Scuola Internazionale Superiore di Studi Avanzati

International School for Advanced Studies

Anyon Physics on the Torus

Thesis submitted for the degree of

"Doctor Philosophiæ"

CANDIDATE

Kurt Lechner

SUPERVISOR

Prof. Roberto Iengo

April 1991

An Evi

Acknowledgements

First and most of all I would like to thank my supervisor Roberto Iengo who teached me so much during the period in which we worked together. All the results presented in this thesis arose from a very close collaboration with him which was very fruitful for me. I learned from him how to do computations in Physics and how to do Physics without computations. He convinced me that Physics *can be* an "honest handicraft", not only a pleasant amusement. I would like to thank him also for his friendship which accompanied our common work.

Thanks go to P.A. Marchetti and A. Schwimmer for fruitful and illuminating discussions.

This Ph.D. thesis concludes formally my years of study. In this occasion I would like to remember M. Tonin, the supervisor of my "tesi di laurea", whom I encountered at the beginning, thanking him for his continuous encouragement and for guiding me during the "first tender steps".

I would like to thank the friends whom I met at the International School in Trieste for all the hours we passed together and for their consolation when everything went wrong (I hope they did not have much to do). I would like to remember in particular Fabio Bagarello, Daniele Tommasini and Enrico Nardi who were for me more than simple colleagues. For its consolation when everything went wrong I have to thank also the "Brigantino" for the best pizza in the town.

I would like to thank my friends Paola B. and Flavio S., and Daniela C. and Paolo L. for their moral support from afar, for believing in me more than me and for keeping their fingers crossed for me many times.

I would like to thank Evi L. (to whom this thesis is dedicated) and Meinhard M. for their continuous support from my dear South–Tyrol and for softening my homesickness during these years.

Finally I would like to thank my parents, my sisters and my brother and all my nieces and nephews for enduring the presence of a "long–term student" in the family.

Contents

1. Introduction and Summary

The notion of statistics plays a fundamental role in the theory of elementary particles, its importance being related with the even more simple concept of "indistinguishable particles". This concept does not seem to play any important role in classical physics, but gives rise to far–reaching consequences at the quantum level. The dynamics of an assembly of identical particles in Quantum Mechanics is influenced not only by "conventional" forces but also by the particle statistics, which in some sense can be seen as an additional interaction. In (Quantum) Statistical Mechanics a system of fermions acts like a classical gas with repulsive interactions, while a system of bosons acts like a classical gas with attractive interactions. The significant role of these "statistical forces" in physics can be inferred remembering, for example, Pauli's exclusion principle which forms the basis for the explanation of the periodic table of the atoms, and the concepts of Bose condensation and Fermi surface in the theories of superfluidity and superconductivity.

In three spatial dimensions the only possibilities are Bose and Fermi statistics, the (covering of the) rotation group being $SU(2)$. But in two spatial dimensions particles can carry any real spin, the rotation group being $U(1)$, and due to a generalized spin–statistics connection they can obey arbitrary statistics: under interchange of two particles the quantum mechanical wave function can pick–up an arbitrary phase. The idea of fractional statistics in $D = 2 + 1$ dimensions appeared rather recently in physics[1,2] and it became popular only through the work of Wilczek[3,4], who called particles with strange spin and statistics *anyons* in that they can carry *any* spin.

Apart from opening a new research channel the introduction of anyons in two–dimensional physics created also a new point of contact between Solid State Physics and the Theory of Elementary Particles. The role of anyons[5,6] in the explanation of the Fractional Quantum Hall Effect[7] is now widely accepted: the vortex excitations of Laughlin's wave function, which represents the ground state of the $1/m$ effect, carry fractional statistics, and the ground states of the general fractional effect result from a condensation of those vortices[8] (hierarchy).

The theory of anyonic superconductivity presents a candidate for the explanation of the superconducting properties observed in certain materials at high temperature for which up to now no satisfactory explanation exists. It was argued by Laughlin [9] (one should habituate to the frequent recurrence of his name in the physics related to anyons) that a system of particles obeying fractional statistics becomes superconducting by itself. This expectation has been confirmed in refs. [10,11] which are based on a Mean field approximation. Although the phenomenological relevance of anyonic superconductivity is still unclear the theory merits the granted interest in that it presents a new mechanism which is not based on the "Fermi liquid" concept *as a matter of principle* in Solid State Physics.

This thesis should clarify some of the aspects of fractional statistics physics through the study of a system of anyons on a torus. We found that the most convenient (and evidently consistent) way to introduce anyons is through a fictitious gauge field [12] whose dynamics is governed by an (abelian) $U(1)$– Chern–Simons action. The resulting theory has attracted a lot of attention not only because of its possible relevance for condensed matter physics, as said above, but also as an interesting case of a topological field theory, see for example refs. [13–18].

The motivations for considering anyons on a toroidal surface are many, in a certain sense it is the most natural choice: the torus is compact, boundaryless and translation invariant. Its compactedness gives rise to a discrete spectrum of energy and momentum providing thus a discrete counting of states and multiplicities; this allows for a detailed analysis of many properties of the system (many treatments in the literature are implicitly based on a finite surface). One of the distinctive features of the torus with respect to Riemann surfaces of different genus, for example with respect to the sphere, is its translation invariance. Any sensible theory on a torus should respect this intrinsic symmetry and allow therefore for the construction of a covariant conserved total momentum operator whose components commute among them and also with the Hamiltonian. This program can be indeed implemented and during its implementation we discovered the remarkable role played by the topological components of the Chern–Simons field with this respect: their presence is *needed* to restore the commutativity of the Hamiltonian and the two components of the momentum operator. This fact, together with the absence of boundary of the toroidal surface, permits us then to analyse autonomous currents along the handles and we can investigate in particular the possibility of the existence of *persistent* currents, which would constitute the distinctive features of a superconducting system. Moreover, the so constructed theory admits naturally a truly *translation invariant* Mean Field Approximation, while the usual treatments on the plane give rise to a Mean Field approximation in which translations are only projectively presented[11].

Our theory is described by a first quantized non–relativistic matter field interacting with a statistical abelian Chern–Simons gauge field at a certain adimensional coupling constant k. Ultimately we deal with a (rather intriguing) quantum mechanical problem. The first "intrigue" comes from the introduction of a *rational* coupling constant k (which is needed if one wants to analyse, for example, Laughlin's vortices) because of the appearance of a global anomaly at the quantum level: the generators of global gauge transformations along the two handles of the torus commute only up to a phase, while classically they commute of course, the gauge group being $U(1)$. This anomaly does not destroy the consistency of the theory[18,19] but gives rather rise to an enlarged Hilbert space to represent the algebra of global gauge transformations. In this "large" Hilbert space we can finally invoke a superselection rule making a corresponding projection of the Hilbert space, which can be viewed also as a gauge fixing of the global gauge transformations, and we can show that the expectation values of the physical observables are independent of the gauge fixing. These results can be viewed as generalizations of the ones obtained in refs. [18,19] for the pure Chern–Simons theory.

The second "intrigue" is Modular invariance. We view our torus as a physical object immersed in three–dimensional space and from this point of view it is not obvious to us to what this invariance corresponds. However, as it stands our Hamiltonian *is* indeed modular co–variant; moreover, for each rational k we are able to construct a unique modular invariant large Hilbert space. That also the gauge fixed physical Hilbert space carries a representation of the modular group is due to a non trivial factorization property of the representation of the modular group in the large Hilbert space.

As we said above, the two components of the momentum commute among them and with the Hamiltonian. This allows us to diagonalize simultaneously the Hamiltonian and the momentum and to search in particular for the minimal energy eigenvalue at fixed total momentum. This happens also in our translation invariant Mean Field Approximation of a system of anyons at *integer* k, which on the plane is supposed to exhibit a superconducting ground state with "k Landau levels filled". We find that on the torus a ground state with (a translation invariant version of) "k Landau levels filled" exists only for macroscopically quantized values of the total momentum, while for generic momenta one has to excite also the $(k+1)$–th Landau level which is separated by an energy gap. So the energy has a sharp minimum for those particular momenta and the ground state is not degenerate. Trying to change slightly the momentum would require an amount of energy larger than the gap and therefore the corresponding macroscopically quantized currents are protected against external perturbations which try to reduce them, a property which characterizes superconducting states. With this respect it is interesting to note that we find that also the Laughlin–like ground state which arises in our treatment of anyons in an external magnetic field, relevant for the fractional Hall effect hierarchy, exists only for macroscopically quantized values of the total momentum, and hence the just described mechanism gives rise to zero diagonal resistance. In this case, however, although the result concerning macroscopically quantized momenta is the same, the way it comes out looks completely different; the energy gap, for example, is provided in this case by the Coulomb interaction.

The results presented in this thesis are based on refs. [20,21] and on work done in collaboration with R. Iengo. We will also describe further developments which are not contained in those papers.

The thesis is organized as follows. In Chapter one we recall how the notions of fractional spin and statistics appeared in two–dimensional physics and expose briefly the role played by anyons in the explanation of the Fractional Quantum Hall Effect and the mechanism of anyonic superconductivity.

In chapter two we treat in some detail the canonical quantization of the *pure* abelian Chern–Simons theory at arbitrary rational Chern–Simons coupling constant k: our formulation of the theory of anyons on the torus in the next chapter is based on this theory. We use an algebraic approach to investigate the main properties of this system concentrating on the construction of the Hilbert space and on its modular properties. We explain in detail how to manage consistently the global anomaly mentioned earlier. The structure of the Hilbert space which results, in the coherent state representation, constitutes the starting frame for the coupling to matter in chapter three.

In chapter three we introduce a (non relativistic) matter field and develop its coupling to the Chern–Simons gauge field, which induces the statistics flip, in first quantization. Due to the presence of an integer number of anyons and to the Dirac quantization condition on a torus k is forced to be rational. We construct the Hamiltonian and the total momentum operator, evidencing the crucial role played by the topological components of the Chern–Simons field with respect to their commutativity, and determine the conditions which define the Hilbert space. These conditions are then completely solved to obtain an explicit basis for the whole Hilbert space. We define then a gauge–fixed Hilbert space proving that it carries a representation of the modular group and that, a necessary consistency check, the physics is independent of the gauge–fixing. In section 4.6 we find the exact ground state solutions of a "self–dual" Hamiltonian [22,23].

Throughout this thesis we work in a "gauge" where the wave function obeys ordinary statistics and the fractional statistics is presented by a (non trivial) Hamiltonian which contains the Chern–Simons field. On the plane there exists a singular gauge transformation which transforms the Hamiltonian in the free one and the wave function in a "function" which picks up a phase if two particles are interchanged. In section 4.7 we determine the singular gauge transformation on the torus which transforms the Hamiltonian in an "essentially free" Hamiltonian and the (bosonic or fermionic) wave function in an anyonic one. We find that the "essentially free" Hamiltonian can be further *reduced* to the free one at the expense of introducing a *multi–component* wave function, in agreement with general braid group analysis results on non simply connected surfaces [24].

In chapter five we consider a system of anyons at *integer* coupling constant k with the purpose of investigating its superconducting properties. We consider in particular its Mean Field Approximation which turns out to be translation invariant. In this approximation the Hamiltonian problem of the many–body system can be completely solved; the many–body energy eigenstates at fixed total momentum turn out to constitute a kind of translation invariant Landau–levels, with a collective degeneracy which turns out to be somewhat smaller than the one obtained on the plane by taking direct products of single–particle Landau–levels. In particular, our many–body momentum eigenstates *can not* be factorized into one–particle states. We derive explicitly the antisymmetric many–body ground state at fixed momentum and find the macroscopic quantization of momenta and the corresponding superconductivity mechanism, mentioned before. These protected states generate a real magnetic field inside the cavity of the torus, which we compute, and whose flux turns out to be quantized as $\frac{1}{k}$ times the fundamental unit of flux. It is interesting to note that this is precisely the amount of the elementary fluxoid excitation entering the discussion in refs. [9,11].

Chapter six is devoted to the analysis of a system of anyons on a torus in a (real) external magnetic field, which we think of as one of the vortex excitation components appearing in the ground state of Haldane's hierarchy of the Fractional Quantum Hall Effect. Said in other words, we consider the Hall Effect of anyons. Our treatment differs, however, from the usual ones in that we impose the vanishing of the Lorentz force, as is appropriate for the classical Hall effect, by means of an effective Lagrangian (the Lorentz force has to be cancelled by the electric field). Accordingly this Lagrangian has then to contain a Chern–Simons action also for the real electromagnetic field. Then topological components of the electromagnetic field appear naturally and, as one can expect, the overall translation invariance, which is broken by the introduction of the external magnetic field, is restored. In summary, the introduction of an electromagnetic Chern–Simons action in the Lagrangian imposes the vanishing of the Lorentz force and restores, at the same time, translation invariance. Using the results of the previous chapters we construct the Hilbert space, and we find, moreover, the exact ground state at fixed momentum (minimizing the Coulomb repulsion a la Laughlin). This Laughlin–like ground state turns out to exist only for particular values of the momentum of the total system (electrons plus vortices) and this explains, as said above, the vanishing of the diagonal resistance. Finally we repeat the steps, which lead to the fractional Hall hierarchy, on the torus.

2. Fractional Statistics and Anyon Physics

In this introductory chapter we recall some of the fundamental notions regarding fractional statistics in $2+1$ dimensions and their appearance in effectively two–dimensional physical systems. We do not pretend to be exhaustive from the historical point of view nor to present all the developments in various directions which occurred since the discovery of this interesting "new" physics (for instance, the important relationship between Chern–Simons Theories and Conformal Field theories, see for example refs. [13–18], and the interesting feature of P and T violation in the presence of anyons, see for instance refs. [25,26], are not discussed all). We will in particular concentrate on those aspects which enter crucially in the theory presented in this thesis. For more details about the role of excitations with fractional quantum numbers in the quantum Hall Effect see refs. [27,28]; for their role in two dimensional models of high–T_c superconductivity see ref. [29].

2.1 Fractional statistics and spin in $D = 2 + 1$

In rotation invariant quantum systems in D space–time dimensions the spin, S, of massive particles labels the irreducible unitary representations of the covering group $\widetilde{SO}(D-1)$ [30,31]. For $D = 4$ these representations are labelled by integers and half–integers, but in three space–time dimensions the rotation group is $SO(2) \sim U(1)$, whose covering group is isomorphic to the real line. Therefore the spin S can be *any* real number: $S \in R$. This is the reason why Wilczek [4] called particles with arbitrary real spin S in $2+1$ dimensions *anyons*.

In analogy with the Spin-Statistics theorem in four dimensions it is then expected that those particles obey fractional statistics. That this is indeed the case is also well established from an axiomatic point of view, especially in Relativistic Quantum Field Theories (see [32,33] and references therein). If under interchange of two particles one gets a phase $e^{i\theta}$, the corresponding spin is

$$S = \frac{\theta}{2\pi} \; mod \; Z. \tag{2.1}$$

$\theta = 0$ corresponds to bosons and $\theta = \pi$ corresponds to fermions. This $U(1)$–spin should not be confused with the three–dimensional spin.

The paper by Leinaas and Myrheim [2] presents the first explicit and particularly lucid account of the possibility of fractional statistics. It should be noted that also in ref. [34], independently of Leinaas and Myrheim, the possibility of fractional statistics particles and many of their main properties have been discovered. The importance of topology for the quantization of extended objects and its role in giving rise to the phenomenon of "strange" quantum numbers have already been established in 1967 by Finkelstein and Rubinstein [1]. Several of the results of the quoted references were rediscovered and made popular by Wilczek [3,4]. Following him we can explain the principal ideas in a simple physical situation. Let us imagine to have a very thin and very long solenoid disposed along the z–axis. The singular limit of this situation is a delta–function flux tube, realized by the vector potential

$$A_i = -\frac{\Phi}{2\pi} \frac{\varepsilon_{ij} x_j}{|\vec{x}|^2}. \tag{2.2}$$

For what follows it is not essential to take this limit, (2.2) permits however to check everything directly. Away from the origin the corresponding magnetic field, $\vec{B} = \vec{\nabla} \times \vec{A}$, is zero, as is appropriate for a solenoid of course, while its flux across a surface perpendicular to the z–axis is

$$\int \vec{B} \cdot d\sigma = \oint \vec{A} \cdot d\vec{x} = \Phi.$$

This can be seen most easily by writing the components of $\vec{A}$ in polar coordinates: $A_r = 0$, $A_\varphi = \frac{\Phi}{2\pi}$. Classically this vector potential has no effect since it gives vanishing magnetic field strength, but it gives rise to a non trivial quantum mechanics, as pointed out by Aharonov and Bohm [35]. We consider a particle of charge q which is bound to move in a surface perpendicular to the solenoid. Clearly, if no current flows in the solenoid then, due to the facts that the wave function is periodic in φ and that rotations around the z–axis are generated by $l_z = \frac{1}{i}\partial_\varphi$, the angular momentum is quantized in units of integers: $l_z = m$. If, on the other hand, $\Phi \neq 0$ rotations around the z–axis are generated by the covariant angular momentum $l_z = \frac{1}{i}\partial_\varphi - qA_\varphi$. If the azimuthal dependence of the wave function is $\Psi \sim e^{im\varphi}$, with m integer for continuity, then the angular momentum is quantized as

$$l_z = m - \frac{q\Phi}{2\pi} \tag{2.3}$$

which in general is neither integer nor half integer. Things can be presented in a different way eliminating the gauge potential via a (singular) gauge transformation

$$A_i' = A_i - \partial_i \Lambda = 0, \quad \Lambda = \frac{\Phi}{2\pi} \cdot \varphi$$

Then the wave function, $\Psi' = e^{-iq\Lambda}\Psi$, is no longer single valued, but behaves as

$$\Psi'(\varphi + 2\pi) = e^{-iq\Phi}\Psi'(\varphi). \tag{2.4}$$

There is no vector potential and the angular momentum is identified as usual by $l_z = \frac{1}{i}\partial_\varphi$, and the same spectrum (2.3) of fractional spin is gotten. Wilczek expected, according to a generalized spin–statistics connection, that these "flux–tube–charged–particle composites", which he called anyons, obey fractional statistics, interpolating between bosons and fermions. If we take, in fact, two anyons and transport the first of them covariantly in the vector potential of the second along a closed loop, which contains the second anyon inside, we get a phase $e^{-2iq\Phi}$ (a contribution $e^{-iq\Phi}$ comes from rotating the first anyon around the second and the other contribution $e^{-iq\Phi}$ comes from rotating the second anyon around itself). However, if the two anyons are *identical* particles we have to get a definite phase also if we *interchange* them; for consistency this phase has then clearly to be half of the phase we have just computed, i.e. $e^{-iq\Phi}$. Comparing with (2.3) we realize that the spin–statistics relation (2.1) is indeed satisfied. We should notice that the "charges" and "fluxes" entering the discussion are fictitious objects which should not be confused with the corresponding "real" electromagnetic counterparts.

The scenario presented here, a part from explaining the consistency of the notions of fractional spin and statistics in two dimensions, constitutes also a proposal, even if somehow implicit, of how to introduce objects with fractional statistics in quantum mechanics, i.e. via the use of fictitious gauge potentials. The Chern–Simons construction for fractional statistics, which extends naturally the above ideas to a many body system, appeared clearly for the first time in the paper by Arovas, Schrieffer, Wilczek and Zee [12] on the "Statistical Mechanics of Anyons". They introduced a $(2+1)$–dimensional fictitious gauge potential A_μ and a matter current J_μ together with the action

$$S = \frac{k}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \int d^3x \, A_\mu \, J^\mu .$$

The first term in this expression is called Chern–Simons action. The zero–*th* component of the equations of motion gives the constraint $\partial_x A_y - \partial_y A_x = -\frac{2\pi}{k} J_0$, which states that the magnetic field is proportional to the matter density. In quantum mechanics J_0 becomes a sum of δ–functions and one can solve this equation for $\vec{A}$. The l–*th* particle feels then a vector potential $\vec{A_l} = \frac{1}{k}\vec{\nabla}_l \times \sum_{j\neq l} \ln|\vec{x}_l - \vec{x}_j|$. It is not difficult to realize that, as before, also this gauge field can be eliminated by a gauge transformation at the expense of introducing a multivalued wave function (see section 4.7 for more details)

$$\Psi^{an} = \prod_{i<j} \frac{(z_i - z_j)^{1/k}}{|z_i - z_j|^{1/k}} \, \Psi \tag{2.5}$$

where Ψ obeys ordinary statistics and $z = x + iy$. This wave function gives under the interchange of two particles a phase $\exp\left(\frac{i\pi}{k}\right)$. In section 4.7 we will in particular see in which sense this "anyon gauge", in which the Hamiltonian is the free one and the wave function is multivalued, is accessible also on the torus.

Let us also state the composition law for fractional spins. If we have a composite of n identical anyons, each of which carries spin S, the composite carries a spin S_n which is given by

$$S_n = n^2 \cdot S \;\; mod \; Z. \tag{2.6}$$

It is easily checked that for the ordinary cases $S = 0, 1/2, 1, ...$ this gives the usual composition law.

In this thesis we will concentrate primarily on non relativistic Quantum Mechanics of anyons; for a non relativistic second quantized formulation on the plane see for instance [23].

For completeness we remark also that in the classical non linear sigma model the introduction of the so called Hopf term in the action [36] gives rise, at the quantum level, to fractional spin and statistics too.

2.2 Anyons and Quantum Hall Effect

Laughlin's explanation of the Integer Quantum Hall Effect [37] (IE) is based on very fundamental considerations. Relying on the hypothesis of the existence of an energy gap and using gauge invariance Laughlin showed that the Hall conductivity is quantized as

$$\sigma_H = n \cdot \frac{e^2}{h} \tag{2.7}$$

where n is an integer, and only the fundamental constants e and h show up. This is the conductivity which would arise from n Landau levels (see below) being completely filled. Two peculiar features of the (IE) are the appearance of plateaus of the conductivity as a function of the filling factor (classically one would obtain a straight line) and the corresponding vanishing of the longitudinal resistance [27,28]. If the system were described by the ideal one–particle "free" Hamiltonian given below, i.e. in the absence of disorder, both these phenomena would not occur. The fundamental importance of the presence of *impurities* for the existence of the (IE), i.e. plateaus and zero diagonal resistance, was noted by Prange in ref. [38]. In the presence of impurities localized states are created, in contrast to the extended states corresponding to (perturbed) Landau levels, and the Fermi level can ly in a mobility gap in a band of localized states. Roughly speaking, this gap insures the vanishing of the diagonal resistance and, as shown in [38], the current carried by the remaining extended states is appropriately enhanced to give just back the conductivity (2.7) of "n filled Landau levels" (with respect to this "conservation law" see also the review [28] and references therein).

To summarize the essential features of the explanation of the Fractional Quantum Hall Effect [7,39] (FE) let us recall the form of the "free" one–particle Hamiltonian (we neglect the spin):

$$H_0 = \frac{2}{m} D^\dagger D + \frac{eB}{2m}. \tag{2.8}$$

Here B is the external magnetic field, orthogonal to the plane, and m and e are the mass and charge of the electron respectively. In Laughlin's treatment of the (FE) the Coulomb repulsion is taken into account indirectly[7], see below, and the disorder is treated as in the (IE). The covariant derivative (in complex notation) is given by

$$D = \partial + \frac{1}{4}\, eB\bar{z}$$

and satisfies $[D, D^\dagger] = \frac{eB}{2}$. We define the magnetic length l, which determines the scale of the problem, as $l^2 = \frac{1}{e\,B}$. The energy levels (Landau levels) are given by

$$\varepsilon_n = \left(n + \frac{1}{2}\right) \frac{eB}{m}$$

and the related eigenfunctions are

$$\psi_n = \left(D^\dagger\right)^n \psi_0$$

where the ground state is determined through the equation $D\,\psi_0 = 0$. This condition is solved by

$$\psi_0 = g(\bar{z}) \exp\left(-\frac{|z|^2}{4l^2}\right)$$

where g is an arbitrary antiholomorphic function. So, on the plane each Landau level is infinitely degenerate. If the surface has a finite area ν, however, this degeneracy Ω becomes finite in that it is proportional to the total flux of the magnetic field according to

$$\Omega = e\,\frac{B \cdot \nu}{2\pi}.$$

Given a system of N particles the "filling" factor f is defined by $f = \frac{N}{\Omega}$. If $f \leq 1$, according to the (FE), all the electrons can be accommodated in the ground state, and, in the ideal case under investigation here, the many–particle antisymmetric ground state exhibits a large degeneracy (only for $f = 1$, i.e. $N = \Omega$ this ground state is unique). This degeneracy is clearly removed if one takes into account the Coulomb interaction. Supposing that the introduction of the Coulomb repulsion does not lead to the excitation of the higher Landau levels (as is reasonable if the magnetic field is large) one has to minimize it among the states

$$g(\bar{z}_1, ..., \bar{z}_N) \exp\left(-\frac{1}{4l^2}\sum_{i=1}^{N} |z_i|^2\right)$$

where g is an antiholomorphic function. For fractional fillings of the form $f = \frac{1}{2J+1}$ with J integer Laughlin [7] proposed as minimizing solution, according to his "one–component plasma analogy", the state

$$\psi^J = \prod_{i<j} (\bar{z}_i - \bar{z}_j)^{2J+1} e^{-\frac{1}{4l^2}\sum |z_i|^2}. \tag{2.9}$$

which describes a circular, incompressible [7] droplet of fluid. This state, which, as emphasized by Laughlin, is an angular momentum eigenstate, is antisymmetric and has a zero of the highest possible order as $z_i \to z_j$ as is appropriate to minimize the repulsive Coulomb interaction. The projection of this state on the one computed numerically, with various kinds of repulsive potentials, gave 1 with a precision of one per thousand [7], meaning that ψ^J reproduces the correct ground state with very high precision and that the solution in (2.9) has universal character in that it is largely independent of the form of the interaction. The analog of (2.9) on the torus was found by Haldane and Rezayi in ref. [40]. In this case it follows from the theory of holomorphic sections on the torus that the highest order of the zero for $z_i \to z_j$ is precisely $2J+1$. This rederivation of the Laughlin wave function (2.9) emphasizes that it is the correct short distance behavior of the wave function rather than angular momentum considerations that lie behind the explanation of the (FE).

In ref. [7] Laughlin gave also an (approximate) analytic expression for the elementary excitations of ψ^J, localized in z_0, which correspond to piercing the fluid at z_0 with an infinitely thin solenoid and passing through it a flux quantum $\Delta\Phi = \frac{2\pi}{e}$ adiabatically. Accordingly approximate quasi–holes and quasi–particles are given respectively by:

$$\begin{aligned} \psi_-^J &= e^{-\frac{1}{4l^2}\sum |z_i|^2} \prod_i (\bar{z}_i - \bar{z}_0) \prod_{i<j} (\bar{z}_i - \bar{z}_j)^{2J+1} \\ \psi_+^J &= e^{-\frac{1}{4l^2}\sum |z_i|^2} \prod_i \left(\frac{\partial}{\partial \bar{z}_i} - \frac{z_0}{2l^2} \right) \prod_{i<j} (\bar{z}_i - \bar{z}_j)^{2J+1} \end{aligned} \tag{2.10}$$

Also the projection of these states onto the ones computed numerically shows that they represent the excitations of the Laughlin wave function with a high accuracy. The quasi–hole state ψ_-^J represents a charge deficiency in z_0 (the wave function goes to zero if one of the particles approaches z_0) whose amount Laughlin derived already within his "one–component plasma approach". He obtained that the charge of the quasi–hole/electron state is given by

$$e^* = \pm \frac{e}{2J+1} \tag{2.11}$$

and is thus fractional.

If the filling is not of the particular form $\frac{1}{2J+1}$ a state of the form (2.9), which gives rise to a particularly low Coulomb energy, does not exist; hence this state is protected by an energy gap and particularly stable, explaining the drop in the diagonal resistance in the (FE) at those fillings. The presence of a corresponding "small" plateau in the conductivity is explained like in the (IE)[7].

The Hall effect at more general rational fillings $f = p/q < 1$, where q is an odd integer, cannot be explained in this way. For these fillings Haldane[8], using a formalism based on a spherical geometry, proposed a hierarchical scheme of ground states. Within this proposal for fillings $f = p/q$ the ground state is derived from a parent state of the form (2.9) with an imbalance of quasi–particle or quasi–hole excitations. In the simplest case we have a two–component fluid with an *electron* component and one fractionally charged say *quasi–particle* component. If the effective filling (see chapter six) of the quasi–particle component is of the form $\frac{1}{2J'+1}$, for some integer J', the collective ground state of the excitation fluid is analogous to the ground state of the electron fluid (2.9) and hence particularly stable. In this way Haldane deduced that the filling factors at which a (FE) can occur can be expressed by a continued fraction

$$f = \cfrac{1}{2J+1+\cfrac{\alpha_1}{2J_1+\cfrac{\alpha_2}{\cdots + \cfrac{\alpha_n}{2J_n}}}} \tag{2.12}$$

where $\alpha_j = \pm 1$ according to whether the corresponding component is made out of quasi–holes or quasi–particles. The fillings given in this formula are all rational with an *odd* denominator.

Halperin realized in ref. [5] the important fact that the quantization rule (2.12) is just the one which arises from a set of identical charged particles that obey *fractional statistics*, i.e. such that the wave function changes by a complex phase factor when two particles are interchanged. He got this result on the plane in the "anyon gauge", i.e. using a representation in which the Hamiltonian for the excitations contains only the external magnetic field while the wave function is multivalued. In our treatment of the hierarchy on the torus (see chapter six) in order to solve certain phase ambiguities it is necessary to work in a gauge in which the wave function is well defined and the statistics is represented by a fictitious Chern–Simons gauge potential. For the details on the derivation of the allowed fillings within the framework of the hierarchy we refer to chapter six, where the extension to the torus is given too.

Let us also state the important result obtained by Arovas, Schrieffer and Wilczek in ref. [6] who computed explicitly the quantum numbers of the quasi–holes and quasi–particles given by Laughlin (2.10), using a Berry–phase technique[41]. According to Berry, given a Hamiltonian $H(z_0)$ which depends on a parameter z_0, if z_0 transverses slowly a loop, then in addition to the usual phase $\int^t E(t')dt'$, where $E(t')$ is the adiabatic energy, an extra phase γ occurs in $\psi(t)$ which is independent on how slowly the path is transversed. This phase is given by

$$\gamma = -i \oint \left\langle \psi_-^J(t), \frac{d\psi_-^J}{dt} \right\rangle dt. \tag{2.13}$$

A particular case is given by the Aharonov–Bohm phase discussed in section 2.1. Let us consider a path in which z_0 moves adiabatically around a circle of area A enclosing magnetic flux $\Phi = -B \cdot A$. From (2.4) we see that the wave function acquires a phase which is given by

$$e^* B \cdot A \tag{2.14}$$

where e^* is the electric charge of the quasi–hole ψ_-^J. On the other hand eq. (2.13) can be easily evaluated[6] to give

$$\gamma = -2\pi\, A\, \rho = -\frac{eB \cdot A}{2J+1} \tag{2.15}$$

where ρ is the electron's density. Comparing with (2.14) we get back Laughlin's result (2.11) for the charge of his excitations.

With the same method we can compute the phase we get if one excitation moves around another. Eq. (2.15) is in fact very general; it says that the phase is -2π times the matter density enclosed by the loop times the area, i.e. -2π times the number of electrons enclosed. But we have just learned that Laughlin's (hole)excitations carry electron number $-\frac{1}{2J+1}$. Taking into account that an interchange is half a winding we get that the statistical phase of a quasi–hole is given by

$$\exp\left(\frac{i\pi}{2J+1}\right).$$

This shows explicitly that the elementary excitations of Laughlin's wave function obey fractional statistics and confirms therefore Halperin's results about the origin of the fractional Hall hierarchy.

2.3 Anyons and Superconductivity

In refs. [9,42], based on a previous paper by Kalmeyer and Laughlin [43] on the "Equivalence of the Resonating–Valence–Bond and Fractional Quantum Hall States" Laughlin argued that a system of two–dimensional quasi–particles with fractional statistics exhibits a superconducting ground state, and that this mechanism could account for high–T_c superconductivity. Based on Andersons resonating–valence–bond idea [44,45] Kivelson, Rokhsar and Sethna [46] proposed an explanation of this phenomenon which relied on a Bose condensation of "holons", i.e. charged particles with zero three–dimensional spin, which can be thought of as formed by a charged fermion and a "spinon", a neutral spin 1/2 excitation. Since holons have zero spin one can imagine that they are bosons and that they can undergo a direct bose condensation without any pairing. If this would be so then the flux quantum would be h/e. Experimentally it appears, however, to be $h/2e$, at least in the regimes where it has been studied so far, indicating the presence of a charge–2 condensate. Laughlin suggested [9] then that spinons as well as holons obey 1/2 (semionic) fractional statistics. If this is so then holons are no longer bosons, they cannot undergo Bose condensation, but *pairs* of them are bosons, see (2.6), and Laughlin found this to be a good reason to suspect that a gas of particles obeying 1/2 statistics might actually be a superconductor with a charge–2 order parameter (see also ref. [47]).

There is an extremely naive argument, which suggests that in general – excluding fermions – an anyon gas will be superfluid (or, if the anyons are electromagnetically charged, superconducting) at zero temperature. Fermions with arbitrarily weak attractive forces are, in fact, known to form superfluids at zero temperature; on the other hand, the analysis about the statistical mechanics of anyons in ref. [12] revealed that anyons interpolate between fermions (which act like a classical gas with *repulsive* interactions) and bosons (which act like a classical gas with *attractive* interactions). Therefore anyons can be seen as fermions with an attractive force among them.

Following Laughlin one is therefore led to consider a system of N_A particles with fractional statistics $e^{\frac{i\pi}{k}}$, whose wave function is given by (2.5) where we take Ψ to be fermionic (see [11] for a discussion of the bosonic case). The dynamics of this system is governed by the free Hamiltonian $H_{free} = \sum_{j=1}^{N_A} \frac{p_j^2}{2m}$. It is convenient to present the system in the "fermion gauge" where the Hamiltonian becomes

$$H = -\frac{1}{2m} \sum_{j=1}^{N_A} \left(\vec{\partial}_j - ie\vec{A}_j \right)^2 \tag{2.16}$$

and acts on a wave function Ψ which is antisymmetric under $\vec{x}_i \leftrightarrow \vec{x}_j$. Here the vector potential (we introduced the fictitious charge e) is

$$\vec{A}_j = \frac{1}{ke} \vec{\partial}_j \times \sum_{i \neq j} \ln|\vec{x}_j - \vec{x}_i|. \tag{2.17}$$

The (pointlike) fictitious flux associated to each particle is therefore given by $-\frac{2\pi}{ek}$, i.e., restoring the fundamental constants, $-\frac{hc}{ek}$ (the sign is of course only a matter of convention). The Hamiltonian problem can unfortunately not be solved exactly; in ref. [48] the Hartree-Fock approximation is used to solve for the ground state, showing also that the method gives sensible results for the case $k = 1$, which corresponds to free bosons and whose solution is explicitly known. In that paper it is also realized that the Hartree–Fock approximation is equivalent to the mean field approximation; we will now expose briefly the results of the latter[9,11,49].

We saw that the statistical interaction can be implemented by attaching fictitious charge and flux to fermions, which give rise to long range interactions, see (2.16). We would like to replace the effect of many distant particles by a mean field; the deviations from the mean interaction should be represented by residual weak or short–range interactions which one hopes to can treat as a perturbation. In our case it corresponds to replace the many singular flux tubes by a *uniform* fictitious magnetic field with the same total flux. If the total area is ν, and hence the average particle density $\rho = \frac{N_A}{\nu}$, the magnitude of this mean magnetic field, tied to the density of the particles, is

$$b = -\frac{2\pi}{ke}\rho.$$

Classically the particles move then along cyclotron orbits with radius

$$r = -\frac{mv}{eb}.$$

Taking for the velocity the one of the nominal Fermi surface

$$v = \frac{\sqrt{4\pi\rho}}{m}$$

one finds[11] that a typical cyclotron orbit contains $\rho\pi r^2 = k^2$ particles on the average. If this number is large with respect to unity, we should expect that the mean field approximation gives sensible results in that each particle "sees" a large number of point–like fluxes which on the average can then likely be replaced by their constant mean value. From this point of view this approximation corresponds to an expansion around fermionic statistics since as $k \to \infty$ the wave function in the anyon gauge becomes antisymmetric. The ultimate goal would be to hope that the results obtained in this way can be extrapolated down to small values of k, e.g. $k = 2$. In ref. [50], based on exact numerical calculations with a small number of particles, it is found that the statistical fluxes and the external constant magnetic fluxes are to some extent interchangeable. This is can be viewed as a signal for the fact that the mean field approximation gives indeed sensible results even if the number of particles and k are not extremely large.

Let us state the most important results of the mean field theory. The system corresponds to N_A particles which move in a constant magnetic field and is solved exactly by the Landau levels (see sections 2.2 and 5.2). The one–body energy spectrum is given by

$$E_n = -\left(n+\frac{1}{2}\right)\frac{eb}{m} = \left(n+\frac{1}{2}\right)\frac{2\pi}{mk}\rho$$

and the degeneracy of each Landau level is determined by the total flux of the magnetic field

$$-\frac{e}{2\pi}\,b\nu = \frac{N_A}{k}. \tag{2.18}$$

If the statistics determining parameter k is an *integer* then the ground state of the system is constituted by k exactly filled Landau levels. So, at these values of the statistical parameter the ground state will have a particularly favorable energy. Exactly filling the top band ought to be analogous to complete a shell in an atom, or filling a band in a solid. The ground state should therefore exhibit a certain rigidity with respect to external perturbations and an energy gap, which is nominally the difference between two Landau levels $\Delta E = \frac{2\pi}{mk}\,\rho$, and which is also the energy needed to create a particle–hole excitation. In ref. [11] the effect of adding, or subtracting, a *real* small magnetic field to the system in the ground state is investigated and it is found that in both cases the energy increases. This suggests that the anyon gas will strive to exclude an external magnetic field, which points in the direction of a Meissner effect, a phenomenon which is typical in superconducting systems (see also refs. [51,52]), and induced the authors of [11] to identify the charged quasi–particles excitations with the flux carrying vortex excitations. Via this identification the value of the flux quantum can also be inferred. Adding, in fact, a single fundamental unit $-\frac{2\pi}{e}$ of real flux increases the number of available states per Landau level by one, see (2.18). Thus for k filled Landau levels this corresponds to create k holes. But this is not the most elementary excitation; the most elementary excitation corresponds to the creation of just *one* hole and the elementary fluxoid is therefore $\frac{1}{k}$ of the fundamental unit:

$$-\frac{2\pi}{ke}.$$

Another important aspect of the system under investigation here, which is not exhibited by the mean field approximation, is the existence of a sharp isolated Goldstone mode in the spectrum, i.e. the existence of an excitation with the dispersion relation $\omega \sim v_0|\vec{k}|$ at low frequencies. This was discovered by Fetter, Hanna and Laughlin [10] who, taking into account the residual interactions which are neglected in the mean field approximation, found the necessary pole in the current–current correlator (their computation does, however, unfortunately not reveal the origin of this massless mode). This points on one hand in the direction of the existence of long–wavelength sound waves which have nothing they can possibly decay into and present therefore dissipationless superflows (the anyon gas is *compressible* in contrast to the ground state of the Quantum Hall effect which is indeed *incompressible*, due to the fact that the external magnetic field is fixed). In ref. [29] this longitudinal mode is viewed as an essential ingredient for the full explanation of the Meissner effect. On the other hand the presence of this bosonic zero mode should be related to some spontaneously broken symmetry, which is exact in the microscopic theory, but violated in the effective mean field theory. It is not obvious at all, at least to me, which is the symmetry that is broken; in refs. [11,29] it is argued that what is broken is not a symmetry, but rather a "fact", namely the generators of the magnetic translations, which commute with the Hamiltonian, do *not* commute among themselves (their commutator is a c–number) meaning that in the mean field theory the group of translations is *projectively* represented (see for instance [53]). This implies in particular that there are no states which are invariant under translations. From this point of view it is interesting to notice that in chapter five we will be able to construct a mean field theory for anyons on the torus which is *truly* translation invariant, i.e. where H, P_x and P_y commute. Still we find evidence of a superconducting quantum state.

The unsatisfactory aspects of the theory of anyonic superconductivity in its present stage, apart from its still unclear relevance from the phenomenological point of view, are the elusiveness of the order parameter, the unclear role of the spontaneous symmetry breaking, which is of fundamental importance in the "classical" theories of superfluidity [54] and superconductivity [55], and, perhaps most importantly, the fact that the mean field theory and the related self–consistent field approximation (see for instance refs. [49,56]), on which many theoretical derivations rely heavily, are so little under control (it is sufficient to remember that one replaces very singular mathematical objects, the point–like fluxes, with very smooth ones, actually constant magnetic fields).

3. Pure Chern-Simons Theory at rational coupling

3.1 Canonical quantization

The pure three–dimensional $U(1)$ Chern–Simons theory constitutes an interesting example of a topological field theory [13], with the peculiarity that it does not exhibit dynamical degrees of freedom describing the propagation of point particles, like ordinary Quantum Field Theories, see for example ref. [57]. Rather, the phase space variables of the theory are the integrals of the gauge field around the non contractible loops C of the underlying two–dimensional surface. The gauge invariant observables are the exponentials of those integrals when appropriately weighted with a charge q, that is, the Wilson lines

$$W(C,q) = \exp\left\{ i\ q \oint_C A \right\} \tag{3.1}$$

where A is the gauge potential one–form. Moreover, in the models we will consider, the Hilbert space, which is entirely spanned by those physical observables, will be finite dimensional, which excludes the possibility of the propagation of ordinary particles.

We consider a three–dimensional Chern–Simons theory living on a space–time manifold which is the direct product $R \times \Sigma$, where Σ is a compact two–dimensional Riemann surface of genus 1, i.e. a torus. Actually, many features of this theory carry over to the case of higher genus, but here we are manly interested in the specific case of the torus because of our applications in the subsequent chapters. At the end of this chapter we will briefly review the distinctive features of the canonical quantization of the Chern–Simons action on surfaces with genus $g > 1$.

We write the action as

$$S_{\rm CS} = \frac{k}{4\pi} \int d^3x\, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda = \frac{k}{4\pi} \int AdA \tag{3.2}$$

where the gauge potential one–form is given by $A = dx^\mu A_\mu$ and normalized so that under a gauge transformation it changes by

$$A \rightarrow A + i\ gdg^{-1} \tag{3.3}$$

where d is the differential and g is a map from $R \times \Sigma$ into $U(1)$. Here we assume that the $U(1)$ is compact, i.e. equivalent to a circle (see however ref. [58] for non compact $U(1)$ and also for the effect of the introduction of the Maxwell term in the Lagrangian). The type of values which the positive coupling constant k in (3.3) assumes triggers the structure of the Hilbert space in a very sensible way. Let us anticipate that, at least in the present case in which the Chern–Simons potential is *not* coupled to a matter current, the coupling constant k of the *abelian* theory, in consideration here, is not quantized [18]. What actually happens is that the Hilbert space assumes very different structures according to integer, rational and irrational values of k. For *rational* values the Hilbert space becomes finite dimensional, carrying a finite dimensional irreducible representation of the group of gauge transformations with non zero winding numbers around the two periods of the torus (in the following we will call these transformations *large* or *global* gauge transformations). If, on the other hand, k is *irrational* there are no finite dimensional irreducible representations of the group of global gauge transformations and the Hilbert space becomes infinite dimensional. When coupling the pure Chern–Simons theory to matter with non vanishing total charge, as we will do in the next chapter in order to describe anyons, the Dirac quantization condition will actually restrict the parameter k to rational values. In this Chapter we describe therefore in some detail the quantization of the pure Chern–Simons action on the torus at rational coupling k, our further developments in this thesis being heavily based on it. For the discussion of irrational k, which force the total charge of the matter system to be zero, see however refs. [18,19].

We expand the action in (3.2)

$$S_{\mathrm{CS}} = \frac{k}{4\pi} \int d^3x \left(A_2 \dot{A}_1 - A_1 \dot{A}_2 + 2 A_0 F_{12} \right) \tag{3.4}$$

The time derivative of A_0 does not appear in the action and so variation with respect to A_0 gives the constraint

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 = 0 \tag{3.5}$$

which says that at fixed time the spatial gauge potential is pure gauge. We choose to quantize the theory by first solving the constraint and then quantizing the independent degrees of freedom[58]. Note that in refs. [14,57] a different procedure is followed, in which the constraint (3.5) is imposed on the states of the Hilbert space.

The Lagrangian is first order in time derivatives and canonical quantization of the spatial components of A gives the commutation relation

$$[A_1(x), A_2(y)] = \frac{2\pi i}{k}\delta^2(x-y) \tag{3.6}$$

So A_1 and A_2 are already phase space variables (and span not only the configuration space). The Hamiltonian turns out to be zero, which is of course a consequence of the metric–independence of the Chern–Simons action. We can therefore conclude that a complete set of observables is given by the Wilson loops of eq. (3.1) for all loops C lying in Σ. For the case of compact $U(1)$ we are considering here, q has to be integer to insure invariance of $W(C,q)$ under (3.3). In fact, if we parametrize the gauge–functions g, which have to be well defined on the torus, as $g = \exp(i\alpha(\vec{x}))$ where now $\alpha(\vec{x})$ has to be periodic on the torus up to integer multiples of 2π, we have that under (3.3) $iq\oint_C A$ changes by

$$2\pi iq(N_1 + N_2) \tag{3.7}$$

where the N_i are the (integer) winding numbers of g along the two handles of the torus. Due to the fact that $F_{12} = 0$ Wilson loops that can be continuously deformed into each other are equal. Therefore there are as many independent observables as there are independent non trivial loops on Σ; in the case of the torus these are precisely *two*.

We choose the usual cohomology basis and set, in order to simplify writings, the cycles's lengths equal to one, imagining the torus as a rectangle with identified edges. This corresponds to set in the Fuchsian torus the modular parameter $\tau = \tau_x + i\nu = i$. In section 3.4 we will re-introduce the modular parameter in that in the coherent state representation it permits us to keep conveniently track of modular covariance.

The two basic (unexponentiated) Wilson line operators are then given by

$$\begin{aligned} a_1 &= \int_0^1 dx_1 A_1(\vec{x}) \\ a_2 &= \int_0^1 dx_2 A_2(\vec{x}) \end{aligned} \tag{3.8}$$

which, due to (3.6), satisfy the algebra

$$[a_1, a_2] = \frac{2\pi i}{k} \tag{3.9}$$

and change under large gauge transformations by

$$a_i \longrightarrow a_i + 2\pi N_i \tag{3.10}$$

We can therefore regard a_1 as canonical coordinate and a_2 as canonical conjugate momentum. The gauge invariant Wilson loops are given by

$$\begin{aligned} W_1(p) &= \mathrm{e}^{-ipa_2} \\ W_2(q) &= \mathrm{e}^{+iqa_1} \end{aligned} \tag{3.11}$$

The different signs and the association of indices in (3.11) are only a matter of convention, p and q being arbitrary positive or negative integers. Summarizing, the operators $W_1(1)$ and $W_2(1)$ constitute the full set of observables and span thus the full phase space of the theory, not just the configuration space.

We want also to implement the global gauge transformations (3.10) through unitary operators; these operators have simply to shift the a_i variables by multiples of 2π, so it is not difficult to write them down:

$$\begin{aligned} U_1(p) &= \mathrm{e}^{-ipka_2} \\ U_2(q) &= \mathrm{e}^{+iqka_1} \end{aligned} \tag{3.12}$$

Taking into account (3.9) one sees that U_1 (U_2) shifts a_1 (a_2) by $2\pi p$ ($2\pi q$) and leaves the W_i invariant:

$$[U_i(m), W_j(n)] = 0 \qquad \text{for each } i \text{ and } j \tag{3.13}$$

as it must happen, as the Wilson lines are gauge invariant. For completeness we report also the commutation relation between the Wilson lines:

$$W_1(p)\, W_2(q) = \mathrm{e}^{-2\pi i \frac{pq}{k}}\, W_2(q)\, W_1(p). \tag{3.14}$$

In order to realize the algebra of all these operators in a Hilbert space one has to specify the value of k and search for a list of its irreducible representations. This will be done in the next section via the individuation of suitable Casimir operators.

3.2 The Hilbert space and irreducible representations

We will present the irreducible representations in an "abstract" form without giving an explicit realization. This allows us to establish the structure of the Hilbert space in a very clear form and to investigate its basic properties, such as modular invariance and gauge invariance, in a rather simple way. When we couple the pure Chern–Simons theory to matter these properties will actually continue to hold, but in the language (i.e. the coherent state representation), which is necessary [17] and very appropriate [14,57] to describe this coupling, the investigation of those properties is rather complicated from a technical point of view. The abstract treatment of the present chapter, besides clarifying very much things, will serve us almost directly to understand relevant features in the next chapter. In this context one should also remark that the usual (q,p) representation of the algebra (3.9) is *not* appropriate in this case in that states are not normalizable (see section 3.4).

Let us state at the beginning the relation which determines crucially the structure of the Hilbert space. This is the commutation relation between the two operators which generate the gauge transformations with non trivial windings along the two handles. One has in fact

$$U_1(p)\,U_2(q) = \mathrm{e}^{-2\pi i\,k\,p\,q}\,U_2(q)\,U_1(p) \tag{3.15}$$

For the reasons said in the preceding section we take a rational coupling constant

$$k = \frac{r}{s} \tag{3.16}$$

where we define r and s to be relative prime positive integer numbers. Eq. (3.15) tells us then that in general, for arbitrary rational k, except the case $s = 1$, the operators which implement the large gauge transformations along the two handles of the torus do not commute among them. This means that, at the quantum level, the $U(1)$ gauge group on the torus is plagued by a type of global anomaly if k is not an integer. This does not imply that we have to force k to be integer to save the consistency of the theory, it implies that the states of the Hilbert space can not be invariant under all gauge transformations. The Hilbert space has rather to *represent* the algebra (3.15) together with the algebra of the Wilson lines. The space which we find in this way appears actually as an "enlarged" Hilbert space, where the enlarging happened in order to accommodate a representation of (3.15)[18,19]. The "physical" Hilbert space will be obtained by imposing a suitable superselection rule on this "large" space, which corresponds to a gauge fixing of the theory. At the end one can show that all physical observables are actually independent of the particular gauge fixing one chooses and this, in turn, insures the gauge invariance of the theory. Let us now illustrate this construction in some detail. The results for integer k can of course easily be obtained as a particular case of the present procedure by setting $s = 1$.

We start by observing that the representations of the algebra of the operators W_i and U_i, given above, can be classified by the eigenvalues of the following Casimir operators

$$\begin{aligned} C_1(p) &\equiv U_1(s \cdot p) = \mathrm{e}^{-i r p a_2} \\ C_2(q) &\equiv U_2(s \cdot q) = \mathrm{e}^{+i r q a_1} \end{aligned} \tag{3.17}$$

These unitary operators commute with the Wilson lines, being products of U_i–operators, and commute with the U_i due to (3.15). Clearly C_1 commutes also with C_2. Therefore, in each irreducible representation of the algebra at hand, these operators have to be c–number constant phases:

$$\begin{aligned} C_1(p) &= \mathrm{e}^{2\pi i\, prs\varphi_1} \\ C_2(q) &= \mathrm{e}^{2\pi i\, qrs\varphi_2} \end{aligned} \tag{3.18}$$

where the φ_i are now real fixed numbers which play the role of vacuum angles labeling the representations: different phases will give rise to inequivalent representations.

To find the structure of the Hilbert space it is convenient to diagonalize simultaneously the operators C_1, C_2, U_1, W_1 which commute among them (clearly any other complete set does the same job). First we impose the eigenvalue equation for U_1 and W_1:

$$\begin{aligned} U_1(p)|\ \rangle &= \mathrm{e}^{2\pi i\, p\lambda}|\ \rangle \\ W_1(p)|\ \rangle &= \mathrm{e}^{2\pi i\, p\sigma}|\ \rangle \end{aligned} \tag{3.19}$$

where λ and σ are phases, and observe that we have the operatorial identities $U_1(p)^s = W_1(p)^r = C_1(p) = \mathrm{e}^{2\pi i\, prs\varphi_1}$. This relation allows us to determine the possible eigenvalues of W_1, U_1:

$$\lambda = r\varphi_1 - n\frac{r}{s}$$
$$\sigma = s\varphi_1 - m\frac{s}{r} \tag{3.20}$$

where m and n are integers in the interval $1 \leq n \leq s,\ 1 \leq m \leq r$. The additional powers of r and s in the numerators of the second terms at the r.h.s. of (3.20) are introduced only for a convenient labeling, r and s being coprime and λ and σ being defined modulo integers. Let us call the corresponding eigenvectors $|m,n\rangle$. Due to the commutation relations (3.14) and (3.15) it is clear that W_2 and U_2 act as shift operators on the indices m and n respectively: $U_2(q)\,|m,n\rangle \propto |m,n+q\rangle$, $W_2(q)|m,n> \propto |m+q,n\rangle$. If we fix the phases of the states such that $|m+Mr,n+Ns\rangle = |m,n\rangle$ for all integers M,N, the operatorial identity $U_2(q)^s = W_2(q)^r = C_2(q) = e^{2\pi i\, qrs\varphi_2}$ allows one to determine the proportionality phases. Then we can summarize the action of the operators W_i, U_i on the "large" Hilbert space (whose dimension is $r \cdot s$), spanned by the states $|m,n\rangle$, as follows:

$$\begin{aligned}
W_1(p)\big|m,n\big\rangle &= e^{2\pi i\, p\left(s\,\varphi_1 - \frac{m}{k}\right)}\big|m,n\big\rangle \\
W_2(q)\big|m,n\big\rangle &= e^{2\pi i\, q\, s\, \varphi_2}\big|m+q,n\big\rangle \\
U_1(p)\big|m,n\big\rangle &= e^{2\pi i\, p(r\,\varphi_1 - n\,k)}\big|m,n\big\rangle \\
U_2(q)\big|m,n\big\rangle &= e^{2\pi i\, q\, r\, \varphi_2}\big|m,n+q\big\rangle
\end{aligned} \tag{3.21}$$

The corresponding eigenvalues of a_2 are given by

$$a_2\big|m,n\big\rangle = \frac{2\pi}{r}(ms + nr - rs\,\varphi_1)\big|m,n\big\rangle. \tag{3.22}$$

The preceding relations describe completely the irreducible representations of the algebra at hand and we see that each representation is characterized by a pair of real vacuum phases $\varphi_{1,2}$.

Let us now comment on the duality properties of these representations which, are cloosely related to the modular invariance properties discussed in the next section. Notice, to this purpose, that we could also have chosen to diagonalize W_2 and U_2 instead of W_1 and U_1; this would correspond to a different choice of basis in the *same* representation, which is, in fact, the Dual basis, given by a discrete Fourier Transform:

$$\big|j,l\big\rangle_D = \frac{1}{\sqrt{rs}}\sum_{m=1}^{r}\sum_{n=1}^{s} e^{-2\pi i\left(\frac{1}{k}mj + k\,nl\right)}\big|m,n\big\rangle \tag{3.23}$$

The normalization is chosen in the usual way such that ${}_D\langle j,l|j,l\rangle_D = 1$ if, as a convention, $\langle m,n|m,n\rangle = 1$. It is easy to check that on the dual basis $|j,l\rangle_D$ the operators W_2 and U_2 are diagonal while W_1 and U_1 act as raising/lowering operators, and one gets relations which are completely analogous to (3.21). Here we report the eigenvalues of a_1:

$$a_1|j,l\rangle_D = \frac{2\pi}{r}(js + lr + rs\,\varphi_2)|j,l\rangle_D \tag{3.24}$$

where $1 \leq j \leq r$, $1 \leq l \leq s$.

Referring to the basis given in (3.21), we see that the index m spans the "observable" space on which the gauge invariant Wilson lines act non trivially, while the index n spans the "internal" gauge space on which the large gauge transformations act: as a consequence of (3.13), which is the basic statement of gauge invariance, the two spaces do not mix. This allows us to identify states with different values of n as a unique physical ray and the "physical" Hilbert space H_{ph} becomes then r–dimensional. Formally one can impose a gauge–fixing restricting the large Hilbert space to a set of representative vectors

$$|m\rangle^{(0)} \equiv \sum_{n=1}^{s} c_n^{(0)}|m,n\rangle \tag{3.25}$$

where the set of constants $c_n^{(0)}$, with $\sum_n |c_n^{(0)}|^2 = 1$, define the particular gauge choice. The space H_{ph} is then spanned by the set $|m\rangle^{(0)}$. It is clear from (3.21) that the Wilson lines operators act on the states $|m\rangle^{(0)}$ in the same way they act on $|m,n\rangle$ (upon ignoring the index n) and that therefore the matrix elements of the gauge invariant observables are independent on the gauge choice $\{c_n^{(0)}\}$, these constants being *unobservable*. This is our statement of gauge invariance.

To summarize, the inequivalent irreducible Hilbert spaces are labeled by the (observable) vacuum angles $\varphi_{1,2}$ and by the gauge fixing constants $\{c_n^{(0)}\}$, and are r–dimensional.

Let us briefly comment on the interesting particular case (see chapter five) of integer k, i.e. $s = 1$. In this case the relation (3.15) tells us that the generators of global gauge transformations commute and that therefore states can be gauge invariant up to phases. The irreducible representations are k–dimensional and there is no need to introduce an internal gauge space to represent the algebra of the U_i, which are actually Casimir operators themselves. Inequivalent representations are again labeled by two vacuum angles.

Up to now the vacuum angles were arbitrary. In the next section we will see that modular invariance determines their values uniquely.

3.3 Modular invariance

Up to now we worked explicitly in the (a_1, a_2) basis, corresponding to $a-$ and $b-$loops around the handles. Actually, any other basis is also appropriate and should yield the same results. We can take as a general basis $a_1' = \alpha a_1 + \beta a_2$, $a_2' = \gamma a_1 + \delta a_2$ where the coefficients $\alpha, \ldots, \delta$ have to be such that the knowledge of $a_{1,2}$ modulo 2π determines uniquely $a_{1,2}'$ modulo 2π and viceversa. This is the case if $\alpha, \ldots, \delta$ are all integers such that $\alpha\delta - \beta\gamma = 1$, meaning that the matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

defines a modular transformation. The new variables satisfy the same commutation relations $[a_1', a_2'] = \frac{2\pi i}{k}$, and thus modular transformations are canonical transformations that, quantum mechanically, should be implemented through unitary operators in the Hilbert space.

As is well known the whole group of modular transformations can be generated by two particular transformations, and it turns out that a convenient choice is given by the two (unitary) generators

$$\begin{aligned} A &= \mathrm{e}^{-\frac{ik}{4\pi} a_1^2} \\ B &= \mathrm{e}^{-\frac{ik}{4\pi} a_2^2} \end{aligned} \tag{3.26}$$

which act on the Wilson lines as

$$\begin{aligned} A\,\mathrm{e}^{+ima_1} A^{-1} &= \mathrm{e}^{+ima_1} \\ A\,\mathrm{e}^{-ima_2} A^{-1} &= \mathrm{e}^{-im(a_1+a_2)} \\ B\,\mathrm{e}^{+ima_1} B^{-1} &= \mathrm{e}^{+im(a_1-a_2)} \\ B\,\mathrm{e}^{-ima_2} B^{-1} &= \mathrm{e}^{-ima_2}. \end{aligned} \tag{3.27}$$

A and B constitute therefore a basis for the group of modular transformations. Notice that these operators *do not* correspond to the usual choice of generators on the Fuchsian torus, which send $\tau \to -\frac{1}{\tau}$ and $\tau \to \tau + 1$, they correspond rather to $\tau \to \frac{\tau}{\tau+1}$ and $\tau \to \tau + 1$ respectively.

Let us now discuss modular invariance, which, as discussed at the beginning of this section, should constitute a physical symmetry of the theory. The question can be stated in the following terms: the Hilbert spaces, found in the previous section, are modular invariant, if they carry a representation of the operators A and B. To this order it is necessary (and sufficient) to check whether or not A and B commute with the Casimir operators defined in (3.18). A straightforward calculation gives:

$$\begin{aligned} C_1(p)\, A\, C_1(p)^{-1} &= A\left[C_2(p)\, \mathrm{e}^{-i\,p\,r\,s\,\pi}\right] \\ C_2(q)\, B\, C_2(q)^{-1} &= B\left[C_1(q)\, \mathrm{e}^{-i\,q\,r\,s\,\pi}\right] \end{aligned} \tag{3.28}$$

the remaining commutators being trivial. Modular invariance is therefore insured only in those representations in which the quantities between square brackets at the right hand side of (3.28) are equal to one. This fixes the vacuum angles to

$$\varphi_1 = \frac{1}{2} = \varphi_2 \tag{3.29}$$

and the resulting irreducible representation is unique. From now on we work in this representation. In summary, the (unitary) implementation of modular invariance as a physical symmetry fixes the vacuum angles $\varphi_{1,2}$ uniquely.

Up to now the discussion was relative to the "large" Hilbert spaces which we know, from the preceding section, to contain states which correspond to the same physical ray. The physical Hilbert space H_{ph} is obtained after a suitable gauge fixing. We want now to implement modular invariance also in the physical space H_{ph}. To this order we need to make the action of the operators A, B on the "large" Hilbert space more explicit. From (3.22), (3.24) and (3.29) we get immediately:

$$\begin{aligned} A\big|j,l\big\rangle_D &= \mathrm{e}^{-i\frac{\pi}{rs}\left(sj + rl + \frac{rs}{2}\right)^2}\big|j,l\big\rangle_D \\ &= \mathrm{e}^{-i\pi\frac{rs}{4}}\,\mathrm{e}^{\frac{-i\pi}{k}j(j+r)}\,\mathrm{e}^{-i\pi k\, l(l+s)}\big|j,l\big\rangle_D \\ B\big|m,n\big\rangle &= \mathrm{e}^{-i\frac{\pi}{rs}\left(sm + rn + \frac{rs}{2}\right)^2}\big|m,n\big\rangle \end{aligned} \tag{3.30}$$

while to compute the action of say B on $|j,l\rangle_D$ we have to work a little bit more. We apply B to eq. (3.23), and compute $B\,|m,n\rangle$ through to (3.30). At the end we use again eq. (3.23), inverting the Fourier transform, to express everything in terms of $|j,l\rangle_D$. The computation can be carried out explicitly and gives:

$$ {}_D\langle j',l'|B|j,l\rangle_D = \mathcal{N}\, e^{\frac{i\pi}{k}\left(j-j'-\frac{r}{2}\right)^2} e^{i\pi k\left(l-l'-\frac{s}{2}\right)^2} \tag{3.31} $$

where $\mathcal{N}$ is a normalization constant which is independent on j,l,j',l'. This equation gives directly $B\,|j,l\rangle_D$, expressed in terms of $|j,l\rangle_D$. The unitarity of the matrix given in (3.31) can be easily inferred directly via an explicit computation, and, most importantly one observes that this matrix factorizes in a product of two unitary matrices, $B^{Ph}_{j,j'}\cdot B^{G}_{l,l'}$, one which represents the modular transformation in the physical space and one which represents it in the internal gauge space. The same happens obviously for the action of A, see (3.30). Due to the fact that the operators A,B generate the whole modular group, we can conclude that for an arbitrary modular transformation $\mathcal{M}$ we have

$$ \mathcal{M}\,|j,l\rangle_D = \sum_{j'=1}^{r}\sum_{l'=1}^{s} \mathcal{M}^{Ph}_{j,j'}\,\mathcal{M}^{G}_{l,l'}\,|j',l'\rangle_D \tag{3.32} $$

where the unitary matrices $\mathcal{M}^{Ph}$ and $\mathcal{M}^{G}$ represent the action of the modular group independently. If we consider now our physical Hilbert space H_{ph}, spanned by the vectors given in (3.25) (actually, by their duals) we see that the transformation of eq. (3.32) acts observably on the physical index j, while the action in the internal space (index l) amounts to a change of the gauge–fixing constants $c_n^{(0)}$ which, on the other hand, are not observable. So we can *define* the action of a modular transformation in H_{ph} by:

$$ \mathcal{M}|j\rangle_D^{(0)} \equiv \sum_{j'=1}^{r} \mathcal{M}^{Ph}_{j,j'}\,|j'\rangle_D^{(0)} \tag{3.33} $$

This relation concretizes our statement that the physical Hilbert space is modular invariant. In the next chapter we will see that the definition of the projected Hilbert space and of the modular transformation defined in eq. (3.33) can be applied also if we couple the Chern–Simons field to matter. The importance of this physical symmetry is due to the fact that the resulting Hamiltonian will be modular *covariant*, transforming, in fact, in a very simple way; this will allow us, for example, to construct new energy eigenstates starting from given ones. Also, the analysis of currents can be constrained to currents in one direction only, while currents in the other direction can be obtained by a modular transformation.

Let us finally make a comment on the case of integer k which has been widely discussed in the literature[14,17,18,19,57,58]. In this case ($s = 1$) the Casimir operators C_i coincide with the generators of global gauge transformations and modular invariance means that the gauge transformations have to commute with the modular operators, see eqs. (3.28). From that equation one sees also the importance of the introduction of the vacuum angles $\varphi_{1,2}$ when imposing modular invariance. In refs. [14,57], which deal with a generic compact surface of arbitrary genus g, it was claimed that modular invariance forces k to be an *even* integer and the phases $\varphi_{1,2}$ to be zero. Here we proved that, at least in the case of the torus ($g = 1$), this conclusion can be evaded, and there exists for any integer k, even or odd, a modular invariant choice of the vacuum angles. It is not clear to us, however, in which sense the relation between the Chern–Simons theory and conformal blocks, found for even k[14,17,57] can be extended to the case of *odd* k.

3.4 (Q,P) – and coherent state representation

The algebra we are considering in this chapter is based on the canonical commutation relations, given in (3.9), about which almost everything is known. At first sight the most convenient (explicit) representation would be the (Q, P) – or configuration space – representation. In this representation one introduces a wave function which depends on one of the *real* variables, for example $a_1 \equiv x$, and the other variable, in this case a_2, becomes a derivative operator

$$a_2 \equiv -\frac{2\pi i}{k}\frac{\partial}{\partial x} \tag{3.34}$$

which acts on the wave function $f(x)$. In this representation the relevant operators become $C_1(p) = \mathrm{e}^{-2\pi p s \frac{\partial}{\partial x}}$, $C_2(q) = \mathrm{e}^{irqx}$, $W_1(p) = \mathrm{e}^{\frac{-2\pi p}{k}\frac{\partial}{\partial x}}$, $U_1(p) = \mathrm{e}^{-2\pi p \frac{\partial}{\partial x}}$. The Hilbert space is made out of all those functions $f(x)$ which satisfy the relations (3.18) identically (we keep here the phases $\varphi_{1,2}$ arbitrary):

$$f(x - 2\pi s p) = e^{2\pi i p r s\, \varphi_1} f(x)$$
$$e^{i\, r\, q\, x} f(x) = e^{2\pi i q r s\, \varphi_2} f(x) \tag{3.35}$$

The first equation has the general solution

$$f(x) = e^{-i\varphi_1 r x} \sum_{u=-\infty}^{+\infty} f_u \, e^{i\frac{u}{s}x} \tag{3.36}$$

where the sum is over integers and the f_u are arbitrary coefficients. The second equation in (3.35) implies then on the f_u the relation

$$f_{u - rs\cdot q} = e^{2\pi i q r s\, \varphi_2} f_u \tag{3.37}$$

This means that the linearly independent solutions of the eqs. (3.35) are precisely $r \cdot s$, as expected, and a convenient basis for them can be given by

$$f_l(x) = e^{-i\varphi_1 r x} \sum_u e^{i(l - rs\cdot u)\frac{x}{s}} e^{-2\pi i (l - rs\cdot u)\varphi_2} \tag{3.38}$$

where l is defined modulo rs, i.e. $f_l(x) = f_{l+rs}(x)$. Moreover, parametrizing the index l as $l = m \cdot s + n \cdot r$, where m (n) is defined modulo r (s), (3.38) translates into

$$f_{m,n}(x) = e^{-i\varphi_1 r x} \sum_u e^{i(m + kn - ru)x} e^{-2\pi i\, s(m + kn - ru)\varphi_2} \tag{3.39}$$

The identities $f_{m+r,n}(x) = f_{m,n+s}(x) = f_{m,n}(x)$ hold, and one can also check that the relations of eqs. (3.21) are satisfied, so that the identification $f_{m,n}(x) \equiv |m,n\rangle$ is completely justified.

This explicit and rather simple realization of the Hilbert space has actually an important drawback. This is due to the fact that the most natural choice of the scalar product in this space would be $\langle f, g\rangle \equiv \int_0^{2\pi s} f^*(x)\, g(x) dx$. This gives the true orthogonality relations between different elements of the basis, but the *norm* of a state $f_{m,n}$ is ill–defined:

$$\langle f_{m,n} | f_{m,n} \rangle = 2\pi s \sum_{u,u'} \delta_{u,u'}$$

This is clearly a consequence of the bad convergence properties of the definition (3.39) of $f_{m,n}$ itself.

In the pure Chern–Simons theory one could actually introduce a scalar product "by hand" (the Hilbert space is finite dimensional), but in the coupling to matter the topological components of the gauge field will play a role which admits a spatial interpretation, they are in fact correlated with the center–of–mass motion. It is therefore convenient to work in a realization in which the scalar product is defined by an integral; in our case the right choice turns out to be the coherent state representation [57,14], which is also appropriate for the complex language which we will adopt. In this language the torus is described by a complex parameter τ, with $\nu \equiv \mathrm{Im}\tau > 0$, and with the periodicity relation $z \simeq z + m + n\tau$. ν corresponds here to the area of the torus. The introduction of the parameter τ may seem a little bit un–natural at this point, but it permits us to keep track of modular invariance in the usual convenient way.

The canonical commutation relations (3.9) become then $[a_1, a_2] = \frac{1}{\nu}\frac{2\pi i}{k}$. Then one introduces a complex topological component of the gauge field via

$$a = \frac{\nu}{2\pi i}\left(a_1 - i a_2\right) \tag{3.40}$$

with the commutation relations:

$$[\bar{a}, a] = \frac{\nu}{k\pi}. \tag{3.41}$$

Due to the fact that the coordinate on the torus is now defined modulo $m + n\tau$ the large gauge transformations amount in this language to the shift

$$a \longrightarrow a + p + q\bar{\tau}. \tag{3.42}$$

The wave function becomes an *analytic* function, $\Psi(a)$, of the complex variable a alone. The operator $\bar{a}$ becomes the derivative with respect to a:

$$a^{\dagger} \equiv \bar{a} = \frac{\nu}{\pi k}\frac{\partial}{\partial a}$$

To insure that this operator is indeed the hermitian conjugate of a, one has to introduce a gaussian measure, $d\mu(a, \bar{a}) = \exp\left(-\frac{k\pi}{\nu}\bar{a}a\right) d\bar{a}da$, in the scalar product of the Hilbert space:

$$\langle \Psi | \Phi \rangle = \int d^2 a\, \mathrm{e}^{-\frac{k\pi}{\nu}\bar{a}a}\, \overline{\Psi(a)}\, \Phi(a). \tag{3.43}$$

The operators $U_{1,2}$ in (3.12) have now to implement large gauge transformations with non trivial windings around the **1** and τ–cycles respectively, according to (3.42). In the coherent state representation the operators in (3.12) become:

$$U_1(p) = \exp\left(p\left(\frac{\partial}{\partial a} - \frac{\pi k}{\nu}a\right)\right) = \exp\left(-\frac{\pi k}{\nu}\left(\frac{1}{2}p^2 + ap\right)\right)\exp\left(p\frac{\partial}{\partial a}\right)$$
$$U_2(q) = \exp\left(q\left(\bar{\tau}\frac{\partial}{\partial a} - \frac{\pi k}{\nu}a\tau\right)\right) = \exp\left(-\frac{\pi k}{\nu}\left(\frac{1}{2}q^2|\tau|^2 + aq\tau\right)\right)\exp\left(q\bar{\tau}\frac{\partial}{\partial a}\right) \tag{3.44}$$

Notice the appearance of terms quadratic in q and p at the right hand side of these equations. These are precisely the ones which account for the non trivial transformation of the gaussian measure under the shifts in (3.42). One can, in fact, check that

$$\mathrm{e}^{-\frac{k\pi}{\nu}\bar{a}a}\,\overline{U_1(p)\Phi(a)}\,U_1(p)\Psi(a) = \mathrm{e}^{-\frac{k\pi}{\nu}\overline{(a+p)}(a+p)}\,\overline{\Phi(a+p)}\Psi(a+p) \tag{3.45}$$

with an analogous relation for U_2. The Hilbert space can now be determined by the usual relation (note that C_1 and C_2 commute):

$$C_1(p)\,C_2(q)\,\Psi(a) = \mathrm{e}^{2\pi i\,r\,s(p\varphi_1 + q\varphi_2)}\,\Psi(a) \tag{3.46}$$

which reads as:

$$\Psi(a + s\cdot(p+q\bar{\tau})) = \mathrm{e}^{\frac{rs\pi}{\nu}\left(\frac{|p+q\tau|^2}{2} + \frac{a}{s}(p+q\tau)\right) + i\pi\,r\,s(p\,q + 2\varphi_1 p + 2\varphi_2 q)}\,\Psi(a) \tag{3.47}$$

To find the solutions of this equation one uses the scaled variable

$$b \equiv \frac{a}{s}$$

and sets

$$\Psi(b) = \exp\left(\frac{rs\pi}{2\nu}b^2\right)\Lambda(b) \tag{3.48}$$

Then (3.47) becomes:

$$\Lambda(b + p + q\bar{\tau}) = \exp\left(i\pi rs\left(q^2\bar{\tau} + 2bq + 2\varphi_1 p + 2\varphi_2 q\right)\right)\Lambda(b) \tag{3.49}$$

This means that Λ is a quasi periodic function on the torus[59] of weight $r \cdot s$ whose solutions, $r \cdot s$ in number, can be given in terms of θ–functions[20]. Substituting back in (3.48) one gets finally that the linearly independent solutions are given by

$$\Psi_l(a) = \exp\left(\frac{\pi k}{2\nu} a^2\right) \overline{\theta\begin{bmatrix} \frac{l}{rs} - \varphi_1 \\ rs \cdot \varphi_2 \end{bmatrix}(r\bar{a}|rs\tau)} \tag{3.50}$$

where $l = 1, ..., r \cdot s$. Setting, as before, $l = s \cdot m + r \cdot n$ one gets the canonical basis of (3.21). Also in this formalism one can see that modular invariance forces the phases to $\varphi_{1,2} = \frac{1}{2}$: only in this case the modular transformed of solutions of (3.47) are again solutions of (3.47). We will see this in some detail in the next chapter when we discuss the modular invariance of the Chern-Simons theory coupled to non relativistic matter.

The scalar product defined in eq. (3.43) (the integral being extended over the s-torus) turns out to be the standard inner product for θ–functions[57] and one gets the right orthonormality relations

$$\langle \Psi_l | \Psi_{l'} \rangle = c\, \delta_{l,l'}$$

where c is a constant. The scalar product defined in (3.43) can be naturally extended to the case when also matter is present (see next chapter).

Higher genus. For $g > 1$[18] the phase space variables of the theory can be chosen to be the holonomies, a_i, b_i, of the gauge field around the elements of the canonical cohomology basis, the a– and b–cycles, which satisfy the commutation relations $[a_i, b_j] = \frac{2\pi}{k}\delta_{ij}$, all other commutators being zero. For an arbitrary set of vacuum angles (one for each handle) to each couple of handles now there corresponds an $r \cdot s$ dimensional Hilbert space and the dimension of the total Hilbert space is $(rs)^g$. However, if one imposes modular invariance, in ref. [58] it is found that for even r there exists a unique choice of vacuum angles which insures modular invariance, while for odd r no such choice exists, and in order to represent the modular transformations one has to enlarge the dimensionality of the Hilbert space by a factor of 2^{2g}. With this respect the torus plays an exceptional role in that modular invariance is insured for *every* rational value of k in the irreducible representation of large gauge transformations and Wilson–lines. The main distinctive feature of the torus with respect to other Riemann surfaces is, however, its translation invariance. This property will permit us to define a conserved momentum operator and to analize therefore currents and eventually supercurrents. This is one of the main reasons why we work on the torus, see however ref. [8] for treatments on the sphere.

4. Coupling to non relativistic matter

As we saw in the preceding chapter the pure abelian Chern-Simons theory does not describe any propagating degrees of freedom. In this sense the interest in this theory is due more to its topological properties, which are related to the concepts of links, nodes and Jones polynomials[13,14,57,58] which we did not touch in the previous chapter. From the dynamical point of view, a gauge invariant minimal coupling of the Chern-Simons action to a conserved current is the most interesting and natural procedure to investigate the effects of an "anomalous" kinetic term for a gauge field in a system of interacting charges. It turns out that, on the plane, the gauge field can be eliminated from the equations of motion to give rise to an effective non–local self interaction between point like charges, which can be interpreted saying that to each particle an infinitesimally thin flux tube is attached and that the other particles move in the field of those flux tubes. Actually, we will see that on the torus there are additional degrees of freedom, the topological components of the gauge field, identified in the previous chapter as Wilson line integrals, which can not be eliminated and which constitute independent degrees of freedom.

To promote the gauge field to a true dynamical degree of freedom, in which there exist photons, in addition to the Chern-Simons term one would have to introduce the usual kinetic Maxwell term[49,51,58,60,61] in the action, $\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}$. Therefore, from a "phenomenological" point of view, a theory, in which a matter field is minimally coupled to a Chern–Simons potential, corresponds to a low energy approximation of this more general theory, where the lower–derivative Chern-Simons term dominates the higher-derivative Maxwell term.

The most important motivation for considering only the Chern-Simons action as kinetic term for the gauge field, as explained in chapter two, is of course the statistics flip induced automatically by this interaction: on the plane, a singular gauge transformation transforms the (complicated) Hamiltonian in the free one and the wave function with ordinary statistics in a function with fractional statistics, as explained in chapter two. Especially on the torus it is very convenient to work in the representation in which the wave function has ordinary statistics and, in this sense, there are no phase ambiguities. In this representation there are two complications left. First, the wave function transforms non trivially under large gauge transformations. Second, going from one coordinate patch to an other (due to the non trivial topology of the torus this patching is unavoidable) the wave function has to satisfy an appropriate quasi–periodicity condition due to the presence of an (effective) monopole inside the surface. It is important to notice that these two complications have not to be confused with the fractional statistics inherent in the problem, which, in this representation is actually encoded in the Hamiltonian: under interchange of two particles the wave function is well defined, picking up either a plus or a minus sign.

4.1 On the quantization of the abelian Chern-Simons coupling constant

The particular value of the Chern–Simons coupling constant k, in a theory which lives on a space–time manifold with trivial topology, does not affect the principal properties of the theory itself. If, on the other hand, the underlying surface has a non trivial topology, the particular value of k determines heavily the structure of the theory.

To see this let us consider an action in which the Chern-Simons field is minimally coupled to a matter field Ψ, relativistic or non relativistic, or, more generally, to a conserved current

$$S[A,J] = S_{\rm CS} + S_{\rm mat}[A,J] \tag{4.1}$$

where we recall that $S_{\rm CS}$ is given by (3.2). It is known that the coupling constant of the non abelian Chern-Simons action (pure or not) has to be quantized as an integer, in order that the quantum theory, defined by the functional integral of the phase exponential of the action

$$\Gamma = \int \{\mathcal{D}A\} \exp\left(iS_{n.ab.}\right) \tag{4.2}$$

be well defined and non vanishing[61,62] (here $S_{n.ab.}$ is the non abelian analog of (3.2)). This is due to the presence of a term in the gauge transformation of $S_{n.ab.}$ which is non homogenous in the gauge potential A_μ and which depends only on the winding number of the large gauge transformation, being actually proportional to it. The functional integral sums over all gauge transformations and becomes actually ill–defined unless k is integer.

The situation is different in the abelian case as it has been pointed out by Polychronakos in ref. [19]. The quantum theory is again defined by the path integral

$$\Gamma = \int \{\mathcal{D}A\} \exp\left(iS[A, J]\right) \tag{4.3}$$

If we assume that the matter part of the action $S_{\mathrm{mat}}[A, J]$ is gauge invariant, under a gauge transformation (see (3.3)) the action in equation (4.1) changes by

$$\Delta S[A, J] = i\frac{k}{4\pi}\int gdg^{-1}F \tag{4.4}$$

where the field strength is given by $F = dA$. We see that this transformation depends on F, meaning that the integrality of ΔS depends on the particular gauge field configurations which are coupled to the matter field, i.e. on those which appear really in the theory. For concreteness, take the three-dimensional space-time manifold to be $T^2 \times S^1$, compactifying the time coordinate on a circle, and consider a gauge transformation g which winds N times around S^1, where N is an integer:

$$i\int gdg^{-1} = 2\pi N$$

Moreover, take the gauge field to be in a configuration which corresponds to an integer monopole number M (Dirac quantization condition) with associated flux

$$\int_{T^2} F = 2\pi M. \tag{4.5}$$

Then the integral in (4.4) could be calculated by formally factorizing it, were it not for a subtlety with a factor of *two*[63]. The point is that the gauge potential one–form A can not be globally defined on $T^2 \times S^1$, if we admit monopole configurations, as we do; it can only be defined by an appropriate patching. Therefore also the integral defining the Chern-Simons action in (3.2) is meaningless as it stands; it has to be defined by a rather complicated procedure[64] taking properly into account the contributions of the intersections of the coordinate patches (it is not sufficient to *subtract* simply the contributions of the double/triple intersections). The appropriate treatment of ref. [63] reveals, with respect to our "naive" counting, the presence of an additional factor of two. In conclusion we get:

$$\Delta S[A,J] = 2 \cdot \left(\frac{k}{4\pi}\right)(2\pi N)(2\pi M) = 2\pi k M N \tag{4.6}$$

If one wants this to be a multiple of 2π for *all* M then clearly k has to be an integer. The same conclusion can be reached by defining the Chern-Simons action as an integral of the square of the first Chern class over a four dimensional manifold, M_4, which bounds the corresponding three dimensional manifold [13,65]

$$S_{\mathrm{CS}} = \pi k \int_{M_4} \left(\frac{F}{2\pi}\right)^2$$

The ambiguity in this definition is given by πk times the integral over a closed manifold of the square of the first Chern class, which is in general an integer. However, for a three dimensional orientable manifold this integer is always *even* [65] and the same conclusion is reached.

However, and this is the key observation of ref. [19], even if k is not an integer the path integral in (4.3) does not vanish and continues to make sense in general: the global gauge anomaly in (4.6) is harmless. What happens is that the path integral decomposes in sectors of different integer fluxes M. For sectors, for which (4.6) is not a multiple of 2π the path integral will indeed vanish, upon summation over all N. If k is *irrational* only the zero flux sector contributes, if k is *integer* the theory contains states of all fluxes, while for *rational* k, $k = \frac{r}{s}$, the theory contains only states for which the flux is an integer multiple of $2\pi \cdot s$. If the charges one wants to couple to the Chern-Simons field are all equal, as it is appropriate to induce the statistics flip, the total flux of F has to be different from zero, as we will see in the following. This determines k to be rational. We choose therefore

$$k = \frac{r}{s} \tag{4.7}$$

where we define r and s to be positive coprime integers. In what we are interested in throughout this thesis is a quantum mechanical theory of a Chern-Simons field coupled to non relativistic matter on a torus. In the construction of this theory we will indeed encounter in many points the need of a rational coupling constant in order to get a well defined theory: the consistency of our quantum mechanical theory, as we will see, relies actually crucially on (4.7) and on the Dirac quantization condition eq. (4.5).

4.2 Hamiltonian formulation on the plane

The minimal coupling of a Chern–Simons field to a non relativistic matter field $\Psi(x)$, commuting or anticommuting, can be described by the following action[11,23,61,66]:

$$S = \frac{k}{4\pi} \int d^3x \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \int d^3x \left(i\Psi^\dagger \mathcal{D}_t \Psi + \frac{1}{2m} \Psi^\dagger \left(\mathcal{D}_x^2 + \mathcal{D}_y^2\right) \Psi \right) \tag{4.8}$$

The covariant derivative is defined as usual by $\mathcal{D}_\mu = \partial_\mu - iA_\mu$, where μ goes from zero to two. We have absorbed the (fictitious) electric charge q in the definition of the vector potential so that the new coupling constant should actually be $\frac{k}{q^2}$, which we continue to call k for simplicity. Notice, however, that what we said in the previous section about the quantization of k can be regarded also as a quantization condition for the fictitious charge q. As long as we couple the gauge field to a matter system with only a *single* type of charges what is quantized is the ratio $\frac{k}{q^2}$.

Our goal is to set up a quantum mechanical treatment of the system described by this action on a two dimensional surface with periodic boundary conditions, i.e. on a torus. To this order we derive first the equations of motion and the Hamiltonian staying on the plane, and specialize in the next section to the torus. Quantum mechanics is obtained by projecting the second quantized theory on a subspace with a fixed number of particles, i.e. of anyons. In this section we follow mainly ref. [23].

Varying (4.8) with respect to A_μ we obtain ($F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$)

$$\varepsilon^{\mu\nu\lambda} F_{\nu\lambda} = -\frac{4\pi}{k} J^\mu \tag{4.9}$$

where the charge density is $\rho = J^0 = \Psi^\dagger\Psi$, and the current density is given by

$$J_n = \frac{1}{2m\,i} \left(\left(\mathcal{D}_n \Psi\right)^\dagger \Psi - \Psi^\dagger \mathcal{D}_n \Psi \right). \tag{4.10}$$

On one hand, due to the Bianchi identity for the Chern–Simons field strength, eq. (4.9) implies immediately current conservation and, on the other hand, it implies that classically the field strength $F_{\mu\nu}$ is confined to the particle's worldlines. This implies also that there are no *classical* Lorentz forces between the particles. This can also be seen more directly by observing that the Lorentz force is proportional to $F_{\mu\nu}J^{\nu}$, which is zero due to (4.9). Nevertheless, a non trivial dynamics will arise at the quantum mechanical level due to the need of introducing a vector potential describing the field strength, in analogy to the Aharonov–Bohm effect [35].

The zero–*th* component of eq. (4.9) gives the constraint

$$F_{mn} = \partial_m A_n - \partial_n A_m = -\varepsilon_{mn}\, \frac{2\pi}{k}\, \Psi^\dagger \Psi \tag{4.11}$$

where $\varepsilon_{12} = +1 = -\varepsilon_{12}$. Eq. (4.11) determines the space components A_i of the gauge potential as a function of the density (we leave the time–dependence implicit):

$$A_m(x) = \varepsilon_{mn}\, \frac{1}{2k} \int d^2y\, \partial_n G(x-y) \Psi^\dagger(y)\Psi(y) \tag{4.12}$$

This is the solution of eq. (4.11) in the Coulomb gauge $\partial_n A_n = 0$. G is the Green function for the Laplacian on the plane, satisfying $\nabla^2 G(x) = 4\pi\delta^2(x)$, which is given by

$$G(x-y) = \ln|\vec{x}-\vec{y}|^2 \tag{4.13}$$

On the torus eq. (4.12) will be modified due to the presence of topological components of the gauge field, as we saw in the previous chapter, as well as for the presence of zero modes of the Laplacian operator. The spatial part of (4.9) gives the equation

$$F_{0n} = \partial_0 A_n - \partial_n A_0 = \frac{2\pi}{k} \varepsilon_{nm} J_m \tag{4.14}$$

which, due to current conservation, permits to compute A_0 in terms of the spatial current [60,66] given in equation (4.10): the gauge potential A_μ becomes therefore a dependent degree of freedom.

Variation with respect to $\Psi^\dagger$ gives the equation of motion for the matter field

$$\left(i\frac{\partial}{\partial t} + A_0 \right) \Psi = -\frac{1}{2m} \vec{\mathcal{D}} \cdot \vec{\mathcal{D}}\, \Psi. \tag{4.15}$$

On the other hand, noting that the momentum conjugate to Ψ is $\Pi_\Psi = i\Psi^\dagger$, one can deduce the Hamiltonian of the system, which assumes the simple form

$$H = \frac{1}{2m} \int d^2x\, \Pi_n^\dagger(x) \Pi_n(x) \tag{4.16}$$

where

$$\Pi_m(x) = (\partial_m - iA_m(x))\,\Psi(x) \tag{4.17}$$

and A_m is given in (4.12).

In deriving this equation we used the fact that the pure Chern–Simons action, being intrinsically independent of the metric tensor, does not contribute to the Hamiltonian, and we took advantage of the constraint (4.11). This Hamiltonian may seem somewhat puzzling at first sight, in that it seems the Hamiltonian for a charged particle interacting with an electromagnetic field, in a gauge where $A_0 = 0$, and one could ask what happens with gauge invariance. Also, it does not seem to reproduce correctly the equations of motion for Ψ, obtained by taking the Poisson bracket of the Hamiltonian with respect to $\Psi^\dagger$. To explain these apparent problems it is sufficient to observe that there is a (hidden) contribution to the Poisson bracket between H and $\Psi^\dagger$, coming from (4.12), which equals precisely the expression for A_0 which is obtained[66] by solving eq. (4.14) for A_0. The final result is indeed the gauge covariant equation of motion (4.15).

The second quantized version of the theory is obtained by imposing the canonical commutation relations (we choose here bosonic ones)

$$\begin{aligned} \left[\Psi(x_1), \Psi^\dagger(x_2)\right] &= \delta^2(x_1 - x_2) \\ \left[\Psi(x_1), \Psi(x_2)\right] &= 0 = \left[\Psi^\dagger(x_1), \Psi^\dagger(x_2)\right], \end{aligned} \tag{4.18}$$

with a suitable normal ordering prescription for the operators Π given in (4.17). One has in fact

$$[\Psi(y), A_m(x)] = \frac{1}{2k}\varepsilon_{mn}\,\partial_n G(x-y)\Psi(y). \tag{4.19}$$

$\varepsilon_{mn}\partial_n G(x-y)$ is ill–defined at the origin; following ref. [23] we define it to vanish there to preserve the antisymmetry of $\varepsilon_{mn}\partial_n G(x-y)$ under space–reflection. Therefore no ordering ambiguity afflicts Π and $\Pi^\dagger$.

Following Jackiw and Pi[66,23], one can also consider a more general Hamiltonian, obtained by adding to (4.16) a local attractive quartic interaction of strength g:

$$H_g = \frac{1}{2m}\int d^2x\, \Pi_n^\dagger(x)\Pi_n(x) - \frac{g}{2}\int d^2x\; :\left(\Psi^\dagger(x)\Psi(x)\right)^2: \tag{4.20}$$

In view of (4.11) this interaction can also be regarded as a magnetic field–charge density interaction. The interesting peculiarity of this Hamiltonian is that classically it describes a self–dual system [23], in the sense that the equations governing their ground states are self–dual equations, provided one chooses for g the particular value

$$g = \frac{2\pi}{km} \tag{4.21}$$

Also, for this value of g the system turns out to be related to the two–dimensional Pauli interaction. For more details about this Hamiltonian, and for classical self–dual soliton solutions of (4.20) see refs. [23,66]. For further comparison with our formulation of the first quantization on the torus we consider now H_g as our Hamiltonian. It is also clear that this Hamiltonian can be obtained by modifying correspondingly the action (4.8).

The first quantized Hamiltonian is now straightforwardly obtained observing that the number operator

$$\mathcal{Q} = \int d^2x \, \rho(x) \tag{4.22}$$

commutes with the Hamiltonian, and that one can diagonalize simultaneously $\mathcal{Q}$ and H_g.

$$\begin{aligned} H_g \, |E, N_A\rangle &= E \, |E, N_A\rangle \\ \mathcal{Q} \, |E, N_A\rangle &= N_A \, |E, N_A\rangle \end{aligned} \tag{4.23}$$

We project these equations on states with a fixed number, N_A, of anyons in the configuration space representation:

$$|\vec{x}_i\rangle \equiv \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_{N_A}) \, |0\rangle \tag{4.24}$$

The wave function is defined as $\Psi(x_1, ..., x_{N_A}) \equiv \langle \vec{x}_i | E, N_A\rangle$ and the projection of (4.23) on the states in (4.24) gives

$$\langle \vec{x}_i | H_g | E, N_A\rangle = \langle 0 \, | [\Psi(x_1) \cdots \Psi(x_{N_A}), H_g] | \, E, N_A\rangle = E \, \Psi(x_1, ..., x_{N_A}) \tag{4.25}$$

The commutators are easily evaluated, no divergences are encountered, and one gets the first quantized Hamiltonian

$$H_I \, \Psi(x_1, ..., x_{N_A}) \equiv \left[-\frac{1}{2m} \sum_{l=1}^{N_A} \left(\vec{\partial}_l - i\vec{A}_l \right)^2 - g \sum_{i<j} \delta^2(x_i - x_j) \right] \Psi(x_1, ..., x_{N_A}) \tag{4.26}$$

where

$$\vec{A}_l \equiv \frac{1}{2k} \vec{\nabla}_l \times \sum_{j \neq l} \ln|\vec{x}_l - \vec{x}_j|^2 \tag{4.27}$$

$((\vec{\nabla}_\times)_m = \varepsilon_{mn}\partial_n)$. Let us make a few comments about this result. First we observe that the one-particle problem is free, i.e. there are no self–interactions, the corresponding Hamiltonian being the free one $-\frac{1}{2m}\nabla^2$. Second, the expression (4.27) for the vector potential in the Hamiltonian H_I could also have been obtained by substituting for the density operator in (4.12) its first quantized counterpart

$$\Psi^\dagger(y)\Psi(y) \to \sum_{i=1}^{N_A} \delta^2(y - x_i) \tag{4.28}$$

and by evaluating the so obtained $\vec{A}(x)$ in the point $x = x_l$. In doing so, on the right hand side of (4.12) the term $(\varepsilon_{mn}\partial_n G)(0)$ appears, which has to be set to zero according to our normal ordering prescription, and one gets back eq. (4.27). Third, we should remark that, at the second quantized level, the eq. (4.15) acquires an additional contribution coming from re–ordering[23], in that one has to compute $i\dot{\Psi}(x) = [\Psi(x), H]$. This additional contribution in the commutator is actually needed in the evaluation of (4.25) to restore the full covariant Laplacians in (4.26). The wave function of eq. (4.26) is symmetric in the coordinates, but it is clear that the same expression is obtained for the Hamiltonian if we consider the Ψ's as anticommuting fields, the wave function becoming antisymmetric: due to the fact that we are in the non relativistic regime both choices are allowed.

In the complex language, $z = x + iy$, the Hamiltonian becomes

$$\begin{aligned} H_I &= -\frac{1}{m}\sum_{i=1}^{N_A}\left(\mathcal{D}_i\overline{\mathcal{D}}_i + \overline{\mathcal{D}}_i\mathcal{D}_i\right) - g\sum_{i<j}\delta^2(z_i - z_j) \\ &= -\frac{2}{m}\sum_{i=1}^{N_A}\overline{\mathcal{D}}_i\mathcal{D}_i + \left(\frac{2\pi}{km} - g\right)\cdot\sum_{i<j}\delta^2(z_i - z_j) \end{aligned} \tag{4.29}$$

where the covariant derivatives are

$$\begin{aligned} \mathcal{D}_l &= \partial_l + \frac{1}{2k}\partial_l\sum_{j\neq l} G(z_l - z_j) \\ \overline{\mathcal{D}}_l &= \bar{\partial}_l - \frac{1}{2k}\bar{\partial}_l\sum_{j\neq l} G(z_l - z_j). \end{aligned} \tag{4.30}$$

In equation (4.29) we note a δ–function ambiguity in the definition of the Hamiltonian H_I itself, due to the fact that in the absence of the gauge field one has $\nabla^2 = 4\bar{\partial}\partial = 4\partial\bar{\partial}$, while the same is not true for the covariant derivatives. In the following we will decide to consider a "normal ordered" Hamiltonian[22], $H_I = -\frac{2}{m}\sum_i \overline{\mathcal{D}}_i\mathcal{D}_i$, where we do not consider the δ–function contributions at all.

Alternatively we can also say that we choose for g the particular value $g = \frac{2\pi}{km}$ to get this Hamiltonian. Notice that this is precisely the value of eq. (4.21), for which the classical system becomes self–dual and for which in ref. [66] explicit classical ground state solutions have been found. It is therefore not to surprising that we will be able to find all the ground state solutions of the corresponding quantum mechanical system. It is, however, obvious, by direct inspection, that the quantum mechanical ground state solutions, i.e. the solutions of

$$\left(\sum_i \overline{\mathcal{D}}_i\mathcal{D}_i\right)\Psi = 0$$

are not normalizable on the *plane*. This is one more reason for abandoning now the plane and to switch to the torus in the next section.

On wave functions which vanish for $z_i \to z_j$ the δ–function contributions are in any case irrelevant.

4.3 Hamiltonian formulation on the torus

The most convenient description of the torus is given by the Fuchsian representation, determined by a complex modular parameter τ, which we will use in the rest of this thesis. At the beginning of this section, however, we start with a real torus, with dimensions L_1 and L_2 respectively, and with area $\nu = L_1 \cdot L_2$, to make contact with the results of the previous section.

We begin reconsidering the constraint (4.11) on the torus[20,21]. Remembering that we defined the number operator $Q = \int d^2x \Psi^\dagger(x)\Psi(x)$, integration over the torus yields

$$\frac{1}{2\pi}\int F_{xy} = -\frac{Q}{k}$$

Now the eigenvalues of the number operator Q are integers, and the Dirac quantization condition tells us that the left hand side is an integer number too. This permits us to conclude that k has to be rational, as anticipated in section 4.1. We will derive the Dirac quantization condition in the quantum mechanical treatment below.

We know from the previous chapter that at the torus the constraint (4.11) leaves the constant topological components of the gauge field undetermined. We can solve it writing

$$A_m(x) = \frac{a_m}{L_m} + \frac{\pi}{\nu k}\varepsilon_{mn}x_n Q + \frac{1}{2k}\varepsilon_{mn}\int d^2y\,\partial_n P(x-y)\Psi^\dagger(y)\Psi(y). \qquad (4.31)$$

Here $P(x-y)$ is the Greens function for the Laplacian on the torus, which, due to the presence of a zero mode, satisfies

$$\nabla^2 P(x-y) = 4\pi\left(\delta^2(x-y) - \frac{1}{\nu}\right)$$

and is a single valued and modular invariant function on the torus. The explicit expression for P will be given below. Eq. (4.31) is the solution in the Coulomb gauge $\partial_m A_m = 0$, where the gauge potential transforms by a constant as $x_m \to x_m + L_m N_m$.

The Hamiltonian is again given by (4.16) and (4.17) (or (4.20)), but there is now an additional normal ordering ambiguity in the Hamiltonian due to the fact that instead of (4.19) one has now

$$[\Psi(y), A_m(x)] = \frac{1}{2k}\varepsilon_{mn}\,\partial_n P(x-y)\Psi(y) + \frac{\pi}{\nu k}\varepsilon_{mn}x_n\Psi(y) \qquad (4.32)$$

Even if we define now $\varepsilon_{mn}\,\partial_n P(0)$ to vanish, as on the plane, there is a *finite* ordering ambiguity in the definition of $\Pi(x)$, due to the last term in this equation, which is finite as $x \to y$. As a short cut, to find the first quantized Hamiltonian, in which we are interested in, we proceed according to (4.28). Before doing that we switch now definitively to the complex formulation. The derivatives become $\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar\partial = \frac{\partial}{\partial \bar z} = \frac{1}{2}(\partial_x + i\partial_y)$ and the complex components of the vector potential are $A = \frac{1}{2}(A_x - iA_y)$, $\bar A = \frac{1}{2}(A_x + iA_y)$. The covariant derivatives are $\mathcal{D} = \partial - iA$, $-\mathcal{D}^\dagger = \overline{\mathcal{D}} = \bar\partial - i\bar A$. The complex topological components of the gauge field are conveniently normalized as

$$A_{top} = -i\frac{\pi}{\nu}a \qquad (4.33)$$

according to (3.40). We recall that z is defined modulo $m+n\tau$ and that the area of the torus is $\nu = \text{Im}\tau$. Eq. (4.31) becomes then

$$-iA(z) = -\frac{\pi}{\nu}\,a + \frac{\pi}{2\nu k}\,\bar{z}\,Q + \frac{1}{2k}\,\partial \int d^2w\, P(z-w)\Psi^\dagger(w)\Psi(w)$$

and $\bar{A}$ is the complex conjugate. The Greens function P of the Laplacian, satisfying

$$\frac{1}{4}\nabla^2 P(z) = \bar{\partial}\partial P(z) = \pi\left(\delta^2(z) - \frac{1}{\nu}\right)$$

is given by

$$P(z) = \ln\left|\frac{\theta\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](z)}{\theta'\left[\begin{smallmatrix}1/2\\1/2\end{smallmatrix}\right](0)}\right|^2 + \frac{\pi}{2\nu}(z-\bar{z})^2. \tag{4.34}$$

This is nothing else then the scalar propagator in string theory, expressed in terms of the standard θ–functions with characteristics. To derive the first quantized theory we fix the number of particles to be $N_{\mathcal{A}}$ and enforce the substitution (4.28) (Q becomes $N_{\mathcal{A}}$)

$$-iA(z) = -\frac{\pi}{\nu}\,a + \frac{\pi}{2\nu k}\,\bar{z}\,N_{\mathcal{A}} + \frac{1}{2k}\sum_{j=1}^{N_{\mathcal{A}}}\partial P(z-z_j)$$

The vector potential A_l, analogous to (4.27), is obtained by setting in this equation $z = z_l$ and by defining, as before, $\partial P(0)$ to be zero:

$$\begin{aligned}-iA_l &= -\frac{\pi}{\nu}a + \frac{\pi}{2\nu k}\,N_{\mathcal{A}}\,\bar{z}_l + \frac{1}{2k}\sum_{j\neq l}\partial_l P(z_l - z_j)\\ -i\bar{A}_l &= \frac{\pi}{\nu}\bar{a} - \frac{\pi}{2\nu k}\,N_{\mathcal{A}}\,z_l - \frac{1}{2k}\sum_{j\neq l}\bar{\partial}_l P(z_l-z_j).\end{aligned} \tag{4.35}$$

We turn now to the topological components of the gauge field, i.e. to the Wilson lines. In the Schrödinger representation, which we are adopting here they are time independent dual quantum variables, as we saw already in the previous chapter. Indeed, substituting (4.33) in (4.8) we get a contribution to the action of the form

$$-i\frac{k\pi}{\nu}\int \bar{a}\frac{d}{dt}a\,dt \tag{4.36}$$

and canonical quantization gives

$$[\bar{a}, a] = \frac{\nu}{\pi k} \tag{4.37}$$

in agreement with eq. (3.41) of chapter three. There we saw already everything about the coherent state representation of this algebra, which turned out to be very convenient, and which we will adopt also here. We recall that $\bar{a}$ becomes the derivative operator with respect to a, $\bar{a} = a^\dagger = \frac{\nu}{\pi k}\frac{\partial}{\partial a}$ and that the wave function depends on a (not on $\bar{a}$). Therefore we get on the torus the following Hamiltonian

$$\begin{aligned} H_T &= -\frac{1}{m}\sum_{i=1}^{N_A}\left(\mathcal{D}_i\overline{\mathcal{D}}_i + \overline{\mathcal{D}}_i\mathcal{D}_i\right) - g\sum_{i<j}\delta^2(z_i - z_j) \\ &= -\frac{2}{m}\sum_{i=1}^{N_A}\overline{\mathcal{D}}_i\mathcal{D}_i + \left(\frac{2\pi}{km} - g\right)\cdot\sum_{i<j}\delta^2(z_i - z_j) \end{aligned} \tag{4.38}$$

where the covariant derivatives are given by

$$\begin{aligned} \mathcal{D}_i &= \partial_i - \frac{\pi}{\nu}a + \frac{\pi}{2\nu k}N_A\,\bar{z}_i + \frac{1}{2k}\sum_{j\neq i}\partial_i P(i,j) \\ -\mathcal{D}_i^\dagger = \overline{\mathcal{D}}_i &= \bar{\partial}_i + \frac{1}{k}\frac{\partial}{\partial a} - \frac{\pi}{2\nu k}N_A\,z_i - \frac{1}{2k}\sum_{j\neq i}\bar{\partial}_i P(i,j) \end{aligned} \tag{4.39}$$

with $P(i,j) \equiv P(z_i - z_j)$. Note that *formally* (4.38) coincides with the corresponding expression on the plane (4.29), the covariant derivatives being different in the two cases. What is actually surprising is that the δ–function contributions proportional to $\frac{2\pi}{km}$ turn out to be the same in both cases. Those contributions come from the commutators $[\overline{\mathcal{D}}_i, \mathcal{D}_i]$ at the same i, which turn out to be equal. That this happens is due to a contribution in the commutator coming from the topological components, which cancels a corresponding contribution from the z–derivatives. This tells us that we are on the right way. This interplay between the large components of the gauge field and the space-variables will appear in many points of the theory, and in a crucial way in the implementation of translation invariance. Thus also in this case we can choose for the coupling constant g of the attractive hard core interaction[22] the value $g = \frac{2\pi}{km}$ to "normal order" our Hamiltonian. Choosing the anyon's mass m equal to 2 we will, in fact, consider in the following the Hamiltonian

$$H = \sum_{i=1}^{N_A}\mathcal{D}_i^\dagger\mathcal{D}_i. \tag{4.40}$$

This Hamiltonian acts on wave functions which depend on all the coordinates of the anyons z_i, $i = 1, ..., N_A$, as well as analytically on the complex variable a, as said above:

$$\Psi = \Psi(a, z_1, \ldots, z_{N_A}, \bar{z}_1, \ldots, \bar{z}_{N_A}, t)$$

The scalar product in this Hilbert space is given by

$$(\Psi_1, \Psi_2) = \int \prod_{l=1}^{N_A} d^2 z_l \int d^2 a \exp\left(-\frac{k\pi}{\nu}\bar{a}a\right) \overline{\Psi_1}\,\Psi_2 \tag{4.41}$$

(see chapter three for the occurrence of the coherent state measure). The z_i–variables in (4.41) have to be integrated over the fundamental domain of the torus, while the integration domain for a will be specified in section 4.5.

Not all functions of a and z_i belong to the Hilbert space, the set of functions which stay in it is actually heavily constrained by the symmetries of the theory. Thus the next section is devoted to an analysis of these symmetries and to the derivation of the conditions on the physical states, implied by them.

4.4 The basic symmetries

Among the symmetries which we treat in this section there is a dynamical symmetry, which we begin with: translation invariance. This is actually an intrinsic property of the torus, not shared with Riemann surfaces of different genus, and a theory which lives on the torus should exhibit this symmetry. To see that this is indeed the case in our theory we rewrite the covariant derivatives of (4.39) as follows:

$$\begin{aligned}
\mathcal{D}_i &= \Delta_i + \sum_{j\neq i} f(i,j) \\
\Delta_i &= \partial_i - \frac{\pi}{\nu}a + \frac{\pi}{2\nu k}\sum_j \bar{z}_j \\
f(i,j) &= \frac{\pi}{2\nu k}(\bar{z}_i - \bar{z}_j) + \frac{1}{2k}\partial_i P(i,j)
\end{aligned} \tag{4.42}$$

Notice that the $f(i,j)$ are functions which depend only on the differences $z_i - z_j$ and that $f(i,j) = -f(j,i)$. Moreover

$$[\Delta_i, \Delta_j^\dagger] = 0 = [\Delta_i, \Delta_j] \quad \text{for all } i, j. \tag{4.43}$$

The (complex) total momentum operator is given by

$$\begin{aligned} \mathcal{P} &= \sum_i \mathcal{D}_i = \sum_i \Delta_i \\ \mathcal{P}^\dagger &= \sum_i \mathcal{D}_i^\dagger = \sum_i \Delta_i^\dagger, \end{aligned} \tag{4.44}$$

and taking into account (4.42) one gets easily

$$[H, \mathcal{P}] = 0 = [H, \mathcal{P}^\dagger]. \tag{4.45}$$

This means that both components of the total momentum are conserved operators and that the theory is translation invariant. But there is even more: as a direct consequence of (4.43) one has also

$$[\mathcal{P}, \bar{\mathcal{P}}] = 0 \tag{4.46}$$

These commutation relations hold actually also in the corresponding theory on the plane (see section 4.2), which is translation invariant too. Here, however, we note the peculiar role, played by the large components of the gauge field, to restore these commutation relations. It is finally due to their presence, if in the next chapter we will be able to formulate a translation invariant Mean Field Approximation of the theory.

These commutation relations imply also that we can diagonalize simultaneously the two components of the momentum operator and the Hamiltonian. Before we can do that we have to determine the Hilbert space, $\mathcal{H}$, in which those operators act. The essential ingredients for this purpose are the following two:

I) Covariance under large gauge transformations, i.e. under one–valued mappings from the torus into $U(1)$ with non trivial windings around the handles.

II) The quasi–periodicity condition of the wave function under shifts of $z_i \to z_i + m + n\tau$. The cocycle property for the corresponding transition function will imply a quantization condition on the total flux, the Dirac quantization condition.

We deal first with the large gauge transformations. They are given by the $U(1)$–mappings

$$g_{p,q}(z) = \exp\left(\frac{\pi}{\nu}(p(\bar{z} - z) + q(\tau\bar{z} - \bar{\tau}z))\right) \tag{4.47}$$

which induce on the gauge potential one–form $\mathcal{A} \equiv Adz + \bar{A}d\bar{z}$ the transformation $\mathcal{A} \to \mathcal{A} + ig_{p,q}\,\mathrm{d}\,g_{p,q}^{-1}$, where d is the differential. Comparison with (4.33) shows then that large gauge transformations correspond to discrete shifts of the variables a:

$$a \to a + p + q\bar{\tau} \tag{4.48}$$

where p and q are integers, as already seen in section 3.4. We can now proceed in complete analogy to that section and construct operators $U_1(p)$ and $U_2(q)$ on $\mathcal{H}$, which implement transformations with non trivial windings along the $\mathbf{1}$ and τ directions respectively. We derived these operators already for the pure Chern–Simons theory in eq. (3.44); it remains to multiply them by the operator which generates the shift for the matter, given in (4.47):

$$\begin{aligned} U_1(p) &= \prod_i g_{p,0}(z_i)\exp\left(-\frac{\pi k}{\nu}\left(\frac{1}{2}p^2 + ap\right)\right)\exp\left(p\frac{\partial}{\partial a}\right) \\ U_2(q) &= \prod_i g_{0,q}(z_i)\exp\left(-\frac{\pi k}{\nu}\left(\frac{1}{2}q^2|\tau|^2 + aq\tau\right)\right)\exp\left(q\bar{\tau}\frac{\partial}{\partial a}\right) \end{aligned} \tag{4.49}$$

The following commutation relations hold:

$$[U_i, \mathcal{D}_j] = 0 = [U_i, \mathcal{D}_j^\dagger], \tag{4.50}$$

which insure the invariance of the Hamiltonian under large gauge transformations. Also, the fundamental relation (3.15) continues to hold:

$$U_1(p)\,U_2(q) = \mathrm{e}^{-2\pi i\,k\,p\,q}\,U_2(q)\,U_1(p), \tag{4.51}$$

whose interpretation and representations have already been discussed in section 3.2. The operators

$$\begin{aligned} C_1(p) &\equiv U_1(s\cdot p) \\ C_2(q) &\equiv U_2(s\cdot q) \end{aligned} \tag{4.52}$$

are again seen to be Casimir operators which then in any irreducible representation have to be constants, i.e. phases. Each vector $\Psi(a, z_i, \bar{z}_i)$ which belongs to the Hilbert space $\mathcal{H}$ has then to satisfy the relation

$$C_1(p)C_2(q)\Psi(a) = \mathrm{e}^{2\pi i\,r\,s(p\varphi_1 + q\varphi_2)}\Psi(a) \tag{4.53}$$

for a fixed pair of real phases $\varphi_{1,2}$. We know already from our discussion in section 3.3 that modular co–variance imposes $\varphi_1 = \varphi_2 = 1/2$. Actually, in the present complex formulation of our model this fact is related with a curious integer number theoretic problem, as we will see in a moment.

Eq. (4.53) can then be rewritten as

$$\Psi(a + s\cdot(p+q\bar{\tau})) = g^{-1}_{sp,sq}(z)\, e^{\frac{rs\pi}{\nu}\left(\frac{|p+q\tau|^2}{2} + \frac{a}{s}(p+q\tau)\right) + i\pi rs(pq+p+q)}\, \Psi(a) \tag{4.54}$$

where z is the center–of–mass coordinate, $z \equiv \sum_{i=1}^{N_A} z_i$, and summarizes our requirement of invariance under large gauge transformations: every state $\Psi \in \mathcal{H}$ has to fulfill (4.54).

We address now the issue of modular invariance in the coupled system – in the pure Chern–Simons theory it has already been discussed extensively in section 3.3 – and turn afterwards to point II (the quasi periodicity condition of the wave function under shifts around a handle of the torus).

The group of modular transformations corresponds to the equivalent ways of parametrizing the non trivial cohomology cycles on the torus. In the complex language it amounts to send

$$\begin{aligned}
\tau \longrightarrow \tau^{\mathcal{M}} &= \frac{\alpha\tau+\beta}{\gamma\tau+\delta}\\
\nu \longrightarrow \nu^{\mathcal{M}} &= \frac{\nu}{|\gamma\tau+\delta|^2}\\
z \longrightarrow z^{\mathcal{M}} &= \frac{z}{\gamma\tau+\delta}\\
a \longrightarrow a^{\mathcal{M}} &= \frac{a}{\gamma\bar{\tau}+\delta}
\end{aligned} \tag{4.55}$$

where $\alpha, .., \delta$ are integers such that

$$\alpha\delta - \beta\gamma = 1. \tag{4.56}$$

If we think of the torus as a donut immersed in R^3, as we do, these transformations do actually not admit an immediate *physical* interpretation; we observe, however, that our Hamiltonian and momentum operators *are* indeed covariant under this group, more precisely they change by an overall factor

$$\begin{aligned}
H(z_i^{\mathcal{M}}, a^{\mathcal{M}}, \tau^{\mathcal{M}}) &= |\gamma\tau+\delta|^2\, H(z_i, a, \tau)\\
\mathcal{P}(z_i^{\mathcal{M}}, a^{\mathcal{M}}, \tau^{\mathcal{M}}) &= (\gamma\tau+\delta)\, \mathcal{P}(z_i, a, \tau).
\end{aligned} \tag{4.57}$$

This implies that if $\Psi(z_i, a, \tau)$ is a simultaneous eigenfunction of H and $\mathcal{P}$ with eigenvalues $E(\tau)$ and $P(\tau)$ rispectively, the modular transformed state

$$\Psi^{\mathcal{M}}(z_i, a, \tau) \equiv \Psi(z_i^{\mathcal{M}}, a^{\mathcal{M}}, \tau^{\mathcal{M}}) \tag{4.58}$$

would be a simultaneous eigenfunction too,

$$\begin{aligned} H\,\Psi^{\mathcal{M}}(z_i, a, \tau) &= \frac{E(\tau^{\mathcal{M}})}{|\gamma\tau+\delta|^2}\,\Psi^{\mathcal{M}}(z_i, a, \tau) \\ \mathcal{P}\,\Psi^{\mathcal{M}}(z_i, a, \tau) &= \frac{P(\tau^{\mathcal{M}})}{\gamma\tau+\delta}\,\Psi^{\mathcal{M}}(z_i, a, \tau), \end{aligned} \tag{4.59}$$

provided it belongs to $\mathcal{H}$. To ensure this property we have to require the modular co–variance of (4.54), that is, we have to require that, if a state Ψ satisfies (4.54) for all integers p, q, then the state $\Psi^{\mathcal{M}}$ should satisfy the *same* relation (4.54) for all p, q. That this is actually the case is easily verified by direct inspection. The unique non trivial point is the invariance of the last phase exponential

$$\begin{aligned} \mathrm{e}^{i\pi rs(pq+p+q)} &\to \mathrm{e}^{i\pi rs[(\alpha p+\beta q)(\gamma p+\delta q)+\alpha p+\beta q+\gamma p+\delta q)]} \\ &= \mathrm{e}^{i\pi rs[pq+(\alpha+\gamma+\alpha\gamma)p+(\beta+\delta+\beta\delta)q]} \end{aligned} \tag{4.60}$$

We used (4.56) and the fact that for all integers M, N $\exp(i\pi MN^2) = \exp(i\pi MN)$. Moreover, due again to (4.56), the quantities between round brackets in the last row can be seen to be always odd. This can be checked, for example, by considering the two generators of the modular group $\tau \to \tau+1$, $\tau \to -\frac{1}{\tau}$. Thus one gets back the original phase $\mathrm{e}^{i\pi rs(pq+p+q)}$ for all modular transformations. It is then also clear that any other choice for the phases $\varphi_{1,2}$ would break modular invariance!

Before giving the general solution of (4.54) we state now the restriction on Ψ coming from the quasi–periodicity of the wave function on the torus (point II)). After translation along one of the two homology cycles, $z_i \to z_i + m_i + n_i\tau$, the wave function should return to its original value up to a gauge transformation, according to the transformation property of the covariant derivative in (4.39):

$$\mathcal{D}_i \to \mathcal{D}_i + \frac{\pi}{2\nu k} N_A\,(m_i + n_i\bar{\tau})$$

The most general transformation law, compatible with the group property, is given by $(Q \equiv N_A/k)$:

$$
\begin{aligned}
V_1(m)\Psi &\equiv \prod_i e^{-\frac{\pi Q}{2\nu} m_i(\bar{z}_i - z_i) - 2\pi i Q m_i \alpha_1} e^{m_i(\partial_i + \bar{\partial}_i)} \Psi = \Psi \\
V_2(n)\Psi &\equiv \prod_i e^{-\frac{\pi Q}{2\nu} n_i(\tau\bar{z}_i - \bar{\tau} z_i) - 2\pi i Q n_i \alpha_2} e^{n_i(\tau\partial_i + \bar{\tau}\bar{\partial}_i)} \Psi = \Psi
\end{aligned}
\tag{4.61}
$$

where the $\alpha_{1,2}$ are fixed phases. Notice that $[V_i, \mathcal{D}_j] = 0 = [V_i, \bar{\mathcal{D}}_j]$, implying that the Hamiltonian is a well defined operator in the Hilbert space, and that $[V_j, U_i] = 0$, as is required by invariance under large gauge transformations. The cocycle property of the transition functions on the torus can be stated in the form

$$[V_1(m), V_2(n)] = 0$$

and implies the flux quantization (Dirac quantization condition):

$$Q \equiv \frac{N_A}{k} = \text{integer.} \tag{4.62}$$

Notice that eq. (4.62) implies the rationality of k, as anticipated several times: $k = \frac{r}{s}$. Actually, due to the fact that r and s are relative prime integers, N_A has to be an integer multiple of r:

$$N_A = r \cdot u, \qquad Q = s \cdot u \tag{4.63}.$$

If we rely again on modular invariance we have to put $\alpha_1 = \alpha_2 = \frac{1}{2}$, and the conditions (4.61) can then be brought in the form

$$\Psi(z_i + m_i + n_i\tau) = \prod_i e^{\frac{\pi Q}{2\nu}(m_i(\bar{z}_i - z_i) + n_i(\tau\bar{z}_i - \bar{\tau} z_i)) + i\pi Q(m_i n_i + m_i + n_i)} \Psi(z_i) \equiv L_{m_i,n_i}\Psi(z_i) \tag{4.64}$$

In summary, the Hilbert space $\mathcal{H}$ is made out of all those functions $\Psi(z_i, \bar{z}_i, a)$ which satisfy the conditions (4.54) and (4.64), and carries a representation of the group of modular transformations: so this group corresponds to a physical symmetry of our model. In the next section we will determine the general solutions of these two equations, to find an explicit basis for $\mathcal{H}$, and we will also show in which sense modular invariance is saved when we project the so determined "large" Hilbert space to the *physical* Hilbert space, i.e. when we introduce a gauge fixing for the large gauge transformations. The procedure and its consistency have already been shown in sections 3.2 and 3.3, in the next section we have only to generalize slightly this procedure due to the presence of matter.

4.5 The Hilbert space

We give first the general solutions of (4.54). Recalling that $z \equiv \sum_i z_i$ we use the variable

$$b = \frac{1}{s}\left(a - \frac{\bar{z}}{k}\right)$$

and define the function

$$\Psi(b) = \exp\left(\frac{rs\pi}{2\nu} b^2 + s\,\frac{\pi}{\nu} b\, z\right) \Lambda(b).$$

Then (4.54) becomes

$$\Lambda(b + p + q\bar{\tau}) = \exp\left(i\pi rs\left(q^2\bar{\tau} + 2bq + p + q\right)\right) \Lambda(b)$$

But this functional equation has already been solved in section 3.4, see eqs. (3.49) and (3.50). Substituting back everything we get the $r \cdot s$ independent solutions

$$\phi_{m,n}(a, z) = \exp\left(\frac{\pi}{\nu}\left(a - \frac{\bar{z}}{k}\right) z + \frac{\pi k}{2\nu}\left(a - \frac{\bar{z}}{k}\right)^2 + \frac{\pi |z|^2}{2\nu k}\right) \overline{\theta\begin{bmatrix} \frac{m}{r} + \frac{n}{s} + \frac{1}{2} \\ \frac{rs}{2} \end{bmatrix}(s(k\bar{a} - z)|rs\tau)} \tag{4.65}$$

These functions are generalizations to the coupling with matter of the ones given in (3.50) for the pure Chern–Simons theory, which determine the allowed a–dependence of the states in the Hilbert space. Notice that the coordinate dependence is only through the center-of-mass coordinate z. We included some additional z–dependence for later convenience.

Again m and n are defined modulo r and s respectively, and therefore we can restrict them to $1 \leq m \leq r$ and $1 \leq n \leq s$. The action of the large gauge transformations $U_{1,2}$, which are given in (4.49), on this basis turns out to be

$$\begin{aligned} U_1(p)\,\phi_{m,n} &= \mathrm{e}^{-2\pi i p n k + i\pi p r}\,\phi_{m,n} \\ U_2(q)\,\phi_{m,n} &= \mathrm{e}^{+i\pi q r}\,\phi_{m,n+q} \end{aligned} \tag{4.66}$$

These is formally identical to (3.21) and permits us again to conclude that the index n of $\phi_{m,n}$ spans an internal space, on which the large gauge transformations act, while the index m spans the physical space which is not touched by those transformations. In the following we will see that this is in fact the appropriate interpretation. It is also clear from (4.54) and (4.66) that, in order to implement the operators U_i unitarily, the integration region for a in the scalar product defined in (4.41) has to be the s–torus (If we write $a = u + v\bar{\tau}$ the integration region is $0 \leq u, v \leq s$). It can then be shown that one has

$$\int d^2 a \exp\left(-\frac{k\pi}{\nu}\bar{a}a\right) \overline{\phi_{m_1,n_1}}\,\phi_{m_2,n_2} = s^2\sqrt{\frac{\nu}{2\,rs}} \cdot \delta_{m_1 m_2} \cdot \delta_{n_1 n_2} \tag{4.67}$$

The generic state in $\mathcal{H}$ can now be written as

$$\Psi = \sum_{m=1}^{r}\sum_{n=1}^{s} \phi_{m,n}\Psi_{m,n}(z_i, \bar{z}_i) \tag{4.68}$$

where the $\Psi_{m,n}$ depend only on the anyon's coordinates, but not on a. Due to (4.67) the scalar product turns then out to be

$$(\Psi, \Psi') = s^2\sqrt{\frac{\nu}{2rs}} \int \prod_{l=1}^{N_A} d^2 z_l \sum_{m=1}^{r}\sum_{n=1}^{s} \overline{\Psi}_{m,n}\,\Psi'_{m,n}. \tag{4.69}$$

It remains to find a convenient basis for the $\Psi_{m,n}$ which span the complete Hilbert space in agreement with (4.64). This basis is obviously not unique and we have to decide the operators which we want to diagonalize. It is convenient to analyse the Hilbert space in terms of the total momentum which we know to be a conserved operator:

$$\begin{aligned} \mathcal{P}\Psi_P &= P\Psi_P \\ \mathcal{P}^\dagger\Psi_P &= \bar{P}\Psi_P \end{aligned} \tag{4.70}$$

The Hilbert space $\mathcal{H}$ will then become a direct sum of Hilbert spaces at fixed momentum P:

$$\mathcal{H} = \sum \oplus \mathcal{H}_P. \tag{4.71}$$

The eigenfunctions can be written as

$$\Psi_P = \exp\left(\frac{P}{N_A}z - \frac{\bar{P}}{N_A}\bar{z}\right) \cdot \sum_{m,n} \phi_{m,n} h_{m,n}(z_i, \bar{z}_i) \tag{4.72}$$

and with the help of the orthogonality relations (4.67) and of the identities $\mathcal{P}\phi_{m,n} = \bar{\mathcal{P}}\phi_{m,n} = 0$, the eigenvalue equations (4.70) become $\left(\sum_i \partial_i\right) h_{m,n} = \left(\sum_i \bar{\partial}_i\right) h_{m,n} = 0$. This means that the $h_{m,n}$'s are translation invariant and depend thus only on the differences of the coordinates:

$$h_{m,n} = h_{m,n}(z_{ij}, \bar{z}_{ij})$$

where $z_{ij} \equiv z_i - z_j$, with the condition $h_{m+r,n} = h_{m,n+s} = h_{m,n}$. From now on we work at fixed momentum P.

To impose now the constraint (4.64) we need the identity

$$\begin{aligned}\phi_{m,n}(z_i + m_i + n_i\tau) =& L_{m_i,n_i} \prod_{i<j} \mathrm{e}^{i\pi \frac{m_{ij} n_{ij}}{k} + \frac{\pi}{2\nu k}(z_{ij}(m_{ij}+n_{ij}\bar{\tau}) - \bar{z}_{ij}(m_{ij}+n_{ij}\tau))} \\ & \mathrm{e}^{-\pi i(Q+s)\left(\sum_i m_i + \sum_i n_i\right)} \mathrm{e}^{2\pi i \frac{m}{k}\sum m_i} \phi_{m-\sum n_i, n}(z_i)\end{aligned} \tag{4.73}$$

where we remember that Q is the integer defined by $Q = N_{\mathcal{A}}/k$. If we take this identity into account and write the complex momentum as

$$P = \frac{\pi}{\nu}(P_1 + P_2\bar{\tau}) \tag{4.74}$$

where the P_i are real, (4.64) becomes

$$\begin{aligned}h_{m,n}(z_i + m_i + n_i\tau) =& \prod_{i<j} \mathrm{e}^{-i\pi \frac{m_{ij} n_{ij}}{k} - \frac{\pi}{2\nu k}(z_{ij}(m_{ij}+n_{ij}\bar{\tau}) - \bar{z}_{ij}(m_{ij}+n_{ij}\tau))} \\ & \mathrm{e}^{2\pi i \sum m_i \frac{\tilde{P}_2}{N_{\mathcal{A}}} - 2\pi i \sum n_i \frac{\tilde{P}_1}{N_{\mathcal{A}}}} \mathrm{e}^{-2\pi i \frac{m}{k}\sum m_i} h_{m-\sum n_i, n}(z_i)\end{aligned} \tag{4.75}$$

Here we defined

$$\tilde{P}_i \equiv P_i + rs\frac{u(u+1)}{2} \tag{4.76}$$

and u is the integer defined in (4.62). As a consequence of the translation invariance of the $h_{m,n}$ this relation implies immediately that the $\tilde{P}_i$, and therefore the P_i, have to be integer, as can be seen by setting in eq. (4.75) $m_i = M$, $n_i = N$ for all i. This is of course the expected quantization condition for the total momentum on a compact space. One can check that the so obtained spectrum of $\mathcal{P}$ is modular invariant. From (4.59) we read the modular transformed eigenvalue

$$P^{\mathcal{M}} = \frac{1}{\gamma\tau + \delta} \cdot \frac{\pi}{\nu^{\mathcal{M}}}(P_1 + P_2\bar{\tau}^{\mathcal{M}}) = \frac{\pi}{\nu}\left[(\delta P_1 + \beta P_2) + (\gamma P_1 + \alpha P_2)\bar{\tau}\right]$$

which is again of the form (4.74) where also the new $P_{1,2}$ are integers.

A set of functions $h_{m,n}$, which depend only on the differences of the coordinates z_{ij}, and fulfill the set of relations (4.75) identically, identify a state in our Hilbert space. Actually we can make the important observation that the transformation property (4.75) operates only in the physical space, labeled by the index m, while in the internal gauge space, labeled by n, it acts as the identity operator. This is the reason for why also in the coupled case we can make a consistent projection of the Hilbert space, as we did for the pure Chern–Simons theory in section 3.2. In other words, the relations (4.66) and (4.75) allow us to restrict the Hilbert space consistently to the states

$$\Psi_P^{(0)} = \exp\left(\frac{P}{N_A}z - \frac{\bar{P}}{N_A}\bar{z}\right) \sum_{m=1}^{r} \phi_m^{(0)}\, h_m(z_{ij}, \bar{z}_{ij}) \tag{4.77}$$

where

$$\phi_m^{(0)} \equiv \sum_n c_n^{(0)}\, \phi_{m,n} \tag{4.78}$$

and the constants $c_n^{(0)}$, with $\sum_{n=1}^{s} |c_n^{(0)}|^2 = 1$, define the gauge fixing constants, as in section 3.2. The physics is contained in the r (translation invariant) functions h_m, which, under shifts of the coordinates, have to transform cyclically among themselves according to (4.75) (the index n in that equation can be ignored in this context). It is important to notice that, due to the fact that all our observables are gauge invariant, and therefore do not see the internal index n, the matrix elements of the observables are *independent* of our gauge choice $\{c_n^{(0)}\}$, see also ref. [19]. The analogy with the unobservability of the gauge fixing parameter ξ in QCD may be appropriate. To conclude, our theory is invariant under global gauge transformations in the sense that a gauge transformation, see eqs. (4.66), does not change the form of the state (4.77), sending unobservabley the gauge fixing constants from $c_n^{(0)}$ to $c_n^{\prime(0)} = u_{nm}c_m^{(0)}$ where u_{mn} is a unitary matrix.

Let us now discuss what happens about modular invariance for what concerns the projected Hilbert space, spanned by (4.77). Remember that the large Hilbert space, given by (4.68), is modular invariant, as a consequence of the modular invariance of the defining equations (4.54) and (4.64). This means that if a state $\Psi(z_i, a, t)$ satisfies those equations the state $\Psi^{\mathcal{M}}$ given in (4.58) satisfies those equations too. To compute the action of a generic modular transformation (4.55) on Ψ we consider the two generating transformations $\tau \to \tau + 1$ and $\tau \to -\frac{1}{\tau}$. For $\tau \to \tau + 1$ one has [67]

$$\phi^{\mathcal{M}}_{m,n} = \mathrm{e}^{\frac{-i\pi}{rs}\left(ms+nr+\frac{rs}{2}\right)^2} \phi_{m,n} \tag{4.79}$$

while for $\tau \to -\frac{1}{\tau}$ one gets

$$\phi^{\mathcal{M}}_{m,n} = \mathcal{N} \sum_{m'=1}^{r} \sum_{n'=1}^{s} \mathrm{e}^{-2\pi i\left(\frac{mm'}{k}+knn'\right)} \phi_{m',n'} \tag{4.80}$$

where $\mathcal{N}$ is a constant which does not depend on m and n. Notice that the last relation is nothing else then the discrete Fourier transform of $\phi_{m,n}$; this is, of course, what one expects due to the fact that the corresponding modular transformation exchanges the two cohomology cycles. The transformed basis of eq. (4.80) corresponds to the dual basis of eq. (3.23). We note the remarkable fact that, as in the pure Chern–Simons theory, the two modular transformations given in eqs. (4.79) and (4.80) factorize in a tensor product of two matrices, one for the internal space (index n) and one for the physical space (index m). For a generic modular transformation $\mathcal{M}$ we have therefore, see (4.68):

$$\Psi^{\mathcal{M}} = \sum_{m,n,m',n'} \mathcal{M}^{Ph}_{mm'} \mathcal{M}^{G}_{nn'} \phi_{m',n'} \Psi_{m,n}(z_i^{\mathcal{M}}, \bar{z}_i^{\mathcal{M}}, \tau^{\mathcal{M}}) \tag{4.81}$$

where the matrices $\mathcal{M}^{Ph}$ and $\mathcal{M}^{G}$ represent the modular group separately in the physical and gauge spaces respectively. We know already that this state, being the modular transformed of a physical state, stays in the Hilbert space. However, due to (4.73), in which the index n does not enter at all, the state in (4.81) stays also in the Hilbert space, if the indices n and n' are not summed over but kept fixed, and if the matrix $\mathcal{M}^G$ is suppressed. This allows us to *define* a representation of the modular group also in the projected space, spanned by (4.77):

$$\Psi_P^{(0)\mathcal{M}} = \exp\left(\frac{P^{\mathcal{M}}}{N_A} z - \frac{\bar{P}^{\mathcal{M}}}{N_A}\bar{z}\right) \sum_{m,m'} \mathcal{M}^{Ph}_{mm'} \phi^{(0)}_{m'} h_m(z_{ij}^{\mathcal{M}}, \bar{z}_{ij}^{\mathcal{M}}, \tau^{\mathcal{M}}) \tag{4.82}$$

This state has the right transformation properties and constitutes obviously a representation of the modular group. Notice that the momentum eigenvalue has also been changed according to (4.59): modular transformations mix the currents flowing along the two handles. In summary, the physical projected Hilbert space,

$$\mathcal{H}^{(0)} = \sum \oplus \mathcal{H}_P^{(0)} \tag{4.83},$$

spanned by the vectors (4.77), where the h_m have to satisfy (4.75), carries a representation of the modular group and is gauge invariant in the sense specified above.

As a last achievement of this section we would like to give an explicit basis for the spaces $\mathcal{H}_P^{(0)}$ by finding all the solutions of (4.75).

The identity $\phi^{(0)}_{m+Mr} = \phi^{(0)}_m$ implies the restriction $h_{m+Mr} = h_m$, and so it is necessary to find an h_0 such that:

$$\begin{aligned} h_0(z_i + m_i + n_i\tau) = \prod_{i<j} & \mathrm{e}^{-i\pi\frac{m_{ij}n_{ij}}{k} - \frac{\pi}{2\nu k}(z_{ij}(m_{ij}+n_{ij}\bar\tau) - \bar z_{ij}(m_{ij}+n_{ij}\tau))} \\ & \exp\left(2\pi i \sum m_i \frac{\tilde P_2}{N_A} - 2\pi i \sum n_i \frac{\tilde P_1}{N_A}\right) h_0(z_i) \end{aligned} \tag{4.84}$$

for all m_i and for all sets n_i such that $\sum_i n_i = Nr$ for some integer N. Let us suppose to have found such an h_0. Then we can determine all the other components of h_m in terms of h_0 via

$$h_m(z_i) \equiv \exp\left(-2\pi i m \frac{\tilde P_1}{N_A}\right) \prod_{i<j} \mathrm{e}^{-\frac{\pi}{2\nu k} n_{ij}(z_{ij}\bar\tau - \bar z_{ij}\tau)} h_0(z_i - n_i\tau) \tag{4.85}$$

where the set $\{n_i\}$ has to be such that $\sum_i n_i = m \pmod r$. The property (4.84) insures then, first, that the r.h.s. of (4.85) is independent of the particular set $\{n_i\}$ one chooses, and second, that the vector h_m defined in (4.85) satisfies in turn eq. (4.75) for arbitrary (m_i, n_i). So we are now completely reduced to the problem of finding the general solutions of (4.84). It is convenient to parametrize them in the form

$$h_0 = W(z_{ij}, \bar z_{ij}) \prod_{i<j} \mathrm{e}^{\frac{\pi}{2\nu k}(\bar z_{ij}^2 - |z_{ij}|^2)} G \tag{4.86}$$

where the first factor W is a single valued function on the torus and G, from (4.84) and (4.86), has to satisfy

$$G(z_i + m_i + n_i\tau) = \mathrm{e}^{2\pi i \sum m_i \frac{\tilde P_2}{N_A} - 2\pi i \sum n_i \frac{\tilde P_1}{N_A}} \prod_{i<j} e^{\frac{\pi i \bar\tau}{k}(n_{ij})^2 + \frac{2\pi i}{k} n_{ij}\bar z_{ij}} G(z_i) \tag{4.87}$$

for $\sum_i n_i = Nr$. The general solutions of this equation[20] are parametrized in terms of an arbitrary function C of the imaginary parts of z_{ij} and of a given N_A–tuple of integers r_i, $i = 1, ..., N_A$, such that

$$\sum_i r_i = -\tilde{P}_2.$$

In terms of these data they are given by

$$G^{\{r_i\}} = \sum_{\{n_j\}} \sum_{N=1}^{u} \delta\left(Nr - \sum_i n_i\right) \prod_{i<j} e^{-\frac{i\pi\bar{r}}{k}\left(n_{ij} - \frac{r_{ij}}{Q}\right)^2 - \frac{2\pi i}{k}\left(n_{ij} - \frac{r_{ij}}{Q}\right)\bar{z}_{ij}} . \\ e^{2\pi i \tilde{P}_1 \frac{N}{u}} C\left(z_{ij} - \bar{z}_{ij} + 2i\nu\left(n_{ij} - \frac{r_{ij}}{Q}\right)\right) \tag{4.88}$$

The sets $\{r_i\}$ are defined modulo the equivalence relation

$$r_i \sim r_i + s(N - un_i), \qquad \sum_i n_i = rN$$

in the sense that r_i and $r_i + s(N - un_i)$ give the same solution at fixed C. That these expressions are solutions of (4.84) can be easily verified by explicit computation; in the next chapter we will give a proof of this formula (sect. 5.2) and an interpretation of the sets $\{r_i\}$ as well as of the functions C in the context of the mean–field approximation. There it will also become clear that in general the parametrization we gave above is an over–parametrization of the Hilbert sub–space $\mathcal{H}_P^{(0)}$, but it will turn out to be suitable for our purposes.

This concludes the kinematical part of our analysis, i.e. the determination of the Hilbert space, and we turn now in the next section of this chapter to the dynamics, i.e. to the diagonalization of the Hamiltonian, as far as it is possible.

Let us make finally a comment on the one-particle problem, i.e. the case $N_A = 1$, $k = \frac{1}{s}$, $Q = s$. Setting $z_i = z$ the Hamiltonian becomes

$$H_1 = \Delta^\dagger \Delta$$

where $\Delta = \partial - \frac{\pi}{\nu} a + \frac{\pi}{2\nu k}\bar{z} = \mathcal{P}$ and

$$[\Delta^\dagger, \Delta] = 0$$

The Hilbert space is given simply by the functions (see (4.77))

$$\Psi_P^{(0)} = \exp\left(Pz - \bar{P}\bar{z}\right)\phi_1^{(0)}$$

since there is only one h_m $(r = 1)$ which has to be constant. These functions are already eigenstates of the Hamiltonian

$$H_1 \Psi_P^{(0)} = |P|^2\, \Psi_P^{(0)}$$

and constitute a kind of plane wave basis. Therefore the one–body problem is free, exactly as on the plane, as it should be of course. With respect to this result we note again the crucial role played by the topological components of the gauge field.

4.6 Exact ground states

What we determined in section five are the eigenspaces $\mathcal{H}_P^{(0)}$ of the momentum operator $\mathcal{P}$, and we try now to find eigenstates of the (normal ordered) Hamiltonian, given in (4.40), in each of the $\mathcal{H}_P^{(0)}$. For some solutions on the plane see ref. [22].

To this purpose we note the identity

$$H = \sum_i \left(\mathcal{D}_i - \frac{P}{N_A}\right)^\dagger \left(\mathcal{D}_i - \frac{P}{N_A}\right) - \bar{\mathcal{P}}\frac{P}{N_A} + \frac{\bar{P}}{N_A}\mathcal{P} - \frac{|P|^2}{N_A} \tag{4.89}$$

which on $\mathcal{H}_P^{(0)}$ becomes

$$H\,\Psi_P^{(0)} = \left[\sum_i \left(\mathcal{D}_i - \frac{P}{N_A}\right)^\dagger \left(\mathcal{D}_i - \frac{P}{N_A}\right) + \frac{|P|^2}{N_A}\right]\Psi_P^{(0)} \tag{4.90}$$

If we insert for $\Psi_P^{(0)}$ the general expression (4.77), the eigenvalue equation $H\Psi_P^{(0)} = E\Psi_P^{(0)}$ becomes

$$\left(\sum_i D_i^\dagger D_i\right) h_m = \left(E - \frac{|P|^2}{N_A}\right) h_m \tag{4.91}$$

for $m = 1, ..., r$, where we defined the reduced derivatives

$$D_i = \partial_i + \frac{\pi}{2\nu k}\sum_j \bar{z}_{ij} + \frac{1}{2k}\sum_{j\neq i} \partial_i P(i,j) \tag{4.92}$$

together with the reduced Hamiltonian $H_R = \sum_i D_i^\dagger D_i$. In deriving (4.91) we used the identity $\mathcal{D}_i \phi_{m,n} = \phi_{m,n} D_i$. Actually, the defining relation (4.85) insures that it is necessary and sufficient to solve (4.91) for h_0 only. So we are reduced to

$$H_R\, h_0 = \left(E - \frac{|P|^2}{N_A} \right) h_0 . \tag{4.93}$$

Let us now search for the ground states of our Hamiltonian in $\mathcal{H}_P^{(0)}$, i.e. for the states with minimal energy at fixed momentum P. The smallest possible energy is $E = \frac{|P|^2}{N_A}$ and the corresponding eigenvectors, if they exist, have to satisfy

$$D_i\, h_0 = 0 \tag{4.94}$$

for $i = 1, ..., N_A$. The equations (4.94) make sense in that the covariant derivatives D_i commute among them:

$$[D_i, D_j] = 0 \quad \text{for all } i \text{ and } j.$$

Recalling the parametrization (4.86) of $\mathcal{H}_P^{(0)}$, with the choice

$$W = \prod_{i<j} e^{-\frac{1}{2k} P(i,j)} \tag{4.95}$$

(4.94) becomes

$$\partial_i G = 0$$

for all i, meaning that G has to be an antiholomorphic function of the coordinates, and so the function C in (4.88) has simply to be a constant. We conclude that a basis for the ground states, with ground state energy $E = \frac{|P|^2}{N_A}$, is given by

$$\begin{aligned} h_0^{\{r_i\}} &= \prod_{i<j} \mathrm{e}^{-\frac{1}{2k} P(i,j)} \prod_{i<j} \mathrm{e}^{\frac{\pi}{2\nu k}(\bar{z}_{ij}^2 - |z_{ij}|^2)} . \\ \sum_{\{n_j\}} \sum_{N=1}^{u} & \delta\left(Nr - \sum_i n_i \right) \mathrm{e}^{2\pi i \tilde{P}_1 \frac{N}{u}} \prod_{i<j} e^{-\frac{i\pi\bar{\tau}}{k}\left(n_{ij} - \frac{r_{ij}}{Q}\right)^2 - \frac{2\pi i}{k}\left(n_{ij} - \frac{r_{ij}}{Q}\right)\bar{z}_{ij}} \end{aligned} \tag{4.96}$$

where we remember that the sets $\{r_i\}$, with $\sum_i r_i = -\tilde{P}_2$, are defined modulo the equivalence relation $r_i \sim r_i + s(N - un_i)$ if $\sum_i n_i = rN$, in the sense that r_i and $r_i + s(N - un_i)$ give the same solution, a part from a constant. So we get the important result that the degeneracy of the multi–particle ground state is finite. An analysis of the above mentioned equivalence relation reveals, in particular, that this degeneracy is given by

$$\Omega = s \cdot \left(\frac{N_A}{k}\right)^{N_A - 2} . \tag{4.97}$$

It is important to mention that the normalizability of the elements of the basis given in (4.96) requires that

$$k > 1$$

since

$$P(i,j) \longrightarrow \ln |z_i - z_j|^2 \qquad \text{as} \;\; z_i \longrightarrow z_j .$$

We conclude that states with the minimal energy $E = \frac{|P|^2}{N_A}$ exist if $k > 1$. We will see in a moment, however, that also for $k < 1$ there exist particular linear combinations of the states given in (4.96) which are normalizable too.

Up to now, in fact, we did not specify the intrinsic statistics of the anyons, and we come now to the more specific, and also more interesting, issue of the derivation of the ground state(s) under the condition that the anyons are either bosons or fermions. We succeed in the following three cases:

I. Case. If we take $k > 1$ and we consider the anyons to be bosons we get all the solutions by symmetrizing the states of (4.96) in the $z_i^{'}$s. It is clear that these ground states exhibit a very large degeneracy, Ω_B, whose precise expression, depending also on the momenta P_i, can not be computed easily. In the limit of large N_A one gets however [20]

$$\Omega_B \longrightarrow \frac{1}{u} \cdot \begin{pmatrix} Q + N_A - 2 \\ N_A - 1 \end{pmatrix} \approx \frac{1}{N_A} e^{cN_A} \tag{4.98}$$

where c is given by $c = (r+s)\ln(r+s) - r\ln r - s\ln s$.

II. Case. We consider the anyons as fermions and take $k \leq 1$ and such that the integer part of $\frac{1}{k}$ is *odd*. Then we can write $s = (2J+1)r + l$ where $0 < l < r$, and J is an integer. Observe that now the ground state wave functions (4.96) exhibit a singularity $\frac{1}{|z_{ij}|^{2J+1+\frac{l}{r}}}$ for $z_i \rightarrow z_j$ and so the normalizability of the wave function requires that G in (4.86) should behave as $(\bar{z}_{ij})^{2J+1}$. We have therefore to factorize from G a product of θ–functions:

$$h_0 = \prod_{i<j} e^{-\frac{1}{2k}P(i,j)} \prod_{i<j} e^{\frac{\pi}{2\nu k}(\bar{z}_{ij}^2 - |z_{ij}|^2)} \left(\prod_{i<j} \overline{\theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_{ij}|\tau)} \right)^{2J+1} G^{'} \tag{4.99}$$

Eq. (4.87) imposes on $G^{'}$ the transformation rule

$$G^{'}(z_i + m_i + n_i\tau) = e^{2\pi i \sum m_i \frac{P_2^{'}}{N_A} - 2\pi i \sum n_i \frac{P_1^{'}}{N_A}} \prod_{i<j} e^{\frac{\pi i \bar{\tau}}{k^{'}}(n_{ij})^2 + \frac{2\pi i}{k^{'}} n_{ij} \bar{z}_{ij}} G^{'}(z_i) \tag{4.100}$$

with

$$k^{'} \equiv \frac{r}{l}, \qquad P_i^{'} \equiv P_i + rs\frac{u(u+1)}{2} + \frac{N_A(N_A+1)}{2}$$

The solutions of (4.100) can be read off from (4.87), (4.88) ($C = 1$), with $k \to k^{'}$ and $P_i \to P_i^{'}$, and to get completely antisymmetric wave functions we have to take symmetrized solutions of (4.88). Again, these wave functions exhibit a degeneracy which grows exponentially with N_A, but much slower than (4.98). Notice, however, that if $l = 0$, i.e. $k = \frac{1}{2J+1}$ and $r = 1$, (4.100) becomes

$$G^{'}(z_i + m_i + n_i\tau) = e^{2\pi i \sum m_i \frac{P_2}{N_A} - 2\pi i \sum n_i \frac{P_1}{N_A}} G^{'}(z_i) \tag{4.101}$$

Remembering that $G^{'}$ has to be antiholomorphic this equation admits a solution only for the exceptional values of the momenta

$$P_i = p_i \cdot N_A, \qquad \text{with } p_i \text{ integer} \tag{4.102}$$

and then this solution is also unique, in fact, $G^{'} = \text{const.}$, which is of course symmetric. Therefore, for $k = \frac{1}{2J+1}$, there exists a unique antisymmetric ground state for the exceptional momenta given in (4.102). If one of the P_i is not a multiple of N_A we have to excite the system and the energy would be larger than $E = \frac{|P|^2}{N_A}$ by a certain amount of energy. The states with those particular momenta are therefore protected by an energy gap and particularly stable against external disturbances. We will see that this kind of superconductivity mechanism constitutes a general feature of our theory which is not confined to the particular values of k considered here, but shows up in many circumstances (see also chapter five).

III. Case. In analogy to the previous case we consider the anyons as bosons and $k \leq 1$ and such that the integer part of $\frac{1}{k}$ is *even*. Then we can write $s = 2Jr + l$ and repeat the same procedure as before to get the solutions, which are easily obtained from (4.99) with the substitutions $2J + 1 \to 2J$, $P_i' \to P_i + rs\frac{u(u+1)}{2}$. Also here if k is of the form $k = \frac{1}{2J}$, we get a unique symmetric solution if the momenta P_i are integer multiples of N_A, and also in this case, these particular momentum and energy eigen states are protected by the above mentioned mechanism.

In general, if we except the case $k = 1$, it is a difficult task to determine explicitly which linear combinations of the states, given in (4.96), are the states we derived above. But it is also clear that for $k > 1$ there exist no *antisymmetric* linear combinations of those states. In fact, for $k > 1$ we have $Q \equiv \frac{N_A}{k} < N_A$ and so we can not accommodate all the particles in the ground state by the Pauli principle (remember that roughly speaking $1 \leq r_i \leq Q$). In this case one has to analyze also excited states and this will be done by means of the Mean Field approximation in the next chapter.

4.7 Anyons in the anyon gauge

On the plane there are two ways of formulating the problem of Chern–Simons induced anyons. Either the wave function has ordinary statistics, and the (complicated) Hamiltonian is given by $H = \sum_i \mathcal{D}_i^\dagger \mathcal{D}_i$ where $\mathcal{D}_i = \partial_i + \frac{1}{2k}\partial_i \sum_{j \neq i} \ln|z_i - z_j|^2$ and one has the eigenvalue problem

$$H\,\Psi = E\,\Psi,$$

or one can invoke a (unitary) singular gauge transformation

$$\Psi = \prod_{i<j} \left(\frac{\bar{z}_i - \bar{z}_j}{z_i - z_j} \right)^{\frac{1}{2k}} \cdot \Psi^{an} \equiv U \cdot \Psi^{an}$$

to transform the Hamiltonian in the free one

$$\left(\sum_i \partial_i^\dagger \partial_i \right) \Psi^{an} = E\,\Psi^{an}$$

and to remain with a wave function with exotic statistics, which behaves as

$$\Psi^{an} \simeq (\bar{z}_{ij})^{-1/k} \tag{4.103}$$

under the interchange of z_i and z_j. Here we disregard the δ–function ambiguities discussed in chapter four, which are irrelevant with this respect. Let us explain in which sense an analogous statement, even if not quit identical, can be made on the torus.

Here the Hamiltonian is given by eqs. (4.39), (4.47) and the wave function Ψ with ordinary statistics depends also on a according to (4.68). For definiteness let us consider its gauge fixed version

$$\Psi = \sum_{i=1}^{r} \phi_m^{(0)} \Psi_m \tag{4.104}$$

where we have absorbed the momentum carrying exponential in Ψ_m, see (4.77). We consider again the eigenvalue equation

$$H\Psi = E\,\Psi \tag{4.105}$$

It turns out that in this case the appropriate singular gauge transformation is given by

$$U = \prod_{i<j} \left(\frac{\overline{\theta\left[{}^{1/2}_{1/2}\right](z_{ij})}}{\theta\left[{}^{1/2}_{1/2}\right](z_{ij})} \exp\left(\frac{\pi}{2\nu}\left(\bar{z}_{ij}^2 - z_{ij}^2\right)\right)\right)^{\frac{1}{2k}} \tag{4.106}$$

which gives

$$\Psi^{an} = U^{-1} \cdot \Psi \equiv \sum_{m=1}^{r} \phi_m^{(0)} \Psi_m^{an} \tag{4.107}$$

Eq.(4.105) becomes then

$$\left(\sum_i \Delta_i^\dagger \Delta_i\right) \Psi^{an} = E\,\Psi^{an} \tag{4.108}$$

where Δ_i is defined in (4.42). Notice that

$$[\Delta_i, \Delta_j^\dagger] = 0 = [\Delta_i, \Delta_j]$$

for all i, j, meaning that eq. (4.108) is the torus version of the free Hamiltonian on the plane. This remaining "interaction" can actually not be eliminated by a *unitary* transformation, although it can be *reduced* to the free N_A–particle problem. In fact, if we plug eq. (4.107) into (4.108) we obtain a set of m free eigenvalue equations

$$\left(\sum_i \partial_i^\dagger \partial_i\right) \Psi_m^{an} = E\,\Psi_m^{an},$$

which can be further reduced, as we know from our previous discussion, to a unique equation, say the one for Ψ_0^{an}. The components of the vector Ψ_m^{an} are related among them by the relation $(Q = N_A/k)$

$$\Psi_m^{an}(z_i + m_i + n_i\tau) = \mathrm{e}^{+\pi i(Q+s)(\sum_i m_i + \sum_i n_i)}\,\mathrm{e}^{-2\pi i \frac{m}{k}\sum m_i}\,\Psi^{an}_{m-\sum n_i}(z_i) \qquad (4.109)$$

with the condition $\Psi_{m+rN}^{an} = \Psi_m^{an}$ for all integers N.

So we see that on the torus, even if formally one can reduce the anyon's problem to a system of N_A non interacting particles, one remains with a wave function which has r components. This is actually consistent with the results of the general braid group analysis on the torus [24] which show that fractional statistics on a torus is consistent only with multi–component wave functions. The same formal result has been obtained in a lattice version on the torus [68].

It is interesting to see what in this gauge the ground state solutions at fixed momentum, $E = \frac{|P|^2}{N_A}$, become, which we found in the previous section (eq. (4.96)):

$$\begin{aligned} h_0^{\{r_i\}\,an} &= U^{-1}h_0^{\{r_i\}} = \overline{\prod_{i<j}\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}^{-1/k}(z_{ij})}\cdot \\ &\sum_{\{n_j\}}\sum_{N=1}^{u}\delta\left(Nr - \sum_i n_i\right)\mathrm{e}^{2\pi i\tilde{P}_1\frac{N}{u}}\prod_{i<j}e^{-\frac{i\pi\bar{\tau}}{k}\left(n_{ij}-\frac{r_{ij}}{Q}\right)^2 - \frac{2\pi i}{k}\left(n_{ij}-\frac{r_{ij}}{Q}\right)\bar{z}_{ij}} \end{aligned} \qquad (4.110)$$

These solutions behave locally as (4.103) under interchange of two coordinates, as expected. The formula (4.110) has to be compared with eq. (4.96) – the second rows are actually identical – which exhibits ordinary statistics.

The expressions we wrote in this section, like (4.109) and (4.110), are somewhat formal, not only because of the fractional statistics ambiguity, but also because of their ill-defined transformation properties under the shifts $z \to z+m+n\tau$. To get (4.109), for example, we had to choose a convention for the transformation properties of the rational powers of θ–functions appearing in (4.106). This is one more reason for continuing to work in the gauge we adopted in this chapter: in that frame both of those ambiguities are absent.

To conclude, on the torus there exists a singular unitary gauge transformation which transforms the wave function with ordinary statistics into one with fractional statistics and the Hamiltonian into the one which describes the free motion of the particles, although not being formally the free Hamiltonian, see (4.108). This system can be further *reduced* to a *true* free system in which the wave function with fractional statistics however, has r components, in agreement with general results about fractional statistics on closed, multiply connected, orientable manifolds[24].

5. Mean Field Theory and Superconductivity

5.1 Translation invariant Mean Field Approximation

There are strong reasons to think that a system of anyons, as the one under investigation in the previous chapter, at integer Chern–Simons coupling constant k behaves like a superconductor[9,10,11,13]. The motivation for this believe relies essentially on the following occurrence: in the mean field or random phase approximation (RPA)[9,11] the single particle spectrum becomes that of the Landau levels of a particle in a fictitious constant magnetic field, orthogonal to the surface, the number of particles being such that in the fermionic many–body ground state the lowest k Landau levels are exactly filled. The energy to create a separated particle–hole pair corresponds just to the energy necessary to excite an anyon into the lowest empty Landau level band. So there is a finite energy gap which protects the ground state and may lead to the existence of supercurrents.

These considerations are made in a "static" system, i.e. in a system in which the states do not carry momentum, while superconducting properties are inferred indirectly, proving for example, the existence of a gap. Actually, the mean field approximation on the plane breaks translation invariance and it is not possible at all to define a "good" momentum operator. To be more precise, what breaks down is the commutativity between the two components of the momentum operator[11]:

$$[P_x, P_y] \neq 0.$$

In this section we formulate a translation invariant mean field approximation, which arises actually automatically in our theory on the torus, in which P_x and P_y are conserved and commute among them. This allows us to analize dynamical states with non vanishing total momentum in both directions and to derive a translation invariant version of the theory of Landau levels. It turns out that the notion of a degenerate one–particle Landau level is replaced by a many–body degeneracy which, due to full translation invariance, can *not* be reduced to a tensor product of one–particle states.

Nevertheless it happens that the fermionic many–body ground state reflects, for integer k, the structure of k exactly filled Landau levels, although with some important differences with respect to what happens on the plane. For example, the just mentioned ground state will exist, and be superconducting, only for macroscopically quantized values of the momentum.

We attribute to the anyons an intrinsic fermionic[11] statistics with total statistical angle

$$\theta = \pi\left(1 - \frac{1}{k}\right) \tag{5.1}$$

with respect to the coordinate $\bar{z}$ according to the conventions followed in the previous chapter. This means that, with respect to other authors, we exchanged the roles of z and $\bar{z}$, which is of course only a matter of conventions.

In the next chapter the mean field problem enters the discussion of a system of anyons in a constant external magnetic field, so for the moment we take k to be an arbitrary rational number and consider in section 5.3 integer values in the context of the superconducting system.

The mean field approximation constitutes a suitable approximation scheme for large values of k, as we saw in chapter two. It corresponds to replace the point–like fluxes, generated by each particle, by their integrated mean value over the surface. This picture is expected to be more and more sensible as $N_A \to \infty$. Looking at the covariant derivatives given in (4.39) we see that the potential A_i is already decomposed in a term linear in $\bar{z}_i$, $\frac{\pi}{2\nu k} N_A \bar{z}_i$, to which a constant magnetic flux is associated and which carries the main effect, and terms proportional to $\partial P(i,j)$ which describe the fluctuations and whose related total flux is, in fact, zero. Therefore in our theory the mean field approximation corresponds to drop in the covariant derivatives (4.39) the terms proportional to $\partial P(i,j)$. Notice that those terms, which are derivatives of the Green's function of the Laplacian on the torus, are well defined functions on the torus and, moreover, modular invariant. This means that the transformation properties of the wave function as $z \to z + m + n\tau$ remain the same as in the exact case, and that all what we said about modular invariance holds through also in the mean field theory.

The mean field Hamiltonian is given by

$$H_{MF} = \sum_i \nabla_i^\dagger \nabla_i \tag{5.2}$$

where the mean field covariant derivatives are given by

$$\begin{aligned}\nabla_i &= \partial_i - \frac{\pi}{\nu} a + \frac{\pi}{2\nu k} \sum_j \bar{z}_j + \frac{\pi}{2\nu k} \sum_j (\bar{z}_i - \bar{z}_j) \\ \nabla_i^\dagger &= -\bar{\partial}_i - \frac{1}{k}\frac{\partial}{\partial a} + \frac{\pi}{2\nu k} \sum_j z_j + \frac{\pi}{2\nu k} \sum_j (z_i - z_j)\end{aligned} \tag{5.3}$$

Due to the fact that the commutator $[\nabla_i, \nabla_j^\dagger]$ is a constant, in this case there are no δ–function ambiguities, and the Hamiltonian is well defined up to an additive constant. The total momentum is still given by

$$\mathcal{P} = \sum_i \nabla_i$$

and turns actually out to be the same as the one in the exact theory. It is straightforward to check that again

$$\begin{aligned}[][\mathcal{P}, H_{MF}] &= 0 = [\mathcal{P}^\dagger, H_{MF}] \\ [\mathcal{P}, \mathcal{P}^\dagger] &= 0,\end{aligned} \tag{5.4}$$

meaning that the mean field approximation is translation invariant and that we can diagonalize simultaneously the Hamiltonian and the two components of the momentum operator. In this theory this can actually be done exactly (see the next section). The Hilbert space can also here be written as a direct sum of Hilbert spaces at fixed total momentum: $\mathcal{H}^{(0)} = \sum_P \oplus \mathcal{H}_P^{(0)}$. The states of the spaces $\mathcal{H}_P^{(0)}$ can be written as in (4.77)

$$\Psi_P^{(0)} = \exp\left(\frac{P}{N_A} z - \frac{\bar{P}}{N_A} \bar{z}\right) \sum_{m=1}^{r} \phi_m^{(0)} h_m \tag{5.5}$$

and the eigenvalue problem

$$\begin{aligned}\mathcal{P}\, \Psi_P^{(0)} &= P\, \Psi_P^{(0)} \\ \mathcal{P}^\dagger\, \Psi_P^{(0)} &= \bar{P}\, \Psi_P^{(0)} \\ H_{MF}\, \Psi_P^{(0)} &= E\, \Psi_P^{(0)}\end{aligned} \tag{5.6}$$

is mapped in the system

$$H_R\, h_m = \left(E - \frac{|P|^2}{N_A}\right) h_m \tag{5.7}$$

with the condition that the h_m depend only on the differences of the coordinates $z_{ij}, \bar{z}_{ij}$. The reduced Hamiltonian is given by

$$H_R = \sum_i \alpha_i^\dagger \alpha_i \tag{5.8}$$

where

$$\begin{aligned} \alpha_i &\equiv \partial_i + \frac{\pi}{2\nu k}\sum_j \bar{z}_{ij} \\ \alpha_i^\dagger &\equiv -\bar{\partial}_i + \frac{\pi}{2\nu k}\sum_j z_{ij} \end{aligned} \tag{5.9}$$

Again one can define the h_m in terms of h_0 through eq. (4.85) and then also here it can be immediately checked that it is sufficient to solve (5.7) for h_0 only. To complete the statement of the problem we have to remember the transformation properties of h_0 given in (4.84). This transformation property, which defines the (reduced) Hilbert space of h_0, can be conveniently rewritten if we define the following operators:

$$\begin{aligned} \beta_i &\equiv \partial_i - \frac{\pi}{2\nu k}\sum_j \bar{z}_{ij} \\ \beta_i^\dagger &\equiv -\bar{\partial}_i - \frac{\pi}{2\nu k}\sum_j z_{ij}. \end{aligned} \tag{5.10}$$

Notice the appearance of a minus sign with respect to eq. (5.9). Then (4.84) becomes

$$A(m_i)\, h_0 \equiv e^{\sum_j m_j(\beta_j - \beta_j^\dagger)}\, h_0 = \exp\left(2\pi i \frac{\tilde{P}_2}{N_A}\sum_j m_j\right) h_0$$

$$B(n_i)\, h_0 \equiv e^{\sum_j n_j(\tau\beta_j - \bar{\tau}\beta_j^\dagger)}\, h_0 = \exp\left(-2\pi i \frac{\tilde{P}_1}{N_A}\sum_j n_j\right) h_0 \quad \text{for } \sum_i n_i = 0 \ mod\ r. \tag{5.11}$$

Notice that the operators $B(n_i)$ are only defined for $\sum_i n_i = 0$ *mod* r. Here we see the importance of this condition in that it implies that A commutes with B. In fact, one has the algebra

$$\begin{aligned} [\alpha_i, \alpha_j] &= 0 = [\alpha_i^\dagger, \alpha_j^\dagger] \\ [\beta_i, \beta_j] &= 0 = [\beta_i^\dagger, \beta_j^\dagger] \\ [\alpha_i, \alpha_j^\dagger] &= \frac{\pi}{\nu k}(N_A\delta_{ij} - 1) \\ [\beta_i, \beta_j^\dagger] &= -\frac{\pi}{\nu k}(N_A\delta_{ij} - 1) \\ [\alpha_i, \beta_j] &= 0 = [\alpha_i, \beta_j^\dagger] \end{aligned} \tag{5.12}$$

which implies

$$[A(m_i), B(n_i)] = 0 \tag{5.13}$$

This is a compatibility condition for the eqs. (5.11) which can, in fact, be read as a simultaneous eigenvector problem. The algebra (5.12) implies also that the covariant derivatives α and $\alpha^\dagger$, and hence also the Hamiltonian, commute with A and B and are thus well defined operators in the reduced Hilbert space. Based on (5.12) and on the condition for the Hilbert space (5.11) we will solve in the next section completely the spectral problem for H_R.

5.2 Spectrum and degeneracy of the many–body system

Let us summarize the problem: we reduced our original dynamical mean field theory to the problem of diagonalizing the reduced Hamiltonian in eq. (5.8) which lives in the reduced Hilbert space V defined by (5.11). The overall momentum has already been separated using the true translation invariance of the system and so we work at fixed momentum. Traces of the particular momenta chosen survive, however, through eq. (5.11). This dependence will actually influence heavily the structure and properties of the ground state at integer k, as we will see in the next section. The problem at hand looks actually very similar to the problem of N_A particles in a constant external magnetic field, in the symmetric gauge, on a torus, whose spectrum we know to be given by Landau–levels, each of which exhibits a degeneracy which is the total flux of the magnetic field divided by 2π. There is however the crucial difference that our model is translation invariant and so we were allowed to factorize the overall momentum. As a consequence the remaining many–body Hilbert space is actually somehow smaller then the one corresponding to the Landau levels. In particular, the many–body wave function will not be a product of single particle wave functions and the many–body degeneracy will in general be smaller then the one resulting from ordinary Landau levels. This is essentially due to the particular dependence of the wave function on the center–of–mass coordinate implied by (5.5). Nevertheless for particular values of the total momenta the many–body ground state fits in the picture of ordinary Landau levels.

Let us first recall that the states of V are translation invariant by definition, and so the operators α and β, defined only on V, satisfy:

$$\sum_i \alpha_i = 0 = \sum_i \beta_i. \tag{5.14}$$

The eigenvalues of H_R can be found observing that the algebra of the α–operators corresponds essentially to an algebra of off–diagonal harmonic oscillators, which can be easily diagonalized. We look for an orthogonal transformation $\gamma_i = \sum_j M_{ij}\alpha_j$ such that $[\gamma_i, \gamma_j] = E_i \delta_{ij}$, where the E_i are the eigenvalues of the matrix

$$C_{ij} = \frac{\pi}{\nu k}(N_A \delta_{ij} - 1)$$

The eigenvalue problem is easily solved and one gets the eigenvalue $E_1 = 0$ with eigenvector

$$\gamma_1 = \frac{1}{\sqrt{N_A}} \sum_i \alpha_i = 0 \quad \text{on } V,$$

and

$$E_i = \frac{\pi N_A}{\nu k}$$

for $i > 1$, which is $(N_A - 1)$ times degenerate. The corresponding eigenvectors γ_i span a basis for the differences $\alpha_i - \alpha_j$. The Hamiltonian becomes then

$$H_R = \sum_{i=2}^{N_A} \gamma_i^\dagger \gamma_i$$

where $[\gamma_i, \gamma_j] = \frac{\pi}{\nu k} N_A \delta_{ij}$, which is a sum of $N_A - 1$ free harmonic oscillators. The Hilbert space is then obtained by the usual procedure. We determine the ground states via

$$\gamma_i \big| * \big\rangle = 0 \quad \text{for } i = 2, \ldots, N_A,$$

while the excited states, and hence the entire Hilbert space, are given by

$$\prod_{i=1}^{N_A - 1} \left(\gamma_i^\dagger\right)^{m_i} | * \rangle. \tag{5.15}$$

Due to (5.14), however, the problem can be stated in the equivalent form

$$\begin{aligned} \alpha_i \big| * \big\rangle &= 0 \quad \text{for } i = 1, \ldots, N_A \\ \big|\{m_i\}, *\big\rangle &\equiv \prod_{i=1}^{N_A} \left(\alpha_i^\dagger\right)^{m_i} \big| * \big\rangle \\ H_R \big|\{m_i\}, *\big\rangle &= \left(\frac{\pi}{\nu} \cdot \frac{N_A}{k} \cdot \sum_i m_i\right) \big|\{m_i\}, *\big\rangle \end{aligned} \tag{5.16}$$

It is clear that also the states $|\{m_i\}, *\rangle$ span the Hilbert space; the difference w.r.t. eq. (5.15) is that the states given there form an *orthogonal* basis in each eigenspace of H_R, while the states of eq. (5.16) do not. It is however more convenient to work in the "symmetric" basis spanned by the latter states. Notice that if the state $|*\rangle$ stays in V i.e. satisfies (5.11), then, due to (5.12), also the states $|\{m_i\}, *\rangle$ stay in V.

To conclude, the spectrum of the Hamiltonian is given in (5.16) and what remains is to determine the degeneracy of each energy eigenvalue, and to find the ground states $|*\rangle$. It is clear that there is a trivial degeneracy due to the invariance of the energy eigenvalues in eq. (5.16) under interchange of the m_i. But there is an additional intrinsic degeneracy of each eigenvalue, due to the fact that the ground states $|*\rangle$ are not unique: this Landau–level–like degeneracy, which, as we will see in a moment is due to a one–body discrete translation invariance which arises in the mean field approximation and is absent in the exact problem, is carried by all energy eigenvalues and is the same for all of them. To determine this degeneracy we need the unitary operators which implement this symmetry. To find them we observe that the operators β_i and $\beta_i^\dagger$ commute indeed with H_R, and constitute thus dynamical symmetries; more precisely, they generate translations of the i–th anyon in the two directions of the torus. They take us, however, out of the Hilbert space in that they do not commute with the defining operators A and B given in (5.11). But the corresponding Weyl operators, when appropriately weighted, commute with A and B. They are given by:

$$\begin{aligned} U(s_i) &= \prod_j \exp\left[\frac{s_j}{Q}\left(\tau\beta_j - \bar{\tau}\beta_j^\dagger\right)\right] \qquad \sum_i s_i = 0 \ mod \ \ N_A \\ \tilde{U}(q_i) &= \prod_j \exp\left[\frac{q_j}{Q}\left(\beta_j - \beta_j^\dagger\right)\right] \qquad \sum_i q_i = 0 \ mod \ \ u \end{aligned} \tag{5.17}$$

where the s_i and q_i are integers. These operators are dynamical symmetries of the mean field theory, in the sense that they commute also with H_R. Moreover, the complete set of physical phase space variables is now given by

$$\alpha_i,\ \alpha_i^\dagger,\ U(s_i),\ \tilde{U}(q_i). \tag{5.18}$$

The generators of discrete translations satisfy the following algebra ($s_{ij} \equiv s_i - s_j$ etc.):

$$U(s_i)\,\tilde{U}(q_i) = \prod_{i<j} \exp\left(\frac{2\pi i}{QN_A}\, q_{ij} s_{ij}\right) \cdot \tilde{U}(q_i)\, U(s_i). \tag{5.19}$$

Each eigenspace of the Hamiltonian has then to represent this algebra and to carry, in particular, an irreducible representation of it in that we expect that no other dynamical symmetries are present (a part from the trivial permutation symmetry we mentioned above). To find the degeneracy of the eigenspaces it is then sufficient to search for irreducible representations of (5.18) and (5.19).

We adopt a standard technique and observe that the relevant Casimir operators, which commute with the entire phase space in (5.19), coincide actually with the operators A and B:

$$\begin{aligned} B(n_i) &= U(Q \cdot n_i) = \exp\left(-2\pi i \frac{\tilde{P}_1}{N_A} \sum_i n_i\right) \quad \text{for} \quad \sum_i n_i = 0 \quad mod\ r \\ A(m_i) &= \tilde{U}(Q \cdot m_i) = \exp\left(2\pi i \frac{\tilde{P}_2}{N_A} \sum_i m_i\right) \end{aligned} \tag{5.20}$$

The algebra in (5.19) is formally very similar to the one of large gauge transformations in chapter three, see eq. (3.15). Also here we can diagonalize only one of the two U–operators, say $\tilde{U}(q_i)$, and then the $U(s_i)$ connect different eigenvectors of $\tilde{U}(q_i)$ according to (5.19). Due to (5.14) we can write the eigenvalue equations as

$$\tilde{U}(q_i)\big|\{r_i\}\big\rangle = \prod_{i<j} \exp\left(\frac{2\pi i}{QN_A} q_{ij} r_{ij}\right) \big|\{r_i\}\big\rangle \tag{5.21}$$

where the real r_i determine the eigenvalues and are defined modulo an overall additive constant because only their differences r_{ij} appear in (5.21). Comparing with the second relation of (5.20) we get

$$e^{2\pi i\left(\sum_j m_j r_j - \frac{1}{N_A}\sum_j m_j \sum_i r_i\right)} = e^{2\pi i \frac{\tilde{P}_2}{N_A}\sum_i m_i}$$

which has to hold for all integers m_i and permits us to conclude that the r_i have to be *integers* such that

$$\sum_{i=1}^{N_A} r_i = -\tilde{P}_2. \tag{5.22}$$

Moreover (5.19) and (5.20) imply the relation

$$U(Qn_i)\big|\{r_i\}\big\rangle = \big|\{r_i + s(N - un_i)\}\big\rangle = \mathrm{e}^{-2\pi i \frac{\tilde{P}_1}{N_A}\sum_i n_i}\big|\{r_i\}\big\rangle \tag{5.23}$$

where N is the integer such that $\sum_i n_i = rN$. This relation means that the irreducible representations of the algebra at hand (at fixed momentum there is actually only one of them) are spanned by the set of vectors

$$\big|\{r_i\}\big\rangle,$$

where the integers r_i have to fulfill (5.22), and are determined modulo the equivalence relation

$$\big|\{r_i\}\big\rangle \cong \big|\{r_i + s(N - un_i)\}\big\rangle \;\text{ for } \sum_i n_i = rN \tag{5.24}$$

where the difference between these two states is actually only a phase according to (5.23). Taking this equivalence relation into account one can count the number of linearly independent states

$$\Omega = s \cdot Q^{N_A - 2}, \tag{5.25}$$

and this is then also the dimension of the irreducible representation, and hence of the degeneracy of each energy eigenvalue. Notice that the corresponding degeneracy of a system of N_A particles in a constant external magnetic field with the same total flux $2\pi\, Q$, and therefore with a one–particle Landau level degeneracy equal to Q, would have produced an N_A–body degeneracy

$$D = Q^{N_A} > \Omega.$$

This could in principle spoil the degeneracy counting which, for integer k, leads on the plane to a ground state with an integer number of filled Landau levels and to the superconductivity effect (if the anyons have an intrinsic fermionic statistics). In the next section we will see that for particular values of the momenta $\tilde{P}_i$ there will exist a completely antisymmetric ground state which remembers the structure of k filled Landau levels, although in the present model the notion of one–particle Landau levels does not even make sense. All what we need to make this analysis has already been derived in this section on a purely algebraic basis, and there would be no need to determine explicitly the states $|*\rangle = |\{r_i\}\rangle$. We determine them here for completeness and also to show that they are not direct products of one particle states. As a by–product we will also be able to determine an explicit basis for the Hilbert space of the exact theory and hence prove eq. (4.88), as promised in chapter four.

We search for the ground–states

$$\alpha_i \big| * \big\rangle = 0 \tag{5.26}$$

which satisfy (5.11). We set, as in the previous chapter,

$$\big| * \big\rangle = \prod_{i<j} \exp\left(\frac{\pi}{2\nu k}(\bar{z}_{ij}^2 - |z_{ij}|^2)\right) G(z_{ij}, \bar{z}_{ij}) \tag{5.27}$$

and eq. (5.26) tells us then that G has to depend only on $\bar{z}_{ij}$, i.e to be an antiholomorphic translation invariant function of the z_i. The relations (5.11) imply then that G has to satisfy the equation (4.87) of chapter four. To solve (4.87) we observe that for $n_i = 0$ it becomes

$$G(\bar{z}_i + m_i) = e^{2\pi i \frac{\tilde{P}_2}{N_A} \sum_i m_i} G(\bar{z}_i)$$

which is solved by a Fourier series:

$$G(\bar{z}_i) = \sum_{\{l_j\}} e^{2\pi i \sum_j \left(l_j + \frac{\tilde{P}_2}{N_A}\right) \bar{z}_j} C(l_j). \tag{5.28}$$

To insure that G depends only on the differences $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$ only those coefficients $C(l_j)$ have to be different from zero for which $\sum_j l_j = -\tilde{P}_2$. Now we impose eq. (4.87) with $m_j = 0$ on (5.28) (remember $\sum_i n_i = Nr$):

$$\sum_{\{l_j\}} e^{2\pi i \sum_j \bar{z}_j (l_j + \frac{\tilde{P}_2}{N_A}) - 2\pi i \sum_j n_j \frac{\tilde{P}_1}{N_A}} \prod_{i<j} e^{\frac{i\pi\bar{\tau}}{k}(n_{ij})^2} C(l_j + s(N - un_j)) =$$

$$\sum_{\{l_j\}} e^{2\pi i \sum_j \bar{z}_j (l_j + \frac{\tilde{P}_2}{N_A}) + 2\pi i \sum_j \left(l_j + \frac{\tilde{P}_2}{N_A}\right) n_j \bar{\tau}} C(l_j).$$

Equating the Fourier–coefficients we get:

$$C(l_j + s(N - un_j)) \prod_{i<j} e^{\frac{i\pi\bar{\tau}}{k}\left(n_{ij} - \frac{l_{ij}}{Q}\right)^2} = C(l_j) \prod_{i<j} \left(e^{\frac{i\pi\bar{\tau}}{k}\left(\frac{l_{ij}}{Q}\right)^2} \right) e^{2\pi i \frac{\tilde{P}_1}{N_A} \sum_j n_j} \tag{5.29}$$

which means that the independent solutions are determined by a basic set of integers $r_i = l_i^0$ such that $\sum_i r_i = -\tilde{P}_2$. The solutions can then be written as

$$\begin{gathered} \left|\{r_i\}\right\rangle = \prod_{i<j} \exp\left(\frac{\pi}{2\nu k}(\bar{z}_{ij}^2 - |z_{ij}|^2)\right) \cdot \\ \sum_{\{n_j\}} \sum_{N=1}^{u} \delta\left(Nr - \sum_i n_i\right) e^{2\pi i \tilde{P}_1 \frac{N}{u}} \prod_{i<j} e^{-\frac{i\pi\bar{\tau}}{k}\left(n_{ij} - \frac{r_{ij}}{Q}\right)^2 - \frac{2\pi i}{k}\left(n_{ij} - \frac{r_{ij}}{Q}\right)\bar{z}_{ij}} \end{gathered} \tag{5.30}$$

and one can verify explicitly that all the algebraic relations we wrote in this section, and in particular the equivalence relation (5.23), are indeed satisfied. The excited states are now given by

$$\left|\{m_i\},\{r_i\}\right\rangle = \prod_{i=1}^{N_A} \left(\alpha_i^\dagger\right)^{m_i} \left|\{r_i\}\right\rangle \tag{5.31}.$$

Eq. (5.30) shows also that the many–body ground states are not factorized into single particle states. One can wonder what form assume the excited states given in (5.31). Due to (5.14), to obtain the first excited states it is sufficient to compute

$$\begin{gathered} \left(\alpha_k^\dagger - \alpha_l^\dagger\right) \left|\{r_i\}\right\rangle = \prod_{i<j} \exp\left(\frac{\pi}{2\nu k}(\bar{z}_{ij}^2 - |z_{ij}|^2)\right) \cdot \\ \sum_{\{n_j\}} \sum_{N=1}^{u} \delta\left(Nr - \sum_i n_i\right) e^{2\pi i \tilde{P}_1 \frac{N}{u}} \prod_{i<j} e^{-\frac{i\pi\bar{\tau}}{k}\left(n_{ij} - \frac{r_{ij}}{Q}\right)^2 - \frac{2\pi i}{k}\left(n_{ij} - \frac{r_{ij}}{Q}\right)\bar{z}_{ij}} . \\ \frac{\pi}{Q\nu}\left(z_{kl} - \bar{z}_{kl} + 2i\nu\left(n_{kl} - \frac{r_{kl}}{Q}\right)\right) \end{gathered} \tag{5.32}$$

So we see that the first excited states are obtained by introducing in the Fourier series essentially a periodized power of $z_{ij} - \bar{z}_{ij}$. This is much similar to what happens in the ordinary Landau levels on the plane, where the first excited Landau level corresponds to add a power of z to the gaussian exponential [27]. It is not difficult to convince one self that the higher excited states, obtained by applying more creation operators to $|\{r_i\}\rangle$, give in general rise to the insertion of a polynomial of $\left(z_{kl} - \bar{z}_{kl} + 2i\nu\left(n_{kl} - \frac{r_{kl}}{Q}\right)\right)$ in (5.32). The so obtained states span the complete Hilbert space V, i.e. they are a basis of all the solutions of the boundary conditions in eq. (5.11). Due to the fact that the boundary conditions for the physical states in the exact theory and in the mean field theory are the same, this proves also that the Hilbert space in the exact theory can be represented by the eqs. (4.86) and (4.88).

5.3 The fermionic ground state for integer k

In this section we specialize to the case of integer k and analize the superconducting properties of such a system [20]:

$$k = r, \quad s = 1.$$

The main qualitative difference to the general case is that now the global gauge transformations along the two cohomology cycles commute among them (see (4.51))

$$U_1(p)\,U_2(q) = U_2(q)\,U_1(p)$$

and that therefore states can be gauge invariant. There is no internal gauge space and there is no need of a gauge fixing for this internal space. The wave functions at fixed total momentum are given by

$$\Psi_P = \exp\left(\frac{P}{N_A}z - \frac{\bar{P}}{N_A}\bar{z}\right)\sum_{m=1}^{r}\phi_m\, h_m \tag{5.33}$$

where the ϕ_m are obtained from the formula (4.65) by setting $n = 0$ and $s = 1$. Modular invariance is in this case realized directly by enforcing the substitutions (4.55) and the resulting new wave function stays again in the Hilbert space spanned by (5.33).

We consider again the reduced dynamical problem, i.e. the one for h_0. In this case the Hilbert space is spanned by the states $|\{m_i\},\{r_i\}\rangle$, given in (5.31) and (5.30), with the equivalence relation

$$\big|\{m_i\},\{r_i+(N-Qn_i)\}\big\rangle = \exp\left(-2\pi i\frac{\tilde{P}_1}{N_A}\sum_i n_i\right)\big|\{m_i\},\{r_i\}\big\rangle \;\text{ for } \sum_i n_i = kN \tag{5.34}$$

and with the constraint

$$\sum_{i=1}^{N_A} r_i = -\tilde{P}_2. \tag{5.35}$$

We want to determine the many–body ground state at fixed momentum assuming that the anyons have an intrinsic fermionic statistics. This means that we have to determine the completely antisymmetric eigenstate of the Hamiltonian with the lowest energy. That is, we have to find sets $\{m_i\}$ and $\{r_i\}$ such that the completely antisymmetrized state

$$\frac{1}{N_A!}\sum_{\mathcal{P}}(-)^{\mathcal{P}}\left|\{m_i\},\{r_i\}\right\rangle_{\mathcal{P}} \tag{5.36}$$

minimizes the energy given in (5.16). The antisymmetrization operation is on the coordinates $(z_i, \bar{z}_i)$, but looking at (5.31) and (5.30) we see that interchanging z_i with z_j corresponds to make the simultaneous interchange $m_i \leftrightarrow m_j$ and $r_i \leftrightarrow r_j$. The m_i are completely independent positive integers, while the r_i are subject to the equivalence relation (5.34) and to the constraint (5.35). The relation (5.34) tells us that the r_i are defined modulo Q and modulo an overall constant shift. Therefore we can restrict any r_i to

$$1 \leq r_i \leq Q$$

where it is understood that one of the r_i, we can take by convention r_{N_A}, has in general to ly outside of that interval, because of (5.35). Then there is a residual equivalence related to an overall shift which tells that

$$r_i \sim r_i + N \quad mod\ Q. \tag{5.37}$$

So every quantum number r_i can assume at most Q values, and it is therefore obvious that we have to excite at least k energy levels putting Q anyons in the ground state, Q anyons in the second level, and so on up to fill up the k–th level with Q anyons too. The proposed sets $\{m_i^0\}, \{r_i^0\}$, which should produce this ground state upon antisymmetrization, are then given by:

$$\begin{array}{llllll} i = & \underbrace{1,\ldots,Q} & \underbrace{Q+1,\ldots,2Q} & \ldots & \underbrace{(k-1)Q+1,\ldots,N_A = kQ} \\ m_i^0 = & 0,\ldots,0 & 1,\ldots\ldots\ldots,1 & \ldots & k-1,\ldots\ldots\ldots\ldots,k-1 \\ r_i^0 = & 1,\ldots,Q & 1,\ldots\ldots\ldots,Q & \ldots & 1,\ldots\ldots\ldots\ldots,Q-1,r_{N_A}^0 \end{array} \tag{5.38}$$

where (see (4.76))

$$r_{N_A}^0 = -\sum_{i=1}^{N_A-1} r_i^0 - \tilde{P}_2 = -P_2 + (k(Q+1)+1)Q \tag{5.39}$$

in order to satisfy (5.35). The question is if the corresponding state is antisymmetrizable or not, i.e. if under interchange of two arbitrary couples, $(m_i^0, r_i^0) \leftrightarrow (m_j^0, r_j^0)$, we get really a different state. This is obvious, by construction, if we interchange couples belonging to different Q–tuples in (5.38). On the other hand, if $P_2 \neq 0 \; mod \; Q$, say $P_2 = M \; mod \; Q$ with $1 \leq M < Q$, then in the last Q–tuple of the r_i^0 there are actually two entries, the M-th one and the last one, which are equal modulo Q. Therefore the interchange of the corresponding couples

$$(m^0_{(k-1)Q+M}, r^0_{(k-1)Q+M}) \leftrightarrow (m^0_{N_A}, r^0_{N_A})$$

would produce the same state; actually, due to (5.34), one gets even the same phase. As a consequence, under antisymmetrization the corresponding state goes then clearly to zero. This implies that a necessary condition for the antisymmetrizability of (5.38) is that P_2 is a multiple of Q:

$$P_2 = p_2 \cdot Q. \tag{5.40}$$

This is actually not sufficient to insure that (5.36), with $r_i \to r_i^0$, is different from zero. We have in fact still to investigate those permutations which correspond to an overall shift according to (5.37) and leave therefore, apart from a phase, the state invariant. Let us compute this phase. The relevant permutations, $\mathcal{P}_N$, are cyclic permutations of the same order N in each of the Q–tuples of (5.38). The sign of these permutations, appearing in (5.36), is given by

$$(-)^{\mathcal{P}_N} = (-)^{N \cdot k(Q-1)} = e^{i\pi k N(Q-1)}. \tag{5.41}$$

Furthermore there is a phase arising from (5.34) which is

$$e^{-2\pi i \frac{\hat{P}_1}{Q} N} = e^{-2\pi i \frac{P_1}{Q} N - \pi i k N(Q+1)}$$

(see (4.76) with $s = 1$, $u = Q$, $k = r$). The total phase becomes therefore $e^{-2\pi i \frac{P_1}{Q} N}$ which in (5.36) sums up to

$$\sum_{N=1}^{Q} e^{-2\pi i \frac{P_1 N}{Q}} = Q \cdot \delta(P_1 mod \; Q).$$

This is different from zero only if also P_1 is a multiple of Q. We can conclude that the state $\left|\{m_i^0\},\{r_i^0\}\right\rangle$ gives rise to a completely antisymmetric ground state only if both components of the momenta assume the exceptional values

$$P_i = p_i \cdot Q \quad \text{for } i = 1,2, \tag{5.42}$$

and the fermionic ground state is then given by

$$\frac{1}{N_A!}\sum_{\mathcal{P}}(-)^{\mathcal{P}}\left(\prod_i^{N_A}\left(\alpha_i^{\dagger}\right)^{m_i^0}\left|\{r_i^0\}\right\rangle\right)_{\mathcal{P}} \tag{5.43}$$

We could have reached the same conclusion already from (5.40) alone, due to modular invariance. In fact, the modular transformed state of (5.43) under $\tau \to -\frac{1}{\tau}$, gives the completely antisymmetric ground state with $P_1 \to P_2, P_2 \to -P_1$. This implies already that also P_1 has to be a multiple of Q. Notice that the spectrum of exceptional momenta (5.42) closes under an arbitrary modular transformation.

Eq. (5.43), with the m_i^0, r_i^0 specified in (5.38), is the state of minimal energy with fixed momentum

$$P = \frac{\pi}{\nu}\cdot Q \cdot (p_1 + p_2\bar{\tau}) \tag{5.44}$$

the energy being (see (5.7))

$$\begin{aligned} E &= \frac{|P|^2}{N_A} + \frac{\pi}{\nu}\cdot\frac{N_A}{k}\cdot Q\sum_{j=1}^{k}(j-1) \\ &= \frac{|P|^2}{N_A} + \frac{\pi}{\nu}\cdot\frac{k-1}{2k}\cdot N_A^2. \end{aligned} \tag{5.45}$$

and the ground state is unique.

As an example we construct explicitly the corresponding state for the case $k = 2$, $Q = 2$ ($N_A = 4$). The exceptional momenta are then $P_2 = 2p_2, P_1 = 2p_1$. In this case $r_i = 1,2$ and the set r_i^0 is given by $(1,2,1,2)$ *mod* 2, while the m_i^0 are $(0,0,1,1)$. We denote the corresponding ground state by $|1212\rangle$ and according to (5.31) we have to antisymmetrize the state $\alpha_3^{\dagger}\alpha_4^{\dagger}\,|1212\rangle$. In the process of antisymmetrization we produce states $|\cdots\rangle$ which are different from $|1212\rangle$, but it is easy to see that due to (5.34) the resulting expression contains the three independent states:

$$\left|1212\right\rangle = \left|2121\right\rangle \qquad \left|2112\right\rangle = \left|1221\right\rangle \qquad \left|1122\right\rangle = \left|2211\right\rangle.$$

The completely antisymmetric state is then

$$\frac{1}{2}\sum_{\mathcal{P}}(-)^{\mathcal{P}}\left(\alpha_3^\dagger\alpha_4^\dagger|1212\rangle\right)_{\mathcal{P}} = (\alpha_3^\dagger-\alpha_1^\dagger)(\alpha_4^\dagger-\alpha_2^\dagger)|1212\rangle + (\alpha_1^\dagger-\alpha_4^\dagger)(\alpha_3^\dagger-\alpha_2^\dagger)|2112\rangle + (\alpha_1^\dagger-\alpha_2^\dagger)(\alpha_4^\dagger-\alpha_3^\dagger)|1122\rangle$$

If instead the momenta P_i are not exceptional ones, i.e they are not integer multiples of Q, then one has to excite one of the anyons once more applying an additional creation operator to the state $|\{m_i^0\},\{r_i^0\}\rangle$ in eq. (5.43). If we have, for example, $P_2 = M \neq 0 \ mod \ Q$ and $P_1 = 0$ and take for the r_i^0 the same set as in (5.38), then we have to increase the power of $\alpha_{N_A}^\dagger$ by one unit to get a completely antisymmetric state:

$$\frac{1}{N_A!}\sum_{\mathcal{P}}(-)^{\mathcal{P}}\left(\alpha_{N_A}^\dagger\prod_i^{N_A}\left(\alpha_i^\dagger\right)^{m_i^0}|\{r_i^0\}\rangle\right)_{\mathcal{P}}.$$

The energy of this state is

$$E = \frac{|P|^2}{N_A} + \frac{\pi}{\nu}\cdot\frac{k-1}{2k}\cdot N_A^2 + \frac{\pi}{\nu}\cdot\frac{N_A}{k}$$

which is much higher then the corresponding minimal energy at exceptional momentum (5.45). In this case it is also clear that the ground state is not unique in that the anyon which has to be excited in the k–th level can essentially stay in Q different states.

To summarize, for generic fixed integer momenta $P_{1,2}$ the minimal energy eigenvalue for the many–body system is

$$E_{min}(P) = \frac{1}{N_A}\left|\frac{\pi}{\nu}(P_1 + P_2\bar{\tau})\right|^2 - \frac{\pi}{\nu}\frac{N_A}{k}\cdot\delta_{P_1,p_1Q}\cdot\delta_{P_2,p_2Q} + \text{const.} \qquad (5.46)$$

We see that for the particular values $P = \frac{\pi}{\nu}Q(p_1 + p_2\bar{\tau})$ the ground state energy is particularly low and the corresponding ground states are therefore "protected" with respect to external perturbations by the macroscopic gap

$$\Delta E = \frac{\pi}{\nu k}N_A.$$

If the system stays in these states, then any change of the total momentum to nearby values would require a cost in energy equal to this gap, and therefore an excitation which changes the momentum is highly unfavored from an energetic point of view. The resulting quantum motion exhibits thus superconducting properties due to the flow of collective persistent currents. To jump from one of these protected states with exceptional momenta to another protected state the total momentum should indeed change by an integer multiple of $\frac{\pi}{\nu k} \cdot N_A$, which means that the momentum of *each* anyon has to change by a multiple of $1/k$ (in unit of the cycle's length of the torus). This shows also a certain rigidity and coherence of the system in that it is very difficult to change the momentum of only one anyon, which would correspond to excite the system from a protected state to a non protected state across the gap; the anyon fluid is rather moving as a whole in which all the anyons move together. This superconductivity mechanism should be compared with the superfluidity mechanism[54], induced by phononic elementary excitations, and with the BCS theory[55], based on an energy gap in the elementary excitation spectrum: it stays in some sense in between.

5.4 The magnetic field in the quantum state

In this section we consider the semiclassical coupling of the anyons with the true electromagnetic field, calling e the anyon's electric charge (we will still neglect the Coulomb interaction, as it is appropriate to a Mean Field treatment).

We expect that the motion along a handle (for instance in the x-direction, taking for simplicity $\tau_x = 0$, corresponding to $P_1 = 0$) will generate a magnetic field in the 3-dimensional cavity inside the torus, like in a solenoid. We introduce the coupling with the e.m. field by replacing the covariant derivative $\mathcal{D}_x$ with the one including $A_x^{e.m.}$:

$$\mathcal{D}_x \to \mathcal{D}_x - ieA_x^{e.m.}$$

We compute the e.m. current in an eigenstate with total momentum $P = -\frac{P_2 \pi}{\nu}\bar{\tau}$ by integrating over d^2a and over all the anyons' positions but one. It follows from the translation properties of the state that the current is constant over the surface

$$J_x^{e.m.} = \frac{eN_A}{\nu}\left(2\pi\frac{P_2}{N_A} - eA_x^{e.m.}\right).$$

In three dimensions the torus surface is seen as the external "skin" of a donut with radii R, along the x-direction, and νR along the y-direction. We imagine the torus (for large ν) as a cylinder in the y-direction, with identified ends. We call m the anyon's mass. From the three-dimensional point of view the e.m. current is:

$$J_x^{e.m.} = \frac{e\rho}{m}\left(\frac{P_2}{RN_A} - eA_x^{e.m.}\right)\delta(r-R)$$

where r is the distance from the cylinder axis, and ρ is the anyon density

$$\rho = \frac{N_A}{\nu(2\pi R)^2}$$

From the Maxwell equation

$$-\partial_r \frac{1}{r}\partial_r r A_x^{e.m.} = J_x^{e.m.}$$

we find

$$A_x^{e.m.}(r) = \frac{1}{1+\frac{e^2\rho R}{2m}}\left(\frac{r}{2}\theta(R-r) + \frac{R^2}{2r}\theta(r-R)\right)\frac{e\rho P_2}{mN_A R}$$

and the magnetic field is $B_y = \frac{1}{r}\partial_r r A_x^{e.m.}$.

We conclude that the anyon system (in the case of Fermions) exhibits quantum states, which are protected with respect to external perturbations as discussed in the previous section, giving rise to quantized values of the magnetic field, that is for $P_1 = 0$ and $P_2 = nN_A/k$

$$B_y = n \cdot \frac{1}{1+\frac{e^2\rho R}{2m}} \cdot \frac{e\rho}{m\,R\,k}$$

for $r \leq R$, and zero outside the donut (in units $\hbar = c = 1$.)

Since $A_x^{e.m.}$ is constant over the torus surface and the Hamiltonian including the electromagnetic field is obtained by replacing (see (5.3))

$$\nabla_i \to \nabla_i - i\frac{e}{2}A_x^{e.m.}$$

we get in particular in the mean field approximation that the reduced eigenvalue equation (5.7) becomes

$$H_R\, h_m = \left(E - \frac{|P'|^2}{N_A}\right) h_m$$

where the shifted total momentum is given by

$$P' = P - i\frac{e}{2}N_A \cdot A_x^{e.m.}.$$

Therefore the eigenstates are the same as those discussed in the previous section, the corresponding energy eigenvalues (5.45) being simply modified by a shift in the total momentum.

We conclude this section with an interesting observation. We can evaluate the flux of the magnetic field B_y given above across the section of the donut:

$$\Phi = n \cdot \frac{1}{1 + \frac{e^2 \rho R}{2m}} \cdot \frac{e\rho\pi R}{m\,k}.$$

Notice that in the thermodynamical limit, $R \to \infty$, ρ fixed, this becomes

$$\Phi = n \cdot \frac{1}{k} \cdot \frac{2\pi}{e}$$

which is thus an integer multiple of the elementary flux excitations appearing in the ground state of ref. [11]. These excitations behave as if their charge would be $k \cdot e$, corresponding to bosonic droplets of k particles with charge e, in analogy to Laughlin's pairing of holons [9].

6. Anyons in the Fractional Quantum Hall Effect

6.1 Anyons in an external Magnetic Field

The ground state of the fractional Quantum Hall Effect at a simple fractional filling $f = \frac{1}{m}$, where m is an odd integer, is well described by Laughlin's wave function [7]. The excitations of this state, which Laughlin gave already in that paper, appear to have fractional charge and statistics [6,7]. If the filling is a composed fractional number the ground state turns out to be decomposed in a component which stays at a simple fractional filling with respect to the external magnetic field, and in an excited anyonic component which carries in addition to the external magnetic flux a fictitious statistical flux: the two fluxes sum up to produce for the anyons an effective total filling which turns out to be simple; with respect to this total flux the anyonic component can then stay in a Laughlin–like wave function too. In general there can be more then one anyonic components, each of which represents the excitations of the underlying component, and they give rise to the so called fractional Hall effect hierarchy [5,8].

Thus we are led to consider in this chapter a system of anyons which feel, in addition to their statistical flux, a constant external magnetic field orthogonal to the surface, i.e. we study the Hall effect of one of the just mentioned anyonic components on the torus (see refs. [69,70] for treatments and ground state solutions on the plane). The statistical interaction between the anyons is again conveniently induced by a Chern–Simons field, but in addition to this we will also introduce a dynamically dependent electric field, tangent to the surface, to complete the description of the Hall effect, where a uniform motion of the charged particles occurs in a direction orthogonal to both the electric and the magnetic field, which are in turn orthogonal to each other. Classically, this motion occurs for a particular value of the electric field, related to the current and the magnetic field by:

$$\frac{N_A}{\nu} \cdot e\, E_i = B_0 \cdot \varepsilon_{ij}\, J_j \tag{6.1}$$

Here E_i is the electric field (in the x, y–directions), e the electric charge of the anyons, B_0 the external magnetic field taken to be antiparallel to the z–direction , N_A the number of anyons and therefore, in our units, N_A/ν their superficial density. We assume $B_0 > 0$. Notice that the relation (6.1) corresponds to impose the vanishing of the total Lorentz force.

Because of the Dirac quantization condition the flux of the magnetic field out of the surface must be equal to (2π) times an integer over e, and therefore we can write

$$e\,\frac{B_0 \cdot \nu}{N_A} = 2\pi \cdot \frac{q}{p} \tag{6.2}$$

where p and q are coprime integers, and N_A must be an integer multiple of p. The ratio $\frac{p}{q}$ is called the "filling" because it is the number of particles divided by the degeneracy of a Landau level.

We will not discuss possible microscopic origins of the so constrained Hall electric field, we will rather impose (6.1) as a constraint in an effective Lagrangian (see also [71]). This Lagrangian describes the behavior of a system of N_A non relativistic particles, the anyons, interacting with the statistical Chern-Simons field A_μ^{cs} and the electromagnetic field A_μ^{em}. It is useful to work in the gauge where $A_0^{em} = 0$. The anyons are also here described in first quantization by means of a wave function.

Our discussion contains therefore as a particular case the more standard problem of the (static) Landau levels on a torus, which has been previously discussed for the case of particles of ordinary statistics, see refs. [40,72]. The picture of the (static) Landau levels of anyons on a torus can be formally obtained from our results omitting the electric field: of course then one can no longer study the Hall currents.

We assume here that the anyons are fermions interacting with the Chern-Simons field at a rational coupling $k = r/s$, see chapter four. So the total statistical angle θ is given by

$$\theta = \pi\left(1 - \frac{s}{r}\right) \tag{6.3}$$

and the wave function in the anyon gauge, see section 4.7, behaves locally, i.e. for $z_i - z_j \to 0$ as

$$\Psi^{an} \sim \left(\bar{z}_{ij}\right)^{-s/r} \cdot P_o(z_i, \bar{z}_i) \tag{6.4}$$

where $P_o(z_i, \bar{z}_i)$ has to be *odd* under the interchange $z_i \leftrightarrow z_j$. To pass to our standard gauge, in which the wave function is well defined, we have to multiply (6.4) with the phase factor $(\bar{z}_{ij}/z_{ij})^{s/2r}$ to get the local behavior

$$\Psi \sim |z_{ij}|^{-\frac{s}{r}} \cdot P_o(z_i, \bar{z}_i) \tag{6.5}$$

This holds actually for for $e > 0$, the case of $e < 0$ will be discussed below.

We noticed already that eq. (6.1) amounts to impose the vanishing of the Lorentz force. On the other hand we know that the Chern–Simons action implies precisely the vanishing of this force. It is therefore natural to impose this constraint through the introduction of an *electromagnetic* Chern–Simons term in the Lagrangian. This leads us to consider the following (effective) action:

$$\begin{aligned} S = \frac{1}{4\pi}\frac{r}{s}\int d^3x\, \epsilon^{\mu\nu\lambda} A^{cs}_\mu \partial_\nu A^{cs}_\lambda - \frac{1}{4\pi}\frac{p}{q}\, e^2 \int d^3x\, \epsilon^{ij} A^{em}_i \dot{A}^{em}_j + \\ + \int d^3x \left(\Psi^\dagger \left(i\partial_t + A^{cs}_0\right) \Psi + \frac{1}{4}\, \Psi^\dagger \left(\mathcal{D}^2_x + \mathcal{D}^2_y\right) \Psi \right) \end{aligned} \tag{6.6}$$

Here Ψ is the matter field, and the spatial covariant derivatives are

$$\mathcal{D}_n = \partial_n - iA^{cs}_n - i\, e\, A^{em}_n . \tag{6.7}$$

The anyon mass has been set equal to two. Notice that the coefficient of the electromagnetic Chern–Simons action has been chosen such that variation of (6.6) with respect to A^{em}_i gives precisely the constraint of eq.(6.1). Let us also observe that gauge invariance is safe in that formally one can complete the e.m. Chern–Simons action in (6.6), introducing a scalar electric potential $A^{em}_0(t)$, which depends only on the time coordinate as we still neglect the Coulomb interaction, and adding also the term

$$e \int d^3x\, \Psi^\dagger A^{em}_0(t) \Psi .$$

Variation of the resulting action with respect to $A^{em}_0(t)$ gives then (we are considering a constant magnetic field $F^{em}_{xy} = \text{const.}$)

$$\frac{p}{q}\frac{e}{2\pi}\left(\partial_x A^{em}_y - \partial_y A^{em}_x\right) = -\frac{N_A}{\nu}, \tag{6.8}$$

which is clearly nothing else then (6.2). This equation determines only the "small" components of the electromagnetic gauge potential, leaving its large components a^{em}, which are constant over the surface, undetermined. So we have, in complex notation, the decomposition

$$-i\,e\,A^{em} = -\frac{\pi}{\nu}a^{em} + \frac{\pi}{2\nu}\frac{q}{p}N_A\bar{z}$$

and an analogous expression for $\bar{A}^{em}$.

Nothing changes with respect to the treatment of the Chern–Simons field, we get for it the expressions in (4.35) and change only the name of its topological components, which we call a^{cs} to distinguish them from their electromagnetic counterparts a^{em}.

By canonical quantization of the above action we find that in the Schroedinger representation the variables a^{cs} and $\bar{a}^{cs}$ like a^{em} and $\bar{a}^{em}$ are time independent canonically conjugated variables obeying

$$\begin{gathered}[\bar{a}^{cs}, a^{cs}] = \frac{\nu}{\pi}\frac{s}{r} \qquad [\bar{a}^{em}, a^{em}] = \frac{\nu}{\pi}\frac{q}{p} \\ [a^{cs}, a^{em}] = 0 = [\bar{a}^{cs}, a^{em}]\end{gathered} \tag{6.9}$$

As in chapter four we use the coherent state formalism where the wave function depends on a^{cs} and on a^{em} besides the anyons coordinates: $\Psi = \Psi(a^{cs}, a^{em}, z_1, ..., z_{N_A})$ and the scalar product is defined by

$$(\Psi_1, \Psi_2) = \int \prod_{j=1}^{N_A} d^2 z_j \int d\mu^{cs}\, d\mu^{em}\, \bar{\Psi}_1 \Psi_2 \tag{6.10}$$

where $d\mu^{cs} = d\bar{a}^{cs} da^{cs} \exp\left(-\frac{r}{s}\frac{\pi}{\nu}\bar{a}^{cs}a^{cs}\right)$ and $d\mu^{em} = d\bar{a}^{em} da^{em} \exp\left(-\frac{p}{q}\frac{\pi}{\nu}\bar{a}^{em}a^{em}\right)$, see section 3.4. The action of $\bar{a}$ on the wave function is represented by

$$\bar{a} = a^{\dagger} = \frac{\nu}{\pi k}\frac{\partial}{\partial a} \tag{6.11}$$

where $k = r/s$ for a^{cs}, and $k = p/q$ for a^{em}. The Hamiltonian is then (see section 4.3 for details on the derivation)

$$H = \sum_{i=1}^{N_A} \mathcal{D}_i^{\dagger}\mathcal{D}_i \tag{6.12}$$

where

$$\begin{aligned}\mathcal{D}_i &= \partial_i - \frac{\pi}{\nu}(a^{cs} + a^{em}) + \frac{\pi}{2\nu f}N_A\bar{z}_i + \frac{1}{2}\frac{s}{r}\sum_{j\neq i}\partial_i P(i,j) \\ \mathcal{D}_i^{\dagger} &= -\bar{\partial}_i + -\frac{s}{r}\frac{\partial}{\partial a^{cs}} - \frac{q}{p}\frac{\partial}{\partial a^{em}} + \frac{\pi}{2\nu f}N_A z_i + \frac{1}{2}\frac{s}{r}\sum_{j\neq i}\bar{\partial}_i P(i,j)\end{aligned} \tag{6.13}$$

We defined

$$\frac{1}{f} = \frac{s}{r} + \frac{q}{p} \tag{6.14}$$

and we remind that $P(i,j)$ is the standard scalar propagator on the torus (4.34).

Notice that possible normal ordering contact terms to be added to the Hamiltonian (6.12) are in our case irrelevant because of our assumption that the anyons are described as *fermions* interacting through the Chern-Simons field and therefore we have to look for a completely antisymmetric wave function, vanishing for $z_{ij} = z_i - z_j = 0$.

We notice also that defining in analogy with eqs. (4.42),(4.44)

$$\begin{aligned} \Delta_i &= \partial_i - \frac{\pi}{\nu}(a^{cs} + a^{em}) + \frac{\pi}{2\nu f}\sum_j \bar{z}_j \\ \Delta_i^\dagger &= -\bar{\partial}_i - \frac{s}{r}\frac{\partial}{\partial a^{cs}} - \frac{q}{p}\frac{\partial}{\partial a^{em}} + \frac{\pi}{2\nu f}\sum_j z_j \end{aligned} \tag{6.15}$$

due to (6.9) we have

$$[\Delta_i, \Delta_j] = 0 = [\Delta_i, \Delta_j^\dagger], \tag{6.16}$$

and that defining the total momentum

$$\begin{aligned} \mathcal{P} &= \sum_i \mathcal{D}_i = \sum_i \Delta_i \\ \mathcal{P}^\dagger &= \sum_i \mathcal{D}_i^\dagger = \sum_i \Delta_i^\dagger \end{aligned} \tag{6.17}$$

we have

$$[\mathcal{P}, \mathcal{P}^\dagger] = [\mathcal{P}, H] = [\mathcal{P}^\dagger, H] = 0 \tag{6.18}$$

Therefore the system is translation invariant and we can look for simultaneous eigenstates of $\mathcal{P}, \mathcal{P}^\dagger$ and H:

$$\mathcal{P}\Psi_P = P\Psi_P \qquad \mathcal{P}^\dagger\Psi_P = \bar{P}\Psi_P \qquad H\Psi_P = E(P)\Psi_P \tag{6.19}$$

The determination of the Hilbert space results from a generalization of the construction we performed in chapter four. Under shifts of the cs topological components the wave function has to transform as in (4.54), while under shifts of the em topological components

$$a^{em} \to a^{em} + q \cdot (M + N\bar{\tau})$$

the wave function has to transform in an analogous manner, which is obtained from (4.54) through the substitutions $s \to q$, $r \to p$. The solutions of these equations are both of the form (4.65) and the general state in the theory is a linear combination of

$$\phi^{cs}_{M_1,N_1} \cdot \phi^{em}_{M_2,N_2} \tag{6.20}$$

where the $\phi^{cs}_{M_1,N_1}$ depend only on a^{cs}, while the $\phi^{em}_{M_2,N_2}$ depend only on a^{em} according to (4.65). Moreover, we have the restrictions: $1 \leq M_1 \leq r$, $1 \leq N_1 \leq s$, $1 \leq M_2 \leq p$, $1 \leq N_2 \leq q$. Also here we invoke a gauge fixing in the two internal gauge spaces, labeled by $N_{1,2}$, while the indices $M_{1,2}$ span the physical space, and we can factorize from the wave function a momentum carrying exponential. The states in the physical Hilbert space become then

$$\Psi_P = \exp\left(\frac{P}{N_A}z - \frac{\bar{P}}{N_A}\bar{z}\right) \sum_{M_1=1}^{r} \sum_{M_2=1}^{p} \phi^{(0)cs}_{M_1} \cdot \phi^{(0)em}_{M_2} \cdot h_{M_1,M_2} \tag{6.21}$$

The first two equations of (6.19) imply that the h_{M_1,M_2} depend only on the differences of the coordinates.

Under shifts of the coordinates, $z_i \to z_i + m_i + n_i\tau$, the covariant derivatives in (6.13) transform as (remember that $P(i,j)$ is single valued on the surface)

$$\mathcal{D}_i \to \mathcal{D}_i + \frac{\pi}{2\nu f} N_A \left(m_i + n_i\bar{\tau}\right),$$

meaning that their transformation properties are uniquely determined by the quantity $\frac{N_A}{f}$. In the mean field approximation, where one neglects the terms in $P(i,j)$ on the r.h.s. of (6.13), f represents actually the number of particles divided by the degeneracy of the mean field Landau level: therefore we call f the "effective filling".

Accordingly the wave function has to transform under $z_i \to z_i + m_i + n_i\tau$ as in (4.64) where now the total flux, i.e. the real external flux plus the mean fictitious one, is given by

$$2\pi\, Q = 2\pi\, \frac{N_A}{f}. \tag{6.22}$$

Also here one could compute the transformations properties of (6.20) under shifts of z and deduce then the resulting transformation properties of the h_{M_1,M_2} to get a basis of the Hilbert space. This would however give a rather massy representation of the Hilbert space, which is not so illuminating, depending also heavily on the particular values of p and r. For example, if p and r are *coprime* integers, due to the fact that

$$\phi^{(0)cs}_{M_1}(z_i+m_i+n_i\tau)\cdot\phi^{(0)em}_{M_2}(z_i+m_i+n_i\tau) = \mathcal{K}(m_i,n_i)\cdot\phi^{(0)cs}_{M_1-\sum n_i}\cdot\phi^{(0)em}_{M_2-\sum n_i} \quad (6.23)$$

where $\mathcal{K}(m_i,n_i)$ is a suitable prefactor (see (4.73)), all the functions h_{M_1,M_2}, and therefore the state in the Hilbert space, are determined by say $h_{0,0}$ only. If r and p, however, are arbitrary integers which are not coprime, then the situation is more complicated and the structure of the Hilbert space is much more involved[73].

In the next section, on the other hand, we will determine the Laughlin–like ground state solutions of the Hamiltonian at fillings corresponding to the fractional Hall hierarchy, which implies in particular $r=p$, and verify that it fulfills the right transformation properties, but only for exceptional values of the momenta.

6.2 Laughlin–like solutions and hierarchy of fractional Hall states

We can exactly solve for the fermionic ground state(s) at fixed momentum provided that the effective filling f satisfies

$$0 < f \leq 1. \quad (6.24)$$

This corresponds to a magnetic field

$$eB_0 = 2\pi\left(\frac{1}{f}-\frac{s}{r}\right)\frac{N_A}{\nu}. \quad (6.25)$$

To find the eigenstates of minimal energy at fixed momentum P we adopt the procedure outlined in section 4.6. In particular also in this case the minimal energy at fixed momentum is

$$E(P) = \frac{|P|^2}{N_A}. \quad (6.26)$$

Imposing the eigenvalue equations (6.19) on (6.21) with this energy eigenvalue, we get the ground state conditions (compare with eqs. (4.92),(4.94))

$$D_i\, h_{M_1,M_2} \equiv \left(\partial_i + \frac{\pi}{2\nu f}\sum_j \bar{z}_{ij} + \frac{1}{2}\cdot\frac{s}{r}\sum_{j\neq i}\partial_i P(i,j)\right) h_{M_1,M_2} = 0 \quad (6.27)$$

In deriving this equation from (6.19) we used the identities

$$\begin{aligned}\mathcal{D}_i\left(\phi^{(0)cs}_{M_1}\cdot\phi^{(0)em}_{M_2}\right)&=\left(\phi^{(0)cs}_{M_1}\cdot\phi^{(0)em}_{M_2}\right)D_i\\ \mathcal{D}^\dagger_i\left(\phi^{(0)cs}_{M_1}\cdot\phi^{(0)em}_{M_2}\right)&=\left(\phi^{(0)cs}_{M_1}\cdot\phi^{(0)em}_{M_2}\right)D^\dagger_i\end{aligned}\tag{6.28}$$

Eq. (6.27) is solved by

$$h_{M_1,M_2}=\prod_{i<j}\exp\left(-\frac{s}{2r}P(i,j)\right)\prod_{i<j}\exp\left(\frac{\pi}{2\nu f}(\bar{z}^2_{ij}-|z_{ij}|^2)\right)G_{M_1,M_2}(\bar{z}_{ij})$$

where the G_{M_1,M_2} are antiholomorphic translation invariant functions. As said before, the boundary conditions for this set of functions are in general technically involved; the solutions are also here labeled essentially by integers $1\leq r_i\leq Q$, $i=1,...,N_A$, with an equivalence relation analogous to (5.34), where Q is defined in (6.22). Due to (6.24) there exist actually in general a lot of antisymmetric ground states, which can all be explicitly determined in this way[73]. Eq. (6.24) says, in fact, that there is enough degeneracy to accommodate all the particles in the ground state.

We will not further pursue the analysis of the general model, but discuss now the explicit solution for a particularly interesting case, which we mentioned at the beginning of this chapter, and which makes contact with the so called "fractional Hall effect hierarchy". The anyonic components of the fractional Hall effect ground state feel, in fact, a total (=fictitious + external) flux, such that their total filling is precisely the inverse of an integer odd number. In our theory this corresponds to the particular choice

$$f=\frac{1}{2J+1},\qquad J\ \text{integer.}\tag{6.29}$$

Notice that this implies

$$p=r,\qquad s+q=r\,(2J+1),\qquad N_A=u\cdot r.$$

From the eqs. (6.21) and (6.23) we deduce that the Hilbert space decomposes in this case into a direct sum of r Hilbert spaces, $\mathcal{H}=\sum_{M_0=1}^{r}\oplus\mathcal{H}_{M_0}$, where each of the components $\mathcal{H}_{M_0}$ is spanned by the states

$$\begin{aligned}\Psi_P=&\exp\left(\frac{P}{N_A}z-\frac{\bar{P}}{N_A}\bar{z}\right)\prod_{i<j}\exp\left(-\frac{s}{2r}P(i,j)\right)\\ &\prod_{i<j}\exp\left(\frac{\pi}{2\nu f}(\bar{z}^2_{ij}-|z_{ij}|^2)\right)\sum_{M=1}^{r}\phi^{(0)cs}_{M}\cdot\phi^{(0)em}_{M+M_0}\cdot G_M\end{aligned}\tag{6.30}$$

This is a general parametrization of the Hilbert space, but we are interested in the ground states which are obtained for antiholomorphic functions G_M, as we saw above. Proceeding in the usual way, we recover that these functions have in any case to satisfy the transformation properties (for a given M_0)

$$\begin{aligned} G_M(z_i + m_i + n_i\tau) =& \mathrm{e}^{-2\pi i M_0 \frac{q}{r} \sum m_i} \mathrm{e}^{2\pi i \sum m_i \frac{\tilde{P}_2}{NA} - 2\pi i \sum n_i \frac{\tilde{P}_1}{NA}} \\ & \prod_{i<j} \mathrm{e}^{(2J+1)\left(i\pi\bar{\tau}(n_{ij})^2 + 2\pi i\, n_{ij}\, \bar{z}_{ij}\right)} G_{M-\sum n_i}(z_i). \end{aligned} \tag{6.31}$$

where (notice that $q+s$ has the same parity as r)

$$\tilde{P}_i \equiv P_i + r^2 \cdot \frac{u(u+1)}{2} \tag{6.32}$$

and $G_{M+r} = G_M$. The momenta P_i, and therefore also the $\tilde{P}_i$ have still to be integer.

In the determination of the ground state of this system at the plane the Coulomb interaction, which we are not taking into account explicitly, plays a crucial role. The solution which minimizes the Coulomb repulsion is in fact known to be locally of the Laughlin form [5,7]

$$\lim_{i \to j} |\Psi_P| \sim |\bar{z}_{ij}|^{q/r}$$

The wave function Ψ_P in eq. (6.30) carries already a factor of $|z_{ij}|^{-s/r}$ through the $P(i,j)$ terms; this means that the G_M (we remember that they have to be antiholomorphic in the ground state) should behave locally as

$$G_M \sim (\bar{z}_{ij})^{2J+1}.$$

Due to (anti)holomorphicity, the largest power of $\bar{z}_{ij}$, compatible with the transformation rule (6.31), is indeed $2J+1$. Therefore, to minimize the Coulomb interaction on the torus we set

$$G_M = \prod_{i<j} \left(\overline{\theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z_{ij}|\tau)} \right)^{2J+1} \cdot g_M \tag{6.33}$$

Let us now see if this is consistent with the transformation rule (6.31). The g_M should transform as

$$g_M(z_i + m_i + n_i\tau) = \mathrm{e}^{-2\pi i M_0 \frac{q}{r} \sum m_i} \mathrm{e}^{2\pi i \sum m_i \frac{\tilde{P}_2}{NA} - 2\pi i \sum n_i \frac{\tilde{P}_1}{NA}} g_{M-\sum n_i}(z_i) \tag{6.34}$$

with $g_{M+r} = g_M$, where

$$\hat{P}_i \equiv P_i + u\frac{r(r+1)}{2}. \tag{6.35}$$

For analiticity reasons eq. (6.34) is consistent only with a z_i–independent i.e. constant set of g_M and reads then simply

$$g_M = \exp\left(-2\pi i M_0 \frac{q}{r} L\right) \exp\left(2\pi i L \frac{\hat{P}_2}{N_A} - 2\pi i N \frac{\hat{P}_1}{N_A}\right) g_{M-N}$$

for all integers L, N. This recursive relation admits solutions only if the momenta $\hat{P}_i$ are integer multiples of u:

$$\hat{P}_i = u \cdot \hat{p}_i = \frac{N_A}{r} \cdot \hat{p}_i$$

with, in particular

$$\hat{P}_2 = u \cdot (q M_0 \ mod\ r). \tag{6.36}$$

Conversely, it is clear that for all $\hat{P}_2$ which are multiples of u it exists an M_0 such that (6.36) is true. The g_M are then simply given by:

$$g_M = \exp\left(-2\pi i\, M \frac{\hat{P}_1}{N_A}\right) = \exp\left(-2\pi i\, M \left(\frac{P_1}{N_A} + \frac{r+1}{2}\right)\right) \tag{6.37}$$

Noting now that P_i and $\hat{P}_i$ differ by an integer multiple of u, we can conclude that the Laughlin–like ground state exists only for the exceptional momenta

$$P = \frac{\pi}{\nu}\frac{N_A}{r}(n_1 + n_2\bar{\tau}) \tag{6.38}$$

where $n_{1,2}$ are integers. It is important to notice that this spectrum closes under modular transformations, which shows once more that our gauge–fixed physical Hilbert space is indeed modular invariant. In ref. [21] we gave the explicit solution for the particular case in which $n_{1,2}$ are multiples of r (for r *odd*). Then, from eqs. (6.36), (6.37) one gets simply $g_M = 1$ for all M, and $M_0 = 0$, and we get back the result of [21]. In general the complete ground state wave function, which for every exceptional momentum (6.38) is unique, becomes

$$\Psi_P = \prod_{i<j} \exp\left(-\frac{s}{2r} P(i,j)\right) \cdot \exp\left(\frac{P}{N_A} z - \frac{\bar{P}}{N_A}\bar{z}\right) \cdot \prod_{i<j}\left(\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z_{ij}|\tau)\right)^{2J+1}$$
$$\prod_{i<j} \exp\left(\frac{\pi}{2\nu f}(\bar{z}_{ij}^2 - |z_{ij}|^2)\right) \sum_{M=1}^{r} \exp\left(-2\pi i M \frac{\hat{P}_1}{N_A}\right) \phi_M^{(0)cs} \cdot \phi_{M+M_0}^{(0)em} \tag{6.39}$$

When the particles have ordinary statistics one has to put $s = 0$, remove $\phi_M^{(0)cs}$ and restrict the sum to $M = 1$ in the above formula. In this case $q = 2J + 1$ and $r = 1$. In this way one gets a translation invariant version of the Laughlin wave function on the torus. In the case of the (static) Landau levels on a torus, previously discussed in the literature[40,72], the variable a^{em} is not introduced and therefore the $2J + 1$ wave functions obtained by taking $\phi^{em}_{M_1=1,N_1}$ of section 4.5, formula (4.65), $N_1 = 1, ..., 2J+1$, evaluated for $a^{em} = 0$, are not each other related by a gauge transformation and give rise to a $(2J + 1)$–fold degeneracy in agreement with ref. [40]. In our formulation this degeneracy is absent: the state (6.39) is unique.

Since a solution of the Laughlin form is possible only for the momenta of eq. (6.38), the corresponding quantum state is protected against small disturbances which would alter the momentum by a few units (in units of the inverse of the homology cycles' length), since the energy must then jump above the gap due to the Coulomb interaction, which separates the state of the Laughlin form from other configurations. The corresponding motion is then expected to be superfluid, i.e. to have zero "diagonal" resistance.

The solution Ψ_P behaves locally for $z_i \to z_j$ as $\bar{z}_{ij}^{2J+1} \cdot |z_{ij}|^{-\frac{s}{r}}$. To compare with the anyon gauge (section 4.7), which is often used on the plane, multiply locally the wave function by the phase factor $(z_{ij}/\bar{z}_{ij})^{\frac{s}{2r}}$ to get the behavior

$$\sim \bar{z}_{ij}^{2J+1-s/r}. \tag{6.40}$$

Let us finally show how the fractional Hall effect hierarchy can be established on the torus. Following Halperin, ref. [5], we may imagine that the anyons are excitations of an underlying system of particles of charge e_0 (which eventually could also be anyons of a previous hierarchical level), and write the statistical angle of eq. (6.3) as

$$\frac{\theta}{\pi} = -\frac{\alpha}{\mu_0} \qquad i.e. \qquad \frac{s}{r} = 1 + \frac{\alpha}{\mu_0} \tag{6.41}$$

where μ_0 is in general fractional (to be recursively determined) and $\alpha = \pm 1$ according to whether we consider "particle" or "hole" excitations. This is the case for excitations of an underlying system whose wave function behaves locally on the plane for $w_i \to w_j$

(calling w_l the complex coordinates of the underlying particles) as

$$\sim (\bar{w}_{ij})^{\mu_0}. \tag{6.42}$$

Comparing with (6.40) we can say that the anyons also behave according to a wave function of the form (6.42) with an exponent μ given by

$$\mu = 2J - \frac{\alpha}{\mu_0} \tag{6.43}$$

Notice that for $\mu_0 \geq 1$ and $J \geq 1$, the case to be discussed in the following, then also $\mu \geq 1$ and therefore, in particular, the wave function vanishes for $z_{ij} \to 0$. Note also that from (6.41) $s/r \geq 0$.

The charge e of the anyon is related to e_0 by

$$e = \frac{\alpha}{\mu_0} e_0 \tag{6.44}$$

When $e < 0$ then eq. (6.2) gives $q/p < 0$. In this case it is convenient to put

$$s/r = -\left(1 + \frac{\alpha}{\mu_0}\right) < 0$$

so that also $f^{-1} = -(2J+1) < 0$. We then interchange the role of $\mathcal{D}$ and $\bar{\mathcal{D}}$, and we get the same wave function with $z_l \leftrightarrow \bar{z}_l$, and $a^{cs,em} \leftrightarrow -\bar{a}^{cs,em}$. In particular the local behavior in the anyon gauge will be now instead of (6.4)

$$\Psi \sim (z_{ij})^{s/r} P_o(z_i, \bar{z}_i) \tag{6.45}$$

where P_o is odd, and for our ground state solution we get

$$\Psi \sim (z_{ij})^{s/r+2J+1}.$$

Like before $\frac{\theta}{\pi} = -\frac{\alpha}{\mu_0}$.

So, modulo a complex conjugation, we can use the previous results both for $e > 0$ and for $e < 0$ and in particular we have from eq. (6.25)

$$N_A = |e| \frac{1}{\mu} \cdot \frac{B_0 \nu}{2\pi} \tag{6.46}$$

It is interesting to note that to get this equation we do not need to introduce a "classical plasma" like it is done on the plane[7]. Since eq. (6.42) corresponds to a filling $1/\mu_0$ we have for the density of the underlying system

$$\frac{N^0}{\nu} = |e_0| \frac{1}{\mu_0} \cdot \frac{B_0}{2\pi}$$

We can then compute the total charge of the total system, i.e. the underlying one plus the anyons excitations one,

$$e_0 N_T = e_0 N^0 + e N_A \tag{6.47}$$

from which we get for the total particle number density

$$\frac{N_T}{\nu} = \frac{N^0}{\nu} + \frac{e|e|}{e_0 \mu} \frac{B_0}{2\pi} \tag{6.48}$$

Of course, the picture of the anyons as excitations of an underlying system makes sense only in the limit of a very large torus and very large B_0 and N_T (finite density). On a large but finite torus the picture is only approximate and strictly speaking it requires values of B_0 such that N_T and N_A are integers. In the following we assume that this is true at each step of the hierarchy and we will only concentrate on the filling.

One can then proceed one step further and consider the excitations of the anyons system, and so on. One can then relate the step $(s+1)$ to the step (s) by iterating eqs. (6.43), (6.44) and (6.48) in the following way

$$\begin{aligned}
\mu_{s+1} &= 2J_{s+1} - \frac{\alpha_{s+1}}{\mu_s} \\
e_{s+1} &= \frac{\alpha_{s+1}}{\mu_s} e_s \\
N_T^{(s+1)} &= N_T^{(s)} + \frac{e_{s+1}|e_{s+1}|}{\mu_{s+1} e_0} \frac{B_0 \nu}{2\pi}
\end{aligned}$$

By defining the total filling by

$$F^{(s)} = N_T^{(s)} \cdot \frac{2\pi}{|e_0| B_0 \nu}$$

we get that at the $(s+1)$–th hierarchical level we are describing a filling

$$F^{(s+1)} = F^{(s)} + \frac{\alpha_{s+1}}{\mu_{s+1}} \frac{1}{\mu_s^2} \frac{e_s|e_s|}{e_0|e_0|}$$

(the notation is slightly changed with respect to ref. [5], in particular for what concerns the notation for the charge).

As examples, we get, like in ref. [5], that if one starts with $e_0 = \mu_0 = \alpha_1 = 1$ and $F^{(0)} = 0$, one has at the first step of the hierarchy $e_1 = 1$ and $\mu_1 = 2J_1 - 1$ which corresponds to electrons with filling

$$F^{(1)} = 1, 1/3, 1/5, ... \qquad \text{for } J_1 = 1, 2, 3, ...$$

Then at the next step one can have excitations with $e_2 = \frac{\alpha_2}{\mu_1}$ with statistics $\mu_2 = 2J_2 - \frac{\alpha_2}{\mu_1}$ corresponding to fillings

$$F^{(2)} = 2/3, 4/5, ... \quad \text{for} \quad \mu_1 = 1 \quad \text{and} \quad \alpha_2 = -1$$
$$F^{(2)} = 2/7, 4/13, ... \quad \text{for} \quad \mu_1 = 3 \quad \text{and} \quad \alpha_2 = -1$$
$$F^{(2)} = 2/5, 4/11, ... \quad \text{for} \quad \mu_1 = 3 \quad \text{and} \quad \alpha_2 = +1$$

Let us now discuss the total momentum of our system. We use a reasoning similar to the one followed for computing the total number, see eqs. (6.47) and (6.48). We first compute the total e.m. current

$$J_T = e_0 N_T v = e_0 N^0 v_0 + e N_A v_{AL} \tag{6.49}$$

where v_{AL} is the velocity of the excitations as seen in the Lab. reference frame, i.e. $v_{AL} = v_0 + v_r$ where v_r is the velocity of the excitations relative to the underlying system. The total momentum is then

$$P_T = M_0 N_T v = P_0 \left(1 + \frac{e}{e_0}\frac{N_A}{N^0}\right) + P_A \frac{e}{e_0}\frac{M_0}{M_A} \tag{6.50}$$

where M_0 is the mass of the particles of the underlying system, M_A is the mass of the anyon excitations and $P_A = M_A N_A v_r$ is the momentum of the anyon system relative to the underlying one. Since we do not know how to compute precisely M_0/M_A, let us consider in particular the case $P_A = 0$, where the anyons are at rest with respect to the underlying system, at each level of the hierarchy. The iteration of eq. (6.50) gives then

$$P_T^{(s+1)} = \frac{N_T^{(s+1)}}{N_T^{(s)}} \cdot P_T^{(s)}$$

Since at the first level we have ordinary electrons and therefore $r = 1$ in eq. (6.38), we have $P_T^{(1)} = n\,N_T^{(1)}$ (in units of 2π times the inverse of the cycles' length). We see then that the exceptional momenta are

$$P_T^{(s+1)} = n\,N_T^{(s+1)}$$

i.e. integer multiples of the total number of electrons in units of 2π times the inverse of the cycles' length. It is only for these values, in the case of zero relative velocity, that the wave functions of the anyon systems at each level of the hierarchy can be in the ground state with respect to the Coulomb repulsion.

REFERENCES

[1] D. Finkelstein and Julio Rubinstein, J. Math. Phys. **9** (68) 1762.

[2] J.M. Leinaas and J. Myrheim, Nuovo Cimento **37B** (77) 1.

[3] F. Wilczek, Phys. Rev. Lett. **48** (82) 1144.

[4] F. Wilczek, Phys. Rev. Lett. **49** (82) 957.

[5] B.I. Halperin, Phys. Rev. Lett. **52** (84) 1583.

[6] D. Arovas, J.R. Schrieffer and F. Wilczek, Phys. Rev. Lett. **53** (84) 722.

[7] R.B. Laughlin, Phys. Rev. Lett. **50** (83) 1395.

[8] D.M. Haldane, Phys. Rev. Lett. **51** (83) 605.

[9] R.B. Laughlin, Phys. Rev. Lett. **60** (88) 2677.

[10] A.L. Fetter, C.B. Hanna and R.B. Laughlin, Phys. Rev. **B39** (89) 9679.

[11] Y.H. Chen, F. Wilczek, E. Witten and B.I. Halperin, Int. J. Mod. Phys. **B3** (89) 1001.

[12] D.P. Arovas, R. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. **B251** [FS13] (85) 117.

[13] E. Witten, Comm. Math. Phys. **121** (89) 351.

[14] M. Bos and V.P. Nair, Int. J. Mod. Phys. **A5** (90) 959.

[15] J.M.F. Labastida and A.V. Ramallo, Phys. Lett. **B227** (89) 92.

[16] J.M.F. Labastida and A.V. Ramallo, Phys. Lett. **B228** (89) 214.

[17] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. **B326** (89) 108.

[18] A.P. Polychronakos, *Abelian Chern–Simons Theories and Conformal Blocks*, preprint, UFIFT –89–9.

[19] A.P. Polychronakos, Phys. Lett. **B241** (90) 37.

[20] R. Iengo and K. Lechner, Nucl. Phys. **B346 [FS]** (90) 551.

[21] R. Iengo and K. Lechner, *Exact Results for Anyons on a Torus*, SISSA 182/90/EP, Trieste, submitted to Nucl. Phys. B.

[22] S.M. Girvin, A.H. MacDonald, M.P.A. Fisher, S.J. Rey and J.P. Sethna, Phys. Rev. Lett. **65** (90) 1671.

[23] R. Jackiw and S.Y. Pi, Phys. Rev. **D42** (90) 3500.

[24] T. Einarsson, Phys. Rev. Lett. **64** (90) 1995.

[25] X.G. Wen, F. Wilczek and A. Zee, Phys. Rev. **B39** (89) 11413.

[26] B.I. Halperin, J. March–Russell and F. Wilczek, Phys. Rev. **B40** (89) 8726.

[27] *The Quantum Hall Effect*, edited by R.E. Prange and S.M. Girvin, Graduate Texts in Contemporary Physics, New York-Berlin-Heidelberg, Springer Verlag (1987).

[28] Hideo Aoki, Rep. Prog. Phys. **50** (87) 655.

[29] *Fractional Statistics and Anyon Superconductivity*, edited by F. Wilczek, World Scientific, Singapore–New Jersey–London–Hong Kong (1990).

[30] E. Wigner, Ann. of Math. **40** (39) 149.

[31] V. Bargmann, Ann. of Math. **48** (47) 568.

[32] J. Fröhlich and P.A. Marchetti, Lett. Math. Phys. **16** (88) 347.

[33] J. Fröhlich and P.A. Marchetti, *Spin–Statistics Theorem and Scattering in planar Quantum Field Theories with braid statistics*, DFPD/TH/10/90, Padua, May (1990).

[34] G.A. Goldin, R. Menikoff and D.H. Sharp, J. Math. Phys. **22** (81) 1664.

[35] Y. Aharonov and D. Bohm, Phys. Rev. **115** (59) 485.

[36] F. Wilczek and A. Zee, Phys. Rev. Lett. **51** (83) 2250.

[37] R.B. Laughlin, Phys. Rev. **B23** (81) 5632.

[38] R.E. Prange, Phys. Rev. **B23** (81) 4802.

[39] R.B. Laughlin, Phys. Rev. **B27** (83) 3383.

[40] F.D.M. Haldane and E.H. Rezayi, Phys. Rev. **B31** (85) 2529.

[41] M.V. Berry, Proc. R. Soc. **A392** (84) 45.

[42] R.B. Laughlin, Sciences **242** (88) 525.

[43] V. Kalmeyer and R.B. Laughlin, Phys. Rev. Lett. **59** (87) 2095.

[44] P.W. Anderson, Mater. Res. Bull. **8** (73) 153.

[45] P.W. Anderson, Science **236** (87) 1196.

[46] S.A. Kivelson, D.S. Rokhsar and J.P. Sethna, Phys. Rev. **B35** (87) 8865.

[47] G.S. Canright, S.M. Girvin and A. Brass, Phys. Rev. Lett. **63** (89) 2295.

[48] C.B. Hanna, R.B. Laughlin and A.L. Fetter, Phys. Rev. **B40** (89) 8745.

[49] Y. Hosotani and S. Chakravarty, Phys. Rev. **B42** (90) 342.

[50] G.S. Canright, S.M. Girvin and A. Brass, Phys. Rev. Lett. **63** (89) 2291.

[51] J. Lykken, J. Sonnenschein and N. Weiss, Phys. Rev. **D42** (90) 2161.

[52] S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. **B340** (90) 403.

[53] S. Fubini, Int. J. Mod. Phys. **A5** (90) 3533.

[54] L.D. Landau, J. Phys. USSR **5** (41) 71, **11** (47) 91.

[55] J. Bardeen, L. Cooper and J. Schrieffer, Phys. Rev. **108** (57) 1175.

[56] J. Hetrick, Y. Hosotani and B. Lee, *Magnetic and Thermal Properties of the Anyon Superconductor*, UMN–TH–831/90.

[57] M. Bos and V.P. Nair, Phys. Lett. **B223** (89) 61.

[58] A.P. Polychronakos, Ann. Phys. **203** (90) 231.

[59] D. Mumford, *Tata Lectures on Theta*, Birkhauser Vols. 2 (1983).

[60] Y. Hosotani, Phys. Rev. Lett. **62** (89) 2785.

[61] R. Jackiw, "Topics in planar Physics", lectures given at the XXIX Schladming School (Austria), March 1990.

[62] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48** (82) 975; Ann. Phys. (NY) **140** (82) 372.

[63] A.P. Polychronakos, Nucl. Phys. **B281** (87) 241.

[64] O. Alvarez, Comm. Math. Phys. **100** (85) 279.

[65] R. Dijkgraaf and E. Witten, Comm. Math. Phys. **129** (90) 393.

[66] R. Jackiw and S.Y. Pi, Phys. Rev. Lett. **64** (90) 2969.

[67] K. Lechner, unpublished.

[68] X.G. Wen, E. Dagotto and E. Fradkin, *How to put anyons on a torus*, P/90/4/49, Santa Barbara.

[69] M.D. Johnson and G.S. Canright, Phys. Rev. **B41** (90) 6870.

[70] G. Dunne, A. Lerda and C. Trugenberger, *Landau levels and Vertex Operators for Anyons*, CTP/1938/January 1991.

[71] J. Fröhlich and T. Kerler, *Universality in Quantum Hall Systems*, ETH–TH/90–26.

[72] A. Mielke, Z. Physik **B80** (90) 57.

[73] K. Lechner, unpublished.