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A THESIS SUBMITTED FOR THE DEGREE OF
"DOCTOR PHILOSOPHIAE"

COVARIANT STRING FIELD THEORY

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CONTENTS :

INTRODUCTION:

The sucsess of the field theory for the point particle suggests that it could also be formulated for the strings. The first step in this direction is to introduce a string field which is a functional of the space-time variables which are functions on the string world sheet. Indeed it is this fact that makes the string field theory become quite complicated.

Due to the reparametrization invariance of the world sheet, some of the string coordinates are unphysical. Elimination of them results in a unitary gauge which is called a light cone gauge. In fact the string field theory is formulated for the first time in this gauge[1] by making use of the path integral approach which was developed in [2]. Afterwards in [3] and [4,5] a more rigorous formulation in operator approach was developed. These methods of formulating a bosonic string field theory were then utilized to formulate superstring field theories in [6].

Due to the loss of manifest Lorentz invariance in the light cone gauge, most of the charming aspects of the field theories are absent in the above theories. In a covariant string field theory compactification of the d-4 dimensions would be easily perceived, the underlying symmetry would be revealed, it might be efficent to find the higher loop amplitudes in perturbative approach and the nonperturbative calculations would be allowed.

An important step in this direction was made by W. Siegel [7] who formulated a free covariant string field theory (we only deal with the open bosonic strings) by making use of the Becchi-Rouet-Stora-Tyutin (BRST)-charge of Kato and Ogawa[8]. He also proposed to introduce the interactions by incorporating the string length parameters by hand.

The BRST-charge can also be used to define a gauge invariant, covariant free string field theory[9] which has an infinite number of ghost fields due to the nilpotency of the BRST-charge (for the free case see also [10] and the references there in). Indeed fewer ghost fields leads to a wrong counting of the physical degrees of freedom.

For the interacting gauge invariant, covariant string field theory there are some different approachs. The one which is due to E. Witten [11] is based on the non-commutative algebra and the selection of interaction points as the mid-point of the strings. Another one consists of generalizing the interactions of the light cone gauge string field theory to the covariant one by incorporating string lengths by hand[12,13]. But this led to an over counting problem. Besides this, due to not having a variable like the proper time, the number of loop diagrams was increased[14] and also the relation between the scattering amplitudes of the light cone gauge and the covariant theory became obscure after one loop level. Through some underlying symmetry principles K. Kaku proposed a "geometric string field theory" and he claimed that the theories which are mentioned above are some gauge fixed variations of this [15].

In this thesis we will study a covariant string field theory which is a natural generalization of light cone gauge one. In this approach the problems of incorporating the string lengths by hand and introducing a proper time are resolved in [16]. A. Neveu and P. West developed a gauge fixed action which has Osp(26,2 | 2) in the zero mode sector and Osp(25,1 | 2) invariance in the other mass levels. Then the Parisi-Sourlas mechanism[17] guarantees the unitarity of the theory. In [18] we have also shown that even the physical subspace of the Hilbert space is spanned by the DDF states[19], the resulting scattering amplitudes being equivalent to the light cone gauge string field theory ones at all levels of perturbation expansion.

In fact Osp(d,2 | 2) invariant gauge fixed string field theory action was introduced by W.Siegel[20] but it differs from the above case in zero modes. However by introducing another ghost zero mode W. Siegel and his colloboraters have also developed an Osp(d,2 | 2) invariant free string theory independently[21].

In [16] A. Neveu and P. West proposed a gauge invariant theory but they fixed the gauge only up to the first ghost level for the free part of the action. In [22] we performed the gauge fixing of the whole action, which includes the three string interaction, to reach a Osp(d,2 | 2) invariant action by some suitable gauge fixing conditions. The main difference between the free case and the interacting one comes from the fact that latter does not possess an off-shell nilpotent gauge generator as the free case. Indeed it is the nilpotency property of the gauge generator that yields an infinite number of ghost fields which are essential to have the correct count of the physical degrees of freedom [9] in the free case. This problem is resolved by using the Batalin-Vilkovisky method of quantization of gauge theories [23], which is first noted in [24] in the context of string fieldtheory.

In all of the above-mentioned theories the kinetic part of the action somehow related to the BRST-charge which follows from the first quantization of strings. So that they are explicitly dependent on the background metric. In [25,26] a purely cubic action is proposed. As it does not have a kinetic term, it is free of the background and due to a different condensation of the string field, it is possible to create the kinetic term in any background.

The thesis is organized as follows. In the first chapter we review briefly the light cone gauge string field theory (we only deal with the open bosonic strings) whose knowledge is essential to understand the Osp(d,2 | 2) invariant theory.

Chapter II is a review of the Batalin-Vilkovisky method of quantization of gauge theories, which will prove useful in the subsequent chapters. Due to the fact that some of the concepts which they use are derived from the first quantization of the constrained systems, to present the method in a more comprehensible way we begin from the first quantization. To clarify the techniques used we give the Yang-Mills case as an example.

The point particle, even if it has quite different features from the string, always provides a good preperation to the string case. So in chapter III we present the first quantization of a point particle which leads to an Osp(d,2 | 2) invariant field theory. It is shown that the Osp(d,2 | 2) invariant theory can be achieved after a gauge fixing of a gauge invariant action which describes the point particles when the field function is physical and on-shell.

In chapter IV first of all we show that first quantization of a string in a gauge leads naturally to a superspace in which it is natural to formulate an Osp(26,2 | 2) string field theory. It is shown that a gauge invariant action of Neveu-West after a suitable gauge fixing, by making use of the Batalin-Vilkovisky formalism, leads to the Osp(26,2 | 2) invariant string field theory. The physical subspace of the Hilbert space is shown to be spanned by the DDF states, so that the scattering amplitude at tree level is the same as the one which results from the light cone gauge string field theory due to the following property of the Fourier components of Neumann functions[12]

$$\sum_{s=1}^{N} N_{no}^{rs} \, p_{+}^{s} = 0 .$$

However, we demonstrate that at all orders of perturbation the above equivalence which was mentioned for tree level is satisfied.

Chapter V is devoted to the perturbative calculations in string field theory. Using the operator approach to search for it provides more efficent way of calculating string scattering amplitudes. We presented the calculation of the four photon scattering amplitude at tree level as a sample to show the techniques which one uses in the operator approach. In the second part of this chapter we give some preliminaries of the first loop calculation in the operator approach in string theorycontext.

In the last chapter we discuss the results of the previous chapters and also some open problems in the Osp(d,2 | 2) string field theory.

I. LIGHT CONE GAUGE STRING FIELD THEORY :

A string is described, in terms of world sheet parameters σ and τ, with the coordinates $x^\mu(\sigma,\tau)$ where $0<\tau<\infty$, $0<\sigma<\pi\alpha$ and $\alpha=2p^+$. The Greek indices run from 0 to d-1 and the light cone gauge coordinates of a vector v^μ are defined as $v^+=(1/\sqrt{2})(v^0+v^{d-1})$, $v^-=(1/\sqrt{2})(-v^0+v^{d-1})$ and v^i where the Latin indices denote the transversal coordinates. To define a string field theory we introduce a field functional $\Phi(x)$. If we divide the σ interval into an infinite set of points $(\sigma_1,\sigma_2,\ldots,\sigma_N)$ this functional can be written as[3]

$$\Phi[x^\mu(\sigma_1),x^\mu(\sigma_2),\ldots,x^\mu(\sigma_N)]$$

by suppressing the proper time dependence. Obviously this functional does not depend explicitly on σ. When one tries to second quantize the system canonicaly,he has to introduce the canonical momentum which is defined in terms of a master Lagrangian $\mathfrak{L}$ which is a suitable field theory Lagrangian:

$$\Pi[x,\sigma]=\delta\,\mathfrak{L}\,/\,\delta[\partial\Phi(x)/\partial x^0(\sigma)]$$

But now the canonical momentum has an explicit dependence on σ. Thus there is no one to one correspondence between the field functional and the canonical momentum. This technical problem can be resolved by fixing a common time for the entire string as[27]

$$x^+=i\,\tau=\text{time} \tag{1.1}$$
$$(dx^\mu/d\tau)^2-(dx^\mu/d\sigma)^2=(dx^\mu/d\tau)(dx^\mu/d\sigma)=0.$$

In this gauge, namely in the light cone gauge, there is one to one correspondence between the string functional and the canonical momentum which is now given as follows

$$\Pi[x]=\delta\mathfrak{L}/\delta(\partial\Phi(x)/\partial x^{+})$$

In the light cone gauge only the transversal coordinates and the zero modes of the longitudinal components are physically relevant so that the field functional is defined to depend only on these variables. Now by using this field functional we want to write a master Lagrangian which ensures that we reproduce the usual spectrum of states and Hamiltonian in the first quantized approach. This will be achieved if the free part of the master Lagrangian generates the following Schrodingerequation

$$\int d\sigma\, \alpha^{-1} [-\partial^{2}/\partial x^{i\,2} + (\partial x^{i}/\partial\sigma)^{2}]\, \Phi(x) = i\partial\, \Phi(x)/\partial x^{+}$$

Because written in terms of normal mode operators of x^i and p^i ($= -i\,\partial / \partial x^i$ inconfiguration space)it yields

$$i\,\partial\Phi(x)/\partial x^{+} = \alpha^{-1}\, \{ i\sum_{n=1}^{\infty} \alpha^{i}{}_{n}{}^{\dagger}\, \alpha^{i}{}_{n} + p_i^{2} - 1\}\ \Phi(x) \tag{1.2}$$

which ensures that the time translations will be produced by the Hamiltonian of the first-quantized approach. As it is obvious we have already fixed the ambiguity which appears in the normal ordering of zero modes. At $\tau = 0$ the normal mode expansions of $x^i(\sigma)$ and $p^i(\sigma)$ are

$$x^{i}(\sigma) = x^{i} - i\sum_{n=1}^{\infty} n^{-1}\, (\alpha^{i}{}_{n}{}^{\dagger} - \alpha^{i}{}_{n})\, \cos(n\sigma/\alpha)$$

$$p^{i}(\sigma) = p^{i} + \sum_{n=1}^{\infty} (\alpha^{i}{}_{n}{}^{\dagger} + \alpha^{i}{}_{n})\, \cos(n\sigma/\alpha) \tag{1.3}$$

where $\alpha^{i}_{n}{}^{\dagger} = \alpha^{i}_{-n}$ and they satisfy the following algebra

$$[\alpha^{i}_{n} \, , \alpha^{j}_{m}{}^{\dagger}] = n \, \delta_{n,m} \, \delta^{i,j} \tag{1.4}$$

Now (1.2) can easly be derived by extending the range of σ as $-\pi < \sigma / \alpha < \pi$ by defining $x^{i}(-\sigma) = x^{i}(\sigma)$ for $0 \leq \sigma/\alpha \leq \pi$.

Thus the free part of the master Lagrangian is [3] (when it doesn't create confusion we will use x^{i} , p^{i} instead of $x^{i}(\sigma)$, $p^{i}(\sigma)$ for the sake of simplicity of the notation)

$$\mathcal{L}_{0} = \int d\alpha \, d\sigma \, Dx^{i} \, \{ \Phi^{\dagger}(x) \, i \, \partial \Phi(x) / \partial x^{+} - \Phi^{\dagger}(x) \, [-\partial^{2} / \partial x^{i\,2} + (\partial x^{i} / \partial\sigma)^{2}] \, \Phi(x) \} \tag{1.5}$$

In this approach the interactions are introduced as joining or splitting of strings at their end points. For example three string interaction is given such that two incoming strings join at the end points to give an outgoing string at a time T_{int} or an incoming string splits at a time T_{int} into two to give two outgoing strings. This can be represented as a string diagram which is given in figure 1.

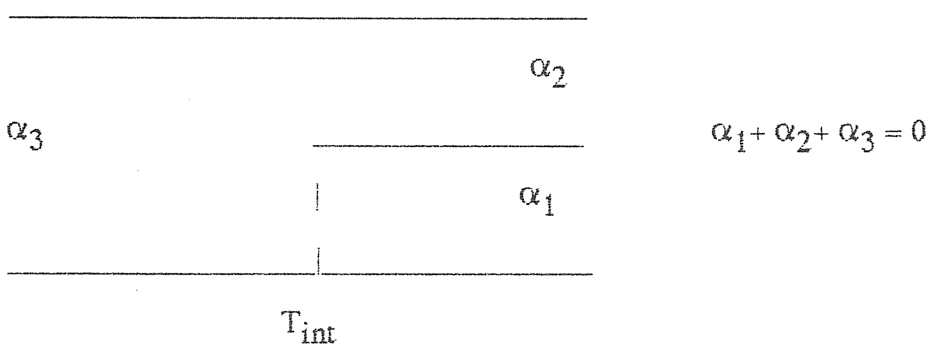

Figure 1.

Thus by introducing a coupling constant g , the three string interaction part of the master Lagrangian can be written as

$$\mathfrak{L}_1 = g \int \prod_{r=1,2,3} d\alpha_r (\alpha_r)^{-1} Dx^i_r \, \delta(\alpha_1 + \alpha_2 + \alpha_3) \, \Phi^\dagger(x_1) \, \Phi^\dagger(x_2) \, \Phi(x_3)$$

$$\prod_{\sigma_3} \delta\big(x^i_3(\sigma_3) - x^i_1(\sigma_1)\, \theta(\pi|\alpha_1| - \sigma_3) - x^i_2(\sigma_2)\, \theta(\sigma_3 - \pi|\alpha_1|) \big) + \text{h.c.} \tag{1.6}$$

where the parametrization is shown in figure 1 and $0 < \sigma_i < \pi |\alpha_i|$. It is shown in [3] that the interactions which are introduced through the overlap δ-functions can be described equivalently using the path integrals which are taken over the appropriate Riemann surfaces. We will not repeat this proof here, but nevertheless for later convenience we need to review the path integral method which is developed in [1,2]:

N string interaction which is illustrated in figure 2 leads to the following S matrix

$$S = \int d\tau_2 \, d\tau_3 \ldots d\tau_{N-2} \, A(\tau_2, \tau_3 \ldots, \tau_{N-2}) \tag{1.7}$$

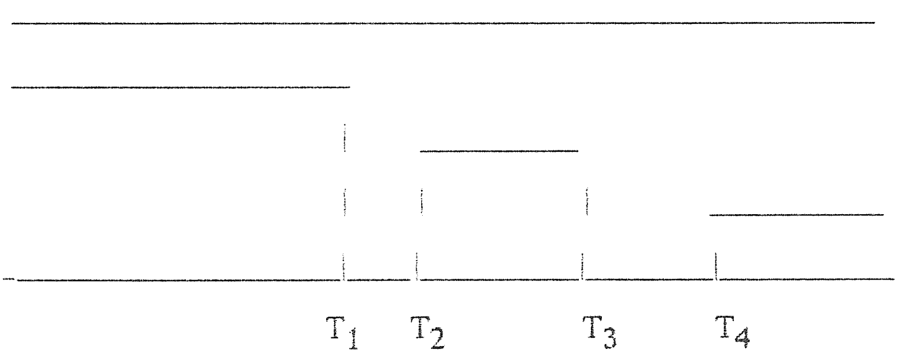

Figure 2.

where N-2 interaction times of the external strings are indicated with τ_r and the last integral $\int d\tau_1$

is not performed since the integration over it leads to the momentum conservation. In momentum space where the modes of the momenta are enumerated with n. A can be written as

$$A(\tau_2, \tau_3 \ldots, \tau_{N-2}) = g^{N-2} M(\tau_2, \tau_3 \ldots, \tau_{N-2}) \prod_{k=1}^{N} \prod_{n,i} dp^i_{n,k} \, \Psi_k(p^i_{n,k}) \, B \tag{1.8}$$

where $\Psi_k(p^i_{n,k})$ is the momentum space wave function of the k^{th} string and M is a normalization factor. B is given with the following path integral, which must be taken over the appropriate Riemannsurface,

$$B = \int \prod_i Dx^i(\sigma,\tau) \exp\{ i\sum_k (\pi\alpha_k)^{-1} \int d\sigma_k p^i_k(\sigma_k) \, x^i(\sigma,\tau_a) - \int d\sigma d\tau \, \mathfrak{L}(\sigma,\tau) - \sum_k p^-_k \tau_a \}$$

where $\tau_a \rightarrow \infty$ for the outgoing strings and $\tau_a \rightarrow -\infty$ for the incoming ones and $\sigma = \sigma_k$ when it is on the k^{th} string. $\mathfrak{L}(\sigma,\tau)$ is the usual string Lagrangian which is given as

$$\mathfrak{L}(\sigma,\tau) = (1/4\pi) \{ (dx/d\tau)^2 + (dx/d\sigma)^2 \} \tag{1.9}$$

To get rid of the linear term in the functional integral one can perform a change of variables:

$$x^i(\sigma,\tau) \rightarrow x^{i\,\prime}(\sigma,\tau) = x^i(\sigma,\tau) + i\sum_k (\pi\alpha_k)^{-1} \int d\sigma' \, N(\sigma,\tau\,;\sigma',\tau_a) \, p^i_k(\sigma'_k)$$

by making use of the Neumann function N which is defined as follows

$$\{(\partial/\partial\tau)^2 + (\partial/\partial\sigma)^2\} N(\sigma,\tau\,;\sigma',\tau') \equiv \Delta N(\sigma,\tau\,;\sigma',\tau') = 2\pi\,\delta(\sigma-\sigma')\,\delta(\tau-\tau')$$

$$\partial N(\sigma,\tau\,;\sigma',\tau') / \partial n = f(\sigma,\tau) \tag{1.10}$$

where the latter is valued on the boundary. $(\partial/\partial n)$ denotes the normal derivative and f is an arbitrary but σ', τ' independent function. Symbolically this change of variables effects the exponential part of the integral as

$$\int (\mathcal{L} + P X)\, d\sigma d\tau = \int (X \Delta X + PX)\, d\sigma d\tau$$

$$= \int \{X + (2\pi)^{-1} \int NP\, d\sigma' d\tau'\} \Delta \{X + (2\pi)^{-1} \int NP\, d\sigma' d\tau'\} + (2\pi)^{-1} \int P N P\, d\sigma d\sigma'$$

$$= \int X' \Delta X'\, d\sigma d\tau + (2\pi)^{-1} \int P N P\, d\sigma\, d\sigma' d\tau\, d\tau'$$

where P should be read as $(\pi\alpha)^{-1}\, p^i(\sigma)\, \delta(\tau - \tau_a)$. Thus by defining a new normalization factor we may write B as [5]

$$B = |\Delta|^{-(d-2)/2} \exp\{(½) \Sigma_{r,s}\, \pi^{-2}\, \alpha_r^{-1}\, \alpha_s^{-1} \int d\sigma\, d\sigma'\; p^i_r(\sigma_r)\, p^i_s(\sigma_s)\, N(\sigma, \tau_a ; \sigma', \tau_a') - \Sigma_r p_r^- \tau_a\} \qquad (1.11)$$

By observing that τ can be written in terms of Neumann coefficents as

$$\tau = (2\pi)^{-1}\, \Sigma_r \int_r d\sigma'\; N((\sigma, \tau ; \sigma', \tau_a)$$

the terms of B can be combined to yield

$$B = |\Delta|^{-(d-2)/2} \exp\{(½) \Sigma_{r,s}\, \pi^{-2}\, \alpha_r^{-1}\, \alpha_s^{-1} \int d\sigma\, d\sigma'\; p^i_r(\sigma_r)\, p^i_s(\sigma_s)\, N(\sigma, \tau_a ; \sigma', \tau_a') +$$

$$+ (½) \Sigma_{r,s}\, \pi^{-2}\, \alpha_r^{-1}\, \alpha_s^{-1} \int d\sigma\, d\sigma'\; p^\mu_r\, p^\mu_s\, N(\sigma, \tau_a ; \sigma', \tau_a')\} \qquad (1.12)$$

where now $p^1_r(\sigma_r)$ does not possess zero mode operator.

The Neumann function is conformally invariant, so that we may write it down in the σ,τ plane through a conformal map once we know it for the upper half complex plane. In the latter plane the Neumann function which is suitable at tree level can be found easly as

$$N = \ln|z - z'| + \ln|z - z'^*|. \tag{1.13}$$

The half plane can be mapped into the $\rho = \tau + i\sigma$ plane at tree level with the following transformation

$$\rho = \sum_1^N \alpha_r \ln(z - Z_r) \qquad \text{with} \qquad \sum_1^N \alpha_r = 0 \tag{1.14}$$

where Z_r is the point on the real axis onto which the point $\tau = \tau_a$ on the r^{th} string is mapped. The interaction points are defined as the points which maximize $\rho(z)$. In the string scattering amplitudes three of the Z_r will be fixed due to the Mobious invariance.

Fourier components of the Neumann functions are defined by the following mode expansion [1,13]

$$N(\sigma,\tau;\sigma',\tau') = -\delta_{r,s}\left(\sum_{n=1}^{\infty}(2/n)\exp(-n|\xi_r-\xi_s|)\ \cos n\eta_r\ \cos n\eta_s\ -2\max(\xi_r,\xi_s)\right) +$$

$$+ 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} N^{rs}_{mn}\ \exp(m\,\xi_r - n\,\xi_s)\cos n\eta_r\ \cos n\eta_s$$

where σ,τ and σ',τ' belong respectively to the strings r and s. The variables η_r and ξ_r are related to ρ_r (the world sheet variable ρ will be labeled as ρ_r when it is taken on the r^{th} string) as

$$\rho_r = \alpha_r (\xi_r + i \eta_r) + \tau_o^r + i \beta_r \qquad ; \qquad \xi_r \leq 0 \qquad ; \qquad 0 \leq \eta_r \leq \pi$$

τ_o^r is the interaction time of the r^{th} string, $\tau_o^r = \mathrm{Re}\, \rho_r(z_o)$, where at z_o, $d\rho(z)/dz = 0$ and β_r is defined in terms of string length parameters as

$$\beta_r = \sum_{s=1}^{r-1} \alpha_s$$

The Fourier components of the Neumann functions, N^{rs}_{mn}, at tree level can be given in an integral representation asfollows[5]

$$N^{rs}_{oo} = \ln | Z_r - Z_s | \qquad \text{when } r \neq s$$

$$N^{rr}_{oo} = - \sum_{s \neq r} (\alpha_s / \alpha_r) \ \ln | Z_r - Z_s | \tag{1.15}$$

$$N^{rs}_{on} = N^{sr}_{no} = n^{-1} (2\pi i)^{-1} \oint_{Z_r} dz \ (z - Z_s)^{-1} \exp(- n \zeta_r(z)) \qquad n > 0$$

$$N^{rs}_{nm} = N^{sr}_{mn} = n^{-1} m^{-1} (2\pi i)^{-2} \oint_{Z_r} dz \oint_{Z_s} dz' \ (z - z')^{-2} \ \exp (- n \zeta_r(z) - m \zeta_s(z')) \qquad n,m > 0$$

where $\zeta_r(z) = \xi_r + i \eta_r$ .Now it is possible to write the three point interaction vertex , which is given in terms of the overlap δ-functions in (1.6), by making use of the Neumann coefficents which would be found for the Riemann surface drawn in Fig. 1 as follows

$$|V(1,2,3)\rangle = (2\pi)^d \delta(\Sigma\, p_r) \exp\{(1/2) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \Sigma_{r,s} N^{rs}_{nm}\, \alpha_{n,r}^{\dagger}\, \alpha_{m,s}^{\dagger}\}\, |0\rangle$$

where the ground state is defined as $\alpha_{n,r} |0\rangle = 0$ for $n > 0$ and $\alpha_{0,r} = p_r$.

The Lorentz invariance of the theory restricts the dimension to be d=26 even at three string vertex case[1]. When four string scattering amplitude is taken into account the duality and the Lorentz invariance will be regained by introducing a four string interaction vertex [3,5]. At one loop level one has to introduce also the open-closed string interactions due to the same motivations[3]. So all the properties of the first quantized strings are also present in the field theory case.

II. BATALIN-VILKOVISKY METHOD OF QUANTIZATION OF GAUGE THEORIES :

Let us consider a physical system which has m_+ bosonic and m_- fermionic, first class constraints:

$$\Phi_\alpha = 0 \qquad \alpha=1,2,\ldots,m_+ + m_- = m \tag{2.1}$$

which satisfy the following generalized Poisson bracket algebra

$$\{ \Phi_\alpha , \Phi_\beta \} = \Phi_\gamma T^\gamma{}_{\alpha\beta} \qquad ; \qquad \{ Ho, \Phi_\alpha \} = \Phi_\beta V^\beta{}_\alpha \tag{2.2}$$

where $T^\gamma{}_{\alpha\beta}$, $V^\beta{}_\alpha$ depend on the canonical coordinates and H_o is the canonical Hamiltonian. Generalized Possion brackets are defined as [29]

$$\{ A , B \} = (\partial_r A/\partial q_i) (\partial_l B/\partial p_i) - (-1)^{\varepsilon(A)\,\varepsilon(B)} (\partial_r B/\partial q_i) (\partial_l A/\partial p_i) \tag{2.3}$$

where $\partial_r/\partial q_i$ and $\partial_l/\partial p_i$ indicate right and left derivatives with respect to 2n canonical variables ($i=1, 2, \ldots, n_+ + n_- = n$). $\varepsilon(A) = 0$ when A is bosonic and $\varepsilon(A) = 1$ when it is fermionic. Thus the system is described with the following action

$$S = \int_0^T dt \, \{ p_i \, dq_i/dt - H_o \} \tag{2.4}$$

and has 2m unphysical degrees of freedom which can be eliminated using the constraint equations (2.1) with some gauge conditions:

$$\tilde{\chi}_\alpha = 0 \qquad : \qquad \det\{\Phi_\alpha , \tilde{\chi}_\beta\} = 0 \tag{2.5}$$

to reach a unitary action:

$$S_{phys} = \int_0^T dt \, \{p^a_{phys} \, dq^a_{phys}/dt \; - \; H_{phys}\} \tag{2.6}$$

where $a = 1, 2, \ldots, n\text{-}m$. Generally (2.6) is not any more manifestly covariant.

Instead of considering the constraints separately we may put them into the original action via some Lagrange multipliers, λ_α, as

$$S = \int_0^T dt \, \{ p_i \, dq_i/dt \; - \; H_o \; - \lambda_\alpha \Phi_\alpha \} \tag{2.7}$$

(Of course still the gauge conditions have to be taken into consideration to reach (2.6)). Let us see how this action changes under the transformations which are generated by the constraints as follows[30]

$$\delta_\Lambda F = \{ F , \Lambda^\alpha \Phi_\alpha \} = (-)^{\varepsilon_\alpha \varepsilon_F} \Lambda^\alpha \{ F , \Phi_\alpha \}$$

where F represents the canonical coordinates and $\varepsilon_\alpha = \varepsilon(\Phi^\alpha)$ and $\varepsilon_F = \varepsilon(F)$. Due to this definition we have:

$$\delta_\Lambda p^i = -\Lambda^\alpha \, \partial_r \Phi_\alpha / \partial q^i \qquad \delta_\Lambda q^i = (-1)^{\varepsilon_\alpha \varepsilon_i} \Lambda^\alpha \, \partial_l \Phi_\alpha / \partial p^i$$

$$\delta_\Lambda \Phi_\alpha = -\Lambda^\beta \, \Phi_\gamma T^\gamma{}_{\alpha\beta} \qquad \delta_\Lambda H_o = \Lambda^\alpha \Phi_\beta V_\alpha{}^\beta \tag{2.8}$$

where $\varepsilon_i = \varepsilon(q^i) = \varepsilon(p^i)$. Now the change in the action is

$$\delta_\Lambda S = \int_0^T dt \{ -\Lambda^\alpha \partial_r \Phi_\alpha / \partial q^i \ \partial q^i / \partial t + p^i \partial [(-1)^{\varepsilon_\alpha \varepsilon_i} \Lambda^\alpha \partial_l \Phi_\alpha / \partial p^i] / \partial t - \Lambda^\alpha \Phi_\beta V_\alpha{}^\beta + \tag{2.9}$$

$$+ \lambda^\alpha \Lambda^\beta \Phi_\gamma T^\gamma{}_{\alpha\beta} - \Phi_\alpha (\delta_\Lambda \lambda^\alpha) \}$$

If we transform also the Lagrange multipliers as

$$\delta_\Lambda \lambda^\alpha = \partial \Lambda^\alpha / \partial t + \lambda^\gamma \Lambda^\beta T^\alpha{}_{\gamma\beta} + \Lambda^\beta V_\beta{}^\alpha \tag{2.10}$$

$\delta_\Lambda S$ leads to

$$\delta_\Lambda S = \int_0^T dt \{ - \Lambda^\alpha \partial_r \Phi_\alpha / \partial q^i \ \partial q^i / \partial t + \partial [\Lambda^\alpha p^i \partial_l \Phi_\alpha / \partial p^i] / \partial t - \Lambda^\alpha \partial p^i / \partial t \, \partial_l \Phi_\alpha / \partial p^i - \partial \Lambda^\alpha / \partial t \Phi_\alpha \} \tag{2.11}$$

For any function of the canonical coordinates it easy to see that the following equalities are satisfied

$$\partial_r \Phi_\alpha / \partial q^i \ \partial q^i / \partial t + \partial p^i / \partial t \ \partial_l \Phi_\alpha / \partial p^i = d\Phi_\alpha / dt \tag{2.12}$$

$$p^i \partial_l \Phi_\alpha / \partial p^i = N_\alpha \Phi_\alpha \quad \text{(without summation over } \alpha)$$

where N_α is a number which will be determined by he momentum dependence of the constraint. Hence the change in the action is reduced to

$$\delta_\Lambda S = [(N_\alpha - 1) \Lambda^\alpha \Phi_\alpha] \Big|_0^T \tag{2.13}$$

When the constraint is linear in $p_i q_i$, $N_\alpha = 1$ but in the other cases the gauge parameter must depend on time and vanish on the boundary, according to have a gauge invariant action under the transformations (2.8)and (2.10) which are generated by the constraints.

Now it is convenient to treat the Lagrange multipliers on the same footing as the dynamical variables q_i. Thus associate them an equal number of real momenta π_α with the same Grasmann parity and which obey the following generalized Poisson brackets

$$\{\pi_\alpha , \lambda_\beta \} = - \delta_{\alpha\beta} .$$

In order to not to change the physical concept of the theory we constrain π_α to vanish classically. In fact these constraints generate the following transformations

$$\delta\lambda^\alpha = \{ \lambda^\alpha , \kappa_\beta \pi^\beta \} = (-1)^{\varepsilon_\alpha} \kappa^\alpha$$

expressing the arbitrariness of the Lagrange multipliers.

Thus the above procedure has led to an extended phase space which is defined in terms of the canonical variables $\{q_i , p_i ; \lambda_\alpha , \pi_\alpha \}$ and the constraints $G_a = \{\pi_\alpha , \Phi_\alpha\}$ where $a=1, \ldots ,2m$. The new Poisson bracket relations can be written as

$$\{ G_a , G_b \} = G_c U^c{}_{ab}$$

$$\{ H_o , G_a \} = G_b V^b{}_a . \qquad (2.14)$$

The canonical action now reads

$$S = \int dt \{ p^i \partial q^i/\partial t + \pi_\alpha \partial\lambda_\alpha/\partial t - H_o - \lambda_\alpha \Phi_\alpha + \pi_\alpha \chi_\alpha \} \qquad (2.15)$$

where the Lagrange multipliers χ_α are interpreted as gauge fixing conditions. We enlarge this phase space by adding a new canonical pair η_a ,φ_a which have opposite statistics to G_a. In a unified notation where $q_A = \{ q^i , \lambda_\alpha \}$ and $p_A = \{p^i , \pi_\alpha \}$ so A=1, . . . , n+m, a theorem due toFradkin-Vilkovisky [29] can be postulated as follows: The functional integral

$$Z = \int dq_A \, dp_A \, d\eta_a \, d\varphi_a \exp\{ i \int dt[p_A \, \partial q_A/\partial t + \varphi_a \, \partial\eta_a/\partial t - H_\psi \} \tag{2.16}$$

does not depend on the choice of ψ (it is arbitrary modulo the restrictions imposed on it by the requirment of non-degeneracy), where H_ψ is defined in terms of the BRST charge:

$$Q = G_a\eta^a + (½)(-1)^{\varepsilon_b} \varphi_a U^a{}_{bc}\eta^c\eta^b \quad , \tag{2.17}$$

as follows

$$H_\psi = H_o + \varphi_a V^a{}_b \, \eta^b - \{ \psi , Q \} \tag{2.18}$$

We are mostly interested in the systems where the canonical Hamiltonian vanishs, so that in the proof of the the above theorem we set:

$$H_o=0 \quad , \; V^a{}_b = 0.$$

The proof of the theorem is easy to achive if we can show that the functional integral is invariant under the transformations which are generated by BRST charge as

$$\delta F = \{F, \lambda Q\}. \tag{2.19}$$

The BRST invariance of H_ψ follows directly from the Jacobi identity:

$$(1 - (-1)^{\varepsilon_Q}) \{ \{\psi , Q \} , Q \} + \{ \{ Q , Q \} , \psi \} = 0 \ ,$$

(where the Grasmann parity of BRST charge is $\varepsilon_Q = 1$) and the nilpotency of Q:

$$\{ Q, Q\} = 0 \qquad (2.20)$$

(2.20) can be proved by making use of the following relations, which are derived from (2.14),

$$\{G_a\eta^a , G_b\eta^b\} = -(-1)^{\varepsilon_b} G_c U^c{}_{ba} \eta^a \eta^b$$

$$\{ (-1)^{\varepsilon_b} U^a{}_{bc} \eta^c \eta^b , G_c\eta^c\} = -(-1)^{\varepsilon_b} (-1)^{\varepsilon_c} U^a{}_{bd} \eta^d U^b{}_{ce} \eta^e \eta^c$$

$$\{U^c{}_{ba} \eta^a \eta^b , U^d{}_{ef} \eta^f \eta^e\} = 0$$

The other part of the action transforms under Q as follows, which is derived by using a generalization of (2.12) to G_a and $U^a{}_{bc}$,

$$\int_0^T dt \{ Q , p_A \partial q_A/\partial t + \varphi_a \partial\eta_a/\partial t \} = (½)(-1)^{\varepsilon_b} N_U \int_0^T dt\, d[\varphi_a U^a{}_{bc}\eta^c\eta^b]/dt$$

where N_U is defined as

$$p_A\, \partial U^a{}_{bc} /\partial p_A = N_U U^a{}_{bc}$$

Thus when $N_U \neq 0$ by putting the condition that $\eta^c=0$ on the boundary (this is in accord with the condition which we put on the gauge parameter to have (2.13)=0) gives the desired result. i. e. The action of the functional integral (2.16) is BRST invariant. The ψ independence of the action follows from the BRST invariance by taking the parameter of the transformation as

$$\lambda = \int (\psi' - \psi)\, dt .$$

This transformation yields a Jacobian

$$dq_A\, dp_A\, d\eta_a\, d\varphi_a = dq'_A\, dp'_A\, d\eta'_a\, d\varphi'_a \left(1 + \int \{\psi' - \psi , Q\}\, dt\right)$$

For small $(\psi' - \psi)$ it leads to

$$Z_\psi = Z_{\psi'} .$$

This completes the proof of the Fradkin-Vilkovisky theorem.

Once we proved this theorem we may take

$$\psi = \varphi_a\, \Theta^a \qquad ; \qquad \Theta^a = \{\, \chi_\alpha , -\lambda_\alpha \}$$

so that the original part of the action becomes equal to (2.15). Above fermionic function ψ which will play an inportant role in formulating the quantization of the gauge theories with open gauge algebras, is called "gaugefermion".

The unitarty of the S matrix is a corollary of this theorem: By taking ψ as above we can fix the gauge such that

$$\chi_\alpha + \partial\lambda_\alpha/\partial t = 0 \quad \text{and} \quad \det\{\, \chi_\alpha + \partial\lambda_\alpha/\partial t, \Phi_\alpha \} \neq 0$$

which leads to the unitary theory† which has the action (2.6) after performing the integration over the Lagrange multipliers λ_α.

Quantization of the system can be achived either in the unitary gauge or in the covariant description by replacing the Poisson brackets of the canonical variables of the related theory with the commutators or anticommutators: $\{\ ,\ \} \to (-i/\hbar)[\ ,\ ]_\pm$. In the covariant description the additional 2m coordinates, which have the opposite statistics of the original phase space variables, will play the role of the ghosts after quantization.

The formalism which we illustrated above can also be applied to the field theory, so that also in this case the concepts like "gauge fermion" will make sense, but now it will be a functional of the field functions. Before representing the Batalin-Vilkovisky method of quantization of the gauge theories let us classify the gauge theories and also give the conditions which have to be fulfiled to use this method:

If $A(\phi)$ represents a classical gauge action in terms of the fields ϕ^i, $i=1, \ldots, n=n_+ + n_-$ (n_+ and n_- are the numbers of the bosonic and fermionic fields), we assume that there exists at least one stationary point ϕ_0 of it where

$$\delta_r A(\phi)/\delta\phi^i \Big|_{\phi_0} = 0.$$

---

† Recently it is shown that [31], in the case of having only bosonic constraints, it is possible to construct a local change of variables to realize an Osp(1,1|2) invariance in the phase space which leads, via Parisi-Sourlas mechanism, to the equivalance of the system which is defined with the functional integral (2.16) to the unitary one.

The infinitesimal gauge transformations are represented as

$$\delta\phi^i = R^i_{\alpha_o}(\phi)\,\delta\theta^{\alpha_o}$$

in terms of the gauge parameters $\delta\theta^{\alpha_o}$ with Grassmann parity $\varepsilon(\delta\theta^{\alpha_o}) = \varepsilon_{\alpha_o}$ and generators $R^i_{\alpha_o}$, $\varepsilon(R^i_{\alpha_o}) = \varepsilon_i + \varepsilon_{\alpha_o}$, where $\varepsilon_i = \varepsilon(\phi^i)$. It is assumed that in a neighborhood of ϕ_o, m_{o+} bosonic and m_{o-} fermionic Noether identities hold:

$$(\delta_r A(\phi)/\delta\phi^i)\,R^i_{\alpha_o} = 0 \qquad ; \qquad \alpha_o = 1, \dots, m = m_{o+} + m_{o-}$$

If the rank of $R^i_{\alpha_o}$ is maximal at the stationary point or in other terms, if the vectors $R^i_{\alpha_o}$ enumerated by α_o are linearly independent in the neighborhood of the stationary point the theory is called <u>irreducible</u> or <u>zero-stage</u> theory [23,29].

Let us suppose that there exist some vectors $Z_1{}^{\alpha_o}{}_{\alpha_1}$ which are enumerated by α_1, such that

$$R^i_{\alpha_o} Z_1{}^{\alpha_o}{}_{\alpha_1}\Big|_{\phi_o} = 0$$

$$\varepsilon(Z_1{}^{\alpha_o}{}_{\alpha_1}) = \varepsilon_{\alpha_o} + \varepsilon_{\alpha_1} \qquad \alpha_1 = 1, \dots, m_1 = m_{1+} + m_{1-}$$

and they are linearly independent in a neighborhood of the stationary point or equivalently

$$\text{rank } Z_1{}^{\alpha_o}{}_{\alpha_1}\Big|_{\phi_o} = m_1$$

where now

$$\text{rank } R^{i}_{\alpha_0}\Big|_{\phi_0} = m_0 - m_1 \quad ; \qquad m_0 > m_1 .$$

The gauge theories which satisfy the above properties are called first-stage theories

In the second stage theories , in a neighborhood of the stationary point also $Z_1{}^{\alpha_0}{}_{\alpha_1}$ have some zero eigenvalue eigenvectors, $Z_2{}^{\alpha_1}{}_{\alpha_2}$, which are enumerated by $\alpha_2=1, \ldots,$ $m_2=m_{2+} + m_{2-}$ which are linearly independent in a neighborhood of the stationary point. The Grassmann parity of it is $\varepsilon(Z_2{}^{\alpha_1}{}_{\alpha_2}) = \varepsilon_{\alpha_1} + \varepsilon_{\alpha_2}$. In this case:

$$\text{rank } R^{i}{}_{\alpha_0}\Big|_{\phi_0} = m_0 - (m_1 - m_2) \quad ; \qquad m_0 > m_1 - m_2$$

$$\text{rank } Z_1{}^{\alpha_0}{}_{\alpha_1}\Big|_{\phi_0} = m_1 - m_2 \quad ; \qquad m_1 > m_2$$

$$\text{rank } Z_2{}^{\alpha_1}{}_{\alpha_2}\Big|_{\phi_0} = m_2 .$$

The higher stage theories are defined in a similar way.

In all of these theories the number of admissible gauge conditions, needed to remove the degeneracy of the action, must be equal to the number of the independent gauge generators, i.e. The gauge conditions must eliminate only (but all) the unphysical degrees of freedom. This condition is known as the completenesscondition†.

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† Even Batalin and Vilkovisky claimed that the formalism is applicable only to the finite-stage theories, we will see in chapter IV that this is not necessarly true if one can show that the completeness conditionholds.

To quantize a gauge theory (reducible or irreducible) introduce a set of fields Φ^A which include the original gauge fields ϕ^i (the rules of deciding the content of this set will be clear in the following and it will be seen that the fields other than the original ones correspond to the ghosts). The functional integral of the theory can be expressed in the following form

$$Z_\psi = \int \prod d\Phi^A \exp[i\hbar^{-1} W_\Sigma(\Phi)] \tag{2.21}$$

where we suppose that $W_\Sigma(\Phi)$ generates the correct Feynman rules. ψ represents the gauge fermion and Σ specifies the gauge fixing. As we have seen, gauge fixing conditions can be introduced into the action by making use of the gauge fermion so that $W_\Sigma(\Phi)$ can equivalently be written as

$$W_\Sigma(\Phi) = W(\Phi, \delta_r\psi/\delta\Phi)$$

By taking the ψ dependence of W as above we may attribute to each Φ^A an antifield $\Phi^*{}_A$ such that the gauge fixing condition is $\Phi^*{}_A = \delta_r\psi/\delta\Phi^A$. Then $\Phi^*{}_A$ has the opposite statistics of Φ^A, we will see that the antifields correspond to the antighost fields (for example $\Phi^*{}_i$ will be related to the antighost of the first ghost field one has to introduce, so its statistics is correct). Therefore we can summarize the above discussion as

$$W_\Sigma(\Phi) = W(\Phi,\Phi^*)\Big|_{\Phi^*=\delta_r\psi/\delta\Phi} .$$

Let us define a transformation, in fact it is the most general version of the BRST transformation, as

$$\delta\Phi^A = (\Phi^A , W) . \mu \Big|_{\Phi^* = \delta_r \psi / \delta\Phi} = \delta_l W / \delta\Phi^*_A . \mu \Big|_{\Sigma} \qquad (2.22)$$

where μ is an anticommuting parameter and the <u>antibracket</u>of two operators P and Q is defined as follows

$$(P , Q) = (\delta_r P/\delta\Phi^A)(\delta_l Q/\delta\Phi^*_A) - (\delta_r P/\delta\Phi^*_A)(\delta_l Q/\delta\Phi^A)$$

The antibrackets have the following properties

$$\varepsilon[(P , Q)] = \varepsilon (P) + \varepsilon (Q) + 1$$

$$(P , Q) = -(-1)^{[\varepsilon(P) + 1] [\varepsilon(Q) + 1]} (Q , P)$$

$$(-1)^{[\varepsilon(P) + 1] [\varepsilon(Q) + 1]} (Q , (S , P)) + \text{cyclic permutation of } Q,S,P. = 0$$

For a bosonic operator B

$$(B , B) = 2 (\delta_r B/\delta\Phi^A)(\delta_l B/\delta\Phi^*)$$

which is generally different from zero, while for a fermionic operator F

$$(F , F) = 0$$

and for any operator P

$((P, P), P) = (P, (P, P)) = 0.$

The change of the functional integral (2.21) under the transformations (2.22) is

$$\delta_\mu Z_\psi = \int \prod d\Phi^A \{ i\hbar^{-1} (\delta_r W_\Sigma(\Phi)/\delta\Phi^A) (\delta_l W(\Phi,\Phi^*) / \delta\Phi^*{}_A)|_\Sigma$$

$$+ (\delta_r/\delta\Phi^A) (\delta_l/\delta\Phi^*{}_A) W(\Phi,\Phi^*) |_\Sigma \} \mu \exp [i\hbar^{-1} W_\Sigma(\Phi)]$$

Then the functional integral is invariant under the BRST transformations (2.22) if

$$(½)(W, W) = i\hbar^{-1} \Delta W \qquad \text{where } \Delta = (\delta_r/\delta\Phi^A) (\delta_l/\delta\Phi^*{}_A) \qquad (2.23)$$

Now the proof of the independence of the functional integral from the choice of the gauge fermion follows if one takes the gauge parameter in (2.21) as

$$\mu = i\hbar^{-1}\delta\psi.$$

Of course the gauge fermion is arbitrary modulo the restrictions imposed by the requirment of nondegeneracy.

Let us expand W in powers of $\hbar$ as

$$W = S + \sum_{n=1}^{\infty} \hbar^n M_n. \qquad (2.24)$$

Use of it in (2.23) yields the following set of equations

$$n = 0 \qquad (S, S) = 0 \qquad (2.25)$$

$$n = 1 \qquad (M_1 , S) = i \Delta S \tag{2.26}$$

$$n \geq 2 \qquad (M_n , S) = i \Delta M_{n-1} - (½) \sum_{m=1}^{n-1} (M_m , M_{n-m}) \tag{2.27}$$

(2.25) is called the <u>masterequation</u>. Except the classical part of W, namely S, the terms in (2.24) give the quantum measure of the functional integral. Apart from the measure part the unique equation which we have to solve is the master equation (2.25). Its solution, when restricted on Σ, will give the full action.

Correctness of the classical limit dictates that the original gauge fields should be included in Φ^A as well as the following boundary condition holds

$$S(\Phi,\Phi^*)|_{\Phi^*=0} = A(\phi). \tag{2.28}$$

To get rid of all the unphysical degrees of freedom (nondegenarcy of the action) the rank of the Hessian of $S(\Phi,\Phi^*)$ must be maximal at the stationary point of $S(\Phi,\Phi^*)$. <u>Proper solution</u> of (2.25) is the solution which satisfies the above condition. Let us clarify this requirment:

By introducing the following collective notation

$$z^a = (\Phi^A ; \Phi^*_A) \qquad ; \quad A = 1, ... , N \qquad a = 1, ... , 2N$$

and the matrix

$$\eta_{ab} = \begin{pmatrix} 0 & \delta_{AB} \\ -\delta_{AB} & 0 \end{pmatrix}$$

it is possible to write the antibrackets as

$$(P , Q) = (\delta_r P/\delta z^a)\, \eta^{ab}\, (\delta_l Q/\delta z^b).$$

Now the derivative of the master equation leads to:

$$\delta_r (S , S) / \delta z^a = (S , \delta_r S/ \delta z^a) = (\delta_r S/ \delta z^b)\, \mathfrak{R}^a{}_b = 0 \qquad (2.29)$$

where $\mathfrak{R}^a{}_b$ is defined as follows:

$$\mathfrak{R}^a{}_b = \eta^{ac}\, \delta_l\, \delta_r S / \delta z^c\, \delta z^b , \qquad (2.30)$$

so that any solution of the master equation is a gauge invariant action. Let us assume that S has a stationary point z_o:

$$\delta_r S / \delta z^a \Big|_{z_o} = 0$$

so that the derivative of (2.29) yields the nilpotency of the Hessian at the stationary point:

$$\mathfrak{R}^a{}_b\, \mathfrak{R}^b{}_c \Big|_{z_o} = 0 . \qquad (2.31)$$

Generally a symmetric (2N x 2N) type matrix when it is nilpotent has at most N independent elements, hence from (2.31) it follows that

$$\text{rank}\ \delta_l\ \delta_r S / \delta z^c\ \delta z^b \Big|_{z_o} = r \leq N.$$

A solution of the master equation is called proper when r=N. Then all of the elements of the Hessian of S, $\delta_l\ \delta_r S / \delta z^c\ \delta z^b$, can be written in terms of N independent elements of it at z_o. Hence if all of the zero eigenvalues of the Hessian of S are included in itself, at the stationary point, the solution of the master equation is called proper.

We can expand the solution of the master equation in terms of the antifields as

$$S(\Phi\ ,\Phi^*) = A(\phi) + \sum_{n=1}^{\infty} \Phi^*_{A_n}\ .\ .\ .\ \Phi^*_{A_1}\ S^{A_1\ .\ .\ .\ A_n}(\Phi) \qquad (2.32)$$

The coefficents of this expansion, $S^{A_1\ .\ .\ .\ A_n}(\Phi)$, are called the structure functions and the equations which relate them, by using (2.32) in (2.25), are called structure equations.

The minimal content of Φ^A depends on the reducibility properties of the theory and it will become clear in the following. However let us suppose that the minimal set Φ^A_{min} contains only the original gauge fields ϕ^i. Then the master equation as well as the correctness of the classical limit will be satisfied if we set

$$S^{A_1\ .\ .\ .\ A_n}(\Phi) = 0.$$

But, due to the gauge invariance of the action

$$\text{rank}\ \delta_l\ \delta_r\ A(\phi) / \delta\phi^i\ \delta\phi^j \Big|_{\phi_o} = n - m$$

where m is the rank of the gauge generator matrix at the stationary point ϕ_0. Thus this solution is not proper and the minimal set must always contain some new fields.

Zero-stage theories:

The derivative of the Noether identities with respect to the fields ϕ^i, at the stationary point yields

$$[\delta_l \delta_r A(\phi) / \delta\phi^i \delta\phi^j] R^i_{\alpha_0} \Big|_{\phi_0} = 0. \qquad (2.33)$$

On the other hand the nilpotency relation (2.31) leads to

$$[\delta_l \delta_r S(\Phi, \Phi^*) / \delta\Phi^A \delta\Phi^B] [\delta_l \delta_r S(\Phi, \Phi^*) / \delta\Phi^*_B \delta\Phi^C] \Big|_{\delta S/\delta\Phi = \delta S/\delta\Phi^* = 0} + (-1)^{\varepsilon(A)\varepsilon(C)} \{ A \leftrightarrow C \} = 0$$

which gives the following equation for the first term in the right hand side of (2.32)

$$[\delta_l \delta_r A(\phi) / \delta\phi^i \delta\phi^j] [\delta_l \delta_r S(\Phi, \Phi^*) / \delta\phi^*_i \delta\Phi^{C'}] \Big|_{\delta S/\delta\Phi = \delta S/\delta\Phi^* = 0} = 0 \qquad (2.34)$$

where $\Phi^{C'}$ includes only the new fields one has to introduce. Comparison of (2.33) with (2.34) leads to the following expression for the gauge generators, by introducing some new fields $C_0^{\alpha_0}$,

$$R^i_{\alpha_0} = \delta_l \delta_r S(\Phi, \Phi^*) / \delta\phi^*_i \delta C_0^{\alpha_0} \Big|_{\Phi^* = 0}.$$

Let us attribute the following ghost numbers (or gauge ghost numbers) to the fields which we have introduced,

$$gh\{ \phi^i, \phi^*_i, C_o^{\alpha_o}, C^*_{o\alpha_o} \} = \{ 0, -1, 1, -2 \}. \tag{2.35}$$

Ghost number of S must be zero. In fact ghost number conservation is an essential ingredient to find the structure functions which makes S a proper solution of the master equation.

Now the Hessian of S is

$$\begin{array}{c|cccc|} & \phi & \phi^* & C_o & C_o^* \\ \hline \phi & \frac{\delta^2 A(\phi)}{\delta\phi^i\,\delta\phi^j}\Big|_{\phi_o} & & & \\ \phi^* & & & R^i_{\alpha_o}\big|_{\phi_o} & \\ C_o & & R^i_{\alpha_o}\big|_{\phi_o} & & \\ C_o^* & & & & \\ \hline \end{array}$$

so that the rank of the Hessian, r, reads

$$r = \text{rank}\ \delta_l\, \delta_r\, A(\phi) / \delta\phi^i\, \delta\phi^j \Big|_{\phi_o} + 2\ \text{rank}\ R^i_{\alpha_o} \Big|_{\phi_o} = (n - m_o) + 2\, m_o = n + m_o,$$

where m_o is the rank of $R^i_{\alpha_o}$. The rank of the Hessian of S is equal to the number of fields so that the minimal set of fields which we need is

$$\Phi^A{}_{min} = \{ \phi^i , C_o{}^{\alpha_o} \}. \tag{2.36}$$

In the irreducible theories the number of gauge conditions must be equal to the number of the gauge generators. This restricts the gauge fermion. One way of introducing the gauge condition is enlarging the content of Φ^A by $2m_o$ new fields:

$$B_{o\alpha_o} , \quad \pi_{o\alpha_o} ; \qquad \varepsilon(B_{o\alpha_o}) = \varepsilon_{\alpha_o} + 1 , \quad \varepsilon(\pi_{o\alpha_o}) = \varepsilon_{\alpha_o} .$$

The following S is still a proper solution of the master equation

$$S(\Phi, \Phi^*) = S(\Phi_{min}, \Phi^*{}_{min}) + B_o{}^{*\alpha_o} \pi_{o\alpha_o} . \tag{2.37}$$

We take the gauge fermion such that it is independent of $\pi_{o\alpha_o}$, however dependent on $B_{o\alpha_o}$ such that

$$\delta\psi / \delta B_{o\alpha_o} = 0 \tag{2.38}$$

are the gauge conditions. As we defined before gauge fixed action is achived by setting

$$\Phi^*{}_A = \delta\psi / \delta\Phi^A.$$

Therefore by integrating out the Lagrange multipliers in the functional integral, we arrive to the gauge fixed action, which will generate the Feynman rules of the theory,

$$S_{g.f.} = S(\Phi^A, \delta\psi / \delta\Phi^A) \Big|_{\delta\psi/\delta B=0} . \tag{2.39}$$

As it is obvious the fields $C_o^{\alpha_o}$ and $B_{o\alpha_o}$ are the Fadeev-Popov ghost and antighost fields. The requirment of fixing the gauge completely restricts the gauge fermion as follows

$$\text{rank}\,\{[\delta_l\delta_r\,\psi / \delta B_{o\alpha_o}\,\delta\phi^i\,]\,R^i{}_{\beta_o}\}\Big|_{\substack{\phi=\phi_o \\ C_o=B_o=0}} = m_o.$$

Now to illustate the method let us apply it to the Yang-Mills case. By taking into consideration the attributed ghost numbers of the fields which appear in the minimal set, it is possible to find the most general form of the structure equations [29]. But it can be easly seen that the solution of them will lead to the following S in the case of Yang-Mills theory

$$S(\Phi_{min}, \Phi^*{}_{min}) = A(\phi) + \phi^*{}_i R^i{}_{\alpha_o} C_o{}^{\alpha_o} + C^*{}_{o\alpha_o} T^{\alpha_o}{}_{\beta_o\gamma_o} C_o{}^{\beta_o} C_o{}^{\gamma_o} \qquad (2.40)$$

where

$$\phi^i = A_\mu{}^a, \quad R^i{}_{\alpha_o} = D^\mu{}_{ab} = \delta_{ab}\partial_\mu + f_{ab}{}^c A^\mu{}_c, \quad T^{\alpha_o}{}_{\beta_o\gamma_o} = (\tfrac{1}{2})\, f^a{}_{bc}, \; C_o{}^{\alpha_o} = \eta^a,$$

$$A(\phi) = F^2, \quad F^a{}_{\mu\upsilon} = \partial_\mu A_\upsilon{}^a - \partial_\upsilon A_\mu{}^a - f^a{}_{bc} A_\mu{}^b A_\upsilon{}^b.$$

It can be directly shown that it satisfies the master equation by using the algebra of structure constants $f^a{}_{bc}$ and the gauge invariance of $A(\phi)$. To fix the gauge we enlarge the set of fields so that S reads

$$S(\Phi, \Phi^*) = S(\Phi_{min}, \Phi^*{}_{min}) + \bar{\eta}^*{}_a \pi^a.$$

The following choice of gauge fermion

$\psi = \bar{\eta}_a \partial_\mu A_\mu{}^a$

generatesthegaugecondition

$\delta\psi / \delta\bar{\eta}_a = \partial_\mu A_\mu{}^a = 0.$

Indeed $\bar{\eta}_a$ is the antighost and the gauge fixed action takes the following familiar form

$$S_{g.\ f.} = \int dx^\mu \{F^2 + \bar{\eta}_a \partial_\mu D_\mu{}^{ab} \eta_b + \pi_a \partial_\mu A_\mu{}^a \}.$$

First-stagetheories:

When the theory is first-stage introduction of $C_o{}^{\alpha_o}$ is not sufficent to have a proper solution of the master equation. Because the rank of $R^i{}_{\alpha_o}$ is smaller than m_o. In fact now the ghost field $C_o{}^{\alpha_o}$ has become a gauge field so that we have to introduce another field, which we can call ghost of ghost, such that

$$(\delta_l/\delta C^*{}_{o\alpha_o}) (\delta_r/\delta C_1{}^{\alpha_1}) S(\Phi, \Phi^*) \Big|_{\Phi^*=0} = Z_1{}^{\alpha_o}{}_{\alpha_1}.$$

So that the rank of the Hessian now reads

$$r = \text{rank}\ \delta_l \delta_r A(\phi) / \delta\phi^i \delta\phi^j \Big|_{\phi_o} + 2\ \text{rank}\ R^i{}_{\alpha_o} \Big|_{\phi_o} + 2\ \text{rank}\ Z_1{}^{\alpha_o}{}_{\alpha_1} \Big|_{\phi_o}$$

$$= [n - (m_o - m_1)] + 2(m_o - m_1) + 2 m_1 = n + m_o + m_1.$$

Thus the minimal set is

$$\Phi^A{}_{min} = \{ \phi^i , C_o{}^{\alpha_o}, C_1{}^{\alpha_1} \}.$$

The ghost numbers of the new fields are defined as

$$gh\ (C_1{}^{\alpha_1}, C^*{}_{1\alpha_1}) = (2, -3).$$

The requirement of ghost number conservation leads to

$$S(\Phi_{min}, \Phi^*{}_{min}) = A(\phi) + \phi^*{}_i R^i{}_{\alpha_o} C_o{}^{\alpha_o} + C^*{}_{o\alpha_o} (Z_1{}^{\alpha_o}{}_{\alpha_1} C_1{}^{\alpha_1} + T^{\alpha_o}{}_{\beta_o \gamma_o} C_o{}^{\beta_o} C_o{}^{\gamma_o}) +$$

$$+ \phi^*{}_i \phi^*{}_j (B^{ji}{}_{\alpha_1} C_1{}^{\alpha_1} + E^{ji}{}_{\alpha_o\beta_o} C_o{}^{\beta_o} C_o{}^{\alpha_o}) + \ldots \quad (2.41)$$

and use of this in (2.24) gives the following structure equations

$$(\delta_r A(\phi)/\delta\phi^i) R^i{}_{\alpha_o} C_o{}^{\alpha_o} = 0$$

$$R^i{}_{\alpha_o} Z_1{}^{\alpha_o}{}_{\alpha_1} C_1{}^{\alpha_1} - 2 (\delta_r A(\phi)/\delta\phi^j) B^{ji}{}_{\alpha_1} C_1{}^{\alpha_1} (-1)^{\varepsilon_i} = 0$$

$$(2.42)$$

$$(\delta_r R^i{}_{\alpha_o} C_o{}^{\alpha_o} /\delta\phi^j) R^j{}_{\beta_o} C_o{}^{\beta_o} + R^i{}_{\alpha_o} T^{\alpha_o}{}_{\beta_o \gamma_o} C_o{}^{\beta_o} C_o{}^{\gamma_o} -$$

$$- 2 (\delta_r A(\phi)/\delta\phi^j) E^{ji}{}_{\alpha_o\beta_o} C_o{}^{\beta_o} C_o{}^{\alpha_o}) (-1)^{\varepsilon_i} = 0$$

and so on.

Gauge fixing can be obtained by introducing some new fields and defining the gauge fermion suitably as before. In this case we enlarge the set of fields by introducing three pairs of new fields to write the proper solution of the master equation as

$$S(\Phi, \Phi^*) = S(\Phi_{min}, \Phi^*_{min}) + B_o^{*\alpha_o} \pi_{o\alpha_o} + B_1^{*\alpha_1} \pi_{1\alpha_1} + C'_1{}^*{}_{\alpha_1} \pi'_1{}^{\alpha_1} \tag{2.42}$$

These new fields have the following Grassmann parities and the ghost numbers

$B_{o\alpha_o}$	$\pi_{o\alpha_o}$	$B_{1\alpha_1}$	$\pi_{1\alpha_1}$	$C'_1{}^{\alpha_1}$	$\pi'_1{}^{\alpha_1}$	
$\varepsilon_{\alpha_o}+1$	ε_{α_o}	ε_{α_1}	$\varepsilon_{\alpha_1}+1$	ε_{α_1}	$\varepsilon_{\alpha_1}+1$	(2.43)
-1	0	-2	-1	0	1	

We require that the ghost number of the gauge fermion is -1 in agreement with the assigned ghost numbers to the fields which take part in the minimal set. From this requirment one can derive the ghost numbers of the antifields of the new fields. Gauge conditions are generated by the gauge fermion as follows

$$\delta\psi / \delta B_{o\alpha_o} = 0 , \qquad \delta\psi / \delta B_{1\alpha_1} = 0 , \qquad \delta\psi / \delta C'_1{}^{\alpha_1} = 0$$

which will follow by integration over the Lagrange multipliers, after restricting (2.42) toΣ. Again the requirment of fixing the gauge completely restricts the gauge fermion at the stationary point of S, denoted by Φ_o, as follows

$$\text{rank } [\delta_l\delta_r \psi / \delta B_{o\alpha_o} \delta\phi^i]\Big|_{\Phi=\Phi_o} = m_o - m_1$$

$$\text{rank } \{[\delta_l\delta_r \psi / \delta B_{o\alpha_o} \delta\phi^i] R^i{}_{\beta_o}\}\Big|_{\Phi=\Phi_o} = m_o - m_1$$

$$\text{rank } \{[\delta_l\delta_r \psi / \delta B_{1\alpha_1} \delta C_o{}^{\alpha_o}] Z_1{}^{\alpha_o}{}_{\beta_1}\}\Big|_{\Phi=\Phi_o} = m_1$$

$$\mathrm{rank}\,\{\,\overline{Z}_1{}^{\alpha_1}{}_{\alpha_o}\,[\delta_l\delta_r\,\psi\,/\,\delta B_{0\alpha_0}\,\delta C'_1{}^{\beta_1}\,]\}\Big|_{\Phi=\Phi_o} = m_1$$

In the above equation $\overline{Z}_1{}^{\alpha_1}{}_{\alpha_o}$ are the left zero-eigenvalue eigenvectors of $\delta_l\delta_r\,\psi\,/\,\delta B_{0\alpha_o}\,\delta\phi^i$ and have the following properties

$$\mathrm{rank}\,\overline{Z}_1{}^{\alpha_1}{}_{\alpha_o}\Big|_{\Phi=\Phi_o} = m_1\,, \qquad \varepsilon(Q_1{}^{\alpha_1}{}_{\alpha_o}) = \varepsilon_{\alpha_o} + \varepsilon_{\alpha_1}.$$

At each higher stage the formalisim becomes more complicated, but the essence of the procedure of finding the proper solution of the master equation and also the gauge fixing does not change.

III. POINT PARTICLE:

A) FIRST QUANTIZATION:

In a flat configuration space a relativistic point particle is described through the following action [32]

$$A = \int_0^T L\, d\tau = -m \int_0^T d\tau\, [-(dx^\mu / d\tau)\, (dx_\mu / d\tau)]^{1/2}.$$

The canonical momentum which results from this action

$$p^\mu = m\, [-(dx^\mu / d\tau)\, (dx_\mu / d\tau)]^{-1/2}\, (dx^\mu / d\tau)$$

implies the following constraint

$$\Phi = p^\mu p_\mu + m^2 = 0$$

and the canonical Hamiltonian, which is defined as

$$H_o = p_\mu\, dx^\mu / d\tau - L$$

yields zero. As we have seen in chapter II, we may equivalently describe this system with the following action

$$A = \int_0^T (p_\mu\, dx^\mu / d\tau - e^\omega \Phi)d\tau$$

where the Lagrange multiplier, e^ω, depends on the proper time τ. As it is pointed out in the previous chapter, A is invariant under the following transformations [33]

$$\delta x^\mu = 2\,\varepsilon\, e^\omega p^\mu \quad ; \quad \delta p^\mu = 0 \quad ; \quad \delta\omega = e^\omega\, d(\varepsilon\, e^\omega) / d\tau$$

if the gauge parameter satisfies

$$\varepsilon\,(T) = \varepsilon\,(0) = 0\,. \tag{3.1}$$

A legitimate gauge fixing condition must fix the gauge completely. This requires that a gauge preserving transformation should yield $\varepsilon\,(\tau) = 0$ for all τ [32]. In fact the following gauge condition

$$\frac{d\omega}{d\tau} = 0 \tag{3.2}$$

is preserved by the transformations where $\partial^2\varepsilon(\tau) / \partial\tau^2 = 0$ which leads to $\varepsilon\,(\tau) = 0$ for all τ, due to the boundary conditions (3.1). If the gauge condition is ω=constant, the gauge preserving transformations have the solution $\varepsilon(\tau)$ = constant, where the constant is not necessarily equal to zero. The gauge condition (3.2) leads to the following BRST action [33,34]

$$A = \int_0^T \{\, p_\mu\ dx^\mu / d\tau\ -\ e^\omega\,(p^\mu p_\mu + m^2) + k\, d\omega/d\tau - db/d\tau\,\rho - \eta\, dc/d\tau\ + \eta\,\rho\, e^\omega \,\}\, d\tau \tag{3.3}$$

where the last two terms define $dc/dt = \rho\ e^\omega$, which we need for not having second time derivatives in the action. The action is Hermitian since x^μ, p^μ, c, ρ, ω, k are Hermitian while b, η are anti-Hermitian. These variables obey the following Poisson bracket relations

$$\{p_\mu , x_\upsilon\} = \delta_{\mu\upsilon} \qquad \{k , \omega\} = 1$$

$$\{\rho , b\} = 1 \qquad \{\eta , c\} = -1 \tag{3.4}$$

and the others are zero(3.3) is invariant under the following transformations

$$\delta x^\mu = 2\,\lambda c\, e^\omega p^\mu \quad ; \quad \delta p^\mu = 0 \quad ; \quad \delta\omega = \lambda\rho\, e^\omega \quad ; \quad \delta\rho = 0 \quad ;$$

$$\delta b = -\lambda\, e^\omega (k - c\,\eta) \quad ; \quad \delta c = \lambda\, c e^\omega \rho \quad ; \quad \delta\eta = \lambda\, e^\omega (p^\mu p_\mu + m^2 - \eta\,\rho) \quad ;$$

$$\delta k = -\lambda\, e^\omega [\, c\, (p^\mu p_\mu + m^2 - \eta\,\rho) + \rho\, k\,].$$

The above transformations are generated by the BRST charge

$$Q = e^\omega [\, c\, (p^\mu p_\mu + m^2 - \eta\,\rho) + \rho\, k\,] \tag{3.5}$$

as $\delta F = \{\lambda Q , F\}$. The Hamiltonian , Δ, can be read off from (3.3) :

$$\Delta = \alpha^{-1} (p^\mu p_\mu + m^2 - \eta\,\rho) \tag{3.6}$$

where we defined $e^\omega \equiv \alpha^{-1}$ and it is the remnant of string length parameter. The Hamiltonian can also be written as

$$\Delta = -\{ Q , \eta \}. \tag{3.7}$$

Now there is also an anti-BRST charge which is given as

$\overline{Q} = e^{\omega}[\, b\,(\, p^{\mu}p_{\mu} + m^2 - \eta\,\rho) + \eta\, k\,]\,.$

Q and $\overline{Q}$ obey the Poisson bracket relations

$$\{Q\,,Q\} = \{\overline{Q}\,,\overline{Q}\} = \{\,Q\,,\overline{Q}\,\} = 0\;. \tag{3.8}$$

The first quantization can be achived as usual by making use of the rule

$$\{\;\;,\;\;\} \rightarrow \;\; -i\hbar^{-1}\,[\;\;,\;\;]_{\pm}\;.$$

Thus by the following replacements

$x^{\mu} \rightarrow x^{\mu}\;\;,\;\; c \rightarrow c\;\;,\;\; b \rightarrow b\;\;,\;\; \omega \rightarrow \omega\;\;,\;\; p^{\mu} \rightarrow -i\partial/\partial x^{\mu}\;\;,\;\; \rho \rightarrow i\partial/\partial b\;\;,$

$\eta \rightarrow -i\partial/\partial c\;\;,\;\; k \rightarrow -i\partial/\partial\omega,$

we may write in configuration space the following normal ordered, first quantized operators

$$\Delta = \alpha^{-1}\,[\,-\partial^2/\partial\, {x_{\mu}}^2 + m^2 \;\; -(\partial/\partial c)\,(\partial/\partial b)\,] \tag{3.9}$$

$$Q = (\tfrac{1}{2})\,\{\,c\,,\Delta\,\} - (\partial/\partial\alpha\,)\,(\partial/\partial b\,) + (2\alpha)^{-1}\partial/\partial b \tag{3.10}$$

B) GAUGE INVARIANT FIELD THEORY AND GAUGE FIXING[16,22]:

We may use the BRST-charge (3.7) to formulate a point particle field theory. For this purpose we introduce a function

$$\chi = \chi(x^{\mu}, c, b, \tau, \alpha) \equiv \chi (z, \tau, \alpha).$$

To this field, which lives in a superspace which has d+2 bosonic and 2 anticommuting coordinates, we may attribute an algebraic ghost number, which is defined by the following operator

$$N = b\, \partial /\partial b - c\, \partial /\partial c. \tag{3.11}$$

By making use of the BRST charge and some field functionals it is possible to write down a gauge invariant free action as

$$A_{free} = \int dz\, d\tau\, (d\alpha/\alpha) \; \{ \chi (z, \tau, -\alpha)\, i\, d \chi (z, \tau, \alpha) /d\tau \; - \; \chi (z, \tau, -\alpha)\, Q\, \tilde{\chi} (z, \tau, \alpha) \; -$$

$$- \; \tilde{\chi} (z, \tau, -\alpha)\, Q\, \chi (z, \tau, \alpha) \} \tag{3.12}$$

where the range of α is

$$-\infty < \alpha < \infty \qquad \text{and} \qquad \Phi(z, \tau, -\alpha) = \Phi^{\dagger}(z, \tau, \alpha) \qquad \text{for } \alpha > 0. \tag{3.13}$$

$\chi (z, \tau, \alpha)$ and $\tilde{\chi} (z, \tau, \alpha)$ are two independent fields which behave under the algebraic ghost number operator, (3.8), as

$$N\chi(z,\tau,\alpha)=0$$

$$N\tilde{\chi}(z,\tau,\alpha)=\tilde{\chi}(z,\tau,\alpha) \tag{3.14}$$

and Q decreases the algebraic ghost number by one. (3.9) is invariant under the gauge transformations

$$\delta\chi(z,\tau,\alpha)=-Q\Lambda(z,\tau,\alpha)$$

$$\delta\tilde{\chi}(z,\tau,\alpha)=-i\,d\Lambda(z,\tau,\alpha)/d\tau \tag{3.15}$$

where the anticommuting gauge parameter $\Lambda(z,\tau,\alpha)$ has algebraic ghost number 1 and it is an anti-Hermitian field. Later in this chapter, we wil see that this system is equivalent to the point particle by a suitable gauge fixing and with some physical state conditions.

We may add an interaction term to (3.9) by making use of the following manifestly Osp(d-1,1|2) invariant vertex (see the next section for the orthosymplectic group)

$$v(1,2,3)=\delta(c_1-c_2)\,\delta(c_2-c_3)\,\delta(b_1-b_2)\,\delta(b_2-b_3)\,\delta(x^{\mu}{}_1-x^{\mu}{}_2)\,\delta(x^{\mu}{}_2-x^{\mu}{}_3)\,.$$

$$.\,\delta(\alpha_1+\alpha_2+\alpha_3) \tag{3.16}$$

which is also invariant under the cyclic permutations of 1, 2, 3. Thus the interaction part of the actionis

$$A_1=g\prod_{r=1}^{3}\int dz_r\,(d\alpha_r/\alpha_r)\,d\tau\,v(1,2,3)\,c_1\tilde{\chi}(1)\chi(2)\chi(3) \tag{3.17}$$

where $X(r) = X(z_r, \tau, \alpha_r)$ and $\tilde{X}(r) = \tilde{X}(z_r, \tau, \alpha_r)$. Due to the definition (3.13) the free part of the action, where the α dependence is explicitly shown is obviously Hermitian. However in (3.17) α dependence is not explicit, but due to $\delta(\alpha_1 + \alpha_2 + \alpha_3)$ term in the vertex its Hermiticityisguaranteed.

The whole action, $A = A_{free} + A_I$, is invariant under the following gauge transformations

$$\delta X(z_1, \tau, \alpha_1) = -Q(1)\Lambda(z_1, \tau, \alpha_1) + g\int dz_2 dz_3 (d\alpha_2/\alpha_2)(d\alpha_3/\alpha_3) f(1,2,3) X(2)\Lambda(3)$$

$$\equiv R_{(1)}\Lambda \tag{3.18}$$

$$\delta \tilde{X}(z_1, \tau, \alpha_1) = -i\, d\Lambda(z_1, \tau, \alpha_1)/d\tau - g\int dz_2 dz_3 (d\alpha_2/\alpha_2)(d\alpha_3/\alpha_3) f(1,2,3) \tilde{X}(2)\Lambda(3)$$

$$\equiv R_{(2)}\Lambda \tag{3.19}$$

where the anticommuting local gauge parameter Λ has algebraic ghost number 1 and it is an anti-Hermitian field. f is defined in terms of the vertex operators and c_1 as

$$f(1,2,3) = [v(1,2,3) + v(1,3,2)]\, c_1 \tag{3.20}$$

$R_{(1)}$ and $R_{(2)}$ are the gauge generators which are symbolically written as

$$R_{(1)} = -Q(1) + g\, F(1,2,3)\, X(2)$$

$$R_{(2)} = -i\, d/d\tau - g\, F(1,2,3)\, \tilde{X}(2) \tag{3.21}$$

in a notation which is self-explaining.

By using the cyclic invariance of the vertex the change in the free part of the action under (3.18) and (3.19) is

$$\delta A_{Free} = g \prod_{r=1}^{3} \int dz_r\, (d\alpha_r/\alpha_r)\, d\tau\, v(1,2,3)\, c_1 \Big\{ d[\Lambda(1) \times (2) \times (3)] / d\tau \;+$$

$$+\, d\Lambda(1)/d\tau \times (2) \times (3) \;-\; \tilde{\chi}(1)[Q(2) \times (2)\, \Lambda(3) \;+\; Q(3)\, \Lambda(2) \times (3)] \;-$$

$$-\, \tilde{\chi}(1) Q(1)\, [\times (2)\, \Lambda(3) \;+\; \Lambda(2) \times (3)] \Big\}. \qquad (3.22)$$

By observing the following property of the vertex

$$[\Sigma_{r=1,2,3}\, Q(r)]\, v(1,2,3)\, c_1 = 0. \qquad (3.23)$$

and making use of the fact that the total time derivative in (3.22) vanishs due to the boundary conditions which we suppose to hold, one can write (3.22) as follows

$$\delta A_{Free} = g \prod_{r=1}^{3} \int dz_r\, (d\alpha_r/\alpha_r)\, d\tau\, v(1,2,3)\, c_1 \Big\{ d\Lambda(1)/d\tau \times (2) \times (3) \;+$$

$$+\; \tilde{\chi}(1)\, Q(2)\, \Lambda(2) \times (3) + \tilde{\chi}(1) \times (2) Q(3)\, \Lambda(3) \Big\} \qquad (3.24)$$

which is indeed equal to the change of the interaction part which results from the Abelian gauge transformations up to a minus sign, so that the whole action is invariant under the gauge transformations (3.18) and (3.19).

Now we want to fix the gauge such that the resulting action would have Osp(d,2 | 2) invariance. Let us write (3.18) and (3.19) in a unified notation, which allows us to use the Batalin-Vilkovisky method as formulated in the chapter II, as

$$\delta\phi^i = R^i_{\alpha_o}(\phi)\,\delta\theta^{\alpha_o}$$

where $\phi^1 = X$ and $\phi^2 = \tilde{X}$. There are two different ways of enumerating $R^i_{\alpha_o}(\phi)$ by α_o. One of them is taking

$$\delta\theta^{\alpha_o} = \Lambda \qquad \text{so} \qquad R^i_{\alpha_o} = R_{(i)}\,,$$

and the second one consists of setting $\alpha_o = \{z, z'\}$ so that

$$R^i_z = R_{(1)}\,\delta^{i1} \quad , \quad R^i_{z'} = R_{(2)}\,\delta^{i2} \quad ;$$

$$\delta\theta^z = \delta^z_{z_0}\,\Lambda(z_o) \quad , \quad \delta\theta^{z'} = \delta^{z'}_{z_0}\,\Lambda(z_o) \quad .$$

These two different ways of enumeration have quite different features:

In the former case the theory is a zero stage one due to the fact that $R^i_{\alpha_o}$ does not have any zero eigenvalue eigenvector.

However the latter enumeration leads to an infinite stage theory due to the fact that

$$[R_{(1)}]^2 \Big|_{\phi_o} = 0,$$

i. e. $Z_1{}^z{}_{z1} = Z_2{}^{z1}{}_{z2} = \ldots = R_{(1)}$; $\quad Z_1{}^{z'}{}_{z'1} = Z_2{}^{z'1}{}_{z'2} = \ldots = 0.$

But in both of the cases we have to introduce only one ghost field $C = C(z, \alpha, \tau)$. Indeed this is the normal procedure for the zero stage theory. In the other case since $R_{(1)}$ changes the algebraic ghost number by -1 at each stage we must introduce a new ghost field which has an algebraic ghost number one more than the precedent, however algebraic ghost number of a field can not be more than 1 and less than -1.

Therefore the minimal content of Φ^A is

$$\Phi^A{}_{min} = (\chi , \tilde{\chi} , C)$$

and we attribute them the following gauge ghost numbers

$$(g.\, g.\, n.) (\chi , \tilde{\chi} , C) = (0 , 0 , 1).$$

Introduce also the antifields

$$\Phi^*{}_{min\,A} = (\chi^* , \tilde{\chi}^* , C^*) \qquad ; \qquad (g.\, g.\, n.)\Phi^*{}_{min\,A} = (-1, -1, -2).$$

Thus S can be written as

$$S(\Phi_{min}, \Phi^*{}_{min}) = A + (\chi^* R_{(1)} C + \tilde{\chi}^* R_{(2)} C + h.\, c.\,)$$

$$= A - \int dz\, d\tau\, (d\alpha/\alpha) \{ \tilde{\chi}^* (z, \tau, -\alpha)\, i\, d\, C(z, \tau, \alpha) /d\tau +$$

$$+ \tilde{\chi}^* (z, \tau, -\alpha)\, Q(1)\, C (z, \tau, \alpha) \} + g \prod_{r=1}^{3} \int dz_r\, (d\alpha_r/\alpha_r)\, d\tau \qquad (3.25)$$

$$f(1, 2, 3) \{ \chi^*(1)\, \chi (2)\, C(3) - \tilde{\chi}^*(1)\, \tilde{\chi} (2)\, C(3) \} + h.\, c.$$

The Noether identities guarantee that it satisfies the master equation.

The suitable gauge fermion which will lead to an Osp(d,2|2) invariant gauge fixed action is

$$\psi = -\int dz\, d\tau\, (d\alpha/\alpha)\ \{ B(z,\tau,-\alpha)\,[\tilde{X}^*(z,\tau,\alpha) - \partial X(z,\tau,\alpha)/\partial c] + h.c. \} \quad (3.26)$$

where B is the antighost field which has the same algebraic and gauge ghost numbers, namely -1. Making use of (3.25) and (3.26) in (2.37) yields the gauge fixed action

$$S = S(\Phi^A, \delta\psi/\delta\Phi^A) - \int dz\, d\tau\, (d\alpha/\alpha)\ \{ \Sigma(z,\tau,-\alpha)\,[\tilde{X}(z,\tau,\alpha) - \partial X(z,\tau,\alpha)/\partial c] + h.c. \} \quad (3.27)$$

Integration over the Lagrange multiplier Σ sets $\tilde{X}(z,\tau,\alpha) = \partial X(z,\tau,\alpha)/\partial c$ in A. Hence due to

$$\Delta(1) = \{ \partial/\partial c_1\ ,\ Q(1) \}$$

we find the gauge fixed action as

$$S_{g.f.} = \int dz\, d\tau\, (d\alpha/\alpha)\, \{ X(z,\tau,-\alpha)(id/d\tau - \Delta)X(z,\tau,\alpha) + B(z,\tau,-\alpha)(id/d\tau - \Delta)C(z,\tau,\alpha) + C(z,\tau,-\alpha)(id/d\tau - \Delta)B(z,\tau,\alpha) \} + g \prod_{r=1}^{3} \int dz_r\, (d\alpha_r/\alpha_r)\, d\tau\, v(1,2,3)\{ X(1)X(2)X(3) + B(1)X(2)C(3) + B(1)C(2)X(3) \}. \quad (3.28)$$

As before the interaction part of the action is not explicitly Hermitian. However it should be read such that when explicitly written all of the possible α configuration terms are present. By taking the algebraic ghost numbers of the fields into account, one can see that they have the following expansion in terms of anticommuting coordinates

$$X = \phi_o + cb\,\phi_3 , \qquad C = b\,\phi_1 , \qquad B = c\,\phi_2 .$$

Thus if we define a superfield as

$$\phi = \phi_o + b\,\phi_1 + c\phi_2 + cb\,\phi_3$$

(3.28) can be written as follows:

$$S_{g.f.} = \int dz\, d\tau\, (d\alpha/\alpha)\, \phi(z, \tau, -\alpha)\, (id/d\tau - \Delta)\, \phi(z, \tau, \alpha) +$$

$$+ \quad g \prod_{r=1}^{3} \int dz_r\, (d\alpha_r/\alpha_r)\, d\tau\, v(1, 2, 3)\ \phi(1)\, \phi(2)\, \phi(3) \qquad (3.29)$$

We will examine the symmetries and their consequences of this action in the next section.

C) PROPERTIES OF THE GAUGE FIXED ACTION:

Before inquiring about the properties of the gauge fixed action (3.29), let us clarify the relation between the Osp(d|2) invariance [35,36] and the Parisi-Sourlas symmetry [17].

Let us define a graded manifold with $x^{\alpha} = (x^{\mu}, \theta_1, \theta_2)$ where x^{μ} is bosonic, $\mu = 0, 1, \ldots, d-1$, and

$$\theta_1^2 = \theta_2^2 = \{\theta_1, \theta_2\} = 0 . \tag{3.30}$$

On this manifold Osp(d|2) is equivalent to super-rotations or in other words Osp(d|2) is group of transformations which preserves the distance

$$(x - y)^2 = \eta_{\alpha\beta} (x - y)^{\alpha} (x - y)^{\beta},$$

$$\eta_{\alpha\beta} = \begin{pmatrix} \eta_{\mu\upsilon} & 0 \\ 0 & \varepsilon^{ab} \end{pmatrix}$$

where ε^{ab} is completely antisymmetric unit matrix, and a, b=1, 2. osp(d|2) algebra is given as

$$[J_{\mu\upsilon}, J_{\rho\sigma}] = i(\eta_{\mu\sigma} J_{\upsilon\rho} + \eta_{\upsilon\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\upsilon\sigma} - \eta_{\upsilon\sigma} J_{\mu\rho}),$$

$$[J_{ab}, J_{cd}] = \varepsilon_{ad} J_{bc} + \varepsilon_{bc} J_{ad} + \varepsilon_{ac} J_{bd} + \varepsilon_{bd} J_{ac},$$

$$[J_{\mu\upsilon}, P_{\rho}] = i(\eta_{\upsilon\rho} P_{\mu} - \eta_{\mu\rho} P_{\upsilon}),$$

$$[J_{\mu\upsilon}, J_{\rho a}] = i(\eta_{\upsilon\rho} J_{\mu a} - \eta_{\mu\rho} J_{\upsilon a}),$$

$$[J_{ab}, P_c] = \varepsilon_{bc} P_a + \varepsilon_{ac} P_b ,$$

$$[J_{ab}, J_{\rho c}] = \varepsilon_{bc} J_{\rho a} + \varepsilon_{ac} J_{\rho b} , \quad (3.31)$$

$$\{J_{\mu a}, P_b\} = -i\, \varepsilon_{ab} P_\mu ,$$

$$[P_\mu , J_{\upsilon a}] = i\, \eta_{\mu\upsilon} P_a ,$$

$$\{J_{\mu a}, J_{\upsilon b}\} = -\eta_{\mu\upsilon} J_{ab} + i\, \varepsilon_{ab} J_{\mu\upsilon} ,$$

In this superspace the osp(d|2) algebra has a differential representation

$$J_{\mu\upsilon} = -i(x_\mu\, \partial/\partial x^\upsilon \ - \ x_\upsilon\, \partial/\partial x^\mu) \ , \ J_{ab} = \theta_a\, \partial/\partial\theta_b \ + \ \theta_b\, \partial/\partial\theta_a ,$$

$$J_{\mu a} = \theta_a\, \partial/\partial x^\mu + x_\mu\, \partial/\partial\theta_a \ , P_\mu = -i\partial/\partial x^\mu \quad , P_a = \partial/\partial\theta_a . \quad (3.32)$$

Let us take a combination of $J_{\mu a}$ with a constant, anticommuting tensor $\lambda_{\mu a}$ as follows

$$\lambda_{\mu a} J_{\mu a} = -\, \theta_a \lambda_{\mu a}\, \partial/\partial x^\mu \ + \lambda_{\mu a} x^\mu\, \partial/\partial\theta_a \equiv A + B$$

where the operators A and B are defined as

$$A = -\theta_1 \lambda_{\mu 1}\, \partial/\partial x^\mu + \lambda_{\mu 2}\, x^\mu\, \partial/\partial\theta_2$$

$$B = -\theta_2 \lambda_{\mu 2}\, \partial/\partial x^\mu + \lambda_{\mu 1}\, x^\mu\, \partial/\partial\theta_1 .$$

Let us expand a superfield in terms of the anticommuting coordinates as

$$\Phi = \Phi_0 + \theta_1 \Phi_1 + \theta_2 \Phi_2 + \theta_1\theta_2 \Phi_3. \tag{3.33}$$

Transformations of the coefficents under the operator A , $\delta_A\Phi = [A , \Phi]$, lead to

$$\delta_A\Phi_0 = \lambda_{\mu 2}\, x^\mu\, \Phi_2 , \qquad \delta_A\Phi_1 = -\lambda_{\mu 1}\, \partial\Phi_0/\partial x^\mu + \lambda_{\mu 2}\, x^\mu\, \Phi_3 ,$$

$$\delta_A\Phi_2 = 0 , \qquad \delta_A\Phi_3 = \lambda_{\mu 1}\, \partial\Phi_2/\partial x^\mu , \tag{3.34}$$

and similarly under B:

$$\delta_B\Phi_0 = \lambda_{\mu 1}\, x^\mu\, \Phi_1 , \qquad \delta_B\Phi_2 = -\lambda_{\mu 2}\, \partial\Phi_0/\partial x^\mu - \lambda_{\mu 1}\, x^\mu\, \Phi_3 ,$$

$$\delta_B\Phi_2 = 0 , \qquad \delta_B\Phi_3 = -\lambda_{\mu 2}\, \partial\Phi_1/\partial x^\mu . \tag{3.35}$$

We regognize the Parisi-Sourlas symmetry in (3.34) by choosing the parameters as

$$\lambda_{\mu 1} = 2\alpha\, \varepsilon_\mu, \qquad \lambda_{\mu 2} = -\alpha\, \varepsilon_\mu. \tag{3.36}$$

Thus the Parisi-Sourlas symmetry is a part of the Osp(d|2) group transformations and the generators of it are given by

$$J_{\mu\theta_1} = \theta_1\, \partial/\partial x^\mu + x^\mu\, \partial/\partial\theta_2. \tag{3.37}$$

Now we are ready to examine the symmetries of the gauge fixed action. Let us rewrite the free part of the action (3.29) by introducing the Fourier transformed field

$$\phi(\alpha) = (\alpha/2\pi)\int_{-\infty}^{\infty} dy^- \exp(i\alpha y^-)\tilde{\varphi}(y^-) \tag{3.38}$$

where $\tilde{\varphi}(y^-)$ is a real function. Then the free action yields

$$S_{free} = \int dy^{\tilde{\alpha}}\, db\, dc\, \tilde{\varphi}\,(\square + m^2)\,\tilde{\varphi}, \tag{3.39}$$

where

$$y^{\tilde{\alpha}} = x^{\mu} \quad \text{for} \quad \tilde{\alpha}=\mu=0,1,\ldots,d-1, \qquad y^+ = y^{d+1} + y^d = \tau, \qquad y^- = y^{d+1} - y^d$$

$$\square = (\partial/\partial y^{\tilde{\alpha}})\,\partial/\partial y_{\tilde{\alpha}} - (\partial/\partial c)\,\partial/\partial b. \tag{3.40}$$

(3.39) is manifestly Osp(d,2|2) invariant. One of the generators of the Parisi-Sourlas symmetry

$$J_{-c} = c\,\partial/\partial y^- + y^-\,\partial/\partial b$$

when written in the momentum space and in the light cone gauge where $p^- = \Delta$, leads to

$$J_{-c} = (\tfrac{1}{2}\alpha)\{c, p_{\mu}^2 + m^2 - (\partial/\partial c)(\partial/\partial b)\} - (\partial/\partial\alpha)(\partial/\partial b) + (1/2\alpha)\,\partial/\partial b = Q \tag{3.41}$$

Similarly anti-BRST charge can be shown to be generated by J_{-b}. Consequently BRST and anti-BRST charges are contained in the Osp(d,2|2) group which means that they are contained in the same group as the Poincare group.

The interaction part of the action (3.29), can be written as $\int dy^{\tilde{\alpha}} dc\, db\, \phi^3(y^{\tilde{\alpha}}, c, b)$, by taking into consideration the delta functions which are present in v(1, 2, 3). So that it is invariant under the Parisi-Sourlas transformations.

BRST invariance of the free part of the gauge fixed action follows directly from its manifest Osp(d,2|2) invariance in the superspace ($y^{\tilde{\alpha}}$, c, b) and the fact that BRST charge is one of the generators of this group. However even without writing it in the superspace one can see that the free part of (3.29) is BRST invariant. The BRST transformation

$$\delta\, \phi(z, \tau, -\alpha) = \kappa\, Q\, \phi(z, \tau, -\alpha) \tag{3.42}$$

where κ is a constant, anticommuting, anti-Hermitian parameter and leaves the free part of (3.29) invariant because

$$[Q\,,\Delta] = [Q\,, d/d\tau] = 0.$$

Similarly one can show that it is also anti-BRST invariant, by using $\overline{Q}$ instead of Q.

Interaction part is not invariant under the Abelian transformations, indeed variation of it under (3.42) yields

$$\delta S_I = -g \prod_{r=1}^{3} \int dz_r\, (d\alpha_r/\alpha_r)\, d\tau\, v(1, 2, 3)\{\kappa[c_1\Delta(1) + \alpha_1{}^{-1}\partial/\partial c_1 \;-\; (\partial/\partial\alpha_1)\partial/\partial b_1\; \phi(1)]\, \phi(2)\, \phi(3) + \\ + \text{cyclic}\;\} \tag{3.43}$$

By using the boundary condition

$$\phi(z, \tau, \pm\infty) \;=\; 0 \tag{3.44}$$

we may perform a partial integration over α_1 in the third term of (3.43) to obtain

$$-g \prod_{r=1}^{3} \int dz_r\, (d\alpha_r/\alpha_r)\, d\tau\, \nu(1,2,3)\, \delta'(\alpha_1+\alpha_2+\alpha_3)\kappa\Big[\sum_r \partial/\partial b_r\Big]\, \phi(1)\,\phi(2)\,\phi(3)$$

where $\nu(1, 2, 3)$ is the vertex which does not include $\delta(\alpha_1+\alpha_2+\alpha_3)$ and $\delta'(\alpha_1+\alpha_2+\alpha_3)$ is the derivative of the delta function with respect to its argument. Now the partial integration over the first quantization antighost b_r yields

$$-g \prod_{r=1}^{3} \int dz_r\, (d\alpha_r/\alpha_r)\, d\tau\, \delta(c_1 - c_2)\, \delta(c_2 - c_3)\, \delta(x^\mu{}_1 - x^\mu{}_2)\, \delta(x^\mu{}_2 - x^\mu{}_3)$$

$$\delta'(\alpha_1+\alpha_2+\alpha_3)\kappa\Big[\Big(\sum_r \partial/\partial b_r\Big)\, \delta(b_1 - b_2)\, \delta(b_2 - b_3)\Big]\, \phi(1)\,\phi(2)\,\phi(3) \;=\; 0\,.$$

Similarly the second term of (3.43) gives a vanishing contribution so that the change in the interaction part is

$$\delta S_I = -g \prod_{r=1}^{3} \int dz_r\, (d\alpha_r/\alpha_r)\, d\tau\, v(1,2,3)\kappa\Big[\sum_r c_r \Delta(r)\Big]\;\; \phi(1)\phi(2)\,\phi(3)$$

Now if we add a ϕ^2 term to (3.42) as follows

$$\delta_g\phi(z, \tau, -\alpha) = \int dz_2\, dz_3 (d\alpha_2/\alpha_2)\, (d\alpha_3/\alpha_3)\, v(1,2,3)\kappa c_1\, \phi(2)\,\phi(3) \qquad (3.45)$$

the kinetic part of (3.29) changes such that the total action is invariant. This follows from two facts. The time derivative becomes a total derivative so that it does not contribute due to the vanishing of the fields at the τ-boundary and also the following property of Δ

$$\int dz\, d\tau\, (d\alpha/\alpha)\ X(z,\tau,-\alpha)\,\Delta\,\Lambda(z,\tau,\alpha) = -\int dz\, d\tau\, (d\alpha/\alpha)\,[\Delta X(z,\tau,-\alpha)]\ \Lambda(z,\tau,\alpha).$$

It can be easly checked that S is invariant under the full BRST-transformations to all orders in g.

Thus in the interacting field theory case the BRST-charge gets a contribution which is linear in the field functional. The physical state condition is still given as

$$Q\phi_{phys.} = 0 \quad ; \qquad N\phi_{phys.} = 0. \tag{3.46}$$

Expansion in the anticommuting coordinates leads to the following equation for the components of $\phi_{phys.}$

$$(-\partial^2 + m^2)\,\phi_o + (½)\phi_2 - \partial\phi_2/\partial\omega = 0. \tag{3.47}$$

The two states which are related as

$$\phi' = \phi + Q\,\Lambda \tag{3.48}$$

are equivalent due to the fact that $Q\Lambda$, where Λ is a field which is given in components as $\Lambda = b\Lambda_1$, has a vanishing norm. i. e. it is a supirious state. In components (3.48) leads to

$$\phi'_o = \phi_o + \alpha^{-1}\partial\Lambda_1/\partial\omega + (½)\,\alpha^{-1}\Lambda_1\ , \quad \phi'_2 = \phi_2 + \alpha^{-1}(-\partial^2 + m^2)\,\Lambda_1. \tag{3.49}$$

A_1 can be used to set $\phi_2 = 0$, so that ϕ_0 satisfies the Klein-Gordon equation

$$(-\partial^2 + m^2)\,\phi_0 = 0.$$

ϕ_0 can not be removed, because all the fields so also A_1 must vanish at $\alpha=\pm\infty$ [see (3.44)]. But it is still possible to set the α dependence of the physical state arbitrarily. Therefore the free, physical, on-shell states satisfy the following equations which result from the above gauge fixing and from the equations of motion

$$d\,\phi_{phys.}/d\tau = 0 \qquad , \qquad (-\partial^2 + m^2)\,\phi_{phys.} = 0 \tag{3.50}$$

and they may have an arbitrary α dependence. But since the theory is invariant under $Osp(d,2\,|\,2)$ (or equivalently has Parisi-Sourlas symmetry), in perturbative expansion the Green functions, which result when we use the free physical on-shell states as the external states, will be in the following form

$$\int dx^\mu\, dy^-\, dy^+\, db\, dc\, F(\, Y\cdot x + Y^+ y^-\,,\ x\cdot x + y^+ y^- + cb) \tag{3.51}$$

where Y is an arbitrary vector which has d+2 bosonic components. If we go to the Euclidean coordinates on y^- and y^+ we may change the coordinates as

$$y^- = \rho\,\exp(-i\theta) \qquad y^+ = \rho\,\exp(i\theta) \tag{3.52}$$

so that the integration over the θ variable effectively sets $Y^+ = 0$. Thus (3.51) yields

$$\int dx^\mu\, d\rho^2\, db\, dc\, G(\, Y\cdot x\ ,\ x\cdot x + \rho^2 + cb) \;=\; \int dx^\mu\, G(\, Y\cdot x\ ,\ x\cdot x\,) \tag{3.53}$$

which follows from the fact that anticommuting coordinates behave like negative dimensional bosoniccoordinates:

$$\int db\, dc\, f(bc) = -df(z)/dz|_{z=0} = \lim_{d\to -2} \int d^d r\, f(r^2) = \lim_{d\to -2} S_d \int r^{d-1} dr\, f(r^2) \qquad (3.54)$$

where S_d is the surface of the unit sphere in d dimensions:

$$S_d = 2\pi^{d/2} / \Gamma(d/2).$$

Thus the theory due to its invariances leads to the same scattering amplitudes of the point particle if the external states are taken on-shell as well as physical.

IV. STRINGS:

A) FIRST QUANTIZATION[16]:

The canonical procedure, when one writes the string Lagrangian as the area of the world sheet, leads to a vanishing canonical Hamiltonian and two constraints. In terms of these constraints, as we have seen in the second chapter, the string (we deal only with the open bosonic strings) can be described with the following action

$$A = \int_0^T d\tau \int_0^\pi d\sigma \{ p_\mu \ \partial x^\mu / \partial\tau \ - \lambda^+ [p_\mu{}^2 + (\partial x^\mu / \partial\sigma)^2] \ - \ 2\lambda^- p_\mu \partial x^\mu / \partial\sigma \} \tag{4.1}$$

where λ^+ and λ^- are Lagrange multipliers. By making use of the usual Poisson bracket relations of p and x, one can see that the transformations of them generated by the constraints do not leave the action invariant. The invariance can be regained by transforming also the Lagrange multipliers. Hence the following set of transformations

$$\delta x^\mu = 2\{ \kappa^+ p^\mu + \kappa^- \partial x^\mu / \partial\sigma \}$$

$$\delta p^\mu = 2\partial\{ \kappa^- p^\mu + \kappa^+ \partial x^\mu / \partial\sigma \} / \partial\sigma$$

$$\delta\lambda^+ = \partial\kappa^+/\partial\tau - 2\lambda^-\partial\kappa^+/\partial\sigma + 2\,\partial\lambda^-/\partial\sigma\,\kappa^+ - 2\,\lambda^+\partial\kappa^-/\partial\sigma + 2\,\partial\lambda^+/\partial\sigma\,\kappa^- \tag{4.2}$$

$$\delta\lambda^- = \partial\kappa^-/\partial\tau - 2\lambda^-\partial\kappa^-/\partial\sigma + 2\,\partial\lambda^-/\partial\sigma\,\kappa^- - 2\,\lambda^+\partial\kappa^+/\partial\sigma + 2\,\partial\lambda^+/\partial\sigma\,\kappa^+$$

leave the action invariant if the following boundary conditions are satisfied

$$\int_0^{\pi} d\sigma \,[\, p_\mu{}^2 - (\partial x^\mu / \partial\sigma)^2]\, \kappa^+ \Big|_0^T = 0 \qquad (4.3)$$

$$\int_0^T d\tau \,\{\, \kappa^+ [\, (\partial x^\mu / \partial\sigma)\, \partial x^\mu / \partial\tau \; - \lambda^- [\, p_\mu{}^2 + (\partial x^\mu / \partial\sigma)^2] \; - 2\,\lambda^+ (\partial x^\mu / \partial\sigma)\, p_\mu] \; +$$

$$+ \; \kappa^- [\, p^\mu\, \partial x^\mu / \partial\tau \; - \lambda^+ [\, p_\mu{}^2 + (\partial x^\mu / \partial\sigma)^2] \; - 2\,\lambda^- (\partial x^\mu / \partial\sigma)\, p_\mu]\} \Big|_0^{\pi} = 0 \qquad (4.4)$$

In the gauge $\lambda^- = 0$ due to the open string boundary conditions $\partial x^\mu / \partial\sigma = 0$ at $\sigma = 0, \pi$, (4.4) will be satisfied if we put the following conditions on the gauge parameters

$$\kappa^- (\tau, 0) = \kappa^- (\tau, \pi) = 0$$

$$\kappa^+ (T, \sigma) = \kappa^+ (0, \sigma) = 0. \qquad (4.5)$$

Now we will show that if $\lambda^- = 0$ and $\lambda^+ = a$, where a is σ, τ independent, the most general gauge condition can be reached by a general coordinate transformation. Demonstrating this for the infinitisimal case is sufficent. So take

$$\lambda^- = \varepsilon^- , \qquad \lambda^+ = a + \varepsilon^+$$

where ε^- and ε^+ represents the small deviations from the above values of Lagrange multipliers. When we use them in (4.2), by neglecting the second order terms, they yield

$$\delta\lambda^{+} = \partial \kappa^{+} / \partial\tau + 2a \ \partial \kappa^{-} / \partial\sigma$$

(4.6)

$$\delta\lambda^{-} = \partial \kappa^{-} / \partial\tau - 2a \ \partial \kappa^{+} / \partial\sigma$$

Integration of the first equation of (4.6) over σ and using (4.5) lead to

$$\delta \left(\int_0^{\pi} \lambda^{+} \, d\sigma \right) = d\left(\int_0^{\pi} \kappa^{+} \, d\sigma \right) / d\tau. \tag{4.7}$$

$\kappa(\tau) = \int_0^{\pi} \kappa^{+} \, d\sigma$ depends on τ due to the boundary conditions (4.5). As we have already seen in point particle case a gauge preserving transformation should lead to $\kappa(\tau) = 0$ for al τ, which can be satisfied if the gauge fixing includes time derivatives of the gauge parameter.

Let us set in (4.6), without the loss of generality , $a = ½$. In this case we may solve those equations for arbitrary $\delta\lambda^{-}$, $\delta\lambda^{+}$ if the following condition is satisfied

$$\int_0^T d\tau \int_0^{\pi} d\sigma \ \delta\lambda^{+} = 0. \tag{4.8}$$

In fact for a given point $z=\tau+i\sigma$ define $z_1 = z^* + 2\pi i$, $z_2 = -z^* + 2T$, $z_3 = -z + 2\pi i + 2T$ as it is shown in the figure 3.

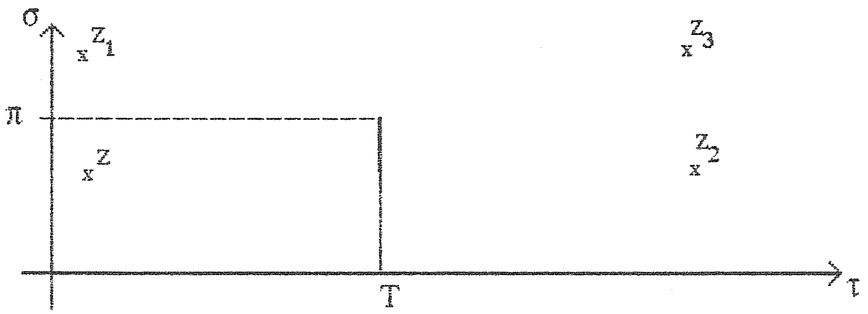

Figure 3.

By making use of the Weierstrass zeta function $\zeta(z)$ with a real period 2T and an imaginary period $2\pi i$ one can see that the solution of (4.6) with $\sigma = ½$ is given as follows

$$\kappa^+(z) = (1/4\pi) \int_0^T d\tau' \int_0^\pi d\sigma' \, [\zeta(z-z') + \zeta(z^*-z'^*) + \zeta(z-z'_1) + \zeta(z^*-z'_1{}^*) + \zeta(z-z'_2) + \\ + \zeta(z^*-z'_2{}^*) + \zeta(z-z'_3) + \zeta(z^*-z'_3{}^*)] \, \delta\lambda^+(z') + \\ + (1/4\pi i) \int_0^T d\tau' \int_0^\pi d\sigma' [\, \zeta(z-z') - \zeta(z^*-z'^*) - \zeta(z-z'_1) + \zeta(z^*-z'_1{}^*) - \zeta(z-z'_2) + \\ + \zeta(z^*-z'_2{}^*) + \zeta(z-z'_3) - \zeta(z^*-z'_3{}^*)] \, \delta\lambda^-(z') \tag{4.9}$$

$$\kappa^-(z) = (-1/4\pi i) \int_0^T d\tau' \int_0^\pi d\sigma' \, [\zeta(z-z') - \zeta(z^*-z'^*) + \zeta(z-z'_1) - \zeta(z^*-z'_1{}^*) + \zeta(z-z'_2) + \\ - \zeta(z^*-z'_2{}^*) + \zeta(z-z'_3) - \zeta(z^*-z'_3{}^*)] \, \delta\lambda^+(z') + \\ + (1/4\pi) \int_0^T d\tau' \int_0^\pi d\sigma' [\, \zeta(z-z') + \zeta(z^*-z'^*) - \zeta(z-z'_1) - \zeta(z^*-z'_1{}^*) - \zeta(z-z'_2) + \\ - \zeta(z^*-z'_2{}^*) + \zeta(z-z'_3) + \zeta(z^*-z'_3{}^*)] \, \delta\lambda^-(z') \tag{4.10}$$

The following property of the zeta function [37]

$$\zeta(z + 2w) = \zeta(\, z\,) + 2\,\zeta(\, w\,)$$

where 2w is a primitive period of the Weierstrass zeta function, leads directly to the requirment (4.8).

Now we are ready to fix the gauge. For this purpose we write the Lagrange multipliers as

$$\lambda^{+} = e^{\omega}(1 + \tilde{\lambda}^{+}), \qquad \lambda^{-} = e^{\omega}\tilde{\lambda}^{-} \tag{4.11}$$

where ω is σ independent. A legitimate gauge condition is to choose

$$d\omega/dt = 0, \quad \tilde{\lambda}^{-} = \tilde{\lambda}^{+} = 0. \tag{4.12}$$

To find the BRST action let us decompose also the gauge parameters:

$$\kappa^{+} = e^{\omega}(\kappa + \tilde{\kappa}^{+}), \qquad \kappa^{-} = e^{\omega}\tilde{\kappa}^{-} . \tag{4.13}$$

As usual the BRST transformations can be found through the replacement

$$\kappa = \Omega c, \quad \tilde{\kappa}^{+} = \Omega C^{+}, \quad \tilde{\kappa}^{-} = \Omega C^{-} \tag{4.14}$$

where Ω is an anticommuting, constant parameter and c, C^{+}, C^{-} are the anticommuting ghost coordinates. By using these replacements one can obtain the following variations

$$\delta\omega = e^{-\omega} d(e^{\omega}\Omega c)/d\tau + \dots ; \quad \delta\tilde{\lambda}^{+} = e^{-\omega} d(e^{\omega}\Omega C^{+})/d\tau + \dots ;$$
$$\delta\tilde{\lambda}^{-} = e^{-\omega} d(e^{\omega}\Omega C^{-})/d\tau + \dots \tag{4.15}$$

where " + ... " denotes the terms which are propotional to $\tilde{\lambda}^{-}$ and $\tilde{\lambda}^{+}$.

To reach the BRST action we have to take into account the gauge fixing conditions which can be achived by adding " $k d\omega/dt + K^{+}\tilde{\lambda}^{+} + K^{-}\tilde{\lambda}^{-}$ " to the action. When the variations (4.15) are performed we may absorb the terms which are propotional to $\tilde{\lambda}^{\pm}$ in $K^{\pm}$. So if we redefine the terms which we have to add the original action as $K^{+'}\tilde{\lambda}^{+} + K^{-'}\tilde{\lambda}^{-}$ it is sufficent to take the variations as if they only consist of the the terms which are written explicitly in (4.15). Now $K^{+'}\tilde{\lambda}^{+} + K^{-'}\tilde{\lambda}^{-}$ term can be droped, since in effect it is disconnected from the rest of the action.

The terms which do not depend on the σ variable can be treated similar to the point particle. The variations of p^{μ} and x^{μ} and ω can be read off from (4.2) and (4.14)-(4.15)

$$\delta x^{\mu} = \Omega e^{\omega} (c p^{\mu} + C^{+} p^{\mu} + C^{-} \partial x^{\mu} / \partial\sigma)$$

$$\delta p^{\mu} = \Omega e^{\omega} \partial (p^{\mu} C^{-} + C^{+} \partial x^{\mu} / \partial\sigma + c \partial x^{\mu} / \partial\sigma) / \partial \sigma \qquad (4.16)$$

$$\delta\omega = (\tfrac{1}{2}) \Omega e^{\omega} \rho$$

where $\rho = e^{-\omega} dc/d\tau$. Due to the gauge fixing conditions (4.12) the original action which now reads $p_{\mu} \partial x^{\mu} / \partial\tau - e^{\omega} [p_{\mu}^{2} + (\partial x^{\mu} / \partial\sigma)^{2}]$, is not invariant under the above variations. We define the canonical partners of the ghost fields by adding the following term to the action

$$B^{-} dC^{-}/d\tau + B^{+} dC^{+}/d\tau . \qquad (4.17)$$

Now the original action is invariant under the transformations (4.16) in accord with (4.17) if we add a term like

$$-2e^{\omega} (B^{-} \partial C^{+}/\partial\sigma + B^{+} \partial C^{-}/\partial\sigma) \qquad (4.18)$$

and define the BRST transformation of the new fields as

$$\delta B^- = -\Omega e^{\omega} p^{\mu} \partial x_{\mu}/\partial\sigma \ + \dots$$

$$\delta B^+ = -(1/2)\Omega e^{\omega} [p_{\mu}^2 + (\partial x^{\mu}/\partial\sigma)^2] \ + \dots \tag{4.19}$$

where " + ... " indicates some other terms which can be present. In fact we define the BRST action as

$$A = \int d\tau \int_0^{\pi} d\sigma \{ p_{\mu} \ \partial x^{\mu}/\partial\tau + k d\omega/d\tau - db/d\tau \ \rho - \eta dc/d\tau + B^- dC^-/d\tau + B^+ dC^+/d\tau -$$

$$- e^{\omega} \{ p_{\mu}^2 + (\partial x^{\mu}/\partial\sigma)^2 - \eta\rho + 2 (B^-\partial C^+/\partial\sigma + B^+\partial C^-/\partial\sigma)\} \tag{4.20}$$

which is invariant under the variations (4.16) if we define the variations of the ghost fields as follows (in the following it is not explicitly shown but the σ integrals are normalized to 1)

$$\delta\rho = 0 \ ; \quad \delta\eta = (1/2)\,\Omega e^{\omega} \int_0^{\pi} d\sigma \{[p_{\mu}^2 + (\partial x^{\mu}/\partial\sigma)^2] - \eta\rho + 2 (B^-\partial C^+/\partial\sigma + B^+\partial C^-/\partial\sigma)\}$$

$$\delta c = (1/2)\,\Omega e^{\omega} \int_0^{\pi} d\sigma \{ 2C^+\partial C^-/\partial\sigma + 2C^-\partial C^+/\partial\sigma + c\rho \}$$

$$\delta b = (1/2)\,\Omega e^{\omega} \int_0^{\pi} d\sigma \{ B^+C^+ + B^-C^- + c\eta - k \}$$

$$\delta C^- = \Omega e^{\omega} \{ c\,\partial C^+/\partial\sigma - (1/2)\rho\, C^- - C^+\partial C^+/\partial\sigma + C^-\partial C^-/\partial\sigma \}$$

$$\delta C^+ = \Omega e^{\omega} \{ c\,\partial C^-/\partial\sigma - (1/2)\rho\, C^+ - C^+\partial C^-/\partial\sigma + C^-\partial C^+/\partial\sigma \} \tag{4.21}$$

$$\delta B^- = \Omega\, e^{\omega} \{ - p_\mu \partial x^\mu / \partial\sigma + c\, \partial B^+/\partial\sigma + (½)\rho\, B^- + 2\,\eta\, \partial C^+/\partial\sigma +$$

$$+ \partial[B^+C^+ + B^-C^-]/\partial\sigma + B^+\partial C^+/\partial\sigma + B^-\partial C^-/\partial\sigma \}$$

$$\delta B^+ = \Omega\, e^{\omega} \{ - (½)[p_\mu{}^2 + (\partial x^\mu / \partial\sigma)^2] + c\, \partial B^-/\partial\sigma + (½)\rho\, B^+ + 2\,\eta\, \partial C^-/\partial\sigma +$$

$$+ \partial[B^+C^- + B^-C^+]/\partial\sigma + B^+\partial C^-/\partial\sigma + B^-\partial C^+/\partial\sigma \}$$

$$\delta k = - (½)\, \Omega\, e^{\omega} \int_0^{\pi} d\sigma \, \{ c[p_\mu{}^2 + (\partial x^\mu / \partial\sigma)^2 - \eta\rho + 2\, (B^-\partial C^+/\partial\sigma + B^+\partial C^-/\partial\sigma] +$$

$$+ C^+ [p_\mu{}^2 + (\partial x^\mu / \partial\sigma)^2] + 2\, C^- p_\mu \partial x^\mu / \partial\sigma + \rho k - \rho\, [B^+C^+ + B^-C^-] +$$

$$+ 2\, B^+(C^+\partial C^-/\partial\sigma + C^-\partial C^+/\partial\sigma) + 2B^-(C^+\partial C^+/\partial\sigma + C^-\partial C^-/\partial\sigma) +$$

$$+ 2\eta(C^+\partial C^-/\partial\sigma + C^-\partial C^+/\partial\sigma) \}.$$

The BRST charge which follows from these transformations can be seen to be

$$Q = (½)\; e^{\omega} \int_0^{\pi} d\sigma \, \{ c[p_\mu{}^2 + (\partial x^\mu / \partial\sigma)^2 - \eta\rho + 2\, (B^-\partial C^+/\partial\sigma + B^+\partial C^-/\partial\sigma] + C^+ [p_\mu{}^2 +$$

$$+(\partial x^\mu / \partial\sigma)^2] + 2\, C^- p_\mu \partial x^\mu / \partial\sigma + \rho k - \rho\, [B^+C^+ + B^-C^-] + 2\, B^+(C^+\partial C^-/\partial\sigma +$$

$$+C^-\partial C^+/\partial\sigma) + 2B^-(C^+\partial C^+/\partial\sigma + C^-\partial C^-/\partial\sigma) + 2\eta(C^+\partial C^-/\partial\sigma + C^-\partial C^+/\partial\sigma) \}. \quad (4.22)$$

This can also be verified by showing that the transformations (4.21) are generated by the charge given in (4.22), due to the following Poisson bracket relations

$$\{c\,,\eta\} = -1 \;;\; \{b\,,\rho\} = 1 \;;\; \{\omega\,,k\} = 1 \;;\; \{p^{\mu}\,,x^{\nu}\} = \eta^{\nu\mu}\,\delta(\sigma-\sigma') \;;$$

$$\{C^{+}(\sigma)\,,B^{+}(\sigma')\} = \delta(\sigma-\sigma') \;;\quad \{C^{-}(\sigma)\,,B^{-}(\sigma')\} = \delta(\sigma-\sigma')\,. \tag{4.23}$$

The fields b, $B^{\pm}$, η, Ω are anti-Hermitian while c, $c^{\pm}$, x^{μ}, p^{μ}, b, ρ, ω are Hermitian.

The Hamiltonian which we read off from (4.20) is

$$\Delta = e^{\omega}\int_0^{\pi} d\sigma\,[\,p_{\mu}^{2} + (\partial x^{\mu}/\partial\sigma)^{2} - \eta\rho + 2\,(B^{-}\partial C^{+}/\partial\sigma + B^{+}\partial C^{-}/\partial\sigma)\,] \tag{4.24}$$

and it can be written in terms of the BRST-charge as

$$\Delta = -2\,\{\eta\,,Q\} \tag{4.25}$$

Replacing the Poisson brackets with commutators or anticommutators as

$$\{\;,\;\} \rightarrow -i\,\hbar^{-1}\,[\;,\;]_{\pm}\;,$$

leads to the first quantization of the string. These relations can be realized by the following replacements of the classical fields

$$x^{\mu} \rightarrow x^{\mu} \;,\quad p^{\mu} \rightarrow -i\partial/\partial x^{\mu} \;,\quad c \rightarrow c \;,\quad b \rightarrow b \;,\quad \omega \rightarrow \omega \;,$$

$$\rho \rightarrow i\partial/\partial b \;,\quad \eta \rightarrow -i\partial/\partial c \;,\quad k \rightarrow -i\partial/\partial\omega \;,\quad C^{+} \rightarrow C(\sigma)\;, \tag{4.26}$$

$$B^{+} \rightarrow \delta/\delta C(\sigma) \;,\quad B^{-} \rightarrow B(\sigma) \;,\quad C^{-} \rightarrow \delta/\delta B(\sigma)\,.$$

By making use of the following boundary conditions, which are consistent with the BRST variations,

$$\partial x^\mu(\sigma)/\partial\sigma \Big|_{\sigma=0,\,\pi} = \partial C(\sigma)/\partial\sigma \Big|_{\sigma=0,\,\pi} = B(\sigma)\Big|_{\sigma=0,\,\pi} = 0$$

one can expand the phase space variables in their normal modes. For the following it is useful to extend the range of σ as $-\pi \le \sigma \le \pi$, by defining

$$x^\mu(-\sigma) = x^\mu(\sigma)\ , \qquad C(-\sigma) = C(\sigma)\ , \qquad B(-\sigma) = -B(\sigma) \quad \text{for} \quad 0 \le \sigma \le \pi\ . \tag{4.27}$$

So the normal mode expansion reads

$$\begin{aligned} p^\mu(\sigma) - \partial x^\mu(\sigma)/\partial\sigma &= \sum_{n=-\infty}^{\infty} \alpha_n^\mu\, e^{in\sigma} \\ \delta/\delta C(\sigma) + iB(\sigma) &= \sum_{n=-\infty}^{\infty} \bar\beta_n\, e^{in\sigma} \\ -i\delta/\delta B(\sigma) + C(\sigma) &= \sum_{n=-\infty}^{\infty} \beta_n\, e^{in\sigma} \end{aligned} \tag{4.28}$$

where the following Hermiticity relations hold

$$\alpha_{-n}^\mu = \alpha^{\dagger}{}_n^\mu \ ; \quad \beta_{-n} = \beta^{\dagger}{}_n \quad ; \quad \bar\beta_{-n} = -\bar\beta^{\dagger}{}_n \qquad n>0$$

$$\alpha_0^\mu = p^\mu \quad ; \quad \beta_0 = \partial/\partial c \quad ; \quad \bar\beta_0 = \partial/\partial b.$$

The normal mode operators obey the following (anti) commutator relations

$$\{\bar{\beta}_n , \beta_m\} = \delta_{n+m,0} \quad , [\, \alpha_n{}^{\mu} , \alpha_m{}^{\nu} \,] = n\, \delta^{\mu\nu}\, \delta_{n+m,0} \quad \text{for} \quad n > 0. \tag{4.29}$$

B) GAUGE INVARIANT FIELD THEORY AND GAUGE FIXING[16, 22]:

For formulating the field theory we may use either an oscillator representation which can be achived by making use of the expansions (4.28) or a functional representation. To define the latter representation let us introduce the following variables

$$P^{\mu}(\sigma) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha_n{}^{\mu}\, e^{in\sigma} \; ; \; P_{\bar{\beta}}(\sigma) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |n|^{1/2}\, \bar{\beta}_n\, e^{in\sigma} \; ; \; P_{\beta}(\sigma) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |n|^{1/2} \beta_n\, e^{in\sigma} \tag{4.30}$$

where they have the usual representation in coordinate space in terms of their canonical partners:

$$X^{\mu}(\sigma) \quad , \quad \bar{\beta}\,(\sigma) \quad , \quad \beta\,(\sigma) \tag{4.31}$$

The field theory can be formulated by introducing a field functional

$$\chi = \chi\,(\, X^{\mu}(\sigma), \bar{\beta}\,(\sigma), \beta\,(\sigma), x^{\mu} , b, c, \alpha, \tau\,) \tag{4.32}$$

which is written in the functional representation. By making use of the BRST-charge and the field functional the free field theory can be described via the following action

$$A_{free} = \int dZ(\sigma)\, dz\, d\tau\, (d\alpha/\alpha)\, \{\, \chi(\, Z(\sigma), z, \tau, -\alpha\,)\, i d\chi(\, Z(\sigma), z, \tau, \alpha\,) / d\tau \;\; - \tag{4.33}$$

$$- \tilde{X}(Z(\sigma), z, \tau, -\alpha)\, Q\, X(Z(\sigma), z, \tau, \alpha) - X(Z(\sigma), z, \tau, -\alpha)\, Q\, \tilde{X}(Z(\sigma), z, \tau, \alpha)\}$$

where

$$Z(\sigma) = \{ X^{\mu}(\sigma), \bar{\beta}(\sigma), \beta(\sigma) \}, \quad z = \{x^{\mu}, b, c\}$$

or more rigoursly $Z(\sigma)$ indicates all the non-zero mode operators in the oscillator representation. In the oscillator representation the normal ordered Q is given as

$$Q = e^{\omega}\{ cK - 2\, (\partial/\partial c)\, M + d + D + (½)(\partial/\partial b)\, [-c\partial/\partial c + N' + \partial/\partial\omega]\} \tag{4.34}$$

where the entries of it are defined as follows

$$K = (½)\, (\alpha^{\mu}{}_{0})^{2} + \sum_{n=1}^{\infty}[\alpha^{\mu}{}_{-n}\alpha^{\mu}{}_{n} + n\beta_{-n}\bar{\beta}_{n} + n\bar{\beta}^{\dagger}{}_{n}\beta_{n}] - 1$$

$$M = \sum_{n=1}^{\infty} n\, \beta_{-n}\, \beta_{n}$$

$$d = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n\, \beta_{-n}\, [\, L_{n} + (m+n)\, \bar{\beta}^{\dagger}{}_{m-n}\beta_{m} + (½)\, (m-n)\beta_{-m}\bar{\beta}_{m+n}] \tag{4.35}$$

$$D = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n\, \beta_{n}\, [\, L_{-n} + (m+n)\, \beta_{-m}\bar{\beta}_{m-n} + (½)(m-n)\, \bar{\beta}^{\dagger}{}_{m+n}\beta_{m}\,]$$

$$N' = \sum_{n=1}^{\infty} [\bar{\beta}^{\dagger}{}_{n}\beta_{n} + \beta^{\dagger}{}_{n}\bar{\beta}_{n}] + ½$$

where

$$L^{x}{}_{n} = (1/4) \int_{-\pi}^{\pi} [p^{\mu}(\sigma) - \partial x^{\mu}(\sigma)/\partial\sigma]^{2}\, \exp(-in\sigma)\, d\sigma.$$

As it is obvious we have already fixed the ambiguity in the normal ordering of the zero mode operators $\alpha^\mu{}_o$. The BRST-charge can be written as

$$Q = \alpha^{-1} Q_{old} + \alpha^{-1} \partial/\partial b (- c\partial/\partial c + N' + \partial/\partial\omega)$$

where Q_{old} is the BRST-charge of Kato-Ogawa [8], which results when one introduces only one pair of ghost zero mode operators. Nilpotency of the BRST-charge:

$$Q^2 = 0 \qquad (4.36)$$

follows from the relations

$$(Q_{old})^2 = 0, \qquad [N , d] = -d, \qquad [N , D] = -D, \qquad [N , M] = -2M,$$

which are valid in 26 dimensional space-time.

In the operator representation we introduce a ground state which has the following properties

$$(\alpha_n{}^\mu, \beta_n, \bar\beta_n) |0\rangle = 0 \quad \text{for} \quad n>0 \qquad (4.37)$$

to define a string field functional in this representation as

$$|X\rangle = \Sigma (1 / \sqrt{N!M!})\beta_{-n_1} \dots \beta_{-n_N} \bar\beta_{-m_1} \dots \bar\beta_{-m_M} \Phi_{n_1 \dots n_N m_1 \dots m_M}(\alpha_{-n}, \alpha, \tau, c, b)|0\rangle . \qquad (4.38)$$

To the fields which were used in (4.33) we assign the following algebraic ghost numbers

$N \chi = 0, \quad N \tilde{\chi} = \tilde{\chi}$

where N is given as follows

$$N = \sum_{n=1}^{\infty} \bar{\beta}^{\dagger}{}_{n}\beta_{n} + \beta^{\dagger}{}_{n}\bar{\beta}_{n}] + b\,\partial/\partial b + c\,\partial/\partial c . \tag{4.39}$$

As it can easly be seen Q decreases the algebraic ghost number by 1.

(4.33) is invariant under the following gauge transformations

$$\delta \chi (Z(\sigma), z, \tau, \alpha) = - Q\Lambda (Z(\sigma), z, \tau, \alpha)$$

$$\delta \tilde{\chi} (Z(\sigma), z, \tau, \alpha) = - i d\Lambda(Z(\sigma), z, \tau, \alpha) / d\tau + Q \Lambda' (Z(\sigma), z, \tau, \alpha) \tag{4.40}$$

where Λ and Λ' carry algebraic ghost numbers 1 and 2 respectively and they are anti-Hermitian.

Interactions can be introduced via three string vertex V which is obtained as a generalization of the light cone vertex [13]:

$$V(1, 2, 3) = \delta(Y^{3}(\eta_3) - \theta(-\eta_3 + \rho_2\pi)\, Y^{2}(\eta_2) - \theta(\eta_3 - \rho_2\pi)\, Y^{1}(\eta_1))\delta(\alpha_1 + \alpha_2 + \alpha_3). \tag{4.41}$$

In the functional representation which we have written (4.41), we kept he α_r part in momentum space instead of writing it in the configuration space as the other part and

$$Y^{r} = (x^{\mu}(\eta_r), B(\eta_r), C(\eta_r), x^{\mu}, b, c)$$

$$- \pi \leq \eta_r \leq \pi \quad ; \quad \rho_2 = \alpha_2 / \alpha_3.$$

The coefficents $\overline{N}_{nm}^{rs}$ and $\overline{N}_n^{rs}$:

$$\overline{N}_{nm}^{rs} = (-\alpha_1\, \alpha_2\, \alpha_3 / m\, \alpha_r + n\, \alpha_s)\, (m\, n / \alpha_r\, \alpha_s)\, f_m(\gamma_r)\, f_n(\gamma_s) \quad ; \qquad \gamma_r = -\alpha_{r+1} / \alpha_r$$

$$\overline{N}_n^{rs} = (1/\alpha_r)\, f_n(\gamma_r)\, (\alpha_r\, \delta_{r+1,s} - \alpha_{r+1} \delta_{r,s})$$

$$f_n(\gamma) = (1/n!)\, (n\gamma - 1)\, (n\gamma - 2) \, . \; . \; . \; . \; (n\gamma - n + 1),$$

can be used to write the three string vertex in oscillator representation as follows

$$V(1,2,3) = {}_1\langle 0|\; {}_2\langle 0|\; {}_3\langle 0|\; \exp\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{r,s=1,2,3} [\overline{N}_{nm}^{rs}\, (\alpha_n^{\mu r}\, \alpha_m^{\mu s}/2 - n\beta_n^r \bar{\beta}_m^s) +$$

$$+ \overline{N}_n^{rs}\, (\alpha_n^{\mu r}\, \alpha_0^{\mu s}/2 - n\beta_n^r \bar{\beta}_0^s - n\beta_0^r \bar{\beta}_n^s) \}$$

$$\exp\{[\alpha_1 \ln(-\alpha_1/\alpha_3) + \alpha_2 \ln(-\alpha_2/\alpha_3)] \sum_r (L_0^r - 1)/\alpha_r \} \qquad (4.42)$$

$$\delta(\sum_r \alpha_0^{\mu r})\, \delta(\sum_r \alpha_r)\, \delta(\sum_r \beta_0^r)\, \delta(\sum_r \bar{\beta}_0^r)$$

where

$$L_n = \sum_{m=-\infty}^{\infty} [\alpha_{n+m}^{\mu}\, \alpha_{-m}^{\mu}/2 - n\bar{\beta}_{n+m}\, \beta_{-m}] + \bar{\beta}_n\, \beta_0. \qquad (4.43)$$

The full action is obtained by adding the following term to (4.33)

$$A_I = g \prod_{r=1}^{3} \int dZ_r(\sigma)\, dz_r\, d\tau\, (d\alpha_r/\alpha_r)\, V(1,2,3)\, C(\sigma_{int})\, \tilde{X}(1) X(2) X(3) \qquad (4.44)$$

where now $C(\sigma_{int})$ possesses also the zero mode component c. $\chi(-\alpha) = \chi^\dagger(\alpha)$ for $\alpha > 0$, so that A_{free} is explicitly Hermitian but because of using a compact notation in (4.44) this is not explicit in A_I. Nevertheless the delta function of $\Sigma\alpha_r$'s guarantees the Hermiticity of the interaction part of theaction.

A is invariant under the following gauge transformations up to order g,

$$\delta \chi(1) = -Q(1)\Lambda(1) + g \prod_{r=1}^{3} \int dZ_r(\sigma)\, dz_r\, (d\alpha_r/\alpha_r)\, F(1, 2, 3)\, \chi(2)\, \Lambda(3)$$

$$\delta \tilde{\chi}(1) = -i d\Lambda(1)/d\tau - g \prod_{r=1}^{3} \int dZ_r(\sigma)\, dz_r\, (d\alpha_r/\alpha_r)\, F(1, 2, 3)\, \tilde{\chi}(2)\, \Lambda(3) + \tag{4.45}$$

$$+ Q(1)\Lambda'(1) - g \prod_{r=1}^{3} \int dZ_r(\sigma)\, dz_r\, (d\alpha_r/\alpha_r)\, F(1, 2, 3)\, \chi(2)\, \Lambda'(3)$$

where F is defined in terms of V and $C(\sigma_{int})$ as follows

$$F(1, 2, 3) = [\, V(1, 2, 3) + V(1, 3, 2)\,]\, C(\sigma_{int}) . \tag{4.46}$$

The invariance follows from the fact that integration of total time derivative is zero and also [1, 12, 38]

$$[\, \sum_{r=1}^{3} Q(r)\,] V(1, 2, 3)\, C(\sigma_{int}) = 0 \tag{4.47}$$

When the gauge generators are written symbolically as

$$R_1 = -Q + F'(1,2,3)\,\chi(2)$$

$$R_2 = -i d/d\tau - F'(1,2,3)\,\tilde{\chi}(2) \qquad (4.48)$$

$$R_3 = Q - F'(1,2,3)\,\chi(2)$$

they allow us to give the gauge transformations (4.45) in the following compact form

$$\begin{pmatrix} \chi \\ \tilde{\chi} \end{pmatrix} = R \begin{pmatrix} \Lambda \\ \Lambda' \end{pmatrix} \equiv \begin{pmatrix} R_1 & 0 \\ R_2 & R_3 \end{pmatrix} \begin{pmatrix} \Lambda \\ \Lambda' \end{pmatrix}$$

In the same notation we may also write the matrices Z_r, which were introduced in the chapter II as follows

$$Z_r = \begin{pmatrix} Z_{r1} & 0 \\ Z_{r2} & Z_{r3} \end{pmatrix}$$

Now it is easy to see that

$$R_1 = Z_{r1} \;;\quad R_2 = Z_{r2} \;;\quad R_3 = Z_{r3} \quad;\quad r = 1, 2, \ldots$$

Thus the theory is an infinite stage gauge theory. We have to introduce an infinite number of ghost fields G_r. At each stage we need two ghost fields which differ in algebraic ghost number. Hence we enumerate them as $G_r^{\alpha_r}$ and $G_r^{\alpha_{r+1}}$. Their algebraic and gauge ghost numbers are listed below.

	$G_0{}^{\alpha_0}$	$G_0{}^{\alpha_1}$	$G_1{}^{\alpha_1}$	$G_1{}^{\alpha_2}$	$G_2{}^{\alpha_2}$	......
gauge ghost # :	1	1	2	2	3	......
algebraic ghost # :	1	2	2	3	3	......

To use the Batalin-Vilkovisky method of quantization we also introduce $G^*{}_{r\,\alpha_r}$ and $G^*{}_{r\,\alpha_{r+1}}$ and attribute them the following ghost numbers

	$G^*{}_{0\,\alpha_0}$	$G^*{}_{0\,\alpha_1}$	$G^*{}_{1\,\alpha_1}$	$G^*{}_{1\,\alpha_2}$	$G^*{}_{2\,\alpha_2}$	....
gauge ghost # :	- 2	- 2	- 3	- 3	- 4	....
algebraic ghost #:	- 1	- 2	- 2	-3	- 3	....

To reach a unified notation which will be useful in the sequel, let us define

$$G_{-1}{}^{\alpha_{-1}} = \chi, \qquad G^*{}_{-1\,\alpha_{-1}} = \chi^*, \qquad G_{-1}{}^{\alpha_0} = \tilde{\chi}, \qquad G^*{}_{-1\,\alpha_0} = \tilde{\chi}^*$$

where $G^*{}_{-1\,\alpha_{-1}}$ and $G^*{}_{-1\,\alpha_0}$ have the same gauge ghost number, namely -1 but different algebraic ghost numbers: the former has 0 and the latter has -1. All $G_r{}^{\alpha_r}$, $G^*{}_{r\,\alpha_{r+1}}$ and $G_r{}^{\alpha_{r+1}}$, $G^*{}_{r\,\alpha_r}$ are, respectively, commuting and anticommuting fields.

Appling the Batalin-Vilkovisky method naively consists of solving the structure equations, to find the structure functions which allows one to find the proper solution of the master equation. In this case due to having a large number of gauge generators this is quite cumbersome. Instead we will propose for $S(\Phi_{min} , \Phi^*{}_{min})$ the following expression, which reproduces the classical action A when one sets $G^* = 0$, and demonstrate that it satisfies the master equation up to order g,

$$S(\Phi_{min}, \Phi^*_{min}) = G_{-1}{}^{\alpha_{-1}}(1)\, i\, d\, G_{-1}{}^{\alpha_{-1}}(1) / d\tau \;-\; G_{-1}{}^{\alpha_{-1}}(1)\, Q(1)\, G_{-1}{}^{\alpha_0}(1) \;-$$

$$- G_{-1}{}^{\alpha_0}(1)\, Q(1)\, G_{-1}{}^{\alpha_{-1}}(1) + gV(1, 2, 3)\, C(\sigma_{int})\, G_{-1}{}^{\alpha_0}(1)\, G_{-1}{}^{\alpha_{-1}}(2)\, G_{-1}{}^{\alpha_{-1}}(3) +$$

$$+\sum_{r=-1}^{\infty} \{ - G^*_{r\,\alpha_{r+1}}(1)\, i\, d\, G_{r+1}{}^{\alpha_{r+1}}(1) / d\tau \;-\; G^*_{r\,\alpha_r}(1)\, Q(1)\, G_{r+1}{}^{\alpha_{r+1}}(1) \;+$$

$$+ G^*_{r\,\alpha_{r+1}}(1)\, Q(1)\, G_{r+1}{}^{\alpha_{r+2}}(1) \;+\; h.\,c.\, \} +$$

$$+ \; g\sum_{\substack{r,s=-1 \\ r+s\geq -1}} F(1, 2, 3)\, \{ (1/2)\, G^*_{r+s\;\alpha_{r+s}}(1)\;\; G_r{}^{\alpha_r}(2)\, G_s{}^{\alpha_s}(3) \;-$$

$$- \;\; G^*_{r+s\;\alpha_{r+s+1}}(1)\;\; G_r{}^{\alpha_{r+1}}(2)\, G_s{}^{\alpha_s}(3) \;\} + \qquad (4.49)$$

$$+ \; g\sum_{r,s=-1} F(1, 2, 3)\, \{ (1/2)\, G^*_{r\,\alpha_{r+1}}(1)\, G^*_{s\,\alpha_{s+1}}(2)\, G_{r+s+3}{}^{\alpha_{r+s+4}}(3) \;-$$

$$- \;\; G^*_{r\alpha_r}(1)\;\; G^*_{s\alpha_{s+1}}(2)\;\; G_{r+s+3}{}^{\alpha_{r+s+3}}(3) \}.$$

In the above equation as well as in the following ones the integration variables are suppressed. It is easy to demonstrate that S has both ghost numbers equal to zero and also reproduces all of the Z_r's correctly. The master equation now can be written as follows

$$(1/2)\,(S, S) = \sum_{s=-1}^{\infty} \{ [\, \delta_r S / \delta\, G_s{}^{\alpha_s}(1)]\, [\, \delta_l S / dG^*_{s\,\alpha_s}(1)] \;+$$

$$+ \;\; [\, \delta_r S / \delta\, G_s{}^{\alpha_{s+1}}(1)]\, [\, \delta_l S / \delta G^*_{s\,\alpha_{s+1}}(1)] + \; h.\,c.\, \} \qquad (4.50)$$

Now let us examine the terms which we get when (4.49) is used in (4.50). Without the

h. c. part the master equation has the following entries:

A) The terms which have time derivative: It can be easly seen that order g^0 terms cancel each other. At order g the G^* independent terms lead to

$$(½)F(1, 2, 3)\{G_{-1}{}^{\alpha}{}_{-1}(1)\, d[G_{-1}{}^{\alpha}{}_{-1}(2)G_0{}^{\alpha}{}_0(3)]/d\tau + G_{-1}{}^{\alpha}{}_{-1}(1)\, d[G_0{}^{\alpha}{}_0(2)\, G_{-1}{}^{\alpha}{}_{-1}(3)]/d\tau\} -$$

$$- V(1, 2, 3)\, C(\sigma_{int})\, G_{-1}{}^{\alpha}{}_{-1}(1)\, G_{-1}{}^{\alpha}{}_{-1}(2) dG_0{}^{\alpha}{}_0(3)/d\tau,$$

which vanishs when we use the cyclic symmetry of the vertex and also due to the fact that in this notation total time derivative is equal to zero.

The terms which include only one G^* can be written as

$$F(1, 2, 3)\{-\sum_1^{\infty} G_{-1}{}^{\alpha}{}_{-1}(1)\, d[G^*{}_{r\alpha_{r+1}}(2)\;\; G_{r+2}{}^{\alpha}{}_{r+2}(3)]d\tau + \sum_0^{\infty} G^*{}_{r-1\alpha_r}(1)\, G_r{}^{\alpha}{}_r(3)\, d\, G_0{}^{\alpha}{}_0(2)/d\tau -$$

$$- (½)\sum_{\substack{r,s=-1 \\ r+s\geq 0}} G^*{}_{r+s-1\;\alpha_{r+s}}(1)\, d[\, G_r{}^{\alpha}{}_r(2)\, G_s{}^{\alpha}{}_s(3)]d\tau\; +$$

$$+ \sum_{\substack{r=0 \\ s=-1}} G^*{}_{r+s\;\alpha_{r+s+1}}(1)\, G_s{}^{\alpha}{}_s(3)\, dG_{r+1}{}^{\alpha}{}_{r+1}(2)/dt\}$$

where the terms which include $G_{-1}{}^{\alpha}{}_{-1}$ and also the ones which have $G^*{}_{-1\alpha_{-1}}$ cancel separately and the rest, which tends to zero, can be written as follows

$$F(1, 2, 3)\{\, - (½)\sum_{r,s=1} G^*{}_{r+s-1\;\alpha_{r+s}}(1)\, d[\, G_r{}^{\alpha}{}_r(2)\, G_s{}^{\alpha}{}_s(3)]d\tau\; +$$

$$+ \sum_{\substack{r=0\\ s=-1}} G^*_{r+s\ \alpha_{r+s+1}}(1)\, G_s{}^{\alpha_s}(3)\, dG_{r+1}{}^{\alpha_{r+1}}(2)/dt\}.$$

It is easy to see that the G^*G^* terms yield zero because their contribution can be written as follows

$$(½)\, F(1, 2, 3)\sum_{r,s=1} \{G^*_{r\ \alpha_{r+1}}(1)\,(d/d\tau)\, G^*_{s\ \alpha_{s+1}}(2) + G^*_{r\ \alpha_{r+1}}(2)\,(d/d\tau)\, G^*_{s\ \alpha_{s+1}}(1) +$$

$$+\ d[G^*_{r\ \alpha_{r+1}}(1)\ \ G^*_{s\ \alpha_{s+1}}(2)]/d\tau\ \}\ G_{r+s+4}{}^{\alpha_{r+s+4}}(3)$$

where in this expression and also in the above ones cyclic symmetry of the vertex is used.

B) The terms which do not have time derivative:

The G^* independent terms can be written, by making use of the cyclic symmetry of V and the property F(1, 2, 3) = F(1, 3, 2), as

$$F(1, 2, 3)\, [-\sum_{s=1}^{3} Q(s)]\ \ G_{-1}{}^{\alpha_0}(1)\, G_{-1}{}^{\alpha_{-1}}(2)\, G_0{}^{\alpha_0}(3) +$$

$$+\ V(1, 2, 3)\, C(\sigma_{int})[\sum_{s=1}^{3} Q(s)]\ (1)\, G_{-1}{}^{\alpha_{-1}}(1)\, G_{-1}{}^{\alpha_{-1}}(2)\, G_0{}^{\alpha_1}(3)$$

When we calculate the terms which include only one G^*, taking into consideration $G^*_{r\alpha_r}$ and $G^*_{r\ \alpha_{r+1}}$ type terms seperately simplifies the calculation. $G_{-1}{}^{\alpha_{-1}}$ and $G_{-1}{}^{\alpha_0}$ including ones of the $G^*_{r\ \alpha_{r+1}}$ type terms lead to

$$F(1, 2, 3)\, [-\sum_{s=1}^{3} Q(s)]\ [\, G_{-1}{}^{\alpha_0}(1) \sum_{r=-1}^{\infty} G^*_{r\ \alpha_{r+1}}(2)\, G_{r+2}{}^{\alpha_{r+2}}(3) +$$

$$+ G_{-1}{}^{\alpha_{-1}}(1) \sum_{r=-1}^{\infty} G^*_{r\,\alpha_{r+1}}(2)\, G_{r+2}{}^{\alpha_{r+3}}(3)]$$

and the rest reads

$$F(1,2,3)\,[-\sum_s Q(s)] \sum_{s=0,\; r=-1} G^*_{r+s\,\alpha_{r+s+1}}(1)\, G_s{}^{\alpha_{s+1}}(2)\, G_{r+1}{}^{\alpha_{r+1}}(3).$$

In the $G^*_{r\,\alpha_r}$ dependent part we seperate only the $G_{-1}{}^{\alpha_{-1}}$ terms to write all of them as

$$F(1,2,3)\,[\sum_s Q(s)][\, G_{-1}{}^{\alpha_{-1}}(1) \sum_{r=-1}^{\infty} G_{r+2}{}^{\alpha_{r+2}}(2)\, G^*_{r\,\alpha_r}(3) +$$

$$+ \sum_{r,s=0}^{\infty} G^*_{r+s-1\,\alpha_{r+s-1}}(1)\, G_r{}^{\alpha_r}(2)\, G_s{}^{\alpha_s}(3)].$$

G^*G^* terms lead to the following expression

$$F(1,2,3)\,[\sum_s Q(s)] \sum_{r,s=-1} \{- G^*_{r\,\alpha_r}(1)\, G^*_{s\,\alpha_{s+1}}(2)\, G_{r+s+4}{}^{\alpha_{r+s+4}}(3) +$$

$$+ (1/2)\, G^*_{r\,\alpha_{r+1}}(1)\, G^*_{s\,\alpha_{s+1}}(2)\, G_{r+s+4}{}^{\alpha_{r+s+5}}(3)\}.$$

All of the above terms vanishes due to (4.47). This completes the demonstration of

$$(\, S(\Phi_{min}, \Phi^*_{min}),\, S(\Phi_{min}, \Phi^*_{min})\,) = 0$$

where $S(\Phi_{min}, \Phi^*_{min})$ is given in (4.49).

Let us take the gauge fermion as follows

$$\psi(\Phi) = -\sum_{r=-1}^{\infty} \overline{G}_{r+1\,\alpha_{r+1}}(1)\,[G_r^{\alpha_{r+1}}(1) - (\partial/\partial c_1)\, G_r^{\alpha_r}(1)] \tag{4.51}$$

where $\overline{G}_{r+1\,\alpha_{r+1}}$ are the antighost fields and their ghost numbers can be read off by taking into consideration the ghost numbers of the ghost fields and the gauge fermion. Then the gauge fixed actionis

$$S = S(\Phi_{min}, \delta\psi/\delta\Phi_{min}) - \sum_{r=-1}^{\infty} \Omega_{r\,\alpha_r}(1)\,[G_r^{\alpha_{r+1}}(1) - (\partial/\partial c_1)\, G_r^{\alpha_r}(1)]. \tag{4.52}$$

After integrating over the Lagrange multipliers $\Omega_{r\,\alpha_r}$ we find the following gauge fixed action, where again we suppressed the integral variables and α_r dependence,

$$S_{g.\,f.} = G_{-1}^{\alpha_{-1}}(1)\, i d\, G_{-1}^{\alpha_{-1}}(1)/d\tau + \sum_{r=0} \overline{G}_{r\,\alpha_r}(1)\, i d\, G_r^{\alpha_r}(1)/d\tau - G_{-1}^{\alpha_{-1}}(1)\,\Delta(1)\, G_{-1}^{\alpha_{-1}}(1) -$$

$$- \sum_{r=0} \overline{G}_{r\,\alpha_r}(1)\,\Delta(1)\, G_r^{\alpha_r}(1) + \sum_{\substack{r,s=-1 \\ r+s\geq -1}} V(1,2,3)\, \overline{G}_{r+s+1\,\alpha_{r+s+1}}(1)\, G_r^{\alpha_r}(2)\, G_s^{\alpha_s}(3) +$$

$$+ \sum_{r,s=-1} V(1,2,3)\, \overline{G}_{r+1\,\alpha_{r+1}}(1)\, \overline{G}_{s+1\,\alpha_{s+1}}(2)\, G_{r+s+3}^{\alpha_{r+s+3}}(3) + h.\,c. \tag{4.53}$$

where the Hamiltonian density, in functional representation, is given as

$$\alpha\Delta = \alpha\{Q, \partial/\partial c\} = (4\pi)^{-1}\int_{-\pi}^{\pi} d\sigma\{\, p^2 - \partial/\partial c\ \partial/\partial b\ + :P_\mu{}^2(\sigma): +\ :P_\beta(\sigma)\,P_{\bar\beta}(\sigma):\} . \quad (4.54)$$

Let us introduce a new field φ :

$$\varphi = X + \sum_{r=0}^{\infty} \left(\overline{G}_{r\alpha_r} + G_r{}^{\alpha_r} \right) . \quad (4.55)$$

In terms of φ we may write the gauge fixed action (4.53) as follows

$$S_{g.\,f.} = \int dZ(\sigma)\, dz\, d\tau\, (d\alpha/\alpha)\, \{\ \varphi\,(\,Z(\sigma),\, z,\, \tau,\, -\alpha\,)[\,id\,/\,d\tau\ - \Delta\,]\ \varphi\,(\,Z(\sigma),\, z,\, \tau,\, \alpha\,) +$$

$$+\, g \prod_{r=1}^{3} \int dZ_r(\sigma)\, dz_r\, d\tau\, (d\alpha_r/\alpha_r)\, V(1,2,3)\ \ \varphi(1)\varphi(2)\varphi\,(3) \quad (4.56)$$

where the free part of it leads to the following partition function [9]

$$\mathrm{STr}\ x^{\sum_{n=1}^{\infty}(\,\alpha^\dagger{}_n{}^\mu\,\alpha_n{}^\mu\, +\, n\,\beta^\dagger{}_n\bar\beta_n\, +\, n\,\bar\beta^\dagger{}_n\,\beta_n\,)}$$

which leads to the partition function of the string theory in the light cone gauge:

$$\prod_1^{\infty} (1 - x^n)^{-24}$$

so that the completeness relation holds.

C) SYMMETRIES OF THE GAUGE FIXED ACTION:

Let us introduce the Fourier transformed field $\tilde{\varphi}$ by the following definition

$$\varphi\,(Z(\sigma),\,z,\,\tau,\,\alpha\,) = (\alpha/2\pi)\int_{-\infty}^{\infty} dy^-\,\exp(i\alpha y^-)\,\tilde{\varphi}(\,Z(\sigma),\,z,\,\tau,y^-)\,. \qquad (4.57)$$

Then by defining $y^+ = \tau$ and making use of (4.57), the free part of the gauge fixed action (4.56) can be written as

$$(½)\int dy^{\overline{\mu}}\;db\;dcDX^{\mu}(\sigma)\,D\overline{\beta}(\sigma)\,D\beta(\sigma)\;\tilde{\varphi}\,\Box\,\tilde{\varphi} \qquad (4.58)$$

where $\overline{\mu} = 0,\,1,\,.\,.\,.\,,27$ and

$$\Box = (\partial^2/\partial y_{\mu}{}^2)\;-\;\partial/\partial c\,\partial/\partial b\;+\;\int_{-\pi}^{\pi} d\sigma\,[\,\partial^{\,2}/\partial X_{\mu}{}^2(\sigma)\;+\partial\,/\partial\beta(\sigma)\,\partial\,/\partial\overline{\beta}(\sigma)\,]. \qquad (4.59)$$

Therefore the free part of the gauge fixed action is manifestly invariant under Osp(26,2 | 2) in the zero mode sector and Osp(25,1 | 2) invariant in the excited mode sectors.

Instead of working in the manifestly orthosymplectic invariant form let us carry on our calculations in the light cone gauge in which the gauge fixed action (4.56) was written. For this porpose it is useful to remember the proof of the Poincare invariance of string theory in the light cone gauge [27]. As it is well known, the closure of the Poincare algebra can easly be shown except the commutator of J^{i-} by itself which must be zero for the closure of the algebra. Let us write this generator in the notation of [27]

$$J^{i-} = (1/2\sqrt{2}\,)\{(\,q_o{}^i/a_o{}^+)[L^{\perp}{}_o - \alpha(0)] + (1\,/a_o{}^+)[\,L^{\perp}{}_o - \alpha(0)]\,q_o{}^i - (1/\sqrt{2}\,)\,q_o{}^-a_o{}^i\;-$$

$$-(i/\,/a_o{}^+)\sum_{n=1}^{\infty}(1/n)[\,a_{-n}{}^iL^{\perp}{}_n\;-\;L^{\perp}{}_{-n}\,a_n{}^i] \qquad (4.60)$$

where $L^{\perp}_n$ are the Virasoro generators which include the d-2 transverse coordinates. When d=26 and $\alpha(0)=1$ the commutator of it gives

$$[J^{i-}, J^{j-}]=0 \qquad (4.61)$$

by making use of the commutators of the normal mode operators which we give the nonzero ones below

$$[a^i_n, a^j_{-m}]=n\,\delta_{n,m}\,\delta^{i,j} \quad , \quad [q^i_o, a^j_o]=-i\sqrt{2}\,\delta^{i,j} \quad , \quad [q^-_o, a^+_o]=-i\sqrt{2} \qquad (4.62)$$

Now by rescaling $b\rightarrow b/2$ we may write Q which was given in (4.34) as follows

$$Q=-(1/2\alpha)[c(L_o-1)+(L_o-1)c]+\alpha^{-1}\sum_{n=1}^{\infty}[L_{-n}\beta_n-\beta_{-n}L_n]+(\tfrac{1}{2})\partial/\partial b(e^{\omega}\partial/\partial\omega+\partial/\partial\omega\, e^{\omega}) \qquad (4.63)$$

where L_n is given in (4.43) and it satisfies the following Virasoro algebra

$$[L_n, L_m]=(n-m)L_{n+m}+2n(n^2-1)\delta_{n+m,0} \qquad (4.64)$$

So that it is the same algebra which $L^{\perp}_n$ satisfy. This is due to the fact that effect of the two commuting α_μ are canceled by the anticommuting coordinates.

Comparision of (4.63) with (4.60) gives the formal identity betwen Q and one of the light cone generators of the Poincare group J^{i-}. So Q is one of the generators of Osp(26,2 | 2) which is the symmetry group of the zero mode sector. Indeed it mixes the τ and c coordinates. Anti-BRST charge $\overline{Q}$ which can be found by commuting Q with one of the Sp(2) generators (see eq.3.31) is the generator which mixes τ and b. Therefore due to the simplicity of the other generators in the light cone gauge and the relation

$\{ Q , \overline{Q} \} = 0,$

the BRST invariance of the free part of the gauge fixed action which follows from the relations

$[Q , \Delta] = [Q , id/d\tau] = 0$

is sufficent to show the Osp(26,2 | 2) invariance of it when written as in (4.56).

Similar to the free case invariance of the whole action (4.56) under Osp(26,2 | 2) will follow if we can show that it is BRST-invariant. Now the interaction term will give some new contributions to the BRST-charge. In fact (4.56) is not invariant under Q but it is invariant under the folowing transformations up to order g

$$\delta\varphi(1) = \Lambda Q(1)\,\varphi(1) - g\,\Lambda \prod_{r=1} \int dZ_r(\sigma)\, dz_r\, (d\alpha_r/\alpha_r)\, V(1,2,3)C(\sigma_{int})\, \varphi(2)\varphi(3) \qquad (4.65)$$

This can be proved in the same manner which we demonstrated the BRST-invariance of the point particle and by using the relation [1, 13, 16]

$$V(1,2,3)\sum_{r=1}^{3} Q(r) = V(1,2,3)\, C(\sigma_{int}) \sum_{r=1}^{3} \Delta(r) \qquad (4.66)$$

The physical states are defined with the conditions

$$N\Phi = 0 \qquad (4.67a)$$

$$Q\Phi = 0 \qquad (4.67b)$$

so that the physical fields Φ and Φ' which are related as

$$\Phi' = \Phi + Q\,\Gamma \tag{4.68}$$

are in the same equivalence class, where $N\Gamma = \Gamma$. If we expand Φ in terms of the zero mode operators as

$$\Phi = \Phi_o + \Phi_1\, c + \Phi_2\, b + \Phi_3\, cb \tag{4.69}$$

(4.67) gives

$$2M\Phi_1 - (-\tfrac{1}{2} + \partial/\partial\omega)\Phi_2 + (d + D)\Phi_o = 0, \tag{4.70a}$$

$$K\Phi_o - (-\tfrac{1}{2} + \partial/\partial\omega)\Phi_3 + (d + D)\Phi_1 = 0\,, \tag{4.70b}$$

$$-2M\Phi_3 + (d + D)\Phi_2 = 0\,, \tag{4.70c}$$

$$-K\Phi_2 + (d + D)\Phi_3 = 0. \tag{4.70d}$$

By writing $\Phi' = \Phi + \delta\Phi$ and expanding Γ in components as in (4.69) we may write (4.68) in component form as

$$\delta\Phi_o = - 2M\Gamma_1 + (\tfrac{1}{2} + \partial/\partial\omega)\Gamma_2 + (d + D)\Gamma_o\,, \tag{4.71a}$$

$$\delta\Phi_1 = - K\Gamma_o + (\tfrac{1}{2} + \partial/\partial\omega)\Gamma_3 + (d + D)\Gamma_1\,, \tag{4.71b}$$

$$\delta\Phi_2 = 2M\Gamma_3 + (d + D)\Gamma_2 , \tag{4.71c}$$

$$\delta\Phi_3 = K\Gamma_2 + (d + D)\Gamma_3 \tag{4.71d}$$

where $\delta\Phi$ is shifted by α.

By making use of the operator representation of the fields (4.38) one can see that in (4.71c) Γ_3 can be used to eliminate all Φ_2 so that we take $\Phi_2 = 0$ and now Γ_2 satisfies $(d+D)\Gamma_2=0$. Γ_2 satisfying this equation can be used to set $\Phi_3 = 0$ because from (4.70c)-(4.70d) now Φ_3 satisfies $2M\Phi_3 = (d + D)\Phi_3 = 0$. A transformation which leaves $\Phi_2 = \Phi_3 = 0$ leads to the following equations for Γ_2 and Γ_3,

$$K\Gamma_2 + (d + D)\Gamma_3 = 2M\Gamma_3 + (d + D)\Gamma_2 = 0. \tag{4.72}$$

But now from (4.70a)-(4.70b) we see that

$$2M\Phi_1 + (d + D)\Phi_o = K\Phi_o + (d + D)\Phi_1 = 0 ,$$

which is in the same form with (4.72). Thus we may use remnant freedom in Γ_2 and Γ_3 in (4.71a)-(4.71b) to set the α dependence of Φ_o and Φ_1 to be prescribed. Γ_2 and Γ_3 can not be used to set Φ_o and Φ_1 to zero because as we have seen there is not any freedom on the boundary due to the vanishing of all the fields at $\alpha = \pm\infty$.

Thus the physical state condition (4.67b) has led to

$$Q_{old}(\Phi_o + c\Phi_1) = 0 \tag{4.73}$$

where Φ_o and Φ_1 have some prescribed α dependence. By making use of (4.71b) one can choose a state from the equivalence class whose first component vanishes $\Phi_1=0$ by an appropriate choice of Γ_2 . Now from (4.70a) and (4.70b) Φ_o is subject to

$$(d + D)\Phi_o = K\Phi_o = 0. \tag{4.74}$$

When combined with the free equations of motion (4.74) yields

$$d\Phi_o / d\tau = 0. \tag{4.75}$$

The projector E onto the DDF states, in the Hilbert space which is spanned by the fields Φ_o which are subject to $Q_{old}\Phi_o=0$, can be written as [39]

$$E = \{ Q_{old} , S \}$$

where S is an operator which can be written in terms of the mode operators as follows

$$S = \sum_{m=-\infty}^{\infty} \bar{\beta}_{-m}\{ \oint(dz/2\pi i)\, z^m \; [k^\mu . p^\mu \; - \; k^\mu \sum_{n=1}^{\infty}(\alpha_n{}^\mu z^{-n} \; + \; \alpha_{-n}{}^\mu z^n)] \; - \; \delta_{m,o}\}$$

where k^μ is a light-like momentum satisfying

$$k^2 = 0, \qquad k\cdot p = 1.$$

The action of E on Φ_o is

$$E\Phi_o = Q_{old}[S\Phi_o]$$

so (4.71a) can be used to set $E\Phi_0 = 0$. Therefore the physical on-shell states of this theory are the DDF-states [19]:

$$A^i_n = (2\pi i)^{-1} \oint dz\, z^{n-1} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \alpha^i_m\, z^{-m} \exp\{-2n\alpha^{-1} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} m^{-1} \alpha^+_m\, z^{-m}\} \qquad (4.76)$$

where $A^{i\,\dagger}_n = A^i_{-n}$.

D) EQUIVALENCE OF THE SCATTERING AMPLITUDES OF THE Osp(26,2 | 2) INVARIANT AND LIGHT CONE GAUGE STRING FIELD THEORIES [18]:

By following [40] let us write down a generating functional of Green functions in terms of the light cone coordinates as

$$Z_\lambda[J] = \int D\tilde{\varphi}\, \exp\{-\int dZ\{[\lambda + (1-\lambda)\, \delta(y^+)\delta(x^- - y^-)\delta(c)\delta(b) \prod_1^\infty \delta(X_n^-)\delta(X_n^+)\, \delta(\beta_n)\delta(\bar{\beta}_n)]$$

$$S_1(\tilde{\varphi}) + S_2(\tilde{\varphi}) + J\tilde{\varphi}\, \delta(x^- - y^-)\} \qquad (4.77)$$

where $\tilde{\varphi}$ was defined in (4.57) and Z denotes all the variables. The integral over the anticommuting variables is normalized such that

$$\int dc\, db\, d\bar{\beta}(\sigma)\, d\beta(\sigma)\, bc\bar{\beta}(\sigma)\, \beta(\sigma) = -1.$$

The support of J is restricted to the subspace which is spanned by the DDF states, (4.76), i. e.

$J = J(x^i, x^-, X^i(\sigma), X^-(\sigma))$. The other entries of (4.86) are defined as follows

$$S_1(\tilde{\varphi}) = \langle\tilde{\varphi}|\{-(½)p^2 - \sum_{n=1}^{\infty} \alpha_{-n}{}^i \alpha_n{}^i\}|\tilde{\varphi}\rangle + \int dZ_1\, dZ_2 V_F(1,2,3)\, \tilde{\varphi}(1)\, \tilde{\varphi}(2)\, \tilde{\varphi}(3) + \text{h. c.} \tag{4.78}$$

$$S_2(\tilde{\varphi}) = \langle\tilde{\varphi}|\{ (½)[-\partial/\partial y^- \partial/\partial y^+ + \partial/\partial b \partial/\partial c] + \sum_{n=1}^{\infty}(\alpha_{-n}{}^+\alpha_n{}^- + \alpha_{-n}{}^-\alpha_n{}^+ - n\, \beta_{-n}\bar{\beta}_n - n\, \bar{\beta}_{-n}\, \beta_n)\}|\tilde{\varphi}\rangle \tag{4.79}$$

where $V_F(1,2,3)$ is the three string interaction vertex (it includes also g) which is written in terms of y^- and in (4.78) $Z_3 = Z$.

When one sets $\lambda=1$ the resultant generating functional describes a covariant string field theory which has Osp(26,2 | 2) invariance in the zero mode sector and Osp(25,1 | 2) invariance in the other mode sectors up to the first order in g. By introducing some new interactions, which we will specify later, one can obtain an action which is invariant to all orders in g.

Now we want to show that when $\lambda=0$ S_2 decouples from S_1 so that, up to a change in the normalization factor, $Z_0[J]$ describes the system which has only bosonic coordinates where in the zero modes it has Lorentz invariance and has SO(24) invariance in the other mode sectors:

When the δ functions are taken into account the S_1 part of the exponential term in $Z_0[J]$ reads

$$\int dx^\mu \{\langle\Phi|[(½)(\partial/\partial x^\mu)^2 + \sum_{n=1}^{\infty}\alpha_{-n}{}^i\alpha_n{}^i]|\Phi\rangle - \int dx^\mu{}_1 dx^\mu{}_2 \langle v(1,2,3)|\, |\Phi_1\rangle\, |\Phi_2\rangle\, |\Phi_3\rangle$$

where $\langle\Phi|$ is the N=M=0 component of $\tilde{\varphi}$, when written in oscillator representation as in (4.38) and it depends only on $(X^i{}_n, x^i, x^-)$. Now one can easily see that the fermionic part of (4.79) decouples due to the fact that it cannot have $\Phi(X^i_n, x^i, x^-)$ dependence. The longitudinal part of (4.79) can be written as

$$(1/4\pi)\int_{-\pi}^{\pi} d\sigma\, \tilde{\varphi}\, :\partial/\partial X^-(\sigma)\partial/\partial X^+(\sigma):\, \tilde{\varphi}$$

so it also decouples. The $\partial/\partial y^-\partial/\partial y^+$ part will lead to an expression like

$$\int dy^- dy^+ \;\; f(\, y^- y^+ + Ay^- + By^+ \,) \tag{4.80}$$

which does not depend on A and B because going Euclidean on the y^-, y^+ variables we may write $y^- = \rho e^{-i\theta}$, $y^+ = \rho e^{-i\theta}$ and the θ integral effectively sets A=B=0[33] , so that this part also decouples from S_1 (an action which is invariant to all orders in g will still satisfy these properties so that the results which we will represent are valid to all orders in g).

Since the derivatives of $\ln Z_\lambda[J]$ with respect to J generate the Green functions, independence of $\partial \ln Z_\lambda[J]/\partial\lambda$ from J will lead to the desired result. So let us examine the properties of:

$$\partial \ln Z_\lambda[J]/\partial\lambda = -\int dZ\, [\, 1 - \delta(y^+)\delta(x^- - y^-)\delta(c)\delta(b)\prod_1^\infty \delta(X_n^-)\delta(X_n^+)\; \delta(\beta_n)\delta(\bar{\beta}_n)]\; \langle S_1\rangle_{J,\lambda} \tag{4.81}$$

Due to the Osp(2 | 2) invariance in y^-, y^+; c, b and $X^-(\sigma)$, $X^+(\sigma)$; $\bar{\beta}(\sigma)$, $\beta(\sigma)$ coordinates:

$$\langle S_1\rangle_{J,\lambda} = F_\lambda[\, J(\, x^i, X^i(\sigma)\,)\, ;\, cb + y^- y^+ + (1/4\pi)\int_{-\pi}^{\pi} d\sigma : X^-(\sigma)\, X^+(\sigma) + \bar{\beta}(\sigma)\,\beta(\sigma) :\,]$$

where we have already set $y^- = X^+(\sigma) = 0$ with the same argument which was used to show the independence of (4.79) from A and B. So by performing the integration over $z \equiv y^-$, y^+, c, b, $X^-(\sigma)$, $X^+(\sigma)$, $\bar{\beta}(\sigma)$, $\beta(\sigma)$ we get

$$\int dz\, F_\lambda = -F_\lambda[\, J(x^i, X^i(\sigma))\, ;\ \infty] + F_\lambda[\, J(x^i, X^i(\sigma))\, ;\ 0] \qquad (4.82)$$

where the following property of the integral [17]

$$\int dp^2 dc\bar{c}b\, f(p^2 + cb\,) \;=\; -\int dp^2\, \partial f(p^2)/\partial p^2$$

is used. Since the support of $J(x^i, X^i(\sigma))$ is restricted to z=0, by cluster decomposition

$$F_\lambda[\, J(x^i, X^i(\sigma))\, ;\ \infty] = F_\lambda[\, 0\, ;\ \infty].$$

Therefore (4.81) reads

$$\partial \ln Z_\lambda(J)/\partial\lambda = F_\lambda[\, 0\, ;\ \infty]$$

This completes the proof of the independence of the Green functions from λ, so that the Green functions of the two theories which result when we set λ=1 and λ=0 are same. As it is mentioned above the former of these theories corresponds to Osp(26,2 | 2) invariant covariant string field theory and the latter one leads to the following generating functional by changing the normalization factor,

$$Z_0[J] = \int D\Phi(X^i(\sigma), X^-(\sigma), x^\mu)\, \exp\{-\int dX^i(\sigma) dX^-(\sigma) dx^\mu\, [\delta(X^-(\sigma))\, S_1(\Phi) +$$

$$+ \;\; \delta(x^+)\, J(\, X^i(\sigma), X^-(\sigma), x^i, x^-)\Phi(X^i(\sigma), X^-(\sigma), x^\mu)]\}$$

This theory leads to the scattering amplitudes which can be given in terms of DDF states, and some polarizationtensors $\varepsilon^{i_1 \cdots i_N; n_1 \cdots n_N}$ as

$$A = \langle 0 | \sum \varepsilon^{i_1 \cdots i_N ; n_1 \cdots n_N} A^{i_1}_{-n_1} \cdots A^{i_N}_{-n_N} f(x^\mu, \alpha^i_{-n}) | 0 \rangle$$

but now since $f(x^\mu, \alpha^i_n)$ is independent of α^+_n we may substitute the DDF states with α^i_{-n} operators which leads to the following amplitude

$$A = \langle 0 | \sum \varepsilon^{i_1 \cdots i_N ; n_1 \cdots n_N} \alpha^{i_1}_{-n_1} \cdots \alpha^{i_N}_{-n_N} f(x^\mu, \alpha^i_{-n}) | 0 \rangle$$

Indeed this is the scattering amplitude which one would find in the light cone gauge string field theory. Therefore at all orders of perturbation the scattering amplitudes of the Osp(26,2 | 2) invariant string field theory are equal to the ones which one finds in the light cone gauge.

IV. PERTURBATIVE CALCULATION OF SCATTERING AMPLITUDES:

A) FOUR PHOTON SCATTERING AMPLITUDE AT TREE LEVEL[41]:

Before begining to the calculation of the four photon scattering amplitude let us summarize briefly the procedure to find the general form of the string scattering amplitudes at tree level in terms of the Neumann coefficents [5], as a preperation to the next section.

Without the loss of generality let us consider only the four string amplitude. Generally the four string scattering amplitude is given as

$$A=\int_{-\infty}^{\infty} dT \int d5d6 \, \langle 4| \langle 3| \langle V(1,2,5)| \exp[T(L_o-1)] \, |V(6,3,4)\rangle \, |1\rangle \, |2\rangle$$

where strings 1and 2 join at a time T_1 to give the string 5 which propogates to string 6 and string 6 at a time T_2 separates to give the strings 3 and 4. The vertex V(1, 2, 3) is defined in (4.42) or equivalently can be written as follows, by making use of the Fourier coefficents of the Neumann functions which were given in (1.15),

$$\langle V(1,2,3)| = \mu \langle 0|_1 \langle 0|_2 \langle 0|_3 \exp\{\frac{1}{2}\sum_{n,m\geq 0} N^{rs}_{mn} \alpha^r_n \alpha^s_m + \sum_{n,m\geq 0} N^{rs}_{mn} \theta^r_n \overline{\theta}^s_m \tag{5.1}$$

where $\theta_n=\sqrt{n}\,\beta_n$, $\overline{\theta}_n=\sqrt{n}\,\overline{\beta}_n$ for $n\neq 0$ and $\theta_o=\beta_o$, $\overline{\theta}_o=\overline{\beta}_o$ and $\mu=\exp(-\tau_o\sum_r \alpha_r^{-1})$. L_o can be found by setting n=0 in (4.43). d5 and d6 denote the zero mode variables of the strings 5 and 6, respectively. By writing T as the difference of interaction times, $T=T_2-T_1$, one can combine the vertices and the exponential term to yield

$$A=\int_{-\infty}^{\infty} dT \int d5d6 <4|<3|<V_{T_1}(1,2,5)||V_{T_2}(6,3,4)> |1> |2> \qquad (5.2)$$

where $<V_{Ti}(1, 2, 3)|$ is given as $<V(1, 2, 3)|$ but with some new Neumann coefficents which now depend on T_i.

Let us derive the momentum dependent and independent parts of A explicitly. This can be achived by making use of the relations [42]

$$<0|\exp\{(\tfrac{1}{2})\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_m \cdot M_{mn}\alpha_n + \sum_{m=1}^{\infty} L_m \cdot \alpha_m\}$$

$$\exp\{(\tfrac{1}{2})\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\alpha_{-m} \cdot N_{mn}\alpha_{-n} + \sum_{m=1}^{\infty} K_m \cdot \alpha_{-m}\}|0> =$$

$$= [\det(1-\overline{M}\,\overline{N})]^{-d/2} \exp\{(\tfrac{1}{2})\overline{L}\,\overline{N}(1-\overline{M}\,\overline{N})^{-1}\overline{L} + \overline{K}(1-\overline{M}\,\overline{N})^{-1}\overline{L} +$$

$$+ \overline{K}(1-\overline{M}\,\overline{N})^{-1}\overline{M}\,\overline{K}\} \qquad (5.3)$$

where on the right hand side of the equality we have suppressed the vecor products and the indices which are summed over. The barred quantities are defined as follows

$$\overline{M}_{mn} = \sqrt{m}\sqrt{n}\,M_{mn} \qquad , \qquad \overline{L}^{\mu}{}_m = \sqrt{m}\,L^{\mu}{}_m$$

and similar relations for $\overline{N}$ and $\overline{K}$.

For the anticommuting modes we have the following relation

$$<0|\exp\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta_m U^{(1)}{}_{mn}\bar{\beta}_n + \sum_{m=1}^{\infty}\bar{\beta}_m V^{(1)}{}_m + \sum_{m=1}^{\infty}\beta_m W^{(1)}{}_m\}$$

$$\exp\{-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta^{\dagger}{}_m U^{(2)}{}_{mn}\bar{\beta}^{\dagger}{}_n - \sum_{m=1}^{\infty}\bar{\beta}^{\dagger}{}_m V^{(2)}{}_m - \sum_{m=1}^{\infty}\beta^{\dagger}{}_m W^{(2)}{}_m\}|0> =$$

$$= [\det(1 - U^{(2)}\,U^{(1)})]^{\frac{1}{2}} \exp\{ - V^{(2)}\,(1 - U^{(1)}\,U^{(2)})^{-1}\,W^{(1)} + V^{(1)}\,(1 - U^{(2)}\,U^{(1)})^{-1}\,W^{(2)} +$$

$$+ V^{(2)}\,U^{(1)}\,(1 - U^{(2)}\,U^{(1)})^{-1}\,W^{(2)} - V^{(1)}\,U^{(2)}\,(1 - U^{(1)}\,U^{(2)})^{-1}\,W^{(1)}\,\} \qquad (5.4)$$

where again we suppressed the indices which are summed over.

Making use of (5.3) and (5.4) in (5.2) to take the trace over the non-zero modes of the intermediate strings, lead to a momentum independent measure part and a momentum dependent exponential part which will be still multiplied with the external states. Now the external states can be taken as the light cone gauge ones due to the following relation [12]

$$\sum_{s} N^{rs}{}_{no}\, P^{s}{}_{+} = 0\,. \qquad (5.5)$$

The momentum dependent part must be same with the one which one finds by making use of the mapping of the world sheet onto the upper half complex plane as it was summarized in chapter I. Thus it is equale to the exponent part of (1.12) with the Neumann coefficents of four string interaction diagram which is drawn as in figure 3.

For the measure part the unique information is the relation of it with the momentum dependent part which is discovered by Cremmer and Gervais [5] :

$$-4w\, d\{ w\, d\ln \det (I - (1/4)\overline{N^T}{}_{155}\,\overline{N^T}{}_{266}\,) / dw\,\} / dw =$$

$$= (\alpha_1\alpha_2 / \alpha_3\alpha_4)[d(w\, da_{oo}/ dw) / dw]^2\, [db_{oo}/ dw]^{-2} \qquad (5.6)$$

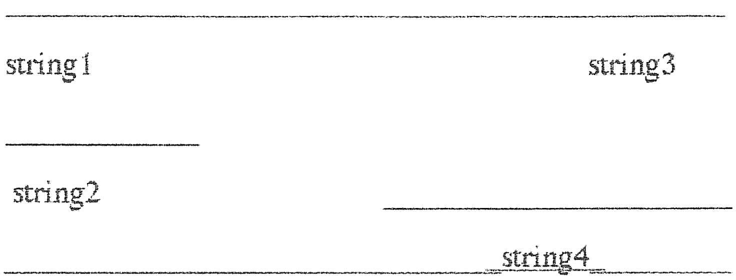

Figure 3.

where $w = \exp[T/\alpha_6]$ and $\overline{N^T}_{rs}$ are related to the Neumann coefficents as follows

$$\overline{N^T}_{rs} = C^{-1/2}\ N^T_{rs}\ C^{-1/2}$$

where $C_{mn} = n^{-1}\,\delta_{mn}$. a_{oo} and b_{oo} are related to the zero mode part of the exponential term and their explicit form can be found in [5]. The solution of (5.6) by expressing T_1 and T_2 in terms of the Koba-Nielsen variables and the interaction points in the upper half plane leads to

$$|\alpha_5|^{-1}\, dT\, \det\left(I - (1/4)\overline{N^T}_{255}\ \overline{N^T}_{166}\ \right)^{-12} \exp\{ \sum_i \alpha_i^{-1}\, [T_{(i)} - \theta_{(i)}] =$$

$$= \prod_{r<s} | Z_r - Z_s|^{(\alpha_r/\alpha_s + \alpha_s/\alpha_r)} \prod dZ_r / dV_{abc} \qquad (5.7)$$

where the sum over i runs from 1 to 6 and

$$T_{(1)} = T_{(2)} = T_{(5)} = T_1 \quad , \qquad T_{(3)} = T_{(4)} = T_{(6)} = T_2 \quad ,$$

$$\theta_1 = \theta_2 = \theta_5 = \alpha_1 \ln|\alpha_1| + \alpha_2 \ln|\alpha_2| + \alpha_5 \ln|\alpha_5| \,,$$

$$\theta_3 = \theta_4 = \theta_6 = \alpha_3 \ln|\alpha_3| + \alpha_4 \ln|\alpha_4| + \alpha_6 \ln|\alpha_6| \,.$$

Z_r are the Koba-Nielsen variables and the invariant volume element is given as follows, in terms of the variables $Z_{a,b,c}$ which one has to fix due to the Mobious invariance of the string amplitude,

$$dV_{abc} = |Z_a - Z_b|^{-1} |Z_b - Z_c|^{-1} |Z_c - Z_a|^{-1} \, dZ_a \, dZ_b \, dZ_c \,. \tag{5.8}$$

Thus the four point amplitude at tree level is, up to some overall constants and δ-functions,

$$A_{(4)} = \int [\prod dZ_r / dV_{abc}] \prod_{r<s} | Z_r - Z_s|^{[p_r \cdot p_s + (\alpha_r / \alpha_s)(1 - p_s \cdot p_s/2) + (\alpha_s / \alpha_r)(1 - p_r \cdot p_r/2)]}$$

$$\exp[\sum \alpha_r^{-1} T_{(r)} (p_s \cdot p_s/2 - 1) \langle \text{ext. states}| \exp\{ (1/2) \sum_{r,s} \alpha^r_{-m} \cdot N^{rs}_{mn} \alpha^s_{-n} +$$

$$+ \; (1/2) \sum_{r,s} \alpha^r_o \cdot N^{rs}_{on} \alpha^s_{-n} - \sum_{r,s} \sqrt{mn} \, \beta^{r\dagger}_m N^{rs}_{mn} \bar{\beta}^{s\dagger}_n$$

$$- \sum_{r,s} \sqrt{n} \, \beta^{r\dagger}_n N^{rs}_{no} \bar{\beta}^s_o - \sum_{r,s} \sqrt{n} \, \beta^r_o N^{rs}_{on} \bar{\beta}^{s\dagger}_n \} \,|\, 0\rangle \tag{5.9}$$

where the four point Neumann coefficents can be found by making use of (1.15).

To calculate the four photon amplitude we take the external states as

$$\langle \text{external states}| = \langle 0| \prod \varepsilon_r^{i_r}(p) \, \alpha_1^{r\, i_r} \quad , \tag{5.10}$$

where the polarization vectors $\varepsilon_r^{i_r}(p)$ depend on the momenta and the i_r run from 1 to 24 (the summation on them are suppressed). The polarization vectors and the momenta satisfy the following equalities

$$p_r \cdot p_r = \varepsilon_r \cdot p_r = 0 \qquad \text{(no summation over r).}$$

Due to not possessing neither world sheet ghost nor longitudinal bosonic mode operator dependence, use of (5.10) in (5.9) leads to the amplitude which would result from the calculation in the light cone gauge. Thus the four photon scattering amplitude can be written as follows

$$A = \phi \int [\prod dZ_r / dV_{abc}] \prod | Z_r - Z_s |^{(p_r \cdot p_s + \alpha_r / \alpha_s + \alpha_s / \alpha_r)} \{A_1 + A_2 + A_3\} \qquad (5.11)$$

where ϕ is a phase factor which will be specified later. Use of the commutation relations of $\alpha_1{}^r$ yield

$$A_1 = (½) [\prod \varepsilon_1{}^{i_r}(p_r)] \sum_{r'<s';r''<s''} N^{r's'}{}_{11} N^{r''s''}{}_{11} \qquad (5.12a)$$

$$A_2 = [\prod \varepsilon_1{}^{i_r}(p_r)] \sum_{r,s;r'<s';r''<s''} N^{r''s''}{}_{11} N^{r'r}{}_{10} N^{s's}{}_{10} p_s{}^{i_{s'}} p_r{}^{i_{r'}} d^{i_{r''}, i_{s''}} \qquad (5.12b)$$

$$A_3 = [\prod \varepsilon_1{}^{i_r}(p_r)] \prod \sum_s N^{rs}{}_{10} \; p_s{}^{i_{s'}} \qquad (5.12c)$$

where the primed indices should be taken different from each other. A_a appear with different string tension factors but due to our convention $\alpha' = ½$ this is not explicit in (5.12). The Fourier coefficents of the Neumann functions can be found by making use of (1.15) as follows (in the following equations the definition of the Neumann coefficents are different from (1.15) up to some phase factors)

$$N^{rs}_{11} = (Z_r - Z_s)^{-2} \prod_{r \neq r'} (Z_r - Z_{r'})^{-\alpha_{r'}/\alpha_r} \prod_{s \neq s'} (Z_s - Z_{s'})^{-\alpha_{s'}/\alpha_s} , \qquad r \neq s$$

$$N^{rs}_{10} = [- \sum_{r \neq s} (\alpha_s/\alpha_r)(Z_r - Z_s)^{-1}] \prod_{r \neq s} (Z_r - Z_s)^{-\alpha_s/\alpha_r} \tag{5.13}$$

$$\sum_{r \neq s} N^{rs}_{10}\, p^{ir}_s = [\sum_{r \neq s} p^{ir}_s (Z_r - Z_s)^{-1}] \prod_{r \neq s} (Z_r - Z_s)^{-\alpha_s/\alpha_r}$$

Now our task is to use (5.13) in (5.11) and perform the integration. This direct but cumbersome calculation can be simplified by observing the fact that all of the A_a have a common overall factor which cancels the following part of the measure

$$\prod_{r<s} | Z_r - Z_s |^{\alpha_r/\alpha_s + \alpha_s/\alpha_r} . \tag{5.14}$$

Let us exhibit this for A_2. By suppressing the polarization vector part, (5.12b) can be written as a combination of the following terms

$$N^{r\,r+1}_{11}\, N^{r+2\,s}_{10}\, N^{r+3\,s'}_{10}\, p^{ir+2}_{s}\, p^{ir+3}_{s'} \tag{5.15a}$$

$$N^{r\,r+2}_{11}\, N^{r+3\,s}_{10}\, N^{r+1\,s'}_{10}\, p^{ir+3}_{s}\, p^{ir+1}_{s'} \tag{5.15b}$$

$$N^{r\,r+3}{}_{11}\, N^{r+1\,s}{}_{10}\, N^{r+2\,s'}{}_{10}\, p^{ir+1}{}_{s}\, p^{ir+2}{}_{s'} \qquad (5.15c)$$

where r+a is defined modulo 4. By giving some different values to r the above terms generate A_2. Making use of (5.13) in (5.15a) yields, up to a phase,

$$(Z_r - Z_{r+1})^{-2} \{ -\sum_{s\neq r+2} p^{ir+2}{}_{r+2}\, (\alpha_s/\alpha_{r+2})\, (Z_{r+2} - Z_s)^{-1} + \sum_{s\neq r+2} p^{ir+2}{}_{s}\, (Z_{r+2} - Z_s)^{-1} \} \times$$

$$\times \{ r+2 \rightarrow r+3 \} \prod_{r<s} | Z_r - Z_s |^{-\alpha_r/\alpha_s - \alpha_s/\alpha_r} \quad . \qquad (5.16)$$

The calculation of the other two (5.15b) and (5.15c) are similar to the above one and they lead, respectively, to

$$(Z_r - Z_{r+2})^{-2} \{ r+2 \rightarrow r+3 \} \{ r+2 \rightarrow r+1 \}$$

$$(Z_r - Z_{r+2})^{-2} \{ r+2 \rightarrow r+3 \} \{ r+2 \rightarrow r+1 \}$$

where $\{ r+2 \rightarrow s \}$ is same with the first paranthesis of (5.16) except r+2 is replaced with s.

We fix three of Z_r, by still keeping one of them as an arbitrary constant, as follows

$$Z_1 = 0 \quad , \qquad Z_3 = 1 \quad , \qquad Z_4 = c > 1$$

and the one which is unfixed is taken suitable to calculate the s-channel diagram:

$Z_2 = x \qquad\qquad 0 < x < 1.$

When we keep c arbitrary the integral over x will yield hypergeometric functions which depend on c. In principle the recursion relations of the hypergeometric functions must lead to the independence of the amplitude from c. But due to having a large number of terms this method is not desirable. Instead we will take the limit $c \to \infty$ in the integrand so that the integration will lead to the beta functions only.

Without keeping the track of the overall sign factors the remaning part of the measure after the cancelation of (5.14) is

$$c^{p_1 \cdot p_4 + 1} (1 - c)^{p_3 \cdot p_4 + 1} x^{p_1 \cdot p_2} (1 - x)^{p_2 \cdot p_3} (c - x)^{p_2 \cdot p_4} .$$

In the $c \to \infty$ limit due to momentum conservation it behaves like

$$x^{\alpha_s} (1 - x)^{\alpha_t} c^2 [1 + O(c^{-1})]$$

where

$$\alpha_s = (½) (p_1 + p_2)^2 = p_1 \cdot p_2, \quad \alpha_t = (½) (p_3 + p_2)^2 = p_3 \cdot p_2.$$

Before calculating the whole amplitude let us specify the overall phase factor ϕ which was introduced in (5.11). It has two different entries: (a) All of the overall sign factors coming from the measure as well as from A_a. (b) The phases which we have not taken into account when we found the Neumann coefficents as in (5.13) from (1.15). Indeed this phase factor is same for all of the terms.

run from 1 to 3 and the external states must be taken suitable to the three photon case. Up to some constant overall factors the amplitude reads

$$\varepsilon^{i_1}{}_1\, \varepsilon^{i_2}{}_2\, \varepsilon^{i_3}{}_3 \{ \sum_{r<s,\, s\neq k\neq r} N^{rs}{}_{11}\, N^{ks'}{}_{10}\, p^{i_k}{}_{s'}\, \delta^{i_r i_s} + \sum_{r,\, s,\, k} N^{1r}{}_{10} N^{2k}{}_{10}\, N^{3s}{}_{10}\, p^{i_1}{}_r\, p^{i_2}{}_k\, p^{i_3}{}_s \}$$

$$c^{p_1 \cdot p_3 - \alpha_1/\alpha_3 - \alpha_3/\alpha_1 + 1}\, (1-c)^{p_2 \cdot p_3 + \alpha_2/\alpha_3 + \alpha_3/\alpha_2 + 1}\, \delta(\sum_r p_r).$$

Similar to the four photon case the α_r dependence of the measure cancels and the rest of it behaves like c^2 in the $c \to \infty$ limit due to the momentum conservation. For the Neumann coefficents $N^{rs}{}_{11}$ and $N^{rs}{}_{10}$ (5.13) is still valid if we restrict the string number to run from 1 to 3. Making use of these the first term of the amplitude leads to

$$\lim_{c \to \infty} c^2 [1 + O(c^{-1})]\ \varepsilon^i{}_1\, \varepsilon^j{}_2\, \varepsilon^k{}_3 \{ [c^{-1} p^k{}_1 + (c^{-1} - 1)^{-1} p^k{}_2 - (\alpha_1/\alpha_3)\, c^{-1} p^k{}_3 -$$

$$- (\alpha_2/\alpha_3)(c-1)^{-1} p^k{}_3]\, \delta^{ij} + c^{-2} [p^j{}_1 + (1-c)^{-1} p^j{}_3 - (\alpha_1/\alpha_2)\, p^j{}_2 -$$

$$- (\alpha_3/\alpha_2)(1-c)^{-1} p^j{}_2]\, \delta^{ik} + (1-c)^{-2} [-p^i{}_2 - c^{-1} p^i{}_3 + (\alpha_2/\alpha_1)\, p^i{}_1 +$$

$$+ (\alpha_3/\alpha_1)\, c^{-1} p^i{}_1]\, \delta^{jk} \}.$$

O(c) terms again cancel due to the momentum conservation and the resultant form of the contribution of the first term reads

$$\varepsilon^i_1\, \varepsilon^i_2\, \varepsilon^k_3\, [\, p^k_2 - (\alpha_2/\alpha_3)\, p^k_3\,] + \varepsilon^i_1\, \varepsilon^i_3\, \varepsilon^j_2\, [\, p^j_1 - (\alpha_1/\alpha_2)\, p^j_2\,] -$$

$$- \varepsilon^j_2\, \varepsilon^j_3\, \varepsilon^i_1\, [\, p^i_2 - (\alpha_2/\alpha_1)\, p^i_1\,] .$$

Now it is obvious that the to achieve a covariant form one has to set

$$\varepsilon^i_r\, \varepsilon^i_s = \varepsilon^\mu_r\, \varepsilon^\mu_s\ , \qquad \varepsilon^i_r\, p^i_r = \varepsilon^-_r\, p^+_r \tag{5.20}$$

which is equivalent to set $\varepsilon^+_r = 0$. Thus by making use of (5.20) the scattering amplitude of three photons yields

$$\varepsilon_1 \cdot \varepsilon_2\ \varepsilon_3 \cdot p_2 + \varepsilon_1 \cdot \varepsilon_3\ \varepsilon_2 \cdot p_1\ - \varepsilon_3 \cdot \varepsilon_2\ \varepsilon_1 \cdot p_2 + \varepsilon_1 \cdot p_2\ \varepsilon_2 \cdot p_3\ \varepsilon_3 \cdot p_1 \tag{5.21}$$

which is the same amplitude given in [43] for three photon scattering amplitude at tree level.

We may use the same machinery to write the four photon amplitude covariantly. For example F_1 which is defined in (5.17)-(5.19) reads to

$$F_1 = \varepsilon_1 \cdot \varepsilon_3\ \varepsilon_2 \cdot p_3\ \varepsilon_4 \cdot p_2$$

when we set $\varepsilon^+_r = 0$ for all r.

We calculated also the other terms of the scattering amplitude in the lines of the above calculation but it is quite lengthy so that we present only the part which results from A_3, which we call $A^{(3)}$,

$$
\begin{aligned}
A^{(3)} = -(\tfrac{1}{2})\,(2\pi)^{26}\,\delta\Big(\sum_r p^{\mu}{}_r\Big)\,\varepsilon_1\,\varepsilon_2\,\varepsilon_3\,\varepsilon_4\,\{ & B(\alpha_s\,,\,\alpha_t+1)\,[p_2\,p_1\,p_1\,p_2 + p_3\,p_1\,p_1\,p_3\,] + \\
& + B(\alpha_s+1\,,\,\alpha_t+1\,)\,p_3\,p_1\,p_1\,p_2 - B(\alpha_s+1\,,\,\alpha_t\,)\,[p_2\,p_3\,p_1\,p_2 - p_3\,p_1\,p_2\,p_2 + p_3\,p_3\,p_1\,p_3\,] - \\
& - B(\alpha_s+2\,,\,\alpha_t\,)\,p_3\,p_3\,p_1\,p_2 + B(\alpha_s\,,\,\alpha_t\,)\,[p_2\,p_1\,p_2\,p_2 - p_2\,p_3\,p_1\,p_3 + p_3\,p_1\,p_2\,p_3\,] - \\
& - B(\alpha_s+1\,,\,\alpha_t-1)\,[p_2\,p_3\,p_2\,p_2 + p_3\,p_3\,p_2\,p_3\,] - B(\alpha_s+2\,,\,\alpha_t-1)\,p_3\,p_3\,p_1\,p_2 + \qquad (5.22) \\
& + B(\alpha_s-1\,,\,\alpha_t+1\,)\,p_2\,p_1\,p_1\,p_3 + B(\alpha_s-1\,,\,\alpha_t\,)\,p_2\,p_1\,p_2\,p_3 + B(\alpha_s\,,\,\alpha_t-1)\,p_2\,p_3\,p_2\,p_3
\end{aligned}
$$

which should be read such that ε_1 is multiplied with the first momentum, ε_2 with the second, ε_3 with the third and ε_4 with the fourth one in the sequence.

The comparision of our results with [44] shows that the Osp(26,2 | 2) invariant string field theory which has the same scattering amplitude with the light cone gauge field theory, reproduces the expected result which was derived by using the covariant operator approach for the four photon scattering at tree level.

B)PLANAR LOOP:

Let us take into consideration the simplest loop diagram, as shown in the figure 4.

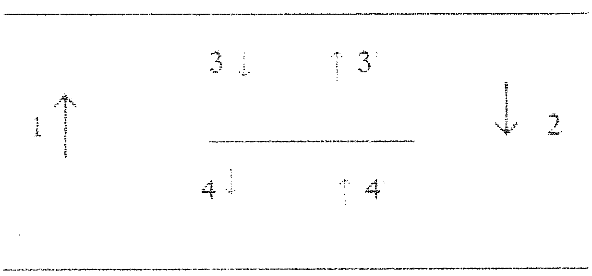

Figure 4.

In the figure the arrows indicate the parametrization of α_r which follows from the δ-functions of the vertices. 3 (4) and 3' (4') have the same kind of normal mode operators but different parametrization. To calculate this diagram the three string vertex (5.1) is sufficent.

The amplitude reads, up to sum overall constants,

$$I_p = \langle 2 | \int dT\, dp^\mu\, d\alpha\, d\beta_o\, d\bar{\beta}_o \langle V(3,4,1) | e^{T(L_{o(3)}-1)(L_{o(4)}-1)} | V(3',4',2) \rangle | 1 \rangle$$

where the external states $\langle 1 |$ and $\langle 2 |$ are DDF states (4.76), $L_{o(3)}$ and $L_{o(4)}$ are given in terms of the mode operators of string 3 and 4, the integration variables are the zero mode variables of the one of strings which construct the planar loop.

As in the tree level case we write the integration variable T as $T = T_2 - T_1$ and we combine the exponential factor with the vertices which lead to

$$I_p = \langle 2 | \int dT\, dp^\mu\, d\alpha\, d\beta_o\, d\bar{\beta}_o \langle V_{T_1}(3,4,1) | \, | V_{T_2}(3',4',2) \rangle | 1 \rangle \qquad (5.23)$$

where the new Neumann coefficents which build up $\langle V_T(1,2,3)|$ are now different from the ones which we foynd at tree level.

To take the trace over the nonzero mode operators of intermediate strings we may still use (5.3) and (5.4) by defining the following column vectors

$$\overline{\alpha}_a = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha_n^{(3)} \\ \alpha_m^{(4)} \end{pmatrix} \qquad\qquad B_a = \frac{1}{\sqrt{2}}\begin{pmatrix} \beta_n^{(3)} \\ \overline{\beta}_m^{(4)} \end{pmatrix}$$

which satisfy the following commutation (anticommutation) relations

$$[\,\overline{\alpha}_a\,,\,\overline{\alpha}^\dagger{}_b\,] = a\,\delta_{ab} \qquad , \qquad \{\,B_a\,,\,B^\dagger{}_b\,\} = \delta_{ab}$$

where we have not written the transpose operations explicitly for the sake of the simplicity of the notation but they are obvious. Now the matrices M and N in (5.3) are

$$M = \begin{pmatrix} N_{T_2}^{3'3'} & N_{T_2}^{3'4'} \\ N_{T_2}^{4'3'} & N_{T_2}^{4'4'} \end{pmatrix} \qquad\qquad N = \begin{pmatrix} N_{T_1}^{33} & N_{T_1}^{34} \\ N_{T_1}^{43} & N_{T_1}^{44} \end{pmatrix}$$

where we have suppressed the m and n indices. L and K now read

$$L = \begin{pmatrix} N_{T_2}^{1'3'}\,\alpha_{(1)} \\ N_{T_2}^{1'4'}\,\alpha_{(1)} \end{pmatrix} \qquad\qquad K = \begin{pmatrix} N_{T_1}^{23}\,\alpha^\dagger_{(2)} \\ N_{T_1}^{24}\,\alpha^\dagger_{(2)} \end{pmatrix}$$

So that the trace over the nonzero bosonic oscillators leads to

$$[\det(1-\overline{M}\,\overline{N})]^{-13} \exp\{ (½)\, \overline{L}\,\overline{N}\,(1-\overline{M}\,\overline{N})^{-1}\,\overline{L} + \overline{K}\,(1-\overline{M}\,\overline{N})^{-1}\,\overline{L} + \overline{K}\,(1-\overline{M}\,\overline{N})^{-1}\,\overline{N}\,\overline{K} \}$$

where the barred quantities are defined as before. Now it is easy to see that the trace over the fermionic operators effect only the measure part due to the fact that the external states have only α_n dependence. Thus the measure part yields

$$[\det(1-\overline{M}\,\overline{N})]^{-12} \quad .$$

For simplicity let us take the axternal states as tachyons, so that the planar loop amplitude reads

$$I_p = \delta(k_1 - k_2) \int dT\, dp^\mu\, d\alpha\, d\beta_o\, d\overline{\beta}_o\, A\, [\det(1-\overline{M}\,\overline{N})]^{-12} \exp\{ C_1 k_1^2 + C_2 k_2^2 +$$

$$+ (C_3 k_1 + C_4 k_2)\cdot p + B\,(p^2 + \overline{\beta}_o \beta_o) \} \qquad (5.24)$$

where A, B, C_r can be found easly but for the moment we don't need their explicit forms. After the integration over p^μ and $\overline{\beta}_o$, β_o and making use of $k_1^2 = k_2^2 = 1$ the planar loop amplitude reads

$$I_p = \delta(k_1 - k_2) \int dT\, d\alpha\, A\, B^{-24}\, [\det(1-\overline{M}\,\overline{N})]^{-12} \exp(C\, k_1 \cdot k_2) \qquad (5.25)$$

which is indeed the planar loop amplitude which would result from the light cone gauge string field theory.

Now our task is to write (5.25) in terms of Koba-Nielsen variables by mapping the planar loop diagram onto the upper half plane. This is achieved by the following map [28] (see Fig.5)

$$\rho(z) = \tau(z) + i\,\sigma(z) = \alpha_1 \ln \psi(Z_1/z, w) / \psi(Z_2/z, w) \quad + \text{ const.} \tag{5.26}$$

where $\alpha_2 = -\alpha_1$ is used. Due to the Mobious invariance we may fix three points in the upper half plane. In (5.26) 0 and ∞ are already fixed so that we may fix one of Z_r:

$Z_2 = 1$, $Z_1 = x$.

Figure 5: The cut is mapped onto the arcs and $w = r_2/r_1$.

ψ can be written in terms of Jacobi Θ-functions as follows

$$\psi(y,w) = -2\pi i\, [\exp i\pi\xi^2/\tau]\, \Theta_1(\xi \mid \tau) / \Theta'_1(0 \mid \tau) \tag{5.27}$$

where prime denotes the derivative with respect to the argument and the new variables are defined as follows

$$\xi = (2\pi i)^{-1} \ln y \quad , \qquad \tau = (2\pi i)^{-1} \ln w. \tag{5.28}$$

Equivalently ψ can be written as a power series in w

$$\ln \psi(Z_r/z, w) = \ln(1 - Z_r/z) - (½)\ln Z_r/z + \sum_{n=1}^{\infty} \{ \ln(1 - w^n Z_r/z) + \ln(1 - w^n z/Z_r) -$$

$$- 2\ln(1 - w^n) \} - \ln z \ln Z_r / \ln w + \text{const.} \tag{5.29}$$

Interaction times defined as $T_r = \text{Re}\rho(z^r_o)$, where z^r_o are the points which maximize $\rho(z)$:

$$\frac{d\rho(z)}{dz}\Big|_{z=z^r_o} = 0 \ . \tag{5.30}$$

Making use of (5.27) in (5.30) yields

$$\frac{2\pi i\, \xi_x}{\tau} + \frac{\Theta_1'(-\xi_z|\tau)}{\Theta_1(-\xi_z|\tau)} - \frac{\Theta_1'(-\xi_z + \xi_x|\tau)}{\Theta_1(-\xi_z + \xi_x|\tau)}\Big|_{z = z^r_o} = 0 \tag{5.31}$$

where ξ_x and ξ_z are defined as in (5.28) by setting y=x and y=z, respectively.

Weierstrass zeta functions are not elliptic functions. They behave under a shift of their primitive periods 2w and 2w' as follows

$$\zeta(z + 2w) = \zeta(z) + 2\zeta(w),$$

(5.32)

$$\zeta(z + 2w') = \zeta(z) + 2\zeta(w')$$

and they are related to the Θ-functions as

$$2w\,\zeta(2wz) = 4w\,z\,\zeta(w)\;\frac{\Theta_1'(z\,|\,\tau)}{\Theta_1(z\,|\,\tau)}\;, \qquad \tau = \frac{w}{w'}\;.$$

Now we may write (5.31) as follows

$$\text{const.} - \zeta(\xi'_z) - \zeta(\xi'_z - \xi'_x)\Big|_{z=z^r_o} = 0 \tag{5.33}$$

where

$$\text{const.} = \pi i / w\tau + 2\,\zeta(w) \qquad \xi' = -2w\,\xi.$$

Now due to (5.32) the difference of two zeta functions, $\zeta(\xi'_z) - \zeta(\xi'_z - \xi'_x)$, is a doubly periodic elliptic function so there exist two solutions of (5.33), z^1_o, z^2_o. It is possible to write the interaction times $T_1 = \mathrm{Re}\,\rho(z^1_o)$ and $T_2 = \mathrm{Re}\,\rho(z^2_o)$ formally [45] but it is not useful for the explicit calculations. At first sight it seems that it is possible to solve (5.33) by making use of (5.29) and the iteration techniques, as a power series in w due to the fact that $0 \le w \le 1$, but after the third iteration it becomes so complicated that it does not allow the generalization to reach to the complete solution. Thus we know that (5.33) has two solutions but we are not able to write them explicitly. So that even if we can write α as

$$\alpha = \alpha_2\,\frac{\ln x}{\ln w} \tag{5.34}$$

we are not able to demonstrate explicitly that the integration region $\{\, 0 \le \alpha \le \alpha_2 \;,\; 0 < T < \infty \,\}$, which is suitable for the s-channel, is mapped onto the integration region in the upper half plane, $\{0 < w < 1 \ , \ w < x < 1 \}$.

The next step is to show if the Jacobian of the transformation is different from zero. Fortunately this has already calculated in [28] and given as

$$\left|\frac{\partial(T,\alpha)}{\partial(x,w)}\right| = \alpha_2 \, (w \, | \ln w \, |)^{-1} \Big\{ \prod_{z_0^1, z_0^2} \left| \frac{\partial^2 \rho(z)}{\partial z^2} \right| \Big\}^{1/2} \tag{5.35}$$

so that it is different from zero, due to the fact that $z^1{}_o$ and $z^2{}_o$ maximize $\rho(z)$ by definition.

The exponential term in (5.25) can be taken equal to the momentum dependent part of the amplitude when written in the upper half plane. Thus the momentum dependent part of (5.23) is

$$\exp\,(C\, k_1 \cdot k_2\,) = [\, \psi(x,w)\,]^{-2k_1 \cdot k_2} \quad . \tag{5.36}$$

Now as it was done in the tree level case we have to find a resolvable relation between C and the measure part to write the latter in terms of x and w. The resultant amplitude must coincide with the one which one finds by making use of the other techniques. Unfortunately the relation between the measure and C is quite hard to achive, since the former is very complicated. The method which led to (5.6) at tree level can not be followed because it is closely related to the explicit form of the time dependent Neumann function coefficents and we have found different time dependent Neumann function coefficents in the planar loop case, (5.23).

VI. DISCUSSION:

The gauge conditions which respect the boundary conditions lead to a superspace for the point particle as well as for the string. The string field theory which is formulated in this superspace has an Osp(d,2 | 2) symmetry which allows us to incorporate string lengths naturally. Then the unitarity of the theory follows directly by the Parisi-Sourlas mechanism.

The Batalin-Vilkovisky method of quantization, which usually seems like "cracking nuts with a hammer" for most of the known point particle gauge field theories, reveals its power in the quantization of the gauge invariant string field theory.

Gauge invariance of the string field theory is up to the order of g. The invariance at order g^2 will follow by adding the following four string interaction term to the action

$$\prod_{r=1}^{4} \int dZ_r\, dz_r \frac{d\alpha_r}{\alpha_r}\ d\tau\ V(1,2,3,4)\ \tilde{X}(1)\, X(2)\, X(3)\, X(4)$$

where V(1,2,3,4) is a function of $C(\sigma_{int})$. The gauge generators will now have a quadratic term. Unfortunately this will complicate the gauge fixing procedure. At the higher levels one would have to introduce also the closed string interactions for keeping the gauge invariance. It is obvious that these follow from the fact that the proof of the Osp(d,2 | 2) invariance of the gauge fixed action is almost equivalent to the demonstration of the Lorentz invariance of the light cone gauge string field theory.

In principle there is no obstacle in formulating a closed string field theory which has the properties of the open string field theory which we describe. The condition

$$[L_0 - \overline{L_0}]\, \Phi = 0,$$

where L_0 and $\overline{L}_0$ are defined in terms of right and left mode operators and Φ is a closed string field functional, results after an integration over a Lagrange multiplier, but in any case one has to use a projection operator to satisfy this condition. It is not clear if some other gauge fixings can allow one not to treat the above condition separately as well as lead to an Osp(d,2|2) invariant theory.

In chapter V we try to give an idea about the perturbative calculations in string field theory by making use of the operator approach. At tree level even if it was complicated to do so, we were able to calculate the scattering amplitudes. At first loop level the calculation became quite complicated so that for the time being in this approach we have not been able to calculate the scattering amplitudes. Thus it seems that using the path integral approach is more efficent than the operatoral approach. For demonstrating that an open string field theory provides a single cover of the moduli space, one can also find some other techniques as was done for the closed string case in [46].

Obviously the next step is to formulate a superstring field theory which results from a first quantization which leads naturally to an Osp(d,2|2) invariance. To this aim in [47] BRST quantization of the spinning particle in a gauge similar to the point particle case is performed.

One hopes to utilize the covariant string field theory to discover the nonperturbative aspects of strings. But until now there is no information to support this so that the problem remains open.

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