVV equations are reformulated as geometrical conditions on M. Frobenius manifolds arise as geometric structures in many branches of mathematics, like enumerative geometry and quantun cohomology [33] [36]. Moreover there is a FM structure on the space of orbits of a Coxeter group (the solutions (1.5), (1.6), (1.7) correspond to the groups A_3, B_3, H_3 respectively) and on the universal unfolding of simple singularities [48] [49] [19] [16] (see also [5]). We will return later to the examples.

1.3 Deformed flat connection

The connection ∇ can be deformed by a complex parameter z. We introduce a *deformed* connection $\tilde\nabla$ on $M \times \mathbf{C}$: for any $u, v \in T_tM$, depending also on z, we define

$$\tilde\nabla_u v := \nabla_u v + z u \cdot v,$$

$$\tilde\nabla_{\frac{d}{dz}} v := \frac{\partial}{\partial z} v + E \cdot v - \frac{1}{z}\hat\mu v,$$

$$\tilde\nabla_{\frac{d}{dz}} \frac{d}{dz} = 0, \qquad \tilde\nabla_u \frac{d}{dz} = 0$$

where E is the Euler vector field and

$$\hat\mu := I - \frac{d}{2} - \nabla E$$

is an operator acting on v. In coordinates t:

$$\hat\mu = \mathrm{diag}(\mu_1, ..., \mu_n), \qquad \mu_\alpha = q_\alpha - \frac{d}{2},$$

provided that ∇E is diagonalizable. From iv) of section 1.2 it follows that

$$\eta\hat\mu + \hat\mu^T\eta = 0$$

Theorem [16]: $\tilde\nabla$ *is flat.*

To find a flat coordinate $\tilde t(t, z)$ we impose $\tilde\nabla d\tilde t = 0$, which becomes the linear system

$$\partial_\alpha \xi = z C_\alpha(t)\xi, \tag{1.8}$$

$$\partial_z \xi = \left[\mathcal{U}(t) + \frac{\hat\mu}{z}\right]\xi, \tag{1.9}$$

where ξ is a column vector of components $\xi^\alpha = \eta^{\alpha\mu}\partial\tilde t/\partial t^\mu$, $\alpha = 1, ..., n$ (sum omitted), and $C_\alpha(t) = \left(c^\beta_{\alpha\gamma}(t)\right)$, $\mathcal{U} := \left(E^\mu c^\beta_{\mu\gamma}(t)\right)$. From the definition we have $\mathcal{U}^T\eta = \eta\mathcal{U}$. The compatibility of the system is equivalent to the fact that the curvature for $\tilde\nabla$ is zero.

We stress that the deformed connection is a natural structure on a Frobenius manifold. Roughly speaking, a manifold with a flat connection is a Frobenius manifold if the deformed connection is flat. More precisely, suppose that M is a (smooth/analytic) manifold such that T_tM is a commutative algebra with unit element and a bilinear form $< .,. >$ invariant w.r.t. the product. Take an arbitrary vector field E and define the deformation $\tilde{\nabla}$ as above. Then the following is true:

$\tilde{\nabla}$ is flat if and only if $\partial_\delta c_{\alpha\beta\gamma}$ is completely symmetric, $\nabla\nabla E = 0$, the product is associative and $Lie_E\ c = c$.

This statement simply translates the conditions of compatibility of (1.8) (1.9).

Therefore if M is Frobenius, then $\tilde{\nabla}$ is flat. Conversely, if $\tilde{\nabla}$ is flat, then M is a Frobenius manifold (with $q_1 = 0$) provided that we also require iii), iv) of section 1.2 and $\nabla(e) = 0$.

1.4 Semisimple Frobenius manifolds

Definition: A commutative, associative algebra A with unit element is *semisimple* if there is no element $a \in A$ such that $a^k = 0$ for some $k \in \mathbf{N}$.

Any semisimple Frobenius algebra of dimension n over $\mathbf{C}$ is isomorphic to $\mathbf{C} \oplus \mathbf{C} \oplus ...\mathbf{C}$ n times. The direct sum is orthogonal w.r.t. $< .,. >$.

Therefore, A has a basis $\pi_1,...,\pi_n$ such that

$$\pi_i \cdot \pi_j = \delta_{ij}\pi_i \quad \text{no sum}$$

$$< \pi_i, \pi_j >= \eta_{ii}\delta_{ij}$$

$\pi_1,...,\pi_n$ are called *idempotents*, determined up to permutation.

Definition: A Frobenius manifold is *semisimple* if T_tM is semisimple at generic t. Such t is called a semisimple point.

If A is semisimple, the vector $\mathcal{E} := \sum_i u_i\pi_i$, $u_i \in \mathbf{C}$, $u_i \neq u_j$ for $i \neq j$, has n distinct eigenvalues u_i with eigenvectors π_i. Conversely, consider an algebra admitting a vector $\mathcal{E}$ having distinct eigenvalues $u_1,...,u_n$ and eigenvectors $e_1,...,e_n$; from associativity and commutativity it follows that $(\mathcal{E} \cdot e_i) \cdot e_j = e_i \cdot (\mathcal{E} \cdot e_j)$, namely $(u_i - u_j)e_i \cdot e_j = 0 \Rightarrow e_i \cdot e_j = 0$ for $i \neq j$. Thus the algebra is semisimple. We have proved that:

A is semisimple if and only if there exists a vector $\mathcal{E}$ such that the multiplication $\mathcal{E}\cdot$ has n distinct eigenvalues.

If follows that semisimplicity is an "open property" in a Frobenius manifold M: if $t_0 \in M$ is semisimple, T_tM is still semisimple for any t is a neighbourhood of t_0.

Theorem [16]: *Let $t \in M$ be a semisimple point and $\pi_1(t),...,\pi_n(t)$ a basis of idempotents in T_tM. The commutator $[\pi_i, \pi_j] = 0$ and there exist local coordinates $u_1, ..., u_n$ around t such that*

$$\pi_i = \frac{\partial}{\partial u_i}$$

The local coordinates u_i are determined up to shift and permutation. Let $\mathcal{S}_n$ be the symmetric group of n elements. Let $\mathbf{C}^n\backslash$ diagonals $:= \{(u_1, ..., u_n) \in \mathbf{C}^n$ such that $u_i \neq u_j$ for $i \neq j\}$. Finally, let $[(u_1(t), ..., u_n(t))]$ be an equivalence class in $\mathcal{M} \to \frac{(\mathbf{C}^n\backslash \text{ diagonals})}{\mathcal{S}_n}$.

Theorem: *Let $\mathcal{U}(t)$ be the matrix of multiplication by the Euler vector field. Let*

$$\mathcal{M} := \{t \in M \text{ such that } \det(\mathcal{U}(t) - \lambda) = 0 \text{ has } n \text{ distinct eigenvalues } u_1(t), ..., u_n(t)\}$$

The map

$$\mathcal{M} \to \frac{(\mathbf{C}^n\backslash \text{ diagonals})}{\mathcal{S}_n}$$

defined by $t \mapsto [(u_1(t), ..., u_n(t))]$ is a local diffeomorphism, i.e. $u_1(t), ..., u_n(t)$ are local coordinates. Moreover

$$\frac{\partial}{\partial u_i(t)} \cdot \frac{\partial}{\partial u_j(t)} = \delta_{ij}\frac{\partial}{\partial u_i(t)}, \qquad E(t) = \sum_{i=1}^n u_i(t)\frac{\partial}{\partial u_i(t)}$$

A point in $\mathcal{M}$ is semisimple, but in principle there may be semisimple points t such that $u_i(t) = u_j(t)$. u_1,...,u_n are called *canonical coordinates*. They are defined up to permutation.

We introduce the orthonormal basis $f_i := \frac{\pi_i}{\sqrt{<\pi_i,\pi_i>}}$ and we define the matrix $\Psi = (\psi_{i\alpha})$ by

$$\partial_\alpha = \sum_{i=1}^{n} \psi_{i\alpha} f_i.$$

From definitions and simple computations it follows that

$$\sqrt{<\pi_i,\pi_i>} = \psi_{i1}, \qquad <\partial_i,\partial_j> = \delta_{ij}\psi_{i1}^2,$$

$$\partial_\alpha = \sum_{i=1}^{n} \frac{\psi_{i\alpha}}{\psi_{i1}}\partial_i, \qquad \partial_i = \psi_{i1}\sum_{\alpha=1}^{n} \psi_{i\alpha}\eta^{\alpha\beta}\partial_\beta,$$

$$c_{\alpha\beta\gamma} = \sum_{i=1}^{n} \frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}},$$

where $\partial_i := \frac{\partial}{\partial u_i}$. Furthermore, the relation

$$\Psi^T\Psi = \eta,$$

holds true (we will prove it later). Finally we define,

$$U := \text{diag}(u_1(t),...,u_n(t)) = \Psi\mathcal{U}(t)\Psi^{-1}, \qquad V(u) := \Psi\hat{\mu}\Psi^{-1}$$

$V(u)$ is skew-symmetric, because $\eta\hat{\mu} + \hat{\mu}\eta = 0$.

In the following we restrict to analytic *semisimple* Frobenius manifolds. The matrix $\mathcal{U}$ can be diagonalized with *distinct eigenvalues* on the open dense subset $\mathcal{M}$ of M. Later it will be convenient to introduce an alternative notation for Ψ, namely we will often denote Ψ with the name ϕ_0. The systems (1.8) and (1.9) become:

$$\frac{\partial y}{\partial u_i} = [zE_i + V_i]\ y \tag{1.10}$$

$$\frac{\partial y}{\partial z} = \left[U + \frac{V}{z}\right] y, \tag{1.11}$$

where the row-vector y is $y := \phi_0\ \xi$ and E_i is a diagonal matrix such that $(E_i)_{ii} = 1$ and all the other entries are 0; we have also defined

$$V_i := \frac{\partial\phi_0}{\partial u_i}\ \phi_0^{-1}$$

The compatibility of the two systems is equivalent to

$$[U, V_k] = [E_k, V] \quad \Longrightarrow \quad (V_k)_{ij} = \frac{\delta_{ki} - \delta_{kj}}{u_i - u_j}\ V_{ij}$$

$$\frac{\partial V}{\partial u_i} = [V_i, V]$$

At $z = 0$ we have a *fundamental matrix solution*

$$Y_0(z,u) = \left[\sum_{p=0}^{\infty} \phi_p(u)\ z^p\right] z^{\hat{\mu}} z^R, \qquad \phi_0(u) = \Psi(u) \tag{1.12}$$

where $R_{\alpha\beta} = 0$ if $\mu_\alpha - \mu_\beta \neq k > 0$, $k \in \mathbf{N}$. We will return later to this point. The compatibility of the systems implies $\frac{\partial R}{\partial u_i} = 0$ (we discuss this point in section 1.7) and then, by plugging Y_0 into (1.10) we get

$$\frac{\partial\phi_{p},}{\partial u_i} = E_i\phi_{p-1} + V_i\phi_p \tag{1.13}$$

Finally, let $\Phi(z,u) := \sum_{p=0}^{\infty} \phi_p(u)\ z^p$. The condition $\Phi(-z,u)^T\Phi(-z,u) = \eta$ holds (we prove it is section 1.7) and it implies

$$\phi_0^T\phi_0 = \eta, \qquad \sum_{p=0}^{m} \phi_p^T\phi_{m-p} = 0 \qquad \text{for any } m > 0 \tag{1.14}$$

- Conversely, if we start from a (local) solution $V(u)$, $\Psi(u)$ of

$$\frac{\partial V}{\partial u_i} = [V_i, V], \tag{1.15}$$

$$\frac{\partial \Psi}{\partial u_i} = V_i\Psi \tag{1.16}$$

such that

$$\det\Psi(u) \neq 0, \qquad \Pi_{i=1}^{n}\psi_{i1}(u) \neq 0.$$

for u in a neighbourhood of some u_0, we can reconstruct a Frobenius manifold locally from the formualae

$$\eta = \Psi^T\Psi,$$

$$\partial_\alpha = \sum_{i=1}^{n} \frac{\psi_{i\alpha}}{\psi_{i1}}\partial_i, \qquad \partial_i = \psi_{i1}\sum_{\alpha=1}^{n}\psi_{i\alpha}\eta^{\alpha\beta}\partial_\beta,$$

$$< \partial_i, \partial_j >= \delta_{ij}\psi_{i1}^2, \qquad c_{\alpha\beta\gamma} = \sum_{i=1}^{n}\frac{\psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma}}{\psi_{i1}}.$$

$$E = \sum_i u_i\partial_i, \qquad e = \frac{\partial}{\partial t^1} = \sum_{i=1}^{n}\frac{\partial}{\partial u_i}$$

The structure of $V(u)$ is as follows: we fix $V(u_0) = V_0$ at u_0. Then we take an invertible solution Ψ of (1.16) and we define

$$V(u) = \Psi(u)\ V_0\ \Psi(u)^{-1}.$$

This solves the equation for V, as it is verified by direct substitution. It is the unique solution such that the initial value is V_0. Let $\hat{\mu}$ be its Jordan form and let μ_1 be the first eigenvalue (i.e. the first column of $\hat{\mu}$ is $(\mu_1, 0, 0, ..., 0)^T$.

$$V_0 = C\ \hat{\mu}\ C^{-1}$$

where C is an invertible matrix independent of $u = (u_1, .., u_n)$. Now we observe that we can re-scale $\Psi(u) \mapsto \Psi(u)\ C$ and therefore a solution of (1.15) is

$$V(u) = \Psi(u)\ \hat{\mu}\ \Psi(u)^{-1}$$

The dimension of the manifold is

$$d := -2\mu_1$$

This corresponds to the choice of the first column $(\psi_{11}, ..., \psi_{n1})^T$ of $\Psi(u)$.

1.5 Inverse Reconstruction of a Semisimple FM (I)

In this section we show that it is possible to construct a local parametric solution of the WDVV equations in terms of the coefficients of (1.12). The result is discussed in [17] and it is the main formula which allows to reduce the problem of solving the WDVV equations to problems of isomonodromic deformations of linear systems of differential equations. In chapter 6 we will present a more explicit (and computable!) parametric solution of the WDVV eqs. for $n = 3$ as a consequence of the results of this section.

As a first step we note that the condition $\tilde{\nabla} d\tilde{t} = 0$ is satisfied both by a flat coordinate $\tilde{t}^\alpha$ and by $\tilde{t}_\alpha := \eta_{\alpha\beta}\tilde{t}^\beta$. Thus, we choose a fundamental matrix solution of (1.8), (1.9) of the form:

$$\Xi = (\partial^\alpha\tilde{t}_\beta) \equiv \left(\eta^{\alpha\gamma}\frac{\partial\tilde{t}_\beta}{\partial t^\gamma}\right) = \left[\sum_{p=0}^{\infty} H_p(t)z^p\right]\ z^{\hat{\mu}}\ z^R, \qquad H_0 = I,$$

close to $z = 0$. If we restrict to the system (1.8) only, we can choose as a fundamental solution

$$H(z,t) := \sum_{p=0}^{\infty} H_p(t) z^p$$

and so the flat coordinates of $\tilde{\nabla}$ *on* M (not on $M \times \mathbf{C}$) are

$$\tilde{t}_\alpha = \sum_{p=0}^{\infty} h_{\alpha,p}(t) z^p$$

where

$$h_{\alpha 0} = t_\alpha \equiv \eta_{\alpha\beta} t^\beta \tag{1.17}$$

$$\partial_\gamma \partial_\beta h_{\alpha,p+1} = c^\epsilon_{\gamma\beta} \partial_\epsilon h_{\alpha,p}, \qquad p = 0,1,2,... \tag{1.18}$$

We stress that the normalization $H_0 = I$ is precisely what is necessary to have $\tilde{t}_\alpha(z=0) = t_\alpha$ and it corresponds exactly to $Y = \phi_0 \Xi$ in (1.12).

Observe that $h_{\alpha 0} = t_\alpha \equiv \eta_{\alpha\beta} t^\beta$ implies

$$\partial_\beta h_{\alpha,0} = \eta_{\beta\alpha} \equiv c_{\beta\alpha 1} \tag{1.19}$$

Denote by $\nabla f := \left(\eta^{\alpha\beta}\partial_\beta f\right)\partial_\alpha$ the gradient of the function f. The following are a choice for the flat coordinates and for the solution of the WDVV equations

$$t_\alpha = < \nabla h_{\alpha,0}, \nabla h_{1,1} > \equiv \eta^{\mu\nu} \partial_\mu h_{\alpha,0} \partial_\nu h_{1,1} \tag{1.20}$$

$$\begin{aligned} F(t) = \frac{1}{2} [&< \nabla h_{\alpha,1}, \nabla h_{1,1} > \eta^{\alpha\beta} < \nabla h_{\beta,0}, \nabla h_{1,1} > \\ &- < \nabla h_{1,1}, \nabla h_{1,2} > - < \nabla h_{1,3}, \nabla h_{1,0} >] \end{aligned} \tag{1.21}$$

To prove it, it is enough to check by direct differentiation that $\partial_\alpha t_\beta = \eta_{\alpha\beta}$ and $\partial_\alpha\partial_\beta\partial_\gamma F(t) = c_{\alpha\beta\gamma}(t)$, using (1.17), (1.18), (1.19) and... some patience.

In the following, we denote the entry (i,j) of a matrix A_k by $A_{ij,k}$. We recall that $\Psi \equiv \phi_0$ and we observe that

$$\partial_\mu = \sum_{i=1}^n \frac{\phi_{i\mu,0}}{\phi_{i1,0}} \partial_i, \qquad Y_{i\alpha} = \frac{1}{\phi_{i1,0}} \partial_i \tilde{t}_\alpha$$

where $\partial_i = \frac{\partial}{\partial u_i}$. From this it follows that

$$\frac{1}{\phi_{i1,0}} \partial_i h_{\alpha,p} = \phi_{i\alpha,p}$$

and thus:

$$t_\alpha(u) = \sum_{i=1}^n \phi_{i\alpha,0}\phi_{i1,1} \tag{1.22}$$

$$F(t(u)) = \frac{1}{2}\left[\eta^{\alpha\beta} \sum_{i=1}^n \phi_{i\alpha,1}\phi_{i1,1} \sum_{j=1}^n \phi_{j\beta,0}\phi_{j1,1} - \sum_{i=1}^n \phi_{i1,1}\phi_{i1,2} - \sum_{i=1}^n \phi_{i1,3}\phi_{i1,0}\right]$$

Equivalently

$$F(t(u)) = \frac{1}{2}\left[t^\alpha t^\beta \sum_{i=1}^n \phi_{i\alpha,0}\phi_{i\beta,1} - \sum_{i=1}^n (\phi_{i1,1}\phi_{i1,2} + \phi_{i1,3}\phi_{i1,0})\right] \tag{1.23}$$

It is now clear that we can locally reconstruct a Frobenius manifold from $\phi_0(u)$, $\phi_1(u)$, $\phi_2(u)$, $\phi_3(u)$ of (1.12). It is enough to know the local solutions $\phi_0(u) \equiv \Psi(u)$ and $V(u)$ of (1.15) and (1.16) in order to construct the system (1.11) and is fundamental matrix (1.12).

The equations (1.15) and (1.16) express the fact that the dependence on $u = (u_1, ..., u_n)$ of the coefficients system (1.11) is *isomonodromic*. We are going to describe this property and to show how it is possible to characterize locally the matrix coefficient $V(u)$ and the matrix $\phi_0(u)$ of such a system from its monodromy data. We will also explain that the problem of solving (1.15) and (1.16) is equivalent to solving a *boundary value problem.*

As a consequence, any analytic semisimple Frobenius manifold can be locally parametrized by a set of monodromy data.

1.6 Monodromy and Isomonodromic deformation of a linear system

We briefly review some basic notions about the monodromy of a solution of a linear system of differential equations and about isomonodromic deformations. We make use of the results of [2] and [29].

1.6.1 Monodromy

Consider a system of differential equations whose coefficients are $N \times N$ matrices, meromorphic on $\mathbf{C}$ with a finite number of poles $a_1, ..., a_n, \infty$. By Liouville theorem, the most general form of such a system is

$$\frac{dY}{dz} = A(z)\, Y,$$

$$A(z) = z^{r_\infty - 1}\left[A_0^{(\infty)} + \frac{A_1^{(\infty)}}{z} + ... + \frac{A_{r_\infty}^{(\infty)}}{z^{r_\infty}}\right] +$$

$$+ \sum_{k=1}^{n}\left\{\frac{1}{(z-a_k)^{r_k+1}}\left[A_0^{(k)} + A_1^{(k)}(z-a_k) + ... + A_{r_k}^{(k)}(z-a_k)^{r_k+1}\right]\right\}$$

$$A_0^{(\infty)} \text{ diagonal}, \quad r_\nu \geq 0 \text{ integers }, \quad \nu = 1, ..., n, \infty.$$

In the case $r_\nu \geq 1$ we suppose that $A_0^{(\nu)}$ has *distinct eigenvalues*. For z sufficiently big we can find $G(z)$ holomorphic at ∞ such that

$$G^{-1}(z)A(z)G(z) = z^{r_\infty - 1}\left[\Lambda_0^{(\infty)} + \frac{\Lambda_1^{(\infty)}}{z} + + ... + \frac{\Lambda_{r_\infty}^{(\infty)}}{z^{r_\infty}} + ...\right] \quad \text{diagonal}, \quad \Lambda_0^{(\infty)} = A_0^{(\infty)}$$

For $z - a_k$ sufficiently small we can find $G_k(z)$ holomorphic at a_k such that

$$G_k^{-1}(z)A(z)G_k(z) = \frac{1}{(z-a_k)^{r_k+1}}\left[\Lambda_0^{(k)} + \Lambda_1^{(k)}(z-a_k) + ... + \Lambda_{r_k}^{(k)}(z-a_k)^{r_k+1} + ...\right], \quad \text{diagonal}$$

The matrices $\Lambda_l^{(\nu)}$, $\nu = 1, ..., n, \infty$ are uniquely determined by $A(z)$.

• **Representation of the solutions**

It is a standard result that if z_0 is none of the singular points $a_1, ..., a_n, \infty$, there exists an invertible matrix $Y(z)$ holomorphic in a neighbourhood of z_0 which solve the system. It is called a *fundamental solution.* Any other fundamental solution in the neighbourhood of z_0 is $Y(z)\ M$, where M is an invertible matrix independent of z.

As it is well known, $Y(z)$ has analytic continuation along any path σ not containing the singular points. The analytic continuation depends on the homotopy class of the path in $CP^1\backslash\{a_1, ..., a_n, \infty\}$. We fix a base point z_0 in $CP^1\backslash\{a_1, ..., a_n, \infty\}$ and a base of loops $\gamma_1, ..., \gamma_n$ in the fundamental group $\pi(CP^1\backslash\{a_1, ..., a_n, \infty\}; z_0)$, starting at z_0 and encircling $a_1, ..., a_n$. Consider a loop γ: the solution $Y(z)$ obtained by analytic continuation along a path σ and the solution $Y'(z)$ obtained by analytic continuation along $\gamma \cdot \sigma$ are connected by a constant matrix M_γ. Namely $Y'(z) = Y(z)\ M_\gamma$. M_γ is called *monodromy matrix* for the loop γ. Observe that $\gamma \mapsto M_\gamma$ is an anti-homomorphism. The map

$$\pi(CP^1\backslash\{a_1, ..., a_n, \infty\}; z_0) \to GL(N, \mathbf{C})$$

is called *monodromy representation.* All the monodromy matrices are obtained as products of the matrices $M_1, ..., M_n$ corresponding to $\gamma_1, ..., \gamma_n$ (note that $\gamma_1\gamma_2...\gamma_n = \gamma_\infty \Rightarrow M_\infty = M_n...M_2M_1$).

Remark: i) If $\pi(CP^1\backslash\{a_1, ..., a_n, \infty\}; z_0)$ is fixed but we change normalization from $Y(z)$ to $Y(z)\ C$, $\det C \neq 0$, the monodromy matrices change by conjugation $M_i \mapsto C^{-1}M_iC$.

ii) The monodromy matrices change by conjugation also if we change the base point and the basis of loops.

Therefore $A(z)$ determines the monodromy representation up to conjugation.

In the following, a fundamental solution $Y(z)$ will be a a branch, namely the analytic continuation of a fundamental solution defined in a neighbourhood of z_0 along a path from z_0 to z.

We can choose fundamental solutions $Y_l^{(\infty)}(z)$ such they have a specific asymptotic behaviour on the sectors

$$\mathcal{S}_l^{(\infty)}=\{|z|>R \text{ such that } \frac{\pi}{r_\infty}(l-1)-\delta<\arg(z)<\frac{\pi}{r_\infty}l\},\quad l=1,...,2r_\infty$$

defined for R sufficiently big and for δ sufficiently small. The behaviour is

$$Y_l^{(\infty)}(z)=F_l^{(\infty)}(z)\ e^{T^{(\infty)}(z)}$$

$$F_l^{(\infty)}(z)\sim I+\frac{F_1^{(\infty)}}{z}+\frac{F_2^{(\infty)}}{z^2}+...\qquad z\to\infty \text{ in } \mathcal{S}_l^{(\infty)},$$

$$T^{(\infty)}=\frac{\Lambda_0^{(\infty)}}{r_\infty}\ z^{r_\infty}+\frac{\Lambda_1^{(\infty)}}{r_\infty-1}\ z^{r_\infty-1}+...+\Lambda_{r_\infty-1}^{(\infty)}z+\Lambda_{r_\infty}\ln(z)$$

In the following we denote $Y_1^{(\infty)}$ with the name $Y(z)$:

$$Y(z):=Y_1^{(\infty)}(z).$$

For ϵ small we consider the sectors

$$\mathcal{S}_l^{(k)}=\{|z-a_k|<\epsilon \text{ such that } \frac{\pi}{r_k}(l-1)-\delta<\arg(z-a_k)<\frac{\pi}{r_k}l\},\quad l=1,...,2r_k$$

where we can choose fundamental solutions

$$Y_l^{(k)}=F_l^{(k)}(z)\ e^{T^{(k)}(z)}$$

$$F_l^{(k)}(z)\sim F_0^{(k)}+F_1^{(k)}(z-a_k)+F_1^{(k)}(z-a_k)^2+...,\qquad z\to a_k \text{ in } \mathcal{S}_l^{(k)}$$

$$T^{(k)}(z)=-\left(\frac{\Lambda_0^{(k)}}{r_k(z-a_k)^{r_k}}+\frac{\Lambda_1^{(k)}}{(r_k-1)(z-a_k)^{r_k-1}}+...+\frac{\Lambda_{r_k-1}^{(k)}}{(z-a_k)}\right)+\Lambda_{r_k}^{(k)}\ln(z-a_k)$$

In the case $r_k=0$, we suppose that the $A_0^{(k)}$ are diagonalizable, with eigenvalues $\lambda_1^{(k)},...,\lambda_m^{(k)}$. Therefore $\exp\{T^{(k)}(z)\}=(z-a_k)^{\Lambda_0^{(k)}}\ (z-a_k)^{R^{(k)}}$, where the entry $R_{ij}^{(k)}\neq 0$ only if $\lambda_i^{(k)}-\lambda_j^{(k)}$ is an integer greater than zero. In the same way, if $r_\infty=0$, $\exp\{T^{(\infty)}(z)\}=z^{\Lambda_0^{(\infty)}}\ z^{R^{(\infty)}}$, where $R_{ij}^{(\infty)}\neq 0$ only if $\lambda_j^{(\infty)}-\lambda_i^{(\infty)}$ is an integer greater than zero. The general case of non diagonalizability may be treated with more technicalities, which are not necessary here. If $r_\nu=0$, the asymptotic series above are convergent in a neighbourhood of the points a_ν, $\nu=1,...,n,\infty$ ($a_\infty:=\infty$).

The matrices $F_l^{(\nu)}$, $l=0,1,2,...$ and $\nu=1,...,n,\infty$, are uniquely constructed from the coefficients of $A(z)$ by direct substitution of the formal series into the system (note that $A(z)$ must be expanded close to a_ν [where $a_\infty:=\infty$] before substituting).

• **Stokes' rays**

Let $r_\infty\geq 1$ and $\lambda_1^{(\infty)}$,..., $\lambda_n^{(\infty)}$ be the *distinct* eigenvalues of $A_0^{(\infty)}$. We define the *Stokes rays* to be the the half lines where the real part of $(\lambda_i^{(\infty)}-\lambda_j^{(\infty)})z^{r_\infty}$ is zero. Therefore, if

$$\lambda_i^{(\infty)}-\lambda_j^{(\infty)}=|\lambda_i^{(\infty)}-\lambda_j^{(\infty)}|\ \exp\{i\alpha_{ij}\},$$

the Stokes' rays are

$$R_{ij,h}:=\left\{z=\rho e^{i\theta_{ij,h}} \text{ such that } \theta_{ij,h}=\frac{1}{r_\infty}\left(\frac{\pi}{2}-\alpha_{ij}\right)+h\ \frac{\pi}{r_\infty}\right\},\quad h=0,1,...,2r_\infty-1$$

The following properties are easily proved from the very definition of Stokes rays:

i) If a fundamental solution has the asymptotic behaviour we gave above as $z\to\infty$ in some sector $\alpha<\arg(z)<\beta$, then the the solution has the same behaviour on the extension of the sector up to the nearest stokes rays (not included).

ii) If a solution has the asymptotic behaviour above, for $z \to \infty$, in a sector of angular amplitude greater than $\frac{\pi}{r_\infty}$, then such solution is unique.

As a consequence, we can extend the sectors $\mathcal{S}_l^{(\infty)}$ up to the nearest Stokes rays. Suppose we count the distinct rays in clockwise (or counter-clockwise) order, so that we can label the distinct rays as R_1, R_2,... etc (if some rays coincide, they have the same label). This means that the maximal extension of $\mathcal{S}_l^{(\infty)}$ goes from one ray R_m to the ray R_{m+1} plus an angle $\frac{\pi}{r_\infty}$ (rays are not included in the sector!). In the following, by $\mathcal{S}_l^{(\infty)}$ we mean the already extended sector.

Since two fundamental matrices are connected by the multiplication of an invertible matrix to the right, we have

$$Y_{l+1}^{(\infty)}(z) = Y_l^{(\infty)}(z)\ S_l^{(\infty)}, \quad z \in \mathcal{S}_l^{(\infty)} \cap \mathcal{S}_{l+1}^{(\infty)} \qquad l = 1, ..., 2r_\infty - 1$$

$$Y_1(z) = Y_{2r_\infty}(z)\ S_{2r_\infty}^{(\infty)}, \quad z \in \mathcal{S}_{2r_\infty}^{(\infty)} \cap \mathcal{S}_1^{(\infty)}$$

The matrices $S_1^{(\infty)}$,...,S_{2r_∞} are called *Stokes' matrices.*

The same holds true at any a_k, with obvious modification of the above definitions; therefore we have a set of Stokes' matrices $S_1^{(\nu)}$,...,$S_{2r_\nu}^{(\nu)}$, $\nu = 1, ..., n, \infty$. It's not our purpose to give a full description of the structure of these matrices. We will construct explicitly some Stokes matrices later. Here we just remark that

a) The element on the diagonals are all equal to 1, except for

$$\text{diag}(S_{2r_\nu}^{(\nu)}) = \exp\{-2\pi i \Lambda_{r_\nu}^{(\nu)}\}, \qquad \nu = 1, 2, ..., n, \infty$$

b) If the entry $(S_l^{(\nu)})_{ij} \neq 0$, the $(S_l^{(\nu)})_{ji} = 0$.

The connection between $Y(z) := Y_1^{(\infty)}(z)$ and the other solutions $Y_1^{(k)}(z)$ is again given by invertible matrices $C^{(k)}$:

$$Y(z) = \begin{cases} Y_1^{(\infty)}(z) \\ Y_1^{(k)}(z)\ C^{(k)}, \quad k = 1, ..., n \end{cases}$$

The above formula is to be intended as the analytic continuation of $Y = Y_1^{(\infty)}$ along a path from a neighbourhood of ∞ to z in a neighbourhood of a_k (imagine the path passing through the base-point z_0!).

• **Monodromy**

From the above definitions it follows that for the counter-clockwise loop γ_∞ defined by $z \mapsto ze^{2\pi i}$ for $|z|$ as big as to encloses all the singularities a_1,...,a_n, the monodromy M_∞ of $Y(z)$ is

$$Y(z) \mapsto Y(z) M_\infty, \qquad M_\infty = (S_1^{(\infty)} ... S_{2r_\infty}^{(\infty)})^{-1}$$

Note: $M_\infty = \exp\{2\pi i \Lambda_0^{(\infty)}\}$ if $r_\infty = 0$

For a counter-clockwise loop γ_k defined by $(z - a_k) \mapsto (z - a_k)e^{2\pi i}$ in a neighbourhood of a_k not containing other singularities

$$Y(z) \mapsto Y(z) M_k, \qquad M_k = {C^{(k)}}^{-1} (S_1^{(k)} ... S_{2r_k}^{(k)})^{-1} C^{(k)}$$

Note: $M_k = {C^{(k)}}^{-1} \exp\{2\pi i \Lambda_0^{(k)}\}\ C^{(k)}$ if $r_k = 0$.

1.6.2 Isomonodromic deformations

Here we quote the results of the famous paper by Jimbo, Miwa and Ueno [29]. Suppose that $A(z)$ depends on some additional parameters $t = (t_1, ..., t_q)$. This means that

$$a_k = a_k(t), \quad A_l^{(k)} = A_l^{(k)}(t), \quad A_l^{(\infty)} = A_l^{(\infty)}(t)$$

Again we assume $A_0^{(\infty)}$ to be diagonal. We expect that the $\Lambda_l^{(k)}$'s and $F_l^{(\nu)}$'s depend on t. The fundamental matrix $Y(z)$ now depends on t, namely it is $Y(z,t) \equiv Y_1^{(\infty)}(z,t)$.

Let t vary in a (small) open set V . We say that the deformation is *isomonodromic* if

$$\begin{array}{ccccc} \Lambda_{r_\infty}^{(\infty)}, & S_1^{(\infty)}, & \dots, & S_{2r_\infty}^{(\infty)}, & C^{(\infty)} = I \\ \Lambda_{r_1}^{(1)}, & S_1^{(1)}, & \dots, & S_{2r_1}^{(1)}, & C^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ \Lambda_{r_n}^{(n)}, & S_1^{(n)}, & \dots, & S_{2r_n}^{(n)}, & C^{(n)} \end{array}$$

are independent of $t \in V$. In the case $r_\nu = 0$ we also require that $R^{(\nu)}$ is independent of t. This implies that the monodromy matrices are also independent of t and the differential

$$\Omega(z,t) := dY(z,t)\ Y(z,t)^{-1} = \sum_{j=1}^{q} \frac{\partial Y(z,t)}{\partial t_j} Y(z,t)^{-1}\ dt_j,$$

is single-valued and meromorphic in z. We have not enough space here to describe the details. It is enough to say that

$$\Omega(z,t) = \sum_{l=0}^{r_\infty} \Omega_l^{(\infty)}(t)\ z^l + \sum_{k=1}^{n} \left\{ \sum_{l=1}^{r_k+1} \frac{\Omega_l^{(k)}(t)}{(z-a_k)^l} \right\}.$$

The forms $\Omega_l^{(\nu)}(t)$ $\nu = 1, ..., n, \infty$ are uniquely and explicitly determined by $Y(z,t)$ and therefore by $A(z,t)$. More precisely,

$$F^\infty(z,t) dT^{(\infty)}(z,t) F^\infty(z,t)^{-1} = \sum_{l=0}^{r_\infty} \Omega_l^{(\infty)}(t)\ z^l + O\left(\frac{1}{z}\right)$$

$$F^{(k)}(z,t) dT^{(k)}(z,t) F^{(k)}(z,t)^{-1} = \sum_{l=1}^{r_k+1} \frac{\Omega_l^{(k)}(t)}{(z-a_k)^l} + O(1)$$

Theorem [29]: *The deformation $t \in V$ of $\frac{dY}{dz} = A(z,t)Y$ is isomonodromic if and only if Y satisfies*

$$dY = \Omega(z,t) Y$$

where $\Omega(z,t)$ is uniquely determined by $A(z,t)$.

If we expand both sides of $\Omega(z,t) = dY(z,t)\ Y(z,t)^{-1}$ at a_k, we take into account that $Y(z,t) = \left[F_0^{(k)}(t) + O(z-a_k)\right]\ e^{T^{(k)}(z,t)}\ C^{(k)}$ in the right hand-side and we equate the zero-order terms $(z-a_k)^0$, we find the following equation:

$$dF_0^{(k)} = \theta^{(k)}(t)\ F_0^{(k)},$$

where $\theta^{(k)}$ is a form determined by $A(z)$ and $F_0^{(k)}$.

In the above discussion, the asymptotic expansions and the Stokes' phenomenon are supposed to be uniform in $t \in V$. This makes it possible to exchange asymptotic expansions and differentiation "d".

The systems

$$\frac{\partial Y}{\partial z} = A(z,t)\ Y,$$
$$dY = \Omega(z,t)\ Y,$$

are compatible if and only if

$$dA = \frac{\partial \Omega}{\partial z} + [\Omega, A] \quad (\text{it is } d\partial_z = \partial_z d),$$

(the second compatibility condition $d \wedge d = 0$, namely $d\Omega = \Omega \wedge \Omega$, follows from the above (see [29])).

Finally, we construct a non linear system of differential equations for the (entries of) $A_l^{(\nu)}(t)$, for $a_k(t)$ and for $F_0^{(k)}(t)$ which ensures that the deformation is isomonodromic.

If we consider the entries of the $A_l^{(\nu)}$ and the singular points a_k as parameters, it is proved [51] [29] that the fundamental matrices are holomorphic and the asymptotic expansions are uniform in a small open set of the parameters.

Theorem [29]: *The deformation (for t in a small open set V) is isomonodromic if and only if $A(z,t)$ and the $F_0^{(k)}(t)$'s are solutions of*

$$dA = \frac{\partial \Omega}{\partial z} + [\Omega, A],$$

$$dF_0^{(k)} = \theta^{(k)}(t)\ F_0^{(k)}. \tag{1.24}$$

for $t \in V$. Here the forms Ω and $\theta^{(k)}$ are given above as functions of $F_0^{(k)}$ and $A_l^{(\nu)}$.

Finally, we recall that in [29] it is proved that the maximum number of independent parameters t can be chosen to be

$$\begin{array}{cccc} & \Lambda_0^{(\infty)}, & \dots, & \Lambda_{r_\infty-1}^{(\infty)} \\ a_1 & \Lambda_0^{(1)}, & \dots, & \Lambda_{r_1-1}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_n & \Lambda_0^{(n)}, & \dots, & \Lambda_{r_n-1}^{(n)} \end{array}$$

where by $\Lambda_l^{(k)}$ we mean its eigenvalues.

1.7 Monodromy data of a Semisimple Frobenius Manifold

We apply the results of the previous section to a Frobenius manifold M. In the point $t = (t^1, ..., t^n) \in M$ we consider the system (1.9). For simplicity, we suppose that $\hat{\mu}$ is diagonalizable with eigenvalues μ_α, $\alpha = 1, 2, ..., n$. Namely $\hat{\mu} = \text{diag}(\mu_1, ..., \mu_n)$. Let t be fixed. Since $z = 0$ is a fuchsian singularity (1.9) has a fundamental matrix solution

$$\Xi(x,t) = H(z,t)\ z^{\hat{\mu}}\ z^R, \quad H(z,t) = \left[\sum_{p=0}^{\infty} H_l(t) z^p\right], \quad H_0 = I,$$

$$R = R_1 + R_2 + ..., \qquad (R_k)_{\alpha\beta} \neq 0 \text{ only if } \mu_\alpha - \mu_\beta = k,$$

where the series $\left[\sum_{p=0}^{\infty} H_l(t) z^p\right]$ is convergent in a neighbourhood of $z = 0$. R is not uniquely determined. The ambiguity is [17],

$$R \mapsto G\ R\ G^{-1},$$

where

$$G = 1 + \Delta_1 + \Delta_2 + ...,$$

$$(1 - \Delta_1 + \Delta_2 - \Delta_3 + ...)\ \eta\ (1 + \Delta_1 + \Delta_2 + \Delta_3 + ...), \qquad (\Delta)_{\alpha\beta} \neq 0 \text{ only if } \mu_\alpha - \mu_\beta = k$$

Let's call $[R]$ such an orbit.

In [17] it is proved that $R \in [R]$ is independent of $t \in M$. In the semisimple case this property of isomonodromicity follows from our general considerations about isomonodromic deformations. Actually, R is independent of *any* $t \in M$, not only for t in a small neighbourhhod of a given $t_0 \in M$. Since R and $\hat{\mu}$ are independent of $t \in M$, the following definition makes sense:

Definition: $\hat{\mu}$ and $[R]$ are called *monodromy data of the Frobenoius manifold* at $z = 0$.

We turn to the semisimple case through the gauge $y = \Psi\xi$. The system (1.9) becomes (1.11). A fundamental matrix solution is

$$Y^{(0)}(z,u) = \Phi(z,u)\ z^{\hat{\mu}}\ z^R, \quad \Phi(z,u) = \sum_{p=0}^{\infty} \phi_p(u) z^p, \quad \phi_0(u) = \Psi(u),$$

(the ϕ_p's are the matrices $F_p^{(0)}$ of section 1.6). In $\pi(CP^1 \backslash \{0, \infty\}, z_0)$ we consider the basic loop $z_0 \mapsto z_0 e^{2\pi i}$. The monodromy of $Y^{(0)}$ is $e^{2\pi i \hat{\mu}} e^{2\pi i R}$.

Remark: The "symmetries" $\eta\hat{\mu}+\hat{\mu}^T\eta=0$, $\mathcal{U}^T\eta=\eta\mathcal{U}$ imply that $\xi_1(-z,t)^T\eta\xi_2(z,t)$ is independent of z for any two solutions $\xi_1(z,t)$, $\xi_1(z,t)$ of (1.9). In particular $H(-z,t)^T\eta H(z,t)=\eta$. Now, $\Phi(z,u)=\Psi(u)H(z,t(u))$, therefore

$$\Phi(-z,u)^T\ \Phi(z,u)=\eta.$$

(1.11) has a formal solution

$$Y_F=\left[I+\frac{F_1}{z}+\frac{F_2}{z^2}+...\right]\ e^{z\ U},\qquad z\to\infty$$

where F_j's are $n\times n$ matrices (the $F_j^{(\infty)}$ in our general discussion). We explained that fundamental matrix solutions exist which have $\tilde{Y}_F$ as asymptotic expansion for $z\to\infty$. We choose the Stokes' rays

$$R_{rs}=\{z=-i\rho(\bar{u_r}-\bar{u_s}),\quad \rho>0\},\quad r\neq s.$$

Note that $\Re\ ((u_r-u_s)z)=0$ on R_{rs}, therefore the definition of Stokes' rays is satisfied. Let l be an oriented line not containing Stokes' rays and passing through the origin, with a positive half-line l_+ and a negative l_-. It is characterize also by the angle φ between l_+ and the positive real axis. We call Π_R and Π_L the half planes to the right and left of l w.r.t its orientation. Π_L can be extended in Π_R and Π_R in Π_L respectively, up to the first Stokes rays we encounter performing that extension. The resulting extended *overlapping* sectors will be called $\mathcal{S}_L$ and $\mathcal{S}_R$. There exist unique fundamental matrix solutions Y_L and Y_R having the asymptotic expansion in $\mathcal{S}_L$ and $\mathcal{S}_R$ respectively [2]. They are related by the *Stokes' matrix* matrix S, such that

$$Y_L(z,u)=Y_R(z,u)S$$

in the overlapping of $\mathcal{S}_L$ and $\mathcal{S}_R$ containing l_+. In the opposite overlapping region containing l_- one can prove (as a consequence of the skew-symmetry of V, see [16]) that the corresponding Stokes matrix is S^T: namely, $Y_L(z,u)=Y_R(z,u)S^T$. The Stokes' matrix S has entries

$$s_{ii}=1,\qquad s_{ij}=0\quad \text{if } R_{ij}\subset\Pi_R$$

This follows from the fact that on the overlapping region $0<\arg z<\frac{\pi}{k}$ there are no Stokes' rays and

$$e^{zU}\ S\ e^{-zU}\sim I,\qquad z\to\infty,\qquad \text{then}\quad e^{z(u_i-u_j)}\ s_{ij}\to\delta_{ij}$$

Moreover, $\Re\,(z(u_i-u_j))>0$ to the left of the ray R_{ij}, while $\Re\,(z(u_i-u_j))<0$ to the right (the natural orientation on R_{ij}, from $z=0$ to ∞ is understood). This implies

$$|e^{zu_i}|>|e^{zu_j}|\qquad \text{and } e^{z(u_i-u_j)}\to\infty\quad \text{as } z\to\infty$$

on the left, while on the right

$$|e^{zu_i}|<|e^{zu_j}|\qquad \text{and } e^{z(u_i-u_j)}\to 0\quad \text{as } z\to\infty$$

We call *central connection matrix* the connection matrix C such that

$$Y^{(0)}(z,u)=\tilde{Y}_R(z,u)C\quad z\in\Pi_R,$$

(here C is the inverse of the connection matrix $C^{(0)}$ we introduced in the general discussion).

The monodromy of

$$Y(z,u):=Y_R(z,u)$$

at $z=0$ is therefore

$$M_0=S^TS^{-1}\equiv C\ e^{2\pi i\hat{\mu}}e^{2\pi iR}\ C^{-1}$$

The "symmetries" $\Phi(-z,u)^T\Phi(z,u)=\eta$, $Y_R(z,u)^TY_L(z,u)=I$ (in the overlapping region containing l_+), $Y_R(ze^{-2\pi i},u)^TY_L(z,u)=I$ (in the overlapping region containing l_-) imply

$$S=C\ e^{-i\pi R}\ e^{-i\pi\hat{\mu}}\ \eta^{-1}\ C^T,$$

$$S^T=C\ e^{i\pi R}\ e^{i\pi\hat{\mu}}\ \eta^{-1}\ C^T,$$

We note that if C' is such that $z^{\hat{\mu}}z^R\, C' z^{-R}z^{-\hat{\mu}} = C_0 + C_1 z + C_2 z^2 + \ldots$ is a convergent series at $z=0$, then $Y^{(0)}C'$ has again the form $[\sum_p \phi'_p(u)]\; z^{\hat{\mu}}z^R$. In particular, $C_0\hat{\mu} = \hat{\mu}C_0$ and therefore $\phi'_0 = \phi_0 C_0$ is again a matrix such that ${\phi'_0}^{-1} V \phi'_0 = \hat{\mu}$. Also, $C'^{-1}\; e^{2\pi i\hat{\mu}}e^{2\pi i R}\; C' = e^{2\pi i\hat{\mu}}e^{2\pi i R}$. Such matrices form a normal subgroup $C_0(\hat{\mu}, R)$ in the group of matrices commuting with $e^{2\pi i\hat{\mu}}e^{2i\pi R}$.

Theorem [17]: *If two systems*

$$\frac{dy^{(i)}}{dz} = \left[U + \frac{V^{(i)}}{z}\right]\; y^{(i)}, \quad {V^{(i)}}^T = -V^{(i)}, \quad i = 1,2,$$

have the same $\hat{\mu}$ (diagonal from of $V^{(i)}$), R, S (w.r.t. the same line) and the same C (up to $C \mapsto CC'$, $C' \in C_0(\hat{\mu}, R)$), then $V^{(1)} = V^{(2)}$.

The theorem is important to our purpose of classification of Frobenius manifolds. It states that $\hat{\mu}$, R, S (w.r.t. a line) and C determine uniquely the system (1.11) at a fixed $u = (u_1, \ldots, u_n)$.

In the discussion above $u = (u_1, \ldots, u_n)$ was fixed. Now we let if vary.

Definition: Let e be the unit vector. We call e, $\hat{\mu}$, R, S, C the *monodromy data* of the FM in a neighbourhood of the semisimple point $(u_1, \ldots, u_n)$ where they are computed.

The definition makes sense because the dependence on u in the coefficients of (1.11) is *isomonodromic*, as we are going to show. We included e in the definition because the eigenvector of V with eigenvalue $\mu_1 = -d/2$ must be marked and it corresponds to th unity in M. In canonical coordinates it is the first column of $\Psi = \phi_0$.

We explain why the dependence on u is isomonodromic. We know that $\hat{\mu}$ and R are independent of $t \in M$. Therefore, for the local change of coordinates $t = t(u)$ they are *locally* independent of u. Namely, they are constant if u varies in a small open set. The system (1.11) is a particular example of the general systems of section 1.6, Let

$$Y(z,u) := Y_R(z,u).$$

a) Suppose that the deformation u is isomonodromic. From

$$Y(z,u) = \left[I + \frac{F_1}{z} + O\left(\frac{1}{z}\right)\right]\; e^{zU}, \quad z \to \infty \tag{1.25}$$

$$= \sum_{p=0}^{\infty}\; [\phi_p(u) z^p]\; z^{\hat{\mu}}z^R\; C^{-1}, \quad z \to 0.$$

we construct $\Omega(z,u)$:

$$\Omega(z,u) := \frac{\partial Y(z,u)}{u_i}\; Y(z,u)^{-1} = \begin{cases} zE_i + [F_1, E_i] + O\left(\frac{1}{z}\right), & z \to \infty \\ \frac{\partial \phi_0}{\partial u_i}\; \phi_0^{-1} + O(z), & z \to 0 \end{cases}$$

Therefore, $\Omega(z,u) - (zE_i + [F_1, E_i]) \to 0$ as $z \to \infty$ and it is holomorphic at $z = 0$. By Liouville theorem:

$$\Omega(z,u) = zE_i + [F_1(u), E_i]$$

We expand the two sides of $\Omega_0(z,u) = \frac{\partial Y(z,u)}{u_i}\; Y(z,u)^{-1}$ near $z = 0$:

$$zE_i + [F_1, E_i] = \frac{\partial \phi_0}{\partial u_i}\; \phi_0^{-1} + O(z)$$

At $z = 0$:

$$\frac{\partial \phi_0}{\partial u_i} = [F_1, E_i]\; \phi_0$$

This is precisely the equation (1.24). Finally, F_1 is computed from the coefficients of (1.11) by substituting (1.25):

$$\left(-\frac{F_1^2}{z^2} + \ldots\right) e^{zU} + \left(I + \frac{F_1}{z} + \ldots\right) U e^{zU} = \left[U + \frac{V}{z}\right] \left(I + \frac{F_1}{z} + \ldots\right) U e^{zU}$$

From the term $\frac{1}{z}$ we have

$$(F_1)_{ij} = \frac{V_{ij}}{u_j - u_i}$$

and from the term $\frac{1}{z^2}$ we have

$$(F_1)_{ii} = \sum_{j \neq i} \frac{V_{ij}^2}{u_j - u_i}$$

If we put

$$V_k := [F_1, E_k]$$

we obtain

$$(V_k)_{ij} = \frac{\delta_{ki} - \delta_{kj}}{u_i - u_j} V_{ij}$$

which is precisely equivalent to $[U, V_k] = [E_k, V]$.

b) From the general theory of section 1.6 we conclude that the deformation u of the system (1.11) is isomonodromic if and only if

$$\frac{\partial Y}{\partial u_i} = [zE_i + V_i]\, Y$$

where V_i is uniquely determined by

$$[U, V_k] = [E_k, V]$$

In particular, the matrix $\phi_0(u)$ satisfies

$$\frac{\partial \phi_0}{\partial u_i} = V_i\ \phi_0.$$

Here we recognize precisely the properties of a semisimple Frobenius manifold!

In section 1.6 we learnt that the deformation is isomonodromic if and only if $dA = \partial_z \Omega + [\Omega, z]$, $dF_0^{(k)} = \theta^{(k)} F_0^{(k)}$. In our case, the two conditions become respectively the condition of compatibility:

$$\frac{\partial V}{\partial u_i} = [V_i, V], \quad [U, V_i] = [E_i, V]$$

and

$$\frac{\partial \phi_0}{\partial u_i} = V_i\ \phi_0.$$

We conclude that for a semisimple Frobenius manifold, u is an isomonodromic deformation, then $\hat{\mu}$, R, S (for a fixed line), C are independent of u, if u varies in a sufficiently small open set.

1.8 Inverse Reconstruction of a Semisimple FM (II)

Let's fix $u = u^{(0)} = (u_1^{(0)}, ..., u_n^{(0)})$ such that $u_i^{(0)} \neq u_j^{(0)}$ for $i \neq j$. Suppose we give μ, R, an admissible line l, S and C such that

$$s_{ij} \neq 0 \text{ only if the Stokes' ray } R_{ij} \in \Pi_L$$
$$S^T S^{-1} \equiv C\ e^{2\pi i \hat{\mu}} e^{2\pi i R}\ C^{-1}$$
$$S = C\ e^{-i\pi R}\ e^{-i\pi \hat{\mu}}\ \eta^{-1}\ C^T,$$
$$S^T = C\ e^{i\pi R}\ e^{i\pi \hat{\mu}}\ \eta^{-1}\ C^T.$$

Let D be a disk specified by $|z| < \rho$ for some small ρ. Let P_L and P_R be the intersection of the external part of the disk with Π_L and Π_R respectively. We denote by ∂D_R and ∂D_L the lines on the boundary of D on the side of P_R and P_L respectively; we denote by $\tilde{l}_+$ and $\tilde{l}_-$ the portion of l_+ and l_- respectively, on the common boundary of P_R and P_L. Let's consider the following (discontinuous) boundary value problem (b.v.p.): "construct a piecewise holomorphic matrix function

$$\Phi(z) = \begin{cases} \Phi_R(z), & z \in P_R \\ \Phi_L(z), & z \in P_L \\ \Phi_0(z), & z \in D \end{cases},$$

continuous on the boundary of P_R, P_L, D respectively, such that

$$\Phi_L(\zeta) = \Phi_R(\zeta)\ e^{\zeta U} S e^{-\zeta U}, \quad \zeta \in \tilde{l}_+$$

$$\Phi_L(\zeta) = \Phi_R(\zeta)\ e^{\zeta U} S^T e^{-\zeta U}, \quad \zeta \in \tilde{l}_-$$

$$\Phi_0(\zeta) = \Phi_R(\zeta)\ e^{\zeta U} C \zeta^{-R} \zeta^{-\hat{\mu}}, \quad \zeta \in \partial D_R$$

$$\Phi_0(\zeta) = \Phi_L(\zeta)\ e^{\zeta U} S^{-1} C \zeta^{-R} \zeta^{-\hat{\mu}}, \quad \zeta \in \partial D_L$$

$$\Phi_{L/R}(z) \to I \text{ if } z \to \infty \text{ in } P_{L/R}\text{" .}$$

The reader may observe that $\tilde{Y}_{L/R}(z) := \Phi_{L/R}(z) e^{zU}$, $\tilde{Y}^{(0)}(z) := \Phi_0(z,u) z^{\hat{\mu}} z^R$ have precisely the monodromy properties of the solutions of (1.11).

Theorem [40][35][17]: *If the above boundary value problem has solution for a given* $u^{(0)} = (u_1^{(0)}, ..., u_n^{(0)})$ *such that* $u_i^{(0)} \neq u_j^{(0)}$ *for* $i \neq j$*, then:*

i) it is unique.

ii) The solution exists and it is analytic for u *in a neighbourhood of* $u^{(0)}$*.*

iii) The solution has analytic continuation as a meromorphic function on the universal covering of $\mathbf{C}^n \backslash \{diagonals\} := \{(u_1, ..., u_n) \mid u_i \neq u_j \text{ for } i \neq j\}$.

Consider the solutions $\tilde{Y}_{L/R}$, $\tilde{Y}^{(0)}$ of the b.v.p.. We have $\Phi_R(z) = I + \frac{F_1}{z} + O\left(\frac{1}{z^2}\right)$ as $z \to \infty$ in P_R. We also have $\Phi_0(z) = \sum_{p=0}^{\infty} \phi_p z^p$ as $z \to 0$. Therefore

$$\frac{\partial \tilde{Y}_R}{\partial z}\ \tilde{Y}_R^{-1} = \left[U + \frac{[F_1, U]}{z} + O\left(\frac{1}{z^2}\right)\right], \quad z \to \infty,$$

$$\frac{\partial \tilde{Y}^{(0)}}{\partial z}\ (\tilde{Y}^{(0)})^{-1} = \frac{1}{z}\left[\phi_0 \hat{\mu} \phi_0^{-1} + O(z)\right], \quad z \to 0.$$

Since C is independent of u the right hand-side of the two equalities above are equal. Also S is independent of u, therefore the matrices $\tilde{Y}$ above satisfy

$$\frac{\partial y}{\partial z} = \left[U + \frac{V}{z}\right] y, \quad V(u) := [F_1(u), U] \equiv \phi_0 \hat{\mu} \phi_0^{-1}.$$

In the same way

$$\frac{\partial \tilde{Y}_R}{\partial u_i}\ \tilde{Y}_R^{-1} = zE_i + [F_1, E_i] + \left(\frac{1}{z}\right), \quad z \to \infty$$

$$\frac{\partial \tilde{Y}^{(0)}}{\partial u_i}\ (\tilde{Y}^{(0)})^{-1} = \frac{\partial \phi_0}{\partial u_i}\ \phi_0^{-1} + O(z), \quad z \to 0.$$

The right hand-sides are equal, therefore the $\tilde{Y}$'s satisfy

$$\frac{\partial y}{\partial u_i} = [zE_i + V_i]\ y, \quad V_i(u) := [F_1(u), E_i] \equiv \frac{\partial \phi_0}{\partial u_i}\ \phi_0^{-1}.$$

We conclude that from the solution of the b.v.p. we obtain solutions to (1.11), (1.10). This means that we can locally reconstruct a Frobenius structure from the local solution of the b.v.p. and we can do the analytic continuation of such a structure by analytically continuing the solution of the b.v.p to the universal covering of $\mathbf{C}^n \backslash$diagonals. In order to do this, it is enough to use the solution of the b.v.p. $\Phi(z) = \sum_{p=0}^{\infty} \phi_p z^p$ as $z \to 0$ and the formulae (1.22), (1.23), provided that at the given initial point $u^{(0)}$ ($u_i^{(0)} \neq u_j^{(0)}$ for $i \neq j$) the condition $\Pi_{i=1}^n \phi_{i1,0}(u^{(0)}) \neq 0$. is satisfied. If it is not, there is a singularity in the change of coordinates; actually $c_{\alpha\beta\gamma} = \sum_{i=1}^n \frac{\phi_{0,i\alpha}\phi_{0,i\beta}\phi_{0,i\gamma}}{\phi_{0,i1}}$ may diverge and $dt^\alpha = \eta^{\alpha\beta} \sum_{i=1}^n \phi_{i\beta,0}(u) \phi_{i1,0}(u) du_i$ are not independent.

1.8.1 Analytic continuation

The analytic continuation of the solution of the b.v.p. to the universal covering of $\mathbf{C}^n\backslash$diagonals gives the analytic continuation of a Frobenius structure. Since $(u_1, ..., u_n)$ are local coordinates, they are defined up to permutation. Therefore, the analytic continuation is described by the fundamental group $\pi((\mathbf{C}^n\backslash\text{diagonals})/S_n, u^{(0)})$, where S_n is the symmetric group of n elements. This group is called *Braid group* $\mathcal{B}_n$ [6].

The local solution of the b.v.p. is obtained from the monodromy data. Whereas $\hat{\mu}$ and R are constant on the manifold, S and C are constant only for small deformations. We fix the line l *once and for all* and let u vary, so that also the Stokes' rays move. From the definition of S, it follows that whenever a Stokes' ray crosses l some entries of S which where zero may become non zero, and other entries must vanish. This is a discrete jump, described by an action of the braid group.

In order to describe this action we first note what is the effect of permutations on (1.11), S and C. Let $U = \text{diag}(u_1,\ u_2,\ ...,\ u_n)$. Let $\sigma:\ (1,2,..,n) \mapsto (\sigma(1), \sigma(2), ..., \sigma(n))$ be a permutation. It is represented by an invertible matrix P which acts as the gauge $y \mapsto Py$. The new system has matrix

$$P\ U\ P^{-1}\ =\ \text{diag}(u_{\sigma(1)}, u_{\sigma(2)}, ..., u_{\sigma(n)}).$$

S and C are then transformed in PSP^{-1} and PC. For a suitable P, PSP^{-1} is upper triangular. As a general result [2], the good permutation is the one which puts $u_{\sigma(1)}$, ..., $u_{\sigma(n)}$ in lexicographical order w.r.t. the oriented line l. P corresponds to a change of coordinates in the given chart, consisting in the permutation σ of the coordinates.

We recall that the braid group is generated by $n-1$ elementary braids β_{12}, β_{23}, ..., $\beta_{n-1,n}$, with relations:

$$\beta_{i,i+1}\beta_{j,j+1} = \beta_{j,j+1}\beta_{i,i+1} \ \text{ for } i+1 \neq j,\ j+1 \neq i$$

$$\beta_{i,i+1}\beta_{i+1,i+2}\beta_{i,i+1} = \beta_{i+1,i+2}\beta_{i,i+1}\beta_{i+1,i+2}$$

Let's start from $u = u^{(0)}$. If we move sufficiently far away from $u^{(0)}$, some Stokes' rays cross the *fixed* admissible line l. Then, we must change Y_L and Y_R, S and C. The motions of the points u_1, ..., u_n represent transformations of the braid group. Actually, a braid $\beta_{i,i+1}$ can be represented as an "elementary" deformation consisting of a permutation of u_i, u_{i+1} moving counter-clockwise (clockwise or counter-clockwise is a matter of convention).

Suppose u_1, ..., u_n are already in lexicographical order w.r.t. l, so that S is upper triangular (recall that this configuration can be reached by a suitable permutation P). The effect on S of the deformation of u_i, u_{i+1} representing $\beta_{i,i+1}$ is the following [17]:

$$S \mapsto S^{\beta_{i,i+1}} := A^{\beta_{i,i+1}}(S)\ S\ A^{\beta_{i,i+1}}(S)$$

where

$$\left(A^{\beta_{i,i+1}}(S)\right)_{kk} = 1 \qquad k = 1, ..., n \quad n \neq i,\ i+1$$
$$\left(A^{\beta_{i,i+1}}(S)\right)_{i+1,i+1} = -s_{i,i+1}$$
$$\left(A^{\beta_{i,i+1}}(S)\right)_{i,i+1} = \left(A^{\beta_{i,i+1}}(S)\right)_{i+1,i} = 1$$

and all the other entries are zero. For the inverse braid $\beta_{i,i+1}^{-1}$ (u_i and u_{i+1} move clockwise) the representation is

$$\left(A^{\beta_{i,i+1}^{-1}}(S)\right)_{kk} = 1 \qquad k = 1, ..., n \quad n \neq i,\ i+1$$
$$\left(A^{\beta_{i,i+1}^{-1}}(S)\right)_{i,i} = -s_{i,i+1}$$
$$\left(A^{\beta_{i,i+1}^{-1}}(S)\right)_{i,i+1} = \left(A^{\beta_{i,i+1}^{-1}}(S)\right)_{i+1,i} = 1$$

and all the other entries are zero. We remark that S^{β} is still upper triangular.

The effect on C is

$$C \mapsto A^{\beta_{i,i+1}}C$$

Not all the braids are actually to be considered. Suppose we do the following gauge $y \mapsto Jy$, $J = \text{diag}(\pm 1, ..., \pm 1)$, on the system (1.11). Therefore $JUJ^{-1} \equiv U$ but S is transformed to JSJ^{-1}, where some entries change sign. The formulae which define a local chart of the manifold in terms of monodromy data (i.e. in terms of ϕ_p, $p = 0, 1, 2, 3$) are not affected by this transformation. The analytic continuation of the local structure on the universal covering of $(\mathbf{C}^n\backslash\text{diagonals})/\mathcal{S}_n$ is therefore described by the elements of the quotient group

$$\mathcal{B}_n/\{\beta \in \mathcal{B}_n \mid S^\beta = JSJ\} \tag{1.26}$$

Therefore (also recall that the analytic continuation is meromorphic) we conclude [17]:

For the given monodromy data (e, $\hat{\mu}$, R, S, C) the local Frobenius structure obtained from the solution of the b.v.p. extends to a dense open subset of the manifold given by the covering of $(\mathbf{C}^n\backslash diagonals)/\mathcal{S}_n$ w.r.t. the covering transformations in the quotient (1.26).

Let's start from a Frobenius manifold M. Let $\mathcal{M}$ be the open sub-manifold of points t such that $\mathcal{U}(t)$ has distinct eigenvalues. If we compute its monodromy data (e, $\hat{\mu}$, R, S, C) at a point $u^{(0)} \in \mathcal{M}$ and we construct the Frobenius structure from the analytic continuation of the corresponding b.v.p. on the covering of $(\mathbf{C}^n\backslash diagonals)/\mathcal{S}_n$ w.r.t. the quotient (1.26), then there is an equivalence of Frobenius structures between this last manifold and $\mathcal{M}$.

1.9 Intersection Form and Monodromy Group of a Frobenius Manifold

The deformed flat connection was introduced as a natural structure on a Frobenius manifold and allows to transform the problem of solving the WDVV equations to a problem of isomonodromic deformations. There is a further natural structure on a Frobenius manifold which makes it possible to do the same. It is the intersection form.

There is a natural isomorphism $\varphi : T_tM \to T_t^*M$ induced by $<.,.>$. Namely, let $v \in T_tM$ and define $\varphi(v) :=< v,.>$. This allow us to define the product in T_t^*M as follows: for $v, w \in T_tM$ we define $\varphi(v)\cdot\varphi(w) :=< v\cdot w,.>$. In flat coordinates $t^1, ..., t^n$ the product is

$$dt^\alpha \cdot dt^\beta = c_\gamma^{\alpha\beta}(t)\ dt^\gamma, \qquad c_\gamma^{\alpha\beta}(t) = \eta^{\beta\delta} c_{\delta\gamma}^{\alpha}(t),$$

(sums over repeated indices are omitted).

Definition: The *intersection form* at $t \in M$ is a bilinear form on T_t^*M defined by

$$(\omega_1, \omega_2) := (\omega_1 \cdot \omega_2)(E(t))$$

where $E(t)$ is the Euler vector field. In coordinates

$$g^{\alpha\beta}(t) := (dt^\alpha, dt^\beta) = E^\gamma(t) c_\gamma^{\alpha\beta}.$$

Recall that $c_\gamma^{\alpha\beta}$ is independent of t^1. Let $\tilde{t} := (t^2, ..., t^n)$. Also note that $c_1^{\alpha\beta} = \eta^{\alpha\beta}$; then

$$g^{\alpha\beta}(t) = t^1\eta^{\alpha\beta} + \tilde{g}^{\alpha\beta}(\tilde{t})$$

where $\tilde{g}^{\alpha\beta}(\tilde{t})$ depends only on $\tilde{t}$. Therefore

$$\det((g^{\alpha\beta}(t))) = \det(\eta^{-1})\ (t^1)^n + c_{n-1}\ (\tilde{t})(t^1)^{n-1} + c_{n-2}\ (\tilde{t})(t^1)^{n-2} + ... + c_0(\tilde{t})$$

This proves that in an analytic Frobenius manifold the intersection form is non-degenerate on an open dense subset $M\backslash\Sigma$, where

$$\Sigma := \{t \in M \mid \det((g^{\alpha\beta}(t))) = 0\}$$

is called the *discriminant locus*. On $M\backslash\Sigma$ we define $(\ g_{\alpha\beta}(t)\ ) := (\ g^{\alpha\beta}(t)\ )^{-1}$. It is a result of [16] [17] that

The metric $g_{\alpha\beta}dt^\alpha dt^\beta$ is flat on $M\backslash\Sigma$ and its Christoffel coefficients in flat coordinates for $<.,.>$ are

$$\Gamma_\gamma^{\alpha\beta} := -g^{\alpha\delta}\Gamma_{\delta\gamma}^{\beta} = \left(\frac{d+1}{2} - q_\beta\right)\ c_\gamma^{\alpha\beta}$$

Let's denote by $< .,. >^*$ the bilinear form on T_t^*M defined by $\eta^{-1} = (\eta^{\alpha\beta})$. Let $(.,.) - \lambda < .,. >^*$ be a family of bilinear forms on T^*M, parametrized by $\lambda \in \mathbf{C}$. In flat coordinates for η:

$$g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta} = \eta^{\alpha\beta}(t^1 - \lambda) + \tilde{g}^{\alpha\beta}((\tilde{t})$$

Therefore

$$\det(g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta}) = \det(\eta^{-1})\ (t^1 - \lambda)^n + c_{n-1}\ (\tilde{t})(t^1 - \lambda)^{n-1} + ... + c_0(\tilde{t})$$

and $(.,.) - \lambda < .,. >^*$ is not degenerate on the open dense subset $M \backslash \Sigma_\lambda$, where

$$\Sigma_\lambda := \{t \mid det(g^{\alpha\beta}(t) - \lambda\eta^{\alpha\beta}) = 0\}.$$

The n roots of the determinant, considered as a polynomial in $(t^1 - \lambda)$, are:

$$t^1 - \lambda = f_1(\tilde{t}),\ ...,\ t^1 - \lambda = f_n(\tilde{t}), \tag{1.27}$$

where f_1, ..., f_n are functions of $\tilde{t} = (t^2, ..., t^n)$.

Theorem [16]: $(.,.) - \lambda < .,. >^*$ *is a flat pencil of metrics, namely it is flat and its Christoffel coefficients are the sum of the Christoffel coefficients of* g *and* η*:*

$$\left(\Gamma_{(.,.)-\lambda<.,.>^*}\right)_\gamma^{\alpha\beta} = \left(\Gamma_{(.,.)}\right)_\gamma^{\alpha\beta} + 0 = \left(\frac{d+1}{2} - q_\beta\right)\ c_\gamma^{\alpha\beta}.$$

In order to find a flat coordinate x for the pencil we impose $\hat{\nabla} dx = 0$, where $\hat{\nabla}$ is the connection for the metric induced by the pencil. We skip details (see [16] [17]) and we give the final result:

$$(\mathcal{U}(t) - \lambda)\ \frac{\partial \xi}{\partial \lambda} = \left(\frac{1}{2} + \hat{\mu}\right) \xi \tag{1.28}$$

$$(\mathcal{U}(t) - \lambda)\ \partial_\beta \xi + C_\beta \left(\frac{1}{2} + \hat{\mu}\right) \xi = 0 \tag{1.29}$$

where the column vector $\xi = (\xi^1, ..., \xi^n)^T$ is defined by $\xi^\alpha = \eta^{\alpha\beta}\frac{\partial x}{\partial t^\beta}$. $\mathcal{U}$ and C_β have already been defined.

In the semisimple case, let $u_1, ..., u_n$ be local canonical coordinates, equal to the distinct eigenvalues of $\mathcal{U}(t)$. From the definitions we have

$$du_i \cdot du_j = \frac{1}{\eta_{ii}}\delta_{ij} du_i, \quad g^{ij}(u) = (du_1, du_j) = \frac{u_i}{\eta_{ii}}\delta_{ij}, \quad \eta_{ii} = \psi_{i1}^2$$

Then $g^{ij} - \lambda\eta^{ij} = \frac{u_i - \lambda}{\eta_{ii}}\delta_{ij}$ and

$$\det((g^{ij} - \lambda\eta^{ij})) = \frac{1}{\det((\eta_{ij}))}\ (u_1 - \lambda)(u_2 - \lambda)...(u_n - \lambda).$$

Namely, the roots λ of the above polynomial are the canonical coordinates. Now we recall that $\mathcal{M} \subset M$ was defined as the sub-manifold of semisimple points such that $u_i \neq u_j$ for $i \neq j$. If λ is fixed, then the discriminant Σ_λ is:

$$\Sigma_\lambda \cap \mathcal{M} := \bigcup_{i=1}^n \{t \in \mathcal{M} \mid u_i(t) = \lambda\}$$

Of course, on the component $u_i(t) = \lambda$ we must have $u_j(t) \neq \lambda$ for any $j \neq i$. As a consequence of (1.27) we have a representation for the canonical coordinates:

$$u_i(t) = t^1 - f_i(\tilde{t})$$

Now we perform the gauge $\phi(\lambda, u) := \Psi(u(t))\xi(\lambda, t)$, where $t = t(u)$ *locally*. The system (1.28), (1.29) becomes

$$(U - \lambda)\frac{\partial \phi}{\partial \lambda} = \left(\frac{1}{2} + V(u)\right) \phi,$$

$$(U-\lambda)\frac{\partial \phi}{\partial u_i} = \left[(U-\lambda)V_i(u) - E_i\left(\frac{1}{2}+V(u)\right)\right]\phi.$$

The matrix V_i is defined by

$$V_i := \frac{\partial \Psi}{\partial u_i}\,\Psi^{-1}$$

Equivalently:

$$\frac{\partial \phi}{\partial \lambda} = \sum_{i=1}^{n} \frac{B_i}{\lambda - u_i}\phi, \qquad B_i := -E_i\left(\frac{1}{2}+V\right), \tag{1.30}$$

$$\frac{\partial \phi}{\partial u_i} = \left(V_i - \frac{B_i}{\lambda - u_i}\right)\phi. \tag{1.31}$$

• **Isomonodromy and Inverse Reconstruction**

The compatibility of the two systems (it is enough to take $\frac{\partial}{\partial\lambda}\frac{\partial}{\partial u_i} = \frac{\partial}{\partial u_i}\frac{\partial}{\partial\lambda}$ which implies $\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j} = \frac{\partial}{\partial u_j}\frac{\partial}{\partial u_i}$) is

$$\partial_i B_j = [V_i, B_j] + \frac{[B_i, B_j]}{u_i - u_j}, \quad i \neq j \tag{1.32}$$

$$\sum_{j=1}^{n} (\partial_i B_j + [B_j, V_i]) = 0 \tag{1.33}$$

The first is equivalent to

$$[U, V_i] = [E_i, V]$$

and the second to

$$\partial_i V = [V_i, V]$$

The compatibility conditions and $\frac{\partial \Psi}{\partial u_i} = V_i\ \Psi$ are the conditions of isomonodromicity of section 1.6. Therefore, the problem of the WDVV equations is transformed into an isomonodromy deformation problem for (1.30) which is equivalent to the same problem for (1.11). Incidentally, we anticipate that the systems (1.30) and (1.11) are connected by a Laplace transform (see section 4.8). The boundary value problem for the fuchsian system (1.30) is the classical *Riemann Hilbert Problem* which we are going to partly solve in section 5 for $n = 3$ in terms of Painlevé transcendents.

• **Monodromy Group**

Let $x_1(t,\lambda) = x_1(t^1-\lambda, \tilde{t}), ..., x_n(t,\lambda) = x_n(t^1-\lambda,\tilde{t})$ be flat coordinates obtained from a fundamental matrix solution of the system (1.30) (1.31). They are multi-valued functions of λ and of t; for $\lambda = 0$ they represent the flat coordinates for the intersection form. For a loop γ around the discriminant Σ the monodromy of the coordinates is linear, namely:

$$(x_1(t), ..., x_n(t)) \to (x_1(t), ..., x_n(t))M_\gamma, \qquad M_\gamma \in GL(n, \mathbf{C})$$

The image of the representation

$$\pi(M\backslash\Sigma, t_0) \to GL(n, \mathbf{C})$$

is called the *Monodromy group of the Frobenius manifold.* To show this we restrict to the semisimple case. We choose a particular loop around Σ starting and ending at $u_0 = u(t_0)$, defined by the conditions that $\tilde{t}_0$ is fixed and only t^1 varies according to the rule $t^1 = t_0^1 - \lambda$. The discriminant locus Σ is reached when one of the $u_i(t^1, \tilde{t}_0)$ becomes zero:

$$u_i(t) = t^1 - f_i(\tilde{t}) \equiv (t_0^1 - \lambda) + f_i(\tilde{t}_0) = 0$$

This means that the discriminant is reached if λ equals one of the canonical coordinates $u_1(t_0) = t_0^1 - f_1(\tilde{t}_0), ..., u_n(t_0) = t_0^1 - f_n(\tilde{t}_0)$. On the other hand, the coordinates $x_a(t) = x_a(t_0^1 - \lambda, t_0^2, ..., t_0^n)$, $a = 1, ..., n$, come from the solution of (1.30) when the poles are $u_1 = u_1(t_0)$, ..., $u_n = u_n(t_0)$. Hence the monodromy group of the system (1.30), which is independent of $u_0 = u(t_0)$ for small deformations of u in a neighbourhood of u_0, precisely describes the monodromy group of the Frobenius manifold. Actually, if $\Phi = [\phi^{(1)}, ..., \phi^{(n)}]$ is a fundamental matrix of (1.30), where $\phi^{(a)}$, $a = 1, ..., n$ are independent columns, then

$$\partial_i x_a = \psi_{i1} \phi_i^{(a)},$$

This follows from the gauge $\phi = \Psi \xi$, the definition of ξ and the formula $\frac{\partial}{\partial u_i} = \psi_{i1} \sum_\epsilon \psi_{i\epsilon} \partial^\epsilon$. If λ describes a loop around $u_i(t_0)$, Φ is changed by a monodromy matrix R_i:

$$\Phi \to \Phi' := \Phi R_i$$

Therefore

$$\partial_i x'_b(u_0) = \partial_i x_a(u_0) \ (R_i)_{ab}$$

which can be integrated because R_i is independent of u:

$$(x'_1(u_0), ..., x'_n(u_0)) = (x_1(u_0), ..., x_n(u_0)) \ R_i + (c_1, ..., c_n)$$

where the c_i's are constants. We can put them equal to zero. Actually, one can verify [17] that

$$x_a(u, \lambda) = \frac{2\sqrt{2}}{1-d} \sum_{i=1}^n (u_i - \lambda) \psi_{i1}(u) \phi_i^{(a)}(u, \lambda)$$

If u is subject to a "big" deformation, one can find an action of the braid group on the R_i's, and the new monodromy matrices generate the same group. We will return to this point and to the concrete description of the R_i's (which are orthogonal transformations w.r.t $g^{\alpha\beta}$) in section 4.8.

Finally, we give another formula for $g^{\alpha\beta}$ by differentiating twice the expression

$$E^\gamma \partial_\gamma F = (2-d)F + \frac{1}{2} A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C$$

which is the quasi-homogeneity of F up to quadratic terms. By recalling that $E^\gamma = (1 - q_\gamma) t^\gamma + r_\gamma$ and that $\partial_\alpha \partial_\beta \partial_\gamma F = c_{\alpha\beta\gamma}$ we obtain

$$g^{\alpha\beta}(t) = (1 + d - q_\alpha - q_\beta) \partial^\alpha \partial^\beta F(t) + A^{\alpha\beta} \tag{1.34}$$

where $\partial^\alpha = \eta^{\alpha\beta} \partial_\beta$, $A^{\alpha\beta} = \eta^{\alpha\gamma} \eta^{\beta\delta} A_{\gamma\delta}$.

1.10 Reduction of the Equations of Isomonodromic Deformation to a Painlevé 6 Equation

The compatibility conditions (1.32),(1.33) are equivalent to the differential equations for V, V_i, Ψ whose solution allows us to do the inverse local reconstruction of the manifold. Solving such equations is equivalent to solving the boundary value problem (b.v.p.), provided that we find a mean to parameterize the integration constants in terms of the monodromy data defining the b.v.p..

For $n = 3$ such equations were reduced [16] to a particular form of the VI Painlevé equation. This is the first step towards the solution of the b.v.p. and the inverse reconstruction. In chapter 5 we'll see how to parameterize the solutions of the Painlevé equation in terms of monodromy data. Finally, in chapter 6 we'll apply the results to the explicit inverse reconstructions of some 3-dimensional Frobenius manifolds.

Let

$$\hat{\mu} = \text{diag}(\mu, 0, -\mu)$$

We consider the auxiliary system

$$\frac{\partial \Phi}{\partial \lambda} = \sum_{i=1}^n \frac{\mathcal{B}_i}{\lambda - u_i} \Phi$$

$$\frac{\partial \Phi}{\partial u_i} = \left(V_i - \frac{\mathcal{B}_i}{\lambda - u_i} \right) \Phi$$

where

$$\mathcal{B}_i := -E_i \ V$$

The compatibility conditions expressed in terms of B_i are (1.32), (1.32) where $\mathcal{B}_i$ is substituted by B_i. They are again equivalent to $[U, V_i] = [e_i, V]$ and $\partial_i V = [V_i, V]$, plus the condition $\partial_i \phi_0 = V_i \phi_0$. Therefore, we can study the isomonodromy dependence on u of the auxiliary system instead of (1.31) (1.31).

Let $n = 3$. We observe that

$$\sum_{i=1}^{n} \frac{\mathcal{B}_i}{\lambda - u_i} = -(\lambda - U)^{-1}\, V = -(\lambda - U)^{-1}\, \phi_0 \hat{\mu} \phi_0^{-1}$$

We put $X(\lambda) := \phi_0^{-1}\, \Phi(\lambda)$ and rewrite:

$$\frac{\partial X}{\partial \lambda} = -\mu\, v(\lambda, u)\, \mathrm{diag}(1, 0, -1)\, X$$

where

$$v(\lambda, u) := \phi_0^{-1} (\lambda - U)^{-1} \phi_0 = -\mu \sum_{i=1}^{3} \frac{\mathcal{A}_i}{\lambda - u_i},$$

$$\mathcal{A}_i := \phi_0^{-1} E_i \phi_0$$

Let X be the column vector

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \quad \Rightarrow \quad \mathrm{diag}(1, 0, -1)\, X = \begin{pmatrix} X_1 \\ 0 \\ -X_3 \end{pmatrix}$$

Therefore

$$\frac{\partial}{\partial \lambda} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = -\mu \begin{pmatrix} v_{11} & -v_{13} \\ v_{31} & v_{33} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = -\mu \sum_{i=1}^{3} \frac{A_i}{\lambda - u_i} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{1.35}$$

where

$$A_i := \begin{pmatrix} \phi_{i1,0}\phi_{i3,0} & -\phi_{i3,0}^2 \\ \phi_{i1,0}^2 & \phi_{i1,0}\phi_{i3,0} \end{pmatrix}, \quad A_1 + A_2 + A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.36}$$

To write the A_i's, we have taken into account the relations $\phi_0^T \phi_0 = \eta$ and the choice

$$(\phi_{12,0}, \phi_{22,0}, \phi_{32,0}) = \pm i(\phi_{21,0}\phi_{33,0} - \phi_{23,0}\phi_{31,0}, \phi_{13,0}\phi_{31,0} - \phi_{11,0}\phi_{33,0}, \phi_{11,0}\phi_{23,0} - \phi_{13,0}\phi_{21,0})$$

We also obtain:

$$\frac{\partial}{\partial u_i} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \mu \frac{A_i}{\lambda - u_i} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{1.37}$$

The second component of X is obtained by quadratures:

$$\frac{\partial}{\partial \lambda} X_2 = \mu\, (v_{21}\, X_1 - v_{23}\, X_3)$$

$$\frac{\partial}{\partial u_i} X_2 = \frac{\mu}{\lambda - u_i} [\phi_{i1,0}\phi_{i2,0}\, X_1 - \phi_{i2,0}\phi_{i3,0}\, X_3]$$

The reduced 2×2 system (1.35) (1.37) is solved (see [30] [16]) by introducing the following coordinates $q(u)$, $p(u)$ in the space of matrices A_i modulo diagonal conjugation: q is the root of

$$\left(\sum_{i=1}^{3} \frac{A_i}{\lambda - u_i} \right)_{12} = 0$$

and

$$p := \left(\sum_{i=1}^{3} \frac{A_i}{q - u_i} \right)_{11}$$

The entries of the A_i's are re-expressed as follows:

$$\phi_{i1,0}\, \phi_{i3,0} = -\frac{q - u_i}{2\mu P'(u_i)} \left[P(q)p^2 + \frac{2\mu}{q - u_i} P(q)p + \mu^2 (q + 2u_i - \sum_{j=1}^{3} u_j \right]$$

$$\phi_{13,0}^2 = -k\frac{q-u_i}{P'(u_i)}$$

$$\phi_{i1,0}^2 = -\frac{q-u_i}{4\mu P'(u_i)\;k}\left[P(q)p^2 + \frac{2\mu}{q-u_i}P(q)p + \mu^2(q+2u_i - \sum_{j=1}^{3} u_j\right]^2$$

Here k is a parameter, $P(z) = (z-u_1)(z-u_2)(z-u_3)$. The compatibility of (1.35) (1.37) is

$$\frac{\partial q}{\partial u_i} = \frac{P(q)}{P'(u_i)}\left[2p + \frac{1}{q-u_i}\right] \tag{1.38}$$

$$\frac{\partial p}{\partial u_i} = -\frac{P'(q)p^2 + (2q+u_i - \sum_{j=1}^3 u_j)p + \mu(1-\mu)}{P'(u_i)} \tag{1.39}$$

$$\frac{\partial \ln k}{\partial u_i} = (2\mu-1)\frac{q-u_i}{P'(u_i)}$$

In the variables

$$x = \frac{u_3-u_1}{u_2-u_1}, \quad y = \frac{q-u_1}{u_2-u_1}$$

the system (1.38), (1.39) becomes a special case of the VI Painlevé equation, with the following choice of the parameters (in the standard notation of [26]):

$$\alpha = \frac{(2\mu-1)^2}{2}, \quad \beta=\gamma=0, \quad \delta=\frac{1}{2}$$

Namely:

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right]\left(\frac{dy}{dx}\right)^2 - \left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right]\frac{dy}{dx}$$
$$+\frac{1}{2}\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[(2\mu-1)^2 + \frac{x(x-1)}{(y-x)^2}\right], \qquad \mu\in\mathbf{C} \tag{1.40}$$

In the following, this equation will be referred to as PVI_μ. From a solution of (PVI_μ) one can reconstruct

$$q = (u_2-u_1)\,y\left(\frac{u_3-u_1}{u_2-u_1}\right) + u_1$$

$$p = \frac{1}{2}\frac{P'(u_3)}{P(q)}\,y'\left(\frac{u_3-u_1}{u_2-u_1}\right) - \frac{1}{2}\frac{1}{q-u_3}$$

From the very definition of q we have:

$$y(x) = \frac{xR}{x(1+R)-1}, \qquad R := \frac{(A_1)_{12}}{(A_2)_{12}} \equiv \left(\frac{\phi_{13,0}}{\phi_{23,0}}\right)^2$$

1.11 Conclusion

A semisimple Frobenius manifold can be locally parametrized by a set of monodromy data. The local parameterization is given by the formulae of section 1.4 and by (1.23) (1.22). The matrices ϕ_p are obtained solving a boundary value problem or, equivalently, the equations of isomonodromic deformation $\frac{\partial V}{\partial u_i} = [V_i, V]$, $\frac{\partial \phi_0}{\partial u_i} = V_i\phi_0$. When $n=3$ the problem is reduced to a Painlevé 6 equation.

Chapter 2

Quantum Cohomology of Projective spaces

In this chapter we introduce the FM called quantum cohomology of the projective space CP^d and we describe its connections to enumerative geometry. In the last section we will prove that the radius of convergence of the famous Kontsevich's solution of WDVV equations for CP^2 corresponds to a singularity in the change from flat to canonical coordinates.

We start by introducing a structure of Frobenius algebra on the cohomology $H^*(X, \mathbf{C})$ of a closed oriented manifold X of dimension d such that

$$H^i(X, \mathbf{C}) = 0 \quad \text{for } i \text{ odd}$$

then

$$H^*(X, \mathbf{C}) = \otimes_{i=0}^{d} H^{2i}(X, \mathbf{C}).$$

For brevity we omit $\mathbf{C}$ in H. We realize $H^*(X)$ by classes of closed differential forms. The unit element is a 0-form $e_1 \in H^0(X)$. Let denote by ω_α a form in $H^{2q_\alpha}(X)$, where $q_1 = 0$, $q_2 = 1$, ..., $q_{d+1} = d$. The product of two forms ω_α, ω_β is defined by the wedge product $\omega_\alpha \wedge \omega_\beta \in H^{2(q_\alpha + q_\beta)}(X)$ and the bilinear form is

$$< \omega_\alpha, \omega_\beta > := \int_X \omega_\alpha \wedge \omega_\beta \neq 0 \iff q_\alpha + q_\beta = d$$

It is not degenerate by Poincaré duality and of course only $q_\alpha + q_{d-\alpha+1} = d$.

Let $X = CP^d$. Let $e_1 = 1 \in H^0(CP^d)$, $e_2 \in H^2(CP^d)$, ..., $e_{d+1} \in H^{2d}(CP^d)$ be a basis in $H^*(CP^d)$. For a suitable normalization we have

$$(\eta\alpha\beta) := (< e_\alpha, e_\beta >) = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

The multiplication is

$$e_\alpha \wedge e_\beta = e_{\alpha+\beta-1}.$$

We observe that it can also be written as

$$e_\alpha \wedge e_\beta = c^\gamma_{\alpha\beta} e_\gamma, \quad \text{sums on } \gamma$$

where

$$\eta_{\alpha\delta} c^\delta_{\beta\gamma} := \frac{\partial^3 F_0(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$

$$F_0(t) := \frac{1}{2}(t^1)^2 t^n + \frac{1}{2} t^1 \sum_{\alpha=2}^{n-1} t^\alpha t^{n-\alpha+1}$$

F_0 is the trivial solution of WDVV equations. We can construct a trivial Frobenius manifold whose points are $t := \sum_{\alpha=1}^{d+1} t^\alpha e_\alpha$. It has tangent space $H^*(CP^d)$ at any t. By *quantum cohomology* of CP^d (denoted by $QH^*(CP^d)$ we mean a Frobenius manifold whose structure is specified by

$$F(t) = F_0(t) + \text{ analytic perturbation}$$

This manifold has therefore tangent spaces $T_t QH^*(CP^d) = H^*(CP^d)$, with the same $< .,. >$ as above, but the multiplication is a deformation, depending on t, of the wedge product (this is the origin of the adjective "quantum").

2.1 The case of CP^2

We restrict to CP^2. In this case

$$F_0(t) = \frac{1}{2}\left[(t^1)^2 t^3 + t^1 (t^2)^2\right]$$

which generates the product for the basis $e_1 = 1 \in H^0$, $e_2 \in H^2$, $e_3 \in H^4$. The deformation was introduced by Kontsevich [33].

2.1.1 Kontsevich's solution

The WDVV equations for $n = 3$ variables have solutions

$$F(t_1, t_2, t_3) = F_0(t_1, t_2, t_3) + f(t_2, t_3)$$

where

$$f_{222} f_{233} + f_{333} = (f_{223})^2 \tag{2.1}$$

with the notation $f_{ijk} := \frac{\partial^3 f}{\partial t_i \partial t_j \partial t_k}$. As for notations, the variables t_j are flat coordinates in the Frobenius manifold associate to F. They should be written with upper indices, but we use the lower for convenience now.

Let N_k be the number of rational curves $CP^1 \to CP^2$ of degree k through $3k - 1$ generic points. Kontsevich [33] found the solution

$$f(t_2, t_3) = \frac{1}{t_3}\varphi(\tau), \qquad \varphi(\tau) = \sum_{k=1}^{\infty} A_k \tau^k, \qquad \tau = t_3^3 \, e^{t_2} \tag{2.2}$$

where

$$A_k = \frac{N_k}{(3k-1)!}$$

We note that this solution has precisely the form of the general solution of the WDVV eqs. for $n = 3$, $d = 2$ and $r_2 = 3$. If we put $\tau = e^X$ we rewrite (2.1) as follows

$$\Phi(X) := \varphi(e^X) = \sum_{k=1}^{\infty} A_k \, e^{kX},$$

$$-6\Phi + 33\Phi' - 54\Phi'' - (\Phi'')^2 + \Phi''' \left(27 + 2\Phi' - 3\Phi''\right) = 0 \tag{2.3}$$

The prime stands for the derivative w.r.t X. If we fix A_1, the above (2.3) determines the A_k uniquely. Since $N_1 = 1$, we fix

$$A_1 = \frac{1}{2}.$$

Then (2.3) yields the recurrence relation

$$A_k = \sum_{i=1}^{k-1} \left[\frac{A_1 A_{k-i} \; i(k-i)\big((3i-2)(3k-3i-2)(k+2) + 8k - 8\big)}{6(3k-1)(3k-2)(3k-3)}\right] \tag{2.4}$$

2.1.2 Convergence of Kontsevich solution

The convergence of (2.2) was studied by Di Francesco and Itzykson [12]. They proved that

$$A_k = b\, a^k\, k^{-\frac{7}{2}} \left(1 + O\left(\frac{1}{k}\right)\right), \qquad k \to \infty$$

and numerically computed

$$a = 0.138, \qquad b = 6.1 \quad .$$

The result implies that $\varphi(\tau)$ converges in a neighbourhood of $\tau = 0$ with radius of convergence $\frac{1}{a}$. We will return later on the numerical evaluation of these numbers.

The proof of [12] is divided in two steps. The first is based on the relation (2.4), to prove that

$$A_k^{\frac{1}{k}} \to a \text{ for } k \to \infty, \qquad \frac{1}{108} < a < \frac{2}{3}$$

a is real positive because the A_k's are such. It follows that we can rewrite

$$A_k = b a^k\, k^{\omega} \left(1 + O\left(\frac{1}{k}\right)\right), \qquad \omega \in \mathbf{R}$$

The above estimate implies that $\varphi(\tau)$ has the radius of convergence $\frac{1}{a}$. The second step is the determination of ω making use of the differential equation (2.3). Let's write

$$A_k := C_k\, a^k$$

$$\Phi(X) = \sum_{k=1}^{\infty} A_k\, e^{kX} = \sum_{k=1}^{\infty} C_k\, e^{k(X - X_0)}, \qquad X_0 := \ln \frac{1}{a}$$

The inequality $\frac{1}{108} < a < \frac{2}{3}$ implies that $X_0 > 0$. The series converges at least for $\Re X < X_0$. To determine ω we divide $\Phi(X)$ into a regular part at X_0 and a singular one. Namely

$$\Phi(X) = \sum_{k=0}^{\infty} d_k (X - X_0)^k + (X - X_0)^{\gamma} \sum_{k=0}^{\infty} e_k (X - X_0)^k, \qquad \gamma > 0, \quad \gamma \notin \mathbf{N},$$

d_k and e_k are coefficients. By substituting into (2.3) we see that the equation is satisfied only if $\gamma = \frac{5}{2}$. Namely:

$$\Phi(X) = d_0 + d_1 (X - X_0) + d_2 (X - X_0)^2 + e_0 (X - X0)^{\frac{5}{2}} + \ldots$$

This implies that $\Phi(X)$, $\Phi'(X)$ and $\Phi''(X)$ exist at X_0 but $\Phi'''(X)$ diverges like

$$\Phi'''(X) \asymp \frac{1}{\sqrt{X - X_0}}, \qquad X \to X_0 \tag{2.5}$$

On the other hand $\Phi'''(X)$ behaves like the series

$$\sum_{k=1}^{\infty} b\, k^{\omega + 3}\, e^{k(X - X_0)}, \qquad \Re(X - X_0) < 0$$

Suppose $X \in \mathbf{R}$, $X < X_0$. Then $\Delta := X - X_0 < 0$ and the above series is

$$\frac{b}{|\Delta|^{3+\omega}} \sum_{k=1}^{\infty} (|\Delta| k)^{3+\omega} e^{-|\Delta| k} \sim \frac{b}{|\Delta|^{3+\omega}} \int_0^{\infty} dx\, x^{3+\omega} e^{-x}$$

It follows from (2.5) that this must diverge like $\Delta^{-\frac{1}{2}}$, and thus $\omega = -\frac{7}{2}$ (the integral remains finite).

As a consequence of (2.3) and of the divergence of $\Phi'''(X)$

$$27 + 2\Phi'(X_0) - 3\Phi''(X_0) = 0$$

2.1.3 Small Cohomology

We realize $QH^*(CP^2)$ as the set of points $t = t^1e_1 + t^2e_2 + t^3e_3 \in H^*(CP^2)$ (we restore the upper indices only in this formula) such that the tangent space at t is again $H^*(CP^2)$ with the product $e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(t)e_\gamma$, where

$$c_{\alpha\beta\gamma} = \eta_{\alpha\delta}c^\delta_{\beta\gamma} = \partial_\alpha\partial_\beta\partial_\gamma F(t)$$

$F(t)$ being Kontsevich solution. We also observe that $F(t_1, t_2 + 2\pi i, t_3)$ differs from $F(t_1, t_2, t_3)$ by quadratic terms:

$$F(t_1, t_2 + 2\pi i, t_3) = F(t_1, t_2, t_3) + 2\pi i t_1 t_2 - 2\pi^2 t_1.$$

Thus $QH^*(CP^2)$ is a sub-manifold of the quotient $H^*(CP^2)/2\pi i H^2(CP^2, \mathbf{Z})$ because $e_2 \in H^2(CP^2, \mathbf{Z})$.

Suppose $t_3 \to 0$ and $t_2 \to -\infty$. From Kontsevich solution we get

$$e_2 \cdot e_2 = (t_3e^{t_2} + ...)\ e_1 + \left(\frac{1}{2}t_3^2e^{t_2} + ...\right)\ e_2 + e_3,$$

$$e_2 \cdot e_3 = (e^{t_2} + ...)\ e_1 + (t_3e^{t_2} + ...)\ e_2,$$

$$e_3 \cdot e_3 = \left(\frac{t_3^2}{2}e^{2t_2} + ...\right)\ e_1 + (e^{t_2} + ...)\ e_2.$$

It follows that for $t_2 \to -\infty$ we recover the "classical" wedge product. On the other hand, for $t_3 = 0$:

$$e_2 \cdot e_2 = e_3, \quad e_2 \cdot e_3 = q\ e_1, \quad e_3 \cdot e_3 = q\ e_2, \qquad q := e^{t_2}$$

The limit $t_3 = 0$ is called *small cohomology*. By the identification $e_1 \mapsto 1$, $e_2 \mapsto x$, $e_3 \mapsto x^2$, the algebra $T_{(t_1,t_2,0)}QH^*(CP^2)$ is isomorphic to $\mathbf{C}[x]/(x^3 = q)$.

2.2 The case of CP^d

For $d = 1$ the deformation is given by

$$F(t)\frac{1}{2}t_1^2t_2 + e^{t_2}$$

For any $d \geq 2$, the deformation is given by the following solution of the WDVV equations [33] [36]:

$$F(t) = F_0(t) + \sum_{k=1}^{\infty}\left[\sum_{n=2}^{\infty}\tilde{\sum_{\alpha_1,...,\alpha_n}}\frac{N_k(\alpha_1, ..., \alpha_n)}{n!}\ t_{\alpha_1}...t_{\alpha_n}\right]e^{kt_2}$$

where

$$\tilde{\sum_{\alpha_1,...,\alpha_n}} := \sum_{\alpha_1+...+\alpha_n = 2n+d(k+1)+k-3}$$

Here $N_k(\alpha_1, ..., \alpha_n)$ is the number of rational curves $CP^1 \to CP^d$ of degree k through n projective subspaces of codimensions $\alpha_1 - 1, ..., \alpha_n - 1 \geq 2$ in general position. In particular, there is one line through two points, then

$$N_1(d+1, d+1) = 1$$

Note that in Kontsevich solution $N_k = N_k(d+1, d+1)$.

In flat coordinates the *Euler vector field* is

$$E = \sum_{\alpha\neq 2}(1 - q_\alpha)t^\alpha\frac{\partial}{\partial t^\alpha} + k\frac{\partial}{\partial t^2}$$

$$q_1 = 0,\ q_2 = 1,\ q_3 = 2,\ ...,\ q_k = k - 1$$

and

$$\hat{\mu} = \text{diag}(\mu_1, ..., \mu_k) = \text{diag}(-\frac{d}{2}, -\frac{d-2}{2}, ..., \frac{d-2}{2}, \frac{d}{2}), \qquad \mu_\alpha = q_\alpha - \frac{d}{2}$$

2.3 Nature of the singular point X_0

We ask the question whether the singularity X_0 in Kontsevich solution for CP^2 corresponds to the fact that two canonical coordinates u_1, u_2, u_3 merge. Actually, we know that the structure of the semisimple manifold may become singular in such points because the solutions of the boundary value problem (or, equivalently, of the the equations of isomonodromic deformation) are meromorphic on the universal covering of $\mathbf{C}^n\backslash$diagonals and are multivalued if $u_i - u_j$ $(i \neq j)$ describes a loop around zero. In this section we restore the upper indices for the flat coordinates t^α.

The canonical coordinates can be computed from the intersection form. We recall that the flat metric is

$$\eta = (\eta^{\alpha\beta}) := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The intersection form is given by the formula (1.34):

$$g^{\alpha\beta} = (d+1-q_\alpha-q_\beta)\, \eta^{\alpha\mu}\eta^{\beta\nu}\partial_\mu\partial_\nu F + A^{\alpha\beta}, \quad \alpha,\beta = 1,2,3,$$

where $d=2$ and the *charges* are $q_1 = 0$, $q_2 = 1$, $q_3 = 2$. The matrix $A^{\alpha\beta}$ appears in the action of the Euler vector field

$$E := t^1\partial_1 + 3\partial_2 - t^3\partial_3$$

on $F(t^1,t^2,t^3)$:

$$E(F)(t^1,t^2,t^3) = (3-d)F(t^1,t^2,t^3) + A_{\mu\nu}t^\mu t^\nu \equiv F(t^1,t^2,t^3) + 3t^1t^2$$

Thus

$$(A^{\alpha\beta}) = (\eta^{\alpha\mu}\eta^{\beta\nu}A_{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$

After the above preliminaries, we are able to compute the intersection form:

$$(g^{\alpha\beta}) = \begin{pmatrix} \frac{3}{[t^3]^3}[2\Phi - 9\Phi' + 9\Phi''] & \frac{2}{[t^3]^2}[3\Phi'' - \Phi'] & t^1 \\ \frac{2}{[t^3]^2}[3\Phi'' - \Phi'] & t^1 + \frac{1}{t^3}\Phi'' & 3 \\ t^1 & 3 & -t^3 \end{pmatrix}$$

The canonical coordinates are roots of

$$\det((g^{\alpha\beta} - u\eta) = 0$$

This is the polynomial

$$u^3 - \left(3t^1 + \frac{1}{t^3}\Phi''\right) u^2 - \left(-3[t^1]^2 - 2\frac{t^1}{t^3}\Phi'' + \frac{1}{[t^3]^2}(9\Phi'' + 15\Phi' - 6\Phi)\right) u + P(t,\Phi)$$

where

$$P(t,\Phi) = \frac{1}{[t^3]^3}\left(-9t^1t^3\Phi'' + 243\Phi'' - 243\Phi' + 6\Phi\Phi'\right.$$

$$\left.-9(\Phi'')^2 + 6t^1t^3\Phi + [t^1]^2[t^3]^2\Phi'' - 3\Phi'\Phi'' + [t^1]^3[t^3]^3 - 4(\Phi')^2 + 54\Phi - 15t^1t^3\Phi'\right)$$

It follows that

$$u_i(t^1,t^3,X) = t^1 + \frac{1}{t^3}\mathcal{V}_i(X)$$

$\mathcal{V}_i(X)$ depends on X through $\Phi(X)$ and derivatives. We also observe that

$$u_1 + u_2 + u_3 = 3t^1 + \frac{1}{t^3}\Phi''(X)$$

We verify numerically that $u_i \neq u_j$ for $i \neq j$ at $X = X_0$. In order to do this we need to compute $\Phi(X_0)$, $\Phi'(X_0)$, $\Phi''(X_0)$ in the following approximation

$$\Phi(X_0) \cong \sum_{k=1}^{N} A_k \frac{1}{a^k}, \quad \Phi'(X_0) \cong \sum_{k=1}^{N} k\ A_k \frac{1}{a^k}, \qquad \Phi''(X_0) \cong \sum_{k=1}^{N} k^2\ A_k \frac{1}{a^k},$$

In our computation we fixed $N = 1000$ and we computed the A_k, $k = 1, 2, ..., 1000$ exactly using the relation (2.4). Then we computed a and b by the least squares method. For large k, say for $k \geq N_0$, we assumed that

$$A_k \cong b a^k k^{-\frac{7}{2}} \tag{2.6}$$

which implies

$$\ln(A_k\ k^{\frac{7}{2}}) \cong (\ln a)\ k + \ln b$$

The corrections to this law are $O\left(\frac{1}{k}\right)$. This is the line to fit the data $k^{\frac{7}{2}} A_k$. Let

$$\bar{y} := \frac{1}{N - N_0 + 1} \sum_{k=N_0}^{N} \ln(A_k\ k^{\frac{7}{2}}), \qquad \bar{k} := \frac{1}{N - N_0 + 1} \sum_{N_0}^{N} k.$$

By the least squares method

$$\ln a = \frac{\sum_{k=N_0}^{N} (k - \bar{k})(\ln(A_k\ k^{\frac{7}{2}}) - \bar{y})}{\sum_{k=N_0}^{N} (k - \bar{k})^2}, \quad \text{with error } \left(\frac{1}{\bar{k}^2}\right)$$

$$\ln b = \bar{y} - (\ln a)\ \bar{k} \quad \text{with error } \left(\frac{1}{\bar{k}}\right)$$

For $N = 1000$, A_{1000} is of the order 10^{-840}. In our computation we set the accuracy to 890 digits. Here is the results, for three choices of N_0. The result should improve as N_0 increases, since the approximation (2.6) becomes better.

$$N_0 = 500, \quad a = 0.138009444..., \quad b = 6.02651...$$
$$N_0 = 700, \quad a = 0.138009418..., \quad b = 6.03047...$$
$$N_0 = 900, \quad a = 0.138009415..., \quad b = 6.03062...$$

It follows that (for $N_0 = 900$)

$$\Phi(X_0) = 4.268908...\ , \qquad \Phi'(X_0) = 5.408...\ , \qquad \Phi''(X_0) = 12.25...$$

With these values we find

$$27 + 2\Phi'(X_0) - 3\Phi''(X_0) = 1.07...,$$

But the above should vanish! The reason why this does not happen is that $\Phi''(X_0) = \sum_{k=1}^{N} k^2\ A_k\ \frac{1}{a^k}$ converges slowly. To obtain a better approximation we compute it numerically as

$$\Phi''(X_0) = \frac{1}{3}(27 + 2\Phi'(X_0)) = \frac{1}{3}(27 + 2\sum_{k=1}^{N} k\ A_k\ \frac{1}{a^k}) = 12.60...$$

Substituting into $g^{\alpha\beta}$ and setting $t^1 = t^3 = 1$ we find

$$u_1 \approx 22.25...\ , \quad u_2 \approx -(3.5...) - (2.29...)i\ , \quad u_3 = \bar{u}_2$$

Here the bar means conjugation. Thus, with a sufficient accuracy, we have proved that $u_i \neq u_j$ for $i \neq j$.

Finally, we prove that the singularity is a singularity for the change of coordinates

$$(u_1, u_2, u_3) \mapsto (t^1, t^2, t^3)$$

Recall that

$$\frac{\partial u_1}{\partial t^\alpha} = \frac{\psi_{i\alpha}}{\psi_{i1}}$$

This may become infinite if $\psi_{i1} = 0$ for some i. In our case

$$u_1 + u_2 + u_3 = 3t^1 + \frac{1}{t^3}\Phi(X)'', \quad \frac{\partial X}{\partial t^1} = 0, \quad \frac{\partial X}{\partial t^2} = 1, \quad \frac{\partial X}{\partial t^3} = \frac{3}{t^3}$$

and we compute

$$\frac{\partial}{\partial t^1}(u_1 + u_2 + u_3) = 3,$$

$$\frac{\partial}{\partial t^2}(u_1 + u_2 + u_3) = \frac{1}{t^3}\Phi(X)''',$$

$$\frac{\partial}{\partial t^3}(u_1 + u_2 + u_3) = -\frac{1}{[t^3]^2}\Phi(X)'' + \frac{3}{[t^3]^2}\Phi(X)'''.$$

Thus, we see that the change of coordinates is singular because both $\frac{\partial}{\partial t^2}(u_1+u_2+u_3)$ and $\frac{\partial}{\partial t^3}(u_1+u_2+u_3)$ behave like $\Phi(X)''' \asymp \frac{1}{\sqrt{X-X_0}}$ for $X \to X_0$.

I thank P. Bleher for suggesting me to try the computations of this section.

Chapter 3

Inverse Reconstruction of 2-dimensional FM

This chapter explains in a didactic way the process of inverse reconstruction of a semisimple FM for $n = 2$ through the formulae (1.22), (1.23).

3.1 Exact Solution and Monodromy data

The coefficients of the system $dY/dz = [U + V/z]Y$ are *necessarily*:

$$V(u) = \begin{pmatrix} 0 & i\frac{\sigma}{2} \\ -i\frac{\sigma}{2} & 0 \end{pmatrix}, \qquad U = \text{diag}(u_1, u_2)$$

Here $u = (u_1, u_2)$, and V is independent of u. It has the diagonal form

$$\Psi^{-1} V \Psi = \text{diag}\left(\frac{\sigma}{2}, -\frac{\sigma}{2}\right) =: \hat{\mu},$$

where

$$\Psi(u) = \begin{pmatrix} \frac{1}{2f(u)} & f(u) \\ \frac{1}{2if(u)} & if(u) \end{pmatrix}, \qquad \Psi^T \Psi = \eta := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The description of the Stokes' phenomenon for

$$\frac{dY}{dz} = \left[\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + \begin{pmatrix} 0 & i\frac{\sigma}{2} \\ -i\frac{\sigma}{2} & 0 \end{pmatrix}\right] Y \tag{3.1}$$

requires the stokes rays

$$R_{12} = -i\rho(u_1 - u_2), \quad \rho > 0; \qquad R_{21} = -R_{12}$$

Let l be an oriented line through the origin, not containing the Stokes' rays and having R_{12} to the left. $l = l_+ + l_-$. Then

$$Y_L(z, u) = Y_R(z, u)\ S \quad \text{from the side of } l_+$$

$$Y_L(z, u) = Y_R(ze^{-2\pi i}, u)\ S^T \quad \text{from the side of } l_-$$

$$S = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \qquad s \in \mathbf{C}$$

Let Π_R be the half plane to the right of l. At the origin

$$Y(z, u) = \Phi(z, u) z^{\hat{\mu}} z^R, \qquad \Phi(z, u) = \sum_{k=0}^{\infty} \phi_k(u) z^k, \qquad \phi_0 = \Psi, \tag{3.2}$$

$$Y_0(z, u) = Y_R(z, u)\ C \quad \text{in } \Pi_R$$

C is the central connection matrix. Then

$$S^T S^{-1} = C e^{2\pi i \begin{pmatrix} \frac{\sigma}{2} & 0 \\ 0 & -\frac{\sigma}{2} \end{pmatrix}} e^{2\pi i R} C^{-1}$$

From the trace

$$s^2 = 2(1 - \cos(\pi\sigma))$$

The above monodromy data R, $\hat{\mu}$, S define the boundary value problem of section 1.8. The standard technique to solve a 2-dimensional boundary value problem is to reduce it to a system of differential equations, which is (3.1) in our case, and then to reduce the system to a second order differential equation. It turns out that the equation is (after a change of dependent and independent variables) a Whittaker equation. Therefore, the solution of the b.v.p. is given in terms of Whittaker functions. We skip the details. Let $H := u_1 - u_2$; the fundamental solutions are

$$Y_R(z,u) = \begin{pmatrix} e^{i\frac{\pi}{2}} (H\ z)^{-\frac{1}{2}} e^{z\frac{u_1+u_2}{2}} W_{\frac{1}{2},\frac{\sigma}{2}}\left(e^{-i\pi} H\ z\right) & -i\frac{\sigma}{2} (H\ z)^{-\frac{1}{2}} e^{z\frac{u_1+u_2}{2}} W_{-\frac{1}{2},\frac{\sigma}{2}}(H\ z) \\ i\frac{\sigma}{2} e^{i\frac{\pi}{2}} (H\ z)^{-\frac{1}{2}} e^{z\frac{u_1+u_2}{2}} W_{-\frac{1}{2},\frac{\sigma}{2}}\left(e^{-i\pi} H\ z\right) & (H\ z)^{-\frac{1}{2}} e^{z\frac{u_1+u_2}{2}} W_{\frac{1}{2},\frac{\sigma}{2}}(H\ z) \end{pmatrix}$$

$$\arg(R_{12}) < \arg(z) < \arg(R_{12}) + 2\pi$$

where

$$\arg(R_{12}) := -\frac{\pi}{2} - \arg(u_1 - u_2)$$

$$Y_L(z,u) = \begin{pmatrix} (Y_R(z,u))_{11} & i\frac{\sigma}{2} (H\ z)^{-\frac{1}{2}} e^{z\frac{u_1+u_2}{2}} W_{-\frac{1}{2},\frac{\sigma}{2}}\left(e^{-2i\pi} H\ z\right) \\ (Y_R(z,u))_{12} & -(H\ z)^{-\frac{1}{2}} e^{z\frac{u_1+u_2}{2}} W_{\frac{1}{2},\frac{\sigma}{2}}\left(e^{-2i\pi} H\ z\right) \end{pmatrix}$$

$$\arg(R_{12}) + \pi < \arg(z) < \arg(R_{12}) + 3\pi.$$

$W_{\kappa,\mu}$ are the Whittaker functions.

For the choice of Y_R and Y_L above, also the sign of s can be determined. According to our computations it is

$$s = 2\sin\left(\frac{\pi\sigma}{2}\right)$$

It is computed from the expansion of Y_R and Y_L at $z = 0$.

We stress that *the only monodromy data* are σ and the non-zero entry of R. The purpose of this didactic chapter is to show that the inverse reconstruction of the manifold through (1.22) and (1.23) brings solutions $F(t)$ of the WDVV eqs. *explicitly parametrized by σ and R.*

3.2 Preliminary Computations

The functions $\phi_p(u)$ to be plugged into (1.22), (1.23) may be derived from the above representations in terms of Whittaker functions. We prefer to proceed in a different way, namely by imposing the conditions of isomonodromicity (1.13) to the solution (3.2).

The function $f(u)$ in $\Psi(u) \equiv \phi_0(u)$ is arbitrary, but subject to the condition of isomonodromicity (1.13) for $p = 0$, namely

$$\partial_i \phi_0 = V_i \phi_0$$

where

$$V_1 = \frac{V}{u_1 - u_2}, \quad V_2 = -V_1,$$

Let

$$\mathcal{U} := \Psi^{-1} U \Psi, \quad \mu_1 := \frac{\sigma}{2}, \quad \mu_2 = -\frac{\sigma}{2}, \quad d = -\sigma$$

We will use $h(u)$ to denote an arbitrary functions of u. Let's also denote the entry (i,j) of a matrix A_k by $A_{ij,k}$ or by $(A_k)_{ij}$ according to the convenience. Let $R = R_1 + R_2 + R_3 + \ldots$, $R_{ij,k} \neq 0$ only if

$\mu_i - \mu_j = k > 0$ integer. In order to compute $\phi_p(u)$ of (3.2) we decompose it (and define $H_p(u)$) as follows:

$$\phi_p(u) := \Psi \; H_p(u), \qquad p = 0, 1, 2, ...$$

Plugging the above into (3.1) we obtain

1**

$$\phi_0 = \Psi$$

2**

$$\mu_1 \neq \pm\frac{1}{2}, \qquad H_{ij,1} = \frac{\mathcal{U}_{ij}}{1 + \mu_j - \mu_i}, \qquad R = 0$$

$$\mu_1 = \frac{1}{2}, \qquad H_{12,1} = h_1(u), \qquad R_{12,1} = \mathcal{U}_{12}$$

$$\mu_1 = -\frac{1}{2}, \qquad H_{21,1} = h_1(u), \qquad R_{21,1} = \mathcal{U}_{21}$$

3**

$$\mu \neq \pm 1, \qquad H_{ij,2} = \frac{(\mathcal{U}H_1 - H_1 R_1)_{ij}}{2 + \mu_j - \mu_i}, \qquad R_2 = 0$$

$$\mu = 1, \qquad H_{12,2} = h_2(u), \qquad R_{12,2} = (\mathcal{U}H_1)_{12}, \qquad R_1 = 0$$

$$\mu = -1, \qquad H_{21,2} = h_2(u), \qquad R_{21,2} = (\mathcal{U}H_1)_{21}, \qquad R_1 = 0$$

4**

$$\mu \neq \pm\frac{3}{2}, \qquad H_{ij,3} = \frac{(\mathcal{U}H_2 - H_1 R_2 - H_2 R_1)_{ij}}{3 + \mu_j - \mu_i}, \qquad R_3 = 0$$

$$\mu = \frac{3}{2}, \qquad H_{12,3} = h_3(u), \qquad R_{12,3} = (\mathcal{U}H_2)_{12}, \qquad R_1 = R_2 = 0$$

$$\mu = -\frac{3}{2}, \qquad H_{21,3} = h_3(u), \qquad R_{21,3} = (\mathcal{U}H_2)_{21}, \qquad R_1 = R_2 = 0.$$

We compute $t = t(u)$, $F = F(t(u))$ from the formulae

$$t^1 = \sum_{i=1}^{2} \phi_{i2,0}\phi_{i1,1}, \qquad t^2 = \sum_{i=1}^{2} \phi_{i1,0}\phi_{i1,1} \tag{3.3}$$

$$F = \frac{1}{2}\left[t^\alpha t^\beta \sum_{i=1}^{2} \phi_{i\alpha,0}\phi_{i\beta,1} - \sum_{i=1}^{2} (\phi_{i1,1}\phi_{i1,2} + \phi_{i1,3}\phi_{i1,0}) \right] \tag{3.4}$$

F is defined up to quadratic terms.

In the following we will compute $F = F(t)$ $(t = (t^1, t^2))$ in closed form, obtaining a solution of the WDVV equations. The only needed ingredients are (3.3), (3.4), the conditions of isomonodromicity (1.13) and the symmetries (1.14) for $p = 0, 1, 2, 3$.

For any value of σ the isomonodromicity condition $\partial_i \phi_0 = V_i \phi_0$ reads

$$\frac{\partial f(u)}{\partial u_1} = -\frac{\sigma}{2} \frac{f(u)}{u_1 - u_2}$$

$$\frac{\partial f(u)}{\partial u_2} = \frac{\sigma}{2} \frac{f(u)}{u_1 - u_2}$$

In other words $\frac{\partial f(u)}{\partial u_1} = -\frac{\partial f(u)}{\partial u_2}$ and thus $f(u) \equiv f(u_1 - u_2)$. Let

$$H := u_1 - u_2$$

Therefore

$$\frac{f(H)}{dH} = -\frac{\sigma}{2}\frac{f(H)}{H} \quad \Longrightarrow \quad f(H) = C \; H^{-\frac{\sigma}{2}}, \qquad C \text{ a constant}$$

3.3 The generic case

We start from the generic case: σ not integer. The result of the application of formulae (3.3), (3.4) is

$$t^1 = \frac{u_1+u_2}{2}$$

$$t^2 = \frac{1}{4(1+\sigma)}\frac{u_1-u_2}{f(u)^2}$$

and

$$F(t(u)) = \frac{1}{2}(t^1)^2t^2 + \frac{2(1+\sigma)^3}{(1-\sigma)(\sigma+3)}(t^2)^3 f(u)^4$$

But now observe that

$$f(u)^2 = \frac{u_1-u_2}{4(1+\sigma)t^2}$$

$$u_1-u_2 = H$$

$$f(u)^2 \equiv f(H)^2 = C^2H^{-\sigma}$$

The above three expressions imply

$$H = C_1\ (t^2)^{\frac{1}{1+\sigma}}, \quad C_1 = [4(1+\sigma)C^2]^{\frac{1}{1+\sigma}}$$

Therefore

$$f(u)^4 = C_2(t^2)^{-2\frac{\sigma}{1+\sigma}}$$

and

$$F(t) = \frac{1}{2}(t^1)^2t^2 + C_3(t^2)^{\frac{\sigma+3}{\sigma+1}}$$

as we wanted. Here C_3 (or C, or C_1, or C_2) is an arbitrary constant.

3.4 The cases $\mu_1 = \frac{3}{2}$, $\mu_1 = 1$, $\mu_1 = -1$

1) Case $\mu_1 = \frac{3}{2}$, $\sigma = 3$

Formula (3.3) gives the same result of the generic case (with $\sigma = 3$) because $h_3(u)$ appears only in $\phi_3(u)$ and does not affect t:

$$t^1 = \frac{u_1+u_2}{2}$$

$$t^2 = \frac{1}{4(1+\sigma)}\frac{u_1-u_2}{f(u)^2}\Big|_{\sigma=3}$$

Although $h_3(u)$ appears in $\phi_3(u)$ it is not in F:

$$F(t(u)) = \frac{1}{2}(t^1)^2t^2 + \frac{2(1+\sigma)^3}{(1-\sigma)(\sigma+3)}(t^2)^3 f(u)^4\Big|_{\sigma=3}$$

We may proceed as in the generic case. Actually, now the computation of $f(u)$ is straightforward because

$$R_3 = \begin{pmatrix} 0 & -\frac{1}{16}(u_1-u_2)^3f(u)^2 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, \qquad r = \text{constant}$$

Namely

$$-\frac{1}{16}(u_1-u_2)^3f(u)^2 = r$$

On the other hand, from t^2 we have

$$u_1-u_2 = 16t^2f(u)^2$$

and thus

$$f(u)^4 = \frac{(-r)^{\frac{1}{2}}}{16(t^2)^{\frac{3}{2}}}$$

and finally

$$F(t) = \frac{1}{2}(t^1)^2 t^2 - \frac{3}{2}(-r)^{\frac{1}{2}}(t^2)^{\frac{3}{2}} \equiv \frac{1}{2}(t^1)^2 t^2 + C(t^2)^{\frac{3}{2}}$$

where C is an arbitrary constant.

2) Case $\mu_1 = 1$, $\sigma = 2$.

Again, the arbitrary function $h_2(u)$ does not appear in $t(u)$ and $F(t(u))$:

$$t^1 = \frac{u_1 + u_2}{2}$$

$$t^2 = \frac{1}{4(1+\sigma)} \frac{u_1 - u_2}{f(u)^2}\Big|_{\sigma=2},$$

$$F(t(u)) = \frac{1}{2}(t^1)^2 t^2 + \frac{2(1+\sigma)^3}{(1-\sigma)(\sigma+3)}(t^2)^3 f(u)^4\Big|_{\sigma=2}.$$

Now we proceed like in the generic case and we find the generic result with $\sigma = 2$.

3) Case $\mu_1 = -1$, $\sigma = -2$

Now the formulae (3.3), (3.4) yield

$$t^1 = \frac{u_1 + u_2}{2}$$

$$t^2 = \frac{1}{4} \frac{u_2 - u_1}{f(u)^2}$$

$$F(t(u)) = \frac{3}{2}(t^1)^2 t^2 - \frac{2}{3}(t^2)^3 f(u)^4 - t^1 h_2(u)$$

The condition

$$0 = \phi_0^T \phi_2 - \phi_1^T \phi_1 + \phi_2^T \phi_0 = \begin{pmatrix} 2h_2(u) + \frac{u_1^2 - u_2^2}{4f(u)^2} & 0 \\ 0 & 0 \end{pmatrix}$$

implies

$$h_2(u) = \frac{1}{8} \frac{u_2^2 - u_1^2}{f(u)^2} \equiv t^1 t^2$$

Therefore

$$F(t(u)) = \frac{1}{2}(t^1)^2 t^2 - \frac{2}{3}(t^2)^3 f(u)^4$$

Now we proceed as in the generic case, using $f(H) = CH^{-\frac{\sigma}{2}} = CH$ and we find the generic result with $\sigma = -2$.

3.5 The case $\mu_1 = -\frac{1}{2}$

We analyze the case $\mu_1 = -\frac{1}{2}$, $\sigma = -1$. The formula (3.3) gives

$$t^1 = \frac{u_1 + u_2}{2}$$

$$t^2 = h_1(u)$$

By putting $h_1(u) = t^2$ we get from (3.4)

$$F(t(u)) = \frac{1}{2}(t^1)^2 t^2 + \frac{1}{16} \frac{(u_1 - t^1)^3}{f(u)^2} = \frac{1}{2}(t^1)^2 t^2 + \frac{1}{16} \frac{\left(\frac{u_1 - u_2}{2}\right)^3}{f(u)^2}$$

From the condition $\partial_{u_i} \Psi = V_i \Psi$ we computed the differential equation for $f(u_1 - u_2) = f(H)$ and we got $f(H) = CH^{-\frac{\sigma}{2}} = CH^{\frac{1}{2}}$. But it is straightforward to obtain $f(u)$ from

$$R_1 = \begin{pmatrix} 0 & 0 \\ \frac{u_1 - u_2}{4f(u)^2} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}$$

from which

$$f(u)^2 = \frac{u_1 - u_2}{4r} = \frac{1}{4r}H$$

The last thing we need is to determine H as a function of t^1, t^2. We can't use the condition $\Phi(-z)^T\Phi(z) = \eta$, because direct computation shows that the following are identically satisfied: $\phi_0^T\phi_1 - \phi_1^T\phi_0 = 0$, $\phi_0^T\phi_2 - \phi_1^T\phi_1 + \phi_2^T\phi_0 = 0$, $\phi_0^T\phi_3 - \phi_1^T\phi_2 + \phi_2^T\phi_1 - \phi_3^T\phi_0 = 0$. We make use of the isomonodromicity conditions

$$\frac{\partial \phi_1}{\partial u_1} = E_1\phi_0 + V_1\phi_1, \quad \frac{\partial \phi_1}{\partial u_2} = E_2\phi_0 + V_2\phi_1$$

which become

$$\frac{\partial h_1(u)}{\partial u_1} = \frac{1}{4f(u)^2}, \quad \frac{\partial h_1(u)}{\partial u_2} = -\frac{\partial h_1(u)}{\partial u_1}$$

Thus

$$h_1(u) \equiv h_1(u_1 - u_2),$$

and

$$\frac{dh_1(H)}{dH} = \frac{r}{H} \implies t^2 \equiv h_1(H) = r\ln(H) + D$$

D a constant.

$$f(u)^4 = \frac{H^2}{16r^2} = Ce^{2\frac{t^2}{r}}$$

C a constant ($C = \exp(-2D)$) We get the final result

$$F(t) = \frac{1}{2}(t^1)^2t^2 + Ce^{2\frac{t^2}{r}}$$

3.6 The case $\mu_1 = \frac{1}{2}$

Let $\mu_1 = \frac{1}{2}$, $\sigma = 1$. t is like in the generic case

$$t^1 = \frac{u_1 + u_2}{2}$$

$$t^2 = \frac{u_1 - u_2}{8f(u)^2}$$

while F contains $h_1(u)$

$$f(t(u)) = \frac{1}{2}(t^1)^2t^2 + \frac{1}{2}(t^2)^2h_1(u) - 3(t^2)^3f(u)^4$$

We determine $f(u)$ as in the generic case, or better we observe that

$$R_1 = \begin{pmatrix} 0 & (u_1 - u_2)f(u)^2 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}$$

Thus

$$f(u)^2 = \frac{r}{u_1 - u_2}$$

We determine $h_1(u)$. The condition $\Phi(-z)^T\Phi(z) = \eta$ does not help, because it is automatically satisfied. We use the isomonodromicity conditions

$$\frac{\partial \phi_1}{\partial u_1} = E_1\phi_0 + V_1\phi_1, \quad \frac{\partial \phi_1}{\partial u_2} = E_2\phi_0 + V_2\phi_1$$

which become

$$\frac{\partial h_1(u)}{\partial u_1} = f(u)^2, \quad \frac{\partial h_1(u)}{\partial u_2} = -\frac{\partial h_1(u)}{\partial u_1}$$

Therefore

$$h_1(u) \equiv h_1(u_1 - u_2)$$

We keep into account that $f(u)^2 = r/H$:

$$\frac{dh_1(H)}{dH} = \frac{r}{H} \implies h_1(H) = r\ln(H) + D$$

D a constant. Finally, recall that

$$t^2 = \frac{H}{8f(u)^2} \equiv \frac{H^2}{8r},$$

hence

$$f(u)^4 = \frac{r}{8t^2},$$

which contributes a linear term to $F(t)$, and

$$h_1(u) = \frac{r}{2}\ln(t^2) + B$$

$B = \frac{r}{2}\ln(8r) + C$ is an arbitrary constant. Finally

$$F(t) = \frac{1}{2}(t^1)^2 t^2 + \frac{r}{4}(t^2)^2\ln(t^2)$$

as we wanted.

3.7 The case $\mu_1 = -\frac{3}{2}$

Finally, let's take $\mu_1 = -\frac{3}{2}$, $\sigma = -3$. From (3.3), (3.4) we have

$$t^1 = \frac{u_1 + u_2}{2}$$

$$t^2 = \frac{u_2 - u_1}{8f(u)^2}$$

$$F(t(u)) = \frac{3}{4}(t^1)^2 t^2 + (t^2)^3 f(u)^4 - \frac{1}{2}h_3(u)$$

$f(u)$ is obtainable as in the generic case, but it is more straightforward to use

$$R_3 = \begin{pmatrix} 0 & 0 \\ \frac{(u_2-u_1)^3}{64f(u)^2} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix} \implies f(u)^2 = \frac{(u_2-u_1)^3}{64r}$$

To obtain $h_3(u)$ we can't rely on $\Phi(-z)^T\Phi(z) = \eta$, which turns out to be identically satisfied. We use again the conditions

$$\frac{\partial \phi_3}{\partial u_i} = E_i\phi_2 + V_i\phi_3 \tag{3.5}$$

It is convenient to introduce

$$G(u) := \frac{1}{4}(t^1)^2 t^2 - \frac{1}{2}h_3(u)$$

The above (3.5) becomes

$$\frac{\partial G}{\partial u_1} = \frac{r}{2(u_2-u_1)}, \qquad \frac{\partial G}{\partial u_2} = -\frac{\partial G}{\partial u_1}$$

which implies $G(u) = G(u_1 - u_2)$ and

$$\frac{dG}{dH} = -\frac{r}{2H} \implies G(H) = -\frac{r}{2}\ln(H) + C$$

C a constant. Finally, recall that

$$t^2 = \frac{H^2}{8r} \implies G(H(t^2)) = \frac{r}{4}\ln(t^2) + C_1$$

Thus

$$F(t) = \frac{1}{2}(t^1)^2 t^2 + \frac{r}{4}\ln(t^2)$$

having dropped the constant terms.

3.8 Conclusions

The solution of the boundary value problem for the monodromy data σ and the non zero entry r of the matrix R (Stokes matrices and other monodromy data depend on σ) was obtained solving the equations (1.13) with the constraints (1.14).

We have obtained the solutions of the WDVV eqs. which we already derived from elementary considerations in chapter 1. They are parametrized explicitly by the monodromy data:

$$\text{For } \sigma \neq \pm 1, -3 \qquad F(t) = \frac{1}{2}(t^1)^2 t^2 + C(t^2)^{\frac{3+\sigma}{1+\sigma}}$$

where C is a constant.

$$\sigma = -1, \qquad F(t) = \frac{1}{2}(t^1)^2 t^2 + Ce^{2\frac{t^2}{r}}$$

$$\sigma = 1, \qquad F(t) = \frac{1}{2}(t^1)^2 t^2 + C(t^2)^2 \ln(t^2)$$

$$\sigma = -3, \qquad F(t) = \frac{1}{2}(t^1)^2 t^2 + C \ln(t^2)$$

Chapter 4

Stokes Matrices and Monodromy of $QH^*(CP^d)$

4.1 Introduction

In this chapter we compute Stokes' matrices and monodromy group for the Frobenius manifold given by the quantum cohomology of the projective space $\mathbf{CP}^d$. The first motivation is to know the monodromy data of $QH^*(CP^d)$, in view of the solution of the inverse problem. Actually, the global analytic properties of the solution of the WDVV eqs. obtained by Kontsevich-Manin (see section 2.2) are unknown and the boundary value problem may shed light on them.

The second motivation is to study the links between quantum cohomology and the theory of coherent sheaves, in order to prove a long-lasting conjecture explained in the introduction of the thesis.

The main result of this chapter is the proof (theorems 2, 2′) that the conjecture about coincidence of the Stokes matrix for quantum cohomology of $\mathbf{CP}^d$ and the Gram matrix $\chi(E_i, E_j)$ of a full exceptional collection in $Der^b(Coh(\mathbf{CP}^d))$ is true. We prove that the Stokes'matrix can be reduced to the canonical form $S = (s_{ij})$ where

$$s_{ii} = 1, \quad s_{ij} = \binom{d+1}{j-i}, \quad s_{ji} = 0, \quad i < j,$$

by the action of the braid group. This form is equal to the Gram matrix $\chi(\mathcal{O}(i-1), \mathcal{O}(j-1))$ (modulo the action of the braid group). See the introduction to the thesis for further details.

In this way, we generalize to any d the result obtained in [17] for $d = 2$.

We also study the structure of the monodromy group of the quantum cohomology of $\mathbf{CP}^d$. We prove (theorem 3 of this chapter) that for $d = 3$ the group is isomorphic to the subgroup of orientation preserving transformations in the hyperbolic triangular group $[2, 4, \infty]$. In [17] it was proved that for $d = 2$ the monodromy group is isomorphic to the direct product of the subgroup of orientation preserving transformations in $[2, 3, \infty]$ and the cyclic group of order 2, $C_2 = \{\pm\}$. Our numerical calculations also suggest that for any d even the monodromy group may be isomorphic to the orientation preserving transformations in $[2, d+1, \infty]$, and for any d odd to the direct product of the orientation preserving transformations in $[2, d+1, \infty]$ by C_2.

4.2 The system corresponding to $\mathbf{CP}^{k-1}$

We introduce here the linear system of differential equations whose Stokes matrices are the Stokes matrices for the quantum cohomology of $\mathbf{CP}^{k-1}$. We use the more convenient choice $k = d + 1$. First of all we recall that

$$\hat{\mu} = \text{diag}(\mu_1, ..., \mu_k) = \text{diag}(-\frac{k-1}{2}, -\frac{k-3}{2}, ..., \frac{k-3}{2}, \frac{k-1}{2}), \qquad \mu_\alpha = q_\alpha - \frac{d}{2}$$

Consider the system of differential equations determining deformed flat coordinates:

$$\partial_z \xi = (\mathcal{U}(t) + \frac{1}{z}\mu)\xi \tag{4.1}$$

$$\partial_\alpha \xi = z C_\alpha(t)\xi$$

Let us compute $C_2(t)$ and $\mathcal{U}(t)$ at the semi-simple point $(0, t^2, 0, ..., 0)$. In order to do this the reader should look back at section 2.2

$$E \cdot \partial_\beta = E^\gamma c^\alpha_{\gamma\beta}\ \partial_\alpha = E^2 c^\alpha_{2\beta}\ \partial_\alpha = k\ c^\alpha_{2\beta}\partial_\alpha \equiv \mathcal{U}^\alpha{}_\beta \partial_\alpha$$

Moreover $c_{2\alpha\beta}(0, t^2, 0, ..., 0) = \partial_2\partial_\alpha\partial_\beta F(0, t^2, ..., 0)$. This immediately yields

$$C_2(0, t^2, 0, ..., 0) = \begin{pmatrix} 0 & & & & e^{t^2} \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix}, \quad \mathcal{U}(0, t^2, 0, ..., 0) = \begin{pmatrix} 0 & & & & ke^{t^2} \\ k & 0 & & & \\ & k & 0 & & \\ & & \ddots & \ddots & \\ & & & k & 0 \end{pmatrix}$$

We warn the reader that for convenience we use slightly different notations and normalizations than in chapter 1. Let $y_\alpha := \eta_{\alpha\beta}\xi^\beta \equiv \partial_\alpha \tilde{t}$. It satisfies

$$\partial_z y = \left(\hat{\mathcal{U}}(t^2) - \frac{1}{z}\hat{\mu}\right) y \tag{4.2}$$

$$\partial_2 y = z\hat{C}_2(t^2) y \tag{4.3}$$

where

$$\hat{C}_2(t^2) := \eta\ C_2(0, t^2, ..., 0)\ \eta^{-1} = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ e^{t^2} & & & & & 0 \end{pmatrix}$$

$$\hat{\mathcal{U}}(t^2) := \eta\ \mathcal{U}(0, t^2, ..., 0)\ \eta^{-1} = \begin{pmatrix} 0 & k & & & & \\ & 0 & k & & & \\ & & 0 & k & & \\ & & & \ddots & \ddots & \\ & & & & 0 & k \\ ke^{t^2} & & & & & 0 \end{pmatrix}$$

Lemma 1: *Let $y(t^2, z) = (y_1(t^2, z), ..., y_k(t^2, z))^T$ be a column vector solution of the above system (4.2) (4.3). With the following substitution*

$$y_\alpha(t^2, z) = \frac{1}{k^{\alpha-1}}\ z^{\frac{k-1}{2}-\alpha+1}\ (z\partial_z)^{(\alpha-1)}\ \varphi(t^2, z) \tag{4.4}$$

$$\equiv z^{\frac{k-1}{2}-\alpha+1}\ \partial_2^{\alpha-1}\varphi(t^2, z), \quad \alpha = 1, 2, ..., k \tag{4.5}$$

the above system is equivalent to the equations

$$(z\partial_z)^k \varphi = (kz)^k e^{t_2}\ \varphi \tag{4.6}$$

$$\partial_2^k \varphi = z^k e^{t_2}\ \varphi \tag{4.7}$$

The proof is a simple calculation we leave to the reader. The substitution of the lemma implies $\partial_2\varphi = \frac{1}{k} z\partial_z\varphi$. Then

$$e^{\frac{t^2}{k}} \frac{\partial}{\partial e^{\frac{t^2}{k}}}\varphi(t^2, z) = z\frac{\partial}{\partial z}\varphi(t^2, z)$$

which implies (with abuse of notation)

$$\varphi(t^2, z) \equiv \varphi(ze^{\frac{t^2}{k}})$$

Namely, φ (at $(0, t^2, ..., 0)$) depends on one argument $w = ze^{\frac{t^2}{k}}$ and satisfies the *generalized hypergeometric equation*

$$\left(w\frac{d}{dw}\right)^k \varphi(w) = (kw)^k \varphi(w) \tag{4.8}$$

The equation is equivalent to the system

$$\frac{dY}{dw} = \left[\hat{\mathcal{U}} + \frac{\mu}{w}\right] Y \tag{4.9}$$

where

$$Y_n(w) = \frac{1}{k^{n-1}} w^{\frac{k-1}{2}-n+1} (w\partial_w)^{(n-1)} \varphi(w) \quad n = 1, 2, ..., k \tag{4.10}$$

$$\hat{\mathcal{U}} := \hat{\mathcal{U}}(0) = \begin{bmatrix} 0 & k & & & & \\ & 0 & k & & & \\ & & 0 & k & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & k \\ k & & & & & 0 \end{bmatrix}$$

$$\mu := -\hat{\mu} = \operatorname{diag}\left(\frac{k-1}{2}, \frac{k-3}{2}, \frac{k-5}{2}, ..., -\frac{k-3}{2}, -\frac{k-1}{2}\right)$$

The system (4.9) may also be interpreted as the system (4.2) with $t^2 = 0$. We will return later on the connection between its monodromy data and the monodromy data of the system (4.2).

Let us study system (4.9). We change notation and choose the more familiar letter z instead of w. So, the system (4.9) is re-written as

$$\frac{dY}{dz} = \left[\hat{\mathcal{U}} + \frac{\mu}{z}\right] Y \tag{4.11}$$

and (4.10) (4.8) become

$$Y_n(z) = \frac{1}{k^{n-1}} z^{\frac{k-1}{2}-n+1} (z\partial_z)^{(n-1)} \varphi(z) \quad n = 1, 2, ..., k \tag{4.12}$$

$$\left(z\frac{d}{dz}\right)^k \varphi(z) = (kz)^k \varphi(z) \tag{4.13}$$

The point $z = 0$ is e fuchsian singularity, and $z = \infty$ is a singularity of the second kind. (4.11) has a fundamental matrix solution $Y_0(z)$ whose behaviour at $z = 0$ is

$$Y_0(z) = (I + O(z))z^{\hat{\mu}}z^R \qquad R = \begin{bmatrix} 0 & k & & & & \\ & 0 & k & & & \\ & & 0 & k & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & k \\ & & & & & 0 \end{bmatrix}$$

and the monodromy for a counter-clockwise loop around the origin is $e^{2\pi i\mu}e^{2\pi iR}$.

The characteristic polynomial of the matrix $\hat{\mathcal{U}}$ is $0 = \det(\hat{\mathcal{U}} - u) = (-u)^k + (-1)^{k+1}k^k$. It has k distinct eigenvalues $u_n = k\, e^{\frac{2\pi i(n-1)}{k}}$, $n = 1, ..., k$. The equations for the eigenvector $\mathbf{x}_n$ corresponding to u_n, namely $\hat{\mathcal{U}}\mathbf{x}_n = u_n\, \mathbf{x}_n$, written for the components ${x^1}_n$, ...,${x^k}_n$ of the column vector $\mathbf{x}_n$ are

$${x^{l+1}}_n = e^{\frac{2\pi i(n-1)}{k}} {x^l}_n \quad l = 1, 2, ..., k-1, \quad {x^1}_n = e^{\frac{2\pi i(n-1)}{k}} {x^k}_n$$

With the choice $x^1{}_n = e^{\frac{i\pi(n-1)}{k}}$ we get $\mathbf{x}_n = (e^{\frac{i\pi(n-1)}{k}}, e^{\frac{i3\pi(n-1)}{k}}, e^{\frac{i5\pi(n-1)}{k}}, ..., e^{-\frac{i\pi(n-1)}{k}})^T$. The matrix

$$X = \frac{1}{\sqrt{k}} [\mathbf{x}_1|\mathbf{x}_2|...|\mathbf{x}_k] = \frac{1}{\sqrt{k}} (x^j{}_n) \qquad x^j{}_n = e^{(2j-1)i\pi\frac{n-1}{k}} \quad j,n = 1,2,...,k$$

puts $\hat{\mathcal{U}}$ in diagonal form:

$$U = X^{-1}\hat{\mathcal{U}}X = \text{diag}(u_1, u_2, ..., u_n, ..., u_k) \qquad u_n = k\ e^{\frac{2\pi i(n-1)}{k}}$$

We stress that $u_i \neq u_j$ for $i \neq j$. The system (4.11) is transformed by the gauge X in an equivalent form

$$\frac{d\tilde{Y}}{dz} = \left[U + \frac{V}{z}\right]\tilde{Y} \tag{4.14}$$

$$\tilde{Y} = X^{-1}Y, \quad U = X^{-1}\hat{\mathcal{U}}X, \quad V = X^{-1}\mu X$$

Observe that $\eta\mu + \mu\eta = 0$, $XX^T = \eta^{-1}$, and therefore $V + V^T = 0$. Anyway, V here is not exactly the matrix V in chapter 1, but its complete permutation.

4.3 Asymptotic Behaviour and Stokes' Phenomenon

Our aim is to explicitly compute a Stokes' matrix for the above system (4.14), or for the system (4.11). The system (4.14) has formal solution

$$\tilde{Y}_F = \left[I + \frac{F_1}{z} + \frac{F_2}{z^2} + ...\right]\ e^{z\ U}$$

where F_j's are $k \times k$ matrices. A possible choice for the labelling of the rays is the following: we call R_{rs} the Stokes' ray

$$R_{rs} = \{z = -i\rho(\bar{u_r} - \bar{u_s}), \quad \rho > 0\} \qquad r \neq s$$

Lemma 2: *For $r < s$ the Stokes' rays of the system (4.14) are*

$$R_{rs} = \left\{z = \rho\ \exp\left(i\left[\frac{2\pi}{k} - \frac{\pi}{k}(r+s)\right]\right), \quad \rho > 0\right\}$$

$$R_{sr} = -R_{rs}$$

Proof: Just compute

$$-i(\bar{u_r} - \bar{u_s}) = -i(e^{-i\frac{2\pi}{k}(r-1)} - e^{-i\frac{2\pi}{k}(s-1)}) =$$
$$= 2\sin\left(\frac{\pi}{k}(s-r)\right)\ e^{\left(i\left[\frac{2\pi}{k} - \frac{\pi}{k}(r+s)\right]\right)}$$

Then we note that $\sin\left(\frac{\pi}{k}(s-r)\right)$ is positive because $0 < s - r \leq k - 1$.

□

Remark 1: $R_{rs} = R_{pq}$ for $r + s = p + q$. R_{12} is at $\arg z = -\frac{\pi}{k}$, R_{13} is at $\arg z = -\frac{2\pi}{k}$, and so on. For $r + s = k + 2$ the corresponding R_{rs}'s are at $\arg z = -\pi$ and the R_{sr}'s are at $\arg z = 0$. $R_{k-1,k}$ is at the angle $-2\pi + \frac{3\pi}{k}$ or, equivalently, at $\frac{3\pi}{k}$. See figure 1.

We choose two admissible overlapping sectors in a canonical way. Let l be an admissible oriented line through the origin, namely a line not containing Stokes' rays. For our purposes we take

$$l = \{z \mid z = \rho e^{i\epsilon}, \quad \rho \in \mathbf{R}, \quad 0 < \epsilon < \frac{\pi}{k}\}$$

l has the orientation inherited from $\mathbf{R}$. We call Π_R and Π_L the half planes to the right/left of l w.r.t its orientation.

$$\Pi_R = \{-\pi + \epsilon < \arg z < \epsilon\} \quad \Pi_L = \{\epsilon < \arg z < \pi + \epsilon\}$$

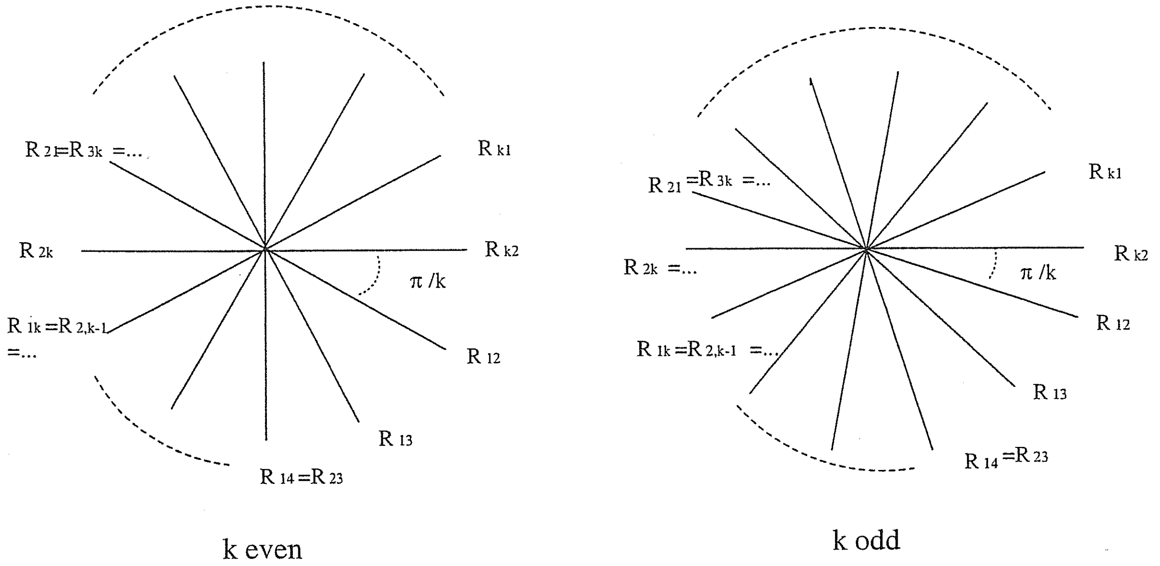


Figure 4.1: Stokes' rays

We then define two different "admissible" sectors $\mathcal{S}_L$, $\mathcal{S}_R$ which contain l

$$\mathcal{S}_L = \{z \in \mathbf{C} \mid 0 < argz < \pi + \frac{\pi}{k}\} \supset \Pi_L$$

$$\mathcal{S}_R = \{z \in \mathbf{C} \mid -\pi < argz < \frac{\pi}{k}\} \supset \Pi_R$$

We call che corresponding solutions $\tilde{Y}_L(z)$ and $\tilde{Y}_R(z)$. The *Stokes' matrix* of the system (4.14) with respect to the admissible line l is the connection matrix S such that

$$\tilde{Y}_L(z) = \tilde{Y}_R(z) S \qquad 0 < \arg z < \frac{\pi}{k}$$

On the opposite overlapping region one can prove (as a consequence of the skew-symmetry of V) that

$$\tilde{Y}_L(z) = \tilde{Y}_R(z\, e^{-2\pi i}) S^T \qquad \pi < \arg z < \pi + \frac{\pi}{k}$$

We call *central connection matrix* the connection matrix C such that $\tilde{Y}_0(z) = \tilde{Y}_R(z) C, \quad z \in \Pi_R$.

It is clear that the system (4.11) has solutions $Y_0(z) = X\tilde{Y}_0(z)$, and $Y_L(z) = X\tilde{Y}_L(z)$, $Y_R(z) = X\tilde{Y}_R(z)$ asymptotic to $X\tilde{Y}_F(z)$ as $z \to \infty$ in $\mathcal{S}_L$ and $\mathcal{S}_R$ respectively, which are connected by the same S and C.

In order to compute the entries of S explicitly, we use the reduction of (4.11) to the generalized hypergeometric equation (4.13). If $\varphi^{(1)}(z)$, ..., $\varphi^{(k)}(z)$ is a basis of k linearly independent solutions of (4.13), then the matrix $Y(z)$ of entries (n, j) defined by

$$Y_n^{(j)}(z) := \frac{1}{k^{n-1}}\; z^{\frac{k-1}{2}-n+1}\; (z\partial_z)^{(n-1)}\; \varphi^{(j)}(z) \tag{4.15}$$

is a fundamental matrix for (4.11).

Lemma 3: *The generalized hypergeometric equation (4.13) has two bases of linearly independent solutions $\varphi_{L/R}^{(1)}(z)$, ..., $\varphi_{L/R}^{(k)}(z)$ having asymptotic behaviours*

$$\varphi_{L/R}^{(n)} = \frac{1}{\sqrt{k}} \frac{e^{i\frac{\pi}{k}(n-1)}}{z^{\frac{k-1}{2}}} \exp\left[k\, e^{i\frac{2\pi}{k}(n-1)}\, z\right] \left(1 + O\left(\frac{1}{z}\right)\right), \qquad z \to \infty$$

in $\mathcal{S}_L$ and $\mathcal{S}_R$ respectively. Let $\Phi(z)$ denote the row vector $[\varphi^{(1)}(z), ..., \varphi^{(k)}(z)]$. The fundamental matrices $Y_L(z)$, $Y_R(z)$ of (4.11) are expressed through formula (4.15) in terms of $\Phi_L(z)$ and $\Phi_R(z)$ and

$$Y_L(z) = Y_R(z)\; S \qquad 0 < \arg z < \frac{\pi}{k}$$

if and only if

$$\Phi_L(z) = \Phi_R(z)S \qquad 0 < \arg z < \frac{\pi}{k}$$

Proof: Simply observe that for a fundamental solution in $\mathcal{S}_L$ or $\mathcal{S}_R$ (we omit subscripts L, R)

$$Y(z) = \begin{bmatrix} z^{\frac{k-1}{2}}\varphi^{(1)} & \dots & z^{\frac{k-1}{2}}\varphi^{(k)} \\ \vdots & \dots & \vdots \end{bmatrix} = X\tilde{Y}$$

which is asymptotic, for $z \to \infty$, to

$$\sim \begin{bmatrix} 1 & e^{i\frac{\pi}{k}} & e^{i\frac{2\pi}{k}} & e^{i\frac{3\pi}{k}} & \dots & e^{i\frac{k-1}{k}\pi} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix} \begin{bmatrix} \exp(kz) & & & \\ & \exp(ke^{\frac{2\pi i}{k}}z) & & \\ & & \ddots & \\ & & & \exp(ke^{\frac{2\pi i(k-1)}{k}}) \end{bmatrix}$$

Now , the first row of $Y(z)$ is $z^{\frac{k-1}{2}}\Phi(z)$

□

We recall that the Stokes' matrix S has entries $s_{ii} = 1$, $s_{ij} = 0$ if $R_{ij} \subset \Pi_R$.

Lemma 4 : *S has a column whose entries are all zero but one. More precisely:*
For k even

$$s_{i,\frac{k}{2}+1} = 0 \quad \forall i \neq \frac{k}{2}+1, \qquad s_{\frac{k}{2}+1,\frac{k}{2}+1} = 1$$

For k odd

$$s_{i,\frac{k+1}{2}} = 0 \quad \forall i \neq \frac{k+1}{2}, \qquad s_{\frac{k+1}{2},\frac{k+1}{2}} = 1$$

Proof: Let us determine n such that $s_{in} = 0$ for any $i \neq n$ and $s_{nn} = 1$. We need to find all rays in Π_R. We start with R_{rs} with $r < s$. We know that for $r + s = k + 2$ the ray is the negative real half-line (at angle $-\pi$). Then $R_{rs} \subset \Pi_R$ for $r + s \leq k + 1$ ($r < s$). Then, in Π_R we have

$$\begin{matrix} R_{12} & R_{13} & \dots & R_{1k} \\ R_{23} & R_{24} & \dots & R_{2,k-1} \\ R_{34} & R_{35} & \dots & R_{3,k-2} \\ \vdots & \vdots & & \\ R_{ab} & & & \end{matrix}$$

where $R_{ab} = R_{\frac{k}{2},\frac{k}{2}+1}$ for k even, and $R_{\frac{k-1}{2},\frac{k+3}{2}}$ for k odd. In Π_R we have also R_{rs} with $r + s \geq k + 2$ and $r > s$. For fixed n we require $R_{in} \subset \Pi_R$ for any i. Namely,

$$\forall i < n \quad i + n \leq k + 1 , \qquad \forall i > n \quad i + n \geq k + 2$$

This yields $n = \frac{k}{2} + 1$ for k even, $n = \frac{k+1}{2}$ for k odd.

□

Let $n(k)$ be $\frac{k}{2} + 1$, or $\frac{k+1}{2}$. Lemma 4 implies that the $n(k)^{th}$ columns of Y_L and Y_R coincide. In particular, their asymptotic representation holds for $-\pi < \arg z < \pi + \frac{\pi}{k}$. Actually, this domain can be further enlarged, up to

$$-\frac{\pi}{k} - \pi < \arg z < \pi + \frac{\pi}{k} \qquad k \text{ even}$$

$$-\pi < \arg z < \pi + \frac{2\pi}{k} \qquad k \text{ odd}$$

To see this recall that $|e^{zu_i}| < |e^{zu_j}|$ on the right of R_{ij}, and conversely on the left. Then it is easy to see that for k even $\left|\exp(z\ u_{\frac{k}{2}+1})\right|$ dominates all exponentials in the sector $-\frac{\pi}{k} - \pi < \arg z < \frac{\pi}{k} - \pi$, while for k odd $\left|\exp(z\ u_{\frac{k+1}{2}})\right|$ dominates all exponentials in the sector $\pi < \arg z < \pi + \frac{2\pi}{k}$.

The first entry of the $n(k)^{th}$ column is $\varphi_L^{(n(k))}(z) \equiv \varphi_R^{(n(k))}(z)$ times $z^{\frac{k-1}{2}}$. Then $\varphi^{(n(k))}$ has the established asymptotic behaviour on the enlarged domains above.

We now introduce an integral representation for a solution $\varphi(z)$ of the generalized hypergeometric equation which will allow us to compute the entries of S.

Lemma 5: *The function*

$$g^{(n)}(z) = \frac{1}{(2\pi)^{\frac{k+1}{2}}\ e^{i\pi(\frac{k}{2}-n-1)}} \int_{-c-i\infty}^{-c+i\infty} ds\ \Gamma^k(-s)\ e^{-i\pi ks}\ e^{i2(n-1)\pi s}\ z^{ks}$$

defined for $\frac{\pi}{2} - 2(n-1)\frac{\pi}{k} < \arg z < \frac{3\pi}{2} - 2(n-1)\frac{\pi}{k}$, $z \neq 0$ *and for any positive number* $c > 0$, *is a solution of the generalized hypergeometric equation (4.13) (the path of integration is a vertical line through* $-c$*). It has asymptotic behaviour*

$$g^{(n)}(z) \sim \frac{1}{\sqrt{k}}\ \frac{e^{i\frac{\pi}{k}(n-1)}}{z^{\frac{k-1}{2}}}\ \exp\left(ke^{i\frac{2\pi}{k}(n-1)}z\right) \qquad z \to \infty$$

In particular, for $n(k) = \frac{k}{2}+1$ *(k even), or* $n(k) = \frac{k+1}{2}$ *(k odd), the analytic continuation of* $g^{(n(k))}(z)$ *has the above asymptotic behaviour in the domains*

$$-\pi - \frac{\pi}{k} < \arg z < \pi + \frac{\pi}{k} \qquad k\ even$$

$$-\pi < \arg z < \pi + \frac{2\pi}{k} \qquad k\ odd$$

and coincides with the solution $\varphi_L^{(n(k))} \equiv \varphi_R^{(n(k))}$ *appearing in the first rows of the fundamental matrices* Y_L *and* Y_R *of the system (4.11).*

The following identity holds

$$\sum_{m=0}^{k} (-1)^{m-k} \binom{k}{m}\ g^{(n)}(z\ e^{i\frac{2\pi}{k}m}) = 0 \tag{4.16}$$

The proof is omitted. The reader is referred to the paper [25] by the author of this thesis, especially to the preprint.

Remark 2: Observe that for basic solutions of the hypergeometric equation $\Phi_{L/R} = [\varphi_{L/R}^{(1)},...,\varphi_{L/R}^{(k)}]$,

$$\varphi^{(n)}(z) \sim \frac{1}{\sqrt{k}} \frac{e^{i\frac{\pi}{k}(n-1)}}{z^{\frac{k-1}{2}}} \exp(ke^{i\frac{2\pi}{k}(n-1)}z)$$

on some sector, and

$$\varphi^{(n)}(ze^{\frac{2\pi i}{k}}) \sim (-1)\ \frac{1}{\sqrt{k}} \frac{e^{i\frac{\pi}{k}([n+1]-1)}}{z^{\frac{k-1}{2}}} \exp(ke^{i\frac{2\pi}{k}([n+1]-1)}z)$$

like $-\varphi^{(n+1)}$, on the sector rotated by $\frac{-2\pi}{k}$. Note however that $\varphi^{(k)}(ze^{\frac{2\pi i}{k}}) \sim (\sqrt{k}z^{\frac{k-1}{2}})^{-1}\ e^{kz}$, like $\varphi^{(1)}(z)$.

Also, note that $g^{(n)}(ze^{\frac{2\pi i}{k}}) = -g^{(n+1)}(z)$ (we mean analytic continuations), with asymptotic behaviour on rotated domain.

4.4 Monodromy Data of the Quantum Cohomology of $\mathbf{CP}^{k-1}$

Let us return to the system (4.2).

$$\partial_z y = \left(\hat{\mathcal{U}}(t^2) + \frac{1}{z}\mu\right) y \tag{4.17}$$

In this section we use the original notation $w = ze^{\frac{t^2}{k}}$. The system has a fundamental matrix

$$y_0(t^2, z) = (I + H_1(t^2)z + H_2(t^2)z^2 + ...)\ z^{\hat{\mu}}\ z^R, \qquad z \to 0$$

where R is the same of system (4.11). The series appearing in the solutions converges near $z=0$. The matrix $\hat{\mathcal{U}}(t^2)$ has eigenvalues and eigenvectors

$$u_n(t^2) = e^{i\frac{2\pi}{k}(n-1)}\, e^{\frac{t^2}{k}} \equiv u_n e^{\frac{t^2}{k}} \qquad n=1,...,k$$

$$\mathbf{x}_n(t^2) \text{ of entries } x^j{}_n(t^2) = e^{i(2j-1)(n-1)\frac{\pi}{k}}\, e^{\frac{2j-1-k}{2k}t^2} \equiv x^j{}_n e^{\frac{2j-1-k}{2k}t^2}$$

Let $X(t^2) = (x^j{}_n(t^2))$. With the gauge $y = X(t^2)\ \tilde{y}(t^2,z)$ we obtain the equivalent system

$$\partial_z \tilde{y} = \left[U(t^2) + \frac{V(t^2)}{z}\right]\tilde{y} \tag{4.18}$$

$$U(t^2) = X^{-1}(t^2)\ \hat{\mathcal{U}}(t^2)\ X(t^2) = \mathrm{diag}(u_1(t^2),...,u_k(t^2))$$
$$V(t^2) = X^{-1}(t^2)\ \mu\ X(t^2), \quad V(t^2)^T + V(t^2) = 0$$

Let us fix an initial point $t_0 = (0, t_0^2, 0, ..., 0)$. The system (4.18) has fundamental matrices $y_R(t_0^2, z)$, $y_L(t_0^2, z)$, which are asymptotic to the formal solution

$$\tilde{y}_F(t_0^2, z) = \left[I + \frac{F_1(t_0^2)}{z} + \frac{F_2(t_0^2)}{z^2} + ...\right] e^{z\ U(t_0^2)}$$

in the sectors

$$\mathcal{S}_L(t_0) = \{z \in \mathbf{C} \mid 0 < \arg\left[z\exp\left(\frac{t_0^2}{k}\right)\right] < \pi + \frac{\pi}{k}\}$$

$$\mathcal{S}_R(t_0) = \{z \in \mathbf{C} \mid -\pi < \arg\left[z\exp\left(\frac{t_0^2}{k}\right)\right] < \frac{\pi}{k}\}$$

and

$$\tilde{y}_L(t_0^2, z) = y_R(t_0^2, z)\ S \qquad 0 < \arg\left[z\exp\left(\frac{t_0^2}{k}\right)\right] < \frac{\pi}{k}$$

with respect to the admissible line

$$l_{t_0} := \{z \mid z = \rho\exp\left(i\epsilon - \frac{\Im m\ t_0^2}{k}\right),\ \ \rho > 0\}$$

The Stokes' matrix is precisely the matrix S of system (4.11) with respect to the admissible line l_{t_0}. Also the central connection matrix defined by

$$y_0(t_0^2, z) = y_R(t_0^2, z)\ C \qquad -\pi < \arg\left[z e^{\frac{t_0^2}{k}}\right] < \frac{\pi}{k}$$

is the the same of the system (4.11).

Definition: C and S, together with $\hat{\mu}$, R, and $e = \frac{\partial}{\partial t^1}$ are the *monodromy data* of the quantum cohomology of $\mathbf{CP}^{k-1}$ in a local chart containing t_0.

Recall that we fixed a point $t_0 = (0, t_0^2, ..., 0)$. When we consider a point t away from t_0, the system (4.17) acquires the general form

$$\partial_z y = \left[\hat{\mathcal{U}}(t) + \frac{\mu}{z}\right] y \tag{4.19}$$

where $\hat{\mathcal{U}}(t^1,..,t^k) = \eta\ \mathcal{U}(t)\ \eta^{-1}$ and $y_\alpha^{(j)}(t^1,..,t^k;z) = \partial_\alpha \tilde{t}^j(t,z)$.

The admissible line l_{t_0} must be considered *fixed once and for all.* Instead, the Stokes' rays change. This is because they are functions of the eigenvalues $u_1(t)$, ..., $u_k(t)$ of the matrix $\hat{\mathcal{U}}(t^1,..,t^k)$. For example, if just t^2 varies, while $t^1 = t^3 = ... = t^k = 0$, the system (4.17) has Stokes' rays

$$R_{rs}(t^2) = \{z \mid\ = \rho\ \exp\left(i\frac{2\pi}{k} - i\frac{\pi}{k}(r+s) - i\frac{\Im m\ t^2}{k}\right),\ \ \rho > 0\}$$

The dependence of the coefficients of the system (4.19) on t is isomonodromic. Then μ and R are the same for any t. S and C do not change if we move in a sufficiently small neighbourhood of t_0. Problems arise when some Stokes' rays cross l_{t_0}. S and C must be modified by an action of the braid group. We will return to this point later.

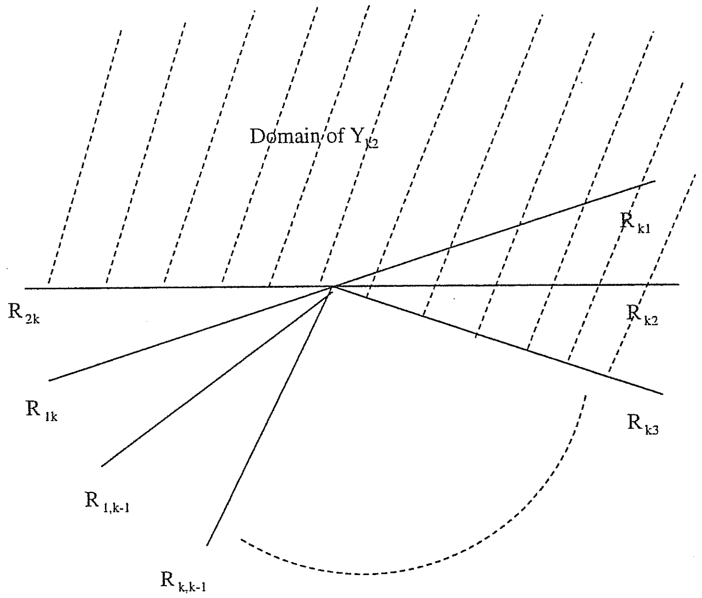


Figure 4.2: Sector where Y_{k2} has the asymptotic behaviour $X\ e^{zU}$ for $z \to \infty$

4.5 Computation of S

To compute S, we factorize it in "Stokes' factors". Our fundamental matrix Y_L has the required asymptotic form on the sector between R_{k2} $(\arg z = 0)$ and R_{1k} $(\arg z = \pi + \frac{\pi}{k})$. Y_R has the same behaviour between R_{2k} $(\arg z = -\pi)$ and R_{k1} $(\arg z = \frac{\pi}{k})$.

Of course, we can consider fundamental matrices with the same asymptotic behaviour on other sectors of angular width less then $\pi + \frac{\pi}{k}$ and bounded by two Stokes' rays. We introduce the following notation: consider a fundamental matrix of (4.11) having the required asymptotic behaviour on such a sector. If we go all over the sector clockwise we meet Stokes rays belonging to the sector at each displacement of $\frac{\pi}{k}$. Let R_{ij} be the last ray we meet before reaching the boundary (the boundary is still a Stokes ray *not* belonging to the sector). Then we will call the fundamental matrix Y_{ij}. For example, $Y_L = Y_{k1}$ and $Y_R = Y_{1k}$. See figure 2.

Sometimes, a different labelling is used in the literature. The rays must be enumerated as in figure 3. The numeration refers to the line l: the rays are labelled in counter-clockwise order starting from the first one in Π_R (which will be R_0; then R_0, R_1, ..., R_{k-1} are in Π_R, and R_k, ..., R_{2k-1} are in Π_L). For our particular choice of l , $R_0 \equiv R_{1k}$ (at $\arg z = -\pi + \frac{\pi}{k}$); $R_1 \equiv R_{1,k-1}$ follows counter-clockwise... Then we proceed until we reach $R_{k-1} \equiv R_{k2}$ before crossing l, and so on. The fundamental matrices are labelled as we prescribed above, namely Y_j if its sector contains R_j as the last ray met going all over the sector clockwise before the boundary. The sector itself is denoted by $\mathcal{S}_j$. See figure 3.

We define *Stokes' factors* to be the connection matrices K_j such that

$$Y_{j+1}(z) = Y_j(z)K_j$$

on the overlapping region of width π. We warn the reader that also the Stokes' factors will be labelled with both conventions above, according to our convenience (for example $K_0 \equiv K_{1k}$).

As a consequence of the above definitions, we can factorize S as follows

$$Y_L = Y_{k1} = Y_{k2}K_{k2} = Y_{k3}K_{k3}K_{k2} = ...$$

$$... = Y_{1k}K_{1k}K_{1,k-1}K_{k,k-1}K_{k,k-2}...K_{k3}K_{k2} \equiv Y_R S$$

Then

$$S = K_{1k}K_{1,k-1}K_{k,k-1}K_{k,k-2}...K_{k3}K_{k2} \tag{4.20}$$

We observe that, being the first row of $Y(z)$ equal to $z^{\frac{k-1}{2}}\Phi(z)$, the following holds:

$$\Phi_{j+1}(z) = \Phi_j K_j$$

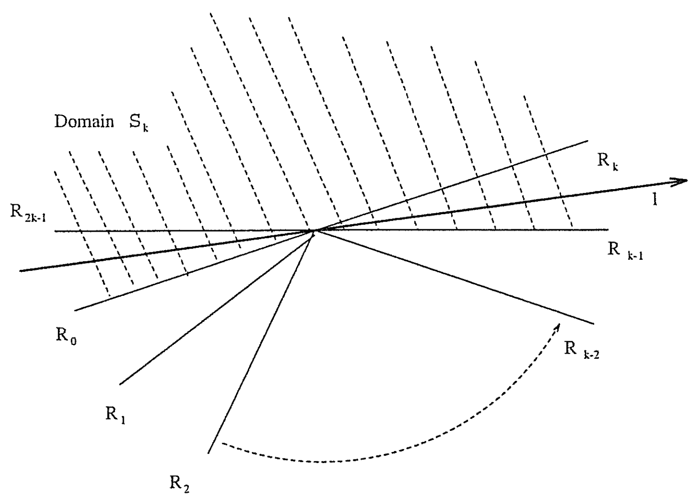


Figure 4.3: Sector $\mathcal{S}_k$ and labelling of Stokes' rays

Remark 3: The Stokes' factors of the system (4.11) and of the gauge-equivalent system (4.14) are the same. From the skew-symmetry of V it follows that $K_{ji} = (K_{ij}^{-1})^T$.

Before computing the Stokes factors explicitly, we show that just two of them are enough to compute all the others. Let

$$F(z) := \left(\frac{1}{\sqrt{k}} \frac{1}{z^{\frac{k-1}{2}}} \exp(kz),\ \frac{1}{\sqrt{k}} \frac{e^{\frac{i\pi}{k}}}{z^{\frac{k-1}{2}}} \exp(ke^{\frac{2\pi i}{k}} z),\ ...,\ \frac{1}{\sqrt{k}} \frac{e^{\frac{i\pi}{k}(k-1)}}{z^{\frac{k-1}{2}}} \exp(ke^{\frac{2\pi i}{k}(k-1)} z) \right)$$

$$= (F^{(1)}(z),\ F^{(2)}(z),\ ...,\ F^{(k)}(z))$$

be the row vector whose entries are the first terms of the asymptotic expansions of an actual solution $\Phi(z)$ of the generalized hypergeometric equation. By a straightforward computation we see that

$$F(ze^{\frac{2\pi i}{k}}) = F(z) T_F\ , \qquad T_F = \begin{pmatrix} 0 & & & & \dots & 1 \\ -1 & 0 & & & & \\ & -1 & 0 & & & \\ & & -1 & & & \\ & & & \ddots & \ddots & \vdots \\ & & & & -1 & 0 \end{pmatrix}$$

We use now the convention of enumeration of Stokes' rays R_0, R_1, ... starting from l (see above). Let $\Phi_m(z)$ be an actual solution of the hypergeometric equation having asymptotic behaviour $F(z)$ on $\mathcal{S}_m$.

$$\Phi_m(z) \sim F(z) \quad z \to \infty \quad z \in \mathcal{S}_m$$

Then

$$\Phi_{m+2}(ze^{\frac{2\pi i}{k}}) \sim F(ze^{\frac{2\pi i}{k}}) = F(z) T_F\ , \qquad z \in \mathcal{S}_m$$

Namely,

$$\Phi_{m+2}(ze^{\frac{2\pi i}{k}}) T_F^{-1} \sim F(z)\ , \qquad z \in \mathcal{S}_m$$

then, since the solution having asymptotic behaviour $F(z)$ in a sector wider then π is unique, we have

$$\Phi_{m+2}(ze^{\frac{2\pi i}{k}}) = \Phi_m(z) T_F\ , \qquad z \in \mathcal{S}_m$$

Lemma 6: *For any* $p \in \mathbf{Z}$

$$K_{m+2p} = T_F^{-p} K_m T_F^p$$

Proof: For $z \in \mathcal{S}_m \cap \mathcal{S}_{m+1}$

$$\Phi_{m+1}(z) = \Phi_m(z)\ K_m = \Phi_{m+2}(ze^{\frac{2\pi i}{k}})\ T_F^{-1} K_m =$$

$$= \Phi_{m+3}(ze^{\frac{2\pi i}{k}})\ K_{m+2}^{-1}T_F^{-1}K_m = \Phi_{m+1}(z)\ \ T_F\ K_{m+2}^{-1}\ T_F^{-1}\ K_m$$

Then $K_{m+2} = T_F^{-1}\ K_m\ T_F$. By induction we prove the lemma.

□

¿From the lemma it follows that just two Stokes' factors are enough to compute all the others. We are ready to give a concise formula for S:

Theorem 1: *Let l be an admissible line (i.e. not containing Stokes' rays), and let us enumerate the rays in counter-clockwise order starting from the first one in Π_R (which will be R_0, then R_0, R_1, ..., R_{k-1} are in Π_R, and R_k, ..., R_{2k-1} are in Π_L). Then the Stokes' matrix for (4.11), (4.14), (4.13) and $k > 3$ is*

$$S = \begin{cases} (K_0\ K_1\ T_F^{-1})^{\frac{k}{2}}\ T_F^{\frac{k}{2}}\ \equiv\ T_F^{\frac{k}{2}}\ (T_F^{-1}\ K_{k-2}\ K_{k-1})^{\frac{k}{2}}, & k\ \textit{even} \\ (K_0\ K_1\ T_F^{-1})^{\frac{k-1}{2}}\ K_0\ T_F^{\frac{k-1}{2}}\ \equiv\ T_F^{\frac{k-1}{2}}\ K_{k-1}\ (T_F^{-1}\ K_{k-2}\ K_{k-1})^{\frac{k-1}{2}}, & k\ \textit{odd} \end{cases} \tag{4.21}$$

Proof: $S = K_0\ K_1\ K_2\ ...\ K_{k-1}$, $K_{2p} = T_F^{-p}\ K_0\ T_F^p$ and $K_{2p+1} = T_F^{-p}\ K_1\ T_F^p$. Then

$$S = K_0 K_1\ (T_F^{-1} K_0 T_F)\ (T_F^{-1} K_1 T_F)\ K_4 K_5\ ...\ K_{k-1}$$

$$= (K_0 K_1 T_F^{-1})\ K_0 K_1\ T_F\ K_4 K_5\ ...\ K_{k-1}$$

$$= (K_0 K_1 T_F^{-1})\ K_0 K_1\ T_F\ (T_F^{-2} K_0 T_F^2)\ (T_F^{-2} K_1 T_F^2)\ K_6\ ...\ K_{k-1}$$

$$= (K_0 K_1 T_F^{-1})(K_0 K_1 T_F^{-1})\ K_0 K_1 T_F^2\ K_6\ ...\ K_{k-1}$$

Now observe that $K_{k-1} = T_F^{-(\frac{k}{2}-1)}\ K_1\ T_F^{\frac{k}{2}-1}$ for k even, while $K_{k-1} = T_F^{-(\frac{k-1}{2})}\ K_0\ T_F^{\frac{k-1}{2}}$ for k odd. Then, for k even

$$S = (K_0 K_1 T_F^{-1})^{\frac{k}{2}-2}\ K_0 K_1 T_F^{\frac{k}{2}-2}\ K_{k-2} K_{k-1}$$

$$= (K_0 K_1 T_F^{-1})^{\frac{k}{2}-2}\ K_0 K_1 T_F^{\frac{k}{2}-2}\ T_F^{-(\frac{k}{2}-1)} K_0 T_F^{\frac{k}{2}-1}\ T_F^{-(\frac{k}{2}-1)} K_1 T_F^{\frac{k}{2}-1} = (K_0\ K_1\ T_F^{-1})^{\frac{k}{2}}\ T_F^{\frac{k}{2}}$$

For k odd:

$$S = (K_0 K_1 T_F^{-1})^{\frac{k-3}{2}}\ K_0 K_1 T_F^{\frac{k-3}{2}}\ K_{k-1}$$

$$= (K_0 K_1 T_F^{-1})^{\frac{k-3}{2}}\ K_0 K_1 T_F^{\frac{k-3}{2}}\ T_F^{-(\frac{k-1}{2})} K_0 T_F^{\frac{k-1}{2}} = (K_0\ K_1\ T_F^{-1})^{\frac{k-1}{2}}\ K_0\ T_F^{\frac{k-1}{2}}$$

If instead we write the Stokes' factors in term of K_{k-2} and K_{k-1} we obtain the other two formulas in the same way.

□

Remark 4: For our particular choice of l, $K_0 \equiv K_{1k}$, $K_1 \equiv K_{1,k-1}$, $K_{k-2} \equiv K_{k3}$ and $K_{k-1} \equiv K_{k2}$. For $k = 3$

$$S = T_F K_{32}\ (T_F^{-1} K_{12} K_{32}) = K_{13} K_{12} K_{32}$$

It is now worth deriving some properties of the monodromy of $Y(z)$ (for (4.11)) and $\Phi(z)$ (for (4.13)), which will be useful later. Consider $\Phi_m(z)$ with asymptotic behaviour $F(z)$ on $\mathcal{S}_m$. Then

$$\Phi_m(z) = \Phi_{m-2}(z)\ K_{m-2}\ K_{m-1} \equiv \Phi_m(ze^{\frac{2\pi i}{k}})\ T_F^{-1}\ K_{m-2}\ K_{m-1}$$

On the other hand

$$\Phi_m(z) = \Phi_{m+2}(z)\ K_{m+1}^{-1}\ K_m^{-1} \equiv \Phi_m(ze^{-\frac{2\pi i}{k}})\ T_F\ K_{m+1}^{-1}\ K_m^{-1}$$

This proves the following

Lemma 7: *The basic solution $\Phi_m(z)$ of the generalized hypergeometric equation (4.13) with asymptotic behaviour $F(z)$ on $\mathcal{S}_m$, satisfies the identity*

$$\Phi_m(ze^{\frac{2\pi i}{k}}) = \Phi_m(z)\ T_m$$

where

$$T_m := K_{m-1}^{-1}\ K_{m-2}^{-1}\ T_F\ =\ T_F\ K_{m+1}^{-1}\ K_m^{-1}$$

Corollary 1: *The monodromy (at $z=0$) of $\Phi_m(z)$ is*

$$\Phi_m(ze^{2\pi i}) = \Phi_m(z)\ (T_m)^k$$

Now, for our particular choice of the line l and for $m=k$, $\Phi_m(z) = \Phi_L(z)$. For the solution $Y_L(z)$ of (4.11), the relations $Y_R(z) = Y_L(z)\ S^{-1}$ $(0 < \arg z < \frac{\pi}{k})$, $Y_L(z) = Y_R(ze^{-2\pi i})\ S^T$ $(\pi < \arg z < \pi + \frac{\pi}{k})$ immediately imply

$$Y_L(ze^{2\pi i}) = Y_L(z)\ S^{-1}\ S^T$$

Recall that the (n,j)-th entry of $Y(z)$ is $Y_{n,j}(z) \equiv Y_n^{(j)}(z) = \frac{1}{k^{n-1}}\ z^{\frac{k-1}{2}-n+1}\ (z\partial_z)^{(n-1)}\ \varphi^{(j)}(z)$, and observe that $(ze^{2\pi i})^{\frac{k-1}{2}} = (-1)^{k-1}\ z^{\frac{k-1}{2}}$ Then, from Corollary 1 we get the following:

Corollary 2: *Let T be the k-monodromy matrix of Φ_L (namely, $T \equiv T_k$ for our choice of l). Then*

$$T^k = (-1)^{k-1}\ S^{-1}\ S^T$$

Our formula (4.21) allows us to easily compute S. The recipe is simply to take K_{k3}, K_{k2} (which we are going to compute explicitly) and substitute them into

$$S = T_F^{\frac{k}{2}}\ (T_F^{-1}\ K_{k3}\ K_{k2})^{\frac{k}{2}} = T_F^{\frac{k}{2}}\ T^{-\frac{k}{2}} \tag{4.22}$$

for k even, or into

$$S = T_F^{\frac{k-1}{2}}\ K_{k2}\ (T_F^{-1}\ K_{k3}\ K_{k2})^{\frac{k-1}{2}} = T_F^{\frac{k-1}{2}}\ K_{k2}\ T^{-\frac{k-1}{2}} \tag{4.23}$$

for k odd.

Computation of Stokes' factors: We need to distinguish between k odd and even. In the following $g(z)$ will mean $g^{(n(k))}(z)$ ($n(k) = \frac{k}{2}+1$ or $\frac{k+1}{2}$ for k even or odd respectively).

k odd:

$$g(z) = \varphi_L^{(\frac{k+1}{2})}(z) \equiv \varphi_R^{(\frac{k+1}{2})}(z)$$

with asymptotic behaviour on

$$-\pi < \arg z < \pi + \frac{2\pi}{k}$$

If we iterate the map $z \mapsto ze^{\frac{2\pi i}{k}}$ for $m = 1, 2, ..., \frac{k+1}{2}$ times, the domain of $g(ze^{\frac{2\pi i}{k}m})$ for each m covers $\mathcal{S}_R$. When we reach $m = \frac{k+1}{2}$ a new iteration (i.e. a new rotation of the domain of $-\frac{2\pi}{k}$) will live the sector $-\frac{\pi}{k} < \arg z < \frac{\pi}{k}$ of $\mathcal{S}_R$ uncovered. The same, if we do $z \mapsto ze^{-\frac{2\pi i}{k}}$ the sector $-\pi < \arg z < -\pi + \frac{2\pi}{k}$ of $\mathcal{S}_R$ remains uncovered. Then, by remark 2:

$$\Phi_{1k}(z) \equiv \Phi_R(z) = \Big((-1)^{\frac{k-1}{2}} g(ze^{i\frac{2\pi}{k}(\frac{k+1}{2})}),\ \dots\ \tfrac{k-3}{2}\ \text{unknown terms}\ \dots,$$

$$g(z),\ -g(ze^{\frac{2\pi i}{k}}),\ g(ze^{\frac{4\pi i}{k}}),\ \dots,\ (-1)^{\frac{k-1}{2}} g(ze^{i\frac{2\pi}{k}(\frac{k-1}{2})})\Big)$$

In the same way we see that

$$\Phi_{k1}(z) \equiv \Phi_L(z) = \Big((-1)^{\frac{k-1}{2}} g(ze^{-i\frac{2\pi}{k}(\frac{k-1}{2})}),\ -(-1)^{\frac{k-1}{2}} g(ze^{-i\frac{2\pi}{k}(\frac{k-3}{2})}),\ \dots,-g(ze^{-\frac{2\pi i}{k}}),$$

$$g(z),\ \dots\ \tfrac{k-1}{2}\text{ unknown terms }\dots\Big)$$

and similar expressions for $\Phi_{1,k-1}$, $\Phi_{k,k-1}$, $\Phi_{k,k-2}$, ..., Φ_{k3},Φ_{k2}. The unknown terms are computed using the identity (4.16) and simple considerations on the dominant exponentials $|e^{zu_i}|$ on the sectors which remain uncovered in the iterations of $z \mapsto ze^{\pm i\frac{2\pi}{k}}$.

Example: A simple example will clarify this procedure. Let $k = 7$; then $g = \varphi^{(4)}$,

$$\Phi_{17} = \Phi_R = (-g(ze^{\frac{8\pi i}{7}}),\ ?,\ ?,\ g(z),\ -g(ze^{\frac{2\pi i}{7}}),\ g(ze^{\frac{4\pi i}{7}}),\ -g(ze^{\frac{6\pi i}{7}}))$$

We look for $\varphi_R^{(3)}$. If we take $(-1)g(ze^{\frac{-2\pi i}{k}})$ we miss to cover $-\pi < \arg z < -\pi + \frac{2\pi}{7}$ in $\mathcal{S}_R$. On $-\pi < \arg z < -\pi + \frac{\pi}{7}$ (between two nearby Stokes' rays) we have the relations $|e^{zu_4}| > |e^{zu_5}| > |e^{zu_3}|$. On $-\pi + \frac{\pi}{7} < \arg z < -\pi + \frac{2\pi}{7}$, $|e^{zu_4}| > |e^{zu_3}|$ (later on we will simply write $4 > 3$). Then

$$\varphi_R^{(3)}(z) = (-1)g(ze^{\frac{-2\pi i}{7}}) + c_4\varphi_R^{(4)}(z) + c_5\varphi_R^{(5)}(z)$$

To find c_4, c_5 we need another representation for $\varphi_R^{(3)}$. We consider $(-1)g(ze^{\frac{12\pi i}{7}})$, which has the correct asymptotic behaviour, but on a domain which leaves uncovered $-\frac{3\pi}{7} < \arg z < \frac{\pi}{7}$. The relations are: on $0 < \arg z < \frac{\pi}{7}$, $1 > 7 > 2 > 6 > 3$; on $-\frac{\pi}{7} < \arg z < 0$, $1 > 2 > 7 > 3$; on $-\frac{2\pi}{7} < \arg z < -\frac{\pi}{7}$, $2 > 1 > 3$; on $-\frac{3\pi}{7} < \arg z < -\frac{2\pi}{7}$, $2 > 3$. Then

$$\varphi_R^{(3)}(z) = (-1)g(ze^{\frac{12\pi i}{7}}) + d_1\varphi_R^{(1)}(z) + d_2\varphi_R^{(2)}(z) + d_6\varphi_R^{(6)}(z) + d_7\varphi_R^{(7)}(z)$$

In the same way one finds

$$\varphi_R^{(2)}(z) = \begin{cases} g(ze^{\frac{-4\pi i}{7}}) + a_3\varphi_R^{(3)}(z) + a_4\varphi_R^{(4)}(z) + a_5\varphi_R^{(5)}(z) + a_6\varphi_R^{(6)}(z) \\ g(ze^{\frac{10\pi i}{7}}) + b_1\varphi_R^{(1)}(z) + b_2\varphi_R^{(4)}(z) \end{cases}$$

$\varphi^{(i)}$'s are known for $i = 1, 4, 5, 6, 7$. Using the identity (4.16) we compute a, b, c, d. We get

$$\varphi_R^{(2)}(z) = g(ze^{\frac{-4\pi i}{7}}) - \binom{7}{1} g(ze^{\frac{-2\pi i}{7}}) + \binom{7}{2} g(z) - \binom{7}{3} g(ze^{\frac{2\pi i}{7}}) + \binom{7}{4} g(ze^{\frac{4\pi i}{7}})$$

$$\varphi_R^{(3)}(z) = -g(ze^{\frac{-2\pi i}{7}}) + \binom{7}{1} g(z) - \binom{7}{2} g(ze^{\frac{2\pi i}{7}})$$

A similar computation gives $\Phi_{71} = \Phi_L$.

$$\Phi_{71}^T = \begin{bmatrix} -g(ze^{-\frac{6\pi i}{7}}) \\ g(ze^{-\frac{4\pi i}{7}}) \\ -g(ze^{\frac{-2\pi i}{7}}) \\ g(z) \\ -g(ze^{\frac{2\pi i}{7}}) + \binom{7}{6} g(z) \\ g(ze^{\frac{4\pi i}{7}}) - \binom{7}{6} g(ze^{\frac{2\pi i}{7}}) + \binom{7}{5} g(z) - \binom{7}{4} g(ze^{-\frac{2\pi i}{7}}) \\ -g(ze^{\frac{6\pi i}{7}}) + \binom{7}{6} g(ze^{\frac{4\pi i}{7}}) - \binom{7}{5} g(ze^{\frac{2\pi i}{7}}) + \binom{7}{4} g(z) - \binom{7}{3} g(ze^{-\frac{2\pi i}{7}}) + \binom{7}{2} g(ze^{-\frac{4\pi i}{7}}) \end{bmatrix}$$

and

$$\Phi_{72}^T = \begin{bmatrix} -g(ze^{-\frac{6\pi i}{7}}) \\ g(ze^{-\frac{4\pi i}{7}}) \\ -g(ze^{\frac{-2\pi i}{7}}) \\ g(z) \\ -g(ze^{\frac{2\pi i}{7}}) \\ g(ze^{\frac{4\pi i}{7}}) - \binom{7}{6} g(ze^{\frac{2\pi i}{7}}) + \binom{7}{5} g(z) \\ -g(ze^{\frac{6\pi i}{7}}) + \binom{7}{6} g(ze^{\frac{4\pi i}{7}}) - \binom{7}{5} g(ze^{\frac{2\pi i}{7}}) + \binom{7}{4} g(z) - \binom{7}{3} g(ze^{-\frac{2\pi i}{7}}) \end{bmatrix}$$

Notice that in each of the last three entries of Φ_{72} there is a term missing w.r.t the corresponding entries of Φ_{71}. This immediately implies

$$K_{72} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \binom{7}{2} \\ 0 & 0 & 1 & 0 & 0 & \binom{7}{4} & 0 \\ 0 & 0 & 0 & 1 & \binom{7}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The next step is the computation of Φ_{73} and K_{73}, through $\Phi_{72} = \Phi_{73}K_{73}$. It is done in the same way...

The above procedure is extended to the general case. In Appendix 2 we give, for example, the general expressions of Φ_R and Φ_L. The factors of interest are:

k odd:

$(K_{k2})_{j,\ k-j+2} = \binom{k}{2(j-1)}$ for $j = 2, ..., \frac{k+1}{2}$. $(K_{k2})_{j,j} = 1$ for $j = 1, ..., k$. All the other entries are zero.

$(K_{k3})_{2,1} = -\binom{k}{1}$; $(K_{k3})_{j,\ k-j+3} = \binom{k}{2j-3}$ for $j = 3, ..., \frac{k+1}{2}$. $(K_{k3})_{j,j} = 1$ for $j = 1, ..., k$. All the other entries are zero. Namely:

$$K_{k2} = \begin{pmatrix} 1 & & & & & & & & & & 0 \\ & 1 & & & & & & & & & \binom{k}{2} \\ & & 1 & & & & & & & \binom{k}{4} & \\ & & & 1 & & & & & \binom{k}{6} & & \\ & & & & \ddots & & & \cdots & & & \\ & & & & & 1 & \binom{k}{k-1} & & & & \\ & & & & & & 1 & & & & \\ & & & & & & & \ddots & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & 1 & \\ & & & & & & & & & & 1 \end{pmatrix}$$

$$K_{k3} = \begin{pmatrix} 1 & & & & & & & & & & & & 0 \\ -\binom{k}{1} & 1 & & & & & & & & & & & 0 \\ & & 1 & & & & & & & & & & \binom{k}{3} \\ & & & 1 & & & & & & & & \binom{k}{5} & \\ & & & & 1 & & & & & & \binom{k}{7} & & \\ & & & & & \ddots & & & & \cdots & & & \\ & & & & & & 1 & 0 & \binom{k}{k-2} & & & & \\ & & & & & & & 1 & 0 & & & & \\ & & & & & & & & 1 & & & & \\ & & & & & & & & & \ddots & & & \\ & & & & & & & & & & 1 & & \\ & & & & & & & & & & & 1 & \\ & & & & & & & & & & & & 1 \end{pmatrix}$$

k even:

$(K_{k2})_{j,\ k-j+2} = \binom{k}{2(j-1)}$ for $j = 2, ..., \frac{k}{2}$. $(K_{k2})_{j,j} = 1$ for $j = 1, ..., k$. All the other entries are zero.

$(K_{k3})_{2,1} = -\binom{k}{1}$; $(K_{k3})_{j,\ k-j+3} = \binom{k}{2j-3}$, for $j = 3, ..., \frac{k}{2} + 1$. $(K_{k3})_{j,j} = 1$ for $j = 1, ..., k$. All the other entries are zero. Namely

$$K_{k2} = \begin{pmatrix} 1 & & & & & & & & & & 0 \\ & 1 & & & & & & & & \binom{k}{2} & \\ & & 1 & & & & & & \binom{k}{4} & & \\ & & & 1 & & & & \binom{k}{6} & & & \\ & & & & \ddots & & \cdot^{\cdot^{\cdot}} & & & & \\ & & & & & 1 \quad 0 \quad \binom{k}{k-2} & & & & & \\ & & & & & 1 \quad 0 & & & & & \\ & & & & & 1 & & & & & \\ & & & & & & \ddots & & & & \\ & & & & & & & 1 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & & 1 \end{pmatrix}$$

$$K_{k3} = \begin{pmatrix} 1 & & & & & & & & & 0 \\ -\binom{k}{1} & 1 & & & & & & & & 0 \\ & & 1 & & & & & & & \binom{k}{3} \\ & & & 1 & & & & & \binom{k}{5} & \\ & & & & 1 & & & \binom{k}{7} & & \\ & & & & & \ddots & \cdot^{\cdot^{\cdot}} & & & \\ & & & & & 1 \quad \binom{k}{k-1} & & & & \\ & & & & & 1 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1 & \\ & & & & & & & & & 1 \end{pmatrix}$$

Our equation is a particular case of general confluent hypergeometric equations, whose Stokes' factors were computed by A. Duval and C. Mitschi in [20]

4.6 Reduction of S to "Canonical" Form

Some examples of computations of S are at the end of this chapter. The reader may observe that S is not in a nice upper triangular form (see also Lemma 4) and quite strange numbers (complicated combinations of sum and products of binomial coefficients) appear.

We recall that some natural operations are allowed on the Stokes matrices of a Frobenius manifold:

a) Permutations. Let us consider the system

$$\frac{d\tilde{Y}}{dz} = \left[U + \frac{1}{z}V\right]\ \tilde{Y}$$

where $U = \text{diag}(u_1,\ u_2,\ ...,\ u_k)$. Let $\sigma :\ (1,2,..,k) \mapsto (\sigma(1), \sigma(2), ..., \sigma(k))$ be a permutation. It is represented by an invertible matrix P such that

$$P\ U\ P^{-1}\ =\ \text{diag}(u_{\sigma(1)}, u_{\sigma(2)}, ..., u_{\sigma(k)})$$

S and C are then transformed in PSP^{-1} and PC. For a suitable P, PSP^{-1} is upper triangular. The good permutation is the one which puts $u_{\sigma(1)}$, ..., $u_{\sigma(k)}$ in lexicographical ordering w.r.t. the oriented line l. P corresponds to a change of coordinates in the given chart, consisting in the permutation σ of the coordinates.

b) Sign changes of the entries. The construction at point a) is repeated, but now a diagonal matrix J with 1's or -1's on the diagonal takes the place of P. In JSJ^{-1} some entries change sign. Note that $JUJ^{-1} \equiv U$. Moreover, the formulae of chapter 1 which define a local chart of the manifold in terms of monodromy data are not affected by $S \mapsto JSJ$, $C \mapsto JC$.

c) Action of the braid group.
In section 4.4 we proved that the Stokes' rays of the system (4.18), and more generally the rays of the system (4.19), depend on the point t of the manifold. This is equivalent to the fact that the eigenvalues $u_1(t)$, ..., $u_k(t)$ of $\hat{\mathcal{U}}(t)$ depend on t. Let us start from the point $t_0 = (0, t_0^2, ..., 0)$. If we move sufficiently

far away from t_0 (we move to another chart), some Stokes' rays cross the *fixed* admissible line l_{t_0}. Then, we must change "Left" and "Right" solutions of (4.19). Then S and C change. The motions of the points $u_1(t)$, ..., $u_k(t)$ as t changes represent transformations of the braid group. We already described the action of the braid group on S in chapter 1. We recall that S must be set in upper triangular form by a permutation before acting with a braid.

In figure 4 we have drawn some lines $L_j = \{\lambda = u_j + \rho e^{i(\frac{\pi}{2}-\epsilon)},\ \ \rho > 0\}$ $(0 < \epsilon < \frac{\pi}{k}$ is the angle of $l)$, which help us to visualize the topological effect of the braids action (they are the branch cuts for the fuchsian system which will be introduced in section 8). We are going to prove that the braid whose effect is to set the deformed points in cyclic order and the cuts in the configuration of figure 4 (namely, the last two cuts remain unchanged, the others are alternatively "inverted"), brings S in a *canonical form*:

$$s_{ii} = 1, \quad s_{ij} = \begin{pmatrix} k \\ j-i \end{pmatrix}, \quad s_{ji} = 0, \qquad i < j$$

To be precise, we find that the last column is negative. The "canonical form" is reached after the conjugation $S \mapsto \mathcal{I}S\mathcal{I}$, where $\mathcal{I} := \text{diag}(1, 1, ..., -1)$.

Lemma 8: *Let the points u_j $(j = 1, .., k)$ be in lexicographical order w.r.t the admissible line l. Then the braid*

$$\beta := (\beta_{k-5,k-4}\ \beta_{k-6,k-5}\ \cdots\ \beta_{12})\ (\beta_{k-6,k-5}\ \beta_{k-7,k-6}\ \cdots\ \beta_{23})\ (\beta_{k-7,k-6}\ \cdots\ \beta_{34})\ \cdots$$

$$\cdots\ \beta_{\frac{k}{2}-2,\frac{k}{2}-1}\ (\beta_{k-3,k-2}\ \beta_{k-4,k-3}\ \cdots\ \beta_{12})$$

for k even, or

$$\beta := (\beta_{k-5,k-4}\ \beta_{k-6,k-5}\ \cdots\ \beta_{12})\ (\beta_{k-6,k-5}\ \beta_{k-7,k-6}\ \cdots\ \beta_{23})\ (\beta_{k-7,k-6}\ \cdots\ \beta_{34})\ \cdots$$

$$\cdots\ (\beta_{\frac{k-3}{2},\frac{k-1}{2}}\ \beta_{\frac{k-5}{2},\frac{k-3}{2}})\ (\beta_{k-3,k-2}\ \beta_{k-4,k-3}\ \cdots\ \beta_{12})$$

for k odd, brings the points in cyclic counter-clockwise order, u_1 being the first point in Π_L (figure 4, right side, or figure 5).

Note that we have collected the braids in $\frac{k}{2} - 1$ (k even), or $\frac{k-3}{2}$ (k odd) sequences (...).

Proof: Let k be even. The first braid $\beta_{k-5,k-4}$ interchanges u_{k-4} and u_{k-5}. The second braid interchanges u_{k-5} and u_{k-6}. One easily sees that the effect of the first sequence of braids $(\beta_{k-5,k-4}\ \beta_{k-6,k-5} \cdots \beta_{12})$ is to bring u_1 in the (old) position of u_{k-4}, u_{k-4} in the position of u_{k-5}, u_{k-5} in the position of u_{k-6}, ..., u_4 in the position of u_3 and u_2 in the position of u_1 (figure 5). u_k, u_{k-1}, u_{k-2}, u_{k-3} are not moved.

The second sequence of braids $(\beta_{k-6,k-5}\ \beta_{k-7,k-6}\ \cdots\ \beta_{23})$ acts in a similar way, bringing u_2 in u_{k-5}, u_{k-5} in u_{k-6}, ..., u_3 in u_2. u_k, u_{k-1}, u_{k-2}, u_{k-3}, u_{k-4} are not moved.

We go on in this way. After the action of

$$(\beta_{k-5,k-4}\ \beta_{k-6,k-5}\ \cdots\ \beta_{12})\ (\beta_{k-6,k-5}\ \beta_{k-7,k-6}\ \cdots\ \beta_{23})\ (\beta_{k-7,k-6}\ \cdots\ \beta_{34})\ \cdots\ \beta_{\frac{k}{2}-2,\frac{k}{2}-1}$$

the points are as in figure 5: u_k is on the positive real axis, u_{k-2} is the first point met in counter-clockwise order, u_1 is the second, u_2 is the third; the points are in cyclic order up to u_{k-3}; finally, u_{k-1} is the last point before reaching again the positive real axis from below.

Then, $(\beta_{k-3,k-2}\ \beta_{k-4,k-3}\ \cdots\ \beta_{12})$ brings u_1 in u_{k-2}, u_{k-2} in u_{k-3}, u_{k-3} in u_{k-4}, and so on. The cyclic order is reached.

For k odd the proof is similar.

□

A careful consideration of the effect of the braid β on the lines L_j (which we leave as an exercise for the reader) shows that they are alternatively inverted as in figure 4. To reconstruct uniquely this configuration we just need to know the oriented line l, namely, its angle ϵ w.r.t the positive real axis. The points u_{k-1}, u_k and the lines L_{k-1} and L_k are unchanged (angle $\frac{\pi}{2} - \epsilon$). The line at u_1 starts in the opposite direction, it goes around $u_2, ..., u_{k-2}$ without intersecting other cuts, and then goes to ∞ with the original asymptotic direction $\frac{\pi}{2} - \epsilon$. Moving in the direction opposite to that of l we meet u_{k-2}. Its line has the original direction $\frac{\pi}{2} - \epsilon$. Then we meet u_2, and the corresponding line starts with opposite direction, goes around $u_3, ..., u_{k-3}$ and then goes to ∞ with asymptotic direction $\frac{\pi}{2} - \epsilon$. And so on.

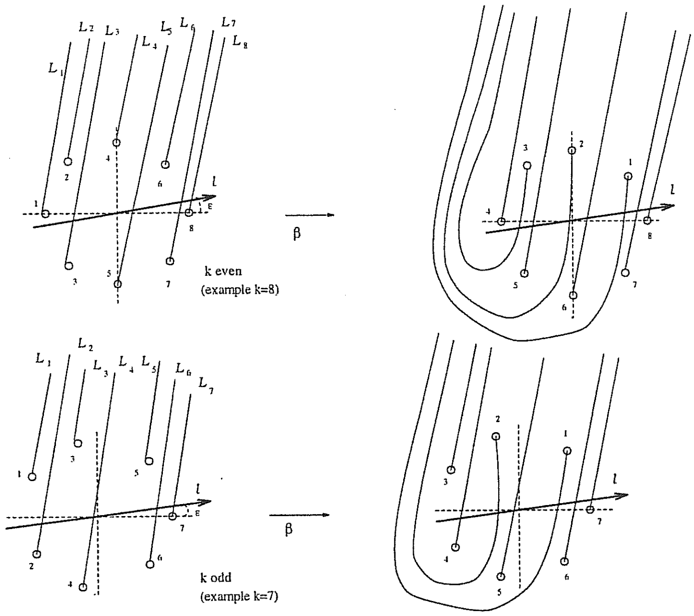


Figure 4.4: Effect of the braid which brings S to the canonical form on the lines L_j.

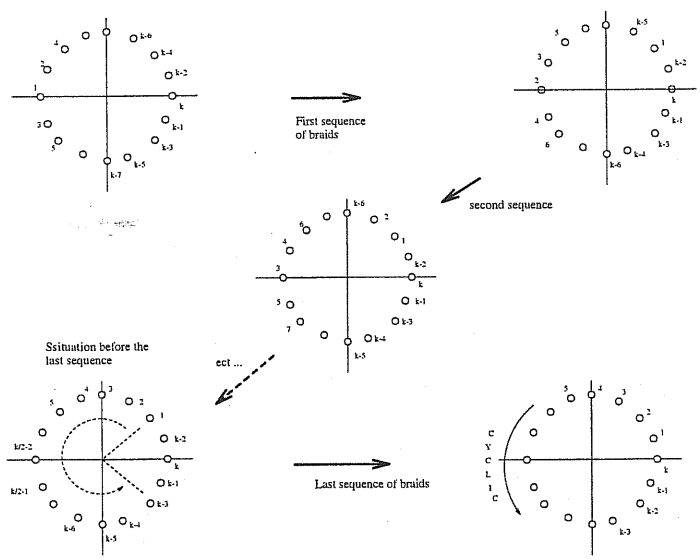


Figure 4.5: Effect of the sequence of braids which brings S to the canonical form (the figure refers to k even).

Now we find the matrix representation for the braid β.

Proposition 1: *The braid β of Lemma 8 has the following matrix representation:*

$$A^\beta(S) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & 0 & 0 & 1 & 0 & \binom{k}{1} & 0 & 0 \\
 & & & & & & 1 & 0 & \binom{k}{1} & 0 & \binom{k}{2} & 0 & 0 \\
 & & & & & & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \binom{k}{3} & 0 & 0 \\
 & & & & & \cdots & * & . & * & . & * & . & . \\
0 & 0 & 0 & 1 & 0 & & * & . & * & . & * & . & . \\
0 & 1 & 0 & \binom{k}{1} & 0 & \cdots & * & . & * & . & * & . & . \\
1 & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \cdots & * & . & * & . & * & . & . \\
0 & 0 & 1 & \binom{k}{3} & 0 & & * & . & * & . & * & . & . \\
0 & 0 & 0 & 0 & 1 & & * & . & * & . & * & . & . \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & * & . & * & . & * & . & . \\
 & & & & & & \binom{k}{k-7} & 0 & \binom{k}{k-6} & 0 & \binom{k}{k-5} & 0 & 0 \\
 & & & & & & 0 & 1 & \binom{k}{k-5} & 0 & \binom{k}{k-4} & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 1 & \binom{k}{k-3} & 0 & 0 \\
 & & & & & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

for k even.

$$A^\beta(S) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & 0 & 0 & 1 & 0 & \binom{k}{1} & 0 & 0 \\
 & & & & & & & 1 & 0 & \binom{k}{1} & 0 & \binom{k}{2} & 0 & 0 \\
 & & & & & & & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \binom{k}{3} & 0 & 0 \\
 & & & & & & \cdots & * & . & * & . & * & . & . \\
0 & 0 & 0 & 0 & 1 & 0 & & * & . & * & . & * & . & . \\
0 & 0 & 1 & 0 & \binom{k}{1} & 0 & \cdots & * & . & * & . & * & . & . \\
1 & 0 & \binom{k}{1} & 0 & \binom{k}{2} & 0 & \cdots & * & . & * & . & * & . & . \\
0 & 1 & \binom{k}{2} & 0 & \binom{k}{3} & 0 & & * & . & * & . & * & . & . \\
0 & 0 & 0 & 1 & \binom{k}{4} & 0 & & * & . & * & . & * & . & . \\
0 & 0 & 0 & 0 & 0 & 1 & & * & . & * & . & * & . & . \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \cdots & * & . & * & . & * & . & . \\
 & & & & & & & \binom{k}{k-7} & 0 & \binom{k}{k-6} & 0 & \binom{k}{k-5} & 0 & 0 \\
 & & & & & & & 0 & 1 & \binom{k}{k-5} & 0 & \binom{k}{k-4} & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 1 & \binom{k}{k-3} & 0 & 0 \\
 & & & & & & & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & & & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

for k odd.

The "$*$" means $\begin{pmatrix} k \\ j \end{pmatrix}$, and j increases by one when we move downwards row by row.

Proof: It is quite long and technical. We refer to the paper of the author of this thesis [25].

We are ready to prove the main result:

Theorem 2: *Consider the Stokes matrix $S = T_F^{\frac{k}{2}}\ T^{-\frac{k}{2}}$ (k even) or $S = T_F^{\frac{k-1}{2}} K_{k2} T^{-\frac{k-1}{2}}$ (k odd) and set it in the upper triangular form $S_{upper} = P\ S\ P^{-1}$ by the permutation P. Then, there exists a braid β (Lemma 8), represented by a matrix A^β (Proposition), which sets S_{upper} in the canonical form (after conjugation by* diag(1,1,...,-1) *):*

$$s_{ij} = \begin{pmatrix} k \\ j-i \end{pmatrix}, \quad i<j$$

Another conjugation by $\mathrm{diag}(-1,1,-1,1,-1,\ldots)$ *brings the matrix in the equivalent canonical form*

$$s_{ij} = (-1)^{j-i}\begin{pmatrix} k \\ j-i \end{pmatrix}, \quad i<j$$

Finally, by the action of the braid group, the last matrix can be put in the form

$$s_{ij} = \begin{pmatrix} k-1+j-i \\ j-i \end{pmatrix}, \qquad i<j$$

In all the above matrices

$$s_{ii} = 1, \qquad s_{ij} = 0 \quad i>j$$

Proof: First, we want to explain which is the braid which brings the upper triangular matrix with entries $s_{ij} = (-1)^{j-i}\begin{pmatrix} k \\ j-i \end{pmatrix}$ in the matrix $s_{ij} = \begin{pmatrix} k-1+j-i \\ j-i \end{pmatrix}$. We make use of the following known result [55]:

Consider the upper triangular Stokes' matrix S, the braid $\beta = \beta_{12}\ (\beta_{23}\beta_{12})\ (\beta_{34}\beta_{23}\beta_{12})\ \ldots\ (\beta_{n-1,n}\beta_{n-2,n-1}$ $\ldots\ \beta_{23}\beta_{12})$ and the permutation $\hat{P}$ of entries $\hat{P}_{i,k-i+1} = 1$ for $i = 1,\ldots,k$, and zero otherwise. Then, the relation

$$\left[S^{-1}\right]^{\beta} = \hat{P} S^T \hat{P} \tag{4.24}$$

holds.

Observe that for the matrix S, whose upper triangular part has entries $s_{ij} = \begin{pmatrix} k-1+j-i \\ j-i \end{pmatrix}$, we have $\hat{P}S^T\hat{P} \equiv S$. Moreover, S^{-1} is upper triangular with entries $s_{ij} = (-1)^{j-i}\begin{pmatrix} k \\ j-i \end{pmatrix}$. This proves that S and S^{-1} are equivalent w.r.t the action of the braid group.

Let us now prove the theorem staring from k even. We have to prove that $A^{\beta}\ P\ T_F^{\frac{k}{2}}\ T^{-\frac{k}{2}}\ P^{-1}[A^{\beta}]^T$ is in "canonical form". The proof "reduces" to the computation of products of matrices explicitly given. We do the products in an shrewd way. First we rewrite

$$S^{\beta} = A^{\beta}\ \left(P\ T_F\ P^{-1}\right)^{\frac{k}{2}}\ \left(P\ T^{-1}\ P^{-1}\right)^{\frac{k}{2}}\ [A^{\beta}]^T$$

and we quite easily compute $(P\ T_F\ P^{-1})^{\frac{k}{2}}$ (here P is the permutation introduced at the end of the proof of proposition 1). The result is $(P\ T_F\ P^{-1})^{\frac{k}{2}}_{i,k-i+1} = \pm 1$, with $+$ for i even and $-$ for i odd. The other entries are zero. Then, using the expression of A^{β} from the proposition:

$$F := A^{\beta}\ \left(P\ T_F\ P^{-1}\right)^{\frac{k}{2}} = \begin{pmatrix}
0 & 0 & 1 & & & & & & & & & \\
 & 0 & \binom{k}{1} & 0 & 1 & & & & & & & \\
 & & \binom{k}{2} & 0 & \binom{k}{1} & & & & & & & \\
 & & \binom{k}{3} & 0 & \binom{k}{2} & 0 & 1 & & & & & \\
 & & \vdots & & \vdots & & \vdots & & & & & \\
 & & & & & & & \cdots & 1 & & & \\
 & & * & & * & & * & \cdots & \binom{k}{1} & 0 & 1 & \\
 & & * & & * & & * & \cdots & \binom{k}{2} & 0 & \binom{k}{1} & -1 \\
 & & * & & * & & * & \cdots & \binom{k}{3} & -1 & 0 & \\
 & & \vdots & & \vdots & & \vdots & & & & & \\
 & & \binom{k}{k-4} & 0 & \binom{k}{k-5} & & & & & & & \\
 & & \binom{k}{k-3} & -1 & 0 & & & & & & & \\
0 & -1 & 0 & & & & & & & & & \\
-1 & 0 & & & & & & & & & &
\end{pmatrix}$$

(the -1 on the last column is on the $\frac{k}{2}$-th row). Using the explicit expressions for K_{k3} and K_{k2} of section

5 we compute $T^{-1} = T_F^{-1}\ K_{k3}\ K_{k2}$ and then

$$
P\,T^{-1}\,P^{-1} = \left(\begin{array}{ccccccccccccc}
0 & 0 & -1 & & & & & & & & & & \\
-1 & 0 & -\binom{k}{k-1} & & & & & & & & & & \\
0 & 0 & 0 & 0 & -1 & & & & & & & & \\
0 & -1 & -\binom{k}{k-2} & 0 & -\binom{k}{k-3} & 0 & & & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & -1 & & & & & & \\
 & & & -1 & -\binom{k}{k-4} & 0 & -\binom{k}{k-5} & & & & & & \\
 & & & & & & & \ddots & & & & & \\
 & & & & & & & & 0 & 0 & 0 & -1 & 0 \\
 & & & & & & & & -1 & -\binom{k}{4} & 0 & -\binom{k}{3} & 0 \\
 & & & & & & & & & 0 & 0 & 0 & 1 \\
 & & & & & & & & & 0 & -1 & -\binom{k}{2} & \binom{k}{1}
\end{array}\right)
$$

After this we computed $F\ \left[P\,T^{-1}\,P^{-1}\right]$, $F\ \left[P\,T^{-1}\,P^{-1}\right]\ \left[P\,T^{-1}\,P^{-1}\right]$, $F\ \left[P\,T^{-1}\,P^{-1}\right]^3$, ..., $F\ \left[P\,T^{-1}\,P^{-1}\right]^{\frac{k}{2}}$. We omit the intermediate computations and we give the final result:

$$
F_1 := F\ \left(P\,T^{-1}\,P^{-1}\right)^{\frac{k}{2}} = \left(\begin{array}{ccccccccccccc}
 & & & & & & & & & & 1 & \binom{k}{2} & -\binom{k}{1} \\
 & & & & & & & & & \binom{k}{4} & 0 & \binom{k}{3} & -\binom{k}{2} \\
 & & & & & & & & & \binom{k}{5} & 0 & \binom{k}{4} & -\binom{k}{3} \\
 & & & & & & & \cdots & & \vdots & \vdots & \vdots & \vdots \\
 & & & & & 1 & \binom{k}{k-6} & \cdots & \cdots & * & 0 & * & * \\
 & & & 1 & \binom{k}{k-4} & 0 & \binom{k}{k-5} & \cdots & \cdots & * & 0 & * & * \\
 & 1 & \binom{k}{k-2} & 0 & \binom{k}{k-3} & 0 & \binom{k}{k-4} & \cdots & \cdots & * & 0 & * & * \\
1 & 0 & \binom{k}{k-1} & 0 & \binom{k}{k-2} & 0 & \binom{k}{k-3} & \cdots & \cdots & * & 0 & * & * \\
 & & 1 & 0 & \binom{k}{k-1} & 0 & \binom{k}{k-2} & \cdots & \cdots & * & 0 & * & * \\
 & & & & 1 & 0 & \binom{k}{k-1} & \cdots & \cdots & * & 0 & * & * \\
 & & & & & & 1 & \cdots & \cdots & * & 0 & * & * \\
 & & & & & & & & & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & \cdots & \cdots & \binom{k}{k-1} & 0 & \binom{k}{k-2} & -\binom{k}{k-4} \\
 & & & & & & & & & & 1 & \binom{k}{k-1} & -\binom{k}{k-2} \\
 & & & & & & & & & & & & -\binom{k}{k-1} \\
 & & & & & & & & & & & & 1
\end{array}\right)
$$

Now, multiplying $F_1\ \left[A^\beta\right]^T$, we obtain precisely the "canonical form".

For k odd, we did a similar computation. We omit the detail, but we indicate the order of multiplications which yielded the most simple expressions to multiply step by step. Our aim is to compute $A^\beta\ P\ T_F^{\frac{k-1}{2}}\ K_{k2}\ (T_F^{-1}\ K_{k3}\ K_{k2})^{\frac{k-1}{2}}\ P^{-1}\ [A^\beta]^T$. First, we computed PT_FP^{-1}, then $(PT_FP^{-1})^{\frac{k-1}{2}}$, then $F := A^\beta\ (PT_FP^{-1})^{\frac{k-1}{2}}$. After this, we computed $PK_{k2}P^{-1}$, and $F_1 := F\ PK_{k2}P^{-1}$. Finally, we calculated $m := P\ T_F^{-1}\ K_{k3}\ K_{k2}\ P^{-1}$ and $F_1\ m$, $F_1\ m\ m$, ..., $F_2 := F_1\ m^{\frac{k-1}{2}}$. The matrix $F_2\ [A^\beta]^T$ proved to be in "canonical form".

□

4.7 Canonical form of S^{-1}

The matrix S^{-1} such that $Y_R(z) = Y_L(z)\ S^{-1}$ can be put in the same canonical form of S, as a consequence of the relation (4.24). The only remarks we want to add concern the braid which brings S^{-1} to the canonical form, because it arranges the lines L_j in a "beautiful" shape.

Lemma 8′: *Let the points u_j be in lexicographical order w.r.t the admissible line l. Let us denote $\sigma_{i,i+1} := \beta_{i,i+1}^{-1}$. Then, the following braid arranges the points in cyclic clockwise order, u_1 being the first point in Π_L for k even, or the last in Π_R for k odd (w.r.t the clockwise order) (see figure 6):*

$$
\beta' := \Big[\ (\sigma_{34}\sigma_{56}\sigma_{78}\ \dots\ \sigma_{k-3,k-2}\sigma_{k-1,k})\ (\sigma_{45}\sigma_{67}\sigma_{89}\ \dots\ \sigma_{k-4,k-3}\sigma_{k-2,k-1})\ \dots
$$

$$
\dots\ (\sigma_{\frac{k}{2},\frac{k}{2}+1}\sigma_{\frac{k}{2}+2,\frac{k}{2}+3})\ \sigma_{\frac{k}{2}+1,\frac{k}{2}+2}\Big]
$$

$$
\Big[\ (\sigma_{\frac{k}{2}+2,\frac{k}{2}+3}\sigma_{\frac{k}{2}+3,\frac{k}{2}+4}\ \dots\ \sigma_{k-2,k-1}\sigma_{k-1,k})\ (\sigma_{\frac{k}{2}+2,\frac{k}{2}+3}\sigma_{\frac{k}{2}+3,\frac{k}{2}+4}\ \dots\ \sigma_{k-2,k-1})\ \dots
$$

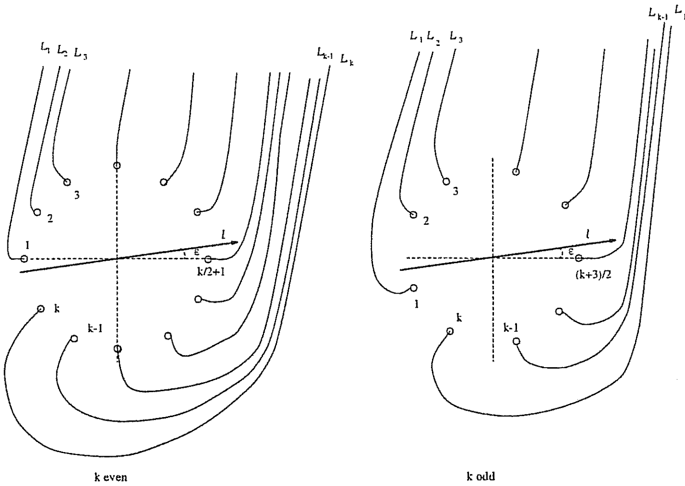


Figure 4.6: Lines L_j after the braid which brings S^{-1} to canonical form.

$$\dots \left(\sigma_{\frac{k}{2}+2,\frac{k}{2}+3}\sigma_{\frac{k}{2}+3,\frac{k}{2}+4}\right)\, \sigma_{\frac{k}{2}+2,\frac{k}{2}+3}\Big]$$

for k even, and

$$\beta' := \beta_{12}\ \Big[\ \left(\sigma_{45}\sigma_{67}\sigma_{89}\ \dots\ \sigma_{k-3,k-2}\sigma_{k-1,k}\right)\ \left(\sigma_{56}\sigma_{78}\ \dots\ \sigma_{k-4,k-3}\sigma_{k-2,k-1}\right)\ \dots$$

$$\dots \left(\sigma_{\frac{k+1}{2},\frac{k+3}{2}}\sigma_{\frac{k+5}{2},\frac{k+7}{2}}\right)\ \sigma_{\frac{k+3}{2},\frac{k+5}{2}}\Big]$$

$$\Big[\ \left(\sigma_{\frac{k+5}{2},\frac{k+7}{2}}\sigma_{\frac{k+7}{2},\frac{k+9}{2}}\ \dots\ \sigma_{k-2,k-1}\sigma_{k-1,k}\right)\ \left(\sigma_{\frac{k+5}{2},\frac{k+7}{2}}\sigma_{\frac{k+7}{2},\frac{k+9}{2}}\ \dots\ \sigma_{k-2,k-1}\right)\ \dots$$

$$\dots \left(\sigma_{\frac{k+5}{2},\frac{k+7}{2}}\sigma_{\frac{k+7}{2},\frac{k+9}{2}}\right)\ \sigma_{\frac{k+5}{2},\frac{k+7}{2}}\Big]$$

for k odd.

A careful consideration of the topological effect of the braid on the lines L_j shows that they are arranged as in figure 6. To reconstruct the configuration it is enough to know the admissible line l (at angle ϵ w.r.t. the positive real axis). In fact, u_1 is the first point in Π_L (in clockwise order) for k even, or the last in Π_R for k odd. The lines come out of the points in centrifugal directions. They go to infinity, without intersections (so preserving their lexicographical order w.r.t l) with the original asymptotic direction $\frac{\pi}{2} - \epsilon$.

$A^{\beta'}$ can be computed as in proposition 1, and the analogous of theorem 2 can be proved using the braid β'.

4.8 Relation between Irregular and Fuchsian systems

Let us consider the fuchsian system (1.30)

$$(U-\lambda)\ \frac{d\phi}{d\lambda} = \left(\frac{1}{2} + V\right)\ \phi \tag{4.25}$$

which can also be written

$$\frac{d\phi}{d\lambda} = \sum_{j=1}^{k} \frac{B_j}{\lambda - u_j}\ \phi, \quad B_j = -E_j\ \left(\frac{1}{2} + V\right)$$

Its dependence on u is isomonodromic. Around the point u_j a fundamental matrix has the form

$$\left[Q_0 + O\left(\lambda - u_j\right)\right]\ \left(\lambda - u_j\right)^M$$

where M =diag$(-\frac{1}{2},0,...,0)$ and the columns of Q_0 are the eigenvectors of B_j; in particular, the first column is $(0,..,0,1,0,...,0)^T$, and 1 occurs at the j^{th} position. Then, the system has k independent vector solutions, of which $k-1$ are regular near u_j and the last is

$$\phi^{(j)}(\lambda) = \frac{1}{\sqrt{\lambda - u_j}} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} + O\left(\sqrt{\lambda - u_j}\right) \quad \lambda \to u_j$$

where 1 occurs at the j^{th} row. For any u_j we can construct such a basis of solutions. The branch of $\sqrt{\lambda - u_j}$ is chosen as follows: let us consider an angle η with a range of 2π, for example $-\frac{\pi}{2} \leq \eta < \frac{3\pi}{2}$, such that $\eta \neq \arg(u_i - u_j)$, $\forall i \neq j$. Then consider the cuts $L_j = \{\lambda = u_j + \rho e^{i\eta},\ \ \rho > 0\}$. Actually, the cuts have two sides, $L_j^+ = \{\lambda = u_j + \rho e^{i\eta},\ \ \rho > 0\}$ and $L_j^- = \{\lambda = u_j + \rho e^{i(\eta - 2\pi)},\ \ \rho > 0\}$. The branch is determined by the choice $\log(\lambda - u_j) = \log|\lambda - u_j| + i\eta$ on L_j^+ and $\log(\lambda - u_j) = \log|\lambda - u_j| + i(\eta - 2\pi)$ on L_j^-. On $\mathbf{C}\backslash \bigcup_j L_j$, $\sqrt{\lambda - u_1}$, ..., $\sqrt{\lambda - u_k}$ are single valued.

For any two (column) vector solutions $\phi(\lambda)$, $\phi'(\lambda)$ we define the symmetric bilinear form:

$$((\ \phi, \phi'\ )) := \phi(\lambda)^T(\lambda - U)\phi'(\lambda)$$

which is independent of λ and u_1, ...,u_k. Let G be the matrix whose entries are $G_{ij} = ((\ \phi^{(i)}, \phi^{(j)}\ ))$. In particular, we normalize in such a way that $G_{ii} = 1$. Then, it can be proved (see [3] and also [17]) that near u_j

$$\phi^{(i)}(\lambda) = G_{ij}\phi^{(j)}(\lambda) + r_{ij}(\lambda)$$

where $r_{ij}(\lambda)$ is regular near u_j. For a counter-clockwise loop around u_j the monodromy of $\phi^{(i)}$ is

$$\phi^{(i)} \mapsto R_j\phi^{(i)} := \phi^{(i)} - 2\frac{(\phi^{(i)}, \phi^{(j)})}{(\phi^{(j)}, \phi^{(j)})}\ \phi^{(j)} \equiv \phi^{(i)} - 2G_{ij}\ \phi^{(j)}$$

Then, the monodromy group of (4.25) acts on $\phi^{(1)}$, ..., $\phi^{(k)}$ as a reflection group whose Gram matrix is $2G$. In particular, $\phi^{(1)}$, ..., $\phi^{(k)}$ are linearly independent (and then a basis) if and only if $\det G \neq 0$.

In general, given two linearly independent (column) vector solutions $\phi^{(1)}$, $\phi^{(2)}$ we have

$$G_{12} = ((\ \phi^{(1)}(u,\lambda), \phi^{(2)}(u,\lambda)\ )) = (dx_1(u,\lambda), dx_2(u,\lambda)) - \lambda < dx_1(u,\lambda), dx_2(u,\lambda) >^*$$

$$\equiv (dx_1(u,0), dx_2(u,0))$$

where $(.,.)$ is the intersection form g. This means that in the case $\det G \neq 0$ the monodromy group of the manifold is $O(n,g)$.

Now consider an oriented line l of argument $\varphi = \frac{\pi}{2} - \eta$, and for any j define the following vector

$$Y^{(j)} = -\frac{\sqrt{z}}{2\sqrt{\pi}} \int_{\gamma_j} d\lambda\ \phi^{(j)}(\lambda)\ e^{\lambda z} \tag{4.26}$$

which is a Laplace transform of $\phi^{(j)}$. The path γ_j comes from infinity near L_j^+, encircles u_j and returns to infinity along L_j^-. We can define $\Pi_L = \{\varphi < \arg z < \varphi + \pi\}$ and $\Pi_R = \{\varphi - \pi < \arg z < \varphi\}$. $\lambda = \infty$ is a regular singularity for (4.25), then the integrals exist for $z \in \Pi_L$, and the non-singular matrix $Y(z) := \left[Y^{(1)}|...|Y^{(k)}\right]$ has the asymptotic behaviour

$$Y(z) \sim \left(I + O\left(\frac{1}{z}\right)\right)\ e^{zU} \quad z \to \infty,\ \ z \in \Pi_L$$

and satisfies the system

$$\frac{dY}{dz} = \left[U + \frac{V}{z}\right]Y.$$

Then it is the fundamental matrix Y_L. Note that l is admissible, since it does not contain Stokes' rays.

It is a fundamental result [17] that the Stokes' matrix of (4.14) satisfies

$$S + S^T = 2G$$

Finally, we recall [17] that if u is subject to a "big" deformation, the monodromy group does not change: actually, the action of the elementary braid $\beta_{i,i+1}$ is given by

$$(R_1, ..., R_i, R_{i+1}, ..., R_n) \to (R_1, ..., R_{i+1}, R_{i+1}R_iR_{i+1}, ..., R_n)$$

and the group remains the same.

4.9 Monodromy Group of the Quantum Cohomology of $\mathbf{CP}^{k-1}$

The monodromy group is generated by the monodromy of the solutions of (4.25) when λ describes loops around $u_1(t), ..., u_k(t)$.

From (4.26)

$$\partial_\alpha \tilde{t}^j(t,z) = -\frac{\sqrt{z}}{2\sqrt{\pi}} \int d\lambda \; \partial_\alpha x^j(t,\lambda) \; e^{\lambda z} \tag{4.27}$$

and therefore we must add the effect of the displacement $t^2 \mapsto t^2 + 2\pi i$ in order to obtain the full monodromy. In fact, in this case

$$[\varphi^{(1)}(ze^{\frac{t^2}{k}}), ..., \varphi^{(k)}(ze^{\frac{t^2}{k}})] \mapsto [\varphi^{(1)}(ze^{\frac{t^2}{k}}), ..., \varphi^{(k)}(ze^{\frac{t^2}{k}})] \; T$$

and the same holds for $\tilde{t}(z, (0, t^2, ..., 0))$.

Then, the monodromy group of the quantum cohomology of $\mathbf{CP}^{k-1}$ is generated by the transformations R_1, R_2, ..., R_k, T introduced in the preceding sections.

We are going to study the structure of the monodromy group of $\mathbf{CP}^{k-1}$ for any $k \geq 3$. Recall that the matrix S for (4.14) is not upper triangular, because in U the order of u_1, ..., u_k is not lexicographical w.r.t. the line l. Then, Coxeter identity is $-S^{-1}S^T =$ product of the R_j's in the order referred to l. For example, for $k = 3$, $S^{-1}S^T = -R_2R_3R_1$, since the lexicographical ordering would be u_2, u_3, u_1. From the identity $S^{-1}S^T = (-1)^{k-1}T^k$ it follows a first general relation in the group

$$T^k = (-1)^k \text{ product of } R_j\text{'s in suitable order}$$

Two cases must now be distinguished.

k odd: As a general result [3], $\det G = 0$ if and only if $V + \frac{1}{2}$ has an integer eigenvalue. The eigenvalues of V are $\frac{k-1}{2}$, $\frac{k-3}{2}$, ..., $-\frac{k-1}{2}$. Then, for k odd, $\det G \neq 0$, and $\phi^{(1)}$, ..., $\phi^{(k)}$ are a basis. The matrices R_j are

$$R_j = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ -2G_{j1} & -2G_{j2} & \dots & -1 & \dots & -2G_{jk} \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where $S^T + S = 2G$.

In concrete examples, we have "empirically" found other relations like

$$R_2 = p_1(T, R_1)$$

$$R_2 = p_2(T, R_1)$$

$$\vdots$$

$$R_k = p_k(T, R_1)$$

where $p_j(T, R_1)$ means a product of the elements T and R_1. We have also found the relation

$$(TR_1)^k = -I$$

We investigated the following cases:

$\mathbf{CP}^2$ $(k=3)$

$$R_2 = TR_1T^{-1}, \quad R_3 = T\ (R_1R_2R_1)\ T^{-1}$$
$$(TR_1)^3 = -I$$
$$T^3 = -R_2R_3R_1$$

$\mathbf{CP}^4$ $(k=5)$

$$\begin{cases} R_2 = TR_1T^{-1}, & R_3 = TR_2T^{-1} \\ R_4 = T\ (R_2R_3R_2)\ T^{-1}, & R_5 = T^{-1}\ (R_2R_1R_2)\ T \end{cases}$$
$$(TR_1)^5 = -I$$
$$T^5 = -R_3R_4R_2R_5R_1$$

$\mathbf{CP}^6$ $(k=7)$

$$\begin{cases} R_2 = TR_1T^{-1}, \quad R_3 = TR_2T^{-1}, \quad R_4 = TR_3T^{-1} \\ R_5 = T\ (R_3R_4R_3)T^{-1}, \quad R_6 = T(TR_1)^3\ R_2\ [T(TR_1)^3]^{-1} \\ R_7 = T^{-2}(R_3R_2R_3)T^2 \end{cases}$$
$$(TR_1)^7 = -I$$
$$T^7 = -R_4R_5R_3R_6R_2R_7R_1$$

Note that one relation, for example that for R_k, can be derived from the others, and that just $R_1, T, -I$ are enough to generate the monodromy group in each of the examples. They satisfy (*in the examples*) the relations:

$$R_1^2 = (-I\ T\ R_1)^k = (-I)^2 = I$$

$$R_1(-I)\ ((-I)R_1)^{-1} = I, \quad T(-I)\ ((-I)T)^{-1} = I$$

The last two relations mean simply the commutativity of $-I$ with R_1 and T. The relations are not only satisfied, but also "fulfilled" (namely, $(-I\ T\ R_1)^n \neq I$ for $n < k$). Now call

$$X := R_1, \quad Y := -ITR_1, \quad Z = -I$$

These elements generate the monodromy group of $\mathbf{CP}^{k-1}$ with *at least* the relations

$$X^2 = Y^k = Z^2 = 1$$

$$(ZX)(XZ)^{-1} = 1, \quad (ZY)(YZ)^{-1} = 1$$

Note that Z generates the cyclic group C_2 of order 2.

If there were no other relations (which we did not find "empirically"), we would conclude that the monodromy group of the quantum cohomology of $\mathbf{CP}^k$ (*in the examples*) is isomorphic to the direct product

$$< X, Y \mid X^2 = Y^k = 1 > \times C_2$$

where $< X, Y \mid X^2 = Y^k = 1 >$ means the group generated by X, Y with relations $X^2 = Y^k = 1$.

k **even:** Now $\det G = 0$, since $V + \frac{1}{2}$ has integer eigenvalues. G has rank $k-1$ and the eigenspace of its eigenvalue 0 has dimension 1. Let $(z^1, ..., z^k)^T$ be an eigenvector of eigenvalue 1. The vector $v := \sum_{j=1}^k z^j \phi^{(j)}$ is zero, because

$$\left(v, \phi^{(i)}\right) = \sum_{j=1}^{k} z^j G_{ji} = 0 \quad \forall i$$

then

$$z^1\phi^{(1)} + z^{(2)}\phi^{(2)} + ... + z^k\phi^{(k)} = 0$$

and $k-1$ of the $\phi^{(j)}$'s are linearly independent. The fuchsian system (4.25) has a regular (vector) solution $\phi_0(\lambda) = \sum_{n=0}^d \phi_n\lambda^n$, where ϕ_n are constant (column) vectors, and ϕ_d is the eigenvector of $V + \frac{1}{2}$ relative to the largest integer eigenvalue less or equal to zero; this eigenvalue is precisely $-d$ (see

[3]). In our case, $d = 0$ and $\phi_o(\lambda) = \phi_0$, a constant vector. $\phi^{(1)}$, $\phi^{(2)}$, ..., $\phi^{(k-1)}$, ϕ_0 is then a possible choice for a basis of solutions.

Observe that in the gauge equivalent form $\psi = X\phi$, ψ_0 is the eigenvector of $\frac{1}{2} + \hat{\mu}$ with eigenvalue zero. Then

$$\psi_0 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \partial_1 x \\ \partial_2 x \\ \vdots \\ \vdots \\ \partial_k x \end{pmatrix}$$

where all the entries are zero but the one at position $\frac{k}{2} + 1$. x is the flat coordinate for $(\, , \,) - \lambda < \, , \, >$ corresponding to ψ_0. Then, we can chose the following flat coordinates:

$$x^1(\lambda, t), \; x^2(\lambda, t), \; \ldots \; , \; x^{k-1}(\lambda, t), \; t^{\frac{k}{2}+1}$$

The monodromy group then acts on a $k - 1$ dimensional space.

Let us determine the reduction of R_1, R_2, ..., R_k, T to the $k - 1$ dimensional space. The entries of T on the vectors $\phi^{(j)}$ are: $T\phi^{(i)} = \sum_{j=1}^{k} T_{ji}\phi^{(j)}$, $i = 1, ..., k$. On the new basis $\phi^{(1)}$, ..., $\phi^{(k-1)}$, ϕ_0 the matrices are rewritten

$$\begin{aligned} R_j\phi^{(i)} &= \phi^{(i)} - 2G_{ij}\phi^{(j)} \quad i = 1, .., k-1 \quad j \neq k \\ R_j\phi_0 &= \phi_0 \qquad\qquad\qquad\qquad\qquad j \neq k \\ R_k\phi^{(i)} &= \phi^{(i)} - 2G_{ik}\left(-\frac{1}{z^k}\sum_{j=1}^{k-1} z^j\phi^{(j)}\right) \quad i \neq k \\ T\phi^{(i)} &= \sum_{j=1}^{k-1} T_{ji}\phi^{(j)} + T_{ki}\left(-\frac{1}{z^k}\sum_{j=1}^{k-1} z^j\phi^{(j)}\right) \end{aligned}$$

Then the matrices assume a reduced form $\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$

We studied two examples; besides Coxeter identity T^k =product of R_j's, we found relations similar to the case k odd:

$$\begin{aligned} R_2 &= p_1(T, R_1) \\ R_2 &= p_2(T, R_1) \\ &\vdots \\ R_k &= p_k(T, R_1) \end{aligned}$$

and

$$(TR_1)^k = I$$

Namely:

$\mathbf{CP}^3$ $(k = 4)$

$$\begin{cases} R_2 = TR_1T^{-1}, \quad R_3 = TR_2T^{-1} \\ R_4 = T^{-1}\,(R_2R_1R_2)\,T \end{cases}$$

$$(TR_1)^4 = I$$

$$T^4 = R_3R_2R_4R_1$$

$\mathbf{CP}^5$ $(k = 6)$

$$\begin{cases} R_2 = TR_1T^{-1}, \quad R_3 = TR_2T^{-1} \\ R_4 = TR_3T^{-1}, \quad R_5 = T\,(R_2R_3R_4R_3R_2)\,T^{-1} \\ R_6 = T^{-1}\,(R_2R_1R_2)\,T \end{cases}$$

$$(TR_1)^6 = I$$

$$T^6 = R_4R_3R_5R_2R_6R_1$$

The same remarks of k odd hold here. Call

$$X := R_1, \quad Y := R_1 T$$

then, if there were no other hidden relations, the monodromy group of the quantum cohomology of $\mathbf{CP}^k$ (*in the examples*) would be isomorphic to

$$< X, Y, \mid X^2 = Y^k = 1 >$$

Note that $< X, Y, \mid X^2 = Y^k = 1 >$ is (isomorphic to) the subgroup of orientation preserving transformations of the hyperbolic triangular group $[2, k, \infty]$.

Lemma 10: *The subgroup of the orientation preserving transformations of the hyperbolic triangular group* $[2, k, \infty]$ *is isomorphic to the subgroup of* $PSL(2, \mathbf{R})$ *generated by*

$$\tau \mapsto -\frac{1}{\tau}$$

$$\tau \mapsto \frac{1}{2\cos\frac{\pi}{k} - \tau}$$

$\tau \in H := \{z \in \mathbf{C} \mid \Im z > 0\}$

Proof: Consider three integers m_1, m_2, m_3 such that

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1$$

In the Bolyai-Lobatchewsky plane H, the triangular group $[m_1, m_3, m_3]$ of hyperpolic reflections in the sides of hyperbolic triangles of angles $\frac{\pi}{m_1}$, $\frac{\pi}{m_2}$, $\frac{\pi}{m_3}$ is generated by three reflections r_1, r_2, r_3 satisfying the relations

$$r_1^2 = r_2^2 = r_3^2 = (r_2 r_3)^{m_1} = (r_3 r_1)^{m_2} = (r_2 r_1)^{m_3} = 1$$

and the subgroup of orientation preserving transformation is generated by $X = r_2 r_3$, $Y = r_3 r_1$, Then

$$X^{m_1} = Y^{m_2} = (XY)^{m_3} = 1$$

For $m_1 = 2$, $m_2 = k$, $m_3 = \infty$, a fundamental triangular region is $\{0 < \Re z < \cos\frac{\pi}{k}\} \cap \{|z| > 1\}$. Then

$$r_1(\tau) = -\bar{\tau}, \quad r_2(\tau) = \frac{1}{\bar{\tau}}, \quad r_3(\tau) = 2\cos\frac{\pi}{k} - \bar{\tau}$$

The bar means complex conjugation. Then

$$X(\tau) = -\frac{1}{\tau}, \quad Y(\tau) = \frac{1}{2\cos\frac{\pi}{k} - \tau}$$

□

Remark : The orientation preserving transformations of $[2, 3, \infty]$ are the *modular group* $PSL(2, \mathbf{Z})$.

Theorem 3: *The monodromy group of the quantum cohomology of* $\mathbf{CP}^2$ *is isomorphic to*

$$< X, Y, \mid X^2 = Y^3 = 1 > \times C_2 \cong PSL(2, \mathbf{Z}) \times C_2 \tag{4.28}$$

The monodromy group of the quantum cohomology of $\mathbf{CP}^3$ *is isomorphic to*

$$< X, Y, \mid X^2 = Y^4 = 1 > \cong \textit{ orient. preserv. transf. of } [2, 4, \infty] \tag{4.29}$$

The theorem for the case of $\mathbf{CP}^2$ is already proved in [17].

Proof : a) $\mathbf{CP}^2$:

$$R_1 = \begin{pmatrix} -1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 3 & 3 \\ 0 & -1 & 0 \end{pmatrix}$$

and $X = R_1$, $Y = -ITR_1$ and $Z = -I$ satisfy the relations of (4.28). They act on the column vector $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The quadratic form $q(x,y,z) = \mathbf{x}^T G \mathbf{x}$ is R_1 and T - invariant. Then T, R_1 act on two dimensional invariant subspaces $q(x,y,z) =$ constant. On each of these subspaces we introduce new coordinates $\chi \in \mathbf{R}$ and $\varphi \in [0, 2\pi)$. Let $\tau = e^{\chi}\, e^{i\varphi}$ and

$$x = \frac{a}{2}(\tau\bar{\tau} - \frac{3}{2}(\tau + \bar{\tau}) + 1)\ \frac{i}{\tau - \bar{\tau}}$$

$$y = \frac{a}{2}(\tau\bar{\tau} - \frac{1}{2}(\tau + \bar{\tau}) - 1)\ \frac{i}{\tau - \bar{\tau}}$$

$$z = \frac{a}{2}(-\tau\bar{\tau} - \frac{1}{2}(\tau + \bar{\tau}) + 1)\ \frac{i}{\tau - \bar{\tau}}$$

$a \in \mathbf{R}$, $a \neq 0$. Note that $q(x,y,z) = a^2 > 0$. Then, it is easily verified that

$$\mathbf{x}(-\frac{1}{\tau}) = -X\ \mathbf{x}(\tau)$$

$$\mathbf{x}(\frac{1}{1-\tau}) = Y\ \mathbf{x}(\tau)$$

$$\mathbf{x}(\tau, -a) = Z\ \mathbf{x}(\tau, a)$$

. This implies the 1 to 1 correspondence between the generators of the modular group and X and Y.

b) Case of $\mathbf{CP}^3$.

$$R_1 = \begin{pmatrix} -1 & 4 & -10 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrices are already written on $\phi^{(1)}$, $\phi^{(2)}$, $\phi^{(3)}$, ϕ_0. Recall that the monodromy acts only on x^1, x^2, x^3, because the last flat coordinate is t^3. This action is given by the following three dimensional matrices, acting on a three dimensional space of vectors $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$r_1 = \begin{pmatrix} -1 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad t := \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 3 \\ 0 & -1 & 3 \end{pmatrix}$$

We redefine $X = r_1$ and $Y = tr_1$, which satisfy the relations (4.29). We proceed as above, defining

$$x = a(\tau\bar{\tau} - \frac{1}{\sqrt{2}}(\tau + \bar{\tau}) + \frac{1}{3})\ \frac{i}{\tau - \bar{\tau}}$$

$$y = a(-\frac{2}{3}\tau\bar{\tau} - \frac{2\sqrt{2}}{3}(\tau + \bar{\tau}) + \frac{2}{3})\ \frac{i}{\tau - \bar{\tau}}$$

$$z = a(-\frac{1}{3}\tau\bar{\tau} - \frac{\sqrt{2}}{6}(\tau + \bar{\tau}) + \frac{1}{3})\ \frac{i}{\tau - \bar{\tau}}$$

$a \neq 0$. Note that $\mathbf{x}^T g \mathbf{x} = (8/9)a^2$, where g is the 3×3 reduction of G. It is easily verified that

$$\mathbf{x}(-\frac{1}{\tau}) = -X\ \mathbf{x}(\tau)$$

$$\mathbf{x}(\frac{1}{\sqrt{2} - \tau}) = -Y\ \mathbf{x}(\tau)$$

which proves the theorem.

□

4.10 Some numerical examples

First we give Φ_R and Φ_L. k **odd:**

$$\Phi_R(z)^T = \begin{bmatrix} (-1)^{\frac{k-1}{2}} \left[g(ze^{-\frac{2\pi i}{k}(\frac{k-1}{2})}) - \binom{k}{1} g(ze^{-\frac{2\pi i}{k}(\frac{k-3}{2})}) + \ldots + \binom{k}{k-1} g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})}) \right] \\ \vdots \\ g(ze^{-\frac{4\pi i}{k}}) - \binom{k}{1} g(ze^{-\frac{2\pi i}{k}}) + \binom{k}{2} g(z) - \binom{k}{3} g(ze^{\frac{2\pi i}{k}}) + \binom{k}{4} g(ze^{\frac{4\pi i}{k}}) \\ -g(ze^{-\frac{2\pi i}{k}}) + \binom{k}{1} g(z) - \binom{k}{2} g(ze^{\frac{2\pi i}{k}}) \\ g(z) \\ -g(ze^{\frac{2\pi i}{k}}) \\ g(ze^{\frac{4\pi i}{k}}) \\ \vdots \\ (-1)^{\frac{k-1}{2}} g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})}) \end{bmatrix}$$

$$\Phi_L(z)^T = \begin{bmatrix} (-1)^{\frac{k-1}{2}} g(ze^{-\frac{2\pi i}{k}(\frac{k-1}{2})}) \\ -(-1)^{\frac{k-1}{2}} g(ze^{-\frac{2\pi i}{k}(\frac{k-3}{2})}) \\ \vdots \\ -g(ze^{-\frac{2\pi i}{k}}) \\ g(z) \\ -g(ze^{\frac{2\pi i}{k}}) + \binom{k}{k-1} g(z) \\ g(ze^{\frac{4\pi i}{k}}) - \binom{k}{k-1} g(ze^{\frac{2\pi i}{k}}) + \binom{k}{k-2} g(z) - \binom{k}{k-3} g(ze^{-\frac{2\pi i}{k}}) \\ \vdots \\ (-1)^{\frac{k-1}{2}} \left[g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})}) - \binom{k}{k-1} g(ze^{\frac{2\pi i}{k}(\frac{k-3}{2})}) + \ldots - \binom{k}{2} g(ze^{-\frac{2\pi i}{k}(\frac{k-3}{2})}) \right] \end{bmatrix}$$

k **even:**

$$\Phi_R(z)^T = \begin{bmatrix} (-1)^{\frac{k}{2}} \left[g(ze^{-i\pi}) - \binom{k}{1} g(ze^{-i\pi+i\frac{2\pi}{k}}) + \ldots + \binom{k}{k-1} g(ze^{i(\pi-\frac{2\pi}{k})}) \right] \\ \vdots \\ g(ze^{-\frac{4\pi i}{k}}) - \binom{k}{1} g(ze^{-\frac{2\pi i}{k}}) + \binom{k}{2} g(z) - \binom{k}{3} g(ze^{\frac{2\pi i}{k}}) \\ -g(ze^{-\frac{2\pi i}{k}}) + \binom{k}{1} g(z) \\ g(z) \\ -g(ze^{\frac{2\pi i}{k}}) \\ g(ze^{\frac{4\pi i}{k}}) \\ \vdots \\ (-1)^{\frac{k}{2}-1} g(ze^{\frac{2\pi i}{k}(\frac{k}{2}-1)}) \end{bmatrix}$$

$$\Phi_L(z)^T = \begin{bmatrix} (-1)^{\frac{k}{2}} g(ze^{-i\pi}) \\ \vdots \\ g(ze^{-i\frac{4\pi}{k}}) \\ -g(ze^{-\frac{2\pi i}{k}}) \\ g(z) \\ -g(ze^{\frac{2\pi i}{k}}) + \binom{k}{k-1} g(z) - \binom{k}{k-2} g(ze^{-i\frac{2\pi}{k}}) \\ g(ze^{\frac{4\pi i}{k}}) - \binom{k}{k-1} g(ze^{\frac{2\pi i}{k}}) + \binom{k}{k-2} g(z) - \binom{k}{k-3} g(ze^{-\frac{2\pi i}{k}}) + \binom{k}{k-4} g(ze^{-i\frac{4\pi}{k}}) \\ \vdots \\ (-1)^{\frac{k}{2}-1} \left[g(ze^{i\pi-i\frac{2\pi}{k}}) - \binom{k}{k-1} g(ze^{i\pi-i\frac{4\pi}{k}}) + \ldots - \binom{k}{2} g(ze^{-i\pi+i\frac{2\pi i}{k}}) \right] \end{bmatrix}$$

We give all the matrices of interest for $k = 2, 3, 7$. Many more examples are in the preprint version of [25]. S_{upper} is PSP^{-1}. A stands for A^{β}, A' for $A^{\beta'}$. $S^{\beta} = AS_{upper}A^T$, $S^{\beta'} = A'S_{upper}^{-1}[A']^T$. $\sigma_{i,i+1} := \beta_{i,i+1}^{-1}$

$\mathbf{CP}^2$

$$K_{12} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad K_{13} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad K_{32} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T := \begin{bmatrix} 0 & 0 & 1 \\ -1 & 3 & 3 \\ 0 & -1 & 0 \end{bmatrix}$$

$$S := \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 3 \\ -3 & 0 & 1 \end{bmatrix} \quad PSP^{-1} = \begin{bmatrix} 1 & 3 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{CP}^3$

$$K_{42} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad K_{43} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T := \begin{bmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 6 & 4 \\ 0 & -1 & 4 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad T_1 := \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S := \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 6 \\ 10 & -4 & 1 & -20 \\ -4 & 0 & 0 & 1 \end{bmatrix}$$

$$P := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad S_{upper} = \begin{bmatrix} 1 & -4 & -20 & 10 \\ 0 & 1 & 6 & -4 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad S^{\beta} = S^{\beta'} := \begin{bmatrix} 1 & 4 & 6 & -4 \\ 0 & 1 & 4 & -6 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\beta = \beta_{12}$

$\mathbf{CP}^6$

$$K_{72} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 21 \\ 0 & 0 & 1 & 0 & 0 & 35 & 0 \\ 0 & 0 & 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad K_{73} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 35 \\ 0 & 0 & 0 & 1 & 0 & 21 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 21 & 7 \\ 0 & -1 & 0 & 0 & 35 & 35 & 0 \\ 0 & 0 & -1 & 7 & 21 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \quad S := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & 1 & 0 & 0 & 0 & 0 & 21 \\ 28 & -7 & 1 & 0 & 0 & 35 & -112 \\ -84 & 28 & -7 & 1 & 7 & -224 & 378 \\ -378 & 112 & -21 & 0 & 1 & -728 & 1638 \\ 224 & -35 & 0 & 0 & 0 & 1 & -728 \\ -7 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 7 & 0 & 0 \\ 1 & 0 & 7 & 0 & 21 & 0 & 0 \\ 0 & 1 & 21 & 0 & 35 & 0 & 0 \\ 0 & 0 & 0 & 1 & 35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad A' := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -35 & 1 & 0 & 0 \\ 0 & 0 & 0 & -21 & 0 & -7 & 1 \\ 0 & 0 & 0 & 7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$S^{\beta} := \begin{bmatrix} 1 & 7 & 21 & 35 & 35 & 21 & -7 \\ 0 & 1 & 7 & 21 & 35 & 35 & -21 \\ 0 & 0 & 1 & 7 & 21 & 35 & -35 \\ 0 & 0 & 0 & 1 & 7 & 21 & -35 \\ 0 & 0 & 0 & 0 & 1 & 7 & -21 \\ 0 & 0 & 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad S^{\beta'} := \begin{bmatrix} 1 & 7 & 21 & 35 & 35 & -21 & -7 \\ 0 & 1 & 7 & 21 & 35 & -35 & -21 \\ 0 & 0 & 1 & 7 & 21 & -35 & -35 \\ 0 & 0 & 0 & 1 & 7 & -21 & -35 \\ 0 & 0 & 0 & 0 & 1 & -7 & -21 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\beta = (\beta_{23}\beta_{12})(\beta_{45}\beta_{34}\beta_{23}\beta_{12})$, $\beta' = \beta_{12}[(\sigma_{45}\sigma_{67})\sigma_{56}]\sigma_{67}$.

Chapter 5

Connection problem and Critical Behaviour for PVI_μ

In this chapter we study the critical behaviour and the connection problem for the solutions of PVI_μ (1.10). The purpose is to fix some results which will be used in the next chapters when we solve the inverse problem for the associated Frobenius manifold in terms of Painlevé transcendents.

We refer to the Introduction of the thesis for a general discussion of the results of this chapter.

5.1 Branches of PVI_μ and monodromy data - known results

The equation PVI_μ is equivalent to the equations of isomonodromy deformation of the fuchsian system obtained in section 1.10

$$\frac{dY}{dz}=\left[\frac{A_1(u)}{z-u_1}+\frac{A_2(u)}{z-u_2}+\frac{A_3(u)}{z-u_3}\right]\ Y:=A(z;u)\ Y \tag{5.1}$$

$$u:=(u_1,u_2,u_3),\qquad \mathrm{tr}(A_i)=\det A_i=0,\qquad \sum_{i=1}^3 A_i=-\mathrm{diag}(\mu,-\mu)$$

If u is deformed, the monodromy matrices of a solution of the system (5.1) do not change, provided that the deformation is "small" (see below). The connection to PVI_μ is given by

$$x=\frac{u_3-u_1}{u_2-u_1},\quad y(x)=\frac{q(u)-u_1}{u_2-u_1}$$

where $q(u_1,u_2,u_3)$ is the root of

$$[A(q;u_1,u_2,u_3)]_{12}=0\qquad \text{if } \mu\neq 0$$

The case $\mu=0$ is disregarded, because $PVI_{\mu=0}\equiv PVI_{\mu=1}$.

The system (5.1) has fuchsian singularities at u_1, u_2, u_3. Let us fix a branch $Y(z,u)$ of a fundamental matrix solution by choosing branch cuts in the z plane and a basis of loops in $\pi(\mathbf{C}\backslash\{u_1,u_2,u_3\};z_0)$, where z_0 is a base-point. Let γ_i be a basis of loops encircling counter-clockwise the point u_i, $i=1,2,3$. See figure 5.1. Then

$$Y(z,u)\mapsto Y(z,u)M_i,\quad i=1,2,3,\qquad \det M_i\neq 0,$$

if z describes a loop γ_i. Along the loop $\gamma_\infty:=\gamma_1\cdot\gamma_2\cdot\gamma_3$ we have $Y\mapsto YM_\infty$. M_i are the *monodromy matrices*, and they give a representation of the fundamental group. Of course $M_\infty=M_3M_2M_1$. The transformations $Y'(z,u)=Y(z,u)B\quad \det(B)\neq 0$ yields all the possible fundamental matrices, hence the monodromy matrices of (5.1) are defined up to conjugation

$$M_i\mapsto M_i'=B^{-1}M_iB.$$

From the standard theory of fuchsian systems it follows that we can choose a fundamental solution behaving as follows

$$Y(z;u) = \begin{cases} (I + O(\frac{1}{z}))\ z^{-\hat{\mu}} z^R\ C_\infty, & z \to \infty \\ G_i(I + O(z-u_i))\ (z-u_i)^J\ C_i, & z \to u_i, \quad i = 1,2,3 \end{cases}$$

where $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\hat{\mu}$ =diag$(\mu, -\mu)$, $G_i J G_i^{-1} = A_i$ $(i = 0, x, 1)$ and

$$R = \begin{cases} 0\ , & \text{if } 2\mu \notin \mathbf{Z} \\ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, & \mu > 0 \\ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, & \mu < 0 \end{cases} \quad \text{if } 2\mu \in \mathbf{Z}$$

$b \in \mathbf{C}$ being determined by the matrices A_i. Then $M_i = C_i^{-1} e^{2\pi i J} C_i$, $M_\infty = C_\infty^{-1} e^{-2\pi i \hat{\mu}} e^{2\pi i R} C_\infty$.

The dependence of the fuchsian system on u is isomonodromic. This means that for small deformations of u the monodromy matrices do not change [30] [26] and then

$$\partial_i R = \partial_i C_1 = \partial_i C_2 = \partial_i C_3 = \partial_i C_\infty = 0$$

"Small" deformation means that $x = (u_2 - u_1)/(u_3 - u_1)$ can vary in the x-plane provided it does not describe complete loops around $0, 1, \infty$ (in other words, we fix branch cuts, like $\alpha < \arg(x) < \alpha + 2\pi$ and $\beta < \arg(1-x) < \beta + 2\pi$ for $\alpha, \beta \in \mathbf{R}$). For "big" deformations, the monodromy matrices change according to an action of the pure braid group. This point will be discussed later.

Suppose we have a branch $y(x)$. We associate to it the fuchsian system (we stress that we actually have a branch defined by the cuts, thus the fuchsian system has monodromy matrices independent of x). Therefore, to any branch of a Painlevé transcendent there corresponds a monodromy representation.

Conversely, the problem of finding the branches of the Painlevé transcendents of PVI_μ for given monodromy matrices (up to conjugation) is the problem of finding a fuchsian system (5.1) having the given monodromy matrices. This problem is called *Riemann-Hilbert problem*, or 21^{th} *Hilbert problem.* For a given PVI_μ (i.e. for a fixed μ) there is a one-to-one correspondence between a monodromy representation and a branch of a transcendent if and only if the Riemann-Hilbert problem has a unique solution.

• **Riemann-Hilbert problem (R.H.)**: find the coefficients $A_i(u)$, $i = 1,2,3$ from the following monodromy data:

a) the matrices

$$\hat{\mu} = \text{diag}(\mu, -\mu), \qquad \mu \in \mathbf{C} \backslash \{0\}$$

$$R = \begin{cases} 0\ , & \text{if } 2\mu \notin \mathbf{Z} \\ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, & \mu > 0 \\ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, & \mu < 0 \end{cases} \quad \text{if } 2\mu \in \mathbf{Z}$$

$b \in \mathbf{C}$.

b) three poles u_1, u_2, u_3, a base-point and a base of loops in $\pi(\mathbf{C} \backslash \{u_1, u_2, u_3\}; z_0)$. See figure 5.1.

c) three monodromy matrices M_1, M_2, M_3 relative to the loops (counter-clockwise) and a matrix M_∞ similar to $e^{-2\pi i \hat{\mu}} e^{2\pi i R}$, and satisfying

$$\text{tr}(M_i) = 2, \quad \det(M_i) = 1, \qquad i = 1,2,3$$

$$M_3\ M_2\ M_1 = M_\infty$$

$$M_\infty = C_\infty^{-1}\ e^{-2\pi i \hat{\mu}} e^{2\pi i R}\ C_\infty \tag{5.2}$$

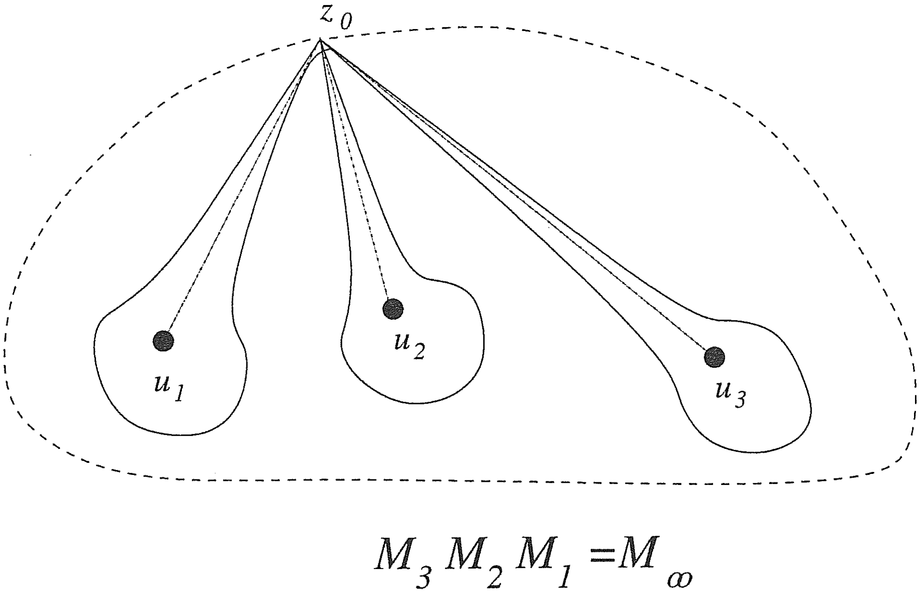


Figure 5.1: Choice of a basis in $\pi_0(\mathbf{C}\backslash\{u_1, u_2, u_3\})$

where C_∞ realizes the similitude. We also choose the indices of the problem, namely we fix $\frac{1}{2\pi i}\log M_i$ as follows: let

$$J := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

We require there exist three *connection matrices* C_1, C_2, C_3 such that

$$C_i^{-1} e^{2\pi i J} C_i = M_i, \qquad i = 1, 2, 3 \tag{5.3}$$

and we look for a matrix valued meromorphic function $Y(z;u)$ such that

$$Y(z;u) = \begin{cases} G_\infty(I + O(\frac{1}{z}))\ z^{-\hat{\mu}} z^R\ C_\infty, & z \to \infty \\ G_i(I + O(z - u_i))\ (z - u_i)^J\ C_i, & z \to u_i, \quad i = 1, 2, 3 \end{cases} \tag{5.4}$$

G_∞ and G_i are invertible matrices depending on u. The coefficient of the fuchsian system are then given by $A(z; u_1, u_2, u_3) := \frac{dY(z;u)}{dz} Y(z;u)^{-1}$.

Recall that a 2×2 R.H. is always solvable [1]. The monodromy matrices are considered up to the conjugation

$$M_i \mapsto M_i' = B^{-1} M_i B, \qquad \det B \neq 0, \quad i = 1, 2, 3, \infty \tag{5.5}$$

and the coefficients of the fuchsian system itself are considered up to conjugation $A_i \mapsto F^{-1} A_i F$ ($i = 1, 2, 3$), by an invertible matrix F. Actually, two conjugated fuchsian systems admit fundamental matrix solutions with the same monodromy, and a given fuchsian system defines the monodromy up to conjugation (depending on the choice of the fundamental solution).

On the other hand, a triple of monodromy matrices M_1, M_2, M_3 may be realized by two fuchsian systems which are not conjugated. This corresponds to the fact that the solutions C_∞, C_i of (5.2), (5.3) are not unique, and the choice of different particular solutions may give rise to fuchsian systems which are not conjugated. If this is the case, there is no one-to-one correspondence between monodromy matrices (up to conjugation) and solutions of PVI_μ. It is easy to prove that:

The R.H. has a unique solution, up to conjugation, for $2\mu \notin \mathbf{Z}$ or for $2\mu \in \mathbf{Z}$ and $R \neq 0$. [1]

[1] The proof is done in the following way: consider two solutions C and $\tilde{C}$ of the equations (5.2), (5.3). Then

$$(C_i \tilde{C}_i^{-1})^{-1} e^{2\pi i J} (C_i \tilde{C}_i^{-1}) = e^{2\pi i J}$$

$$(C_\infty \tilde{C}_\infty^{-1})^{-1} e^{-2\pi i \hat{\mu}} e^{2\pi i R} (C_\infty \tilde{C}_\infty^{-1}) = e^{-2\pi i \hat{\mu}} e^{2\pi i R}$$

Once the R.H. is solved, the sum of the matrix coefficients $A_i(u)$ of the solution $A(z;u_1,u_2,u_3) = \sum_{i=1}^3 \frac{A_i(u)}{z-u_i}$ must be diagonalized (to give $-$ diag$(\mu,-\mu)$). [2] After that, a branch y(x) of PVI_μ can be "computed" from $[A(q;u_1,u_2,u_3)]_{12} = 0$. The fact that the R.H. has a unique solution for the given monodromy data (if $2\mu \notin \mathbf{Z}$ or $2\mu \in \mathbf{Z}$ and $R \neq 0$) means that there is a one-to-one correspondence between the triple M_1, M_2, M_3 and the branch $y(x)$.

We review some known results [21] [37].

1) One $M_i = I$ if and only if the Schlesinger equations yield $q(u) \equiv u_i$. This does not correspond to a solution of PVI_μ.

2) If the M_i's, $i = 1,2,3$, commute, then μ is integer (as it follows from the fact that the 2×2 matrices with 1's on the diagonals commute if and only if they can be simultaneously put in upper or lower triangular form). There are solutions of PVI_μ only for

$$M_1 = \begin{pmatrix} 1 & i\pi a \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & i\pi \\ 0 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & i\pi(1-a) \\ 0 & 1 \end{pmatrix}, \qquad a \neq 0,1$$

In this case $R = 0$ and $M_\infty = I$. For $\mu = 1$ the solution is

$$y(x) = \frac{ax}{1-(1-a)x}$$

and for other integers μ the solution is obtained from $\mu = 1$ by a birational transformation [21] [37]. In particular, these solutions are rational.

3) Non commuting M_i's.

The parameters in the space of the monodromy representation, independent of conjugation of the M_i, are

$$2 - x_1^2 := \operatorname{tr}(M_1M_2), \quad 2 - x_2^2 := \operatorname{tr}(M_2M_3), \quad 2 - x_3^2 := \operatorname{tr}(M_1M_3)$$

We find

$$C_i\tilde{C}_i^{-1} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \qquad a,b \in \mathbf{C}, \quad a \neq 0$$

Note that this matrix commutes with J, then

$$(z-u_i)^J C_i = (z-u_i)^J \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \tilde{C}_i = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} (z-u_i)^J \tilde{C}_i$$

We also find

$$C_\infty \tilde{C}_\infty^{-1} = \begin{cases} i) \quad \operatorname{diag}(\alpha,\beta), \quad \alpha\beta \neq 0; & \text{if } 2\mu \notin \mathbf{Z} \\ ii) \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} (\mu > 0), \quad \begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix} (\mu < 0), \quad \alpha \neq 0, & \text{if } 2\mu \in \mathbf{Z}, R \neq 0 \\ iii) \text{ Any invertible matrix} & \text{if } 2\mu \in \mathbf{Z}, R = 0 \end{cases}$$

Then

$$i)\ z^{-\hat{\mu}} C_\infty = z^{-\hat{\mu}} \operatorname{diag}(\alpha,\beta) \tilde{C}_\infty = \operatorname{diag}(\alpha,\beta) z^{\hat{\mu}} \tilde{C}_\infty$$

$$ii)\ z^{-\hat{\mu}} z^{-R} C_\infty = \ \dots\ = \left[\alpha I + \frac{1}{z^{|2\mu|}} Q\right] z^{-\hat{\mu}} z^{-R} \tilde{C}_\infty$$

where $Q = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$, or $Q = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}$.

$$iii)\ i)\ z^{-\hat{\mu}} C_\infty = \ \dots\ = \left[\frac{Q_1}{z^{|2\mu|}} + Q_0 + Q_{-1} z^{|2\mu|}\right] z^{-\hat{\mu}} \tilde{C}_\infty$$

where $Q_0 = \operatorname{diag}(\alpha,\beta)$, $Q_{\pm 1}$ are respectively upper and lower triangular (or lower and upper triangular, depending on the sign of μ), and $C_\infty \tilde{C}_\infty^{-1} = Q_1 + Q_0 + Q_{-1}$

This implies that the two solutions $Y(z;u)$, $\tilde{Y}(z;u)$ of the form (5.4) with C and $\tilde{C}$ respectively, are such that $Y(z;u)\ \tilde{Y}(z;u)^{-1}$ is holomorphic at each u_i, while at $z = \infty$ it is

$$Y(z;u)\ \tilde{Y}(z;u)^{-1} \to \begin{cases} i)\ G_\infty \operatorname{diag}(\alpha,\beta) G_\infty^{-1} \\ ii)\ \alpha I \\ iii)\ \text{divergent} \end{cases}$$

Thus the two fuchsian systems are conjugated only in the cases $i)$ and $ii)$, because in those cases $Y\tilde{Y}^{-1}$ is holomorphic everywhere on $\mathbf{P}^1$, and then it is a constant. In other words *the R.H. has a unique solution, up to conjugation, for $2\mu \notin \mathbf{Z}$ or for $2\mu \in \mathbf{Z}$ and $R \neq 0$.*

[2] Note however that if $G_\infty = C_\infty = I$, then $\sum_{i=1}^3 A_i$ is already diagonal. Moreover, for $2\mu \notin \mathbf{Z}$, M_1, M_2, M_3 and the choice of normalization $Y(z;u)z^{\hat{\mu}} \to I$ if $z \to \infty$ determine uniquely A_1, A_2, A_3. Actually, for any diagonal invertible matrix D, the matrices $M_1' = D^{-1}M_1D$, $M_2' = D^{-1}M_2D$, $M_3' = D^{-1}M_3D$ determine the coefficients $D^{-1}A_iD$, whose sum is still diagonal (the normalization of Y is the same).

The triple (x_0, x_1, x_∞) in the introduction and in the remaining part of this paper corresponds to (x_1, x_2, x_3).

3.1) If at least two of the x_j's are zero, then one of the M_i's is I, or the matrices commute. We return to the case 1 or 2. Note that in case 2 $(x_1, x_2, x_3) = (0, 0, 0)$.

3.2) At most one of the x_j's is zero. Namely, the triple (x_1, x_2, x_3) is *admissible*. In this case it is possible to fully parameterize the monodromy using the triple (x_1, x_2, x_3) : namely, there exists a basis such that:

$$M_1 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 + \frac{x_2 x_3}{x_1} & -\frac{x_2^2}{x_1} \\ \frac{x_3^2}{x_1} & 1 - \frac{x_2 x_3}{x_1} \end{pmatrix},$$

if $x_1 \neq 0$. If $x_1 = 0$ we just choose a similar parameterization starting from x_2 or x_3. The relation

$$M_3 M_2 M_1 \text{ similar to } e^{-2\pi i \hat{\mu}} e^{2\pi i R}$$

implies

$$x_1^2 + x_2^2 + x_3^2 - x_1 x_2 x_3 = 4 \sin^2(\pi\mu)$$

The signs of the x_i's must be chosen in such a way that the above relation is satisfied. The conjugation (5.5) changes the triple by two signs. Thus the true parameters for the monodromy data are classes of equivalence of triples (x_1, x_2, x_3) defined by the change of two signs.

We distinguish three cases:

i) $2\mu \notin \mathbf{Z}$. Then there is a one to one correspondence between monodromy data (x_0, x_1, x_∞) and the branches of transcendents of PVI_μ. The solutions of [21] are included in this case: the connection problem was solved for the class of transcendents having the following local behaviour at the critical points $x = 0, 1, \infty$:

$$y(x) = a^{(0)} x^{1-\sigma^{(0)}} (1 + O(|x|^\delta)), \qquad x \to 0, \tag{5.6}$$

$$y(x) = 1 - a^{(1)} (1-x)^{1-\sigma^{(1)}} (1 + O(|1-x|^\delta)), \qquad x \to 1, \tag{5.7}$$

$$y(x) = a^{(\infty)} x^{-\sigma^{(\infty)}} (1 + O(|x|^{-\delta})), \qquad x \to \infty, \tag{5.8}$$

where $a^{(i)}$ and $\sigma^{(i)}$ are complex numbers such that $a^{(i)} \neq 0$ and $0 \leq \Re\sigma^{(i)} < 1$. δ is a small positive number. This behaviour is true if x converges to the critical points inside a sector in the x-plane with vertex on the corresponding critical point and finite angular width. The *connection problem* was solved finding the relation among the three couples $(\sigma^{(i)}, a^{(i)})$, $i = 0, 1, \infty$ and the triple (x_0, x_1, x_∞). In [21] all the algebraic solutions are classified and related to the finite reflection groups A_3, B_3, H_3.

ii) For any μ half integer there is an infinite set of *Picard type solutions* (see [37]), in one to one correspondence to triples of monodromy data ($R \neq 0$) not in the equivalence class of $(2, 2, 2)$. These solutions form a two parameter family, behave asymptotically as the solutions of the case i), and comprise a denumerable subclass of algebraic solutions. For any half integer $\mu \neq \frac{1}{2}$ there is also a one parameter family of *Chazy solutions*. For them the one to one correspondence with monodromy data is lost. In fact, they form an infinite family but any element of the family corresponds to a triple (x_1, x_2, x_3) in the orbit (of the braid group) of the triple $(2, 2, 2)$ (this orbit is simply obtained by changing two signs in all possible ways). They appear in the case $R = 0$ (no other solutions of PVI_μ correspond to $R = 0$ and μ half integer). The result of our paper applies to the Picard's solutions with $x_i \neq \pm 2$.

iii) μ integer. In this case $R \neq 0$ ($R = 0$ only in the case 2) of commuting monodromy matrices and μ integer). There is a one to one correspondence between monodromy data (x_0, x_1, x_∞) and the branches. To our knowledge, this case has not yet been studied. There are relevant examples of Frobenius manifolds where these solutions must appear, like the case of Quantum Cohomology of CP^2. In this case $\mu = -1$, the triple $(x_1, x_2, x_3) = (3, 3, 3)$ (the monodromy data coincide with the elements of the Stokes'matrix of the corresponding Frobenius manifold [17] [25]) and the real part of σ is equal to 1.

In this chapter we find the critical behaviour and we solve the connection problem for almost all the triples and for any $\mu \neq 0$, the only restriction required being

$$x_i \neq \pm 2 \implies \sigma^{(i)} \neq 1, \qquad i = 0, 1, \infty$$

5.2 Local Behaviour – Theorem 1

A solution $y(x)$ of PVI_μ is a meromorphic function of the point x belonging to the universal covering of $\mathbf{P}^1\backslash\{0,1,\infty\}$. Let σ and a denote two complex numbers, with the restrictions

$$\sigma \in \Omega := \mathbf{C}\backslash\{(-\infty,0)\cup[1,+\infty)\} \qquad a\neq 0$$

Our first aim is to show the existence of solutions of PVI_μ which behave like $a\ x^{1-\sigma}$, as $x\to 0$ along a suitable path. Thus, we concentrate on a small punctured neighbourhood of $x=0$, and the point x can be read as a point in the universal covering of $\mathbf{C}_0 := \mathbf{C}\backslash\{0\}$ with $0<|x|<\epsilon$ ($\epsilon<1$). Namely, $x=|x|e^{i\arg(x)}$, where $-\infty<\arg(x)<+\infty$.

In order to specify the way x may tend to zero, we introduce a domain contained in the universal covering of $\mathbf{C}_0$ (we denote the universal covering by $\widetilde{\mathbf{C}_0}$). If $\sigma=0$ we define the domain

$$D(\epsilon;\sigma=0)=\{x\in\widetilde{\mathbf{C}_0} \text{ s.t. } |x|<\epsilon\}$$

If $\sigma\neq 0$ we observe that

$$|x^\sigma|=|x|^{\sigma'(x)}, \quad \text{where } \sigma'(x):=\Re\sigma-\frac{\Im\sigma\ \arg(x)}{\log|x|}$$

We try to define a domain where the exponent $\sigma'(x)$ satisfies the restriction $0\le\sigma'(x)<1$ for $x\to 0$. Let $\theta_1,\ \theta_2\in\mathbf{R}$, $0<\tilde{\sigma}<1$. The desired domain is:

$$D(\epsilon;\sigma;\theta_1,\theta_2,\tilde{\sigma}) := \{x\in\widetilde{\mathbf{C}_0} \text{ s.t. } |x|<\epsilon,\ \ e^{-\theta_1\Im\sigma}|x|^{\tilde{\sigma}}\le|x^\sigma|\le e^{-\theta_2\Im\sigma}|x|^0,\quad 0<\tilde{\sigma}<1\},\quad 0<\epsilon<1.$$

The domain can be written as

$$|x|<\epsilon, \qquad \Re\sigma\log|x|+\theta_2\Im\sigma\le\Im\sigma\arg(x)\le(\Re\sigma-\tilde{\sigma})\log|x|+\theta_1\Im\sigma$$

Figure 5.2 shows the domains. Note that if $0\le\Re\sigma<1$ the domain contains, for $|x|$ sufficiently small, any sector $\alpha<\arg(x)<\alpha+2\pi$ (α a real number); but for $\Re\sigma<0$ and $\Re\sigma\ge 1$ it is not possible to include a sector in it when $x\to 0$.

Theorem 1: *Let $\mu\neq 0$. For any $\sigma\in\Omega$, for any $a\in\mathbf{C}$, $a\neq 0$, for any $\theta_1,\theta_2\in\mathbf{R}$ and for any $0<\tilde{\sigma}<1$, there exists a sufficiently small positive ϵ such that the equation PVI_μ has a transcendent $y(x;\sigma,a)$ with behaviour*

$$y(x;\sigma,a)=ax^{1-\sigma}\left(1+O(|x|^\delta)\right) \qquad ,0<\delta<1, \tag{5.9}$$

as $x\to 0$ in $D(\epsilon;\sigma;\theta_1,\theta_2,\tilde{\sigma})$.

The above local behaviour is valid with an exception which occurs if $\Im\sigma\neq 0$ and $x\to 0$ along the special paths $\Im\sigma\arg(x)=\Re\sigma\log|x|+d$, where d is a constant such that the path is contained in $D(\epsilon;\sigma;\theta_1,\theta_2)$: the local behaviour becomes:

$$y(x;\sigma,a)=a(x)\ x^{1-\sigma}(1+O(|x|^\delta)), \qquad \sigma\neq 0,$$

$$a(x)=a\ \left(1+\frac{1}{2a}\ Ce^{i\alpha(x)}+\frac{1}{16a^2}\ C^2e^{2i\alpha(x)}\right)=O(1),\ \textit{for } x\to 0$$

where

$$x^\sigma=Ce^{i\alpha(x)}, \qquad C:=e^{-d}, \quad \alpha(x):=\Re\sigma\arg(x)+\Im\sigma\ln|x|\Big|_{\Im\sigma\arg(x)=\Re\sigma\log|x|+d}$$

The small number ϵ depends on $\tilde{\sigma}$, θ_1 and a. In the following we may sometimes omit ϵ, $\tilde{\sigma}$, θ_i and write simply $D(\sigma)$.

In figure 5.2 we draw the possible paths along which $x\to 0$. Any path is allowed if $\Im\sigma=0$. If $\Im\sigma\neq 0$, an example of allowed paths is the following:

$$|x|<\epsilon, \qquad \Im\sigma\arg(x)=(\Re\sigma-\Sigma)\ln|x|+b, \tag{5.10}$$

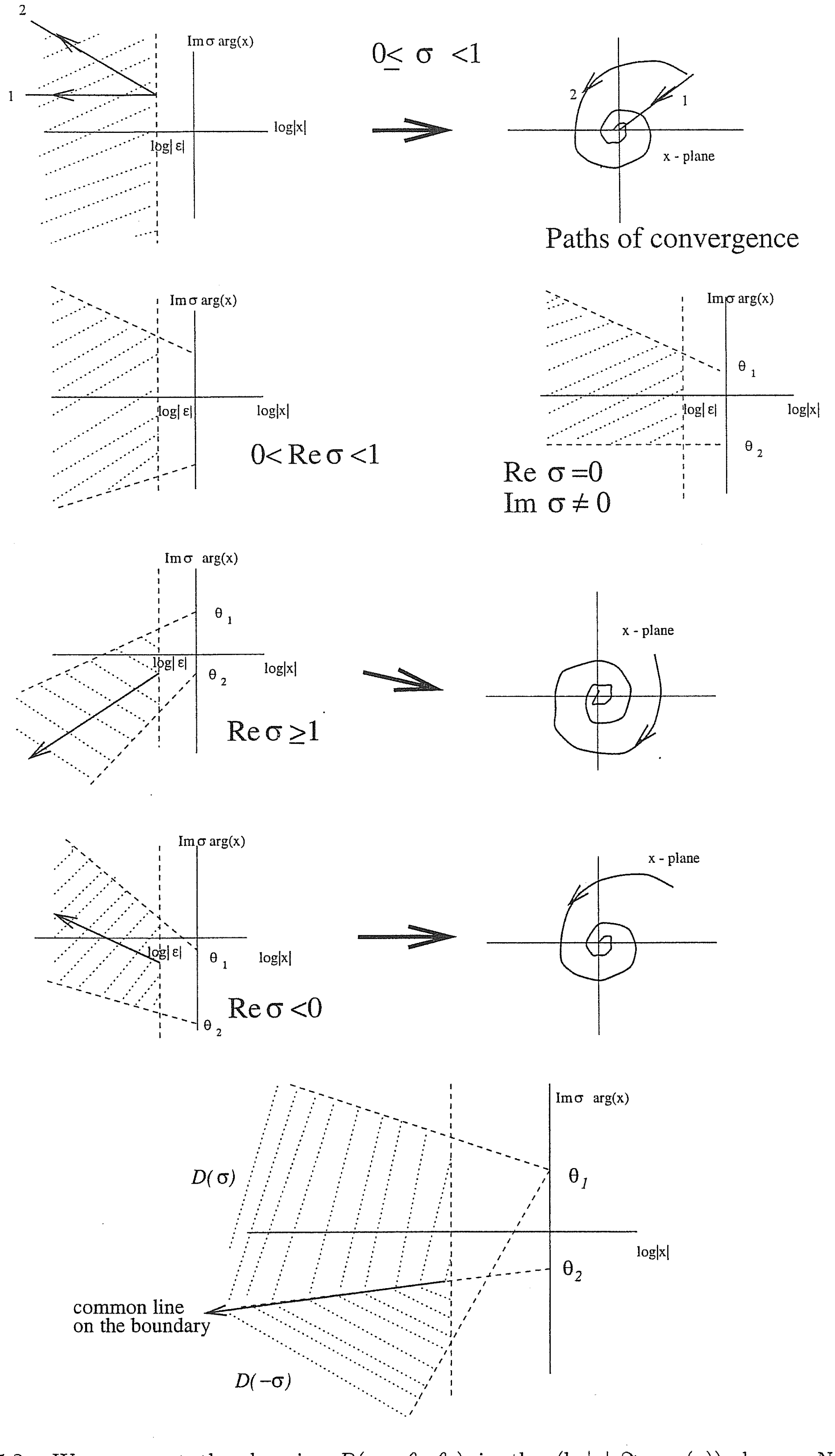


Figure 5.2: We represent the domains $D(\epsilon;\sigma;\theta_1,\theta_2)$ in the $(\ln|x|, \Im\sigma\arg(x))$-plane. Note that $D(\epsilon;\sigma;\theta_1,\theta_2) = D(\epsilon;\sigma = 0)$ for real $0 \leq \sigma < 1$. We also represent some lines along which x converges to 0. These lines are also represented in the x-plane: they are radial paths or spirals.

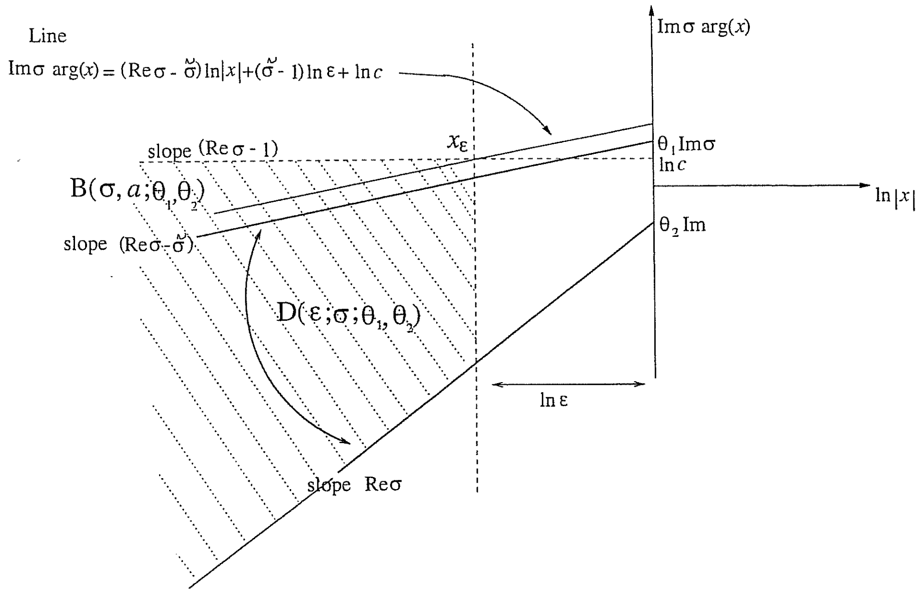


Figure 5.3: Construction of the domain $B(\sigma, a; \theta_2, \tilde{\sigma})$ for $\Re\sigma = 1$.

for a suitable choice of the additive constant b and for $0 \leq \Sigma \leq \tilde{\sigma}$. In general, these paths are spirals, represented in figure 5.2. They include radial paths if $0 \leq \Re\sigma < 1$ and $\Sigma = \Re\sigma$, because in this case $\arg(x) =$ constant. But there are only spiral paths whenever $\Re\sigma < 0$ and $\Re\sigma \geq 1$. The special paths $\Im\sigma \arg(x) = \Re\sigma \log|x| + b$, are parallel to one of the boundary lines of $D(\sigma)$ in the plane $(\ln|x|, \arg(x))$. The boundary line is $\Re\sigma \ln|x| + \Im\sigma\theta_2$ and it is shared by $D(\sigma)$ and $D(-\sigma)$ (with the same θ_2).

• **Restrictions on the domain** $D(\epsilon;\sigma;\theta_1,\theta_2)$: In theorem 1 we can choose θ_1 arbitrarily. Apparently, if we increase $\theta_1\Im\sigma$ the domain $D(\epsilon;\sigma;\theta_1,\theta_2)$ becomes larger. But ϵ itself depends on θ_1. In the proof of theorem 1 (section 5.7) we have to impose

$$\epsilon^{1-\tilde{\sigma}} \leq c\, e^{-\theta_1\Im\sigma}$$

where c is a constant (depending on a). Equivalently, $\theta_1\Im\sigma \leq (\tilde{\sigma}-1)\ln\epsilon + \ln c$. This means that if we increase $\Im\sigma\theta_1$ we have to decrease ϵ. Therefore, for $x \in D(\epsilon;\sigma;\theta_1,\theta_2)$ we have:

$$\Im\sigma \arg(x) \leq (\Re\sigma - \tilde{\sigma})\ln|x| + \theta_1\Im\sigma \leq (\Re\sigma - \tilde{\sigma})\ln|x| + (\tilde{\sigma}-1)\ln\epsilon + \ln c$$

We advise the reader to visualize a point x in the plane $(\ln|x|, \Im\sigma\arg(x))$. With this visualization in mind, let x_ϵ be the point $\{(\Re\sigma - \tilde{\sigma})\ln|x| + (\tilde{\sigma}-1)\ln\epsilon + \ln c\} \cap \{|x| = \epsilon\}$ (see figure 5.3). Namely,

$$\arg x_\epsilon = (\Re\sigma - 1)\ln\epsilon + \ln c$$

This means that the union of the domains $D(\epsilon(\theta_1);\sigma;\theta_1,\theta_2)$ on all values of θ_1 for given $\sigma, a, \tilde{\sigma}, \theta_2$ is

$$\bigcup_{\theta_1} D\big(\epsilon(\theta_1);\sigma;\theta_1,\theta_2,\tilde{\sigma}\big) \subseteq B(\sigma,a;\theta_2,\tilde{\sigma})$$

where

$$B(\sigma,a;\theta_2,\tilde{\sigma}) := \{|x|<1 \text{ such that } \Re\sigma\ln|x| + \theta_2\Im\sigma < \Im\sigma\arg(x) \leq (\Re\sigma-1)\ln|x| + \ln c\} \qquad (5.11)$$

The dependence on a of the domain B defined above is motivated by the fact that c depends on a (but not on θ_1, θ_2).

If $0 \leq \Re\sigma < 1$, the above result is not a limitation on the values of $\arg(x)$ of the points x that we want to include in a given $D(\sigma;\epsilon)$ provided that $|x|$ is sufficiently small. Also in the case $\Re\sigma < 0$ there is no limitation, because we can always decrease $\Im\sigma\theta_2$ without affecting ϵ in order to include in $D(\epsilon;\sigma;\theta_1,\theta_2)$ a point x such that $|x| < \epsilon$. But this is not the case if $\Re\sigma \geq 1$. Actually, if x (in the $(\ln|x|, \Im\sigma\arg(x))$-plane) lies above the set $B(\sigma,a;\theta_2,\tilde{\sigma})$ it never can be included in any $D(\epsilon;\sigma;\theta_1,\theta_2)$. See figure 5.4

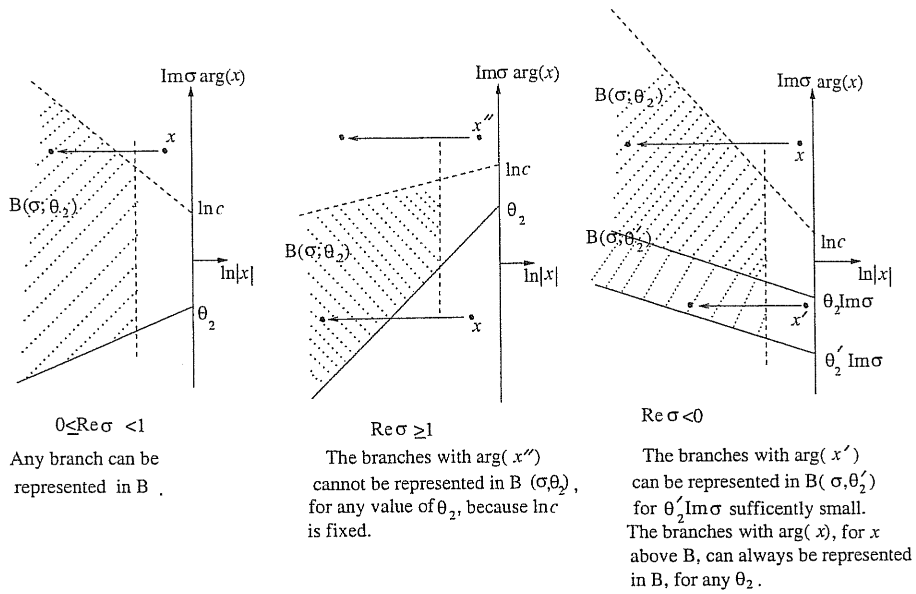


Figure 5.4: For $\Re\sigma \geq 1$ we can not include all values of $\arg(x)$ in B

5.3 Parameterization of a branch through Monodromy Data – Theorem 2

We are going to consider the fuchsian system (5.1) for the special choice

$$u_1 = 0, \quad u_2 = 1, \quad u_3 = x$$

The labels $i = 1, 2, 3$ will be substituted by the labels $i = 0, 1, x$, and the system becomes

$$\frac{dY}{dz} = \left[\frac{A_0(x)}{z} + \frac{A_x(x)}{z-x} + \frac{A_1(x)}{z-1}\right] Y$$

Also, the triple (x_1, x_2, x_3) will be denoted by (x_0, x_1, x_∞), as in [21].

We consider only admissible triples and $x_i \neq \pm 2$, $i = 0, 1, \infty$. We recall that two admissible triples are equivalent if their elements differ just by the change of two signs and that

$$x_0^2 + x_1^2 + x_\infty^2 - x_0 x_1 x_\infty = 4 \sin^2(\pi\mu) \tag{5.12}$$

We denote by $y(x; x_0, x_1, x_\infty)$ a branch in one to one correspondence with (x_0, x_1, x_∞). Since we operate close to $x = 0$, the branch is specified by $\alpha < \arg(x) < \alpha + 2\pi$, $\alpha \in \mathbf{R}$.

Theorem 2: *Let μ be any non zero complex number.*

For any $\sigma \in \Omega$ and for any $a \neq 0$ there exists a triple of monodromy data (x_0, x_1, x_∞) uniquely determined (up to equivalence) by the following formulae:

i) $\sigma \neq 0, \pm 2\mu + 2m$ for any $m \in \mathbf{Z}$.

$$\begin{cases} x_0 = 2\sin(\frac{\pi}{2}\sigma) \\ x_1 = i\left(\frac{1}{f(\sigma,\mu)G(\sigma,\mu)}\ \sqrt{a} - G(\sigma,\mu)\ \frac{1}{\sqrt{a}}\right) \\ x_\infty = \frac{1}{f(\sigma,\mu)G(\sigma,\mu)e^{-\frac{i\pi\sigma}{2}}}\ \sqrt{a} + G(\sigma,\mu)e^{-i\frac{\pi\sigma}{2}}\ \frac{1}{\sqrt{a}} \end{cases}$$

where

$$f(\sigma,\mu) = \frac{2\cos^2(\frac{\pi}{2}\sigma)}{\cos(\pi\sigma) - \cos(2\pi\mu)}, \qquad G(\sigma,\mu) = \frac{1}{2}\frac{4^\sigma \Gamma(\frac{\sigma+1}{2})^2}{\Gamma(1-\mu+\frac{\sigma}{2})\Gamma(\mu+\frac{\sigma}{2})}$$

Any sign of $\sqrt{a}$ is good (changing the sign of $\sqrt{a}$ is equivalent to changing the sign of both x_1, x_∞).

ii) $\sigma = 0$

$$\begin{cases} x_0 = 0 \\ x_1^2 = 2\sin(\pi\mu)\ \sqrt{1-a} \\ x_\infty^2 = 2\sin(\pi\mu)\ \sqrt{a} \end{cases}$$

We can take any sign of the square roots

iii) $\sigma = \pm 2\mu + 2m$.

iii1) $\sigma = 2\mu + 2m,\ m = 0, 1, 2, \ldots$

$$\begin{cases} x_0 = 2\sin(\pi\mu) \\ x_1 = -\frac{i}{2}\frac{16^{\mu+m}\Gamma(\mu+m+\frac{1}{2})^2}{\Gamma(m+1)\Gamma(2\mu+m)}\ \frac{1}{\sqrt{a}} \\ x_\infty = i\ x_1\ e^{-i\pi\mu} \end{cases}$$

iii2) $\sigma = 2\mu + 2m,\ m = -1, -2, -3, \ldots$

$$\begin{cases} x_0 = 2\sin(\pi\mu) \\ x_1 = 2i\frac{\pi^2}{\cos^2(\pi\mu)}\frac{1}{16^{\mu+m}\Gamma(\mu+m+\frac{1}{2})^2\Gamma(-2\mu-m+1)\Gamma(-m)}\ \sqrt{a} \\ x_\infty = -ix_1 e^{i\pi\mu} \end{cases}$$

iii3) $\sigma = -2\mu + 2m,\ m = 1, 2, 3, \ldots$

$$\begin{cases} x_0 = -2\sin(\pi\mu) \\ x_1 = -\frac{i}{2}\frac{16^{-\mu+m}\Gamma(-\mu+m+\frac{1}{2})^2}{\Gamma(m-2\mu+1)\Gamma(m)}\ \frac{1}{\sqrt{a}} \\ x_\infty = ix_1 e^{i\pi\mu} \end{cases}$$

iii4) $\sigma = -2\mu + 2m,\ m = 0, -1, -2, -3, \ldots$

$$\begin{cases} x_0 = -2\sin(\pi\mu) \\ x_1 = 2i\frac{\pi^2}{\cos^2(\pi\mu)}\frac{1}{16^{-\mu+m}\Gamma(-\mu+m+\frac{1}{2})^2\Gamma(2\mu-m)\Gamma(1-m)}\ \sqrt{a} \\ x_\infty = -ix_1 e^{-i\pi\mu} \end{cases}$$

In all the above formulae the relation $x_0^2 + x_1^2 + x_\infty^2 - x_0x_1x_\infty = 4\sin^2(\pi\mu)$ *is automatically satisfied. Note that* $\sigma \neq 1$ *implies* $x_0 \neq \pm 2$. *Equivalent triples (by the change of two signs) are also allowed.*

Let $x \in D(\epsilon;\sigma)$. *The branch at* x *of* $y(x;\sigma,a)$ *coincides with* $y(x;x_0,x_1,x_\infty)$. [3]

Conversely, for any set of monodromy data $(x_0,\ x_1,\ x_\infty)$ *such that* $x_0^2 + x_1^2 + x_\infty^2 - x_0x_1x_\infty = 4\sin^2(\pi\mu)$, $x_i \neq \pm 2$, *there exist parameters* σ *and* a *obtained as follows:*

I) Generic case

$$\cos(\pi\sigma) = 1 - \frac{x_0^2}{2}$$

[3]Note that we have picked up a point $x \in D(\sigma)$, therefore $\alpha < \arg(x) < \alpha + 2\pi$ for a suitable α. In other words, the branch is specified by $\alpha < \arg(x) < \alpha + 2\pi$. If we pick up a new point $x' = e^{2\pi i}x \in D(\sigma)$ (provided this is possible), then $y(x';\sigma,a)$ is again equal to the branch $y(x';x_0,x_1,x_\infty)$ at x', the branch being specified by $\alpha + 2\pi < \arg(x) < \alpha + 4\pi$. In section 5.5 we will describe in detail the problem of analytic continuation. We anticipate that for the loop $x' = xe^{2\pi i}$ we have the continuation $y(x';\sigma,a) = y(x';x_0,x_1,x_\infty) \equiv y(x;x_0',x_1',x_\infty')$.

$y(x;x_0',x_1',x_\infty')$ is a new branch of $y(x;x_0,x_1,x_\infty)$ corresponding to the continuation above at the same point x. In other words, we put branch cuts in the x-plane, then $\arg(x)$ can not increase by 2π and the analytic continuation of a branch yields a new branch with the same $\arg(x)$ and new monodromy data (x_0',x_1',x_∞').

$y(x';x_0,x_1,x_\infty)$ is the continuation of the branch $y(x;x_0,x_1,x_\infty)$ in the universal covering of $C_0 \cap \{|x| < \epsilon\}$. It has new $\arg(x)$, i.e $\arg(x) \mapsto \arg(x') = \arg(x) + 2\pi$ and the same monodromy data. If x' still lies in $D(\sigma)$ we can represent the continuation as $y(x';\sigma,a) = y(x';x_0,x_1,x_\infty)$, where the branch $y(x';x_0,x_1,x_\infty)$ has the branch cut specified by $\alpha + 2\pi < \arg(x') < \alpha + 4\pi$, while $y(x;x_0,x_1,x_\infty)$ has branch cut $\alpha < \arg(x) < \alpha + 2\pi$.

$$a=\frac{iG(\sigma,\mu)^2}{2\sin(\pi\sigma)}\left[2(1+e^{-i\pi\sigma})-f(x_0,x_1,x_\infty)(x_\infty^2+e^{-i\pi\sigma}x_1^2)\right]\ f(x_0,x_1,x_\infty)$$

where

$$f(x_0,x_1,x_\infty):=f(\sigma(x_0),\mu)=\frac{4-x_0^2}{2-x_0^2-2\cos(2\pi\mu)}=\frac{4-x_0^2}{x_1^2+x_\infty^2-x_0x_1x_\infty}$$

Any solution σ of the first equation must satisfy the restriction $\sigma\neq\pm2\mu+2m$ for any $m\in\mathbf{Z}$, otherwise we encounter the singularities in $G(\sigma,\mu)$ and in $f(\sigma,\mu)$. If $x_0^2=4$ the system has solutions $\sigma=1+2n$, $n\in\mathbf{Z}$, which do not belong to Ω.

II) $x_0=0$.

$$\sigma=0,$$

$$a=\frac{x_\infty^2}{x_1^2+x_\infty^2}.$$

provided that $x_1\neq0$ and $x_\infty\neq0$, namely $\mu\notin\mathbf{Z}$.

III) $x_0^2=4\sin^2(\pi\mu)$. Then (5.12) implies $x_\infty^2=-x_1^2\ \exp(\pm2\pi i\mu)$. Four cases which yield the values of σ non included in I) and II) must be considered

III1) $x_\infty^2=-x_1^2e^{-2\pi i\mu}$

$$\sigma=2\mu+2m,\qquad m=0,1,2,...$$

$$a=-\frac{1}{4x_1^2}\frac{16^{2\mu+2m}\Gamma(\mu+m+\frac{1}{2})^4}{\Gamma(m+1)^2\Gamma(2\mu+m)^2}$$

III2) $x_\infty^2=-x_1^2e^{2\pi i\mu}$

$$\sigma=2\mu+2m,\qquad m=-1,-2,-3,...$$

$$a=-\frac{\cos^4(\pi\mu)}{4\pi^4}16^{2\mu+2m}\Gamma(\mu+m+\frac{1}{2})^4\Gamma(-2\mu-m+1)^2\Gamma(-m)^2\ x_1^2$$

III3) $x_\infty^2=-x_1^2e^{2\pi i\mu}$

$$\sigma=-2\mu+2m,\qquad m=1,2,3,...$$

$$a=-\frac{1}{4x_1^2}\frac{16^{-2\mu+2m}\Gamma(-\mu+m+\frac{1}{2})^4}{\Gamma(m-2\mu+1)^2\Gamma(m)^2}$$

III4) $x_\infty^2=-x_1^2e^{-2\pi i\mu}$

$$\sigma=-2\mu+2m,\qquad m=0,-1,-2,-3,...$$

$$a=-\frac{\cos^4(\pi\mu)}{4\pi^4}16^{-2\mu+2m}\Gamma(-\mu+m+\frac{1}{2})^4\Gamma(2\mu-m)^2\Gamma(1-m)^2\ x_1^2$$

If $x_0^2\neq4$ ($\sigma\neq1+2n$, $n\in\mathbf{Z}$) we can always choose (from I), II), III)) $\sigma\in\Omega$.

Let $x\in D(\epsilon;\sigma)$ (then there exists $\alpha\in\mathbf{R}$ such that $\alpha<\arg(x)<\alpha+2\pi$). The branch $y(x;x_0,x_1,x_\infty)$ coincides at x with the transcendent $y(x;\sigma,a)$ of theorem 1.

We stress that the proof of the theorem is valid also for the *resonant* case $2\mu\in\mathbf{Z}\backslash\{0\}$. To read the formulae, it is enough to just substitute an integer for 2μ in the above formulae i) or I). Actually, we note that ii), iii); II), III) cannot occur. Note that for μ integer the case ii), II) degenerates to $(x_0,x_1,x_\infty)=(0,0,0)$ and a arbitrary. This is the case in which the triple is not a good parameterization for the monodromy (not admissible triple). Anyway, we know that in this case there is a one-parameter family of rational solutions [37], which are all obtained by a birational transformation from the family

$$y(x)=\frac{ax}{1-(1-a)x},\qquad \mu=1$$

At $x=0$ the the behaviour is $y(x)=ax(1+O(x))$, and then the limit of theorem 2 for $\mu\to n\in\mathbf{Z}\backslash\{0\}$ and $\sigma=0$ yields the above one-parameter family. Recall that $R=0$ in this case.

Remark 1: The equation

$$\cos(\pi\sigma)=1-\frac{x_0^2}{2}$$

determines σ up to $\sigma \mapsto \pm\sigma + 2n$, $n \in \mathbf{Z}$. [4] These different values give different coefficients a and different domains $D(\epsilon_n, \pm\sigma + 2n)$. Thus the same branch $y(x; x_0, x_1, x_\infty)$ has analytic continuations on different domains with different asymptotic behaviours prescribed by theorem 1. In particular, note that if $\Im\sigma \neq 0$ it is always possible to choose

$$0 \leq \Re\sigma \leq 1.$$

Observe however that for $0 \leq \sigma < 1$ the operation $\sigma \mapsto \pm\sigma + 2n$ is not allowed, because σ leaves Ω. Then, our paper does not give any information about what happens for $\sigma = 1$.

We also note that $\sigma = \sigma(x_0)$ and $a = a(\sigma; x_0, x_1, x_\infty)$, therefore $a(\sigma; x_0, x_1, x_\infty) \neq a(\pm\sigma+2n; x_0, x_1, x_\infty)$.

Remark 2: The domains $D(\sigma)$ and $D(-\sigma)$, with the same θ_2, intersect along the common boundary $\Im\sigma \arg(x) = \Re\sigma \log|x| + \theta_2\Im\sigma$ (see figure 5.2). The asymptotic behaviour of the analytic continuation of the branch $y(x; x_0, x_1, x_\infty)$ at x belonging to the common boundary is given in terms of $(\sigma(x_0), a(\sigma; x_0, x_1, x_\infty))$ and $(-\sigma(x_0), a(-\sigma; x_0, x_1, x_\infty))$ respectively.

This implies that the two different asymptotic representations of theorem 1 on $D(\sigma)$ and $D(-\sigma)$ must become equal along the boundary. Actually, from theorem 1 it is clear that along the boundary of $D(\sigma)$, the behaviour of $y(x)$ is

$$y(x) = A(x; \sigma, a(\sigma))\ x\ \left(1 + O(|x|^\delta)\right)$$

where δ is a small number between 0 and 1 and

$$A(x; \sigma, a(\sigma)) = a(Ce^{i\alpha(x;\sigma)})^{-1} + \frac{1}{2} + \frac{1}{16a} Ce^{i\alpha(x;\sigma)}$$

$$x^\sigma = Ce^{i\alpha(x;\sigma)}, \qquad C = e^{-\theta_2\Im\sigma}, \qquad \alpha(x;\sigma) = \Re\sigma \arg(x) + \Im\sigma \ln|x|\big|_{\Im\sigma \arg(x) = \Re\sigma \log|x| + \theta_2\Im\sigma}$$

We observe that $\alpha(x; -\sigma) = -\alpha(x; \sigma)$. After the proof of theorem 2 we'll see that $a(\sigma) = \frac{1}{16a(-\sigma)}$: this immediately implies that

$$A(x; -\sigma, a(-\sigma)) = A(x; \sigma, a(\sigma))$$

Therefore, the asymptotic behaviour, as prescribed by theorem 1 in $D(\sigma)$ and $D(-\sigma)$, is the same along the common boundary of the two domains.

We end the section with the following

Proposition: *Let $y(x) \sim ax^{1-\sigma}$ as $x \to 0$ in a domain $D(\epsilon, \sigma)$. Then, $y(x)$ coincides with $y(x; \sigma, a)$ of theorem 1*

Proof: see section 5.8.

5.4 Alternative Representations of the Transcendents

We study the critical point $x = 0$ (the points $x = 1, \infty$ will be discussed in section 5.6).

If $\Im\sigma \neq 0$, the freedom $\sigma \mapsto \pm\sigma + 2n$ allows us to always reduce the exponent to $0 \leq \Re\sigma \leq 1$. So, the values of σ we may restrict to are

$$0 \leq \Re\sigma \leq 1 \text{ for } \Im\sigma \neq 0,$$

$$0 \leq \sigma < 1 \text{ for } \sigma \in \mathbf{R}.$$

In the $(\ln|x|, \Im\sigma \arg(x))$-plane we draw the domains $D(\sigma)$, $D(-\sigma)$, $D(-\sigma+2)$, $D(2-\sigma)$, etc - see figure 5.5 (left). Here θ_1 and θ_2 are the same (ϵ may not be the same, so we choose the smallest). We suppose that the domains and the corresponding transcendents of theorem 1 are associated to the same triple. So we have different critical behaviours predicted by theorem 1 in the different domains for the same transcendent. Some "small sectors" remain uncovered by the union of the domains (figure 5.5 (right)). If $x \to 0$ inside these sectors, we do not know the behaviour of the transcendent. If $\Re\sigma = 1$, a radial path converging to $x = 0$ will end up in a forbidden "small sector" (see also figure 5.7 for the case $\Re\sigma = 1$).

If we draw, for the same θ_2, the domains $B(\sigma)$, $B(-\sigma)$, $B(-\sigma + 2)$, etc, defined in (5.11) we obtain strips in the $(\ln|x|, \Im\sigma \arg(x))$-plane which are *certainly forbidden* to theorem 1 (see figure 5.6). In the

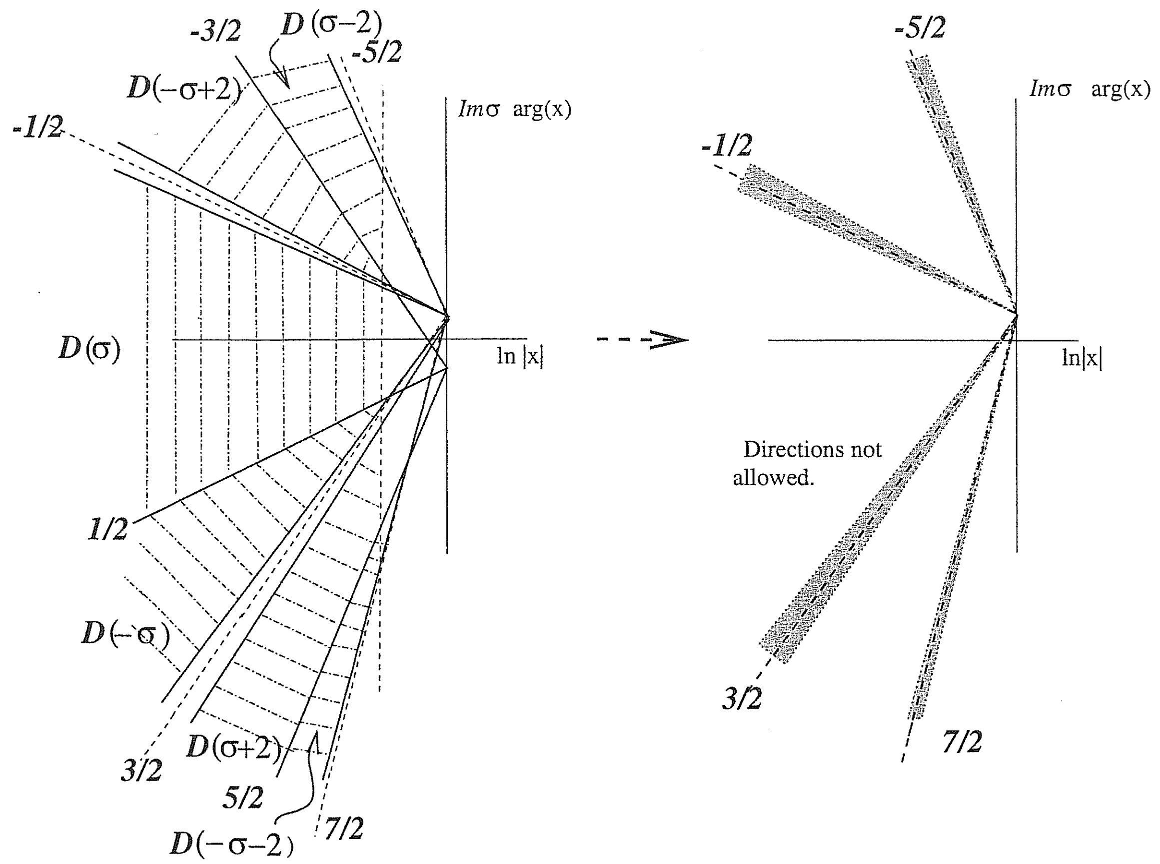


Figure 5.5: Domains for $\sigma = \frac{1}{2} + i\Im\sigma$. The numbers close to the lines are their slopes. The "small sectors" around the dotted lines represented in the right figure are not contained in the union of the domains. If $x \to 0$ along a direction which ends in one of these sectors, we do not know the behaviour of the transcendent.

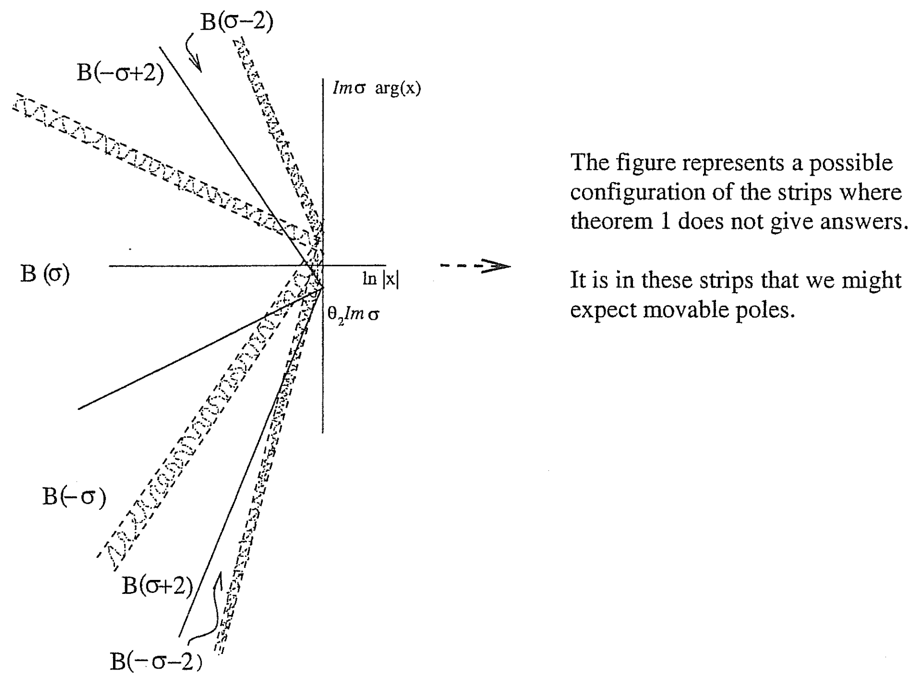


Figure 5.6:

strips we know nothing about the transcendent. We guess that there might be poles there, as we verify in one example later.

What is the behaviour along the directions not described by theorem 1? In the very particular case $(x_0, x_1, x_\infty) \in \{(2,2,2), (2,-2,-2), (-2,-2,2), (-2,2,-2)\}$ (and so $x_0 = \pm 2$!), it is known that $PVI_{\mu=-1/2}$ has a 1-parameter family of *classical* solutions [37]. The asymptotic behaviour of a branch for *radial* convergence to the critical points 0, 1 ,∞ was computed in [37]:

$$y(x) = \begin{cases} -\ln(x)^{-2}(1+O(\ln(x)^{-1})), & x \to 0 \\ 1+\ln(1-x)^{-2}(1+O(\ln(1-x)^{-1})), & x \to 1 \\ -x\ln(1/x)^{-2}(1+O(\ln(1/x)^{-1}), & x \to \infty \end{cases}$$

The branch is specified by $|\arg(x)| < \pi$, $|\arg(1-x)| < \pi$. This behaviour is completely different from $\sim a(x)\tilde{x}^{1-\sigma}$ as $x \to 0$. Intuitively, as x_0 approaches the value 2, $1-\sigma$ approaches 0 and the decay of $y(x) \sim ax^{1-\sigma}$ becomes logarithmic. These solutions were called *Chazy solutions* in [37], because they can be computed as functions of solutions of the Chazy equation.

This section is devoted to the investigation of the local behaviour at $x = 0$ of the analytic continuation of a branch in the regions not described in theorem 1.

5.4.1 Elliptic Representation

The transcendents of PVI_μ can be represented in the elliptic form [22]

$$y(x) = \mathcal{P}\left(\frac{u(x)}{2}; \omega_1(x), \omega_2(x)\right)$$

where $\mathcal{P}(z;\omega_1,\omega_2)$ is the Weierstrass elliptic function of half-periods ω_1, ω_2. u(x) solves the non-linear differential equation

$$\mathcal{L}(u) = \frac{\alpha}{x(1-x)}\frac{\partial}{\partial u}\left(\mathcal{P}\left(\frac{u}{2};\omega_1(x),\omega_2(x)\right)\right), \quad \alpha = \frac{(2\mu-1)^2}{2} \tag{5.13}$$

where the differential linear operator $\mathcal{L}$ applied to u is

$$\mathcal{L}(u) := x(1-x)\,\frac{d^2u}{dx^2} + (1-2x)\,\frac{du}{dx} - \frac{1}{4}\,u.$$

The half-periods are two independent solutions of $\mathcal{L}(u) = 0$ normalized as follows:

$$\omega_1(x) := \frac{\pi}{2}\,F(x), \qquad \omega_2(x) := -\frac{i}{2}[F(x)\ln x + F_1(x)]$$

where $F(x)$ is the hypergeometric function

$$F(x) := F\left(\frac{1}{2},\frac{1}{2},1;x\right) = \sum_{n=0}^{\infty}\frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{(n!)^2}x^n,$$

and

$$F_1(x) := \sum_{n=0}^{\infty}\frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{(n!)^2}2\left[\psi(n+\frac{1}{2}) - \psi(n+1)\right]x^n$$

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z), \quad \psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2, \quad \psi(1) = -\gamma, \quad \psi(a+n) = \psi(a) + \sum_{l=0}^{n-1}\frac{1}{a+l}.$$

The solutions u of (5.13) are in general unknown. Before trying to study them in general terms, we solve it in a special case:

[4] In the case $\sigma = 1 + i\nu$, $\nu \in \mathbf{R}\backslash\{0\}$, the freedom $\sigma \mapsto -\sigma + 2$ is equivalent to $\sigma = 1 + i\nu \mapsto \sigma - 2i\nu = 1 - i\nu$.

Example: The equation $PVI_{\mu=1/2}$ has a two parameter family of solutions discovered by Picard [44] [42] [37]. It is easily obtained from (5.13). Since $\alpha = 0$, u solves the hypergeometric equation $\mathcal{L}(u) = 0$ and has the general form

$$\frac{u(x)}{2} := \nu_1\omega_1(x) + \nu_2\omega_2(x), \quad \nu_i \in \mathbf{C}, \quad 0 \leq \Re\nu_i < 2, \quad (\nu_1, \nu_2) \neq (0,0),$$

A branch of $y(x)$ is specified by a branch of $\ln x$. The monodromy data computed in [37] are

$$x_0 = -2\cos\pi r_1, \quad x_1 = -2\cos\pi r_2, \quad x_\infty = -2\cos\pi r_3,$$

$$r_1 = \frac{\nu_2}{2}, \quad r_2 = 1 - \frac{\nu_1}{2}, \quad r_3 = \frac{\nu_1 - \nu_2}{2}, \qquad \text{for } \nu_1 > \nu_2$$

$$r_1 = 1 - \frac{\nu_2}{2}, \quad r_2 = \frac{\nu_1}{2}, \quad r_3 = \frac{\nu_2 - \nu_1}{2}, \qquad \text{for } \nu_1 < \nu_2.$$

The modular parameter is now a function of x. Since we are interested in it when $x \to 0$ we give its expansion:

$$\tau(x) = \frac{\omega_2(x)}{\omega_1(x)} = \frac{1}{\pi}(\arg x - i \ln|x|) + \frac{4i}{\pi}\ln 2 + O(x), \quad x \to 0.$$

We see that $\Im\tau > 0$ as $x \to 0$. Now, if

$$\left|\Im\frac{u(x)}{4\omega_1}\right| < \Im\tau, \tag{5.14}$$

we can expand the Weierstrass function in Fourier series. Condition (5.14) becomes

$$\frac{1}{2}\left|\Im\nu_1 + \frac{\Im\nu_2}{\pi}\arg(x) - \frac{\Re\nu_2}{\pi}\ln|x| + \frac{4\ln 2}{\pi}\Re\nu_2\right| < -\frac{\ln|x|}{\pi} + \frac{4\ln 2}{\pi} + O(x), \qquad \text{as } x \to 0$$

namely,

$$(\Re\nu_2+2)\ln|x| - \pi\Im\nu_1 - 4\ln 2\,(\Re\nu_2+2) < \Im\nu_2\arg(x) < (\Re\nu_2-2)\ln|x| - \pi\Im\nu_1 - 4\ln 2\,(\Re\nu_2-2). \tag{5.15}$$

or

$$\text{Any value of } \arg(x) \text{ if } \Im\nu_2 = 0.$$

The Fourier expansion is

$$y(x) = \frac{x+1}{3} + \frac{1}{F(x)^2}\left[\frac{1}{\sin^2\left(-\frac{1}{2}[i\nu_2(\ln(x) + \frac{F_1(x)}{F(x)}) - \pi\nu_1]\right)} - \frac{1}{3} + \right.$$

$$\left. +8\sum_{n=1}^{\infty}\frac{x^{2n}}{e^{-2n\frac{F_1(x)}{F(x)}} - x^{2n}}\sin^2\left(-\frac{n}{2}[i\nu_2(\ln(x) + \frac{F_1(x)}{F(x)}) - \pi\nu_1]\right)\right]$$

$$= \frac{x}{2} + (1 - \frac{x}{2} + O(x^2))\left[\frac{1}{\sin^2\left(-\frac{1}{2}[i\nu_2(\ln(x) + \frac{F_1(x)}{F(x)}) - \pi\nu_1]\right)} + \right.$$

$$\left. -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1} x^{2-\nu_2} + O(x^2 + x^{3-\nu_2} + x^{4-\nu_2})\right], \qquad x \to 0 \text{ in the domain (5.15)}$$

As far as *radial* convergence is concerned, we have:

a) $0 < \Re\nu_2 < 2$,

$$\frac{1}{\sin^2(\ldots)} = -\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right] x^{\nu_2}\,(1 + O(|x^{\nu_2}|)),$$

and so

$$y(x) = \left\{-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right] x^{\nu_2} + \frac{1}{2}x - \frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1} x^{2-\nu_2}\right\}\,(1 + O(x^\delta)), \quad \delta > 0, \tag{5.16}$$

in accordance with theorem 1. We can identify $1-\sigma$ with ν_2 for $0<\Re\nu_2<1$, or with $2-\nu_2$ for $1<\Re\nu_2<2$. In the case $\Re\nu_2=1$ the three terms x^{ν_2}, x, $x^{2-\nu_2}$ have the same order and we find again the behaviour of theorem 1

$$y(x)=\left\{ax^{\nu_2}+\frac{x}{2}+\frac{1}{16a}x^{2-\nu_2}\right\}(1+O(x^\delta))=ax^{\nu_2}\left\{1+\frac{1}{2a}x^{-i\Im\nu_2}+\frac{1}{16a^2}x^{-2i\Im\nu_2}\right\}(1+O(x^\delta)),$$

where $a=-\frac{1}{4}\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2-1}}\right]$.

b) $\Re\nu_2=0$. Put $\nu_2=i\nu$ (namely, $\sigma=1-i\nu$). The domain (5.15) is now (for sufficiently small $|x|$):

$$2\ln|x|-\pi\Im\nu_1-8\ln 2<\Im\nu_2\arg(x)<-2\ln|x|-\pi\Im\nu_1+8\ln 2, \tag{5.17}$$

or

$$2\ln|x|+\pi\Im\nu_1-8\ln 2<\Im\sigma\arg(x)<-2\ln|x|+\pi\Im\nu_1+8\ln 2.$$

For radial convergence we have

$$y(x)=\frac{1+O(x)}{\sin^2(\frac{\nu}{2}\ln(x)+\frac{\nu}{2}\frac{F_1(x)}{F(x)}+\pi\nu_1)}+O(x).$$

This is an oscillating functions, and it may have poles. Suppose for example that ν_1 is real. Since $F_1(x)/F(x)$ is a convergent power series ($|x|<1$) with real coefficients and defines a bounded function, then $y(x)$ has a sequence of poles on the positive real axis, converging to $x=0$.

If $\nu=0$, namely $\nu_2=0$ (and then $x_0=2$)

$$y(x)=\frac{1}{\sin^2(\pi\nu_1)}(1+0(|x|)).$$

In the domain (5.17) spiral convergence of x to zero is also allowed. In that case, the non-constancy of $\arg(x)$ still gives a behaviour (5.16).

The case b) in the above example is good to understand the limits of theorem 1 in giving a complete description of the behaviour of Painlevé transcendents. Actually, theorem 1 (together with the transformation $\sigma\to-\sigma$) yields the behaviour (5.16) in the domain $D(\sigma)\cup D(-\sigma)$ ($\Re\sigma=1$):

$$(1+\tilde\sigma)\ln|x|+\theta_1\Im\sigma\le\Im\sigma\arg x\le(1-\tilde\sigma)\ln|x|+\theta_1\Im\sigma,$$

where radial convergence to $x=0$ is not allowed. On the other hands, the transformations $\sigma\to\pm(\sigma-2)$, gives a further domain $D(\sigma-2)\cup D(-\sigma+2)$:

$$(-1+\tilde\sigma)\ln|x|+\theta_1\Im\sigma\le\Im\sigma\arg x\le-(1+\tilde\sigma)\ln|x|+\theta_1\Im\sigma.$$

but again it is not possible for x to converge to $x=0$ along a radial path. Figure 5.7 shows $D(\sigma)\cup D(-\sigma)\cup D(2-\sigma)\cup D(\sigma-2)$. Note that a radial path would be allowed if it were possible to make $\tilde\sigma\to1$ and the interior of the set obtained as the limit for $\tilde\sigma\to1$ of $D(\sigma)\cup D(-\sigma)\cup D(2-\sigma)\cup D(\sigma-2)$ has the shape of (5.17). Actually, the intersection of the two sets is never empty. On (5.17) the elliptic representation predicts an oscillating behaviour and poles along the paths not allowed by theorem 1. So now it must be definitely clear that the "limit" of theorem 1 for $\tilde\sigma\to1$ is not trivial.

Remark on the example: For μ half integer all the possible values of (x_0,x_1,x_∞) such that $x_0^2+x_1^2+x_\infty^2-x_0x_1x_\infty=4$ are covered by Chazy and Picard's solutions, with the warning that for $\mu=\frac{1}{2}$ the image (through birational transformations) of Chazy solutions is $y=\infty$. See [37].

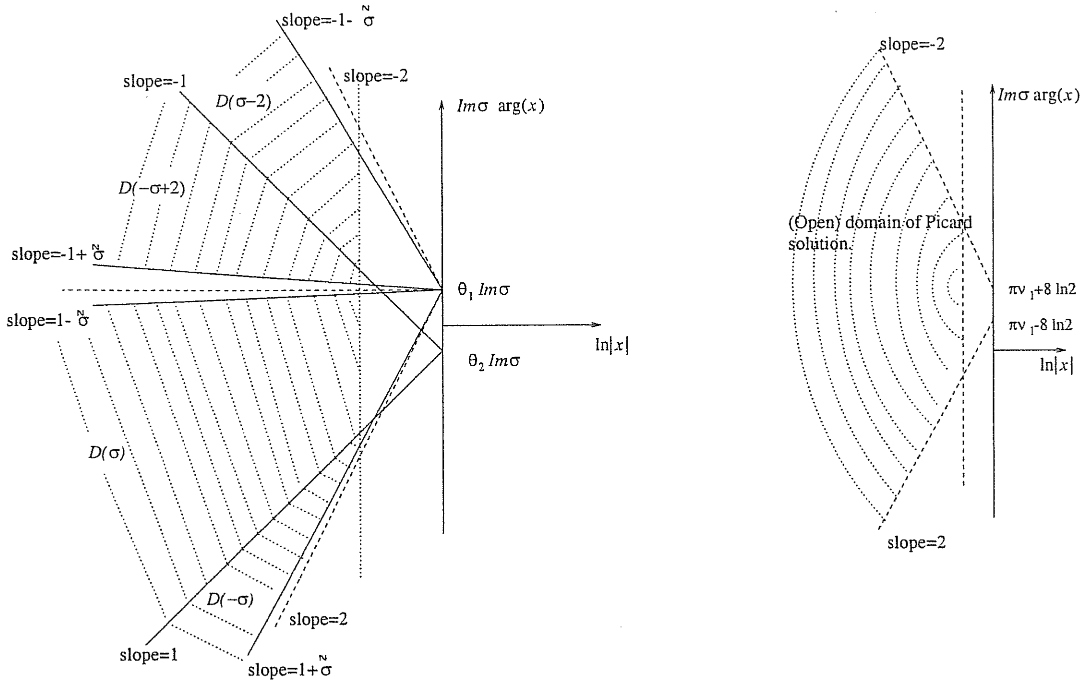
We turn to the general case. In section 5.9 we will prove the following theorem:

Theorem 3: *For any complex* ν_1, ν_2 *such that*

$$\nu_2\not\in(-\infty,0]\cup[2,+\infty)$$

there exists a sufficiently small r such that

$$y(x)=\mathcal{P}(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x);\omega_1(x),\omega_2(x))$$



Domain $D(\sigma) \cup D(-\sigma) \cup D(\sigma-2) \cup D(-\sigma+2)$ for $\sigma = 1 + i\, Im\, \sigma$. Comparison with the domain where Piccard solution is expanded (picture above).

Below we represent the domain $\mathcal{D}(r, \nu_1, \nu_2)$ of theorem 3 for immaginary ν_2, and we compare it to the domain D(σ) with the identification $\nu_2 = 1 - \sigma$ (and for suitable θ_1, θ_2).

The numbers close to the boundary lines are their slopes ($\tilde{\varepsilon} = 1 - \tilde{\sigma}$ is arbitrarily small)

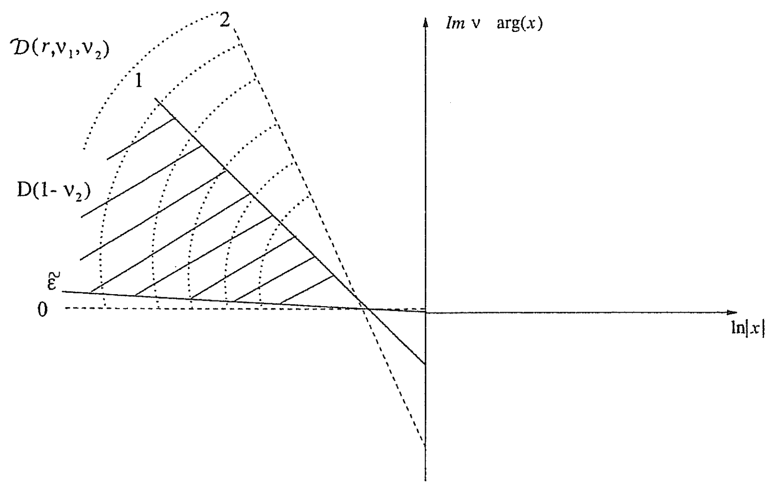


Figure 5.7:

in the domain

$$\mathcal{D}(r;\nu_1,\nu_2):=\left\{x\in\tilde{\mathbf{C}}_0 \quad such\ that\ |x|<r,\left|\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right|<r,\left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right|<r\right\}$$

The function $v(x)$ is holomorphic in $\mathcal{D}(r;\nu_1,\nu_2)$ and has convergent expansion

$$v(x)=\sum_{n\geq 1}a_nx^n+\sum_{n\geq 0,\ m\geq 1}b_{nm}x^n\left(\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right)^m+\sum_{n\geq 0,\ m\geq 1}c_{nm}x^n\left(\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right)^m \tag{5.18}$$

where a_n, b_{nm}, c_{nm} are rational functions of ν_2. Moreover, there exists a constant $M(\nu_2)$ depending on ν_2 such that $v(x)\leq M(\nu_2)\left(|x|+\left|\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right|+\left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right|\right)$ in $\mathcal{D}(r;\nu_1,\nu_2)$.

The domain $\mathcal{D}(r;\nu_1,\nu_2)$ is

$$|x|<r,\quad \Re\nu_2\ln|x|+C_1-\ln r<\Im\nu_2\arg x<(\Re\nu_2-2)\ln|x|+C_2+\ln r, \tag{5.19}$$

$$C_1:=-\left[4\ln 2\Re\nu_2+\pi\Im\nu_1\right],\qquad C_2:=C_1+8\ln 2.$$

If ν_2 is real (therefore $0<\nu_2<2$), the domain is simply $|x|<r$. The critical behaviour is obtained expanding $y(x)$ in Fourier series:

$$\mathcal{P}\left(\frac{u}{2};\omega_1,\omega_2\right)=-\frac{\pi^2}{12\omega_1^2}+\frac{2\pi^2}{\omega_1^2}\sum_{n=1}^{\infty}\frac{ne^{2\pi in\tau}}{1-e^{2\pi in\tau}}\left(1-\cos\left(n\frac{\pi u}{2\omega_1}\right)\right)+\frac{\pi^2}{4\omega_1^2}\frac{1}{\sin^2\left(\frac{\pi u}{4\omega_1}\right)} \tag{5.20}$$

The expansion can be performed if $\Im\tau(x)>0$ and $\left|\Im\left(\frac{u(x)}{\omega_1(x)}\right)\right|<\Im\tau$; these conditions are satisfied in $\mathcal{D}(r;\nu_1,\nu_2)$. Let's put $F_1/F=-4\ln 2+g(x)$. It follows that $g(x)=O(x)$. Taking into account (5.20) and theorem 3, the expansion of $y(x)$ for $x\to 0$, $x\in\mathcal{D}(r;\nu_1,\nu_2)$, is

$$y(x)=\left[\frac{1+x}{3}-\frac{\pi^2}{12\omega_1(x)^2}\right]+\frac{\pi^2}{\omega_1(x)^2}\sum_{n=1}^{\infty}\frac{n}{1-\left(\frac{e^{g(x)}}{16}\right)^{2n}x^{2n}}\left\{2\left(\frac{e^{g(x)}}{16}\right)^{2n}x^{2n}\right.$$

$$\left.-e^{n(\nu_2+2)g(x)}\left[\frac{e^{i\pi\nu_1}}{16^{2+\nu_2}}x^{2+\nu_2}\right]^n e^{in\pi\frac{v(x)}{\omega_1(x)}}-e^{n(2-\nu_2)g(x)}\left[\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right]^n e^{-i\pi n\frac{v(x)}{\omega_1(x)}}\right\}$$

$$+\frac{\pi^2}{4\omega_1(x)^2}\frac{1}{\sin^2\left(-i\frac{\nu_2}{2}\ln x+i\frac{\nu_2}{2}\ln 16+\frac{\pi\nu_1}{2}-i\frac{\nu_2}{2}g(x)+\frac{\pi v(x)}{2\omega_1(x)}\right)}$$

where

$$\frac{1}{\sin^2(\ldots)}=-\frac{4}{e^{\nu_2 g(x)}\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}e^{i\pi\frac{v(x)}{\omega_1(x)}}+e^{-\nu_2 g(x)}\frac{e^{-i\pi\nu_1}}{16^{-\nu_2}}x^{-\nu_2}e^{-i\pi\frac{v(x)}{\omega_1(x)}}-2}$$

We also observe that $\omega_1(x)\equiv\frac{\pi}{2}F(x)=\frac{\pi}{2}(1+\frac{1}{4}x+O(x^2))$, $\frac{1+x}{3}-\frac{\pi^2}{12\omega_1(x)^2}\equiv\frac{1+x}{3}-\frac{1}{3F(x)}=\frac{1}{2}x(1+O(x))$, $e^{g(x)}=1+O(x)$ and

$$e^{\pm i\pi\frac{v(x)}{\omega_1(x)}}=1+O\left(|x|+\left|\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right|+\left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right|\right)$$

In order to single out the leading terms, we observe that we are dealing with the powers x, $x^{2-\nu_2}$, x^{ν_2} in $\mathcal{D}(r;\nu_1,\nu_2)$. If $0<\nu_2<2$ (the only allowed real values of ν_2) $|x^{\nu_2}|$ is leading if $0<\nu_2<1$ and $|x^{2-\nu_2}|$ is leading if $1<\nu_2<2$. We have

$$\frac{1}{\sin^2(\ldots)}=-4\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\left[1+O(|x|+|x^{\nu_2}|+|x^{2-\nu_2}|)\right]$$

Thus, there exists $0<\delta<1$ (explicitly computable in terms of ν_2) such that

$$y(x)=\left[\frac{1}{2}x-\frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]x^{\nu_2}-\frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1}x^{2-\nu_2}\right](1+O(x^\delta))$$

$$= \begin{cases} \left[\frac{1}{2} - \frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right] - \frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1}\right] x\,(1+O(x^\delta)), & \text{if } \nu_2 = 1 \\ -\frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right] x^{\nu_2}(1+O(x^\delta)) & \text{if } 0<\nu_2<1 \\ -\frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1} x^{2-\nu_2}(1+O(x^\delta)) & \text{if } 1<\nu_2<2 \end{cases}$$

The behaviour is the one of theorem 1 for $\sigma = 0$ in the first case, $\sigma = 1-\nu_2$ in the second, $\sigma = \nu_2 - 1$ in the third.

We turn to the case $\Im\nu_2 \neq 0$. We consider a path contained in $\mathcal{D}(r;\nu_1,\nu_2)$ of equation

$$\Im\nu_2 \arg(x) = (\Re\nu_2 - \mathcal{V})\ln|x| + b, \qquad 0 \le \mathcal{V} \le 2 \tag{5.21}$$

with a suitable constant b. Thus $|x^{2-\nu_2}| = |x|^{2-\mathcal{V}}e^b$, $|x^{\nu_2}| = |x|^{\mathcal{V}}e^{-b}$ and

$$|x^{\nu_2}| \;\text{ is leading for }\; 0 \le \mathcal{V} < 1,$$

$$|x^{\nu_2}|,\ |x|,\ |x^{2-\nu_2}| \;\text{ have the same order for }\; \mathcal{V} = 1,$$

$$|x^{2-\nu_2}| \;\text{ is leading for }\; 1 < \mathcal{V} \le 2.$$

If $\mathcal{V} = 0$,

$$\left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right| < r, \qquad \text{but }\; x^{\nu_2} \not\to 0 \;\text{ as }\; x \to 0.$$

If $\mathcal{V} = 2$,

$$\left|\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right| < r \qquad \text{but }\; x^{2-\nu_2} \not\to 0 \;\text{ as }\; x \to 0.$$

This also implies that $v(x) \not\to 0$ as $x \to 0$ along the paths with $\mathcal{V} = 0$ or $\mathcal{V} = 2$. Therefore we conclude that:

a) If $x \to 0$ in $\mathcal{D}(r;\nu_1,\nu_2)$ along (5.21) for $\mathcal{V} \neq 0, 2$, then

$$y(x) = \left[\frac{1}{2}x - \frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]x^{\nu_2} - \frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1}x^{2-\nu_2}\right](1+O(x^\delta)), \qquad 0<\delta<1.$$

The term $-\frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]x^{\nu_2}(1+$ higher orders) is $\frac{1}{\sin(\dots)^2}$. The three leading terms have the same order if the convergence is along a path asymptotic to (5.21) with $\mathcal{V} = 1$. Otherwise

$$y(x) = -\frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]x^{\nu_2}(1+O(x^\delta))$$

or

$$y(x) = -\frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]^{-1}x^{2-\nu_2}(1+O(x^\delta))$$

according to the path. This is the behaviour of theorem 1 with $1-\sigma = \nu_2$ or $2-\nu_2$.

Let $\nu_2 = 1-\sigma$ and consider the intersection $\mathcal{D}(r;\nu_1,\nu_2) \cap \mathcal{D}(\sigma)$ in the $(\ln|x|, \Im\nu_2\arg(x))$-plane. See figure 5.7. We choose ν_1 such that $a = -\frac{1}{4}\left[\frac{e^{\pi\nu_1}}{16^{\nu_2-1}}\right]$. According to the proposition in section 5.3, on the intersection we identify the transcendents of the elliptic representation to those of theorem 1, (we do not need to specify θ_1, θ_2, because the intersection is never empty).

Equivalently, we can choose the identification $1-\sigma = 2-\nu_2$ and repeat the argument.

The identification makes it possible to investigate the behaviour of the transcendents of theorem 1 along a path (5.10) with $\Sigma = 1$. (5.10) coincides with (5.21) for $\mathcal{V} = 0$ if we define $1-\sigma := \nu_2$, or (5.21) for $\mathcal{V} = 2$ if we define $1-\sigma = 2-\nu_2$. We discuss the problem in the following two points:

b) If $\mathcal{V} = 0$ the term

$$\frac{1}{\sin^2\left(-i\frac{\nu_2}{2}\ln x + \left[i\frac{\nu_2}{2}\ln 16 + \frac{\pi\nu_1}{2}\right] - i\frac{\nu_2}{2}g(x) + \frac{\pi v(x)}{2\omega_1(x)}\right)}$$

$$= -4\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}e^{\nu_2 g(x)+i\pi\frac{v(x)}{\omega_1(x)}} \frac{1}{1-2\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}e^{\nu_2 g(x)+i\pi\frac{v(x)}{\omega_1(x)}}+\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}e^{\nu_2 g(x)+i\pi\frac{v(x)}{\omega_1(x)}}\right]^2}$$

is *oscillating as* $x \to 0$ and does not vanish. Note that there are no poles because the denominator does not vanish in $\mathcal{D}(r;\nu_1,\nu_2)$ since $\left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right| < r < 1$. We prefer to keep the trigonometric notation and write

$$y(x) = O(x) + \frac{1}{F(x)^2}\frac{1}{\sin^2\left(-i\frac{\nu_2}{2}\ln x + \left[i\frac{\nu_2}{2}\ln 16 + \frac{\pi\nu_1}{2}\right] - i\frac{\nu_2}{2}g(x) + \frac{v(x)}{F(x)}\right)}$$

$$= \frac{1+O(x)}{\sin^2\left(-i\frac{\nu_2}{2}\ln x + \left[i\frac{\nu_2}{2}\ln 16 + \frac{\pi\nu_1}{2}\right] + \sum_{m=1}^{\infty} c_{0m}(\nu_2)\left[\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right]^m\right)} + O(x)$$

The last step is obtained taking into account the non vanishing term in (5.18) and $\frac{\pi v(x)}{2\omega_1(x)} = \frac{v(x)}{F(x)} = v(x)(1+O(x))$.

c) If $\mathcal{V} = 2$ the series

$$-\sum_{n=1}^{\infty}\frac{n}{1-\left(\frac{e^{g(x)}}{16}\right)^{2n}x^{2n}}e^{n(2-\nu_2)g(x)}\left[\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right]^n e^{-i\pi n\frac{v(x)}{\omega_1(x)}}$$

which appears in $y(x)$ is oscillating. Again, $y(x)$ does not vanish.

We apply the result to the problem of radial convergence when $\Re\sigma = 1$. We identify $\nu_2 = 1-\sigma$ and choose $\nu_2 = i\nu$, $\nu \neq 0$ real. Let $x \to 0$ in $\mathcal{D}(r;\nu_1,i\nu)$ along the line arg(x)= constant (it is the line with $\mathcal{V}=0$). We have

$$y(x) = O(x) + \frac{1}{F(x)^2}\frac{1}{\sin^2\left(\frac{\nu}{2}\ln x - \nu\ln 16 + \frac{\pi}{2}\nu_1 + \frac{\nu}{2}g(x) + \frac{\pi v(x)}{2\omega_1(x)}\right)}$$

We use again (5.18), $\frac{1}{F(x)} = 1+O(x)$, $g(x) = O(x)$ and $\frac{\pi v(x)}{2\omega_1(x)} = \frac{v(x)}{F(x)} = v(x)(1+O(x))$. We have

$$= y(x) = O(x) + \frac{1+O(x)}{\sin^2\left(\frac{\nu}{2}\ln x - \nu\ln 16 + \frac{\pi\nu_1}{2} + \sum_{m=1}^{\infty} c_{om}(\nu)\left[\left(\frac{e^{i\pi\nu_1}}{e^{i\nu}}\right)x^{i\nu}\right]^m + O(x)\right)}$$

$$\equiv O(x) + \frac{1+O(x)}{\sin^2\left(\frac{\nu}{2}\ln x - \nu\ln 16 + \frac{\pi\nu_1}{2} + \sum_{m=1}^{\infty} c_{om}(\nu)\left[\left(\frac{e^{i\pi\nu_1}}{e^{i\nu}}\right)x^{i\nu}\right]^m\right)}$$

The last step is possible because $\sin(f(x)+O(x)) = \sin(f(x)) + O(x) = \sin(f(x))(1+O(x))$ if $f(x) \not\to 0$ as $x \to 0$; this is our case for $f(x) = \frac{\nu}{2}\ln x - \nu\ln 16 + \frac{\pi\nu_1}{2} + \sum_{m=1}^{\infty} c_{om}(\nu)\left[\left(\frac{e^{i\pi\nu_1}}{e^{i\nu}}\right)x^{i\nu}\right]^m$ in $\mathcal{D}$.

Thanks to the identification at point a), we have extended the result of theorem 1 when $\Re\sigma = 1$ and the convergence to $x = 0$ is along a radial path, provide that the limitation on arg(x) imposed in $\mathcal{D}(r;\nu_1,i\nu)$ is respected, namely

$$-\pi\Im\nu_1 - \ln r < \nu\arg(x)$$

The above limitation is the analogous of the limitation imposed by $B(\sigma,a;\theta_2,\tilde{\sigma})$ of (5.11).

As a last remark we observe that the coefficients in the expansion of $v(x)$ can be computed by direct substitution of v into the elliptic form of PVI_μ, the right hand-side being expanded in Fourier series.

5.4.2 Shimomura's Representation

In [52] and [27] S Shimomura proved the following statement for the Painlevé VI equation with any value of the parameters $\alpha,\beta,\gamma,\delta$.

For any complex number k and for any $\sigma \in \mathbf{C} - (-\infty,0] - [1,+\infty)$ there is a sufficiently small r, depending on σ, such that the equation $PVI_{\alpha,\beta,\gamma,\delta}$ has a holomorphic solution in the domain

$$\mathcal{D}_s(r;\sigma,k) = \{x \in \tilde{C}_0 \mid |x| < r,\ |e^{-k}x^{1-\sigma}| < r,\ |e^k x^\sigma| < r\}$$

with the following representation:

$$y(x;\sigma,k) = \frac{1}{\cosh^2(\frac{\sigma-1}{2}\ln x + \frac{k}{2} + \frac{v(x)}{2})},$$

where

$$v(x) = \sum_{n\geq 1} a_n(\sigma)x^n + \sum_{n\geq 0,\ m\geq 1} b_{nm}(\sigma)x^n(e^{-k}x^{1-\sigma})^m + \sum_{n\geq 0,\ m\geq 1} c_{nm}(\sigma)x^n(e^k x^\sigma)^m,$$

$a_n(\sigma), b_{nm}(\sigma), c_{nm}(\sigma)$ are rational functions of σ (and they may actually be computed recursively by direct substitution into the equation), and the series defining $v(\tilde{x})$ is convergent (and holomorphic) in $\mathcal{D}(r;\sigma,k)$. Moreover, there exists a constant $M = M(\sigma)$ such that

$$|v(x)| \leq M(\sigma)\ \left(|x| + |e^{-k}x^{1-\sigma}| + |e^k x^\sigma|\right). \tag{5.22}$$

The domain $\mathcal{D}(r;\sigma,k)$ is specified by the conditions:

$$|x| < r, \quad \Re\sigma \ln|x| + [\Re k - \ln r] < \Im\sigma \arg(x) < (\Re\sigma - 1)\ln|x| + [\Re k + \ln r]. \tag{5.23}$$

This is an open domain in the plane $(\ln|x|, \arg(x))$. It can be compared with the domain $D(\epsilon;\sigma,\theta_1,\theta_2)$ of theorem 1 (figure 5.8). Note that (5.23) imposes a limitation on $\arg(x)$. The situation is analogous to the elliptic representation. For example, if $\Re\sigma = 1$ we have

$$\Im\sigma \arg(x) < [\Re k + \ln r], \qquad (\ln r < 0)$$

After the identification of the Shimomura's transcendent with those of theorem 1 (see point a.1) below), the above limitation turns out to be the analogous of the limitation imposed to $D(\epsilon;\sigma;\theta_1,\theta_2)$ by $B(\sigma, a;\theta_2,\tilde{\sigma})$ of (5.11).

Like the elliptic representation, Shimomura's allows us to investigate what happens when $x \to 0$ along a path (5.10) with $\Sigma = 1$, contained in $\mathcal{D}_s(r;\sigma,k)$. It is a radial path if $\Re\sigma = 1$. Along (5.10) we have $|x^\sigma| = |x|^\Sigma e^{-b}$. We suppose $\Im\sigma \neq 0$.

a) $0 \leq \Sigma < 1$. We observe that $|x^{1-\sigma}e^{-k}| \to 0$ as $x \to 0$ along the line. Then:

$$y(x;\sigma,k) = \frac{1}{\cosh^2(\frac{\sigma-1}{2}\ln x + \frac{k}{2} + \frac{v(x)}{2})} = \frac{4}{x^{\sigma-1}e^k e^{v(x)} + x^{1-\sigma}e^{-k}e^{-v(x)} + 2},$$

$$= 4e^{-k}e^{-v(x)}x^{1-\sigma}\ \frac{1}{(1+e^{-k}e^{-v(x)}x^{1-\sigma})^2} = 4e^{-k}e^{-v(x)}x^{1-\sigma}\ \left(1 + e^{-v(x)}O(|e^{-k}x^{1-\sigma}|)\right).$$

Two sub-cases:

a.1) $\Sigma \neq 0$. Then $|x^\sigma e^k| \to 0$ and $v(x) \to 0$ (see (5.22)). Thus

$$y(x;\sigma,k) = 4e^{-k}x^{1-\sigma}\ \left(1 + O(|x| + |e^k x^\sigma| + |e^{-k}x^{1-\sigma}|)\right)$$

Following the proposition in section 5.2, we identify $y(x;;\sigma,k)$ and $y(x;\sigma,a)$ $(a = 4e^{-k})$ on $D_s(r;\sigma,k) \cap D(\epsilon;\sigma;\theta_1,\theta_2)$, which is not empty for any θ_1, θ_2. See figure 5.8.

a.2) $\Sigma = 0$. $|x^\sigma e^k| \to$ constant$< r$, so $|v(x)|$ does not vanish. Then

$$y(x) = a(x)x^{1-\sigma}\ \left(1 + O(|e^{-k}x^{1-\sigma}|)\right), \qquad a(x) = 4e^{-k}e^{-v(x)},$$

which must coincide with the result of theorem 1:

$$y(x) = a\left(1 + \frac{C}{2a}e^{i\alpha(x)} + \frac{C^2}{16a^2}e^{2i\alpha(x)}\right)x^{1-\sigma}\ \left(1 + O(|x|^{1-\sigma_1})\right).$$

b) $\Sigma = 1$. In this case theorem 1 fails. Now $|x^{1-\sigma}e^{-k}| \to$ (constant$\neq 0$) $< r$. Therefore $y(x)$ does not vanish as $x \to 0$. We keep the representation

$$y(x;\sigma,k) = \frac{1}{\cosh^2(\frac{\sigma-1}{2}\ln x + \frac{k}{2} + \frac{v(x)}{2})} \equiv \frac{1}{\sin^2(i\frac{\sigma-1}{2}\ln x + i\frac{k}{2} + i\frac{v(x)}{2} - \frac{\pi}{2})}$$

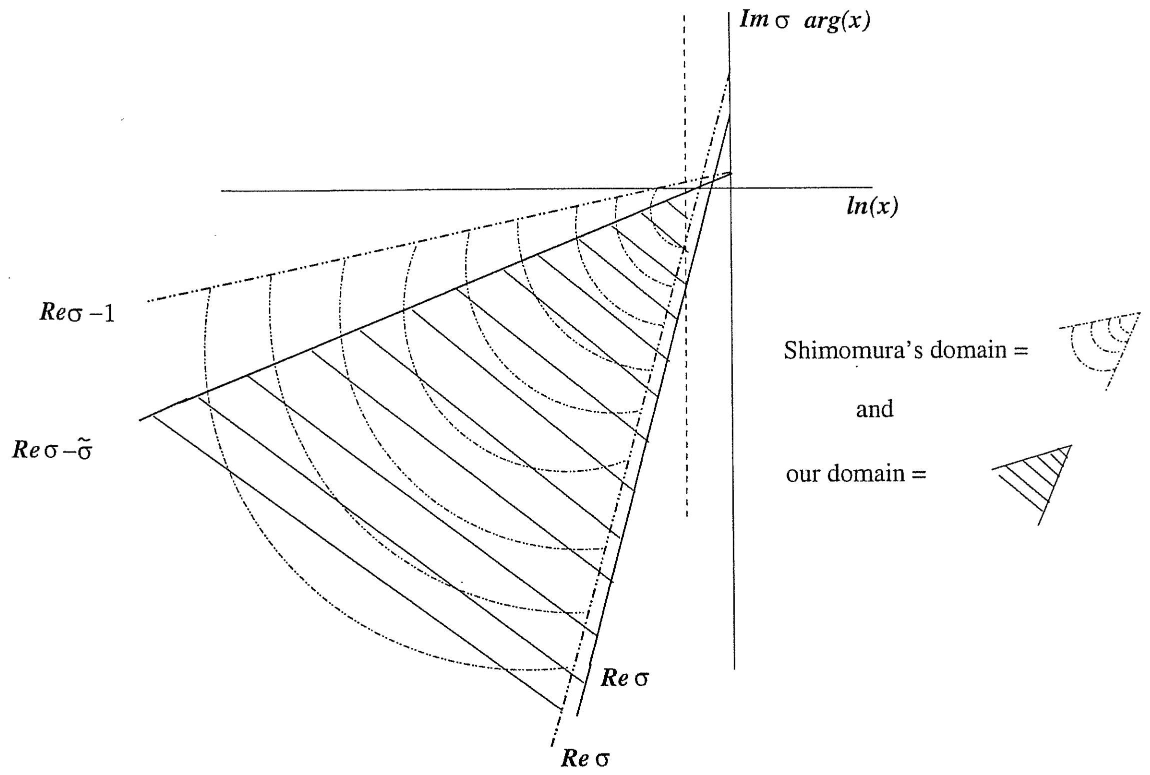


Figure 5.8: The domains $\mathcal{D}_s(r;\sigma,k)$ and $D(\epsilon;\sigma;\theta_1,\theta_2,\tilde{\sigma})$

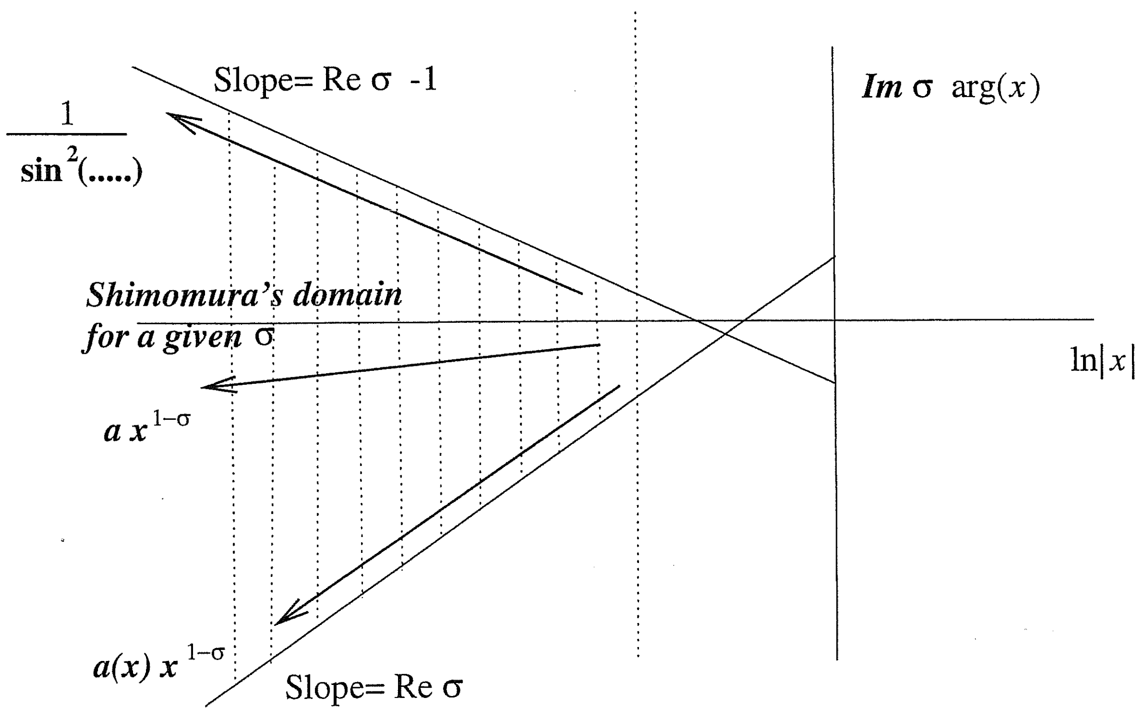


Figure 5.9: Critical behaviour of $y(x;\sigma,k)$ along different lines in $\mathcal{D}_s(r;\sigma,k)$

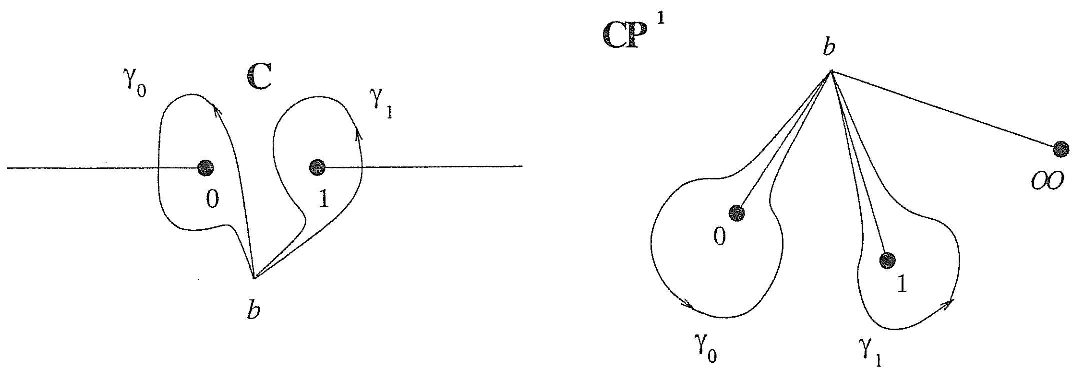


Figure 5.10: Base point and loops in $\mathbf{C}\backslash\{0,1\}$ and in $\mathbf{CP}^1\backslash\{0,1,\infty\}$.

$v(x)$ does not vanish and $y(x)$ is oscillating as $x \to 0$, with no limit. We remark that like in the elliptic representation, $\cosh^2(...)$ does not vanish in $\mathcal{D}_s(r;\sigma,k)$, so we do not have poles. Figure 5.9 synthesizes points a.1), a.2), b).

As an application, we consider the case $\Re\sigma = 1$, namely $\sigma = 1 - i\nu$, $\nu \in \mathbf{R}\backslash\{0\}$. Then, the path corresponding to $\Sigma = 1$ is a *radial* path in the x-plane and

$$y(x;1-i\nu,k) = \frac{1}{\sin^2\left(\frac{\nu}{2}\ln(x) + \frac{ik}{2} - \frac{\pi}{2} + i\frac{v(x)}{2}\right)}$$

$$= \frac{1+O(x)}{\sin^2\left(\frac{\nu}{2}\ln(x) + \frac{ik}{2} - \frac{\pi}{2} + \frac{i}{2}\sum_{m\geq 1} b_{0m}(\sigma)(e^{-k}\tilde{x}^{1-\sigma})^m\right)}$$

5.5 Analytic Continuation of a Branch

We describe the analytic continuation of the transcendent $y(x;\sigma,a)$.

We fix a basis in the fundamental group $\pi(\mathbf{P}^1\backslash\{0,1,\infty\},b)$, where b is the base-point (figure 5.10), and we choose a basis γ_0, γ_1 of two loops around 0 and 1 respectively. The analytic continuation of a branch $y(x;x_0,x_1,x_\infty)$ along paths encircling $x=0$ and $x=1$ (a loop around $x=\infty$ is homotopic to the product of γ_0, γ_1) is given by the action of the group of the pure braids on the monodromy data (see [21]). Namely, for a counter-clockwise loop around 0 we have to transform (x_0,x_1,x_∞) by the action of the braid β_1^2, where

$$\beta_1: \quad (x_0,x_1,x_\infty) \mapsto (-x_0, x_\infty - x_0x_1, x_1)$$

$$\beta_1^2: \quad (x_0,x_1,x_\infty) \mapsto (x_0,\ x_1 + x_0x_\infty - x_1x_0^2,\ x_\infty - x_0x_1)$$

For a counter-clockwise loop around 1 we need the braid β_2^2, where

$$\beta_2: \quad (x_0,x_1,x_\infty) \mapsto (x_\infty, -x_1, x_0 - x_1x_\infty)$$

$$\beta_2^2: \quad (x_0,x_1,x_\infty) \mapsto (x_0 - x_1x_\infty,\ x_1,\ x_\infty + x_0x_1 - x_\infty x_1^2)$$

A generic loop $\mathbf{P}^1\backslash\{0,1,\infty\}$ is represented by a braid β, which is a product of factors β_1 and β_2. Let σ and a be associated to (x_0,x_1,x_∞) and let $x \in D(\sigma)$. At x, the branch $y(x;x_0,x_1,x_\infty)$ coincides with $y(x;\sigma,a)$. The braid β acts on (x_0,x_1,x_∞) and produces a new triple $(x_0^\beta,x_1^\beta,x_\infty^\beta)$. We plug the new triple into the formulae of theorem 2 and we obtain the new parameters σ^β, a^β for the new branch $y(x;x_0^\beta,x_1^\beta,x_\infty^\beta)$ which coincides, at *the same* point x, with $y(x;\sigma^\beta,a^\beta)$.

To further clarify the concept, consider the transcendent $y(x;\sigma,a)$ in the point $x \in D(\sigma)$. Let us start at x, we perform the loop γ_1 around 1 and we go back to x. The transcendent becomes $y(x;\sigma^{\beta_2^2},a^{\beta_2^2})$. Namely

$$\gamma_1: \ y(x;\sigma,a) \longrightarrow y(x;\sigma^{\beta_2^2},a^{\beta_2^2}).$$

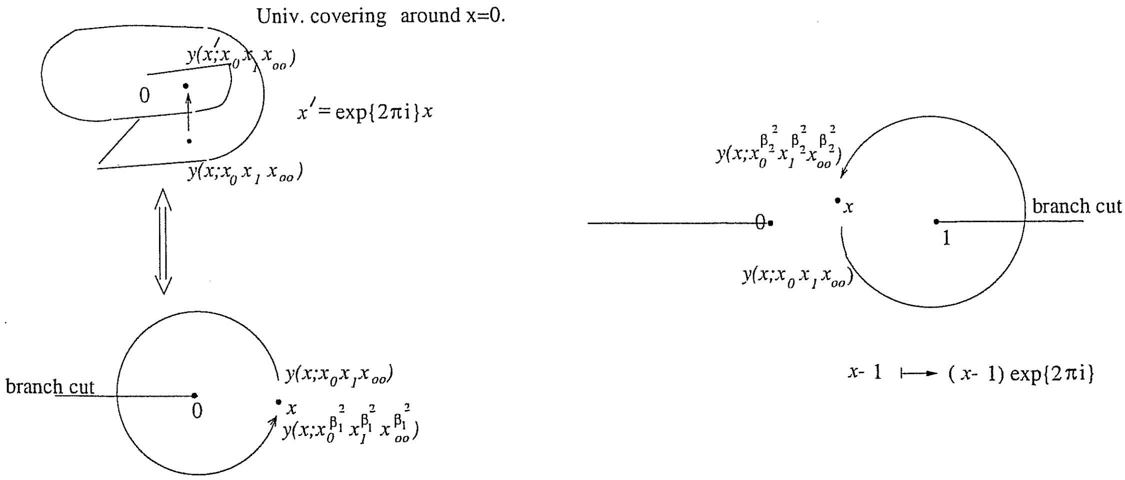


Figure 5.11: Analytic continuation of a branch for a loop around $x = 0$ and a loop around $x = 1$. We also draw the analytic continuation on the universal covering

Now let $x \in D(\sigma)$. Suppose that $x' := e^{2\pi i}x \in D(\sigma)$ (this assumption is always possible if $0 \leq \Re\sigma < 1$; if $\Re\sigma = 1$ we may need to consider $D(\sigma) \cup D(-\sigma) \cup D(\sigma-2) \cup D(2-\sigma)$). The loop γ_0 transforms $x \mapsto x'$ and according to theorem 1 we have

$$y(x;\sigma,a) \longrightarrow y(x';\sigma,a) = a[x']^{1-\sigma}\left(1+O\left(|x'|^{\delta}\right)\right)$$

$$= ae^{-2\pi i\sigma}x^{1-\sigma}(1+O(|x|^{\delta})) \equiv y(x;\sigma,ae^{-2\pi i\sigma})$$

This means that, *if we fix a branch cut for* x, the analytic continuation of $y(x;\sigma,a)$ starting at x, going around 0 with the loop γ_0 and returning back to *the same* x is

$$\gamma_0 : \; y(x;\sigma,a) \longrightarrow y(x;\sigma,ae^{-2\pi i\sigma}) \tag{5.24}$$

On the other hand, if x is considered as apoint on the universal covering of $\mathbf{C}_0 \cap \{|x| < \epsilon\}$ we simply have

$$\gamma_0 : \; y(x;\sigma,a) \longrightarrow y(x';\sigma,a)$$

Now we note that the transformation of (σ,a) according to the braid β_1 is

$$\beta_1^2 : \; (\sigma,a) \mapsto (\sigma,ae^{-2\pi i\sigma}) \tag{5.25}$$

as it follows from the fact that x_0 is not affected, then σ does not change, and from the explicit computation of $a(x_0^{\beta_1^2},x_1^{\beta_1^2},x_\infty^{\beta_1^2})$ through theorem 2 (we will do it at the end of section 5.8). Therefore (5.24) is

$$\gamma_0 : \; y(x;\sigma,a) \longrightarrow y(x;\sigma^{\beta_1^2},a^{\beta_1^2})$$

Thus theorem 1 is in accordance with the analytic continuation obtained by the action of the braid group.

5.6 Singular Points $x = 1$, $x = \infty$ (Connection Problem)

In this subsection we use the notation $\sigma^{(0)}$ and $a^{(0)}$ to denote the parameters of theorem 1 near the critical point $x = 0$. We describe now the analogous of theorem 1 near $x = 1$ and $x = \infty$. The three critical points 0, 1, ∞ are equivalent thanks to the symmetries of the PVI_μ equation.

a) Let

$$x = \frac{1}{t} \quad y(x) := \frac{1}{t}\,\hat{y}(t) \tag{5.26}$$

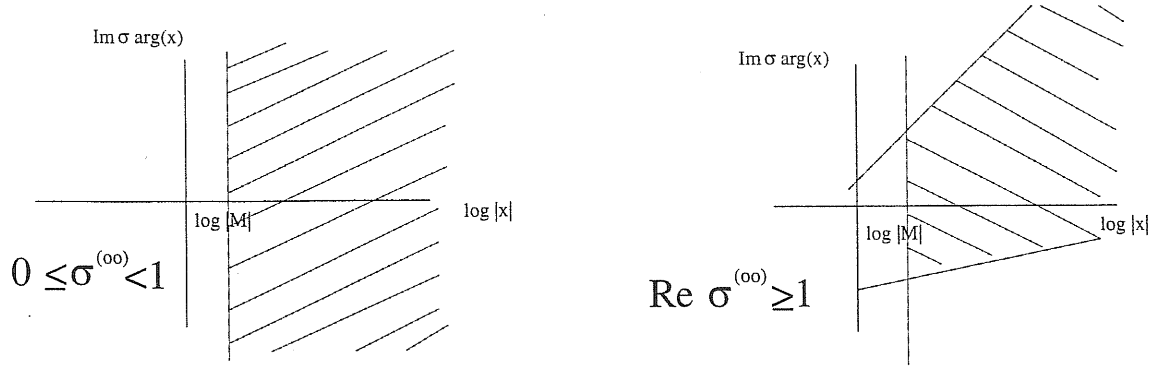


Figure 5.12: Some examples of the domain $D(M;\sigma;\theta_1,\theta_2,\tilde{\sigma})$

Then $y(x)$ is a solution of PVI_μ (variable x) if and only if $\hat{y}(t)$ is a solution of PVI_μ (variable t). The singularities 0 and ∞ are exchanged. Now, we can prove theorem 1 near $t = 0$. We go back to $y(x)$ and find a transcendent $y(x;\sigma^{(\infty)},a^{(\infty)})$ with behaviour

$$y(x;\sigma^{(\infty)},a^{(\infty)}) = a^{(\infty)}x^{\sigma^{(\infty)}}\left(1+O(\frac{1}{|x|^\delta})\right) \qquad x \to \infty \tag{5.27}$$

in

$$D(M;\sigma^{(\infty)};\theta_1,\theta_2,\tilde{\sigma}) := \{x \in \widetilde{\mathbf{C}\backslash\{\infty\}} \text{ s.t. } |x| > M,\ e^{-\theta_1\Im\sigma^{(\infty)}}|x|^{-\tilde{\sigma}} \le |x^{-\sigma^{(\infty)}}| \le e^{-\theta_2\Im\sigma^{(\infty)}}$$
$$0 \le \tilde{\sigma} < 1\}$$

where $M > 0$ is sufficiently big and $0 < \delta < 1$ is small (figure 5.12).

b) Let

$$x = 1-t, \quad y(x) = 1-\hat{y}(t) \tag{5.28}$$

$y(x)$ satisfies PVI_μ if and only if $\hat{y}(t)$ satisfies PVI_μ. Theorem 1 holds for $\hat{y}(t)$ near $t = 0$ with coefficients $\sigma^{(1)}$ and $a^{(1)}$. Going back to $y(x)$ we obtain a transcendent $y(x;\sigma^{(1)},a^{(1)})$ such that

$$y(x,\sigma^{(1)},a^{(1)}) = 1 - a^{(1)}(1-x)^{1-\sigma^{(1)}}(1+O(|1-x|^\delta)) \qquad x \to 1 \tag{5.29}$$

in

$$D(\epsilon;\sigma^{(1)};\theta_1,\theta_2,\tilde{\sigma}) := \{x \in \widetilde{\mathbf{C}\backslash\{1\}} \text{ s.t. } |1-x| < \epsilon,\ e^{-\theta_1\Im\sigma}|1-x|^{\tilde{\sigma}} \le |(1-x)^{\sigma^{(1)}}| \le e^{-\theta_2\Im\sigma},$$
$$0 \le \tilde{\sigma} < 1\}$$

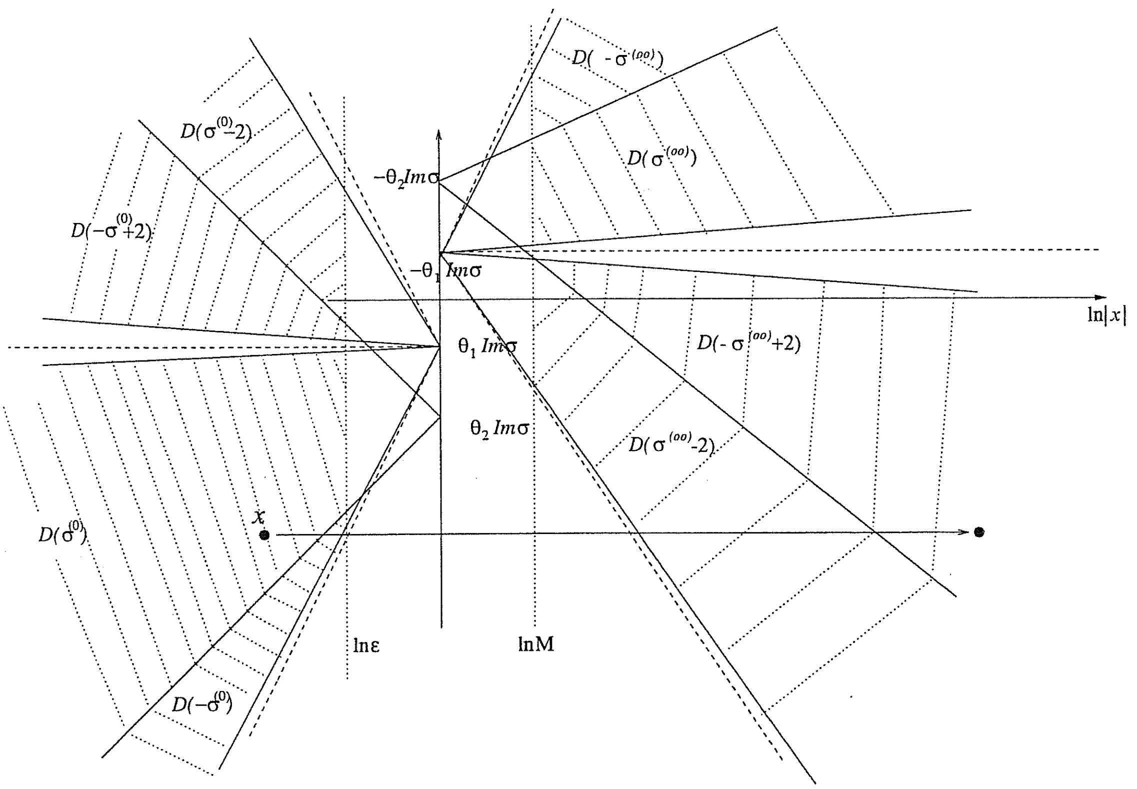
We choose a triple of monodromy data (x_0,x_1,x_∞) and we compute the corresponding $\sigma^{(0)} = \sigma^{(0)}(x_0)$, $0 \le \Re\sigma^{(0)} \le 1$, and $a^{(0)}$. We recall that $a^{(0)}$ depends on the triple but also on $\sigma^{(0)}$, namely for the same triple the change $\sigma^{(0)} \mapsto \pm\sigma^{(0)} + 2n$ changes $a^{(0)}$; so we'll write $a^{(0)}(\pm\sigma^{(0)} + 2n)$.

Let $x \in D(\sigma^{(0)}) \cup D(-\sigma^{(0)}) \cup D(2-\sigma^{(0)}) \cup D(\sigma^{(0)}-2)$ [5]. At x there exists a unique branch $y(x;x_0,x_1,x_\infty)$ whose analytic continuation is $y(x;\sigma^{(0)},a(\sigma^{(0)}))$ if $x \in D(\sigma^{(0)})$, or $y(x;-\sigma^{(0)},a(-\sigma^{(0)}))$ if $x \in D(-\sigma^{(0)})$, etc.

The branch $y(x;x_0,x_1,x_\infty)$ is also defined for $x \in D(M;\sigma^{(\infty)}) \cup D(M;-\sigma^{(\infty)}) \cup D(M;-\sigma^{(\infty)}) \cup D(M;\sigma^{(\infty)}-2)$ [6]. As it is proved in [21]

$$y(x;x_0,x_1,x_\infty) = \frac{1}{t}\hat{y}(t;x_\infty,-x_1,x_0-x_1x_\infty), \qquad x = \frac{1}{t}$$

From the data $(x_\infty,-x_1,x_0-x_1x_\infty)$ we compute $\sigma^{(\infty)}$ and $a^{(\infty)}(\sigma^{(\infty)})$, $a^{(\infty)}(-\sigma^{(\infty)})$, $a^{(\infty)}(\sigma^{(\infty)}-2)$, etc. The analytic continuation of the branch $y(x;x_0,x_1,x_\infty)$ is then $y(x;\sigma^{(\infty)},a^{(\infty)}(\sigma^{(\infty)}))$ if $x \in D(M_1;\sigma^{(\infty)})$, it is $y(x;-\sigma^{(\infty)},a^{(\infty)}(-\sigma^{(\infty)}))$ if $x \in D(M_2;-\sigma^{(\infty)})$, etc.



In the figure the special case Re $\sigma^{(0)}$=Re $\sigma^{(\infty)}$=1 (Im σ=0) is considered.

Figure 5.13: Connection problem for the points $x = 0$, $x = \infty$

The above discussion solves the connection problem between $x = 0$ and $x = \infty$.

In the same way we solve the connection problem between $x = 0$ and $x = 1$ by recalling that in [21] it is proved that a branch $y(x; x_0, x_1, x_\infty)$ is

$$y(x; x_0, x_1, x_\infty) = 1 - \hat{y}(t; x_1, x_0, x_0 x_1 - x_\infty), \qquad x = 1 - t$$

We repeat the same argument. We remark again that for $\Re\sigma^{(0)} = \Re\sigma^{(1)} = 1$ it is necessary to consider the union of $D(\sigma^{(1)})$, $D(-\sigma^{(1)})$, $D(2 - \sigma^{(1)})$, $D(\sigma^{(1)} - 2)$ to include all possible values of $\arg(1 - x)$.

5.7 Proof of Theorem 1

In order to prove theorem 1 we have to recall the connection between PVI_μ and Schlesinger equations for 2×2 matrices $A_0(x)$, $A_x(x)$, $A_1(x)$

$$\begin{aligned} \frac{dA_0}{dx} &= \frac{[A_x, A_0]}{x} \\ \frac{dA_1}{dx} &= \frac{[A_1, A_x]}{1-x} \\ \frac{dA_x}{dx} &= \frac{[A_x, A_0]}{x} + \frac{[A_1, A_x]}{1-x} \end{aligned} \tag{5.30}$$

They are the analogous of (1.32) (CoMp2). We look for solutions satisfying

$$A_0(x) + A_x(x) + A_1(x) = \begin{pmatrix} -\mu & 0 \\ 0 & \mu \end{pmatrix} := -A_\infty \quad \mu \in \mathbf{C}, \quad 2\mu \notin \mathbf{Z}$$

$$\text{tr}(A_i) = \det(A_i) = 0$$

Now let

$$A(z, x) := \frac{A_0}{z} + \frac{A_x}{z - x} + \frac{A_1}{z - 1}$$

We explained that $y(x)$ is a solution of PVI_μ if and only if $A(y(x), x)_{12} = 0$.

The system (5.30) is a particular case of the system

$$\begin{aligned} \frac{dA_\mu}{dx} &= \sum_{\nu=1}^{n_2} [A_\mu, B_\nu] \; f_{\mu\nu}(x) \\ \frac{dB_\nu}{dx} &= -\frac{1}{x} \sum_{\nu'=1}^{n_2} [B_\nu, B_{\nu'}] + \sum_{\mu=1}^{n_1} [B_\nu, A_\mu] \; g_{\mu\nu}(x) + \sum_{\nu'=1}^{n_2} [B_\nu, B_{\nu'}] \; h_{\nu\nu'}(x) \end{aligned} \tag{5.31}$$

where the functions $f_{\mu\nu}$, $g_{\mu\nu}$, $h_{\mu\nu}$ are meromorphic with poles at $x = 1, \infty$ and $\sum_\nu B_\nu + \sum_\mu A_\mu = -A_\infty$ (here the subscript μ is a label, not the eigenvalue of A_∞ !). System (5.30) is obtained for $f_{\mu\nu} = g_{\mu\nu} = b_\nu / (a_\mu - x b_\nu)$, $h_{\mu\nu} = 0$, $n_1 = 1$, $n_2 = 2$, $a_1 = b_2 = 1$, $b_1 = 0$ and $B_1 = A_0$, $B_2 = A_x$, $A_1 = A_1$.

We prove the analogous result of [50], page 262, for the domain $D(\epsilon; \sigma)$:

Lemma 1: *Consider matrices B_ν^0 ($\nu = 1, .., n_2$), A_μ^0 ($\mu = 1, .., n_1$) and Λ, independent of x and such that*

$$\sum_\nu B_\nu^0 + \sum_\mu A_\mu^0 = -A_\infty$$

$$\sum_\nu B_\nu^0 = \Lambda, \qquad \textit{eigenvalues}(\Lambda) = \frac{\sigma}{2}, \; -\frac{\sigma}{2}, \qquad \sigma \in \Omega$$

Suppose that $f_{\mu\nu}$, $g_{\mu\nu}$, $h_{\mu\nu}$ are holomorphic if $|x| < \epsilon'$, for some small $\epsilon' < 1$.

For any $0 < \tilde{\sigma} < 1$ and θ_1, θ_2 real there exists a sufficiently small $0 < \epsilon < \epsilon'$ such that the system (5.31) has holomorphic solutions $A_\mu(x)$, $B_\nu(x)$ in $D(\epsilon; \sigma; \theta_1, \theta_2, \tilde{\sigma})$ satisfying:

$$||A_\mu(x) - A_\mu^0|| \leq C \; |x|^{1-\sigma_1}$$

[5] This is always possible for any $\arg(x)$ if $|x|$ is small enough

[6] The necessity of considering the union of the four domains comes from the fact that if $\Re\sigma^{(\infty)} = \Re\sigma^{(0)} = 1$ it is not obvious that we can move from $x \in D(\sigma^{(0)})$ to $x \in D(M, \sigma^{(\infty)})$ keeping $\arg(x)$ fixed, therefore keeping the same branch of the transcendent.

$$||x^{-\Lambda}B_\nu(x)\ x^{\Lambda}-B_\nu^0|| \le C\ |x|^{1-\sigma_1}$$

Here C is a positive constant and $\tilde{\sigma}<\sigma_1<1$

Important remark: There is no need to assume here that $2\mu\not\in \mathbf{Z}$. The theorem holds true for any value of μ. If in the system (5.31) the functions $f_{\mu\nu}$, $g_{\mu\nu}$, $h_{\mu\nu}$ are chosen in such a way to yield Schlesinger equations for the fuchsian system of PVI_μ, the assumption $2\mu\not\in\mathbf{Z}$ is still not necessary, provided the matrix R is considered as a monodromy datum independent of the deformation parameter x.

Proof: Let $A(x)$ and $B(x)$ be 2×2 matrices holomorphic on $D(\epsilon;\sigma)$ (we understand $\theta_1,\theta_2,\tilde{\sigma}$ which have been chosen) and such that

$$||A(x)||\le C_1,\qquad ||B(x)||\le C_2\qquad \text{on } D(\epsilon;\sigma)$$

Let $f(x)$ be a holomorphic function for $|x|<\epsilon'$. Let σ_2 be a real number such that $\tilde{\sigma}<\sigma_2<1$. Then, there exists a sufficiently small $\epsilon<\epsilon'$ such that for $x\in D(\epsilon;\sigma)$ we have:

$$||x^{\pm\Lambda}\ A(x)\ x^{\mp\Lambda}||\le C_1|x|^{-\sigma_2}$$

$$||x^{\pm\Lambda}\ B(x)\ x^{\mp\Lambda}||\le C_2|x|^{-\sigma_2}$$

$$\left|\left|x^{-\Lambda}\int_{L(x)} ds\ A(s)\ s^{\Lambda}\ B(s)\ s^{-\Lambda}\ f(s)\ x^{\Lambda}\right|\right|\le C_1C_2\ |x|^{1-\sigma_2}$$

$$\left|\left|x^{-\Lambda}\int_{L(x)} ds\ s^{\Lambda}\ B(s)\ s^{-\Lambda}\ A(s)\ f(s)\ x^{\Lambda}\right|\right|\le C_1C_2\ |x|^{1-\sigma_2}$$

where $L(x)$ is a path in $D(\epsilon;\sigma)$ joining 0 to x. To prove the estimates, we observe that

$$||x^{\Lambda}||=||x^{\mathrm{diag}(\frac{\sigma}{2},-\frac{\sigma}{2})}||=\max\{|x^\sigma|^{\frac{1}{2}},|x^\sigma|^{-\frac{1}{2}}\}\le e^{\frac{\theta_1}{2}\Im\sigma}\ |x|^{-\frac{\tilde{\sigma}}{2}},\qquad \text{in } D(\epsilon;\sigma)$$

Then

$$||x^{\Lambda}\ A(x)\ x^{-\Lambda}||\le ||x^{\Lambda}||\ ||A(x)||\ ||\ x^{-\Lambda}||\le e^{\theta_1\Im\sigma}\ C_1\ |x|^{-\tilde{\sigma}}$$
$$=\left(e^{\theta_1\Im\sigma}\ |x|^{\sigma_2-\tilde{\sigma}}\right)\ C_1\ |x|^{-\sigma_2}$$

Thus, if ϵ is small enough (we require $\epsilon^{\sigma_2-\tilde{\sigma}}\le e^{-\theta_1\Im\sigma}$) we obtain $||x^{\Lambda}\ A(x)\ x^{-\Lambda}||\le C_1|x|^{-\sigma_2}$.

We turn to the integrals. The integrands are holomorphic on $D(\epsilon;\sigma)$, then they do not depend on the choice of the path, but only on the point x. We choose a real number σ^* such that $0\le\sigma^*\le\tilde{\sigma}$ and we choose the path

$$\arg(s)=a\log|s|+b,\qquad a=\frac{\Re\sigma-\sigma^*}{\Im\sigma},\qquad \text{or } \arg(s)=b\equiv\arg(x)\ \text{ if } \Im\sigma=0$$

where b is chosen appropriately such that $L(x)$ stays in $D(\epsilon;\sigma)$. See figure 5.14. Then we compute

$$\left|\left|x^{-\Lambda}\int_{L(x)} ds\ A(s)\ s^{\Lambda}\ B(s)\ s^{-\Lambda}\ f(s)\ x^{\Lambda}\right|\right|=\left|\left|\int_{L(x)} ds\ x^{-\Lambda}\ A(s)\ x^{\Lambda}\ \left(\frac{s}{x}\right)^{\Lambda}\ B(s)\ \left(\frac{s}{x}\right)^{-\Lambda}\ f(s)\right|\right|$$

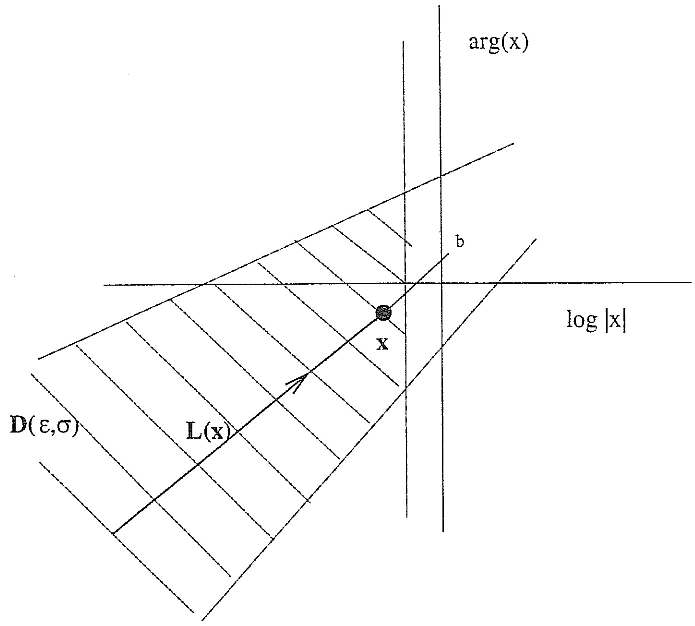
$$\le e^{2\theta_1\Im\sigma}x^{-\tilde{\sigma}}\ C_1C_2\ \max_{|x|<\epsilon}|f(x)|\ \int_{L(x)}|ds|\ \frac{|s|^{-\sigma^*}}{|x|^{-\sigma^*}}$$

where by $|ds|$ I mean $d\vartheta\ |ds(\vartheta)/d\vartheta|$, and ϑ is a parameter on the curve $L(x)$. The last step in the inequality follows from

$$\left|\left|\left(\frac{s}{x}\right)^{\Lambda}\right|\right|=\left|\left|\mathrm{diag}\left(\frac{s^{\frac{\sigma}{2}}}{x^{\frac{\sigma}{2}}},\frac{s^{-\frac{\sigma}{2}}}{x^{-\frac{\sigma}{2}}}\right)\right|\right|=\max_L\left\{\frac{|s^{\frac{\sigma}{2}}|}{|x^{\frac{\sigma}{2}}|},\frac{|s^{-\frac{\sigma}{2}}|}{|x^{-\frac{\sigma}{2}}|}\right\}$$

and the observation that, on L, $|s^\sigma|=|s|^{\tilde{\sigma}}e^{-b\Im\sigma}$. Thus

$$\max\left\{\frac{|s^{\frac{\sigma}{2}}|}{|x^{\frac{\sigma}{2}}|},\frac{|s^{-\frac{\sigma}{2}}|}{|x^{-\frac{\sigma}{2}}|}\right\}=\max\left\{\frac{|s|^{\frac{\sigma^*}{2}}}{|x|^{\frac{\sigma^*}{2}}},\frac{|s|^{-\frac{\sigma^*}{2}}}{|x|^{-\frac{\sigma^*}{2}}}\right\}=\frac{|s|^{-\frac{\sigma^*}{2}}}{|x|^{-\frac{\sigma^*}{2}}},\qquad |s|\le|x|$$



Path of integration.

Figure 5.14:

The parameter s on $L(x)$ is

$$s(\vartheta) := e^{\frac{\vartheta - b}{a}}\ e^{i\vartheta}$$

where $\vartheta \in (-\infty, \arg(x)]$ or $\in [\arg(x), +\infty)$. Then

$$\int_{L(x)} |ds|\ |s|^{-\sigma^*} = \int_{L(x)} d\vartheta \left|\frac{ds}{d\vartheta}\right|\ |s(\vartheta)|^{-\sigma^*} = \frac{|a|}{1-\sigma^*}\left|i + \frac{1}{a}\right| e^{\frac{\arg(x)-b}{a}(1-\sigma^*)} = \frac{|a|}{1-\sigma^*}\left|i + \frac{1}{a}\right|\ |x|^{1-\sigma^*}$$

Then, the initial integral is less or equal to

$$e^{\theta_1 \Im\sigma}\ \max_{|x|<\epsilon} |f(x)|\ C_1 C_2 \text{ constant } |x|^{1-\tilde{\sigma}}$$

Now, we write $|x|^{1-\tilde{\sigma}} = |x|^{\sigma_2 - \tilde{\sigma}}\ |x|^{1-\sigma_2}$ and we obtain, for sufficiently small ϵ:

$$e^{\theta_1 \Im\sigma}\ \max_{|x|<\epsilon} |f(x)|\ C_1 C_2 \text{ constant } |x|^{1-\tilde{\sigma}} \leq C_1 C_2\ |x|^{1-\sigma_2}$$

We remark that for $\sigma = 0$ the above estimates are still valid. Actually $||x^{\Lambda}|| \equiv ||x^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}||$ diverges like $|\log x|$, $||x^{\Lambda} A(x) x^{-\Lambda}||$ are less or equal to $C_1\ |\log(x)|^2$, and finally $||x^{-\Lambda}\ \int_{L(x)} ds\ A(s)\ s^{\Lambda}\ B(s)\ s^{-\Lambda}\ f(s)\ x^{\Lambda}||$ is less or equal to $C_1 C_2 \max|f|\ |\log(x)|^2 \int_{L(x)} |ds|\ |\log s|^2$. We can choose $L(x)$ to be a radial path $s = \rho \exp(i\alpha)$, $0 < \rho < |x|$, α fixed. Then the integral is $|x|(\log|x|^2 - 2\log|x| + 2 + \alpha^2)$. The factor $|x|$ does the job, because we rewrite it as $|x|^{\sigma_2}\ |x|^{1-\sigma_2}$ (here σ_2 is any number between 0 and 1) and we proceed as above to choose ϵ small enough in such a way that $\left(\max|f|\ |x|^{\sigma_2} \times \text{ function diverging like } \log^2|x|\right) \leq 1$.

The estimates above are in a sense enough to prove the lemma.

We solve the Schlesinger equations by successive approximations, as in [50]: let $\tilde{B}_\nu(x) := x^{-\Lambda} B_\nu(x) x^{\Lambda}$. The Schlesinger equations are re-written as

$$\frac{dA_\mu}{dx} = \sum_{\nu=1}^{n_2} [A_\mu, x^{\Lambda} \tilde{B}_\nu x^{-\Lambda}]\ f_{\mu\nu}(x)$$

$$\frac{d\tilde{B}_\nu}{dx} = \frac{1}{x}[\tilde{B}_\nu, \sum_\mu x^{-\Lambda}(A_\mu(x) - A_\mu^0)x^\Lambda] + \sum_{\mu=1}^{n_1}[\tilde{B}_\nu, x^{-\Lambda}A_\mu x^\Lambda]\ g_{\mu\nu}(x) + \sum_{\nu'=1}^{n_2}[\tilde{B}_\nu, \tilde{B}_{\nu'}]\ h_{\nu\nu'}(x)$$

Then, by successive approximations:

$$A_\mu^{(k)}(x) = A_\mu^0 + \int_{L(x)} ds\ \sum_\nu [A_\mu^{(k-1)}(s), s^\Lambda \tilde{B}_\nu^{(k-1)}(s)s^{-\Lambda}]\ f_{\mu\nu}(s)$$

$$\tilde{B}_\nu^{(k)}(x) = B_\nu^0 + \int_{L(x)} ds\ \left\{\frac{1}{s}[\tilde{B}_\nu^{(k-1)}(s), \sum_\mu s^{-\Lambda}(A_\mu^{(k-1)}(s) - A_\mu^0)s^\Lambda] + \right.$$

$$\left. + \sum_\mu [\tilde{B}_\nu^{(k-1)}(s), s^{-\Lambda}A_\mu^{(k-1)}(s)x^\Lambda]\ g_{\mu\nu}(s) + \sum_{\nu'}[\tilde{B}_\nu^{(k-1)}(s), \tilde{B}_{\nu'}^{(k-1)}(s)]\ h_{\nu\nu'}\right\}$$

The functions $A_\mu^{(k)}(x)$, $B_\nu^{(k)}(x)$ are holomorphic in $D(\epsilon;\sigma)$, by construction. Observe that $||A_\mu^0|| \leq C$, $||B_\nu^0|| \leq C$ for some constant C. We claim that for $|x|$ sufficiently small

$$\begin{gathered} ||A_\mu^{(k)}(x) - A_\mu^0|| \leq C|x|^{1-\sigma_1} \\ \left|\left|x^{-\Lambda}\left(A_\mu^{(k)}(x) - A_\mu^0\right)x^\Lambda\right|\right| \leq C^2|x|^{1-\sigma_2} \\ ||\tilde{B}_\nu^{(k)}(x) - B_\nu^0|| \leq C|x|^{1-\sigma_1} \end{gathered} \tag{5.32}$$

where $\tilde{\sigma} < \sigma_2 < \sigma_1 < 1$. Note that the above inequalities imply $||A_\mu^{(k)}|| \leq 2C$, $||\tilde{B}_\nu^{(k)}|| \leq 2C$. Moreover we claim that

$$\begin{gathered} ||A_\mu^{(k)}(x) - A_\mu^{(k-1)}(x)|| \leq C\ \delta^{k-1}\ |x|^{1-\sigma_1} \\ \left|\left|x^{-\Lambda}\left(A_\mu^{(k)}(x) - A_\mu^{(k-1)}(x)\right)x^\Lambda\right|\right| \leq C^2\ \delta^{k-1}\ |x|^{1-\sigma_2} \\ ||\tilde{B}_\nu^{(k)}(x) - B_\nu^{(k-1)}(x)|| \leq C\ \delta^{k-1}|x|^{1-\sigma_1} \end{gathered} \tag{5.33}$$

where $0 < \delta < 1$.

For $k = 1$ the above inequalities are proved using the simple methods used in the estimates at the beginning of the proof. Then we proceed by induction, still using the same estimates. We leave this technical point to the reader, but we give at least one example of how to proceed. As an example, we prove the $(k+1)^{th}$ step of the first of (5.33):

$$||A_\mu^{(k+1)}(x) - A_\mu^{(k)}(x)|| \leq C\ \delta^k\ |t|^{1-\sigma_1}$$

supposing the k^{th} step of (5.33) is true. Let us proceed using the integral equations:

$$||A_\mu^{(k+1)}(x) - A_\mu^{(k)}(x)|| = \left|\left|\int_{L(x)} ds\ \sum_{\nu=1}^{n_2}\left(A_\mu^{(k)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda} - A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda} + \right.\right.\right.$$

$$\left.\left.\left. + s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda}A_\mu^{(k-1)} - s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda}A_\mu^{(k)}\right)\ f_{\mu\nu}(s)\right|\right| \leq$$

$$\leq \int_{L(x)} d|s|\ \sum_{\nu=1}^{n_2}\left|\left|A_\mu^{(k)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda} - A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda}\right|\right|\ |f_{\mu\nu}(s)| +$$

$$+ \int_{L(x)} d|s|\ \sum_{\nu=1}^{n_2}\left|\left|s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda}A_\mu^{(k-1)} - s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda}A_\mu^{(k)}\right|\right|\ |f_{\mu\nu}(s)|$$

Now we estimate

$$\begin{gathered} \left|\left|A_\mu^{(k)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda} - A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda}\right|\right| \leq \\ \leq \left|\left|A_\mu^{(k)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda} - A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda}\right|\right| + \\ + \left|\left|A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k)}s^{-\Lambda} - A_\mu^{(k-1)}s^\Lambda\tilde{B}_\nu^{(k-1)}s^{-\Lambda}\right|\right| \end{gathered}$$

$$\leq ||A_\mu^{(k)} - A_\mu^{(k-1)}|| \; ||s^\Lambda \tilde{B}_\nu^{(k)} s^{-\Lambda}|| + ||A_\mu^{(k-1)}|| \; ||s^\Lambda|| \; ||\tilde{B}_\nu^{(k)} - \tilde{B}_\nu^{(k-1)}|| \; ||s^{-\Lambda}||$$

By induction then:

$$\leq \left(C\ \delta^{k-1}\ |s|^{1-\sigma_1}\right)\ 2C\ e^{\theta_1 \Im\sigma}|s|^{-\tilde{\sigma}} + 2C\ \left(C\ \delta^{k-1}\ |s|^{1-\sigma_1}\right)\ e^{\theta_1\Im\sigma}|s|^{-\tilde{\sigma}}$$

The other term is estimated in an analogous way. Then

$$||A_\mu^{(k+1)} - A_\mu^{(k)}|| \leq \text{constant } 8n_2 C^2 \max|f_{\mu\nu}|\ \delta^{k-1}\ e^{\theta_1\Im\sigma}|x|^{1-\tilde{\sigma}}\ |x|^{1-\sigma_1}$$

We choose ϵ small enough to have constant $8n_2 C \max|f|\ e^{\theta_1\Im\sigma}|x|^{1-\tilde{\sigma}} \leq \delta$. Note that the choice of ϵ is independent of k. In the case $\sigma = 0$, $|x|^{1-\tilde{\sigma}}$ is substituted by $|x|(\log^2|x| + O(\log|x|))$.

The inequalities (5.32) (5.33) imply the convergence of the successive approximations to a solution of the Schlesinger equations, satisfying the assertion of the lemma, plus the additional inequality

$$||x^{-\Lambda}(A_\mu(x) - A_\mu^0)x^\Lambda|| \leq C^2|x|^{1-\sigma_2}$$

□

We observe that we imposed

$$\epsilon^{\sigma_2-\tilde{\sigma}} \leq ce^{-\theta_1\Im\sigma}, \quad \epsilon^{1-\tilde{\sigma}} \leq ce^{-\theta_1\Im\sigma}, \tag{5.34}$$

where c is a constant constructed in the theorem (c is proportional to $\frac{1-\tilde{\sigma}}{8n_2C}$ and $C = \max\{||A_\mu^0||, ||B_\nu^0||\}$).

We turn to the case in which we are concerned: we consider three matrices A_0^0, A_x^0, A_1^0 such that

$$A_0^0 + A_x^0 = \Lambda, \quad A_0^0 + A_x^0 + A_1^0 = \text{diag}(-\mu, \mu)$$

$$\text{tr}(A_i^0) = \det(A_i^0) = 0, \quad i = 0, x, 1$$

Lemma 2: *Let r and s be two complex numbers not equal to 0 and ∞. Let T be the matrix which brings Λ to the Jordan form:*

$$T^{-1}\Lambda T = \begin{cases} diag(\frac{\sigma}{2}, -\frac{\sigma}{2}), & \sigma \neq 0 \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \sigma = 0 \end{cases}$$

The general solution of

$$A_0^0 + A_x^1 + A_1^0 = \begin{pmatrix} -\mu & 0 \\ 0 & \mu \end{pmatrix}, \quad \text{tr}(A_i) = \det(A_i) = 0, \quad A_0^0 + A_x^0 = \Lambda$$

is the following:

For $\sigma \neq 0, \pm 2\mu$:

$$\Lambda = \frac{1}{8\mu}\begin{pmatrix} -\sigma^2 - (2\mu)^2 & (\sigma^2 - (2\mu)^2)r \\ \frac{(2\mu)^2 - \sigma^2}{r} & \sigma^2 + (2\mu)^2 \end{pmatrix} \quad A_1^0 = \frac{\sigma^2 - (2\mu)^2}{8\mu}\begin{pmatrix} 1 & -r \\ \frac{1}{r} & -1 \end{pmatrix}$$

$$A_0^0 = T\begin{pmatrix} \frac{\sigma}{4} & \frac{\sigma}{4}s \\ -\frac{\sigma}{4}\frac{1}{s} & -\frac{\sigma}{4} \end{pmatrix} T^{-1}, \quad A_x^0 = T\begin{pmatrix} \frac{\sigma}{4} & -\frac{\sigma}{4}s \\ \frac{\sigma}{4}\frac{1}{s} & -\frac{\sigma}{4} \end{pmatrix} T^{-1}$$

where

$$T = \begin{pmatrix} 1 & 1 \\ \frac{(\sigma+2\mu)^2}{\sigma^2-(2\mu)^2}\frac{1}{r} & \frac{(\sigma-2\mu)^2}{\sigma^2-(2\mu)^2}\frac{1}{r} \end{pmatrix}$$

For $\sigma = -2\mu$: A_0^0 and A_x^0 as above, but

$$\Lambda = \begin{pmatrix} -\mu & r \\ 0 & \mu \end{pmatrix} \quad A_1^0 = \begin{pmatrix} 0 & -r \\ 0 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & \frac{2\mu}{r} \end{pmatrix} \tag{5.35}$$

or

$$\Lambda = \begin{pmatrix} -\mu & 0 \\ r & \mu \end{pmatrix} \quad A_1^0 = \begin{pmatrix} 0 & 0 \\ -r & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ -\frac{r}{2\mu} & 1 \end{pmatrix} \tag{5.36}$$

For $\sigma = 2\mu$: A_0^0 and A_x^0 as above, but

$$\Lambda = \begin{pmatrix} -\mu & r \\ 0 & \mu \end{pmatrix} \quad A_1^0 = \begin{pmatrix} 0 & -r \\ 0 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 1 \\ \frac{2\mu}{r} & 0 \end{pmatrix} \tag{5.37}$$

or

$$\Lambda = \begin{pmatrix} -\mu & 0 \\ r & \mu \end{pmatrix} \quad A_1^0 = \begin{pmatrix} 0 & 0 \\ -r & 0 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{r}{2\mu} \end{pmatrix} \tag{5.38}$$

For $\sigma = 0$:

$$A_0^0 = T \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} T^{-1} \quad A_x^0 = T \begin{pmatrix} 0 & 1-s \\ 0 & 0 \end{pmatrix} T^{-1}$$

$$\Lambda = \begin{pmatrix} -\frac{\mu}{2} & -\frac{\mu^2}{4} r \\ \frac{1}{r} & \frac{\mu}{2} \end{pmatrix} \quad A_1^0 = \begin{pmatrix} -\frac{\mu}{2} & \frac{\mu^2}{4} r \\ -\frac{1}{r} & \frac{\mu}{2} \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ -\frac{2}{\mu r} & -2\frac{\mu+2}{\mu^2}\frac{1}{r} \end{pmatrix}$$

We leave the proof as an exercise for the reader. □

We are ready to prove theorem 1:

Theorem 1: *The solutions of PVI_μ, corresponding to the solutions of Schlesinger equations (5.30) obtained in lemma 1, have the following behaviour for $x \to 0$ along a path $\Im\sigma \arg(x) = (\Re\sigma - \Sigma)\log|x| + b\Im\sigma$, $0 \le \Sigma \le \tilde{\sigma}$, contained in $D(\epsilon; \sigma, \theta_1, \theta_2)$:*

$$y(x) = a(x)\, x^{1-\sigma}(1 + O(|x|^\delta)), \qquad \sigma \neq 0$$

$$y(x) = sx\,(1 + O(|x|^\delta)), \qquad \sigma = 0$$

where $0 < \delta < 1$ is a small number, and $a(x)$ can be computed as a function of s. Namely

$$a(x) = -\frac{1}{4s}$$

along any path, except for the paths $\Im\sigma \arg(x) = \Re\sigma \log|x| + b\Im\sigma$, along which $x^\sigma = Ce^{i\alpha(x)}$ (C is a constant $=|x^\sigma|$ and $\alpha(x)$ is the (real) phase). In this case

$$a(x) = -\frac{1}{4s}\left(1 - 2\, s\, Ce^{i\alpha(x)} + s^2\, C^2 e^{2i\alpha(x)}\right) = O(1), \ \textit{for}\ x \to 0$$

Proof: $y(x)$ can be computed in terms of the $A_i(x)$ from $A(y(x), x)_{12} = 0$:

$$y(x) = \frac{x(A_0)_{12}}{(1+x)(A_0)_{12} + (A_x)_{12} + x(A_1)_{12}} \equiv \frac{x(A_0)_{12}}{x(A_0)_{12} - (A_1)_{12} + x(A_1)_{12}}$$

$$= -x\frac{(A_0)_{12}}{(A_1)_{12}}\,\frac{1}{1 - x(1 + \frac{(A_0)_{12}}{(A_1)_{12}})}$$

As a consequence of lemmas 1 and 2 it follows that $|x\,(A_1)_{12}| \le c\,|x|\,(1 + O(|x|^{1-\sigma_1}))$ and $|x\,(A_0)_{12}| \le c\,|x|^{1-\tilde{\sigma}}\,(1 + O(|x|^{1-\sigma_1}))$, where c is a constant. Then

$$y(x) = -x\frac{(A_0)_{12}}{(A_1)_{12}}\,(1 + O(|x|^{1-\tilde{\sigma}}))$$

From lemma 2 we find, for $\sigma \neq 0, \pm 2\mu$:

$$(A_0)_{12} = -r\frac{\sigma^2 - 4\mu^2}{32\mu}\left[\frac{x^{-\sigma}}{s}(1 + O(|x|^{1-\sigma_1})) + s\, x^\sigma\,(1 + O(|x|^{1-\sigma_1})) - 2(1 + O(|x|^{1-\sigma_1}))\right]$$

$$(A_1)_{12} = -r\frac{\sigma^2 - 4\mu^2}{8\mu}(1 + O(|x|^{1-\sigma_1}))$$

Then (recall that $\tilde{\sigma} < \sigma_1$)

$$y(x) = -\frac{x}{4}\left[\frac{x^{-\sigma}}{s}(1 + O(|x|^{1-\sigma_1})) + s\ x^{\sigma}\ (1 + O(|x|^{1-\sigma_1})) - 2(1 + O(|x|^{1-\sigma_1}))\right]\ (1 + O(|x|^{1-\sigma_1}))$$

Now $x \to 0$ along a path

$$\Im\sigma \arg(x) = (\Re\sigma - \Sigma)\log|x| + b\Im\sigma$$

for a suitable b and $0 \leq \Sigma \leq \tilde{\sigma}$. Along this path we rewrite x^σ in terms of its absolute value $|x^\sigma| = C|x|^\Sigma$ ($C = e^{-b\Im\sigma}$) and its real phase $\alpha(x)$

$$x^\sigma = C\ |x|^\Sigma\ e^{i\alpha(x)}, \quad \alpha(x) = \Re\sigma \arg(x) + \Im\sigma \ln|x|\Big|_{\Im\sigma \arg(x) = (\Re\sigma - \Sigma)\log|x| + b\Im\sigma}.$$

Then

$$y(x) = -\frac{x^{1-\sigma}}{4}\left[\frac{1}{s} - 2Ce^{i\alpha(x)}|x|^\Sigma(1 + O(|x|^{1-\sigma_1})) + sC^2e^{2i\alpha(x)}\ |x|^{2\Sigma}\ (1 + O(|x|^{1-\sigma_1}))\right]\ (1 + O(|x|^{1-\sigma_1}))$$

For $\Sigma \neq 0$ the above expression becomes

$$y(x) = -\frac{1}{4s}x^{1-\sigma}\ \ (1 + O(|x|^{1-\sigma_1}) + O(|x|^\Sigma))$$

We collect the two $O(..)$ contribution in $O(|x|^\delta)$ where $\delta = \min\{1 - \sigma_1, \Sigma\}$ is a small number between 0 and 1. We take the occasion here to remark that in the case of real $0 < \sigma < 1$, if we consider $x \to 0$ along a radial path (i.e. $\arg(x) = b$), then $\Sigma = \tilde{\sigma} = \sigma$ and thus:

$$y(x) = \begin{cases} \frac{1}{4s}x^{1-\sigma}(1 + O(|x|^\sigma)) \text{ for } 0 < \sigma < \frac{1}{2} \\ \frac{1}{4s}x^{1-\sigma}(1 + O(|x|^{1-\sigma_1})) \text{ for } \frac{1}{2} < \sigma < 1 \end{cases}$$

Finally, along the path with $\Sigma = 0$ we have:

$$y(x) = -\frac{x^{1-\sigma}}{4}\ \left(\frac{1}{s} - 2Ce^{i\alpha(x)} + s\ C^2e^{2i\alpha(x)}\right)\ \ (1 + O(|x|^{1-\sigma_1}))$$

We let the reader verify that also in the cases $\sigma = \pm 2\mu$ the behaviour of $y(x)$ is as above (use the matrices (5.35)and (5.37) – the reason why we disregard the matrices (5.36), (5.38) will be clarified at the end of the proof of theorem 2) and that for $\sigma = 0$

$$y(x) = s\ x\ (1 + O(|x|^{1-\sigma_1}))$$

For $\sigma = 0$, we recall that $0 < \sigma_1 < 1$ is arbitrarily small. □

In the proof of lemma 1 we imposed (5.34). Hence, the reader may observe that ϵ depends on $\tilde{\sigma}$, θ_1 and on $||A_0^0||$, $||A_x^0||$, $||A_1^0||$; thus it depends also on s ($\Rightarrow$ on a). The second inequality in (5.34) has been used in section 5.2 to construct the domain (5.11).

5.8 Proof of theorem 2

We are interested in lemma 1 when

$$f_{\mu\nu} = g_{\mu\nu} = \frac{b_\nu}{a_\mu - xb_\nu}, \quad h_{\mu\nu} = 0$$

$$a_\mu, b_\nu \in \mathbf{C}, \quad a_\mu \neq 0 \quad \forall\mu = 1, ..., n_1$$

Equations (5.31) are the isomonodromy deformation equations for the fuchsian system

$$\frac{dY}{dz} = \left[\sum_{\mu=1}^{n_1}\frac{A_\mu(\tilde{x})}{z - a_\mu} + \sum_{\nu=1}^{n_2}\frac{B_\nu(\tilde{x})}{z - xb_\nu}\right]\ Y$$

As a corollary of lemma 1, for a fundamental matrix solution $Y(z,x)$ of the fuchsian system the limits

$$\hat{Y}(z) := \lim_{x\to 0} Y(z,x)$$

$$\tilde{Y}(z) := \lim_{x\to 0} x^{-\Lambda} Y(xz,x)$$

exist when $x \to 0$ in $D(\epsilon;\sigma)$. They satisfy

$$\frac{d\hat{Y}}{dz} = \left[\sum_{\mu=1}^{n_1} \frac{A_\mu^0}{z-a_\mu} + \frac{\Lambda}{z}\right] \hat{Y}$$

$$\frac{d\tilde{Y}}{dz} = \sum_{\nu=1}^{n_2} \frac{B_\nu(x)}{z-b_\nu} \tilde{Y}$$

In our case, the last three systems reduce to

$$\frac{dY}{dz} = \left[\frac{A_0(x)}{z} + \frac{A_x(x)}{z-x} + \frac{A_1(x)}{z-1}\right] Y \tag{5.39}$$

$$\frac{d\hat{Y}}{dz} = \left[\frac{A_1^0}{z-1} + \frac{\Lambda}{z}\right] \hat{Y} \tag{5.40}$$

$$\frac{d\tilde{Y}}{dz} = \left[\frac{A_0^0}{z} + \frac{A_x^0}{z-1}\right] \tilde{Y} \tag{5.41}$$

Before taking the limit $x \to 0$, let us choose

$$Y(z,x) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-A_\infty} z^R, \qquad z\to\infty \tag{5.42}$$

and define as above

$$\hat{Y}(z) := \lim_{x\to 0} Y(z,x), \quad \tilde{Y}(z) := \lim_{x\to 0} \tilde{x}^{-\Lambda} Y(xz,x)$$

For the system (5.40) we choose a fundamental matrix solution normalized as follows

$$\hat{Y}_N(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{-A_\infty} z^R, \qquad z\to\infty \tag{5.43}$$

$$= (I+O(z))\, z^\Lambda\, \hat{C}_0, \qquad z\to 0$$

$$= \hat{G}_1(I+O(z-1))\,(z-1)^J\, \hat{C}_1, \qquad z\to 1$$

Where $\hat{G}_1^{-1} A_1^0 \hat{G}_1 = J$, $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $\hat{C}_0$, $\hat{C}_1$ are *connection matrices.* Note that R is the same of (5.42), since it is independent of x. For (5.41) we choose a fundamental matrix solution normalized as follows

$$\tilde{Y}_N(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^\Lambda, \qquad z\to\infty \tag{5.44}$$

$$= \tilde{G}_0(I+O(z))\, z^J\, \tilde{C}_0, \qquad z\to 0$$

$$= \tilde{G}_1(I+O(z-1))\,(z-1)^J\, \tilde{C}_1, \qquad z\to 1.$$

Here $\tilde{G}_0^{-1} A_0^0 \tilde{G}_0 = J$, $\tilde{G}_1^{-1} A_x^0 \tilde{G}_1 = J$.

Now we prove that

$$\hat{Y}(z) = \hat{Y}_N(z)$$

$$\tilde{Y}(z) = \tilde{Y}_N(z)\, \hat{C}_0 \tag{5.45}$$

The proof we give here uses the technique of the proof of Proposition 2.1. in [28]. The (isomonodromic) dependence of $Y(z,x)$ on x is given by

$$\frac{dY(z,x)}{dx} = -\frac{A_x(x)}{z-x} Y(z,x) := F(z,x) Y(z,x)$$

Then

$$Y(z,x) = \hat{Y}(z) + \int_0^x dx_1 \; F(z,x_1)Y(z,x_1)$$

The integration is on a path $\arg(x) = a\log|x| + b$, $a = \frac{\Re\sigma - \sigma^*}{\Im\sigma}$ $(0 \leq \sigma^* \leq \tilde{\sigma})$, or $\arg(x) = 0$ if $\Im\sigma = 0$. The path is contained in $D(\sigma)$ and joins 0 and x, like $L(x)$ in the proof of theorem 1 (figure 10). By successive approximations we have:

$$Y^{(1)}(z,x) = \hat{Y}(z) + \int_0^x dx_1 F(z,x_1)\hat{Y}(z)$$

$$Y^{(2)}(z,x) = \hat{Y}(z) + \int_0^x dx_1 F(z,x_1)Y^{(1)}(z,x_1)$$

$$\vdots$$

$$Y^{(n)}(z,x) = \hat{Y}(z) + \int_0^x dx_1 F(z,x_1)Y^{(n-1)}(z,x_1)$$

$$= \left[I + \int_0^x dx_1 \int_0^{x_1} dx_2 ... \int_0^{x_{n-1}} dx_n F(z,x_1)F(z,x_2)...F(z,x_n)\right] \; \hat{Y}(z)$$

Performing integration like in the proof of theorem 1 we evaluate $||Y^{(n)}(z,x) - Y^{(n-1)}(z,x)||$. Recall that $\hat{Y}(z)$ has singularities at $z = 0$, $z = x$. Thus, if $|z| > |x|$ we obtain

$$||Y^{(n)}(z,x) - Y^{(n-1)}(z,x)|| \leq \frac{MC^n}{\Pi_{m=1}^n (m - sigma^*)} |x|^{n-\sigma^*},$$

where M and C are constants. Then $Y^{(n)} = \hat{Y} + (Y^{(1)} - \hat{Y}) + ... + (Y^{(n)} - Y^{(n-1)})$ converges for $n \to \infty$ uniformly in z in every compact set contained in $\{z \mid |z| > |x|\}$ and uniformly in $x \in D(\sigma)$. We can exchange limit and integration, thus obtaining $Y(z,x) = \lim_{n\to\infty} Y^{(n)}(z,x)$. Namely

$$Y(z,x) = U(z,x)\hat{Y}(z),$$

$$U(z,x) = I + \sum_{n=1}^{\infty} \int_0^x dx_1 \int_0^{x_1} dx_2 ... \int_0^{x_{n-1}} dx_n F(z,x_1)F(z,x_2)...F(z,x_n)$$

being the convergence of the series uniformly in $x \in D(\sigma)$ and in z in every compact set contained in $\{z \mid |z| > |x|\}$. Of course

$$U(z,x) = I + O\left(\frac{1}{z}\right) \quad \text{for } x \to 0 \text{ and } Y(z,x) \to \hat{Y}(z)$$

But now observe that

$$\hat{Y}(z) = U(z,x)^{-1}Y(z,x) = \left(I + O\left(\frac{1}{z}\right)\right) \; \left(I + O\left(\frac{1}{z}\right)\right) z^{-A_\infty} z^R, \qquad z \to \infty$$

Then

$$\hat{Y}(z) \equiv \hat{Y}_N(z)$$

Finally, for $z \to 1$,

$$Y(z,x) = U(x,z)\hat{Y}_N(z) = U(x,z) \; \hat{G}_1(I + O(z-1))(z-1)^J \hat{C}_1$$

$$= G_1(x)(I + O(z-1))(z-1)^J \hat{C}_1$$

This implies

$$C_1 \equiv \hat{C}_1$$

and then

$$M_1 = \hat{C}_1^{-1} e^{2\pi i J} \hat{C}_1 \tag{5.46}$$

Here we have chosen a monodromy representation for (5.39) by fixing a base-point and a basis in the fundamental group of $\mathbf{P}^1$ as in figure 5.15. M_0, M_1, M_x, M_∞ are the monodromy matrices for the solution (5.42) corresponding to the loops γ_i $i = 0, x, 1, \infty$. $M_\infty M_1 M_x M_0 = I$. The result (5.46) may also be proved simply observing that M_1 becomes $\hat{M}_1$ as $x \to 0$ in $D(\sigma)$ because the system (5.40) is obtained from (5.39) when $z = x$ and $z = 0$ merge and the singular point $z = 1$ does not move. x may converge to 0 along spiral paths (figure 5.15). We recall that the braid $\beta_{i,i+1}$ changes the monodromy matrices of $\frac{dY}{dz} = \sum_{i=1}^{n} \frac{A_i(u)}{z-u_i} Y$ according to $M_i \mapsto M_{i+1}$, $M_{i+1} \mapsto M_{i+1} M_i M_{i+1}^{-1}$, $M_k \mapsto M_k$ for any $k \neq i, i+1$; therefore, if $\arg(x)$ increases of 2π as $x \to 0$ in (5.39) we have

$$M_0 \mapsto M_x, \quad M_x \mapsto M_x M_0 M_x^{-1}, \quad M_1 \mapsto M_1$$

If follows that M_1 does not change and then

$$M_1 \equiv \hat{M}_1 = \hat{C}_1^{-1} e^{2\pi i J} \hat{C}_1 \tag{5.47}$$

where $\hat{M}_1$ is the monodromy matrix of (5.43) for the loop $\hat{\gamma}_1$ in the basis of figure 5.15.

Now we turn to $\tilde{Y}(z)$. Let $\tilde{Y}(z,x) := x^{-\Lambda} Y(xz, x)$, and by definition $\tilde{Y}(z,x) \to \tilde{Y}(z)$ as $x \to 0$. In this case

$$\frac{d\tilde{Y}(z,x)}{dx} = \left[\frac{x^{-\Lambda}(A_0 + A_x)x^\Lambda - \Lambda}{x} + \frac{x^{-\Lambda} A_1 x^\Lambda}{x - \frac{1}{z}} \right] \tilde{Y}(z,x) := \tilde{F}(z,x)\tilde{Y}(z,x)$$

Proceeding by successive approximations as above we get

$$\tilde{Y}(z,x) = V(z,x)\tilde{Y}(z),$$

$$V(z,x) = I + \sum_{n=1}^{\infty} \int_0^x dx_1 ... \int_0^{x_{n-1}} dx_n \tilde{F}(z,x_1)...\tilde{F}(z,x_n) \to I \text{ for } x \to 0$$

uniformly in $x \in D(\sigma)$ and in z in every compact subset of $\{z \mid |z| < \frac{1}{|x|}\}$.

Let's investigate the behaviour of $\tilde{Y}(z)$ as $z \to \infty$ and compare it to the behaviour of $\tilde{Y}_N(z)$. First we note that

$$x^{-\Lambda} \hat{Y}_N(xz) = x^{-\Lambda}(I + O(xz))(xz)^\Lambda \hat{C}_0 \to z^\Lambda \hat{C}_0 \text{ for } x \to 0.$$

Then

$$\left[x^{-\Lambda} Y(xz,x)\right] \left[x^{-\Lambda} \hat{Y}_N(xz)\right]^{-1} = x^{-\Lambda} U(xz,x) x^\Lambda \to \tilde{Y}(z) \hat{C}_0^{-1} z^{-\Lambda}.$$

On the other hand, from the properties of $U(z,x)$ we know that $x^{-\Lambda} U(xz,x) x^\Lambda$ is holomorphic in every compact subset of $\{z \mid |z| > 1\}$ and $x^{-\Lambda} U(xz,x) x^\Lambda = I + O\left(\frac{1}{z}\right)$ as $z \to \infty$. Thus

$$\tilde{U}(z) := \lim_{x \to 0} x^{-\Lambda} U(xz,x) x^\Lambda$$

exists uniformly in every compact subset of $\{z \mid |z| > 1\}$ and

$$\tilde{U}(z) = I + O\left(\frac{1}{z}\right), \quad z \to \infty$$

Then

$$\tilde{Y}(z) = \tilde{U}(z) z^\Lambda \hat{C}_0 \equiv \tilde{Y}_N(z) \hat{C}_0,$$

as we wanted to prove. Finally, the above result implies

$$Y(z,x) = x^\Lambda V(\frac{z}{x}, x) \tilde{Y}_N(\frac{z}{x}) \hat{C}_0$$

$$= \begin{cases} x^\Lambda V(\frac{z}{x}, x) \tilde{G}_0 (I + O(z/x)) x^{-J} z^J \tilde{C}_0 \hat{C}_0 = G_0(x)(I + O(z)) z^J \tilde{C}_0 \hat{C}_0, & z \to 0 \\ x^\Lambda V(\frac{z}{x}, x) \tilde{G}_1 \left(O\left(\frac{z}{x} - 1\right)\right) \left(\frac{z}{x} - 1\right)^J \tilde{C}_1 \hat{C}_0 = G_x(x)(I + O(z-x))(z-x)^J \tilde{C}_1 \hat{C}_0, & z \to x \end{cases}$$

Let $\tilde{M}_0$, $\tilde{M}_1$ denote the monodromy matrices of $\tilde{Y}_N(z)$ in the basis of figure 13. The above result implies:

$$M_0 = \hat{C}_0^{-1} \tilde{C}_0^{-1} e^{2\pi i J} \tilde{C}_0 \hat{C}_0 = \hat{C}_0^{-1} \tilde{M}_0 \hat{C}_0 \tag{5.48}$$

$$M_x = \hat{C}_0^{-1} \tilde{C}_1^{-1} e^{2\pi i J} \tilde{C}_1 \hat{C}_0 = \hat{C}_0^{-1} \tilde{M}_1 \hat{C}_0 \tag{5.49}$$

The same result may be obtained observing that from

$$\frac{d(x^{-\Lambda}Y(xz,x))}{dz}=\left[\frac{x^{-\Lambda}A_0x^{\Lambda}}{z}+\frac{x^{-\Lambda}A_xx^{\Lambda}}{z-1}+\frac{x^{-\Lambda}A_1x^{\Lambda}}{z-\frac{1}{x}}\right]x^{-\Lambda}Y(xz,x) \tag{5.50}$$

we obtain the system (5.41) as $z=\frac{1}{x}$ and $z=\infty$ merge (figure 5.15). The singularities $z=0$, $z=1$, $z=1/x$ of (5.50) correspond to $z=0$, $z=x$, $z=1$ of (5.39). The poles $z=0$ and $z=1$ of (5.50) do not move as $x\to 0$ and $\frac{1}{x}$ converges to ∞, in general along spirals. At any turn of the spiral the system (5.50) has new monodromy matrices according to the action of the braid group

$$M_1\mapsto M_\infty,\quad M_\infty\mapsto M_\infty M_1 M_\infty^{-1}$$

but

$$M_0\mapsto M_0,\quad M_x\mapsto M_x$$

Hence, the limit $\tilde{Y}(z)$ still has monodromy M_0 and M_x at $z=0,x$. Since $\tilde{Y}=\tilde{Y}_N\hat{C}_0$ we conclude that M_0 and M_x are (5.48) and (5.49).

In order to find the parameterization $y(x;\sigma,a)$ in terms of (x_0,x_1,x_∞) we have to compute the monodromy matrices M_0, M_1, M_∞ in terms of σ and a, and then take the traces of their products. In order to do this we use the formulae (5.47), (5.48),(5.49). In fact, the matrices $\tilde{M}_i$ $(i=0,1)$ and $\hat{M}_1$ can be computed explicitly because a 2×2 fuchsian system with three singular points can be reduced to the hypergeometric equation, whose monodromy is completely known.

Lemma 3: *The Gauss hypergeometric equation*

$$z(1-z)\,\frac{d^2y}{dz^2}+[\gamma_0-z(\alpha_0+\beta_0+1)]\,\frac{dy}{dz}-\alpha_0\beta_0\;y=0 \tag{5.51}$$

is equivalent to the system

$$\frac{d\Psi}{dz}=\left[\frac{1}{z}\begin{pmatrix}0 & 0\\ -\alpha_0\beta_0 & -\gamma_0\end{pmatrix}+\frac{1}{z-1}\begin{pmatrix}0 & 1\\ 0 & \gamma_0-\alpha_0-\beta_0\end{pmatrix}\right]\;\Psi \tag{5.52}$$

where $\Psi=\begin{pmatrix}y\\ (z-1)\frac{dy}{dz}\end{pmatrix}$.

Lemma 4: *Let B_0 and B_1 be matrices of eigenvalues $0,1-\gamma$, and $0,\gamma-\alpha-\beta-1$ respectively, such that*

$$B_0+B_1=\ \mathrm{diag}(-\alpha,-\beta),\quad \alpha\neq\beta$$

Then

$$B_0=\begin{pmatrix}\frac{\alpha(1+\beta-\gamma)}{\alpha-\beta} & \frac{\alpha(\gamma-\alpha-1)}{\alpha-\beta}\;r\\ \frac{\beta(\beta+1-\gamma)}{\alpha-\beta}\;\frac{1}{r} & \frac{\beta(\gamma-\alpha-1)}{\alpha-\beta}\end{pmatrix}$$

$$B_1=\begin{pmatrix}\frac{\alpha(\gamma-\alpha-1)}{\alpha-\beta} & -(B_0)_{12}\\ -(B_0)_{21} & \frac{\beta(\beta+1-\gamma)}{\alpha-\beta}\end{pmatrix}$$

for any $r\neq 0$.

We leave the proof as an exercise. The following lemma connects lemmas 3 and 4:

Lemma 5: *The system (5.52) with*

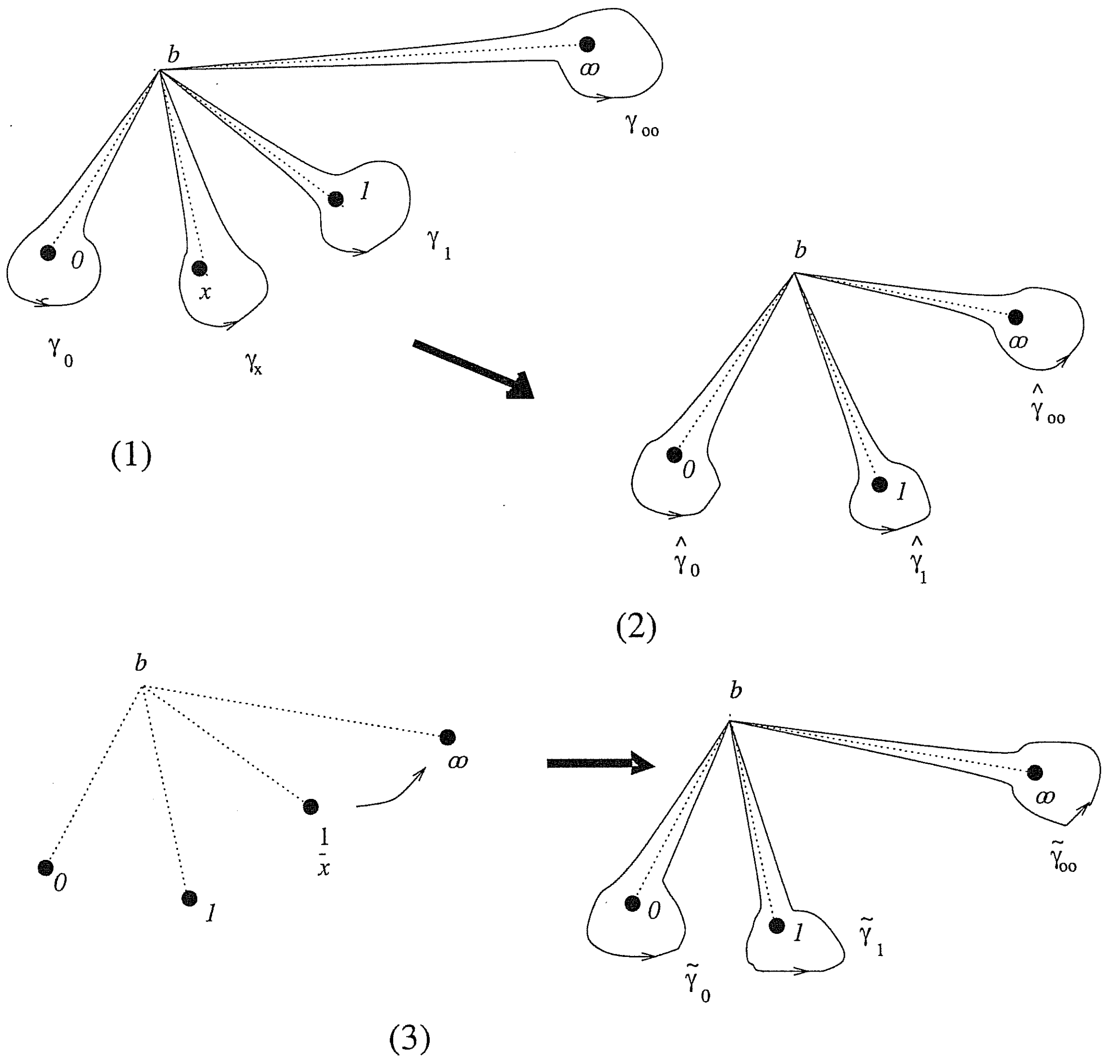
$$\alpha_0=\alpha,\quad \beta_0=\beta+1,\quad \gamma_0=\gamma,\quad \alpha\neq\beta$$

is gauge-equivalent to the system

$$\frac{dX}{dz}=\left[\frac{B_0}{z}+\frac{B_1}{z-1}\right]\;X \tag{5.53}$$

where B_0, B_1 are given in lemma 4. This means that there exists a matrix

$$G(z):=\begin{pmatrix}1 & 0\\ \frac{\alpha-\beta}{(\beta+1-\gamma)\beta}\left[\alpha\;z+\frac{\alpha(\beta+1-\gamma)}{\alpha-\beta}\right]\;\frac{1}{r} & z\;\frac{\alpha-\beta}{(\beta+1-\gamma)\beta}\;\frac{1}{r}\end{pmatrix}$$



(1): Branch cuts and loops for the fuchsian system associated with PVI_μ

(2): Branch cuts and loops when $x \to 0$

(3): Branch cuts and loops for the rescaled system before and after $x \to 0$

Figure 5.15:

such that $X(z) = G(z)\ \Psi(z)$. It follows that (5.53) and the corresponding hypergeometric equation (5.51) have the same fuchsian singularities 0,1,∞ and the same monodromy group.

Proof: By direct computation. □

Note that the form of $G(z)$ ensures that if y_1, y_2 are independent solutions of the hypergeopmetric equation, then a fundamental matrix of (5.53) may be chosen to be $X(z) = \begin{pmatrix} y_1(z) & y_2(z) \\ * & * \end{pmatrix}$

Now we compute the monodromy matrices for the systems (5.40), (5.41) by reduction to an hypergeometric equation. We first study the case $\sigma \notin \mathbf{Z}$. Let us start with (5.40). With the gauge

$$Y^{(1)}(z) := z^{-\frac{\sigma}{2}}\ \hat{Y}(z)$$

we transform (5.40) in

$$\frac{dY^{(1)}}{dz} = \left[\frac{A_1^0}{z-1} + \frac{\Lambda - \frac{\sigma}{2}I}{z}\right] Y^{(1)} \tag{5.54}$$

We identify the matrices B_0, B_1 with $\Lambda - \frac{\sigma}{2}I$ and A_1^0 with eigenvalues $0, -\sigma$ and $0, 0$ respectively. Moreover $A_1^0 + \Lambda - \frac{\sigma}{2}I = \text{diag}(-\mu - \frac{\sigma}{2}, \mu - \frac{\sigma}{2})$. Thus:

$$\alpha = \mu + \frac{\sigma}{2}, \quad \beta = -\mu + \frac{\sigma}{2}, \quad \gamma = \sigma + 1$$

$$\alpha - \beta = 2\mu \neq 0 \quad \text{by hypothesis}$$

The parameters of the correspondent hypergeometric equation are

$$\begin{cases} \alpha_0 = \mu + \frac{\sigma}{2} \\ \beta_0 = 1 - \mu + \frac{\sigma}{2} \\ \gamma_0 = \sigma + 1 \end{cases}$$

From them we deduce the nature of two linearly independent solutions at $z = 0$. Since $\gamma_0 \notin \mathbf{Z}$ ($\sigma \notin \mathbf{Z}$) the solutions are expressed in terms of hypergeometric functions. On the other hand, the effective parameters at $z = 1$ and $z = \infty$ are respectively:

$$\begin{cases} \alpha_1 := \alpha_0 = \mu + \frac{\sigma}{2} \\ \beta_1 := \beta_0 = 1 - \mu + \frac{\sigma}{2} \\ \gamma_1 := \alpha_0 + \beta_0 - \gamma_0 + 1 = 1 \end{cases}$$

$$\begin{cases} \alpha_\infty := \alpha_0 = \mu + \frac{\sigma}{2} \\ \beta_\infty := \alpha_0 - \gamma_0 + 1 = \mu - \frac{\sigma}{2} \\ \gamma_\infty = \alpha_0 - \beta_0 + 1 = 2\mu \end{cases}$$

Since $\gamma_1 = 1$, at least one solution has a logarithmic singularity at $z = 1$. Also note that $\gamma_\infty = 2\mu$, therefore logarithmic singularities appear at $z = \infty$ if $2\mu \in \mathbf{Z}\backslash\{0\}$.

For the derivations which follows, we use the notations of the fundamental paper by Norlund [41]. To derive the connection formulae we use the paper of Norlund when logarithms are involved. Otherwise, in the generic case, any textbook of special functions (like [34]) may be used.

First case: $\alpha_0, \beta_0 \notin \mathbf{Z}$. This means

$$\sigma \neq \pm 2\mu + 2m, \qquad m \in \mathbf{Z}$$

We can choose the following independent solutions of the hypergeometric equation:

At $z = 0$

$$y_1^{(0)}(z) = F(\alpha_0, \beta_0, \gamma_0; z)$$

$$y_2^{(0)}(z) = z^{1-\gamma_0}\ F(\alpha_0 - \gamma_0 + 1, \beta_0 - \gamma_0 + 1, 2 - \gamma_0; z) \tag{5.55}$$

where $F(\alpha, \beta, \gamma; z)$ is the well known hypergeometric function (see [41]).

At $z = 1$

$$y_1^{(1)}(z) == F(\alpha_1, \beta_1, \gamma_1; 1 - z)$$

$$y_2^{(1)}(z) = g(\alpha_1, \beta_1, \gamma_1; 1-z)$$

Here $g(\alpha, \beta, \gamma; z)$ is a logarithmic solution introduced in [41], and $\gamma \equiv \gamma_1 = 1$.

At $z = \infty$, we consider first the case $2\mu \notin \mathbf{Z}$, while the resonant case will be considered later. Two independent solutions are:

$$y_1^{(\infty)} = z^{-\alpha_0}\ F(\alpha_\infty, \beta_\infty, \gamma_\infty; \frac{1}{z})$$

$$y_2^{(\infty)} = z^{-\beta_0}\ F(\beta_0, \beta_0 - \gamma_0 + 1, \beta_0 - \alpha_0 + 1; \frac{1}{z})$$

Then, from the connection formulas between $F(...;z)$ and $g(...;z)$ of [34] and [41] we derive

$$[y_1^{(\infty)}, y_2^{(\infty)}] = [y_1^{(0)}, y_2^{(0)}]\ C_{0\infty}$$

$$C_{0\infty} = \begin{pmatrix} e^{-i\pi\alpha_0}\frac{\Gamma(1+\alpha_0-\beta_0)\Gamma(1-\gamma_0)}{\Gamma(1-\beta_0)\Gamma(1+\alpha_0-\gamma_0)} & e^{-i\pi\beta_0}\frac{\Gamma(1+\beta_0-\alpha_0)\Gamma(1-\gamma_0)}{\Gamma(1-\alpha_0)\Gamma(1+\beta_0-\gamma_0)} \\ e^{i\pi(\gamma_0-\alpha_0-1)}\frac{\Gamma(1+\alpha_0-\beta_0)\Gamma(\gamma_0-1)}{\Gamma(\alpha_0)\Gamma(\gamma_0-\beta_0)} & e^{i\pi(\gamma_0-\beta_0-1)}\frac{\Gamma(1+\beta_0-\alpha_0)\Gamma(\gamma_0-1)}{\Gamma(\beta_0)\Gamma(\gamma_0-\alpha_0)} \end{pmatrix}$$

$$[y_1^{(0)}, y_2^{(0)}] = [y_1^{(1)}, y_2^{(1)}]\ C_{01}$$

$$C_{01} = \begin{pmatrix} 0 & -\frac{\pi\sin(\pi(\alpha_0+\beta_0))}{\sin(\pi\alpha_0)\sin(\pi\beta_0)}\frac{\Gamma(2-\gamma_0)}{\Gamma(1-\alpha_0)\Gamma(1-\beta_0)} \\ -\frac{\Gamma(\gamma_0)}{\Gamma(\gamma_0-\alpha_0)\Gamma(\gamma_0-\beta_0)} & -\frac{\Gamma(2-\gamma_0)}{\Gamma(1-\alpha_0)\Gamma(1-\beta_0)} \end{pmatrix}$$

We observe that

$$Y^{(1)}(z) = \left(I + \frac{F}{z} + O\left(\frac{1}{z^2}\right)\right)\ z^{\mathrm{diag}(-\mu-\frac{\sigma}{2}, \mu-\frac{\sigma}{2})}, \quad z \to \infty$$

$$= \hat{G}_0(I + O(z))\ z^{\mathrm{diag}(0,-\sigma)}\ \hat{G}_0^{-1}\hat{C}_0, \quad z \to 0$$

$$= \hat{G}_1(I + O(z-1))\ (z-1)^J\ \hat{C}_1, \quad z \to 1$$

where $\hat{G}_0 \equiv T$ of lemma 2; namely $\hat{G}_0^{-1}\Lambda\hat{G}_0 = \mathrm{diag}(\frac{\sigma}{2}, -\frac{\sigma}{2})$. By direct substitution in the differential equation we compute the coefficient F

$$F = -\begin{pmatrix} (A_1^0)_{11} & \frac{(A_1^0)_{12}}{1-2\mu} \\ \frac{(A_1^0)_{21}}{1+2\mu} & (A_1^0)_{22} \end{pmatrix}, \quad \text{where } A_1^0 = \frac{\sigma^2 - (2\mu)^2}{8\mu}\begin{pmatrix} 1 & -r \\ \frac{1}{r} & -1 \end{pmatrix}$$

Thus, from the asymptotic behaviour of the hypergometric function ($F(\alpha, \beta, \gamma; \frac{1}{z}) \sim 1$, $z \to \infty$) we derive

$$Y^{(1)}(z) = \begin{pmatrix} y_1^{(\infty)}(z) & r\frac{\sigma^2-(2\mu)^1}{8\mu(1-2\mu)}\ y_2^{(\infty)} \\ * & * \end{pmatrix}$$

From

$$Y^{(1)}(z) \sim \begin{pmatrix} 1 & z^{-\sigma} \\ * & * \end{pmatrix}\ \hat{G}_0^{-1}\hat{C}_0, \quad z \to 0 \tag{5.56}$$

we derive

$$Y^{(1)}(z) = \begin{pmatrix} y_1^{(0)}(z) & y_2^{(0)}(z) \\ * & * \end{pmatrix}\hat{G}_0^{-1}\hat{C}_0$$

Finally, observe that $\hat{G}_1 = \begin{pmatrix} a & \frac{a}{\omega} + br \\ \frac{a}{r} & b \end{pmatrix}$ for arbitrary $a, b \in \mathbf{C}$, $a \neq 0$, and $\omega := \frac{\sigma^2-(2\mu)^2}{8\mu}$. We recall that $y_2^{(1)} = g(\alpha_1, \beta_1, 1; 1-z) \sim \psi(\alpha_1) + \psi(\beta_1) - 2\psi(1) - i\pi + \log(z-1)$, $|\arg(1-z)| < \pi$, as $z \to 1$. We can choose $a = 1$ and a suitable b, in such a way that the asymptotic behaviour of $Y^{(1)}$ for $z \to 1$ is precisely realized by

$$Y^{(1)}(z) = \begin{pmatrix} y_1^{(1)}(z) & y_2^{(1)}(z) \\ * & * \end{pmatrix}\ \hat{C}_1$$

Therefore we conclude that the connection matrices are:

$$\hat{C}_0 = \hat{G}_0\begin{pmatrix} (C_{0\infty})_{11} & r\ \frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{12} \\ (C_{0\infty})_{21} & r\ \frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{22} \end{pmatrix}$$

$$\hat{C}_1 = C_{01}\ (\hat{G}_0^{-1}\hat{C}_0) = C_{01}\begin{pmatrix} (C_{0\infty})_{11} & r\ \frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{12} \\ (C_{0\infty})_{21} & r\ \frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{22} \end{pmatrix}$$

It's now time to consider the more complicated resonant case $2\mu \in \mathbf{Z}\backslash\{0\}$. The behaviour of $Y^{(1)}$ at $z = \infty$ is

$$Y^{(1)}(z) = \left(I + \frac{F}{z} + O\left(\frac{1}{z^2}\right)\right)\ z^{\mathrm{diag}(-\mu-\frac{\sigma}{2},\mu-\frac{\sigma}{2})}z^R$$

$$R = \begin{pmatrix} 0 & R_{12} \\ 0 & 0 \end{pmatrix}, \qquad \text{for } \mu = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots$$

$$R = \begin{pmatrix} 0 & 0 \\ R_{21} & 0 \end{pmatrix}, \qquad \text{for } \mu = -\frac{1}{2}, -1, -\frac{3}{2}, -2, -\frac{5}{2}, \dots$$

and the entry R_{12} is determined by the entries of A_1^0. For example, if $\mu = \frac{1}{2}$ we can compute $R_{12} = (A_1^0)_{12} = -r\frac{\sigma^2-1}{4}$ (and F_{12} arbitrary); if $\mu = -\frac{1}{2}$ we have $R_{21} = (A_1^0)_{21} = -\frac{1}{r}\frac{\sigma^2-1}{4}$ (and F_{21} arbitrary); if $\mu = 1$ we have $R_{12} = -r\frac{\sigma^2(\sigma^2-4)}{32}$.

Since $\sigma \notin \mathbf{Z}$, $R \neq 0$. This is true for any $2\mu \in \mathbf{Z}\backslash\{0\}$. Note that the R computed here coincides (by isomonodromicity) to the R of the system(5.39).

There is a logarithmic solution at ∞. Only $C_{0\infty}$ and thus $\hat{C}_0$ and $\hat{C}_1$ change with respect to the non-resonant case. We will see in a while that such matrices disappear in the computation of $\mathrm{tr}(M_iM_j)$, $i,j = 0,1,x$. Therefore, it is not necessary to know them explicitly. Actually, it was not necessary to compute them also in the non resonant case, the only important matrix to know being C_{01}, which is not affected by resonance of μ. This is the reason why the formulae of theorem 2 hold true also in the resonant case.

Second case: $\alpha_0, \beta_0 \in \mathbf{Z}$, namely

$$\sigma = \pm 2\mu + 2m, \qquad m \in \mathbf{Z}$$

The formulae are almost identical to the first case, but C_{01} changes. To see this, we need to distinguish four cases.

i) $\sigma = 2\mu + 2m$, $m = -1, -2, -3, \dots$. We choose

$$y_2^{(1)}(z) = g_0(\alpha_1, \beta_1, \gamma_1; 1 - z)$$

Here $g_0(z)$ is another logarithmic solution of [41]. Thus

$$C_{01} = \begin{pmatrix} \frac{\Gamma(-m)\Gamma(-2\mu-m+1)}{\Gamma(-2\mu-2m)} & 0 \\ 0 & -\frac{\Gamma(1-2\mu-2m)}{\Gamma(1-m-2\mu)\Gamma(-m)} \end{pmatrix}$$

As usual, the matrix is computed from the connection formulas between the hypergeometric functions and g_0 that the reader can find in [41].

ii) $\sigma = 2\mu + 2m$, $m = 0, 1, 2, \dots$. We choose

$$y_1^{(2)} = g(\alpha_1, \beta_1, \gamma_1; 1 - z)$$

Thus

$$C_{01} = \begin{pmatrix} 0 & \frac{\Gamma(m+1)\Gamma(2\mu+m)}{\Gamma(2\mu+2m)} \\ -\frac{\Gamma(2\mu+2m+1)}{\Gamma(2\mu+m)\Gamma(m+1)} & 0 \end{pmatrix}$$

iii) $\sigma = -2\mu + 2m$, $m = 0, -1, -2, \dots$. We choose

$$y_2^{(1)}(z) = g_0(\alpha_1, \beta_1, \gamma_1; 1 - z)$$

Thus

$$C_{01} = \begin{pmatrix} \frac{\Gamma(1-m)\Gamma(2\mu-m)}{\Gamma(2\mu-2m)} & 0 \\ 0 & -\frac{\Gamma(1+2\mu-2m)}{\Gamma(2\mu-m)\Gamma(1-m)} \end{pmatrix}$$

iv) $\sigma = -2\mu + 2m$, $m = 1, 2, 3, \ldots$. We choose

$$y_2^{(1)}(z) = g(\alpha_1, \beta_1, \gamma_1; 1 - z)$$

Thus

$$C_{01} = \begin{pmatrix} 0 & \frac{\Gamma(m)\Gamma(m+1-2\mu)}{\Gamma(2m-2\mu)} \\ -\frac{\Gamma(2m+1-2\mu)}{\Gamma(m+1-2\mu)\Gamma(m)} & 0 \end{pmatrix}$$

Note that this time $F = \begin{pmatrix} 0 & \frac{r}{1-2\mu} \\ 0 & 0 \end{pmatrix}$ in the case $\sigma = \pm 2\mu$ (i.e. $m = 0$) because A_1^0 has a special form in this case. Then in $\hat{C}_0$ the elements $\frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{12}$, $\frac{\sigma^2-(2\mu)^2}{8\mu(1-2\mu)}(C_{0\infty})_{22}$ must be substituted, for $m = 0$, with $\frac{1}{1-2\mu}(C_{0\infty})_{12}$, $\frac{1}{1-2\mu}(C_{0\infty})_{22}$.

We turn to the system (5.41). Let $\tilde{Y}$ be the fundamental matrix (5.44). With the gauge

$$Y^{(2)}(z) := \hat{G}_0^{-1}\left(\tilde{Y}(z)\hat{G}_0\right)$$

we have

$$\frac{dY^{(2)}}{dz} = \left[\frac{\tilde{B}_0}{z} + \frac{\tilde{B}_1}{z-1}\right] Y^{(2)}$$

$$\tilde{B}_0 = \hat{G}^{-1} A_0^0 \hat{G}_0 = \begin{pmatrix} \frac{\sigma}{4} & \frac{\sigma}{4}s \\ -\frac{\sigma}{4s} & -\frac{\sigma}{4} \end{pmatrix}$$

$$\tilde{B}_1 = \hat{G}^{-1} A_x^0 \hat{G}_0 = \begin{pmatrix} \frac{\sigma}{4} & -\frac{\sigma}{4}s \\ \frac{\sigma}{4s} & -\frac{\sigma}{4} \end{pmatrix}$$

This time then

$$\begin{cases} \alpha_0 = -\frac{\sigma}{2} \\ \beta_0 = \frac{\sigma}{2} + 1 \\ \gamma_0 = 1 \end{cases}$$

$$\begin{cases} \alpha_1 = -\frac{\sigma}{2} \\ \beta_1 = \frac{\sigma}{2} + 1 \\ \gamma_1 = 1 \end{cases}$$

$$\begin{cases} \alpha_\infty = -\frac{\sigma}{2} \\ \beta_\infty = \frac{\sigma}{2} \\ \gamma_\infty = \sigma \end{cases}$$

If follows that both at $z = 0$ and $z = 1$ there are logarithmic solutions. We skip all the derivation of the connection formulae, which is done as in the previous cases, with some more technical complications. Before giving the results we observe that

$$Y^{(2)}(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{\mathrm{diag}(\frac{\sigma}{2}, -\frac{\sigma}{2})}, \quad z \to \infty$$

$$= \hat{G}_0^{-1}\tilde{G}_0 \, (1 + O(z)) z^J \, C_0', \quad z \to 0$$

$$= \hat{G}_0^{-1}\tilde{G}_1 \, (1 + O(z-1))(z-1)^J \, C_1', \quad z \to 1$$

where

$$C_i' := \tilde{C}_i \hat{G}_0, \qquad i = 0, 1$$

Then

$$\tilde{M}_0 = \hat{G}_0 \, (C_0')^{-1} \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} C_0' \, \hat{G}_0^{-1}$$

$$\tilde{M}_1 = \hat{G}_0 \, (C_1')^{-1} \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix} C_1' \, \hat{G}_0^{-1}$$

Then, the connection problem may be solved computing C_i'. The result is

$$C_0' = \begin{pmatrix} (C_{0\infty}')_{11} & \frac{\sigma}{\sigma+1}\frac{s}{4}(C_{0\infty}')_{12} \\ (C_{0\infty}')_{21} & \frac{\sigma}{\sigma+1}\frac{s}{4}(C_{0\infty}')_{22} \end{pmatrix}$$

$$C_1' = C_{01}' C_0'$$

where

$$(C_{0\infty}')^{-1} = \begin{pmatrix} \frac{\Gamma(\beta_0-\alpha_0)}{\Gamma(\beta_0)\Gamma(1-\alpha_0)} e^{i\pi\alpha_0} & 0 \\ \frac{\Gamma(\alpha_0-\beta_0)}{\Gamma(\alpha_0)\Gamma(1-\beta_0)} e^{i\pi\beta_0} & -\frac{\Gamma(1-\alpha_0)\Gamma(\beta_0)}{\Gamma(\beta_0-\alpha_0+1)} e^{i\pi\beta_0} \end{pmatrix}$$

$$C_{01}' = \begin{pmatrix} 0 & -\frac{\pi}{\sin(\pi\alpha_0)} \\ -\frac{\sin(\pi\alpha_0)}{\pi} & -e^{-i\pi\alpha_0} \end{pmatrix}$$

The case $\sigma \in \mathbf{Z}$ interests us only if $\sigma = 0$ (otherwise $\sigma \notin \Omega$). We observe that the system (5.40) is precisely the system for $Y^{(2)}(z)$ with the substitution $\sigma \mapsto -2\mu$. In the formulae for x_i^2, $i = 0, 1, \infty$ we only need C_{01}, which is obtained from C_{01}' with $\alpha_0 = \mu$.

As for the system (5.41), the gauge $Y^{(2)} = \hat{G}_0^{-1}\tilde{Y}\hat{G}_0$ yields $\tilde{B}_0 = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}$, $\tilde{B}_1 = \begin{pmatrix} 0 & 1-s \\ 0 & 0 \end{pmatrix}$. Here $\hat{G}_0$ is the matrix such that $\hat{G}_0^{-1}\Lambda\hat{G}_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The behaviour of $Y^{(2)}(z)$ is now:

$$Y^{(2)}(z) = (I + O(\frac{1}{z}))\ z^J \qquad z \to \infty$$

$$= \tilde{\tilde{G}}_0^{-1}\ (1 + O(z))z^J\ C_0', \quad z \to 0$$

$$= \tilde{\tilde{G}}_1\ (1 + O(z-1))(z-1)^J\ C_1', \quad z \to 1$$

Here $\tilde{\tilde{G}}_i$ is the matrix that puts $\tilde{B}_i$ in Jordan form, for $i = 0, 1$. $Y^{(2)}$ can be computed explicitly:

$$Y^{(2)}(z) = \begin{pmatrix} 1 & s\log(z) + (1-s)\log(z-1) \\ 0 & 1 \end{pmatrix}$$

If we choose $\tilde{\tilde{G}}_0$ =diag$(1, 1/s)$, then

$$C_0' = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$$

In the same way we find

$$C_1' = \begin{pmatrix} 1 & 0 \\ 0 & 1-s \end{pmatrix}$$

To prove theorem 2 it is now enough to compute

$$2 - x_0^2 = \operatorname{tr}(M_0 M_x) \equiv \operatorname{tr}(e^{2\pi i J}(C_{01}')^{-1}e^{2\pi i J}C_{01}')$$

$$2 - x_1^2 = \operatorname{tr}(M_x M_1) \equiv \operatorname{tr}((C_1')^{-1}e^{2\pi i J}C_1' C_{01}^{-1}e^{2\pi i J}C_{01})$$

$$2 - x_\infty^2 = \operatorname{tr}(M_0 M_1) \equiv \operatorname{tr}((C_0')^{-1}e^{2\pi i J}C_0' C_{01}^{-1}e^{2\pi i J}C_{01})$$

Note the remarkable simplifications obtained from the cyclic property of the trace (for example, $\hat{C}_0$, $\hat{C}_1$ and $\hat{G}_0$ disappear). The fact that $\hat{C}_0$ and $\hat{C}_1$ disappear implies that the formulae of theorem 2 are derived for any $\mu \neq 0$, including the resonant cases. Thus, the connection formulae in the resonant case $2\mu \in \mathbf{Z}\backslash\{0\}$ are the same of the non-resonant case. The final result of the computation of the traces is:

I) Generic case:

$$\begin{cases} 2(1 - cos(\pi\sigma)) = x_0^2 \\ \frac{1}{f(\sigma,\mu)}\left(2 + F(\sigma,\mu)\ s + \frac{1}{F(\sigma,\mu)\ s}\right) = x_1^2 \\ \frac{1}{f(\sigma,\mu)}\left(2 - F(\sigma,\mu)e^{-i\pi\sigma}\ s - \frac{1}{F(\sigma,\mu)e^{-i\pi\sigma}\ s}\right) = x_\infty^2 \end{cases} \tag{5.57}$$

where

$$f(\sigma,\mu)=\frac{2\cos^2(\frac{\pi}{2}\sigma)}{\cos(\pi\sigma)-\cos(2\pi\mu)}\equiv\frac{4-x_0^2}{x_1^2+x_\infty^2-x_0x_1x_\infty},\qquad F(\sigma,\mu)=f(\sigma,\mu)\frac{16^\sigma\Gamma(\frac{\sigma+1}{2})^4}{\Gamma(1-\mu+\frac{\sigma}{2})^2\Gamma(\mu+\frac{\sigma}{2})^2}$$

II) $\sigma\in 2\mathbf{Z}$, $x_0=0$.

$$\begin{cases}2(1-cos(\pi\sigma))=0\\ 4\sin^2(\pi\mu)\ (1-a)=x_1^2\\ 4\sin^2(\pi\mu)\ a=x_\infty^2\end{cases}$$

III) $x_0^2=4\sin^2(\pi\mu)$. Then (5.12) implies $x_\infty^2=-x_1^2\ \exp(\pm 2\pi i\mu)$. Four cases which yield the values of σ non included in I) and II) must be considered

III1) $x_\infty^2=-x_1^2e^{-2\pi i\mu}$

$$\sigma=2\mu+2m,\qquad m=0,1,2,...$$
$$s=\frac{\Gamma(m+1)^2\Gamma(2\mu+m)^2}{16^{2\mu+2m}\Gamma(\mu+m+\frac{1}{2})^4}\ x_1^2$$

III2) $x_\infty^2=-x_1^2e^{2\pi i\mu}$

$$\sigma=2\mu+2m,\qquad m=-1,-2,-3,...$$
$$s=\frac{\pi^4}{\cos^4(\pi\mu)}\left[16^{2\mu+2m}\Gamma(\mu+m+\frac{1}{2})^4\Gamma(-2\mu-m+1)^2\Gamma(-m)^2\ x_1^2\right]^{-1}$$

III3) $x_\infty^2=-x_1^2e^{2\pi i\mu}$

$$\sigma=-2\mu+2m,\qquad m=1,2,3,...$$
$$s=\frac{\Gamma(m-2\mu+1)^2\Gamma(m)^2}{16^{-2\mu+2m}\Gamma(-\mu+m+\frac{1}{2})^4}\ x_1^2$$

III4) $x_\infty^2=-x_1^2e^{-2\pi i\mu}$

$$\sigma=-2\mu+2m,\qquad m=0,-1,-2,-3,...$$
$$s=\frac{\pi^4}{\cos^4(\pi\mu)}\left[16^{-2\mu+2m}\Gamma(-\mu+m+\frac{1}{2})^4\Gamma(2\mu-m)^2\Gamma(1-m)^2\ x_1^2\right]^{-1}$$

To compute σ and s in the generic case I), with $x_0^2\neq 4$, we solve the system (5.57). It has two unknowns and three equations and we need to prove that it is compatible. Actually, the first equation $2(1-cos(\pi\sigma))=x_0^2$ has always solutions. Let us choose a solution σ_0 ($\pm\sigma_0+2n$, $\forall n\in\mathbf{Z}$ are also solutions). Substitute it in the last two equations. We need to verify they are compatible. Instead of s and $\frac{1}{s}$ write X and Y. We have the linear system in two variable X, Y

$$\begin{pmatrix}F(\sigma_0) & \frac{1}{F(\sigma_0)}\\ F(\sigma_0)\ e^{-i\pi\sigma_0} & \frac{1}{F(\sigma_0)}e^{-i\pi\sigma_0}\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix}=\begin{pmatrix}f(\sigma_0)\ x_1^2-2\\ 2-f(\sigma_0)\ x_\infty^2\end{pmatrix}$$

The system has a unique solution if and only if $2i\sin(\pi\sigma_0)=\det\begin{pmatrix}F(\sigma_0) & \frac{1}{F(\sigma_0)}\\ F(\sigma_0)\ e^{-i\pi\sigma_0} & \frac{1}{F(\sigma_0)}e^{-i\pi\sigma_0}\end{pmatrix}\neq 0$. This happens for $\sigma_0\notin\mathbf{Z}$. The condition is not restrictive, because for σ even we turn to the case II) of the theorem 2, and σ odd is not in Ω. The solution is then

$$X=\frac{2(1+e^{-i\pi\sigma_0})-f(\sigma_0)(x_1^2+x_\infty^2e^{-i\pi\sigma_0})}{F(\sigma_0)(e^{-2\pi i\sigma_0}-1)}$$

$$Y=F(\sigma_0)\ \frac{f(\sigma_0)e^{-i\pi\sigma_0}(e^{-i\pi\sigma_0}x_1^2+x_\infty^2)-2e^{-i\pi\sigma_0}(1+e^{-i\pi\sigma_0})}{e^{-2\pi i\sigma_0}-1}$$

Compatibility of the system means that $X\ Y \equiv 1$. This is verified by direct computation:

$$XY = \frac{e^{-i\pi\sigma}\ \left[2(1+e^{-i\pi\sigma}) - (x_1^2 + x_\infty^2 e^{-i\pi\sigma})f(\sigma)\right]\ \left[(x_1^2 e^{-i\pi\sigma} + x_\infty^2)f(\sigma) - 2(1+e^{-i\pi\sigma})\right]}{(e^{-2i\pi\sigma}-1)^2}$$

$$= \frac{8\cos^2(\frac{\pi\sigma}{2})(x_1^2+x_\infty^2)f(\sigma) - 4(4-\sin^2(\frac{\pi\sigma}{2})) - ((x_1^2+x_\infty^2)^2 - x_0^2x_1^2x_\infty^2)f(\sigma)^2}{-4\sin^2(\pi\sigma)}$$

Using the relations $\cos^2(\frac{\pi\sigma}{2}) = 1 - x_0^2/4$, $\cos(\pi\sigma) = 1 - x_0^2/2$ and $f(\sigma) = \frac{4-x_0^2}{x_1^2+x_\infty^2-x_0x_1x_\infty}$ we obtain

$$XY = \frac{1}{x_0^2}\ \left(-2(x_1^2+x_\infty^2)f(\sigma) + 4 + \frac{(x_1^2+x_\infty^2)^2 - (x_0x_1x_\infty)^2}{x_1^2+x_\infty^2 - x_0x_1x_\infty}f(\sigma)\right)$$

$$= \frac{1}{x_0^2}\ (4 - (x_1^2+x_\infty^2 - x_0x_1x_\infty)f(\sigma)) = \frac{1}{x_0^2}\ (4-(4-x_0^2)) = 1$$

It follows from this construction that for any σ solution of the first equation of (5.57), there always exists a unique s which solves the last two equations. Recall that

$$a = -\frac{1}{4s}$$

To complete the proof of theorem 2 (points *i*), *ii*), *iii*)), we just have to compute the square roots of the x_i^2 $(i = 0, 1, \infty)$ in such a way that (5.12) is satisfied. For example, the square root of I) satisfying (5.12) is

$$\begin{cases} x_0 = 2\sin(\frac{\pi}{2}\sigma) \\ x_1 = \frac{1}{\sqrt{f(\sigma,\mu)}}\left(\sqrt{F(\sigma,\mu)\ s} + \frac{1}{\sqrt{F(\sigma,\mu)\ s}}\right) \\ x_\infty = \frac{i}{\sqrt{f(\sigma,\mu)}}\left(\sqrt{F(\sigma,\mu)\ s}\ e^{-i\frac{\pi\sigma}{2}} - \frac{1}{\sqrt{F(\sigma,\mu)\ s}\ e^{-i\frac{\pi\sigma}{2}}}\right) \end{cases}$$

which yields i), with $F(\sigma,\mu) = f(\sigma,\mu)(2G(\sigma,\mu))^2$. .

We remark that in case II) only $\sigma = 0$ is in Ω. If μ integer in II), the formulae give $(x_0, x_1, x_\infty) = (0,0,0)$. The triple is not admissible, and direct computation gives $R = 0$ for the system (5.54). This is the case of commuting monodromy matrices with a 1-parameter family of rational solutions of PVI_μ. □ [7]

Proof of remark 2

We prove that $a(\sigma) = \frac{1}{16a(-\sigma)}$, namely $s(\sigma) = \frac{1}{s(-\sigma)}$ for $a = -\frac{1}{4s}$. Given monodromy data (x_0, x_1, x_∞) the parameter s corresponding to σ is uniquely determined by

$$\frac{1}{f(\sigma)}\left(2 + F(\sigma)\ s + \frac{1}{F(\sigma)\ s}\right) = x_1^2$$

$$\frac{1}{f(\sigma)}\left(2 - F(\sigma)e^{-i\pi\sigma}\ s - \frac{1}{F(\sigma)e^{-i\pi\sigma}\ s}\right) = x_\infty^2$$

[7] *Remark:* In order to solve the R.H. for the monodromy data (x_0, x_1, x_∞) we choose branch cuts in the x-plane. When x is small we just need to fix $\alpha < \arg(x) < \alpha + 2\pi$, $\alpha \in \mathbf{R}$. Let $A(z, x; x_0, x_1, x_\infty)$ be the matrix solution of the R.H. Consider the loop $x \mapsto x' = xe^{2\pi i}$; the analytic continuation of A along the loop is $A(z, x'; x_0, x_1, x_\infty) \equiv A(z, x; x_0^{\beta_1^2}, x_1^{\beta_1^2}, x_\infty^{\beta_1^2})$. In other words, $A(z, x'; x_0, x_1, x_\infty)$ coincides with the solution of the R.H. obtained with the same branch cut $\alpha < \arg(x) < \alpha + 2\pi$ and new monodromy data $(x_0^{\beta_1^2}, x_1^{\beta_1^2}, x_\infty^{\beta_1^2})$ transformed by the action of the braid group. When we write $A(z, x'; x_0, x_1, x_\infty)$ we are considering A as a function on the universal covering of $\mathbf{C}_0 \cap \{|x| < \epsilon\}$; when we write $A(z, x; x_0^{\beta_1^2}, x_1^{\beta_1^2}, x_\infty^{\beta_1^2})$ we are considering the solution of the R.H. as a "branch".

In the proof of theorem 2 we start from a point $x \in D(\sigma)$ and we take the limits $x \to 0$ of the system and of the rescaled system. At x we assign the monodromy M_0, M_1, M_x characterized by (x_0, x_1, x_∞) and then we take the limit proving the theorem. If we had start from another point $x' = xe^{2\pi i} \in D(\sigma)$ (provided that this is possible for the given $D(\sigma)$ and x) we take the same monodromy M_0, M_1, M_x, because what we were doing is the limit, for $x \to 0$ in $D(\sigma)$, of $A(z, x; x_0, x_1, x_\infty)$ considered as a function defined on the universal covering of $\mathbf{C}_0 \cap \{|x| < \epsilon\}$.

We observe that $f(\sigma) = f(-\sigma)$ and that the properties of the Gamma function

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \qquad \Gamma(z+1) = z\Gamma(z)$$

imply

$$F(-\sigma) = \frac{1}{F(\sigma)}$$

Then the value of s corresponding to $-\sigma$ is (uniquely) determined by

$$\frac{1}{f(\sigma)}\left(2 + \frac{s}{F(\sigma)} + \frac{F(\sigma)}{s}\right) = x_1^2$$

$$\frac{1}{f(\sigma)}\left(2 - \frac{s}{F(\sigma)e^{-i\pi\sigma}} - \frac{F(\sigma)e^{-i\pi\sigma}}{s}\right) = x_\infty^2$$

We conclude that $s(-\sigma) = -\frac{1}{s(\sigma)}$.

The last remark concerns the choice of (5.35), (5.37) instead of (5.36), (5.38). The reason is that at $z = 0$ the system (5.54) has solution corresponding to (5.55). This is true for any $\sigma \neq 0$ in Ω, also for $\sigma \to \pm 2\mu$. This is equivalent to the behaviour (5.56), which is obtainable from the $\hat{G}_0 = T$ of (5.35), (5.37) but not of (5.36), (5.38).

□

Proof of formula (5.25)

We are ready to prove formula (5.25), namely:

$$\beta_1^2 : \quad (\sigma, a) \mapsto (\sigma, a e^{-2\pi i \sigma})$$

For $\sigma = 0$ we have $x_0 = 0$ and $\beta_1^2 : (0, x_1, x_\infty) \mapsto (0, x_1, x_\infty)$. Thus

$$a = \frac{x_\infty^2}{x_1^2 + x_\infty^2} \mapsto \frac{x_\infty^2}{x_1^2 + x_\infty^2} \equiv a$$

For $\sigma = \pm 2\mu + 2m$, we consider the example $\sigma = 2\mu + 2m$, $m = 0, 1, 2, \ldots$. The other cases are analogous. We have $s = x_1^2\, H(\sigma) = -x_\infty^2 H(\sigma) e^{2\pi i \mu}$, where the function $H(\sigma)$ is explicitly given in theorem 2, *III*). Then

$$\beta_1 : \quad s = -x_\infty^2 H(\sigma) e^{2\pi i \mu} \mapsto -x_1^2 H(\sigma) e^{2\pi i \mu} = -s e^{2\pi i \mu}$$

Then

$$\beta_1^2 : \quad s \mapsto s e^{4\pi i \mu} \quad \Longrightarrow \quad a \mapsto a e^{-4\pi i \mu} \equiv a e^{-2\pi i \sigma}$$

For the generic case *I*) ($\sigma \notin \mathbf{Z}$, $\sigma \neq \pm 2\mu + 2m$) recall that

$$\begin{cases} F(\sigma)\, s + \frac{1}{F(\sigma)\, s} = x_1^2 f(\sigma) - 2 \\ F(\sigma) e^{-i\pi\sigma}\, s + \frac{1}{F(\sigma) e^{-i\pi\sigma}\, s} = 2 - x_\infty^2 f(\sigma) \end{cases}$$

has a unique solution s. Also observe that $\beta_1 : x_\infty \mapsto x_1$. Then the transformed parameter $\beta_1 : s \mapsto s^{\beta_1}$ satisfies the equation

$$F(\sigma) e^{-i\pi\sigma}\, s^{\beta_1} + \frac{1}{F(\sigma) e^{-i\pi\sigma}\, s^{\beta_1}} = 2 - x_1^2 f(\sigma)$$

$$\equiv -\left(F(\sigma)\, s + \frac{1}{F(\sigma)\, s}\right)$$

Thus $s^{\beta_1} = -e^{i\pi\sigma}\, s$. This implies

$$\beta_1^2 : \; s \mapsto s e^{2\pi i \sigma} \quad \Longrightarrow \quad a \mapsto a\, e^{-2\pi i \sigma}$$

□

We still have to prove the following

Proposition: *Let $y(x) \sim a x^{1-\sigma}$ as $x \to 0$ in a domain $D(\epsilon, \sigma)$. Then, $y(x)$ coincides with $y(x;\sigma,a)$ of theorem 1*

Proof: Observe that both $y(x)$ and $y(x;\sigma,a)$ have the same asymptotic behaviour for $x \to 0$ in $D(\sigma)$. Let $A_0(x)$, $A_1(x)$, $A_x(x)$ be the matrices constructed from $y(x)$ and $A_0^*(x)$, $A_1^*(x)$, $A_x^*(x)$ constructed from $yx;\sigma,a)$ by means of the formulae (1.36) of section 1.10 and the formulae which give ϕ_0 in terms of $y(x)$ in section 6.1. It follows that $A_i(x)$ and $A_i^*(x)$, $i = 0, 1, x$, have the same asymptotic behaviour as $x \to 0$. This is the behaviour of lemma 1 of section 5.7 (adapted to our case). From the proof of theorem 2 if follows that $A_0(x)$, $A_1(x)$, $A_x(x)$ and $A_0^*(x)$, $A_1^*(x)$, $A_x^*(x)$ produce the same triple (x_0, x_1, x_∞). The solution of the Riemann-Hilbert problem for such a triple is unique and therefore $A_i(x) \equiv A_i^*(x)$, $i = 0, 1, x$. We conclude that $y(x) \equiv y(x;\sigma,a)$. □

5.9 Proof of Theorem 3

To start with, we derive the elliptic form for the general Painlevé 6 equation. We follow [22]. We put

$$u = \int_\infty^y \frac{d\lambda}{\sqrt{\lambda(\lambda-1)(\lambda-x)}} \tag{5.58}$$

We recall that

$$\frac{du}{dx} = \frac{\partial u}{\partial y}\frac{dy}{dx} + \frac{\partial u}{\partial x} = \frac{1}{\sqrt{y(y-1)(y-x)}}\frac{dy}{dx} + \frac{\partial u}{\partial x}$$

from which we compute

$$\frac{d^2u}{dx^2} + \frac{2x-1}{x(x-1)}\frac{du}{dx} + \frac{u}{4x(x-1)} =$$

$$= \frac{1}{\sqrt{y(y-1)(y-x)}}\left[\frac{d^2y}{dx^2} + \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\frac{dy}{dx} - \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\left(\frac{dy}{dx}\right)^2\right]$$

$$+\frac{\partial^2 u}{\partial x^2} + \frac{2x-1}{x(x-1)}\frac{\partial u}{\partial x} + \frac{u}{4x(x-1)}$$

By direct calculation we have:

$$\frac{\partial^2 u}{\partial x^2} + \frac{2x-1}{x(x-1)}\frac{\partial u}{\partial x} + \frac{u}{4x(x-1)} = -\frac{1}{2}\frac{\sqrt{y(y-1)(y-x)}}{x(x-1)}\frac{1}{(y-x)^2}$$

Therefore, $y(x)$ satisfies the Painlevé 6 equation if and only if

$$\frac{d^2u}{dx^2} + \frac{2x-1}{x(x-1)}\frac{du}{dx} + \frac{u}{4x(x-1)} = \frac{\sqrt{y(y-1)(y-x)}}{2x^2(1-x)^2}\left[2\alpha + 2\beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \left(\delta - \frac{1}{2}\right)\frac{x(x-1)}{(y-x)^2}\right] \tag{5.59}$$

We invert the function $u = u(y)$ by observing that we are dealing with an elliptic integral. Therefore, we write

$$y = f(u, x)$$

where $f(u,x)$ is an elliptic function of u. This implies that

$$\frac{\partial y}{\partial u} = \sqrt{y(y-1)(y-x)}$$

The above equality allows us to rewrite (5.59) in the following way:

$$x(1-x)\frac{d^2u}{dx^2} + (1-2x)\frac{du}{dx} - \frac{1}{4}u = \frac{1}{2x(1-x)}\frac{\partial}{\partial u}\psi(u,x), \tag{5.60}$$

where

$$\psi(u,x) := 2\alpha f(u,x) - 2\beta\frac{x}{f(u,x)} + 2\gamma\frac{1-x}{f(u,x)-1} + (1-2\delta)\frac{x(x-1)}{f(u,x)-x}$$

The last step concerns the form of $f(u,x)$. We observe that $4\lambda(\lambda-1)(\lambda-x)$ is not in Weierstrass canonical form. We change variable:

$$\lambda = t + \frac{1+x}{3},$$

and we get the Weierstrass form:

$$4\lambda(\lambda-1)(\lambda-x) = 4t^3 - g_2 t - g_3, \quad g_2 = \frac{4}{3}(1-x+x^2), \quad g_3 := \frac{4}{27}(x-2)(2x-1)(1+x)$$

Thus

$$\frac{u}{2} = \int_\infty^{y-\frac{1+x}{3}} \frac{dt}{\sqrt{4t^3-g_2t-g_3}}$$

which implies

$$y(x) = \mathcal{P}\left(\frac{u}{2};\omega_1,\omega_2\right) + \frac{1+x}{3}$$

We still need to explain what are the *half periods* ω_1, ω_2. In order to do that, we first observe that the Weierstrass form is

$$4t^3 - g_2t - g_3 = 4(t-e_1)(t-e_2)(t-e_3)$$

where

$$e_1 = \frac{2-x}{3}, \quad e_2 = \frac{2x-1}{3}, \quad e_3 = -\frac{1+x}{3}.$$

Therefore

$$g := \sqrt{e_1-e_2} = 1, \quad \kappa^2 := \frac{e_2-e_3}{e_1-e_3} = x, \quad \kappa'^2 := 1-\kappa^2 = 1-x$$

and the half-periods are

$$\omega_1 = \frac{1}{g}\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\kappa^2\xi^2)}} = \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-x\xi^2)}} = \mathbf{K}(x)$$

$$\omega_2 = \frac{i}{g}\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\kappa'^2\xi^2)}} = i\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-(1-x)\xi^2)}} = i\mathbf{K}'(1-x)$$

The elliptic integrals $\mathbf{K}(x)$ and $\mathbf{K}'(1-x)$ are known:

$$\mathbf{K}(x) = \frac{\pi}{2}\, F\left(\frac{1}{2},\frac{1}{2},1;x\right)$$

$$\mathbf{K}'(1-x) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2},1;1-x\right)$$

where $F(x)$ is the hypergeometric function

$$F\left(\frac{1}{2},\frac{1}{2},1;x\right) = \sum_{n=0}^{\infty}\frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{(n!)^2}x^n,$$

$\mathbf{K}(x)$ and $\mathbf{K}'(1-x)$ are two linearly independent solutions of the hypergeometric equation

$$x(1-x)\omega'' + (1-2x)\omega' - \frac{1}{4}\omega = 0.$$

Observe that for $|\arg(x)| < \pi$:

$$-\pi F\left(\frac{1}{2},\frac{1}{2},1;1-x\right) = F\left(\frac{1}{2},\frac{1}{2},1;x\right)\ln(x) + F_1(x)$$

where

$$F_1(x) := \sum_{n=0}^{\infty}\frac{\left[\left(\frac{1}{2}\right)_n\right]^2}{(n!)^2}2\left[\psi(n+\frac{1}{2}) - \psi(n+1)\right]x^n, \quad \psi(z) = \frac{d}{dz}\ln\Gamma(z).$$

Therefore $\omega_2(x) = -\frac{i}{2}[F(x)\ln(x) + F_1(x)]$ where $F(x)$ is a abbreviation for $F\left(\frac{1}{2},\frac{1}{2},1;x\right)$. The series of $F(x)$ and $F_1(x)$ converge for $|x|<1$. Incidentally, we observe that

$$y(x) = \mathcal{P}\left(u(x)/2;\omega_1(x),\omega_2(x)\right) - e_3 = \frac{1}{\mathrm{sn}^2\left(u(x)/2,\kappa^2 = x\right)}$$

Proof of theorem 3: We let $x \to 0$; if $\Im\tau > 0$ and

$$\left|\Im\left(\frac{u}{4\omega_1}\right)\right| < \Im\tau \tag{5.61}$$

we expand the elliptic function in Fourier series (5.20). The first condition $\Im\tau>0$ is always satisfied for $x \to 0$ because

$$\Im\tau(x) = -\frac{1}{\pi}\ln|x| + \frac{4}{\pi}\ln 2 + O(x), \qquad x\to 0.$$

We look for a solution $u(x)$ of (5.13) of the form

$$u(x) = 2\nu_1\omega_1(x) + 2\nu_2\omega_2(x) + 2v(x)$$

where $v(x)$ has to be determined from (5.13). We look for a holomorphic solutions $v(x)$, bounded if $x \to 0$. We observe that

$$\frac{u(x)}{4\omega_1(x)} = \frac{\nu_1}{2} + \frac{\nu_2}{2}\tau(x) + \frac{v(x)}{2\omega_1(x)}$$

$$= \frac{\nu_1}{2} + \frac{\nu_2}{2}\left[-\frac{i}{\pi}\ln x - \frac{i}{\pi}\frac{F_1(x)}{F(x)}\right] + \frac{v(x)}{2\omega_1(x)}, \qquad \left(\text{note that } \frac{F_1(x)}{F(x)} = -4\ln 2 + O(x) \text{ as } x\to 0\right).$$

Thus, if $x \to 0$ the condition (5.61) becomes

$$(2+\Re\nu_2)\ln|x| - \mathcal{C}(x,\nu_1,\nu_2) - 8\ln 2 < \Im\nu_2 \arg(x) < (\Re\nu_2 - 2)\ln|x| - \mathcal{C}(x,\nu_1,\nu_2) + 8\ln 2. \tag{5.62}$$

where $\mathcal{C}(x,\nu_1,\nu_2) = [\Im\frac{\pi v}{2\omega_1} + 4\ln 2\Re\nu_2 + \pi\Im\nu_1 + O(x)]$. We expand the derivative of $\mathcal{P}$ appearing in (5.13)

$$\frac{\partial}{\partial u}\mathcal{P}\left(\frac{u}{2};\omega_1,\omega_2\right) = \left(\frac{\pi}{\omega_1}\right)^3 \sum_{n=1}^{\infty}\frac{n^2 e^{2\pi i n\tau}}{1-e^{2\pi i n\tau}}\sin\left(\frac{n\pi u}{2\omega_1}\right) - \left(\frac{\pi}{2\omega_1}\right)^3 \frac{\cos\left(\frac{\pi u}{4\omega_1}\right)}{\sin^3\left(\frac{\pi u}{4\omega_1}\right)}$$

$$= \frac{1}{2i}\left(\frac{\pi}{\omega_1}\right)^3 \sum_{n=1}^{\infty}\frac{n^2 e^{2\pi i n\tau}}{1-e^{2\pi i n\tau}}\left(e^{in\frac{\pi u}{2\omega_1}} - e^{-in\frac{\pi u}{2\omega_1}}\right) + 4i\left(\frac{\pi}{2\omega_1}\right)^3 \frac{e^{i\frac{\pi u}{4\omega_1}} + e^{-i\frac{\pi u}{4\omega_1}}}{\left(e^{i\frac{\pi u}{4\omega_1}} - e^{-i\frac{\pi u}{4\omega_1}}\right)^3}$$

Now we come to a crucial step in the construction: we collect $e^{-i\frac{\pi u}{4\omega_1}}$ in the last term, which becomes

$$4i\left(\frac{\pi}{2\omega_1}\right)^3 \frac{e^{4\pi i\frac{u}{4\omega_1}} + e^{2\pi i\frac{u}{4\omega_1}}}{\left(e^{2\pi i\frac{u}{4\omega_1}} - 1\right)^3}.$$

The denominator *does not vanish* if $\left|e^{2\pi i\frac{u}{4\omega_1}}\right| < 1$. From now on, this condition is added to (5.61) and reduces the domain (5.62). The expansion of $\frac{\partial}{\partial u}\mathcal{P}$ becomes

$$\frac{\partial}{\partial u}\mathcal{P}\left(\frac{u}{2};\omega_1,\omega_2\right) = \frac{1}{2i}\left(\frac{\pi}{\omega_1}\right)^3 \sum_{n=1}^{\infty}\frac{n^2 e^{i\pi n\left[-\nu_1+(2-\nu_2)\tau - \frac{v}{2\omega_1}\right]}}{1-e^{2\pi i n\tau}}\left(e^{i\pi n\left[\nu_1+\nu_2\tau+\frac{v}{2\omega_1}\right]} - 1\right)$$

$$+4i\left(\frac{\pi}{2\omega_1}\right)^3 \frac{e^{2\pi i\left[\nu_1+\nu_2\tau+\frac{v}{2\omega_1}\right]} + e^{\pi i\left[\nu_1+\nu_2\tau+\frac{v}{2\omega_1}\right]}}{\left(e^{\pi i\left[\nu_1+\nu_2\tau+\frac{v}{2\omega_1}\right]} - 1\right)^3}$$

Let's write $\frac{F_1(x)}{F(x)} = -4\ln 2 + g(x)$, where $g(x) = O(x)$ is a power series starting with x. We have

$$e^{i\pi C\tau} = \frac{x^C}{16^C}\, e^{C\, g(x)} = \frac{x^C}{16^C}(1+O(x)), \quad x\to 0, \qquad \text{for any } C \in \mathbf{C}.$$

Hence

$$\frac{\partial}{\partial u}\mathcal{P}\left(\frac{u}{2};\omega_1,\omega_2\right) = \mathcal{F}\left(x, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}e^{-i\pi\frac{v}{2\omega_1}}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}e^{i\pi\frac{v}{2\omega_1}}\right)$$

where

$$\mathcal{F}(x,y,z) = \frac{1}{2i}\left(\frac{\pi}{\omega_1(x)}\right)^3 \sum_{n=1}^{\infty}\frac{n^2 e^{n(2-\nu_2)g(x)}}{1-\left[\frac{1}{16}e^{g(x)}\right]^{2n}x^{2n}}y^n(z^n-1) + 4i\left(\frac{\pi}{2\omega_1(x)}\right)^3\frac{z^2+z}{(z-1)^3}$$

The series converges for sufficiently small $|x|$ and for $|y|<1$, $|yz|<1$; this is precisely (5.61). However, we require that the last term is holomorphic, so we have to further impose $|z|<1$. On the resulting domain $|x|<r<1$, $|y|<1$, $|z|<1$ $\mathcal{F}(x,y,z)$ is holomorphic and satisfies

$$\mathcal{F}(0,0,0) = 0.$$

The condition $|y|<1$, $|z|<1$ is $\left|\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}e^{-i\pi\frac{v}{2\omega_1}}\right|<1$, $\left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}e^{i\pi\frac{v}{2\omega_1}}\right|<1$, namely

$$\Re\nu_2\ln|x| - \mathcal{C}(x) < \Im\nu_2\arg(x) < (2-\Re\nu_2)\ln|x| - \mathcal{C}(x) + 8\ln 2, \tag{5.63}$$

which is more restrictive that (5.62).

The function $\mathcal{F}$ can be decomposed as follows:

$$\mathcal{F} = \mathcal{F}\left(x, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right) +$$

$$+\left[\mathcal{F}\left(x, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}e^{-i\pi\frac{v}{2\omega_1}}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}e^{i\pi\frac{v}{2\omega_1}}\right) - \mathcal{F}\left(x, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right)\right]$$

$$=: \mathcal{F}\left(x, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right) + \mathcal{G}\left(x, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}, v(x)\right)$$

The above defines $\mathcal{G}(x,y,z,v)$. As a function of its arguments it is holomorphic for $|x|$, $|y|$, $|z|$, $|v|$ less then some $r'<1$. Moreover

$$\mathcal{G}(0,0,0,v) = \mathcal{G}(x,y,z,0) = 0.$$

Let us put $u = u_0 + 2v$, where $u_0 = 2\nu_1\omega_1 + 2\nu_2\omega_2$. Therefore $\mathcal{L}(u_0) = 0$ and $\mathcal{L}(u_0+2v) = \mathcal{L}(u_0) + \mathcal{L}(2v) \equiv 2\mathcal{L}(v)$. Hence (5.13) becomes

$$\mathcal{L}(v) = \frac{\alpha}{2x(1-x)}(\mathcal{F}+\mathcal{G}). \tag{5.64}$$

We put

$$w := xv' \quad (\text{where } v' = \frac{dv}{dx}),$$

and the equation (5.64) becomes

$$w' = \frac{1}{x}\left[\frac{\alpha}{2(1-x)^2}\mathcal{F} + \frac{x(w+\frac{1}{4}v)}{1-x} + \frac{\alpha}{2(1-x)^2}\mathcal{G}\right]$$

Now, let us define

$$\Phi(x,y,z) := \frac{\alpha}{2(1-x)^2}\mathcal{F}(x,y,z),$$

$$\Psi(x,y,z,v,w) := \frac{x(w+\frac{1}{4}v)}{1-x} + \frac{\alpha}{2(1-x)^2}\mathcal{G}(x,y,z,v).$$

They are holomorphic for $|x|, |y|, |z|, |v|, |w|$ small (say less then $r'<1$) and they are such that

$$\Phi(0,0,0) = 0, \quad \Psi(0,0,0,v,w) = \Psi(x,y,z,0,0) = 0.$$

Our initial equation (5.13) becomes the system

$$x\frac{dv}{dx} = w,$$

$$x\frac{dw}{dx} = \Phi(x, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}) + \Psi(x, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}, v(x), w(x)).$$

A system having the structure of the system above, with some slight changes in the arguments of Φ and Ψ, has been studied by S. Shimomura in [27]. He reduced it to a system of integral equations

$$w(x) = \int_{L(x)} \frac{1}{t}\left\{\Phi(t, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}t^{2-\nu_2}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}t^{\nu_2}) + \Psi(t, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}t^{2-\nu_2}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}t^{\nu_2}, v(t), w(t))\right\} dt$$

$$v(x) = \int_{L(x)} \frac{1}{s}\int_{L(s)} \frac{1}{t}\left\{\Phi(t, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}t^{2-\nu_2}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}t^{\nu_2}) + \Psi(t, \frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}t^{2-\nu_2}, \frac{e^{i\pi\nu_1}}{16^{\nu_2}}t^{\nu_2}, v(t), w(t))\right\} dtds$$

and he solved it by successive approximations, with the initial condition $v_0 = w_0 = 0$. The path $L(x)$ is a path connecting x to 0 in (5.63), like the path considered in the proof of theorem 1.

We refer the reader to [27] and to the last of [52]; for reasons of space we just take the result:

For any complex ν_1, ν_2 such that

$$\nu_2 \not\in (-\infty, 0] \cup [2, +\infty)$$

there exists a sufficiently small $r < 1$ such that the system has a solution $v(x)$ holomorphic in

$$\mathcal{D}(r;\nu_1,\nu_2) := \left\{x \in \tilde{\mathbf{C}}_0 \quad \text{such that } |x| < r, \left|\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right| < r, \left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right| < r\right\}$$

with an expansion convergent in $\mathcal{D}(r;\nu_1,\nu_2)$

$$v(x) = \sum_{n\geq 1} a_n x^n + \sum_{n\geq 0,\ m\geq 1} b_{nm}x^n\left(\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right)^m + \sum_{n\geq 0,\ m\geq 1} c_{nm}x^n\left(\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right)^m$$

where a_n, b_{nm}, c_{nm} are rational functions of ν_2. Moreover, there exists a constant $M(\nu_2)$ depending on ν_2 such that $v(x) \leq M(\nu_2)\left(|x| + \left|\frac{e^{-i\pi\nu_1}}{16^{2-\nu_2}}x^{2-\nu_2}\right| + \left|\frac{e^{i\pi\nu_1}}{16^{\nu_2}}x^{\nu_2}\right|\right)$ in $\mathcal{D}(r;\nu_1,\nu_2)$.

We conclude that:

Theorem 3: *for any ν_1 ν_2 such that*

$$\nu_2 \in (\mathbf{C} - \{(-\infty, 0] \cup [2, +\infty)\}),$$

there exists a sufficiently small r such that

$$y(x) = \mathcal{P}(\nu_1\omega_1(x) + \nu_2\omega_2(x) + v(x); \omega_1(x), \omega_2(x))$$

in the domain $\mathcal{D}(r;\nu_1,\nu_2)$, where $v(x)$ is given above.

$\mathcal{D}(r;\nu_1,\nu_2)$ is more explicitly written in the form 5.19 which makes it evident that it is contained in (5.63).

Chapter 6

Reconstruction of 3-dimensional FM

Chapter 3 was a didactic exposition of the procedure of inverse reconstruction of $F(t)$ through (1.22), (1.23). Monodromy data were assigned, the functions ϕ_p were computed directly from (3.1) and the conditions of isomonodromicity (1.13). This procedure is an alternative to the direct solution of the boundary value problem.

Here we follow the same procedure, but the situation is highly non-trivial. The solutions of

$$\partial_i \phi_0 = V_i \phi_0$$
$$\partial V_i = [V_i, V]$$

where $V = \phi_0 \ \mathrm{diag}(\mu, 0, -\mu) \ \phi_0^{-1}$ and $(V_k)_{ij} = ([\delta_{ki} - \delta_{kj}] \ V_{ij})/(u_i - u_j)$, was partly given in section 1.10 in terms of Painlevé trascendents; in section 6.1 we'll give the explicit, computable solution of the above system in terms of the transcendent. Its dependence on the Stokes' matrix is as follows: the Painlevé transcendents are parametrized by a triple (x_0, x_1, x_∞) of monodromy data of a 2×2 fuchsian system (see chapter 5) and therefore $\phi_0(u) = \phi_0(u; x_0, x_1, x_\infty)$ locally. Since the fuchsian system is the 2×2 reduction of a 3×3 fuchsian system (see section 1.10) which is connected to (1.11) by Laplace transform (see section 4.8), the Stokes' matrix of the corresponding (1.11) is expressed in terms of the triple (x_0, x_1, x_∞) itself (see [21]):

$$S = \begin{pmatrix} 1 & x_\infty & x_0 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_0^2 + x_1^2 + x_\infty^2 - x_0 x_1 x_\infty = 4\sin^2(\pi\mu).$$

After the determination of ϕ_0, we compute $\phi_1(u)$, $\phi_2(u)$ and $\phi_3(u)$ by direct substitution of the solution (1.12) into the differential equation (1.11). In generic cases this is enough, but for resonant values of μ the matrix R in (1.12) does not vanish. Some entries of ϕ_p are indeterminate and we can fix them thanks to the higher order conditions of isomonodromicity (1.13) and the condition (1.14), for $p = 1, 2, 3$. The non zero entries of R appear as parameters in the ϕ_p's.

The final hard problem is to obtain the closed form $F = F(t)$ from the parametric equations (1.22), (1.23).

At the end of the procedure we get $F = F(t)$ in terms of the monodromy data S, μ, R.

6.1 Computation of ϕ_0 and V in terms of Painlevé transcendents

Let $n = 3$. We can bring η to the form:

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Let

$$V(u) = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$

which is similar to

$$\hat{\mu} = \text{diag}(\mu, 0, -\mu), \qquad \mu = \sqrt{-(\Omega_1^2+\Omega_2^2+\Omega_3^2)} \text{ constant}, \qquad \mu = -\frac{d}{2}.$$

By simple linear algebra we find the eigenvalues and eigenvectors of V. ϕ_0 is precisely the matrix whose columns are the eigenvectors; imposing also the condition

$$\phi_0^T \phi_0 = \eta$$

we find

$$\phi_0 = \begin{pmatrix} \frac{i}{\sqrt{2}\mu}\frac{\Omega_1\Omega_2-\mu\Omega_3}{(\Omega_1^2+\Omega_3^2)^{\frac{1}{2}}}\, G(u) & \frac{\Omega_1}{i\mu} & \frac{i}{\sqrt{2}\mu}\frac{\Omega_1\Omega_2+\mu\Omega_3}{(\Omega_1^2+\Omega_3^2)^{\frac{1}{2}}}\frac{1}{G(u)} \\ -\frac{i}{\sqrt{2}\mu}(\Omega_1^2+\Omega_3^2)^{\frac{1}{2}}\, G(u) & \frac{\Omega_2}{i\mu} & -\frac{i}{\sqrt{2}\mu}(\Omega_1^2+\Omega_3^2)^{\frac{1}{2}}\frac{1}{G(u)} \\ \frac{i}{\sqrt{2}\mu}\frac{\Omega_2\Omega_3+\mu\Omega_1}{(\Omega_1^2+\Omega_3^2)^{\frac{1}{2}}}\, G(u) & \frac{\Omega_3}{i\mu} & \frac{i}{\sqrt{2}\mu}\frac{\Omega_2\Omega_3-\mu\Omega_1}{(\Omega_1^2+\Omega_3^2)^{\frac{1}{2}}}\frac{1}{G(u)} \end{pmatrix}$$

where $G(u)$ is so far an arbitrary function of $u = (u_1, u_2, u_3)$. To determine it we impose the condition

$$\frac{\partial \phi_0}{\partial u_i} = V_i(u)\phi_0 \tag{6.1}$$

We observe that $\phi_{i2,0} = \Omega_i/(i\mu)$, $i = 1, 2, 3$, and that (6.1) for the $\phi_{i2,0}$'s is equivalent to the equation $\partial_i V = [V_i, V]$. In particular, the last equation implies $\sum_i \partial_i V = \sum_i u_i \partial_i V = 0$. Thus $V(u_1, u_2, u_3) \equiv V(x)$, where

$$x = \frac{u_3 - u_1}{u_2 - u_1}$$

Finally, $\partial_i V = [V_i, V]$ becomes:

$$\begin{aligned} \frac{d\Omega_1}{dx} &= \frac{1}{x}\,\Omega_2\Omega_3 \\ \frac{d\Omega_2}{dx} &= \frac{1}{1-x}\,\Omega_1\Omega_3 \\ \frac{d\Omega_3}{dx} &= \frac{1}{x(x-1)}\,\Omega_1\Omega_2 \end{aligned} \tag{6.2}$$

The equations (6.1), (6.2) are reduced to PVI_μ. The product and squares of the entries of ϕ_0 are expressed in terms of a Painlevé transcendent $y(x)$ in section 1.10, where we followed [16]. Now we work out the explicit expressions for the entries. Let

$$H := u_2 - u_1.$$

The reader may verify that the following entries of ϕ_0 satisfy (6.1), provided that $y = y(x)$ is a Painlevé transcendent and

$$k = k(x, H) := \frac{k_0 \exp\left\{(2\mu - 1) \int^x d\zeta\, \frac{y(\zeta)-\zeta}{\zeta(\zeta-1)}\right\}}{H^{2\mu-1}}, \qquad k_0 \in \mathbf{C}\backslash\{0\}.$$

ϕ_0 is a function of (x, H):

$$\phi_{13,0} = i\frac{\sqrt{k}\sqrt{y}}{\sqrt{H}\sqrt{x}}$$

$$\phi_{23,0} = i\frac{\sqrt{k}\sqrt{y-1}}{\sqrt{H}\sqrt{1-x}}$$

$$\phi_{33,0} = -\frac{\sqrt{k}\sqrt{y-x}}{\sqrt{H}\sqrt{x}\sqrt{1-x}}$$

$$\phi_{12,0} = \frac{1}{\mu}\frac{\sqrt{y-1}\sqrt{y-x}}{\sqrt{x}}\left[\frac{A}{(y-1)(y-x)} + \mu\right]$$

$$\phi_{22,0} = \frac{1}{\mu}\frac{\sqrt{y}\sqrt{y-x}}{\sqrt{1-x}}\left[\frac{A}{y(y-x)} + \mu\right]$$

$$\phi_{32,0} = \frac{i}{\mu}\frac{\sqrt{y}\sqrt{y-1}}{\sqrt{x}\sqrt{1-x}}\left[\frac{A}{y(y-1)}+\mu\right]$$

$$\phi_{11,0} = \frac{i}{2\mu^2}\frac{\sqrt{H}\sqrt{y}}{\sqrt{k(x)}\sqrt{x}}\left[A\left(B+\frac{2\mu}{y}\right)+\mu^2(y-1-x)\right]$$

$$\phi_{21,0} = \frac{i}{2\mu^2}\frac{\sqrt{H}\sqrt{y-1}}{\sqrt{k(x)}\sqrt{1-x}}\left[A\left(B+\frac{2\mu}{y-1}\right)+\mu^2(y+1-x)\right]$$

$$\phi_{31,0} = -\frac{1}{2\mu^2}\frac{\sqrt{H}\sqrt{y-x}}{\sqrt{k(x)}\sqrt{x}\sqrt{1-x}}\left[A\left(B+\frac{2\mu}{y-x}\right)+\mu^2(y-1+x)\right]$$

where

$$A = A(x) := \frac{1}{2}\left[\frac{dy}{dx}x(x-1)-y(y-1)\right], \quad B = B(x) := \frac{A}{y(y-1)(y-x)}$$

An equivalent way to write is

$$\phi_0 = \begin{pmatrix} \frac{E_{11}}{f} & E_{12} & E_{13}f \\ \frac{E_{21}}{f} & E_{22} & E_{23}f \\ \frac{E_{31}}{f} & E_{32} & E_{33}f \end{pmatrix}$$

where

$$f = f(x,H) := i\,\frac{\sqrt{k}\sqrt{y-1}}{\sqrt{H}\sqrt{1-x}} \equiv \frac{\sqrt{k}\sqrt{y-1}}{\sqrt{H}\sqrt{x-1}}$$

$$E_{i2} = \frac{\Omega_i}{i\mu}, \qquad i = 1,2,3$$

$$E_{11} = \frac{\Omega_1\Omega_2-\mu\Omega_3}{2\mu^2}, \quad E_{13} = -\frac{\Omega_1\Omega_2+\mu\Omega_3}{\Omega_1^2+\Omega_3^2}$$

$$E_{21} = -\frac{\Omega_1^2+\Omega_3^2}{2\mu^2}, \qquad E_{23} = 1$$

$$E_{31} = \frac{\Omega_2\Omega_3+\mu\Omega_1}{2\mu^2}, \quad E_{33} = -\frac{\Omega_2\Omega_3-\mu\Omega_1}{\Omega_1^2+\Omega_3^2}$$

and

$$\Omega_1 = i\frac{\sqrt{y-1}\sqrt{y-x}}{\sqrt{x}}\left[\frac{A}{(y-1)(y-x)}+\mu\right]$$

$$\Omega_2 = i\frac{\sqrt{y}\sqrt{y-x}}{\sqrt{1-x}}\left[\frac{A}{y(y-x)}+\mu\right]$$

$$\Omega_3 = -\frac{\sqrt{y}\sqrt{y-1}}{\sqrt{x}\sqrt{1-x}}\left[\frac{A}{y(y-1)}+\mu\right]$$

The branches (signs) in the square roots above are arbitrary. A change of the sign of one root (for example of $\sqrt{H}$) implies a change of two signs in $(\Omega_1,\Omega_2,\Omega_3)$, or the change $(\phi_{i,0},\phi_{i3,0}) \mapsto -(\phi_{i,0},\phi_{i3,0})$. The reader may verify that all these changes do not affect the equations for ϕ_0 and V. We only remark that in the definition of $f(x,H)$ we *chose* $\sqrt{1-x} = i\sqrt{x-1}$.

Conversely, given a solution $(\Omega_1,\Omega_2,\Omega_3)$ of (6.2), a corresponding solution of PVI_μ is

$$y(x) = \frac{xR(x)}{x\,[1+R(x)]\,-1}, \quad R(x) := \left(\frac{\phi_{13,0}}{\phi_{23,0}}\right)^2 = \left(\frac{\Omega_1\Omega_2+\mu\Omega_3}{\mu^2+\Omega_2^2}\right)^2 \tag{6.3}$$

The reader may verify directly that the above formulae solve (6.1), (6.2) and satisfy the equations of section 1.10.

6.2 Explicit Computation of the Flat Coordinates and of F for $n = 3$

Let $t = (t^1, t^2, t^3)$ with higher indices. We compute the parametric form $t = t(x, H)$ and $F = F(x, H))$ using

$$t^1 = \sum_{i=1}^{3} \phi_{i3,0}\phi_{i1,1}, \quad t^2 = \sum_{i=1}^{3} \phi_{i2,0}\phi_{i1,1}, \quad t^3 = \sum_{i=1}^{3} \phi_{i1,0}\phi_{i1,1}, \tag{6.4}$$

$$F = \frac{1}{2}\left[t^\alpha t^\beta \sum_{i=1}^{3} \phi_{i\alpha,0}\phi_{i\beta,1} - \sum_{i=1}^{3} (\phi_{i1,1}\phi_{i1,2} + \phi_{i1,3}\phi_{i1,0})\right] \tag{6.5}$$

The final pourpose is to obtain a closed form $F = F(t)$. We recall that $\mu_1 = \mu$, $\mu_2 = 0$, $\mu_3 = -\mu$. Let us compute ϕ_1, ϕ_2, ϕ_3. We decompose

$$\phi_p := \phi_0 \; H_p, \quad p = 0, 1, 2, ...$$

namely, H_i appears in the fundamental matrix

$$\Xi(z, u) = (I + H_1 \; z + H_2 \; z^2 + H_3 \; z^3 + ...)z^{\hat{\mu}} z^R, \qquad z \to 0$$

which is solution to the equation

$$\frac{d\xi}{dz} = \left[\mathcal{U} + \frac{\hat{\mu}}{z}\right] \xi, \qquad \mathcal{U} = \phi_0^{-1} \; U \; \phi_0$$

By plugging $\Xi(z, u)$ into the equation we find the H_i's. We give now the explicit expression for the entries of the H_i's. The "generic" expression is valid whenever μ does not have one of the special values listed below; in the following $h_{ij}^{(k)}(u)$ are arbitrary functions of $u = (u_1, u_2, u_3)$ to be determined later:

H_1; generic case:

$$H_{ij,1} = \frac{\mathcal{U}_{ij}}{1 + \mu_j - \mu_i}, \qquad R_1 = 0$$

$\mu = \frac{1}{2}$:

$$H_{13,1} = h_{13}^{(1)}(u), \qquad R_{13,1} = \mathcal{U}_{13},$$

$$H_{ij,1} = \frac{\mathcal{U}_{ij}}{1 + \mu_j - \mu_i} \text{ if } (i, j) \neq (1, 3)$$

$\mu = -\frac{1}{2}$:

$$H_{31,1} = h_{31}^{(1)}(u), \qquad R_{31,1} = \mathcal{U}_{31},$$

$$H_{ij,1} = \frac{\mathcal{U}_{ij}}{1 + \mu_j - \mu_i} \text{ if } (i, j) \neq (3, 1)$$

$\mu = 1$:,

$$H_{12,1} = h_{12}^{(1)}(u), \qquad R_{12,1} = \mathcal{U}_{12},$$

$$H_{23,1} = h_{23}^{(1)}(u), \qquad R_{23,1} = \mathcal{U}_{23},$$

$$H_{ij,1} = \frac{\mathcal{U}_{ij}}{1 + \mu_j - \mu_i} \text{ if } (i, j) \notin \{(1, 2), \; (2, 3)\}$$

$\mu = -1$:,

$$H_{21,1} = h_{21}^{(1)}(u), \qquad R_{21,1} = \mathcal{U}_{21},$$

$$H_{32,1} = h_{32}^{(1)}(u), \qquad R_{32,1} = \mathcal{U}_{32},$$

$$H_{ij,1} = \frac{\mathcal{U}_{ij}}{1 + \mu_j - \mu_i} \text{ if } (i, j) \notin \{(2, 1), \; (3, 2)\}$$

H_2; let $\mathcal{U}_2 := \mathcal{U} \; H_1 - H_1 \; R_1$.

Generic case:

$$H_{ij,2} = \frac{\mathcal{U}_{ij,2}}{2+\mu_j-\mu_i}, \qquad R_2 = 0$$

$\mu = 1$:

$$H_{13,2} = h^{(2)}_{13}(u), \qquad R_{13,2} = \mathcal{U}_{13,2}$$

$$H_{ij,2} = \frac{\mathcal{U}_{ij,2}}{2+\mu_j-\mu_i} \text{ if } (i,j) \neq (1,3)$$

$\mu = -1$:

$$H_{31,2} = h^{(2)}_{31}(u), \qquad R_{31,2} = \mathcal{U}_{31,2}$$

$$H_{ij,2} = \frac{\mathcal{U}_{ij,2}}{2+\mu_j-\mu_i} \text{ if } (i,j) \neq (3,1)$$

$\mu = 2$:

$$H_{12,2} = h^{(2)}_{12}(u), \qquad R_{12,2} = \mathcal{U}_{12,2}$$

$$H_{23,2} = h^{(2)}_{23}(u), \qquad R_{23,2} = \mathcal{U}_{23,2}$$

$$H_{ij,2} = \frac{\mathcal{U}_{ij,2}}{2+\mu_j-\mu_i} \text{ if } (i,j) \notin \{(1,2),\ (2,3)\}$$

$\mu = -2$:

$$H_{21,2} = h^{(2)}_{21}(u), \qquad R_{21,2} = \mathcal{U}_{21,2}$$

$$H_{23,2} = h^{(2)}_{32}(u), \qquad R_{32,2} = \mathcal{U}_{32,2}$$

$$H_{ij,2} = \frac{\mathcal{U}_{ij,2}}{2+\mu_j-\mu_i} \text{ if } (i,j) \notin \{(2,1),\ (3,2)\}$$

H_3; let $\mathcal{U}_3 := \mathcal{U}\ H_2 - H_2\ R_1 - H_1\ R_2$.
Generic case:

$$H_{ij,3} = \frac{\mathcal{U}_{ij,3}}{3+\mu_j-\mu_i}, \qquad R_3 = 0$$

$\mu = \frac{3}{2}$;

$$H_{13,3} = h^{(3)}_{13}(u), \qquad R_{13,3} = \mathcal{U}_{13,3}$$

$$H_{ij,3} = \frac{\mathcal{U}_{ij,3}}{3+\mu_j-\mu_i} \text{ if } (i,j) \neq (1,3)$$

$\mu = -\frac{3}{2}$:

$$H_{31,3} = h^{(3)}_{31}(u), \qquad R_{31,3} = \mathcal{U}_{31,3}$$

$$H_{ij,3} = \frac{\mathcal{U}_{ij,3}}{3+\mu_j-\mu_i} \text{ if } (i,j) \neq (3,1)$$

$\mu = 3$:

$$H_{12,3} = h^{(3)}_{12}(u), \qquad R_{12,3} = \mathcal{U}_{12,3}$$

$$H_{23,3} = h^{(3)}_{23}(u), \qquad R_{23,3} = \mathcal{U}_{23,3}$$

$$H_{ij,3} = \frac{\mathcal{U}_{ij,3}}{3+\mu_j-\mu_i} \text{ if } (i,j) \notin \{(1,2),\ (2,3)\}$$

$\mu = -3$:

$$H_{21,3} = h^{(3)}_{21}(u), \qquad R_{21,3} = \mathcal{U}_{21,3}$$

$$H_{32,3} = h^{(3)}_{32}(u), \qquad R_{32,3} = \mathcal{U}_{32,3}$$

$$H_{ij,3} = \frac{\mathcal{U}_{ij,3}}{3+\mu_j-\mu_i} \text{ if } (i,j) \notin \{(2,1),\ (3,2)\}$$

6.2.1 The generic case $\mu \neq \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2},\ \pm 2,\ \pm 3$

Let $\mu \neq \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2},\ \pm 2,\ \pm 3$ and let

$$\phi_0 = \begin{pmatrix} \frac{E_{11}}{f} & E_{12} & E_{13} f \\ \frac{E_{21}}{f} & E_{22} & E_{23} f \\ \frac{E_{31}}{f} & E_{32} & E_{33} f \end{pmatrix}$$

$E_{ij} = E_{i,j}(x)$ and $f(x,H)$ have been previously defined. A hard computation gives us the entries of H_1, H_2, H_3, then ϕ_1, ϕ_2, ϕ_3 and finally t and F from (6.4), (6.5). Being the computation very hard and long, we omit it and just give the result:

$$t^1 = u_1 + a(x)\ H$$

$$t^2 = \frac{1}{1+\mu}\ b(x)\ \frac{H}{f(x,H)}$$

$$t^3 = \frac{1}{1+2\mu}\ c(x)\ \frac{H}{f(x,H)^2}$$

$$F = F_0(t) + \left[\frac{a_1(x)\ c(x)^2}{2(1-2\mu)(3+2\mu)} + \frac{(\mu+4)\ b(x)\ b_1(x)\ c(x)}{2(1-\mu)(2+\mu)(3+2\mu)} + \frac{b(x)^2\ (b_2(x)-a(x))}{(2+\mu)(3+2\mu)}\right] \frac{H^3}{f(x,H)^2}$$

$$F_0(t) := \frac{1}{2} t^1 (t^2)^2 + \frac{1}{2} (t^1)^2 t^3$$

where

$$a(x) := E_{21}E_{23} + x\ E_{31}E_{33}$$

$$b(x) := E_{22}E_{21} + x\ E_{32}E_{31}$$

$$b_1(x) := E_{23}E_{22} + x\ E_{33}E_{32}$$

$$a_1(x) := E_{23}^2 + x\ E_{33}^2$$

$$b_2(x) := E_{22}^2 + x\ E_{32}^2$$

$$c(x) := E_{21}^2 + x\ E_{31}^2$$

They depend rationally on x, $y(x)$, $\frac{dy(x)}{dx}$. Note that $F - F_0$ is independent of u_1, namely it is independent of t^1.

6.2.2 The case of the Quantum Cohomology of projective spaces: $\mu = -1$

Let $\mu = -1$. This is a non-generic case, corresponding to the Frobenius Manifold called the *Quantum Cohomology* of CP^2. In this case the unknown functions $h_{21}^{(1)}$, $h_{32}^{(1)}$, $h_{31}^{(2)}$ have to be determined. It is known that

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \qquad R_2 = 0$$

The direct computation gives

$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ b(x)\ H\ f^{-1} & 0 & 0 \\ 0 & b(x)\ H\ f^{-1} & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b(x)\ H\ f^{-1}(h_{21}^{(1)} - h_{32}^{(1)}) & 0 & 0 \end{pmatrix}$$

which implies

$$f(x,H) = \frac{H}{3}\ b(x), \tag{6.6}$$

$$h_{21}^{(1)} = h_{32}^{(1)}$$

$h_{32}^{(1)}$ is determined using the differential equation :

$$\frac{\partial \phi_1}{\partial u_i} = E_i \phi_0 + V_i \phi_1$$

which implies:

$$\frac{\partial h_{32}^{(1)}}{\partial u_1} = \frac{E_{12}E_{11}}{f}, \quad \frac{\partial h_{32}^{(1)}}{\partial u_2} = \frac{E_{22}E_{21}}{f}, \quad \frac{\partial h_{32}^{(1)}}{\partial u_3} = \frac{E_{32}E_{31}}{f},$$

and thus

$$\frac{\partial h_{32}^{(1)}}{\partial u_1} + \frac{\partial h_{32}^{(1)}}{\partial u_2} + \frac{\partial h_{32}^{(1)}}{\partial u_3} = 0$$

because:

$$E_{12}E_{11} + E_{22}E_{21} + E_{32}E_{31} = 0$$

as it follows from $\phi_0^T \phi_0 = \eta$. Therefore $h_{32}^{(1)}$ is a function of $x = (u_3 - u_1)/(u_2 - u_1)$ and $H = u_2 - u_1$. Keeping into account (6.6) and the relations:

$$\frac{\partial x}{\partial u_1} = \frac{x-1}{H}, \quad \frac{\partial x}{\partial u_2} = -\frac{x}{H}, \quad \frac{\partial x}{\partial u_3} = \frac{1}{H}$$

$$\frac{\partial H}{\partial u_1} = 0, \quad \frac{\partial H}{\partial u_2} = 1, \quad \frac{\partial H}{\partial u_3} = -1,$$

we obtain

$$\frac{\partial h_{32}^{(1)}}{\partial x} = \frac{3}{x + \frac{E_{21}E_{22}}{E_{31}E_{32}}}, \quad \frac{\partial h_{32}^{(1)}}{\partial H} = 3$$

which are integrated as follows:

$$h_{32}^{(1)} = 3\ln(H) + 3\int^x d\zeta \frac{1}{\zeta + \frac{E_{21}E_{22}}{E_{31}E_{32}}}. \tag{6.7}$$

Before determining $h_{31}^{(2)}$ it is worth computing t through 6.4. Explaining the details is too long and tedious, so we give just the final result:

$$t^1 = u_1 + a(x)\ H,$$

$$t^2 = h_{32}^{(1)}, \tag{6.8}$$

$$t^3 = -c(x)\ \frac{H}{f^2} = -9\ \frac{c(x)}{b(x)^2}\ \frac{1}{H} \tag{6.9}$$

We observe that $h_{31}^{(2)}$ does not appear in t. We also observe that both t^1 and t^3 coincide with the limits for $\mu \to -1$ of the same coordinates computed in the generic case. Instead, such a limit does not exist for t^2.

Now we turn to the differential equation

$$\frac{\partial \phi_2}{\partial u_i} = E_i \phi_1 + V_i \phi_2$$

which gives three differential equations for $h_{31}^{(2)}$. Since we already know t, I write the coefficients of the equation in terms of t:

$$\frac{\partial h_{31}^{(2)}}{\partial u_i} = t^1\ \frac{\partial t^3}{\partial u_i} + t^2\ \frac{\partial t^2}{\partial u_i} + t^3\ \frac{\partial t^1}{\partial u_i}, \qquad i = 1, 2, 3.$$

which are immediately integrated:

$$h_{31}^{(2)} = \frac{1}{2}(t^2)^2 + t^1 t^3$$

.

Finally, I give the result of the (hard) computation of F through 6.5, without explaining further details:

$$F = F_0(t) + \left[\frac{1}{6}a_1(x)c(x)^2 + \frac{3}{4}b(x)b_1(x)c(x) + (b_2(x) - a(x))b(x)^2\right]\ \frac{H^3}{f^2} \tag{6.10}$$

Remarkably, this coincides with the limit, for $\mu \to -1$, of the generic case.

6.3 $F(t)$ in closed form

1)Generic case $\mu \neq \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2},\ \pm 2,\ \pm 3$

If we keep into account the dependence of $f(x,H)$ and $k(x,H)$ on H, we see that both t and $F-F_0$ can be factorized in a part depending only on x and another one depending only on H

$$t^2(x,H)=\tau_2(x)\ H^{1+\mu},$$

$$t^3(x,H)=\tau_3(x)\ H^{1+2\mu}$$

$$F(x,H)=F_0(t)+\mathcal{F}(x)\ H^{3+2\mu}$$

where $\tau_2(x)$, $\tau_3(x)$ and $\mathcal{F}(x)$ are explicitly given as rational functions of x, $y(x)$, $\frac{dy(x)}{dx}$ by the formulae of the previous sections. Hence the ratio

$$\frac{t^2}{(t^3)^{\frac{1+\mu}{1+2\mu}}}$$

is independent of H. This is actually the crucial point, because now the closed form $F=F(t)$ must be:

$$F(t)=F_0(t)+(t^3)^{\frac{3+2\mu}{1+2\mu}}\varphi\left(\frac{t^2}{(t^3)^{\frac{1+\mu}{1+2\mu}}}\right)$$

where the function $\varphi(\zeta)$ has to be determined.

Remark: Of course, also

$$F(t)=F_0(t)+(t^2)^{\frac{3+2\mu}{1+\mu}}\ \varphi_1\left(\frac{t^2}{(t^3)^{\frac{1+\mu}{1+2\mu}}}\right)$$

or

$$F(t)=F_0(t)+\frac{(t^2)^4}{t^3}\ \varphi_2\left(\frac{t^2}{(t^3)^{\frac{1+\mu}{1+2\mu}}}\right)$$

are okay.

Remark: The above forms of F can also be obtained by imposing quasi-homogeneity (namely, (1.3)).

We'll obtain closed forms $F=F(t)$ in the following way: Suppose that the entries of S are a triple (x_0,x_1,x_∞).

i) First we choose a critical point $x=0,1,\infty$ of PVI_μ and we expand $y(x;x_0,x_1,x_\infty)$ close to it up to any desired order: this is the transcendent $y(x;\sigma,a)$ of theorem 1 of chapter 5. The coefficients of the expansion, which are rationals in a and σ, are therefore classical functions of the triple (actually, a and σ are rational, trigonometric or Γ functions of the monodromy data; we refer to chapter 5 for details). The most efficient way to do the expansion is to start by computing the expansions of $\Omega_1(x)$, $\Omega_2(x)$, $\Omega_3(x)$ (we have already given the explicit connection between $y(x)$ and the Ω_i's). The algorithm used is an expansion of the Ω_i's in a small parameter [see the appendix A]. It turns out that the effective variable in the expansion is a variable $s\to 0$

$$s:=\begin{cases} x \text{ if } x\to 0\\ 1-x \text{ if } x\to 1\\ \frac{1}{x} \text{ if } x\to\infty\end{cases}$$

ii) We plug the above expansions into $\tau_i(x)$ and $\mathcal{F}(x)$, obtaining an expansion in s. In particular

$$\frac{t^2}{(t^3)^{\frac{1+\mu}{1+2\mu}}}\equiv\frac{\tau_2(x(s))}{\tau_3(x(s))^{\frac{1+\mu}{1+2\mu}}}$$

is expanded.

iii) One of the following cases may occur

$$\frac{\tau_2}{\tau_3^{\frac{1+\mu}{1+2\mu}}} \to \begin{cases} 0 \\ \infty \\ \zeta_0 \\ \text{no limit} \end{cases} \quad \text{for } s \to 0$$

where ζ_0 is a non-zero complex number. If the limit does not exist, the problem becomes complicated. This may actually occur for particular values of the monodromy data (we'll see later that this is the case of the Quantum Cohomology of CP^2, provided that we take $e^{t^2}\ (t^3)^3$ instead of $t^2\ (t^3)^{-\frac{1+\mu}{1+2\mu}}$). If the limit exists, we have a small quantity $X = X(s) \to 0$ as $s \to 0$; in the three cases above X is

$$X := \begin{cases} \dfrac{\tau_2}{\tau_3^{\frac{1+\mu}{1+2\mu}}} \\ \left(\dfrac{\tau_2}{\tau_3^{\frac{1+\mu}{1+2\mu}}}\right)^{-1} \\ \dfrac{\tau_2}{\tau_3^{\frac{1+\mu}{1+2\mu}}} - \zeta_0 \end{cases}$$

iv) We invert the series $X = X(s)$ and find a series $s = s(X)$ for $X \to 0$. Thus we can rewrite $\tau_2 = \tau_2(X)$, $\tau_3 = \tau_3(X)$, $\mathcal{F} = \mathcal{F}(X)$.

v) We compute H as a series in X and as a function of t^3. Now t^3 becomes the variable; namely:

$$H = H(X, t^3) = \left[\frac{t^3}{\tau_3(X)}\right]^{\frac{1}{1+2\mu}}$$

vi) By substituting $H(X, t^3)$ into $F - F_0 = \mathcal{F}(X) H^{3+2\mu}$ we obtain a power series for $F - F_0$ in the small variable X. In other words, we obtain $\varphi(\zeta)$ as a power series in ζ or $\frac{1}{\zeta}$ or $\zeta - \zeta_0$.

vii) Finally, we simply re-express X in term of the *variables* t^2 and t^3 and that's all. We get the closed form $F(t)$ as a power series whose coefficients are classical functions of the monodromy data.

6.4 $F(t)$ from Algebraic Solutions of PVI_μ

We refer to [21] for the algebraic solutions of PVI_μ. The Stokes' matrix of the manifold is

$$S = \begin{pmatrix} 1 & x_\infty & x_0 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix}$$

and in [21] branches of the algebraic solutions of PVI_μ are reconstructed from the above monodromy data. The construction fits into the general framework of chapter 5, but in [21] the monodromy data corresponding to algebraic solutions are carefully analyzed and parametric simple forms for the transcendents are given. In particular, the Stokes' matrices coincide with the Stokes' matrices of the Coxeter groups A_3, B_3, H_3 (or to their images with respect to the action of the braid group).

Algebraic solutions are defined up to the equivalence relations given by symmetries of PVI_μ. There are five equivalence classes. We compute $F(t)$ in closed form for the representatives of classes given below.

I) Tetrahedron (A_3), $\mu = -\frac{1}{4}$

i) $x \to 0$

$$(x_0, x_1, x_\infty) = (0, -1, -1)$$

$$y(x) = \frac{1}{2}x + O(x^2), \quad x \equiv s \to 0$$

We apply the procedure above. I computed y up to order x^{15}. The small variable is

$$X = \frac{t^2}{(t^3)^{\frac{3}{2}}} \to 0 \text{ for } x \to 0$$

and the final result is

$$F - F_0 = \frac{4}{15}k_0^4 \, (t^3)^5 - k_0^2 \, (t^3)^5 \, X^2 + O(X^m), \quad X \to 0$$

$$= \frac{4}{15}k_0^4 \, (t^3)^5 - k_0^2 \, (t^2)^2(t^3)^2 + O\left(\left[\frac{t^2}{(t^3)^{\frac{3}{2}}}\right]^m\right)$$

k_0 is the arbitrary integration constant in $k(x, H)$. I checked the result up to $m = 16$. Note that different solutions $F(t)$ corresponding to different values of k_0 are connected by symmetries of the WDVV equations [16].

ii) $x \to 1$

$$(x_0, x_1, x_\infty) = (-1, 0, -1)$$

$$y(x(s)) = 1 - \frac{1}{2}s + O(s^2), \quad s = 1 - x \to 0$$

X is like in i) and

$$F - F_0 = \frac{4}{15}k_0^4 \, (t^3)^5 + k_0^2 \, (t^3)^5 \, X^2 + O(X^m), \quad X \to 0$$

$$= \frac{4}{15}k_0^4 \, (t^3)^5 + k_0^2 \, (t^2)^2(t^3)^2 + O\left(\left[\frac{t^2}{(t^3)^{\frac{3}{2}}}\right]^m\right)$$

Here k_0^2 has the opposite sign w.r.t. the previous case. The result is checked up to $m = 16$.

ii) $x \to \infty$

$$(x_0, x_1, x_\infty) = (-1, -1, 0)$$

$$y(x(s)) = \frac{1}{2}\left[1 + O(s)\right], \quad s = \frac{1}{x} \to 0$$

Again, X is as in *i)* and *ii)*, and the result is precisely as in *i)*.

iii) Now one example for different monodromy data x_0, x_1, x_∞ and $x \to 0$

$$(x_0, x_1, x_\infty) = (1, 1, 1)$$

$$y(x) = \frac{4^{\frac{2}{3}}}{50} \, x^{\frac{2}{3}}(1 + O(x^\delta)), \quad 0 < \delta < 1, \quad x = s \to 0$$

determined by the formulae in chapter 5 or by the explicit parametric form for $y(x)$ in [21]. This time the computation of the expansion of $y(x)$ is harder than before, because of the fractional exponent. I omit any detail. The final result is:

$$\frac{t^2}{(t^3)^{\frac{3}{2}}} \to \zeta_0 = -\frac{72}{25}\sqrt{2} \, k_0, \quad x \to 0$$

$$X = \left[\frac{t^2}{(t^3)^{\frac{3}{2}}} - \zeta_0\right], \quad x \to 0$$

$$F - F_0 = \frac{119751372}{1953125}k_0^4 \, (t^3)^5 - \frac{314928\sqrt{2}}{15625}k_0^3 \, (t^3)^5 \, X^2 + \frac{2187}{625}k_0^2 \, (t^3)^5 \, X^4 + O(X^m), \quad X \to 0$$

I've checked it up to $m = 14$. Substituting X as a function of t^2 and t^3 we obtain

$$F - F_0 = \frac{4}{15}\alpha^2 \, (t^3)^5 - \alpha \, (t^2)^2(t^3)^2 + O(X^m), \quad \alpha = -\frac{2187}{625}k_0^2.$$

II) Cube (B_3), $\mu = -\frac{1}{3}$.

The computations are similar to those for the case A_3. I just give a few details, namely only the case $x \to 0$ and

$$(x_0, x_1, x_\infty) = (0, -1, -\sqrt{2})$$

$$y(x) = \frac{2}{3}x + O(x^2), \qquad x \to 0$$

Now

$$X = \frac{t^2}{(t^3)^2}$$

and the final result is

$$F - F_0 = \frac{512}{8505}k_0^6\ (t^3)^7 - \frac{16}{27}k_0^3\ (t^2)^2(t^3)^3 - \frac{2i\sqrt{2}}{9}k_0^{\frac{3}{2}}\ (t^2)^3t^3 + O(X^m)$$

for any m I have checked.

III) Icosahedron (H_3), $\mu = -\frac{2}{5}$. We give one example:

$$(x_0, x_1, x_\infty) = (0, 1, \frac{1+\sqrt{5}}{2})$$

$$y(x) = \frac{3+\sqrt{5}}{5+\sqrt{5}}x + O(x^2), \qquad x \to 0$$

Now

$$X = \frac{t^2}{(t^3)^3}$$

The final result is

$$F - F_0 = \frac{18}{55}\alpha^4\ (t^3)^{11} + \frac{9}{5}\alpha^2\ (t^2)^2(t^3)^5 + \alpha\ (t^2)^3(t^3)^2 + O(X^m)$$

where $\alpha = -512\sqrt{5}/[3(\sqrt{5}-5)^{\frac{5}{2}}(\sqrt{5}+5)^{\frac{5}{2}}]\ k_0^{\frac{5}{2}}$.

Remark: In I), II), III) we have recovered the polynomial solutions (1.5), (1.6), (1.7).

IV) Great dodecahedron (H_3), $\mu = -\frac{1}{3}$. Just the final result, which is a power series:

$$F - F_0 = (t^3)^7 \left[A_0 + \sum_{k=2}^{\infty} A_k \left(\frac{t^2}{(t^3)^2}\right)^k\right]$$

$$= \frac{512}{8505}k_0^6\ (t^3)^7 - \frac{16}{27}k_0^3\ (t^2)^2(t^3)^3 + \frac{4i\sqrt{5}}{9}k_0^{\frac{3}{2}}\ t^3(t^2)^3 + \frac{1}{8}\frac{(t^2)^4}{t^3} + \frac{i\sqrt{5}}{80k_0^{\frac{3}{2}}}\frac{(t^2)^5}{(t^3)^3} - \frac{1}{64k_0^3}\frac{(t^2)^6}{(t^3)^5} + \dots$$

the series can be computed up to any order in $X = t^2/(t^3)^2 \to 0$. We obtained it for $x \to 0$ and $(x_0, x_1, x_\infty) = (0, \frac{1+\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2})$, for $x \to 1$ and $(x_0, x_1, x_\infty) = (\frac{1+\sqrt{5}}{2}, 0, \frac{\sqrt{5}-1}{2})$, for $x \to \infty$ and $(x_0, x_1, x_\infty) = (\frac{1+\sqrt{5}}{2}, \frac{\sqrt{5}-1}{2}, 0)$.

V) Great icosahedron (H_3), $\mu = -\frac{1}{5}$. The final result is a power series:

$$F - F_0 = (t^3)^{\frac{13}{3}} \left[A_0 + \sum_{k=2}^{\infty} A_k \left(\frac{t^2}{(t^3)^{\frac{4}{3}}}\right)^k\right]$$

$$= \frac{54}{284375}\alpha^4\ (t^3)^{\frac{13}{3}} - \frac{3}{125}\alpha^2\ (t^2)^2(t^3)^{\frac{5}{3}} + \frac{i}{15}\alpha\ (t^2)^3(t^3)^{\frac{1}{3}} + \frac{1}{72}\frac{(t^2)^4}{t^3} + \frac{i}{108\alpha}\frac{(t^2)^5}{(t^3)^{\frac{7}{3}}} + \dots$$

for $X = t^2/(t^3)^{\frac{4}{3}} \to 0$. Here $k_0 = 270^{\frac{1}{5}}/30\ \alpha^{\frac{6}{5}}$. We checked this expansion for $x \to 0$ and $(x_0, x_1, x_\infty) = (0, -1, \frac{1-\sqrt{5}}{2})$, $x \to \infty$ and $(x_0, x_1, x_\infty) = (-1, \frac{1-\sqrt{5}}{2}, 0)$.

6.5 Closed form for $QH^*(CP^2)$

In this case the factorization is $t^3 = \tau_3(x)H^{-1}$ and $F - F_0 = \mathcal{F}(x)H$, but

$$t^2 = h_{32}^{(1)} = \ln(H^3) + \int^x (...)$$

This implies that

$$e^{t^2}(t^3)^3$$

is independent of H and that

$$F(t) = F_0(t) + \frac{1}{t^3}\,\varphi\left(e^{t^2}(t^3)^3\right)$$

or

$$F(t) = F_0(t) + e^{\frac{t^2}{3}}\,\varphi_1\left(e^{t^2}(t^3)^3\right)$$

The situation is more complicated now, because the behaviour of $y(x)$ for $x \to 0$ is not like $a\,x^{1-\sigma}(1 +$ higher order terms) as in the previous section. The same holds for $x \to 1$ and $x \to \infty$. The reason for this is that the monodromy data are in the orbit w.r.t the action of the braid group of the triple $(x_0, x_1, x_\infty) = (3, 3, 3)$. Hence, the monodromy data are real and their absolute value is greater than 2, so $\Re\sigma^{(i)} = 1$. For the data $(3,3,3)$ we have $\sigma^{(i)} = 1 - i\nu$, $\nu = -\frac{2}{\pi}\ln\left(\frac{3+\sqrt{5}}{2}\right)$. This case corresponds to an oscillatory transcendent if x converges to the critical points along radial directions. Moreover, we do not even know the behaviour of the transcendent and if it has poles for some values of $\arg(x)$. We recall that the effective parameter is $s = x$ or $1 - x$ or $\frac{1}{x}$. It turns out that $e^{t^2}\;(t^3)^3$ has no limit as $s \to 0$.

In the following, we compute $F(t)$ in closed form starting from the expansion of $y(x)$ and the $\Omega_i(x)$'s close to the non-singular point $x_c = \exp\{-i\frac{\pi}{3}\}$ and we obtain the expansion of $F(t)$ due to Kontsevich

$$F(t) = F_0(t) + \frac{1}{t^3}\sum_{k=1}^{\infty}\frac{N_k}{(3k-1)!}\,\left[(t^3)^3 e^{t^2}\right]^k. \tag{6.11}$$

Our result is interesting because it allows us to obtain such a relevant expression starting from the isomonodromy deformation theory applied to Frobenius manifolds.

On the other hand, it is not completely satisfactory. Since the Frobenius manifold can in principle be reconstructed from its monodromy data, we should be able to express the coefficients N_k as functions of the monodromy data. The critical behaviour of $y(x)$ close to a critical point depends upon two parameters which are classical functions of the monodromy data (actually, they depend on (x_0, x_1, x_∞) through algebraic operations and trigonometric and Γ functions). Thus, we should compute $F(t)$ from the local behaviour of $y(x)$ close to a critical point. The choice of a non-singular point x_c is not satisfactory because the expansion of $y(x)$ close to x_c depends on two parameters (initial data $y(x_c)$, $y'(x_c)$), which in general are not classical (known) functions of the two parameters upon which the critical behaviour close to a critical point depends. Thus, they are not classical functions of the monodromy data. This is due to the fact that in general the Painlevé transcendents are not classical functions.

We now compute $F(t)$ in closed form. We expand $y(x)$ close to

$$x_c = e^{-i\frac{\pi}{3}}$$

This choice comes from the knowledge of the structure of $QH^*(CP^2)$ at the "classical" point $t^1 = t^3 = 0$ [17] (se also [25]). Namely, we know that

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & 3q \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \qquad q := e^{t^2}$$

with eigenvalues

$$u_1 = 3q^{\frac{1}{3}}, \quad u_2 = 3q^{\frac{1}{3}}e^{-i\frac{2\pi}{3}}, \quad u_3 = 3q^{\frac{1}{3}}e^{i\frac{2\pi}{3}}.$$

The matrix ϕ_0 is

$$\phi_0 = \begin{pmatrix} q^{-\frac{1}{3}} & 1 & q^{\frac{1}{3}} \\ q^{-\frac{1}{3}}e^{-i\frac{\pi}{3}} & -1 & q^{\frac{1}{3}}e^{i\frac{\pi}{3}} \\ q^{-\frac{1}{3}}e^{i\frac{\pi}{3}} & -1 & q^{\frac{1}{3}}e^{-i\frac{\pi}{3}} \end{pmatrix}$$

Thus

$$x_c = \frac{u_3 - u_1}{u_2 - u_1} = e^{-i\frac{\pi}{3}}$$

Remark: x_c lies at the intersection of the "spherical" neighourhhods of the critical points $x = 0$, $x = 1$, $x = \infty$, defined by the condition that each neighbourhood does not contain another critical point in its interior. Also the point $\bar{x}_c = e^{i\frac{\pi}{3}}$ is at the other intersection. Any permutation of u_1, u_2, u_3 yields x_c or $\bar{x}_c$. The analysis which follows can be repeated at the point $\bar{x}_c$.

Remark: We understand that the choice of x_c is not satisfactory, because we have to rely on the knowledge of $QH^*(CP^2)$ at the point $t^1 = t^3 = 0$, and not only on the monodromy data (which actually were computed in chapter 4 from this knowledge itself!).

In order to compute $y(x)$ we start from $\Omega_1(x)$, $\Omega_2(x)$, $\Omega_3(x)$. We look for a regular expansion

$$\Omega_i(x) = \sum_{k=0}^{\infty} \Omega_i^{(k)}\ (x - x_c)^k, \qquad i = 1, 2, 3.$$

We need the initial conditions $\Omega_i^{(0)}$. We can compute them using

$$\Omega_i = i\mu\ \phi_{i2,0}$$

which implies

$$\Omega_1^{(0)} = -\frac{i}{\sqrt{3}}, \quad \Omega_2^{(0)} = \frac{i}{\sqrt{3}}, \quad \Omega_3^{(0)} = \frac{i}{\sqrt{3}},$$

Then we plug the expansion into (6.2) and we compute the coefficients at any desired order. Finally, we obtain $y(x)$ from (6.3). We skip the details and we just give the first terms of the expansions, the "effective" small variable being $s := x - x_c \to 0$:

$$\Omega_1 = -\frac{i\sqrt{3}}{3} - \left(\frac{1}{6} + \frac{1}{6}i\sqrt{3}\right) s + \frac{1}{9}i\sqrt{3}s^2 + \left(\frac{1}{18} - \frac{1}{18}i\sqrt{3}\right) s^3 - \left(\frac{5}{36} - \frac{5}{108}i\sqrt{3}\right) s^4 + ...$$

$$\Omega_2 = \frac{i\sqrt{3}}{3} + \left(\frac{1}{6} - \frac{1}{6}i\sqrt{3}\right) s - \frac{1}{9}i\sqrt{3}s^2 - \left(\frac{1}{18} + \frac{1}{18}i\sqrt{3}\right) s^3 - \left(\frac{5}{36} + \frac{5}{108}i\sqrt{3}\right) s^4 + ...$$

$$\Omega_3 = \frac{i\sqrt{3}}{3} - \frac{1}{3}s + \frac{2}{9}i\sqrt{3}s^2 + \frac{4}{9}s^3 - \frac{13}{54}i\sqrt{3}s^4 + ...$$

$$y(x) = \frac{1}{2} - \frac{1}{6}i\sqrt{3} + \frac{1}{3}s - \frac{1}{3}i\sqrt{3}s^2 - \frac{1}{3}s^3 + \frac{1}{9}i\sqrt{3}s^4 + \frac{13}{45}s^5 - \frac{37}{135}i\sqrt{3}s^6 - \frac{17}{27}s^7 + ...$$

Once we have Ω_1, Ω_2, Ω_3, we can compute the E_{ij}'s and finally the flat coordinates $t(x, H)$ and $F(x, H)$. At low orders:

$$t^1 = u_1 + \left[\frac{1}{2} - \frac{1}{6}i\sqrt{3} + \frac{1}{3}s + O(s^2)\right] H$$

$$t^3 = \left[-9s + O(s^2)\right]\ H^{-1}$$

$$q = \exp(t^2) = \frac{1}{143}i\sqrt{3}\ q_0\ \left[1 + i\sqrt{3}s + O(s^2)\right]\ H^3$$

where q_0 is an arbitrary integration constant (recall that t^2 is obtained by integration).

$$F = \frac{1}{6}i\sqrt{3}s^2 + \frac{1}{6}s^3 - \frac{1}{18}i\sqrt{3}s^4 + O(s^5)$$

The following quantity is independent of H

$$X := t^3 q^{\frac{1}{3}}$$

For example, if we take the cubic root $\left(-\frac{1}{6}+\frac{1}{18}i\sqrt{3}\right)\, q_0^{\frac{1}{3}}$ of $\frac{1}{143}i\sqrt{3}\; q_0$ we compute

$$X = q_0^{\frac{1}{3}}\;\left[\left(\frac{3}{2}-\frac{1}{2}i\sqrt{3}\right)s-\left(\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)s^2+O(s^3)\right]$$

Another choice of the cubic root does not affect the final result (actually, we will see that $F-F_0$ is a series in X^3). X is the small parameter that tends to 0 as $s\to 0$. We invert the series and find $s=s(X)$, and then we find $H=\tau_3(X)/t^3$ as a series in $X\to 0$. Finally, the non cubic term in F is computed:

$$F-F_0=\frac{1}{t^3}\left[\frac{1}{2\;q_0}X^3+\frac{1}{120\;q_0^2}X^6+\frac{1}{3360\;q_0^3}X^9+\frac{31}{1995840\;q_0^4}X^{12}+\frac{1559}{1556755200\;q_0^5}X^{15}+O(X^{17})\right]$$

We obtained this expansion through the expansions of the Ω_i's and of $y(x)$ at order 16. If we put $q_0=1$, this is exactly Kontsevich's solution, with

$$N_1=1,\quad N_2=1,\quad N_3=12,\quad N_4=620,\quad N_5=87304.$$

Though not completely satisfactory to our theoretical purposes, the above is a procedure to compute Gromov-Witten invariants, which is alternative to the usual procedure consisting in the direct substitution of the expansion (6.11) in the WDVV equations.

Remark: We observe that for the very special case of $QH^*(CP^2)$

$$\frac{\partial^2}{\partial(t^2)^2}\,(F-F_0)=u_1+u_2+u_3-3t^1$$

This follows from the computation of the intersection form of the Frobenius manifold $QH^*(CP^2)$ in terms of F and by recalling that its eigenvalues are u_1, u_2, u_3 (see section 2.3). Therefore,

$$\frac{\partial^2}{\partial(t^2)^2}\,(F-F_0)=u_1+u_2+u_3-3u_1-3\;a(x)\;H$$

$$=H(1+x-3a(x)),\qquad x=\frac{u_3-u_1}{u_2-u_1},\qquad H=u_2-u_1.$$

The above formula allows to compute $F-F_0$ faster than (6.10).

The formulae (6.7), (6.8), (6.9) and (6.10) are completely explicit as rational functions of $y(x)$ and $\frac{dy}{dx}$. Of $y(x)$ we partly know the behaviour close to a singular point (chapter 5), but the information we have on it when x tends to the point along a radial path is not complete. We know that

$$y(x)=\mathcal{P}(\nu_1\omega_1(x)+\nu_2\omega_2(x)+v(x);\omega_1(x),\omega_2(x))$$

and we know the form of $v(x)$ for a limited domain of x which include radial paths with some limitations on $\arg(x)$ (namely $\arg(x)$ must be greater or less of some angle if $x\to 0$). Therefore, for a limited range of $\arg(x)$ we may write the asymptotic expansion for the parametric solution (1.23) (1.22) of the WDVV eqs. Still, the problem of inversion to obtain a closed form $F(t)$ is very hard.

Appendix A

We present a procedure to compute the expansion of the painlevé transcendents of PVI_μ and of the solutions of

$$\frac{d\Omega_1}{ds} = \frac{1}{s}\,\Omega_2\Omega_3$$
$$\frac{d\Omega_2}{ds} = \frac{1}{1-s}\,\Omega_1\Omega_3$$
$$\frac{d\Omega_3}{ds} = \frac{1}{s(s-1)}\,\Omega_1\Omega_2 \tag{A.1}$$

close to the critical point $s=0$. Here we use the notation s instead of x. The system (A.1) determines the matrix $V(u_1,u_2,u_3) \equiv V(s)$

$$V(s) = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}, \qquad s = \frac{u_3-u_1}{u_2-u_1}. \tag{A.2}$$

The corresponding solution of the PVI_μ equation is

$$y(s) = \frac{-sR(s)}{1-s(1+R(s))}, \qquad R(s) := \left[\frac{\Omega_1\Omega_2+\mu\Omega_3}{\mu^2+\Omega_2^2}\right]^2 \tag{A.3}$$

A.1 Expansion with respect to a Small Parameter

We want to study the behaviour of the solution of (A.1) for $s \to 0$. Let

$$s := \epsilon\, z$$

where ϵ is the small parameter. The system (A.1) becomes:

$$\frac{d\Omega_1}{dz} = \frac{1}{z}\,\Omega_2\Omega_3$$
$$\frac{d\Omega_2}{dz} = \frac{\epsilon}{1-\epsilon z}\,\Omega_1\Omega_3$$
$$\frac{d\Omega_3}{dz} = \frac{1}{z(\epsilon z-1)}\,\Omega_1\Omega_2 \tag{A.4}$$

The coefficient of the new system are holomorphic for $\epsilon \in E := \{\epsilon \in \mathbf{C} \mid |\epsilon| \leq \epsilon_0\}$ and for $0 < |z| < \frac{1}{|\epsilon_0|}$, in particular for $z \in D := \{z \in \mathbf{C} |\ R_1 \leq |z| \leq R_2\}$, where R_1 and R_2 are independent of ϵ and satisfy $0 < R_1 < R_2 < \frac{1}{\epsilon_0}$.

We will use the small parameter expansion as a formal way to compute the expansions of the $\Omega_j's$ for $s \to 0$, the only justification being that in the cases we apply it we find expansions in s which we already know they are convergent. To our knowledge, there is no rigorous justification of the (uniform) convergence of the expansions for the Ω_j's in terms of the variable s restored after the small parameter expansion in powers of ϵ.

For $\epsilon \in E$ and $z \in D$ we can expand the fractions as follows:

$$\frac{d\Omega_1}{dz} = \frac{1}{z}\,\Omega_2\Omega_3$$

$$\frac{d\Omega_2}{dz} = \epsilon\,\sum_{n=0}^{\infty} z^n\epsilon^n\,\Omega_1\Omega_3$$

$$\frac{d\Omega_3}{dz} = -\frac{1}{z}\,\sum_{n=0}^{\infty} z^n\epsilon^n\,\Omega_1\Omega_2 \tag{A.5}$$

and we look for a solution expanded in powers of ϵ:

$$\Omega_j(z,\epsilon) = \sum_{n=0}^{\infty} \Omega_j^{(n)}(z)\,\epsilon^n\,, \qquad j = 1,2,3. \tag{A.6}$$

We find the $\Omega_j^{(n)}$'s substituting (A.6) into (A.5). Et order ϵ^0 we find

$$\Omega_2^{(0)\prime} = 0 \Longrightarrow \Omega_2^{(0)} = \frac{i\sigma}{2}$$

$$\Omega_1^{(0)\prime} = \frac{1}{z}\,\Omega_2^{(0)}\Omega_3^{(0)}$$

$$\Omega_3^{(0)\prime} = -\frac{1}{z}\,\Omega_2^{(0)}\Omega_1^{(0)}$$

where σ is so far an arbitrary constant, and the prime denotes the derivative w.r.t. z. Then we solve the linear system for $\Omega_1^{(0)}$ and $\Omega_3^{(0)}$ and find

$$\Omega_1^{(0)} = \tilde{b}\,z^{-\frac{\sigma}{2}} + \tilde{a}\,z^{\frac{\sigma}{2}}$$

$$\Omega_3^{(0)} = i\tilde{b}\,z^{-\frac{\sigma}{2}} - i\tilde{a}\,z^{\frac{\sigma}{2}}$$

where $\tilde{a}$ and $\tilde{b}$ are integration constants. The higher orders are

$$\Omega_2^{(n)}(z) = \int^z d\zeta\,\sum_{k=0}^{n-1}\zeta^k\sum_{l=0}^{n-1-k}\Omega_1^{(l)}(\zeta)\,\Omega_3^{(n-1-k-l)}(\zeta)$$

$$\Omega_1^{(n)\prime} = \frac{1}{z}\,\Omega_2^{(0)}\Omega_3^{(n)} + A_1^{(n)}(z)$$

$$\Omega_3^{(n)\prime} = -\frac{1}{z}\,\Omega_2^{(0)}\Omega_1^{(n)} + A_3^{(n)}(z)$$

where

$$A_1^{(n)}(z) = \frac{1}{z}\sum_{k=1}^{n}\,\Omega_2^{(k)}(z)\,\Omega_3^{(n-k)}(z)$$

$$A_3(z) = -\frac{1}{z}\left[\sum_{k=1}^{n}\,\Omega_2^{(k)}(z)\,\Omega_1^{(n-k)}(z) + \sum_{k=1}^{n} z^k\sum_{l=0}^{n-k}\,\Omega_1^{(l)}(z)\,\Omega_2^{(n-k-l)}(z)\right]$$

The system for $\Omega_1^{(n)}$, $\Omega_3^{(n)}$ is closed and non-homogeneous. By variation of parameters we find the particular solution

$$\Omega_1^{(n)}(z) = \frac{z^{\sigma/2}}{\sigma}\int^z d\zeta\,\zeta^{1-\frac{\sigma}{2}}R_1^{(n)}(\zeta)\; - \frac{z^{-\sigma/2}}{\sigma}\int^z \zeta^{1+\frac{\sigma}{2}}R_1^{(n)}(\zeta)$$

$$\Omega_3^{(n)}(z) = \frac{z}{i\sigma/2}\left(\Omega_1^{(n)}(z)' - A_1^{(n)}(z)\right)$$

where

$$R_1^{(1)}(z) = \frac{1}{z}A_1^{(n)}(z) + \frac{i\sigma}{2z}A_3^{(n)}(z) + A_1^{(n)}(z)'$$

Result:

$$\Omega_j(s) = s^{-\frac{\sigma}{2}} \sum_{k,\ q=0}^{\infty} b_{kq}^{(j)}\ s^{k+(1-\sigma)q} + s^{\frac{\sigma}{2}} \sum_{k,\ q=0}^{\infty} a_{kq}^{(j)}\ s^{k+(1+\sigma)q}\ , \qquad j=1,3$$

$$\Omega_2 = \sum_{k,\ q=0}^{\infty} b_{kq}^{(2)}\ s^{k+(1-\sigma)q} + \sum_{k,\ q=0}^{\infty} a_{kq}^{(2)}\ s^{k+(1+\sigma)q} \tag{A.7}$$

The coefficients $a_{kq}^{(j)}$ and $b_{kq}^{(j)}$ contain ϵ. In fact, they are functions of $a := \tilde{a}\epsilon^{-\frac{\sigma}{2}}$, $b := \tilde{b}\epsilon^{\frac{\sigma}{2}}$.

A.2 Solution by Formal Computation

Consider the system (A.1) and expand the fractions as $s \to 0$. We find

$$\frac{d\Omega_1}{ds} = \frac{1}{s}\ \Omega_2\Omega_3$$

$$\frac{d\Omega_2}{ds} = \sum_{n=0}^{\infty} s^n\ \Omega_1\Omega_3$$

$$\frac{d\Omega_3}{ds} = -\frac{1}{s}\sum_{n=0}^{\infty} s^n\ \Omega_1\Omega_2 \tag{A.8}$$

We can look for a solution written in formal series

$$\Omega_j(s) = s^{-\frac{\sigma}{2}} \sum_{k,\ q=0}^{\infty} b_{kq}^{(j)}\ s^{k+(1-\sigma)q} + s^{\frac{\sigma}{2}} \sum_{k,\ q=0}^{\infty} a_{kq}^{(j)}\ s^{k+(1+\sigma)q}\ , \qquad j=1,3$$

$$\Omega_2 = \sum_{k,\ q=0}^{\infty} b_{kq}^{(2)}\ s^{k+(1-\sigma)q} + \sum_{k,\ q=0}^{\infty} a_{kq}^{(2)}\ s^{k+(1+\sigma)q}$$

Plugging the series into the equation we find solvable relations between the coefficients and we can determine them. For example, the first relations give

$$\Omega_2 = \frac{i\sigma}{2} + \left(\frac{i[b_{00}^{(1)}]^2}{1-\sigma} s^{1-\sigma} + ...\right) - \left(\frac{i[a_{00}^{(1)}]^2}{1+\sigma} s^{1+\sigma} + ...\right)$$

$$\Omega_1 = (b_{00}^{(1)} s^{-\frac{\sigma}{2}} + ...) + (a_{00}^{(1)} s^{\frac{\sigma}{2}} + ...)$$

$$\Omega_3 = (ib_{00}^{(1)} s^{-\frac{\sigma}{2}} + ...) + (-ia_{00}^{(1)} s^{\frac{\sigma}{2}} + ...)$$

All the coefficients determined by successive relations are functions of σ, $b_{00}^{(1)}$, $a_{00}^{(1)}$. These are the three parameters on which the solution of (A.1) must depend. We can identify $b_{00}^{(1)}$ with b and $a_{00}^{(1)}$ with a.

A.3 The Range of σ

The above computations make sense if σ is not an *odd* integer, otherwise some coefficients of the expansions for the Ω_j's diverge (see for example the first terms of Ω_2 in the preceding section).

Moreover, the expansion in the small parameter yields the following approximation at order 0 for Ω_2:

$$\Omega_2 \approx \frac{i\sigma}{2} \equiv \text{constant}$$

The approximation at order 1 contains powers $z^{1-\sigma}$, $z^{1+\sigma}$. If we assume that the approximation at order 0 in ϵ is actually the limit of Ω_2 as $s = \epsilon\ z \to 0$, than we need

$$-1 < \Re\sigma < 1$$

Of course, this makes sense if $s \to 0$ along a radial path (i.e. within a sector of amplitude less than 2π).

The ordering of the expansion (A.7) is somehow conventional: namely, we could transfer some terms multiplied by $s^{\frac{\sigma}{2}}$ in the series multiplied by $s^{-\frac{\sigma}{2}}$, and conversely. I report the first terms:

$$\Omega_1(s) = bs^{-\frac{\sigma}{2}}\left(1-\frac{b^2}{(1-\sigma)^2}s^{1-\sigma}+\frac{\sigma^2}{4(1-\sigma)}s+\frac{a^2}{(1+\sigma)^2}s^{1+\sigma}+...\right)$$

$$+as^{\frac{\sigma}{2}}\left(1+\frac{b^2}{(1-\sigma)^2}s^{1-\sigma}+\frac{\sigma^2}{4(1+\sigma)}s-\frac{a^2}{(1+\sigma)^2}s^{1+\sigma}+...\right)$$

$$\Omega_3(s) = ibs^{-\frac{\sigma}{2}}\left(1-\frac{b^2}{(1-\sigma)^2}s^{1-\sigma}+\frac{\sigma(\sigma-2)}{4(1-\sigma)}s+\frac{a^2}{(1+\sigma)^2}s^{1+\sigma}+...\right)$$

$$-ias^{\frac{\sigma}{2}}\left(1+\frac{b^2}{(1-\sigma)^2}s^{1-\sigma}+\frac{\sigma(\sigma+2)}{4(1+\sigma)}s-\frac{a^2}{(1+\sigma)^2}s^{1+\sigma}+...\right)$$

$$\Omega_2(s) = i\,\frac{\sigma}{2}+i\frac{b^2}{(1-\sigma)^2}s^{1-\sigma}-i\frac{a^2}{1+\sigma}s^{1+\sigma}+...$$

Note that the dots do not mean higher order terms. There may be terms bigger than those written above (which are computed through the expansion in the small parameter up to order ϵ) depending on the value of $\Re\sigma$ in $(-1,1)$.

Finally, we note that we can always assume:

$$0 \le \Re\sigma < 1,$$

because that would not affect the expansion of the solutions but for the change of two signs. With this in mind, the expansions above are:

$$\Omega_1 = bs^{-\frac{\sigma}{2}}(1+O(s^{1-\sigma}))+as^{\frac{\sigma}{2}}(1+O(s))$$

$$\Omega_3 = ibs^{-\frac{\sigma}{2}}(1+O(s^{1-\sigma}))-ias^{\frac{\sigma}{2}}(1+O(s))$$

$$\Omega_2 = \frac{i\sigma}{2}(1+O(s^{1-\sigma}))$$

They give a Painlevé transcendent with the behaviour of theorem 1 of chapter 5.

A.4 Small Parameter Expansion in one Non-generic Case: "C-hazy Solution"

We need to investigate what happens if $\Re\sigma = 1$.

We do it only for $\sigma = 1$. If we perform the small parameter expansions as before, we find the same $\Omega_j^{(0)}$ than before. But due to the exponent $z^{-1/2}$ the integration for $\Omega_2^{(1)}$ gives

$$\Omega_2^{(1)} = -\frac{i}{2}\tilde{a}^2 z^2 + i\tilde{b}^2\ln(z)$$

In this way, we find for Ω_1 and Ω_3 an expansion in power of ϵ with coefficients which are polynomials in $\ln(z)$; also the powers $z^{-1/2}$, $z^{1/2}$,... , $z^{n/2}$, $n>0$ appear in the coefficients. Ω_2 is an expansion in power of ϵ with coefficients which are polynomials in $\ln(z)$ and z.

It is not obvious how to recombine z and ϵ when logarithms appear. We can put $\epsilon = 1$. Anyway, we see that the first correction to the constant $\frac{i}{2}$ in Ω_2 is $\ln(s)$, which is *not a correction* to the constant when $s \to 0$, because it diverges.

This makes us not trust the validity of the expansion (A.6) for $\sigma = 1$.

We can try an expansion which already contains logarithms of ϵ. The experience with Chazy solutions to the Painlevé VI eq. suggests to choose:

$$\Omega_j(z,\epsilon) = \sum_{k=-1}^{+\infty}\sum_{n=0}^{+\infty}\Omega_{j,n}^{(k)}(z)\,\frac{\epsilon^{\frac{2k+1}{2}}}{(\ln\epsilon)^n}, \qquad j=1,3 \tag{A.9}$$

$$\Omega_2 = \sum_{k=0}^{+\infty}\sum_{n=0}^{+\infty} \Omega_{2,n}^{(k)}(z)\ \frac{\epsilon^k}{(\ln\epsilon)^n} \tag{A.10}$$

Then we substitute in (A.5) and we equate powers of ϵ and $\ln\epsilon$. I omit long computations. The requirement that we could re-compose the powers of $\ln\epsilon$ and $\ln z$ appearing in the expansion in the form $\ln(z\epsilon)$ imposes very strong relations on the integration constants. The result to which we are led when we solve the equations for the coefficients $\Omega_{j,n}^{(k)}$ equating powers up to $\frac{1}{(\ln\epsilon)^n}$ and $\epsilon^{-1/2}$ is :

$$s = z\epsilon, \qquad \epsilon \to 0$$

$$\Omega_1 = \frac{i}{s^{\frac{1}{2}}(\ln(s)+C)} + O\left(\frac{1}{(\ln\epsilon)^n}\right) + O(\epsilon^{\frac{1}{2}}),$$

$$\Omega_3 = \frac{-1}{s^{\frac{1}{2}}(\ln(s)+C)} + O\left(\frac{1}{(\ln\epsilon)^n}\right) + O(\epsilon^{\frac{1}{2}}),$$

$$\Omega_2 = \frac{i}{2} + \frac{i}{\ln s + C} + O\left(\frac{1}{(\ln\epsilon)^n}\right) + O(\epsilon^{\frac{1}{2}}).$$

The substitution of these formulas in (A.3) gives the asymptotic behaviour for $s \to 0$ of the Chazy solutions. We remark that the above Ω_j's imply

$$\Omega_1^2 + \Omega_2^2 + \Omega_3^2 = -\frac{1}{4} + o(1)$$

therefore, $\mu^2 = \frac{1}{4}$, an then we have only Chazy solutions!

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