Aspects of large N analysis for the Yang-Mills-Higgs model and matrix models

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1 Introduction

The main argument of this thesis is the analysis of some nonperturbative features of the Yang-Mills-Higgs (YMH) model in three dimensions for gauge group $SU(N)$ and some implication for the large N limit.

Along the way we have also stopped to analyze the Eguchi-Kawai reduction method which is a tool for investigating the large N limit of any model of matrices.

Our YMH model is the direct generalization to $SU(N)$ of the Georgi-Glashow model [1]. This was one of the first models providing a spontaneous breaking of a gauge symmetry, later incorporated in the celebrated Weinberg-Salam model of electroweak interactions.

While in the WS model the symmetry breaking is implemented in the weak isospin group, it was soon realized that the same phenomenon in the color group could provide a new set of interesting physical phenomena. Among them the quantization of charges of quarks and the existence and stability of nontrivial classical "soliton" solutions to the static equations of motions [2, 3]. They are then three dimensional objects.

This latter new field configurations were shown to possess a particle interpretation as magnetically charged objects, and thus were named *monopoles*. They realized with a smooth field configuration the older, singular, Dirac monopole [4].

Strictly the origin of the 't Hooft Polyakov monopole is exactly in the YMH model, because the Higgs potential provides a length scale and the size for such configurations, providing their stability.

It was however argued in the celebrated paper [5] on oblique confinement that such configurations should exists also in pure Yang-Mills theory of strong interactions, and play a major role in the mechanism of quark confinement with their condensation.

Already the Georgi Glashow model in two dimensions shows analogous configurations, *vortices* [6], which account for the BCS theory of superconductivity [7].

It was an idea from the early ages of QCD [8, 9, 10] that quark confinement could be explained with a picture analogous to that of type II superconductors but with the role of electric and magnetic charges interchanged (and for late issues like abelian dominance in confinement see [11, 12]).

For nonabelian groups $SU(N)$ the vortex solutions are not stable, mainly due to the triviality of the fundamental group, and this complicates the matter in that it requires the breaking to some subgroup with abelian factors.

This is exactly realized in the YMH model, and indeed the ideas that were around, about confinement and the infrared behavior of gauge theories with compact gauge group [13], were explicitly shown to hold in the context of the Coulomb gas approach to the ensemble of widely separated monopole solutions [14].

This treatment is based on the fact that widely separated monopoles interact via a Coulomb interaction, despite the exact form of the energy is not known explicitly for generic parameters and also the space of moduli of the generic solutions has a nontrivial geometric structure [15].

After these early years, lattice simulations have been largely employed to test the occurrence and condensation of monopoles in correspondence to the deconfinement-confinement transition, in the pure Yang-Mills theory [16].

Because of the absence of the Higgs field, monopoles are of no fixed size, and thus some mechanism of generation of mass is necessary if monopoles are to be relevant for the confinement mechanism. This is indeed found on the lattice and there are arguments in the three dimensional continuum theory [17].

Let's recall also that the *static* four dimensional theory is equivalent to the YMH model with the limit of no potential. This is one of the main reasons of why to study the YMH model. We choose to include also the potential which is hoped to arise from the nonperturbative effects of the four dimensional theory.

Up to now this is the standard lore on the monopole confinement scenario. The theory can be generalized to arbitrary gauge groups, and in particular to $SU(N)$, not just for academic interest, but because the presence of a further parameter, N, can be used to approximate the physical case of $N = 3$.

This method dates back to early seventies, starting with 't Hooft [18] applying new ideas about large N to gauge theory and generic adjoint matrix fields, and is based on the possibility to extract the leading order in N of interesting quantities.

Many interesting features are domain of this expansion, mainly: the classification of Feynman graphs according to the surfaces where they can be embedded gives also the order of $1/N$, actually $1/N^2$, and this allows the contact with the string interpretation of gauge theories on one hand, and on the other with the random surface interpretation of zero dimensional matrix models; for suitable operators [19] the correlation functions "factorize", that is, are given by the disconnected part at leading order; finally N is not renormalized, in that it is a dimensionless external fixed parameter.

Of course a special mention is due to the pure $N = \infty$ case, where one implicitly lets $N \to \infty$ *before* removing any cutoff. This can bring the theory to have a different phase structure. There is a specific example in one dimension that is the Kazakov-Migdal phase transition.

The second property above, factorization, shows that for the operators which follow it, the large N limit is a kind of semiclassical limit. Their fluctuations are suppressed, and $1/N$ plays the role of $\hbar$ for them.

One of the interesting phenomena, and of recent investigation, is the emergence of a new dynamic in the $N = \infty$ limit of some theories. These include affine Toda models, principal chiral fields in one and two dimensions, two dimensional QCD, not to mention the matrix models.

In all these theories the effect of taking N large is to generate an infinity of states which coalesce to form collective excitations of some other, higher dimensional, theory.

The emergence of a new dimension is the remnant of the matrix index of the diagonal fields. In these theories angular variables play an important role but are integrated, more or less explicitly, to leave an effective interaction for the eigenvalues.

An example is the theory of the principal chiral field one the line[20], which is equivalent, for some boundary conditions, to two dimensional QCD with its interpretation in terms of two dimensional string.

We come thus to our case of Coulomb gas of magnetic monopoles. It was constructed as a sum on the classical dilute configurations of monopoles, weighted with the determinant of gaussian fluctuations around them.

With the Coulomb gas there is, via a Sine-Gordon transform, a dual representation and the possibility to achieve an estimate about the *string tension* for Wilson loops of large area.

The string tension is related to the mass of the monopoles by an exponential relation, which shows, like in the dual Landau-Ginzburg theory, that it is related to the density of monopoles.

For $SU(N)$ we have $N(N-1)$ kind of monopoles, with magnetic charges in different couples of $U(1)$ sectors, and the coulomb gas can be generalized to this case [21].

Also the Sine-Gordon transform is easily constructed, provided one takes into account the different species of monopoles that necessarily exist. This was not done in [21].

One also has to consider the determinant of quantum fluctuations around the different monopole backgrounds.

Up to this point the analysis is valid for any N and provides no new features.

A possible new behavior comes instead from the large N limit, because there necessarily appears a distribution of masses which are present in the theory.

These masses can be of order, a priori, in the interval from 0 to N, and the physics of course has to be very different from case to case.

Because from the monopole construction the masses are related to differences of eigenvalues of the Higgs field at infinity, ϕ^{∞}, all the model depends on its distribution of eigenvalues.

In the standard picture of symmetry breaking this can not be changed by any fluctuation, once the universe has formed. One simply fixes it. The modulus of ϕ^∞ gets renormalized and possibly shifted as with the arguments of effective Higgs potential, but there is no indication on its direction in the Cartan space.

In particular it is not possible to derive the distribution of eigenvalues from any effective action like, for example, the partition function of the monopole gas.

Indeed all processes which are considered in perturbation theory end up with an effective potential that is flat in Cartan rotations.

Nevertheless in the course of this analysis one is tempted to use the *unitary gauge*, because there the physical degrees of freedom are explicit and the Higgs eigenvalues too.

The unitary gauge is somewhat singular, because its Faddeev Popov determinant is not defined in the continuum. It turns out to be the product at each point of the Van-der-Monde determinant constructed with the eigenvalues of ϕ^∞.

This factor in the functional integral seems to provide measure zero for all configurations where some eigenvalues coincide, giving a sort of repulsion of eigenvalues.

This immediately faces with the problem that for monopole configurations there is necessarily some point where the eigenvalues coincide, thus giving zero weight to all these configurations.

Fortunately as the analysis is carried on, and still thinking that the theory *is* renormalizable, we can show that once the quantum fluctuations of the massive gauge fields are taken into account, the Van-der-Monde ultralocal determinant is canceled in part. What remains is just the Van-der-Monde determinant of the eigenvalues of ϕ^∞!

Even this term does not authorize us to think to some quantum lifting of the degeneracy in Cartan directions, because again ϕ^∞ is fixed at the "beginning of the universe".

However one can think that at the epoch of its formation, the system *is* sensible to this term, and thus chooses the distribution which maximizes it.

We have found this distribution, which determines back the distribution of masses of gauge bosons and monopoles in the system.

It shows a peculiar shape, and should bring peculiar consequences in the properties of the system [82].

It is worth noting that the same distribution of eigenvalues of 'Higgs field' and gauge boson masses, is found [22] in the recent nonperturbative solution of $SU(N)$ supersymmetric Yang-Mills theory in four dimensions (the $\mathcal{N} = 1$ case), once the complete confinement is required. It is striking that the same distribution appears in the large N, and as its very origin is not known in that case, it can arise from just geometrical arguments like in our case.

Coming back to our problem of the monopole gas, we have now a distribution to analyze the system and we could proceed.

Unfortunately the very difficulty of the program is still there, because the semiclassical sum needs the evaluation of the determinant in the external field of a monopole.

This problem is still unsolved despite many efforts [23, 24, 25, 26] and is non-trivial.

We would like to extract some information at least in the large N limit, and for this we thought to study the Eguchi Kawai reduction method which carries drastical simplifications. The application of the EK method to the monopole determinant is yet to be done, and I plan to study it soon.

Because we have gauge configurations with nontrivial boundary conditions, it is necessary to be careful with the EK reduction, because it reduces all theories to a single lattice site.

In fact models with finite volume ("hot" models) have been constructed but with much more effort in the context of the Twisted Eguchi-Kawai reduction. We promise to study also this possibility in the future.

We try to analyze thus the cases of EK reduction applied to the first models that one encounters, before the theories with diagrammatical perturbative expansion: the gaussian matrix model and the chiral field.

From the lesson of matrix models, the angular evolution plays an important role in many models, and it happens that the EK reduction in some sense 'mistreats' it.

Already from the gaussian model the prescription of *uniform* quenched momenta does not work to reproduce the partition function of a matrix oscillator. This model has in no way a diagrammatical expansion, and the uniform quenching gives the wrong result.

For what regards the Principal chiral field, we analyze it on the finite time interval, to find the known partition function. The model with fixed boundary conditions, which would show the nice feature of large N phase transition, gives rise to a non solvable matrix model, but the one with integrated boundary conditions reduces to the Itzykson Zuber unitary matrix integral and can be solved.

After some steps, the uniform distribution of momenta is shown to reproduce the *weak coupling* phase of the discretized version of the model, and thus the wrong phase in the continuum limit.

We show however that if the one dimensional chiral field is treated as a phase factor and quenching is done on its "connections", the quenching gives the correct result.

Let's summarize our path in this work.

The first part of this work deals with some new large N ideas for the YMH model in three dimensions. Needless to say there is a large historical and scientific background and it is of course difficult to say something really new on these subjects. Nevertheless some latest ideas on matrix models, subject on which I have worked in the first part of my period here at SISSA, are a valuable tool and should find applications in otherwise 'slow' fields. The study of large N model of monopole gas are not investigated, to our knowledge, for example. This work wants to be a starting point for this investigation.

Along this analysis we have found many and different problems to think about, the principal is the reason how the Eguchi Kawai works, an issue that also has not been completely clarified, despite of the volume of numerical calculations.

After one finds a reliable method for the functional determinant, it will be possible to draw definite conclusions on the large N monopole gas, which seems promising some interesting feature, due to the competition of factors which takes place in the large N limit. This work will surely be continued in the near future.

I have of course to thank prof. D. Boulatov for posing problems and for discussing their development during this year; I have surely learnt much interesting physics.

Then I would like to thank Prof. L. Bonora for the work made on multi matrix models, and for having introduced me to the world of classical integrable hyerarchies on which I will surely continue some of my activity. At the same time I thank E. Vinteler for the collaboration, for many discussions in general and on the one-dimensional matrix model, and for the atmosphere.

SISSA itself must be reminded for the nice and stimulating environment, and so all the people that I have met in these years. Among them A. Valleriani, V. Antonelli, A. Fabbri, L. Sbano, D. Gouthier (the formula of double radicals), P. dall'Aglio and P. Siniscalco also for the collaboration in the field of Hopf algebras. To F. Vissani a special thank for the kindness he has demonstrated in human and physical discussions, and a final thank to D. Amadori for more than psychological support.

2 Preliminaries for the YMH model

We start by considering the Yang-Mills/Higgs $_3$ model built on the gauge group $SU(N)$, so it is a 3-dimensional euclidean theory of a gauge field $A_\mu = A_\mu^a T_a = \mathbf{A}_\mu \cdot \mathbf{T}$ and a matter field $\phi = \phi^a T_a = \boldsymbol{\phi} \cdot \mathbf{T}$ both living in the adjoint representation.

Both are arranged in $N \times N$ matrix fields living in the algebra of $SU(N)$. In all the following we will always denote with plain symbols such fields, like A_μ, ϕ, and with bold symbols the vector of their components along the algebra generators, like $\mathbf{A}_\mu$, $\boldsymbol{\phi}$ above. Whenever the field will be diagonalized, its vector will have only the Cartan components.

To fix the notation we take the normalization such that

$$\mathrm{tr}(T_a T_b) = \frac{1}{2}\delta_{ab}$$

in the fundamental representation. We will occasionally mention also the theory with T in the adjoint representation, in next chapter, because they give rise to very different monopole scenario.

We have thus the following notations:

$$2\mathrm{tr}\phi^2 = \boldsymbol{\phi} \cdot \boldsymbol{\phi} = |\boldsymbol{\phi}|^2.$$

The configuration space Γ is the space of maps from R^3 to the couple (A, ϕ) with finite action $\mathcal{S}$.

In this configuration space acts a continuous $SU(N)$ gauge group:

$$A_\mu(x) \rightarrow \omega^{-1}(x)\partial_\mu \omega(x) + g\omega^{-1}(x)A_\mu(x)\omega(x) \tag{2.1}$$
$$\phi(x) \rightarrow \omega^{-1}(x)\phi(x)\omega(x) \tag{2.2}$$

which leaves the action invariant.

Because we will focus also on the large N expansion, it is necessary to adapt the parameters of the theory to this limit.

It is the standard remark [27] that the large N limit is nontrivial only if the perturbative series remains finite, and this requires the coupling constants to be suppressed with powers of N. Rescaling the fields one can require all terms to be of the same order.

The action is then:

$$\mathcal{S} = \int \mathrm{d}^3x \left\{ \frac{N}{2g^2}\mathrm{tr}(G_{\mu\nu}G_{\mu\nu}) + \frac{N}{2}\mathrm{tr}(\mathcal{D}_\mu\phi\mathcal{D}_\mu\phi) + V(\phi) \right\} \tag{2.3}$$

with $\mathcal{D}_\mu = \partial_\mu + g[A_\mu, \cdot]$, $G_{\mu\nu} = [D_\mu, D_\nu] = +g(\partial_\mu A_\nu - \partial_\nu A_\mu) + g^2[A_\mu, A_\nu]$ and $V(\phi)$ a scalar potential.

This YMH theory in three dimensions can be seen as the static version of the four dimensional minkowsky YMH, or even of self-dual pure YM theory, where the A_0 component assumes the part of the Higgs field and a potential is argued to appear as a nonperturbative effect [?]. It has thus also some direct phenomenological interest.

So, to make contact with the four dimensional theories, one respects four dimensional renormalizability, and takes the potential to be a quartic polynomial in the traces of the Higgs field:

$$V(\phi) = \lambda[\mathrm{tr}(\phi^2) - \mu^2]^2 \tag{2.4}$$

(Where μ is of order N). The choice of a function which is symmetric in the different Cartan directions is not the only one [28] but is the more natural.

$V(\phi)$ induces a spontaneous shift of the Higgs vacuum value from zero to some ϕ^∞, and thus also a spontaneous breaking of the non abelian group G down to the subgroup H which leaves ϕ^∞ invariant. This is called the *little group.*

2.1 Spontaneous breaking of G

The last statements follow from the constraint of finite action, which also is derived from the four dimensional constraint of finite energy. One has two results:

• the potential V, being positive, has to vanish at infinity in all directions, so that one must have always $\mathrm{tr}\phi^2 \to \mu$ at infinity (as, at least, $\frac{1}{r}$). This shows that Γ is divided in disjoint sets classified by the winding of the two-sphere at infinity into the coset G/H (a unit vector in internal space modulo symmetry around it): $\pi_2(G/H)$. This group is isomorphic to $\pi_1(H)$ because $\pi_1(G)$ is trivial.

In case the vacuum Higgs field has all different eigenvalues the little group H is the maximal abelian subgroup $U^{N-1}(1)$, and the classification has $N-1$ topological quantum numbers: Z^{N-1}. Moreover thanks to the vanishing of $\pi_2(G)$, this classification is gauge invariant, because gauge transformations of ϕ_∞ are homotopic to the identity.

• then, in the spirit of the semiclassical approach, one considers the ϕ's with $|\phi| = \mu$ as a Higgs vacuum and expands the action in fluctuations around that configuration. One has a pointwise breaking of G down to the little group, which leaves $\phi(x)$ invariant. All gauge fields not belonging to H acquire a mass, while those in the little group remain massless.

At points where the Higgs field has all different eigenvalues, the breaking is maximal and the little group is $U^{N-1}(1)$. As 't Hooft shows in [5], the manifold of points where two eigenvalues of ϕ coincide has dimension $d-3$, that is, for us, consists of isolated points. We will return on this points in the next section.

In the next section we will see also, in the case of $SU(N)$, how the topological number which classifies the Higgs field represents the magnetic charge of the total configuration (Higgs and gauge fields) under the broken symmetry group (modulo equivalence under the Weyl discrete symmetry).

The little group H is always of the form $T' \times G'$, where T' is some abelian torus and G' is a simple subgroup. In any case it always includes a $U(1)$ subgroup. It is called the electromagnetic group and is generated by the Higgs field itself.

We have still to consider the other terms in the action, namely the Higgs kinetic term and the pure gauge term. They can in principle forbid some of the previous topological sectors, as happens in the gaugeless limit*.

In our case [29] the coupling to the gauge fields allows for different internal directions of the Higgs field at infinity, absorbing the magnitude of its kinetic term. It allows in fact for smooth configurations with nontrivial winding.

For example for any Higgs configuration of generic winding at infinity, Φ, one can define a finite action configuration in all the space as follows:

$$\begin{cases} \phi(x) = \Phi(x/|x|) \\ A(x) = \frac{g^{-1}}{|\phi|^2}[\phi, \partial\phi] + a_\mu \phi \end{cases} \quad \text{for } |x| > 1 \text{ and some smooth field inside.} \tag{2.5}$$

From this ansatz one can construct finite action solutions with nontrivial windings (for example [30]).

This is however just a topological existence argument and does not pay attention to whether they are a minimum of the action or not. We wanted to show it to make explicit the configurations which belong to the various classes.

Before embarking in the next section in the analysis of the classical solutions, let us make a final remark valid for all the configurations of nontrivial winding.

Following ϕ smoothly in all the space, one necessarily meets a point where it has at least two coinciding eigenvalues, because otherwise the winding would have disappeared.

This statement is obviously gauge invariant so it is true even in non regular gauges like the unitary gauge.

*In fact with no coupling to the gauge fields the Higgs kinetic term forces ϕ to have the same direction at infinity, and we end up necessarily with winding number zero.

3 Classical monopoles

In this section we introduce the classification of monopoles [4, 31, 32, 33, 34, 35, 36, 37] that arise in 3 dimensions for the YM_3 with group $SU(N)$:

$$\mathcal{S} = \int d^3x \left[\frac{N}{2}\mathrm{tr}(G_{\mu\nu}G_{\mu\nu}) + \frac{N}{2}\mathrm{tr}(\mathcal{D}_\mu\phi\mathcal{D}_\mu\phi) + \lambda(\mathrm{tr}(\phi^2) - \mu^2)^2 \right] \tag{3.6}$$

First of all one introduces what are called *point monopoles*, singular configurations that are the solutions of the equations of YMH plus the requirement of minimum action:

$$\mathcal{D}_\mu G_{\mu\nu} = 0 \tag{3.7a}$$

$$\frac{\partial}{\partial\phi^a}V(\phi) = 0 \tag{3.7b}$$

$$\mathcal{D}_\mu\phi^a = 0 \tag{3.7c}$$

The last two equations impose that one gets everywhere zero contribution to the action from the Higgs sector, so that we are in what is called *Higgs vacuum.* Some gauge fields will the have to be zero and the others will allow for nontrivial solutions localized near isolated points.

To see this it is useful to choose the abelian "unitary" gauge $\phi \in H$ (where H will be the Cartan subalgebra of SU(N)). We will write $\phi = \boldsymbol{\phi} \cdot \mathbf{T}$, where $\mathbf{T}$ are the $(N-1)$ commuting generators of H in some faithful representation and $\boldsymbol{\phi}$ is a vector in R^{N-1}.

In this gauge the above equations imply that the Higgs field is a constant in all space $\phi(x) = \phi_\infty$ and that A_μ has nonzero components only in the algebra of the *little group* of ϕ_∞, i.e. only if $[A, \phi_\infty] = 0$.

For ϕ_∞ with generic eigenvalues (all different) the little group is simply $U(1)^{N-1}$, so that only the Cartan gauge fields survive in the vacuum.

In the vacuum ϕ_∞ induces perturbatively a mass term for each gauge field that is charged with respect to ϕ_∞. For each root $\boldsymbol{\alpha}$ of SU(N) the mass of the charged $A^{\pm}_{\boldsymbol{\alpha}}$ fields is $m_{W_\alpha} = g|\boldsymbol{\phi}_\infty \cdot \boldsymbol{\alpha}|$. At the same time the Cartan gauge fields decouple from the Higgs fields.

As $V(\phi)$ is flat in all gauge directions, the $V'(\phi) = 0$ constraint only imposes a fixes the modulus of the vacuum Higgs field: $\mathrm{tr}\phi^2_\infty = \mu^2$.

We will assume at this point a vacuum ϕ_∞ with all different eigenvalues. This is a gauge invariant statement and we will see that this vacuum configurations will be preferred by the system itself, in the large N limit. Alternatively it could be imposed by some gauge invariant external source.

3.1 Point monopoles

The $G_{\mu\nu}$ field equation now allows for abelian $U(1)^{N-1}$ solutions with a singular Dirac string [4]:

$$A_\mu = \mathbf{T}\cdot\mathbf{q}D_\mu \qquad D_\mu = (1 - z/r)\partial_\mu \tan^{-1}(y/x) \tag{3.8}$$
$$G_{\mu\nu} = g\mathbf{T}\cdot\mathbf{q}\frac{\epsilon_{\mu\nu\lambda}x_\lambda}{|x|^3} \qquad \mathbf{q}\in \mathrm{R}^{N-1}.$$

$\mathbf{q}$ is the nonabelian charge of the monopole, but as we are in the unitary gauge, it belongs to $U(1)^{N-1}$: it is for now an arbitrary vector in R^{N-1}.

As $z \to -\infty$ we have the asymptotic form:

$$A_\mu = 2\mathbf{T}\cdot\mathbf{q}\partial_\mu\Phi \qquad (0 \le \Phi = \tan^{-1}(y/x) \le 2\pi). \tag{3.9}$$

Observing the phase of a loop around the string we get a realization of $\pi_1(U(1)^{N-1})$ and obtain the admissible monopoles: the generalized Dirac condition

$$e^{4\pi i\mathbf{q}\cdot\mathbf{T}} = \mathbf{1}. \tag{3.10}$$

This condition restricts the possible charges $\mathbf{q}$ to belong to a *lattice* in R^{N-1}, in fact $\mathbf{q}$ has to be reciprocal to each weight of the representation chosen for the $\mathbf{T}$: for every weight $\mathbf{m}_i$,

$$\mathbf{q}\cdot\mathbf{m}_i = \frac{n_i}{2} \qquad n\in \mathrm{Z}. \tag{3.11}$$

The lattice of charges depends thus on the representation chosen for the $\mathbf{T}$'s, calling in the game also the global properties of the representation of the gauge group.

It can be more or less dense depending on the modulus of the highest weight of the representation.

One now introduces the (dual) *co-roots*, $\boldsymbol{\alpha}_i^* = \boldsymbol{\alpha}_i/\boldsymbol{\alpha}_i\cdot\boldsymbol{\alpha}_i$, (where $\boldsymbol{\alpha}_i$ are the simple roots). For each weight they satisfy the relation $\mathbf{m}\cdot\boldsymbol{\alpha}^* = n/2$, so that they are reciprocal to the weight lattice. The coroot system defines what [31] have called the *dual group*. For $SU(N)$ the dual group is isomorphic to it, denoted $SU^*(N)$. Moreover for roots normalized to unity the coroot lattice coincides with the root lattice.

An immediate consequence of the relation with the weight lattice is that the coroots (and also their multiples) are always between the possible magnetic charges $\mathbf{q}$. Monopoles in the adjoint representation of the dual group are thus always present.

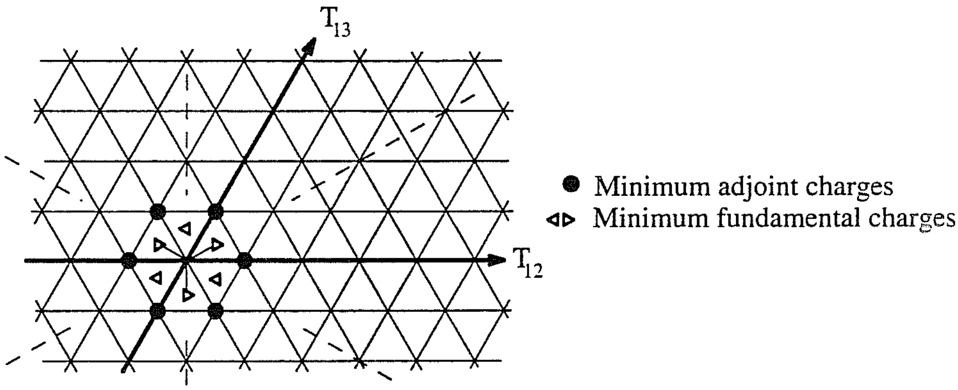
Usually one studies the simplest cases of the fundamental and adjoint representations, for the gauge field.

For $\mathbf{T}$ in the fundamental representation ($\mathbf{T} = \frac{1}{2}\boldsymbol{\lambda}$) the weights are the fundamental ones and the magnetic lattice is just the coroot lattice

$$\mathbf{q} = \sum_{i=1\ldots N-1} n_i \boldsymbol{\alpha}_i^* \qquad n_i \in Z \tag{3.12}$$

The monopoles of minimum charge transform in this adjoint representation.

The picture represent the coroot lattice for $SU^*(3)$, its generators as black circles. The small triangles represent the fundamental monopoles that arise for adjoint gauge generators†. There is thus a nice duality: for gauge variables with fundamental (adjoint) generators, the minimum monopoles transform in the adjoint (fundamental) representation of the dual group.



A remark is due for the Weyl group which acts in the weight lattice and thus also on the magnetic charges. It is generated by the reflection with respect to the planes orthogonal to the roots, and sends every lattice that we have considered into itself. The action can be seen as a reflection also on the magnetic charges.

Seen on the Cartan generators, it simply exchanges the diagonal entries (as can be easily seen in the (overcomplete) basis $2(T_{ij})_{kl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = \begin{pmatrix} \ddots & & & \\ & & 1 & \\ & & \ddots & \\ & -1 & & \ddots \end{pmatrix}$).

†In fact an other case mentioned in the literature [21] is that of gauge variables with generators in the adjoint representation. The nonzero weights are in this case the simple roots, so that the reciprocal lattice coincides with the weight lattice of the dual group ($\mathbf{m}^* \cdot \boldsymbol{\alpha} = n/2$):

$$\mathbf{q} = \sum_{i=1\ldots N-1} n_i \mathbf{m}_i^* \qquad n_i \in Z. \tag{3.13}$$

One can say that the minimum charge monopoles transform now in the fundamental representation of the dual group. They are shown in the picture as small triangles. The monopoles that arise in this case include also the previous adjoint charges as combinations of minimal monopoles, although the lattice generated by them is not shown.

On the fields, any Weyl action is equivalent to a *global* gauge rotation with respect to the generator $E_\alpha + E_{-\alpha}$, where α is the relative root, and this shows that monopole configurations related by Weyl symmetry are gauge equivalent.

This has implications on the type of monopoles: for the usual case of gauge variables in the fundamental representation the minimum charge monopoles are all related by Weyl transformations.

In the adjoint representation instead monopoles are classified by the dual fundamental weights which are not all Weyl-equivalent: they are divided in classes according to the $N-1$ (nontrivial) elements of $\pi_1(SU(N)/Z_N)$. Weyl reflections only act within these classes.

An example for the dual $SU(3)$ is shown in the picture, the fundamental coweights are represented by the the small triangles, the right pointing giving the [3] representation, the left the [3*]. The Weyl transformations are the reflections with respect to the long-dashed lines.

One sees that the fundamental monopoles come in two triplets invariant under Weyl, while the usual adjoint monopoles come in a whole sextet.

3.2 Regular monopoles

Let's notice first from (3.10) that if the $U^{N-1}(1)$ path $e^{2\mathbf{q}\cdot\mathbf{T}\Phi}$ can be continuously gauged away *in SU(N)* to the identity, then the Dirac string will decrease of intensity and disappear. The gauge transformation needed to do that is necessarily noncostant, so that one will end up with a noncostant Higgs field. If this can be done, the point monopole is the basis for a regular one with finite energy.

It is clear that there is a great freedom to construct these regular solutions of the equations of motion. And it is really a nontrivial problem.

The *spherically symmetric* solutions have been studied in detail in [32] which have classified all the possible charges from which one can determine a finite energy configuration.

The charges which admit spherical solutions are given by $\mathbf{q} = \mathbf{q}' - \mathbf{q}''$ where $\mathbf{q}'$ and $\mathbf{q}''$ are the roots of two embeddings of SU(2) in SU(N) and $\mathbf{q}''$ must also be in the little group. One SU(2) $\mathbf{q}'$ is needed to rotate the Higgs field to a radial gauge, and the other is a remaining freedom to define the spherical gauge configuration.

Since the factor in (3.10) is also a loop in the chosen representation of $SU(N)$, and since $SU(N)$ is simply connected, the process is clearly possible for any charge only if the generators are in a faithful representation.

In the case for example of the adjoint representation, there are N inequivalent loops which join the identity to the elements of the center of $SU(N)$. So, for $N-1$ kind of point monopoles, the string is impossible to remove and they are genuine Dirac monopoles. An other accident of this case is that the minimum charge monopoles are in this case N(N-1) (the weights of all fundamental representations), while the nontrivial elements of the center Z_N are N-1. The minimum charges can thus be divided in (N-1) sets of N elements, and the Weyl group acts only within each set.

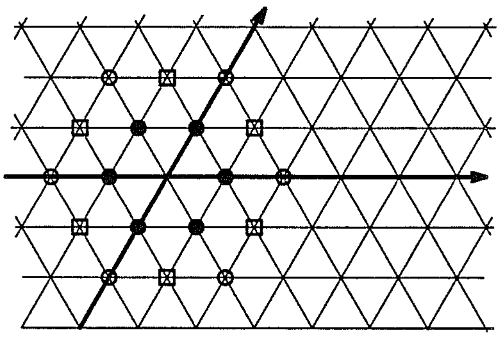
All this fundamental monopoles, associated to the nontrivial paths in SU(N) around the Dirac string, cannot be made regular.

The Weyl group relates monopoles of different charges by a global gauge transformation. This is a difference with the abelian case where different charges classify gauge inequivalent monopoles. This peculiarity of non-abelian theories follows mainly from the simple-connectedness of SU(N), but also from the fact that colored flux lines are not gauge invariant (but covariant) and can thus be deformed or changed of color by gauge transformation.

Another important notice is that the charge quantization condition (3.10) does not depend on the vacuum Higgs field ϕ_∞, and this is an advantage of the unitary gauge.

Regular monopoles instead are constructed changing gauge transforming the Dirac string into a varying Higgs field and this process depends on the ϕ_∞ boundary condition.

As we are going to assume ϕ_∞ to have all different eigenvalues, the charge $\mathbf{q}''$, belonging to the little group, has to be necessarily zero, so that the magnetic charges of spherically symmetric monopoles will coincide with $\mathbf{q}$ roots of SU(2) embeddings. The figure shows them for $SU(3)$, for $N > 3$ the pattern is much more complicated.



Possible spherical monopoles for SU(3)

● 3 -> 2+1 embedding Q=(1/2,-1/2, 0)

○ 3 -> 3 embedding Q=(1, 0, -1)

□ 3 -> 3 only with degenerate ϕ Q=(1,-1/2,-1/2)

Only the first two cases are possible for nondegenerate ϕ_∞. For this cases the solution is well-known: $T_q = \mathbf{q}\cdot\mathbf{T}$ with some $E_\mathbf{q}$ and $E_{-\mathbf{q}}$, give the SU(2) subalgebra and

$$\begin{aligned} \hat{A}_\mu(\mathbf{r},\mathbf{q}) &= T_q D_\mu + K(m_W \mathbf{r})\left[E_\mathbf{q} e^{-i\phi}(i\hat{\theta}+\hat{\phi})_\mu + E_{-\mathbf{q}} e^{i\phi}(-i\hat{\theta}+\hat{\phi})_\mu\right] \\ \hat{\phi}(\mathbf{r},\mathbf{q}) &= \phi_\infty\cdot(\mathbf{T}-\mathbf{q}T_q) + H(m_H \mathbf{r})(\phi_\infty\cdot\mathbf{q})T_q \end{aligned} \quad (3.14)$$

Changing gauge to the regular, radial one one can "smear out" the string D_μ leaving an isolated singularity at the origin.

3.2.1 Masses

In the last expressions, m_W is the mass of the two charged gauge bosons in the chosen sector of the monopole, function of the Higgs vacuum: $m_W = g|\phi^\infty \cdot \mathbf{q}| = g|\phi_i^\infty - \phi_j^\infty|$; m_H is instead the mass of the Higgs field "in the sector of T_q": $m_H^2 = 2\lambda((\phi_i^\infty)^2 + (\phi_j^\infty)^2)$.

They regulate the exponential decay of the massive field components of the monopole as well as its total classical action. We have in fact

$$\mathcal{S}_{cl} = N|\mathbf{q}|\frac{m_W(\mathbf{q})}{g^2}C(\frac{\lambda}{Ng^2}). \tag{3.15}$$

The function C is found [38] to approach the value 4π when the argument goes to zero, which for $SU(2)$ is called BPS limit.

In the large N limit we will take this value, but we stress that the BPS limit for $SU(2)$, $\lambda \to 0$, is physically different because also the Higgs mass $\lambda\mu^2$ vanishes.

Considering the $N(N-1)$ unit charge monopoles, we see that, fixed ϕ^∞, their (pseudo) mass ranges in the interval $0 \leq m_W \leq \mu/g$.

We see that most of the properties of the various objects are ruled by the m_W of the gauge bosons in the relative $SU(2)$ sector, so that all depends on the Higgs vacuum eigenvalues ϕ_i^∞.

The constraint $|\phi^\infty| = \mu \simeq O(N^{1/2})$ has important consequences on the hierarchy of masses that are present in the model in the large N limit, because, as we will see, necessarily there are masses that become small at least as $O(1/N)$. We will also find masses of order $O(1/N^2)$.

All this affects the physics in the large N limit.

The higher charge monopoles are not realizable as single spherical configurations in three dimensions, although the topological argument indicates the existence of some minimum of the action.

Some can be constructed as multimonopole-like configurations which possess discrete symmetry groups, with stability given by the Higgs attraction. Such configurations have been found to exist with tetrahedral, octahedral (but not icosahedral symmetry, for a late reference, see [39] and therein).

All these, as well as the spherical monopoles of charge greater than 1, have higher mass proportional to their charge and they are expected to dissociate into smaller constituents. Hence their contribution to the infrared region is negligible and one can discard them.

We will instead use approximate solutions built by superimposing an arbitrary number of minimal monopoles at large distances.

They are constructed easily in the unitary gauge and are then regularized by means of a procedure similar to the one for the single monopole. We just need the proof of existence [40] of the gauge transformation needed to do that, because we will work in the unitary gauge.

$$A_\mu^{(n)}(x) \equiv \sum_{a=1}^{n} \hat{A}_\mu(x - x_a, \mathrm{q}_a)$$
$$\phi^{(n)}(x) \equiv \sum_{i=a}^{n} \hat{\phi}(x - x_a, \mathrm{q}_a) \tag{3.16}$$

They depend only on the parameters of the n single monopoles $\{x_a, \mathrm{q}_a\}$ and are good solution to the equations of motions for distances much bigger than the monopole sizes. This approximation is called *dilute gas.*

The moduli space of generic configurations have been studied intensively, starting from the already nontrivial case of two monopoles [15, 39], and is found to possess a nontrivial geometry.

Fortunately the interaction of monopoles simplifies drastically for large distances compared to the monopole size and remains function of the relative distances only.

In fact the action for such dilute multimonopole configurations is found to be approximated by the self-action of each monopole plus a monopole-monopole interaction term in the form of a Coulomb interaction [13].

In the semiclassical quantization, and via a dual formulation, it is this interaction which will account for the confinement mechanism.

4 Quantum fluctuations

At this point one would like to start the much more ambitious program of quantizing the theory.

Even the semiclassical treatment at one loop order is a nontrivial task, because involves the calculation of functional determinants in classical backgrounds.

There are some simplifications in the BPS limit [38, 40] because the three-dimensional configurations represent a four dimensional selfdual background and there are useful relations between fermion and boson determinants in this case, but we stress that the physics in this limit is drastically different. The main reason for this is that the Higgs field is massless and thus gives a further long range interaction. We thus want to consider $\lambda > 0$.

A major progress was made by Polyakov, in analogy with other simpler models he applied [14] the semiclassical quantization to the gaussian fluctuations around the monopole solutions and carried the program to evaluate the Wilson loop resumming the semiclassical expansion.

His treatment, for the compact QED, shows that the area law for the Wilson loop emerges from this semiclassical treatment exactly because of the condensation of monopoles.

Later in [21] Das and Wadia have reached the same conclusion for the problem with $SU(2)$ gauge group. Recently also for the case of *pure* YM_3 they have argued that confinement arises from the gas of monopoles, using nonperturbative results from the theory of the three dimensional Coulomb gas [17].

So we have, at our disposal, the minima of the classical action that are supposed to contribute mainly to the functional integral, and after taking into account the gaussian fluctuations around them, one can introduce the semiclassical sum.

We will see that important nonperturbative features of the model are reproduced by this approach.

4.1 Semiclassical program

In [14, 21] the semiclassical quantization of a system of monopoles is approached through a grand canonical ensemble of magnetic particles. The sum on all configurations gives the partition function in a nonperturbative way, and after a generalized Poisson transform, the saddle point technique can be applied.

The gauge fields are integrated perturbatively at one loop, taking into account the regular monopole backgrounds as nontrivial minima of the action.

In this approach many approximations are to be taken carefully, and the spontaneously broken Higgs field plays an important game.

First of all the gas of monopoles is assumed dilute, thus considering just the coulomb part of the monopole-monopole interaction. The classical and one-loop action of a single monopole configuration is taken into account also assuming the diluteness.

The size of the monopole configuration can be kept under control thanks to the presence of the Higgs field that fixes its magnitude through its vacuum expectation value.

After the generalized Poisson& Sine-Gordon transform has been used, usually one is limited to the minimum charge monopoles, assuming the higher ones dissociate rapidly.

We could try to consider also the higher charges. To this aim we extract, from the Wilkinson Goldhaber analysis, that spherical monopoles have limited charge (for limited N, $|\mathrm{q}| \leq N-1$). Moreover their mass grows linearly with the charge while their size gets linearly shrink-ed.

4.2 Grand canonical ensemble

The partition function of *pure* YM in 2+1 dimensions is transformed into the sum on all configurations made of any number of monopoles of charges $\{\mathrm{q}_a\}$ and locations $\{x_a\}$. For $SU(2)$ it is:

$$Z = \sum_{n=0}^{\infty} \frac{\xi^n}{n!} Q_n$$

$$Q_n = \sum_{\{\mathrm{q}_1 \cdots \mathrm{q}_n\}} \int \prod_a \mathrm{d}^3 x_a \, \mathrm{e}^{\frac{-2\pi}{g^2} \sum_{a \neq b} \frac{q_a q_b}{|x_a - x_b|}} \qquad (q_a = \pm 1, \pm 2, \ldots)$$

where ξ is the classical and one loop contribution to the action of the one monopole configuration [14].

For $SU(N)$ this expression is no longer good because the weight of each monopole depends on the charge, $\xi = \xi(\mathrm{q})$. Hence:

$$Z = \sum_{n=0}^{\infty} \frac{1}{n!} Q_n \tag{4.17}$$

$$Q_n = \sum_{\{\mathrm{q}_1 \cdots \mathrm{q}_n\}} \int \prod_{a=1}^{n} \mathrm{d}^3 x_a \, \xi(\mathrm{q}_a) \cdot \mathrm{e}^{\frac{-2\pi}{g^2} \sum_{a \neq b} \frac{\mathrm{q}_a \cdot \mathrm{q}_b}{|x_a - x_b|}} \qquad (\mathrm{q}_a \textit{ in the root lattice}) \tag{4.18}$$

The partition function can be nicely reexpressed as a functional integral using the standard Sine-Gordon transform [41]:

$$Z = \int \mathcal{D}\chi \mathrm{e}^{-\frac{g^2}{32\pi^2} \int \mathrm{d}^3 x \left[(\partial \chi)^2 - \sum_i M^2(\mathrm{q}_i) \mathrm{e}^{i \mathrm{q}_i \cdot \chi} \right]}. \tag{4.19}$$

here the mass M^2 comes from the weight ξ: $M^2 = 32\pi^2\xi/g^2$. χ is a $N-1$ components scalar field whose propagator is just the coulomb potential, and the sum on i is the sum on all the possible magnetic charges of one monopole (the magnetic lattice). The last term in the integral is the functional generator of the multi-charge configurations.

For the symmetry of the minimum charges, the roots $\mathrm{q}_i = -\mathrm{q}_{-i}$, the last term becomes a cosine that gives the name to the transform.

This representation of the coulomb gas has to be understood in a perturbative sense, because it is just the perturbative expansion which reproduces, diagram by diagram, the dilute gas. In this spirit the χ field configurations have to vanish at infinity, although there is formally an infinite 'zero-point' energy of the vacuum $\chi = 0$. This problem disappears with the normalization.

4.3 Wilson loop

One can also succeed to evaluate gauge invariant operators like the Wilson loop W_C:

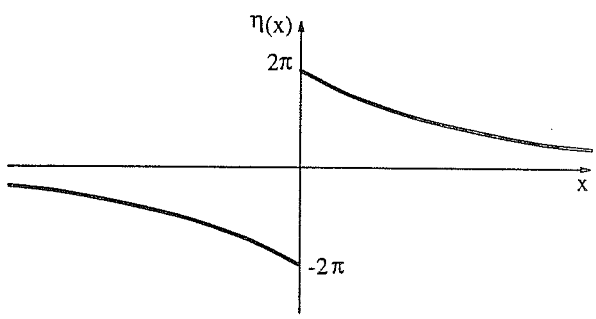
$$W_C \;=\; \frac{1}{N}\left\langle \mathrm{Tr}\, \mathrm{e}^{i\int_C A\cdot \mathrm{dx}} \right\rangle \qquad (4.20)$$

for any contour C in three dimensional space-time.

If we take into account the form of A_μ in the unitary gauge (3.8) for each monopole of charge q_a and location x_a, (and after using the Stokes theorem) we can rewrite the Wilson loop as an external source for the configuration of charges $\{\mathrm{q}_a\}$:

$$\begin{aligned} W_C &= \frac{1}{N}\left\langle \mathrm{Tr}\, \mathrm{e}^{i\mathrm{T}\cdot\sum_a \mathrm{q}_a \int_S \mathrm{d}^2\sigma_\mu(y)\frac{(x_a-y)_\mu}{|x_a-y|^3}} \right\rangle \qquad \eta(x) = \int \mathrm{d}^2\sigma_\mu(y)\frac{(x-y)_\mu}{|x-y|^3} \\ &= \frac{1}{N}\left\langle \mathrm{Tr}\, \mathrm{e}^{i\mathrm{T}\cdot\sum_a \mathrm{q}_a \eta(x_a)} \right\rangle \end{aligned} \qquad (4.21)$$

The nice interpretation of this formula is that instead of the flux through the loop of the magnetic field produced by the q_a charges, one thinks to a potential ($\eta(x)$, in the picture) which acts on the charges q_a at points x_a, produced by a dipole layer on the surface S spanning the loop. The problem of evaluating the Wilson loop in the functional integral is reduced to the average of the canonical ensemble under the action of this external potential.



The dipole density is unitary and in direction of $\mathrm{d}^2\sigma_\mu$ orthogonal to the surface. The potential is in practice the solid angle of the loop seen by the charge at x_a.

In the limit of very large loop this potential is constant on the two sides leaving just the discontinuity, which is of crucial importance.

We can see all this by rewriting

$$W_C = \frac{1}{N}\left\langle \mathrm{Tr}\, e^{i\mathbf{T}\cdot\sum_a \mathbf{q}_a \eta(x_a)}\right\rangle = \frac{1}{N}\sum_\alpha \left\langle e^{i\sum_a \mathbf{m}_\alpha\cdot\mathbf{q}_a\eta(x_a)}\right\rangle \tag{4.22}$$

where the sum on α is on the fundamental weights $\mathbf{m}_\alpha$ of $SU(N)$.

It is straightforward to evaluate this operator in the coulomb gas ensemble (4.17), and the result is, after a shift in the $\boldsymbol{\chi}$ field:

$$W_C = \frac{1}{ZN}\sum_\alpha \int \mathcal{D}\boldsymbol{\chi} e^{-\frac{g^2}{32\pi^2}\int d^3x\left[(\partial\boldsymbol{\chi}-\mathbf{m}_\alpha\partial\eta)^2-\sum_i M^2(\mathbf{q}_i)e^{i\mathbf{q}_i\cdot\boldsymbol{\chi}}\right]} \tag{4.23}$$

4.4 Saddle point solution

The nonlinear above problem is solvable by the saddle point technique, remembering that M^2 is asymptotically small, thus limiting the nonlinearity.

The saddle point is given by the equations, one for each $\mathbf{m}_\alpha$:

$$-\partial^2\boldsymbol{\chi} + \mathbf{m}_\alpha\partial^2\eta = -\sum_{half\ i} M^2(\mathbf{q}_i)\mathbf{q}_i \sin(\mathbf{q}_i\cdot\boldsymbol{\chi}) \tag{4.24}$$

(The sum on i runs now only on one half of the magnetic lattice because the negative symmetric part is included in the sine).

The term $\mathbf{m}_\alpha\partial^2\eta$ imposes a discontinuity of the solution in 'internal' direction $\mathbf{m}_\alpha$ and together with the constraint of perturbativeness $\boldsymbol{\chi}(\pm\infty) = 0$, this leaves only the solutions of constant direction $\boldsymbol{\chi}(x) = \mathbf{m}_\alpha\chi(x)$.

Taking the product with $\mathbf{m}_\alpha$, one gets the scalar equations

$$(\partial^2\chi - \partial^2\eta)\frac{N-1}{2N} = \sum_{half\ i} M^2(\mathbf{q}_i)\mathbf{m}_\alpha\cdot\mathbf{q}_i \sin(\mathbf{q}_i\cdot\mathbf{m}_\alpha\chi) \tag{4.25}$$

(where we used the fact that for the weights of the fundamental repr. $\mathbf{m}_\alpha^2 = \frac{N-1}{2N}$).

At this point one has to specify which magnetic charges $\mathbf{q}_i$ to use, and we see from the Sine-Gordon representation that higher charge monopoles give rise to a generalized Sine-Gordon potential which is of lower magnitude, even if it has shorter periods. This means that higher charges generate only a perturbation of the potential due to minimal charges.

The sum over i is then limited to the minimum magnetic charges, i.e. the co-roots of the first picture.

We have to evaluate the scalar product $\mathbf{m}_\alpha\cdot\mathbf{q}_i$ of a fundamental weight with the adjoint weights.

To this aim we remember that the roots $\mathbf{q}_i$ are $N(N-1)$, and that given a fundamental weight $\mathbf{m}_\alpha$, $(N-1)(N-2)$ of them are orthogonal to it, while $N-1$ have scalar product $1/2$ and the others $(N-1)$ (negative symmetric) have scalar product $-1/2$.

It is then sufficient to limit the sum to the $(N-1)$ cases all giving result $1/2$ for the scalar product:

$$\partial^2\chi = \partial^2\eta + \frac{2N}{N-1}\frac{1}{2}\sin(\chi/2)(N-1)\bar{M}^2 \tag{4.26}$$

where we have introduced the *averaged* $\bar{M}^2$ given by

$$\bar{M}^2 = \frac{1}{(N-1)}\sum_{\mathbf{q}_i:\mathbf{q}_i\cdot\mathbf{m}_\alpha=1/2} M^2(\mathbf{q}_i). \tag{4.27}$$

It is this quantity which carries information on the physics that we obtain in the large N limit. $M^2(\mathbf{q})$ represents, in Coulomb gas language, the *fugacity* of the monopole species $\mathbf{q}$. $\bar{M}^2$ is then directly related to the average density of monopoles, which is strongly believed to be the order parameter for confinement.

Note also that many monopoles are mutually neutral and that $\bar{M}^2$ is the average in the $(N-1)$ sectors that are not orthogonal to $\mathbf{m}_\alpha$. $\bar{M}^2$ depends on α and to evaluate the Wilson loop we will also average on $\mathbf{m}_\alpha$, finally.

The solution of (4.26) is known explicitly:

$$\chi(x) = \begin{cases} 4\tan^{-1}\mathrm{e}^{-\bar{M}\sqrt{\frac{N}{2}}x} & x>0 \\ -4\tan^{-1}\mathrm{e}^{\bar{M}\sqrt{\frac{N}{2}}x} & x<0 \end{cases} \tag{4.28}$$

which consists of two parts of a Sine-Gordon soliton. We remark that this soliton is permitted only because the discontinuity allows the field to vanish at infinity. Otherwise there could be an infinity of other classical solutions.

Inserting this into eq. (4.23), it gives the estimate for the Wilson loop:

$$\begin{aligned} W_C &= \frac{1}{N}\sum_\alpha \exp\left\{-\frac{g^2}{32\pi^2}\int \mathrm{d}^3x \left[\frac{N-1}{2N}(\chi'-\eta')^2 - \bar{M}^2(N-1)(\cos(\chi/2)-1)\right]\right\} \\ &\simeq \frac{1}{N}\sum_\alpha \mathrm{e}^{-\sigma_\alpha A} \end{aligned} \tag{4.29}$$

with string tensions σ_α of

$$\sigma_\alpha = \frac{g^2\bar{M}}{32N\pi^2}\frac{N-1}{2}\sqrt{\frac{2}{N}} = \frac{g}{8\pi}\frac{N-1}{N}\sqrt{\frac{\xi}{N}} \tag{4.30}$$

This shows that confinement of quarks exists in this theory for generic values of the coupling constants and for finite N, but to extract the behavior with large N it is necessary to perform the average for $\bar{M}$ and then for the Wilson loop $\frac{1}{N-1}\sum_\alpha \mathrm{e}^{-\sigma_\alpha A}$.

The N dependence of M, that is of ξ, is not known explicitly. As in [14] ξ is the one loop partition function in a single monopole background, before the integration of the zero mode translation coordinate:

$$\xi(\mathbf{q}) = N^{9/2} m_W^3 \left(\frac{m_W^{3/2}(\mathbf{q})}{g^3} \right) A(\lambda/g^2) e^{-N \frac{m_W(\mathbf{q})}{g^2} C(\lambda/N g^2)} \tag{4.31}$$

The condition of validity of the saddle point approximation, which represents the low Debye density, is $\bar{\xi} << 1$. It can be seen to hold for finite N from the above expression, where the exponential vanishes asymptotically. In the limit of N large we have to know something more precise on the whole average $\bar{\xi}$.

4.5 Determinant

In the previous section, $\xi(\mathbf{q})$ is the statistical-quantum weight of a particular background configuration, so that the higher it is, the higher is the importance of that particular configuration, although it may have large action.

Up to now we assumed ξ to be some fixed quantity. Now, in order to draw some conclusion about the string tension σ, we need to find something more precise on it.

The evaluation of ξ is the problem of calculating the functional determinant of the fluctuating fields around the one monopole solution. It would be a hopeless problem to calculate it exactly in an arbitrary external field, as it is equivalent to the solution of a Schroedinger or Dirac equation in an external potential‡.

We will try to extract some information from the high N analysis of the problem.

The idea comes from the fact that after gauge fixing, and better in the unitary gauge, the Higgs field has only N components, while its effective action, upon integration of the gauge sector, is of order N^2. Hence the saddle point should be applicable, and the Higgs field is a semiclassical quantity with respect to $1/N$, which acts like $\hbar \to 0$ to suppress its fluctuations.

We will treat the fields in one loop approximation around the one monopole configuration $\hat{A}_\mu, \hat{\phi}$ adding the fluctuating fields a_μ, φ, so that: $A_\mu = \hat{A}_\mu + a_\mu$, $\phi = \hat{\phi} + \varphi$.

In doing so, we are faced with the problem of gauge fixing, because there are zero modes of the action. The gauge invariance involves the total field $(\hat{A}_\mu + a_\mu, \hat{\phi} + \varphi)$, and one can split the gauge variation between the fluctuating and the background fields in an arbitrary manner.

Among the possible (infinite) choices, one can assign the whole field variation either to the fluctuations or to the background. The latter choice is of little or no utility, the first, instead, is quite convenient in that it keeps away the gauge invariance problem from the background fields.

So we will keep the background fields in some fixed gauge, and consider the gauge group as acting on the sole fluctuations:

$$\begin{aligned} &\delta_{gauge}(\hat{A}_\mu, \hat{\phi}) = 0 \\ &\delta_{gauge}(a_\mu, \varphi) = \delta_{(A_\mu,\phi)}(a_\mu, \varphi) = (\mathcal{D}_\mu \alpha, -ig[\phi, \alpha]). \end{aligned} \tag{4.32}$$

The gauge for the classical fields is left for now unspecified, even is the radial gauge satisfy automatically the background gauge. In the next section instead we will choose for all fields the unitary gauge.

‡In [23] has calculated numerically the expression with the heat kernel method for N=2, but the estimate of the behavior with N has yet to be done.

The partition function in a one-monopole background, ξ, is a gauge invariant object but not gauge independent, at least in principle. Every gauge fixing could provide different physical insight, as happens already with spontaneous symmetry breaking.

According to 't Hooft and Polyakov ξ it is better calculated in the so called *background, "natural", gauge*, for two reasons: one is the presence of zero modes, to appear in a moment, for the action of fluctuations, and they are best treated in the background gauge; the second is that the candidate with opposite features, the unitary gauge, usually addressed to be non-renormalizable, can be cured only considering with care the ultralocal Faddeev-Popov determinant which introduces a nonpolynomial term in the action.

This can be seen from the Vandermonde determinant appearing in the measure of φ after elimination of the $(SU(N))$ 'angular' part of φ: $\mathcal{D}\varphi \to \mathcal{D}\varphi\Delta^2(\varphi)$.

There are two ways to deal with this Δ^2; it can be reabsorbed in the measure by a (nonlinear but nonsingular) change of φ_i variables, but it yields a nonlinear model on a singular curved target space of the kind (for $SU(2)$ for example):

$$\mathrm{L} = \varphi^{-\frac{4}{3}}(\partial_\mu\varphi)^2 \tag{4.33}$$

Alternatively thinking to a lattice, one can exponentiate the determinant and convert it in the divergent logarithmic potential: $\delta(0)\sum_{i\neq j}\int \log|\phi_i - \phi_j|\mathrm{d}^3x$, but the continuum limit seems problematic due to the ultralocal nature of Δ^2. This procedure can be justified with care starting from the R_ξ kind of gauges ['tH,V] [42] or with other kind of regular gauge fixing.

I can note however, that this divergent extra potential, being exponentiated without the help of ghost fields, carries an $\hbar$ factor, and thus already is a *quantum* correction to the action.

As the theory is renormalizable for every gauge R_ξ, in the ϕ effective action there should be, then, a divergent term from the massive gauge fields which cancels it. In effect it is what we find after the one loop analysis.

Hence for the eventual final saddle point analysis on the effective potential, there shouldn't be a serious problem from this term.

We will adopt in the following paragraph the background gauge and in the next the unitary one.

4.5.1 Background Gauge

Here we take, as gauge fixing:

$$F_\kappa = \hat{D}_\mu a_\mu - i\kappa g[\hat{\phi}, \varphi] = B(x) \tag{4.34}$$

In the limit $\kappa \to \infty$ the second term reproduces the unitary gauge. This is a variant of the pure background gauge (that has $B = 0$, $\kappa = 1$), already considered by [5] or of the "natural gauge" that appears in [14]. It has a long history in the literature, due to its double features of renormalizability together with massive ghost fields.

The usual gaussian averaging of the gauge fixing $\delta(F_k - B)$, to get a quadratic term in the action, is in some contrast with the high N analysis because removes the gauge fixing and leaves N^2 degrees of freedom. On the other hand fixing strictly the gauge in the case $k \neq 1$ presents some subtleties.

We will denote together the fluctuating fields $a_\mu(x)$ and $\varphi(x)$ with $\Phi = \begin{pmatrix} a_\mu \\ \varphi \end{pmatrix}$ and the full quadratic action will be, in matrix form, $\mathcal{S}_{quadr} = {}^t\Phi \mathcal{M} \Phi$. Scalar product $\Phi \cdot \Phi$ will imply integration on space-time.

About the action we just need to say, for now§, that it is annihilated by the following zero modes:

$$\textit{gauge} \qquad \Phi_0(\alpha) = \begin{pmatrix} a_\mu(\alpha) \\ \varphi(\alpha) \end{pmatrix} = \begin{pmatrix} \hat{D}_\mu \alpha \\ -ig[\hat{\phi}, \alpha] \end{pmatrix} \tag{4.35}$$

$$\textit{translation} \qquad \Phi_0^{(i)} = \begin{pmatrix} \bar{a}_\mu^{(i)} \\ \bar{\varphi}^{(i)} \end{pmatrix} = \begin{pmatrix} \frac{1}{g}\hat{G}_{\mu i} \\ \hat{D}_i \hat{\phi} \end{pmatrix} \tag{4.36}$$

The complete field Φ can thus be decomposed in zero modes plus nonzero ones, Φ_n, eigenfunctions of the action:

$$\Phi = R_i \Phi_0^{(i)} + \Phi_0(\alpha) + \xi_n \Phi_n$$

The translational modes (4.36) are written in a gauge so that they satisfy the gauge-fixing (4.34) with $\kappa = 1$, $B = 0$. This is useful because they are orthogonal to the gauge modes. They are normalized to

§Explicitly it is:

$$\mathcal{S}_{quadr} = N\mathrm{tr} \int \left[([\hat{D}_\mu, a_\nu] - [\hat{D}_\nu, a_\mu])^2 + \frac{g}{2}\hat{G}_{\mu\nu}[a_\mu, a_\nu] + (g[a_\mu, \hat{\phi} + \varphi] + [\hat{D}_\mu, \varphi])^2 + {}^t\varphi V''(\varphi)\varphi \right]$$

Together with the proper mass term for gauge fields, $\frac{1}{2}Ng^2[\hat{\phi}, a_\mu]^2$, there is also the bilinear mixing $N([a_\mu, \varphi][\hat{D}_\mu, \hat{\phi}] + [a_\mu, \hat{\phi}][\hat{D}_\mu, \varphi])$.

$$|\Phi_0^{(i)}|^2 = \mathcal{N}_i = N\mathrm{tr}\int \frac{1}{g^2}\hat{G}_{i\mu}^2 + (\hat{D}_i\hat{\phi})^2. \tag{4.37}$$

All these zero modes are treated with the standard Faddeev-Popov method to extract the integration on collective coordinates R_i, $\alpha(x)$:

$$1 = \mathrm{Det}_{ij}\left[\frac{\partial}{\partial R_j}\Phi\cdot\Phi_0^{(i)}\right]\mathrm{Det}\left[\frac{\delta}{\delta\alpha(x)}F_k(\Phi(y))\right]\cdot\int \mathrm{d}^3R\mathcal{D}\alpha\,\delta(F_k - B)\prod_{(i)}\delta\left(\Phi\cdot\Phi_0^{(i)}\right).$$

Insertion of this unity in the functional integral replaces the flat fluctuations with the right variables R_i, $\alpha(x)$ and gives the two determinants. The first Det gives $\prod_i \mathcal{N}_i^{1/2}$, the second is the Faddeev-Popov determinant

$$\mathrm{Det}\left[\frac{\delta}{\delta\alpha(x)}F_k(\Phi(y))\right] = \mathrm{Det}[\mathcal{M}_{FP}] = \mathrm{Det}\left[\hat{D}_\mu D_\mu - \kappa g^2[\hat{\phi},[\phi,\cdot]]\right] \tag{4.38}$$

After elimination of the first delta, the remaining fluctuations represent the functional integral restricted the non translational modes:

$$Z = \mathrm{e}^{-\mathcal{S}_{cl}}\int \mathrm{d}^3R\prod_i \mathcal{N}_i^{1/2}\int \tilde{\mathcal{D}}\Phi\delta\left(F_k(\Phi) - B\right)\mathrm{Det}[\mathcal{M}_{FP}]\mathrm{e}^{-\int {}^t\Phi\mathcal{M}\Phi} \tag{4.39}$$

The Faddeev-Popov determinant can be evaluated, in one loop approximation, in the sole classical background fields.

Then, as seen from (4.37), each factor in $\prod_i \mathcal{N}_i^{1/2}$ is the action in a space-time direction without the potential. Because we consider spherical monopoles only, all $\mathcal{N}_i$ are equal and

$$\mathcal{N}_i = \frac{N}{3}\mathrm{tr}\int \frac{1}{g^2}\hat{G}^2 + (\hat{D}\hat{\phi})^2 \tag{4.40}$$

They coincide with the action in the BPS limit, hence, after the discussion of section **3.2.1**, in the large N limit we also take $\mathcal{N} = N\frac{4\pi}{3}\frac{m_W}{g^2}$.

One can perform a functional integration over B (with $\mathrm{e}^{-\frac{\lambda}{2}\int B^2}$) to remove the $\delta(F^k - B)$:

$$\begin{aligned} Z &= \int \mathrm{d}^3R\mathrm{e}^{-\mathcal{S}_{cl}}\mathcal{N}^{3/2}\mathrm{Det}\left[\hat{D}_\mu\hat{D}_\mu - \kappa g^2[\hat{\phi},[\hat{\phi},\cdot]]\right]\int \tilde{\mathcal{D}}\Phi\mathrm{e}^{-\int {}^t\Phi(\mathcal{M}+\frac{\lambda}{2}\mathcal{M}_{gf})\Phi} \\ &= \int \mathrm{d}^3R\mathrm{e}^{-\mathcal{S}_{cl}}\mathcal{N}^{3/2}\mathrm{Det}\left[\hat{D}_\mu\hat{D}_\mu - \kappa g^2[\hat{\phi},[\hat{\phi},\cdot]]\right]\widetilde{\mathrm{Det}}^{-1/2}\left[\mathcal{M} + \frac{\lambda}{2}\mathcal{M}_{gf}\right] \end{aligned} \tag{4.41}$$

Here with ${}^t\Phi\mathcal{M}_{gf}\Phi = F_k^2(\Phi)$ we denote the standard gauge fixing term arising from F_κ, while with $\tilde{\mathcal{D}}\Phi = \tilde{\mathcal{D}}a\tilde{\mathcal{D}}\varphi$, as with the determinant $\widetilde{\mathrm{Det}}$, we integrate on translation-fixed fluctuations.

This is analogous to the formula of Polyakov [14], but does not show the explicit dependence on κ; one has to go through the loop expansion in gauge bosons and ghosts in the external field, and proceed to resum all contributions of generic order in g if one wants to control the limit $\kappa \to \infty$. Actually in this limit the ghost masses become large so that they decouple, but at the same time the coupling with gauge and Higgs also diverges, so that the resumming is nontrivial. We will see the first order in an other way in the next section.

Now we proceed in a different direction, we eliminate the $\delta(F_\kappa - B)$ by direct integration of the gauge modes.

This point of view has the advantage of showing the physical degrees of freedom Φ_n, together with the explicit dependence on κ that we want to compare with the unitary gauge.

Briefly, instead of $\widetilde{\mathrm{Det}}\left[\mathcal{M} + \frac{\lambda}{2}\mathcal{M}_{gf}\right]$ we get the decoupled product of two determinants $\widetilde{\mathrm{Det}}_{gf}\left[\mathcal{M}\right]\mathrm{Det}^2\left[\sqrt{\frac{\lambda}{2}}\mathcal{M}_{FP}\right]$ plus a measure jacobian dependent on κ. The second determinant cancels formally the FP determinant, and $\lambda/2$ disappears in the normalization. κ of course remains in the measure, but the limit is nonsingular

Let's expand as before the fields as $\Phi = R_i\Phi_0^{(i)} + \Phi_0(\alpha) + \xi_n\Phi_n$. We can eliminates the zero modes, taking into account the jacobian from Φ to (α, R_i).

We get (still ignoring contributions at more than one loop):

$$Z = \int \mathrm{d}^3R\, \mathrm{e}^{-\mathcal{S}_{cl}} \prod_i \mathcal{N}_i^{1/2} \sqrt{\mathrm{Det}\left[\hat{D}_\mu\hat{D}_\mu - g^2[\hat{\phi},[\hat{\phi},\cdot]]\right]} \int \mathcal{D}(\xi_n\Phi_n)\mathrm{e}^{-\xi_n^2\int\Phi_n\mathcal{M}\Phi_n} \quad (4.42)$$

The dependence on κ and B seems disappeared, but the eigenfunctions Φ_n have to satisfy the gauge fixing so that they are sensitive to κ and B.

Remembering that Φ_n was satisfying the natural gauge ((4.34) $B = 1$, $\kappa = 0$), to pass to a choice with $B \neq 0$, $\kappa \neq 1$ we have to perform a gauge transformation of the Φ_n, so that they satisfy the new gauge fixing:

$$\Phi_n \to \Phi_n + \Phi_0(\alpha_n). \quad (4.43)$$

This gauge transformation is:

$$\alpha_n(B, \kappa - 1) = \left[\hat{D}_\mu\hat{D}_\mu - g^2[\hat{\phi},[\hat{\phi},\cdot]]\right]^{-1}\left(ig(\kappa - 1)[\hat{\phi},\varphi_n] + B\right). \quad (4.44)$$

Notice that $\alpha_n(B, \kappa - 1) \to \infty$ when $\kappa \to \infty$.

Because we are just mixing with components along the gauge zero modes, the quadratic form in the action is unaffected.

The new basis Φ_n is still orthogonal to the translational modes. For this reason we can retain the same measure of translations $\mathcal{N}^{3/2}$.

However we have now eigenfunctions which are non normalized and not even mutually orthogonal, and we have to re-normalize the measure.

$$\tilde{\mathcal{D}}a_\mu \tilde{\mathcal{D}}\varphi = \mathcal{D}[\xi_n \Phi_n] = \prod_n \mathrm{d}\xi_n \quad \rightarrow \quad \mathcal{D}[\xi_n (\Phi_n + \Phi_0(\alpha_n))] = \prod_n \mathrm{d}\xi_n \cdot J$$

$$J = \sqrt{\det_{mn}\left[\mathbf{1}_{mn} + \phi_0(\alpha_m)\phi_0(\alpha_n)\right]} \tag{4.45}$$

$$\mathrm{e}^{-\int {}^t\Phi \mathcal{M} \Phi} = \mathrm{e}^{-\mathcal{M}_{nn}\xi_n^2} \quad \rightarrow \quad \mathrm{e}^{-\int ({}^t\Phi + \Phi_0(\alpha))\mathcal{M}(\Phi+\Phi_0(\alpha))} = \mathrm{e}^{-\mathcal{M}_{nn}\xi_n^2}$$

The new jacobian J carries the information about the gauge dependence on κ of the functional integral. J, although a formal expression, is explicitly function of the gauge parameter. The partition function is:

$$\xi = \int \mathrm{d}^3 R \mathrm{e}^{-\mathcal{S}_{cl}} \mathcal{N}^{3/2} \sqrt{\mathrm{Det}\left[\hat{D}_\mu \hat{D}_\mu - g^2[\hat{\phi},[\hat{\phi},\cdot]]\right]} \cdot J \cdot \prod_n \mathrm{d}\xi_n \mathrm{e}^{-\mathcal{M}_{nn}\xi_n^2} \tag{4.46}$$

For example in the case $\kappa = 1$, $B \neq 0$ we have $\alpha_n(B, \kappa - 1) = \alpha(B)$ and $J = \sqrt{\det_{mn}\left[\mathbf{1}_{mn} + |\phi_0(\alpha(B))|\right]}$ It does not depend explicitly on eigenfunctions, but when $\kappa \neq 1$ we have:

$$J = \sqrt{\det\left[\mathbf{1}_{mn} + \Phi_0(\alpha_m) \cdot \Phi_0(\alpha_n)\right]} \tag{4.47}$$

$$\alpha_n(B, \kappa - 1) = \left[\hat{D}_\mu \hat{D}_\mu - g^2[\hat{\phi},[\hat{\phi},\cdot]]\right]^{-1} \left(ig(\kappa - 1)[\hat{\phi}, \varphi_n] + B\right). \tag{4.48}$$

We can rewrite J explicitly as a function of κ as:

$$J = \sqrt{\det\left[\mathbf{1}_{mn} + (\kappa-1)^2 \hat{D}_\mu a_\mu^{(m)} \frac{1}{\hat{D}_\nu \hat{D}_\nu - g^2[\hat{\phi},[\hat{\phi},\cdot]]} \hat{D}_\mu a_\mu^{(n)}\right]} \tag{4.49}$$

and after rescaling $(\kappa - 1)^2$, we can write:

$$J = \sqrt{\det\left[\hat{D}_\mu a_\mu^{(m)} \frac{1}{\hat{D}_\nu \hat{D}_\nu - g^2[\hat{\phi},[\hat{\phi},\cdot]]} \hat{D}_\mu a_\mu^{(n)}\right]} \cdot$$

$$\cdot \sqrt{\det\left[\mathbf{1}_{mn} + \frac{1}{(\kappa-1)^2}\left(\hat{D}_\mu a_\mu^{(m)} \frac{1}{\hat{D}_\nu \hat{D}_\nu - g^2[\hat{\phi},[\hat{\phi},\cdot]]} \hat{D}_\mu a_\mu^{(n)}\right)^{-1}\right]} \tag{4.50}$$

The conclusion that we would like to draw from the present calculation is that in the limit $\kappa \to \infty$ the singular behavior of the FP determinant in (4.41) has been canceled by the gauge bosons.

We pass directly to the unitary gauge, then, and we extract some information about the large N limit.

4.5.2 Unitary gauge

In this section we fix the gauge by requiring the Higgs field to be diagonal. This gauge is called unitary following the fact that unphysical degrees of freedom are absent. Examples of such gauges are the limit $\xi \to \infty$ of the Stueckelberg gauge fixing, gauges constructed ad-hoc to decouple the unphysical part of A_μ [27, 43, 44, 45, 46], the unitary formulation of the Weimberg-Salam model, coinciding with the $\xi \to \infty$ limit of the R_ξ gauges.

Imposing the Higgs field to be diagonal eliminates all the local gauge invariance apart from the aforementioned Weyl group and a subgroup which coincides with the little group of the Higgs field, (that is at least a $U^{N-1}(1)$).

In the case that we consider, we assume the background Higgs field to have all different eigenvalues and the fluctuation to be small with respect to it, so this is precisely the case. We will verify a posteriori whether this vacuum configuration is (locally) stable.

The residual $U^{N-1}(1)$ abelian gauge invariance has to be cured in a second moment by means of a further gauge fixing.

Like all models with a Higgs phenomenon, there are pseudo-Goldstone bosons associated to the broken flat direction in internal space. They are unphysical fields and their degrees of freedom are "eaten up" with the standard mechanism by the relative gauge fields which acquire one polarization more together with the mass. This happens explicitly in the unitary gauge.

At the same time two things happen: first the gauge fixing requires, through proper handling of the integration measure, the introduction of a Faddeev-Popov determinant; second the massive gauge fields have a Proca propagator, which carries bad behavior at large momentum.

For this last peculiarity the unitary gauge is usually addressed as non renormalizable, because the gauge fields produce, even at one loop, a new set of counterterms not present in the original lagrangian.

There is quite a lot of literature on the self-canceling of some of these non-renormalizable divergences, starting from [47], the remaining divergences are found (see [48] and therein) to vanish on the equations of motions, so that on-shell amplitudes do not suffer of this problem. (On the other hand, in the monopole background, we have to calculate the full partition function of fluctuations, which is not a physical quantity).

We will see that the Faddeev-Popov determinant participates exactly to render the theory manifestly renormalizable.

This can already be inferred from the $\xi \to \infty$ limit of the R_ξ gauge (for the charged gauge fields): for any value of ξ, the theory is renormalizable [27, 43, 44, 45, 46], so that the only counterterms needed are of the same form of the lagrangian. The limit $\xi \to \infty$ is well defined for the massive gauge propagator, so that assuming some suitable regularization¶ the cubic divergences cancel order by order between ghosts and massive gauge fields (or, alternatively, between the Faddeev-Popov determinant and the gauge fields). While the charged ghosts acquire an infinite mass, thus decoupling, also their coupling to the Higgs becomes large, leaving a correct counterterm to the gauge divergences. It follows that in the careful limit we do not expect any new effective interactions.

A cancelation of this kind has been proven to happen in an abelian gauge model by Appelquist and Quinn [49] long ago.

Explicitly we impose the unitary gauge by the constraint‖:

$$F(\varphi) \equiv \varphi^{charged} = 0 \tag{4.51}$$

It has no derivatives and acts on a field which transforms locally under the gauge group, hence it requires the pointwise Faddeev-Popov jacobian:

$$\left.\frac{\delta}{\delta\alpha(y)} F(\delta_\alpha \varphi(x))\right|_{F=0} = [\hat{\phi} + \varphi, \cdot]\delta(x-y) \tag{4.52}$$

that gives the following functional determinant:

$$\mathrm{Det}[\mathcal{M}_{FP}] = \prod_x \prod_{i<j} (\hat{\phi}_i + \varphi_i - \hat{\phi}_j - \varphi_j)^2 = \prod_x \Delta^2(\phi(x)). \tag{4.53}$$

Here a regularization has to be implicit to make sense of the infinite product.

$\mathrm{Det}[\mathcal{M}_{FP}]$ can be exponentiated without the help of ghost fields thanks to the relation $\mathrm{Det}A = \mathrm{e}^{\mathrm{Tr}\log A}$ to yield an effective potential for the Higgs field:

$$\mathrm{e}^{\sum_x \sum_{i<j} \log(\phi_i - \phi_j)} = \mathrm{e}^{\delta(0)\int \mathrm{d}^3x \log \Delta^2(\phi(x))} \tag{4.54}$$

and finally because the action is multiplied by a $-1/\hbar$ factor, we need to multiply by a $-\hbar$ factor so that it ends up describing a *one loop* correction to the bare action (in the form of a repulsion of the ϕ eigenvalues, as in matrix models).

In three dimensions it has a cubic divergent coupling constant and a non polynomial structure.

Together with this one loop correction, we have then to consider the other contributions from the propagating fluctuations, namely the one loop diagrams of gauge fields a_μ and diagonal Higgs field φ.

¶Dimensional regularization is not good for this scope because the divergences we have are of the kind $\delta(0)$ which vanish identically: $\int \mathrm{d}^d p = 0$

‖It is the same thing to consider the $\xi \to \infty$ limit of the R_ξ background gauges of the last section, namely $[\phi^0, \phi] = 0$ or $[\hat{\phi}, \varphi] = 0$.

Summing up, we still have to calculate:

$$\xi = \mathcal{N}^{3/2} \int \mathrm{d}^3 R \int \mathcal{D}\varphi \int \mathcal{D}a_\mu \mathrm{e}^{-\frac{1}{\hbar}\left[S_{cl}+S_{quadr}-\hbar\delta(0)\int \mathrm{d}^3x \log \Delta^2(\phi(x))\right]}. \tag{4.55}$$

Now we would like to exploit the fact that in the unitary gauge the Higgs field has only N components, while its effective action is still of order N^2. In the large N limit this implies that the fluctuations φ are suppressed, and ϕ is in all respect a classical field.

In particular it will be possible to apply the saddle point method on its effective action:

$$\mathrm{e}^{-\Gamma[\phi]} = \int \mathcal{D}a_\mu \mathrm{e}^{-\frac{1}{\hbar}\left[S_{cl}+S_{quadr}-\hbar\delta(0)\int \mathrm{d}^3x \log \Delta^2(\phi(x))\right]}. \tag{4.56}$$

Of course this calculation is still not simple, because it depends on the classical field $\hat{\mathbf{A}}$.

To this aim, we recall the action of fluctuations

$$\mathcal{S}_{quadr} = N\mathrm{tr} \int \Big[([\hat{D}_\mu, a_\nu] - [\hat{D}_\nu, a_\mu])^2 + \frac{g}{2}\hat{G}_{\mu\nu}[a_\mu, a_\nu] + (g[a_\mu, \hat{\phi}+\varphi] + [\hat{D}_\mu, \varphi])^2 + \\ + {}^t\varphi V''(\hat{\phi})\varphi\Big] \tag{4.57}$$

understanding that φ is diagonal.

All the effect that we want to discuss arises from the massive gauge fields circulating in a loop, so we calculate the divergent part of it.

Because the gauge propagator is constant at large momentum, the loop contributes with an arbitrary number of insertions of interaction terms.

The cubic interactions in the gauge or Higgs fields do not enter at one loop, and simultaneous interactions of two a_μ with a Higgs field plus a background gauge give perturbative corrections to what we need, so we leave them apart.

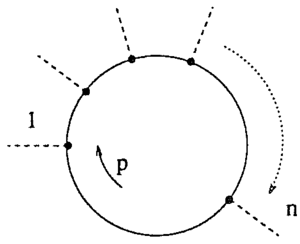
From the above action (4.57) we take the relevant quadratic interaction terms of the fluctuating a_μ^{ij} with the external fields:

$$g^2 a_\mu^{ij} a_\mu^{ji} (2\hat{\phi}_i^\infty - 2\hat{\phi}_j^\infty + \varphi'_i - \varphi'_j)(\varphi'_i - \varphi'_j) \tag{4.58}$$

where the Higgs field $\phi = \hat{\phi} + \varphi$ has been decomposed in a different way: the asymptotic constant field $\hat{\phi}^\infty$ which regulates the gauge bosons mass, plus the remaining classical nonuniform background and fluctuating fields which have to be treated as a total external field φ': $\phi = \hat{\phi}^\infty + \varphi'$.

The loop consists thus of the same charged a_μ^{ij} field running along the loop with propagator

$$\frac{-g_{\mu\nu}}{p^2 - m_{ij}^2} + \frac{1}{m_{ij}^2}\frac{p_\mu p_\nu}{p^2 - m_{ij}^2} \tag{4.59}$$



(with $m_{ij} = g|\hat{\phi}_i^\infty - \hat{\phi}_j^\infty|$) and an arbitrary number of insertions of $v_{ij} = g^2(2\hat{\phi}_i^\infty - 2\hat{\phi}_j^\infty + \varphi'_i - \varphi'_j)(\varphi'_i - \varphi'_j)$.

We insert this v_{ij} at zero external momentum, because we are dealing with the divergent part. We have thus, for n insertions,

$$\sum_{i<j} v_{ij}^n \int \mathrm{d}^3 p \,\mathrm{tr}\left(\frac{-\mathbf{1}}{p^2 - m_{ij}^2} + \frac{1}{m_{ij}^2}\frac{\mathbf{p}\times\mathbf{p}}{p^2 - m_{ij}^2}\right)^n = \delta(0)\sum_{i<j}\left(\frac{v_{ij}}{m_{ij}^2}\right)^n \tag{4.60}$$

There is a combinatorial factor which comes from the $(n-1)!$ ways to insert the interactions compared with the $n!$ ways to attach the resulting counterterm. Hence the factor is $1/n$.

Summing up all these divergent contributions we reconstruct the logarithmic potential:

$$\hbar\delta(0)\sum_{i\neq j}\sum_n \frac{1}{n}\left(\frac{g^2\big((2\hat{\phi}_i^\infty - 2\hat{\phi}_j^\infty + \varphi'_i - \varphi'_j)(\varphi'_i - \varphi'_j)\big)}{m_{ij}^2}\right)^n =$$
$$\hbar\delta(0)\sum_{i\neq j}\left(\log|\phi_i - \phi_j| - \log|\hat{\phi}_i^\infty - \hat{\phi}_j^\infty|\right) \tag{4.61}$$

It clearly cancels only part of the Faddeev-Popov Van-der-Monde determinant above in (4.55), and leaves the second term $-\hbar\delta(0)\int\log\Delta^2(\hat{\phi}^\infty)$ function of the Higgs vacuum only.

Now, on one hand it is just a constant which is the same for all the topological sectors and can be absorbed in the normalization for what regards ξ and all fluctuations, on the other it can be thought as a potential correcting the vacuum constant value $\hat{\phi}^\infty$.

This result needs some discussion. An example of this mechanism is well known from the study of the perturbative corrections to the Higgs potential [42]: the quantum corrections keep a nonzero Higgs v.e.v. also in the limit of no bare breaking $\mu \to 0$.

From this point of view it provides a repulsive potential for the eigenvalues, justifying the assumption of $\hat{\phi}_i \neq \hat{\phi}_j$ in the vacuum value.

It may appear strange that we have obtained an effective potential with a divergent $\delta(0) \simeq \Lambda^3$ constant, because it seems to be stronger than any of the renormalized other terms in the action, in the continuum limit.

Nonetheless the fact that it depends only on $\hat{\phi}^\infty$ does not allows us to treat it like the other terms in the effective action.

The scalar potential $V(\phi)$ is of a very different nature because it depends on the fluctuations also. It can lead the field to attain its minimum, which has to be the vacuum for the system, in order to be stable against local perturbations.

On the other hand this effective potential is expected to play a role let's say, just at the moment of symmetry breaking, when the sources should decide which direction in the Cartan space to choose. It is at this stage that the Vandermonde potential is important and the quantum theory requires one unique direction with nondegenerate Higgs field.

Eventually one must leave to external sources just the discrete choice among the $N(N-1)$ possible vacuums related by Weyl.

The usual potential V, degenerate in Cartan directions, gets renormalized but still requires just $|\phi^\infty| = \mu$.

All this analysis is independent of the large N limit, but as we want to draw some conclusion about the dilute gas of monopoles, we must know ϕ^∞. In the next paragraph we will find it according to the above discussion, in the large N limit.

4.6 The Higgs vacuum

The vacuum field ϕ^∞ plays an important role in the dilute gas picture, because it decides if the monopoles are relevant to confinement.

According to the discussion about the unitary gauge in last section, the vector of eigenvalues $\phi^\infty \in \mathrm{R}^{N-1}$ is defined by the minimum of

$$-\sum_{i<j} \log |\phi_i^\infty - \phi_j^\infty| \tag{4.62}$$

with the constraint

$$\sum_i (\phi_i^\infty)^2 = \mu^2 \tag{4.63}$$

and we recall that μ^2 is of order N, so that the components of ϕ^∞ are of order 1.

The solution for finite N, although existing, is not easily found. We instead turn to the large N limit and introduce the non standard density of eigenvalues:

$$\rho(\phi^\infty) = \left(N \frac{\mathrm{d}\phi_i^\infty}{\mathrm{d}i} \right)^{-1} \tag{4.64}$$

It is of order 1, as a consequence of last constraint.

We want to solve for it to obtain the distribution of eigenvalues of the vacuum Higgs field. The equation for $\rho(x)$ is:

$$N \cdot P \int_{-a}^{a} \frac{\rho(x)}{x-y} \mathrm{d}y = 2\lambda x \tag{4.65}$$

where the right hand side is the Lagrange multiplier for the constraint (4.63) which is equivalent to $\int x^2 \rho(x) \mathrm{d}x = \mu$.

In the large N limit the Lagrange multiplier is negligible (in fact we do not expect an attraction of eigenvalues coming from this constraint, because $\mu^2 \sim N$) and we have to solve:

$$P \int_{-a}^{a} \frac{\rho(x)}{x-y} \mathrm{d}y = 0 \tag{4.66}$$

The by now standard method [50] is to introduce the resolvent of ρ as

$$F(x) = P \int_{-a}^{a} \frac{\rho(x)}{x-y} \mathrm{d}y \tag{4.67}$$

which has the following properties: It is analytic out of the *cut* $[-a, a]$ on the real axis; it goes to zero at infinity as $\frac{1}{|x|}$; It it real on the real axis $[-a, a]$ excluded; near the cut it has zero real part and a discontinuity in the imaginary part given by the unknown $\pi\rho(x)$.

The unique function with these requirements is

$$F(x) = C \frac{1}{\sqrt{x^2 - a^2}} \tag{4.68}$$

from which we finally read (and normalize) the distribution $\rho(x)$:

$$\rho(x) = \frac{1}{\pi} \frac{1}{\sqrt{a^2 - x^2}}. \tag{4.69}$$

The result is thus an *inverted semicircle law.*

Its domain is defined by the constraint (4.63) (we have introduced the fixed scale $\tilde{\mu}^2 = \mu^2/N$):

$$N \int_{-a}^{a} x^2 \rho(x) dx = N\tilde{\mu}^2 \tag{4.70}$$

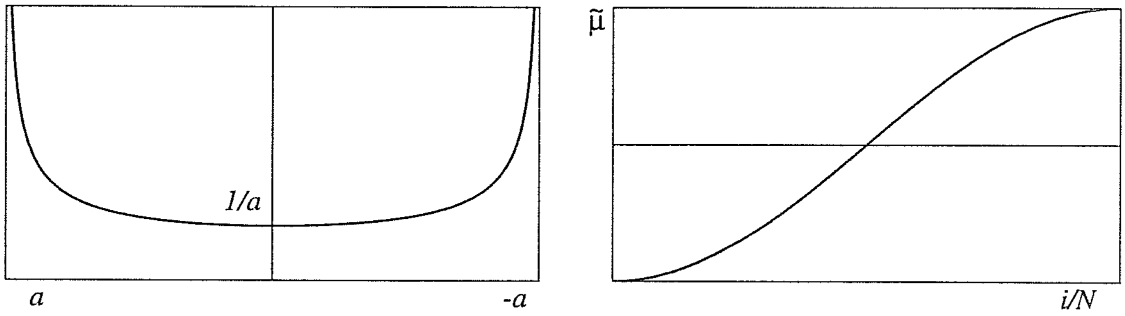
which gives:

$$a^2 = \frac{2}{\pi}\tilde{\mu}^2 \tag{4.71}$$

$\rho(x)$ of (4.69) represents the following Higgs configuration in the large N limit:

$$\phi_i^\infty = \sqrt{\frac{2}{\pi}}\tilde{\mu}\cos(\pi\frac{i}{N}). \quad (4.72)$$

Of course there are $N!$ equivalent configurations related by Weyl permutations. We show the ordered ones in the picture.



Eigenvalue distribution $\rho(\phi)$ Ordered eigenvalues of ϕ

Let's make a few comments on this result:

• First, in the large N limit we find that the Higgs eigenvalues remain of order one, while in the usual picture there is no indication and one could also assign all the vacuum expectation value μ to a single eigenvalue.

• Second, the Weyl degeneracy is completely broken by ϕ^∞, and no symmetry remains apart from the abelian $U(1)^{N-1}$.

• Third, and more important, our vacuum value (4.72) shows a peculiar characteristic, namely that, near the edges of the distribution, differences of eigenvalues are of order $\frac{1}{N^2}$. As a consequence the masses of gauge bosons in the $SU(2)$ sectors near the two ends are vanishing as $1/N^2$.In the middle one has the normally expected $1/N$ masses.

This facts means that there will be an infinite number of very light monopoles, with masses vanishing as $1/N$.

The implications on the semiclassical picture of monopole gas are interesting, and we will deal with them in the next section.

On the other hand, independently of the monopole gas, the presence of an infinity of massless modes can lead to a new phase of the model, of course present only in the $N = \infty$ sector.

Let us mention also a nice correspondence with the solvable cases of $N = 2 \to N = 1$ supersymmetric gauge theories in four dimensions.

In the $SU(N)$ generalization of the $(N = 1)SYM_4$, as analyzed by Douglas and Shenker [22], the ϕ^∞ field is represented by the points in the moduli space of the

$N = 2$ theory which become the vacua of the theory broken to $N = 1$.

The exact solution shows that those points are exactly of the form of our ϕ^∞. In particular the Weyl degeneracy is completely broken, and moreover the different charged sectors will have masses of gauge bosons and of monopoles given again by $\cos(\pi \frac{i}{N}) - \cos(\pi \frac{j}{N})$, for example $\simeq 1/N^2$ near the edges of the distribution.

We remark that in the supersymmetric theory the ground state appears in a full nonperturbative and geometric way, while in our case it must be stressed that we are not dealing exactly with quantum corrections, but with a global factor in the Higgs measure, expected to be important just during the formation of the system.

A main feature, the hierarchy of masses, is the same, and this is at least curious.

4.7 The hierarchy of masses and Monopole degeneracies

Here we come to the observation that in the large N limit there is a hierarchy of masses which arises from the model, once given the distribution of Higgs eigenvalues. As N gets large, many of the objects become very light, thus bringing in a different physics, like often happens in this limit.

We have the gauge bosons and monopole pseudo-masses

$$m_W(\mathbf{q}) = g|\phi^\infty \cdot \mathbf{q}|$$
$$M(\mathbf{q}) = N\frac{|\phi^\infty \cdot \mathbf{q}|}{g}$$

which are function of the $SU(2)$ root $\mathbf{q}$ which specifies the charge of the monopole. Explicitly as a function of the Higgs eigenvalues

$$|\phi^\infty \cdot \mathbf{q}| = g\frac{1}{\sqrt{2}}|\phi_i^\infty - \phi_j^\infty| \tag{4.73}$$

when $\mathbf{q}$ is the charge in the sector (ij).

The possible values are $N(N-1)$ and we see that according to the distribution of eigenvalues in the Higgs field, they range in the interval $(\mu/N^2, 1)$.

Moreover: at the ends of the Higgs domain, $\pm\mu$, where the eigenvalues concentrate, we have masses of order $1/N^2$, while from the standard differences we get $\sim N^2$ masses of order $1/N$ and more.

We can look at the distribution of differences:

$$\begin{aligned} \sigma(d) &= \int\int dx dy \rho(x)\rho(y)\delta(|x-y|-d) \\ &= 4\int_{-a}^{-d/2} dt \frac{1}{\sqrt{(a^2-t^2)(a^2-(t+d)^2)}}. \end{aligned} \tag{4.74}$$

which is logarithmic singular for $d \to 0$ showing the phenomenon:

$$\sigma(d) \simeq \frac{2}{\pi^2}\frac{1}{a}\log a/d. \tag{4.75}$$

because masses are proportional to d this is also the distribution of masses in the model, in the large N limit.

The monopole pseudo-mass M has a N factor which compensates this vanishing for the great part of them, but nevertheless still many monopoles are massless in the limit.

4.8 Dilute gas in the large N limit

We will use these informations in the dilute gas ensemble of section **4.2**.

We have to perform the average of $\xi(\mathbf{q})$ in the subspace of the simple roots which is not orthogonal to a given weight $\mathbf{m}_\alpha$. This is of the form: $\mathbf{q}_i = \mathbf{m}_\alpha \pm \mathbf{m}_\beta$ $(\beta \neq \alpha)$.

Then

$$\bar{\xi}(m_\beta) = \frac{1}{(N-1)} \sum_{\beta \neq \alpha} \xi(\mathbf{m}_\alpha - \mathbf{m}_\beta) \tag{4.76}$$

For large N the average $\bar{\xi}$ has the form:

$$\bar{\xi} \sim N^{9/2} \int dx \rho(x) \frac{m_W^{9/2}}{g^3} \sqrt{A} \mathrm{e}^{-N\frac{m_W}{g^2}} \tag{4.77}$$

which leads to the following average

$$N^{9/2}\sqrt{A} \int_{-a}^{a} dx \frac{1}{\sqrt{a^2 - x^2}} \frac{((x-b)\tilde{\mu})^{9/2}}{g^3} \mathrm{e}^{-N(x-b)\frac{\tilde{\mu}}{g^2}} \tag{4.78}$$

where the large N limit has to be performed.

Here the first observation is that the exponential in the large N limit becomes a delta function (times a $1/N$ factor).

But then the $(x-b)^{9/2}$ prefactor drives the integral to zero. The integral is expanded and as many factors of N come from derivatives of the delta function, as there are in front.

Unfortunately we know that also the factor $\sqrt{A}$ depends on N, so that we ca not draw for now any definite conclusion.

What we can say is that surely the N factor in the mass of monopoles should make important the monopoles near the edges of the distribution, which we know are an integrable infinity.

The preexponential factor should not create problems, because the powers of m_W which appear come from the integration on translation zero modes, which has to be normalized to the size of monopoles, thus bringing as many factors of N as needed. At the same time the distribution should become relevant for the result.

Surely there is much work to be done, starting from the analysis of the fluctuating determinant.

To this aim we introduce the EK method which should allow us this calculation in the large N limit.

4.8.1 A remark on the Gribov problem

Here we want to make some remark on the Gribov ambiguity and its links with the topological sectors that we have encountered in the analysis of the monopole configurations, and with the Faddeev-Popov determinant that we have introduced in the unitary gauge.

The main point is that the Faddeev-Popov determinant carries the information on how good is our choice of gauge. It represents the local jacobian for the change of variables from the connection space to the quotient by the group action.

At points in functional space where the FP determinant vanishes, it means that we are not taking a complete quotient, in fiber bundle language the section we have chosen becomes tangent to the fiber along some gauge direction.

These points mark usually what is called the Gribov horizon, the name because it is the boundary of the maximal region where a section can be continued.

Here, to be concrete, we have in three dimensions a gauge connection and a matter field so the space of fields Γ is that of couples (A_μ, ϕ), functions from R^3 to the gauge algebra. As we have seen Γ is disconnected in components according to the total magnetic charge, as described in paragraph **2**.

In every component acts the gauge group $\mathcal{G} = \{g(x) : \mathrm{R}_3 \to SU(N)\}$, and no boundary conditions have to be imposed at infinity because by homotopic arguments gauge transformations connected to the identity do not change component.

This is best and more appropriately seen in the regular gauges, while in the unitary gauge we know that A_μ becomes singular along some Dirac string (but still $\mathcal{S} < \infty$). Nevertheless in the unitary gauge the FP determinant and the gauge fixing depend only on the Higgs field so that we can draw some conclusion.

The U-gauge has been introduced in the last paragraphs with its relative Faddeev-Popov determinant, which turns out to be

$$\prod_x \prod_{ij} \left(\phi_i(x) - \phi_j(x)\right)^2 . \tag{4.79}$$

Now we recall as remarked in section 2, that in nontrivial sectors of the Higgs winding at infinity, ϕ has necessarily some coinciding eigenvalues at some point.

This is proven in any regular gauge but is valid also in other gauges because it's a gauge invariant statement.

In the semiclassical picture of monopole gas, at each monopole location two (or more) eigenvalues coincide.

Hence we find that in the unitary gauge the FP determinant above seems to vanish identically for any nontrivial configuration. More suggestively one can think that the Gribov horizon is made of monopole configurations. The same happens at each field configuration throughout the whole nontrivial sectors.

This would mean that the unitary gauge is an ill defined section, in that it does not fix the gauge. Explicitly indeed it leaves intact some subgroup at the points in space where we have a monopole. This is so because it's the Higgs SSB which does not break the group invariance at the location of the monopole.

However there are two points which solve this seemingly bad problem.

• The determinant in the form above is ill defined. It requires us to live in a distribution space, whereas we usually consider smooth functions. After this remark it appears evident that no gauge invariance remains unfixed, because the smoothness constrains the gauge variation at the "origin" to follow that in the neighbor. So there is no such thing as the gauge variation at a single point, even if the theory has local invariance.

• In the last paragraph we proved that the Faddeev-Popov determinant and its vanishing is canceled by the gauge loops, so that if the jacobian vanishes it is just because the change of variables is singular, and in fact at the same time the integral takes care of this and diverges by the same amount so to correct the measure.

This is in contrast with standard Gribov phenomenon where the vanishing of the FP determinant is a unavoidable problem.

This result for the unitary gauge shows that it is, in respect to other regular gauges, not really a wrong gauge, it just has different properties. Moreover the absence of unphysical fields, the Higgs field is diagonal and no Goldstone bosons circulate around, makes it a good tool to investigate the quantum theory.

5 Eguchi Kawai reduction

The Eguchi-Kawai and Quenched Eguchi-Kawai reductions (EK and QEK) were invented around 1982 to reproduce, in the large N limit, the nonperturbative features of matrix field theories [51, 52, 53, 54, 55].

QEK has been applied with success to lattice QCD, with some important modifications [53], where it produced nice results especially in the strong phase.

It works to reproduce the sum of all planar Feynman graphs of the continuum theory from a single site matrix model. Taken a theory of a $N \times N$ matrix field $\phi(x) \in \mathcal{A}$ in d dimensions:

$$\mathcal{S} = \int \mathrm{d}^d x \, N \mathrm{tr} \left[L_2(\phi) + g_3 L_3(\phi) + g_4 L_4(\phi) + \cdots \right] \tag{5.80}$$

the quenched EK reduction is introduced with the following rules:

1) The field $\phi(x)$ is reduced to a variable ϕ in zero dimensions via the prescription: $\phi(x) = \mathrm{e}^{-iPx} \phi \mathrm{e}^{iPx}$, where $P = diag(p_1 \ldots p_N)$. The x dependence cancels in the action thanks to locality and translational invariance.

2) The volume integral is replaced by the inverse cutoff $a = \frac{1}{\Lambda}$.

3) The (zero dimensional) Feynman graphs are then averaged over the N momenta P with the measure $\frac{V}{a^d} \int_{-\Lambda/2}^{\Lambda/2} \prod_i \mathrm{d}^d(ap_i)$ or any other measure flat in the origin.

As pointed out by [54] also a (fixed) random distribution of the momenta p_i in the hypercube of side Λ is sufficient to reproduce, in the large N limit, the planar amplitudes. In [52, 56] G. Parisi and I. Bars argued that the uniform distribution should always be used.

Rule 2) above is a version of QEK obtained after scaling the fields by a factor $\sqrt{\frac{a^d}{V}}$, otherwise the prescription would say:

2') The volume integral is replaced by V and all coupling constants are rescaled by $g_r^{EK} = g_r \left(\frac{V}{a^d}\right)^{\frac{r-2}{2}}$.

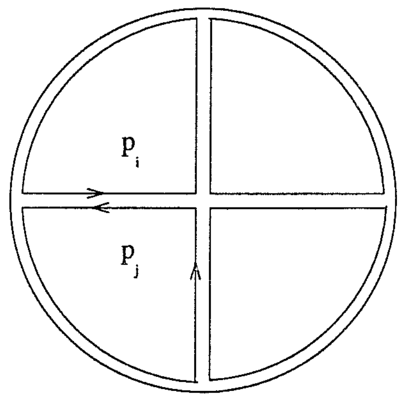
Rules 2) and 2') are equivalent but such rescaling of fields introduces a factor from the measure: $\left(\frac{V}{a^d}\right)^{-dim\mathcal{A}/2}$.

The proof of this rules is strictly perturbative: one looks at the Feynman graphs of the zero dimensional field and checks that they reproduce the *integrands* of the full theory. The average on P gives finally the momentum integration of loops, so that the $N \to \infty$ limit has to be taken before removing the cutoff. We show here how it works with a generic planar diagram.

Taken the above momentum measure, and the following EK action:

$$\mathcal{S} = VN\mathrm{tr}\left[\frac{1}{2}[P,\phi]^2 + \sum_r g_r L_r(\phi)\right] \tag{5.81}$$

one has the following factors in any Feynman diagram of l loops:
from p propagators: $(VN)^{-p}$;
from color traces: N^{l+1};
from $\{n_r\}$ vertices $\prod_r (VNg_r)^{n_r}$;
from integration over $p_1 \dots p_l$: $\frac{V}{a^d} a^l$
while momenta $p_{l+1} \dots p_N$ normalize to one.



Combining the contributions and using $2p = \sum_r r n_r$, $l = p+1-\sum_r n_r$, and to recover the right large N dependence, $N^2 \prod_r g_r^{n_r}$, one has to require:

$$g_r^{EK} = g_r \left(\frac{V}{a^d}\right)^{\frac{r-2}{2}} . \tag{5.82}$$

This proof relies on the existence of a diagrammatical expansion and is valid strictly at $N = \infty$, because only diagrams up to $N-1$ loops are reproduced. This was an early observation [52].

There is another approach to the EK reduction which proves its equivalence in the strong coupling phase, and it is based on the loop equations. We refer for this to the literature in [55, 54].

We would like to show also that the reduced evolution of the charged sector (rule 1 above), which takes place in the quenched Eguchi-Kawai models formulated in terms of group variables, like σ-models or models of lattice link variables, can also be understood from the semiclassical point of view, where $1/N$ play the role of the Planck constant.

Let's see this in more detail by examining the field theory on the group manifold. We take here the action of a Principal Chiral Model in d dimensions with group $SU(N)$, plus a possible potential.

One starts from the decomposition of group variables in Cartan and angular variables, $g = \omega h \omega^{-1}$. In these variables the kinetic term also gets decomposed and the angular evolution decouples:

$$\begin{aligned} L_2 &= \frac{N}{2}\mathrm{tr}(\partial_\mu g \partial_\mu g^{-1}) + V = \\ &= \frac{N}{2}\mathrm{tr}(\partial h \partial h^{-1}) + N\mathrm{tr}\left(h(\omega^{-1}\partial\omega)h^{-1}(\omega^{-1}\partial\omega) - (\omega^{-1}\partial\omega)^2\right) + V = \end{aligned} \tag{5.83}$$

$$= \frac{N}{2}\mathrm{tr}(\partial h \partial h^{-1}) + 2\sum_{i<j} |h_i - h_j||\omega^{-1}\partial\omega|^2_{ij} + V$$

Thus the motion of the ω variables is free, on the group manifold.

So if one formulates the model in terms of the angular connection $a = \omega^{-1}\partial\omega$, the angular dynamic is simply gaussian, and the nonlinearity manifests only in the boundary conditions via $\omega(x) = P\exp(\int a)$.

We recall here that on the line the integration on angular variables leads, at each point of time, to a factor which cancels the Van-der-Monde determinants of the measure, and this happens on all the time interval apart from the two ends, where the boundary conditions have to be treated carefully. It is in this way that the theory of free fermions arises.

The partition function results then given by the sum on all the classical minima of the action satisfying the given boundary conditions.

5.1 One dimension

Let's restrict to the one dimensional time interval, where QEK is more evident.

Fixed b.c. translate, for diagonal and angular variables, in:

$$\begin{aligned} h(0) &= h_0 & \qquad \omega(0) &= \omega_0 \\ h(T) &= h_T & \qquad \omega(T) &= \omega_T. \end{aligned}$$

These variables are somewhat redundant, because each couple (h, ω) is defined at least modulo a Weyl permutation of the eigenvalues of h, so that these b.c. come together with all possible permutation of eigenvalues of $h(0)$ and $h(T)$.

In case h has some coinciding eigenvalues the ω's also have a further ambiguity in the "little group" of h. We shall also sum on these possibilities, then.

We will first analyze the angular motion and then the diagonal one.

5.1.1 Angular boundary conditions

equal b.c.: $\omega(0) = \omega(T) = \omega_0$

In the case of fixed but equal boundary conditions on the angular part, one can always reabsorbe ω_0 in the h field $h(t) \to \omega_0^{-1}h(t)\omega_0$ (no longer diagonal) and consider just the free orbits of $\omega(t)$ starting and ending at the identity.

Such orbits are parameterized as moving in the Cartan directions plus a global group conjugation via an arbitrary Ω:

$$\omega(t) = \Omega^{-1}\mathrm{e}^{iPt}\Omega \tag{5.84}$$

P is a diagonal matrix with discrete entries $p_i = \frac{2\pi}{T}n_i$.

The global integration on Ω, to account for all the equivalent Cartan group directions, will merge together with the h integration to reproduce the EK prescription.

generic b.c.: $\omega(0) = \omega_0$, $\omega(T) = \omega_T$

This is again reduced to: $\omega(0) = \mathbf{1}$, $\omega(T) = \omega_T\omega_0^{-1}$ factorizing global ω_0.

Then, still parameterizing the orbits with $\omega(t) = \Omega^{-1}e^{iPt}\Omega$, one has to satisfy the condition

$$\Omega^{-1}e^{iPt}\Omega = \omega_T\omega_0^{-1}. \tag{5.85}$$

This implies that: P has to coincide with the eigenvalues of $\log(\omega_T\omega_0^{-1})/T$ modulo permutations and plus multiples of $\frac{2\pi}{T}$; Ω has to diagonalize $\omega_T\omega_0^{-1}$, and its freedom is then only rotation in the "little group" of $\omega_T\omega_0^{-1}$.

5.1.2 Diagonal dynamic

Up to now the argument is independent from the large N limit. This latter affects the integration on $h(t)$ and the average on P. For the functional integration on $h(t)$, as we know that it is a diagonal field, at large N its kinetic term of order N^2 calls for a constant solution.

Surely for b.c. $h(0) \neq h(T)$ there is no such constant solution and the dynamic is different and nontrivial: the leading N is not given by the constant field.

To make contact with the EK reduction, however, we are forced to allow only periodic boundary conditions, much in the spirit of the master field.

This is a pity because we are loosing the models with nontrivial b.c., for example also the dynamic with free b.c., equivalent to free fermions on the circle, is due to eigenvalues "moving" on the circle with nonzero momenta, so that a constant h is not sufficient.

Moreover we loose also the angular dynamics which realizes the Weyl permutation of eigenvalues. This latter in fact is subleading in large N with respect to the master field constant solution.

So, for identical eigenvalues at the boundary, we have the large N solution, $h(t) = h$ constant, and we are left to integrate on a field at a single point, given by

$$g_{ek} = \Omega^{-1}\omega_0^{-1}h\omega_0\Omega \tag{5.86}$$

.

Note however that the integration on g_{ek} lives, modulo a conjugation, in the little group of $\omega_0^{-1}\omega_T$, which could be smaller than the whole group. Moreover its eigenvalues are fixed according to the b.c. .

This latter fact is in agreement with the general statement of the QEK reduction that neutral degrees of freedom should not propagate [21, 54], or equivalently that the angular ones suffice for the leading large N part.

All together the prescription amounts to reduce the field to**

$$g(t) = \mathrm{e}^{-iPt} g_{ek} \mathrm{e}^{iPt}, \tag{5.87}$$

integrate on the angular group in g_{ek} and average on the entries of P.

Thanks to factorization, the average is to be performed on the free energy.

Finally we have the sum on global Weyl permutations of the eigenvalues of h together with rotations of ω_0, ω_T. This is of course irrelevant if the model is globally invariant.

However let's notice that all the procedure becomes nontrivial depending on the chosen h: In case of h having coinciding eigenvalues, this translates into P having a continuous domain, instead of a discrete set of values.

This arises because, suppose h_T has coinciding eigenvalues, we are obliged to consider all ω_T defined up to some element Λ of the little group of h_T. Consequently we have to consider all momenta which satisfy (5.85),

$$\Omega^{-1} \mathrm{e}^{iPt} \Omega = \omega_T \Lambda \omega_0^{-1} \tag{5.88}$$

and this amounts to diagonalization of that little group, so that the spectrum of p_i becomes continuous. The integration on the possible Λ has translated into an integration on non integer momenta.

These arguments are independent of the diagrammatical expansion, and a simple example will show better how these QEK reductions work whenever there is no diagrammatical expansion.

5.1.3 Gaussian model

Take just the matrix oscillator in any dimension d:

$$Z = \int \mathcal{D}\phi \mathrm{e}^{-\int d^d x \mathrm{tr}[(\partial\phi)^2 - \omega^2 \phi^2]}. \tag{5.89}$$

In Fourier components of each matrix entry $\phi_{ij}(p_{ij})$ one rewrites as

$$Z = \prod_{i,j} \left(\prod_{p_{ij} \in \frac{2\pi}{T} Z} \frac{1}{p_{ij}^2 + \omega^2} \right) \tag{5.90}$$

**Notice that we have eliminated an $\Omega^{-1} \cdots \Omega$ factor by global invariance of the model.

and the free energy is expressed as a quenched sum to yield the correct result:

$$
\begin{aligned}
F &= N^2 \sum_{p\in\frac{2\pi}{T}Z} \log\left(\frac{1}{p^2+\omega^2}\right) = \\
&= \frac{N^2}{2}\log\frac{T}{\sinh(\omega T)}
\end{aligned}
\tag{5.91}
$$

plus the standard normalization of the Feynman path integral.

In the usual prescriptions on EK reduction one would replace $p_{ij} \to (p_i - p_j)$, and take the sum on $i \neq j$ of some given fixed distribution $\{p_i\}$.

It is easily seen that in this way one sum is lacking, and the result is incorrect.

On the other hand if one averages over the possible distributions $\{p_i\}$ (that we stress represent still classical solutions)

$$
\sum_{i\neq j}\left[\sum_{\{p_i\in\frac{2\pi}{Z}\}} \log\left(\frac{1}{(p_i-p_j)^2}+\omega^2\right)\right]
\tag{5.92}
$$

we get the right large N result apart from some subleading term because the term in square brackets is independent of i, j.

The next model in order of simplicity, but already nontrivial, is the principal chiral field in one dimension.

5.2 Principal Chiral Field

We try here to solve the one dimensional model of a principal chiral field by means of the Eguchi-Kawai reduction, to investigate how the quenched momenta behave to reproduce the planar sector of the theory.

The one dimensional chiral field is known to have a peculiar structure in the planar sector alone, showing a third order phase transition absent in the finite N theory.

It is formulated in terms of a $U(N)$ valued field on a line in the interval $0 < t < T$, with (left and right) invariant action:

$$Z(g(0), g(T)) = \int_{g(0)}^{g(T)} \mathcal{D}g(t) e^{-\frac{N}{\lambda_0}\int_0^T \mathrm{tr}\partial_t g^{-1}\partial_t g} \tag{5.93}$$

It is also the model of a $U(N)$ symmetric top. The boundary conditions are fixed here to be $g(0)$ and $g(T)$.

By now this model is solvable by different techniques in the presence of arbitrary boundary conditions and is shown to be equivalent to QCD_2 on a cylinder with fixed boundary holonomies.

In the planar limit one meets a Kosterlitz-Thouless (or better Douglas Kazakov) third order phase transition at some T_c which is function of the boundary conditions, the transition being present even with $g(0) = g(T) = \mathbf{1}$, at $T_c\lambda_0 = \pi^2$.

Z represents also the partition function for the quantum mechanics on the group manifold, heat kernel at temperature T, which has been proven [57, 58] to be the exact Wilson renormalization group invariant action for two dimensional YM theory.

It has been also shown to possess a stringy formulation in terms of maps between two-dimensional surfaces.

All the equivalences are best proven writing Z as a sum on all irreps of the group (for ex. for QCD_2 on the sphere, $g(0) = g(T) = \mathbf{1}$)

$$Z = \sum_R d_R^2 e^{-\frac{\lambda_0 T}{2N} C_2(R)} \tag{5.94}$$

The large N limit provides interesting features. In that limit this sum is dominated by a single kind of representations (a distribution of components of the highest weight, namely a semicircle law for $T\lambda_0$ small). This distribution in turn, being of discrete variables, is bounded by 1, and passing above the critical coupling this constraint leads to a different phase, much like the condensation of ordinary gases. The transition is third order [Douglas Kazakov, Mihanan, Gross..].

In the equivalent theory QCD_2, the transition is shown to be induced by the instantons. The instanton charges, integers $n_1 \dots n_N$, are dual (practically in Poisson sense) to the weight components [59].

The components of the highest weight have also the interpretation of momenta of free fermions on a circle $[0, T]$, and it is the Pauli exclusion principle which constrains their density to be less than one. As the support of the distribution shrinks to zero, $1/\sqrt{\lambda_0 T} \to 0$, the density increases hitting the bound. This picture arises when looking at the model as a theory of fermions, a strictly two-dimensional point of view.

We come thus to the Quenched Eguchi Kawai approach. As it should reproduce the planar limit of matrix field theories, so it is immediate to apply it here.

According to the general prescriptions of the last section, to each matrix index is assigned a single momentum, and their distribution does not matter for the infrared behavior of the system, as long as it is flat in the origin.

(An other question is whether the EK can be used to calculate the partition function per se, in addiction to planar 'physical' diagrams. For zero external source the free energy is not exactly given by diagrams and may come out incorrectly..)

Identifying $g(t)$ with the holonomy around the compact space dimension, the PCF model above reproduces 2DYM on some surfaces, depending on the boundary conditions:

CYLINDER: $g(0) \sim g_0, g(T) = g_T$, DISC: $g(0) \sim \mathbf{1}, g(T) \sim g_T$,
SPHERE: $g(0) = g(T) = \mathbf{1}$, TORUS: $g(0) \sim g(T) = g$ integrated out.

5.2.1 Reduction

So we come to the PCF, where the matrix degrees of freedom are $g(t)$, and the quenched EK reduction can be implemented assigning a set of one-dimensional momenta to its indices:

$$g(t) = \mathrm{e}^{-iPt} g \mathrm{e}^{iPt} \qquad g_{ij}(t) = \mathrm{e}^{-i(p_i - p_j)t} g_{ij}. \qquad (P = diag(p_1, \dots, p_N)). \tag{5.95}$$

- We give discrete momenta $p_i = \frac{2\pi}{T} n_i$, $(n_i \in \mathbf{Z})$, according to the discussion of the previous section.

The boundary conditions, in any case, are naturally those of a torus: $g(0) \sim g(T) = g$ integrated out in the EK procedure. Hence we expect no phase transition, and the system is always in the strong coupling phase. Also we expect the partition function to vanish, in accordance to the fact that there are no maps from the sphere to the torus.

The replacement yields the following zero dimensional model:

$$\mathrm{Z} = \int dg e^{\frac{2N}{\lambda_0} \frac{1}{\Lambda} \mathrm{tr}\left[Pg^{-1}Pg - P^2\right]}. \tag{5.96}$$

The IZ integral over unitary matrices g gives:

$$Z = e^{-\frac{2NT}{\lambda_0}\sum_i p_i^2} \frac{\det_{ij} e^{\frac{2N}{\lambda_0\Lambda}p_i p_j}}{\Delta^2(\sqrt{\frac{2T}{\lambda_0}}P)} = \frac{\det_{ij} e^{-N\frac{16\pi^3}{\lambda_0 T^2\Lambda}\frac{(n_i-n_j)^2}{2}}}{\Delta^2(\sqrt{\frac{16\pi^3}{\lambda_0 T^2\Lambda}}n)} \tag{5.97}$$

Notice that a saddle point analysis could be performed if the distribution of p_i would reach a smooth limit for large N, according to the recent analysis of Matytsin [60] in terms of the Burgers equation.

Here instead we have discrete variables p_i in a non compact support.

5.2.2 Uniform distribution

We recall that even if this model is essentially gaussian, it has a diagrammatical expansion once one introduces a as: $g = e^a$, so, according to the EK prescriptions, a uniform distribution of momenta in the presence of a cutoff should be sufficient.

One has thus $\frac{\Lambda T}{2\pi}$ groups of $r = N\frac{2\pi}{\Lambda T}$ coinciding integers n_i (where $p_i = 2\pi n_i/T$).

The determinant is unity in the large N limit if all n_i's are different, but there is some cancelation with $\Delta^{-2}(n)$ for any group of coinciding n_i's:

We find, for the coinciding of n_i's††:

$$\lim_{\varepsilon\to 0}\frac{\det_{ij} e^{-N\beta\varepsilon^2(n_i-n_j)^2/2}}{\Delta^2(\varepsilon n)} = \frac{(N\beta)^{r(r-1)/2}}{\Delta_0(r)} \qquad (\varepsilon\to 0 \quad r = \#n_i) \tag{5.98}$$

where a limit $\varepsilon \to 0$ has been introduced to approach the vanishing of $n_i - n_j$.

Out of the square Vandermonde in the denominator of (5.97) remain some non-coinciding factors $(n-m)^r$, and finally we get:

$$\begin{aligned} Z &= \left(\frac{16\pi^3}{\lambda_0 T^2\Lambda}\right)^{-N(N-1)/2}\left[\frac{\left(N\frac{16\pi^3}{\lambda_0 T^2\Lambda}\right)^{N\frac{2\pi}{\Lambda T}(N\frac{2\pi}{\Lambda T}-1)/2}}{\Delta_0(N\frac{2\pi}{\Lambda T})}\right]^{(\frac{\Lambda T}{2\pi})}\frac{\mathrm{Det}_{nm} e^{-\frac{1}{2}N\frac{16\pi^3}{\lambda_0 T^2\Lambda}(n-m)^2}}{\prod_{m\neq n}(n-m)^{N^2(\frac{2\pi}{\Lambda T})^2}} \\ &\simeq \left(\frac{16\pi^3}{\lambda_0 T^2\Lambda}\right)^{-\frac{N^2}{2}}\left(N\frac{16\pi^3}{\lambda_0 T^2\Lambda}\right)^{\frac{N^2}{2}(\frac{2\pi}{\Lambda T})^2(\frac{\Lambda T}{2\pi})}\Delta_0^{-\frac{\Lambda T}{2\pi}}(N\frac{2\pi}{\Lambda T})\Delta_0^{-2N^2(\frac{2\pi}{\Lambda T})^2}(\frac{\Lambda T}{2\pi}) \end{aligned}$$

The free energy is thus, in the large N limit:

$$\mathbf{F} = -\log Z =$$

††where

$$\Delta_0(x) \equiv \prod_{k=1}^{x-1} k! \quad \simeq \quad e^{\frac{1}{2}x^2(\log x - 3/2)}$$

$$
\begin{aligned}
&= \frac{N^2}{2}\left[\log(\frac{16\pi^3}{\lambda_0 T^2\Lambda}) - \frac{2\pi}{\Lambda T}\log(N\frac{16\pi^3}{\lambda_0 T^2 \Lambda})\right] + \\
&\quad + \frac{\Lambda T}{2\pi}\log\Delta_0(N\frac{2\pi}{\Lambda T}) + 2N^2(\frac{2\pi}{\Lambda T})^2 \log\Delta_0(\frac{\Lambda T}{2\pi}) \\
&= N^2\left[\frac{1}{2}\log\left(\frac{4\pi\Lambda}{\lambda_0}\right) - 3/4\right] \qquad (5.99)
\end{aligned}
$$

Some discussion is needed, because after all we do not find zero as a result and the reason is not clear.

The above result was obtained also in infinite volume by Bars [56], and was interpreted as the equivalent model in the presence of a regulator, thus exhibiting the Gross-Witten [61] phase transition.

However the Gross Witten phase transition is a lattice artifact in this case, and as the cutoff is removed this result coincides with the wrong phase of this model, namely the weak coupling limit, whereas either the Bars analysis because of infinite volume, or ours because we believe to be on the torus, should bring to the strong coupling phase of the chiral field and thus give zero partition function.

Of course these are just numbers and can be reabsorbed in the normalization, for what regards planar diagrams, but the presence of the cutoff is what shows the wrong result.

We believe these being others clues which indicate that the uniform distribution of momenta fails to reproduce, contrary to the claim in [56], the full structure of large N matrix theories.

In the next section we instead find the correct result leaving unspecified the distribution of momenta, but in an other reduction of the same model.

5.2.3 Prescription from connection

Here we stick to the prescription coming from the algebra degrees of freedom, $A(t) = \mathrm{e}^{-iPt}A\mathrm{e}^{iPt}$. This reduced time evolution, which is not pure gauge, gives, integrating for $g(t)$:

$$
g(t) = g_0 P\exp[i\int \mathrm{d}t\, A(t)] = g_0 \mathrm{e}^{i(A-P)t}\mathrm{e}^{iPt}. \qquad (5.100)
$$

We have thus:

$$
g(0) = g_0 \qquad\qquad g(T) = g_0\mathrm{e}^{i(A-P)T}\mathrm{e}^{iPT}. \qquad (5.101)
$$

Inserting this in the PCF integral with an inserted group delta function and using $g^{-1}\partial_t g(t) = \mathrm{e}^{-iPt}A\mathrm{e}^{iPt}$:

$$
Z = \sum_R \int \mathrm{d}A\, \mathrm{e}^{-\frac{TN}{\lambda_0}\mathrm{tr}A^2}\chi_R(g_0\mathrm{e}^{i(A-P)T}\mathrm{e}^{iPT})\chi_R(g_0)
$$

$$
\begin{aligned}
&= \sum_R \chi_R(g_0)\mathrm{tr}_R \int \mathrm{d}A^a \, \mathrm{e}^{-\frac{T}{\lambda_0}A^2+T(A^a-p^a)\tau^a} \mathrm{e}^{iPt} \\
&= \left(\frac{\lambda_0}{T}\right)^{\frac{N(N-1)}{4}} \sum_R \chi_R^2(g_0)\mathrm{e}^{-\lambda_0 T C_2(R)}
\end{aligned}
\tag{5.102}
$$

after some steps.

• Alternatively we note that in $g(t)$ the second factor e^{iPt} is a local gauge transform for the measure, and we can take

$$
g(t) = g_0 \mathrm{e}^{i(A-P)t} \qquad\qquad A(t) = A - P \tag{5.103}
$$

(in fact $\mathrm{e}^{-iPt}A\mathrm{e}^{iPt}$ is gauge equivalent to $A-P$).

Inserting the ansatz in the partition function one has:

$$
\begin{aligned}
Z &= \sum_R \int \mathrm{d}A \, \mathrm{e}^{-\frac{TN}{\lambda_0}\mathrm{tr}(A-P)^2} \chi_R(g_0 \mathrm{e}^{i(A-P)T})\chi_R(g_0) \\
&= \sum_R \chi_R(g_0)\mathrm{tr}_R \int \mathrm{d}A^a \, \mathrm{e}^{-\frac{T}{\lambda_0}(A-P)^2+T(A^a-p^a)\tau^a} g_0 \\
&= \left(\frac{\lambda_0}{T}\right)^{\frac{N(N-1)}{4}} \sum_R \chi_R^2(g_0)\mathrm{e}^{-\lambda_0 T C_2(R)}
\end{aligned}
\tag{5.104}
$$

This shows that with this last prescription the momenta are unimportant due to the integral on the gauge connection, and the vacuum is taken to be $A = P$. This is automatically in accordance with the general quenching prescriptions for gauge connections of Gross-Kitazawa.

6 1D Matrix model

The matrix models introduced in the last section to test the EK prescription, although may seem to have just academic interest, possess many features which are interesting from the physical point of view, and serve as powerful tools in the solution of complex models of contemporary physics.

Because in a first part of my stay here at SISSA I have studied this subject, i will review the solution arising from the exact treatment. We will try to get some comparison with the EK approach at the end.

It is already the model in zero dimensions, called 1-matrix model, which achieved great success by predicting the right critical exponents of two-dimensional gravity coupled to matter of unitary conformal models, at large N [62, 63, 64]. It was shown that they possess a nontrivial integrable structure with respect to the flow of coupling constants, precisely as in topological gravity coupled to matter.

The one dimensional matrix model, in the case of *free boundary conditions*, is exactly solvable, in the singlet sector. Its version on the time lattice, which will inspire the following sections, is also found to contain different integrable structures [81], yields a representation in terms of fermions on the line, and is shown to describe correlation functions of loop operators in $c = 1$ string theory [65, 66].

Once the model is discretized, it reduces to a *chain* of matrices coupled by bilinear interactions.

The case when boundary conditions are not free, for example periodic [67], the chain is closed and the model is not solvable in closed form. The reason is that it receives contributions from the constrained evolution of angular variables.

With free boundary conditions instead the chain is open, and is precisely this which renders the model solvable. Also for finite N the solution provides all correlators of traces of matrices in different position of the chain, which reproduces the correct time dependence of the original one dimensional model.

We are going to use the so called method of "Q-matrices", illustrated largely in our paper [81]. There is also shown the emergence of classical integrable hierarchies which arise in the chain model of matrices, in the form of Poisson reductions of the KP hierarchy: they are by now called $2n$-boson KP hierarchies and generalized KdV hierarchies.

6.1 Multi-matrix models

In this section we intend to analyze matrix models made of q Hermitian $N \times N$ matrices with bilinear couplings between different matrices. Unless otherwise specified, by this we mean an open chain of q matrices, each linearly interacting with the nearest neighbors. These models have been already introduced and partially analyzed in [68] (for other approaches to multi–matrix models, see [69],[70],[71],[72], [73],[74]).

We review here some general results concerning q–matrix models, [68]. The partition function of the q–matrix model is given by

$$Z_N(t,c) = \int dM_1 dM_2 \dots dM_q e^{TrU} \tag{6.105}$$

where $M_1, \dots, M_q$ are Hermitian $N \times N$ matrices and

$$U = \sum_{\alpha=1}^{q} V_\alpha + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} M_\alpha M_{\alpha+1}$$

with potentials

$$V_\alpha = \sum_{r=1}^{p_\alpha} \bar{t}_{\alpha,r} M_\alpha^r \qquad \alpha = 1, 2 \dots, q \tag{6.106}$$

The p_α's are finite positive integers.

We denote by $\mathcal{M}_{p_1,p_2,\dots,p_q}$ the corresponding q–matrix model. It has become moreover customary to associate to the generic q–matrix model (6.105) the Dynkin diagram A_q. Occasionally we will stick to this convention and speak about nodes and links.

We are interested in computing the correlation functions of the operators

$$\tau_{\alpha,k} = trM_\alpha^k$$

and possibly of other composite operators (see below). For this reason we complete the above model by replacing (6.106) with the more general potentials

$$V_\alpha = \sum_{r=1}^{\infty} t_{\alpha,r} M_\alpha^r, \qquad \alpha = 1, \dots q \tag{6.107}$$

where $t_{\alpha,r} \equiv \bar{t}_{\alpha,r}$ for $r \leq p_\alpha$.

In other words we have embedded the original couplings $\bar{t}_{\alpha,r}$ into infinite sets of couplings. Therefore we have two types of couplings. The first type consists of those couplings (the barred ones) that define the model: they represent the true *dynamical* parameters of the theory; they are kept non-vanishing throughout the calculations. The second type encompasses the remaining couplings, which are introduced only for computational purposes. In terms of ordinary field theory the former are analogous to the interaction couplings, while the latter correspond to external sources (coupled to composite operators). Any correlation function is obtained by differentiating $\ln Z_N$

with respect to the couplings associated to the operators that appear in the correlator and then setting to zero (only) the external couplings.

From now on we will not make any formal distinction between interacting and external couplings. Case by case we will specify which are the interaction couplings and which are the external ones. Finally, it is sometime convenient to consider N on the same footing as the couplings and to set $t_{\alpha,0} \equiv N$.

6.1.1 Orthogonal polynomials

The most popular procedure to calculate the partition function consists of three steps [75],[76],[77]:
(i). One integrates out the angular parts such that only the integrations over the eigenvalues are left,

$$Z_N(t,c) = \text{const} \int \prod_{\alpha=1}^{q} \prod_{i=1}^{N} d\lambda_{\alpha,i} \Delta(\lambda_1) e^U \Delta(\lambda_q), \tag{6.108}$$

where

$$U = \sum_{\alpha=1}^{q} \sum_{i=1}^{N} V_\alpha(\lambda_{\alpha,i}) + \sum_{\alpha=1}^{q-1} \sum_{i=1}^{N} c_{\alpha,\alpha+1} \lambda_{\alpha,i} \lambda_{\alpha+1,i}, \tag{6.109}$$

and $\Delta(\lambda_1)$ and $\Delta(\lambda_q)$ are Vandermonde determinants.
(ii). One introduces the orthogonal polynomials

$$\xi_n(\lambda_1) = \lambda_1^n + \text{lower powers}, \qquad \eta_n(\lambda_q) = \lambda_q^n + \text{lower powers}$$

which satisfy the orthogonality relations

$$\int d\lambda_1 \ldots d\lambda_q \xi_n(\lambda_1) e^{\mu} \eta_m(\lambda_q) = h_n(t,c) \delta_{nm} \tag{6.110}$$

where

$$\mu \equiv \sum_{\alpha=1}^{q} \sum_{r=1}^{\infty} t_{\alpha,r} \lambda_\alpha^r + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} \lambda_\alpha \lambda_{\alpha+1}. \tag{6.111}$$

(iii). If one expands the Vandermonde determinants in terms of these orthogonal polynomials and using the orthogonality relation (6.110), one can easily calculate the partition function

$$Z_N(t,c) = const\ N! \prod_{i=0}^{N-1} h_i \tag{6.112}$$

Knowing the $h(c,t)$'s amounts to knowing the partition function, up to an N-dependent constant. In turn the information concerning the $h(c,t)$'s can be encoded in suitable *flow equations*, subject to specific conditions, *the coupling conditions.* Before we come to that, however, we recall some necessary notations.

For any matrix M, we define the conjugate $\mathcal{M}$

$$\mathcal{M} = H^{-1}MH, \qquad H_{ij} = h_i\delta_{ij}, \qquad \bar{M}_{ij} = M_{ji}, \qquad M_l(j) \equiv M_{j,j-l}.$$

As usual we introduce the natural gradation

$$deg[E_{ij}] = j - i, \qquad \text{where} \qquad (E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$$

and, for any given matrix M, if all its non–zero elements have degrees in the interval $[a,b]$, then we will simply write: $M \in [a,b]$. Moreover M_+ will denote the upper triangular part of M (including the main diagonal), while $M_- = M - M_+$. We will write

$$\mathrm{Tr}(M) = \sum_{i=0}^{N-1} M_{ii}$$

The latter operation will be referred to as taking the finite trace.

Coupling conditions.

First we introduce the Q–type matrices

$$\int \prod_{\alpha=1}^{q} d\lambda_\alpha \xi_n(\lambda_1) e^{\mu} \lambda_\alpha \eta_m(\lambda_q) \equiv Q_{nm}(\alpha)h_m = \mathcal{Q}_{mn}(\alpha)h_n, \quad \alpha = 1, \ldots, q. \quad (6.113)$$

Among them, $Q(1), \mathcal{Q}(q)$ are Jacobi matrices: their pure upper triangular part is $I_+ = \sum_i E_{i,i+1}$. We will need two P–type matrices, defined by

$$\int \prod_{\alpha=1}^{q} d\lambda_\alpha \Big(\frac{\partial}{\partial\lambda_1}\xi_n(\lambda_1)\Big) e^{\mu}\eta_m(\lambda_q) \equiv P_{nm}(1)h_m \qquad (6.114)$$

$$\int d\lambda_1 \ldots d\lambda_q \xi_n(\lambda_1) e^{\mu} \Big(\frac{\partial}{\partial\lambda_q}\eta_m(\lambda_q)\Big) \equiv P_{mn}(q)h_n \qquad (6.115)$$

The matrices (6.113) we introduced above are not completely independent. More precisely all the $Q(\alpha)$'s can be expressed in terms of only one of them and one matrix P. Expressing the trivial fact that the integral of the total derivative of the integrand in eq.(6.110) with respect to $\lambda_\alpha, 1 \le \alpha \le q$ vanishes, we can easily derive the constraints or *coupling conditions*

$$P(1) + V_1' + c_{12}Q(2) = 0, \qquad (6.116a)$$

$$c_{\alpha-1,\alpha}Q(\alpha-1) + V_\alpha' + c_{\alpha,\alpha+1}Q(\alpha+1) = 0, \qquad 2 \le \alpha \le q-1, \quad (6.116b)$$

$$c_{q-1,q}Q(q-1) + V_q' + \bar{\mathcal{P}}(q) = 0. \qquad (6.116c)$$

where we use the notation

$$V'_\alpha = \sum_{r=1}^{p_\alpha} r t_{\alpha,r} Q^{r-1}(\alpha), \qquad \alpha = 1, 2, \ldots, q$$

These conditions explicitly show that the Jacobi matrices depend on the choice of the potentials. In fact they completely determine the degrees of the matrices $Q(\alpha)$. A simple calculation shows that

$$Q(\alpha) \in [-m_\alpha, n_\alpha], \qquad \alpha = 1, 2, \ldots, q$$

where

$$\begin{aligned} m_1 &= (p_q - 1) \ldots (p_3 - 1)(p_2 - 1) \\ m_\alpha &= (p_q - 1)(p_{q-1} - 1) \ldots (p_{\alpha+1} - 1), \qquad & 2 \leq \alpha \leq q - 1 \\ m_q &= 1 \end{aligned}$$

and

$$\begin{aligned} n_1 &= 1 \\ n_\alpha &= (p_{\alpha-1} - 1) \ldots (p_2 - 1)(p_1 - 1), \qquad & 2 \leq \alpha \leq q - 1 \\ n_q &= (p_{q-1} - 1) \ldots (p_2 - 1)(p_1 - 1) \end{aligned}$$

Throughout this chapter we will refer to the following parameterization coordinates of the Jacobi matrices

$$Q(1) = I_+ + \sum_i \sum_{l=0}^{m_1} a_l(i) E_{i,i-l}, \qquad Q(q) = I_+ + \sum_i \sum_{l=0}^{m_q} b_l(i) E_{i,i-l} \tag{6.117}$$

and for the supplementary matrices

$$Q(\alpha) = \sum_i \sum_{l=-n_\alpha}^{m_\alpha} T_l^{(\alpha)}(i) E_{i,i-l}, \quad 2 \leq \alpha \leq q - 1 \tag{6.118}$$

Flow equations

The flow equations of the q–matrix model can be expressed by means of the following hierarchies of equations for the matrices $Q(\alpha)$.

$$\frac{\partial}{\partial t_{\beta,k}} Q(\alpha) = [Q_+^k(\beta), \;\; Q(\alpha)], \qquad 1 \leq \beta \leq \alpha \tag{6.119a}$$

$$\frac{\partial}{\partial t_{\beta,k}} Q(\alpha) = [Q(\alpha), \;\; Q_-^k(\beta)], \qquad \alpha \leq \beta \leq q \tag{6.119b}$$

These flows commute and define a multi–component Toda lattice hierarchy, [78],[73].

Reconstruction formulae.

The coupling conditions and the flow equations allow us to calculate the matrix elements of $Q(\alpha)$. From the latter we can reconstruct the partition function as follows. We start from the following main formula

$$\frac{\partial}{\partial t_{\alpha,r}}\ln Z_N(t,c) = \mathrm{Tr}\Big(Q^r(\alpha)\Big), \qquad 1 \le \alpha \le q \tag{6.120}$$

It is evident that, by means of the flow equations for $Q(\alpha)$, we can express all the derivatives of $\ln Z_N$ with respect to the couplings $t_{\alpha,k}$ (i.e. all the correlators) as finite traces of commutators of the $Q(\alpha)$'s themselves. In other words, knowing the $Q(\alpha)$'s, we can reconstruct the partition function (up to a constant depending only on N). In particular we can get

$$\frac{\partial^2}{\partial t_{1,1}\partial t_{\alpha,r}}\ln Z_N(t,c) = \Big(Q^r(\alpha)\Big)_{N,N-1}, \qquad 1 \le \alpha \le q \tag{6.121}$$

It was already noticed in [68] that this equation leads to the two-dimensional Toda lattice equation.

6.2 Gaussian q–matrix model

Let us now concentrate on the most general case (6.111) with quadratic potentials at most. In particular μ takes the form

$$\mu = \mu(\lambda_1,\ldots,\lambda_q) = \sum_{\alpha=1}^{q} u_\alpha\lambda_\alpha + \sum_{\alpha=1}^{q} t_\alpha\lambda_\alpha^2 + \sum_{\alpha=1}^{q-1} c_\alpha\lambda_\alpha\lambda_{\alpha+1} \tag{6.122}$$

The coupling conditions are

$$P(1) + u_1 + 2t_1Q(1) + c_1Q(2) = 0 \tag{6.123a}$$

$$u_\alpha + 2t_\alpha Q(\alpha) + c_\alpha Q(l+1) + c_{\alpha-1}Q(\alpha-1) = 0, \ (\alpha = 2,\ldots,q-1) \tag{6.123b}$$

$$\bar{\mathcal{P}}(q) + u_q + 2t_qQ(q) + c_{q-1}Q(q-1) = 0 \tag{6.123c}$$

These coupling conditions imply that $Q(\alpha)$ has only three non–vanishing diagonal lines, the main diagonal and the two adjacent lines. Now let us simplify the coordinatization of such matrix as follows

$$Q(\alpha) = \epsilon_+(\alpha) + \epsilon_0(\alpha) + \epsilon_-(\alpha) \tag{6.124}$$

where

$$\epsilon_-(\alpha) = \sum_n g_\alpha(n)E_{n,n-1}, \quad \epsilon_0(\alpha) = \sum_n s_\alpha(n)E_{n,n}, \quad \epsilon_+(\alpha) = \sum_n h_\alpha(n)E_{n,n+1}$$

with the understanding that $h_1(n) = 1$ and $g_q(n) = R(n)$. In terms of these coordinates the above coupling equations take the form of the following linear system

$$
\begin{aligned}
&2t_1 + c_1 h_2(n) = 0 \\
&2t_1 s_1(n) + c_1 s_2(n) + u_1 = 0 \qquad (6.125a) \\
&n + 2t_1 g_1(n) + c_1 g_2(n) = 0 \\
&2t_\alpha h_\alpha(n) + c_\alpha h_{\alpha+1}(n) + c_{\alpha-1} h_{\alpha-1}(n) = 0, \qquad \alpha = 2, \dots, q-1 \\
&2t_\alpha s_\alpha(n) + c_\alpha s_{\alpha+1}(n) + c_{\alpha-1} s_{\alpha-1}(n) + u_\alpha = 0, \quad \alpha = 2, \dots, q-1 \qquad (6.125b) \\
&2t_\alpha g_\alpha(n) + c_\alpha g_{\alpha+1}(n) + c_{\alpha-1} g_{\alpha-1}(n) = 0, \qquad \alpha = 2, \dots, q-1 \\
&\frac{n+1}{R(n+1)} + 2t_q h_q(n) + c_{q-1} h_{q-1}(n) = 0 \\
&2t_q s_q(n) + c_{q-1} s_{q-1}(n) = 0 \qquad (6.125c) \\
&2t_q R(n) + c_{q-1} g_{q-1}(n) = 0
\end{aligned}
$$

The solution of this system is expressed in terms of the matrices X_α and Y_α, defined as follows

$$
X_\alpha = \begin{pmatrix}
2t_1 & c_1 & 0 & \dots & 0 & 0 \\
c_1 & 2t_2 & c_2 & \dots & 0 & 0 \\
0 & c_2 & 2t_3 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 2t_{\alpha-1} & c_{\alpha-1} \\
0 & 0 & 0 & \dots & c_{\alpha-1} & 2t_\alpha
\end{pmatrix} \qquad (6.126)
$$

and

$$
Y_\alpha = \begin{pmatrix}
2t_\alpha & c_\alpha & 0 & \dots & 0 & 0 \\
c_\alpha & 2t_{\alpha+1} & c_{\alpha+1} & \dots & 0 & 0 \\
0 & c_{\alpha+1} & 2t_{\alpha+2} & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & 0 & \dots & 2t_{q-1} & c_{q-1} \\
0 & 0 & 0 & \dots & c_{q-1} & 2t_q
\end{pmatrix} \qquad (6.127)
$$

Of course $Y_1 \equiv X_q$. One finds

$$
\begin{aligned}
h_\alpha(n) &= (-1)^\alpha (c_1 c_2 \dots c_{\alpha-1})^{-1} \det X_{\alpha-1} \\
R(n) &= (-1)^q n c_1 c_2 \dots c_{q-1} \Big(\det X_q \Big)^{-1} \qquad (6.128) \\
g_\alpha(n) &= (-1)^\alpha n c_1 c_2 \dots c_{\alpha-1} \frac{\det Y_{\alpha+1}}{\det X_q}
\end{aligned}
$$

Moreover, if we denote by S and U the vectors $(s_1, s_2, \ldots, s_q)^t$ and $(u_1, \ldots, u_q)^t$, respectively, we have

$$S = -X_q^{-1} U \tag{6.129}$$

As we have already remarked we can always without loss of generality suppress the linear terms in u_α by constant shifts of M_α. In such a case $S = 0$.

It is now easy to see that, at the cosmological point ($t_\alpha = u_\alpha = 0$), the solution (6.128) is well defined when q is even, while it is singular when q is odd – in the latter case, for example, $\det X_q = 0$.

In the last part of this section we would like to dispel a seemingly obvious objection to the very content of this model. Take the generic quadratic model of q matrices with nearest neighbor interactions

$$U = \sum_{\alpha=1}^{q} t_\alpha M_\alpha^2 + \sum_{\alpha=1}^{q-1} c_{\alpha,\alpha+1} M_\alpha M_{\alpha+1} \equiv \sum_{\alpha,\beta}^{q} M_\alpha A_{\alpha\beta} M_\beta. \tag{6.130}$$

The $q \times q$ matrix A is symmetric, and, for the theory of central quadrics, it can be brought to a canonical diagonal form with all ones or minus ones on the diagonal. The signature of A is of course a characteristic of the potential.

Let us see the consequences of this simple remark as far as the matrix model is concerned. The diagonalization of A can be achieved by integrating in the path integral over suitable linear combinations of the matrices M_α, instead of integrating simply over the M_α's. Of course this gives rise to a Jacobian factor, which is however one if one uses only shifts of the M_α's. In this way one brings A to the diagonal form

$$A = \mathrm{Diag}(f_1, \ldots, f_q) \tag{6.131}$$

but does not rescale its elements to unity. However this form is sufficient for our subsequent discussion. The initial matrix model appears at this point to be equivalent to the decoupled model with potential

$$U' = \sum_\alpha f_\alpha M_\alpha^2.$$

with partition function $Z = \mathrm{const}(N)(f_1 f_2 \ldots f_q)^{-N^2/2}$. We remark however that this procedure is of no help if one has to compute correlation functions of composite operators, in that it screws up the definition of the states and renders the calculation of the correlators practically impossible. The procedure followed in this chapter, i.e. the use of the generalized Toda lattice hierarchy, has precisely the virtue that it allows the calculation of the exact correlators of significant composite operators.

Finally let us remark that we can easily generalize the results of this section to the cases when in the potential are present, beside the terms of (6.130), also interactions of the type $c_{\alpha,\beta}D_\alpha D_\beta$ where $D_\alpha = \text{Diag } \mathrm{M}_\alpha$ and $\beta \neq \alpha-1, \alpha, \alpha+1$. In such cases the method is the same as in the chain models, with the only difference that the matrices X_α and Y_α will have, at the position (α, β), additional non-vanishing entries $c_{\alpha\beta}$ if the latter are present in the potential. The relative operators, called *extra states*, are found to be in correspondence with the string discrete states.

6.3 Schwinger-Dyson equations

The analogous of the above *coupling conditions* are the Schwinger-Dyson equations, arising from the invariance of the matrix integrals under reparametrization of the matrix variables: $M \to f(M)$.

They appear as a set of equations for the correlation functions, and the model can be solved also in this way.

We consider for example the two-matrix model with partition function:

$$Z = \int DM_1 DM_2 \exp(-\mathrm{Tr}(V_1(M_1) + V_2(M_2) + cM_1M_2)) \tag{6.132}$$

where the potentials are:

$$V_1(M_1) = \sum_{k\geq 1} t_k M_1^k, V_2(M_2) = \sum_{k\geq 1} s_k M_2^k$$

The partition function is invariant under the infinitesimal transformations of the matrices:

$$\begin{aligned} M_1 &\to M_1 + \varepsilon_{1,n} M_1^{n+1}, \quad n \geq -1, \varepsilon_{1,n} \text{ infinitesimal} \\ M_2 &\to M_2 + \varepsilon_{2,n} M_2^{n+1}, \quad n \geq -1, \varepsilon_{2,n} \text{ infinitesimal} \end{aligned}$$

Taking into account the contributions from the measure that transforms as:

$$DM_1 \to \begin{cases} DM_1(1+\varepsilon_{1,-1}Nt_1) & (n=-1), \\ DM_1(1+\varepsilon_{1,0}N(N+1)/2) & (n=0), \\ DM_1[1+\varepsilon_{1,n}((N+\frac{n+1}{2}\mathrm{Tr}M_1^n + \sum_{k=0}^n \mathrm{Tr}M_1^k \mathrm{Tr}M_1^{n-k})] & (n\geq 1), \end{cases} \tag{6.133}$$

and the transformed potential term:

$$\exp(V_1(M_1)) \to (1 - \varepsilon_{1,n}\mathrm{Tr}V_1'(M_1)M_1^{n+1})\exp(V_1(M_1))$$

we obtain the Dyson-Schwinger equations (for $c = 0$):

$$\mathcal{L}_n^{[1]}(1) Z(t_k, s_k) = 0 \tag{6.134}$$

with:

$$\mathcal{L}_n^{[1]} = \sum_{k=1}^{\infty} kt_k \frac{\partial}{\partial t_{k+n}} + (N + \frac{n+1}{2})\frac{\partial}{\partial t_n} + \sum_{k=1}^{n-1} \frac{1}{2}\frac{\partial}{\partial t_k \partial t_{n-k}}$$
$$+Nt_1\delta_{n,-1} + N(N+1)/2\ \delta_{n,0}, \quad n \geq -1$$

If we take the analytical transformation of general form we get the W_n constraints for the two-matrix model.

$$W_n^{[r]} Z(t_k, s_k, c) = 0, \tilde{W}_n^{[r]} Z(t_k, s_k, c) = 0, \tag{6.135}$$

with

$$\begin{aligned} W_n^{[r]} &= (-c)^n \mathcal{L}_n^{[r]}(1) - \mathcal{L}_n^{[r+n]}(2), \\ \tilde{W}_n^{[r]} &= (-c)^n \mathcal{L}_n^{[r]}(2) - \mathcal{L}_n^{[r+n]}(1), \end{aligned} \tag{6.136}$$

The partition function is also invariant under mixed infinitesimal transformations of the matrices:

$$M_i \to M_i + \varepsilon_{i,nm} M_1^{n+1} M_2^{m+1}, \quad n, m \geq -1, \qquad \varepsilon_{i,nm}, i = 1,2 \text{ infinitesimal} \tag{6.137}$$

6.3.1 Schwinger-Dyson equations in 2-matrix models

Quadratic potential

For the quadratic action $S = t_2 M_1^2 + s_2 M_2^2 - cM_1M_2$ (we can always make a shift in M_1, M_2 to put $t_1 = s_1 = 0$) we have the following Schwinger-Dyson equations:

$$\begin{aligned} 2t_2 W_n - cW_{n-1,1} &= \sum_{j=0}^{n-2} W_j W_{n-j-2} \\ 2s_2 W_{n,1} - cW_{n+1} &= 0 \\ 2t_2 W_{n,1} - cW_{n-1,2} &= \sum_{j=0}^{n-2} W_j W_{n-j-2,1} \\ 2t_2 W_{n,m} - cW_{n-1,m+1} &= \sum_{j=0}^{n-2} W_j W_{n-j-2,m} \end{aligned} \tag{6.138}$$

(Where $W_{n,m} = \langle \mathrm{Tr} M_1^n M_2^m \rangle$ and $W_n = W_{n,0}$). From the first two equations we get :

$$(4t_2 s_2 - c^2) W_n = 2s_2(W_0 W_{n-2} + \dots W_{n-2} W_0) \tag{6.139}$$

We introduce the generating function for correlation functions W_n:

$$G(t) = \sum_{k=0}^{\infty} W_k t^k$$

Using eq. (6.139) the generating function satisfies (we set $W_0 = 1$):

$$t^2 \frac{2s_2}{4t_2 s_2 - c^2} G(t)^2 = G(t) - 1 \tag{6.140}$$

with the solution:

$$\begin{aligned} G(t) = \frac{1}{2t^2} \frac{4t_2 s_2 - c^2}{2s_2} \left(1 - \sqrt{1 - 4t^2 \left(\frac{2s_2}{4t_2 s_2 - c^2} \right)} \right) = \\ = \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!} \left(\frac{2s_2}{4t_2 s_2 - c^2} \right)^n t^{2n} \end{aligned} \tag{6.141}$$

from which we get the non-zero correlation functions:

$$W_{2k,0} = < \mathrm{Tr} M_1^{2k} > = \frac{(2k)!}{k!(k+1)!} \left(\frac{2s_2}{4t_2 s_2 - c^2} \right)^k \tag{6.142}$$

The same for :

$$W_{0,2k} = < \mathrm{Tr} M_2^{2k} > = \frac{(2k)!}{k!(k+1)!} \left(\frac{2t_2}{4t_2 s_2 - c^2} \right)^k . \tag{6.143}$$

From the second, third and fourth eqns. (6.138) we get the $\langle M_1^n M_2 \rangle$, $\langle M_1^n M_2^2 \rangle$ and $\langle M_1^n M_2^3 \rangle$ correlation functions:

$$\begin{aligned} W_{2k+1,1} &= \frac{c}{2s_2} W_{2k+2} \\ W_{2k,1} &= 0 \end{aligned} \tag{6.144}$$

$$\begin{aligned} W_{2k,2} &= \frac{c^2}{4s_2^2} W_{2k+2} + \frac{1}{2s_2} W_0 W_{2k} \\ &= \left(c^2 \frac{3k}{k+2} + 4t_2 s_2 \right) \frac{(2k)!}{k!(k+1)!} \frac{(2s_2)^{k-1}}{(4t_2 s_2 - c^2)^{k+1}} \\ W_{2k+1,2} &= 0 \end{aligned} \tag{6.145}$$

$$\begin{aligned} W_{2k-1,3} &= \left(\frac{c}{2s_2} \right)^3 W_{2k+2} + \frac{c}{2s_2^2} W_0 W_{2k} \\ &= \left(c^2 \frac{k-1}{k+2} + 4t_2 s_2 \right) \frac{(2k)!}{k!(k+1)!} \frac{(2s_2)^{k-1}}{(4t_2 s_2 - c^2)^{k+1}} \frac{c}{2s_2} \\ W_{2k,3} &= 0 \end{aligned} \tag{6.146}$$

Also the two point functions can be incorporated in a generating function:

$$G(t,s) = \sum W_{k,l} t^k s^l$$

which can solve the fourth eq. (6.138):

$$G(t,s) = \frac{2t_2 sG(s) - ctG(t)}{2t_2 s - ct - t^2 sG(t)} \tag{6.147}$$

The symmetry ($s \leftrightarrow t$) is not manifest but can be proved by means of the identity (6.140) for $G(t)$ and $G(s)$.

We can write down also the Schwinger-Dyson equations for the four-point functions $W_{n,m;n',m'} =< \mathrm{Tr} M_1^n M_2^m M_1^{n'} M_2^{m'} >$:

$$\begin{aligned} 2t_2 W_{n,m;n',m'} - cW_{n-1,m+1;n',m'} &= \sum_{j=0}^{n-2} W_j W_{n-2-j,m;n',m'} + \sum_{j=0}^{n'-1} W_{j,m} W_{n+n'-2-j,m'} \\ 2s_2 W_{n',m+m'+1} - cW_{1,m;n',m'} &= \sum_{j=0}^{m-1} W_j W_{n',m+m'-1-j} + \sum_{j=0}^{m'-1} W_j W_{n',m+m'-1-j} \end{aligned} \tag{6.148}$$

In this case the generating function is:

$$\begin{aligned} G(t,s;t',s') &= \sum W_{n,m;n',m'} t^n s^m (t')^{m'} (s')^{n'} = \qquad (6.149) \\ &= \frac{1}{2t_2 s - ct - t^2 sG(t)} \left[(c - t'sG(t',s)) \frac{tt'G(t',s') - t^2 G(t,s')}{t - t'} - \right. \\ & \left. \quad -2t_2 \frac{ss'G(t',s') - s^2 G(t',s)}{s - s'} \right] \end{aligned}$$

The first four-point non-trivial correlation function is :

$$W_{1,1;1,1} = \langle \mathrm{Tr}(M_1 M_2 M_1 M_2) \rangle = \frac{c}{2t_2} W_{1,3} = \frac{2c^2}{(4t_2 s_2 - c^2)^2} \tag{6.150}$$

to be compared with :

$$W_{2,2} = \left\langle \mathrm{Tr}(M_1^2 M_2^2) \right\rangle = \frac{4t_2 s_2 + c^2}{(4t_2 s_2 - c^2)^2} \tag{6.151}$$

Cubic potential

The action is $S = t_3 M_2^3 + t_2 M_1^2 + s_3 M_1^3 + s_2 M_2^2 - cM_1 M_2$ (we can always make a shift in M_1, M_2 to put $t_1 = s_1 = 0$) we have the following Schwinger-Dyson equations:

$$\begin{aligned} 3t_3 W_{n+1} + 2t_2 W_n - cW_{n-1,1} &= \sum_{j=0}^{n-2} W_j W_{n-j-2} \\ 3s_3 W_{n,2} + 2s_2 W_{n,1} - cW_{n+1} &= 0 \\ 3t_3 W_{n+1,1} + 2t_2 W_{n,1} - cW_{n-1,2} &= \sum_{j=0}^{n-2} W_j W_{n-j-2,1} \\ 3t_3 W_{n+1,m} + 2t_2 W_{n,m} - cW_{n-1,m+1} &= \sum_{j=0}^{n-2} W_j W_{n-j-2,m} \end{aligned} \tag{6.152}$$

Writing in terms of the generating function, we get an algebraic third order equation in $G(t)$:

$$t^6G^3(t) - (6t_3 + 4t_2t + \frac{2s_2c}{3s_3}t^2)t^3G^2(t) + \alpha G(t) - \beta = 0 \qquad (6.153)$$

with:

$$\begin{aligned}
\alpha &= (3t_3 + 2t_2t)(3t_3 + 2t_2t + \frac{2s_2c}{3s_3}t^2) - \frac{c^3}{3s_3}t^3 + (3t_3W_1 + 2t_2)t^4 + 3t_3t^3 \\
\beta &= ((3t_3W_1 + 2t_2)t + 3t_3)(3t_3 + 2t_2t + \frac{2s_2c}{3s_3}t^2) - \frac{c^3}{3s_3}t^2 + \\
&\quad +(3t_3W_{1,1} + 2t_2W_1)ct^3 + 3t_3cW_1t^2
\end{aligned}$$

We recover the generating function for quadratic potential in the limit $3s_3, 3t_3 \to 0$ if we pick up only the singular terms proportional with $1/(3s_3)$.

We can express $W_{1,1}$ in terms of W_2, W_3:

$$cW_{1,1} = 3t_3W_3 + 2t_2W_2 - 1$$

From the fourth eq. 6.153 we get the generating function for the 2-point correlation functions:

$$G(t,s) = \frac{(3t_3 + 2t_2t)sG(s) + 3t_3sG_1(s) - cst^2G(t)}{(3t_3 + 2t_2t)s - ct^2 - t^3sG(t)} \qquad (6.154)$$

where:

$$G_m(t) = \frac{1}{m!}\partial_s^m G(t,s)|_{s=0}, \quad G(t,s) = \sum_{m=0}^{\infty} G_m(t)s^m \qquad (6.155)$$

6.4 The W–constraints

This section is devoted to the derivation of the W–constraints in q–matrix models. From both the coupling equations (6.116c) and consistency conditions, we get the W-constraints in the form: $Tr(Q^{n+r}(\alpha)\partial^r_{\lambda_\alpha}(*)) = 0$ where $*$ are the relations (6.113) (For another approach see [79]):

$$\int \prod_{\alpha=1}^{q} d\lambda_\alpha \xi_n(\lambda_1)e^{\mu}\lambda_\alpha\eta_m(\lambda_q) \equiv Q_{nm}(\alpha)h_m = Q_{mn}(\alpha)h_n, \quad \alpha = 1, \ldots, q. \quad (6.156)$$

W-constraints have the form:

$$W_n^{[r]}(\alpha)Z_N(t,c) = 0, \ r \geq 0, n \geq -r; \quad \alpha = 1, \ldots q. \qquad (6.157)$$

or

$$(\mathcal{L}_n^{[r]}(\alpha) - (-1)^r T_n^{[r]}(\alpha)) Z_N(t,c) = 0.$$

involving the interaction operator $T_n^{[r]}$ which depends only on all the couplings $g_{a_1\ldots a_q}$, except $g_{0,\ldots,0,a_\alpha,0,\ldots,0} = t_{\alpha,a_\alpha}$.

For example $T_n^{[1]}$ and $T_n^{[2]}$ are:

$$\begin{aligned} T_n^{[1]}(\alpha) &= a_\alpha g_{a_1\ldots a_q} \frac{\partial}{\partial g_{a_1\ldots,a_\alpha+n,\ldots a_q}} \\ T_n^{[2]}(\alpha) &= a_\alpha a'_\alpha g_{a_1\ldots a_q} g_{a'_1\ldots a'_q} \frac{\partial}{\partial g_{a_1+a'_1\ldots,a_\alpha+a'_\alpha+n,\ldots a_q+a'_q}} + \\ &+ a_\alpha(a_\alpha - 1) g_{a_1\ldots a_q} \frac{\partial}{\partial g_{a_1\ldots,a_\alpha+n,\ldots a_q}} \end{aligned} \tag{6.158}$$

The operator $\mathcal{L}_n^{[r]}(1)$ has the same form as that of the two-matrix model:

$$\mathcal{L}_n^{[r]}(1) = \int dz : \frac{1}{r+1}(\partial_z + J)^{r+1} : z^{r+n} \tag{6.159}$$

where :: is the normal ordering and $J(z)$ is the $U(1)$ current:

$$J(z) = \sum_{k=1}^{p_1} k t_{1,k} z^{k-1} + N z^{-1} + \sum_{k=1}^{\infty} z^{-k-1} \frac{\partial}{\partial t_{1,k}} \tag{6.160}$$

The same expression holds for $\mathcal{L}_n^{[r]}(q)$.

The expression of $\mathcal{L}_n^{[r]}(\alpha), \alpha = 2, \ldots q-1$ is different due to the absence of the P-matrix term:

$$\mathcal{L}_n^{[r]}(\alpha) = \int dz : \frac{1}{r}(\partial_z + V'_\alpha)^r P_\alpha : z^{r+n} \tag{6.161}$$

with

$$\begin{aligned} V'_\alpha &= \sum_{k=1}^{p_\alpha} k t_{\alpha,k} z^{k-1} + N z^{-1}, \\ P_\alpha &= N z^{-1} + \sum_{k=1}^{\infty} z^{-k-1} \frac{\partial}{\partial t_{\alpha,k}} \end{aligned} \tag{6.162}$$

The explicit expression of the first terms is:

$$\begin{aligned} \mathcal{L}_n^{[1]}(\alpha) &= \sum_{k=1}^{\infty} k t_{\alpha,k} \frac{\partial}{\partial t_{\alpha,k+n}} + N t_{\alpha,1} \delta_{n,-1} \\ \mathcal{L}_n^{[2]}(\alpha) &= \sum_{k=1}^{\infty} k(k-1) t_{\alpha,k} \frac{\partial}{\partial t_{\alpha,k+n}} + \sum_{k_1,k_2} k_1 k_2 t_{\alpha,k_1} t_{\alpha,k_2} \frac{\partial}{\partial t_{\alpha,k+n}} + \\ &+ N^2 t_{\alpha,1} \delta_{n,-1} + N(t_{\alpha,1}^2 + 2 t_{\alpha,2}) \delta_{n,-2} \end{aligned}$$

As an example we write down the $W^{[1]}_{-1}, W^{[1]}_0$ and $W^{[1]}_1$ constraints for the three matrix model.

$W^{[1]}$:

$$\begin{aligned}
\sum kt_k \langle \tau_{k-1} \rangle + Nt_1 + c_{12} \langle \lambda_1 \rangle + c_{13} \langle \sigma_1 \rangle &= 0 \\
\sum ku_k \langle \lambda_{k-1} \rangle + Nu_1 + c_{12} \langle \tau_1 \rangle + c_{23} \langle \sigma_1 \rangle &= 0 \\
\sum ks_k \langle \sigma_{k-1} \rangle + Ns_1 + c_{23} \langle \lambda_1 \rangle + c_{13} \langle \tau_1 \rangle &= 0
\end{aligned}$$

$$\begin{aligned}
\sum kt_k \langle \tau_k \rangle + c_{12} \langle \chi_{110} \rangle + c_{13} \langle \chi_{101} \rangle &= -\frac{N(N+1)}{2} \\
\sum ku_k \langle \lambda_k \rangle + c_{12} \langle \chi_{110} \rangle + c_{23} \langle \chi_{011} \rangle &= 0 \\
\sum kt_k \langle \sigma_k \rangle + c_{13} \langle \chi_{101} \rangle + c_{23} \langle \chi_{011} \rangle &= -\frac{N(N+1)}{2}
\end{aligned}$$

$$\begin{aligned}
\sum kt_k \langle \tau_{k+1} \rangle + (N+1) \langle \tau_1 \rangle + c_{12} \langle \chi_{210} \rangle + c_{13} \langle \chi_{201} \rangle &= 0 \\
\sum ku_k \langle \lambda_{k+1} \rangle \qquad\qquad + c_{12} \langle \chi_{120} \rangle + c_{23} \langle \chi_{021} \rangle &= 0 \\
\sum kt_k \langle \sigma_{k+1} \rangle + (N+1) \langle \sigma_1 \rangle + c_{13} \langle \chi_{102} \rangle + c_{23} \langle \chi_{021} \rangle &= 0
\end{aligned}$$

One easily sees from the second group of identities that the limit of pure chain models does not exists for three–matrix models. The same thing holds for odd–q matrix models. However, writing down the W constraints for even–q matrix models, one can see that such a limit exists. This confirms the results obtained with other methods.

6.5 Discrete $1D$-matrix model

6.5.1 Nonzero momentum correlation functions

We calculate the 1-, 2- and 3-point correlation functions in the discrete $c = 1$ matrix model. We show that the 1-point c.f in the momenta space can be represented as a sum of delta functions of the form $\delta(p + 2k\omega_0)$-this means that the action of the operators $\tau_{\alpha,2r} = \mathrm{Tr} M_\alpha^{2r}$ on the vacuum introduce the particles with integer momenta $p = 2k\omega_0$. The 2-point correlation functions are sums over the δ-functions at integer momenta $p = (l - k)\omega_0$, hence we have extended states labeled by 2 indices which are the discrete states of $c = 1$ matrix model. In the Q-matrix approach [81] we have calculated the n-point correlation functions (for $n = 1, 2, 3$) for the q-multimatrix chain model. The 1-, 2- and 3-point correlation functions are given by:

$$\begin{aligned} < \tau_{\alpha,2r} > &= \mathrm{Tr} Q_\alpha^{2r} \\ < \tau_{\alpha,2r}\tau_{\beta,2s} > &= \mathrm{Tr}[(Q_\alpha^{2r})_+, (Q_\beta^{2s})_-], \quad \alpha \leq \beta \\ < \tau_{\alpha,2r}\tau_{\beta,2s}\tau_{\gamma,2t} > &= \mathrm{Tr}[Q_\alpha^{2r} Q_\beta^{2s} Q_\gamma^{2t} - \quad , \quad \alpha \leq \beta \leq \gamma \\ & -((Q_\alpha^{2r})_- Q_\beta^{2s} Q_\gamma^{2t} + Q_\beta^{2s} Q_\gamma^{2t} (Q_\alpha^{2r})_+ + \text{cyclic perm.}) + \\ & +((Q_\alpha^{2r})_- Q_\beta^{2s} (Q_\gamma^{2t})_+ + \text{perm.}) + 2(Q_\alpha^{2r})_+ (Q_\beta^{2s})_+ (Q_\gamma^{2t})_+] \end{aligned} \tag{6.163}$$

The relations above are valid for arbitrary potentials in the multi-matrix model. In relation with $c = 1$ matrix model we will restrict ourselves to the gaussian potentials.

The Q-matrices for a gaussian model take the simple form:

$$Q_\alpha = h_\alpha I_+ + g_\alpha \epsilon_- \tag{6.164}$$

In the work [81] we have already derived the form of Q_α^{2r} in terms of h_α, g_α :

$$Q_\alpha^{2r} = \sum_{k=0}^{r} \sum_{i=0}^{2r-2k} \frac{(2r)!}{(2r - 2k - i)! i! k!} (-1)^k 2^{-k} g^{i+k} h^{2r-(i+k)} I_+^{2r-2k-i} \epsilon_-^i \tag{6.165}$$

For $(Q_\alpha^{2r})_+$ and $(Q_\alpha^{2r})_-$ we have the same formula but the index i in the sum is restricted to take values only from 0 to $r - k$, respectively from $r - k$ to $2r - 2k$.

To calculate the traces of Q-matrices in the relations (6.164), we need to know the following traces $\mathrm{Tr}(I_+^n \epsilon_-^m)$, $Tr(I_+^n \epsilon_-^m I_+^p \epsilon_-^q)$ and $\mathrm{Tr}(I_+^n \epsilon_-^m I_+^p \epsilon_-^q I_+^r \epsilon_-^s)$.

The permutation relation:

$$I_+^n \epsilon_-^m = \sum_{v=0}^{m} \epsilon_-^v I_+^{n-m+v} A_v^{(n,m)}, \qquad A_v^{(n,m)} = \frac{n! m!}{v!(n - m + v)!(m - v)!} \tag{6.166}$$

can be used to calculate the up-mentioned traces.

We collect the needed formulae:

$$\begin{aligned}
\mathrm{Tr}(I_+^n \epsilon_-^m) &= \delta_{nm} n! \begin{pmatrix} N+n \\ n+1 \end{pmatrix} \\
\mathrm{Tr}(I_+^n \epsilon_-^m I_+^p \epsilon_-^q) &= \sum_{v=0}^{m} F(n,m;p,q|v) = \\
&= \sum_{v=0}^{m} \frac{n!m!(q+v)!}{v!(n-m+v)!(m-v)!} \begin{pmatrix} N+q+v \\ q+v+1 \end{pmatrix}, \\
&\quad \text{with } \ n > m, n+p = m+q
\end{aligned} \tag{6.167}$$

$$\mathrm{Tr}(I_+^n \epsilon_-^m I_+^p \epsilon_-^q I_+^r \epsilon_-^s) = \sum_{0 \le l \le q} \sum_{0 \le l' \le m+l} F(n,m;p,q;r,s|l,l') = \tag{6.168}$$

$$= \sum_{l,l'} \frac{p!q!n!(m+l)!(s+l+l')!}{l!l'!(q-l)!(p-q+l)!(n-m-l+l')!(m+l-l')!} \begin{pmatrix} N \\ s+l+l'+1 \end{pmatrix}$$

$$\text{with } \ n+p+r = m+q+s$$

Inserting the relations (6.165) and (6.168) in the defining expressions (6.164), we write the correlation functions explicitly in terms of h_α, g_α:

$$< \tau_{\alpha,2r} >= \sum_{k=0}^{r} \frac{(2r)!(-1)^k 2^{-k}}{k!(r-k)!} \begin{pmatrix} N+r-k \\ r-k+1 \end{pmatrix} (h_\alpha g_\alpha)^r \tag{6.169}$$

$$< \tau_{\alpha,2r} \tau_{\beta,2s} >= \sum_{k=0}^{r} \sum_{l=0}^{s} \sum_{i=0}^{r-k} \sum_{j=s-l}^{2s-2l} \frac{(2r)!}{i!k!(2r-2k-i)!} \frac{(2s)!}{j!l!(2s-2l-j)!} \tag{6.170}$$

$$(-1)^{k+l} 2^{-(k+l)} (g_\alpha h_\beta)^{i+k} (h_\alpha g_\beta)^{j+l} \left(\frac{h_\alpha}{h_\beta}\right)^{r-s} \Big(\sum_{v=0}^{i} F(2r-2k-i, i; 2s-2l-j, j|v) -$$

$$- \sum_{u=0}^{j} F(2s-2l-j, j; 2r-2k-i, i|u)\Big)$$

with $i + j = r + s - k - l$ and where function F is given by relation (6.168).

We omit to write down the formula for 3-point correlation function because of its length.

To simplify the calculations we will consider in what follows only the genus 0 contribution of the correlation functions.

For the 1-point c.f., the genus 0 contribution has the maximum power of N, which is equivalent to setting $k = 0$ in relation (6.169):

$$< \tau_{\alpha,2r} >_0= \frac{(2r)!}{r!(r+1)!} N^{r+1} (h_\alpha g_\alpha)^r \tag{6.171}$$

For the 2-point c.f. the genus 0 constraint imposes the values $v = i-1$, $u = j-1$ in (6.171) which give contribution to the subleading term proportional to N^{i+j}. The leading term proportional with N^{i+j+1} (when $v = i, u = j$) is zero. For genus zero $i+j$ is maximum when $k = l = 0$ and $i+j = r+s$.We use the notation $i = r-n, j = s+n$.

The above considerations permit to write the genus 0 contribution to the 2-point c.f. :

$$< \tau_{\alpha,2r}\tau_{\beta,2s} >_0 = (2r)!(h_\alpha g_\alpha)^r (2s)!(h_\beta g_\beta)^s (-2)\frac{N^{r+s}}{r+s} \sum_{0\leq n\leq min(r,s)} a_n^{(r,s)} \left(\frac{h_\alpha g_\beta}{g_\alpha h_\beta}\right)^n \quad (6.172)$$

with:

$$a_n^{(r,s)} = \frac{1}{(r-n)!(s-n+1)!}\frac{1}{(r+n+1)!(s+n)!}(n(r+s+1)+(s-r)/2) \quad (6.173)$$

The sum is invariant by the transformations $\alpha \leftrightarrow \beta, r \leftrightarrow s, n \leftrightarrow -n$.

In the same way we can calculate the planar contribution to the 3-point correlation functions.

6.5.2 Comparison with the free fermion approach

After these general considerations we consider the special case of the discrete $c = 1$ matrix model.This will permit to find the dependence of the n-point c.f. in terms of the time coordinates for puncture operators .

The $c = 1$ model with discrete time has the partition function:

$$Z = \int dM_i \exp\left[-\frac{\beta}{2}Tr\left(\sum_{i=1}^{q-1}\frac{(M_{i+1}-M_i)^2}{\epsilon} + \epsilon\sum_{i=1}^{q} V(M_i)\right)\right]$$

with a quartic potential $V(M) = M^2 - gM^4$. However, only the contribution near saddle point $V'(M_c) = 0$, where the potential is quadratic in the fluctuation ΔM, is essential

$$V(M) = \frac{1}{4g} - 2\frac{(\Delta M)^2}{\beta}, M = M_c + \frac{\Delta M}{\sqrt{\beta}} \quad (6.174)$$

The new partition function is (up to the constant $\exp(-N\beta\epsilon/(8g))$:

$$Z = \int dM_i \exp\left[Tr\left(\sum_{i=1}^{q}\Delta M_i^2(2\epsilon - \frac{1}{\epsilon}) + \frac{1}{\epsilon}\sum_{i=1}^{q-1}\Delta M_i \Delta M_{i+1}\right)\right]$$

It represents a string theory on circle with radius $R \sim \frac{1}{\epsilon}$.

The coefficients h_α, g_α can be expressed in terms of the determinant D_n of the $n \times n$ matrix :

$$h_\alpha = (-1)^\alpha D_{\alpha-1}, g_\alpha = (-1)^\alpha \frac{D_{q-\alpha}}{D_q} \tag{6.175}$$

where:

$$D_n = \begin{vmatrix} u & 1 & 0 & \dots & \dots & 0 \\ 1 & u & 1 & 0 & \dots & 0 \\ 0 & 1 & u & \ddots & . & . \\ . & 0 & 1 & \ddots & 1 & 0 \\ 0 & \dots & 0 & \ddots & u & 1 \\ 0 & & \dots & 0 & 1 & u \end{vmatrix}$$

We have introduced the parameter $u = 2(2\epsilon^2 - 1)$. In the region $-2 \leq u \leq 2$ the determinant D_n has the simple representation (outside this region the sin–function is replaced by sinh):

$$D_n = \frac{\sin(n+1)\omega}{\sin\omega} \tag{6.176}$$

where $\omega = \arctan\sqrt{(2/u)^2 - 1}$.

In the limit $\epsilon \to 0$, we have $\cos\omega = -1 + 2\epsilon^2, \sin\omega = 2\epsilon$, hence $\omega \sim m\pi - 2\epsilon \to m\pi$. Instead the discrete variable $\alpha = 1 \dots q$ we define the continuous variable $\frac{\alpha}{q} = t \in [0, T]$. Also we must define the rescaled pulsation ω_0 such that $\alpha\omega \to t\omega_0, (q-\alpha)\omega \to (T-t)\omega_0$.

Due to the limit $q\omega \to T\omega_0 = 2\pi$ we can achieve the continuum limit $q \to \infty, \epsilon \to 0$ and maintaining fixed the product $q\epsilon = \pi$. Hence in the continuum limit the period and the pulsation behaves as $T \sim q \to \infty$, $\omega \sim 2\epsilon \to 0$.

We can now calculate:

$$(h_\alpha g_\alpha)^r = \left(\frac{\sin\alpha\omega \sin((q+1)-\alpha)\omega}{\sin\omega \sin(q+1)\omega} \right)^r \tag{6.177}$$

In the continuum limit $\epsilon \to 0$ we have:

$$(h_\alpha g_\alpha)^r \to \left(\frac{\sin 2t\omega_0}{4\epsilon} \right)^r \tag{6.178}$$

We can make an expansion in periodic functions:

$$(h_\alpha g_\alpha)^r = \sum_{l=0}^{2r} d_l^{(r)} e^{4it\omega_0(r/2-l)} \tag{6.179}$$

$$\text{with } d_l^{(r)} = (8\epsilon)^{-r} \begin{pmatrix} r \\ l \end{pmatrix} (-1)^{r/2-l} \tag{6.180}$$

We perform a Fourier transform to get the 1-point CF in momentum space.

But we have 2 problems. The first problem is that the integration is in the interval $[-T/4, T/4]$ (we assume the contribution from only one top of the inverted harmonic potential), and we have periodic functions which have the period nT. This problem is resolved by observing that in the continuum limit $T = 2\pi/\omega_0 \to \infty$, and that in the continuum limit all functions are periodic in the interval $[-T/4, T/4] \to (-\infty, -\infty)$.

The second problem is that the coefficient $d_l^{(r)} \sim (-1)^{r/2}$ for r odd is complex. Hence we must distinguish to cases when r is odd and even. When r is even, the operator $\tau_{2r}(t)$ is:

$$\tau_{2r}(t) \sim (8\epsilon)^{-r} \sum_{l=0}^{r} \binom{r}{l} (-1)^{r/2-l} e^{4it\omega_0(r/2-l)}$$

with the real part:

$$\tau_{2r}(t) \sim (8\epsilon)^{-r} \sum_{l=0}^{r} \binom{r}{l} (-1)^{r/2-l} \cos(4t\omega_0(r/2-l))$$

Instead for r odd the operator $\tau_{2r}(t)$ is multiplied with i and the real part is:

$$\tau_{2r}(t) \sim (8\epsilon)^{-r} \sum_{l=0}^{r} \binom{r}{l} (-1)^{r/2-1-l} \sin(4t\omega_0(r/2-l))$$

Now we can use the result (6.180) in the relation (6.171), and the 1-point CF in the momentum space is:

$$\begin{aligned} < \tau_{2r} >_0 (p) &= \int_{-T/4}^{T/4} dt < \tau_{\alpha,2r}(t) >_0 e^{ipt} = \\ &= \frac{(2r)!}{r!(r+1)!} N^{r+1} \sum_{l=0}^{r} d_l^{(r)} \delta(p + 4\omega_0(\frac{r}{2} - l)) \end{aligned} \qquad (6.181)$$

This result is correct when r is even. Instead, when r is odd the momentum is shifted by $\pi/2$.

In the free fermion method (see [80, 66]), the $c = 1$ matrix model is considered equivalent with a system of free fermions in the harmonic inverted potential. The Liouville mode is interpreted as the classical time τ of flight variable.

The period of the oscillations of the classical particle moving in the given potential is related with the cosmological constant μ: $T = 2\sqrt{\beta}|\log \mu|$, where β is the string coupling.

The equation of motion at the Fermi surface is:

$$\frac{d\lambda}{d\tau} = \frac{1}{2}\sqrt{g(\mu_F - V(\lambda))}$$

where the potential is quartic $V(\lambda) = \lambda^2 - g\lambda^4$.

Expanding near the saddle point $\lambda = \lambda_c + x/\sqrt{\beta}$ we get the equation:

$$\frac{dx}{d\tau} = \sqrt{g(\frac{2x^2}{\beta} - \mu)}$$

with the solution:

$$x(\tau) = \sqrt{\beta/2}\mu^{1/2} cosh(\tau/\sqrt{g\beta/8})$$

The puncture operator in 2D gravity is in this case:

$$O_r = x^r = (\beta/2)^{r/2}\mu^{r/2} cosh^r(\tau/\sqrt{g\beta/8}) \tag{6.182}$$

We can observe that the operator O_r behaves in the same way as the operator $\tau_r(t)$ because this operator has the correlation function:

$$< \tau_{2r}(t) >= \frac{(2r)!}{r!(r+1)!} x^{r+1} (\frac{\sin 2t\omega_0}{4\epsilon^2})^r \tag{6.183}$$

with $x = n/N$.

We can identify the variables (putting for simplicity $g = 1$):

$$\epsilon \sim 1/\sqrt{\beta}, \mu \sim x \sim N^{-1} \tag{6.184}$$

At the critical point, the cosmological constant $\mu \to 0$, which means that $N \to \infty$.

The time t is related via Lorentz rotation with the flight time $t \sim i\tau$, up to a translation by $T/2$.

To calculate 2-point c.f. we need:

$$h_\alpha^{r+n} g_\alpha^{r-n} = \frac{\sin^{r+n} t\omega_0 \cos^{r-n} t\omega_0}{(2\epsilon)^{r+n}} = \sum_{l=0}^{2r} A_l^{(r,n)} e^{2it\omega_0(r-l)} \tag{6.185}$$

$$A_l^{(r,n)} = (8\epsilon)^{-(r+n)}(\epsilon)^{-n} \sum_{l'=r-n}^{r+n} \begin{pmatrix} r+n \\ l' \end{pmatrix} \begin{pmatrix} r-n \\ l-l' \end{pmatrix} (-1)^{\frac{r+n}{2}-l'}$$

In the same way we can write:

$$h_\beta^{s-n} g_\beta^{s+n} = \sum_{k=0}^{2s} \sum_{k'=s+n}^{s-n} A_k^{(s,-n)} e^{2it'\omega_0(s-k)}$$

Using the expression (6.185) in the relation (6.172) we get the 2-point c.f. :

$$< \tau_{\alpha,2r}\tau_{\beta,2s} > \; = \; -2(2r)!(2s)! \frac{N^{r+s}}{r+s} \sum_{k=0}^{2s} \sum_{l=0}^{2r} \mathcal{A}_{kl} \exp i[2t\omega_0(r-l) + 2t'\omega_0(s-k)] \tag{6.186}$$

where :

$$\mathcal{A}_{kl}^{(n)} = \sum_{0\leq n\leq min(r,s)} a_n^{(r,s)} A_l^{(r,n)} A_k^{(s,-n)} \tag{6.187}$$

with $a_n^{(r,s)}$ given by relation (6.173).

The 2-point c.f. can be rewritten:

$$< \tau_{\alpha,2r}\tau_{\beta,2s} >= -2(2r)!(2s)!\frac{N^{r+s}}{r+s}\sum_{k,l}\mathcal{A}_{kl}e^{i\omega_0\Delta t(r-s-(l-k))}e^{i\omega_0\Delta t(r+s-(l+k))}$$

where $\Delta t = t - t'$.

The 2-point c.f. in the momentum space has the form:

$$< \tau_{\alpha,2r}\tau_{\beta,2s} > (p,P) = \int\int_{-T/2}^{T/2} d(\Delta t)\, d(t+t') < \tau_{\alpha,2r}(t)\tau_{\beta,2s}(t') > e^{ip\Delta t}e^{iP(t+t')} \tag{6.188}$$
$$= 4(2r)!(2s)!\frac{N^{r+s}}{r+s}\sum_{l,k}\mathcal{A}_{kl}\delta(p+(r-s-(l-k))\omega_0)\delta(P+(r+s-(l+k))\omega_0)$$

We have extended states at $p = (r-s-(l-k))\omega_0$ (representing discrete states) where $p = -r-s\ldots r+s$. If $t = 0$ we have extended states at $p = r - l\omega_0$ (representing pure tachyons) where $p = -r\ldots r$.

We considered previously the limit $\epsilon \to 0$, or the plain limit when the radius R goes to infinity.

Now we study the self-dual point where $R \sim 1/R$.This corresponds to the case $\epsilon \to 1$.In this limit $\omega = \pi/2 - 2(\epsilon - 1) \to \pi/2$

Instead of ϵ and ω is better to define ϵ' and ω' as:$\epsilon = 1+\epsilon', \omega = \pi/2 - \omega'$. We can take the continuous limit $q \to \infty$ such that $q\omega' = \frac{\pi}{2} = \omega_0$.The variable $t = \frac{\alpha-1}{q}$ is defined as before.

The 1-point c.f remains unchanged at the new limit–the self-dual point.The 2-point c.f. is still given by the formula (6.186) but with $\mathcal{A}_{k,l}$ given by:

$$\mathcal{A}_{k,l} = \sum_{0\leq n\leq min(r,s)} a_n^{(r,s)} A_l^{(r,(-1)^{\alpha-1}n)} A_k^{(s,-(-1)^{\beta-1}n)} \tag{6.189}$$

This new restriction gives the same poles as before (6.189) at $p = \pm i(l-k)\omega_0$, with $l-k$ odd, but only if α, β are both even or odd. In the cases when α is odd, β is even or vice versa, we have the same poles but with $l-k$ even.

For consistency with the case $t = 0$ where we have the poles $p = \pm ik\omega_0$ with k even, we could conclude that we must have only poles with $l-k$ even. This condition imposes that α is odd, β is even or vice versa. This is in agreement with the fact that exactly at the self-dual point the 2-point c.f without momentum do not vanish only if α is odd and β is even or vice versa. Hence discrete states appear as poles in the 2-point c.f. only if the free particles of $c = 1$ matrix model belong to 2 distinct classes with α odd, respectively even. But this is equivalent to the choosing of a special sl_2 subalgebra embedding of the larger algebra sl_n which characterizes the $c = 1$ matrix model. This also could explain why the discrete states satisfy the sl_2 algebra.

References

[1] H. Georgi and S. Glashow. *Phys. Rev. Lett.*, 28:1494, 1972.

[2] G. 'tHooft. *Nucl. Phys.*, B79:2766, 1974.

[3] A.M. Polyakov. *JEPT Lett.*, 20:894, 1974.

[4] P.A.M. Dirac. *Physical Review*, 74:817, 1948.

[5] G. 't Hooft. Topology of the gauge condition and new confinement phases in non-abelian gauge theories. *Nuclear Physics*, B190:455–478, 1981.

[6] H.B. Nielsen and P. Olesen. *Nucl. Phys.*, B61:45, 1973.

[7] S. Weinberg. *Progr. of Teor. Phys. Suppl.*, 86:43, 1986.

[8] G. 't Hooft. in "high energy physics", eps international conference. Palermo 1975, ed. A. Zichichi.

[9] S. Mandelstam. *Phys. Rep.*, 23C:245, 1976.

[10] G. Parisi. *Phys. Rev.*, D11:971, 1975.

[11] H. Toki, H. Suganuma, S. Sasaki, and H. Ichie. Dual ginzburg-landau theory for quark confinement and dynamical chiral-symmetry breaking. (hep-ph/9502277).

[12] H. Toki, H. Suganuma, S. Sasaki, and H. Ichie. Dual ginzburg-landau theory for nonperturbative qcd. (hep-th/9506366).

[13] A.M. Polyakov. Compact gauge fields and the infrared catastrophe. *Physics Letters*, 59B(1):82, 1975.

[14] A.M. Polyakov. Quark confinement and topology of gauge theories. *Nuclear Physics*, B120:429–458, 1977.

[15] M. Atiyah and N. Hitchin. *The geometry and dynamics of magnetic monopoles.* Princeton University Press, 1988.

[16] Adriano Di Giacomo. Monopole condensation in gauge theory vacuum. (Talk given at Lattice 95?).

[17] Sumit R. Das and Spenta R. Wadia. Quark confinement in 2+1 dimensional pure ym theory. (hep-th/9503184).

[18] G. 't Hooft. *Nucl. Phys.*, B72:461, 1972.

[19] L.G. Jaffe. *Rev. Mod. Phys.*, 54:407, 1982.

[20] V.A. Kazakov and T. Winter. Large n phase transition in the heat kernel on the $u(n)$ group. (LPTENS-94/28), 1994.

[21] Spenta R. Wadia and Sumit R. Das. Topology of quantum gauge fields and duality (i): Yang-mills-higgs system in 2+1 dimensions. *Physics Letters*, 106B:386, 1981.

[22] M.R. Douglas and S.H. Shenker. Dynamics of $su(n)$ supersymmetric gauge theory. (hep-lat/9503163).

[23] M. Oleszczuk. Mass generation in three dimensions. (hep-th/9412049).

[24] C. Callias and C.H. Taubes. Functional determinants in euclidean yang-mills theory. *Comm. Math Phys.*, 77:229, 1980.

[25] D.I. D'yakonov, V.Yu. Petrov, and A.V. Yung. Quasiclassical expansion in external yang mills fields and the approximate calculation of functional determinants. *Sov. J. Nucl. Phys.*, 39:150, 1984.

[26] K. Zarembo. Monopole determinant in ym theory at finite temperature. (hep-th/9510031).

[27] G. t́Hooft and M. Veltman. *Nucl. Phys.*, B44:189, 1972.

[28] Ling-Fong Li. Group theory of the spontaneously broken gauge symmetries. *Physical Review D*, 9(6):1723, 1974.

[29] A. Schwarz. *Topology for Physicists.* Springer Verlag, 1993.

[30] J. Arafune, P.G.O. Freund, and C.J. Goebel. Topology of higgs fields. *Journal of Mathematical Physics*, 16(2):433–437, February 1975.

[31] P. Goddard and D. Olive. Magnetic monopoles in gauge field theories. *Rep. Prog. Phys.*, 41:1357, 1978.

[32] David Wilkinson and Alfred S. Goldhaber. Spherically symmetric monopoles. *Physical Review D*, 16(4):1221, 1977.

[33] Y. Brihaye and J. Nuts. Magnetic monopoles in $su(4)$ gauge theories. *Journal of Mathematical Physics*, 18(11):2177, 1977.

[34] A. Sinha. Su(3) magnetic monopoles. *Physical Review D*, 14(8):2016, 1976.

[35] E. Corrigan, D.I. Olive, D.B. Fairlie, and J. Nuyts. Magnetic monopoles in $su(3)$ gauge theories. *Nuclear Physics*, B106:475, 1976.

[36] Tai Tsun Wu and Chen Ning Yang. Concept of nonintegrable phase factors and global formulation of gauge fields. *Physical Review D*, 12(12):3845, 1975.

[37] F. Englert and P. Windley. Quantization condition for 't hooft monopoles in compact simple lie groups. *Physical Review D*, 14(10):2728, 1976.

[38] M.K. Prasad and C.M. Sommerfeld. Exact classical solution for the 't hooft monopole ad the julia-zee dyon. *Phys. Rev. Lett.*, 35(12):760, 1975.

[39] Conor J. Houghton and Paul M. Sutcliffe. Tetrahedral and cubic monopoles. (hep-th/9601146).

[40] Arthur Jaffe and Clifford Taubes. *Vortices and Monopoles.* Birkhauser, 1980.

[41] A.M. Polyakov. *Gauge fields and strings.* Harwood Academic Publisher, 1987.

[42] M. Sher. Electroweak higgs potentials and vacuum stability. *Physics Reports*, 179:273, 1989.

[43] G. 'tHooft. *Nucl. Phys.*, B33:823, 1971.

[44] G. 'tHooft. *Nucl. Phys.*, B35:167, 1971.

[45] B.W. Lee. *Phys. Rev. D*, 5:823, 1972.

[46] B.W. Lee and J. Zinn-Justin. *Phys. Rev. D*, 5:3121,3137,3155, 1972.

[47] T.D. Lee and C.N. Yang. *Phys. Rev.*, 128:885, 1962.

[48] Giuseppe Bimonte, Roberto Iengo, and G. Lozano. Uv asymptotycally free qed as broken ym theory in the unitary gauge. (hep-th/9502185).

[49] T. Appelqyust and H. Quinn. Divergence cancellations in a simplified weak interaction model. *Phys. Lett.*, 39B(2):229, 1972.

[50] E. Brezin, C. Itzykson, G. Parisi, and J.B. Zuber. Planar diagrams. *Commun. Math. Phys.*, 59:35–51, 1978.

[51] T. Eguchi and H. Kawai. *Phys. Rev. Lett.*, 48:1063, 1982.

[52] G. Parisi. *Phys. Lett.*, 112B:463, 1982.

[53] G. Bahnot, U. Heller, and H. Neuberger. The quenched eguchi-kawai model. *Phys. Lett.*, 113B:47–50, 1982.

[54] D. Gross and Y. Kitazawa. A quenched momentum prescription for large-n theories. *Nuclear Phys.*, B206:440–472, 1982.

[55] S.M. Das. Some aspects of large-n theories. *Review of Modern Physics*, 59(1):235–261, 1987.

[56] I. Bars, M. Gunaydin, and S. Yankielowicz. Reduced chiral $u(n) \times u(n)$ model. *Nucl. Phys.*, B219:81–115, 1983.

[57] Michael R. Douglas. Conformal field theory techniques for large n group theory. (hep-th/9303159).

[58] P. Menotti and E. Onofri. The action of $su(n)$ lattice gauge theory in terms of the heat kernel on the group manifold. *Nucl. Phys.*, B190:288–300, 1981.

[59] J.A. Minahan and A.P. Polychronakos. Classical solutions for two dimensional qcd on the sphere. (hep-th/9309119).

[60] A. Matytsin. On the large n limit of the itzykson-zuber integral. (PUPT-1405), 1996.

[61] D. Gross and E. Witten. Possible third order phase transition in the large n lattice gauge theory. *Phys. Rev.*, 21(2):446, 1980.

[62] F. David, *Nucl. Phys.* B257 (1985) 45.

[63] V.A.Kazakov, *Phys. Lett.* B150 (1985) 282.

[64] J.Fröhlich, in: **Lecture Notes in Physics**, Vol. 216, Springer, Berlin, 1985; J.Ambjørn, B. Durhuus and J.Fröhlich, *Nucl. Phys.* B257[FS14](1985) 433.

[65] D. Gross, Lectures at Jerusalem Winter school, (1990)

[66] P. Ginsparg and Gregory Moore, Lectures on 2D gravity and 2D string theory, *hep-th/9304011*

[67] D. Boulatov and V.A. Kazakov, One-dimensional string theory with vortices and the upside-down matrix oscillator, *LPTENS 91/24, KUNS 1094 HE(TH) 91/14*, August 1991

[68] L.Bonora and C.S.Xiong, *Nucl.Phys.*, **B405** (1993), 191.

[69] M. Douglas, *Phys.Lett.* **B238** (1990) 176.

[70] P.Di Francesco, D.Kutasov, *Nucl.Phys.* **B 342**(1990)589

[71] J.De Boer, *Nucl.Phys.* **B336** (1991) 602.

[72] I.Kostov, *Phys.Lett.* **297B** (1992) 74.

[73] C.Ahn and K.Shigemoto, Multi–matrix model and 2D Toda multi–component hierarchy, *CLNS 91/1054.*

[74] S.Kharchev, A.Marshakov, A.Mironov, A.Morozov, *Nucl.Phys.* **B397**(1993)339

[75] D. Bessis, C. Itzykson, and J.-B. Zuber, *Adv. Appl. Math.* **1**(1980) 109.

[76] C.Itzykson and B.Zuber, *J.Math.Phys.* **21** (1980) 411.

[77] M.L.Mehta, *Commun.Math.Phys.* 79 (1981) 327. S. Chadha, G. Mahoux and M. Mehta, *J. Phys. A: Math. Gen.* **14**(1981)579.

[78] K.Ueno and K.Takasaki, Toda Lattice Hierarchy, *Adv.Stud.Pure Math.* **4** (1984) 1.

[79] H.Itoyama, Y.Matsuo, *Phys.Lett.* **B262**(1991), 233

[80] D.Gross *The* $c = 1$ *matrix model*, Jerusalem Winter School 1990.
I.Klebanov, *String theory in two dimensions*, Proc. Trieste Spring School, 1991.
A.Jevicki, *Developments in 2D string theory*, Trieste lectures, April 1993.

[81] L. Bonora, F. Nesti and E. Vinteler, Multimatrix models: Integrability properties and topological content, *Int.J.Mod.Phys.* **A11** (1996) 1797-1830, hep-th/9506124

[82] F. Nesti, SISSA - 143/96/EP, to be submitted