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On the Compatibility of Quantum Matter and Classical Gravity

Thesis submitted for the degree of
"Doctor Philosophiae"

Astrophysics Sector

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Ai miei genitori

Nissuna cosa è, che più c'inganni, che 'l nostro judizio.

Il massimo inganno delli omini è nelle loro opinioni.

Leonardo da Vinci

Abstract

The formulation of a consistent scheme for treating semiclassical systems is considered, giving particular emphasis to the case of gravity.

A critical review of the semiclassical problem is first given, stressing the conceptual aspects of the topic; the theory usually adopted is found to be unsatisfactory, and the need to have an unambiguous interpretation of quantum mechanics before trying to give an alternative description is realized. Our way to arrive at such an interpretation is rather unusual, but it presents the advantage of being almost compelling in the choice to make: First, we reformulate the Schrödinger equation as a set of hydrodynamical equations involving quantities which formally play the role of mass density, velocity and pressure for a fluid; the problem of interpreting the wave function is thus reduced to that of interpreting these quantities. We then show how they can be derived from the Wigner distribution function exactly as in the usual formalism of kinetic theory, and discuss how this fact provides strong support in favour of the statistical interpretation of quantum theory, according to which the state vector describes only *ensembles*, and not *individual systems*. We also consider some implications of these results on the possible existence of a more fundamental theory underlying quantum mechanics.

It is shown that reconsidering the semiclassical problem in the light of the statistical interpretation, one is led to distinguish between a strongly and a weakly semiclassical regime, which are essentially characterized by the size of the statistical dispersion induced in the observables of the classical subsystem by the coupling to the quantum one. It turns out that in the weakly semiclassical regime, in which this dispersion is not negligible, the concept of coupling equations cannot be successfully applied, and one has rather to define a probability distribution even for the values of the classical observables; an hypothesis which allows to specify such a distribution is enunciated. Several examples of the application of these general principles are considered, and it is shown how the treatment of semiclassi-

cal relativistic fields requires a much more sophisticated treatment of the quantum source.

This is provided reformulating quantum theory in terms of a quasiprobability functional $P[\gamma]$ in the space of the histories of the system. It is shown how such a functional allows to reconstruct the usual phase space distributions when integrated over suitable sets of paths, in a way which clarify the relations between operator ordering, path integration and phase space treatment of quantum theory. The relativistic extension of $p[\gamma]$ is also constructed, and an explicitly covariant version of relativistic quantum theory is discussed in some details. It is shown how the latter allows, formally, to consider superpositions of different mass eigenstates, although such superpositions are not directly observable.

Finally, the application of these new techniques to the treatment of semiclassical electromagnetism and gravity, as well as of a scalar field, are considered. It is shown how the usual semiclassical field equations, suitably reinterpreted in terms of averages of the field, are recovered either in linear cases or in the strongly semiclassical regime, but that they do not hold in general. Finally, some possible extensions and implications of the formalism are discussed.

Contents

Chapter 1

Introduction: Why Semiclassical Gravity?

Gravity, as described by general relativity, is by far the most peculiar among the known fundamental interactions. This status of things has its deepest origin in the geometrization lying at the basis of Einstein's theory, which makes it the most elegant and formally perfect of all the viable theoretical physical constructions. In this picture, gravity is not described by some object (e.g., a field) defined *on* spacetime, but rather it *is* the spacetime itself; the very fundamental structure of the theory can thus be epitomized in the relation

$$\text{geometry} \longleftrightarrow \text{matter} . \tag{1.1}$$

In standard general relativity [1,2,3], the left hand side of (1.1) corresponds to the tensor

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R , \tag{1.2}$$

the concept of gravitational field being reduced to that of riemannian curvature of the spacetime manifold; in (1.2), g_{ab} is the metric of spacetime, R_{ab} is the Ricci tensor and $R \equiv g^{ab} R_{ab}$ is the scalar curvature. The properties of classical matter to which the right hand side of (1.1) refers are represented by the stress-energy-momentum tensor[1] T_{ab}, which can be formally obtained from an action S_m for matter, by functional

[1]We do not care about the possible presence of a cosmological constant, which is not relevant for our discussion.

derivation with respect to g^{ab}:

$$T_{ab} = -\frac{\delta S_m}{\delta g^{ab}} \ . \tag{1.3}$$

The relation (1.1) takes thus the form

$$G_{ab} = \kappa T_{ab} \ , \tag{1.4}$$

where κ is a coupling constant, fixed to the value $\kappa = 8\pi G/c^4$, with G Newton's constant and c the speed of light.

Despite the aesthetic appeal of such a theory, it is pretty clear that its domain of applicability suffers from a serious limitation: The notion (1.3) of stress-energy-momentum tensor is well defined only for classical matter, while the matter's behaviour is empirically known to be ultimately quantum. Hence, Eq. (1.4) cannot be applied to situations in which quantum effects play a relevant role, and we ought to seek for a description of gravity which could take into account such circumstances as well. It is good to stress, at this point af the discussion, that the idea that gravity and quantum mechanics should, somehow, cohabit, is more a consequence of the philosophical belief known as "unity of physics" than of some experimental result: Both classical gravity and quantum theory have been successfully tested separately, but there is total lack of experiments devoted to investigate their reciprocal relations. This shows how much wildly speculative is the territory which we are now going to explore.

There are three levels at which a "quantum principle" can be introduced into general relativity; they lead to theories which can be classified as quantum theory in curved spacetime, semiclassical gravity, and quantum gravity. We shall now briefly analyze the status of these three classes of theories, and the chances they have to provide a reasonable solution to the problem mentioned above.

Quantum theory in curved spacetime [4] is the study of quantum matter in a fixed background gravitational field; trivial (only from the point of view of gravity!) examples of these theories are QED and QCD in Minkowski spacetime. This kind of approach is clearly suitable for the analysis of specific effects, but not for the formulation of a fundamental theory: There is in fact no prescription about the back reaction which quantum matter should exert on spacetime, the latter being supposed fixed *a priori*. Nevertheless, the research in this sector of physics has produced most interesting results (particles creation in cosmological models; evaporation of black holes; criticism of the concept of particle), some of which seem to hint at some features of a deeper theory. As a simple example, let us briefly outline the Bekenstein-Hawking's discovery [5,6] that to a black hole can be associated a physically meaningful temperature and entropy, making therefore possible a well posed study of black holes' thermodynamics. In the simple case of a Schwarzschild black hole of mass M, these quantities read

$$T = \frac{\hbar c^3}{8\pi k G} \cdot \frac{1}{M} \,, \tag{1.5}$$

$$S = \frac{4\pi k G}{c\hbar} \cdot M^2 \,, \tag{1.6}$$

where k is Boltzmann's constant; the appearance of Planck's constant $\hbar$ in the expressions (1.5) and (1.6) emphasizes their quantum origin, which can be clarified on the ground of the equality between entropy increase and information loss, applied to the process of collapse of a matter configuration to the black hole. The key of the argument is the result of relativistic quantum theory, that a particle of mass m cannot be localized more precisely than within its Compton length $\lambda_C \equiv \hbar/mc$ [7]. In order for the particle to fit into the hole during collapse, λ_C must be smaller than the Schwarzschild radius, and this requirement imposes a lower limit on the value of m; consequently, the number of possible configurations which can give origin to the same black hole of given mass M, is limited to a well defined value by the quantum nature of matter, allowing thus to speak meaningfully of the quantities T and S.

The thermodynamics of black holes [8,9] has proved to be so rich, simple and physically interesting, that it has stimulated many speculations about the possibility to use it as the starting point for the construction of a deeper theory [10,11]. It has also been suggested [12] that the transformation of pure states into mixtures associated to the process of black holes' evaporation, provides evidence for the crucial role of gravity in the puzzle of state vector reduction. Personally, we find these arguments, although attractive, too vague and cloudy, and not strong enough to justify the huge efforts which the accomplishment of such programs would require. We rather believe that, in the absence of experimental data, the adoption of any new physical hypothesis should be motivated by a careful analysis of the nature and meaning of the problems which one is trying to solve.

Quantum gravity represents, in a sense, the opposite extreme with respect to quantum theory in curved spacetime. In this theory, in fact, not only should the gravitational back reaction of quantum matter be taken into account, but also the geometry would be quantized. After more than forty years of efforts made by considerable physicists in all the world, such a theory does not yet exist in a satisfactory form; there have been many technical advances[2], but very little conceptual understanding [13,14,15,16]; consequently, the large number of competing ideas contrasts with the lack of strong physical principles which could be used as heuristic guide in the choice among them. This status of things, in which there is no general, reliable theoretical framework, induces to think that it may be wrong to try to extrapolate our current methods of investigation to such an exotic field, and that a critical analysis of the foundations of modern physics should precede any further study [16]. For these reasons, we prefer not to take as granted the need to develop a quantum theory of gravity, but rather we shall quickly review the motivations for its construction, trying to maintain an objective attitude, in which only the experimental results are considered as unquestionable.

Let us examine first the motivations of essentially formal nature. The most serious of all is, in our opinion, the use of quantum gravity as a remedy to cure the diseases which plague other field theories [17,18]. It is well known, in fact, that quantum theories of matter fields lead to ultraviolet divergences when treated in a classical background spacetime; now, it happens that, in a quantum theory of gravity, the one-loop contributions from gravitons would be comparable to the vacuum polarization effects of matter, leaving thus the possibility of a mutual cancellation. This may seem, at first, a rather strong and rigorous reason to justify the quantization of the gravitational field, but it is more fair to admit that, when considered on experimental grounds, it looks very speculative. The formalism of quantum field theories, in fact, is quite

[2] And troubles!

messy and inelegant; the complexity of the mathematical machinery employed in the derivation of experimentally testable results, should make us open to the possibility that another formalism may exist, which leads to nearly the same numerical conclusions, without running into the technical problems due to the appearance of infinities. That such alternative approaches are possible, in the sense that the currently available experimental data do not uniquely fix the theory, can be realized thinking to the case of electrodynamics, where the amount of data is extremely high, in comparison to the situation for other interactions, because of the possibility to perform significant experiments at relatively low energies. It is astonishing that most of the results of QED can be understood and even theoretically predicted by a very simple semiclassical model [19], and that semiclassical linear theories, such as stochastic electrodynamics [20,21] or Barut's theory [22,23], can reproduce all the observed details, like those related to the Casimir effect or to the anomalous magnetic moment of the electron, which are commonly believed to provide indisputable evidence for the quantization of the electromagnetic field. We are not arguing, here, on the correctness of these theories, but we simply claim that they are possible rivals of the standard QED, which do not resort to the second quantization techniques and which are, experimentally, as well tested as QED is. These examples make us very suspicious about the validity of the previously advanced argument for quantizing gravity and, more generally, of all the arguments based on the internal consistence of theories which are not well founded from the experimental point of view.

Another context in which quantum gravity is often invoked as a cure, is that of spacetime singularities. A particularly striking consequence of general relativity, which heavily relies on its geometrical character, is in fact the prediction of the existence of "regions" to which the theory itself cannot be applied. More precisely, theorems have been proved [1,3] asserting that a sufficiently general and causally well behaved spacetime, whose energy content satisfies a positivity condition, and in which a convergence criterion is fulfilled, must admit a singularity, defined in terms of incompleteness of causal geodesics; since these curves represent the world lines of classical freely falling particles, it follows that, in the framework of general relativity, there is a large variety of realistic situations in which particles (and observers) can reach the boundary of spacetime in a finite proper time. The implications of such a result are quite intriguing: If one believes that singularities do really exist, then he/she will have serious troubles in specifying boundary conditions; alternatively, it is possible to look at the singularity theorems as at a *reductio ad absurdum* of some hypothesis underlying their proof: They can therefore be used as evidence for the breakdown of the energy condition or for the replacement of general relativity by another theory of spacetime structure. It is a widespread conviction that the introduction of quantum mechanics into the theory would have an almost miraculous effect in solving these problems; however, even here the attitude is to use handwaving arguments, based on a theory which does not yet exist, to guess what such a theory could do in order to remove the pathologies from the existing models. Until now, these studies have given a negligible contribution both to the understanding of singularities and to the formulation of a quantum theory of gravity; hence, we cannot consider them differently from what they really are, i.e., only intuitive guesses. Moreover, we want to point out that these arguments do not strictly require full quantum gravity, but they may equally well lead to a simpler semiclassical theory, in which the gravitational field is not quantized, although interacting with quantum matter; in fact, quantum fields in curved spacetime are known to violate the positive energy condition [4], and this is sufficient to invalidate the conclusions of the singularity theorems, if a prescription for considering back-reaction is assigned.

On the experimental side, there is no evidence, even indirect, in favour or against quantum gravity; therefore, if the problem has to be tackled (and it *ought* to be, being a fundamental one), it is necessary to rely on some leading principle. The lack of experimental data suggests to try to involve the very basic principles of physics in the discussion, through the use of some *gedanken* experiments. There is a statement by Unruh [24] on this subject, which we believe is worth quoting here: "[Gedanken experiments] serve not to test nature but rather to present the *a priori* prejudices of the theorist in their simplest physical guise. They highlight the beliefs and prejudices the theorist has about the physical world – beliefs which could well be proven wrong by true experiments, but which seem necessary to limit the infinite range of possible theories in the absence of experiment."

Unruh himself suggests a gedanken experiment to support the view that the gravitational field should be quantized. The experimental setup consists of a neutron star oscillating in its fundamental quadrupole mode, with consequent emission of gravitational radiation; this would damp both the amplitude Q and the momentum P of the vibration in a time of the order of a second, with consequent decay to zero of the commutator $[\hat{Q}, \hat{P}]$. Such a conclusion is in contrast with the principles of quantum theory, and the natural conclusion of the argument would be thus the need to quantize gravity, in order to provide an additional force due to the vacuum fluctuations of the gravitational field, which would restore $[\hat{Q}, \hat{P}]$ to the value $i\hbar \neq 0$. This reasoning, which could be used also to prove that the electromagnetic field must be quantized, seems, at a first sight, very convincing; however, a deeper analysis shows that it could well be inexact, so that it is not as compelling as it looks. In fact, it is reasonable to accept that, if there exists a physical system which is not quantum but can interact with other, quantum behaving, systems, then quantum theory has certainly to suffer some changes; therefore, it should not be a surprise that $[\hat{Q}, \hat{P}]$ may be found to be damped to zero by such an interaction. The only thing we must be sure about, is that the damping could not be observed in laboratory systems, which are the only one over which precise tests of quantum mechanics have been performed. The result of this gedanken experiment cannot be used as a proof of the need to quantize the gravitational field, because the experiment has never been carried on for a neutron star: It proves only that a classical field is incompatible with a theory in which $[\hat{Q}, \hat{P}] = i\hbar$ forever. It is easy to check, with a rough calculation, that for the electron in a hydrogen atom, the gravitational damping time would be of the order of 10^{39} seconds, which is about 10^{22} times the age of the universe according to standard cosmology! There is no doubt that such small deviations from quantum theory are perfectly allowed by the present technology, since they would be practically impossible to detect. Another possible counter-argument to this gedanken experiment is based on the existence of stochastic electrodynamics, which we have already mentioned above. In fact, what the experiment predicts is, strictly speaking, not the quantization of the field, but only the presence of a zero point fluctuations background; this is clearly a weaker consequence, and can be accounted for even in a semiclassical context, as shown by the example of stochastic electrodynamics.

Other two interesting attempts at establishing the quantum nature of gravity by means of a gedanken experiment, are due to Page and Geilker [25] and to Eppley and Hannah [26]. However, the Page-Geilker argument can actually be reversed to provide a criticism to the usual semiclassical theory of gravity, but leaving open the problems about the necessity to quantize the gravitational field; we shall comment extensively later on about a similar gedanken experiment, so we shall not discuss it here. Eppley and Hannah consider the scattering of classical gravitational wave packets by a quantum particle prepared in a state with spatial localization Δx. Such scattering can take place in two, mutually exclusive, possible ways: either it produces a collapse of the wave function, and it can thus be considered as a position measurement (scattering by a pointlike object), or it does not produce any collapse at all (scattering by an extended object). In the first case, it is easy to realize that, using gravitational wave packets of sufficiently small width and little amplitude (there is no lower limit on these quantities, because gravity is supposed to behave classically), and starting with a particle with well defined momentum (i.e., with a great value of Δx), we are allowed, detecting the position of classical wave packets after scattering, to infer both the values of position and momentum for the particle, thus violating the uncertainty principle. If, on the other case, scattering does not collapse the state vector, then it provides a way for observing the wave function without reduction; Eppley and Hannah show that this would lead to a violation of causality, in the sense that signals could be transmitted at a speed greater than c. They conclude that the assumption that gravity is classical violates very fundamental principles of physics, and hence that semiclassical theories must be rejected.

The weakest point of this argument relies, in our opinion, in the assumption that the state vector may undergo a collapse. As we shall see in Ch. 3, this is typical of the interpretations of quantum theory which suppose that the wave function describe a single system, and leads to tricky conclusions (measurement paradox). However, these are not the only possible interpretations; for example, in the statistical interpretation [27] the state vector describes ensembles, and has no meaning for single individuals: The concept of collapse loses therefore its meaning in situations like that envisaged in the experiment outlined above, and the entire argument is consequently vitiated. It is thus possible to conclude, remembering the quotation from Unruh, that the gedanken experiment due to Eppley and Hannah only displays their prejudices in favour of a particular interpretation of quantum mechanics.

We believe this situation not to be limited to the few examples here discussed, but to be much more general: It is very difficult to devise a gedanken experiment which, involving only features of the present theories which have already been tested experimentally, could prove in a convincing way something about a subject so far from standard physics as this is. Moreover, as we have seen explicitly in the previous examples, it is extremely easy to draw erroneous conclusions because of conceptual misunderstandings or tacit assumptions which may spoil the arguments in a very subtle way. A clarification, from the physical point of view, of the features of a quantum theory of gravity (e.g., meaning of the quantization procedure when applied to the spacetime geometry), and of the foundations of the theories involved in its construction, should precede, in our opinion, the majority of the technical investigations on the entire topic. In particular, a clear understanding of quantum mechanics is especially important, in order to avoid the conceptual pitfalls to which an improper interpretation would almost certainly lead[3]. We hope that our contributions in Chs. 3, 5 and 6, although developed for different purposes, may turn out to be useful also to this aim.

The lack of definitive arguments in favour of a full quantum theory of gravity can be took as a motivation for constructing a semiclassical theory, in which the gravitational field is still treated classically by means of the geometrical description of general relativity, but the back-reaction of quantum matter on the spacetime manifold is taken into account through some relation of the kind of (1.1). As far as we know, in fact, semiclassical gravity could well be a fundamental theory of nature [29,30].

Another argument supporting semiclassical gravity is of a more pragmatic character, being based on the remark that, even if the gravitational field should be quantized, nevertheless there would exist situations in which a semiclassical theory is a fairly good approximation. In fact,

[3]As an example of the tricky conceptual problems which a quantum theory of gravity must face with, let us mention those related to the interpretation of the wave function of the universe in a quantum cosmological context [28].

the effects of quantum gravity should become important at scales of the order of the Planck length

$$l_P \equiv \left(\frac{G\hbar}{c^3}\right)^{1/2} \approx 1.6 \times 10^{-33}\text{cm} , \tag{1.7}$$

whose ridiculously small value, when compared to the scales of ordinary quantum systems, induces to think that a regime is conceivable which plays an intermediate role between the "rigid" scheme of quantum theory in a fixed background spacetime and the still unknown full quantum gravity. It is possible to envisage situations in which the gravitational field is generated by matter behaving quantum mechanically, but is measured averaging over regions whose typical size is much greater than l_P; in such conditions it is therefore meaningful (and useful), even if a quantum theory of gravity is available, to ask for the formulation of a semiclassical treatment, which would constitute a fairly good approximation.

The accomplishment of such a program is, although simpler that for the case of gravity quantization, still rather difficult. The crux of the theory is represented by the relation (1.1). The different behaviours (classical and quantum) of its two members, in fact, forces us to face directly with the problems arising in general whenever a classical system is coupled to a quantum one; these are in their turn deeply intertwined with the issue of the interpretation of the quantum formalism, which is itself a still open foundational subject. Let us explain this point by means of a simple example, constructed from the theory on which almost all the results obtained so far in semiclassical gravity (mainly in the study of black holes' evaporation and of inflationary cosmology) are based: The right hand side of Eq. (1.4) is simply modified in the expectation value of the stress-energy-momentum tensor operator $\hat{T}_{ab}$ of quantum matter, obtaining

$$G_{ab} = \kappa \langle \psi | \hat{T}_{ab} | \psi \rangle , \tag{1.8}$$

where $|\psi\rangle$ is the state vector of matter, normalized to one.

In this context, we can consider a situation [24,25,31] in which a nonrelativistic particle has the same probability 1/2 to be in two disjoint regions of space, far from each other; moreover, let us suppose that the newtonian limit holds, so that Eq. (1.8) becomes

$$\nabla^2 \Phi = 4\pi G m |\psi|^2 , \tag{1.9}$$

where Φ is the gravitational potential, $\psi(\mathbf{x}, t)$ the Schrödinger wave function, and m the mass of the particle. Then, according to Eq. (1.9), the gravitational field should be the one produced by two particles with the same mass $m/2$, placed in the two regions. Performing now a position measurement into one of these regions, the particle will or will not be found , and each of these results will change abruptly the right hand side of Eq. (1.9); this is easily seen to lead either to acausal behaviours (i.e., faster than c propagation) or to drop Eq. (1.9) (and, consequently, Eq. (1.8)). Hence, we must conclude that a semiclassical theory based on Eq. (1.8) is incompatible with an interpretation of quantum mechanics in which the state vector collapses when a measurement is performed. An alternative possibility is to adopt the more solid statistical interpretation, in which $|\psi\rangle$ is not supposed to describe a single system, but rather an ensemble of similarly prepared copies of it. Then the right hand side of Eq. (1.8) must be regarded as an average over such an ensemble, while, on the contrary, the left hand side is referred to a well precise spacetime, so that it seems that Eq. (1.8) does not make any sense at all!

We find this paradox particularly instructive, because it shows clearly the need to achieve physical understanding and insight in the subject, rather than to rely on a purely formal treatment, which can hardly face conceptual problems. Moreover, it provides a good motivation for undertaking a complete revision not only of semiclassical gravity, but of the theory of semiclassical systems in general.

With this purpose in mind, we shall devote our thesis to the establishment of a framework, self-consistent both from the formal[4] *and* the conceptual point of view, which could allow a physically well posed treatment of problems involving semiclassical systems. Our discussion will be mainly performed keeping in mind the concrete example of gravity; however, it is easy to realize that it can be generalized straightforwardly,

[4]We warn here that, although aiming at self-consistency, we do not pretend to be formally rigorous.

and that most of the conclusions hold for an arbitrary semiclassical system as well. In order not to spoil the logical continuity of the treatment, no application is extensively discussed, although some of them are outlined from time to time, and several specific examples are considered whenever they may help the understanding of the general ideas.

In Ch. 2 we shall give a critique of the semiclassical problem for gravity, putting particular emphasis on the importance that a careful choice of an interpretation of quantum theory has in order to correctly formulate the key ideas. In this spirit, Ch. 3 is mainly a long discussion of the reasons to adopt the statistical interpretation; however, it also contains a "kinetic" representation of quantum mechanics, together with an introduction to the quantum phase space distribution functions, which will be used later on; a discourse about the possibility to account for quantum phenomena in terms of a deeper "subquantum" theory is also given. Although the material presented in this chapter, as well as in Chs. 5 and 6, should consist, from the strictly technical point of view, mainly of "tools" to use in semiclassical theories, they contain much more than that, and are structured in such a way that each of them can be regarded as a more or less self-contained presentation of a particular topic in quantum mechanics. This is justified by the importance we attribute to having a clear and precise idea of the content and implications of the latter; in fact, as we have already stressed, most of the foundational troubles arising in this area of research derive from misunderstanding of quantum theory.

In Ch. 4 we shall come back to semiclassical gravity, reconsidering the problem in the light of the results of Ch. 3; we shall find that a "strongly" and a "weakly" semiclassical regimes must be distinguished, depending on the amount of quantum fluctuations. In the weakly semiclassical regime, the concept of coupling equations cannot be applied any more, and it is replaced by a relation between the probability distributions for the classical and the quantum observables, which preserves the semiclassical character of the description by attributing all the uncertainty to the quantum subsystem. Some difficulties of the theory require the introduction, in Chs. 5 and 6, of a functional $P[\gamma]$ defined over the space of the Feynman histories for a quantum system, and which has the meaning of a quasiprobability. In Ch. 5 it is shown how $P[\gamma]$ can be integrated over different classes of histories, to reproduce the usual phase space distributions, thus providing a check for its reliability, while in Ch. 6 we present the relativistic version of $P[\gamma]$. It turns out that the latter requires, to be conveniently defined, a nonstandard formulation of relativistic quantum mechanics, which introduces in the theory a new parameter (classically linked to the proper time), but presents the advantage of being explicitly covariant. In Ch. 7 these techniques are applied to a formulation of a semiclassical theory for relativistic systems, in particular for the gravitational field. Ch. 8 contains some final remarks, and outlines of possible applications and extensions of the ideas presented in the thesis.

From now on, we shall work in units in which $c = 1$, and we shall adopt the signature $+2$ for the metric of spacetime. In tensors, the indices $a, b, c, \ldots$ run from 0 to 3, while $i, j, k, \ldots$ run from 1 to 3; the Minkowski metric is represented by the symbol η_{ab}. The sum over repeated indices is implicit. Quantum operators are distinguished from classical quantities by a "hat", like in $\hat{q}$. Other notations will be defined in due place.

Chapter 2

Critique of Semiclassical Gravity

As we have discussed in Ch. 1, there is no striking evidence, neither of experimental nor of theoretical nature, for the quantization of gravity. This leads to undertaking the less ambitious semiclassical program, in which a classical gravitational field interacts with (and is reacted on by) quantum matter. In such a context, the main difficulty lies in the way to account for back-reaction. The aim of the present chapter is to provide a critical survey of the model which is currently accepted to provide a solution to this problem.

In Sec. 2.1 we discuss qualitatively the issue of back-reaction both for electromagnetism and for gravity, in order to gain physical insight into the topic, while Sec. 2.2 is devoted to a brief review of the formulation of quantum theory in a curved spacetime, which is the basis for the description of matter in semiclassical gravity. In Sec. 2.3 the semiclassical field equations are written, and their structure is discussed. A detailed criticism of their *physical* reliability is performed in Sec. 2.4, leading to conclude that a deep revision of the subject is necessary.

2.1 The Problem of Back-reaction

When a physical system $\mathcal{S}$ can be considered as composed of two subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, its time evolution can be represented as the separate time evolution of $\mathcal{S}_1$ and $\mathcal{S}_2$ under their mutual interaction; in fact, in general, the subsystem $\mathcal{S}_1$ exerts an influence on the behaviour of $\mathcal{S}_2$, and vice versa. Under some circumstances, it may happen that one of these two influences turns out to be practically negligible, although it is, in principle, still present; this leads to a great simplification of the problem, both from the physical and the mathematical point of view, and is usually called the "external field approximation". Quantum theory in curved spacetime is an example of this reduced treatment.

To achieve complete self-consistency by taking into account the reciprocal interaction of both systems may not be easy at all, as it can be realized considering the case of the electrodynamics of pointlike charges. If a particle with mass m and charge e moves in an external electromagnetic field F^{ab}, and we neglect the effects of the field produced by the particle itself, the motion is described simply by the well known Lorentz equation

$$m\,\dot{u}^a = e\,F^{ab}u_b\;, \tag{2.1.1}$$

u^a being the particle's four-velocity and the dot standing for the derivative with respect to the proper time. However, Eq. (2.1.1) is clearly only an approximation, not taking into account the loss of momentum undergone by the particle when accelerating, through its own radiation field. To incorporate this feature in the theory is clearly an important task from the conceptual point of view, since it allows to achieve self-consistency of the treatment, but the corresponding generalization of Eq. (2.1.1) is not trivial. In fact, in order to account for the radiation reaction effects, the Maxwell equations must be considered for the radiation field of the particle, and a careful treatment is required because of the appearance of some divergences which are eliminated by a process of mass renormalization. The result is the Lorentz-Dirac equation [32]

$$m\,\dot{u}^a = e\,F^{ab}u_b + \frac{2}{3}e^2\left(\ddot{u}^a - u^a\dot{u}_b\dot{u}^b\right)\;, \tag{2.1.2}$$

whose solution also requires some care in order to avoid formal and physical troubles.

In the general theory of relativity, the conceptual necessity of accounting for the matter's reaction on gravity is even stronger, and the corresponding problems are more serious. While the Lorentz equation is independent of Maxwell equations, which only require charge conservation, the equation of motion for test particles in a gravitational field (i.e., the geodesic equation) can be shown to be a consequence of

Einstein equation: This fact does not allow, in principle, to neglect the back reaction of matter. Strictly speaking, it is just the two-ways character of the interaction between matter and geometry which determines the matter's motion, even in the extreme case of pointlike particles; in other words: "Space acts on matter, telling it how to move. In turn, matter reacts back on space, telling it how to curve." [2]. This smaller freedom in imposing constraints on the motion of matter can be traced directly to the peculiar structure of Einstein equation (1.4), which has the remarkable property of transforming the *geometrical identity*

$$\nabla^b G_{ab} = 0 \tag{2.1.3}$$

into the *physical equation*

$$\nabla^b T_{ab} = 0 \ , \tag{2.1.4}$$

expressing the laws of conservation for energy and momentum.

The issue of back-reaction changes drastically if one of the two subsystems behaves quantum mechanically. The problems are now already present at the level of writing the coupling equations between $\mathcal{S}_1$ and $\mathcal{S}_2$ (i.e., the field equations, in the case in which one of the two subsystems is matter and the other is a field). To illustrate this point, let us consider the concrete example from atomic physics, in which one wants to describe in a consistent way the energy eigenstates of nonrelativistic quantum electrons in an external electric field. The situation can be conveniently schematized as follows: If the external field is not too strong, the acceleration of the electrons is small, and their radiation reaction can be consequently neglected; moreover, since they are moving slowly, their magnetic field can be neglected as well; therefore, the problem is reduced to that of determining the behaviour of a system of, say, a number Z of nonrelativistic electrons, each affected by the electrostatic field of all the others in addition to the external one. A satisfactory solution is provided by the Hartree method of self-consistent fields [33]; the key idea is to approximate the total wave function as a product of wave functions $\psi_1(\mathbf{x}_1), \ldots, \psi_Z(\mathbf{x}_Z)$ for the single electrons, and to solve simultaneously the Z stationary Schrödinger equations ($n \in \{1, \ldots, Z\}$

labels the electrons)

$$-\frac{\hbar^2}{2m}\nabla^2\psi_n + (V_n + W)\psi_n = E_n\psi_n \ , \tag{2.1.5}$$

where W is the potential energy due to the external field, together with the Z Poisson equations

$$\nabla^2 V_n = -4\pi e^2 \sum_{m\neq n} |\psi_m|^2 \tag{2.1.6}$$

for the "mean potential energy" V_n due to the other electrons[1]. It is interesting to rewrite Eq. (2.1.6) as[2]

$$V_n = e\sum_{m\neq n}\varphi_m \ , \tag{2.1.7}$$

where φ_m satisfies the equation

$$\nabla^2\varphi_m = -4\pi e|\psi_m|^2 \ . \tag{2.1.8}$$

To Eq. (2.1.8) can be ascribed a particular meaning: We can regard it as the field equation linking the *classical* electric potential φ_m (or, if we prefer, the *classical* electric field $\mathbf{E}_m = -\boldsymbol{\nabla}\varphi_m$), to its *quantum* source, which is represented by the charge density $e|\psi_m|^2$. Eq. (2.1.8) is thus a first example of semiclassical field equation.

The source $e|\psi_m|^2$ in Eq. (2.1.8) can be rewritten in an alternative way which suggests a remarkable generalization. Let us introduce, in the Heisenberg picture, the charge density operator[3]

$$\hat{\rho}_e(\mathrm{x},t) \equiv e\,\delta^3(\mathrm{x}\hat{1} - \hat{\mathrm{x}}(t)) \ , \tag{2.1.9}$$

where $\hat{\mathrm{x}}(t)$ is the position operator at time t, whose eigenstates (in the rigged Hilbert space) we label as $|\mathrm{x},t\rangle$:

$$\hat{\mathrm{x}}(t)|\mathrm{x},t\rangle = \mathrm{x}|\mathrm{x},t\rangle \ . \tag{2.1.10}$$

[1]In atomic physics, one may average the $|\psi_m|^2$ over angles in order to retain spherical symmetry. We do not enter here into these technical details.

[2]Here, e is the electron charge *with sign*, so that $e < 0$.

[3]Our sloppy notation could be avoided working in terms of sequences of well defined functions of the operator $\hat{\mathrm{x}}(t)$; however, there would be no significant change in the treatment, and the physics would be less clear.

The choice (2.1.9) can be motivated by the fact that, applying $\hat{\rho}_e(\mathbf{x},t)$ so defined to the eigenstate $|\mathbf{y},t\rangle$, we get

$$\hat{\rho}_e(\mathbf{x},t)|\mathbf{y},t\rangle = e\,\delta^3(\mathbf{x}-\mathbf{y})|\mathbf{y},t\rangle \ , \qquad (2.1.11)$$

so that $|\mathbf{y},t\rangle$ is an eigenstate of charge density as well, corresponding to the eigenvalue $e\,\delta^3(\mathbf{x}-\mathbf{y})$; this is in agreement with the idea that the state $|\mathbf{y},t\rangle$ should approximate the classical concept of pointlike particle located at $\mathbf{y}$ at time t. The unpleasant occurrence of the delta function of operators in Eq. (2.1.9) can be easily eliminated using the identity

$$\int d^3y\,|\mathbf{y},t\rangle\langle\mathbf{y},t| = \hat{1} \ , \qquad (2.1.12)$$

which trivially leads to

$$\hat{\rho}_e(\mathbf{x},t) = e|\mathbf{x},t\rangle\langle\mathbf{x},t| \ . \qquad (2.1.13)$$

Now, let us calculate the expectation value of charge density when the particle is in the state $|\psi\rangle$; the representation (2.1.13) gives immediately

$$\langle\psi|\hat{\rho}_e(\mathbf{x},t)|\psi\rangle = e|\psi(\mathbf{x},t)|^2 \ , \qquad (2.1.14)$$

where

$$\psi(\mathbf{x},t) \equiv \langle\mathbf{x},t|\psi\rangle \qquad (2.1.15)$$

is the Schrödinger wave function. This suggests to generalize Eq. (2.1.8) as

$$\nabla^2\varphi = -4\pi\langle\psi|\hat{\rho}_e|\psi\rangle \ . \qquad (2.1.16)$$

It is possible to go even further, remembering that the density of current is defined, classically, as

$$\mathbf{j}_e = \rho_e\,\mathbf{v} \ ; \qquad (2.1.17)$$

the corresponding quantum mechanical operator is[4] (App. A)

$$\hat{\mathbf{j}}_e(\mathbf{x},t) = \mathcal{W}\left\{\hat{\rho}_e(\mathbf{x},t)\,\frac{1}{m}\,(\hat{\mathbf{p}}(t) - e\mathbf{A}(\hat{\mathbf{x}}(t),t))\right\} =$$
$$= \frac{e}{2m}(|\mathbf{x},t\rangle\langle\mathbf{x},t|\hat{\mathbf{p}}(t) + \hat{\mathbf{p}}(t)|\mathbf{x},t\rangle\langle\mathbf{x},t|) - \frac{e^2}{m}\mathbf{A}(\mathbf{x},t)|\mathbf{x},t\rangle\langle\mathbf{x},t|, \qquad (2.1.18)$$

[4]In Eq. (2.1.18) we write the ordering as $\mathcal{W}$; however, it is trivial to check that, in this case, the use of $\mathcal{S}$ would not make any difference, so that there is no ambiguity.

where $\mathbf{A}$ is the vector potential of the electromagnetic field; the expression (2.1.18) has the expectation value

$$\langle\psi|\hat{\mathbf{j}}_e(\mathbf{x},t)|\psi\rangle = \frac{e\hbar}{2mi}\,\psi(\mathbf{x},t)^*\overleftrightarrow{\nabla}\,\psi(\mathbf{x},t) - \frac{e^2}{m}\,\mathbf{A}(\mathbf{x},t)|\psi(\mathbf{x},t)|^2\;, \quad (2.1.19)$$

where use has been made of the Schrödinger representation of the operator $\hat{\mathbf{p}}(t)$:

$$\langle\mathbf{x},t|\hat{\mathbf{p}}(t)|\psi\rangle = -i\hbar\nabla\psi(\mathbf{x},t)\;. \quad (2.1.20)$$

The semiclassical Maxwell equations can thus be postulated as:

$$\nabla\cdot\mathbf{E} = 4\pi e|\psi|^2\;, \quad (2.1.21)$$

$$\nabla\times\mathbf{B} = 4\pi\frac{e\hbar}{2mi}\,\psi^*\overleftrightarrow{\nabla}\,\psi - 4\pi\frac{e^2}{m}\,\mathbf{A}|\psi|^2 + \frac{\partial\mathbf{E}}{\partial t}\;, \quad (2.1.22)$$

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t}\;, \quad (2.1.23)$$

$$\mathbf{B} = \nabla\times\mathbf{A}\;. \quad (2.1.24)$$

Of course, the system of Eqs. (2.1.21)–(2.1.24) requires to be completed with the Schrödinger equation for $\psi(\mathbf{x},t)$,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\nabla - \frac{ie}{\hbar}\mathbf{A}\right)^2\psi + e\phi\psi\;. \quad (2.1.25)$$

The covariance of Maxwell equations allows to perform a generalization of Eqs. (2.1.21)–(2.1.25) to the case of a relativistic matter source. Although the expressions (2.1.14) and (2.1.19) are not covariant, they can be respectively regarded as the nonrelativistic limits of the time and space components of the quantity $\langle\psi|\hat{j}_e^a|\psi\rangle$. The operator[5] $\hat{j}_e^a$ is now supposed to be the Noether current associated to the $U(1)$ gauge invariance of some lagrangian density $\mathcal{L}$ describing a quantum field. The pairs (2.1.21)–(2.1.22) and (2.1.23)–(2.1.24) of Maxwell equations become thus, in four-dimensional notation,

$$\partial_b F^{ab} = 4\pi\langle\psi|\hat{j}_e^a|\psi\rangle\;, \quad (2.1.26)$$

$$F^{ab} = \partial^a A^b - \partial^b A^a\;, \quad (2.1.27)$$

[5]The index e is not tensorial! It only reminds that j_e^a is the *electric* current.

A^a being now the four-potential. The Schrödinger equation (2.1.25) for the wave function ψ is replaced by the Heisenberg equation for the operator $\hat{j}^a_e$,

$$i\hbar\frac{d\hat{j}^a_e(\mathbf{x},t)}{dt} = i\hbar\frac{\partial\hat{j}^a_e(\mathbf{x},t)}{\partial t} - [\hat{H}(t), \hat{j}^a_e(\mathbf{x},t)] \ , \qquad (2.1.28)$$

where $\hat{H}(t)$ is the hamiltonian operator of the field at time t, obtained by standard methods of canonical quantization [4,7].

Eqs. (2.1.26)–(2.1.28) form the basis of the usual semiclassical theory of electromagnetism in Minkowski spacetime. They also provide a strong motivation for writing the semiclassical field equation for gravity as in (1.8). It is clear, however, that Eq. (1.8) alone is not sufficient to define completely the problem, but it needs to be coupled with an equation for the quantum evolution of the operator $\hat{T}_{ab}$, of the kind of Eq. (2.1.28). Unfortunately, in the case of gravity, the spacetime structure is not given in advance, and it may turn out to be not trivial at all; consequently, such mathematical objects as the time t, or the hamiltonian operator $\hat{H}(t)$ of Eq. (2.1.28), are not defined, and a trivial transposition of quantum theory from Minkowski spacetime to a generic lorentzian manifold cannot be performed. It is precisely to the problem of describing quantum matter in a curved spacetime, which appears so important for a correct study of semiclassical gravity, that the next section is devoted.

2.2 Quantum Theory in Curved Spacetime

The mathematical framework of quantum mechanics deals with the description of the concepts of *state* and *observable*. In the usual formulation of the theory in Minkowski spacetime [35,36], the *state* at a fixed time can be associated (up to a phase factor) to a unit vector[6] of a Hilbert space $\mathcal{H}$, while an *observable* is represented by a linear self-adjoint operator in $\mathcal{H}$. Another concept, introduced at this level as

[6]Here, we limit ourselves to the case of *pure states*, and we do not enter in the details concerning non-normalizable states, rigged Hilbert space, and so on; all these topics are thoroughly discussed in refs. [35,36].

primitive, is that of *measurement* of an observable on a given state; it is a postulate of the theory that if an observable described by the operator $\hat{A}$ is measured on the state $|\psi\rangle \in \mathcal{H}$, the only possible results of the measurement are the eigenvalues $\{a_n\}$ of $\hat{A}$; moreover, the probability that a_n be the outcome of the measurement is

$$P(a_n) = \sum_r |\langle a_n, r|\psi\rangle|^2 , \tag{2.2.1}$$

where r represents a possible degeneracy index, and $\{|a_n, r\rangle\}$ are orthonormalized eigenvectors of $\hat{A}$.

The results of the measurements of an observable change as time passes; this can be expressed by saying that the probabilities $P(a_n)$ are actually functions of time, which we shall write as[7] $P(a_n|t)$. By Eq. (2.2.1) it is evident that either $|\psi\rangle$, or $|a_n, r\rangle$ (i.e., $\hat{A}$), or both of them must be function of time, too; the dynamical problem in quantum theory is therefore to determine this dependence. In the Schrödinger picture, the time evolution is all encoded in the state vector; it is easy to realize that if t_0 and t are two instants of time, $|\psi(t_0)\rangle$ and $|\psi(t)\rangle$ must be linked by a unitary transformation in order to preserve the normalization. More precisely,

$$|\psi(t)\rangle = \hat{U}(t, t_0)|\psi(t_0)\rangle , \tag{2.2.2}$$

where $\hat{U}(t, t_0)$ is a unitary linear operator in $\mathcal{H}$ such that $\hat{U}(t,t) = \hat{1}$. However, since anything can be measured experimentally is only the statistical frequency $P(a_n|t)$ of a result, Eq. (2.2.1) suggests, as an alternative, to consider the state fixed, allowing only the observables to evolve: This is the Heisenberg picture, in which it is $\hat{A}(t)$ to be linked to $\hat{A}(t_0)$ by

$$\hat{A}(t) = \hat{U}(t, t_0)^\dagger \hat{A}(t_0)\hat{U}(t, t_0) . \tag{2.2.3}$$

Eqs. (2.2.2) and (2.2.3) can be rewritten in the more common differential formalism observing that the unitarity of $\hat{U}(t, t_0)$ allows to represent it as

$$\hat{U}(t, t_0) = \exp(i\hat{\Omega}(t, t_0)) , \tag{2.2.4}$$

[7] We adopt this unusual notation in order to stress that $P(a_n|t)$ is a *conditional* probability [34].

with $\hat{\Omega}(t,t_0)$ a self-adjoint linear operator in $\mathcal{H}$ such that $\hat{\Omega}(t,t) = \hat{0}$. Defining the *hamiltonian* as

$$\hat{H}(t) \equiv -\hbar \left.\frac{d}{dt'}\right|_{t'=t} \hat{\Omega}(t',t) , \tag{2.2.5}$$

Eqs. (2.2.2) and (2.2.3) become formal solutions of the differential equations

$$i\hbar \frac{d|\psi(t)\rangle}{dt} = \hat{H}(t)|\psi(t)\rangle \tag{2.2.6}$$

and

$$i\hbar \frac{d\hat{A}(t)}{dt} = -[\hat{H}(t), \hat{A}(t)] , \tag{2.2.7}$$

holding, respectively, in the Schrödinger and in the Heisenberg pictures. In particular, Eq. (2.1.28) is a case of (2.2.7).

There is now clear evidence for the main problem of compatibility between quantum theory and relativity: The general formalism of the former gives to the concept of time a very peculiar and fundamental role, thus entering in conflict with the relativistic requirement of manifest covariance. There are, apparently, two possible ways to solve this problem. One of them relies on the idea that the spacetime coordinates must appear all on the same footing[8] in the fundamental equations; this has been, historically, the basis of the first attempts to formulate a relativistic quantum theory, but it has led to serious conceptual difficulties [7]. Alternatively, it is possible to think that the time t in the dynamical equations is referred to a particular reference frame, that is, to a congruence [38] of future directed timelike curves with a normalized tangent vector, each of those representing a classical observer: Relativistic quantum mechanics would thus involve the concept of observer in its very basic formulation. This last approach is the one currently adopted in studies of quantum mechanics on a fixed background spacetime [4,39], and it has allowed to achieve remarkable results. We shall therefore present it in more detail.

First of all, we must ask for the class of spacetimes which allow the introduction of a global notion of time; the answer is straightforwardly

[8]Apart from obvious differences due to the metric of spacetime.

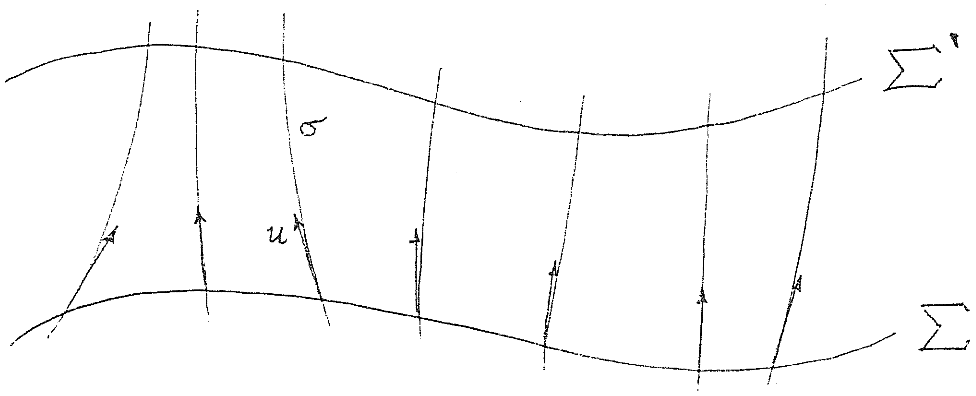

Figure 2.1: The development of the Cauchy hypersurface Σ along the reference frame u^a.

given in the literature [1,3], and indicates that we have to restrict ourselves to a globally hyperbolic spacetime (M, g), which admits Cauchy hypersurfaces. Let Σ_0 be such an hypersurface, with future directed normal vector u^a such that

$$g_{ab}u^a u^b = -1 \; . \tag{2.2.8}$$

Let us choose a congruence of observers $\sigma : I \longrightarrow M$, where I is an open interval of $\mathbb{R}$, whose tangent vector on Σ_0 coincides with u^a (Fig. (2.1)), and such that the reference frame so constructed is proper time synchronizable [40] (the existence of such frames follows from the global hyperbolicity of (M, g), which guarantees the existence of a global function τ such that $u^a = -\nabla^a \tau$ is a future directed timelike vector field). Let us choose, moreover, the origin of the proper time τ of these observers so that $\sigma(\tau_0) \in \Sigma_0$, $\forall \sigma$; at a time $\tau \neq \tau_0$, Σ_0 will be evolved, along the curves σ, into another Cauchy hypersurface $\Sigma(\tau) = \{\sigma(\tau)\}$, orthogonal to the observers and hence representing their transverse space at the proper time τ.

Now it is possible to construct a quantum theory "adapted" to the observers σ, in which states and observables give a description on one hypersurface of the foliation $\{\Sigma(\tau)\}$, and the dynamics is described by a unitary operator $\hat{U}_u(\tau, \tau_0)$ such that $\hat{U}_u(\tau, \tau) = \hat{1}$ and

$$\frac{d\hat{U}_u(\tau, \tau_0)}{d\tau} = -\frac{i}{\hbar}\, \hat{H}_u(\tau)\, \hat{U}_u(\tau, \tau_0) \; , \tag{2.2.9}$$

according to Eqs. (2.2.4) and (2.2.5), where $\hat{H}_u(\tau)$ is the hamiltonian of the system on the hypersurface $\Sigma(\tau)$. It is easy to realize that the analogue of Eqs. (2.2.6) and (2.2.7) are, respectively,

$$i\hbar \frac{d|\psi(\tau)\rangle_u}{d\tau} = \hat{H}_u(\tau)|\psi(\tau)\rangle_u \ , \tag{2.2.10}$$

and

$$i\hbar \frac{d\hat{A}_u(\tau)}{d\tau} = -[\hat{H}_u(\tau), \hat{A}_u(\tau)] \ , \tag{2.2.11}$$

where Eq. (2.2.10) holds in the Schrödinger picture (hence with $\hat{A}_u(\tau) = \hat{A}_u(\tau_0)$), while Eq. (2.2.11) holds in the Heisenberg picture (in which it is $|\psi(\tau)\rangle_u = |\psi(\tau_0)\rangle_u$).

A particular case occurs when the system under study is a quantum field, and the operators $\hat{A}$ are therefore functions of the spacetime point $x \in M$. In this circumstance, the derivative $d/d\tau$ must be replaced by the Lie derivative along u^a, L_u [32,38]. If, in the Schrödinger picture, $L_u \hat{A}^S_u(x) = \hat{0}$, then Eq. (2.2.11) becomes

$$i\hbar \, L_u \hat{A}^H_u(x) = -[\hat{H}_u(\tau(x)), \hat{A}^H_u(x)] \ , \tag{2.2.12}$$

where $\hat{A}^H_u(x)$ is the operator in the Heisenberg picture:

$$\hat{A}^H_u(x) = \hat{U}(\tau(x), \tau_0)^\dagger \hat{A}^S_u(x) \hat{U}(\tau(x), \tau_0) \ . \tag{2.2.13}$$

This method of extending the formalism of quantum theory to the case of a more general reference frame in curved spacetime is quite straightforward and naive, but has led to important theoretical results [4,39]. Nevertheless, it appears somewhat artificial and not sufficiently general, since it relies on some hypothesis, such as the global hyperbolicity, which are not necessarily satisfied in a generic spacetime (and whose validity is very difficult to test in our own universe!). Moreover, from the point of view of semiclassical gravity, these properties could only be verified *a posteriori*, after the problem of finding a spacetime compatible with its quantum matter content has been solved; hence, we are not allowed, in principle, to formulate a semiclassical problem making use of this formalism, unless we restrict our treatment to very specific cases.

There is another possible extension of quantum mechanics to the relativistic domain, which has both the advantages of being explicitly covariant, and of not requiring the spacetime to satisfy too peculiar conditions. In this theory, a distinction is introduced between the *coordinate time* and the *evolutionary time*: While the former is now treated as an observable, just as the space coordinates are, the latter is a parameter, which turns out to be proportional, in the classical limit, to the particle's proper time. Although it appears particularly suitable for the treatment of matter in semiclassical gravity, practically no attention has been paid to this theory within this context. Since our aim, in this chapter, is only to critically review the usually accepted semiclassical equations, we shall postpone a thorough treatment of the subject to Ch. 6, where it will be developed in order to provide a reasonable description of quantum matter in our semiclassical theory.

2.3 The Semiclassical Field Equations

According to the discussion of the previous sections, the usual formulation of the semiclassical problem for gravity can be summarized by the equations [29,41,42]:

$$G_{ab}(x) = \kappa\langle\psi(\tau)|\hat{T}_{ab}(x)|\psi(\tau)\rangle \; ; \tag{2.3.1}$$

$$i\hbar\frac{d|\psi(\tau)\rangle}{d\tau} = \hat{H}(\tau)|\psi(\tau)\rangle \; ; \tag{2.3.2}$$

$$\hat{H}(\tau) = \int\limits_{\Sigma(\tau)} d\Sigma(x)\,\hat{T}_{ab}(x)u^a(x)u^b(x) \; . \tag{2.3.3}$$

In the set of Eqs. (2.3.1)–(2.3.3), the Schrödinger picture has been adopted, and the quantum theory is formulated with respect to the reference frame[9] u^a, as discussed in general in Sec. 2.2; in Eq. (2.3.3), $d\Sigma$ represents the volume element on the hypersurface $\Sigma(\tau)$.

It is important to remark that Eqs. (2.3.1)–(2.3.3) are not complete, since they must be supplemented by a quantum field theory which could express $\hat{T}_{ab}(x)$ in terms of more fundamental field operators acting on the states $|\psi(\tau)\rangle$. This is, however, a completely different subject, so we

[9]To relieve the notations, we have dropped the index u in the states and in the operators.

shall not enter in it , and we shall rather suppose that such a theory is given, without any further problem.

Eq. (2.3.1) seems well motivated, both by the analogy with electromagnetism given in Sec. 2.1, and by the fact that it satisfies the correspondence principle. In fact, it is easy to understand that, when dealing with classically behaving systems, the statistical dispersion of the stress-energy-momentum tensor around the average value $\langle\psi|\hat{T}_{ab}|\psi\rangle$ becomes negligible, and Eq. (2.3.1) reduces to the usual Einstein equation (1.4). Despite these nice features, however, the semiclassical theory based on Eq. (2.3.1) presents some serious conceptual and technical flaws. We shall comment here only about a couple of points, since a detailed criticism of the theory will be performed in the next section.

First of all, it must be pointed out that a straightforward calculation of $\langle\psi|\hat{T}_{ab}|\psi\rangle$ for quantum matter fields would lead, in general, to a divergent expression; the right hand side of Eq. (2.3.1) is therefore the result of complex regularization techniques [4]. This aspect of the problem is complicated further if the semiclassical theory is not regarded as exact, but only as an approximation to quantum gravity, along the line of thought which we have expressed in Ch. 1. In this case the one-loop corrections due to gravitons would produce effects comparable to those of matter fields, and they should be considered as quantum source of classical gravity, too.

Another worrying consequence of Eqs. (2.3.1)–(2.3.3) lies in the fact that they would lead to some strong superselection rule. Their solution, in fact, consists of a spacetime (M,g) *and* a state vector $|\psi(\tau)\rangle$ describing the state of quantum matter with respect to the reference frame u^a; since, in general, different state vectors are compatible with different spacetimes, it follows that the superposition principle is violated [29,30] if semiclassical gravity is described by Eq. (2.3.1). More pragmatically, we can remark that, being the Hilbert space formulation of quantum theory strongly dependent on the global properties of M, as seen in Sec. 2.2, not only does $\hat{T}_{ab}$ contain explicitly (as usual in general relativity) the metric tensor g_{ab}, but also $|\psi(\tau)\rangle$ can be completely characterized only once the entire spacetime structure is known; hence, to find a solution of Eqs. (2.3.1)–(2.3.3) is not an easy task at all, because the theory turns out to be highly nonlinear.

An interesting analysis, which we shall pursue in some detail, concerns the physical structure of the source term in Eq. (2.3.1); this can be conveniently studied by exploring the nonrelativistic limit in flat space-time for the case of a single particle of mass m. The classical momentum is, in this case,

$$p_a \approx m\,\delta_0^a + p_i \delta_a^i \,, \tag{2.3.4}$$

and the classical stress-energy-momentum tensor is, consequently,

$$\mu\,\delta_a^0 \delta_b^0 = \frac{1}{m}\,\mu\, p_i(\delta_a^i \delta_b^0 + \delta_a^0 \delta_b^i) + \frac{1}{m^2}\,\mu\, p_i p_j\, \delta_a^i \delta_b^j \,, \tag{2.3.5}$$

μ being the density of mass, which reads

$$\mu\,(\mathrm{x}, t) = m\,\delta^3(\mathrm{x} - \mathrm{x}(t)) \,, \tag{2.3.6}$$

where $\mathrm{x}(t)$ is the particle's position at time t. The tensor T_{ab} splits thus, in the nonrelativistic limit, into three different objects, namely, the mass density

$$\mu\,(\mathrm{x}, t) \approx T_{00}(x) \,, \tag{2.3.7}$$

the mass current vector

$$j_i(\mathrm{x}, t) = \frac{1}{m}\mu\,(\mathrm{x}, t)\, p_i(t) \approx T_{0i}(x) = T_{i0}(x) \,, \tag{2.3.8}$$

and the stress tensor

$$\Pi_{ij}(\mathrm{x}, t) = \frac{1}{m^2}\mu\,(\mathrm{x}, t)\, p_i(t) p_j(t) \approx T_{ij}(x) \,; \tag{2.3.9}$$

we shall now associate to each of these classical observables a quantum operator.

For the mass density, we have

$$\hat{\mu}\,(\mathrm{x}, t) = m\,\delta^3(\mathrm{x}\hat{1} - \hat{\mathrm{x}}(t)) \,, \tag{2.3.10}$$

which is trivially linked to the charge density operator previously defined in Eq. (2.1.9) by

$$\hat{\mu}\,(\mathrm{x}, t) = \frac{m}{e}\,\hat{\rho}_e(\mathrm{x}, t) \,. \tag{2.3.11}$$

This allows to write immediately

$$\langle \psi | \hat{T}_{00}(x) | \psi \rangle \approx \langle \psi | \hat{\mu}\,(\mathrm{x}, t) | \psi \rangle = m |\psi(\mathrm{x}, t)|^2 \,, \tag{2.3.12}$$

where the Heisenberg picture has been used in the calculation of the expectation value.

Similarly, for the mass current,

$$\hat{\mathbf{j}}(\mathbf{x},t) = \frac{m}{e}\hat{\mathbf{j}}_e(\mathbf{x},t) = \frac{1}{2}(|\mathbf{x},t\rangle\langle\mathbf{x},t|\hat{\mathbf{p}}(t) + \hat{\mathbf{p}}(t)|\mathbf{x},t\rangle\langle\mathbf{x},t|) \; , \tag{2.3.13}$$

where obviously the case $\mathbf{A}(\mathbf{x},t) = \mathbf{0}$ has been considered. Thus

$$\langle\psi|\hat{T}_{oi}(x)|\psi\rangle \approx \langle\psi|\hat{\mathbf{j}}(\mathbf{x},t)|\psi\rangle = \frac{\hbar}{2i}\,\psi(\mathbf{x},t)^* \overleftrightarrow{\boldsymbol{\nabla}}\,\psi(\mathbf{x},t) \; . \tag{2.3.14}$$

It is very interesting to notice that, writing

$$\psi(\mathbf{x},t) = |\psi(\mathbf{x},t)| \;\exp\left(\frac{i}{\hbar}S(\mathbf{x},t)\right) \; , \tag{2.3.15}$$

Eq. (2.3.14) becomes

$$\langle\psi|\hat{\mathbf{j}}(\mathbf{x},t)|\psi\rangle = \langle\psi|\hat{\mu}\,(\mathbf{x},t)|\psi\rangle\,\frac{1}{m}\,\boldsymbol{\nabla}S(\mathbf{x},t) \; , \tag{2.3.16}$$

which, defining

$$\mathbf{v}(\mathbf{x},t) \equiv \frac{1}{m}\,\boldsymbol{\nabla}S(\mathbf{x},t) \; , \tag{2.3.17}$$

takes the form

$$\langle\psi|\hat{\mathbf{j}}(\mathbf{x},t)|\psi\rangle = \langle\psi|\hat{\mu}\,(\mathbf{x},t)|\psi\rangle\,\mathbf{v}(\mathbf{x},t) \; . \tag{2.3.18}$$

It is easy to check that, in virtue of the Schrödinger equation, $\langle\psi|\hat{\mu}|\psi\rangle$ and $\mathbf{v}$ satisfy a continuity equation

$$\frac{\partial\langle\psi|\hat{\mu}|\psi\rangle}{\partial t} + \boldsymbol{\nabla}\cdot(\langle\psi|\hat{\mu}|\psi\rangle\,\mathbf{v}) = 0 \; , \tag{2.3.19}$$

which corresponds to

$$\partial^a\langle\psi|\hat{T}_{0a}(x)|\psi\rangle = 0 \; . \tag{2.3.20}$$

As discussed in detail in App. B, the association of an operator $\hat{\Pi}_{ij}(\mathbf{x},t)$ to the classical stress tensor (2.3.9) is not unique, but produces two different objects $\hat{\Pi}^{W}_{ij}$ and $\hat{\Pi}^{S}_{ij}$ depending on the use of the generalized Weyl's or Rivier's ordering rules, Eqs. (A.19) and (A.16). From Eqs. (B.14) and (B.15) we have

$$\langle\psi|\hat{T}^{W}_{ij}(x)|\psi\rangle \approx \langle\psi|\hat{\Pi}^{W}_{ij}(\mathbf{x},t)|\psi\rangle = p_{ij}(\mathbf{x},t) + \langle\psi|\hat{\mu}\,(\mathbf{x},t)|\psi\rangle\,v_i(\mathbf{x},t)v_j(\mathbf{x},t) \; , \tag{2.3.21}$$

and

$$\langle\psi|\hat{T}^{s}_{ij}(x)|\psi\rangle \approx \langle\psi|\hat{\Pi}^{s}_{ij}(\mathbf{x},t)|\psi\rangle = \langle\psi|\hat{\Pi}^{\mathcal{W}}_{ij}(\mathbf{x},t)|\psi\rangle - \frac{\hbar^2}{4m^2}\partial_i\partial_j\langle\psi|\hat{\mu}\,(\mathbf{x},t)|\psi\rangle, \tag{2.3.22}$$

where

$$p_{ij}(\mathbf{x},t) \equiv -\frac{\hbar^2}{4m^2}\,\langle\psi|\hat{\mu}\,(\mathbf{x},t)|\psi\rangle\,\partial_i\partial_j\ln\langle\psi|\hat{\mu}\,(\mathbf{x},t)|\psi\rangle\ . \tag{2.3.23}$$

Now, the Schrödinger equation implies that

$$\langle\psi|\hat{\mu}|\psi\rangle\frac{Dv_i}{dt} = -\partial_j p_{ij}\ , \tag{2.3.24}$$

where

$$\frac{D}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\ . \tag{2.3.25}$$

Eq. (2.3.24) is, formally, the Euler equation for a fluid with density $\langle\psi|\hat{\mu}|\psi\rangle$, velocity $\mathbf{v}$, and pressure tensor p_{ij}; it corresponds to the equation

$$\partial^a\langle\psi|\hat{T}^{\mathcal{W}}_{ia}(x)|\psi\rangle = 0 \tag{2.3.26}$$

for the expectation value of $\hat{T}_{ab}$. We shall recover Eqs. (2.3.19), (2.3.23) and (2.3.24) within a different context in the next chapter, where they will play a crucial role in orienting us toward the statistical interpretation of quantum theory. Here, we attach to them only a *formal* meaning, to draw the conclusion that, in the nonrelativistic limit, the right hand side of Eq. (2.3.1) has the structure

$$\begin{pmatrix} \langle\psi|\hat{\mu}|\psi\rangle & \langle\psi|\hat{\mu}|\psi\rangle v_i \\ \langle\psi|\hat{\mu}|\psi\rangle v_i & p_{ij} + \langle\psi|\hat{\mu}|\psi\rangle v_i v_j \end{pmatrix}\ . \tag{2.3.27}$$

Hence, the semiclassical theory based on Eq. (2.3.1) assumes that, for what concerns its action as source of gravity, a nonrelativistic quantum particle acts as a fluid whose density, velocity and pressure are given, respectively, by Eqs. (2.3.12), (2.3.17) and (2.3.23). We must remark, however, that the three kinds of terms present in the matrix (2.3.27) are of different order in the velocity v; consequently, their contributions as sources of gravity will be comparable only for a relativistic particle, whose treatment is beyond the approximations performed here, and would thus require a more detailed analysis.

There is still one point which has to be discussed before closing this section: The form (2.3.27) for the expectation value of the stress energy momentum tensor operator clearly depends on the choice of $\mathcal{W}$ as ordering rule for the product of noncommuting operators. Having we chosen the rule $\mathcal{S}$, we would have found again the structure (2.3.27) for $\langle\psi|\hat{T}_{ab}|\psi\rangle$, but with the replacement

$$p_{ij} \longrightarrow p_{ij} - \frac{\hbar^2}{4m^2}\partial_i\partial_j\langle\psi|\hat{\mu}|\psi\rangle \; . \tag{2.3.28}$$

However, it is easy to realize that, while the Weyl-ordered stress energy momentum tensor operator satisfies the equation

$$\partial^b\langle\psi|\hat{T}^{\mathcal{W}}_{ab}|\psi\rangle = 0 \; , \tag{2.3.29}$$

thanks to Eqs. (2.3.20) and (2.3.26), this is not true for the Rivier-ordered one; in fact, being

$$\langle\psi|\hat{T}^{\mathcal{S}}_{ab}|\psi\rangle = \langle\psi|\hat{T}^{\mathcal{W}}_{ab}|\psi\rangle - \frac{\hbar^2}{4m^2}\delta^i_a\delta^j_b\,\partial_i\partial_j\langle\psi|\hat{\mu}|\psi\rangle \; , \tag{2.3.30}$$

one has, as a consequence of Eq. (2.3.29),

$$\partial^b\langle\psi|\hat{T}^{\mathcal{S}}_{ab}|\psi\rangle = -\frac{\hbar^2}{4m^2}\delta^i_a\partial_i\nabla^2\langle\psi|\hat{\mu}|\psi\rangle \; , \tag{2.3.31}$$

not vanishing unless $\langle\psi|\hat{\mu}\,(\mathbf{x},t)|\psi\rangle = \text{const.}$, which is not true in general. The symmetrization rule does not produce, therefore, a right hand side of Eq. (2.3.1) which is consistent with the identity (2.1.3), and has to be discarded in a theory based on such field equation. We shall see in Ch. 3 that there are other reasons, independent from the context of semiclassical gravity although related to the discussion performed in this section, for preferring the Weyl's ordering to the Rivier's one.

2.4 Criticism of the Semiclassical Field Equations

The analysis of Eq. (2.3.1), which we have started in the previous section, could be pursued further, generalizing it to the fully relativistic case and exploring the mathematical structure of the problem so defined [30].

Before carrying on such a sophisticated program, however, we believe that one should be sure that it is *physically* well posed, i.e., that it does not lead to conceptual inconsistencies [31]. For this reason, we shall devote this section to the physical understanding of Eqs. (2.3.1)–(2.3.3).

As mentioned in Sec. 2.3, a technical difficulty of this set of equations is nonlinearity. Usually, the source of this feature is identified in the structure of the Einstein tensor $G_{ab}[g]$, but we want to stress here again that the nonlinearity we are now referring to, has another, more subtle origin, which has to be ascribed to the fact that, in the field equation (2.3.1), the particle is treated as an extended object. This point can be easily understood remembering that the analysis previously performed on the source term $\langle\psi|\hat{T}_{ab}|\psi\rangle$ has shown that, in the nonrelativistic limit, it corresponds to the stress-energy-momentum tensor of a fluid. The subject can be investigated further on, writing Eqs. (2.3.1)–(2.3.3) in the "newtonian" limit, in which the metric is written as

$$g = -(1+2\Phi)dt^2 + (1-2\Phi)d\mathrm{x}^2 \ , \tag{2.4.1}$$

where $\Phi(\mathrm{x}, t)$ is the gravitational potential (assumed to be slowly varying in t), and terms of order Φ^2 are neglected. The condition

$$g^{ab} p_a p_b = -m^2 \tag{2.4.2}$$

leads now to

$$p_a = \left(m + \frac{\mathrm{p}^2}{2m} + m\Phi \right) \delta^0_a + p_i \delta^i_a \ , \tag{2.4.3}$$

which gives[10]

$$\hat{H}(t) = m\hat{1} + \frac{\hat{\mathrm{p}}(t)^2}{2m} + m\Phi(\hat{\mathrm{x}}(t), t) \ . \tag{2.4.4}$$

The semiclassical problem reduces thus to the two equations

$$\nabla^2\Phi = 4\pi G m |\psi|^2 \ , \tag{2.4.5}$$

and[11]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + m\Phi[\psi]\psi \ . \tag{2.4.6}$$

[10]We perform the quantization with respect to the reference frame δ^a_0.

[11]The term $m\psi$ contributes to ψ only by a global phase, and can thus be neglected.

Eq. (2.4.6) is an unusual version of the Schrödinger equation, because it contains the nonlinear term $m\Phi[\psi]\psi$ representing self-interaction of the particle. The origin of this "correction" can be understood thinking to the ψ-wave as propagating in afield created by the particle through Eq. (2.4.5): The potential energy turns out to be a functional of the wave function which, in turn, acts on it. This behaviour represents the remnant, at the nonrelativistic level, of the property of the wave function to propagate on a spacetime whose metric is determined by the wave function itself, as in Eqs. (2.3.1) and (2.3.2). More precisely, since $\hat{H}$ depends on the spacetime metric g_{ab} which, by Eq. (2.3.1), is a functional of $|\psi\rangle$, it follows that $\hat{H}$ will be a functional of $|\psi\rangle$, too, thus leading to a right hand side of Eq. (2.3.2) which is nonlinear in $|\psi\rangle$. The extreme case we have considered in Eqs. (2.4.5) and (2.4.6) shows explicitly that this nonlinearity is increased by, but not due to, the intrinsic nonlinearity of Einstein equation.

If the self-interaction of quantum matter, mediated by the classical gravitational field, would be present in nature at a fundamental level, it would lead to nonlinear corrections to quantum mechanics, which would drastically change the structure of the theory. Theoretical investigations and speculations on this subject are presently performed in different contexts [43,44,45,46,47], but nevertheless, both experiments [48,49] and theory [50,51,52] seems to indicate that, at the moment, there is no need for such revisions of nonrelativistic quantum mechanics, provide the latter is correctly interpreted [27]. The issue appears even more tricky if Eqs. (2.3.1)–(2.3.3) are supposed to describe the semiclassical regime of a fully quantum theory of gravity. It is, in fact, hard to figure out how can nonlinear features emerge in this limit. The situation is complicated further by the fact that exactly the same troubles should plague the semiclassical theory of electromagnetism, based on Eqs. (2.1.21)–(2.1.25); moreover, since for charged elementary particles $e^2 \gg Gm^2$, the nonlinear deviations, if present, would be now much more easy to detect. This disconcerting situation makes us understand that a clarification of the subject is indispensable, in order not to run into trivial mistakes due to a misunderstanding of its physical ground. We shall start by reanalysing the meaning of Eq. (2.1.8) for the Hartree self-consistent electric field.

Let us remind first that the idea underlying the Hartree method consists in realizing that, in order to determine the stationary state of a system of Z atomic electrons, each moving in the electric field created by all the others, it does not make a sensible error to solve the Schrödinger eigenvalue equations (2.1.5), together with Eqs. (2.1.6). The analogy between this approach and that of semiclassical gravity is quite different: While, in semiclassical gravity, one is looking for the *real* spacetime compatible with a single quantum system, in the Hartree theory the electric potentials φ_n are not supposed to correspond to any physically real configuration, but rather to represent a convenient average, useful in the calculation of the electrons' wave functions, which are the only objects of interest. Moreover, two things are clear from the method:

i) The potential φ_n due to the n-th electron is only an effective *mean* potential;

ii) The potential φ_n due to the n-th electron *does not* act on the n-th electron itself.

Both of these properties are not satisfied by Eqs. (2.1.21)–(2.1.25); in particular, Eq. (2.1.25) contains the self-interaction terms mentioned above in our discussion. We must conclude that Eq. (2.1.21)–(2.1.25) are not a straightforward generalization of the Hartree method, but rather that they describe a completely different problem, in which one asks for the electromagnetic field created by a quantum particle. Let us stress again that in the Hatree theory the electric field has only the role of mediating conveniently the interaction between electrons, and that there is no pretence that the fields $\mathbf{E}_n = -\nabla\varphi_n$ are those which really exists in a single copy of the system. A consistent extension of the method would thus not be represented by Eqs. (2.1.21)–(2.1.25), but rather by the set of equations:

$$\nabla \cdot \mathbf{E}_n = 4\pi e |\psi_n|^2 \,, \tag{2.4.7}$$

$$\nabla \times \mathbf{B}_n = 4\pi \frac{e\hbar}{2mi} \psi_n^* \overleftrightarrow{\nabla} \psi_n - 4\pi \frac{e^2}{m} \sum_{m\neq n} \mathbf{A}_m \, |\psi_n|^2 + \frac{\partial \mathbf{E}_n}{\partial t} \,, \tag{2.4.8}$$

$$\mathbf{E}_n = -\nabla\varphi_n - \frac{\partial \mathbf{A}_n}{\partial t} \,, \tag{2.4.9}$$

$$\mathbf{B}_n = \nabla \times \mathbf{A}_n \,, \tag{2.4.10}$$

$$i\hbar\frac{\partial\psi_n}{\partial t} = -\frac{\hbar^2}{2m}\left(\boldsymbol{\nabla} - \frac{ie}{\hbar}\sum_{m\neq n}\mathbf{A}_m\right)^2 \psi_n + e\sum_{m\neq n}\varphi_m\psi_n\ , \qquad (2.4.11)$$

for $n \in \{1, \ldots, Z\}$. Similar considerations hold for the relativistic case.

Having distinguished the semiclassical theories from the Hartree approach, we remain without our original justification for the Eqs. (2.1.21)–(2.1.25), (2.1.26)–(2.1.28) and (2.3.1)–(2.3.3), which are supposed to describe the classical (gravitational or electromagnetic) field due to a quantum matter source. Another point of view which is often adopted in order to motivate the validity of these sets of equations is the comparison with the celebrated Ehrenfest equation

$$m\frac{d^2\langle\psi(t)|\hat{\mathbf{x}}|\psi(t)\rangle}{dt^2} = -\langle\psi(t)|\boldsymbol{\nabla}U(\mathbf{x})|\psi(t)\rangle\ , \qquad (2.4.12)$$

holding for a particle subject to an external potential $U(\mathbf{x})$. However, Eq. (2.4.12) has the same form of the classical equation of motion with the position $\mathbf{x}(t)$ replaced by $\langle\psi(t)|\hat{\mathbf{x}}|\psi(t)\rangle$,

$$m\frac{d^2\langle\psi(t)|\hat{\mathbf{x}}|\psi(t)\rangle}{dt^2} = -\boldsymbol{\nabla}U(\langle\psi(t)|\hat{\mathbf{x}}|\psi(t)\rangle)\ , \qquad (2.4.13)$$

only if the dispersion of $\mathbf{x}$ around $\langle\psi(t)|\hat{\mathbf{x}}|\psi(t)\rangle$ is negligible, a fact depending, ultimately, on the preparation process for the particle at an initial time t_0, and corresponding to the condition for the particle's position to be considered as a classical observable. If this requirement is not satisfied, i.e., if the behaviour of the particle is truly quantum, then Eq. (2.4.13) is inadequate to provide a good description, and it must be corrected by adding extra terms depending on

$$\langle\psi(t)|\hat{\mathbf{x}}^2|\psi(t)\rangle - \langle\psi(t)|\hat{\mathbf{x}}|\psi(t)\rangle^2$$

to the right hand side, according to (2.4.12). Similarly, we expect that Eqs. (2.1.26) and (2.3.1) certainly hold for a state vector corresponding to well defined values of the components of, respectively, $j_e^a(x)$ and $T_{ab}(x)$, i.e., for a state in which matter exhibits a nearly classical behaviour; however, if the state vector represents, for example, a matter configuration which is in a superposition of macroscopically distinguishable states, it seems very unlikely that these equations could describe the physical situation in a reasonable way.

The fact that semiclassical field equations like (2.1.26) and (2.3.1) would make the classical field indifferent to the quantum fluctuations of matter, has been recently emphasized by Boucher and Traschen [53] for the case of gravity, showing how disconcerting it may become when Eq. (2.3.1) is applied to the study of cosmological models whose matter content is a quantum field. In most of the current literature [54] on this topic, the background spacetime is assumed to be initially represented by a spatially homogeneous manifold, satisfying Eq. (2.3.1) with the right hand side constructed out of a scalar field $\phi(x)$, which is, of course, initially homogeneous, too. Usually, it is argued that, as time[12] passes, $\phi(x)$ will develop spatial perturbations, due to quantum fluctuations; these, through the gravitational field equations, will induce perturbations in the spacetime metric, which will act later as centres of condensation for matter, after other physical processes will have taken place. This picturesque scenario, which should provide a physical mechanism for the origin of the inhomogeneities (clusters of galaxies) observed in the present universe, cannot unfortunately rely on Eq. (2.3.1), because this equation do not couple quantum fluctuations of matter to the geometry of spacetime: If the initial conditions are spatially homogeneous both in g_{ab} and in ϕ, they will remain so forever.

Before carrying on the discussion, let us briefly summarize the situation. We have shown that semiclassical field theories are, contrarily to a widespread opinion, substantially different from those based on the "mean field" methods popular in atomic physics; in particular, the similarity presented by Eqs. (2.1.26)- (2.1.28) and (2.3.1)-(2.3.3) with the Hartree equations is only superficial. Moreover, these formulations of the semiclassical problem are unable to take into account the quantum fluctuations occurring in the source, and they seem to involve nonlinear modifications to quantum theory. It seems thus reasonable to ask if they should not be rejected at all, and replaced by a less awkward approach. We shall now provide what we consider as a crucial argument against them, showing that they lead to unacceptable conclusions, whatever (reasonable!) interpretation of quantum theory one may adopt.

[12] Referred to the fundamental observers of the cosmological model.

Our argument is structured around the remark that in the right hand side of Eq. (2.3.1) the wave function ψ seems to be given a "materialistic" meaning, in the sense that $|\psi|^2$ and $\psi^* \overleftrightarrow{\nabla} \psi$ are treated as sources of gravity (see Sec. 2.3), as if the particle were an extended object. This is clearly hard to conciliate with those interpretations in which ψ is not a physical field, but only represents the probability amplitude of a configuration; moreover, it causes serious troubles if associated to an interpretation in which the state vector collapses when a measurement is performed.

Let us analyze the latter point first. The state vector collapse can be written as [35,55]

$$\hat{\rho} \longrightarrow \hat{\rho}' , \tag{2.4.14}$$

where $\hat{\rho}$ is the statistical operator of the system; if the quantity to be measured is an observable with eigenvalues $\{a_n\}$, corresponding to the orthonormal eigenstates[13] $\{|a_n\rangle\}$, we have

$$\hat{\rho} = |\psi\rangle\langle\psi| , \tag{2.4.15}$$

with

$$|\psi\rangle = \sum_n c_n |a_n\rangle , \tag{2.4.16}$$

and

$$\hat{\rho}' = \sum_n |c_n|^2 |a_n\rangle\langle a_n| . \tag{2.4.17}$$

According to Eq. (2.3.1), just before the measurement is performed, the source term is

$$\langle\psi|\hat{T}_{ab}|\psi\rangle , \tag{2.4.18}$$

while, immediately after the measurement, it becomes

$$\langle a_n|\hat{T}_{ab}|a_n\rangle , \tag{2.4.19}$$

with a probability $|c_n|^2$. The various $|a_n\rangle$ can correspond to different macroscopical distributions of the sources; if the measurement has a duration (with respect to the reference frame u^a) $\Delta\tau$, the effect of the state vector reduction is to change the source term from (2.4.18) to (2.4.19) within a time $\Delta\tau$. The inconsistency between this process and

[13]We suppose, for sake of simplicity, that there is no degeneracy.

Eq. (2.3.1) is emphasized in Fig. (2.2), which shows a spacetime diagram of the following situation.

At time τ_1 a nonrelativistic particle of mass m is in a state such that it has the same probability 1/2 to be in any of two disjoint spatial (i.e., on $\Sigma(\tau_1)$) regions A and B; at time τ_2 a position measurement on the particle is performed, which takes a time $\Delta\tau$. After the measurement (e.g., at time τ_3), the particle is known to be in the region B (the other possibility is equivalent), with probability 1. If Eqs. (2.3.1) are assumed to hold, the gravitational field at the spacetime point x will have matter contributions from both regions A and B and, possibly, from the dotted region in the figure, while at the point y it will have contributions only from B and, possibly, from the dotted region. Since the time $\Delta\tau$ can be made reasonably small, and x and y can be thus chosen very close to each other, it is clear that the validity of Eq. (2.3.1) is guaranteed only admitting that matter is present in the dotted region, i.e., that the act of measurement induces an acausal (spacelike) flow of matter from A to B. This physically unacceptable conclusion leads to drop either Eq. (2.3.1) or the hypothesis that the state vector may collapse.

We want to stress the fact that the occurrence of a spacelike matter flow is a straightforward consequence of the mathematical structure of Eq. (2.3.1). In fact, the identity (2.1.3) implies

$$\nabla^b \langle \psi(\tau) | \hat{T}_{ab} | \psi(\tau) \rangle = 0 \; . \tag{2.4.20}$$

Eq. (2.4.20), as known, has the meaning of a conservation law, in the sense that any change of the matter's content[14] inside a closed spatial 2-surface implies a flow through the 2-surface itself. The result of the measurement is therefore accompanied, according to Eq. (2.3.1), by a matter flow from A to B, due to the change of the right hand side from (2.4.18) to (2.4.19). That Eq. (2.3.1) require the spacelike flow, can also be understood in a more physical way, thinking that, if such a flow were absent, then, immediately after time $\tau_2 + \Delta$, near the region B the field would be the one generated by a mass m, while at a distance d from it, it would still be, for a time d/c, the one produced by two particles of mass $m/2$, located in A and B. Such a situation is clearly incompatible

[14]Here we do not care about the intricate problems concerning a rigorous definition of energy in a curved spacetime [56], since they are irrelevant to our discussion.

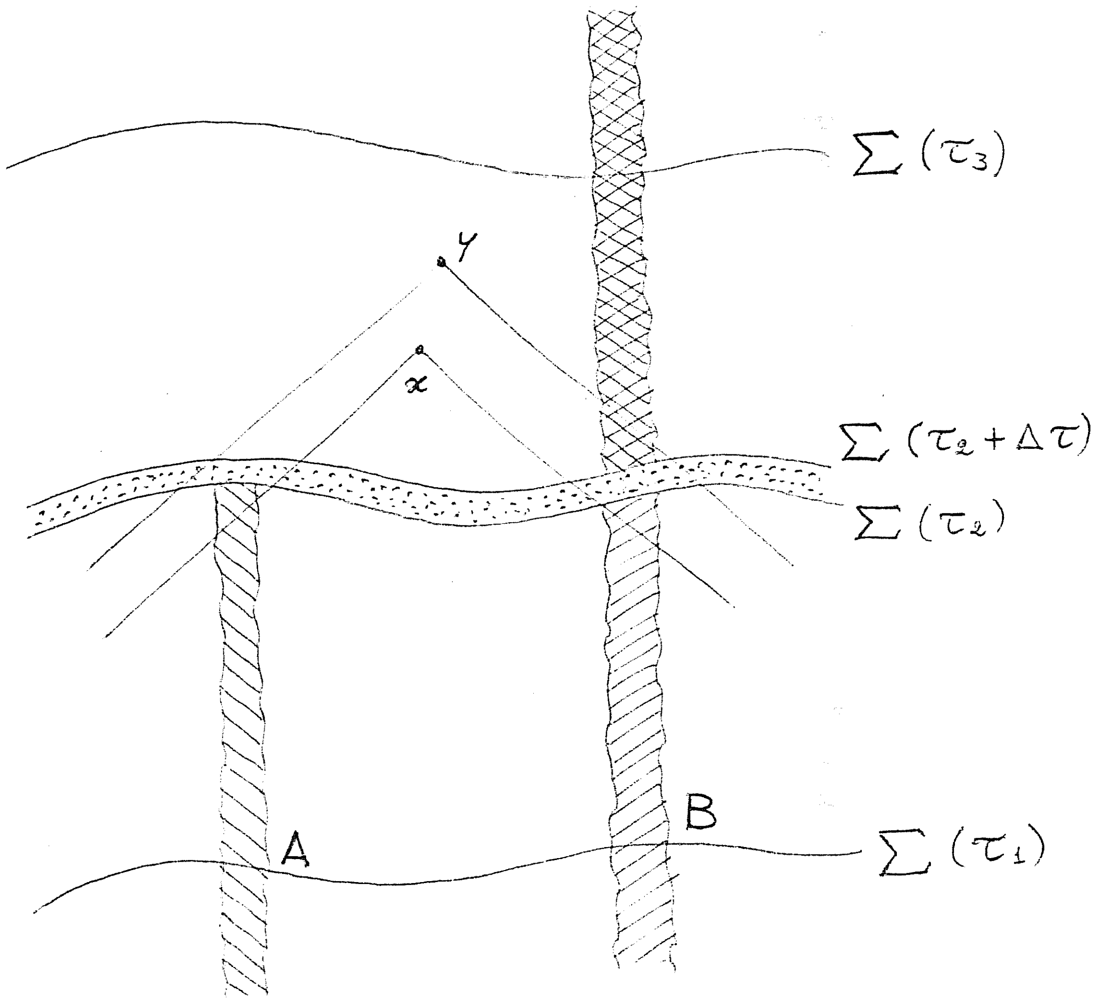

Figure 2.2: A spacetime diagram of the wave function reduction due to a measurement of position.

with the integral formulation of Eq. (2.3.1) in its weak field limit.

These arguments lead naturally to the conclusion that Eq. (2.3.1) could be physically acceptable only if an interpretation of quantum theory is adopted in which the state vector does not collapse. As well known, the most reliable of these are Einstein's [57] (or statistical [27]) and Everett's [58] (or many-worlds [59]): we shall now show that both of them are incompatible with Eq. (2.3.1).

In Everett's formulation, the vector (2.4.16) is supposed to describe faithfully the matter content of space, under the hypothesis of simultaneous existence of different copies of the material system, each of them described by one of the vectors $|a_n\rangle$. When semiclassical gravity is taken into account through Eq. (2.3.1), these copies are allowed to interact with each other, because the gravitational field in a point of spacetime is generated by the source (2.4.18), which takes into account all the copies of the system through the vectors $|a_n\rangle$ contained in $|\psi\rangle$; this situation provides therefore a coupling between different copies, mediated by classical gravity. An observer's state vector, in fact, is contained in one of the components of $|\tau\rangle$, but by a measurement of the gravitational field he/she can be aware of the presence of other components; such a possibility is, however, in conflict with observations, in which the gravitational field due to macroscopic bodies is always found to correspond only to one component of the matter's state vector, against the predictions of Eq. (2.3.1). This result can be interpreted in two ways: If we believe that semiclassical gravity could be a fundamental theory of nature, then it is a proof that Eq. (2.3.1) cannot be the correct field equation; alternatively, it could be considered an evidence for the quantization of gravity [25]. However, even in this second case, it is obvious that Eq. (2.3.1) is inadequate to describe the semiclassical regime, because its left hand side

$$G_{ab}[g] \qquad (2.4.21)$$

only represents an expectation value, and does not correspond to any precise component of the state vector of spacetime.

We run into the same troubles adopting the statistical interpretation of quantum theory. In this case, we must take the view that the state vector $|\psi\rangle$ describes an ensemble $\mathcal{E}_m$ of equally prepared material systems, each of those is supposed to correspond to a gravitational field (spacetime), so that an ensemble $\mathcal{E}_g$ has to be considered for gravity as well. This means again that (2.4.21) is not referred , in general, to any physically realized situation, but only to a fictitious spacetime, which somehow represents an "average" over $\mathcal{E}_g$: Eq. (2.3.1) does not succeed, therefore, in providing us with a reliable formulation of the semiclassical problem, in which one asks for the compatibility between spacetime and its quantum matter content. We must nonetheless remark that to realize that gravity can be described only at the level of an ensemble $\mathcal{E}_g$ amounts, within this context, to recognize the partial failure of the semiclassical program; in fact, it involves the need to give a probabilistic description of spacetime. As we shall discuss extensively in Ch. 4, however, a revision of the subject along these lines allows to give a consistent, although weaker, formulation to the problem; the key idea will be to allow the field to have at least those probabilistic features which are induced by matter. This hypothesis, when combined with the statistical interpretation of quantum theory, removes all the inconsistences and paradoxes discussed previously, and does not require to deal with the most tricky problems of field quantization, since effects of "self-uncertainty" are not taken into account.

One might try to make a last effort for saving Eq. (2.3.1), using "heretic" approaches, such as Bohm's theory [60,61] or stochastic mechanics [62,63]. These theories, however, rely on the hypothesis that systems follow definite trajectories in the configuration space, and this is in direct contrast to the "extended" character of the source in Eq. (2.3.1), as shown, e.g., in (2.3.27). It seems therefore that a careful revision of the entire semiclassical problem is necessary in order to achieve physical consistency.

Chapter 3

Statistical Meaning of Quantum Theory

It is particularly important, when speaking about a physical theory, to distinguish between the mathematical formalism and its interpretation. In the particular case of quantum mechanics, these two aspects are so difficult to relate each other, that the character of the resulting theory has some features reminding schizophrenia. It turns out, however, that in order to deal successfully with most of the applications to microphysics, it is not necessary to enter deeply into the subject: The Born's probabilistic interpretation of $|\psi|^2$ is totally sufficient.

It is not so if one tries to apply quantum theory to more complex situations such as, for example, those involving macroscopic systems; in these cases, a thorough understanding of its conceptual basis is necessary in order to avoid mistakes and/or paradoxes. We encounter the same kind of problems when trying to construct semiclassical theories, of which semiclassical gravity is an example; as seen in Sec. 2.4, in fact, not only does the requirement of physical consistence put severe limits on the kind of description one should adopt, but also the choice of an interpretation of quantum mechanics rather than another one, may involve a complete change in the physical picture. We thus believe that a clarification of the foundations of quantum theory should have high priority within our program of work. This chapter is mainly devoted to provide good reasons for adopting the statistical interpretation.

Our discussion will be rather unusual. We shall not follow the common line of argument which, starting from the classical paradoxes of quantum theory, shows how they find a natural and reasonable solution if one accepts the idea that the state vector does not describe an *individual* system, but only an *ensemble* of similarly prepared copies of it. Since these aspects of the topic have been already discussed very well in the literature [27,64], we prefer not to repeat them here, and we limit ourselves to remind that the statistical interpretation not only provides a viable solution to the paradoxes mentioned above within the standard formalism of quantum theory, but it does not introduce any foreigner metaphysical concept, and it is probably the only one which is consistent with a physical characterization of probability [64]; moreover, it leaves open the possibility of extending quantum mechanics in order to account also for the individuals' behaviour – a very satisfactory feature from a methodological point of view.

Our approach is based on the following, apparently trivial, remark: Although the interpretation of a theory is not uniquely determined by the formalism, it heavily depends on it; a change in the formalism, even if not leading to any new experimental consequence, can induce drastic changes in the interpretation. Perhaps the most dramatic example of this process has been the four dimensional reformulation of special relativity by Minkowski, which led to the introduction of the concept of spacetime, essential in the further development of general relativity. We shall show that there is a formulation of quantum theory, alternative to the Hilbert space one, which almost unavoidably leads to the statistical interpretation; this is the so called "hydrodynamical formulation" [65,66,67].

In Sec. 3.1 we develop the essential features of such formalism, and discuss few of the interpretative problems it raises; only the spinless one-particle case is considered, since it is sufficient for our purpose. Sec. 3.2 is devoted to the interpretation, and to a review of some general concepts of kinetic theory which are extensively used in the discussion. It turns out that not only does the new formalism strongly suggest to adopt the statistical interpretation, but it makes almost impossible to resist to the temptation of trying to "complete" quantum theory, replacing it with a theory of individuals, rather than of ensembles; considerations on this subject are contained in Sec. 3.3.

Our discussion follows closely that of ref. [68].

3.1 Hydrodynamical Formulation of Quantum Mechanics

It is well known [69] that in the nonrelativistic quantum theory of a spinless particle, the Schrödinger equation

$$i\hbar \frac{\partial \psi(\mathbf{x},t)}{\partial t} = H(\mathbf{x}, -i\hbar\boldsymbol{\nabla}, t)\, \psi(\mathbf{x},t) \ , \tag{3.1.1}$$

where

$$H(\mathbf{x}, -i\hbar\boldsymbol{\nabla}, t) = -\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x},t) \tag{3.1.2}$$

is the hamiltonian, with m the particle mass and $U(\mathbf{x},t)$ its potential energy, implies a continuity equation to hold in the form

$$\frac{\partial \mu}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \ , \tag{3.1.3}$$

where[1]

$$\mu\,(\mathbf{x},t) \equiv m\, |\psi(\mathbf{x},t)|^2 \tag{3.1.4}$$

and

$$\mathbf{j}(\mathbf{x},t) \equiv \frac{\hbar}{2i}\, \psi(\mathbf{x},t)^* \overleftrightarrow{\boldsymbol{\nabla}}\, \psi(\mathbf{x},t) \ . \tag{3.1.5}$$

Writing now $\psi(\mathbf{x},t)$ as in Eq. (2.3.15), and defining a velocity field $\mathbf{v}(\mathbf{x},t)$ as in Eq. (2.3.17), Eq. (3.1.5) becomes

$$\mathbf{j}(\mathbf{x},t) = \mu\,(\mathbf{x},t)\, \mathbf{v}(\mathbf{x},t) \ \ : \tag{3.1.6}$$

The continuity equation (3.1.3) can therefore be considered as expressing the conservation of mass for a fluid of density μ and velocity $\mathbf{v}$. It is natural to ask, at this point, if the analogy can be pursued further, extracting also an Euler equation from Eq. (3.1.1), of the form

$$\mu\, \frac{Dv_i}{dt} = -\partial_j p_{ij} - \frac{\mu}{m}\, \partial_i U \ , \tag{3.1.7}$$

where $p_{ij}(\mathbf{x},t)$ is the pressure tensor of the fluid[2] and D/dt is defined as in Eq. (2.3.25). Surprisingly enough, this is possible!

[1] In the definitions (3.1.4) and (3.1.5), a factor m has been introduced in order to make more evident the similarity of Eq. (3.1.7) below with the Euler equation for a fluid.

[2] The p_{ij} here defined has nothing to do, in principle, with the quantity defined in Eq. (2.3.23); however, it will turn out later on that they are the same object. This justifies our abuse of notation.

To prove Eq. (3.1.7), let us first insert Eq. (2.3.15) into the Schrödinger equation (3.1.1), and make use of Eqs. (3.1.3) and (3.1.6) getting

$$\frac{\partial S}{\partial t} + \frac{1}{2m}(\boldsymbol{\nabla} S)^2 + U - \frac{\hbar^2}{2m}\frac{\nabla^2|\psi|}{|\psi|} = 0 \; ; \tag{3.1.8}$$

applying now Eq. (2.3.25) to Eq. (2.3.17), and using Eqs. (3.1.8) and (3.1.4), we find

$$\begin{aligned} \frac{D\mathbf{v}}{dt} &= \frac{1}{m}\frac{D}{dt}\boldsymbol{\nabla} S = \frac{1}{m}\boldsymbol{\nabla}\left(\frac{\partial S}{\partial t} + \frac{1}{2m}(\boldsymbol{\nabla} S)^2\right) = \\ &= -\frac{\hbar^2}{2m\,\mu}\left(\boldsymbol{\nabla}|\psi|\,\nabla^2|\psi| - |\psi|\,\boldsymbol{\nabla}\nabla^2|\psi|\right) - \frac{1}{m}\boldsymbol{\nabla} U \; . \end{aligned} \tag{3.1.9}$$

In order for Eq. (3.1.7) to be satisfied, we are led to write

$$\begin{aligned} \partial_j p_{ij} &= \frac{\hbar^2}{2m}\left(\partial_i|\psi|\,\partial_j\partial_j|\psi| - |\psi|\,\partial_i\partial_j\partial_j|\psi|\right) = \\ &= -\frac{\hbar^2}{4m^2}\,\partial_j(\mu\,\partial_i\partial_j\ln\mu) \; , \end{aligned} \tag{3.1.10}$$

which admits the identification

$$p_{ij} = -\frac{\hbar^2}{4m^2}\,\mu\,\partial_i\partial_j\ln\mu \; . \tag{3.1.11}$$

In principle, to the right hand side of Eq. (3.1.11) could be added an arbitrary symmetric tensor field C_{ij} such that $\partial_j C_{ij} = 0$: For example, in ref. [66] the pressure tensor is identified with

$$P_{ij} = -\frac{\hbar^2}{4m^2}\nabla^2\mu\,\delta_{ij} + \frac{\hbar^2}{4m^2}\,\mu\,\partial_i\ln\mu\,\partial_j\ln\mu \; , \tag{3.1.12}$$

and it is easy to check that

$$\partial_j P_{ij} = \partial_j p_{ij} \; . \tag{3.1.13}$$

Hence, at this level of the treatment, it seems there is no reason to prefer Eq. (3.1.11), or Eq. (3.1.12), or some other expression; nevertheless, we shall continue to use Eq. (3.1.11): This choice will be justified in Sec. 3.3.

We have proved the remarkable result that the system of Eqs. (3.1.3), (3.1.7) and (3.1.11) is equivalent to the Schrödinger equation (3.1.1), once a correspondence between μ and $\mathbf{v}$ on the one side and ψ on the other is established by the Eqs. (3.1.4), (2.3.15) and (2.3.17). It is therefore possible to associate to each quantum mechanical problem a corresponding hydrodynamical one, formulated for a fluid whose pressure tensor is linked to the density by Eq. (3.1.11).

These conclusions are rather exciting, but we must be very careful about their interpretation; it is by no means obvious, in fact, which meaning to attribute to the quantities μ, $\mathbf{v}$ and p_{ij}. We shall therefore try now, to discuss some features and consequences of the formulation developed, which may suggest a possible interpretative line.

Let us begin with a remark of formal character. The quantities $\mu\,(\mathbf{x},t)$, $\mathbf{j}(\mathbf{x},t)$, and

$$\Pi_{ij}(\mathbf{x},t) \equiv p_{ij}(\mathbf{x},t) + \mu\,(\mathbf{x},t)\, v_i(\mathbf{x},t) v_j(\mathbf{x},t) \;, \tag{3.1.14}$$

can all be represented as expectation values of operators. This is a straightforward consequence of the Eqs. (2.3.12), (2.3.14), and (2.3.21), which we rewrite here as[3]:

$$\langle\psi|\,\hat{\mu}\,(\mathbf{x},t)|\psi\rangle = m\,|\psi(\mathbf{x},t)|^2 = \mu\,(\mathbf{x},t) \;; \tag{3.1.15}$$

$$\langle\psi|\,\hat{\mathbf{j}}(\mathbf{x},t)|\psi\rangle = \frac{\hbar}{2i}\,\psi(\mathbf{x},t)^*\,\overleftrightarrow{\nabla}\,\psi(\mathbf{x},t) = \mathbf{j}(\mathbf{x},t) \;; \tag{3.1.16}$$

$$\begin{aligned}\langle\psi|\hat{\Pi}_{ij}(\mathbf{x},t)|\psi\rangle = &-\frac{\hbar^2}{4m^2}\mu\,(\mathbf{x},t)\,\partial_i\partial_j \ln\mu\,(\mathbf{x},t) + \\ &+\mu\,(\mathbf{x},t)\,v_i(\mathbf{x},t)v_j(\mathbf{x},t) = \Pi_{ij}(\mathbf{x},t)\;.\end{aligned} \tag{3.1.17}$$

The existence of operators $\hat{\mu}\,(\mathbf{x},t)$, $\hat{\mathbf{j}}(\mathbf{x},t)$, and $\hat{\Pi}_{ij}(\mathbf{x},t)$, whose expectation values are, respectively, (3.1.4), (3.1.5), and (3.1.11), is rather difficult to justify, if quantum theory is supposed to provide a complete description of a single particle: It is not clear in fact, which observables should they represent in such context. It is one of the requirements that a viable interpretation must fulfill, to clarify this point.

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[3] We drop the subscript in $\hat{\Pi}_{ij}$, since we are obviously referring to the Weyl-ordered stress tensor operator, which is the only one whose expectation value allows to extract a pressure tensor which coincides with p_{ij} defined in Eq. (3.1.11).

Let us now discuss a simple specific example: A particle contained in a rectangular box of sides L_1, L_2, L_3. The wave function [69]

$$\psi(\mathrm{x},t) = e^{-\frac{i}{\hbar}Et}\sqrt{\frac{8}{L_1 L_2 L_3}}\,\sin\left(\frac{n_1\pi}{L_1}x_1\right)\,\sin\left(\frac{n_2\pi}{L_2}x_2\right)\,\sin\left(\frac{n_3\pi}{L_3}x_3\right) \tag{3.1.18}$$

corresponds to an eigenstate of energy with eigenvalue

$$E = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_1^2}{L_1^2}+\frac{n_2^2}{L_2^2}+\frac{n_3^2}{L_3^2}\right) , \tag{3.1.19}$$

where n_i are positive integer numbers. Inserting the expression (3.1.18) into Eqs. (3.1.4) and (3.1.11) we find that p_{ij} is diagonal, with

$$p_{11} = \frac{4\pi^2\hbar^2 n_1^2}{m\,L_1^3\,L_2\,L_3}\sin^2\left(\frac{n_2\pi}{L_2}x_2\right)\,\sin^2\left(\frac{n_3\pi}{L_3}x_3\right) , \tag{3.1.20}$$

and similar results for p_{22} and p_{33}. The component p_{ii} (no summation over i) represents a pressure along the i-th direction which, according to Eq. (3.1.20), does not depend on x_i: On the walls of the box, this pressure is maximum at the centre, and vanishes at the edges; the mean value of, say, p_{11} on a face normal to x_1 is

$$\bar{p}_{11} \equiv \frac{1}{L_2 L_3}\int_0^{L_2} dx_2 \int_0^{L_3} dx_3\, p_{11}(\mathrm{x},t) = \frac{n_1^2\pi^2\hbar^2}{m\,L_1^3\,L_2\,L_3} . \tag{3.1.21}$$

It is striking to notice the complete agreement between Eq. (3.1.21) and the result given by the standard way to calculate pressure on the walls of the box [69, page 67]; it is important to remark, in fact, that the tensor p_{ij} as we have introduced it, had until now only a *formal* role, and there is not therefore any evident reason why it should represent a physically meaningful pressure: The coincidence expressed by the result (3.1.21) is thus something unexpected, and it must find an explanation in the interpretation of the formalism which we are looking for.

Another puzzling consequence can be obtained observing that $\bar{p}_{11}$ given in Eq. (3.1.21), can be also regarded as the average, over the entire volume of the box, of the corresponding component of the pressure tensor; taking into account the analogous expressions for $\bar{p}_{22}$ and $\bar{p}_{33}$ we can write, remembering Eq. (3.1.19),

$$\frac{1}{3}\,\mathrm{tr}\,\bar{\mathsf{p}} = \frac{2}{3}\frac{E}{V} , \tag{3.1.22}$$

where $\bar{\mathsf{p}}$ stands for the entire averaged pressure tensor, and $V \equiv L_1 L_2 L_3$ is the volume of the box. The relation (3.1.22) is strongly reminiscent of the equation of state

$$pV = \frac{2}{3}E \tag{3.1.23}$$

for a perfect gas, in which E represents the total thermal energy. Again, we ask to ourselves: Is this a simple coincidence, or is it a symptom of some subtle analogy existing between these two, apparently totally different, systems?

Another specific example which is worth analyzing is that of a free particle with gaussian wave packet centered in $\mathbf{x} = \mathbf{0}$ at the time t_0:

$$|\psi(\mathbf{x}, t_0)|^2 = \frac{1}{(2\pi)^{3/2} R^3} \exp\left(-\frac{\mathbf{x}^2}{2R^2}\right) , \tag{3.1.24}$$

where R gives an estimate of the gaussian's width. We know that the Schrödinger equation with $U = 0$ predicts that the packet will spread in such a way that it doubles its width in a time

$$t_{Schr} \sim \frac{mR^2}{\hbar} . \tag{3.1.25}$$

In the hydrodynamical formalism, it is straightforward to compute

$$p_{ij} = \frac{\hbar^2 \mu}{4m^2 R^2} \delta_{ij} \equiv p\, \delta_{ij} \ : \tag{3.1.26}$$

The pressure given by Eq. (3.1.26) will cause the gaussian perturbation in the fluid to suffer a "dynamical" spread determined by the Euler equation (3.1.7), which becomes now, at $t = t_0$,

$$\mu \frac{D\mathbf{v}}{dt} = -\frac{\hbar^2}{4m^2 R^2} \boldsymbol{\nabla} \mu . \tag{3.1.27}$$

A rough order of magnitude estimate gives for the acceleration the value

$$\frac{\hbar^2}{m^2 R^3} , \tag{3.1.28}$$

so that the perturbation will increase its size, in the time t, by

$$\frac{\hbar^2 t^2}{m^2 R^3} ; \tag{3.1.29}$$

requiring this length to be equal to R we obtain the time t_{hydr} during which the width of the perturbation is doubled:

$$t_{hydr} \sim \frac{mR^2}{\hbar} \,. \tag{3.1.30}$$

We are thus led, comparing Eq. (3.1.25) and Eq. (3.1.30), to interpret the spreading of wave packets as an effect of pressure; the physical meaning of this attractive statement is nevertheless still obscure, and has to be clarified in an interpretation of the hydrodynamical formalism.

3.2 Interpretation

The few examples now presented, all seem to point toward an interpretation in which the fluid is taken seriously as something really "existing". Yet, such a concept is difficult to conciliate with some features of quantum theory which follow straightforwardly from experience, such as the relation between the wave function ψ and the probability of presence for the particle, and the corpuscular aspects which particles exhibit during their detection.

To understand better the nature of these problems let us consider, for example, the scalar field $\mu(\mathbf{x}, t)$. It is clear that in the hydrodynamical formulation given above, its role is that of density of the fluid; nevertheless, going back to the definition (3.1.4) we see that, except for the coefficient m which fixes the scale, $\mu(\mathbf{x}, t)$ is essentially the squared modulus of the wave function, $|\psi(\mathbf{x}, t)|^2$. In the Copenhagen interpretation of quantum theory [37] this quantity represents the probability density for the particle to be found at the point $\mathbf{x}$ during a measurement of position performed at the time t: It is not clear, therefore, how to conciliate this point of view with the hydrodynamical formulation in a way which may go beyond a mere formal analogy. Alternatively, one would be tempted to adopt an interpretation *à la* Schrödinger [70], in which the particle is seen as a perturbation in a fluid of density μ; as attractive as this picture may look in our context, nevertheless we believe that it could not be maintained, because of the unavoidable difficulties connected to the process of reduction of the wave function [37]. It would be necessary, in fact, to assume that in the course of a measurement of position, the perturbation in the fluid becomes sharply localized at a single point $\mathbf{x}$ of the space, with a probability proportional to $\mu(\mathbf{x}, t)$: Such an assumption looks so unnatural that we find it to be sufficient for abandoning this interpretation, in which the fluid is naively considered to have a real existence in the physical space.

The problems encountered in these attempts of interpreting the hydrodynamical formalism in terms of conventional schemes can be traced to the corpuscular aspects of quantum particles, which are hardly compatible with the concept of fluid. It is well known from kinetic theory, however, that there exist systems, like e.g. gases, which at a macroscopic level are very well described by a fluid model, but reveal a corpuscular structure when analysed on a sufficiently small scale. In these cases it is clear that a hydrodynamical formalism is pertinent only to a description of ensembles, while it turns out to be completely unsuccessful if one tries to apply it to the treatment of situations in which the peculiar behaviour of the individuals is relevant. In order to be more explicit on this point, let us briefly review some features of the classical kinetic theory of gases [71], which will turn out to be useful later, when we shall come back to the problem of interpreting quantum theory.

The system that classical kinetic theory deals with is a gas composed of a very large number N of molecules. According to the laws of mechanics, the state of the gas at a given instant of time is completely specified if the coordinates and the momenta of all the molecules are given at that time; these data correspond to a point $(\mathbf{x}_1, \ldots, \mathbf{x}_N; \mathbf{p}_1, \ldots, \mathbf{p}_N)$ in the $6N$ dimensional Γ-space. As known, different points of the Γ-space can be indistinguishable at a macroscopic level: This leads to refer the concept of a macrostate not to a single copy of the gas, but rather to an entire ensemble of mental copies of it, all corresponding to the same macroscopic conditions even if, at the microscopic scale, their states can be totally different. The mathematical device which describes such an ensemble is the *density function*[4] $\rho(\mathbf{x}_1, \ldots, \mathbf{x}_N; \mathbf{p}_1, \ldots, \mathbf{p}_N|t)$, taking real nonnegative values and representing the probability density in the Γ-space that a generic element of the ensemble be in the microstate $(\mathbf{x}_1, \ldots, \mathbf{x}_N; \mathbf{p}_1, \ldots, \mathbf{p}_N)$ at the time t. If the gas is a hamiltonian sys-

[4]The unusual notation $\rho(\ldots|t)$ has been adopted in order to stress that this is a *conditional* probability. For more details about the role of this concept in quantum theory, see ref. [34].

tem, the function ρ satisfies the Liouville's theorem:

$$\frac{\partial \rho}{\partial t} + \sum_{\alpha=1}^{3N} \left(\dot{x}_\alpha \frac{\partial \rho}{\partial x_\alpha} + \dot{p}_\alpha \frac{\partial \rho}{\partial p_\alpha} \right) = 0 \ , \tag{3.2.1}$$

where the coordinates and the momenta have been relabeled in an obvious way.

The information contained in the function ρ is actually overabundant when dealing with most of the applications of kinetic theory; it is more useful to consider the *distribution function*

$$f(\mathbf{x}, \mathbf{p}|t) \equiv N \int d^3x_2 \dots d^3x_N d^3p_2 \dots d^3p_N$$
$$\rho\left(\mathbf{x}, \mathbf{x}_2, \dots, \mathbf{x}_N; \mathbf{p}, \mathbf{p}_2, \dots, \mathbf{p}_N|t\right) \ , \tag{3.2.2}$$

defined on the six dimensional μ-space under the further assumption that all the molecules are equivalent. The interpretation of f is straightforward: $f(\mathbf{x}, \mathbf{p}|t)/N$ is the probability density in the μ-space that a generic molecule of the gas have position $\mathbf{x}$ and momentum $\mathbf{p}$ at the time t; it is obvious that, if ρ is normalized to one, f turns out to be normalized to N. The distribution function satisfies the equation

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} U \cdot \nabla_{\mathbf{p}} f = \left(\frac{\partial f}{\partial t} \right)_{coll} \ , \tag{3.2.3}$$

where $U(\mathbf{x}, t)$ is the potential energy due to an external field, while the right hand side is the variation of f due to molecular collisions; it is the main problem of kinetic theory to find the function f as a solution of Eq. (3.2.3) for a given molecular dynamics.

The concept of distribution function allows to establish a connection between the microscopic treatment of the gas and its hydrodynamical description; in fact, since f contains no information about the detailed motion of the molecules, it can be successfully used in order to define macroscopic quantities as averages; the simplest of these is the mass density[5]

$$\tilde{\mu}\left(\mathbf{x}, t\right) \equiv m \int d^3p \, f(\mathbf{x}, \mathbf{p}|t) \ . \tag{3.2.4}$$

[5] We denote by a tilde the quantities defined here, in order to distinguish them from the corresponding ones treated in Sec. 3.1.

We can also define a mass current

$$\tilde{\mathrm{j}}(\mathrm{x},t) \equiv \int d^3p\,\mathrm{p}\, f(\mathrm{x},\mathrm{p}|t)\ , \tag{3.2.5}$$

and consequently the macroscopic velocity of the gas

$$\tilde{\mathrm{v}}(\mathrm{x},t) \equiv \frac{\tilde{\mathrm{j}}(\mathrm{x},t)}{\tilde{\mu}\,(\mathrm{x},t)}\ . \tag{3.2.6}$$

Using the velocity (3.2.6), a pressure tensor can be written as

$$\tilde{p}_{ij}(\mathrm{x},t) \equiv m \int d^3p \left(\frac{p_i}{m} - \tilde{v}_i(\mathrm{x},t)\right)\left(\frac{p_j}{m} - \tilde{v}_j(\mathrm{x},t)\right) f(\mathrm{x},\mathrm{p}|t)\ . \tag{3.2.7}$$

The tensor $\tilde{p}_{ij}$ is connected to the stress tensor $\tilde{\Pi}_{ij}$ by the relation

$$\begin{aligned}\tilde{\Pi}_{ij}(\mathrm{x},t) &\equiv \frac{1}{m}\int d^3p\, p_i p_j\, f(\mathrm{x},\mathrm{p}|t) = \\ &= \tilde{p}_{ij}(\mathrm{x},t) + \tilde{\mu}\,(\mathrm{x},t)\,\tilde{v}_i(\mathrm{x},t)\,\tilde{v}_j(\mathrm{x},t)\ .\end{aligned} \tag{3.2.8}$$

Finally, the temperature $\tilde{T}$ and the heat flux vector $\tilde{\mathrm{q}}$ are defined as

$$\frac{3}{2}\frac{k}{m}\,\tilde{\mu}\,(\mathrm{x},t)\,\tilde{T}(\mathrm{x},t) \equiv \frac{m}{2}\int d^3p \left(\frac{\mathrm{p}}{m} - \tilde{\mathrm{v}}(\mathrm{x},t)\right)^2 f(\mathrm{x},\mathrm{p}|t)\ , \tag{3.2.9}$$

where k is the Boltzmann's constant, and

$$\tilde{\mathrm{q}}(\mathrm{x},t) \equiv \frac{m}{2}\int d^3p \left(\frac{\mathrm{p}}{m} - \tilde{\mathrm{v}}(\mathrm{x},t)\right)^2 (\mathrm{p} - m\tilde{\mathrm{v}}(\mathrm{x},t))\ f(\mathrm{x},\mathrm{p}|t)\ . \tag{3.2.10}$$

Let us finish this quick review of the main ideas of kinetic theory observing that to each quantity which is conserved in a collision between two molecules, there corresponds an equation for macroscopic quantities. In particular, the conservation of mass leads to the continuity equation

$$\frac{\partial \tilde{\mu}}{\partial t} + \nabla \cdot \tilde{\mathrm{j}} = 0\ ; \tag{3.2.11}$$

the conservation of momentum to the Euler equation

$$\tilde{\mu}\,\frac{\tilde{D}\tilde{v}_i}{dt} = -\partial_j \tilde{p}_{ij} - \frac{\tilde{\mu}}{m}\,\partial_i U\ ; \tag{3.2.12}$$

while the conservation of energy is associated to

$$k\,\tilde{\mu}\,\frac{\tilde{D}\tilde{T}}{dt} = -\frac{2}{3}\,\nabla \cdot \tilde{\mathrm{q}} - \frac{2}{3}\,\tilde{p}_{ij}\tilde{\Lambda}_{ij}\ , \tag{3.2.13}$$

where now

$$\frac{\tilde{D}}{dt} \equiv \frac{\partial}{\partial t} + \tilde{\mathbf{v}} \cdot \boldsymbol{\nabla} \tag{3.2.14}$$

and

$$\tilde{\Lambda}_{ij} \equiv \frac{m}{2} \left(\partial_i \tilde{v}_j + \partial_j \tilde{v}_i \right) \ . \tag{3.2.15}$$

We have preferred to review the previous material rather than simply to quote the existing literature, in order to establish a terminology which may allow to be as much precise as possible. In fact, we are strongly convinced that, when discussing such delicate topics as the present one, it is necessary to make all the possible efforts in order to avoid semantic misunderstandings, even at the expenses of the compactness of the treatment.

Before we return to the problem of interpretation of the quantum hydrodynamical formalism, we want to make a remark which, although it still deals with the classical kinetic theory, is of great importance in order to fully understand the rest of the paper. Let us suppose that, rather than a gas, we have now a single particle whose behaviour is not exactly known; as an example, we can consider a grain of pollen suspended in water and undergoing a brownian motion. It is possible to define a density function in the Γ-space, which completely describes an ensemble of particles which have been equally prepared from a macroscopic[6] point of view; however, since now the Γ-space coincides with the μ-space, such function may well be identified with the distribution function[7] and consequently denoted as $f(\mathbf{x}, \mathbf{p}|t)$; it is obvious that f is, in this case, normalized to one. We can reformulate these remarks saying that, in the case of a single particle, the distribution function can be considered

[6]The term "macroscopic" is used here in an improper way. What we really mean is that even two "equally prepared" systems may be not identical if considered in great detail. For example, two grains placed at the same point in water with the same initial velocity will be found, in general, at different places after some time: This is due to the fact that the concept of "equal preparation" did not involve the molecules of water in the vessel. It is to this coarse-graining in the preparation of the initial state that we refer using the word "macroscopic" in the present context.

[7]The density function will not necessarily satisfy the Liouville's theorem, because the particle is not necessarily a hamiltonian system (for example, in the case of the grain subjected to brownian collisions, these last cannot be described by a potential); translated in terms of the distribution function, this amounts to the appearance of the collisional term in Eq. (3.2.3).

as the density function describing an ensemble whose members are mental copies, all equally prepared at the macroscopic level, of the particle. A question which arises immediately at this point is the following one: With the distribution function $f(\mathbf{x},\mathbf{p}|t)$ we can calculate, following the definitions (3.2.4)–(3.2.10), the quantities $\tilde{\mu}$, $\tilde{\mathbf{j}}$, $\tilde{\mathbf{v}}$, $\tilde{p}_{ij}$, $\tilde{T}$ and $\tilde{\mathbf{q}}$; what meaning, if any, can be attributed to them in the single particle case which we are now discussing?

There is clearly a qualitative difference between this situation and that of a gas consisting of a very big number of molecules; while for this latter $\tilde{\mu}$, etc. can be regarded as measurable properties of the system, except for statistically small fluctuations, in the former it would be evidently wrong to interpret e.g. $\tilde{\mu}$ as the mass density associated to the particle. It is obvious in fact that $\tilde{\mu}$ could be spread over a large region of space, while the particle is known to be a pointlike object, with a distribution of mass which may be approximated by a delta function. The solution of the riddle is not too difficult, and can be understood as follows: Let us consider a gas made of N *noninteracting* copies of our particle; the distribution function for this new system is

$$f'(\mathbf{x},\mathbf{p}|t) = N\, f(\mathbf{x},\mathbf{p}|t)\ , \tag{3.2.16}$$

defined in the μ-space of the gas, now different from the $6N$ dimensional Γ-space. With the help of (3.2.4)–(3.2.10) it is possible to determine the macroscopic variables for this gas, which we shall denote as $\tilde{\mu}'$, $\tilde{\mathbf{j}}'$, $\tilde{\mathbf{v}}'$, $\tilde{p}'_{ij}$, $\tilde{T}'$ and $\tilde{\mathbf{q}}'$: These have now the usual operational meanings of mass density, mass current, and so on. The corresponding quantities in the case of a single particle are related to these, operationally well defined, observables through the relations $\tilde{\mu}' = N\tilde{\mu}$, $\tilde{\mathbf{j}}' = N\tilde{\mathbf{j}}$, $\tilde{\mathbf{v}}' = \tilde{\mathbf{v}}$, $\tilde{p}'_{ij} = N\tilde{p}_{ij}$, $\tilde{T}' = \tilde{T}$ and $\tilde{\mathbf{q}}' = N\tilde{\mathbf{q}}$, which all follow from Eq. (3.2.16). It is now straightforward to interpret the unprimed quantities as the contribution given by an *average* particle to the primed ones which are, as we have already noticed, physically meaningful in the sense that they can be directly measured on the gas. The example of the grain subjected to brownian motion is particularly useful in order to clarify the meaning of this statement: If we have a vessel containing water, in which a single grain is introduced with known position and momentum at a time t_0, we can calculate the probability density $f(\mathbf{x},\mathbf{p}|t)$ at the time t, and consequently all the quantities $\tilde{\mu}$, $\tilde{\mathbf{j}}$, etc; however, we do not expect these to be meaningful for the single system we are dealing with, but only in a statistical sense, as averages performed over an ensemble of equally prepared systems. For example, there will certainly be no increase in the pressure on the walls of a vessel when a grain is immersed in it: The only measurable consequence could be the detection of discontinuous hits due to the collisions of the grain against the walls; but averaging these hits over a large number of vessels each one containing a suspended grain we shall obtain a "pressure" whose value is in agreement with the prediction which can be extracted from the tensor field $\tilde{p}_{ij}(\mathbf{x}, t)$. However, if we suspend in the water not only one, but rather a very large number N of grains, we shall measure an enhancement of the pressure on the walls which will be due exactly (apart from fluctuations) to the effect of $\tilde{p}'_{ij}(\mathbf{x}, t) = N\tilde{p}_{ij}(\mathbf{x}, t)$; this is the well known phenomenon of *osmotic pressure*. Similar considerations can be performed for the other quantities $\tilde{\mu}$, $\tilde{\mathbf{j}}$, etc.

Our conclusions can be summarized claiming that, while for a gas the macroscopic quantities derived from Eqs. (3.2.4)–(3.2.10) have an operational meaning even for a single copy of the gas (within the limits fixed by the existence of statistical fluctuations), in the case of a single particle they are only appropriate to a description in terms of ensembles, and can in no way be used to describe individuals.

The situation for the quantum particle is very similar to the one now discussed; in both cases we are dealing with a theory which admits a hydrodynamical formalism, although it treats a system composed of a single particle: Hence, one would be tempted to draw the conclusion that quantum mechanics, too, is a theory which successfully describes ensembles, but fails in giving a detailed treatment of the individuals' behaviour. However, while in the classical case the statistical character of the hydrodynamical formalism is justified by the possibility to write $\tilde{\mu}$, $\tilde{\mathbf{j}}$, etc. as averages of quantities related to individuals, this connection is lacking in our treatment of the corresponding variables in quantum theory. In other words, it is not clear, at this point of the discussion, if it is possible to write equations like (3.2.4)–(3.2.7) which could generate the hydrodynamical quantities defined in Eqs. (3.1.4), (3.1.5), (2.3.17) and (3.1.11) for quantum mechanics; moreover, it is not evident if, with the help of equations of the kind of (3.2.9) and (3.2.10), one could define a "temperature" $T(\mathbf{x},t)$ and a "heat" flux vector $\mathbf{q}(\mathbf{x},t)$ which satisfy an analogue of Eq. (3.2.13).

In order to solve these problems, let us try first to evaluate density, current, and so on, with the use of the kinetic equations, and to compare the results so obtained with the expressions given in Sec. 3.1 for the corresponding quantities. In so doing, a function of $\mathbf{x}$, $\mathbf{p}$ and t must be used which play the role of distribution in the μ-space: We shall use for this purpose the Wigner function [36,73]

$$P_W(\mathbf{x},\mathbf{p}|t) \equiv \frac{1}{(\pi\hbar)^3}\int d^3\xi\,\psi(\mathbf{x}+\xi,t)^* e^{2i\mathbf{p}\cdot\xi/\hbar}\psi(\mathbf{x}-\xi,t)\;, \qquad (3.2.17)$$

although it does not satisfy all the requirements necessary in order to be properly called a *distribution*; we postpone a short discussion of the oddities of (3.2.17) to Ch. 5.

Let us compute the mass density first; we have

$$m\int d^3p\,P_W(\mathbf{x},\mathbf{p}|t) = m|\psi(\mathbf{x},t)|^2 = \mu\,(\mathbf{x},t)\;. \qquad (3.2.18)$$

For the mass current we find instead

$$\int d^3p\,\mathbf{p}\,P_W(\mathbf{x},\mathbf{p}|t) = \frac{1}{2\pi^3 i\hbar^2}\int d^3\xi d^3p\,e^{2i\mathbf{p}\cdot\xi/\hbar}\,\psi(\mathbf{x}+\xi,t)^*\,\overleftrightarrow{\nabla}\,\psi(\mathbf{x}-\xi,t)\;, \qquad (3.2.19)$$

where an integration by parts has been performed and the derivatives are done with respect to $\mathbf{x}$; remembering the Fourier representation of the delta function, we get

$$\int d^3p\,\mathbf{p}\,P_W(\mathbf{x},\mathbf{p}|t) = \frac{\hbar}{2i}\,\psi(\mathbf{x},t)^*\,\overleftrightarrow{\nabla}\,\psi(\mathbf{x},t) = \mathbf{j}(\mathbf{x},t)\;. \qquad (3.2.20)$$

The quantities obtained "kinetically" by Eqs. (3.2.18) and (3.2.20) are exactly the same which have already been defined, in Eqs. (3.1.4) and (3.1.5), directly from the Schrödinger equation; consequently, they will satisfy a continuity equation, and the definitions (3.2.6) and (3.1.6) for the velocity will agree with each other. The next step is to work out the expression

$$\frac{1}{m}\int d^3p\,(p_i - mv_i(\mathbf{x},t))(p_j - mv_j(\mathbf{x},t))\,P_W(\mathbf{x},\mathbf{p}|t)\;; \qquad (3.2.21)$$

the rather long calculations are reported in App. C, and establish the identity between (3.2.21) and the pressure tensor (3.1.11): We have thus been able to rederive the hydrodynamical quantities defined in Sec. 3.1 by standard techniques of kinetic theory using, in the place of the distribution function, the Wigner function $P_W(\mathbf{x}, \mathbf{p}|t)$. However, we can now do even more, defining a temperature $T(\mathbf{x}, t)$ and a heat flux $\mathbf{q}(\mathbf{x}, t)$ and checking if they satisfy an equation like (3.2.13) as a consequence of the Schrödinger equation. We find (App. C)

$$\begin{aligned} T(\mathbf{x}, t) &\equiv \frac{1}{3k\,\mu\,(\mathbf{x}, t)} \int d^3p\,(\mathbf{p} - m\mathbf{v}(\mathbf{x}, t))^2 P_W(\mathbf{x}, \mathbf{p}|t) = \\ &= -\frac{\hbar^2}{12km}\,\nabla^2 \ln \mu\,(\mathbf{x}, t)\ , \end{aligned} \tag{3.2.22}$$

and (App. D)

$$\begin{aligned} \mathbf{q}(\mathbf{x}, t) &\equiv \frac{1}{2m} \int d^3p\,(\mathbf{p} - m\mathbf{v}(\mathbf{x}, t))^2 (\mathbf{p} - m\mathbf{v}(\mathbf{x}, t))\, P_W(\mathbf{x}, \mathbf{p}|t) = \\ &= -\frac{\hbar^2}{8m}\,\mu\,(\mathbf{x}, t)\,\nabla^2 \mathbf{v}(\mathbf{x}, t)\ . \end{aligned} \tag{3.2.23}$$

In order to check if the heat transfer equation

$$k\,\mu\,\frac{DT}{dt} = -\frac{2}{3}\boldsymbol{\nabla} \cdot \mathbf{q} - \frac{2}{3} p_{ij} \Lambda_{ij} \tag{3.2.24}$$

holds for the quantities now defined[8], let us first write down the expression of Λ_{ij} for our irrotational velocity field (2.3.17); it is trivial to see that

$$\Lambda_{ij} \equiv \frac{m}{2}(\partial_i v_j + \partial_j v_i) = m\,\partial_i v_j\ . \tag{3.2.25}$$

The left hand side of Eq. (3.2.24) can be transformed with the help of the continuity equation (3.1.3), finding

$$k\,\mu\,\frac{DT}{dt} = \boldsymbol{\nabla} \cdot \left(\frac{\hbar^2 \mu}{12m} \nabla^2 \mathbf{v} \right) + \frac{\hbar^2 \mu}{6m}\,\partial_i v_j\,\partial_i \partial_j \ln \mu\ , \tag{3.2.26}$$

which, reminding Eqs. (3.2.23), (3.2.25) and (3.1.11), turns out to be exactly Eq. (3.2.24).

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[8]It is important to notice that the Eqs. (3.1.3), (3.1.7) and (3.2.24) do not follow straightforwardly from the representation of μ, etc. in terms of P_W, unless an equation for the Wigner function which play a role analogous to that of Eq. (3.2.3) for f is specified. Such an equation would be, however, equivalent to Eq. (3.1.1), and we prefer therefore to use directly this last.

We find this to be a remarkable result, since it shows the powerfulness of the "kinetic" technique now introduced. With the help of this new method, in fact, we have not only succeeded in reproducing the continuity and the Euler equations (3.1.3) and (3.1.7) for the spinless nonrelativistic quantum particle, but we have been able even to define new hydrodynamical variables in a consistent manner, in such a way that the validity of the heat transfer equation (3.2.24) is also established. It is important to remark that Eq. (3.2.24) could well have been obtained directly from the Schrödinger equation, once the expressions

$$T = -\frac{\hbar^2}{12km}\nabla^2 \ln \mu \qquad (3.2.27)$$

and

$$\mathrm{q} = -\frac{\hbar^2}{8m}\mu\,\nabla^2 \mathrm{v} \qquad (3.2.28)$$

for the temperature and the heat flux have been adopted; however, there is no justification for these definitions in such a context, and the task of deriving Eq. (3.2.24) would thus be an extremely difficult one, there being no argument which could help in recognizing the mathematical structure of T and q. This may explain why, although derivations of Eqs. (3.1.3) and (3.1.7) can be found in the literature [65,67], no one has been given for Eq. (3.2.24), at least to the author's knowledge.

The most important consequence of the method discussed in this section is, in our opinion, represented by the fact that it allows to derive quantum mechanics from the Wigner function P_W in exactly the same way in which the macroscopic laws of a gas[9] can be derived from the distribution function f. Let us analyse this point more in detail. In the kinetic theory of gases, the molecules are supposed to behave according to the laws of classical mechanics; this leads to define a distribution function $f(\mathrm{x},\mathrm{p}|t)$ which satisfies Eq. (3.2.3) and allows the hydrodynamical and thermodynamical quantities $\tilde{\mu}$, $\tilde{\mathrm{j}}$, etc. to be defined. As we have already seen, this can be done even when dealing with a single particle, provided we accept that the resulting "macroscopic" theory has only a statistical meaning, in the sense that it fully describes only ensembles, being essentially incomplete when considered at the level of

[9] Or of a dilute solution, or of a plasma, etc.

individuals. Similar considerations can be drawn for quantum theory; as we have shown in Sec. 3.1, the Schrödinger equation for a spinless particle is mathematically equivalent to the hydrodynamical equations for an irrotational perturbation in a fluid whose density, velocity and pressure fields can be related to the wave function through Eqs. (3.1.4), (2.3.17) and (3.1.11). But such a hydrodynamical formalism, when considered in the light of Eqs. (3.2.18), (3.2.20), etc. reveals unquestionably its "macroscopic" character, and its intrinsically statistical meaning if applied to the treatment of a single particle. Being the hydrodynamical formulation equivalent to the Schrödinger equation, we are thus led to conclude that the quantum mechanical description is necessarily a statistical one, and that although it fully represents the behaviour of ensembles of equally prepared systems, it fails in giving a detailed account of the individuals' behaviour. This is essentially the so called *statistical interpretation* of quantum theory, strongly supported by Einstein in the course of his debate with Bohr [37,57,27].

The inconsistencies between the fluid model and the corpuscular aspects of the particles, which we have put in evidence at the beginning of the section, no longer constitute a problem for the theory; it is now evident, in fact, that the hydrodynamical quantities (and consequently the wave function or, if we prefer, the state vector) are appropriate only to the description of ensembles, while the corpuscular features refer to individuals: The origin of the above-mentioned problems is therefore the mistake, whose nature is evidently semantic, of speaking about a single particle referring to concepts which only make sense for ensembles. A clear example which may illustrate this point is that of the single grain of pollen subjected to brownian motion: As we have already pointed out, in this case the hydrodynamical quantities can well be defined in order to provide a statistical description, but to associate them to physical properties of a specific grain would lead to totally incorrect conclusions.

Some of the questions raised in Sec. 3.1 find a natural answer in the context of this interpretation of the formalism. The existence of the operators $\hat{\mu}(\mathbf{x},t)$, $\hat{\mathbf{j}}(\mathbf{x},t)$ and $\hat{\Pi}_{ij}(\mathbf{x},t)$ given by Eqs. (2.3.11), (2.3.13) and (B.1) was hard to understand in the single particle picture, being not clear which observables they were associated to; but if we assume $|\psi\rangle$ to represent an ensemble $\mathcal{E}$ of very many equally prepared copies of the particle, it is evident that $\hat{\mu}$ and $\hat{\jmath}$ can be seen as representing observables on subensembles of $\mathcal{E}$. In order to be more explicit, let us restrict our attention to the case of $\hat{\mu}$; if

$$\mathcal{E} = \bigcup_{\alpha=1}^{n} \mathcal{E}_\alpha \ , \tag{3.2.29}$$

where each subensemble $\mathcal{E}_\alpha$ contains the same large number N of equally prepared copies of the particle, and n is also very large, it will make sense:

i) To consider the mass density as an observable of each $\mathcal{E}_\alpha$, whose value is defined measuring the position of the particle in each copy belonging to $\mathcal{E}_\alpha$, and then defining the ensemble average of them at the point $\mathbf{x}$ at the time t; more precisely, if r labels the copies of the particle in $\mathcal{E}_\alpha$, we define the mass density in $\mathcal{E}_\alpha$ as

$$\mu^{(\alpha)}(\mathbf{x},t) \equiv \frac{1}{N}\sum_{r=1}^{N} \delta^3(\mathbf{x} - \mathbf{x}_r^{(\alpha)}(t)) \ , \tag{3.2.30}$$

$\mathbf{x}_r^{(\alpha)}(t)$ being the position of the r-th copy of the particle in $\mathcal{E}_\alpha$ at the time t;

ii) To assume $|\psi\rangle$ to describe also the ensemble $\bar{\mathcal{E}} \equiv \{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$, whose members are now the subensembles $\mathcal{E}_\alpha$.

The operator $\hat{\mu}(\mathbf{x},t)$ is now clearly associated to the measurements of density in the $\mathcal{E}_\alpha$'s, performed as discussed above; its eigenstates will be those in which the $\mu^{(\alpha)}(\mathbf{x},t)$ are all the same: It is evident that this can happen only in the case of eigenstates of position, consistently with what expressed by Eq. (2.3.10). The average value of the mass density over the ensemble $\bar{\mathcal{E}}$ is given by

$$\frac{1}{n}\sum_{\alpha=1}^{n} \mu^{(\alpha)}(\mathbf{x},t) = \frac{m}{n\,N}\sum_{\alpha,r} \delta^3(\mathbf{x} - \mathbf{x}_r^{(\alpha)}(t)) \ , \tag{3.2.31}$$

which in the limits $n \to +\infty$, $N \to +\infty$ is in agreement with Eq. (3.1.15).

This interpretation of μ, $\mathbf{j}$, p_{ij} gives, as a by-product, further evidence for the unreliability of Eq. (2.3.1) as description of a single system. As we have proved in deriving (2.3.27), in fact, the various components of $\langle\psi|\hat{T}_{ab}|\psi\rangle$ reduce, in the nonrelativistic limit, to the hydrodynamical quantities μ, $\mathbf{j}$, Π_{ij} introduced in Sec. 3.1; since these objects have no meaning for an individual, but only make sense when an ensemble is considered, it follows that the same has to be said for $\langle\psi|\hat{T}_{ab}|\psi\rangle$ and, consequently, for Eq. (2.3.1).

The identity between the value (3.1.21) of the pressure and the result of the more physical calculation given in ref. [69] is no longer mysterious, too: They are in fact two different ways to evaluate the same quantity. It is important to remark, however, that this pressure is exactly on the same foot of the osmotic pressure for a single grain suspended in water: It has only a statistical meaning. After having clarified this point, we can predict that a particle contained in a box, in an eigenstate of energy, will exert discontinuous hits on the walls, and that only an average of these hits will turn out to correspond to the value given by Eq. (3.1.21) for the pressure.

The similarity between Eqs. (3.1.22) and (3.1.23) can be understood noticing that Eq. (3.1.23) can be established even for the case of a single molecule whose motion is "randomized" by some process as, for example, a brownian motion; the only delicate point is that p and E have now to be considered in the sense of statistical averages. Keeping this analogy in mind, and remembering that $\mathrm{tr}\,\bar{\mathsf{p}}$ is already an average quantity, we should look for a reason for identifying the energy E of the quantum particle with some ensemble average. This can be successfully done as follows: The kinetic formalism gives, for the energy density $\varepsilon(\mathbf{x},t)$,

$$\varepsilon(\mathbf{x},t) \equiv \int d^3p \left(\frac{\mathbf{p}^2}{2m} + U(\mathbf{x},t)\right) P_W(\mathbf{x},\mathbf{p}|t) \;, \qquad (3.2.32)$$

which becomes, remembering Eq. (3.2.17),

$$\varepsilon = -\frac{\hbar^2}{4m}(\psi^*\nabla^2\psi+\nabla^2\psi^*\psi)+\frac{\hbar^2}{8m}\boldsymbol{\nabla}\cdot(\psi^*\boldsymbol{\nabla}\psi+\boldsymbol{\nabla}\psi^*\psi)+\psi^*\psi\,U \;. \qquad (3.2.33)$$

It is possible to define also a *local* energy $W(\mathbf{x},t)$ as

$$W(\mathbf{x},t) \equiv \frac{m}{\mu(\mathbf{x},t)}\,\varepsilon(\mathbf{x},t) \;; \qquad (3.2.34)$$

using Eqs. (2.3.17) and (3.2.22) we have

$$W(\mathbf{x},t) = \frac{1}{2}m\mathbf{v}(\mathbf{x},t)^2 + U(\mathbf{x},t) + \frac{3}{2}kT(\mathbf{x},t) \;, \qquad (3.2.35)$$

which is an evocative splitting of W into its "macroscopic" and "microscopic" components. Finally, we shall call *global* energy E the integral

$$E \equiv \int d^3x\, \varepsilon(\mathrm{x}, t) \;, \tag{3.2.36}$$

which can be rewritten as

$$E = \int d^3x\, \psi(\mathrm{x}, t)^*\, H(\mathrm{x}, -i\hbar\nabla, t)\, \psi(\mathrm{x}, t) \;, \tag{3.2.37}$$

the total divergence in Eq. (3.2.33) not contributing to the result: It is thus immediate to identify E with the quantum mechanical expectation value for the particle energy. Combining now Eqs. (3.2.34) and (3.2.35), and remembering Eqs. (3.2.27) and (3.1.11), we get

$$\frac{V}{3}\mathrm{tr}\,\bar{\mathsf{p}} = \frac{2}{3}E - \int d^3x\, \left(\frac{1}{2}\mu\,\mathrm{v}^2 + \frac{\mu}{m}U\right) \;, \tag{3.2.38}$$

where V is the volume of the region over which ψ is different from zero, and $\bar{\mathsf{p}}$ is the pressure tensor averaged over such region. Eq. (3.2.38) reduces to Eq. (3.1.22) for $\mathrm{v} = \mathbf{0}$, $U = 0$, and it is easy to interpret also in the general case; keeping in mind that all the quantities in it have to be considered as ensemble averages, the right hand side can be regarded (apart a numerical factor) as the "thermal" contribution to the energy: The terms which are subtracted to $2E/3$ come in fact from the "ordered" motion.

It is worth noticing that the equation of state for the fluid quantities is, in the case of the quantum particle,

$$\frac{1}{3}\mathrm{tr}\,\mathsf{p} = \frac{k}{m}\,\mu T \;, \tag{3.2.39}$$

as it follows straightforwardly from Eq. (C.6). Eq. (3.2.39) is the same relation holding for a perfect gas, but there is nevertheless a strong difference between these two systems: While for the ideal gas pressure, density and temperature are generally, except for the equation of state, arbitrary, in the case of the quantum particle the pressure tensor and the temperature are determined by the spatial distribution of density, through Eqs. (3.1.11) and (3.2.27). This is a situation which, because of the less freedom involved, reminds more the case of a gas subjected to a defined thermodynamical transformation.

At the end of Sec. 3.1, we noticed that the spreading of wave packets can be alternatively formulated as an effect of pressure. Adopting the point of view of the statistical interpretation, and keeping in mind the analogy with the kinetic theory which was stimulated by the representation of μ, $\mathbf{j}$, etc. in terms of P_W, such a result has nothing extraordinary in it: Actually, it is nothing else than an example of the well known process of *diffusion*. To clarify this point, let us come back to the case of the grains subjected to brownian motion; creating a concentration in a region of the vessel and waiting for some time, the density of the concentration will lower, while its size will increase, simply for the elementary reason that there will be more grains leaving the region of over-density than entering it. The same argument can be applied to the case of a single grain, provided density, pressure, etc. are now considered as quantities characterizing an ensemble, and all the reasoning are performed in a statistical sense. It is thus clear that, in the case of the quantum particle, if we regard (3.1.24) as the probability distribution for position in an ensemble, it is perfectly reasonable to interpret the spread of the wave packet as the result of a diffusion process, even though only at a statistical level.

3.3 Beyond Quantum Theory?

The conclusions reached until now can be summarized saying that quantum theory, in both the hydrodynamical and the standard (Hilbert space) formulations, describes completely an *ensemble* of equally prepared systems, but fails in giving a satisfactory picture of the behaviour of a single *individual*. As we have already noticed, this is essentially the content of the statistical interpretation and is not new at all, dating back to Einstein [57,27]. The point of view that we have developed is, however, richer, since it provides some hints about the character of the processes which may possibly take place beyond the limits of the quantum mechanical description. Our presentation shows in fact a striking similarity between the treatment of particles as given by quantum theory, and the statistical behaviour of molecules which arises in the context of kinetic theory; this makes unavoidable to ask such questions as: Can we represent the quantum particles as following well defined – even if unspecified – trajectories? Can the statistical features of quantum mechanics be reduced to a matter of incomplete description? Can quantum theory be regarded as the "thermodynamical limit" of something more fundamental?

Before tackling the task of trying to answer these questions, let us briefly illustrate the situation for what have been our classical counterpart in Sec. 3.2, i.e., the grain of pollen suspended in water and subjected to brownian motion. In the model which is generally adopted to understand this system, the water molecules collide with each other and with the grain; because of the large size of this latter with respect to the molecules, in general the hits over the grain result in a zero net force on it; however, from time to time this balance is not exact, and the grain changes its momentum very quickly: This gives rise to the zigzag path which is typical of the brownian motion. Both the grain and the molecules of water move, in this model, according to the laws of classical mechanics: In particular, the concept of trajectory has a well defined meaning and, what is even more important, the entire system is a deterministic one. However, it is clearly impossible, in practice, to prepare two copies of the water in the vessel which are in the same microstate: This implies that even preparing, in two distinct experiments, a grain in the same initial state[10], its future behaviour will be ruled by laws which are essentially statistical. Nevertheless, it is important to keep in mind that the statistical features of this description have their origin in deficiencies of the process of state preparation; although such inaccuracy requires a statistical theory in order to account for the experimental results, it is not incompatible with the fundamentally deterministic character of the laws (classical mechanics) which rule the behaviour of the individual systems. As we already know, the mathematical object which encodes all the informations which are relevant in order to describe the statistical behaviour of the grain (or, more precisely, the behaviour of a statistical ensemble of grains), is the distribution function $f(\mathrm{x}, \mathrm{p}|t)$: The knowledge of the microscopic theory underlying brownian motion depends thus on one's ability to derive the peculiar features of f (i.e., Eq. (3.2.3) and the form of the collisional term in it) in terms of classical mechanics and general probability theory.

[10]That is, with the same initial position and momentum.

It is remarkable that such a kinetic model to explain brownian motion can be established *indirectly*, i.e., without observing experimentally the motions of the molecules. The strategy to follow is not particularly difficult: It is sufficient, in fact, to assume the validity of classical mechanics at the molecular level, together with some assumptions about the particular system under consideration: This leads to find the distribution function as a solution of Eq. (3.2.3). The knowledge of $f(\mathbf{x}, \mathbf{p}|t)$ allows then to derive some consequences which are macroscopically observable, and hence testable at the level of the laboratory. Actually, this is just what Einstein did in 1905, in order to provide evidence in favour of the atomic hypothesis [74,71].

The analogy between this well known case and the current situation of quantum theory is quite evident, and we find it enlightening when looking for a nontrivial answer to the third question we have asked above: Is quantum mechanics the "thermodynamics" of some deeper theory?

The possibility to represent the hydrodynamical quantities μ, $\mathbf{j}$, etc. in terms of the Wigner function P_W, makes us rather biased in favour of an affirmative answer. Nevertheless, this only shifts the problem, because in such a case we must be able to give a derivation of the form of $P_W(\mathbf{x}, \mathbf{p}|t)$ in terms of the theory which lies beyond quantum mechanics and governs the behaviour of individuals, in a way analogous to that in which $f(\mathbf{x}, \mathbf{p}|t)$ is deduced from classical mechanics. Contrarily to what happens in this last case, however, the underlying theory is now unknown: The most natural thing to do is therefore to analyse the properties of P_W in order to gather some information about such theory.

Before coming to this point, however, we should like to discuss the concrete example of a theory of such kind which has been suggested almost forty years ago, and is still under debate: Bohm's theory [60,61]. The key idea on which it is based is to assume that the quantum particle possess a well defined trajectory, but that its motion is different from the one predicted by classical mechanics, because of the presence of an extra influence which can be described by a *quantum potential* $Q(\mathbf{x}, t)$. In order to determine explicitly the expression of Q, let us take advantage from the representation (2.3.15) for the wave function; after substitution into the Schrödinger equation (3.1.1) and separation into the real and imaginary parts, one gets the pair of Eqs. (3.1.3) and (3.1.8). The crucial point lies in recognizing that Eq. (3.1.8) has the form of the Hamilton-Jacobi equation, provided $S(\mathbf{x}, t)$ is identified with the Hamilton's principal function and the particle motion is supposed to be affected by the additional potential

$$Q(\mathbf{x}, t) \equiv -\frac{\hbar^2}{2m} \frac{\nabla^2 |\psi|}{|\psi|} . \tag{3.3.1}$$

In this theory, the continuous (fluid) and discrete (corpuscular) aspects of the quantum particle appear therefore unified in the wave function $\psi(\mathbf{x}, t)$, whose modulus plays the role of "pilot field" for a particle described, in a "classical" way, by the Eq. (3.1.8) for the phase $S(\mathbf{x}, t)$: The primary meaning of $|\psi(\mathbf{x}, t)|$ is hence, here, that of a field determining the quantum correction (3.3.1) to the classical equation of motion, and it is only through some assumptions about the random features of the environment that it turns out that $|\psi(\mathbf{x}, t)|^2$ characterizes also the probability density for the presence of the particle in $\mathbf{x}$ at the time t [75,76]; the probabilistic interpretation of ψ is thus, in Bohm's theory, a consequence rather than a postulate.

It might seem, at a first sight, that Bohm's theory could provide us with what we were looking for, i.e., a theory of the individual particles' behaviour which admits ordinary quantum mechanics as the result of some kind of average process over statistical ensembles; we shall now show, however, that this is not the case, and that Bohm's theory itself only involves *average* quantities, being thus so unsuccessful in describing individuals as standard quantum theory is. In order to discuss this point, let us consider first an example from classical theory [77]; for a perfect gas with temperature $\tilde{T}(\mathbf{x}, t)$ and density $\tilde{\mu}(\mathbf{x}, t)$, the pressure tensor is diagonal, with $\tilde{p}_{ij}(\mathbf{x}, t) = \tilde{p}(\mathbf{x}, t)\, \delta_{ij}$. If the gas is subjected to an external field, so that $U(\mathbf{x}, t)$ is the corresponding potential energy per molecule, the velocity field $\tilde{\mathbf{v}}(\mathbf{x}, t)$ must satisfy the Euler equation

$$\tilde{\mu} \frac{\tilde{D}\tilde{\mathbf{v}}}{dt} = -\nabla \tilde{p} - \frac{\tilde{\mu}}{m} \nabla U , \tag{3.3.2}$$

where $\tilde{D}/dt$ is given by Eq. (3.2.14) and m is now the mass of a molecule. For sake of simplicity, let us restrict ourselves to study the irrotational isentropic perturbations; we shall have, by hypothesis,

$$\tilde{\mathbf{v}}(\mathbf{x},t) = \frac{1}{m}\nabla\tilde{S}(\mathbf{x},t) \ , \tag{3.3.3}$$

for some scalar function $\tilde{S}$, and

$$\tilde{p}(\mathbf{x},t) = K\,\tilde{\mu}\,(\mathbf{x},t)^{\gamma} \ , \tag{3.3.4}$$

with K a constant and γ the adiabatic index. Substituting Eqs. (3.3.3) and (3.3.4) into Eq. (3.3.2) we get after some steps

$$\frac{\partial\tilde{S}}{\partial t} + \frac{1}{2m}(\nabla\tilde{S})^2 + U + \frac{\gamma}{\gamma-1}\,mK\,\tilde{\mu}^{\gamma-1} = 0 \ . \tag{3.3.5}$$

If we define now a "thermal potential" $\tilde{Q}(\mathbf{x},t)$ as

$$\tilde{Q} \equiv \frac{\gamma}{\gamma-1}\,mK\,\tilde{\mu}^{\gamma-1} \ , \tag{3.3.6}$$

Eq. (3.3.5) has the form of an Hamilton-Jacobi equation for $\tilde{S}$, with a correction $\tilde{Q}$ whose origin lies in the thermal features of the system[11]. It is clear, however, that Eq. (3.3.5) does not determine the motion of any *physical* particle, but rather it describes the dynamics of the elements of fluid or, if we prefer, it accounts for the *averaged* molecular motion; this can be understood remembering that, from the Hamilton-Jacobi theory, if a particle trajectory crosses the point $\mathbf{x}$ at the time t, its corresponding momentum is

$$\mathbf{p}(t) = \nabla\tilde{S}(\mathbf{x},t) \tag{3.3.7}$$

which, by Eq. (3.3.3), would give

$$\mathbf{p}(t) = m\,\tilde{\mathbf{v}}(\mathbf{x},t) \ ; \tag{3.3.8}$$

but we know from kinetic theory that $\tilde{\mathbf{v}}$ only represents the macroscopic velocity of the gas, which is the average of the molecules' velocities: Therefore Eq. (3.3.5) does not describe the motion of real molecules, but only of fictitious ones which follow the streamlines of the flow. To finish with this example, we want to remark that, even if we have always

[11]Actually, $\tilde{Q}$ can be identified with the enthalpy referred to a molecule of gas [77]; having we instead considered an isothermal, rather than an adiabatic, flow, we would have found the "thermal potential" to be just the Gibbs free energy per molecule.

spoken about a gas, i.e., a many-particles system, the same arguments and conclusions apply to the case of a single particle such as the grain subjected to brownian motion in the water; the only difference, as it is clear from Sec. 3.2, is that $\tilde{T}$, $\tilde{\mu}$ and $\tilde{p}$ have now to be reinterpreted as ensemble averages, rather than concrete quantities of a fluid: The Hamilton-Jacobi equation (3.3.5) describes therefore the dynamics of a fictitious grain, obtained averaging over a large ensemble of grains all prepared with the same initial conditions.

With the experience acquired in working out this classical analogue, let us return to discuss Bohm's theory; since we are examining the possibility that it might provide the "microdynamics" underlying standard quantum theory and describing individual processes, we shall adopt the point of view that Bohm's theory should be consistent with the statistical interpretation developed in Sec. 3.2. In order to see more clearly the analogy with the previous example, let us think to quantum mechanics as expressed in its hydrodynamical formulation, and to Eq. (3.1.8) as if derived from the Euler equation (3.1.7) in a way which is completely similar to the one in which Eq. (3.3.5) has been obtained from Eq. (3.3.2): Formally, this amounts in practice to repeating, backwards, the calculations performed in Sec. 3.1. It is now immediate to interpret (3.1.8) as the equation describing the dynamics of the fluid elements; in fact it is sufficient to notice that, by Hamilton-Jacobi theory and Eq. (2.3.17), it follows

$$\mathbf{p}(t) = \nabla S(\mathbf{x}, t) = m\,\mathbf{v}(\mathbf{x}, t)\ . \qquad (3.3.9)$$

We have now come to the crucial point of our argument: Eq. (3.3.9) shows that what Bohm's theory predicts is that the trajectories of individual particles coincide with the streamlines of the flow in the quantum fluid. Therefore, Bohm's theory can be accepted as a treatment at the "subquantum" level only if $m\,\mathbf{v}(\mathbf{x}, t)$ is the momentum of the particle when passing at the point $\mathbf{x}$ at the time t: Eq. (3.2.20) shows that this is not the case, and that $m\,\mathbf{v}(\mathbf{x}, t)$ should rather be regarded as an *average* momentum. Hence, it would be incorrect to assume Bohm's theory to be a description of physical processes of which quantum mechanics represents, somehow, the "thermodynamics": In fact Eq. (3.1.8) is equivalent to Eq. (3.1.7) exactly in the same way as Eq. (3.3.5) is equivalent to Eq. (3.3.2); this proves that Bohm's theory describes phenomena exactly at the same level of the hydrodynamical formulation[12], and is a statistical theory, too. As such, it does not give a more detailed treatment than standard quantum theory do: The only difference lies in the underlying picture, which admits the concept of particles' trajectories as meaningful. However, our analysis shows that these cannot represent the real behaviour of particles, but have in the best case to be regarded only as results of a statistical average. Such a conclusion seems to point in the direction of theories like the Bohm-Vigier's one [78] or Nelson's stochastic mechanics [62,63], in which the quantum particles are supposed to follow trajectories which are intrinsically irregular, of which those predicted by Bohm's theory are the average.

A further remarkable consequence of our argument is that the quantum potential $Q(\mathbf{x}, t)$ turns out to have no fundamental meaning, as far as the behaviour of a single particle is concerned. In fact, just as the thermal potential $\tilde{Q}(\mathbf{x}, t)$ is not relevant to the molecular motions, but enters only in determining the dynamics of the fluid elements, so $Q(\mathbf{x}, t)$ accounts for the quantum features of the motion at the ensemble level[13], but cannot be meaningfully used in the description of the behaviour of an individual. It follows that the quantum potential is not the right object to consider when dealing with such fundamental topics as nonseparability, since it may contain spurious features coming from its statistical nature; we rather suggest to use the distribution function $P_W(\mathbf{x}, \mathbf{p}|t)$ as a better tool in treating these delicate subjects.

The example now considered shows how difficult and intricate it may be to construct in detail a "subquantum" theory, which could account for quantum mechanics at a statistical level of description. It seems therefore wise not to try to invent explicitly any of such theories, but rather to concentrate the efforts in inferring something about their structure, through a careful analysis of the properties of the phase space distribution function. When starting this program, however, we imme-

[12] And, of course, of the standard Hilbert-space formulation based on the Schrödinger equation.

[13] Under this respect, the dependance of Q on the probability distribution $|\psi|^2$ (or on the density μ, if we prefer), is not a curious feature at all, and could even be expected *a priori*; in fact, if Q has to describe an average motion, it must contain information about the ensemble over which to average, information which resides in $|\psi|^2$. The same argument can be used to argue that $\tilde{Q}$ must depend on $\tilde{\mu}$.

diately run into a serious problem, namely the nonuniqueness of the joint quasiprobability distribution $P(\mathbf{x}, \mathbf{p}|t)$. It is well known, in fact, that there is an entire class of them [79,80], each one correctly reproducing the results of standard quantum theory; this arbitrariness can be traced to the existence of many ordering rules for constructing quantum operators [131], and both are connected to the lack of a unique prescription in the evaluation of "skeletonized" path integrals [81,82,83,84,85,86,87]. On the other hand, if we wish to interpret a distribution function in terms of "subquantum" phenomena, such nonuniqueness appears very tricky, and it would be attractive to guess that, although present in the quantum mechanical description, it would disappear when the underlying theory is considered, leaving a unique distribution $P(\mathbf{x}, \mathbf{p}|t)$. This may seem a completely gratuitous hypothesis, and one would be inclined to think that, even if correct, it could not be proved until the fundamental theory is available; although we have no general results, we nevertheless argue that this is not the case, and that even at our incomplete level of knowledge there is evidence for discarding some distribution functions in favour of others.

To prove this statement, let us return to our derivation of the hydrodynamical quantities μ, $\mathbf{j}$ and p_{ij} from the Wigner function P_W, as given in Sec. 3.3, and let us ask what would have happened having we used, instead of P_W, the Margenau-Hill function [88]

$$P_s(\mathbf{x}, \mathbf{p}|t) \equiv \frac{1}{(2\pi\hbar)^3} \Re \int d^3\xi \, \psi(\mathbf{x} + \boldsymbol{\xi}, t)^* e^{i\mathbf{p}\cdot\boldsymbol{\xi}/\hbar} \psi(\mathbf{x}, t) \, . \tag{3.3.10}$$

It is immediate to check that the results of the calculation of the mass density and current turn out to be exactly the same of those expressed in Eqs. (3.2.18), (3.2.19) and (3.2.20). In fact

$$m \int d^3p \, P_s(\mathbf{x}, \mathbf{p}|t) = m \, |\psi(\mathbf{x}, t)|^2 = \mu\,(\mathbf{x}, t) \, , \tag{3.3.11}$$

and

$$\int d^3p \, \mathbf{p} \, P_s(\mathbf{x}, \mathbf{p}|t) = \Re\,(i\hbar\, \psi(\mathbf{x}, t) \boldsymbol{\nabla} \psi(\mathbf{x}, t)^*) = \mathbf{j}(\mathbf{x}, t) \, ; \tag{3.3.12}$$

consequently, also the velocity field $\mathbf{v}(\mathbf{x}, t)$ will be identical to the one defined in Sec. 3.3. When considering the pressure tensor, however, a discrepancy appears. Let us work out, in fact, the expression

$$\frac{1}{m} \int d^3p \, (p_i - m\, v_i(\mathbf{x}, t)) \, (p_j - m\, v_j(\mathbf{x}, t)) \; P_s(\mathbf{x}, \mathbf{p}|t) \, , \tag{3.3.13}$$

analogous to (3.2.21); following calculations similar to those of App. C, one arrives to identify (3.3.13) with

$$p_{ij} - \frac{\hbar^2}{4m^2}\partial_i\partial_j\mu \ , \tag{3.3.14}$$

where p_{ij} is given by Eq. (3.1.11): Surprisingly enough, the use of the Margenau-Hill function has produced a result which differs from the one obtained using the Wigner distribution.

We could think that the extra term

$$-\frac{\hbar^2}{4m^2}\partial_i\partial_j\mu \tag{3.3.15}$$

is such that it belongs to the class of tensors C_{ij} which can be arbitrarily added to p_{ij}, as discussed in commenting Eq. (3.1.11); however, such tensors must be divergence free, and it is trivial to check that this is not true for (3.3.15). Hence, we are led to reject the Margenau-Hill function as a viable phase space distribution; in fact, its use would lead to the pressure tensor (3.3.14), which does not allow to establish a satisfactory hydrodynamical formalism. Let us notice, to this extent, that the term (3.3.15) does not produce a gradient in Eq. (3.1.9) when inserted into Eq. (3.1.7), and is therefore not equivalent to a simple additive "potential" in Eq. (3.1.8): This would involve major changes in the Schrödinger equation, which we rather prefer not to modify. Nevertheless, one could say that the validity of the Euler equation (3.1.7) relies on the dynamical equation satisfied by the distribution function, and that in the case of the Margenau-Hill function it may be that to Eq. (3.1.7) must be added some counterterms which cancel the extra contributions coming from (3.3.15), thus reproducing exactly the Schrödinger equation (3.1.1). This argument may turn out to be correct and worth investigating; however, we find the modifications it would eventually introduce in the hydrodynamical formalism to be not only unpalatable, but also unnecessary. We thus prefer to drop the Margenau-Hill function, without altering the simplicity and elegance of the formulation.

Further support in favour of this choice comes from a reexamination of the discussion performed at the end of Sec. 2.3 in commenting the differences between Eqs. (2.3.21) and (2.3.22). In the light of the statistical interpretation developed in Sec. 3.2, we can summarize such results saying that the use of the Weyl's ordering in the formation of the operator $\hat{\Pi}_{ij}$ leads to the pressure tensor (3.1.11), while adopting the symmetrization rule one is led to (3.3.14). Since this last, for the reasons discussed above, does not allow to write down a satisfactory hydrodynamical formulation of quantum theory, we are induced to adopt the Weyl's rather than the Rivier's ordering; this conclusion is not surprising at all, if we think that the use of the Margenau-Hill distribution corresponds exactly to that of the ordering S in constructing products of noncommuting operators.

The consequences of these arguments for performing a selection leaving only with $P_{\mathcal{W}}$ and $\mathcal{W}$ are far reaching, because of the existence of the above-mentioned interconnections between the choices of phase space distribution functions, operator ordering and path integration. In the case now considered, we are led not only to prefer the Wigner function to the Margenau-Hill one, and the Weyl's ordering to the Rivier's symmetrization rule, but also the midpoint prescription when dealing with skeletonized path integrals[14]. We want to stress, however, that we do not claim at all that the Wigner function is the *correct* phase space distribution, nor that such a distribution exists: All that we have proved is that, if one wishes to adopt the methods of kinetic theory in reproducing the hydrodynamical formalism of quantum theory starting from a distribution function, then the Margenau-Hill function fails in accomplishing this program in a satisfactorily way, while the Wigner function succeeds. Of course, this is not a proof of the absolute reliability of the Wigner function; it may well be that it fails, too, in accounting for more sophisticated details, leaving room for a better candidate. The really interesting point of our argument consists, in our opinion, in the fact that the use of the kinetic representation of the hydrodynamical quantities allows to remove an arbitrariness which is present at the level of pure quantum mechanics; this makes entirely reasonable the possibility that, at the end, only one distribution function may turn out to be correct, and that the nonuniqueness currently exhibited be due to the incompleteness of the quantum theoretical treatment.

[14]It is worth remarking that claims about a privileged role of the Weyl rule have already been advanced on purely formal grounds [84]; it was pointed out later, however, that such arguments were essentially unjustified [81,86]. Our point of view is completely different, being based on a physical, rather than a formal, discussion.

It should be not surprising that the kinetic viewpoint show itself to be stronger than the hydrodynamical one, since it corresponds to a more accurate level of description of phenomena; after all, the same happens in classical physics. We have an example of this higher effectiveness of the kinetic formalism in the fact that it does not allow for the C_{ij}-arbitrariness mentioned in Sec. 3.1, when defining the pressure tensor; in fact, the expression (3.2.21) turns out to be exactly equal to p_{ij} as given by Eq. (3.1.11), and rules out, for example, the alternative tensor P_{ij} of Eq. (3.1.12), which would be also allowed by the hydrodynamical formalism. This is the reason why we have chosen to use (3.1.11) throughout all our treatment.

We have not been able to give a definitive answer to the questions raised at the beginning of this section. If a "subquantum" theory exists, certainly it is not classical mechanics, nor Bohm's theory; in fact, the former would correspond to phase space distributions different from the Wigner function, while the latter turns out to be just a reformulation of quantum theory holding, as such, at the statistical level as well. Bohm's theory has, nevertheless, the great merit of having explicitly shown that the same results of standard quantum mechanics can be obtained without giving up the concept of definite particles trajectories; the same conclusion can be drawn from Nelson's stochastic mechanics which is, however, a younger theory from a historical viewpoint. This is an important remark, which can be generalized observing that *the formalism of quantum theory does not forbid, in principle, the existence of particles trajectories in a possible underlying theory; it is only its extrapolations based on a positivistic philosophical attitude which do so.* Since we prefer not to adhere to the positivistic belief, which has already proved itself to be quite dangerous when acritically applied to a scientific context (see, e.g., Mach's refutation of the existence of the atoms), we shall remain open with respect to the possibility of thinking in terms of particles trajectories, and we shall not consider them as an unreasonable concept. This attitude will prove useful in treating semiclassical systems.

Chapter 4

The Weakly Semiclassical Regime

The discussion of Ch. 3 sheds new light on the semiclassical theory of gravity. Adopting the statistical interpretation of quantum theory compels to reformulate the problem in such a way that the very concept of field equations turns out to be applicable only to a rather limited kind of semiclassical behaviours; this leads to distinguish between a strongly and a weakly semiclassical regime. The description of the latter is the main subject of the present chapter.

In Sec. 4.1 we discuss qualitatively how the statistical interpretation of quantum mechanics involves a statistical description of classical gravity as well. In Sec. 4.2 a general analysis of semiclassical systems is performed in the light of the statistical interpretation, and the distinction between the strongly and weakly semiclassical behaviours is clearly formulated; moreover, an hypothesis is enunciated, which allows to give a quantitative description of systems in the weakly semiclassical regime. Sec. 4.3 contains some specific examples, illustrating the application of these general principles. Finally, in Sec. 4.4 we show how some serious problems emerge when the classical subsystem is a relativistic field, which require, in order to be solved, a much more sophisticated treatment of the quantum source.

The treatment in this chapter is based essentially on ref. [31].

4.1 Statistical Character of Semiclassical Gravity

As we have seen in the previous chapter, quantum mechanics is an essentially statistical theory; that this statistical character be a mere consequence of the incompleteness of the theory, as suggested in Sec. 3.3, or that it reflect a fundamental feature of the natural laws, is still a matter of debate and speculations. Here, we shall adopt the pragmatic point of view that, since quantum theory provides the best description we have, at the moment, of phenomena occurring at the microscopic level, such a description is necessarily statistical. Hence, any theory of quantum matter turns out to be a theory of ensembles.

This assumption has a profound influence on the formulation of a semiclassical problem, and changes it drastically. In order to investigate this point, let us observe quite generally that there are essentially two possible levels of description of a physical system, which we shall call, for convenience, of first and second kind: Either the detailed state of a copy of the system is described (individual, or first kind, description), or a probability distribution for the values of its observables is given (ensemble, or second kind, description). According to the statistical interpretation, standard quantum theory is of the second kind (although some of its modifications are of the first kind); on the contrary, classical gravitational theory is of the first kind. The problem is: Within a semiclassical context, can we give a first kind description of gravity, if the matter's description is of the second kind? It is not difficult to realize that, in answering to this question, two different situations have to be considered, according to the size of the statistical fluctuations in the quantum observables which act as source of classical gravity.

To make this point clear, let us consider the example of newtonian semiclassical gravity, in which the classical potential $\Phi(\mathrm{x}, t)$ has a nonrelativistic quantum particle of mass m as source. If $|\psi\rangle$ represents the state of an ensemble $\mathcal{E}$ of similarly prepared particles, we can imagine that to each copy in $\mathcal{E}$ correspond a definite $\Phi(\mathrm{x}, t)$. Since the potential Φ depends, classically, on the position of the source, but not on its momentum, we expect that when $|\psi\rangle$ is sharply peaked around a definite point of space, the various $\Phi(\mathrm{x}, t)$ corresponding to the different copies in $\mathcal{E}$ do not differ from each other; in these cases it is therefore meaningful to speak of a definite classical field compatible with quantum matter or, in other words, to obtain a description of the first kind for Φ from a description of the second kind for the particle. However, if $|\psi\rangle$ represents a superposition of position eigenstates, different copies of the particle will correspond, in general, to different $\Phi(\mathbf{x}, t)$; the specification of the state $|\psi\rangle$ of $\mathcal{E}$ does not allow, in this case, to give a description of the first kind of Φ, which has now to be treated on statistical grounds as well. In a more expressive way, we can say that the indeterminacy of matter has been "transmitted" to the field.

We can thus infer that, in the semiclassical regime, it is necessary to distinguish between a "strongly" semiclassical behaviour, in which the classical subsystem admits a description of the first kind, and a "weakly" semiclassical one, in which it is not possible to go beyond a description of the second kind. The criterion for distinguishing between these two regimes consists in calculating the amount of statistical fluctuations in the quantum source. For the case of gravity, this is done by evaluating the quantities [89]

$$\Delta T_{abcd}(x,y)^2 \equiv |\langle\psi|\hat{T}_{ab}(x)\hat{T}_{cd}(y)|\psi\rangle - \langle\psi|\hat{T}_{ab}(x)|\psi\rangle\langle\psi|\hat{T}_{cd}(y)|\psi\rangle| \,, \tag{4.1.1}$$

and studying their limit when $y \to x$; if they are negligible, the behaviour turns out to be strongly semiclassical, and weakly semiclassical otherwise. A much simpler calculation of

$$\Delta\mu\,(\mathbf{x},\mathbf{y})^2 \equiv |\langle\psi|\hat{\mu}\,(\mathbf{x})\hat{\mu}\,(\mathbf{y})|\psi\rangle - \langle\psi|\hat{\mu}\,(\mathbf{x})|\psi\rangle\langle\psi|\hat{\mu}\,(\mathbf{y})|\psi\rangle| \tag{4.1.2}$$

for newtonian gravity, is presented in App. E, and confirms our heuristic arguments in a simple case; moreover, it shows explicitly how careful one has to be in performing this kind of formal manipulations, because of the occurrence of divergent expressions.

These considerations imply that one cannot, in general, recover a description of the first kind for the classical component of a semiclassical system; in particular, it is impossible to have, in the weakly semiclassical regime of gravity, a complete knowledge about the state of a single copy of spacetime. Conveniently rephrased, this means that the hope of obtaining a physically meaningful metric tensor g_{ab} out of $\hat{T}_{ab}$ and $|\psi\rangle$

in a unique way, is purely illusory in the weakly semiclassical regime. This happens because $|\psi\rangle$ does not describe a single quantum system, but an entire ensemble $\mathcal{E}_m$ of similarly prepared copies of it, and each copy in $\mathcal{E}_m$ is compatible, in general, with a different spacetime; the correct formal treatment of gravity has not to be based, therefore, on the concept of a metric g_{ab} of spacetime, but rather on that of a *probability distribution* $P[g]$ for the metrics of the copies of spacetime which belong to an ensemble $\mathcal{E}_g$ compatible with $\mathcal{E}_m$. The concept of semiclassical field equations, providing a unique correspondence between the state of quantum matter and a single spacetime geometry, hold no longer, and have to be replaced by a prescription relating $P[g]$ to $|\psi\rangle$ and $\hat{T}_{ab}$. We must nevertheless remark that, in the strongly semiclassical regime, the dispersion $\Delta T_{abcd}(x,x)$ is small; therefore, the dispersion in g_{ab} will turn out to be correspondingly small, and $P[g]$ will be strongly peaked on some particular metric, thus allowing to speak meaningfully of a definite spacetime compatible with its quantum matter content. Hence, in this case, the semiclassical field equations still make sense; it is only in the weakly semiclassical regime that such a concept cannot be applied any more. It is interesting to observe that the lack of knowledge about the state of the classical component in a weakly semiclassical system should not be surprising, because this is the circumstance closest to the full quantum behaviour that we can conceive.

Of course, even in weakly semiclassical gravity it is possible to define an *average* metric $\langle g_{ab}\rangle$ over $\mathcal{E}_g$, as

$$\langle g_{ab}\rangle \equiv \int_{\mathcal{E}_g} \mathcal{D}g\, P[g]\, g_{ab} \ , \tag{4.1.3}$$

$\mathcal{D}g$ representing a convenient measure; moreover, it is possible to establish a unique correspondence between $\hat{T}_{ab}$ and $|\psi\rangle$, and $\langle g_{ab}\rangle$, thus allowing to resurrect the concept of field equations. However, dealing now with ensemble averages, the field equations lose much of their charm. They would be useful, in fact, only if a collection of similarly prepared, identical matter systems would be available as source of the field; such situation can be easily realized in the domain of atomic physics (and, actually, semiclassical calculations for the electromagnetic field in such a context are often successfully carried on [19]), but in applications of general relativity (particularly in cosmology), the relevant physical system is generally present in a single copy, and this makes therefore the field equation for $\langle g_{ab} \rangle$ uninteresting.

These arguments can be used also to refine some of the criticisms raised in Sec. 2.4 to Eqs. (2.3.1)–(2.3.3). In fact, Eq. (2.3.1) must now be reinterpreted as an equation for ensembles averages, and not for physical quantities; consequently, it would not be very attractive even if it would turn out to be formally correct. However, this change in the interpretation has a drastic effect on the reliability of Eq. (2.3.2), which cannot be considered as meaningful if formulated in a spacetime whose metric satisfies Eq. (2.3.1); as said above, in fact, such a metric is only an average over $\mathcal{E}_g$, and does not describe, in general, any physical gravitational field. This consideration leaves thus the quantum dynamical problem temporary unspecified, giving no prescription for the classical background over which to study the state evolution. It is possible to infer, however, that in the newtonian limit the arbitrariness disappears, leaving with the ordinary Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi \tag{4.1.4}$$

instead of Eq. (2.4.6); this can be understood thinking that the solution Φ of Eq. (2.4.5) is now only an average potential, whose source is the unphysical density of mass $m|\psi|^2$; the real potential for each individual copy of the particle in $\mathcal{E}_m$ does not affect the motion of the particle to this order of approximation, leading to Eq. (4.1.4), and consequently reducing the nonlinearity which we have referred to in commenting Eqs. (2.4.5)–(2.4.6).

4.2 The Weakly Semiclassical Hypothesis

There is a situation in ordinary quantum mechanics which is closely analogue to the one of the gravitational semiclassical problem as described above: The quantum measurement process [37,55]. In fact, let us consider a microscopic system $\mathcal{Q}$, coupled to a macrosystem $\mathcal{C}$ in such a way that a classical observable c of $\mathcal{C}$ acts as a pointer for the measurement of a quantum observable q of $\mathcal{Q}$; then the behaviour of c is driven by that of q: If $\mathcal{Q}$ is in a state corresponding to a well defined value of q, also the value of c will be exactly determined, while if the state of $\mathcal{Q}$ is such that the dispersion of q is not negligible, the repetition of the measurement process will give, in general, different values for c. However, if there is a one-to-one correspondence

$$c = f(q) \tag{4.2.1}$$

between the values of c and q, the distributions for their spectra will be the same, in the sense that

$$P(c) = P(q) \ , \tag{4.2.2}$$

where $P(q)$ is the probability for the outcome q of a measurement of the microobservable, as predicted by the quantum theory, and $P(c)$ is the relative frequency with which the pointer is observed to mark the value $c = f(q)$.

The situation in semiclassical gravity is quite similar to this, and it is easy to realize that it is possible to establish the following correspondences:

- microsystem $\mathcal{Q} \longleftrightarrow$ quantum matter;
- macrosystem $\mathcal{C} \longleftrightarrow$ gravitational field (classical spacetime);
- quantum observable $q \longleftrightarrow$ matter source (T_{ab});
- classical observable $c \longleftrightarrow$ field intensity (g_{ab}).

It is clear, therefore, that to require the validity of an equation of the type (2.3.1) or (2.4.5) would be as absurd as to require, in the quantum measurement process, that the value of c be determined by the expectation value of q; such a relation can only be established, in some cases, for the mean value of c, through Eq. (4.2.2). Let in fact be Eq. (4.2.1) the injective[1] function linking the spectra of c and q; then, repeating many times the measurement, the average value of c will be

$$\langle c \rangle = \sum_c P(c)\, c = \sum_q P(q)\, f(q) \ , \tag{4.2.3}$$

[1]This requirement will be dropped later on.

where Eqs. (4.2.1) and (4.2.2) have been used. Assuming analiticity of $f(q)$, we can expand it as

$$f(q) = a_0 + a_1 q + a_2 q^2 + \cdots ; \tag{4.2.4}$$

moreover, defining the dispersion Δq as

$$\Delta q^2 \equiv \sum_q P(q) q^2 - \langle q \rangle^2 , \tag{4.2.5}$$

we easily get, from Eq. (4.2.3), the relation

$$\langle c \rangle = f(\langle q \rangle) + a_2 \Delta q^2 + \cdots , \tag{4.2.6}$$

where the terms on the right hand side of Eq. (4.2.6) that are different from $f(\langle q \rangle)$

i) depend on Δq and on higher moments of the distribution for the spectrum of q;

ii) depend on coefficients of order higher than that of a_1 in the expansion (4.2.4).

From *i)* and *ii)* it follows straightforwardly that Eq. (4.2.1) implies the corresponding relation between mean values

$$\langle c \rangle = f(\langle q \rangle) \tag{4.2.7}$$

if and only if one of the two following conditions is satisfied:

1. The observable q behaves classically, i.e., its dispersion and higher moments are negligible;

2. The relation (4.2.1) is linear.

In the case of gravity, the metric g_{ab} does not depend linearly on T_{ab}; we expect therefore that, except in the trivial situations in which the source behaves nearly classically, Eqs. (2.3.1) are inappropriate to describe the matter-gravity coupling, even in a statistical sense.

The previous discussion turns out to be very instructive, because it provides us with a model from which the general requirements of a semiclassical theory can be extracted. It is therefore not necessarily restricted to the case of the measurement process, but rather $\mathcal{Q}$ and $\mathcal{C}$ can be two more general systems, interacting with each other by means of a suitable coupling. $\mathcal{Q}$ must be allowed to exhibit *quantum* behaviour in some of its observables, one of those we call q, while $\mathcal{C}$ exhibit *classical* behaviour in at least one observable c; moreover, the interaction between $\mathcal{Q}$ and $\mathcal{C}$ is such that q and c are not independent. This is what we mean, in general, by a semiclassical system; the goal of our theory is to study the related behaviour of q and c.

There is a problem which arises immediately in such a physical situation: To what extent can it be said that c behaves classically, when interaction between the two systems is present? The answer can be given thinking to the specific example of the measurement process, in which c represents the pointer's position. What we ask in this case is that the state of the measuring device (i.e., $\mathcal{C}$) be such that there is a negligible overlapping between different values of c, which, from this point of view, is a classical observable; however, it is pretty obvious that the specific value of c cannot be predicted, being the pointer's position triggered by the quantum observable q, whose behaviour is unpredictable. Therefore, for the most general semiclassical system, we must admit that c displays no interference, but nevertheless its values cannot be predicted: Only a probability can be associated to them.

In general, when speaking of "classical behaviour" of an observable c, we make two distinct requirements:

a) There is no interference between different values of c;

b) Preparing many copies of the system in the same way, the same value of c will be realized.

Property a) takes into account the fact that interference is not observed at a classical level, while b) expresses the "deterministic" character of classical observables. As discussed above, in a generic semiclassical system the observable c can satisfy only a): We thus believe it is justified to call its behaviour "weakly classical", and the resulting theory of $\mathcal{C} + \mathcal{Q}$ "weakly semiclassical".

For a weakly semiclassical system, therefore, only a probability $P(c)$ can be assigned to the values of the observable c; it follows that expressions like

$$c = f(\langle q \rangle) \tag{4.2.8}$$

(which are of the same kind of Eqs. (2.3.1) and (2.4.5)) have no meaning at all; only "statistical" equations like (4.2.7) can be meaningful. This remark is important in order to realize that, if dealing with a single system, then equations of the kind of (4.2.7), even if correct, are useless: In fact their left hand side is an average that can be performed only on several copies, similarly prepared, of the system. All that can be said in this case is about $P(c)$. Conversely, if many copies of the system are available, equations like (4.2.7) can not only be tested, but they can even be used to compute a kind of "mean" value for c, which is physically relevant, in the case these copies act together at the same time; this is the typical situation underlying the experiments of atomic physics in which the electromagnetic field due to quantum-behaving sources is computed through the Maxwell equations [19].

Let us now face the task of investigating in more details the features of a weakly semiclassical system, and how can a theory of it be formulated in general. We shall study again the previous model, consisting of a system $\mathcal{Q}$ with an observable q behaving quantum mechanically, and a system $\mathcal{C}$ with an observable c exhibiting classical behaviour. When the two systems are coupled together, the behaviour of c becomes only weakly classical, since there is still no interference between different values of the observable, but they cannot be predicted any more.

Our starting point will be the hypothesis that, whenever $\mathcal{Q}$ belongs to an ensemble described by a state $|q, \sigma\rangle$ of well defined q (σ takes into account the possible degeneracy), the observable c of $\mathcal{C}$ takes the value given by the "coupling equation"

$$c = f(q) \; ; \tag{4.2.9}$$

we must remind, however, that the function f could be not injective (as it may happen, for example, in the case of a bad measurement). Now, if Q is described by[2]

$$|\psi\rangle = \sum_{q,\sigma} \psi(q,\sigma,t)|q,\sigma,t\rangle \ , \tag{4.2.10}$$

where $\sum$ stands, from now on, for a sum or an integral, accordingly with the discrete or continuous character of the spectrum of the considered observables, then the probability for the value q at time t is

$$P(q|t) = \sum_{\sigma} |\psi(q,\sigma,t)|^2 \ . \tag{4.2.11}$$

Being the function f of Eq. (4.2.9), in general, not injective, a particular value of c can be associated to several values of q: We define, therefore, by

$$P(c,q|t)$$

the probability that the value c, at time t, "come" from the value q of the other observable. In other words, $P(c,q|t)$ can be regarded as a joint probability for the two observables at the same time t. It is well known that we can thus write

$$P(c,q|t) = P(c,t|q,t)\,P(q|t) \ , \tag{4.2.12}$$

where $P(c,t|q,t)$ is the conditional probability for the value c at time t, given the value q for the microobservable at the same time. As usual, the functions in Eq. (4.2.12) will be subject to the following conditions:

$$\sum_{c} P(c,t|q,t) = 1 \ ; \tag{4.2.13}$$

$$\sum_{q} P(c,q|t) = P(c|t) \ ; \tag{4.2.14}$$

the normalization of $P(c|t)$ then follows from that of $P(q|t)$. We are now ready to introduce in our model an analogue of Eq. (4.2.2), which will be based on what we shall call the "*weakly semiclassical hypothesis*" (WSH), whose content we state as follows:

WSH: The probability for the macroobservable to have the value c at time t is the sum of the probabilities $P(q|t)$ over all the distributions of q which are classically compatible with that value.

[2] We work in the Heisenberg picture.

The WSH can be formally written as

$$P(c|t) = \sum_q \alpha(c,q;t)\, P(q|t) \ , \tag{4.2.15}$$

where

$$\alpha(c,q;t) = \begin{cases} 1 & \text{if } c = f(q) \\ 0 & \text{if } c \neq f(q) \ ; \end{cases} \tag{4.2.16}$$

this obviously implies $\alpha(c,q;t) = \delta(c, f(q))$, and, by Eqs. (4.2.12) and (4.2.14),

$$P(c,t|q,t) = \delta(c, f(q)) \ . \tag{4.2.17}$$

This equation holds also in the case of continuous spectra, provided the P are interpreted as probability densities, and δ is a Dirac function rather than a Kronecker symbol. It is interesting to notice that Eq. (4.2.17) reproduces the value given by Eq. (4.2.3) for the average of c; in fact

$$\begin{aligned} \langle c(t) \rangle &\equiv \sum_c P(c|t)c = \sum_{c,q} P(c,q|t)c = \\ &= \sum_{c,q} P(q|t)\,\delta(c,f(q))c = \sum_q P(q|t)f(q) \ . \end{aligned} \tag{4.2.18}$$

The main ideas discussed so far are summarized in Table 4.1, which emphasizes analogies and differences among the possible behaviours of the system $\mathcal{C} + \mathcal{Q}$. Here we want only to make a couple of remarks concerning the physical meaning of the WSH.

First of all, the central role played by the *classical* coupling equations (and, thus, by the classical field equations) in the method has to be stressed. Although no coupling equations can be written in the weakly semiclassical regime, there is still a correspondence between the quantum subsystem and the classical one, which is represented by the relation between $P(q)$ and $P(c)$. The classical coupling equations are essential in selecting those values of q which are associated to a definite value of c; in such a way, they establish a connection between the spectra of the two observables. This connection is "sharp" in the weakly semiclassical regime, because no "self-uncertainty" effects[3] are considered for c, and this allows $P(c)$ to be completely determined by $P(q)$.

[3]Or, if one prefers, "self-dispersion" effects.

CLASSICAL REGIME
• Well defined values of c and q. • The coupling equation $$c = f(q)$$ is perfectly meaningful.
STRONGLY SEMICLASSICAL REGIME
• Well defined values of c; "almost well defined" values of q ($\Delta q \approx 0$). • The semiclassical coupling equation $$\langle c \rangle = f(\langle q \rangle)$$ is meaningful (although only for statistical averages).
WEAKLY SEMICLASSICAL REGIME
• Only $P(c)$ and $P(q)$ can be defined. • The coupling equation is meaningless, but the WSH allows to determine completely $P(c)$ from $P(q)$.
QUANTUM REGIME
• Only $P(c)$ and $P(q)$ can be defined. • $P(q)$ does not determine completely $P(c)$.

Table 4.1: A comparison between the four regimes of behaviour of the compound system $\mathcal{C} + \mathcal{Q}$.

Moreover, it has to be specified that, although not explicitly stated in its formulation, the WSH is hard to justify within the context of an interpretation of quantum theory which deny any possible attribution of a well defined value of the observable q to the subsystem $\mathcal{Q}$, except for the case in which $|\psi\rangle$ is an eigenvalue of $\hat{q}$. On the other hand, the WSH appears entirely reasonable if $|\psi\rangle$ is assumed to describe an ensemble of copies of $\mathcal{Q}$, a fraction $P(q|t)$ of which have, at time t, a value q of the observable at issue. That such a possibility do not violate the uncertainty principle, as it might appear at first, has been carefully discussed by Ballentine [27]; in addition, we would like to remark that, if the weakly semiclassical approach now suggested should turn out to be successful, it would constitute a strong indirect support for such a "realistic" interpretation of quantum mechanics.

4.3 Examples

The previous general analysis can be applied successfully to the particular case of a newtonian gravitational field, whose potential obeys the classical equation

$$\nabla^2\Phi = 4\pi G\mu \, , \tag{4.3.1}$$

where μ is the classical density of mass, which, in the case of a single point particle of mass m, located at y at time t, is

$$\mu\,(\mathrm{x},t) = m\,\delta^3(\mathrm{x}-\mathrm{y}) \, . \tag{4.3.2}$$

In this context, the role of the quantum observable q is played by the particle's position y, while the classical observable c is now the value of Φ at some point x of space, $\Phi(\mathrm{x})$; since there is no degeneracy, the probability density is given directly by

$$P(\mathrm{y}|t) = |\psi(\mathrm{y},t)|^2 \, , \tag{4.3.3}$$

where ψ is the wave function of the particle. Let us notice that $|\mathrm{y},t\rangle$ is not only an eigenstate of position at time t, but also of mass density, with (4.3.2) as eigenvalue:

$$\hat{\mu}\,(\mathrm{x},t)|\mathrm{y},t\rangle = m\,\delta^3(\mathrm{x}-\mathrm{y})|\mathrm{y},t\rangle \; ; \tag{4.3.4}$$

a particle in the state[4] $|\mathbf{y}, t\rangle$ corresponds therefore, by Eq. (4.3.1), to a well definite gravitational potential

$$\Phi(\mathbf{x}, t|\mathbf{y}) = -\frac{Gm}{|\mathbf{x}-\mathbf{y}|} , \tag{4.3.5}$$

under the boundary condition that Φ vanish as $|\mathbf{x}|$ tends to infinity.

We are here in presence of an example of the situation treated previously, when f is not injective; if $\Phi(\mathbf{x})$ is a particular value of the field, then we shall indicate by

$$P(\Phi(\mathbf{x}); \mathbf{y}|t)$$

the joint probability density, at time t, for Φ at $\mathbf{x}$ to have the value $\Phi(\mathbf{x})$ and for the particle position to be $\mathbf{y}$. Using Eq. (4.2.12) we can write $P(\Phi(\mathbf{x}); \mathbf{y}|t)$ in terms of $P(\mathbf{y}|t)$ and of the conditional probability density

$$P(\Phi(\mathbf{x})|\mathbf{y}, t) ,$$

expressing the probability density that, at time t, the field at $\mathbf{x}$ have the value $\Phi(\mathbf{x})$ due to a particle at $\mathbf{y}$.

Accordingly to Eq. (4.2.17) we shall write the WSH for Φ as

$$P(\Phi(\mathbf{x})|\mathbf{y}, t) = \delta\left(\Phi(\mathbf{x}) - \Phi(\mathbf{x}, t|\mathbf{y})\right) . \tag{4.3.6}$$

The probability density that the field at time t be $\Phi(\mathbf{x})$ is, by Eqs. (4.2.14) and (4.3.6),

$$\begin{aligned} P(\Phi(\mathbf{x})|t) &= \int d^3y\, P(\Phi(\mathbf{x}); \mathbf{y}, t) = \\ &= \int d^3y\, \delta\left(\Phi(\mathbf{x}) - \Phi(\mathbf{x}, t|\mathbf{y})\right) P(\mathbf{y}|t) , \end{aligned} \tag{4.3.7}$$

which takes into account the fact that classical particles located at different points can give origin to the same field at $\mathbf{x}$; this point can be better understood working out explicitly the expression (4.3.7). Defining the new variable

$$\eta \equiv \mathbf{x} - \mathbf{y} , \tag{4.3.8}$$

and remembering Eq. (4.3.5) and the properties of the delta function, we find

$$P(\Phi(\mathbf{x})|t) = \frac{Gm}{\Phi(\mathbf{x})^2} \int d^3\eta\, P(\mathbf{x}-\eta|t)\, \delta\left(|\eta| + \frac{Gm}{\Phi(\mathbf{x})}\right) ; \tag{4.3.9}$$

[4] Or, more precisely, in a normalized state strongly picked around $\mathbf{y}$ at time t.

introducing now the unit vector $\mathbf{n}$, collinear with η, such that $\eta = |\eta|\mathbf{n}$, it is easy to get the expression

$$P(\Phi(\mathbf{x})|t) = \frac{G^3 m^3}{\Phi(\mathbf{x})^4} \int_{|\mathbf{n}|=1} d^2 n \, P\left(\mathbf{x} + \frac{Gm}{\Phi(\mathbf{x})}\mathbf{n} \middle| t \right) , \qquad (4.3.10)$$

where the integral is performed over the unit 2-sphere, i.e., only in the angular coordinates. Eq. (4.3.10) has the following physical meaning: Suppose that the value of the field at $\mathbf{x}$ is equal to $\Phi(\mathbf{x})$; this implies that the particle producing it could be everywhere on a sphere of radius $-Gm/\Phi(\mathbf{x})$ around $\mathbf{x}$; therefore, in order to compute the probability for Φ at $\mathbf{x}$ to have that value, we must integrate, according to the WSH, the probability for the particle's position over such a sphere. This argument can be extended to the case of other fields, provided the spherical symmetry is preserved.

In our theory, Eq. (2.4.5) is recovered as an average: In fact, the mean value of Φ in $\mathbf{x}$ at time t is

$$\begin{aligned} \langle \Phi(\mathbf{x}, t) \rangle &\equiv \int d\Phi(\mathbf{x}) \, P(\Phi(\mathbf{x})|t) \, \Phi(\mathbf{x}) = \\ &= \int d^3 y \, |\psi(\mathbf{y}, t)|^2 \, \Phi(\mathbf{x}|\mathbf{y}, t) \, ; \end{aligned} \qquad (4.3.11)$$

remembering that

$$\nabla^2 \Phi(\mathbf{x}, t|\mathbf{y}) = 4\pi G m \, \delta^3(\mathbf{x} - \mathbf{y}) \, , \qquad (4.3.12)$$

we get immediately

$$\nabla^2 \langle \Phi(\mathbf{x}, t) \rangle = 4\pi G m |\psi(\mathbf{x}, t)|^2 = 4\pi G \langle \mu(\mathbf{x}, t) \rangle \, , \qquad (4.3.13)$$

which is just Eq. (2.4.5), but written now for $\langle \Phi \rangle$.

The use of the WSH can be justified, provided it leads, in ordinary situations, to the same results of the usual formalism. Let us therefore investigate a model in which two particles $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively of masses m_1 and m_2, are moving under their reciprocal influence; moreover, let us suppose that $m_2 \gg m_1$. We shall set out this problem first in the standard formulation, then according to the WSH and, finally, in an approximation in the same spirit of Eq. (2.4.5).

1. **Standard theory:** The hamiltonian operator of the system is, in the position (Schrödinger) representation,

$$\hat{H} = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(|\mathbf{x}_1 - \mathbf{x}_2|, t) \ , \tag{4.3.14}$$

where $V(|\mathbf{x}_1 - \mathbf{x}_2|, t)$ is the interaction potential energy between $\mathcal{P}_1$ and $\mathcal{P}_2$; $\hat{H}$ acts on the wave function $\psi(\mathbf{x}_1, \mathbf{x}_2; t)$ in the configuration space. Under the coordinate transformation

$$\mathbf{r} \equiv \mathbf{x}_1 - \mathbf{x}_2 \tag{4.3.15}$$

$$\mathbf{R} \equiv \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} \ , \tag{4.3.16}$$

and with the definition

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2} \ , \tag{4.3.17}$$

the hamiltonian (4.3.14) takes the form

$$\hat{H} = -\frac{\hbar^2}{2(m_1 + m_2)}\nabla_R^2 - \frac{\hbar^2}{2\mu}\nabla_r^2 + V(r, t) \ , \tag{4.3.18}$$

which allows for the separation of the wave function as

$$\psi(\mathbf{x}_1, \mathbf{x}_2; t) = \psi_R(\mathbf{R}, t)\psi_r(\mathbf{r}, t) \ , \tag{4.3.19}$$

where ψ_R and ψ_r obey the equations

$$i\hbar\frac{\partial\psi_R(\mathbf{R}, t)}{\partial t} = -\frac{\hbar^2}{2(m_1 + m_2)}\nabla_R^2\psi_R(\mathbf{R}, t) \tag{4.3.20}$$

$$i\hbar\frac{\partial\psi_r(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2\mu}\nabla_r^2\psi_r(\mathbf{r}, t) + V(r, t)\,\psi_r(\mathbf{r}, t) \ . \tag{4.3.21}$$

Requiring now that $m_2 \gg m_1$, we get

$$m_1 + m_2 \approx m_2$$

$$\mu \approx m_1$$

$$\mathbf{R} \approx \mathbf{x}_2 \ ,$$

so that ψ_R describes completely $\mathcal{P}_2$ as a free particle, and ψ_r describes $\mathcal{P}_1$ as moving in a field due to $\mathcal{P}_2$. The probability density that $\mathcal{P}_1$ and $\mathcal{P}_2$ have, respectively, positions $\mathbf{x}_1$ and $\mathbf{x}_2$ at time t is therefore

$$P(\mathbf{x}_1, \mathbf{x}_2 | t) \approx |\psi_r(\mathbf{x}_1 - \mathbf{x}_2, t)|^2 \cdot |\psi_R(\mathbf{x}_2, t)|^2 \ . \tag{4.3.22}$$

2. **WSH:** Being $m_2 \gg m_1$, we can suppose that the behaviour of $\mathcal{P}_2$ is independent of $\mathcal{P}_1$, and that $\mathcal{P}_2$ is thus described by a wave function $\psi_2(\mathrm{x}_2, t)$ obeying the Schrödinger equation

$$i\hbar \frac{\partial \psi_2}{\partial t} \approx -\frac{\hbar^2}{2m_2} \nabla_2^2 \psi_2 \ . \qquad (4.3.23)$$

Now, if $\mathcal{P}_2$ were fixed at a point x_2 at time t, then $\mathcal{P}_1$ would be described by a wave function $\psi_1(\mathrm{x}_1, t|\mathrm{x}_2)$ subjected to the equation

$$i\hbar \frac{\partial \psi_1}{\partial t} = -\frac{\hbar^2}{2m_1} \nabla_1^2 \psi_1 + V(|\mathrm{x}_1 - \mathrm{x}_2|, t)\psi_1 \ ; \qquad (4.3.24)$$

but the probability to find $\mathcal{P}_2$ in such a configuration is $|\psi_2(\mathrm{x}_2, t)|^2$, and the WSH requires than that the probability for the potential energy in x_1 at time t to be $V(|\mathrm{x}_1 - \mathrm{x}_2|, t)$ is also $|\psi_2(\mathrm{x}_2, t)|^2$. Therefore ψ_1 will obey Eq. (4.3.24) with a probability $|\psi_2(\mathrm{x}_2, t)|^2$, and the probability density to find $\mathcal{P}_1$ at x_1 and $\mathcal{P}_2$ at x_2 at time t will be

$$P(\mathrm{x}_1, \mathrm{x}_2; t) \approx |\psi_1(\mathrm{x}_1, t|\mathrm{x}_2)|^2 \cdot |\psi_2(\mathrm{x}_2, t)|^2 \ . \qquad (4.3.25)$$

Moreover, from Eq. (4.3.24) it is clear that ψ_1 has the form

$$\psi_1(\mathrm{x}_1, t|\mathrm{x}_2) = \tilde{\psi}_1(\mathrm{x}_1 - \mathrm{x}_2, t) \ , \qquad (4.3.26)$$

which, substituted into Eq. (4.3.25), leads to a result in complete agreement with the one obtained by the method 1.

3. **Mean field approximation:** The calculations are the same as in the previous method, but the last term of Eq. (4.3.24) is not proportional to

$$V(|\mathrm{x}_1 - \mathrm{x}_2|, t) \ , \qquad (4.3.27)$$

but rather to

$$\langle \psi_2 | V(|\mathrm{x}_1 \hat{1} - \hat{\mathrm{x}}_2|, t) | \psi_2 \rangle = \int d^3x_2 \, V(|\mathrm{x}_1 - \mathrm{x}_2|, t) |\psi_2(\mathrm{x}_2, t)|^2 \ . \qquad (4.3.28)$$

Eq. (4.3.28) can be easily justified in the case of a newtonian interaction, by noticing that

$$\begin{aligned} &\nabla_1^2 \int d^3x_2 \, V(|\mathrm{x}_1 - \mathrm{x}_2|, t) |\psi_2(\mathrm{x}_2, t)|^2 = \\ &= 4\pi G m_1 m_2 \int d^3x_2 \, \delta^3(\mathrm{x}_1 - \mathrm{x}_2) |\psi_2(\mathrm{x}_2, t)|^2 = \\ &= 4\pi G m_1 m_2 |\psi_2(\mathrm{x}_1, t)|^2 \ , \end{aligned} \qquad (4.3.29)$$

which is a relation of the same kind of Eq. (2.4.5).

It is clear that, being Eq. (4.3.28), in general, quite different from Eq. (4.3.27), this last method will not reproduce, unless in very specific cases, the results of the exact theory, and is thus unreliable.

We interpret the identity of the results obtained by the methods 1 and 2 as evidence for the association to the potential $\Phi(\mathbf{x}, t|\mathbf{y})$, due to a particle located at $\mathbf{y}$, of a probability density $|\psi(\mathbf{y}, t)|^2$. The use of the light test particle $\mathcal{P}_1$ is only motivated by the need to compare the consequences of the two treatments with those of standard quantum theory. This result seems thus to support the WSH, at least in the nonrelativistic case.

4.4 Application to Relativistic Fields

The formulation of the WSH given in Sec. 4.2 is suitable for the treatment of some situations of interest, but nevertheless it presents several drawbacks when applied to the context of relativistic fields.

In order to understand which are the problems and how they can be solved, let us consider, to fix the ideas, a scalar field ϕ obeying the classical inhomogeneous Klein-Gordon equation in the Minkowski spacetime,

$$\eta^{ab}\partial_a\partial_b\phi = J \ , \tag{4.4.1}$$

where J is a matter source for ϕ. It is useful to represent $\phi(x)$ in the form of a Kirchhoff integral by introducing the Green function $D(x, x')$ defined as a solution of the equation

$$\eta^{ab}\partial_a\partial_b D(x - x') = -\delta^4(x - x') \ ; \tag{4.4.2}$$

as well known [32] this amounts to splitting ϕ into two components ϕ_s and ϕ_b,

$$\phi = \phi_s + \phi_b \ , \tag{4.4.3}$$

with

$$\phi_s(x) \equiv -\int_N d^4x'\, D(x - x')J(x') \tag{4.4.4}$$

and

$$\phi_b(x) \equiv \int\limits_{\partial N} d\Sigma(x')\, n^a(x')\, D(x - x')\overleftrightarrow{\nabla}'_a \phi(x') \; ; \qquad (4.4.5)$$

in Eq. (4.4.5) $d\Sigma$ denotes the measure on the hypersurface ∂N, boundary of the spacetime domain N, and $n^a(x')$ is the normal to ∂N at x'. The decomposition (4.4.3) contains an intrinsic arbitrariness in the definitions of ϕ_s and ϕ_b, due to the fact that Eq. (4.4.2) determines $D(x-x')$ only up to a solution of the homogeneous Klein-Gordon equation. This fact does not create any technical problem at a classical level, because only ϕ is operatively meaningful, neither ϕ_s nor ϕ_b being measurable quantities: All the possible splittings (4.4.3) are therefore consistent both formally and physically.

When the source is allowed to behave quantum mechanically, however, such an arbitrariness makes the formulation of the WSH ambiguous. Suppose, in fact, that a choice of the Green function D (and, consequently, of the decomposition (4.4.3)) has been performed; then it is clear that only ϕ_s can inherit a well defined probability distribution from the matter present in N. Let this be $P_s(\phi_s(x))$; the probability distribution for $\phi(x)$ can therefore be written as

$$P(\phi(x)) = \int d\phi_s(x)\, P(\phi(x)|\phi_s(x))\, P_s(\phi_s(x)) \; , \qquad (4.4.6)$$

where $P(\phi(x)|\phi_s(x))$ is the conditioned probability for the field to have value $\phi(x)$ if its source-dependent component value is $\phi_s(x)$. If $P_b(\phi_b(x))$ denotes the probability for the value $\phi_b(x)$ of the component ϕ_b, we can write, by Eq. (4.4.3),

$$P(\phi(x)|\phi_s(x)) = P_b(\phi(x) - \phi_s(x)) \; ; \qquad (4.4.7)$$

Eq. (4.4.6) becomes thus

$$P(\phi(x)) = \int d\phi_s(x)\, P_s(\phi_s(x))\, P_b(\phi(x) - \phi_s(x)) \; . \qquad (4.4.8)$$

When considering the case in which N is the entire spacetime, the behaviour of ϕ_s is still determined by matter, but ϕ_b represents now a totally source-free field. Since the underlying philosophy of the WSH requires the field to have no quantum properties of its own, but only those which are induced by matter, it follows that the probability distribution of ϕ_b must be a completely classical one, i.e.,

$$P_b(\phi_b(x)) = \delta\left(\phi_b(x) - \phi_0(x)\right) , \qquad (4.4.9)$$

where ϕ_0 denotes a well defined free field; introducing Eq. (4.4.9) into Eq. (4.4.8), we get

$$P(\phi(x)) = P_s(\phi(x) - \phi_0(x)) . \qquad (4.4.10)$$

The field ϕ is observable, so $P(\phi(x))$ cannot depend on the arbitrary splitting (4.4.3); however, the right hand side of Eq. (4.4.10) looks as splitting-dependent, and this leads to a contradiction if another decomposition (4.4.3) (i.e., another Green function) is chosen. Therefore, the WSH can be applied in an unambiguous manner to a relativistic field only if a prescription about the choice of D is given; at this stage, this represents a highly unsatisfactory feature of the theory, but we shall not worry too much about it, since similar problems are a typical plague of the classical field theories, and can probably be solved only by suitable assumptions of cosmological nature [90,91]. Hence, from now on we shall work with an unspecified Green function $D(x - x')$, supposing that the ambiguity above has been removed by a convenient prescription. Moreover, we shall make the explicit hypothesis that $\phi_0(x) = 0$; Eq. (4.4.10) is thus simplified in

$$P(\phi(x)) = P_s(\phi(x)) , \qquad (4.4.11)$$

which allows us to drop the suffix s hereafter.

Much more serious problems arise when trying to apply explicitly the WSH to the scalar field ϕ. As it is clear from Eq. (4.4.1), $J(x)$ must be a scalar in the Minkowski spacetime; this suggests to define, for a particle following the world line γ labeled by the proper time $\tau \in \mathbb{R}$,

$$J(x|\gamma) \equiv Q \int_{-\infty}^{+\infty} d\tau\, \delta^4(x - x(\tau)) , \qquad (4.4.12)$$

where Q is a constant representing the "scalar charge" of the particle, and $\tau \mapsto x(\tau)$ defines the particle history in spacetime. Eq. (4.4.4) implies now that the classical field ϕ at the point x due to a particle whose world line is γ is

$$\phi(x|\gamma) = -Q \int_{-\infty}^{+\infty} d\tau \, D(x - x(\tau)) \; . \tag{4.4.13}$$

It is instructive to consider the case of a nonrelativistic source, in which $t(\tau) \approx \tau$; moreover, let us suppose, to fix the ideas, that D is the retarded Green function [32]

$$\begin{aligned} D^{(ret)}(x - x') &\equiv \frac{1}{2\pi} \, \Theta(t - t') \, \delta((x - x')^2) = \\ &= \frac{1}{4\pi|\mathrm{x} - \mathrm{x}'|} \, \delta(|\mathrm{x} - \mathrm{x}'| - (t - t')) \; , \end{aligned} \tag{4.4.14}$$

Θ being the step function. Under these hypothesis, one finds easily

$$\phi(\mathrm{x}, t|\gamma) = -\frac{Q}{4\pi|\mathrm{x} - \mathrm{x}(t_{ret})|} \; , \tag{4.4.15}$$

where the retarded time t_{ret} satisfies the equation

$$t_{ret} = t - |\mathrm{x} - \mathrm{x}(t_{ret})| \; . \tag{4.4.16}$$

In other words, only the intersections of the particles world lines with the past light cone of (t, x) contribute to the field at (t, x); when imposing the WSH, this would require to have a probability distribution normalized over such a light cone; alternatively, one could ask for a probability density *in spacetime*. Both these objects are not specified by the present version of relativistic quantum theory, which only assignes probabilities over a spacelike hypersurface, being based on a 3+1 decomposition of spacetime. One might try to solve this problem by resorting to a 3+1 description of the field as well, thus renouncing to an explicitly covariant treatment; more precisely, it is possible to consider the value of the field at a point of *space*, rather than of *spacetime*, as the physical observable of the theory, and to ask for the probability $P(\phi(\mathrm{x})|t)$ that, at time t, a measurement of ϕ at the point x of *space* give the value $\phi(\mathrm{x})$. However, it is not difficult to realize that even such a treatment does not remove the normalization problem.

Having ascertained the failure of the conventional treatments, let us resort to unconventional ones. In Eq. (4.4.13), the field has been labeled by the greek letter γ in order to express its dependence on the world line of the particle; it may be thus that it is γ, and not the particle position, which plays the fundamental role when writing the WSH explicitly. Let us therefore ask fore the joint probability $P(\phi(x);\gamma)$ that the field at the spacetime point x be $\phi(x)$, and that the particle follow the world line γ; this will be expressed, as usual, by

$$P(\phi(x);\gamma) = P(\phi(x)|\gamma)\,P[\gamma]\;, \tag{4.4.17}$$

where $P(\phi(x)|\gamma)$ is fixed by the WSH as

$$P(\phi(x)|\gamma) = \delta\left(\phi(x) - \phi(x|\gamma)\right)\,. \tag{4.4.18}$$

In Eq. (4.4.17), $P[\gamma]$ stands for a probability functional assigned by quantum theory in the space of all the possible world lines of the particle.

Unfortunately, not only no such object is defined in the literature[5], but its very existence could be seriously questioned, because it might seem that the concept of trajectory is incompatible with the uncertainty principle. However, there are at least three good arguments to reply to these objections: First, as we have already remarked, neither the *formalism* of quantum theory nor its *statistical interpretation* forbid to take into consideration the idea of well defined particles trajectories; it is only the "weakly realistic" philosophy developed by the Copenhagen school to discard it as meaningless on strictly operationalistic grounds. Second, there is already a formulation of quantum theory based on the concept of particle path, which has been developed by Feynman [93] and has proved to be as successful as the standard theory is. Third, as first shown by Wigner [72], it is possible to assign a (sometimes negative) joint quasiprobability $P(\mathbf{x},\mathbf{p}|t)$ to the position and momentum of the particle at the same time t (see Ch. 3 for more details).

For these reasons, we believe that to look for the functional $P[\gamma]$ is not unreasonable, and we shall devote Chs. 5 and 6 to a discussion of this problem. As we shall see, it will turn out that not only $p[\gamma]$ can be effectively constructed, but that in defining its relativistic version, a formulation of quantum theory is recovered which allows to define a probability density $P(x)$ *in spacetime*, thus providing the solution also to the normalization problem mentioned above in this section.

[5]With the remarkable exception of a paper by Dirac [92].

Chapter 5

Quasiprobability Functional Technique in Nonrelativistic Quantum Theory

Some of the problems arising when one tries to apply the WSH to the context of relativistic fields derive from the fact that, in standard quantum theory, no prescription can be assigned about the joint probability of canonically conjugate observables at the same time. However, since the earlier work of Wigner [72], a phase space description of quantum mechanics has been developed [94] which admits the introduction of functions of $\mathbf{x}$ and $\mathbf{p}$ at time t behaving as probability distributions – except for the fact that they are not nonnegative everywhere. One would thus be tempted to apply such a formulation to the description of the matter source in semiclassical fields. It turns out, however, that it is possible to go even further, defining a more fundamental object $P[\gamma]$, which represents the quasiprobability for a quantum system to evolve along the phase space *trajectory* γ. This allows to represent quantum theory in a way which is very close to the classical picture, dealing only with (quasi-) probabilities and not with probability amplitudes; such a technique, conveniently generalized to the relativistic domain in the next chapter, will prove to be extremely powerful for the applications to semiclassical theories.

In Sec. 5.1 we give a more detailed discussion of our purposes, and briefly resume some concepts and notations of the path integral formalism which will turn out to be useful later on. In Sec. 5.2 we show

how to define explicitly a real quasiprobability density functional $P[\gamma]$ for the Feynman paths; in the same section, we give a short review of Dirac's approach to the problem, and compare his and our expressions for $P[\gamma]$. Sec. 5.3 is devoted to showing how, from the quasiprobability $P[\gamma]$, the usual probability distributions for position and momentum can be deduced. In Sec. 5.4 we derive the Wigner and the Margenau and Hill functions as path integrals, over suitably chosen sets of trajectories, of $P[\gamma]$. Finally, in Sec. 5.5, we comment on our results and mention further possible developments of the method here developed.

The treatment follows that of ref. [87].

5.1 Preliminaries

The key idea of Feynman's formulation of quantum mechanics [95,96] is to consider the amplitude for a system to evolve between the times t' and t'', with $t' < t''$, as a sum, with appropriate measure, over all the possible histories satisfying some prescribed boundary conditions at t' and t''. If we are interested, for example, in the case of a single particle in three dimensional space (the system with which we shall deal throughout this chapter, any generalization of it being straightforward), we could ask, e.g., for the amplitude that the particle go from the point $\mathbf{x}'$ at time t', to the point $\mathbf{x}''$ at time t'', thus fixing the boundary conditions for the histories over which the sums will be performed.

For reasons that will become clear later in this section, a history will not be considered as a curve in the extended phase space of the particle, but rather as the pair of curves given by $t \mapsto \mathbf{x}(t)$ and $t \mapsto \mathbf{p}(t)$ in its extended configuration and momentum spaces. With this convention in mind, the amplitude can be formally represented as

$$K(\mathbf{x}'', t''; \mathbf{x}', t') = \int_{\Gamma_1} \mathcal{D}\gamma \, K[\gamma] \, , \tag{5.1.1}$$

where Γ_1 is the set of histories satisfying the boundary conditions mentioned above, i.e., such that $\mathbf{x}(t') = \mathbf{x}'$, $\mathbf{x}(t'') = \mathbf{x}''$; the amplitude $K[\gamma]$ assigned to the history γ is written as

$$K[\gamma] = \exp\left(\frac{i}{\hbar} S[\gamma]\right) \, , \tag{5.1.2}$$

with $S[\gamma]$ the action evaluated along γ:

$$S[\gamma] = \int_{t'}^{t''} dt\, [\mathbf{p}(t) \cdot \dot{\mathbf{x}}(t) - H(\mathbf{x}(t), \mathbf{p}(t))] \ , \qquad (5.1.3)$$

where H is the Hamiltonian, which we suppose, for the sake of simplicity, to be not explicitly dependent on time.

In this chapter, we want to discuss the problem of assigning not only an *amplitude* $K[\gamma]$ as in Eq. (5.1.2), but also a *probability* $P[\gamma]$, to a single trajectory. It is obvious that here, by "probability", we actually mean a probability density in the space of paths: Only quantities like

$$\int_{\Gamma} \mathcal{D}\gamma\, P[\gamma] \ , \qquad (5.1.4)$$

where Γ is some set of histories, can be interpreted as true probabilities.

If such a functional $P[\gamma]$ can be defined, it is clear that not only $K[\gamma]$ must enter in its construction, but also the state ψ. In fact, the amplitudes $K[\gamma]$ are not related to the physical state of the particle, but only to the external conditions that influence its evolution; to get a complete probability amplitude at time t'', which takes into account also the specific conditions (preparation) of the particle at the initial time t', we need to consider [95,96]

$$\psi(\mathbf{x}'', t'') = \int_{C'} d^3x'\, K(\mathbf{x}'', t''; \mathbf{x}', t')\, \psi(\mathbf{x}', t') \ , \qquad (5.1.5)$$

where $\psi(\mathbf{x}, t)$ is the Schrödinger wave function, and C' is the configuration space for the particle at time t'. In other words, we can think of $K(\mathbf{x}'', t''; \mathbf{x}', t')$ as being part of the law that, in a well defined experimental context, rules the evolution of a generic state ψ.

The problem of defining a probability $P[\gamma]$ has already been studied, a long time ago, by Dirac [92]. He found that such a concept can be defined, provided $P[\gamma]$ is allowed to take complex values. At first sight, such a result seems like a disaster! If $P[\gamma]$ is to be interpreted as a probability density, then it has to be not only real, but also nonnegative, by very definition [97]. It seems therefore that there is no reasonable way to assign a probability to each possible history of the particle. However, this would not be the only case in quantum theory in which mathematical objects are defined which behave like probabilities, and yet can attain nonpositive values under some conditions [73,98]. Since these conditions do not correspond to any physical situation which is directly testable, no interpretational problem is involved; in other words, sets Γ of histories such that (5.1.4) turns out to be negative would have no experimental counterpart. From such "quasiprobabilities" it is possible to construct true, experimentally meaningful, positive semidefinite probabilities [73,98,99]. This justifies their use in calculations.

We shall devote the remaining part of this section to a few considerations and remarks which will prove useful in order to justify some procedures used in the chapter. Let us start by remembering that the structure of (5.1.1) can be straightforwardly derived [96,86] from the Hilbert space formulation of quantum theory, writing

$$K(\mathbf{x}'', t''; \mathbf{x}', t') = \langle \mathbf{x}'', t'' | \mathbf{x}', t' \rangle \ , \tag{5.1.6}$$

where $|\mathbf{x}, t\rangle$ is an eigenstate of position at time t, in the Heisenberg picture; dividing the interval $[t', t'']$ into $N+1$ smaller intervals $[t_k, t_{k+1}]$, for $k \in \{0, 1, \ldots, N\}$, with $t' \equiv t_0 < \cdots < t_{N+1} \equiv t''$, one finds, rather trivially,

$$\begin{aligned} K(\mathbf{x}'', t''; \mathbf{x}', t') = \lim_{T \to 0} \int d^3x_1 \ldots d^3x_N \\ \langle \mathbf{x}_{N+1}, t_{N+1} | \mathbf{x}_N, t_N \rangle \cdots \langle \mathbf{x}_1, t_1 | \mathbf{x}_0, t_0 \rangle \ , \end{aligned} \tag{5.1.7}$$

where $T \equiv \max\{\Delta t_k\}$, with $\Delta t_k \equiv t_{k+1} - t_k$. In getting Eq. (5.1.7) from Eq. (5.1.6), the completeness relation for the position has been used N times; applying now $N+1$ completeness relations for the momentum, we get easily

$$\begin{aligned} K(\mathbf{x}'', t''; \mathbf{x}', t') &= \lim_{T \to 0} \frac{1}{\mathcal{N}} \int d^3x_1 \ldots d^3x_N d^3p_0 \ldots d^3p_N \\ &\quad \exp \frac{i}{\hbar} \sum_{k=0}^{N} (\mathbf{p}_k \cdot \Delta\mathbf{x}_k - H_k \Delta t_k) \ , \end{aligned} \tag{5.1.8}$$

where $\Delta\mathbf{x}_k \equiv \mathbf{x}_{k+1} - \mathbf{x}_k$, $\mathcal{N} \equiv (2\pi\hbar)^{3(N+1)}$, and the H_k's are defined, in the limit $T \to 0$, as

$$H_k \equiv \frac{1}{2} \frac{\langle \mathbf{x}_{k+1} | \hat{H} | \mathbf{p}_k \rangle}{\langle \mathbf{x}_{k+1} | \mathbf{p}_k \rangle} + \frac{1}{2} \frac{\langle \mathbf{p}_k | \hat{H} | \mathbf{x}_k \rangle}{\langle \mathbf{p}_k | \mathbf{x}_k \rangle} \ . \tag{5.1.9}$$

We have given in some detail the derivation of Eq. (5.1.8) in order to better understand its relation with the path integral (5.1.1). Identifying the right hand sides of both the expressions, and remembering Eqs. (5.1.2) and (5.1.3), the integral in Eq. (5.1.8) can be interpreted, before taking the limit, as the effective summation of the amplitudes over a set of approximated histories, constructed as follows. Let γ be a path defined by the curves $t \mapsto \mathrm{x}(t)$ and $t \mapsto \mathrm{p}(t)$, satisfying $\mathrm{x}(t') = \mathrm{x}'$ and $\mathrm{x}(t'') = \mathrm{x}''$; performing a partition of the interval $[t', t'']$ as described above, and defining $\mathrm{x}_h \equiv \mathrm{x}(t_h)$, for $h \in \{0, 1, \ldots, N+1\}$, we obtain the set of $N+2$ points in the configuration space

$$\{\mathrm{x}_0, \mathrm{x}_1, \ldots, \mathrm{x}_{N+1}\}\ , \tag{5.1.10}$$

which approximates the image of the curve $t \mapsto \mathrm{x}(t)$ and satisfies the prescribed boundary conditions as $\mathrm{x}_0 = \mathrm{x}'$ and $\mathrm{x}_{N+1} = \mathrm{x}''$. Choosing now, for each $k \in \{0, 1, \ldots, N\}$, a time $\tau_k \equiv (t_k + t_{k+1})/2$, and defining $\mathrm{p}_k \equiv \mathrm{p}(\tau_k)$, we construct a set of $N+1$ points in the momentum space:

$$\{\mathrm{p}_0, \mathrm{p}_1, \ldots, \mathrm{p}_N\}\ . \tag{5.1.11}$$

The sets

$$\{(\mathrm{x}_0, t_0), (\mathrm{x}_1, t_1), \ldots, (\mathrm{x}_{N+1}, t_{N+1})\} \tag{5.1.12}$$

and

$$\{(\mathrm{p}_0, \tau_0), (\mathrm{p}_1, \tau_1), \ldots, (\mathrm{p}_N, \tau_N)\} \tag{5.1.13}$$

in the extended configuration and momentum spaces, constitute a "skeleton" for the history γ; an integration over all the possible sets of positions (5.1.10) and momenta (5.1.11), combined with the limiting procedure $T \to 0$, as in Eq. (5.1.8), corresponds thus to the sum over all the paths with definite positions x' and x'' at times t' and t''. Notice that in (5.1.10) and (5.1.11) the positions and momenta are never considered at the same time: Such sets do not approximate, therefore, any curve in the extended phase space; however, they do approximate, as we can see from (5.1.12) and (5.1.13), a curve in the extended configuration space and one in the extended momentum space. This is the reason for our choice of the representation of the history γ.

Eq. (5.1.8) and the related comments teach us most of what has to be known in order to give an operational, and not only a formal, meaning to an expression containing a sum over histories. It could seem at first glance that Eq. (5.1.8), being peculiar to the boundary conditions typical of the case considered, will be of no general validity; for example, asking for the amplitude $K(\mathbf{p}'', t''; \mathbf{p}', t')$ that the particle, having a momentum $\mathbf{p}'$ at time t', will have a momentum $\mathbf{p}''$ at time t'', one should consider the approximate history given by

$$\{(\mathbf{x}_0, \tau_0), (\mathbf{x}_1, \tau_1), \ldots, (\mathbf{x}_N, \tau_N)\} \tag{5.1.14}$$

and

$$\{(\mathbf{p}_0, t_0), (\mathbf{p}_1, t_1), \ldots, (\mathbf{p}_{N+1}, t_{N+1})\} , \tag{5.1.15}$$

rather than that defined by (5.1.12)–(5.1.13): This would correspond to a different expression for the path integral, and seems to lead to the conclusion that four classes of paths must be considered, accordingly to the four possible specifications for the conditions at t' and t''. However, it is not difficult to understand that this would lead to unnecessary complications, and that the approximation (5.1.12)–(5.1.13) is sufficient for the calculation of all the transition amplitudes; in fact, we can write, using twice the completeness relation for the position,

$$K(\mathbf{p}'', t''; \mathbf{p}', t') = \int d^3x'' d^3x' \frac{e^{-i\mathbf{p}''\cdot\mathbf{x}''/\hbar}}{(2\pi\hbar)^{3/2}} K(\mathbf{x}'', t''; \mathbf{x}', t') \frac{e^{i\mathbf{p}'\cdot\mathbf{x}'/\hbar}}{(2\pi\hbar)^{3/2}} , \tag{5.1.16}$$

expressing $K(\mathbf{p}'', t''; \mathbf{p}', t')$ as a double Fourier transform of the amplitude $K(\mathbf{x}'', t''; \mathbf{x}', t')$; an analogous treatment can be performed in order to obtain $K(\mathbf{p}'', t''; \mathbf{x}', t')$ and $K(\mathbf{x}'', t''; \mathbf{p}', t')$.

A similar situation occurs when quasiprobabilities $P[\gamma]$, rather than amplitudes $K[\gamma]$, are considered. As it will be clear in the next section (see Eq. (??)), the expression for $P[\gamma]$ contains an explicit representation of the state vector $|\psi\rangle$ at both times t' and t''; the character of such representation, i.e., the appearance of the wave function in the configuration space, $\psi(\mathbf{x}, t)$, or in the momentum space, $\phi(\mathbf{p}, t)$, depends clearly on the nature of the boundary conditions for the history considered. However, it will turn out that, as in the present case, a treatment involving only paths approximated by (5.1.12)–(5.1.13) is not restrictive, even when our development of the formalism is considered.

Let us close this section by establishing some useful notations about the sets of histories with definite (even if not necessarily fixed) boundary conditions for the position. By Γ_0 we shall denote the set of all such paths between the times t' and t''. The subset Γ_1 of Γ_0 has been already defined in relation to Eq. (5.1.1); other important subsets are Γ' and Γ'', the sets of all the histories such that, respectively, $\mathrm{x}(t') = \mathrm{x}'$ and $\mathrm{x}(t'') = \mathrm{x}''$. The path integrals of a generic functional $F[\gamma]$ over Γ_0, Γ' and Γ'' will be defined in a natural way as

$$\int_{\Gamma_0} \mathcal{D}\gamma\, F[\gamma] \equiv \int_{C''} d^3x'' \int_{C'} d^3x' \int_{\Gamma_1} \mathcal{D}\gamma\, F[\gamma] \,, \tag{5.1.17}$$

$$\int_{\Gamma'} \mathcal{D}\gamma\, F[\gamma] \equiv \int_{C''} d^3x'' \int_{\Gamma_1} \mathcal{D}\gamma\, F[\gamma] \,, \tag{5.1.18}$$

$$\int_{\Gamma''} \mathcal{D}\gamma\, F[\gamma] \equiv \int_{C'} d^3x' \int_{\Gamma_1} \mathcal{D}\gamma\, F[\gamma] \,. \tag{5.1.19}$$

5.2 Construction of $P[\gamma]$

Let us now face the task of trying to assign a quasiprobability to each history, as discussed in Sec. 5.1. An obvious requirement to impose on the functional $P[\gamma]$ is, of course, that it be normalized; we shall write this condition as

$$\int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma] = 1 \,, \tag{5.2.1}$$

where the integral is defined by Eq. (5.1.17). Eq. (5.2.1) only means that the particle will certainly go, somehow, from somewhere at time t' to somewhere at time t'': It looks therefore like a very reasonable assumption.

In the previous section it was argued that $P[\gamma]$ cannot be constructed only out of $K[\gamma]$, but necessarily must contain also the state ψ; we shall now show how this condition, together with the normalization (5.2.1), allows us to easily determine a reasonable form for the functional $P[\gamma]$.

Combining Eqs. (5.1.5) and (5.1.1), and using Eq. (5.1.19), we come to

$$\psi(\mathrm{x}'', t'') = \int_{\Gamma''} \mathcal{D}\gamma\, K[\gamma]\, \psi(\mathrm{x}(t'), t') \,. \tag{5.2.2}$$

The probability density $P(\mathbf{x}''|t'')$ for the particle to be at the point of space $\mathbf{x}''$ at time t'' reads thus

$$P(\mathbf{x}''|t'') = |\psi(\mathbf{x}'', t'')|^2 = \int_{\Gamma''} \mathcal{D}\gamma\, \psi(\mathbf{x}'', t'')^* \, K[\gamma]\, \psi(\mathbf{x}(t'), t') \; ; \qquad (5.2.3)$$

this is normalized to one [95,100] as

$$1 = \int_{C''} d^3x''\, P(\mathbf{x}''|t'') = \int_{\Gamma_0} \mathcal{D}\gamma\, \psi(\mathbf{x}(t''), t'')^* \, K[\gamma]\, \psi(\mathbf{x}(t'), t') \; . \qquad (5.2.4)$$

A comparison between Eqs. (5.2.1) and (5.2.4) would induce us to write

$$P[\gamma] = \psi(\mathbf{x}(t''), t'')^* \, K[\gamma]\, \psi(\mathbf{x}(t'), t') \; ; \qquad (5.2.5)$$

such a $P[\gamma]$ is obviously normalized, but not necessarily real: Fortunately, this problem can be easily solved. Let us suppose, in fact, that all the reasoning leading to Eq. (5.2.5) would have been performed referring to ψ at t' rather than at t'': Then, Eq. (5.2.2) would have been replaced [95] by

$$\psi(\mathbf{x}', t') = \int_{\Gamma'} \mathcal{D}\gamma\, \psi(\mathbf{x}(t''), t'')\, K[\gamma]^* \; ; \qquad (5.2.6)$$

consequently, we should have written, instead of Eq. (5.2.5),

$$P[\gamma] = \psi(\mathbf{x}(t''), t'')\, K[\gamma]^* \, \psi(\mathbf{x}(t'), t')^* \; , \qquad (5.2.7)$$

which is just its complex conjugate! Therefore we suggest to adopt, for $P[\gamma]$, the real expression

$$P[\gamma] \equiv \Re \left[\psi(\mathbf{x}(t''), t'')^* \, K[\gamma]\, \psi(\mathbf{x}(t'), t') \right] \; , \qquad (5.2.8)$$

where $\Re$ stands for the real part; Eq. (5.2.8) can be seen as a symmetrization of Eqs. (5.2.5) and (5.2.7).

In the derivation of Eq. (5.2.8), we have clearly worked with paths whose boundary conditions are imposed on position; as a consequence, position plays a dominant role in the final expression. However, there is nothing in the general theory forbidding us to use histories with boundary conditions on the curve $t \mapsto \mathbf{p}(t)$ rather than on $t \mapsto \mathbf{x}(t)$. With considerations similar to those performed so far, we find in this case, for the quasiprobability functional associated to the path γ,

$$\tilde{P}[\gamma] = \Re[\phi(\mathbf{p}(t''),t'')^* \tilde{K}[\gamma]\phi(\mathbf{p}(t'),t')] \ , \tag{5.2.9}$$

where the wave function in the momentum space, $\phi(\mathbf{p},t)$, is the Fourier transform of $\psi(\mathbf{x},t)$, and

$$\tilde{K}[\gamma] = \exp\left(\frac{i}{\hbar}\tilde{S}[\gamma]\right) \ , \tag{5.2.10}$$

with

$$\begin{aligned}\tilde{S}[\gamma] &= -\int_{t'}^{t''} dt[\mathbf{x}(t)\cdot\dot{\mathbf{p}}(t) + H(\mathbf{x}(t),\mathbf{p}(t))] = \\ &= S[\gamma] - \mathbf{p}(t'')\cdot\mathbf{x}(t'') + \mathbf{p}(t')\cdot\mathbf{x}(t') \ .\end{aligned} \tag{5.2.11}$$

Notice that the modified action $\tilde{S}$ in Eq. (5.2.11), and the corresponding amplitude $\tilde{K}$ given by Eq. (5.2.10), are just the correct functionals to use if the skeletonization of the histories is performed according to (5.1.14)–(5.1.15); in fact Eq. (5.1.16) can be written in such a case as

$$K(\mathbf{p}'',t'';\mathbf{p}',t') = \int_{\tilde{\Gamma}_1} \tilde{\mathcal{D}}\gamma\, \tilde{K}[\gamma] \ , \tag{5.2.12}$$

where $\tilde{\Gamma}_1$ is the set of all paths with $\mathbf{p}(t') = \mathbf{p}'$ and $\mathbf{p}(t'') = \mathbf{p}''$ as boundary conditions, and $\tilde{\mathcal{D}}\gamma$ is the related measure.

It is easy to check that $P[\gamma]$ and $\tilde{P}[\gamma]$ are different from each other, and it could therefore be thought that the range of applicability for each one of them is rather limited: However, we shall prove that they can both be used in every calculation which does not involve the value of quantities at the extreme times t' or t''. More precisely, if F is a function of $\mathbf{x}$ and $\mathbf{p}$, and $t \in (t',t'')$, then

$$\int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, F(\mathbf{x}(t),\mathbf{p}(t)) = \int_{\tilde{\Gamma}_0} \tilde{\mathcal{D}}\gamma\, \tilde{P}[\gamma]\, F(\mathbf{x}(t),\mathbf{p}(t)) \ . \tag{5.2.13}$$

As we shall see later (in Eq. (5.4.26)), the left hand side of Eq. (5.2.13) can be identified as the expectation value of the quantum operator associated with F in the state $|\psi\rangle$: Eq. (5.2.13) therefore means that the same calculation can be performed breaking the paths as in (5.1.14)–(5.1.15), provided $\tilde{P}$ is used instead of P. We shall postpone the proof of Eq. (5.2.13) to the end of Sec. 5.4, because it must be preceded by some considerations which follow below.

Eq. (5.2.7), which we have obtained, heuristically, from considerations about path integrals, turns out to be just the expression suggested by Dirac [92]. Since, from our standpoint, Eq. (5.2.7) and its complex conjugate (5.2.5) are exactly on the same footing, we shall now critically review Dirac's derivation, in order to understand if there is a compelling reason to prefer Eq. (5.2.7) rather than Eqs. (5.2.5) or (5.2.8).

The method followed by Dirac is based, in essence, on the approximation of a history γ with boundary conditions expressed on position, by the discrete sets of points (5.1.12)–(5.1.13) in the extended configuration and momentum spaces. If a probability $P[\{(\mathbf{x}_h, t_h)\}, \{(\mathbf{p}_k, \tau_k)\}]$ is assigned to the "path" (5.1.12)–(5.1.13), the following obvious identity holds:

$$P[\{(\mathbf{x}_h, t_h)\}, \{(\mathbf{p}_k, \tau_k)\}] = \frac{1}{\mathcal{N}} \int d^3x'_0 \ldots d^3x'_{N+1} d^3p'_0 \ldots d^3p'_N$$
$$P[\{(\mathbf{x}'_h, t_h)\}, \{(\mathbf{p}'_k, \tau_k)\}]\, \mathcal{N} \delta^3(\mathbf{x}'_0 - \mathbf{x}_0) \cdots \delta^3(\mathbf{x}'_{N+1} - \mathbf{x}_{N+1})$$
$$\delta^3(\mathbf{p}'_0 - \mathbf{p}_0) \cdots \delta^3(\mathbf{p}'_N - \mathbf{p}_N) = \mathcal{N} \langle \delta^3_{\mathbf{x}_0} \cdots \delta^3_{\mathbf{x}_{N+1}} \delta^3_{\mathbf{p}_0} \cdots \delta^3_{\mathbf{p}_N} \rangle, \quad (5.2.14)$$

where $\langle \cdots \rangle$ denotes the average. In the quantum mechanical formalism, we can thus write the quasiprobability $P[\{(\mathbf{x}_h, t_h)\}, \{(\mathbf{p}_k, \tau_k)\}]$ as

$$P[\{(\mathbf{x}_h, t_h)\}, \{(\mathbf{p}_k, \tau_k)\}] \equiv \mathcal{N} \langle \psi | \mathcal{O}\{\delta^3(\hat{\mathbf{x}}(t_0) - \mathbf{x}_0 \hat{1}) \cdots$$
$$\delta^3(\hat{\mathbf{x}}(t_{N+1}) - \mathbf{x}_{N+1} \hat{1}) \delta^3(\hat{\mathbf{p}}(\tau_0) - \mathbf{p}_0 \hat{1}) \cdots \delta^3(\hat{\mathbf{p}}(\tau_N) - \mathbf{p}_N \hat{1})\} | \psi \rangle, \quad (5.2.15)$$

where $\hat{\mathbf{x}}(t_h)$, $\hat{\mathbf{p}}(\tau_k)$ are the position and momentum operators of the particle respectively at times t_h and τ_k, in the Heisenberg picture; $\mathcal{O}\{\cdots\}$ denotes an ordering for the product of operators, and represents the crucial point of the treatment. If $\hat{A}_1(t_1), \ldots, \hat{A}_n(t_n)$ are generic operators defined at times $t_1 < \ldots < t_n$, the Dyson chronological product [101] is

$$\mathcal{T}\{\hat{A}_1(t_1) \cdots \hat{A}_n(t_n)\} \equiv \hat{A}_n(t_n) \cdots \hat{A}_1(t_1) \; ; \quad (5.2.16)$$

similarly an "antichronological" product can be defined as

$$\mathcal{T}'\{\hat{A}_1(t_1) \cdots \hat{A}_n(t_n)\} \equiv \hat{A}_1(t_1) \cdots \hat{A}_n(t_n) \; . \quad (5.2.17)$$

The choice of $\mathcal{O}$ that reproduces Dirac's expression corresponds to the use of $\mathcal{T}'$; more explicitly, for $\mathcal{O} = \mathcal{T}'$,

$$\begin{aligned}\mathcal{O}\{\delta^3(\hat{\mathbf{x}}(t_0) - \mathbf{x}_0\hat{1}) \cdots \delta^3(\hat{\mathbf{x}}(t_{N+1}) - \mathbf{x}_{N+1}\hat{1}) \\ \delta^3(\hat{\mathbf{p}}(\tau_0) - \mathbf{p}_0\hat{1}) \cdots \delta^3(\hat{\mathbf{p}}(\tau_N) - \mathbf{p}_N\hat{1})\} = \\ = \delta^3(\hat{\mathbf{x}}(t_0) - \mathbf{x}_0\hat{1})\delta^3(\hat{\mathbf{p}}(\tau_0) - \mathbf{p}_0\hat{1}) \cdots \\ \cdots \delta^3(\hat{\mathbf{x}}(t_N) - \mathbf{x}_N\hat{1})\delta^3(\hat{\mathbf{p}}(\tau_N) - \mathbf{p}_N\hat{1})\delta^3(\hat{\mathbf{x}}(t_{N+1}) - \mathbf{x}_{N+1}\hat{1})\end{aligned} \quad (5.2.18)$$

Substituting Eq. (5.2.18) into Eq. (5.2.15), and inserting $N+2$ completeness relations for position, we get

$$\begin{aligned}P[\{(\mathbf{x}_h, t_h)\}, \{(\mathbf{p}_k, \tau_k)\}] = \mathcal{N}\langle\psi|\mathbf{x}_0, t_0\rangle\langle\mathbf{x}_0, t_0|\delta^3(\hat{\mathbf{p}}(\tau_0) - \mathbf{p}_0\hat{1})|\mathbf{x}_1, t_1\rangle \\ \cdots \langle\mathbf{x}_N, t_N|\delta^3(\hat{\mathbf{p}}(\tau_N) - \mathbf{p}_N\hat{1})|\mathbf{x}_{N+1}, t_{N+1}\rangle\langle\mathbf{x}_{N+1}, t_{N+1}|\psi\rangle.\end{aligned} \quad (5.2.19)$$

Using now the completeness relation for momentum, the k-th factor in Eq. (5.2.19) becomes

$$\langle\mathbf{x}_k, t_k|\delta^3(\hat{\mathbf{p}}(\tau_k) - \mathbf{p}_k\hat{1})|\mathbf{x}_{k+1}, t_{k+1}\rangle = \frac{1}{(2\pi\hbar)^3} e^{-i(\mathbf{p}_k\cdot\Delta\mathbf{x}_k - H_k\Delta t_k)/\hbar} , \quad (5.2.20)$$

with H_k defined by Eq. (5.1.9). Substituting Eq. (5.2.20) into Eq. (5.2.19), and taking the limit $T \to 0$, Eq. (5.2.19) becomes, according to Eq. (5.1.8) above,

$$P[\gamma] = \lim_{T\to 0} P[\{(\mathbf{x}_h, t_h)\}, \{(\mathbf{p}_k, \tau_k)\}] = \psi(\mathbf{x}(t'), t')^* K[\gamma]^* \psi(\mathbf{x}(t''), t'') , \quad (5.2.21)$$

which is exactly expression (5.2.7). It is, however, obvious that the use of $\mathcal{T}$ rather than $\mathcal{T}'$ for the ordering of operators would have led to Eq. (5.2.5): The arbitrariness in the choice of Eqs. (5.2.5) or (5.2.7) as quasiprobability for the history γ reflects therefore the arbitrariness in the ordering of operators according to the chronological or the antichronological rule. In our opinion, the requirement to deal with a *real* quasiprobability is stronger than others which could induce one to prefer the ordering $\mathcal{T}$ or $\mathcal{T}'$ in Eq. (5.2.15): Thus we shall use, from now on, the expression (5.2.8) for $P[\gamma]$, which corresponds to a symmetrized chronological product.

Let us notice, finally, that the expression (5.2.9) for $\tilde{P}[\gamma]$ can be obtained straightforwardly with Dirac's method, simply considering the skeletonized history as (5.1.14)–(5.1.15) rather than as (5.1.12)–(5.1.13).

5.3 Probability Distributions for Position and Momentum

Let us now apply the concept of a quasiprobability functional $P[\gamma]$, as previously defined, to some special cases, in order to check that it does indeed reproduce the results of ordinary quantum theory. We shall start by integrating $P[\gamma]$ over all the trajectories in Γ_1, obtaining easily:

$$P(\mathrm{x}'', t''; \mathrm{x}', t') \equiv \int_{\Gamma_1} \mathcal{D}\gamma \, P[\gamma] = \Re \left[\psi(\mathrm{x}'', t'')^* \, K(\mathrm{x}'', t''; \mathrm{x}', t') \, \psi(\mathrm{x}', t') \right] . \tag{5.3.1}$$

The function $P(\mathrm{x}'', t''; \mathrm{x}', t')$ now derived will be useful later in clarifying the concept of quasiprobability; moreover, it is also connected to the probability densities for position $P(\mathrm{x}'|t')$ and $P(\mathrm{x}''|t'')$, as we shall now show.

The probability that the particle be in x'' at time t'' is, according to our formalism, the integral over the set Γ'' of the functional $P[\gamma]$; thus

$$P(\mathrm{x}''|t'') = \int_{\Gamma''} \mathcal{D}\gamma \, P[\gamma] \, . \tag{5.3.2}$$

Writing the expression for $P[\gamma]$ in Eq. (5.3.2) according to Eq. (5.2.8), and performing the formal integration as in Eq. (5.2.2), we get, at the end, the well known expression

$$P(\mathrm{x}''|t'') = |\psi(\mathrm{x}'', t'')|^2 \, , \tag{5.3.3}$$

in agreement with the results of standard quantum theory [95,100], and consistent with Eq. (5.2.3). $P(\mathrm{x}'|t')$ can be obtained, in a similar way, as an integral over Γ'.

Our method of obtaining the probability distribution for position could be questioned, because we choose to integrate over the paths "arriving" from the past to the point (x'', t'') of the extended configuration space; in principle it could be that the integration over another set of histories, e.g. all those "leaving" (x'', t''), lead to a different value for $P(\mathrm{x}''|t'')$: The derivation of this quantity from $P[\gamma]$ would thus be ill-defined. Fortunately, we shall now prove a lemma from which it follows that this is not the case. Let in fact $t^+ > t''$, and let γ^+ be a path defined for the time interval $[t'', t^+]$ by $t \mapsto \mathrm{x}^+(t)$ and $t \mapsto \mathrm{p}^+(t)$, such that $\mathrm{x}^+(t'') = \mathrm{x}(t'')$, $\mathrm{p}^+(t'') = \mathrm{p}(t'')$. Then it is possible to construct, given a history $\gamma \in \Gamma''$, a new history $\bar{\gamma}$ defined by $t \mapsto \mathrm{x}(t)$, $t \mapsto \mathrm{p}(t)$ for $t \in [t', t'']$, and $t \mapsto \mathrm{x}^+(t)$, $t \mapsto \mathrm{p}^+(t)$ for $t \in [t'', t^+]$. Calling $\bar{\Gamma}''$ the set of all such paths $\bar{\gamma}$ with fixed x'' at time t'', we have the equality

$$\int_{\bar{\Gamma}''} \mathcal{D}\bar{\gamma}\, P[\bar{\gamma}] = \int_{\Gamma''} \mathcal{D}\gamma\, P[\gamma] \,. \tag{5.3.4}$$

In fact, since

$$K[\bar{\gamma}] = K[\gamma^+]\, K[\gamma] \,, \tag{5.3.5}$$

it follows trivially that the left hand side of Eq. (5.3.4) is equal to $|\psi(\mathrm{x}'', t'')|^2$; Eqs. (5.3.2) and (5.3.3) then imply the validity of Eq. (5.3.4). This result ensures that both of the definitions (5.3.1) and (5.3.2) are well posed.

The functions $P(\mathrm{x}'|t')$ and $P(\mathrm{x}''|t'')$ can be derived from the one defined in Eq. (5.3.1) by the following mathematically obvious relations:

$$P(\mathrm{x}'|t') = \int_{C''} d^3x''\, P(\mathrm{x}'', t''; \mathrm{x}', t') \,, \tag{5.3.6}$$

$$P(\mathrm{x}''|t'') = \int_{C'} d^3x'\, P(\mathrm{x}'', t''; \mathrm{x}', t') \,. \tag{5.3.7}$$

Since $P(\mathrm{x}'|t')$ and $P(\mathrm{x}''|t'')$ have the meaning of probability densities for the particle to be, respectively, in x' at time t' or in x'' at time t'', Eqs. (5.3.6)–(5.3.7) suggest that $P(\mathrm{x}'', t''; \mathrm{x}', t')$ be interpreted as the joint quasiprobability density (in $C'' \times C'$) for the particle to be in x' at time t' *and* in x'' at time t''. In fact, with such an interpretation, the right hand side of, e.g., Eq. (5.3.7) would represent the probability density (in C'') that the particle, having been somewhere at time t', be in x'' at time t'': It sounds quite reasonable to equate this to $P(\mathrm{x}''|t'')$. A completely analogous argument can be performed about Eq. (5.3.6).

The interpretation of $P(\mathrm{x}'', t''; \mathrm{x}', t')$ given above, which relies on the well established interpretation of $P(\mathrm{x}|t)$ and agrees, according to Eq. (5.3.1), with the notion of a quasiprobability functional $P[\gamma]$, gives support to the reliability of such a concept; however, we think it is important to stress the fact that, unlike $P(\mathrm{x}|t)$, $P(\mathrm{x}'', t''; \mathrm{x}', t')$ has no direct relation to any real experiment. More precisely, while $P(\mathrm{x}|t)$ represents the probability to *find* the particle at x at time t, $P(\mathrm{x}'',t'';\mathrm{x}',t')$ is not related to the probability to *find* it first (at time t') in x', and then (at time t'') in x'': This latter would rather be

$$|\tilde{\psi}(\mathrm{x}'',t''|\mathrm{x}',t')|^2 \cdot |\psi(\mathrm{x}',t')|^2 , \qquad (5.3.8)$$

where $\tilde{\psi}(\mathrm{x}'',t''|\mathrm{x}',t')$ is the wave function in x'' at time t'' if the particle is known to be in x' at time t'. It is evident that Eqs. (5.3.8) and (5.3.1) differ very much from each other: While Eq. (5.3.8) involves, in its definition, the process of state vector collapse [55], Eq. (5.3.1) makes reference only to the unreduced wave function ψ. This is an important point, as can be understood from the following remark.

Let us decompose the set Γ_1 into two disjoint sets Γ_1^A and Γ_1^B, and define

$$P^I(\mathrm{x}'',t'';\mathrm{x}',t') \equiv \int_{\Gamma_1^I} \mathcal{D}\gamma\, P[\gamma] , \qquad (5.3.9)$$

with $I \in \{A, B\}$. It is clear that

$$P(\mathrm{x}'',t'';\mathrm{x}',t') = P^A(\mathrm{x}'',t'';\mathrm{x}',t') + P^B(\mathrm{x}'',t'';\mathrm{x}',t') , \qquad (5.3.10)$$

with

$$P^I(\mathrm{x}'',t'';\mathrm{x}',t') = \Re\left[\psi^*(\mathrm{x}'',t'')\, K^I(\mathrm{x}'',t'';\mathrm{x}',t')\, \psi(\mathrm{x}',t')\right] , \qquad (5.3.11)$$

and

$$K^I(\mathrm{x}'',t'';\mathrm{x}',t') = \int_{\Gamma_1^I} \mathcal{D}\gamma\, K[\gamma] . \qquad (5.3.12)$$

However, while $K^I(\mathrm{x}'',t'';\mathrm{x}',t')$, from Eq. (5.3.12), depends only on the trajectories in Γ_1^I, this is not true for $P^I(\mathrm{x}'',t'';\mathrm{x}',t')$, which depends also on ψ both at (x',t') and at (x'',t''): Since $\psi(\mathrm{x}'',t'')$ and $\psi(\mathrm{x}',t')$ are linked as in Eq. (5.1.5), which contains $K(\mathrm{x}'',t'';\mathrm{x}',t')$, and $K(\mathrm{x}'',t'';\mathrm{x}',t')$ is evaluated taking into account *all* the trajectories in Γ_1, it follows that $P^I(\mathrm{x}'',t'';\mathrm{x}',t')$ is an object which depends on the entire Γ_1, and not only on Γ_1^I. This is the underlying reason which allows $P[\gamma]$ to incorporate interference effects.

This "global" character of $P[\gamma]$ can be realized even better introducing a new kind of amplitude functional for paths, which has also the effect to compactify the notations. Let $\gamma \in \Gamma''$; we can define a functional over Γ'' as

$$\Phi[\gamma] \equiv K[\gamma]\, \psi(\mathrm{x}(t'), t') , \tag{5.3.13}$$

where $K[\gamma]$ is given by Eq. (5.1.2); $\Phi[\gamma]$ satisfies the relation

$$\int_{\Gamma''} \mathcal{D}\gamma\, \Phi[\gamma] = \psi(\mathrm{x}'', t'') , \tag{5.3.14}$$

and can be interpreted as a "true" probability amplitude for the history γ, in the sense that it contains not only informations about the phase evolution along γ (through $K[\gamma]$), but also about the relative probability between paths with different extremes. Let now $\gamma_1, \gamma_2 \in \Gamma''$, and let us construct a functional Π over $\Gamma'' \times \Gamma''$ as

$$\Pi[\gamma_1, \gamma_2] \equiv \Phi[\gamma_2]^* \Phi[\gamma_1] . \tag{5.3.15}$$

$\Pi[\gamma_1, \gamma_2]$ can be integrated over γ_2, thus obtaining, indicating by $\mathrm{x}_1(t)$ the position along γ_1,

$$\int_{\Gamma''} \mathcal{D}\gamma_2\, \Pi[\gamma_1, \gamma_2] = \psi(\mathrm{x}_1(t''), t'')^* K[\gamma_1] \psi(\mathrm{x}_1(t'), t') , \tag{5.3.16}$$

whose real part is exactly $P[\gamma_1]$. It appears therefore reasonable to consider $\Pi[\gamma_1, \gamma_2]$ as a more fundamental object than $P[\gamma]$, since this latter can be conveniently derived from it; a further conceptual advantage of $\Pi[\gamma_1, \gamma_2]$ over $P[\gamma]$ is that, while $P[\gamma]$ contains, in its very definition (5.2.8) the wave function evaluated at both values t' and t'' of time, thus presupposing an evolution equation to have already been solved, at least formally, $\Pi[\gamma_1, \gamma_2]$ only contains ψ at one single value of t: All the probabilities can be extracted from it simply by a convenient choice of the paths over which to integrate, without the use of any further dynamical principle.

It is interesting to observe, in relation to a possible interpretation of $\Pi[\gamma_1, \gamma_2]$, that Eq. (5.3.15) can be rewritten, using Eqs. (5.3.13) and (5.1.2), as

$$\Pi[\gamma_1, \gamma_2] = \exp i \left(S[\gamma_1] - S[\gamma_2] \right)\, \psi(\mathrm{x}_2(t'), t')^* \psi(\mathrm{x}_1(t'), t') , \tag{5.3.17}$$

and that $\psi(\mathbf{x}_2(t'),t')^*\psi(\mathbf{x}_1(t'),t')$ can be identified with the density matrix $\rho(\mathbf{x}_1(t'),\mathbf{x}_2(t');t')$; therefore the definition of $\Pi[\gamma_1,\gamma_2]$ can be straightforwardly generalized to the case of a mixture, simply defining

$$\Pi[\gamma_1,\gamma_2] \equiv \exp i\,(S[\gamma_1]-S[\gamma_2])\;\rho(\mathbf{x}_1(t'),\mathbf{x}_2(t');t')\;. \tag{5.3.18}$$

The expression (5.3.18) has been already considered, at the level of a kernel for integrals, in studies concerned both with the problem of the influence of a system on another one [102] and with decoherence [103]: This gives further support to the idea that it may play an important role in quantum theory.

The functional $P[\gamma]$ should allow us, of course, to get results that are much more interesting than a simple calculation of $P(\mathbf{x}|t)$; depending on the set Γ, the integral (5.1.4) can produce a large variety of (quasi-)probability functions. An interesting and important example consists in the probability density (in momentum space) $P(\mathbf{p}|t)$ for the particle to have a momentum $\mathbf{p}$ at time t. It is intuitively clear that $P(\mathbf{p}|t)$ should be obtained by integrating $P[\gamma]$ over all the paths with momentum $\mathbf{p}$ at time t: We shall now prove formally that this is indeed the case.

Choosing the partition of $[t',t'']$ in such a way that $\mathbf{p}=\mathbf{p}_M$, $t=\tau_M$, with $M\leq N$, the integration of $P[\gamma]$ over all paths with momentum $\mathbf{p}$ at time t requires the volume element in the "broken" path integral to be changed from

$$d^3x_1\ldots d^3x_N d^3p_0\ldots d^3p_N\;,$$

as in Eq. (5.1.8) to

$$d^3x_0\ldots d^3x_{N+1}d^3p_0\ldots d^3p_{M-1}d^3p_{M+1}\ldots d^3p_N\;\;:$$

This corresponds to summing over all the trajectories in Γ_0, but keeping $\mathbf{p}_M$ fixed. Thus

$$P(\mathbf{p}|t) \equiv \lim_{T\to 0}\frac{1}{\mathcal{N}}\int d^3x_0\ldots d^3x_{N+1}d^3p_0\ldots d^3p_{M-1}d^3p_{M+1}\ldots d^3p_N$$
$$\Re\Big[\psi(\mathbf{x}_{N+1},t_{N+1})^*\exp\tfrac{i}{\hbar}\textstyle\sum_{k=0}^{N}(\mathbf{p}_k\cdot\Delta\mathbf{x}_k-H_k\Delta t_k)\psi(\mathbf{x}_0,t_0)\Big]. \tag{5.3.19}$$

In Eq. (5.3.19) it is easy to perform, formally, most of the integrations, accordingly to Eq. (5.1.8); this simplifies consistently the expression for

$P(\mathbf{p}|t)$, leaving us with

$$P(\mathbf{p}|t) = \frac{1}{(2\pi\hbar)^3} \lim_{T\to 0} \Re \int d^3x_0 d^3x_M d^3x_{M+1} d^3x_{N+1}$$
$$\psi(N+1)^* K(N+1, M+1) e^{i(\mathbf{p}\cdot\Delta\mathbf{x}_M - H_M \Delta t_M)/\hbar} K(M,0)\psi(0), \quad (5.3.20)$$

where an obvious notation for the kernels and the wave functions has been adopted. Using now Eq. (5.1.5), and taking the limit $T \to 0$ under the integrals, Eq. (5.3.20) becomes

$$P(\mathbf{p}|t) = \Re \int d^3x_M d^3x_{M+1} \psi(\mathbf{x}_{M+1}, t)^* \frac{e^{i\mathbf{p}\cdot\mathbf{x}_{M+1}/\hbar}}{(2\pi\hbar)^{3/2}} \cdot \frac{e^{-i\mathbf{p}\cdot\mathbf{x}_M/\hbar}}{(2\pi\hbar)^{3/2}} \psi(\mathbf{x}_M, t), \quad (5.3.21)$$

that is

$$P(\mathbf{p}|t) = |\phi(\mathbf{p}, t)|^2 . \quad (5.3.22)$$

This result agrees perfectly with the predictions of standard quantum theory [95,100], and can therefore be considered as a further successful check on the reliability of $P[\gamma]$.

It is interesting to observe that the conclusion (5.3.22) could have been reached also in the form

$$P(\mathbf{p}''|t'') = \int_{\tilde{\Gamma}''} \tilde{\mathcal{D}}\gamma \, \tilde{P}[\gamma] , \quad (5.3.23)$$

with a much more straightforward procedure. Our derivation based on $P[\gamma]$ emphasizes, however, the fact that both $P[\gamma]$ and $\tilde{P}[\gamma]$ can be used in order to get the same physically meaningful probabilities.

It is worth noticing, in closing this section, that if f is an arbitrary function of position, and γ are paths such that $t \mapsto \mathbf{x}(t)$, then

$$\langle\psi|f(\hat{\mathbf{x}}(t))|\psi\rangle = \int_{\Gamma_0} \mathcal{D}\gamma \, P[\gamma] \, f(\mathbf{x}(t)) , \quad (5.3.24)$$

for all times $t \in (t', t'')$, as it is easy to prove. Similarly, if g is a function of momentum, and $t \mapsto \mathbf{p}(t)$ for the histories γ, then

$$\langle\psi|g(\hat{\mathbf{p}}(t))|\psi\rangle = \int_{\Gamma_0} \mathcal{D}\gamma \, P[\gamma] \, g(\mathbf{p}(t)) . \quad (5.3.25)$$

These relations ensure that $P[\gamma]$ can be used to correctly compute expectation values; a more general result along this line will be given at the end of the next section.

5.4 Phase Space Distributions from $P[\gamma]$

The next logical step in the construction of quasiprobability functions out of $P[\gamma]$ consists in searching for a joint distribution [73,98,79,80] $P(\mathrm{x},\mathrm{p}|t)$ for both x and p at time t. However, since this calculation will require a more delicate treatment than those performed so far, we must first discuss the formal structure of the terms H_k in Eq. (5.1.8).

In the derivation of Eq. (5.1.8) from the Hilbert space formulation of quantum mechanics, the form of the generic factor in Eq. (5.1.7) depends strongly on the prescription used to order the noncommuting operators $\hat{\mathrm{x}}$ and $\hat{\mathrm{p}}$ in the Hamiltonian H. It is possible to invent many of them, but we are interested here mainly in Weyl's and Rivier's [79,80].

Weyl's prescription is, for a same generic component of $\hat{\mathrm{x}}$ and $\hat{\mathrm{p}}$,

$$\mathcal{W}\{\hat{x}^n\hat{p}^m\} \equiv \frac{1}{2^n}\sum_{l=0}^{n}\begin{pmatrix} n \\ l \end{pmatrix}\hat{x}^{n-l}\hat{p}^m\hat{x}^l \; ; \tag{5.4.1}$$

this implies

$$\langle \mathrm{x}_{k+1}|\mathcal{W}\{\hat{x}^n\hat{p}^m\}|\mathrm{x}_k\rangle = \left(\frac{x_{k+1}+x_k}{2}\right)^n \langle \mathrm{x}_{k+1}|\hat{p}^m|\mathrm{x}_k\rangle \; , \tag{5.4.2}$$

leading to

$$\langle \mathrm{x}_{k+1}|\mathcal{W}\{H(\hat{\mathrm{x}},\hat{\mathrm{p}}\}|\mathrm{x}_k\rangle = \langle \mathrm{x}_{k+1}|H(\bar{\mathrm{x}}_k,\hat{\mathrm{p}})|\mathrm{x}_k\rangle \; , \tag{5.4.3}$$

where $H(\mathrm{x},\mathrm{p})$ is the classical hamiltonian, and

$$\bar{\mathrm{x}}_k \equiv \frac{1}{2}(\mathrm{x}_{k+1}+\mathrm{x}_k) \; . \tag{5.4.4}$$

Rivier suggests the symmetrization rule

$$\mathcal{S}\{\hat{x}^n\hat{p}^m\} \equiv \frac{1}{2}(\hat{x}^n\hat{p}^m + \hat{p}^m\hat{x}^n) \; , \tag{5.4.5}$$

which gives

$$\langle \mathrm{x}_{k+1}|\mathcal{S}\{\hat{x}^n\hat{p}^m\}|\mathrm{x}_k\rangle = \left(\frac{x_{k+1}^n+x_k^n}{2}\right) \langle \mathrm{x}_{k+1}|\hat{p}^m|\mathrm{x}_k\rangle \; , \tag{5.4.6}$$

and consequently

$$\langle \mathrm{x}_{k+1}|\mathcal{S}\{H(\hat{\mathrm{x}},\hat{\mathrm{p}})\}|\mathrm{x}_k\rangle = \frac{1}{2}\langle \mathrm{x}_{k+1}|H(\mathrm{x}_{k+1},\hat{\mathrm{p}})|\mathrm{x}_k\rangle + \frac{1}{2}\langle \mathrm{x}_{k+1}|H(\mathrm{x}_k,\hat{\mathrm{p}})|\mathrm{x}_k\rangle \; . \tag{5.4.7}$$

Using the completeness relation for the momentum, it is easy to see that Eq. (5.4.3) leads to

$$H_k = H(\bar{\mathrm{x}}_k, \mathrm{p}_k) \ , \tag{5.4.8}$$

while Eq. (5.4.7) implies

$$H_k = \frac{1}{2} H(\mathrm{x}_{k+1}, \mathrm{p}_k) + \frac{1}{2} H(\mathrm{x}_k, \mathrm{p}_k) \ . \tag{5.4.9}$$

It is therefore clear [86,81] that a path integral formulation of quantum mechanics starting with the midpoint prescription (5.4.8) would lead to the same results of the standard theory in which the Weyl ordering (5.4.1) is adopted; similarly, the use of Eq. (5.4.9) in the path integral corresponds to the ordering (5.4.5) in the Hilbert space formulation. Now, in standard quantum theory, the choice for the ordering for non-commuting operators does not affect the relations (5.3.3) and (5.3.22) for $P(\mathrm{x}|t)$ and $P(\mathrm{p}|t)$, and it is easy to understand that this is also the case in our method, where such equations hold independently of the choice between Eqs. (5.4.8) and (5.4.9). However, when the quantity to be calculated is $P(\mathrm{x},\mathrm{p}|t)$, it is known that the prescriptions (5.4.1) and (5.4.5) lead to inequivalent results [79,80]. More precisely, Eqs. (5.4.1) and (5.4.5) correspond, respectively, to the Wigner [73] function (3.2.17) and to the Margenau-Hill [88] function (3.3.10). We can thus infer that, in our formalism, the calculation of $P(\mathrm{x},\mathrm{p}|t)$ based on Eqs. (5.4.8) or (5.4.9) will also lead, respectively, to Eqs. (3.2.17) or (3.3.10): This is correct, as we shall now formally show.

To get the quasiprobability for the particle to have position x and momentum p at time t, we must obviously integrate $P[\gamma]$ over all the histories which, at time t, cross the point x with a momentum p. According to the prescription (5.4.8), the momentum p_k corresponds to the position $\bar{\mathrm{x}}_k$ given by Eq. (5.4.4); therefore, in the path integral for $P(\mathrm{x},\mathrm{p}|t)$, we must keep constant $\mathrm{p}_M \equiv \mathrm{p}$ and

$$\mathrm{x} \equiv \frac{1}{2}(\mathrm{x}_{M+1} + \mathrm{x}_M) \ , \tag{5.4.10}$$

for $t = \tau_M$, $M \leq N$. Introducing the new variable

$$\xi \equiv \frac{1}{2}(\mathrm{x}_{M+1} - \mathrm{x}_M) \ , \tag{5.4.11}$$

we can write straightforwardly the volume element for the "broken" path integral related to $P(\mathbf{x},\mathbf{p}|t)$ as[1]

$$2^3 d^3x_0 \ldots d^3x_{M-1} d^3\xi d^3x_{M+2} \ldots d^3x_{N+1} d^3p_0 \ldots d^3p_{M-1} d^3p_{M+1} \ldots d^3p_N \; ;$$

thus

$$\begin{aligned} P(\mathbf{x},\mathbf{p}|t) \equiv \lim_{T\to 0} \frac{2^3}{\mathcal{N}} \int & d^3x_0 \ldots d^3x_{M-1} d^3\xi d^3x_{M+2} \ldots d^3x_{N+1} \\ & d^3p_0 \ldots d^3p_{M-1} d^3p_{M+1} \ldots d^3p_N \\ & \Re \left[\psi(\mathbf{x}_{N+1}, t_{N+1})^* \exp \frac{i}{\hbar} \sum_{k=M+1}^{N} (\mathbf{p}_k \cdot \Delta\mathbf{x}_k - H(\bar{\mathbf{x}}_k, \mathbf{p}_k)\Delta t_k) \right. \\ & \exp \frac{i}{\hbar} (2\mathbf{p}\cdot\boldsymbol{\xi} - H(\mathbf{x},\mathbf{p})\Delta t_M) \\ & \left. \exp \frac{i}{\hbar} \sum_{k=0}^{M-1} (\mathbf{p}_k \cdot \Delta\mathbf{x}_k - H(\bar{\mathbf{x}}_k, \mathbf{p}_k)\Delta t_k)\, \psi(\mathbf{x}_0, t_0) \right] , \end{aligned} \tag{5.4.12}$$

which easily becomes, with the help of Eq. (5.1.8),

$$\begin{aligned} P(\mathbf{x},\mathbf{p}|t) = \frac{1}{(\pi\hbar)^3} \lim_{T\to 0} \Re \int & d^3x_0\, d^3\xi\, d^3x_{N+1} \\ & \psi(N+1)^*\, K(N+1, M+1)\, e^{i(2\mathbf{p}\cdot\xi - H(\mathbf{x},\mathbf{p})\Delta t_M)/\hbar}\, K(M,0)\, \psi(0) \end{aligned} \tag{5.4.13}$$

Now, with the same procedure adopted to get Eq. (5.3.21), we obtain, remembering Eqs. (5.4.10) and (5.4.11), the Wigner function (3.2.17).

We need now only show that the alternative prescription (5.4.9) leads to the Margenau and Hill function (3.3.10). To do this, we must decide, first of all, which position $\mathbf{x}$ to keep fixed in the integration, together with the momentum $\mathbf{p} = \mathbf{p}_M$. While, in the previously treated case of Eq. (5.4.8), $\mathbf{x}$ was clearly defined by Eq. (5.4.8) itself, the prescription (5.4.9) is not so transparent about this point. We can notice, however, that Eq. (5.4.9) is equivalent to writing the path integral for $K(\mathbf{x}'', t''; \mathbf{x}', t')$ as

$$\int_{\Gamma_1} \mathcal{D}\gamma\, K[\gamma] = \frac{1}{2} \int_{\Gamma_1} \mathcal{D}_{\mathcal{X}}\gamma\, K[\gamma] + \frac{1}{2} \int_{\Gamma_1} \mathcal{D}_{\mathcal{P}}\gamma\, K[\gamma] \, , \tag{5.4.14}$$

[1]The factor 2^3 comes from the relation $d^3x_M d^3x_{M+1} = 2^3 d^3x d^3\xi$, which is an obvious consequence of Eqs. (5.4.10) and (5.4.11).

where

$$\int_{\Gamma_1} \mathcal{D}_{\mathcal{X}}\gamma \, K[\gamma] \equiv \lim_{T\to 0} \frac{1}{\mathcal{N}} \int d^3x_1 \ldots d^3x_N d^3p_0 \ldots d^3p_N$$

$$\exp \frac{i}{\hbar} \sum_{k=0}^{N} (\mathbf{p}_k \cdot \Delta \mathbf{x}_k - H(\mathbf{x}_{k+1}, \mathbf{p}_k)\Delta t_k), \quad (5.4.15)$$

and

$$\int_{\Gamma_1} \mathcal{D}_{\mathcal{P}}\gamma \, K[\gamma] \equiv \lim_{T\to 0} \frac{1}{\mathcal{N}} \int d^3x_1 \ldots d^3x_N d^3p_0 \ldots d^3p_N$$

$$\exp \frac{i}{\hbar} \sum_{k=0}^{N} (\mathbf{p}_k \cdot \Delta \mathbf{x}_k - H(\mathbf{x}_k, \mathbf{p}_k)\Delta t_k) \, . \quad (5.4.16)$$

In Eqs. (5.4.15) and (5.4.16), the position associated with the momentum $\mathbf{p}_k$ is well defined, and equal, respectively, to $\mathbf{x}_{k+1}$ and $\mathbf{x}_k$. Eq. (5.4.14) can thus be interpreted by saying that half of the trajectories, in the summation, are approximated according to the correspondence rule

$$\mathbf{p}_k \longleftrightarrow \mathbf{x}_{k+1} \, ,$$

while for the other half it is

$$\mathbf{p}_k \longleftrightarrow \mathbf{x}_k \, ;$$

this can also be understood by noticing that the Rivier ordering admits the representation

$$\mathcal{S} = \frac{1}{2}\mathcal{X} + \frac{1}{2}\mathcal{P} \, , \quad (5.4.17)$$

where $\mathcal{X}$ and $\mathcal{P}$ are the orderings

$$\mathcal{X}\{\hat{x}^n \hat{p}^m\} \equiv \hat{x}^n \hat{p}^m \, , \quad (5.4.18)$$

$$\mathcal{P}\{\hat{x}^n \hat{p}^m\} \equiv \hat{p}^m \hat{x}^n \, , \quad (5.4.19)$$

which correspond, respectively, to the integrals (5.4.15) and (5.4.16), and justify the use of the subscripts $\mathcal{X}$ and $\mathcal{P}$ in their measures.

In the light of these considerations, we shall write the quasiprobability $P(\mathbf{x}, \mathbf{p}|t)$ associated with the prescription (5.4.9) as

$$P(\mathbf{x}, \mathbf{p}|t) = \frac{1}{2} P_{\mathcal{X}}(\mathbf{x}, \mathbf{p}|t) + \frac{1}{2} P_{\mathcal{P}}(\mathbf{x}, \mathbf{p}|t) \, , \quad (5.4.20)$$

where $P_{\mathcal{X}}$ and $P_{\mathcal{P}}$ are calculated according to the prescriptions underlying, respectively, Eqs. (5.4.15) and (5.4.16). Therefore, in the integral for $P_{\mathcal{X}}$, we shall keep constant $\mathbf{p}_M \equiv \mathbf{p}$ and $\mathbf{x}_{M+1} \equiv \mathbf{x}$, while in the one for $P_{\mathcal{P}}$ it will be $\mathbf{x}_M$ that plays the role of $\mathbf{x}$.

Let us first compute the form of $P_{\mathcal{X}}$; defining the new variable

$$\boldsymbol{\xi} \equiv \mathbf{x}_M - \mathbf{x}_{M+1} \ , \tag{5.4.21}$$

the volume element for the "broken" path integral is

$$d^3x_0 \dots d^3x_{M-1} d^3\xi d^3x_{M+2} \dots d^3x_{N+1} d^3p_0 \dots d^3p_{M-1} d^3p_{M+1} \dots d^3p_N \ ,$$

and

$$\begin{aligned} P_{\mathcal{X}}(\mathbf{x},\mathbf{p}|t) \equiv & \lim_{T\to 0} \frac{1}{\mathcal{N}} \int d^3x_0 \dots d^3x_{M-1} d^3\xi d^3x_{M+2} \dots d^3x_{N+1} \\ & d^3p_0 \dots d^3p_{M-1} d^3p_{M+1} \dots d^3p_N \\ & \Re \left[\psi(\mathbf{x}_{N+1}, t_{N+1})^* \exp \frac{i}{\hbar} \sum_{k=M+1}^{N} (\mathbf{p}_k \cdot \Delta\mathbf{x}_k - H(\mathbf{x}_{k+1}, \mathbf{p}_k)\Delta t_k) \right. \\ & \exp -\frac{i}{\hbar} (\mathbf{p} \cdot \boldsymbol{\xi} + H(\mathbf{x},\mathbf{p})\Delta t_M) \\ & \left. \exp \frac{i}{\hbar} \sum_{k=0}^{M-1} (\mathbf{p}_k \cdot \Delta\mathbf{x}_k - H(\mathbf{x}_{k+1}, \mathbf{p}_k)\Delta t_k) \, \psi(\mathbf{x}_0, t_0) \right] \ ; \end{aligned} \tag{5.4.22}$$

Eq. (5.4.22) becomes, by Eq. (5.4.15),

$$\begin{aligned} P_{\mathcal{X}}(\mathbf{x},\mathbf{p}|t) = & \frac{1}{(2\pi\hbar)^3} \lim_{T\to 0} \Re \int d^3x_0 d^3\xi d^3x_{N+1} \\ & \psi(N+1)^* K(N+1, M+1) e^{-i(\mathbf{p}\cdot\boldsymbol{\xi} + H(\mathbf{x},\mathbf{p})\Delta t_M)/\hbar} K(M,0)\, \psi(0) \end{aligned} \tag{5.4.23}$$

Using now Eqs. (5.1.5) and (5.4.21), and taking the limit under the integrals, we get

$$P_{\mathcal{X}}(\mathbf{x},\mathbf{p}|t) = \frac{1}{(2\pi\hbar)^3} \Re \int d^3\xi \psi(\mathbf{x},t)^* e^{-i\mathbf{p}\cdot\boldsymbol{\xi}/\hbar} \psi(\mathbf{x}+\boldsymbol{\xi},t) \ , \tag{5.4.24}$$

which is already equal to Eq. (3.3.10). Similarly, for $P_{\mathcal{P}}$, the result is

$$P_{\mathcal{P}}(\mathbf{x},\mathbf{p}|t) = \frac{1}{(2\pi\hbar)^3} \Re \int d^3\xi \psi(\mathbf{x}-\boldsymbol{\xi},t)^* e^{-i\mathbf{p}\cdot\boldsymbol{\xi}/\hbar} \psi(\mathbf{x},t) \ , \tag{5.4.25}$$

which is equal to Eq. (3.3.10) as well. Thus, by relation (5.4.20), we can claim that the prescription (5.4.9) leads to the Margenau-Hill function for the quasiprobability $P(\mathrm{x}, \mathrm{p}|t)$.

We can now generalize the results contained in Eqs. (5.3.24) and (5.3.25) to the case of an observable F which is a function both of position and momentum; if the paths γ are represented, as usual, by $t \mapsto \mathrm{x}(t)$ and $t \mapsto \mathrm{p}(t)$, and $\mathcal{O}$ stands either for the Weyl or for the Rivier ordering, then

$$\langle\psi|\mathcal{O}\{F(\hat{\mathrm{x}}(t), \hat{\mathrm{p}}(t))\}|\psi\rangle = \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, F(\mathrm{x}(t), \mathrm{p}(t)) \;, \qquad (5.4.26)$$

where the integral is performed according to the prescription related to the ordering $\mathcal{O}$, and $t \in (t', t'')$. The proof of Eq. (5.4.26) is rather similar to the others given in this section, and is therefore omitted; we only notice that the key idea is to show that the right hand side is equal to

$$\int d^3x d^3p\, P_{\mathcal{O}}(\mathrm{x}, \mathrm{p}|t)\, F(\mathrm{x}, \mathrm{p}) \;: \qquad (5.4.27)$$

The relation then follows from well known properties of the phase space distributions.

Let us prove, at this point, Eq. (5.2.13) for the case of the prescription (5.4.8) (the case of (5.4.9) is completely analogous); for $t = t_M$, $M \leq N$, the left hand side reads

$$\int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, F(\mathrm{x}(t), \mathrm{p}(t)) = \lim_{T\to 0} \frac{1}{\mathcal{N}} \int d^3x_0 \ldots d^3x_{N+1} d^3p_0 \ldots d^3p_N$$

$$\psi(\mathrm{x}_{N+1}, t_{N+1})^* \exp \frac{i}{\hbar} \sum_{k=0}^{N} (\mathrm{p}_k \cdot \Delta\mathrm{x}_k - H_k \Delta t_k) \psi(\mathrm{x}_0, t_0) F(\bar{\mathrm{x}}_M, \mathrm{p}_M); \quad (5.4.28)$$

the right hand side of Eq. (5.4.28) becomes, remembering that the wave function in the momentum space is the Fourier transform of the one in the configuration space,

$$\lim_{T\to 0} \frac{(2\pi)^3}{\mathcal{N}} \int d^3x_1 \ldots d^3x_N d^3p_0 \ldots d^3p_N\, \phi(\mathrm{p}_N, t_{N+1})^*$$

$$\exp -\frac{i}{\hbar} \sum_{k=1}^{N} (\mathrm{x}_k \cdot \Delta\mathrm{p}_k + H_k \Delta t_k) \phi(\mathrm{p}_0, t_0) e^{-iH_0 \Delta t_0} F(\bar{\mathrm{x}}_M, \mathrm{p}_M), \quad (5.4.29)$$

where $\Delta\mathrm{p}_k \equiv \mathrm{p}_{k+1} - \mathrm{p}_k$; when the limit is taken, (5.4.29) turns out to be equal to the right hand side of Eq. (5.2.13), thus establishing the equality.

To close this section, we believe it is worth noticing that our results, namely that, with the appropriate prescriptions, the quasiprobabilities $P[\gamma]$ sum up to the Wigner and the Margenau-Hill functions (3.2.17) and (3.3.10), give a definitive proof of the statement made in the Introduction, that the $P[\gamma]$ are not, in general, positive semidefinite. In fact, both (3.2.17) and (3.3.10) can attain negative values [73,98], and this is possible only if there are trajectories for which $P[\gamma]$ is negative.

5.5 Conclusions

In the previous sections we defined a quasiprobability functional $P[\gamma]$ on the space of histories for a quantum particle. The introduction of such a concept allows one to reproduce not only the usual probability densities for position and momentum, but also the phase space distributions suggested by Wigner and by Margenau and Hill, in a way which is consistent with the usual connections between path integration and operator ordering. More precisely, a rule for the ordering of noncommuting operators in the Hilbert space corresponds to a prescription on the way one performs the path integrals by approximating the trajectories with sequences of points in the configuration and momentum spaces. We have examined the Weyl and the Rivier orderings which correspond, respectively, to the prescriptions (5.4.8) and (5.4.9): These generate in turn, through our formalism, the Wigner and the Margenau-Hill functions (3.2.17) and (3.3.10). Of course, other rules could be studied.

We believe it is worth stressing that Eqs. (5.2.5), (5.2.7) and (5.2.8) lead all to the same results as regards $P(\mathbf{x}|t)$, $P(\mathbf{p}|t)$ and $P(\mathbf{x},\mathbf{p}|t)$: Therefore these three expressions are, for the concerns of the present chapter, equivalent, apart from matters of taste or prejudice. In choosing Eq. (5.2.8) we have been motivated by the wish to have a *real* quasiprobability for $P[\gamma]$; this led to a "degeneracy" of results for $P(\mathbf{x},\mathbf{p}|t)$, in the sense that the three different rules $\mathcal{S}$, $\mathcal{X}$ and $\mathcal{P}$ all correspond to the same $P(\mathbf{x},\mathbf{p}|t)$, i.e., to the Margenau-Hill function. This could be avoided simply by choosing $P[\gamma]$ according to Eq. (5.2.5): In so doing, all the results would have been the same as ours, with the difference that the use of the orderings $\mathcal{X}$ and $\mathcal{P}$ would have led, respectively, to the functions

$$P'_{\mathcal{X}}(\mathbf{x},\mathbf{p}|t) = \frac{1}{(2\pi\hbar)^3}\int d^3\xi \psi(\mathbf{x},t)^* e^{-i\mathbf{p}\cdot\xi/\hbar}\psi(\mathbf{x}+\xi,t) \; , \qquad (5.5.1)$$

and

$$P'_{\mathcal{P}}(\mathbf{x},\mathbf{p}|t) = P'_{\mathcal{X}}(\mathbf{x},\mathbf{p}|t)^* \; . \qquad (5.5.2)$$

This is in perfect agreement with already performed calculations [79], but since the prescriptions $\mathcal{X}$ and $\mathcal{P}$ do not produce, in general, self adjoint operators, they are not very interesting, and we prefer thus the more symmetric real expression (5.2.8) for $P[\gamma]$; of course, if further investigations give evidence in favour of Eq. (5.2.5) or of its complex conjugate Eq. (5.2.7), this would not create any serious problem for our formalism: Only minor changes in the details would be implied by the replacement of Eq. (5.2.8) with one of those expressions.

There is, however, an argument which we believe is strong enough to rule out both the distributions (5.5.1) and (5.5.2). Let us remember that the probability current $\mathbf{j}(\mathbf{x},t)$ for a quantum particle of mass m can be written as

$$\mathbf{j}(\mathbf{x},t) = \frac{\hbar}{2im}\psi(\mathbf{x},t)^* \overleftrightarrow{\nabla}\, \psi(\mathbf{x},t) \; . \qquad (5.5.3)$$

It is natural to think that a viable phase space distribution $P(\mathbf{x},\mathbf{p}|t)$ should lead to the equation

$$\mathbf{j}(\mathbf{x},t) = \int d^3x' d^3p' \, P(\mathbf{x}',\mathbf{p}'|t)\delta^3(\mathbf{x}-\mathbf{x}')\frac{1}{m}\mathbf{p}' \; : \qquad (5.5.4)$$

We shall now prove that this is true when P is either the Wigner or the Margenau and Hill function, but not for $P = P'_{\mathcal{X}}$ or $P = P'_{\mathcal{P}}$.

Inserting the Wigner function $P_{\mathcal{W}}$ in the right hand side of Eq. (5.5.4), we get, after some trivial steps,

$$\frac{\hbar}{2(\pi\hbar)^3 im}\int d^3p' d^3\xi \, \psi(\mathbf{x}+\xi,t)^* \nabla_\xi e^{2i\mathbf{p}'\cdot\xi/\hbar}\, \psi(\mathbf{x}-\xi,t) \; , \qquad (5.5.5)$$

which, remembering the properties of the delta function, turns out to be equal to (5.5.3). Similarly, using the Margenau-Hill function in Eq. (5.5.4), we find

$$\frac{\hbar}{(2\pi\hbar)^3 im}\Re\int d^3p' d^3\xi \, \psi(\mathbf{x}+\xi,t)^* \nabla_\xi e^{i\mathbf{p}'\cdot\xi/\hbar}\, \psi(\mathbf{x},t) \; , \qquad (5.5.6)$$

again equal to (5.5.3). If, however, we insert Eq. (5.5.1) into Eq. (5.5.4), it is easy to check that the result is not (5.5.3), but rather

$$\frac{\hbar}{im}\psi(\mathbf{x},t)^*\nabla\psi(\mathbf{x},t) \; ; \tag{5.5.7}$$

similarly, $P'_{\mathcal{P}}$ leads to the complex conjugate of (5.5.7). The incompleteness of these results induces us to discard the orderings $\mathcal{X}$ and $\mathcal{P}$; the same conclusions would have been reached by considering

$$\mathbf{j}(\mathbf{x},t) = \langle\psi|\hat{\mathbf{j}}(\mathbf{x},t)|\psi\rangle \; , \tag{5.5.8}$$

for

$$\hat{\mathbf{j}}(\mathbf{x},t) = \mathcal{O}\left\{\delta^3(\hat{\mathbf{x}}(t) - \mathbf{x}\hat{1})\frac{1}{m}\hat{\mathbf{p}}(t)\right\} \; , \tag{5.5.9}$$

with $\mathcal{O}$ equal to $\mathcal{W}$, $\mathcal{S}$, $\mathcal{X}$ or $\mathcal{P}$.

The ideas developed in this chapter appear to be very general, and can be applied to a wide range of circumstances; even if we have concentrated more on the derivation of phase space distributions, we believe our main result to be the reformulation of quantum theory in terms of quasiprobabilities. This technique seems to be quite useful, since it allows to compute physically relevant quantities thinking in terms of sets of well defined paths, each assigned with a given quasiprobability: In such a way it is possible to circumvent the need to use amplitudes, and this turns out to be particularly useful when dealing with semiclassical systems.

Chapter 6

Explicitly Covariant Relativistic Quantum Theory

It has been shown in Ch. 5, how a quasiprobability functional $P[\gamma]$ can be defined on the space of histories for a nonrelativistic quantum particle, as in Eq. (5.2.8). In this chapter we want to extend the notion of the quasiprobability functional $P[\gamma]$ to the case of a relativistic quantum particle. The structure (5.2.8) of the nonrelativistic $P[\gamma]$ makes clear that, in order to be able to construct its relativistic generalization, two preliminary results must first be accomplished:

i) To find a suitable form for the relativistic action $S[\gamma]$;

ii) To find a four dimensional generalization Ψ of the three dimensional wave function ψ.

Both these problems are important and have to be solved; however, from $S[\gamma]$ it is possible to define, through Eq. (5.1.2) and a path integral, a propagator for Ψ which determines, finally, the differential equation that Ψ must satisfy. Therefore, we shall deal first, in Sec. 6.1, with the issue of determining $S[\gamma]$. Sec. 6.2 is devoted to the construction of Ψ, and to the analysis of the equation to which it obeys. In Sec. 6.3 the relativistic quasiprobability $P[\gamma]$ is defined, and its relationship with the quasiprobability distributions obtained straightforwardly from Ψ are investigated. Further details and insight into the structure of the resulting version of quantum theory are discussed in Sec. 6.4.

Throughout all this chapter, we shall deal with a spinless particle in the Minkowski spacetime. The treatment follows that of ref. [104]

6.1 The Action Functional for a Relativistic Particle

The three dimensional form of the action for a particle,

$$^3S[\gamma] = \int_{t'}^{t''} dt \left[p_i(t)\frac{dx^i(t)}{dt} - H(\mathrm{x}(t), \mathrm{p}(t), t)\right] , \tag{6.1.1}$$

is obviously not explicitly covariant, and thus not very helpful in the construction of a relativistic $P[\gamma]$. It is well known, however, that a four dimensional modification of Eq. (6.1.1) exists, which is covariant and from which the equations of motion can be derived by a Hamilton principle. Define, in fact, a path γ as the pair of curves in the extended four dimensional configuration and momentum spaces: $\lambda \mapsto x(\lambda)$, $\lambda \mapsto p(\lambda)$, where $\lambda \in [\lambda', \lambda'']$ is a parameter. Then, for a suitable choice of the superhamiltonian [2] $\mathcal{H}(x, p, \lambda)$, the four dimensional action is

$$S[\gamma] \equiv \int_{\lambda'}^{\lambda''} d\lambda \left[p_a(\lambda)\frac{dx^a(\lambda)}{d\lambda} - \mathcal{H}(x(\lambda), p(\lambda), \lambda)\right] . \tag{6.1.2}$$

It is easy to verify that, denoting by Γ_1 the set of all the histories such that $x(\lambda') = x'$ and $x(\lambda'') = x''$, with fixed x' and x'', the action $S[\gamma]$ is stationary under arbitrary variations of γ belonging to Γ_1 if and only if γ satisfies the four dimensional form of the Hamilton equations:

$$\frac{dx^a}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_a} , \tag{6.1.3}$$

$$\frac{dp_a}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^a} . \tag{6.1.4}$$

The most important problem about the action given by Eq. (6.1.2) is represented by the choice of $\mathcal{H}(x, p, \lambda)$; before discussing this point, however, we want to show how the four dimensional action S can be reduced to the three dimensional action ^3S following a procedure [105] which is well known in classical mechanics [106], and that we shall briefly review here.

Let us consider, as a preliminary example, a particle described by Eq. (6.1.1), with the important feature that the hamiltonian H does not explicitly depend on time. Then the solutions of the Hamilton equations will completely lie on the hypersurface defined by

$$H(\mathbf{x},\mathbf{p}) = E = \text{const.}\ ; \tag{6.1.5}$$

the reduced action

$${}^3\bar{S} \equiv \int_{t'}^{t''} dt\, p_i(t) \frac{dx^i(t)}{dt}\ , \tag{6.1.6}$$

together with the condition (6.1.5), will be thus sufficient to reproduce the equations of motion under a Hamilton principle, since the variation of the additional term $E(t'' - t')$ turns out to be equal to zero. Now, Eq. (6.1.5) can be solved to obtain, e.g., p_3 as a function of the remaining coordinates and momenta,

$$p_3 \equiv -\bar{H}(x^1, x^2; p^1, p^2; \bar{t}; E)\ , \tag{6.1.7}$$

where

$$\bar{t} \equiv x^3\ . \tag{6.1.8}$$

If in the range $[t', t'']$ the function $t \mapsto x^3(t)$ is one to one, Eq. (6.1.6) can be rewritten as

$${}^3\bar{S} = \int_{\bar{t}'}^{\bar{t}''} d\bar{t} \left[p_\alpha(\bar{t}) \frac{dx^\alpha(\bar{t})}{d\bar{t}} - \bar{H} \right]\ , \tag{6.1.9}$$

where $\alpha \in \{1, 2\}$, and $\bar{t}' \equiv x^3(t')$, $\bar{t}'' \equiv x^3(t'')$. It is obvious that ${}^3\bar{S}$ is stationary, under arbitrary variations of $x^\alpha(\bar{t})$ and $p_\alpha(\bar{t})$ such that $\delta x^\alpha(\bar{t}') = \delta x^\alpha(\bar{t}'') = 0$, in correspondence of the history which solves the Hamilton equations; conversely, requiring ${}^3\bar{S}$ to be stationary under the conditions mentioned above, one finds the equations

$$\frac{dx^\alpha}{d\bar{t}} = \frac{\partial \bar{H}}{\partial p_\alpha}\ , \tag{6.1.10}$$

$$\frac{dp_\alpha}{d\bar{t}} = -\frac{\partial \bar{H}}{\partial x^\alpha}\ . \tag{6.1.11}$$

Eqs. (6.1.10) and (6.1.11) do not determine, however, the full history of the particle, but only the shape of its trajectory in the configuration and momentum spaces.

Let us now come back to the four dimensional form of the action, Eq. (6.1.2). If the superhamiltonian $\mathcal{H}$ is not explicitly dependent on λ, the motion takes place on the hypersurface

$$\mathcal{H}(x,p) = \mathcal{E} = \text{const.}\ ; \tag{6.1.12}$$

since the term $\mathcal{E}(\lambda'' - \lambda')$ has a null variation, it is sufficient to consider a reduced action

$$\bar{S} = \int_{\lambda'}^{\lambda''} d\lambda\, p_a(\lambda) \frac{dx^a(\lambda)}{d\lambda}\ . \tag{6.1.13}$$

If in the range $[\lambda', \lambda'']$ the function $\lambda \mapsto x^0(\lambda)$ is one to one, Eq. (6.1.13) can be rewritten as

$$\bar{S} = \int_{t'}^{t''} dt \left[p_i(t) \frac{dx^i(t)}{dt} - H \right]\ , \tag{6.1.14}$$

where H is obtained solving Eq. (6.1.12) with respect to p_0,

$$p_0 \equiv -H(\mathbf{x}, \mathbf{p}, t, \mathcal{E})\ , \tag{6.1.15}$$

and

$$t \equiv x^0\ , \tag{6.1.16}$$

with $t' \equiv x^0(\lambda')$ and $t'' \equiv x^0(\lambda'')$. It is clear that $\bar{S}$ is nothing else than the three dimensional action ${}^3S$ given by Eq. (6.1.1), and that the corresponding Hamilton equations

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i} \tag{6.1.17}$$

and

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \tag{6.1.18}$$

determine the form of the trajectory as $x^i(t)$ and $p_i(t)$, but lack the λ-dependence which can be obtained by the higher dimensional system (6.1.3)–(6.1.4). It is important to notice that the constant $\mathcal{E}$ in

Eq. (6.1.12), like E in Eq. (6.1.5), is determined by the boundary conditions and should not, in principle, have anything to do with any parameter possibly contained into $\mathcal{H}$.

After having made clear these connections between the four dimensional covariant action and its three dimensional reduced form, let us face the problem of the choice of the superhamiltonian $\mathcal{H}$. A necessary requirement is obviously that the Hamilton equations (6.1.3)–(6.1.4) lead to the correct classical equations of motion, for the case, e.g., of a free particle. However, it is well known [107] that this condition is not sufficient: Every superhamiltonian of the form

$$\mathcal{H} = f(\eta^{ab} p_a p_b) , \tag{6.1.19}$$

where f is an arbitrary function, is good for this purpose. In order to identify, therefore, another constraint allowing to define uniquely the function f, let us apply Eqs. (6.1.3) and (6.1.4) to the superhamiltonian (6.1.19); we obtain, respectively,

$$\frac{dx^a}{d\lambda} = 2p^a f'(\eta^{ab} p_a p_b) , \tag{6.1.20}$$

and

$$\frac{dp_a}{d\lambda} = 0 , \tag{6.1.21}$$

where f' denotes the derivative of f with respect to its argument. The relation (6.1.21) has the immediate consequence that $\eta^{ab} p_a p_b$ is a constant, which will be denoted, in general, as $\varepsilon\, m^2$, with $\varepsilon \in \{-1, +1\}$ depending on the initial conditions and m positive. Eq. (6.1.20) becomes thus

$$\frac{dx^a}{d\lambda} = 2p^a f'(\varepsilon\, m^2) . \tag{6.1.22}$$

The arbitrariness in the choice of the parameter λ can be removed by requiring that $dx^a/d\lambda$ be not only proportional, as in Eq. (6.1.22), but exactly equal to p^a. This condition leads to

$$f'(\varepsilon\, m^2) = \frac{1}{2} . \tag{6.1.23}$$

Now, it is clear that, while the universal coefficients present in the superhamiltonian (6.1.19) have to be independent of the particular boundary conditions, ε and m are constants of motion, which do depend on such specific conditions. It is therefore natural to require that f be independent of them; it is easy to see that the only function satisfying this condition and Eq. (6.1.23) at once is

$$f(\xi) = \frac{1}{2}\xi \; . \qquad (6.1.24)$$

The superhamiltonian for a free particle is then

$$\mathcal{H} = \frac{1}{2}\eta^{ab}p_a p_b \; . \qquad (6.1.25)$$

It is important to notice that it is only the additional requirement that f be independent of ε and m to fix uniquely the form (6.1.25) for $\mathcal{H}$: In fact, to Eq. (6.1.25) it is sometimes preferred [105] the superhamiltonian

$$\mathcal{H} = -\sqrt{-\eta^{ab}p_a p_b} \; , \qquad (6.1.26)$$

corresponding to the choice $\varepsilon = -1$ and to a different parametrization; as we have already stressed, both these forms of $\mathcal{H}$ are equally good for the purpose of deriving the equations of motion; however, when discussing first principles, we believe the appearance of m and/or ε in $\mathcal{H}$ to be a disadvantage, being both of them related to the specific boundary conditions, and having not any formal fundamental meaning other than that of constants of motion. Moreover, the condition $\eta^{ab}p_a p_b = \text{const.}$ is a consequence of the classical Hamilton equation (6.1.21), i.e., of the Hamilton principle, which is not imposed at the quantum level. We shall thus use, in the quantization of the relativistic particle, the four dimensional action (6.1.2) where $\mathcal{H}$ reduces, for the case in which the particle is free, to the form (6.1.25).

It is worth noticing, to close this section, that whenever $\mathcal{H}$ does not explicitly depend on λ it is possible to write, according to Eq. (6.1.12),

$$\mathcal{H}(x,p) = \frac{\varepsilon\, m^2}{2} \; , \qquad (6.1.27)$$

and consequently perform the reduction of dimensions as discussed in relation to Eq. (6.1.15). A simple example is that of a particle with charge e in an electromagnetic field described by the potential A_a: The superhamiltonian reads

$$\mathcal{H} = \frac{1}{2}\eta^{ab}(p_a - eA_a)(p_b - eA_b) \; . \qquad (6.1.28)$$

Solving Eq. (6.1.27) with respect to p_0, and remembering Eq. (6.1.15), we get

$$H = \sqrt{(\mathrm{p} - e\mathbf{A})^2 - \varepsilon\, m^2} + eA_0 \ , \tag{6.1.29}$$

which is the hamiltonian usually adopted to describe such a system [105].

6.2 The Relativistic Wave Function

In the previous section we have achieved the first of the two preliminary goals indicated in the introduction to the present chapter, providing an action functional which is sufficiently independent of the specific conditions to allow to be used in Eq. (5.1.2) when performing a path integral quantization of the relativistic particle. The time has now come to find a relativistic generalization of the nonrelativistic wave function $\psi(\mathbf{x}, t)$: This will be the second ingredient from which a relativistic quasiprobability functional can be constructed.

It is obvious, from the expression (6.1.2) for the action and the related definition of a history, that in the relativistic domain the parameter λ plays the same role which, in the nonrelativistic theory, was played by the time t. This can be seen even more clearly by noticing that the path integral

$$K(x'', \lambda''; x', \lambda') = \int_{\Gamma_1} \mathcal{D}\gamma\, K[\gamma] \tag{6.2.1}$$

should be interpreted as the amplitude for the particle to move between the spacetime point x' for a value λ' of the parameter λ, and the point x'' for $\lambda = \lambda''$ (compare Eq. (6.2.1) with the Eq. (5.1.1), which represents the essence of the path integral formulation of quantum mechanics). We expect, therefore, that a relativistic version of the wave function should depend not only on the spacetime point x, but also on the value λ of the parameter labeling the points on the classical histories; such a wave function will be thus written as $\Psi(x, \lambda)$.

An alternative way of looking at $\Psi(x, \lambda)$ is the following: In order to have a covariant formulation, the time t is raised to the level of an observable, like the position $\mathbf{x}$, and its place as a parameter is taken by λ. In the Schrödinger picture, a state of the particle is then denoted as $|\Psi(\lambda)\rangle$, and $|x\rangle$ stands for a state of definite spacetime position x; if

$|y\rangle$ is another such state, corresponding to the event y, we require the normalization

$$\langle x|y\rangle = \delta^4(x-y) \ , \tag{6.2.2}$$

where δ^4 is the delta function over spacetime. Similarly,

$$\Psi(x,\lambda) = \langle x|\Psi(\lambda)\rangle \tag{6.2.3}$$

is the coordinate representation of the state $|\Psi(\lambda)\rangle$, and

$$P(x|\lambda) = |\Psi(x,\lambda)|^2 \tag{6.2.4}$$

represents the probability density (in spacetime) for the particle to be at x when the parameter has the value λ [105,108,109]. Of course,

$$\int d^4x\, P(x|\lambda) = \langle\Psi(\lambda)|\Psi(\lambda)\rangle = 1 \ . \tag{6.2.5}$$

With this interpretation for Ψ, it is straightforward to identify the amplitude $K(x'',\lambda'';x',\lambda')$ as the propagator for Ψ; in other words, if Γ'' is the set of all the paths satisfying the boundary condition $x(\lambda'') = x''$, then

$$\Psi(x'',\lambda'') = \int_{\Gamma''} \mathcal{D}\gamma\, K[\gamma]\, \Psi(x(\lambda'),\lambda') \ . \tag{6.2.6}$$

This equation leads to the four dimensional version of the Schrödinger equation[1]:

$$i\hbar\frac{\partial\Psi(x,\lambda)}{\partial\lambda} = \mathcal{H}\left(x, -i\hbar\frac{\partial}{\partial x},\lambda\right)\Psi(x,\lambda) \ , \tag{6.2.7}$$

where $\mathcal{H}(x,-i\hbar\partial/\partial x,\lambda)$ is the differential operator which is obtained from the superhamiltonian under the substitution

$$p_a \longrightarrow -i\hbar\frac{\partial}{\partial x^a} \ . \tag{6.2.8}$$

Eq. (6.2.7) and the correspondence (6.2.8) can be interpreted in the usual way in terms of the Hilbert space of states for the particle. If $\hat{p}_a$ is the momentum operator acting on such space, its coordinate representation is given by the relation

$$\langle x|\hat{p}_a|\Psi(\lambda)\rangle = -i\hbar\frac{\partial\Psi(x,\lambda)}{\partial x^a} \ ; \tag{6.2.9}$$

[1]The derivation of Eq. (6.2.7) from Eq. (6.2.6) is performed in the same way as in nonrelativistic quantum theory [105,95].

Eq. (6.2.7) is therefore the coordinate version of the abstract Schrödinger equation

$$i\hbar\frac{d|\Psi(\lambda)\rangle}{d\lambda} = \hat{\mathcal{H}}|\Psi(\lambda)\rangle \ , \tag{6.2.10}$$

where, as usual,

$$\hat{\mathcal{H}} \equiv \mathcal{H}(\hat{x},\hat{p},\lambda) \ . \tag{6.2.11}$$

Everything proceeds, therefore, analogously to what happens in nonrelativistic quantum mechanics.

A peculiar feature of this theory is that the particle mass appears in it only as an eigenvalue. In order to better understand this point, let us consider the case in which $\mathcal{H}$ has no explicit dependence on λ, and look for a "stationary" solution of Eq. (6.2.7):

$$\Psi(x,\lambda) = e^{-i\mathcal{E}\lambda/\hbar}\psi(x) \ , \tag{6.2.12}$$

with $\mathcal{E}$ real. Inserting Eq. (6.2.12) into Eq. (6.2.7), and *defining* a positive constant m by

$$\varepsilon\, m^2 \equiv 2\mathcal{E} \ , \tag{6.2.13}$$

where ε is *defined* as

$$\varepsilon \equiv \operatorname{sign}\mathcal{E} \ , \tag{6.2.14}$$

we get the "stationary", i.e., λ-independent, form of the four dimensional Schrödinger equation:

$$\frac{1}{\hbar^2}\left[-2\mathcal{H}\left(x,-i\hbar\frac{\partial}{\partial x}\right)+\varepsilon\, m^2\right]\psi(x) = 0 \ , \tag{6.2.15}$$

which can be identified immediately as a generalized Klein-Gordon equation for a particle with mass m and causal behaviour defined by ε. These features of the particle are thus, in this theory, not imposed *a priori*, but determined by the boundary conditions in the eigenvalue problem corresponding to the stationary version of Eq. (6.2.7). It is worth noticing, at this point, that even though our discussion is restricted, for sake of simplicity, to the case of a spinless particle, for which Ψ is a complex scalar, Eqs. (6.2.7) and (6.2.15) can be generalized to particles with spin, provided Ψ and ψ are considered to be multicomponents columns [110,111,112].

We find the fact that ε and m are not fixed, but take definite values only for some specific states, to be a very strong point in favour of the theory here considered. In fact, let us remember that, at the classical level, both ε and m are fixed as a consequence of the Hamilton equations (6.1.3)–(6.1.4), which imply, for a λ-independent superhamiltonian, $\varepsilon\, m^2$ =const. or, separately:

1. Constancy of the causal type of the motion (ε);

2. Constancy of the particle's mass (m).

The equations of motion (6.1.3)–(6.1.4) have not, however, to be imposed at the quantum level: This amounts, in the path integral formulation, to integrate even on those histories changing their causal character, and/or on which the mass of the particle is not constant. A state $|\Psi(\lambda)\rangle$ will not be, in general, an eigenvector of the operator $\hat{\mathcal{H}}$, corresponding to definite values of ε and m, but it will be a superposition of such eigenstates; in the case of a discrete mass spectrum,

$$|\Psi(\lambda)\rangle = \sum_{n,\varepsilon} c_{n\varepsilon} \exp\left(-\frac{i}{\hbar}\frac{\varepsilon m_n^2}{2}\lambda\right)|n,\varepsilon\rangle \; , \tag{6.2.16}$$

with $|n,\varepsilon\rangle$ normalized vectors satisfying the stationary equation

$$\hat{\mathcal{H}}\,|n,\varepsilon\rangle = \frac{1}{2}\varepsilon\, m_n^2\,|n,\varepsilon\rangle \; , \tag{6.2.17}$$

and $c_{n\varepsilon}$ complex coefficients normalized as

$$\sum_{n,\varepsilon} |c_{n\varepsilon}|^2 = 1 \; . \tag{6.2.18}$$

It is clear that ε and m play here a role analogous to the one played, in the nonrelativistic theory, by the particle's energy. An interesting consequence of such a parallel is the nonlocalizability of particles with a definite mass. In fact, in nonrelativistic quantum mechanics there are no common eigenstates of energy and position; this implies that particles with definite energy cannot be localized in space. Similarly, it is easy to see that in the relativistic theory treated here, it is not possible to conceive a common eigenstate of mass and spacetime position: Particles with definite mass cannot be localized in spacetime. This conclusion is in agreement with the well known analysis of the subject given by Newton and Wigner [113].

In nonrelativistic quantum mechanics, if the hamiltonian is not explicitly dependent on time, the time evolution preserves the energy eigenstates. Similarly, in the relativistic theory here discussed, if the superhamiltonian does not depend explicitly on λ, the λ-evolution carries eigenstates of mass and ε into eigenstates of mass and ε, corresponding to the same eigenvalues. It is interesting to prove this statement using the integral representation (6.2.6); let us first observe that, under the hypothesis considered,

$$K(x'',\lambda'';x',\lambda') \equiv \bar{K}(x'',x',\lambda) \; , \tag{6.2.19}$$

where $\lambda \equiv \lambda'' - \lambda'$. If $\Psi(x',\lambda';m,\varepsilon)$ represents an eigenstate of the superhamiltonian at the parameter value λ', then it will evolve, at $\lambda'' > \lambda'$, into

$$\int d^4x' \, K(x'',\lambda'';x',\lambda') \, \Psi(x',\lambda';m,\varepsilon) \; . \tag{6.2.20}$$

Applying $\mathcal{H}(x'',-i\hbar\partial/\partial x'')$ to Eq. (6.2.20), and noticing that the amplitude $K(x'',\lambda'';x',\lambda')$ satisfies the four dimensional Schrödinger equation in the variables (x'',λ''), we easily get, remembering Eq. (6.2.19),

$$-\,i\hbar \int d^4x' \, \frac{\partial}{\partial\lambda'} K(x'',\lambda'';x',\lambda') \, \Psi(x',\lambda';m,\varepsilon) \; . \tag{6.2.21}$$

A straightforward application of the Leibniz rule, together with the remark that Eq. (6.2.20) does not depend on λ', allows now to transform Eq. (6.2.21) into

$$\int d^4x' \, K(x'',\lambda'';x',\lambda') \, i\hbar \frac{\partial}{\partial\lambda'} \Psi(x',\lambda';m,\varepsilon) \; , \tag{6.2.22}$$

which is clearly equal to

$$\frac{1}{2}\varepsilon m^2 \int d^4x' \, K(x'',\lambda'';x',\lambda') \, \Psi(x',\lambda';m,\varepsilon) \; , \tag{6.2.23}$$

thus proving our assertion.

This feature of the theory is rather important, since it guarantees that, provided $\mathcal{H}$ is independent of λ, a particle with definite mass and causal behaviour will not change them during its evolution. Such a result is not trivial, because $K(x'', \lambda''; x', \lambda')$ contains contributions from all the paths in Γ_1, many of which do not correspond neither to a definite m nor to a definite ε (off mass shell contributions); nevertheless, it turns out that both m and ε are conserved in average. A detailed discussion of the nonrelativistic analogous result concerning energy is contained in ref. [105].

The result now obtained has the important consequence that a particle which is in an eigenstate of $\mathcal{H}$ (i.e., with well defined m and ε) can be fully described by the generalized Klein-Gordon equation (6.2.15). It is therefore interesting to ask if a propagator for Eq. (6.2.15) can be constructed from the amplitude $K(x'', \lambda''; x', \lambda')$. As we shall now show, the answer is affirmative, and will turn out to give just the Green function which is commonly used in the quantum theory of the scalar field .

Let us notice, first of all, that $K^+(x'', \lambda''; x', \lambda')$, defined as

$$K^+(x'', \lambda''; x', \lambda') \equiv \Theta(\lambda'' - \lambda') \, K(x'', \lambda''; x', \lambda') \, , \tag{6.2.24}$$

where Θ is the step function, satisfies the inhomogeneous equation [95]

$$\left[i\hbar \frac{\partial}{\partial \lambda''} - \mathcal{H}\left(x'', -i\hbar \frac{\partial}{\partial x''}, \lambda'' \right) \right] K^+(x'', \lambda''; x', \lambda') = i\hbar \delta(\lambda'' - \lambda') \, \delta^4(x'' - x'). \tag{6.2.25}$$

If $\mathcal{H}$ does not depend explicitly on λ, we can define a function $\bar{K}^+(x'', x', \lambda)$ analogously to what has been done in Eq. (6.2.19), and consider the new function

$$\Delta(x'', x'; m, \varepsilon) \equiv \frac{i\hbar}{2} \int_{-\infty}^{+\infty} d\lambda \, \exp\left(\frac{i}{\hbar} \frac{\varepsilon m^2}{2} \lambda \right) \, \bar{K}^+(x'', x', \lambda) \, , \tag{6.2.26}$$

where the λ-dependence has been integrated over, keeping definite values for m and ε. A simple calculation shows that $\Delta(x'', x'; m, \varepsilon)$ satisfies the equation

$$\frac{1}{\hbar^2} \left[-2\mathcal{H}\left(x'', -i\hbar \frac{\partial}{\partial x''} \right) + \varepsilon m^2 \right] \Delta(x'', x'; m, \varepsilon) = -\delta^4(x'' - x') \, , \tag{6.2.27}$$

i.e., that it is a propagator for Eq. (6.2.15). We have thus only to connect Eq. (6.2.26) with the usual expressions for such a propagator; this can be easily done for the free particle. The amplitude for this case is[2]

$$\bar{K}(x'', x', \lambda) = \langle x''| \exp(-i\hat{\mathcal{H}}\lambda/\hbar)|x'\rangle =$$
$$= \frac{1}{(2\pi\hbar)^4} \int d^4p \, \exp \frac{i}{\hbar} \left[p \cdot (x'' - x') - \frac{1}{2} p^2 \lambda \right] , \qquad (6.2.28)$$

where the completeness relation for momentum has been used twice. By definition (6.2.26), the propagator for the free Klein-Gordon Eq. (6.2.15) reads thus

$$\Delta(x'', x'; m, \varepsilon) = -\frac{1}{(2\pi\hbar)^4} \int d^4p \, e^{ip\cdot(x''-x')/\hbar} \frac{\hbar^2}{-p^2 + \varepsilon m^2 + i0} , \qquad (6.2.29)$$

where we have made use of the formal equality [114]

$$\int_0^{+\infty} d\lambda \, e^{i\xi\lambda} = \frac{i}{\xi + i0} , \qquad (6.2.30)$$

with ξ real and

$$\frac{1}{\xi \pm i0} = P\left(\frac{1}{\xi}\right) \mp i\pi\delta(\xi) , \qquad (6.2.31)$$

P denoting, as usual, the principal part. It is immediate to recognize, in the right hand side of Eq. (6.2.29), the Feynman propagator extended also to the case of spacelike motions as well. A derivation, within our formalism, of the explicit form of $\Delta(x'', x'; m, \varepsilon)$ for a free particle is given in the Appendix.

We find necessary, at this point, to comment on the different physical meanings of $K(x'', \lambda''; x', \lambda')$ and $\Delta(x'', x'; m, \varepsilon)$; while the first of them is a transition amplitude, in the proper meaning, between the states $|x', \lambda'\rangle$ and $|x'', \lambda''\rangle$, as it is evident from the first equality in Eq. (6.2.28), this is not true for the second one. There are, in fact, no states $|x', m, \varepsilon\rangle$ and $|x'', m, \varepsilon\rangle$ whose scalar product could give origin to $\Delta(x'', x'; m, \varepsilon)$: The reason, as we have already noticed, is that no common eigenstate exists for the spacetime position and the superhamiltonian. It would be therefore incorrect to think of $\Delta(x'', x'; m, \varepsilon)$, or of any similarly

[2]Hereafter we shall use the short notations $p_a(x''-x')^a \equiv p\cdot(x''-x')$ and $\eta^{ab}p_ap_b \equiv p^2$.

constructed function, as an amplitude for a particle of mass m and causal type ε to evolve between the spacetime events x' and x''.

The analogy between m and ε in the relativistic theory, and the energy E in nonrelativistic quantum mechanics, is particularly explicit about this fact. The nonrelativistic Schrödinger equation (3.1.1) reduces, for a time-independent hamiltonian and a wave function

$$\psi(\mathbf{x}, t) = e^{-iEt/\hbar}\,\phi(\mathbf{x}) \,, \tag{6.2.32}$$

to the stationary form

$$\frac{1}{\hbar^2}\left[-H(\mathbf{x}, -i\hbar\boldsymbol{\nabla}) + E\right]\phi(\mathbf{x}) = 0 \,. \tag{6.2.33}$$

Defining, from the amplitude [95] $K(\mathbf{x}'', t''; \mathbf{x}', t')$, a function $\bar{K}^{+}(\mathbf{x}'', \mathbf{x}', t)$ as in the relativistic case, with $t \equiv t'' - t'$, it is easy to check that

$$\Delta(\mathbf{x}'', \mathbf{x}'; E) \equiv i\hbar \int_{-\infty}^{+\infty} dt\, e^{iEt/\hbar}\,\bar{K}^{+}(\mathbf{x}'', \mathbf{x}', t) \tag{6.2.34}$$

is a propagator for Eq. (6.2.33), since

$$\frac{1}{\hbar^2}\left[-H(\mathbf{x}, -i\hbar\boldsymbol{\nabla}) + E\right]\Delta(\mathbf{x}'', \mathbf{x}'; E) = -\delta^3(\mathbf{x}'' - \mathbf{x}') \,. \tag{6.2.35}$$

It is clear that, while

$$K(\mathbf{x}'', t''; \mathbf{x}', t') \equiv \langle \mathbf{x}'', t'' | \mathbf{x}', t' \rangle \tag{6.2.36}$$

is, by definition, a transition amplitude, the same cannot be said with regard to $\Delta(\mathbf{x}'', \mathbf{x}'; E)$: Position and energy are incompatible observables, and this implies that $\Delta(\mathbf{x}'', \mathbf{x}'; E)$ has only an essentially mathematical meaning, as well as $\Delta(x'', x'; m, \varepsilon)$.

6.3 Covariant Quasiprobabilities

Having established a covariant version of relativistic quantum theory, we are now in the position to define a quasiprobability (density) for a relativistic history γ, simply following the same arguments which led to Eq. (5.2.8) in the nonrelativistic case; the expression reads

$$P[\gamma] = \Re\left[\Psi(x(\lambda''), \lambda'')^{*}\,K[\gamma]\,\Psi(x(\lambda'), \lambda')\right] \,, \tag{6.3.1}$$

where $K[\gamma]$ is given by Eq. (5.1.2), with $S[\gamma]$ the four dimensional action (6.1.2).

The functional $P[\gamma]$ enjoys the generalizations of all the properties which are satisfied by its nonrelativistic counterpart. It is easy to show, for example, that

$$\int_{\Gamma''} \mathcal{D}\gamma\, P[\gamma] = P(x''|\lambda'') , \tag{6.3.2}$$

where Eq. (6.2.6) has been used. The normalization (6.2.5) guarantees then that

$$\int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma] = 1 . \tag{6.3.3}$$

The proof that the integral of $P[\gamma]$, over all the histories whose four-momentum at the parameter value λ is p, turns out to be equal to

$$P(p|\lambda) = |\Phi(p,\lambda)|^2 , \tag{6.3.4}$$

where

$$\Phi(p,\lambda) = \int d^4x\, \frac{e^{-ip\cdot x/\hbar}}{(2\pi\hbar)^2}\, \Psi(x,\lambda) \tag{6.3.5}$$

is the wave function in the momentum space, is also trivially performed, *mutatis mutandis*, along the same lines of Sec. 5.3. The only significant change is in the expression for the "broken" path integral of $K(x'',\lambda'';x',\lambda')$:

$$K(x'',\lambda'';x',\lambda') = \lim_{\Lambda\to 0} \frac{1}{\mathcal{N}} \int d^4x_1 \ldots d^4x_N$$
$$d^4p_0 \ldots d^4p_N \exp \frac{i}{\hbar} \sum_{k=0}^{N} (p_k \cdot \Delta x_k - \mathcal{H}_k \Delta\lambda_k) . \tag{6.3.6}$$

In Eq. (6.3.6), the interval $[\lambda',\lambda'']$ has been partitioned into $N+1$ subintervals $[\lambda_k,\lambda_{k+1}]$, with $k \in \{0,1,\ldots,N\}$ and $\lambda' \equiv \lambda_0 < \cdots < \lambda_{N+1} \equiv \lambda''$; the length of the $(k+1)$-th of such intervals is denoted as $\Delta\lambda_k \equiv \lambda_{k+1} - \lambda_k$, and $\Lambda \equiv \max\{\Delta\lambda_k\}$; moreover, we have used the notations $\Delta x_k^a \equiv x_{k+1}^a - x_k^a$, $\mathcal{N} \equiv (2\pi\hbar)^{4(N+1)}$ and, in the limit $\Lambda \to 0$,

$$\mathcal{H}_k \equiv \frac{1}{2}\frac{\langle x_{k+1}|\hat{\mathcal{H}}|p_k\rangle}{\langle x_{k+1}|p_k\rangle} + \frac{1}{2}\frac{\langle p_k|\hat{\mathcal{H}}|x_k\rangle}{\langle p_k|x_k\rangle} . \tag{6.3.7}$$

Let us come to an important issue in the formalism here developed: The derivation of phase space distribution functions from the quasiprobability functional $P[\gamma]$. The argument proceeds in the same way as in the nonrelativistic treatment of Sec. 5.4: Since by phase space distribution it is meant a function of x, p and λ which is supposed to represent the (quasi-)probability density for the particle to have spacetime position x and four-momentum p when the parameter value is λ, we should be able to obtain it as an integral (5.1.4), with Γ the set of all the trajectories such that $x(\lambda) = x$ and $p(\lambda) = p$. However, the experience obtained in dealing with the nonrelativistic case, suggests that the result will be dependent on the particular prescription adopted in the explicit calculation of the path integral through the skeletonization of the history and the subsequent limiting procedure; moreover, this arbitrariness will be related to the need to use, in the Hilbert space formulation of the theory, a prescription for the ordering of noncommuting operators. Skipping all the intermediate steps, which do not differ significantly from the calculations contained in Sec. 5.4, we simply claim that, for the prescription related to the Weyl ordering (5.4.1), the phase space distribution function turns out to be

$$P_W(x,p|\lambda) = \frac{1}{(\pi\hbar)^4}\int d^4\xi\, \Psi(x+\xi,\lambda)^* e^{2ip\cdot\xi/\hbar}\Psi(x-\xi,\lambda)\ , \tag{6.3.8}$$

which is the relativistic generalization of the Wigner function [115,116]. On the other hand, considering the prescription related to the Rivier symmetrization rule (5.4.5) one gets

$$P_S(x,p|\lambda) = \frac{1}{(2\pi\hbar)^4}\Re\int d^4\xi\, \Psi(x+\xi,\lambda)^* e^{ip\cdot\xi/\hbar}\Psi(x,\lambda)\ , \tag{6.3.9}$$

which generalizes the Margenau-Hill function [88] to the relativistic domain.

We find important to remark that our results are consistent with those of refs. [115,116], though these last are stated for the λ-independent case. The arbitrariness in the final expression for $P(x,p|\lambda)$ is not a weak point of our derivation, but it rather seems to be an intrinsic feature of quantum theory, which can be probably removed only by the introduction of supplementary requirements (see the discussion about (3.3.14) in Ch. 3). In the formalism of ref. [115], the relativistic phase space distribution is found to correspond to a straightforward generalization of the Wigner function because of the choice, as associated vector, of

$$F^a_W(x,p;\lambda) = \frac{\hbar\eta^{ab}}{2(\pi\hbar)^4 i}\int d^4\xi\, e^{2ip\cdot\xi/\hbar}\,\Psi(x+\xi,\lambda)^*\frac{\overleftrightarrow{\partial}}{\partial x^b}\Psi(x-\xi,\lambda)\ , \quad (6.3.10)$$

where we have adapted the notations to our standards; however, also the vector

$$F^a_S(x,p;\lambda) = -\frac{\hbar\eta^{ab}}{(2\pi\hbar)^4}\Im\int d^4\xi\, e^{ip\cdot\xi/\hbar}\,\frac{\partial\Psi(x+\xi,\lambda)^*}{\partial x^b}\Psi(x,\lambda)\ , \quad (6.3.11)$$

where $\Im$ denotes the imaginary part, is real and satisfies the condition

$$\int d^4p\, F^a_S(x,p;\lambda) = j^a(x,\lambda)\ , \quad (6.3.12)$$

where $j^a(x,\lambda)$ is given by

$$j^a(x,\lambda) \equiv \frac{\hbar\eta^{ab}}{2i}\Psi(x,\lambda)^*\frac{\overleftrightarrow{\partial}}{\partial x^b}\Psi(x,\lambda)\ : \quad (6.3.13)$$

$F^a_S(x,p;\lambda)$ is therefore an equally good candidate as vector associated to the phase space distribution. A simple integration by parts gives in fact

$$F^a_S(x,p;\lambda) = p^a\, P_S(x,p|\lambda)\ , \quad (6.3.14)$$

thus proving the existence of the arbitrariness even in this other formalism.

The quasiprobability functional $P[\gamma]$ allows to write an evocative representation of the current $j^a(x,\lambda)$. Let us define, in fact,

$$j^a(x,\lambda|\gamma) \equiv \delta^4(x-x(\lambda))\,p^a(\lambda)\ , \quad (6.3.15)$$

where $\lambda\mapsto x(\lambda)$ and $\lambda\mapsto p(\lambda)$ define, as usual, the path γ; Eq. (6.3.15) is justified by the relation [32]

$$j^a(x|\gamma) = \int_{-\infty}^{+\infty} d\lambda\, j^a(x,\lambda;\gamma)\ , \quad (6.3.16)$$

which allows to determine the current $j^a(x;\gamma)$ due to a classical particle with history γ: $j^a(x,\lambda;\gamma)$ represents thus a kind of current in the context of the extended spacetime. We shall now show that the average of $j^a(x,\lambda;\gamma)$ over all the possible paths in Γ_0 is just $j^a(x,\lambda)$ defined by Eq. (6.3.13), i.e., that

$$j^a(x,\lambda) = \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, j^a(x,\lambda|\gamma)\,. \tag{6.3.17}$$

In the proof of Eq. (6.3.17) it is necessary to skeletonize the paths, in order to compute explicitly the integral: A prescription about the procedure to follow in this operation is therefore necessary. We shall perform the calculations choosing $\lambda = (\lambda_M + \lambda_{M+1})/2$, $p(\lambda) = p_M$ and $x(\lambda) = (x_M + x_{M+1})/2$, which corresponds to the Weyl ordering (5.4.1) or, equivalently, to the Wigner function (6.3.8); it is easy to check that the use of the prescription associated to the symmetrization rule (5.4.5) (or to the Margenau-Hill function (6.3.9)) would equally lead to the result (6.3.17). Let us thus write the right hand side of Eq. (6.3.17), according to Eqs. (6.3.1) and (6.3.6), as

$$\lim_{\Lambda\to 0} \frac{1}{\mathcal{N}} \Re \int d^4x_0 \ldots d^4x_{N+1} d^4p_0 \ldots d^4p_N \Psi(x_{N+1},\lambda_{N+1})^* \\ \exp\frac{i}{\hbar}\sum_{k=0}^{N}(p_k\cdot\Delta x_k - \mathcal{H}_k\Delta\lambda_k)\,\Psi(x_0,\lambda_0)\,p_M^a\,\delta^4\!\left(x - \frac{x_M+x_{M+1}}{2}\right) \tag{6.3.18}$$

Performing $2N$ integrations and remembering Eq. (6.2.6), the expression (6.3.18) becomes

$$\frac{1}{(2\pi\hbar)^4}\Re\int d^4x_M d^4x_{M+1} d^4p_M\, \Psi(x_{M+1},\lambda)^* \\ e^{ip_M\cdot(x_{M+1}-x_M)/\hbar}\Psi(x_M,\lambda)\,p_M^a\,\delta^4\left(x - \frac{x_M + x_{M+1}}{2}\right)\,, \tag{6.3.19}$$

which, with the change of variables $\xi \equiv x - x_M$, reads

$$\frac{2\hbar}{(2\pi\hbar)^4}\Re\int d^4\xi d^4p_M\,\frac{\eta^{ab}}{2i}\frac{\partial e^{2ip_M\cdot\xi/\hbar}}{\partial\xi^b}\Psi(x+\xi,\lambda)^*\Psi(x-\xi,\lambda)\,. \tag{6.3.20}$$

It is now easy, performing the integration in p_M and remembering few properties of the delta function, to verify that (6.3.20) reduces actually to the right hand side of Eq. (6.3.13), thus establishing the equality in Eq. (6.3.17).

Eq. (6.3.17) acquires a particular interest if we notice that, defining

$$P(x,\lambda|\gamma) \equiv \delta^4(x - x(\lambda)) , \tag{6.3.21}$$

so that

$$j^a(x,\lambda|\gamma) = P(x,\lambda|\gamma)\, p^a(\lambda) , \tag{6.3.22}$$

it follows easily that

$$P(x|\lambda) = \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, P(x,\lambda|\gamma) , \tag{6.3.23}$$

where $P(x|\lambda)$ is given by Eq. (6.2.4). The relations (6.3.21) and (6.3.22) strongly suggest to look for a continuity equation in the extended spacetime, of the kind of

$$\frac{\partial P(x|\lambda)}{\partial \lambda} + \frac{\partial j^a(x,\lambda)}{\partial x^a} = 0 ; \tag{6.3.24}$$

it is easy to check, with the help of Eqs. (6.2.7) and (6.1.25), that Eq. (6.3.24) is really correct, at least in the free particle case[3]. Let us notice that in the stationary case, $j^a(x,\lambda)$ does not depend on λ, and turns out to be equal (apart a factor m) to the Klein-Gordon current

$$j^a(x) \equiv \frac{\hbar \eta^{ab}}{2im} \psi(x)^* \frac{\overleftrightarrow{\partial}}{\partial x^b} \psi(x) , \tag{6.3.25}$$

which obeys a continuity equation in *spacetime*. However, it is clear that, in the context of our theory, it is the Eq. (6.3.24), and not

$$\frac{\partial j^a(x)}{\partial x^a} = 0 , \tag{6.3.26}$$

which expresses the "conservation" of probability; the similarity between Eq. (6.3.26) and the continuity equation of the nonrelativistic quantum theory,

$$\frac{\partial P(\mathrm{x}|t)}{\partial t} + \nabla \cdot \mathrm{j}(\mathrm{x},t) = 0 , \tag{6.3.27}$$

where

$$P(\mathrm{x}|t) = |\psi(\mathrm{x},t)|^2 \tag{6.3.28}$$

[3]For an analogous result in a theory similar to the one presented here, see ref. [117].

and

$$\mathbf{j}(\mathbf{x},t) = \frac{\hbar}{2im}\psi(\mathbf{x},t)^* \overleftrightarrow{\nabla}\, \psi(\mathbf{x},t) \ , \tag{6.3.29}$$

is clearly only apparent: The real analogy has to be drawn between Eqs. (6.3.27) and (6.3.24), while the analogous of Eq. (6.3.26) is the stationary (i.e., t-independent) version of Eq. (6.3.27), i.e.,

$$\nabla \cdot \mathbf{j}(\mathbf{x}) = 0 \ , \tag{6.3.30}$$

with

$$\mathbf{j}(\mathbf{x}) = \frac{\hbar}{2im}\phi(\mathbf{x})^* \overleftrightarrow{\nabla}\, \phi(\mathbf{x}) \ , \tag{6.3.31}$$

and $\phi(\mathbf{x})$ satisfying Eq. (6.2.33).

These considerations, and particularly Eqs. (6.3.17) and (6.3.23), give further support to the main idea underlying the quasiprobability functional technique: *Quantum theory can be represented as a classical statistical theory, provided the space of histories is equipped with the quasiprobability density* $P[\gamma]$.

6.4 Reduction to the Spacetime Level

In the formalism developed so far, the parameter λ plays the role of an "evolutionary time" for the particle, but it is not related in any way to the measurements of time performed by an observer; these latter are rather associated to the coordinate time t which is, as we have already stressed, an observable of the theory. It is clear from the treatment of Sec. 6.1 that, at a classical level, for a free particle of finite mass m and causal behaviour ε, the Hamilton equations imply that the interval $\Delta\lambda$ corresponding to a spacetime displacement Δx^a is, in the limit of "infinitesimal" Δx^a,

$$\Delta\lambda = \frac{1}{m}\sqrt{\varepsilon\, \eta_{ab}\Delta x^a \Delta x^b} \ . \tag{6.4.1}$$

This allows to interpret λ as $1/m$ times the proper length of the particle's trajectory in spacetime, thus establishing a relationship between λ and measurements in spacetime. When passing to the quantum level, however, we have to abandon all the results of the classical theory relying on the Hamilton variational principle, which is not required to hold: Eq. (6.4.1), in particular, will not be correct any more, and will apply only in correspondence of the classical path. As a consequence, we must conclude that an observer making measurements in spacetime has no way to determine the value of λ, or of intervals $\Delta\lambda$: As far as he/she is concerned, λ is a truly unobservable parameter. It is therefore obvious that we must get rid of the λ-dependence in the physical results of the theory, in order to be able to compare them with measurements performed in spacetime.

The probability $P(x|\lambda)$, as said in the comment to Eq. (6.2.4), makes reference to the condition that the parameter have the value λ: As such, it is not directly related to physical measurements, because λ cannot be known by operating in spacetime. A physically meaningful quantity would rather be $P(x)$, i.e., the probability that the particle be at the event x; this can be constructed by "projecting" the extended spacetime $\mathbb{R}^4 \times [\lambda', \lambda'']$ onto the spacetime $\mathbb{R}^4$, and adopting the usual law of probability theory

$$P(x) = \int_{\lambda'}^{\lambda''} d\lambda \, P(x|\lambda) P(\lambda) \ , \tag{6.4.2}$$

where a probability density $P(\lambda)$ has been assigned to each value $\lambda \in [\lambda', \lambda'']$ of the parameter.

The problems are now, quite obviously, which values to assign to the extremes λ' and λ'', and how to determine the function $P(\lambda)$. The most natural choice, if we accept that at a spacetime level there is total ignorance about λ, is

$$P(\lambda) = \frac{1}{\lambda'' - \lambda'} \ , \tag{6.4.3}$$

for $\lambda' \to -\infty$, $\lambda'' \to +\infty$; this assumption corresponds to an equiprobable distribution in λ.

We can now derive the probability distribution $P(x)$ for the general state (6.2.16); in order to avoid technical troubles, let us suppose the mass spectrum to be discrete[4], so that

$$\Psi(x, \lambda) = \sum_{n,\varepsilon} c_{n\varepsilon} \exp\left(-\frac{i}{\hbar}\frac{\varepsilon m_n^2}{2}\lambda\right) \psi_{n\varepsilon}(x) \ , \tag{6.4.4}$$

[4]With some care in the normalization procedures the same treatment, with analogous results, holds also in the case of a continuous spectrum.

where n labels the mass eigenvalues, and

$$\psi_{n\varepsilon}(x) \equiv \langle x|n,\varepsilon\rangle \tag{6.4.5}$$

are eigenstates normalized according to

$$\int d^4x\, \psi_{n\varepsilon}(x)^* \psi_{n'\varepsilon'}(x) = \delta_{nn'}\delta_{\varepsilon\varepsilon'} \,. \tag{6.4.6}$$

The conditional probability $P(x|\lambda)$ is, according to Eq. (6.2.4),

$$P(x|\lambda) = \sum_{n,n',\varepsilon,\varepsilon'} c^*_{n\varepsilon}\, c_{n'\varepsilon'}\, e^{\frac{i}{2\hbar}(\varepsilon m_n^2 - \varepsilon' m_{n'}^2)\lambda}\, \psi_{n\varepsilon}(x)^* \psi_{n'\varepsilon'}(x) \,; \tag{6.4.7}$$

inserting Eq. (6.4.7) into Eq. (6.4.2), with the prescription (6.4.3) and the limits $\lambda' \to -\infty$, $\lambda'' \to +\infty$, one finds

$$P(x) = \sum_{n,\varepsilon} |c_{n\varepsilon}|^2 \cdot |\psi_{n\varepsilon}(x)|^2 \,, \tag{6.4.8}$$

where the identity

$$\lim_{\substack{\lambda' \to -\infty \\ \lambda'' \to +\infty}} \frac{1}{\lambda'' - \lambda'} \int_{\lambda'}^{\lambda''} d\lambda\, e^{\frac{i}{2\hbar}(\varepsilon m_n^2 - \varepsilon' m_{n'}^2)\lambda} = \delta_{\varepsilon\varepsilon'}\delta_{nn'} \tag{6.4.9}$$

has been used. Eq. (6.4.8) is rather important, since it expresses the probability of presence for the particle *in spacetime*, and it worth some comments.

The main difference between $P(x)$ and $P(x|\lambda)$, as it is clear by a comparison of Eqs. (6.4.8) and (6.4.7), is that in the latter, interference between different mass values is present, while $P(x)$ is only a sum of incoherent terms, each one corresponding to a single value of m and ε, to which a probability $|c_{n\varepsilon}|^2$ is assigned. The meaning of this is that interference between different masses and causal behaviours is unobservable as far as spacetime observations are concerned.

We can give a more general proof of this statement by considering, in the Schrödinger picture, the density operator associated to the state (6.2.16):

$$\hat{\rho}(\lambda) = |\Psi(\lambda)\rangle\langle\Psi(\lambda)| = \sum_{n,n',\varepsilon,\varepsilon'} c^*_{n\varepsilon}\, c_{n'\varepsilon'}\, e^{\frac{i}{2\hbar}(\varepsilon m_n^2 - \varepsilon' m_{n'}^2)\lambda}\, |n',\varepsilon'\rangle\langle n,\varepsilon| \,. \tag{6.4.10}$$

If $\hat{\Omega}$ is a self-adjoint operator representing an observable, the probability that this takes the value ω when the parameter is λ is

$$P(\omega|\lambda) = \mathrm{tr}\,(\hat{\rho}(\lambda)|\omega\rangle\langle\omega|)\;, \tag{6.4.11}$$

where $|\omega\rangle$ is the eigenvector of $\hat{\Omega}$ with eigenvalue ω, and possible degeneracy has been neglected for sake of simplicity. For an observer which has no knowledge about λ, however, the interesting quantity is not $P(\omega|\lambda)$, but rather

$$P(\omega) \equiv \lim_{\substack{\lambda'\to-\infty \\ \lambda''\to+\infty}} \int_{\lambda'}^{\lambda''} d\lambda\, P(\omega|\lambda)\, P(\lambda)\;, \tag{6.4.12}$$

with $P(\lambda)$ given by Eq. (6.4.3). The linearity of the trace operation allows to write

$$P(\omega) = \mathrm{tr}\,(\hat{\rho}\,|\omega\rangle\langle\omega|)\;, \tag{6.4.13}$$

where now

$$\hat{\rho} \equiv \lim_{\substack{\lambda'\to-\infty \\ \lambda''\to+\infty}} \int_{\lambda'}^{\lambda''} d\lambda\, \hat{\rho}(\lambda)\, P(\lambda) = \sum_{n,\varepsilon} |c_{n\varepsilon}|^2 \cdot |n,\varepsilon\rangle\langle n,\varepsilon|\;. \tag{6.4.14}$$

Eqs. (6.4.13) and (6.4.14) can be interpreted by saying that, with respect to measurements performed in spacetime, the pure state (6.4.10) and the mixture (6.4.14) are undistinguishable: This is a more precise statement of the conclusion reached in the previous paragraph.

The form of the Eq. (6.4.8) seems to present a problem related to the conservation of probability. Even in the simple case of a single component (λ-stationary case), $P(x) = |\psi_{n\varepsilon}(x)|^2$ is not associated to a continuity equation, because $\psi_{n\varepsilon}(x)$ is a solution of the generalized Klein-Gordon equation (6.2.15). This may look as a serious flaw in the theory, because it brings as consequence the nonconservation of the integral of $P(x)$ over a spacelike hypersurface. An intuitive way to understand this point is to go back to the path integral formulation; what in the extended spacetime is a single path with a unique x for each value of λ appears, when projected on spacetime, as a curly curve whose points are not necessarily in a well defined causal relationship: Under a space+time splitting, it would thus appear as if there were both particles and antiparticles in space [118]. It is thus clear that, being $|\psi_{n\varepsilon}(x)|^2$ the probability density to find either a particle or an antiparticle with mass m_n and causal behaviour ε at the spacetime point x, such nonconservation allows for virtual processes to occur! In the light of the previous discussion, it appears particularly clear that the non positive definiteness of j^0 has nothing mysterious in it: It is simply a consequence of the property of the theory to describe particles and antiparticles at the same time, and does not cause any trouble, since it is not j^0, but rather $|\psi|^2$ which has to be interpreted as a probability density.

The process of reduction to the spacetime level now discussed can be regarded as the quantum counterpart of the reduction of dimensions performed, classically, through the Eqs. (6.1.12)–(6.1.18). It is both remarkable and instructive that a similar treatment can be performed also in the nonrelativistic quantum theory, where it corresponds to a reduction to the *space* level, and turns out to represent the quantum version of Eqs. (6.1.5)–(6.1.11). Writing in fact $P(\mathbf{x}|t)$ as in Eq. (6.3.28), with $\psi(\mathbf{x},t)$ obeying the usual Schrödinger equation (3.1.1) with a time independent hamiltonian, and $t \in [t', t'']$, one can write, using the analogous of Eqs. (6.4.2), (6.4.3) and (6.4.9),

$$P(\mathbf{x}) = \sum_n |c_n|^2 \cdot |\phi_n(\mathbf{x})|^2 \, . \tag{6.4.15}$$

In the derivation of Eq. (6.4.15), the general state $\psi(\mathbf{x},t)$ has been expanded into energy eigenstates (supposed to belong to a discrete spectrum) as

$$\psi(\mathbf{x},t) = \sum_n c_n e^{-iE_n t/\hbar} \phi_n(\mathbf{x}) \, , \tag{6.4.16}$$

where $\phi_n(\mathbf{x})$ are orthonormal functions satisfying the stationary equation

$$H(\mathbf{x}, -i\hbar\nabla)\, \phi_n(\mathbf{x}) = E_n\, \phi_n(\mathbf{x}) \, . \tag{6.4.17}$$

The physical meaning of Eq. (6.4.15) is obvious: $P(\mathbf{x})$ is the probability density that the particle be in $\mathbf{x}$ at some time; such quantity is relevant if we do not (or if we cannot) fix the time at which to ask for the particle's position. This has the remarkable consequence that simple measurements of position, without the knowledge of the time at which they have been performed, do not allow to detect interference between different energy eigenstates. More explicitly, let us suppose that an ensemble of particles has been prepared, at time $t' \to -\infty$, in the superposition (6.4.16); if the position of each member of the ensemble is measured, and the time at which this measurement is performed is controlled and recorded, one obtains a statistical pattern which should approximate the probability distribution $P(\mathbf{x}|t)$, thus exhibiting interference between different values of n. However, if the times of the measurements are randomly distributed, and the only available data are the results of the position measurements, it is clear that the statistical pattern can only be compared with $P(\mathbf{x})$ given by Eq. (6.4.15); but this is the same probability distribution which could have been inferred had the ensemble been prepared in the incoherent mixture described by the density matrix

$$\rho(\mathbf{x},\mathbf{x}') = \sum_n |c_n|^2 \phi_n(\mathbf{x})\phi_n(\mathbf{x}')^* \; : \qquad (6.4.18)$$

Since Eq. (6.4.18) does not show any interference between different values of the energy, we are led to conclude that the same must happen in the experimental procedure described above. A more intuitive argument explaining why interference cannot be observed if the measurements' times are not known is the following: In a superposition of energy eigenstates, the detection of interference is possible by observing oscillations in the populations of different states, with frequencies $|E_n - E_{n'}|/\hbar$; if the times at which the measurements are performed are not recorded, such oscillations cannot be seen, and the pure state (6.4.16) is, under this respect, indistinguishable from the mixture (6.4.18).

The nonrelativistic analogy now presented, though exhibiting many similarities with the relativistic case, contains nevertheless an important difference. In the relativistic theory, the particle's parameter λ is completely unobservable at the spacetime level, and has therefore to be integrated over in the construction of physically relevant quantities such as $P(x)$; the nonrelativistic analogue of λ is, however, the time t (or, more precisely, t/m), whose status in the theory is rather unclear. In a purely formal context, t is also an unobservable parameter, because although it labels the evolution of states and observables, no self-adjoint "time operator" exists which acts in the Hilbert space $L^2(\mathbb{R}^3)$; however, the theory makes the further assumption that t is not only a "particle's time", but that it can be identified with the external "coordinate time" x^0, this latter being accessible to the experimenter: This is the reason why quantities like $P(\mathbf{x}|t)$ are relevant from a physical point of view, rather than $P(\mathbf{x})$.

This puzzling situation can be clarified providing the connection between the relativistic and the nonrelativistic theories, i.e., investigating the nonrelativistic limit of the covariant quantum theory; let us therefore perform this analysis for the general state (6.4.4) in the simple case of the free particle. As we have already discussed in detail, the pure state (6.4.4), which is suitable for a treatment in the extended spacetime $\mathbb{R}^4 \times [\lambda', \lambda'']$, corresponds to the mixture (6.4.14) when the description is restricted to the spacetime $\mathbb{R}^4$. It is clear from Eq. (6.4.14) that at the spacetime level there is no superposition of different mass states: Since the nonrelativistic theory is essentially a space+time treatment, we can work out the case of a single mass component, which is general enough; moreover, the nonrelativistic limit can be performed, by definition, only in the case $\varepsilon = -1$ and $m \neq 0$.

If, in the explicit calculation, we consider the case of a free particle, the solution of Eq. (6.2.15) to start with is therefore

$$\psi(x;m) = \frac{1}{(2\pi\hbar)^2} \int d^4p\, e^{ip\cdot x/\hbar} \delta(p^2+m^2)\, \phi(p) \;, \tag{6.4.19}$$

where $\phi(p)$ is an arbitrary function satisfying the relation

$$\int d^4p\, \delta(p^2+m^2)\, |\phi(p)|^2 = 2m \;, \tag{6.4.20}$$

which guarantees the normalization[5]

$$\int d^4x\, \psi(x;m)^* \psi(x;m') = \delta(m-m') \;. \tag{6.4.21}$$

Applying the usual properties of the delta function and carrying on the integration in p_0, Eq. (6.4.19) can be rewritten as

$$\psi(\mathbf{x},t;m) = \frac{1}{(2\pi\hbar)^2} \int d^3p\, e^{\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{x} - \sqrt{\mathbf{p}^2+m^2}\,t\right)} \phi_P(\mathbf{p};m) +$$

[5]The state we are now dealing with, has no representation in $L^2(\mathbb{R}^4)$: This purely formal inconvenient could be solved by a "normalization in a box" procedure [95]

$$+\frac{1}{(2\pi\hbar)^2}\int d^3p\, e^{\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{x}+\sqrt{\mathbf{p}^2+m^2}t\right)}\,\phi_A(\mathbf{p};m)\;, \qquad (6.4.22)$$

where

$$\phi_P(\mathbf{p};m) \equiv \frac{1}{2\sqrt{\mathbf{p}^2+m^2}}\,\phi\left(\sqrt{\mathbf{p}^2+m^2},\mathbf{p}\right)\;, \qquad (6.4.23)$$

$$\phi_A(\mathbf{p};m) \equiv \frac{1}{2\sqrt{\mathbf{p}^2+m^2}}\,\phi\left(-\sqrt{\mathbf{p}^2+m^2},\mathbf{p}\right)\;; \qquad (6.4.24)$$

Eq. (6.4.22) describes a superposition of a particle state and an antiparticle one. The functions ϕ_P and ϕ_A must satisfy the relation

$$\int d^3p\,\left(|\phi_P(\mathbf{p};m)|^2+|\phi_A(\mathbf{p};m)|^2\right)\frac{\sqrt{\mathbf{p}^2+m^2}}{m}=1\;, \qquad (6.4.25)$$

which guarantees the validity of Eq. (6.4.21). In order to be able to define a probability density for the position in *space* at a given time, let us evaluate the integral of $|\psi(\mathbf{x},t;m)|^2$ over a t =const. hypersurface; a simple calculation gives

$$\begin{aligned}\int d^3x\,|\psi(\mathbf{x},t;m)|^2 &= \frac{1}{2\pi\hbar}\int d^3p\,\left(|\phi_P(\mathbf{p};m)|^2+|\phi_A(\mathbf{p};m)|^2\right)+\\ &+\frac{1}{2\pi\hbar}\int d^3p\,e^{-2\frac{i}{\hbar}\sqrt{\mathbf{p}^2+m^2}t}\,\phi_P(\mathbf{p};m)\,\phi_A(\mathbf{p};m)^*+\\ &+\frac{1}{2\pi\hbar}\int d^3p\,e^{2\frac{i}{\hbar}\sqrt{\mathbf{p}^2+m^2}t}\,\phi_P(\mathbf{p};m)^*\,\phi_A(\mathbf{p};m)\;. \qquad (6.4.26)\end{aligned}$$

The quantity expressed by Eq. (6.4.26) depends on time, showing interference between particle and antiparticle states; however, if we restrict the treatment to particles only, thus requiring $\phi_A(\mathbf{p};m)=0$, Eq. (6.4.26) reduces to

$$\int d^3x\,|\psi(\mathbf{x},t;m)|^2 = \frac{1}{2\pi\hbar}\int d^3p\,|\phi_P(\mathbf{p};m)|^2\;, \qquad (6.4.27)$$

which is indeed a quantity independent of time. Eq. (6.4.27) cannot, however, be regarded as a satisfactory normalization if $|\psi(\mathbf{x},t;m)|^2$ has to be given the meaning of probability density for the position in space at the time t: In fact, its right hand side will depend, in general, on the particular shape of the function ϕ_P, and a renormalization to one is made impossible by Eq. (6.4.25), which becomes now

$$\int d^3p\,|\phi_P(\mathbf{p};m)|^2\frac{\sqrt{\mathbf{p}^2+m^2}}{m}=1\;. \qquad (6.4.28)$$

It is remarkable that these problems are automatically solved in the nonrelativistic limit of the theory. Suppose that $\phi_P(\mathbf{p};m)$ is different from zero only for $|\mathbf{p}| \ll m$; it is then possible to approximate $\sqrt{\mathbf{p}^2+m^2}$ as $m + \mathbf{p}^2/2m$, and Eq. (6.4.28) reduces, in the limit, to

$$\int d^3p\,|\phi_P(\mathbf{p};m)|^2 = 1\ , \tag{6.4.29}$$

which allows to rewrite Eq. (6.4.27) as

$$\int d^3x\,|\psi(\mathbf{x},t;m)|^2 = \frac{1}{2\pi\hbar}\ . \tag{6.4.30}$$

By Eq. (6.4.22), defining

$$\psi_{NR}(\mathbf{x},t;m) \equiv \frac{1}{(2\pi\hbar)^{3/2}}\int d^3p\, e^{\frac{i}{\hbar}\left(\mathbf{p}\cdot\mathbf{x}-\frac{\mathbf{p}^2}{2m}t\right)}\,\phi_P(\mathbf{p};m)\ , \tag{6.4.31}$$

it follows now

$$\psi(\mathbf{x},t;m) \approx \frac{e^{-imt/\hbar}}{(2\pi\hbar)^{1/2}}\,\psi_{NR}(\mathbf{x},t;m)\ , \tag{6.4.32}$$

with

$$\int d^3x\,|\psi_{NR}(\mathbf{x},t;m)|^2 = 1\ . \tag{6.4.33}$$

In the description of the nonrelativistic particle, therefore, we may conveniently adopt the function ψ_{NR} instead of ψ; as it follows from the discussion, to such function can be given the meaning of probability amplitude for the particle's position in space at a given time; moreover, ψ_{NR} satisfies the nonrelativistic Schrödinger equation (3.1.1) for

$$H(\mathbf{x},-i\hbar\boldsymbol{\nabla},t) = -\frac{\hbar^2}{2m}\nabla^2\ . \tag{6.4.34}$$

We are now able to explain the conundrum of the different status of λ and t. In the relativistic theory, performed in the extended spacetime $\mathbb{R}^4 \times [\lambda', \lambda'']$, the state vector belongs to the Hilbert space $L^2(\mathbb{R}^4)$, and its evolution is parametrized by λ – an unobservable quantity; when considering the spacetime level, the dependence on λ is dropped, and the general state is no more represented by a vector of $L^2(\mathbb{R}^4)$, but rather by a density operator $\hat{\rho}$ acting in this space, which is diagonal in m and ε. In the nonrelativistic limit for one of the components of $\hat{\rho}$, we obtain a theory which is formally similar to the relativistic one, but with a configuration space which has one dimension less; the state vector belongs thus to $L^2(\mathbb{R}^3)$, and evolves in t, which is now a parameter (i.e., not an observable) of the theory. However, it is not necessary, now, to project at the *space* level, eliminating the t-dependence; in fact, t *is* a physical observable, since it was so in the relativistic theory we started with. In other words, t happens to play the role of a parameter in the nonrelativistic limit but, owing to the approximate nature of this last, it still preserves his feature of being observable (although *not* in the nonrelativistic theory).

It is worth considering what is the effect of the nonrelativistic limit on the phase space distributions P_W and P_S; we shall perform explicitly these calculations only for the Wigner function P_W, since the case of P_S can be treated in a completely analogous manner. Let us begin noticing that Eqs. (6.2.12), (6.2.13) and (6.4.19) imply, for $\varepsilon = -1$, that

$$\Psi(x \pm \xi, \lambda; m) = \frac{1}{(2\pi\hbar)^2} e^{-\frac{i}{2\hbar}m^2\lambda} \int d^4k \, e^{ik\cdot(x\pm\xi)/\hbar} \delta(k^2 + m^2)\, \phi(k) \,. \tag{6.4.35}$$

Assuming the state vector to describe only particles, and remembering the properties of the delta function, we can write, using Eq. (6.4.23),

$$\delta(k^2 + m^2)\, \phi(k) = \delta(k^0 - \sqrt{\mathbf{k}^2 + m^2})\, \phi_P(\mathbf{k}; m) \,, \tag{6.4.36}$$

which allows to rewrite Eq. (6.4.35) as

$$\Psi(x\pm\xi, \lambda; m) = \frac{1}{(2\pi\hbar)^2} e^{-\frac{i}{2\hbar}m^2\lambda} \int d^3k \, e^{i\left(\mathbf{k}\cdot(\mathbf{x}\pm\boldsymbol{\xi}) - \sqrt{\mathbf{k}^2+m^2}(t\pm\xi^0)\right)/\hbar} \phi_P(\mathbf{k}; m) \,, \tag{6.4.37}$$

where now $\boldsymbol{\xi}$ is the spatial part of the four-vector ξ. Introducing the nonrelativistic condition on $\phi_P(\mathbf{k}; m)$, we can approximate $P_W(x, p|\lambda; m)$ as

$$\frac{1}{(\pi\hbar)^4} \cdot \frac{1}{(2\pi\hbar)^4} \int d\xi^0 d^3\xi d^3k d^3k' \, e^{i\left(2\mathbf{p}\cdot\boldsymbol{\xi} - 2(p^0 - m)\xi^0\right)/\hbar}$$
$$e^{-\frac{i}{\hbar}\left(\mathbf{k}\cdot(\mathbf{x}+\boldsymbol{\xi}) - \frac{\mathbf{k}^2}{2m}t\right)} \phi_P(\mathbf{k}; m)^* \, e^{\frac{i}{\hbar}\left(\mathbf{k}'\cdot(\mathbf{x}-\boldsymbol{\xi}) - \frac{\mathbf{k}'^2}{2m}t\right)} \phi_P(\mathbf{k}'; m); \tag{6.4.38}$$

in Eq. (6.4.38), the terms in $\mathbf{k}^2/2m$ and $\mathbf{k}'^2/2m$ have been neglected with respect to m. Performing now the integration in ξ^0, and remembering

Eq. (6.4.31), we get

$$P_W(\mathbf{x},t,\mathbf{p},p^0|\lambda;m) \approx \frac{1}{2\pi\hbar}\delta(p^0-m)\,P_W^{NR}(\mathbf{x},\mathbf{p}|t;m)\ , \tag{6.4.39}$$

with, as usual,

$$P_W^{NR}(\mathbf{x},\mathbf{p}|t;m) = \frac{1}{(\pi\hbar)^3}\int d^3\xi\,\psi_{NR}(\mathbf{x}+\boldsymbol{\xi},t;m)^*\,e^{2i\mathbf{p}\cdot\boldsymbol{\xi}/\hbar}\,\psi_{NR}(\mathbf{x}-\boldsymbol{\xi},t;m)\ . \tag{6.4.40}$$

The relation (6.4.39) is exactly what we would expect to find, since the delta function has the effect of fixing the value of p^0 to m, thus neglecting all the other contributions due to kinetic energy.

The theory here presented can describe tachions ($\varepsilon = 1$) as well as ordinary particles ($\varepsilon = -1$); this could be regarded as a difficulty, since faster than light motions have never been observed. However, we should like to remark that even in the standard relativistic quantum theory the nonexistence of tachions is not *proved*: They are simply *assumed* not to be present, since the beginning, through the adoption of the relation $\mathbf{p}^2 + m^2 = E^2$ for the asymptotic states. This attitude could be taken also in the context of our theory, *imposing* the condition $\varepsilon = -1$ to be satisfied for the state (6.2.16), i.e., requiring

$$c_{n\varepsilon} = \delta_{\varepsilon,-1}\,c_n\ , \tag{6.4.41}$$

but still allowing histories with an arbitrary causal behaviour to contribute to the amplitude (6.2.1); as it is easy to realize, the condition $\varepsilon = -1$ holds for all the values of λ if it holds for some λ_0: The nonexistence of observable faster than light motions is thus reduced to a matter of boundary conditions.

The initial aim of the present chapter was to perform a relativistic extension of the concept of quasiprobability functional; however, it turns out that the results obtained in the course of the treatment go far beyond such a goal. The most interesting conclusion is, in our opinion, the construction of an explicitly covariant scheme of first quantization: The time coordinate, which in nonrelativistic quantum theory is treated as a parameter, unlike the spatial ones, which represent the observable position of the particle, is now raised to the rank of an observable, too.

The coordinates $\{x^a\}$ represent therefore the *spacetime* position of the relativistic particle, while the role of evolution parameter is now played by a variable λ which, at a classical level, parametrizes the trajectories both of position and momentum.

In this theory, neither the mass m nor the causal type ε of the particle are imposed *a priori*, but they are consequences of the particular boundary conditions: In particular, it is possible to conceive states which correspond to a superposition both of m and ε; only for a λ-independent system, does the concept of definite m and ε make sense, and it is found that in such a case the particle can be described by a Klein-Gordon equation. Moreover, it is easy to construct, starting from the amplitude $\bar{K}(x'', x', \lambda)$ for the particle to evolve between the spacetime points x' and x'' during a parameter lapse λ, the Feynman propagator as a kind of Fourier transform of $\bar{K}$ with respect to λ, as in Eq. (6.2.26). This result is particularly relevant, as it represents a connection between the presentation here developed and more standard formulations of relativistic quantum theory; it suggests that, as far as free particles are considered, our formalism agree with those based on the second quantization of a field. If this is the case even when interactions are taken into account, is still an open question which deserves, in our opinion, further consideration.

Chapter 7

Weakly Semiclassical Relativistic Fields

In Chs. 5 and 6 we have developed the techniques necessary to overcome the troubles mentioned in Sec. 4.4, which occur when a weakly semiclassical relativistic field is described adopting a conventional treatment of the quantum source. In this chapter, we shall apply these new methods to some cases of interest.

In Sec. 7.1 we discuss the formal nature of the sources of relativistic fields, to which the quantum properties are completely attributed, according to the WSH. Sec. 7.2, 7.3 and 7.4 contain, respectively, a semiclassical treatment of the scalar, the electromagnetic and the gravitational fields. The treatment follows mainly ref. [31]

7.1 Sources of Relativistic Fields

The main idea of the WSH is to extract a probability distribution for the values of the classical observables of a system $\mathcal{C}$ from the quantum behaviour of another system $\mathcal{Q}$, interacting with $\mathcal{C}$. We are interested, in this chapter, to the case in which $\mathcal{C}$ is a relativistic field, while $\mathcal{Q}$ is a quantum system coupled to $\mathcal{C}$. In this context, it is of fundamental importance to know exactly which mathematical feature of $\mathcal{Q}$ can be identified as the classical source or the field.

The variational principle provides immediately the recipe. If φ denotes a generic component of a field, while χ stands for a generic variable of matter, the total action of the system matter + field is a functional

$S[\chi, \varphi]$ both of χ and φ; writing $S_f[\varphi]$ for the action of the field in absence of coupling, we have

$$S[\chi, \varphi] = S_f[\varphi] + S_m[\chi, \varphi] \; . \tag{7.1.1}$$

A variation of S with respect to χ amounts, according to Eq. (7.1.1), to a variation of S_m alone, and may be used to compactify the equations of motion for matter in the prescription

$$\frac{\delta S[\chi, \varphi]}{\delta \chi} = 0 \; . \tag{7.1.2}$$

Similarly, the condition

$$\frac{\delta S_f[\varphi]}{\delta \varphi} = 0 \tag{7.1.3}$$

produces the field equations for the free field, while

$$\frac{\delta S[\chi, \varphi]}{\delta \varphi} = 0 \tag{7.1.4}$$

gives the field equations in presence of matter. Rewriting Eq. (7.1.4) as

$$\frac{\delta S_f[\varphi]}{\delta \varphi} = - \frac{\delta S_m[\chi, \varphi]}{\delta \varphi} \; , \tag{7.1.5}$$

allows to identify the source in the term

$$- \frac{\delta S_m[\chi, \varphi]}{\delta \varphi} \; , \tag{7.1.6}$$

which is responsible for the "corrections" to the field equations due to the presence of matter: It is therefore on (7.1.6) that we must concentrate in order to understand which traits of matter influence the behaviour of the field.

Let us now discuss the structure of the classical source for some specific cases. As first example, we consider newtonian gravity, with a pointlike particle of mass m, following the history γ, as source; the action[1] is, by Eq. (5.1.3),

$$S[\gamma] = \int_{-\infty}^{+\infty} dt' \left[\mathrm{p}(t') \cdot \frac{d\mathrm{x}(t')}{dt'} - \frac{1}{2m} \mathrm{p}(t')^2 - m \, \Phi(\mathrm{x}(t'), t') \right] \; , \tag{7.1.7}$$

[1]Since in what follows we are concerned only with the action S_m, we drop the index m hereafter.

where $t' \mapsto \mathbf{x}(t')$ is the particle trajectory in space. Performing the variation

$$\Phi(\mathbf{x}(t'), t') \longrightarrow \Phi(\mathbf{x}(t'), t') + \epsilon\, \delta^3(\mathbf{x} - \mathbf{x}(t'))\delta(t - t') \,, \tag{7.1.8}$$

where ϵ is an arbitrarily small number, the source term is found to be

$$\mu\,(\mathbf{x}, t|\gamma) \equiv -\frac{\delta S[\gamma]}{\delta\Phi(\mathbf{x}, t)} = m\,\delta^3(\mathbf{x} - \mathbf{x}(t)) \,, \tag{7.1.9}$$

in agreement with Eq. (4.3.2). In Eq. (7.1.9), the greek letter γ express the fact that $\mu\,(\mathbf{x}, t|\gamma)$ is the source due to a particle which follows the history γ.

Let us now consider the case of the scalar field in Minkowski spacetime; with a natural choice of the superhamiltonian, Eq. (6.1.2) gives, for the action of a pointlike particle,

$$S[\gamma] = \int_{-\infty}^{+\infty} d\lambda \left[p_a(\lambda)\frac{dx^a(\lambda)}{d\lambda} - \frac{1}{2}\eta^{ab}p_a(\lambda)p_b(\lambda) - Q\,\phi(x(\lambda)) \right] \,, \tag{7.1.10}$$

where Q is the "scalar charge" of the particle. Under the variation

$$\phi(x(\lambda)) \longrightarrow \phi(x(\lambda)) + \epsilon\,\delta^4(x - x(\lambda)) \,, \tag{7.1.11}$$

we have

$$J(x|\gamma) \equiv -\frac{\delta S[\gamma]}{\delta\phi(x)} = Q \int_{-\infty}^{+\infty} d\lambda\, \delta^4(x - x(\lambda)) \,, \tag{7.1.12}$$

which agrees with Eq. (4.4.12), except for the fact that the Q's in the two expressions are not numerically equal, due to the different choices of the world line parametrization.

When dealing with the electromagnetic field, we can insert the superhamiltonian given by Eq. (6.1.28) into Eq. (6.1.2), getting

$$S[\gamma] = \int_{-\infty}^{+\infty} d\lambda \left[p_a(\lambda)\frac{dx^a(\lambda)}{d\lambda} - \frac{1}{2}\eta^{ab}[p_a(\lambda) - eA_a(x(\lambda))]\cdot[p_b(\lambda) - eA_b(x(\lambda))] \right]. \tag{7.1.13}$$

The variation

$$A_a(x(\lambda)) \longrightarrow A_a(x(\lambda)) + \epsilon_a\, \delta^4(x - x(\lambda)) \ , \tag{7.1.14}$$

where ϵ_a is a vector with arbitrarily small components, gives

$$j_e^a(x|\gamma) \equiv -\frac{\delta S[\gamma]}{\delta A_a(x)} = e \int_{-\infty}^{+\infty} d\lambda\, (p^a(\lambda) - eA^a(x(\lambda)))\, \delta^4(x - x(\lambda)) \ , \tag{7.1.15}$$

which, when imposing the classical equations of motion, turns out to coincide with the commonly adopted [32] source for the electromagnetic field,

$$j_e^a(x|\gamma) = e \int_{-\infty}^{+\infty} d\lambda\, \frac{dx^a(\lambda)}{d\lambda}\, \delta^4(x - x(\lambda)) \ . \tag{7.1.16}$$

Classical electromagnetism is known to be invariant under gauge transformations of $A_a(x)$,

$$A_a(x) \longrightarrow A_a(x) + \partial_a \Lambda(x) \ , \tag{7.1.17}$$

where Λ is an arbitrary scalar on Minkowski spacetime. The consequences of this invariance on the source can be extracted requiring that the action (7.1.13) do not change when a transformation (7.1.17) is performed. More formally, we can make the action (7.1.13) Λ-dependent operating in it the substitution (7.1.17), and then require that

$$\frac{\delta S[\gamma]}{\delta \Lambda(x)} = 0 \ . \tag{7.1.18}$$

Since Eq. (7.1.18) is equivalent to considering a variation of A_a of the kind

$$A_a(x(\lambda)) \longrightarrow A_a(x(\lambda)) + \epsilon\, \partial_a \delta^4(x - x(\lambda)) \ , \tag{7.1.19}$$

it is straightforward to realize that it leads to the relation

$$\partial_a j_e^a(x|\gamma) = 0 \ , \tag{7.1.20}$$

expressing the conservation of the electric charge.

It is important to realize that Eq. (7.1.20) is not automatically satisfied, because the history γ is not necessarily classical (we have not imposed the validity of the variational principle for the particle), and consequently

$$p^a(\lambda) - eA^a(x(\lambda))$$

in Eq. (7.1.15) does not coincide, in general, with

$$\frac{dx^a(\lambda)}{d\lambda}$$

(which would be the content of half of the Hamilton equations of motion). Rather, Eq. (7.1.20) has to be considered as a *constraint* which any possible particle history (even a nonclassical one) must satisfy in order to be compatible with a classical electromagnetic field.

Before proceeding to treat the case of gravity, we believe it is important to make a couple of comments about the cases discussed until now. First, let us notice that in Eqs. (7.1.10) and (7.1.13) the particle-field interaction is accounted for by terms containing the field $\phi(x(\lambda))$ or $A_a(x(\lambda))$ is evaluated on the world line of the particle. This makes such expressions of the action suitable for describing either the effect of a fixed external field on the particle motion, or the properties of a particle, whose motion is prescribed, as source of the field; however, both these cases correspond to situations in which the behaviour either of the particle or of the field is *prescribed*. Hence, Eqs. (7.1.10) and (7.1.13) are not suitable for giving a completely self-consistent treatment of the particle + field system, i.e., for taking the radiation reaction into account. We must therefore be aware that any quantum treatment based on them represents necessarily only an approximate version of a fully consistent theory.

These kind of problems are strictly connected to our second remark, which is concerned with the nature of the field source. In the three examples discussed above, we have chosen the source to be a pointlike particle; this is a concept which creates some troubles (including those mentioned above), but it is not impossible to deal with it in the framework of newtonian gravity, as well as of the scalar field theory and of electromagnetism. However, in Einstein's theory there is no room for pointlike particles considered as sources of gravity, because they would lead to an unacceptably singular spacetime metric [119]. Nevertheless, in the following discussion about the source of gravity we deal at first with a pointlike particle, and only later on we extend the treatment to the case of an extended system; the reasons for such a presentation are essentially of pedagogical character.

Let us therefore consider the action for a particle in a gravitational field

$$S[\gamma] = \int_{-\infty}^{+\infty} d\lambda \left[p_a(\lambda) \frac{dx^a(\lambda)}{d\lambda} - \frac{1}{2} g^{ab}(x(\lambda))\, p_a(\lambda) p_b(\lambda) \right] , \qquad (7.1.21)$$

and perform the variation

$$g^{ab}(x(\lambda)) \longrightarrow g^{ab}(x(\lambda)) + \frac{1}{\sqrt{-g(x(\lambda))}}\, \epsilon^{ab}\, \delta^4(x - x(\lambda)) , \qquad (7.1.22)$$

where g is the determinant of the metric g_{ab}, and ϵ^{ab} is a symmetric tensor whose components are arbitrarily small. We get

$$T_{ab}(x|\gamma) \equiv -\frac{\delta S[\gamma]}{\delta g^{ab}(x)} = \frac{1}{\sqrt{-g(x)}} \int_{-\infty}^{+\infty} d\lambda\, \delta^4(x - x(\lambda))\, p_a(\lambda) p_b(\lambda) , \qquad (7.1.23)$$

which is a very pleasant expression of the stress-energy-momentum tensor for a particle with world line γ.

As said above, however, pointlike particles do not fit well into general relativity, and it would be preferable to consider a continuous fluid as source; this can be done as follows[2]. Let us start considering a system of N identical noninteracting relativistic particles in a spacetime with metric g_{ab}; since the parameters λ can be chosen to be the same, the action is

$$S[\gamma] = \sum_{r=1}^{N} \int_{-\infty}^{+\infty} d\lambda \left[p_a^{(r)}(\lambda) \frac{dx^a_{(r)}(\lambda)}{d\lambda} - \frac{1}{2} g^{ab}(x_{(r)}(\lambda))\, p_a^{(r)}(\lambda) p_b^{(r)}(\lambda) \right] , \qquad (7.1.24)$$

where the index r labels the particles. Passing to the continuum limit, r must be replaced by the point ξ of some three-dimensional manifold

[2]The description we present here of the relativistic fluid is very schematic and incomplete. However, for what this thesis is concerned, it is enough to guarantee that such a description *can be given*, and it is not necessary to enter into the details, which we intend to give in some future work.

Σ, and the sum in Eq. (7.1.24) by a corresponding integral over Σ; the action reads now

$$S[\gamma] = \int_{-\infty}^{+\infty} d\lambda \int_{\Sigma} d^3\xi\, n(\xi) \left[p_a(\xi,\lambda) \frac{dx^a(\xi,\lambda)}{d\lambda} - \frac{1}{2} g^{ab}(x(\xi,\lambda)) p_a(\xi,\lambda) p_b(\xi,\lambda) \right], \tag{7.1.25}$$

where $n(\xi)$ is a convenient pseudoscalar function on Σ. With the variation

$$g^{ab}(x(\xi,\lambda)) \longrightarrow g^{ab}(x(\xi,\lambda)) + \frac{1}{\sqrt{-g(x(\xi,\lambda))}} \epsilon^{ab}\, \delta^4(x - x(\xi,\lambda)) \ , \tag{7.1.26}$$

we can calculate the source term from the action (7.1.25):

$$T_{ab}(x|\gamma) \equiv -\frac{\delta S[\gamma]}{\delta g^{ab}(x)} = \frac{1}{\sqrt{-g(x)}} \int_{\Sigma} d^3\xi\, n(\xi)$$

$$\int_{-\infty}^{+\infty} d\lambda\, \delta^4(x - x(\xi,\lambda))\, p_a(\xi,\lambda) p_b(\xi,\lambda) \ . \tag{7.1.27}$$

It is easy to realize that $T_{ab}(x|\gamma)$ expressed by Eq. (7.1.27) corresponds to the stress-energy-momentum tensor of a fluid of dust whose elements are labeled by ξ, and follow the world lines $\lambda \mapsto x(\xi,\lambda)$; the history γ represents now the collection of the histories for each of these fluid elements.

Einstein's theory, as well as electromagnetism, is also characterized by a gauge invariance under some transformations, which are now of the kind [3]

$$g^{ab}(x) \longrightarrow g^{ab}(x) + \nabla^a X^b(x) + \nabla^b X^a(x) \ , \tag{7.1.28}$$

$X^a(x)$ being an arbitrary vector field on spacetime. As we did in the case of electromagnetism, we can explore the consequences of such invariance making the action X-dependent by the substitution (7.1.28), and then requiring that

$$\frac{\delta S[\gamma]}{\delta X^a(x)} = 0 \ . \tag{7.1.29}$$

Let us do this explicitly for the case of the pointlike particle; Eq. (7.1.29) can be sen to correspond to a variation of g^{ab} of the kind

$$g^{ab}(x(\lambda)) \longrightarrow g^{ab}(x(\lambda)) + \frac{1}{\sqrt{-g(x(\lambda))}} [\epsilon^a \nabla^b \delta^4(x - x(\lambda)) + \epsilon^b\, \nabla^a \delta^4(x - x(\lambda))], \tag{7.1.30}$$

which leads to

$$\nabla^b T_{ab}(x|\gamma) = 0 \ , \tag{7.1.31}$$

which is a relation analogous to (2.1.4); the same result would have been obtained starting from the action (7.1.24).

Like for the case of Eq. (7.1.20) in the theory of a particle interacting with the classical electromagnetic field, Eq. (7.1.31) has to be regarded as a constraint which any material system coupled to classical gravity has to satisfy. There is a delicate issue which must be clarified about Eq. (7.1.31): Classically, it is well known [3,2,120] that Eq. (2.1.4), expressing essentially the energy and momentum conservation laws for a system in a gravitational field, contains much information about the equations of motion of such a system. We can show this explicitly for the case of the pointlike particle, in which Eq. (2.1.4) allows to derive the geodesic equation. Let us define the stress tensor

$$\bar{T}^{ab}(x) \equiv \frac{1}{\sqrt{-g(x)}} \int_{-\infty}^{+\infty} d\lambda \, \delta^4(x - x(\lambda)) \, \frac{dx^a(\lambda)}{d\lambda} \frac{dx^b(\lambda)}{d\lambda} \ , \tag{7.1.32}$$

which differs from the contravariant version of (7.1.23) only by the replacement

$$g^{ab}(x(\lambda)) \, p_b(\lambda) \longrightarrow \frac{dx^a(\lambda)}{d\lambda} \ , \tag{7.1.33}$$

corresponding to assume the validity of half of the Hamilton equations. A straightforward application of the rules of covariant differentiation, together with some properties of the delta function, gives

$$\nabla_b \bar{T}^{ab}(x) = \frac{1}{\sqrt{-g(x)}} \int_{-\infty}^{+\infty} d\lambda \, \delta^4(x - x(\lambda)) \left[\frac{d^2 x^a(\lambda)}{d\lambda^2} + \Gamma^a{}_{bc}(x(\lambda)) \frac{dx^b(\lambda)}{d\lambda} \frac{dx^c(\lambda)}{d\lambda} \right] , \tag{7.1.34}$$

where $\Gamma^a{}_{bc}$ are the Christoffel coefficients; requiring now the validity of Eq. (2.1.4), multiplying the equation so obtained by an arbitrary function $F(x)$, and performing an integration over all spacetime, we get

$$\int_{-\infty}^{+\infty} d\lambda \, F(x(\lambda)) \left[\frac{d^2 x^a(\lambda)}{d\lambda^2} + \Gamma^a{}_{bc}(x(\lambda)) \frac{dx^b(\lambda)}{d\lambda} \frac{dx^c(\lambda)}{d\lambda} \right] = 0 \ . \tag{7.1.35}$$

Assuming now that the curve $\lambda \mapsto x(\lambda)$ is injective, i.e., that different values of λ correspond to different points of spacetime (this is a global requirement which rules out, e.g., closed curves, and appears very reasonable if $x(\lambda)$ has to represent a classical world line), and choosing

$$F(x) = \frac{1}{\sqrt{-g(x)}}\,\delta^4(x)\ , \tag{7.1.36}$$

Eq. (7.1.36) gives immediately

$$\frac{d^2x^a(\lambda)}{d\lambda^2} + \Gamma^a{}_{bc}(x(\lambda))\frac{dx^b(\lambda)}{d\lambda}\frac{dx^c(\lambda)}{d\lambda} = 0\ , \tag{7.1.37}$$

which is indeed the equation characterizing the curve $\lambda \mapsto x(\lambda)$ as a geodesic of spacetime.

This result, which can be remarkably generalized, raises the following problem: If the condition expressed by Eq. (2.1.4) is so strong that it contains the equations of motion of the particle, does the consistency requirement (7.1.31) do the same? An affirmative answer would have disastrous consequences on the semiclassical program, because it would imply that a classical gravitational field *forces* matter to follow classical histories! Luckily, the answer is negative. In fact, $T_{ab}(x|\gamma)$ and $\bar{T}_{ab}(x)$ differ, as we have already noticed, by the feature that the former contains the momentum $p_a(\lambda)$, while the latter is expressed in terms of the velocity $dx^a(\lambda)/d\lambda$; while for $\bar{T}_{ab}(x)$ the equations of motion automatically guarantee that Eq. (2.1.4) is fulfilled, and vice versa, this is not true for $t_{ab}(x|\gamma)$, as one can check immediately. It is only when the replacement (7.1.33) is made, that Eq. (7.1.31) implies a classical motion; more formally, one can say that, at least for the case here examined, the gauge invariance (7.1.29), *together* with the Hamilton equations

$$\frac{\delta S[\gamma]}{\delta p_a(\lambda)} = 0\ , \tag{7.1.38}$$

imply the other half of the Hamilton equations:

$$\frac{\delta S[\gamma]}{\delta x^a(\lambda)} = 0\ . \tag{7.1.39}$$

We can conclude that, in the context of the phase space treatment of the particle, the gauge constraint (7.1.31), although strong, does not necessarily require a classical behaviour.

7.2 The Weakly Semiclassical Scalar Field

Let us resume the weakly semiclassical treatment of the scalar field, which we have interrupted in Sec. 4.4 because of the inadequacy of the techniques adopted there.

Eqs. (4.4.17) and (4.4.18) imply

$$\begin{aligned} P(\phi(x)) &= \int_{\Gamma_0} \mathcal{D}\gamma\, P(\phi(x);\gamma) = \\ &= \int_{\Gamma_0} \mathcal{D}\gamma\, \delta(\phi(x) - \phi(x|\gamma))\, P[\gamma] \,, \end{aligned} \tag{7.2.1}$$

where $P[\gamma]$ is now well defined, thanks to the treatment of Chs. 5 and 6, and $\phi(x|\gamma)$ is given by Eq. (4.4.13), which we rewrite here in the new parametrization (remember that the Q in Eq. (7.2.2) is not the same as that in Eq. (4.4.13)!):

$$\phi(x|\gamma) = -Q \int_{-\infty}^{+\infty} d\lambda\, D(x - x(\lambda)) \,. \tag{7.2.2}$$

The probability $P(\phi(x))$ is now normalized, since

$$\int d\phi(x)\, P(\phi(x)) = \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma] = 1 \,, \tag{7.2.3}$$

by Eq. (6.3.3); therefore, it can be conveniently used to define an *ensemble average* of ϕ at the spacetime point x as

$$\begin{aligned} \langle\phi(x)\rangle &\equiv \int d\phi(x)\, P(\phi(x))\, \phi(x) = \\ &= \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, \phi(x|\gamma) \,. \end{aligned} \tag{7.2.4}$$

The average field given by Eq. (7.2.4) satisfies a weakly semiclassical version of Eq. (4.4.1); in fact, remembering Eqs. (7.2.2), (4.4.2) and (7.1.12), we get

$$\eta^{ab}\partial_a\partial_b\langle\phi(x)\rangle = \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, J(x|\gamma) \,. \tag{7.2.5}$$

Comparing now Eqs. (7.1.12), (6.3.21) and (6.3.23), we realize that

$$\int_{\Gamma_0} \mathcal{D}\gamma \, P[\gamma] \, J(x|\gamma) = Q \int_{-\infty}^{+\infty} d\lambda \, P(x|\lambda) \; , \tag{7.2.6}$$

where $P(x|\lambda)$ is given by Eq. (6.2.4); in the Heisenberg picture, defining the source operator $\hat{J}(x)$ as

$$\hat{J}(x) \equiv Q \int_{-\infty}^{+\infty} d\lambda \, |x, \lambda\rangle\langle x, \lambda| = Q \int_{-\infty}^{+\infty} d\lambda \, \delta^4(x\hat{1} - \hat{x}(\lambda)) \; , \tag{7.2.7}$$

we have thus

$$\eta^{ab}\partial_a\partial_b\langle\phi(x)\rangle = \langle\Psi|\hat{J}(x)|\Psi\rangle \; , \tag{7.2.8}$$

as we wanted to prove.

Eq. (7.2.8) is the first semiclassical relativistic field equation that we recover within our formalism, and it worth some comments. Let us start clarifying the meaning of the λ-integration in Eq. (7.2.7); this has an essentially classical origin, which can be understood considering the classical counterpart of Eq. (7.2.7), i.e., Eq. (7.1.12). For sake of clearness, let us make a comparison with the case of newtonian gravity, in which the field Φ at the point $\mathbf{x}$ of space at time t is determined by the particle position at time t. In contrast, in the theory of the scalar field, ϕ at the spacetime point x is determined by the intersections of the particle world line with the support of the Green function D, no matter at which value of λ they may occur: In the calculation of $\phi(x)$, one must take into account all these contributions equally well, and this explains why it is necessary to integrate over λ in order to obtain the sources $J(x|\gamma)$ and $\hat{J}(x)$.

Eq. (7.2.8) has the same form of its classical version (4.4.1), with the only difference that ϕ and J are substituted, respectively, by the average $\langle\phi\rangle$ and by the expectation value $\langle\Psi|\hat{J}|\Psi\rangle$. This is true even if the dispersion in the particle position (and, consequently, in J) is large, thanks to the linearity of the theory, and would not happen in a nonlinear modification of it. To be more explicit on this point, let us consider a scalar field ϕ obeying the classical field equation

$$\eta^{ab}\partial_a\partial_b\phi + f(\phi) = J \; , \tag{7.2.9}$$

where f is a nonlinear function. If the action of the source is chosen again to be given by Eq. (7.1.10), we can define a family of field configurations $\phi(x|\gamma)$, parametrized by γ, satisfying the equation

$$\eta^{ab}\partial_a\partial_b\phi(x|\gamma) + f(\phi(x|\gamma)) = J(x|\gamma) \ , \tag{7.2.10}$$

where $J(x|\gamma)$ is given by Eq. (7.1.12). When considering a quantum source, the WSH can be applied as usual, to obtain Eq. (7.2.1) for $P(\phi(x))$. Until this point, the theory is just analogous to the one developed in the linear case; nevertheless, the semiclassical field equation has not, now, the same form of the classical one. In fact, using Eqs. (7.2.6), (7.2.7) and (6.2.4), we get again

$$\int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, J(x|\gamma) = \langle\Psi|\hat{J}(x)|\Psi\rangle \ ; \tag{7.2.11}$$

inserting Eq. (7.2.10) into the left hand side of Eq. (7.2.11) we get, remembering Eq. (7.2.4),

$$\eta^{ab}\partial_a\partial_b\langle\phi(x)\rangle + \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, f(\phi(x|\gamma)) = \langle\Psi|\hat{J}(x)|\Psi\rangle \ , \tag{7.2.12}$$

which, because of the nonlinearity of f, does not have the same form of Eq. (7.2.9) – except for the "nearly classical" cases in which the statistical dispersion in ϕ is so small that we can make the approximation

$$\int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, f(\phi(x|\gamma)) \approx f(\langle\phi(x)\rangle) \ , \tag{7.2.13}$$

corresponding to a strongly semiclassical regime. We have here an explicit example of the circumstance already mentioned in Chs. 2 and 4, during our criticism of the usual formulation of semiclassical theories. According to these last, in fact, the semiclassical field equation corresponding to Eq. (7.2.9) should be

$$\eta^{ab}\partial_a\partial_b\phi + f(\phi) = \langle\Psi|\hat{J}|\Psi\rangle \ ; \tag{7.2.14}$$

it is perfectly clear, from our treatment, that the range of validity of this equation is very narrow, depending on the possibility to perform the approximation (7.2.13). An equation holding throughout all the semiclassical regime is Eq. (7.2.12), which we can easily rewrite, in the same spirit of Eq. (4.2.6), as

$$\eta^{ab}\partial_a\partial_b\langle\phi(x)\rangle + f(\langle\phi(x)\rangle) + R(\Delta\phi(x),\ldots) = \langle\Psi|\hat{J}(x)|\Psi\rangle \ , \qquad (7.2.15)$$

where the term $R(\Delta\phi(x),\ldots)$ lumps together all the corrections due to the dispersion

$$\Delta\phi(x) \equiv \left(\int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, \phi(x|\gamma)^2 - \langle\phi(x)\rangle^2\right)^{1/2} \ , \qquad (7.2.16)$$

and to higher moments of the distribution of ϕ. It is clear that, unless some specific hypothesis are made about $\Delta\phi(x)$ and the higher moments, Eq. (7.2.15) is very difficult to solve; however, we want to stress once more that any solution of Eq. (7.2.15) or of the simpler Eq. (7.2.8) has, in general, no meaning as referred to a single system, as it is particularly evident from Eq. (7.2.4): This feature reduces considerably the interest of the semiclassical field equations, as already discussed thoroughly in Sec. 4.2.

The path integral treatment of the quasiprobability has shown to be rather powerful and elegant in getting straightforwardly Eq. (7.2.1) for $P(\phi(x))$, together with all its consequences, without running into the troubles mentioned in Sec. 4.4; actually, it seems the *easiest* way for doing that. However, let us remember that in the simple case of newtonian gravity we have been able, in Sec. 4.3, to obtain the expression (4.3.7) for $P(\Phi(\mathbf{x})|t)$ without using such a technique. We shall now check that Eq. (4.3.7) can be exactly recovered making use of the quasiprobability functional method, thus providing an example of the internal consistency of the theory. If $\Phi(\mathbf{x},t|\gamma)$ denotes the value of Φ at the point $\mathbf{x}$ of space at time t, whose source is a particle following the history γ, we have, by Eq. (4.3.5),

$$\Phi(\mathbf{x},t|\gamma) = -\frac{Gm}{|\mathbf{x}-\mathbf{x}(t)|} = \Phi(\mathbf{x},t|\mathbf{x}(t)) \ . \qquad (7.2.17)$$

Defining by

$$P(\Phi(\mathbf{x});\gamma|t)$$

the joint quasiprobability density that the field has, at time t, the value $\Phi(\mathrm{x})$ at the point x, and that the particle follow the history γ, we can write, as usual,

$$P(\Phi(\mathrm{x});\gamma|t) = P(\Phi(\mathrm{x})|\gamma,t)\,P[\gamma]\;, \qquad (7.2.18)$$

where now

$$P(\Phi(\mathrm{x})|\gamma,t)$$

is the probability density that the field has, at time t, the value $\Phi(\mathrm{x})$ at x, *given* that the particle follows the history γ. The WSH gives

$$P(\Phi(\mathrm{x})|\gamma,t) = \delta\left(\Phi(\mathrm{x}) - \Phi(\mathrm{x},t|\gamma)\right)\,. \qquad (7.2.19)$$

It is now straightforward to write

$$P(\Phi(\mathrm{x})|t) = \int_{\Gamma_0} \mathcal{D}\gamma\,\delta\left(\Phi(\mathrm{x} - \Phi(\mathrm{x},t|\gamma)\right)P[\gamma]\;, \qquad (7.2.20)$$

and we have thus only to prove that the right hand side of Eq. (7.2.20) coincide with the second line of Eq. (4.3.7). This can be easily seen as follows: In the path integration in Eq. (7.2.20), the positions and momenta of γ can all be integrated away, except for $\mathrm{x}(t)$, which is the only one appearing in the argument of the delta function, through Eq. (7.2.17). Therefore, using Eqs. (5.3.2) and (5.3.4), we get

$$\int_{\Gamma_0} \mathcal{D}\gamma\,\delta\left(\Phi(\mathrm{x}) - \Phi(\mathrm{x},t|\gamma)\right)P[\gamma] = \int d^3y\,\delta\left(\Phi(\mathrm{x}) - \Phi(\mathrm{x},t|\mathrm{y})\right)P(\mathrm{y}|t)\;, \qquad (7.2.21)$$

where the integration variable $\mathrm{x}(t)$ has been renamed y for convenience; this establishes the desired equality between the $P(\Phi(\mathrm{x})|t)$ defined by Eqs. (4.3.7) and (7.2.20).

We close this section with a brief discussion of the nature of the quantum treatment of the source in the semiclassical theory of the scalar field. Let us start by considering Eq. (7.2.8) for the average $\langle\phi(x)\rangle$. In the right hand side, the state vector $|\Psi\rangle$ is fixed (we work in the Heisenberg picture), and we need only to specify $\hat{J}(x)$; by Eq. (7.2.7), this can be done determining $\hat{x}(\lambda)$ (or, equivalently, $|x,\lambda\rangle$). This operator must be a solution of the Heisenberg equation of motion

$$i\hbar\frac{d\hat{x}(\lambda)}{d\lambda} = -[\mathcal{H}(\hat{x}(\lambda),\hat{p}(\lambda)),\hat{x}(\lambda)]\;, \qquad (7.2.22)$$

which is a straightforward generalization of Eq. (2.2.7) to the explicitly relativistic theory developed in Ch. 6. Which expression has to be chosen for the superhamiltonian $\mathcal{H}$ in Eq. (7.2.22)? Looking at the form of the action (7.1.10), one's immediate answer would be

$$\mathcal{H}(x(\lambda), p(\lambda)) = \frac{1}{2}\eta^{ab} p_a(\lambda) p_b(\lambda) + Q\,\phi(x(\lambda)) \,. \qquad (7.2.23)$$

However, let us suppose that no *external* field is acting on the particle; then $\phi(x(\lambda))$ should be the self-field evaluated on the world line of the particle: Since such a field diverges (see, e.g., Eq. (4.4.15)), we run into the troubles which typically occur when dealing with the radiation reaction of pointlike particles.

An approximate solution, which is the one tacitly adopted in standard quantum theory, is to drop the ϕ-dependence of $\mathcal{H}$, thus recovering the free particle superhamiltonian

$$\mathcal{H}(\hat{x}(\lambda), \hat{p}(\lambda)) = \frac{1}{2}\eta^{ab} \hat{p}_a(\lambda) \hat{p}_b(\lambda) \,; \qquad (7.2.24)$$

this choice amounts to neglect the self-interaction effects, and can be reasonably justified on the basis of the pointlike nature of the particle, which seems to correspond to the absence of an inner structure. However, we want to point out that such a treatment cannot be totally satisfactory; in fact, having we considered an extended source, there would have been necessarily a self-interaction term in the superhamiltonian.

Let us sketch qualitatively how a self-consistent treatment would look like, although we do not enter into the intricate technical details, which have not yet been worked out. The main idea is still that each history γ of the source (which we continue to refer to as *the particle*, for sake of simplicity) is compatible with a field configuration $\phi(x|\gamma)$; moreover, the WSH is expressed again by Eq. (4.4.18), which remains unchanged. What undergoes a conceptually deep modification is the quasiprobability functional $P[\gamma]$: This can be easily understood thinking that the quantum problem is now to describe the behaviour of the particle when its interaction with the classical field is *completely* taken into account, including the possibility of self-reaction; in order to accomplish such a result, the superhamiltonian has to be changed, and this implies a change in $S[\gamma]$ and, by Eqs. (5.1.2) and (6.3.1), in $P[\gamma]$. In spite of these modifications, however, the general structure of the theory as presented in this section does not suffer from any fundamental change.

7.3 Weakly Semiclassical Electromagnetism

The classical theory of the electromagnetic field can be formulated using the four-potential A_a defined by Eq. (2.1.27); the field equation is

$$\eta^{ab}\partial_a\partial_b A^c = -4\pi j_e^c \,, \tag{7.3.1}$$

where j_e^a is the electric current of matter, together with the Lorentz gauge condition

$$\partial_a A^a = 0 \,. \tag{7.3.2}$$

Defining the Green bitensor $D^a_{a'}(x - x')$ by

$$\eta^{ab}\partial_a\partial_b D^c_{c'}(x - x') = -\delta^c_{c'}\,\delta^4(x - x') \,, \tag{7.3.3}$$

one obtains straightforwardly

$$D^a_{a'}(x - x') = \delta^a_{a'}\,D(x - x') \,, \tag{7.3.4}$$

where $D(x - x')$ satisfies Eq. (4.4.2). Neglecting again the contribution from the boundary of spacetime, the Kirchhoff representation of A^a can be written as

$$A^a(x) = 4\pi \int d^4x'\,D(x - x')\,j_e^a(x') \,, \tag{7.3.5}$$

which, for the point particle case (7.1.15), reduces to

$$A^a(x|\gamma) = 4\pi e \int_{-\infty}^{+\infty} d\lambda\,p^a(\lambda)\,D(x - x(\lambda)) \,. \tag{7.3.6}$$

In deriving Eq. (7.3.6), we have not considered the term containing the four-potential $A^a(x(\lambda))$ evaluated on the world line of the particle; this procedure is justified by the assumption we have made, that no other fields are present except for those due to the particle, i.e., that there are no external fields; moreover, we neglect the radiation reaction and suppose that the particle do not interact electromagnetically with itself. These are explicit hypothesis which one should drop in a further development of the theory.

In the weakly semiclassical treatment, we shall denote by

$$P(A(x);\gamma)$$

the joint quasiprobability density that the four-potential at the spacetime point x be $A(x)$, and that the particle follow the world line γ; the WSH is written as

$$P(A^a(x)|\gamma) = \delta\left(A^a(x) - A^a(x|\gamma)\right) . \tag{7.3.7}$$

A few words are necessary in order to explain the notations in Eq. (7.3.7), which has to be read as a relation for *components* of the four-potential: More precisely, $P(A^a(x)|\gamma)$ is a quasiprobability density referred to the a-th component of the potential $A(x)$, and the argument of the delta function is also referred only to the same a-th component. The conditional probability for the entire four-vector $A(x)$ is thus

$$P(A(x)|\gamma) = \delta^4(A(x) - A(x|\gamma)) , \tag{7.3.8}$$

where Eq. (7.3.7) and the definition of δ^4 have been used. We can write now the joint quasiprobability $P(A(x);\gamma)$ as

$$P(A(x);\gamma) = \delta^4(A(x) - A(x|\gamma))\, P[\gamma] , \tag{7.3.9}$$

and the probability that the potential at the spacetime point x be $A(x)$ as

$$P(A(x)) = \int_{\Gamma_0} \mathcal{D}\gamma\, \delta^4(A(x) - A(x|\gamma))\, P[\gamma] . \tag{7.3.10}$$

As expected, $P(A(x))$ is normalized,

$$\int d^4A(x)\, P(A(x)) = 1 , \tag{7.3.11}$$

and allows thus to define the *ensemble average* of A^a at x as

$$\begin{aligned} \langle A^a(x)\rangle &\equiv \int d^4A(x)\, P(A(x))\, A^a(x) = \\ &= \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, A^a(x|\gamma) . \end{aligned} \tag{7.3.12}$$

The average potential $\langle A^a(x)\rangle$ satisfies a weakly semiclassical field equation which has the same form of the classical one, Eq. (7.3.1); this circumstance is due to the linearity of the theory, as explained in details in the previous section with regard to the scalar field. To check it explicitly, let us observe that Eqs. (7.3.12), (7.3.6) and (4.4.2) give

$$\eta^{ab}\partial_a\partial_b\langle A^c(x)\rangle = -4\pi \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, j_e^c(x|\gamma)\ , \tag{7.3.13}$$

where $j_e^a(x|\gamma)$ is given by

$$j_e^a(x|\gamma) = e \int_{-\infty}^{+\infty} d\lambda\, p^a(\lambda)\, \delta^4(x - x(\lambda))\ , \tag{7.3.14}$$

as discussed above. Remembering now Eqs. (6.3.17) and (6.3.13) we have immediately

$$\int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, j_e^a(x|\gamma) = \frac{e\hbar\eta^{ab}}{2i} \int_{-\infty}^{+\infty} d\lambda\, \Psi(x,\lambda)^* \frac{\overleftrightarrow{\partial}}{\partial x^b} \Psi(x,\lambda)\ , \tag{7.3.15}$$

which, defining the electric current operator $\hat{j}_e^a(x)$ as

$$\hat{j}_e^a(x) \equiv \frac{e}{2} \int_{-\infty}^{+\infty} d\lambda\, (|x,\lambda\rangle\langle x,\lambda|\, \hat{p}^a(\lambda) + \hat{p}^a(\lambda)\, |x,\lambda\rangle\langle x,\lambda|)\ , \tag{7.3.16}$$

allows to rewrite Eq. (7.3.13) in the semiclassical form:

$$\eta^{ab}\partial_a\partial_b\langle A^c(x)\rangle = -4\pi\langle\Psi|\hat{j}_e^a(x)|\Psi\rangle\ . \tag{7.3.17}$$

At the classical level, however, Eq. (7.3.17) does not suffice in determining the physical properties of $A^a(x)$, because it has to be coupled to the gauge condition (7.3.2), which guarantees, e.g., the conservation of charge. It is therefore natural to ask if a semiclassical formulation of Eq. (7.3.2) holds for the average $\langle A^a(x)\rangle$. The answer can be obtained using Eqs. (7.3.12) and (7.3.5) to obtain straightforwardly

$$\partial_a\langle A^a(x)\rangle = 4\pi \int d^4x'\, D(x - x') \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, \partial'_a j_e^a(x'|\gamma)\ , \tag{7.3.18}$$

from which one might hastily conclude, applying Eq. (7.1.20), that a semiclassical version of Eq. (7.3.2) holds. There is, however, a subtle point to consider here, which spoils this simple argument. Having not taken self-interaction into account, the quasiprobability distribution $p[\gamma]$ turns out to be independent of the presence of the electromagnetic field; it follows that the right hand side of Eq. (7.3.18) may contain contributions even from histories which do not satisfy Eq. (7.1.20). Physically, this means that if we do not fully consider the interaction particle-field, than there is no reason for the charge to be conserved; more formally, we can notice that nowhere, in Sec. 6.3, it was proved that

$$\partial_a j^a(x|\gamma) = 0 \ . \tag{7.3.19}$$

Hence, neglecting self-interaction of the source, we should expect that

$$\partial_a \langle A^a(x) \rangle \neq 0 \ , \tag{7.3.20}$$

because $\langle A^a(x) \rangle$ is constructed summing also over histories which are not compatible with a classical electromagnetic field. However, in a complete version of the theory, which takes self-interaction into account, the distribution $P[\gamma]$ would change, as explained at the end of Sec. 7.2, and the modified $P[\gamma]$ would be such that only histories γ which are compatible with the presence of the field contribute with a nonvanishing $P[\gamma]$; thus, in this fully consistent treatment, we recover the semiclassical version of Eq. (7.3.2),

$$\partial_a \langle A^a(x) \rangle = 0 \ , \tag{7.3.21}$$

since histories not satisfying the condition (7.1.20) have $P[\gamma] = 0$. We conclude that the violation of Eq. (7.3.21) which appears in Eq. (7.3.18) and in (7.3.20) are unphysical, being due to the incompleteness of the treatment.

7.4 Weakly Semiclassical Gravity

Our final application of the methods and ideas developed in the previous chapters concerns gravity, whose semiclassical treatment was the purpose motivating this thesis. The classical field equation is now Einstein equation (1.4), which we rewrite here for convenience as

$$R_{ab}[g] = \kappa \tilde{T}_{ab}[g] \ , \tag{7.4.1}$$

where

$$\tilde{T}_{ab}[g] \equiv T_{ab}[g] - \frac{1}{2} g_{ab} g^{cd} T_{cd}[g] \ , \tag{7.4.2}$$

and the explicit expression for the Ricci tensor is [3]:

$$R_{ab} = \partial_c \Gamma^c{}_{ab} - \partial_a \Gamma^c{}_{cb} + \Gamma^c{}_{ab} \Gamma^d{}_{cd} - \Gamma^c{}_{db} \Gamma^d{}_{ca} \ , \tag{7.4.3}$$

with $\Gamma^a{}_{bc}$ the Christoffel coefficients given by

$$\Gamma^a{}_{bc} \equiv \frac{1}{2} g^{ad} (\partial_c g_{db} + \partial_b g_{dc} - \partial_d g_{bc}) \ . \tag{7.4.4}$$

Eq. (7.4.1), considered as a set of differential equations for the components of the metric tensor, are extraordinarily complicated, in comparison to Eqs. (4.4.1) and (7.3.1) considered above. Not only, in fact, are these equations nonlinear – which suggests that we shall find again the same problems met when dealing with Eq. (7.2.9) – but the matter source is also inextricably linked to the metric, which appears in the expression for T_{ab}; this can be seen explicitly in Eq. (7.1.27), which contains the factor $(-g(x))^{-1/2}$, and is a consequence of the consistency of the treatment, which now accounts also for self-reaction of the source. These effects cannot be removed, as we did in the cases of the scalar and electromagnetic fields, simply assuming the source to consist of a pointlike particle, and appealing to the structureless of this latter for justifying the absence of self-interaction; in fact, as we have already remarked in Sec. 7.1, pointlike particles have no acceptable status as sources of gravity, because they involve a strongly singular spacetime; they can only be used as a schematization of the concept of test particle, as we did in the discussion following Eq. (7.1.31). Another possibility for not taking into account the dependence of T_{ab} from g_{ab} could be to assume that the self-gravity of the source is small, and to approximate all the $g_{ab}(x)$ appearing in the expression of T_{ab} by the flat metric components η_{ab} plus small corrections $h_{ab}(x)$; this is a reasonable idea, and it presents the advantage of leading to linearized field equations. However, due to the presence of the metric η_{ab}, it is more appropriate for studying the semiclassical perturbations about a classical determined background than for the analysis of a spacetime whose matter content is entirely quantum. It seems therefore that we have to face the problem of formulating a semiclassical theory of gravity without making any

simplifying hypothesis.
As usual, let γ be a history of the matter source which is compatible with the classical field, i.e., such that

$$R_{ab}(x|\gamma) = \kappa \tilde{T}_{ab}(x|\gamma) , \tag{7.4.5}$$

where $R_{ab}(x|\gamma)$ is constructed from a metric $g_{ab}(x|\gamma)$ according to the prescriptions of Eqs. (7.4.3) and (7.4.4); in other words, γ turns out to label pairs of tensors, $T_{ab}(x|\gamma)$ and $g_{ab}(x|\gamma)$, satisfying Einstein equation. It is useful, at this point, to make use of the integral representation of Eq. (7.4.1), which allows to express the metric $g_{ab}(x)$ in the form [121,122,123]

$$\begin{aligned} g_{ab}(x) = & -2\kappa \int_N d\Omega(x')\, D_{ab}{}^{a'b'}(x,x')\, \tilde{T}_{a'b'}(x') - \\ & - \int_{\partial N} d\Sigma(x')\, n^{c'}(x')\, \nabla'_{c'} D_{ab}{}^{a'b'}(x,x')\, g_{a'b'}(x') , \end{aligned} \tag{7.4.6}$$

where N is a region of spacetime with boundary ∂N, $d\Omega$ and $d\Sigma$ are, respectively, the measures on N and ∂N, n^a is the normal vector to ∂N, and $D_{ab}{}^{a'b'}(x,x')$ is a Green bitensor which we shall define more precisely in the following discussion. The existence of the representation (7.4.6) for the Einstein equations is quite surprising, because of their nonlinearity; it turns out, however, that such a nonlinearity is not of a particularly intractable kind, and this is what allows Eq. (7.4.6) to be written. To understand this point, let us define the differential operator

$$E_{ab}{}^{cd}[g] \equiv \frac{1}{2} \delta_{(a}{}^{(c} \delta_{b)}{}^{d)} \Box + R_{(a\ b)}^{\ (c\ d)} , \tag{7.4.7}$$

where $(\ldots)$ denotes complete symmetrization [3], and

$$\Box \equiv g^{ab} \nabla_a \nabla_b . \tag{7.4.8}$$

It is trivial to check that Eq. (7.4.7), when applied to the *same* metric tensor g_{ab} appearing in it, gives

$$E_{ab}{}^{cd}[g]\, g_{cd} = R_{ab}[g] . \tag{7.4.9}$$

Considering a metric differing from g_{ab} by a small perturbation δg_{ab}, one can check that, to the first order in δg_{ab},

$$E_{ab}{}^{cd}[g]\, (g_{cd} + \delta g_{cd}) \approx R_{ab}[g] + \delta R_{ab}[g] , \tag{7.4.10}$$

where $\delta R_{ab}[g]$ is the perturbation in the Ricci tensor induced by the perturbation δg_{ab} of the metric, when the background metric is g_{ab}. Making use of Eq. (7.4.1), and identifying δR_{ab} as due to a small perturbation $\delta \tilde{T}_{ab}$ in the matter distribution, we can write Eqs. (7.4.9) and (7.4.10) as

$$E_{ab}{}^{cd}[g]\, g_{cd} = \kappa\, \tilde{T}_{ab} \ , \tag{7.4.11}$$

$$E_{ab}{}^{cd}[g]\, \delta g_{cd} \approx \kappa\, \delta \tilde{T}_{ab} \ . \tag{7.4.12}$$

Eq. (7.4.12) contains the key idea which lies at the basis of Eq. (7.4.6): Although the correspondence between $\tilde{T}_{ab}$ and g_{ab} is nonlinear, as it is clear by Eqs. (7.4.7) and (7.4.11), the *perturbations* $\delta \tilde{T}_{ab}$ and δg_{ab} are linearly related by Eq. (7.4.12); the left hand side of the latter contains, in fact, the differential operator $E_{ab}{}^{cd}[g]$ evaluated *on the unperturbed* spacetime. This expresses the so-called *quasilinearity* of Einstein equation; physically, we may think to the metric at a point of spacetime as built up adding small contributions due to the various elements of the source, each one propagated throughout the spacetime due to all the other elements. More precisely, if $\delta\Omega(x')$ is a small volume of spacetime at x', contributing to the source by an amount $\tilde{T}_{a'b'}(x')\,\delta\Omega(x')$, its contribution to the metric at x will be

$$\delta g_{ab}(x) \propto D_{ab}{}^{a'b'}(x,x')\,\tilde{T}_{a'b'}(x')\,\delta\Omega(x') \ , \tag{7.4.13}$$

where the propagator $D_{ab}{}^{a'b'}(x,x')$ depends on the metric g_{ab} of spacetime; Eq. (7.4.13) gives an heuristic explanation of Eq. (7.4.6).

A formal derivation of Eq. (7.4.6) require to define, first, the bitensor $D_{ab}{}^{a'b'}(x,x')$; this can be done requiring that

$$E_{ab}{}^{cd}(x)\, D_{cd}{}^{a'b'}(x,x') = -\Gamma_{(a}{}^{(a'}(x,x')\Gamma_{b)}{}^{b')}(x,x')\,\delta\,(x,x') \ , \tag{7.4.14}$$

where $\Gamma_a{}^{a'}(x,x')$ is the bivector of parallel geodesic transport [124,125], and $\delta\,(x,x')$ is the delta function on spacetime. The support of the Green function $D_{ab}{}^{a'b'}(x,x')$ does not consist only of pairs of points which can be connected by a null curve, but contains also diffusive contributions from inside the light cone [123]; the reason for the existence of these latter is the presence of the $R_{(a\ \ b)}^{\ \ (c\ \ d)}$ term in Eq. (7.4.7), which acts as a "mass term" in the propagation of the perturbations δg_{ab}, as it is clear from Eq. (7.4.12). It is not difficult to verify now, following ref. [123], that Eqs. (7.4.10), (7.4.14) and (7.4.1) imply Eq. (7.4.6).

The integral representation (7.4.6) has been used until now mainly as a way to introduce Mach's principle into the general theory of relativity, as an appropriate choice of the boundary conditions [126]; for example, it has been suggested [127] that the boundary term should vanish when N is the entire spacetime. We shall not enter into these arguments here, but we would like to make a remark which we find at least curious. There is a rather close analogy between Mach's principle and the WSH: The former requires that the structure of spacetime be completely determined by its matter content, while the latter expresses the idea that spacetime should have no quantum behaviour of its own, and that the probability distribution for the metric is totally determined by the corresponding one for matter. From this point of view, it is not incorrect to consider the WSH as a kind of "quantum Mach's principle"! It might be interesting to speculate, along these lines, about the status of quantum gravity, as it seems that Mach's principle does not allow the metric to be quantized; in fact, in this case, spacetime would possess physical properties which cannot be ascribed to its matter content. Semiclassical gravity would be thus necessarily a fundamental theory of nature. However, the problem should be analyzed in more details before jumping to any conclusion, and we leave it aside for future investigations, coming back to our main topic, which we left at Eq. (7.4.5).

Thanks to the representation (7.4.6), we can now rewrite Eq. (7.4.5) as

$$g_{ab}(x|\gamma) = -2\kappa \int_M d\Omega(x'|\gamma)\, D_{ab}{}^{a'b'}(x,x'|\gamma)\, \tilde{T}_{a'b'}(x|\gamma) \; , \qquad (7.4.15)$$

where M denotes the entire spacetime, and we have stressed the dependence of the measure and of the Green function from the metric and, consequently from the history of the matter system, labeling them by γ. In Eq. (7.4.15) the boundary integral has been assumed to vanish; this choice is not motivated by Mach's principle, but rather by a wish for simplicity (see Sec. 4.4), and is essentially equivalent to any other for what the following discussion is concerned with.

Introducing the usual joint quasiprobability distribution we can write

$$P(g(x);\gamma) = P(g(x)|\gamma)\, P[\gamma] \; , \qquad (7.4.16)$$

where the WSH fixes the conditional probability as

$$P(g(x)|\gamma) = {}_s\delta_2^0(g(x) - g(x|\gamma)) \ , \tag{7.4.17}$$

with ${}_s\delta_2^0$ the delta function on the space of the symmetric tensors of type $(0\,2)$, which can be defined analogously to δ^4 by products of elementary delta functions. Eqs. (7.4.16) and (7.4.17) allow to write the probability distribution for the metric as

$$P(g(x)) = \int_{\Gamma_0} \mathcal{D}\gamma \ {}_s\delta_2^0(g(x) - g(x|\gamma))\, P[\gamma] \ , \tag{7.4.18}$$

where Γ_0 is the set of all the possible matter histories which are compatible with classical gravity. The ensemble average of g_{ab} at the point x of spacetime can be obtained by integration over the above-mentioned space of tensors; one gets, by Eq. (7.4.18),

$$\begin{aligned} \langle g_{ab}(x)\rangle &\equiv \int dg(x)\, P(g(x))\, g_{ab}(x) = \\ &= \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, g_{ab}(x|\gamma) \ . \end{aligned} \tag{7.4.19}$$

It is interesting to try to recover, from Eq. (7.4.19), a semiclassical version of Eq. (7.4.1); the easiest way for doing that is to apply to Eq. (7.4.19) the differential operator defined in Eq. (7.4.7), but relative to the average metric $\langle g\rangle$. We find, by Eq. (7.4.9),

$$E_{ab}{}^{cd}[\langle g\rangle]\, \langle g_{cd}(x)\rangle = R_{ab}[\langle g(x)\rangle] \ , \tag{7.4.20}$$

and, by Eq. (7.4.19),

$$E_{ab}{}^{cd}[\langle g\rangle]\, \langle g_{cd}(x)\rangle = \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, E_{ab}{}^{cd}[\langle g\rangle]\, g_{cd}(x|\gamma) \ , \tag{7.4.21}$$

and we can conclude that

$$R_{ab}[\langle g(x)\rangle] = \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, E_{ab}{}^{cd}[\langle g\rangle]\, g_{cd}(x|\gamma) \ . \tag{7.4.22}$$

Like in the case of Eq. (7.2.9), we see that, as expected, Eq. (7.4.22) does not reduce to the form

$$R_{ab}[\langle g(x)\rangle] \approx \kappa \int_{\Gamma_0} \mathcal{D}\gamma\, P[\gamma]\, \tilde{T}_{ab}(x|\gamma) \tag{7.4.23}$$

except in the strongly semiclassical regime, in which only histories corresponding to approximately the same metric have a nonvanishing $P[\gamma]$; for these histories we have, in fact,

$$E_{ab}{}^{cd}[\langle g\rangle]\, g_{cd}(x|\gamma) \approx R_{ab}(x|\gamma) \;, \tag{7.4.24}$$

which leads to Eq. (7.4.23). We point out, however, that in the context of gravity equations like Eq. (7.4.21) have a meaning which is essentially of theoretical and formal nature, since the ensemble over which $\langle g_{ab}(x)\rangle$ represents an average consists of a set of spacetimes, which can hardly be endowed with operational properties.

As mentioned in Ch. 1, g_{ab} plays a double role in any metric theory of gravity; it defines what is essentially the generalization of the newtonian potential Φ, and at the same time it determines the metric structure of spacetime. This raises a problem in understanding Eq. (7.4.16): The quasiprobability functional $P[\gamma]$ given by Eq. (6.3.1) contains the wave function $\Psi(x,\lambda)$, which should be normalized, by a generalization of Eqs. (6.2.4) and (6.2.5) to curved spacetime, as

$$\int_M d^4x \sqrt{-g(x)}\, |\Psi(x,\lambda)|^2 = 1 \;; \tag{7.4.25}$$

but, if we accept that $\Psi(x,\lambda)$ define an ensemble of particles, each one of them compatible with a different spacetime, which metric shall we adopt in Eq. (7.4.20)?

Although formulated for the case of a particle, which is, as seen, not a consistent one in general relativity, this problem can be straightforwardly generalized to situations of physical interest, and has therefore to be considered seriously. Our suggestion is to regard the sum-over-histories formalism of quantum theory as more fundamental than the usual one, and susceptible to be applied even to circumstances in which the latter fails. A similar opinion has been expressed by Hartle [128], who has proved, within a more conventional framework than ours, that the path integral approach allows a Schrödinger-Heisenberg formulation on a hypersurface (see Sec. 2.2) only if this one satisfies some specific conditions, which cannot be fulfilled in a generic curved spacetime. If such a viewpoint should turn out to be correct, the use of the concept of state vector, and consequently of the wave function, in the context of semiclassical gravity would be simply inadequate, and the conceptual problem related to Eq. (7.4.20) would be automatically removed. It seems, nevertheless, that the entire problem of formulating quantum mechanics in a curved spacetime is still open; we hope that a generalization of the treatment of Ch. 6, possibly improved with the ideas related to Eqs. (5.3.13)–(5.3.18), could be useful in order to construct a consistent theory.

Chapter 8

Outlooks and Conclusions

In this thesis a scheme has been suggested for treating semiclassical systems in a consistent way, with particular emphasis on the case of gravity. As realized quite soon, the practical achievement of such a program requires to discuss, clarify, and sometimes revise, several topics in physics, apparently far from each other.

Starting from a critical analysis of the usual formulation of semiclassical theories, which we have found essentially ill-posed from a conceptual point of view, we have realized the need to gain physical insight into the foundations of quantum mechanics. Generally, this is done considering the classical paradoxes of the theory, and explaining how they find a solution adopting one or another interpretation; we have followed a completely different approach, reformulating the Schrödinger equation as a set of hydrodynamical equations, and looking for a reasonable interpretation of the corresponding "fluid" quantities. Our main result has been the emergence, from such a treatment, of very strong evidence in favour of the *statistical interpretation*; moreover, some other interesting features have been revealed. Reconsidering the semiclassical problem in the light of these ideas, we have been led to identify, within the the semiclassical regime, a strongly and a weakly semiclassical behaviour; of these two, the latter is the one which is closer to the full quantum behaviour, and corresponds to the breakdown of the concept of coupling equations (of which the field equations represents an example). In the weakly semiclassical regime, in fact, even the classical subsystem can be characterized only by a probability distribution for the values of its observables; nevertheless, these observables are not quantum, and such a distribution is thus totally induced by the interaction with the quantum subsystem.

After having given a prescription (WSH) for specifying quantitatively this relation, we have realized the inadequacy of the standard quantum formalism in treating weakly semiclassical systems. We have thus suggested to adopt an alternative formulation, which makes use of the path integral methods and introduces explicitly a quasiprobability functional for paths – a new concept in quantum mechanics. This formalism allows to recover, in the nonrelativistic theory, the usual expressions for the phase space distributions, and sheds light on the connections between operator ordering, path integration and quantum distribution functions; moreover, it leads to consider a relativistic version of quantum mechanics which is explicitly covariant.

Applying these techniques to the cases of semiclassical newtonian gravity and of semiclassical relativistic fields (scalar field, electromagnetism, gravity), we have found a mathematical confirmation of some previously predicted features; in particular, we have shown how the semiclassical field equations hold only for a statistical average of the field, and how they assume a form analogous to that of the classical equations only in the very special cases either of a linear theory or of a strongly semiclassical behaviour. A semiclassical theory turns out to be, therefore, essentially a theory for the probability distribution of the field values, distribution which can be explicitly determined with the methods suggested in this thesis.

The theory presented is still at a rather preliminary stage, and we expect that it could be further developed. In particular, it should be interesting to try to formulate a weakly semiclassical theory of electromagnetism which takes radiation reaction into account; this might lead to some pleasant surprises, since semiclassical theories which consider self-interaction effects are, in general, rather rich in their phenomenology; for example, Barut's theory [22,23], which is not as accurate as ours (it assumes $|\psi|^2$ as individual source), succeeds in predicting the correct value for the magnetic moment of the electron, as well as the Lamb shift and even the Unruh effect [129].

A possible application which is not of formal character is the use of weakly semiclassical gravity for predicting the spectrum of primordial fluctuations in the early universe; the estimate is essentially reduced to the evaluation of the probability distribution $P(g(x))$ for some cosmological model filled with quantum matter. It is remarkable that this calculation, which cannot be performed at all in the usual theory of semiclassical gravity [53], appears very natural within our formulation. However, we feel that an actual calculation of $P(g(x))$ would turn out to be very complicated, and that the real importance of the theory presented here has therefore to be recognized essentially at the formal level.

Is this theory correct? Of course, only experiments and/or observations could say something definitive about this point. For what concerns the case of gravity, we think that the situation might be considered almost hopeless; however, it is not impossible to invent some practically realizable crucial experiment for the case of electromagnetism, which could carry information about the validity of the general method.

In the absence of experimental data, we can observe that our theory presents, from a structural point of view, at least three advantages:

1. it is logically and formally consistent;
2. it makes a very small number of physical hypothesis;
3. each of the hypothesis of the point 2 is strongly motivated.

It is worth remarking, moreover, that a semiclassical theory would turn out to be useful even if gravity were quantized, as an *effective* treatment describing the semiclassical regime. Our scheme not only seems particularly suitable for this purpose, but it contains also information about some features of the full quantum theory; in particular, it predicts that the concept of field equations, determining the values of the metric and of the matter fields, will be superseded by a prescription assigning the *probability distribution* for those values.

The classical relativistic fields we have considered in Ch. 7 are, from the point of view of a covariant hamiltonian formulation, not dynamical; in fact, as it is clear from Sec. 6.1, the true dynamical variable is not t but λ. However, one can extend the treatment of fields to include their dynamics,simply considering them as the continuum limit of a many-particle system. It would then be straightforward to apply the ideas of Ch. 6 in order to construct an explicitly covariant relativistic quantum field theory, differing from the usual one by the presence of the "evolutionary time" λ; this might be of some use in solving the serious conceptual problems about the role of time in quantum gravity [130].

However, we believe that before considering such exotic extensions of the formalism, we should have the ideas very clear about the meaning of quantum mechanics in general. In fact, if this latter is a theory which makes predictions only at the level of ensembles, such topics as quantum cosmology are essentially meaningless. For these reasons we think it is important to perform further investigations along the directions suggested in Sec. 3.3.

Appendices

A Operator Ordering

The main formal difference between classical and quantum mechanics can be identified in the fact that the operators representing conjugate quantum observables do not commute. In the exemplary case of position and momentum of a particle, we have

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\hat{1} \ . \tag{A.1}$$

This circumstance makes the problem of associating quantum operators to physical observables intrinsically ill-defined. To be more explicit, let us restrict ourselves to consider the one-dimensional case (the generalization being straightforward), and ask for the quantum version of the classical observable x^2p^2. Since the answer must be a self-adjoint operator, one possibility is

$$\hat{Q}_1 = \frac{1}{2}(\hat{x}^2\hat{p}^2 + \hat{p}^2\hat{x}^2) \ . \tag{A.2}$$

However, it is easy to realize that also

$$\hat{Q}_2 = \hat{x}\hat{p}^2\hat{x} \ , \tag{A.3}$$

$$\hat{Q}_3 = \frac{1}{4}(\hat{x}^2\hat{p}^2 + 2\hat{x}\hat{p}^2\hat{x} + \hat{p}^2\hat{x}^2) \ , \tag{A.4}$$

$$\hat{Q}_4 = \frac{1}{3}(\hat{p}^2\hat{x}^2 + \hat{p}\hat{x}^2\hat{p} + \hat{x}^2\hat{p}^2) \ , \tag{A.5}$$

or other operators, could equally well be assumed to represent such an observable. In fact, the commutation relation (A.1) implies

$$\hat{Q}_2 = \hat{Q}_1 + \hbar^2\hat{1} \ , \tag{A.6}$$

$$\hat{Q}_3 = \hat{Q}_1 + \frac{1}{2}\hbar^2 \hat{1} \ , \tag{A.7}$$

and

$$\hat{Q}_4 = \hat{Q}_1 - \hbar^2 \hat{1} \ ; \tag{A.8}$$

in the classical limit, the terms in $\hbar^2$ are negligible, and the different operators $\hat{Q}_1$, $\hat{Q}_2$, $\hat{Q}_3$ and $\hat{Q}_4$ all "degenerate" in the classical observable $x^2 p^2$.

It is a problem of primary importance in quantum theory, to remove this arbitrariness, assigning a well defined prescription for constructing the quantum mechanical operator corresponding to a given classical quantity. Among the rules which have been suggested to this purpose, only three have proved to be essentially free of contradictions [131,80]; they are

a) Weyl's prescription [132]

$$\mathcal{W}\{\hat{x}^n \hat{p}^m\} \equiv \frac{\sum(\text{all possible orders})}{\text{number of all possible orders}} \ ; \tag{A.9}$$

or equivalently, using Eq. (A.1),

$$\mathcal{W}\{\hat{x}^n \hat{p}^m\} = \frac{1}{2^n} \sum_{l=0}^{n} \begin{pmatrix} n \\ l \end{pmatrix} \hat{x}^{n-l} \hat{p}^m \hat{x}^l \ ; \tag{A.10}$$

b) Rivier's symmetrization rule

$$\mathcal{S}\{\hat{x}^n \hat{p}^m\} \equiv \frac{1}{2}(\hat{x}^n \hat{p}^m + \hat{p}^m \hat{x}^n) \ ; \tag{A.11}$$

c) Born-Jordan's prescription

$$\mathcal{B}\{\hat{x}^n \hat{p}^m\} \equiv \frac{1}{m+1} \sum_{l=0}^{m} \hat{p}^{m-l} \hat{x}^n \hat{p}^l \ . \tag{A.12}$$

It is easy to check that the operators $\hat{Q}_1$, $\hat{Q}_3$ and $\hat{Q}_4$ are examples, in the case $n = m = 2$, respectively of the symmetrization, the Weyl's and the Born-Jordan's prescriptions. This shows explicitly that these three rules define operators which are indeed different from each other for generic n and m.

In the main discussion of this thesis, we shall be concerned with the orderings $\mathcal{W}$ and $\mathcal{S}$, mainly because it has been shown [133,79] that they correspond to the use of, respectively, the Wigner and the Margenau-Hill functions in the phase space formulation of quantum theory. It is the purpose of the present appendix to give a slight generalization of Eqs. (A.9) and (A.11), which may allow to define the Weyl's and Rivier's orderings even when operators such as $\delta(x\hat{1} - \hat{x})$ are involved.

The key of our strategy is to expand $\hat{x}$ and $\hat{p}$ making use of the identities

$$\int dx\, \hat{\Pi}(x) = \hat{1} \tag{A.13}$$

and

$$\int dp\, \hat{\Pi}(p) = \hat{1}\ , \tag{A.14}$$

where $\hat{\Pi}(x) \equiv |x\rangle\langle x|$ and $\hat{\Pi}(p) \equiv |p\rangle\langle p|$; one obtains

$$\hat{x}^n \hat{p}^m = \int dx_1 \ldots dx_n dp_1 \ldots dp_m\, \hat{\Pi}(x_1)\cdots\hat{\Pi}(x_n)\hat{\Pi}(p_1)\cdots\hat{\Pi}(p_m)\ . \tag{A.15}$$

The problem of ordering the operators $\hat{x}$ and $\hat{p}$ is thus related to that of ordering the product of the generalized projections $\hat{\Pi}(x_\alpha)$, $\hat{\Pi}(p_\beta)$, for $\alpha \in \{1, \ldots, n\}$, $\beta \in \{1, \ldots, m\}$.

It is easy to see that, choosing

$$\begin{aligned} &\mathcal{S}\{\hat{\Pi}(x_1)\cdots\hat{\Pi}(x_n)\hat{\Pi}(p_1)\cdots\hat{\Pi}(p_m)\} \equiv \\ &\quad \equiv \frac{1}{2}(\hat{\Pi}(x_1)\cdots\hat{\Pi}(x_n)\hat{\Pi}(p_1)\cdots\hat{\Pi}(p_m) + \\ &\quad +\hat{\Pi}(p_1)\cdots\hat{\Pi}(p_m)\hat{\Pi}(x_1)\cdots\hat{\Pi}(x_n))\ , \end{aligned} \tag{A.16}$$

one reproduces, from (A.15), the Rivier's rule (A.11). Similarly, defining

$$\begin{aligned} &\mathcal{W}\{\hat{\Pi}(x_1)\cdots\hat{\Pi}(x_n)\hat{\Pi}(p_1)\cdots\hat{\Pi}(p_m)\} \equiv \\ &\quad \equiv \frac{1}{(n+m)!}\sum \mathrm{Perm}\{\hat{\Pi}(x_1)\cdots\hat{\Pi}(x_n)\hat{\Pi}(p_1)\cdots\hat{\Pi}(p_m)\}\ , \end{aligned} \tag{A.17}$$

where $\sum \mathrm{Perm}\{\cdots\}$ stands for the sum of all the possible permutations of the argument, and observing that

$$[\hat{\Pi}(x_{\alpha_1}), \hat{\Pi}(x_{\alpha_2})] = [\hat{\Pi}(p_{\beta_1}), \hat{\Pi}(p_{\beta_2})] = 0\ , \tag{A.18}$$

for all $\alpha_1, \alpha_2 \in \{1, \dots n\}$, $\beta_1, \beta_2 \in \{1, \dots m\}$, one realizes that

$$\mathcal{W}\{\hat{\Pi}(x_1)\cdots\hat{\Pi}(x_n)\hat{\Pi}(p_1)\cdots\hat{\Pi}(p_m)\} = \frac{\sum(\text{all possible orders})}{\text{number of all possible orders}} ; \tag{A.19}$$

in the numerator of the right hand side of Eq. (A.19), terms differing only by the exchange of generalized projectors of the same observable (position or momentum) are identified. It is easy to check that Eq. (A.19), together with Eq. (A.15), reproduces the Weyl's rule (A.9). Hence, we believe it is justified to regard Eqs. (A.16) and (A.19) as the correct generalizations of the Rivier's and Weyl's ordering rules.

B Calculation of $\langle\psi|\hat{\Pi}_{ij}|\psi\rangle$

In calculating the expectation value $\langle\psi|\hat{\Pi}_{ij}|\psi\rangle$ of the stress tensor operator $\hat{\Pi}_{ij}$, one has to decide first which ordering to adopt in the expression (2.3.21); in this case, in fact, the quantities obtained using $\mathcal{W}$ or $\mathcal{S}$ do not coincide. In this appendix we shall deal only with the technical calculation of both, leaving to the main text a discussion of the possible physical reasons for preferring the one or the other of them.

Let us start writing down, according to Eqs. (A.19) and (A.16), the explicit expressions of $\hat{\Pi}_{ij}(\mathbf{x}, t)$ under the Weyl's and the Rivier's prescriptions; in the Heisenberg picture they are, respectively:

$$\begin{aligned}\hat{\Pi}^{\mathcal{W}}_{ij}(\mathbf{x}, t) &\equiv \mathcal{W}\left\{\hat{\mu}(\mathbf{x}, t)\,\frac{1}{m^2}\hat{p}_i(t)\hat{p}_j(t)\right\} = \\ &= \frac{1}{4m}\left(|\mathbf{x}, t\rangle\langle\mathbf{x}, t|\,\hat{p}_i(t)\hat{p}_j(t) + \hat{p}_i(t)\,|\mathbf{x}, t\rangle\langle\mathbf{x}, t|\,\hat{p}_j(t) + \right.\\ &\left.+\hat{p}_j(t)\,|\mathbf{x}, t\rangle\langle\mathbf{x}, t|\,\hat{p}_i(t) + \hat{p}_i(t)\hat{p}_j(t)\,|\mathbf{x}, t\rangle\langle\mathbf{x}, t|\right) ,\end{aligned} \tag{B.1}$$

and

$$\begin{aligned}\hat{\Pi}^{\mathcal{S}}_{ij}(\mathbf{x}, t) &\equiv \mathcal{S}\left\{\hat{\mu}(\mathbf{x}, t)\,\frac{1}{m^2}\hat{p}_i(t)\hat{p}_j(t)\right\} = \\ &= \frac{1}{2m}\left(|\mathbf{x}, t\rangle\langle\mathbf{x}, t|\,\hat{p}_i(t)\hat{p}_j(t) + \hat{p}_i(t)\hat{p}_j(t)\,|\mathbf{x}, t\rangle\langle\mathbf{x}, t|\right) .\end{aligned} \tag{B.2}$$

It is convenient to define the auxiliary operators

$$\hat{A}_{ij}(\mathbf{x}, t) \equiv |\mathbf{x}, t\rangle\langle\mathbf{x}, t|\,\hat{p}_i(t)\hat{p}_j(t) \tag{B.3}$$

and

$$\hat{B}_{ij}(\mathbf{x},t) \equiv \hat{p}_i(t)\,|\mathbf{x},t\rangle\langle\mathbf{x},t|\,\hat{p}_j(t)\ , \tag{B.4}$$

in terms of which $\hat{\Pi}^{W}_{ij}$ and $\hat{\Pi}^{S}_{ij}$ can be represented as

$$\hat{\Pi}^{W}_{ij}(\mathbf{x},t) = \frac{1}{4m}\left(\hat{A}_{ij}(\mathbf{x},t) + \hat{B}_{ij}(\mathbf{x},t) + \hat{B}_{ij}(\mathbf{x},t)^{\dagger} + \hat{A}_{ij}(\mathbf{x},t)^{\dagger}\right)\ , \tag{B.5}$$

and

$$\hat{\Pi}^{S}_{ij}(\mathbf{x},t) = \frac{1}{2m}\left(\hat{A}_{ij}(\mathbf{x},t) + \hat{A}_{ij}(\mathbf{x},t)^{\dagger}\right)\ ; \tag{B.6}$$

it is thus clear that it suffices to calculate the two expectation values $\langle\psi|\hat{A}_{ij}(\mathbf{x},t)|\psi\rangle$ and $\langle\psi|\hat{B}_{ij}(\mathbf{x},t)|\psi\rangle$. Using Eqs. (2.1.15), (2.1.12) and (2.1.20), one gets

$$\begin{aligned}\langle\psi|\hat{A}_{ij}(\mathbf{x},t)|\psi\rangle &= \hbar^2\,\psi(\mathbf{x},t)^* \int d^3y\,\frac{\partial}{\partial y_i}\langle\mathbf{x},t|\mathbf{y},t\rangle\,\frac{\partial\psi(\mathbf{y},t)}{\partial y_j} = \\ &= -\hbar^2\,\psi(\mathbf{x},t)^*\partial_i\partial_j\psi(\mathbf{x},t)\ ,\end{aligned} \tag{B.7}$$

and

$$\langle\psi|\hat{B}_{ij}(\mathbf{x},t)|\psi\rangle = \hbar^2\,\partial_i\psi(\mathbf{x},t)^*\partial_j\psi(\mathbf{x},t)\ . \tag{B.8}$$

Eqs. (B.7) and (B.8) can be reduced to a more interesting form multiplying and dividing their right hand side by $\psi^*\psi$, and using (for Eq. (B.7)) the identity

$$\frac{\partial_i\partial_j F}{F} = \partial_i\partial_j \ln F + \partial_i \ln F\,\partial_j \ln F\ , \tag{B.9}$$

with F a generic function; one gets, respectively,

$$\langle\psi|\hat{A}_{ij}|\psi\rangle = -\frac{\hbar^2}{m}\langle\psi|\hat{\mu}|\psi\rangle\partial_i\partial_j\ln\psi - \frac{\hbar^2}{m}\langle\psi|\hat{\mu}|\psi\rangle\partial_i\ln\psi\partial_j\ln\psi\ , \tag{B.10}$$

and

$$\langle\psi|\hat{B}_{ij}|\psi\rangle = \frac{\hbar^2}{m}\langle\psi|\hat{\mu}|\psi\rangle\partial_i\ln\psi^*\partial_j\ln\psi\ , \tag{B.11}$$

where Eq. (2.3.12) has been used. Remembering now Eqs. (B.5) and (B.6), we can write

$$\langle\psi|\hat{\Pi}^{W}_{ij}|\psi\rangle = -\frac{\hbar^2}{4m^2}\langle\psi|\hat{\mu}|\psi\rangle\left(\partial_i\partial_j\ln\langle\psi|\hat{\mu}|\psi\rangle + \partial_i\ln\frac{\psi}{\psi^*}\partial_j\ln\frac{\psi}{\psi^*}\right)\ , \tag{B.12}$$

$$\langle\psi|\hat{\Pi}^s_{ij}|\psi\rangle = -\frac{\hbar^2}{2m^2}\langle\psi|\hat{\mu}|\psi\rangle(\partial_i\partial_j \ln\langle\psi|\hat{\mu}|\psi\rangle + \partial_i \ln\psi\,\partial_j \ln\psi + \partial_i \ln\psi^*\partial_j \ln\psi^*), \tag{B.13}$$

which, by Eqs. (2.3.15) and (2.3.17), give finally:

$$\langle\psi|\hat{\Pi}^W_{ij}|\psi\rangle = p_{ij} + \langle\psi|\hat{\mu}|\psi\rangle v_i v_j \;, \tag{B.14}$$

and

$$\langle\psi|\hat{\Pi}^s_{ij}|\psi\rangle = p_{ij} - \frac{\hbar^2}{4m^2}\partial_i\partial_j\langle\psi|\hat{\mu}|\psi\rangle + \langle\psi|\hat{\mu}|\psi\rangle v_i v_j \;, \tag{B.15}$$

where Eq. (2.3.23) has been used.

C Calculation of p_{ij} and T

In this appendix we shall perform explicitly the integrals in Eqs. (3.2.21) and (3.2.22), in order to calculate the quantities $p_{ij}(\mathbf{x},t)$ and $T(\mathbf{x},t)$; let us begin with Eq. (3.2.21). With the change of variables from $\mathbf{p}$ to $\mathbf{p}'$, defined as

$$\mathbf{p}' \equiv \mathbf{p} - m\,\mathbf{v}(\mathbf{x},t) \;, \tag{C.1}$$

and writing explicitly the expression (3.2.17) for the Wigner function, (3.2.21) becomes

$$-\frac{\hbar^2}{4m}\int d^3\xi\,\delta^3(\xi)\,\frac{\partial^2}{\partial\xi_i\,\partial\xi_j}\left[\psi(\mathbf{x}+\xi,t)^*\psi(\mathbf{x}-\xi,t)\,e^{2im\mathbf{v}(\mathbf{x},t)\cdot\xi/\hbar}\right] \;, \tag{C.2}$$

where few properties of the delta function have been used. Performing the derivatives in (C.2) we get

$$-\frac{\hbar^2}{4m}(\partial_i\partial_j\psi^*\psi + \psi^*\partial_i\partial_j\psi - \partial_i\psi^*\partial_j\psi - \partial_j\psi^*\partial_i\psi + \frac{2im}{\hbar}v_i\,\partial_j\psi^*\psi -$$
$$-\frac{2im}{\hbar}v_i\,\psi^*\partial_j\psi + \frac{2im}{\hbar}v_j\,\partial_i\psi^*\psi - \frac{2im}{\hbar}v_j\,\psi^*\partial_i\psi - \frac{4m^2}{\hbar^2}v_iv_j\,\psi^*\psi). \tag{C.3}$$

We can simplify (C.3) multiplying and dividing it by μ; the result is

$$-\frac{\hbar^2\mu}{4m^2}\Bigg(\partial_i\partial_j\ln\mu + \partial_i\ln\frac{\psi^*}{\psi}\,\partial_j\ln\frac{\psi^*}{\psi} +$$
$$+\frac{2im}{\hbar}v_i\,\partial_j\ln\frac{\psi^*}{\psi} + \frac{2im}{\hbar}v_j\,\partial_i\ln\frac{\psi^*}{\psi} - \frac{4m^2}{\hbar^2}v_iv_j\Bigg) \;, \tag{C.4}$$

where the identity (B.9) has been used. Remembering now Eqs. (2.3.15) and (2.3.17), the expression (C.4) reduces to

$$-\frac{\hbar^2}{4m^2}\,\mu\,\partial_i\partial_j\ln\mu\;, \tag{C.5}$$

which coincides with the pressure tensor (3.1.11).

The temperature field $T(\mathbf{x},t)$ can be calculated noticing that, by a comparison between the definition given in the first line of Eq. (3.2.22) and the expression (3.2.21), it follows that

$$T = \frac{m}{3k\,\mu}\,\mathrm{tr}\,\mathsf{p} = -\frac{\hbar^2}{12km}\nabla^2\ln\mu\;, \tag{C.6}$$

which is exactly the result quoted in Eq. (3.2.22).

D Calculation of the Heat Flux Vector

To prove the result stated in Eq. (3.2.23), let us change variables as in Eq. (C.1) and perform a similar treatment; we find first

$$q_i(\mathbf{x},t) = \frac{\hbar^3}{16im}\int d^3\xi\,\delta^3(\boldsymbol{\xi})\,\frac{\partial^3}{\partial\xi_i\partial\xi_j\partial\xi_j}\left[\psi(\mathbf{x}+\boldsymbol{\xi},t)^*\psi(\mathbf{x}-\boldsymbol{\xi},t)e^{2im\mathbf{v}(\mathbf{x},t)\cdot\boldsymbol{\xi}/\hbar}\right], \tag{D.1}$$

and then

$$\begin{aligned} q_i = \frac{\hbar^3}{16im}\,(&\partial_i\partial_j\partial_j\psi^*\psi - \psi^*\partial_i\partial_j\partial_j\psi + \partial_i\psi^*\partial_j\partial_j\psi - \partial_j\partial_j\psi^*\partial_i\psi + \\ &+2\partial_j\psi^*\partial_i\partial_j\psi - 2\partial_i\partial_j\psi^*\partial_j\psi + \frac{2im}{\hbar}v_i\partial_j\partial_j\psi^*\psi + \frac{2im}{\hbar}v_i\psi^*\partial_j\partial_j\psi + \\ &+\frac{4im}{\hbar}v_j\partial_i\partial_j\psi^*\psi + \frac{4im}{\hbar}v_j\psi^*\partial_i\partial_j\psi - \frac{4im}{\hbar}v_j\partial_j\psi^*\partial_i\psi - \frac{4im}{\hbar}v_j\partial_i\psi^*\partial_j\psi - \\ &-\frac{4im}{\hbar}v_i\partial_j\psi^*\partial_j\psi + \frac{8m^2}{\hbar^2}v_iv_j\psi^*\partial_j\psi - \frac{8m^2}{\hbar^2}v_iv_j\partial_j\psi^*\psi - \frac{4m^2}{\hbar^2}v_jv_j\partial_i\psi^*\psi + \\ &+\frac{4m^2}{\hbar^2}v_jv_j\psi^*\partial_i\psi - \frac{8im^3}{\hbar^3}v_iv_jv_j\psi^*\psi)\;. \end{aligned} \tag{D.2}$$

Multiplying and dividing by μ, and using the identities (C.5) and

$$\begin{aligned} \frac{\partial_i\partial_j\partial_k F}{F} = {}&\partial_i\partial_j\partial_k\ln F + \partial_i\ln F\,\partial_j\partial_k\ln F + \partial_j\ln F\,\partial_k\partial_i\ln F + \\ &+\partial_k\ln F\,\partial_i\partial_j\ln F + \partial_i\ln F\,\partial_j\ln F\,\partial_k\ln F\;, \end{aligned} \tag{D.3}$$

(D.2) becomes

$$q_i = \frac{\hbar^3\mu}{16im^2}\left(\partial_i\partial_j\partial_j \ln\frac{\psi^*}{\psi} + 2\,\partial_i\partial_j \ln\mu\,\partial_j \ln\frac{\psi^*}{\psi} + \partial_j\partial_j \ln\mu\,\partial_i \ln\frac{\psi^*}{\psi} + \right.$$
$$+\partial_j \ln\frac{\psi^*}{\psi}\,\partial_j \ln\frac{\psi^*}{\psi}\,\partial_i \ln\frac{\psi^*}{\psi} + \frac{4im}{\hbar}\,v_j\,\partial_i \ln\frac{\psi^*}{\psi}\,\partial_j \ln\frac{\psi^*}{\psi} +$$
$$+\frac{2im}{\hbar}\,v_i\,\partial_j \ln\frac{\psi^*}{\psi}\,\partial_j \ln\frac{\psi^*}{\psi} + \frac{2im}{\hbar}\,v_i\,\partial_j\partial_j \ln\mu + \frac{4im}{\hbar}\,v_j\,\partial_i\partial_j \ln\mu -$$
$$\left. -\frac{8m^2}{\hbar^2}\,v_i v_j\,\partial_j \ln\frac{\psi^*}{\psi} - \frac{4m^2}{\hbar^2}\,v_j v_j\,\partial_i \ln\frac{\psi^*}{\psi} - \frac{8im^3}{\hbar^3}\,v_i v_j v_j\right) \qquad \text{(D.4)}$$

which, remembering Eqs. (2.3.15) and (2.3.17), reduces to

$$q_i = -\frac{\hbar^2}{8m}\,\mu\,\nabla^2 v_i \,. \qquad \text{(D.5)}$$

E Statistical Dispersion of the Source in Newtonian Semiclassical Gravity

Classical newtonian gravity relies on the field equation (4.3.1); the density of mass is thus the matter's observable which is responsible, in this theory, for the gravitational field. In the present appendix, we shall calculate the statistical dispersion of μ when the source is a quantum particle of mass m, working it out explicitly for a particular class of states. We shall perform all the calculations at a fixed time t_0, thus avoiding to be obliged to distinguish between the Schrödinger and the Heisenberg pictures.

Expanding the generic state $|\psi\rangle$ into eigenstates of position we can write, using Eqs. (2.1.12) and (2.1.15),

$$|\psi\rangle = \int d^3x\,\psi(\mathrm{x})|\mathrm{x}\rangle\ , \qquad \text{(E.1)}$$

so that, by Eqs. (2.3.11) and (2.1.13),

$$\hat{\mu}\,(\mathrm{x})|\psi\rangle = m\,\psi(\mathrm{x})|\mathrm{x}\rangle\ , \qquad \text{(E.2)}$$

which allows one to write the expression (4.1.2) as

$$\Delta\mu\,(\mathrm{x},\mathrm{y})^2 = m^2\,|\psi(\mathrm{x})^*\psi(\mathrm{y})\,\delta^3(\mathrm{x}-\mathrm{y}) - \psi(\mathrm{x})^*\psi(\mathrm{y})\psi(\mathrm{x})\psi(\mathrm{y})^*|\ . \qquad \text{(E.3)}$$

In order to see how a superposition of position gives rise to a dispersion in the mass density, let us consider a situation similar to that envisaged in Sec. 2.4, in which the wave function ψ has support in two disjoint spatial regions A and B. We assume the simple form

$$\psi(\mathbf{x}) = \frac{c_A}{\sqrt{V_A}}\chi_A(\mathbf{x}) + \frac{c_B}{\sqrt{V_B}}\chi_B(\mathbf{x}) \ , \qquad \text{(E.4)}$$

where the complex coefficients c_A and c_B satisfy the normalization condition

$$|c_A|^2 + |c_B|^2 = 1 \ , \qquad \text{(E.5)}$$

and χ_A, χ_B are the characteristic functions, respectively, of A and B. Substituting Eq. (E.4) into Eq. (E.3) we find:

$$\begin{aligned}\Delta\mu\,(\mathbf{x},\mathbf{y})^2 = m^2 \Bigg| &|c_A|^2 \frac{\chi_A(\mathbf{x})}{V_A}\,\delta^3(\mathbf{y}-\mathbf{x}) + |c_B|^2 \frac{\chi_B(\mathbf{x})}{V_B}\,\delta^3(\mathbf{y}-\mathbf{x}) - \\ &-|c_A|^4 \frac{\chi_A(\mathbf{x})}{V_A}\frac{\chi_A(\mathbf{y})}{V_A} - |c_B|^4 \frac{\chi_B(\mathbf{x})}{V_B}\frac{\chi_B(\mathbf{y})}{V_B} - \\ &-|c_A|^2|c_B|^2 \frac{\chi_A(\mathbf{x})}{V_A}\frac{\chi_B(\mathbf{y})}{V_B} - |c_A|^2|c_B|^2 \frac{\chi_B(\mathbf{x})}{V_B}\frac{\chi_A(\mathbf{y})}{V_A}\Bigg| \ , \end{aligned} \qquad \text{(E.6)}$$

where the properties

$$\chi_A(\mathbf{x})^2 = \chi_A(\mathbf{x}) \qquad \text{(E.7)}$$

and

$$\chi_A(\mathbf{x})\chi_B(\mathbf{x}) = 0 \ , \qquad \text{for } \mathsf{A} \cap \mathsf{B} = \emptyset \qquad \text{(E.8)}$$

of the characteristic function have been used. Let us now consider the limits in which the regions A and B shrink their volume to zero, respectively around the points $\mathbf{x}_A \in \mathsf{A}$ and $\mathbf{x}_B \in \mathsf{B}$; this allows to use the property

$$\lim_{\mathsf{A}\to\{\mathbf{x}_A\}} \frac{\chi_A(\mathbf{x})}{V_A} = \delta^3(\mathbf{x}-\mathbf{x}_A) \ . \qquad \text{(E.9)}$$

Physically, the wave function (E.4) turns out to correspond, in this limit, to a superposition of two states, each of them representing a complete localization either at $\mathbf{x}_A$ or at $\mathbf{x}_B$. The limit of Eq. (E.6) gives

$$\Delta\mu\,(\mathbf{x},\mathbf{y})^2 = m^2\,|c_A|^2|c_B|^2$$
$$|(\delta^3(\mathbf{x}-\mathbf{x}_A) - \delta^3(\mathbf{x}-\mathbf{x}_B)) \cdot (\delta^3(\mathbf{y}-\mathbf{x}_A) - \delta^3(\mathbf{y}-\mathbf{x}_B))|, \qquad \text{(E.10)}$$

exhibiting correlations for $\mathrm{x}, \mathrm{y} \in \mathsf{A} \cup \mathsf{B}$ only when both c_A and c_B are different from zero. In particular, when $\mathrm{x} = \mathrm{y}$, we obtain

$$\Delta\mu\left(\mathrm{x},\mathrm{x}\right) = m\,|c_A||c_B|\,|\delta^3(\mathrm{x}-\mathrm{x}_A) - \delta^3(\mathrm{x}-\mathrm{x}_B)|\;. \tag{E.11}$$

F Calculation of $\Delta(x'', x'; m, \varepsilon)$

It is interesting to notice that, in our formalism, the explicit form of $\Delta(x'', x'; m, \varepsilon)$ for the free particle case can be straightforwardly derived from Eq. (6.2.26), thus avoiding the long calculations which would occur performing the integral (6.2.29) [114]. In order to prove explicitly this statement, we need to know the function $\bar{K}^+(x'', x', \lambda)$; since Eq. (6.2.28) contains only gaussian integrals, it is easy to find

$$\bar{K}^+(x'', x', \lambda) = -\frac{\Theta(\lambda)}{(2\pi\hbar\lambda)^2}\exp\left(\frac{i\sigma}{2\hbar\lambda}\right)\;, \tag{F.1}$$

where we have defined

$$\sigma \equiv \eta_{ab}(x''-x')^a(x''-x')^b\;. \tag{F.2}$$

Inserting Eq. (F.1) into Eq. (6.2.26), and performing the change of variable

$$\zeta \equiv \frac{1}{2\hbar\lambda}\;, \tag{F.3}$$

we obtain immediately

$$\Delta(x'', x'; m, \varepsilon) = \frac{-i}{(2\pi)^2}\int_0^{+\infty} d\zeta\,\exp\left(i\sigma\zeta + \frac{i\varepsilon m^2}{4\hbar^2\zeta}\right)\;. \tag{F.4}$$

The integral in Eq. (F.4) belongs to the general case [134]

$$\int_0^{+\infty} d\zeta\,\exp\left(-\alpha\zeta - \frac{\beta}{4\zeta}\right) = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2}} K_1\left(\sqrt{\alpha\beta}\right)\;, \tag{F.5}$$

where K_1 is the Bessel function of imaginary argument [134]. Eq. (F.5) is valid provided $\Re\alpha > 0$ and $\Re\beta \geq 0$; since in our case only the second of these conditions is fulfilled, let us add, consistently with Eq. (6.2.30), an "infinitesimal" positive imaginary part to σ, thus obtaining

$$\Delta(x'', x'; m, \varepsilon) = \frac{-im}{(2\pi)^2\hbar}\left(\frac{\varepsilon}{\sigma + i0}\right)^{\frac{1}{2}} K_1\left(\frac{m}{\hbar}\sqrt{-\varepsilon(\sigma + i0)}\right)\;. \tag{F.6}$$

With the usual prescriptions in taking the square roots of complex arguments, Eq. (F.6) generalizes the known expression of the Feynman propagator [114] to the case in which ε can be also equal to $+1$; the massless case is easily recovered observing that, for small $|z|$,

$$K_1(z) = \frac{1}{z} + R(z) , \tag{F.7}$$

where

$$\lim_{|z|\to 0} R(z) = 0 ; \tag{F.8}$$

in the limit $m \to 0$ we obtain therefore

$$\Delta(x'', x'; 0) = \frac{i}{(2\pi)^2} \cdot \frac{1}{\sigma + i0} , \tag{F.9}$$

which is the well known Feynman Green function for a massless Klein-Gordon equation.

For completeness, we present a calculation, along the same lines of the previous one, of the nonrelativistic function $\Delta(\mathbf{x}'', \mathbf{x}'; E)$ for the free particle case. Being

$$\begin{aligned} \bar{K}(\mathbf{x}'', \mathbf{x}', t) &= \langle \mathbf{x}'' | \exp(-i\hat{H}t/\hbar) | \mathbf{x}' \rangle = \\ &= \frac{1}{(2\pi\hbar)^3} \int d^3p \exp \frac{i}{\hbar} \left[\mathbf{p} \cdot (\mathbf{x}'' - \mathbf{x}') - \frac{\mathbf{p}^2}{2m} t \right] , \end{aligned} \tag{F.10}$$

where m is the particle's mass, it follows that

$$\bar{K}^+(\mathbf{x}'', \mathbf{x}', t) = \Theta(t) \left(\frac{m}{2\pi\hbar i t} \right)^{3/2} \exp \left(\frac{im|\mathbf{x}'' - \mathbf{x}'|^2}{2\hbar t} \right) . \tag{F.11}$$

Inserting Eq. (F.11) into Eq. (6.2.34), and defining the new variable

$$\theta \equiv \left(\frac{m}{2\hbar t} \right)^{1/2} , \tag{F.12}$$

we get

$$\Delta(\mathbf{x}'', \mathbf{x}'; E) = e^{-\frac{i\pi}{4}} \frac{\sqrt{2}m}{(2\pi)^{3/2}} \int_0^{+\infty} d\theta \exp \left(i|\mathbf{x}'' - \mathbf{x}'|^2 \theta^2 + \frac{imE}{2\hbar^2\theta^2} \right) . \tag{F.13}$$

The integral in Eq. (F.13) can be evaluated after expressing the exponential in terms of trigonometric functions; the result is [134]

$$\Delta(\mathbf{x}'', \mathbf{x}'; E) = \frac{m}{4\pi|\mathbf{x}'' - \mathbf{x}'|} \exp \left(\frac{2i}{\hbar} \sqrt{\frac{mE}{2}} |\mathbf{x}'' - \mathbf{x}'| \right) . \tag{F.14}$$

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