SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
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Vanishings for the cohomology of stable rank-two vector bundles and reflexive sheaves on the projective space

Thesis submitted for the degree of

"Doctor Philosophiæ"

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September 1994

TRIESTE

Index

0 Introduction

We work over an algebraically closed field with characteristic zero.

One of the most important invariant structures one can attach to a vector bundle E on $\mathbf{P}^r$ is its first cohomology module $M_E = H^1_*(E)$. By theorem B of Serre

$$H^1(E(k)) = 0 \qquad\qquad if \quad k >> 0 \qquad\quad (*),$$

moreover, by the Serre duality,

$$H^1(E(k)) = H^{r-1}(E^*(-k-r-1)) = 0 \qquad if \quad k << 0 \qquad\quad (**)$$

hence M_E is a graded module of finite length. A lot of interesting properties are known for M_E (see [Ho], [BH], [R], [D] and ref. therein), for instance, Horrocks (see [Ho]) proved that M_E completely determine the bundle E and gave a technique to explicitly reconstruct a vector bundle E from its first cohomology module M_E.

One of the fundamental questions raised since the appearance of the Horrocks paper concerns what kinds of properties a finite-length and graded module M should have in order to be the first cohomology module of some vector bundle on a projective space. In the case of rank-two vector bundle on $\mathbf{P}^2$ this problem was solved by Rao who was able to characterize the graded modules which comes from vector bundles in terms of constraints on their minimal free resolutions. Furthermore, similar but weaker results are known for rank-two vector bundles on $\mathbf{P}^3$ (see [R] and [D]). In this case it is known that in order that a graded module M is the first-cohomology module of some vector bundle E on $\mathbf{P}^3$, it is necessary that for any minimal free resolution of M one can construct a very special kind of monad whose cohomology represents the bundle (see chapter III of this thesis for a resume of this construction).

Another fundamental problem concerning the module M_E is the estimate of its "size". In other terms the problem consists in the determination of the least absolute value the number k should have in order to (*) and (**) above hold. More precisely, let E be a vector bundle (more generally, a reflexive sheaf) on $\mathbf{P}^3$, if c_1 and c_2 are the Chern classes of E (c_1, c_2 and c_3 if E is a reflexive sheaf), we ask for the solution to the following

Problem A: find two functions f_1 and f_2 of c_1 and c_2 (of c_1, c_2 and c_3) such that for every E holds

$$H^1(E(k)) = 0 \quad if \quad k > f_1(c_1, c_2) \quad (+)$$
$$H^1(E(k)) = 0 \quad if \quad k < f_2(c_1, c_2) \quad (++)$$

The problem of determination of f_2 was completely solved when Barth-Elencwajg [BE] introduced the notion of spectrum of a stable vector bundle with $c_1 = 0$ and Hartshorne [H] generalized it to reflexive sheaves with any c_1.

Let us recall the following theorems from [H] and [E] (and ref. therein)

Theorem : Let E be a rank two reflexive sheaf with $c_1 = 0$ or -1 on $\mathbf{P}^3$, and assume $H^0(E(-1)) = 0$. Then there exists an unique set of integers $\{k_i\}_{i=1}^{c_2}$ called the spectrum of E, with the following properties (where H denotes the sheaf $\bigoplus O(k_i)$ on $\mathbf{P}^1$):

$$h^1(\mathbf{P}^3, E(k)) = h^0(\mathbf{P}^1, H(l+1)) \quad for \quad k \leq -1$$
$$h^2(\mathbf{P}^3, E(k)) = h^1(\mathbf{P}^1, H(l+1)) \quad for \quad k \geq -3 \quad if \quad c_1 = 0 \quad and \quad l \geq -2 \quad if \quad c_1 = -1$$

Proposition: If E is stable, the spectrum is connected and $\{-k_i\} = \{k_i + c_1\}$.

Combining the two statements it turns out that

$$h^1(E(k)) = 0. \qquad\qquad k < [-\frac{c_2}{2}]$$

and this result is sharp (see sec. 8 of [H]) hence the problem of determining the best f_2 for which (++) holds is solved by setting $f_2 := -[\frac{c_2}{2}]$.

On the contrary, the problem of finding the best function f_1 such that (+) holds, is harder and, as far as we know, widely open.

In this thesis we will try to gain a better understanding of this problem by using one of the most powerful techniques in the study of vector bundles (resp. reflexive sheaves) on $\mathbf{P}^n$ i.e. a method "alla Castelnuovo" which can be described as the attempt of deducing relevant properties of the bundle starting from the behaviour of its restriction to a general hyperplane. Similarly to what happens in the study of space curves, a lot of interesting properties of the bundles on $\mathbf{P}^n$ can be proven with an inductive argument starting from the analysis, which is hopefully simpler, of the two dimensional case.

As we are interested in vanishing theorems for stable vector bundles and reflexive sheaves, the possibility of begin such an inductive argument rests ultimately on the invariance of the stability property when restricting to a general hyperplane. Moreover we need a vanishing theorem, possibly sharp, which covers the two dimensional case.

As regards to the first question, it is completely solved by the so called "Barth's restriction theorem"which says that, with the only exception of vector bundles with $c_1 = 0$ and $c_2 = 1$, the general plane restriction of a stable reflexive sheaf on $\mathbf{P}^3$ is stable:

Theorem (Barth's restriction theorem): Let E be a stable rank two vector bundle on $\mathbf{P}^n$ with $n \geq 3$. Suppose $(c_1, c_2) \neq (0, 1)$. Then the restriction E_H to a general hyperplane H is also stable.

On the other hand, the vanishing we need for vector bundles on $\mathbf{P}^2$ is provided by a theorem of Hartshorne which says that, given a stable rank two vector bundle $\mathcal{E}$ on $\mathbf{P}^2$, we have $h^1(\mathcal{E}(t)) = 0$ if $t \geq c_2 - 2$ (and this bound is sharp).

Theorem (Hartshorne): Let $\mathcal{E}$ be a stable rank two vector bundle on $\mathbf{P}^2$ with Chern classes c_1 and c_2. Set $t := min\{k : \ h^0(\mathcal{E}(k)) \neq 0\}$. Then $h^1(\mathcal{E}(l)) = 0$ for $l \geq c_2 - t^2 - 1 - c_1 t$.

In the first chapter we will give a short proof of these theorems.

The first application of the method alla Castelnuovo described above, was given by Gurrola who proved in [G] the following

Theorem : Let E be a rank two, normalized, stable reflexive sheaf over $\mathbf{P}^3$ with Chern classes c_1, c_2 and c_3. Then

$$h^1(E(l)) = 0 \qquad\qquad if \quad l \geq \frac{1}{2}c_2^2 - \frac{1}{2}(c_1 - 1)c_2 - 1 - \frac{1}{2}c_3$$

This estimate turned out to be optimal in the case of high c_3. Indeed Gurrola was able to prove that the bound above is sharp for stable reflexive shaves with maximal c_3. On the other hand, as pointed out by Ellia in [E], the result of Gurrola is not sharp when $c_3 = 0$ i.e. when the sheaf is indeed a vector bundle. The progress in this direction given by Ellia is contained in the following (see [E])

Theorem: Let E be a stable, rank two vector bundle on $\mathbf{P}^3$, with Chern classes $c_1 = 0$, $c_2 \geq 36$. Then $h^1(E(k)) = 0$ if $k \geq (c_2^2 + c_2 - m^2 + m)/2$, where $m := min\{n : \quad n \geq \frac{c_2^{\frac{1}{2}}}{4}\}$.

In the fourth chapter of this thesis, we will further improve these results by reducing the coefficient of the quadratic part of the last bound and by estending the result to stable reflexive sheaves with any c_1 and $c_3 \leq c_2(\frac{c_2}{3} + 5 + 2c_1)$. Indeed we will be able to prove the following

Theorem I: Let F be a stable rank two reflexive sheaf on $\mathbf{P}^3$ with $c_2 \geq 35$ and $c_3 \leq c_2(\frac{c_2}{3} + 5 + 2c_1)$. Set $m := [\frac{c_2}{3}]$. Then $h^1(F(k)) = 0$ if $k \geq \frac{c_2^2}{2} + \frac{c_2}{2} + \frac{m(1-m)}{2} - c_1(c_2 - m) + \frac{c_1 c_2}{2} - \frac{c_3}{2}$.
$\diamondsuit$

The starting point of our analysis is the following proposition which represents a stronger version of Corollary 2.2 of [G] and which will be proved in the second chapter of this thesis:

Theorem II: Let E be a stable rank-two reflexive sheaf on $\mathbf{P}^3$. Suppose there exists an integer $x > 0$ s.t. $h^1(F_H(k)) = 0$ if $k \geq x$ and H is a general plane. Then $h^1(F(x)) \leq x\{c_2 - \frac{x+3}{2} - c_1\} + c_2(1 + \frac{c_1}{2}) - \frac{c_3}{2}$ and $h^1(F(t)) = 0$ if $t \geq x\{c_2 - \frac{x+1}{2} - c_1\} + c_2(1 + \frac{c_1}{2}) - \frac{c_3}{2}$.
In the light of the proposition above, the following question becomes crucial: let E be a stable, rank two, normalized vector bundle on $\mathbf{P}^3$, and let H be a general plane; what can be said about the cohomology of E_H? In other words, we would like to solve the following

Problem B : Find a function $g(c_2)$ s.t.

$$h^1(E_H(p)) = 0 \qquad if \quad p \geq g(c_2).$$

Combining the Barth's restriction theorem and the result of Hartshorne stated above, we have a first estimate for the function $g(c_2)$: $g(c_2) := c_2 - 2$.

Rather surprisingly, as pointed out by Ellia in [E], the vanishing thus obtained is not sharp. The point is that a set of d points in $\mathbf{P}^2$ which does not impose independent conditions to curves of "high" degree has a particular geometrical configuration. For instance, the worst case is when the d points lie on a line, and the next case is when there are $d - 1$ points on a line. Now, if $h^1(E_H(k)) \neq 0$ for big k and for every plane H, it follows that for most planes, E_H has a section vanishing along a set of points containing many points on a line such that this line is a high order jumping line for E_H. From these "super" jumping lines it is possible to build an unstable plane of high order for E. With this argument Ellia was able to show that $h^1(E_H(k)) = 0$ if $k \geq g(c_2) := c_2 - m$ where m is the round-up of $\frac{c_2^{\frac{1}{2}}}{2}$

In the fourth chapter we will improve the argument above and we will be able to show that all the super jumping lines are constrained to lie in a fixed and very unstable plane. The final result of that chapter will be the Theorem I stated before. Moreover in the course of our analysis we will gain a better estimate for the function g of the problem B by showing that $g(c_2) \leq \frac{2}{3}c_2$ (see theorem I.8 of the chapter 4).

But it is only in the last chapter that we will succeed in the attempt of the determination of the best shape for the function g solving Problem B at least in the case $c_1 = 0$ and $c_2 \geq 20$. Indeed we will prove the following (see the introduction of the last chapter)

Theorem III: let E be a stable rank two vector bundle on $\mathbf{P}^3$ s.t. $c_1(E) = 0$ and $c_2 \geq 20$. Suppose H to be a general plane. Then $h^1(E_H(p)) = 0$ if $p > \frac{c_2(E)-3}{2}$. Moreover, if $c_2(E)$ is odd and $h^1(E_H(\frac{c_2(E)-3}{2})) \neq 0$ for any plane H of $\mathbf{P}^3$, then E belongs to one of the following classes:

a) E is a t'Hooft bundle associated to $c_2 + 1$ skew lines lying on a quadric of $\mathbf{P}^3$;

b) the spectrum of E is maximal, hence $E(1)$ has a section which is a multiplicity 2 structure Y on a degree $\frac{c_2-1}{2}$ plane curve Y_0 s.t. $\omega_{Y^\cdot} \simeq O_{Y^\cdot}(-2)$.

Combining theorems II and III we will get

Theorem IV: Let E be a stable rank two vector bundle on $\mathbf{P}^3$ with $c_1 = 0$ and $c_2 \geq 20$. Then $h^1(E(k)) = 0$ if $k \geq \frac{c_2^2}{2} + \frac{c_2}{2} + \frac{m(1-m)}{2} - c_1(c_2 - m) + \frac{c_1 c_2}{2}$, where $m = [\frac{c_2-1}{2}]$ $\diamondsuit$.

A natural question, now, is how far from giving a sharp answer to Problem A theorem IV is. To understand this, let us briefly review what happens in the case of integral space curves. Let C be an integral and non degenerate curve in $\mathbf{P}^r$ $r \geq 3$, and let I_C be its sheaf of ideals in $\mathbf{P}^r$. If $0 \to I_C \to O_{\mathbf{P}^3} \to O_C \to 0$ is the defining sequence of C, it is clear that the hypersurfaces of degree n in $\mathbf{P}^r$ trace out a complete linear system on C if and only if $h^1(I_C(n)) = 0$. In their paper [GLP], Gruson, Lazarsfeld and Peskine proved the following

Theorem (GLP): Let $C \subset \mathbf{P}^r$ be an integral and non degenerate curve of degree d. Then

i) $h^1(I_C(n)) = 0$ if $n \geq d+1-r$:

ii) $h^1(I_C(d-r)) \neq 0$ iff C is smooth and rational and, either $d = r+1$, or $d > r+1$ and C has a $(d+2-r)$-secant line.

One of the most relevant things this theorem emphasize, is the fact that the non vanishing of $h^1(I_C(n))$ for large n is intimately connected with the existence of a high-order secant line to C.

Coming back to our problem, let us remark that, if $E(n)$ is the least twist of E admitting sections, then the sequence $0 \to O \to E(n) \to I_C(2n+c_1) \to 0$ where C is the zero locus of a section of $E(n)$, shows that $h^1(E(n)) = h^1(I_C(2n+c_1))$ thus Problem A above can be viewed as asking for an analogue for vector bundles of the GLP theorem.

The concept which replace the notion of multisecant in the realm of vector bundles is obviously that of jumping line of high order, thus we can suspect that the non vanishing of $h^1(E(n))$ for large n is a sign of the existence of a high order jumping line for the vector bundle. Because of the fact that the maximal order of a jumping line of a vector bundle is sharply bounded by a well known function of the second Chern class, Chang conjectured that the answer to Problem A is given by

Conjecture :

$$f_1(0, c_2) \leq 2c_2 + 1 - (4c_2+5)^{\frac{1}{2}}$$
$$f_1(-1, c_2) \leq 2c_2 + \tfrac{1}{2} - (c_2 + \tfrac{1}{4})^{\frac{1}{2}}$$

It seems to be very hard to prove the conjecture by Chang or, at least, a vanishing which is linear in c_2. Partial results are available when the spectrum of E has a particular shape or when c_2 is very small. In [H;9.17] is proven that the conjecture holds for stable vector bundles with $c_1 = 0$, $c_2 = 2m+1$ and maximum spectrum, in [E] for stable vector bundles with spectra (0^{c_2}), $(-1, 0^{c_2-2}, 1)$ and for stable vector bundles with $c_1 = 0$, $c_2 \leq 5$.

In the third chapter of this thesis we will proceed a little bit in this direction proving the following

Theorem V the conjecture is true in the following cases:

a) for vector bundles with $c_1 = 0$ and $sp = (-m, .., -1, 0^2, ..., m)$;

b) for vector bundles with $c_1 = -1$ and maximum spectrum;

c) for vector bundles with $c_1 = -1$ and $sp = (-1^a, 0^a)$;

d) for vector bundles with $-1 \leq c_1 \leq 0$, $c_2 \leq 6$.

Let us come to the outline of this thesis.

In the first chapter we will give a short proof of the Barth's restriction theorem and of the vanishing theorem of Hartshorne stated above.

In the second chapter we will prove Theorem II.

In the third chapter we will prove Theorem V; moreover we will achieve the results, interesting in their own, of the complete characterization of the vector bundles of the points a) and b) of Theorem V (see theorems 1.1, 2.1). The vanishing will turn out as a corollary: in the corollary 1.4 we will prove the point a) and in the corollary 2.7 we will prove the point b).

In the fourth chapter we will prove Theorem I and in the last chapter Theorems III and IV.

I heartily thank my thesis advisor Philippe Ellia for suggesting me the problems we will be concerned with, for his help and encouragement.

I also would like to thank Emilia Mezzetti for a lot of interesting discussions.

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1 The Barth's restriction theorem and a vanishing by Hartshorne

As we said in the introduction, one of the most powerful techniques in the study of vector bundles (resp. reflexive sheaves) on $\mathbf{P}^n$ is a method "alla Castelnuovo" which can be described as the attempt of deducing relevant properties of the bundle starting from the behaviour of its restriction to a general hyperplane. Similarly to what happens in the study of space curves, a lot of interesting properties of the bundles on $\mathbf{P}^n$ can be proven with an inductive argument starting from the analysis, which is hopefully simpler, of the two dimensional case.

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Let us come to the outline of this chapter.

In the first section we will give a proof of the Barth's restriction theorem which rests on a famous theorem of Gruson-Peskine-Laudal.

In the second section we will recall some well known properties of the numerical character useful in what follows.

In the third section we will prove the Hartshorne vanishing theorem.

I) A proof of the Barth's restriction thorem

The goal of this section is to give a simple proof of the following

Theorem (Barth's restriction theorem): Let E be a stable rank two vector bundle on $\mathbf{P}^n$ with $n \geq 3$. Suppose $(c_1, c_2) \neq (0,1)$. Then the restriction E_H to a general hyperplane H is also stable.

The proof of the theorem we are giving rests on the following well known result

Theorem (Gruson-Peskine-Laudal): Let $C \subset \mathbf{P}^3$ be an integral curve of degree d, and let H be a general hyperplane. Set $\sigma := min\{k : \ h^1(I_{C\cap H}(k)) \neq 0\}$. If $d > \sigma^2 + 1$, then $h^0(I_C(\sigma)) \neq 0$.

The proof of the Barth's restriction theorem will follow after a lemma and a proposition.

Lemma (Glueing lemma): Let E be a vector bundle on $\mathbf{P}^n$ and let H and K be a general hyperplane and a general codimension two linear space respectively. If $h^0(E_H) = h^0(E_K) = t$ and $h^0(E_K(-1)) = 0$, then $h^0(E) = t$.

Proof. Let us show that, if H is a hyperplane containing K, then $h^0(E_H) = t$. By semicontinuity it is clear that $h^0(E_H) \geq t$. Let us consider the restriction sequence $0 \to E_H(-1) \to E_H \to E_K \to 0$. As $h^0(E_K(-1)) = 0$ we have that the injection $H^0(E_H(-p)) \to H^0(E_H(-p+1))$ is indeed an isomorphism if $p \geq 2$. But $H^0(E_H(-p)) = 0$ if $p >> 0$ hence $H^0(E_H(-1)) = 0$ and the map $f : H^0(E_H(-1)) \to H^0(E_H)$ is injective. By comparing the dimensions it follows that f is an isomorphism. Let us consider the blow up $\mathcal{F}$ of $\mathbf{P}^n$ along K and let $p : \mathcal{F} \to \mathbf{P}^n$ and $q : \mathcal{F} \to L$ the standard projections where L is a line missing K and representing the pencil of hyperplanes through K. From a standard theorem ([H] thm 12.11) we have that q_*p^*E is a vector bundle of rank t on L. Let us now consider the natural morphism $g : q_*p^*E \to q_*(p^*E\mid_Y) \simeq H^0(E_K) \otimes O_L$ where $Y := p^{-1}(K) \simeq K \times L$. As g is an isomorphism fibrewise, it is indeed an isomorphism of vector bundles hence $q_*p^*E \simeq H^0(E_K) \times O_L$ and $h^0(E) \simeq h^0(q_*p^*E) = t$. $\diamondsuit$

Proposition (Maruyama): Let E be a semistable rank two vector bundle on $\mathbf{P}^n$ $(n \geq 3)$. If H is a general hyperplane, then the restriction E_H is also semistable.

Proof. Let us suppose E_H be not semistable for every H. Set $m_H := min\{k : h^0(E_H(k) \neq 0\}$ and $m_0 := sup_H(m_H) < 0$. By semicontinuity $m_H = m_0$ for the general H hence a section of $E_H(m_0)$ vanishes on a codimension two scheme. We have $0 \to O_H \to E_H(m_0) \to I_Z(c_1 + 2m_0) \to 0$ and $0 \to O_K \to E_K(m_0) \to I_{Z \cap K, K}(c_1 + 2m_0) \to 0$ where K is a general hyperplane in H. As $c_1 + 2m_0 < 0$ we have $h^0(E_H(m_0)) = h^0(E_K(m_0)) = 1$ and $h^0(E_K(m_0 - 1)) = 0$. Applying the glueing lemma we conclude $h^(E(m_0) = 1$ in contrast with the semistability of E. $\diamondsuit$

Proof of the Barth's theorem. If m is sufficently large, the general section of $E(m)$ is irreducible and smooth. Let us consider the exact sequence $0 \to O \to E(m) \to I_C(c_1 + 2m_0) \to 0$ where C is a smooth and connected curve of degree $d = c_2 + c_1 m + m^2$. Let us suppose E_H be non stable for a general hyperplane H. By the proposition E_H is semistable. Restricting to H the sequence above we immediately get $\sigma(C) = c_1 + m$ (we keep the same notations as in the theorem of Gruson-Peskine-Laudal). If $c_1 = 0$, we have $\sigma = m$, $d = c_2 + m^2$ hence, if $c_2 \geq 2$, $d > \sigma^2 + 1$ and from the theorem of GPL we find $0 \neq h^0(I_C(\sigma)) = h^0(E)$ in contrast with the stability assumption. Similarly, if $c_1 = -1$, we have $d > \sigma^2 + 1$ for every c_2 and the conclusion follows. $\diamondsuit$

II) The numerical character

In this section we will recall the definition and the principal properties of the numerical character (see [E], [EP] and [GP]).

Let K an algebraically closed field and set $\mathbf{P}^2 := Proj(K[x_0, x_1, x_2])$. Let Z be a zero dimensional subscheme of $\mathbf{P}^2$ and set $A := \frac{K[x_0,x_1,x_2])}{I}$ where I is the omogeneous ideal of Z. The Hilbert function of Z is $h(Z,k) := rkA_k$. Let us suppose the line $x_2 = 0$ miss Z. Then A has a natural structure of $R := K[x_0, x_1]$-graded module of finite type with a resolution of the form

$$0 \longrightarrow \bigoplus_{0}^{\sigma-1} R[-n_i] \longrightarrow \bigoplus_{0}^{\sigma-1} R[-i] \longrightarrow A \longrightarrow 0$$

Definition: the set of integers $\chi_Z(n_0, ..., n_{\sigma-1})$ is called the numerical character of Z.

Let us recall some properties of the numerical character.

i) $n_0 \geq n_1 \geq ... \geq n_{\sigma-1}$;

ii) s is the minimal degree for the curves containing Z;

iii) $h^1(I_Z(k)) := h^1_\chi = degZ - h(Z,k) = \sum_0^{\sigma-1}[(n_i - k - 1)_+ - (i - k - 1)_+]$ where $+$ represents the positive part.

Proposition I ([EP]): If there exists t s.t. $n_{t-1} > n_t + 1$, then there exists a curve T s.t.:

a) if $Z' := Z \cap T$ then $\chi_{Z'} = (n_0, ..., n_{t-1})$;

b) if $Z"$ is the residual scheme of Z' inside Z, then $\chi_{Z"} = (m_0, ...m_{\sigma-t-1})$ with $m_i = n_{t+i} - t$. $\diamondsuit$

Corollary II: If χ_Z is not connected then every curve of degree $\sigma := min\{k : \ h^0(I_Z(k)) \neq 0\}$ is reducible. $\diamondsuit$

Proposition III ([E] pag.53): If $d := degZ$, σ has the same meaning as before and μ, ν are integers s.t. $d = \frac{(\sigma(\sigma+1)}{2} + \frac{\mu(\mu+1)}{2} + \nu < \sigma(\sigma-1)$, and $0 \leq \nu \leq \mu \leq \sigma - 3$, then $h^1_{\chi_Z}(l) \leq h^1_{\Phi_{d,\sigma}}$ with $\Phi_{d,\sigma} := (\sigma+\mu, \sigma+\mu-1, ..., (\sigma+\nu)^2, ..., \sigma+1, \sigma^{\sigma-\mu-1})$. $\diamondsuit$

III) A proof of the Hartshorne's restriction thorem

In this section we will give a short proof, based on the properties of the numerical character listed above, of the following theorem due to Hartshorne (see [H])

Theorem: Let $\mathcal{E}$ be a stable rank two vector bundle on $\mathbf{P}^2$ with Chern classes c_1 and c_2. Set $t := min\{k : \ h^0(\mathcal{E}(k)) \neq 0\}$. Then $h^1(\mathcal{E}(l)) = 0$ for $l \geq c_2 - t^2 - 1 - c_1 t$.

Proof. Let W be the zero locus of a section of $\mathcal{E}(t)$ and Z be the zero locus of a section of $\mathcal{E}(n)$ with $n >> t$. From the sequence $0 \to O \to \mathcal{E}(n) \to I_Z(c_1 + 2n) \to 0$ we have that $t + n + c_1 = \sigma := min\{l : \ h^0(I_Z(l)) \neq 0\}$. Let C be a degree σ curve containing Z. From th following commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & \rightarrow & O & \rightarrow & O & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \cdot C & & \\ 0 & \rightarrow & O(t-n) & \rightarrow & \mathcal{E} & \rightarrow & I_Z(t+n+c_1) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & O(t-n) & \rightarrow & I_W & \rightarrow & I_{Z,C}(t+n+c_1) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

we see that $W \subset C$. But in the construction of the diagram above it is clear that the role of W and Z can be interchanged, thus every curve C' of degree σ containing W does contain Z. Because of the fact that W is locally complete intersection, it follows that, if n is big enough, there exists an irreducible curve of degree $t+n+c_1$ which contains W hence we may suppose the zero dimensional scheme Z be contained in an irreducible curve of degree σ. From the corollary of section II it follows that the numerical character of Z is connected.

What we are going to do now is to show that the theorem easily follows from the bound for $h^1_{\chi_Z}$ given in the last proposition of section II. To keep notations as simple as possible and because of the fact that the cases of even and odd c_1 are very similar, from now on we will assume $c_1 = 0$. With the same notations as before we have $degZ = c_2 + n^2 := d$. We would like to bound $h^1_{\chi_Z}$ with $h^1_{\Phi_{d,\sigma}}$ where $\Phi_{d,\sigma}$ is the maximal character given in proposition III of section II. We have to check that $d < \sigma(\sigma-1)$ (+) and find integers ν, μ such that $d = \frac{\sigma(\sigma+1)}{2} + \frac{\mu(\mu+1)}{2} + \nu$ (*) and $0 \leq \nu \leq \mu \leq \sigma - 3$. The inequality (+) is equivalent to $c_2 < t^2 + (n-1)t + (t-1)n$ which can be assumed if n is sufficently large. Let us set $\mu := \sigma - 2t - 1 = n - t - 1$. Obviously $\mu \leq \sigma - 3$. From (*) we find $\nu = c_2 - t(t+1)$ hence we must check $0 \leq c_2 - t(t+1) \leq n - t - 1$. The first follows from $\chi(\mathcal{E}(t-1)) = t(t+1) - c_2$ (here χ is the Euler characteristic) and from $h^0(\mathcal{E}(t-1)) = h^2(\mathcal{E}(t-1)) = 0$. The second can be assumed if $n >> 0$. From proposition III we get $\Phi_{d,\sigma} = (2n-1, ...(n+c_2-t)^2, ..., t+n+1, (t+n)^{2t})$ and a simple computation shows that

$$h^1_{\Phi_{d,\sigma}}(l) = \begin{cases} 0 & l \geq 2n-2 \\ \frac{(2n-2-l)(2n-1-l)}{2} & 2n-3 \geq l \geq n+c_2-t^2-1. \end{cases}$$

In particular we have $h^1(I_Z(n+c_2-t^2-1)) \leq \frac{(n+t^2-c_2-1)(n+t^2-c_2)}{2} = h^2(O(c_2-n-t^2-1))$ (**). On the other hand, from $0 \to O \to \mathcal{E}(n) \to I_Z(2n) \to 0$, we find $0 \to H^1(\mathcal{E}(c_2-t^2-1)) \to H^1(I_Z(n+c_2-t^2-1)) \to H^1(O(c_2-t^2-1-n)) \to H^2(\mathcal{E}(c_2-t^2-1))$ (++). But $h^2(\mathcal{E}(c_2-t^2-1)) = h^0(\mathcal{E}(-c_2+t^2-2)) = 0$ because $t^2-c_2-2 \leq t-1$ which is equivalent to $t(t+1) \leq c_2+1$ which, as above, follows from the comparison with the Euler characteristic. Finally combining (++) with (**), we have $H^1(I_Z(n+c_2-t^2-1)) \simeq H^1(O(c_2-t^2-1-n))$ hence $h^1(\mathcal{E}(c_2-t^2-1)) = 0$. The same argument shows that $h^1(\mathcal{E}(l)) = 0$ when $l \geq c_2-t^2-1$ and the theorem follows.

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2 A proof of Theorem II

In this brief chapter we will complete the description of our inductive argument by showing how a vanishing theorem for the cohomology of a stable reflexive sheaf on $\mathbf{P}^3$ can be deduced from a vanishing theorem for the cohomology of its general plane restriction.

Our main result will be the proof of Theorem II of the introduction that we restate for the benefit of the reader.

Theorem : Let E be a stable rank-two reflexive sheaf on $\mathbf{P}^3$. Suppose there exists an integer $x > 0$ s.t. $h^1(F_H(k)) = 0$ if $k \geq x$ and H is a general plane. Then $h^1(F(x)) \leq x\{c_2 - \frac{x+3}{2} - c_1\} + c_2(1 + \frac{c_1}{2}) - \frac{c_3}{2}$ and $h^1(F(t)) = 0$ if $t \geq x\{c_2 - \frac{x+1}{2} - c_1\} + c_2(1 + \frac{c_1}{2}) - \frac{c_3}{2}$.

Let us start with the following (see [E], [G])

Lemma : Let E be a stable rank two reflexive sheaf on $\mathbf{P}^3$. Let $x \geq 0$ be an integer. Assume that for a general plane, H, $h^1(E_H(k)) = 0$ if $k \geq x$. Then if $t \geq x + 1$, $h^1(E(t-1)) \neq 0$ implies $h^1(E(t-1)) > h^1(E(t))$.

The proof of the lemma rests on the following well known

Definition-Theorem (Castelnuovo-Mumford): A coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$ is said to be k-regular if $h^i(\mathcal{F}(k-i)) = 0$, if $i > 0$. If $\mathcal{F}$ is k-regular then

i) $\mathcal{F}(k)$ is globally generated;

ii) the natural maps $H^0(\mathcal{F}(k)) \otimes H^0(O(m)) \to H^0(\mathcal{F}(k+m))$ are surjective;

iii) $\mathcal{F}$ is (k+1)-regular.

Proof of the lemma. From the exact sequence

$$0 \longrightarrow E(t-1) \longrightarrow E(t) \longrightarrow E_H(t) \longrightarrow 0$$

we get

$$0 \to H^0(E(t-1)) \to H^0(E(t)) \to H^0(E_H(t)) \to H^1(E(t-1))$$

In particular, for $t \geq x$, we know that $H^0(E_H(t)) = 0$, thus we have exact sequences

$$H^0(E(t)) \xrightarrow{f_t} H^0(E_H(t)) \to H^1(E(t-1)) \xrightarrow{g_t} H^1(E(t)) \qquad (*)$$

We consider the following commutative diagram

$$\begin{array}{ccc} H^0(E(t)) \otimes H^0(O_{\mathbf{P}^3}(1)) & \to & H^0(E(t+1)) \\ f_t \otimes id \downarrow & & \downarrow f_{t+1} \\ H^0(E_H(t)) \otimes H^0(O_H(1)) & \xrightarrow{\alpha} & H^0(E_H(t+1)) \end{array}$$

It is clear that, when α is surjective, f_t is an epimorphism implies f_{t+1} is also an epimorphism. By hypothesis we have $h^1(E_H(t-1)) = 0$ if $t \geq x-1$. Moreover $h^2(E_H(t-2)) = h^0(E_H^*(-t-1)) = h^0(E_H(-t-1-c_1)) = 0$ if H is general because $t \geq 1$ and E_H is a stable vector bundle. From the Castelnuovo-Mumford theorem we have that E_H is t-regular if $t \geq x+1$ hence the map α is surjective there.

Finally, from (*), we see that f_t is surjective iff g_t is an isomorphism; hence, if g_t is an isomorphism for any $t \geq t+1$, then g_m will be an isomorphism for all $m \geq t$. But this can only happen in case $h^1(E(t-1)) = h^1(E(t)) = 0$. $\diamondsuit$

Proof of the Theorem(see [E] Lemma II.2). From the restriction sequence to H

$$0 \longrightarrow E(-1) \longrightarrow E \longrightarrow E_H \longrightarrow 0$$

and from $h^1(E_H(l)) = 0$, $h^2(E_H(l)) = 0$ if $l \geq x$ (the first by hypothesis, the second by stability), we find $H^2(E(l-1)) \simeq H^2(E(l))$ if $l \geq x$ hence both vanish because $h^2(E(p)) = 0$ if $p >> 0$. Furthermore, by stability, we have $H^3(E(l)) = 0$ if $l \geq 0$ hence $h^1(E(x)) = h^0(E(x)) - \chi(E(x)) \leq \Sigma_{0 \leq i \leq x} h^0(F_H(i)) - \chi(F(x))$ (+). From [H1;7.4] we get $h^0(E_H(i)) = \chi(E_H(i)) + h^1(E_H(i)) \leq (t+1)(t+2+c_1) - t - 2 - c_1$ if $i < x$ and $h^1(E_H(x)) = \chi(E_H(x)) = (x+1)(x+2+c_1)$. Moreover, for a coherent sheaf $\mathcal{F}$ of rank r on $\mathbf{P}^3$, the Riemann-Roch theorem says ([H2;2.3])

$$\chi(\mathcal{F}) = r + \binom{c_1+3}{3} - 2c_2 + \frac{1}{2}(c_3 - c_1 c_2) - 1$$

hence

$$\chi(E(l)) = \begin{cases} \frac{1}{3}(l^3 + 6l^2 + 11l + 6) - c_2(l+2) + \frac{c_3}{2} & c_1 = 0 \\ \frac{1}{6}(2l^3 + 9l^2 + 13l + 6) - c_2(l + \frac{3}{2}) + \frac{c_3}{2} & c_1 = -1 \end{cases}$$

Combining with (+) and after some short calculations we find $h^1(F(x)) \leq x\{c_2 - \frac{x+3}{2} - c_1\} + c_2(1 + \frac{c_1}{2}) - \frac{c_3}{2}$. The conclusion follows from the last lemma. $\diamondsuit$.

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3 Vanishings for the cohomology of certain families of stable rank two vector bundles on P^3 (*)

Introduction

One of the motivations of the present paper is provided by the following problem, which, as far as we know, is widely open (see [H2;9.17]):

determine the least integer $m_{c_1}(c_2)$ s.t. $h^1(E(k)) = 0$ if $k \geq m_{c_1}(c_2)$ for every stable, normalized, rank two vector bundle E on $\mathbf{P}^3$.

A similar problem was solved by Gurrola (see [G]) in the context of stable reflexive sheaves. Indeed he was able to show that, given a stable, normalized, rank two reflexive sheaf F with Chern classes c_1, c_2, c_3, there is a function $m(c_1, c_2, c_3)$ s.t. $h^1(F(k)) = 0$ if $k \geq m(c_1, c_2, c_3)$. Moreover that bound turned out to be sharp in the sense that it is realized by reflexive sheaves with maximal c_3. However, in the case $c_3 = 0$, that bound gives a vanishing which should be very far from being sharp for vector bundles. Indeed on one hand the function $m(c_1, c_2, c_3)$ found by Gurrola is quadratic in c_2, on the other hand it was conjectured by Chang ([C]) that it should hold:

$$m_0(c_2) \leq 2c_2 + 1 - (4c_2 + 5)^{\frac{1}{2}}$$

$$m_{-1}(c_2) \leq 2c_2 + \frac{1}{2} - (c_2 + \frac{1}{4})^{\frac{1}{2}}$$

In the paper [E], Ellia was able to improve in the linear part the vanishing by Gurrola but it seems to be very hard to prove the conjecture by Chang or, at least, a vanishing which is linear in c_2. Partial results are available when the spectrum of E has a particular shape

(*) The content of this chapter represents a paper which is going to be published in "Annali di Matematica pura ed applicata" or when c_2 is very small. In [H2;9.17] is proved that the conjecture holds for stable vector bundles with $c_1 = 0$, $c_2 = 2m+1$ and maximum spectrum, in [E] for stable vector bundles with spectra (0^{c_2}), $(-1, 0^{c_2-2}, 1)$ and for stable vector bundles with $c_1 = 0$, $c_2 \leq 5$.

In the present paper we will proceed a little bit in this direction proving the following

Theorem the conjecture is true in the following cases:

a) for vector bundles with $c_1 = 0$ and $sp = (-m, .., -1, 0^2, ..., m)$;

b) for vector bundles with $c_1 = -1$ and maximum spectrum;

c) for vector bundles with $c_1 = -1$ and $sp = (-1^a, 0^a)$;

d) for vector bundles with $-1 \leq c_1 \leq 0$, $c_2 \leq 6$.

In the first two sections of the paper we will achieve the results, interesting in their own, of the complete characterization of the vector bundles of the points a) and b) (see theorems 1.1, 2.1). The vanishing will turn out as a corollary: in the corollary 1.4 we will prove the point a) and in the corollary 2.7 we will prove the point b).

As what concerns for point c) and d) of the proposition, they will be proved in the third section of the paper: point c) in proposition 3.6 and point d) in proposition 3.4.

I) The family of stable rank two vector bundles with spectrum $(-m, .., -1, 0^2, 1, .., m$

In this section we will prove the following

Theorem 1.1: Let E be a stable vector bundle with $c_1 = 0$, $c_2 = 2m+2$ and spectrum $-m$,$-m+1$,...-1,0^2,1, ... $m-1$,m. Then one of the following must happen:

a) $E(1)$ has a section whose zero locus is a curve Y' which is the disjoint union of a line D with a curve Y which is a double structure on a degree $m+1$ plane curve and s.t. $\omega_{Y'} = O_{Y'}(-2)$.

b) $E(2)$ has a section whose zero locus is a curve Y which is the scheme theoretic union of two skew lines L_1 and L_2 with a curve Y_0 which is a double structure on a degree $m+2$ plane curve and s.t. $\omega_{Y_0} = O_{Y_0}(-\Delta)$, where $\Delta := Y_0 \cap (L_1 \cup L_2)$ and $deg\Delta = 4$;

The proof of the theorem will follow after several lemmas.

Lemma 1.2: Let E be as in 1.1, then E has an unstable plane H of order m. Performing the reduction step with respect to H (see [H2;9.1]) gives us E as an extension

$$0 \longrightarrow E' \longrightarrow E \longrightarrow I_{Z,H}(-m) \longrightarrow 0 \qquad (1.1)$$

where the Chern classes of E' are $(-1, m+2, m^2+2m+2)$.

Proof. From well known properties of the spectrum (see section 3 and references therein) we have $h^1(E(-1-m)) = 1$, $h^1(E(-m)) = 3$ therefore there exists a section $h \in H^0(O(1))$ such that $h \in Ker(H^0(O(1)) \otimes H^1(E(-m-1)) \to H^1(E(-m)))$. If H is the plane defined by $h = 0$ we have

$$0 \longrightarrow E(-m-1) \xrightarrow{.h} E(-m) \longrightarrow E_H(-m) \longrightarrow 0 \qquad (+).$$

The exact cohomology sequence shows that $h^0(E_H(-m)) \neq 0$ and H is unstable of order m. The last part of the statement comes from [H2;9.1]. $\diamondsuit$

Lemma 1.3: Let E' be the sheaf coming from the reduction step (1.1), then E' has an unstable plane H' of order m+1. Performing the reduction step with respect to H' we get

$$0 \longrightarrow E''(-1) \longrightarrow E' \longrightarrow I_{Z',H'}(-m-1) \longrightarrow 0 \qquad (1.2).$$

the Chern classes of E'' being $(0, 1, c_3'')$. Moreover the plane H' can be chosen to be the same of H arising in lemma 1.2.

Proof. Dualizing the reduction step sequence (1.1) we get ([H1;9.1])

$$0 \longrightarrow E \xrightarrow{f} E'(1) \longrightarrow O_H(m+1) \longrightarrow 0 \qquad (++)$$

The cohomology long exact sequence shows that $h^2(E'(m-2)) = h^2(E(m-3)) = 1$, $h^2(E'(m-3)) = h^2(E(m-4)) = 3$. Moreover the map f induces a morphism $\hat{f} : H^2_*(E) \to H^2_*(E'(1))$ hence the following diagram commutes

$$\begin{array}{ccc} H^2(E(m-4)) & \xrightarrow{.h} & H^2(E(m-3)) \\ \downarrow \hat{f} & & \downarrow \hat{f} \\ H^2(E'(m-3)) & \xrightarrow{.h} & H^2(E'(m-2)) \end{array}$$

Since $h^2(E_H(m-3)) = h^0(E_H(-m)) = 1$ we have from (+) that the map multiplication by h in the top row is zero. Finally because the vertical rows are isomorphisms, we have that the bottom row is also zero and $h^2(E'_H(m-2)) = h^2(E'(m-2)) = 1$. $\diamondsuit$

Proof of the theorem 1.1. Since E is stable, E'' is semi-stable. It follows ([H2;8.2]) that $c_3'' = 2$ or 0 (c_3'' must be even). If $c_3'' = 2$ since $\chi(E'') > 0$ and $h^2(E'') = 0$ [H2;8.2] we get $h^0(E'') \neq 0$.

If $c_3'' = 0$ then E'' is a vector bundle with $c_1 = 0$, $c_2 = 1$; since it is semi-stable, it is in fact stable.

In conclusion there are two possibilities for E'':

a) $c_3'' = 2$, E'' is semi-stable given by

$$0 \to O \to E'' \to I_D \to 0$$

where D is a line;

b) $c_3'' = 0$, E'' is a null-correlation bundle.

a) From (1.1,1.2) and the last short exact sequence we have $1 = h^0(O) = h^0(E'') = h^0(E'(1)) = h^0(E(1))$. Hence $E'(1)$ has a section which vanishes on a degree $m+2$ curve X' and $E(1)$ has a section which vanishes on a degree $2m+3$ curve Y' s.t. $\omega_{Y'} = O_{Y'}(-2)$. Moreover, since E'', E' and E are isomorphic outside H and D cannot be contained in H (otherwise $h^0(E'(1)) \geq 2$ but $h^0(E'(1)) = h^0(E'') = 1$), we have $Res_H(X') \simeq D$ and $Res_H(Y') \simeq X'$ where $Res_H(C)$ is the residual scheme of the curve C with respect to H. Finally from the first isomorphism we conclude that X' is the union of the line D with a degree $m+1$ curve S lying on H (indeed $degX' = m+2$), and from the second we conclude that Y' is the schematic union of D with a double structure on the plane curve S (this union being disjoin because $\omega_{Y'} = O_{Y'}(-2)$).

b) This time: $H^0(E''(1)) \simeq H^0(E'(2)) \simeq H^0(E(2))$. Arguing as above we get that $E(2)$ has a section vanishing along a curve Y, such that $\omega_Y = O_Y$, $degY = 2m+6$, $Y = Y_0 \cup C$ where $C = L_1 \cup L_2$ is the union of two skew lines and where Y_0 is a double structure on a curve $P \subset H$, (indeed $degP = m+2$ as follows from the computation of the degree of the zero locus of a section of $E'(2)$). We may assume $L_i \not\subset H$ ($E''(1)$ is globally generated). Moreover since $\omega_{Y} = O_{Y}$, L_i intersects Y_0 in a subscheme of length two. This means that L_i is contained in $T_{x_i}(Y_0)$ the Zariski tangent plane to Y_0 at the point $x_i = L_i \cap P$. Set $\Delta := C \cap Y_0$ and $supp\Delta := x_1 \cup x_2$. Let Y_0 be given by the Ferrand construction

$$0 \longrightarrow I_{Y_0} \longrightarrow I_P \longrightarrow L \longrightarrow 0$$

where L is an invertible sheaf on P. Applying the functor $\underline{Hom}(*, \omega_{\mathbf{P}^3})$ to the sequence

$$0 \longrightarrow O_{Y} \longrightarrow O_{Y_0} \oplus O_C \longrightarrow O_\Delta \longrightarrow 0$$

we get the usual long exact sequence of sheaves

$$... \rightarrow \underline{Ext}^2(O_\Delta, \omega_{\mathbf{P}^3}) \rightarrow \underline{Ext}^2(O_{Y_0}, \omega_{\mathbf{P}^3}) \oplus \underline{Ext}^2(O_C, \omega_{\mathbf{P}^3}) \rightarrow \underline{Ext}^2(O_{Y}, \omega_{\mathbf{P}^3}) \rightarrow$$

$$\underline{Ext}^3(O_\Delta, \omega_{\mathbf{P}^3}) \rightarrow \underline{Ext}^3(O_{Y_0}, \omega_{\mathbf{P}^3}) \oplus \underline{Ext}^3(O_C, \omega_{\mathbf{P}^3}) \rightarrow ....$$

From lemma III-7.3 of [H] we have that $\underline{Ext}^2(O_\Delta, \omega_{\mathbf{P}^3}) = 0$. Moreover, as Y_0 and C are locally complete intersection curves (see [B] prop. 1), the last term also vanishes (indeed the structure sheaves of Y_0 and C have resolutions of length 2). Recalling lemma III-7.4 of [H], the sequence above becomes

$$0 \longrightarrow \omega_{Y_0} \oplus \omega_C \longrightarrow \omega_{Y} \simeq O_{Y} \longrightarrow \omega_\Delta \longrightarrow 0 \qquad (*)$$

Tensoring $(*)$ by O_P we get

$$0 \rightarrow \underline{Tor}^1(O_P, \omega_\Delta) \xrightarrow{f} \omega_{Y_0}|_P \oplus (\omega_C \otimes O_P) \xrightarrow{g} O_P \rightarrow O_{\{x_1\} \cup \{x_2\}} \rightarrow 0$$

The sheaves $\underline{Tor}^1(O_P, \omega_\Delta)$ and $\omega_C \otimes O_P$ are supported on the points x_1 and x_2. Furthermore, as Y_0 is locally complete intersection, ω_{Y_0} is locally free on Y_0 hence the first component of the map f and the second of the map g vanish and we can write

$$0 \longrightarrow \omega_{Y_0}|_P \longrightarrow O_P \longrightarrow O_{\{x_1\} \cup \{x_2\}} \longrightarrow 0$$

which shows that $\omega_{Y_0}|_P \simeq O_P(-x_1 - x_2)$. Since $\omega_{Y_0}|_P \simeq \omega_P \otimes L^{-1}$ ([B] prop 1) we deduce $L \simeq \omega_P(x_1 + x_2)$. In particular $h^1(L) = 0$ which implies that the natural restriction map from $Pic(Y_0)$ to $Pic(P)$ is injective [B]; it follows $\omega_{Y_0} = O_{Y_0}(-\Delta)$.

To conclude the proof we need to show that cases a) and b) are effective. For the case a) it suffices to take Y as the disjoint union of a line with a suitable double structure on a plane curve of degree $m+1$ (see [HR] 2.10, 2.11).

b) Let $P \subset H$ be a smooth curve of degree $m+2$. Let x_1, x_2 denote two points of P. Let $L := \omega_P(x_1 + x_2)$. Observe that $N_P \otimes L$ is globally generated. Therefore taking a suitable surjection $N_P^* \to L$ we can construct a double structure, Y_0, on P s.t. $I_{P,Y_0} \simeq L$. We may also assume that the embedded Zariski tangent plane to Y_0 at x_i is different from H (otherwise Y_0 would be a plane curve but this is impossible because $h^1(I_{Y_0}(-1)) = h^0(\omega_P(x_1+x_2)\otimes O(-1)) \neq 0$). Let L_i be a line intersecting P at x_i and such that: $L_i \not\subset H$, $L_i \subset T_{x_i}Y_0$, $L_1 \cap L_2 = \Phi$. We define Y as the scheme theoretic union of Y_0 and $C := L_1 \cup L_2$.

To construct the desired vector bundle E we have to show

i) ω_{Y} has a section which generates almost everywhere;

ii) $h^1(I_Y(-m+1)) \neq 0$.

Indeed Y is a locally Cohen Macaulay and generally locally complete intersection curve, thus a section as in i) will yield an extension: $0 \to O \to E(2) \to I_Y(4) \to 0$ with E a rank two reflexive sheaf. Since $p_a(Y) = p_a(Y_0) + 2$ (because $L_i \subset T_{x_i}Y_0$) we easily see that $c_3(E) = 2p_a(Y) - 2 = 0$ and E is a vector bundle with the required Chern classes.

Finally ii) is equivalent to $h^1(E(-m-1)) \neq 0$. This condition implies $sp(E) = (-m, ..., -1, 0^2, 1, ..$

i) From $h^1(L) = h^1(\omega_P(x_1+x_2)) = 0$ we get the injectivity of the restriction map from $Pic(Y_0)$ to $Pic(P)$. Moreover $\omega_{Y_0}(\Delta)\mid_P \simeq L^{-1} \otimes \omega_P \otimes O(x_1+x_2) \simeq O_P$ implies $\omega_{Y_0}(\Delta) \simeq O_{Y_0}$ hence a generic section of $\omega_{Y_0}(\Delta) \simeq \omega_Y \mid_{Y_0}$ generates everywhere (on Y_0). Since $\omega_Y \mid_C \simeq \omega_C(\Delta) \simeq O_{L_1} \oplus O_{L_2}$ we see that we can find sections of $\omega_Y \mid_{Y_0}$ and $\omega_Y \mid_C$ which glue to give a section of ω_Y generating almost everywhere (in fact everywhere).

ii) To prove that $h^1(I_Y(-m+1)) \neq 0$ we proceed as follows. Firstly we have $h^1(I_Y(-m+1)) = h^0(O_Y(-m+1)) = h^1(O_Y(m-1))$. Moreover from the sequence $0 \to I_{Y_0,Y} \to O_Y \to O_{Y_0} \to 0$ where $I_{Y_0,Y} \simeq O_{L_1}(-2) \oplus O_{L_2}(-2)$, we see that $h^1(O_Y(m-1)) = h^1(O_{Y_0}(m-1))$. Finally from the cohomology exact sequence associated to $0 \to L(m-1) \to$

$O_{Y_0}(m-1) \to O_P(m-1) \to 0$ and from $h^1(L(m-1)) = h^0(L^*(-m+1) \otimes \omega_P) = 0$ we get $h^1(O_Y(m-1)) = h^1(O_{Y_0}(m-1)) = h^1(O_P(m-1)) = h^0(\omega_P(1-m)) = h^0(O_P) = 1$. $\diamondsuit$

Corollary 1.4: Let E be as in theorem 1.1. Then $h^1(E(p)) = 0$ if $p \geq \frac{3c_2}{2} - 1$

Proof. a) The sequence (1.2) reads:

$$0 \longrightarrow E''(-1) \longrightarrow E' \longrightarrow I_{Z',H}(-m-1) \longrightarrow 0 \qquad (\dot{\cdot})$$

with $lengthZ' = \frac{c_3(E'')}{2} = 1$. From the defining sequence of E'' we see that $h^1(E''(l)) = 0$ if $l \in \mathbf{Z}$ thus $h^1(E'(k)) = 0$ if $k \geq m+2$. Dualizing ($\dot{\cdot}$) we have (see the proof of theorem 9.1 of [H2]):

$$0 \longrightarrow E'(1) \longrightarrow E''(1) \longrightarrow I_{W,H}(m+2) \longrightarrow 0 \qquad (\bar{\cdot})$$

where W is a subscheme of Z (see (1.1)) and $lengthW = lengthZ - 1$. From the properties of the spectrum [H2;7.1-5] we have ($sp(E') := \{k_i\}$): $h^2(E'(l)) = h^1(\oplus O(k_i + l + 1)) = 0$ if $l \geq c_2(E') - 2 = m$ and from ($\bar{\cdot}$) we get $h^1(I_{W,H}(r)) = 0$ if $r \geq 2m+1$. This implies $h^1(I_{Z,H}(l)) = 0$ if $l \geq 2m+2$. Indeed from the sequence $0 \to I_{Z,H} \to I_{W,H} \to I_{W,Z} \to 0$ we get $h^1(I_{Z,H}(2m+1)) \leq h^0(I_{W,Z}(2m+1))$ and from $0 \to I_{W,Z} \to O_{Z,H} \to O_{W,H} \to 0$ we find $h^0(I_{W,Z}(2m+1)) = 1$ thus $h^1(I_{Z,H}(2m+1)) \leq 1$. The form of the Hilbert function in terms of the numerical character of Z (see point (iii) of pag. 112 of [EP]) shows that the Hilbert function of Z is strictly increasing until it attains $lengthZ$. This allows us to conclude $h^1(I_{Z,H}(l)) = 0$ if $l \geq 2m+2$. Finally from the sequence (1.1) we find $h^1(E(p)) = 0$ if $p \geq 3m+2 = \frac{3c_2}{2} - 1$.

b) This time the sequence (1.2) reads:

$$0 \longrightarrow E''(-1) \longrightarrow E' \longrightarrow O_H(-m-1) \longrightarrow 0 \qquad (\circ)$$

As E'' is a null-correlation bundle we have $h^1(E''(l)) = 0$ if $l \geq c_2(E'') - 1 = 0$ and $h^1(E'(r)) = 0$ if $r \geq 1$. Dualizing ($\circ$) we get:

$$0 \longrightarrow E'(1) \longrightarrow E''(1) \longrightarrow I_{Z,H}(m+2) \longrightarrow 0$$

where Z is the same scheme of (1.1) ([H2;9.1]). Arguing as above we find $h^1(E(p)) = 0$ if $p \geq \frac{3c_2}{2} - 2$. $\diamondsuit$

II) The family of stable rank two vector bundles with spectrum $(-m+1,...,m-2)$

In this section we will be concerned with the family of rank two stable vector bundles on $\mathbf{P}^3$ with $c_1 = -1$, $c_2 = 2m-2$ and the maximum spectrum $(-m+1,...,m-2)$. As our situation is very similar to that of rank two stable vector bundle with $c_1 = 0$ and maximum spectrum, we will be sketchy and follow sec. 9.3-17 of the paper [H2].

What we are going to prove is the following

Theorem 2.1: For any $m \geq 2$ the stable vector bundles with Chern classes $(-1, 2m-2)$ and maximum spectrum, form an irreducible, non singular, rational family of dimension $3m^2+m-1$. Moreover, given any vector bundle E in the family, then the following are true:

1) $h^0(E(1)) = 1$ and if $s \in H^0(E(1))$, then $(s)_0$ describes a degree $2m-2$ curve Y, which is a multiplicity two structure on a degree $m-1$ plane curve;

2) $h^0(E(m)) \geq \frac{m(m+1)}{2} - 1$ and if $s \in H^0(E(m))$ is general, then it vanishes in codimension 2 and its zero locus is a curve C which is the disjoint union of a degree $2m-2$ plane curve with a complete intersection (m,m-1). $\diamondsuit$

We begin proving the following result:

Proposition 2.2: For any $r \geq 2$, the moduli space of rank 2 semistable reflexive sheaves with Chern classes $(0, r, r^2+r)$ ([H2;8.2] implies that such sheaves are only semistable and not stable and have the maximal c_3 allowed) is irreducible and non singular of dimension r^2+4r+2.

The proof of proposition 2.2 will follow after several lemmas.

Lemma 2.3: Let F be any sheaf as in 2.2, then F has an unstable plane H of order r. The reduction step with respect to H gives us F as an extension

$$0 \longrightarrow O \oplus O(-1) \longrightarrow F \longrightarrow I_{Y,H} \longrightarrow 0 \qquad (2.1)$$

where Y is a degree r plane curve.

Proof. The proof of the existence of H goes along the same lines of lemma 1.2. Reduction step gives

$$0 \longrightarrow F' \longrightarrow F \longrightarrow I_{W,H}(-r) \longrightarrow 0 \qquad (\hat{o})$$

The Chern classes of F' are $(-1, 0, c_3)$. From $(\hat{o})$ and the semistability of F, we see that F' has order of instability -1 (see[S]). Proposition 3.8 of the paper [S] says $c_3 \leq c_2(c_2+1) = 0$, thus F' is a vector bundle. Moreover as $h^0(F') \neq 0, h^0(F'(-1)) = 0$ we see that F' is given by the extension

$$0 \longrightarrow O \longrightarrow F' \longrightarrow I_X(-1) \longrightarrow 0 \qquad (\tilde{o})$$

with X a curve with $degX = c_2' = 0$. But then $X = \Phi$, $I_X(-1) = O(-1)$ and $(\tilde{o})$ splits. Finally putting $s := lengthO_W$, we have ([H2;9.1]) $2s = c_3' = 0$, $s = 0$, $W = \Phi$ and $(\hat{o})$ becomes the extension of the statement. $\diamondsuit$

Lemma 2.4: Every semistable rank 2 reflexive sheaf F with Chern classes $(0, r, r^2+r)$ is given by an extension $0 \to O \to F \to I_Y \to 0$ where Y is a degree r plane curve. Conversely every such an extension gives a reflexive sheaf with the right Chern classes. This construction gives us a family which is irreducible of dimension $r^2 + 4r + 2$.

Proof. [H2;8.2.1] implies that the extension of the statement gives a semistable reflexive sheaf with Chern classes $(0, r, r^2 + r)$. Conversely lemma 2.3 implies F to be given by (2.1) where Y was a degree r plane curve and so F has a section whose zero locus is Y. Dualizing the reduction step sequence (2.1) (see [H2;9.1]) we get $0 \to F \to O \oplus O(1) \to I_{Z,H}(r+1) \to 0$, where Z is a zero dimensional scheme s.t. $lengthZ = r^2+r$ ([H2;9.1]). To prove the statement concerning the family it is sufficient to note that giving F is equivalent to giving a plane H and two plane curves without common components with degrees r and $r+1$. $\diamondsuit$

Proof of the proposition 2.2. To complete the proof we are only concerned with the smoothness. What we have to check is the equality between the dimension of the family above and the dimension of the Zariski tangent space to the moduli space at a sheaf F. The Zariski tangent space at F is $Ext^1(F, F)$ and we have to show $dimExt^1(F, F) = r^2+4r+2$

if F is given by (2.1). To calculate $dimExt^1(F,F)$ one can apply the functor $Hom(\cdot,F)$ to (2.1)

$$0 \to Hom(F,F) \to Hom(O \oplus O(-1),F) \to Ext^1(O_H(-r),F) \to Ext^1(F,F) \to 0$$

$Ext^1(O_H(-r),F)$ can be found applying the functor $Hom(\cdot,F)$ to the resolution sequence of $O_H(r)$, $Ext^1(O(-i),F) = H^1(F(i))$ and $H^0(F(i))$ are easily calculated from (2.1). To find $dimHom(F,F)$ we can proceed as follows. Firstly we have $dimHom(F,F) \geq 2$ because F is not stable. Then applying $Hom(F,\cdot)$ to (2.1) we find $dimHom(F,F) \leq 1 + dimHom(F,O_H(-r))$ and applying $Hom(\cdot,O_H(-r))$ to (2.1) we get $dimHom(F,O_H(-r)) = 1$ so $dimHom(F,F) = 2$ and the proposition follows. ◇

Taking into account 2.3, 2.4, the proof of the following lemmas goes exactly as the one of lemmas 9.11-12 of [H2]:

Lemma 2.5: Let E be as the statement of the theorem. Then E has an unique unstable plane H of order $m-1$. Performing the reduction step with respect to H gives

$$0 \longrightarrow E'(-1) \longrightarrow E \longrightarrow I_{Z,H}(1-m) \longrightarrow 0 \qquad (2.2)$$

where E' is a stable reflexive sheaf with Chern classes $(0, m-1, m^2-m)$; H is also the unique unstable plane of order $m-1$ of E'. ◇

Lemma 2.6: The restriction of E and E' to H is described by

$$0 \longrightarrow O_H(m-2) \longrightarrow E_H \longrightarrow I_{Z,H}(1-m) \longrightarrow 0$$

and $E'_H \simeq I_{Z,H}(m-1) \oplus O_H(1-m)$. ◇

Proof of the theorem 2.1. Theorem 9.1 of [H2] says that the sequence (2.2) has a dual one: $0 \to E(1) \to E'(1) \to O_H(m) \to 0$. Tensoring by O_H the last map and taking into account lemma 2.6 we get a surjection $v : E'_H \simeq I_{Z,H}(m-1) \oplus O_H(1-m) \to O_H(m-1) \to 0$ where v is given by a form f of degree $2m-2$ and not vanishing on Z on the second factor and a scalar on the first. With a simple counting the parameters we conclude the statement concerning the dimension of the family (see [H2;9.13-14]).

Proof of 1). We have $h^0(E(1)) = h^0(E') = h^0(O(-1) \oplus O) = 1$ as follows from the sequences of lemmas 2.3 and 2.5. Giving $s \in H^0(E(1))$ is equivalent to giving $E(1)$ as an extension

$$0 \longrightarrow O \longrightarrow E(1) \longrightarrow I_{Y'}(1) \longrightarrow 0$$

where Y is a degree $2m-2$ scheme of codimension 2 s.t. $\omega_{Y'} = O_{Y'}(-3)$. As the sheaves $E(1)$, E' and $O \oplus O(-1)$ are isomorphic outside the unstable plane H, it is clear that the support of Y is in H. From the sequence of lemma 2.6 we have $H^0(E_H(1)) = H^0(O_H(m-1))$, so the restriction of Y to H is a plane curve Y_0 of degree $m-1$. Because of $\omega_{Y'} = O_{Y'}(-3)$, no component of Y can be contained in H and so the conclusion comes from the degrees.

Proof of 2). From the sequences of Lemmas 2.3 and 2.5 we get $H^0(E(l)) \simeq H^0(E'(l-1)) \simeq H^0(O(l-2) \oplus O(l-1))$ if $l < m$ and $h^0(E(m)) > h^0(O(m-2) \oplus O(m-1))$ (note that the sequence in the middle of the proof of lemma 2.4 shows that Z is a complete intersection $(m, m-1)$ in H). Therefore $E(m)$ has sections vanishing in codimension 2 . Let s be such a section. Its zero locus will be a degree m^2+m-2 curve C s.t. $\omega_C = O_C(2m-5)$. The sequence of lemma 2.6 shows that $H^0(O_H(i+m-2)) \simeq H^0(E_H(i))$ if $i \leq m$. Moreover such isomorphisms are part of an injection $H^0_*(O_H(m-2)) \to H^0_*(E_H)$ of $H^0_*(O_H)$-(graded)modules, hence any section of $E_H(m)$ is a product of the unique section of $E_H(2-m)$ by a form of degree $2m-2$. In particular this is true for the restriction s_H of s to H whose zero locus is then the union of a degree $2m-2$ plane curve P with Z. Arguing as in the proof of lemma 9.16 of the paper [H1], it can be easily seen that $Z \cap P = \Phi$. On the other hand the sequence

$$0 \longrightarrow E' \longrightarrow O \oplus O(-1) \longrightarrow I_{Z,H} \longrightarrow 0$$

obtained dualizing the reduction step (2.1) (see [H1;9.1]) shows that, outside H, s is a complete intersection $(m, m-1)$. $\diamondsuit$

Corollary 2.7: Let E be as in theorem 2.1. Then $h^1(E(p)) = 0$ if $p \geq \frac{3c_2}{2} - 1$

Proof. As the the proof is very similar to that of corollary 1.4 we are going to be very sketchy.

Let E' be the sheaf arising from the sequence (2.2). Dualizing the defining sequence of E' (see (2.1))

$$0 \longrightarrow O \oplus O(-1) \longrightarrow E' \longrightarrow O_H(-m+1) \longrightarrow 0 \qquad (\hat{\cdot})$$

we find ([H2;9.1])

$$0 \longrightarrow E' \longrightarrow O \oplus O(1) \longrightarrow I_{Z,H}(m) \longrightarrow 0 \qquad (\circ)$$

where Z is the same scheme of (2.2). From $(\hat{\cdot})$ we see $h^1(E'(l)) = 0$ if $l \in \mathbf{Z}$. From the properties of the spectrum we find $h^2(E'(l)) = h^1(\oplus O(k_i + l + 1)) = 0$ if $l \geq c_2(E') - 2 = m - 3$ $(sp(E') := \{k_i\})$. Finally from $(\circ)$ and (2.2) we conclude $h^1(I_{Z,H}(r)) = 0$ if $r \geq 2m-3$ and $h^1(E(p)) = 0$ if $p \geq 3m - 4 = \frac{3c_2}{2} - 1$. $\diamondsuit$

III) A vanishing theorem

We start recalling the following facts (see [E],[HR] and ref. therein).

Proposition 3.1: Let E a rank 2 vector bundle on $\mathbf{P}^3$, with Chern classes c_1, c_2, $-1 \leq c_1 \leq 0$. Let M denote the graded module $H^1(E(*))$. If

$$0 \rightarrow L_4 \rightarrow L_3 \rightarrow L_2 \rightarrow L_1 \xrightarrow{\xi} L_0 \rightarrow M \rightarrow 0$$

is a minimal resolution of M, then $rk(L_1) = 2rk(L_0) + 2$, and there exists an isomorphism $\phi : L_1^*(c_1) \rightarrow L_1$ s.t. $\xi\phi\xi^*(c_1) = 0$ and which induces a minimal monad for E:

$$0 \rightarrow \tilde{L}_0^*(c_1) \xrightarrow{\phi\xi^*(c_1)} \tilde{L}_1 \xrightarrow{\xi} \tilde{L}_0 \rightarrow 0.$$

Moreover $\tilde{L}_0^*(c_1)$ is direct summand in L_2 ([HR;3.2]). $\diamondsuit$

In the sequel we will put: $\tilde{L}_0 = \oplus_{1 \leq i \leq r} O(-\alpha_i)$, $\tilde{L}_1 = \oplus_{1 \leq j \leq 2r+2} O(-\beta_j)$, $\alpha_1 \leq ... \leq \alpha_r$; $\beta_1 \leq ... \leq \beta_{2r+2}$.

If moreover E is such that $h^0(E(-1)) = 0$, we denote by $sp(E) = \{k_i\}_{1 \leq i \leq c_2}$, the spectrum of E. Furthermore we set $k^+ := max\{k_i\}$, $s(n) := \#\{k_i | k_i = n\}$. Finally we denote by $r(n)$ the number of minimal generators of degree n of the module M. Then the following properties hold ($m_i := dim M_i$):

1) $r(m) = 0$, $m \geq k^+ - c_1$, $\alpha_r \leq k^+ - 1 - c_1$;

2) $r(m) = 0$, $m \leq -2 - k^+$, $\alpha_1 = -1 - k^+$, $s(k^+) = \#\{i | \alpha_i = -1 - k^+\}$;

3) $\beta_{2r+2} \geq \alpha_r + 1$, $\beta_1 > \alpha_1$;

4)$\{-\beta_j\} = \{\beta_j + c_1\}$;

5) $m_{-1-i} = \sum_{j \geq i} s(j)(j - i + 1)$, $i \geq 0$;

6) $s(i) - 2\sum_{j \geq i+1} s(j) \leq r(-i-1) \leq s(i) - 1$ for $0 \leq i < k^+$, $r(-1-k^+) = s(k)$.

Lemma 3.2: Using the above notations, suppose L_0 has r summands with degrees $\leq l$. Then L_1 must contain at least $r+3$ summands with degrees $\geq 1 - l$ if $c_1 = 0$, $\geq -l$ if $c_1 = -1$ (see [HR] prop 3.3). ◊

Lemma 3.3: Using the above notations, $h^1(E(m)) = 0$, if $m \geq (\beta_r - \alpha_1) + (\beta_{r+1} - \alpha_2) + ... + (\beta_{2r-1} - \alpha_r) + (\beta_{2r} - \alpha_1) + (\beta_{2r+1} - \alpha_1) + (\beta_{2r+2} - 3)$ (see [E] III.5). ◊

The aim of this section is to prove the following improvement of Corollary IV.6 of [E]:

Proposition 3.4: Let E be a stable vector bundle with $c_2 \leq 6$. Then $h^1(E(m)) = 0$ if $m \geq 2c_2 + 2 - (4c_2 + 5)^{1/2}$ and $c_1 = 0$ or $m \geq 2c_2 + 1/2 - (c_2 + 1/4)^{1/2}$ and $c_1 = -1$.

Let us start by proving three lemmas

Lemma 3.5: Let E be a stable vector bundle with Chern classes (-1,6) and spectrum $(-2, -1^2, 0^2, 1)$. Then E is the cohomology of a monad of one of the following types:

$$0 \longrightarrow O(-3) \longrightarrow 2O(-1) \oplus 2O \longrightarrow O(2) \longrightarrow 0$$

$$0 \longrightarrow O(-3) \oplus O(-2) \longrightarrow O(-2) \oplus 2O(-1) \oplus 2O \oplus O(1) \longrightarrow O(1) \oplus O(2) \longrightarrow 0$$

In particular, applying lemma 3.3, we get $h^1(E(m)) = 0$ if $m \geq 8$.

Proof. From the properties of the spectrum we find $m_{-2} = 1$, $m_{-1} = 4$, $r(-2) = 1$, $0 \leq r(-1) \leq 1$.

a) $r(-1) = 0$: none of the α's is equal to -1, moreover the generator in M_{-2} has no relations in degree -1 so none of the β's is equal to -1. By the symmetry property 4) we have $\{\beta_i\} = (0^{a+2}, 1^{a+2})$. From the constraint on the ranks and the property 3) above we have $\{\alpha_i\} = (-2, 0^a)$. Finally lemma 3.2 with $l = 0$ implies $a = 0$;

b) $r(-1) = 1$: now one of the α's is equal to -1 and the generator of M_{-2} has one relation in degree -1 thus property 4) implies $\{\beta_i\} = (-1, 0^c, 1^c, 2)$. By property 3) we have $\{\alpha_i\} = (-2, -1, 0^a, 1^b)$ and from the constraint on the ranks $c = a + b + 2$. Again lemma 3.2 with $l = -1$ gives $b = 0$ and with $l = 0$ gives $a = 0$. $\diamondsuit$

Proposition 3.6: Let E be a stable vector bundle with Chern classes $(-1, c_2)$ and spectrum $(-1^{c_2/2}, 0^{c_2/2})$. Then E is the cohomology of a monad of the following type:

$$0 \to (c_2/2)O(-2) \oplus uO(-1) \to (c_2/2+1+u)O(-1) \oplus (c_2/2+1+u)O \to uO \oplus (c_2/2)O(1) \to 0$$

where $u \leq max(c_2 - 5, \frac{c_2}{2})$. In particular, applying lemma 3.3, we get $h^1(E(m)) = 0$ if $m \geq max(2c_2 - 5, \frac{3c_2}{2})$.

Proof. We have $m_{-1} = c_2/2$, $r(-1) = c_2/2$. From the properties 3), 4) we find $\{\beta_i\} = (0^{c_2/2+1+u}, 1^{c_2/2+1+u})$, $\{\alpha_i\} = (-1^{c_2/2}, 0^u)$. Let us consider the following sequence

$$0 \longrightarrow K \longrightarrow (c_2/2 + 1 + u)O \xrightarrow{j} (c_2/2)O(1) \qquad (3.1)$$

Suppose now the map j to be generically surjective and define $G := coker(uO(-1) \to (c_2/2 + 1 + u)O)$. From the Eagon-Northcott complex associated to $\bar{j} : G \to (c_2/2)O(1)$ ($\bar{j}$ is the map induced by j)

$$0 \longrightarrow O(u) \longrightarrow G \otimes O(\frac{c_2}{2}) \xrightarrow{\bar{j} \otimes det \bar{j}} (c_2/2)O(1) \otimes O(\frac{c_2}{2}) \longrightarrow 0$$

we infer $u \leq c_2/2$ otherwise G would have a section contained in the kernel of $\bar{j}$ and this cannot happen as follows from the sequence $0 \to (c_2/2)O(-2) \to Ker\bar{j} \oplus \mathcal{F} \to E \to 0$ ($\mathcal{F} := Ker((\frac{c_2}{2} + 1 + u)O(-1) \to \tilde{L}_0))$, recalling that $h^0(E) = 0$ by stability. Suppose now the map j of (3.1) to be not generically surjective. Then lemma 3.7 of [HR] implies $h^0(K(1)) \geq 4(c_2/2+1+u) - 10(c_2/2-1)$ and $h^0(E(1)) \geq 4(c_2/2+1+u) - 10(c_2/2-1) - u$. But $h^0(E(1)) \leq 1$ otherwise the zero locus of a section of $E(1)$ would be a plane curve Y with $\omega_Y = O_Y(-3)$ which is impossible. This gives $u \leq c_2 - 5$. $\diamondsuit$

Lemma 3.7: Let E be a stable vector bundle with Chern classes $(-1,6)$ and spectrum $(-2^2,-1,0,1^2)$. Then E is the cohomology of a monad of the following type:

$$0 \longrightarrow 2O(-3) \longrightarrow 3O(-2) \oplus 3O(1) \longrightarrow 2O(2) \longrightarrow 0$$

In particular, applying lemma 3.3, we get $h^1(E(m)) = 0$ if $m \geq 9$.

Proof. We have $m_{-2} = 2$, $m_{-1} = 5$, $r(-1) = 0$. Arguing as above we find $\{\beta_i\} = (-1^3, 0^{a+b}, 1^{a+b}, 2^3)$, $\{\alpha_i\} = (-2^2, 0^a, 1^b)$. From lemma 3.2 with $l = -1$ we get $b = 0$. Let us consider the following sequence

$$0 \longrightarrow K \longrightarrow aO \oplus 3O(1) \xrightarrow{k} 2O(2) \qquad (3.2)$$

The map k cannot have rank < 2 otherwise lemma 3.7 of [HR] would imply $h^0(E) \geq h^0(K) \geq a + 12 - 10 = a + 2$. Setting $G := coker(aO(-1) \to aO \oplus 3O(1))$ we have from the Eagon-Northcott complex associated to $\overline{k} : G \to 2O(2)$ ($\overline{k}$ is the map induced by k)

$$0 \longrightarrow O(a-1) \longrightarrow G \longrightarrow 2O(2) \longrightarrow 0$$

Again, we must have $a - 1 < 0$ otherwise E would have a section. ◇

Proof of proposition 3.4. a) $c_1 = 0$: all the allowed monads are listed in tab. 5.3 of the paper [HR]. For all but the 6(4) it is sufficient to apply lemma 3.3 to the monad. The case 6(4) is treated in corollary 1.4.

b) $c_1 = -1$: the properties of the spectrum listed in section 7 of [H2] and in proposition 5.1 of [H3] show that, if $c_2 \leq 6$, all the possible spectra are covered by lemmas 3.5-7, proposition 3.6 and corollary 2.7. ◇

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4 Some vanishings for the cohomology of rank 2 stable reflexive sheaves on P^3 and for their restriction to a general plane

In this paper we will be concerned with two questions raised by Ellia in the paper [E]. The first part is devoted to an improvement of the results of [E] concerning a vanishing for the cohomology of the restriction to a general plane H of a stable rank two (and normalized) vector bundle E on $\mathbf{P}^3$. By Barth restriction theorem and a result of Hartshorne ([H1;7.4]) it follows that, if H is general, then $h^1(E_H(k)) = 0$ if $k \geq c_2 - 2$. In [E] this result was improved in the sense that $h^1(E_H(k)) = 0$ if $k \geq c_2 - m$ where m is the round-up of $\frac{c_2^{1/2}}{2}$. The starting point of the proof was that if $h^1(E_H(k)) \neq 0$ for a general plane, then E_H would have a section vanishing on a set of points with an high degree intersection with a line L, which turns out to be very unstable. In the Proposition I.4 we will see that all these lines, obtained varying H, have to lie in a fixed plane which turns out to be much unstable. This will give us a contradiction and we will be able to improve the results above showing that $h^1(E_H(k)) = 0$ if $k \geq \frac{2}{3}c_2$ (see theorem I.8). As noted in [E;9.1] this result should not far from being sharp.

In the second part of this paper we will be concerned with the problem of finding the least integer $m_{c_1}(c_2)$ s.t. $h^1(E(k)) = 0$ if $k \geq m_{c_1}(c_2)$. In the paper [G], Gurrola found a function $m(c_1, c_2, c_3)$ s.t. $h^1(F(k)) = 0$ if $k \geq m(c_1, c_2, c_3)$ (+), for any stable rank two reflexive sheaf F with Chern classes c_1, c_2, c_3. This bound is sharp in the sense that it is realized by some reflexive sheaves with $c_3 = c_2^2 - c_2$. However this bound should be very far from being sharp for vector bundles. Indeed in [C] it is suggested that $m_0(c_2) \leq 2c_2 + 1 - (4c_2 + 5)^{\frac{1}{2}}$. In the paper [E] the following improvement of the linear part of (+) was proved: $m_0(c_2) \leq \frac{c_2^2}{2} + \frac{3}{8}c_2 + \frac{c_2^{\frac{1}{2}}}{4}$. In the Theorem II.3 we will improve these results also in the quadratic part, showing that $m_0(c_2) \leq \frac{4c_2^2}{9} + \frac{2}{3}c_2$. Similar results turns out to hold in the case $c_1 = -1$.

Finally we will show (see Theorem III.4) that an improvement of (+) holds also for reflexive sheaves if c_3 is not too big. It turns out that if a stable reflexive sheaf F is s.t. (+) is near to be sharp, then c_3 has to be high and there exists a very unstable plane containing a lot of singular points of F (see Theorem III.2).

I) Vanishing for the restriction to general plane

In this paper all sheaves are assumed to be normalized.

Let $\mathcal{E}$ be a stable rank two vector bundle on $\mathbf{P}^2$. Theorem 7.4 of ref [H1] says that $h^1(\mathcal{E}(m)) = 0$ if $m \geq c_2 - 2$ and $c_1 = 0$ or if $m \geq c_2 - 1$ and $c_1 = -1$. If E is a stable rank two, normalized vector bundle on $\mathbf{P}^3$ with $c_2 \geq 4$ we know by Barth restriction theorem that E_H is stable if H is a general plane of $\mathbf{P}^3$ thus $h^1(E_H(m)) = 0$ if $m \geq c_2 - 1$. What we are going to do is to improve this result (see Theorem I.8). We recall lemma I.1 of [E].

Lemma I.1: Let $\mathcal{E}$ be a semistable rank two vector bundle on $\mathbf{P}^2$ with $c_2 \geq 4$. Set $t := min\{n \in \mathbf{N} : \ h^0(\mathcal{E}(n) \neq 0\}$. Assume $h^1(\mathcal{E}(c_2 - k)) > 0$ and $c_2 + 3 \geq 3k - 2\sqrt{k-1}$ if $c_1 = 0$ or $c_2 + 1 \geq 3k - 2t$ if $c_1 = -1$. Then

i) $t(t + c_1) < k - 1$ and $t \leq k^{\frac{1}{2}}$

ii) consider a section of $\mathcal{E}(t)$: $0 \to O \to \mathcal{E}(t) \to I_Z(2t + c_1) \to 0$, then Z contains a subscheme Z' contained in a line and s.t. $length Z' \geq c_2 + t - k + 2 + c_1$.

Proof. For the case $c_1 = 0$ we refer the reader to [E]. As the case $c_1 = -1$ is very similar we are going to be very sketchy.

i) [H1] 7.4 implies $c_2 - k < c_2 - t^2 + t - 1$ so that $t(t-1) < k - 1$:

ii) set $\tau := max\{m \in \mathbf{N} : \ h^1(I_Z(m)) \neq 0\}$. As $h^1(I_Z(c_2+t-k-1)) = h^1(\mathcal{E}(c_2-k)) \neq 0$ we have $\tau \geq c_2 + t - k - 1$. We would like to use [EP] cor 2 with $s = 2$. We have to check a) $d = degZ = c_2 + t^2 - t \geq 4$; b) $\tau \geq \frac{d}{2} - 1$. a) follows from the hypothesis $c_2 \geq 4$ because $\mathcal{E}$ is stable so $t \geq 1$. To see b) we show that $c_2 + t - k - 1 > \frac{d}{2} - 1$. By i) it is enough to check $c_2 \geq 3k - 1 - 2t$ which is the hypothesis. Finally Z cannot be a complete intersection

$(2, \frac{d}{2})$ because of $\tau > \frac{d}{2} - 1$ and so [EP] Cor2 implies that there exists a subscheme $Z' \subset Z$ contained in a line s.t. $length Z' \geq \tau + 2 \geq c_2 + t - k + 1$. ◇

The proof of the following goes along the same lines of Corollary I.1.2 of [E]:

Corollary I.2: If D is the line containing Z' then

a) $\mathcal{E}_D \simeq O_D(r) \oplus O_D(-r + c_1)$ with $r \geq c_2 - k + 2 + c_1$;

b) if L is different from D then $\mathcal{E}_L \simeq O_L(p) \oplus O_L(-p + c_1)$ with $p < c_2 - k + c_1$. ◇

Remark: the uniqueness at the point b) of the last corollary rests on the fact that the degree of a subscheme of the zero locus of a section of $\mathcal{E}$ is bounded by the second Chern class. As the component of the zero locus of a section of a torsion free sheaf of rank two which miss the singular locus is also bounded by the second Chern class we can conclude:

Corollary I.2': Let $\mathcal{F}$ be a torsion free sheaf on $\mathbf{P}^2$. If $0 < t$ and k have the same meaning as above and the same constraints of lemma I.1 hold, then there can be at most one line missing the singular locus of $\mathcal{F}$ which is unstable of order $> c_2 - k + 2 + c_1$.

Following [E] we will say that a line L is a super-jumping-line (s-j-l), if $\mathcal{E}_L \simeq O_L(r) \oplus O_L(-r + c_1)$ with $r \geq c_2 - k + 2 + c_1$. Corollary I.2 implies that under the hypothesis of Lemma I.1 there is exactly one s-j-l in $\mathbf{P}^2$. Let F be a stable rank two reflexive sheaf on $\mathbf{P}^3$ with $c_2 \geq 4$. Set $t := min\{n \in \mathbf{N};\ h^0(F_H(n)) \neq 0\}$ with H a general plane (t is positive by the restriction theorem of Barth). Suppose now that F_H satisfies the hypothesis of Lemma I.1 for H general. Then we have the following

Proposition I.3: If F satisfies these hypothesis then there exists a plane H containing infinitely many s-j-l.

Proof. Corollary I.2 says that if K is a general plane of $\mathbf{P}^3$ then it contains an unique s-j-l. Let $U \subset \mathbf{P}^{3*}$ be an open set s.t. if $K \in U$ then K contains an unique s-j-l L_K. Let us consider the map s between U and the grassmannian of the lines of $\mathbf{P}^3$ which associates to every plane $K \in U$ the corresponding s-j-l L_K $s : U \rightarrow \mathbf{G}(l, \mathbf{P}^3)$, $s(K) = L_K$. The map s can be thought as a rational map from $\mathbf{P}^{3*}$ to the grassmannian: $s : \mathbf{P}^{3*} - - - \rightarrow \mathbf{G}(l, \mathbf{P}^3)$. What we are going to do is to prove that s cannot be extended to a morphism of $\mathbf{P}^{3*}$. Firstly we give a different interpretation of s. Let $0 \rightarrow \mathcal{Q}_{\mathbf{P}^{3*}} \rightarrow V_{\mathbf{P}^{3*}} \rightarrow O_{\mathbf{P}^{3*}}(1) \rightarrow 0$

($\mathbf{P}(V) \simeq \mathbf{P}^{3*}$) be the tautological short exact sequence of $\mathbf{P}^{3*}$. The fiber $\mathcal{Q}_{K,\mathbf{P}^{3*}}$ of $\mathcal{Q}_{\mathbf{P}^{3*}}$ on the plane K is the hyperplane of V corresponding to K. If $K \in U$ then the line $L_K \subset K$ can be identified with a point lying in $K^* \simeq \mathbf{P}(\mathcal{Q}^*_{\mathbf{P}^{3*}})$, in such a way that s can be thought as a section of the projectivized bundle $\mathbf{P}(\mathcal{Q}^*_{\mathbf{P}^{3*}})$ defined over U. To see that s cannot be extended to a morphism of $\mathbf{P}^{3*}$ to $\mathbf{G}(l, \mathbf{P}^3)$ is equivalent to see that s cannot be extended to a global section of $\mathbf{P}(\mathcal{Q}^*_{\mathbf{P}^{3*}})$. Let us suppose that there exists a global section $\hat{s}$ of $\mathbf{P}(\mathcal{Q}^*_{\mathbf{P}^{3*}})$ extending s. Giving $\hat{s}$ is equivalent to giving a bundle injection $r : 0 \to O_{\mathbf{P}^{3*}} \to \mathcal{Q}^*_{\mathbf{P}^{3*}}(k)$ where k is some suitable integer (see [H;II-7.12]). This implies that $\mathcal{Q}^*_{\mathbf{P}^{3*}}(k)$ has a nonvanishing section and thus $c_3(\mathcal{Q}^*_{\mathbf{P}^{3*}}(k)) = 0$. From the multiplication formula for Chern classes (see [OSS]) it is immediately seen that the only possibility to vanish $c_3(\mathcal{Q}^*_{\mathbf{P}^{3*}}(k))$ is by setting $k = -1$ but this gives a contradiction because $h^0(\mathcal{Q}^*_{\mathbf{P}^{3*}}(-1)) = h^0(T_{\mathbf{P}^{3*}}(-2)) = 0$. This shows that s is a really rational map of $\mathbf{P}^{3*}$. To conclude the proof it is sufficient to take H to be a singular point of s. $\diamondsuit$

Remark: From the description of the proof of proposition I.3 it is clear that every plane containing at least two of the s.j.l. of the image of the map s is singular for s thus it must contain infinitely many s.j.l.. We can thus obtain another proof of the proposition by observing that there are surely planes containing two s.j.l.: it is sufficient to take the s.j.l. contained in two general pencils of planes and get two surfaces (swept out by the s.j.l. of the two pencils) which intersects on a curve each point of which is contained in two s.j.l..

The central fact proved in this paper is contained in the following

Proposition I.4: Let F satisfy the same hypothesis of proposition I.3. Moreover let us suppose $c_2 \geq 3k$, $c_2 \geq 35$. All the s.j.l. are contained in an unique plane K which is the only singular point of the map s of proposition I.3. That plane is unstable of order $\geq c_2 - k + 2 + c_1$.

The proof of the proposition will follow after three lemmas.

Lemma I.5: The general plane of every one-dimensional component V of the singular locus of the map s of the proposition I.3 is unstable of order $\geq \frac{1}{3}c_2 - \frac{1}{4}$.

Proof . Let us call X the closure into the grassmannian of the image of the map s of the proposition. We have $dim X = 2$. As the general plane contains one s-j-l, it is clear that X cannot be the set of lines containing a fixed point of $\mathbf{P}^3$ thus we can suppose that the general line $D \in X$ miss the singular locus of F. Set $H \in V$ a general plane of the family. From Corollary I.2' we know that the number $-a := min\{n : h^0(F_H(n)) \neq 0\}$ is negative. Call Z the zero locus of a section $g \in H^0(F_H(-a))$. By restriction to a general line $D \subset H, D \in X$ of the resolution of the ideal I_Z we find $lengthD \cap Z := x_D \geq c_2 - k + 2 + c_1 - a \geq \frac{2}{3}c_2 + 1 - a(+)$. Because of the fact that H contains infinitely many lines of X it is clear that it must exists a point $p \in Z$ lying in infinitely many s-j-l (otherwise $a \geq \frac{2c_2}{3} + 1$ in which case there is nothing to prove). This implies that the two equations defining Z around p have degree bigger than x_D thus $x_D^2 \leq c_2 + a^2$. Combining with (+) the last inequality and after some short calculations we easily get the proof of the statement. ◇

Lemma I.6: The singular locus of the map s of proposition I.3 is zero-dimensional.

Proof. Let us suppose, by contrast, there is an one dimensional component V in the singular locus of s. Set $t := min\{n : h^0(F(n)) \neq 0\}$. From theorem 0.1 of [H3] we have $t < (3c_2 + 1)^{\frac{1}{2}}$. Set $f \in H^0(F(t))$, call Y the zero locus of f . Lemma I.5 says that the general $H \in V$ is unstable bigger than $\frac{c_2}{3} - \frac{1}{4}$, thus for the general line $L \subset H$ we have $F\mid_L \simeq O(r) \oplus O(-r + c_1)$ with $r \geq \frac{c_2}{3} - \frac{1}{4}$. By restriction to L of the resolution sequence of I_Y and keeping in mind the hypothesis $c_2 \geq 35$, we immediately see that $lengthL \cap Y \geq \frac{1}{3}c_2$ (+). Because of the generality of L, it is clear that $H \in V$ must contain a component of Y. This implies that there is a line $L' \subset Y$ and that V is the pencil through L'. Condition (+) shows that the two equations defining Y at a point $p \in L$ have no terms of degree less than $\frac{c_2}{3}$, thus $(\frac{c_2}{3})^2 \leq degY \leq c_2 + t^2 < 4c_2 + 1$ which gives a contradiction. ◇

Lemma I.7: All the s.j.l. are contained in a surface.

Proof. Suppose by contrast every point of $\mathbf{P}^3$ be contained in some s.j.l.. Let K be a plane which is an isolated singular point of s. If p is a general point then every plane H' containing p is regular for s and contains an unique s.j.l.. Let L be a line containing p and suppose $L \subset H'$. If $p' := L \cap K$ then p' must lie on some s.j.l. $L' \subset K$ (there are infinitely many ones on K) and this leads to an absurd because L and L' should span a plane through p containing two s.j.l.. ◇

Proof of the proposition. Last lemma implies there is a surface containing ∞^2 many s.j.l. thus it must be a plane H. Uniqueness is a consequence of the uniqueness of the s.j.l. contained in the general plane. $\diamondsuit$

Now we are ready to state and prove the main theorem of this section.

Theorem I.8: Let E be a stable rank two vector bundle on $\mathbf{P}^3$ with $c_2 \geq 35$. Let $3 \leq k \leq c_2$ be an integer and H a general plane. Then $H^1(E_H(c_2 - k)) = 0$ if $c_2 \geq 3k$.

Proof. Suppose $H^1(E_H(c_2 - k)) \neq 0$ for every plane H. By proposition I.4 there is a plane K which is unstable for E of order $r \geq c_2 - k + 2 + c_1$. Consider the restriction sequence of E to K: $0 \to E(-1) \to E \to E_K \to 0$. The fact that $h^0(E_K(-r)) \neq 0$ and the stability of E imply $H^1(E(-r-1)) \neq 0$. Theorem 8.1 of [H2] implies that $-\frac{c_2+c_1}{2} - 1 < -r - 1$ and $\frac{c_1+c_2}{2} < k$ which contradicts the hypothesis $k \leq \frac{c_2}{3}$. $\diamondsuit$

II) A vanishing for stable rank two vector bundles on P^3

We recall the following (see [G;2.1-2]):

Lemma II.1: Let E be a stable rank two reflexive sheaf on $\mathbf{P}^3$. Let $x \geq 0$ be an integer. Assume that for a general plane, H, $h^1(E_H(k)) = 0$ if $k \geq x$. Then the module $H^1(E(*))$ is generated in degrees $\leq x - 1$. Furthermore if $t \geq x + 1$, $h^1(E(t-1)) \neq 0$ implies $h^1(E(t-1)) > h^1(E(t))$. $\diamondsuit$

Corollary II.2: Let E be a stable rank two vector bundle on $\mathbf{P}^3$, with $c_2 \geq 35$. Let $m = [\frac{c_2}{3}]$.

i) If H is a general plane then $h^1(E_H(k)) = 0$ if $k \geq c_2 - m$.

ii)For every integer $t \geq c_2 - m + 1$, $h^1(E(t-1)) \neq 0$ implies $h^1(E(t-1)) > h^1(E(t))$.

Proof. i) $m \geq 3$ because $c_2 \geq 35$. Then we may apply Theorem I.8 with $m = k$. ii) Follows from i) and Lemma II.1. $\diamondsuit$

Finally we are ready to state the main theorem of this part.

Theorem II.3: Let E be as in the last corollary. Then $h^1(E(k)) = 0$ if $k \geq \frac{c_2^2}{2} + \frac{c_2}{2} + \frac{m(1-m)}{2} - c_1(c_2 - m) + \frac{c_1 c_2}{2}$, where $m = [\frac{c_2}{3}]$ $\diamondsuit$.

Remark: Setting $c_1 = 0$ and $m = \frac{c_2}{3}$ we have $h^1(E(k)) = 0$ if $k \geq \frac{4c_2^2}{9} + \frac{2c_2}{3}$ so, as anticipated in the introduction, this theorem improves (and extend to the case $c_1 = -1$) the result of [E] in the quadratic part of the bound.

The proof of the theorem rests on the following

Lemma II.4: Let F be a stable rank-two reflexive sheaf on $\mathbf{P}^3$ s.t. $c_2 > 35$. Suppose there exists an integer $x > 0$ s.t. $h^1(F_H(k)) = 0$ if $k \geq x$ and H is a general plane. Then $h^1(F(x)) \leq x\{c_2 - \frac{x+3}{2} - c_1\} + c_2(1 + \frac{c_1}{2}) - \frac{c_3}{2}$ and $h^1(F(t)) = 0$ if $t \geq x\{c_2 - \frac{x+1}{2} - c_1\} + c_2(1 + \frac{c_1}{2}) - \frac{c_3}{2}$.

Proof of the lemma (see [E] Lemma II.2). From the restriction sequence to H we find $h^2(F(l)) = 0$ if $l \geq x$.

By stability we have $h^1(F(x)) = h^0(F(x)) - \chi(F(x)) \leq \Sigma_{0 \leq i \leq x} h^0(F_H(i)) - \chi(F(x))$ (+).

Finally from [H1;7.4] we get $h^0(F_H(i)) = \chi(F_H(i)) + h^1(F_H(i)) \leq (t+1)(t+2+c_1) - t - 2 - c_1$. Combining with (+) and after some short calculations we conclude (see [E] Lemma II.2). $\diamondsuit$

III) A vanishing for stable rank two reflexive sheaves on P^3

We think it may be useful to give a short proof of the following proposition which is probably well known.

Proposition: Let F be a torsionfree sheaf on $\mathbf{P}^2$ with Chern classes c_1, c_2. If n is the length of the singular set of F we have $c_1(F^*) = -c_1$, $c_2(F^*) = c_2 - n$.

Proof of the prop.. As F is torsionfree, we have $codim(supp\underline{Ext}^2(F, O)) \geq 3$ [OSS] so that $\underline{Ext}^2(F, O) = 0$ and the homological dimension of F is 1 . Let $0 \to E_1 \to E_2 \to F \to 0$ be a minimal resolution of F. Dualizing we obtain $0 \to F^* \to (E_2)^* \to (E_1)^* \to \underline{Ext}^1(F, O) \to 0$ and finally $c_t(F^*) = c_{-t}(F) c_t(\underline{Ext}^1(F, O))$. As F is torsionfree, the support of $\underline{Ext}^1(F,O)$ is a zero dimensional scheme S, let n be its length. By induction on n, and starting from the resolution of the coordinate sheaf of a point in $\mathbf{P}^2$ we find $c_t(\underline{Ext}^1(F,O)) = 1 - nt^2$. As the support of $\underline{Ext}^1(F,O)$ is by definition the singular set of F, we have the thesis. $\diamondsuit$

Lemma III.1: Let F be a stable rank two reflexive sheaf on $\mathbf{P}^2$ and let H be an unstable plane for F of order r. Suppose n to be the length of the singular set of F_H. If $0 \to F' \to F \to I_{Z,H}(-r) \to 0$ is the reduction step sequence with $lengthZ = s$, then $s \geq c_2 + r^2 - n + c_1 r$.

Proof. Let $0 \to O_H \xrightarrow{i} F_H^*(-r)$ be the map given by a section of $F_H^*(-r)$. Dualizing and twisting we have a map i^*: $F_H^{**} \to O_H(-r)$ which fails to be surjective on the zero locus, W, of i. As the scheme Z is defined by the image of the composition $i^* \circ f \circ r$: $F \xrightarrow{r} F_H \xrightarrow{f} F_H^{**} \xrightarrow{i^*} O_H(-r)$ where r is the natural restriction and f is the natural inclusion, we have that the degree of Z is bigger than the degree of W. The conclusion comes from the Proposition above because $lengthW = c_2(F_H^*(-r))$. $\diamondsuit$

Theorem III.2: Let F be as in Lemma III.1. Let H be a general plane and k be an integer s.t. $3 \leq k \leq \frac{c_2}{3}$. Assume that $h^1(F_H(c_2 - k)) \neq 0$. Then $c_3 \geq c_2(\frac{c_2}{3} + 5 + 2c_1)$ and the plane of proposition I.4 must contains at least $\frac{1}{2}\{c_3 + c_2(\frac{c_2}{3} + 5 + 2c_1)\}$ singular points of F.

Proof. From Proposition I.4 we know there exists an unstable plane K for F of order $r \geq c_2 - k + 2 + c_1$. Performing the reduction step with respect to K we get

$$0 \longrightarrow F' \longrightarrow F \longrightarrow I_{Z,K} \longrightarrow 0$$

where the Chern classes of F' are $c_1(F') = c_1 - 1$, $c_2(F') = c_2 - r - c_1$, $c_3(F') = c_3 - c_2 - c_1 r - r^2 + 2s$. Let n be the length of the singular locus of F_K. Then Lemma III.1 implies $s \geq c_2 + r^2 - n + c_1 r$ and $c_3(F') \geq c_3 + c_2 + r^2 + c_1 r - 2n$. From now on we suppose $c_1 = 0$ because in the case $c_1 = -1$ the proof is almost identical. As F' is a stable reflexive sheaf, theorem 8.2 of [H2] implies $c_3(F') \leq c_2(F')^2$ (in the case $c_1 = -1$ $F'(1)$ is semistable and the condition is $c_3(F'(1)) \leq c_2(F'(1))^2 + c_2(F'(1))$. Combining with the formulas above we have $c_3 + c_2(1 + 2r) \leq c_2^2 + 2n$. As $2r + 1 \geq 2c_2 - 2k + 5$ we must have

$c_2(c_2-2k+5) \leq 2n-c_3 \leq c_3$ $(c_3 \geq n)$. The hypothesis $c_2 \geq 3k$ implies $c_2-2k+5 \geq \frac{c_2}{3}+5$ so, finally, we get $c_2(\frac{c_2}{3}+5) \leq c_3$, $\frac{1}{2}\{c_3+c_2(\frac{c_2}{3}+5)\} \leq n$ and the theorem follows. $\diamondsuit$

Now we are ready to state the following generalizations of Corollary II.2 and Theorem II.3.

Corollary III.3: Let F be a stable rank two reflexive sheaf on $\mathbf{P}^3$ with $c_2 \geq 35$ and $c_3 \leq c_2(\frac{c_2}{3}+5+2c_1)$. Set $m := [\frac{c_2}{3}]$.

i) If H is a general plane then $h^1(F_H(k)) = 0$ if $k \geq c_2 - m$.

ii) For every integer $t \geq c_2 - m + 1$, $h^1(F(t-1)) \neq 0$ implies $h^1(F(t-1)) > h^1(F(t))$.

$\diamondsuit$

Theorem III.4 : Let F be as in the corollary above. Then $h^1(F(k)) = 0$ if $k \geq \frac{c_2^2}{2} + \frac{c_2}{2} + \frac{m(1-m)}{2} - c_1(c_2-m) + \frac{c_1c_2}{2} - \frac{c_3}{2}$. $\diamondsuit$

The proves of the Corollary and the Theorem follow immediately from Theorem III.2, Lemma II.1 and Lemma II.4. $\diamondsuit$

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5 A vanishing for the cohomology of the general plane restriction of a stable rank 2 vector bundle on P^3

Throughout this paper we will assume all sheaves to be normalized.

This work arise in the attempt of solving the following problem: given a stable rank two vector bundle E on $\mathbf{P}^3$ and a general plane H, what can be said about the cohomology of E_H? In particular, is it possible to determine a function $f(c_2)$ s.t. $h^1(E_H(p)) = 0$ if $p \geq f(c_2)$? The starting point of the analysis, of course, is the Barth's restriction theorem, which says that E_H is stable if H is general. Furthermore, by a result of Hartshorne (see [H1;7.4]), if $\mathcal{E}$ is a stable rank two vector bundle on $\mathbf{P}^2$ s.t. $c_1(\mathcal{E}) = 0$, then $h^1(\mathcal{E}(p)) = 0$ if $p \geq c_2 - 2$. Combining the two results we have a first estimate for the function $f(c_2)$: $f(c_2) := c_2 - 2$. Surprisingly, as pointed out in the papers [E] and [F], the vanishing thus obtained is not sharp. In this paper we succeed in the attempt of finding the better shape for the function f, by proving a vanishing which turns out to be sharp. Moreover, we will be able to classify the "extremal cases" for which the vanishing is optimal. The main result of this paper is encoded in the following

Theorem 0: let E be a stable rank two vector bundle on $\mathbf{P}^3$ s.t. $c_1(E) = 0$, $c_2 \geq 20$. Suppose H to be a general plane. Then $h^1(E_H(p)) = 0$ if $p > \frac{c_2(E)-3}{2}$. Moreover, if $c_2(E)$ is odd and $h^1(E_H(\frac{c_2(E)-3}{2})) \neq 0$ for any plane H of $\mathbf{P}^3$, then E belongs to one of the following classes:

a) E is a t'Hooft bundle associated to $c_2 + 1$ skew lines lying on a quadric of $\mathbf{P}^3$;

b) the spectrum of E is maximal, hence $E(1)$ has a section which is a multiplicity 2 structure Y on a degree $\frac{c_2-1}{2}$ plane curve Y_0 s.t. $\omega_Y \simeq O_Y(-2)$.

The outline of the paper is the following. In the first section we will remark that, if we suppose $\mathcal{E}$ to be a stable rank two vector bundle on $\mathbf{P}^2$ s.t. $h^1(\mathcal{E}(p)) \neq 0$ for p big enough, then the zero locus of a least degree section of such a bundle has a high degree subscheme contained in a plane curve of degree ≤ 2. In the second section we will prove one half of theorem 0, by showing that, if the general plane restriction E_H has an unstable conic of order bigger than $\frac{c_2-1}{2}$, then E is a t'Hooft bundle. In the third section we conclude the proof of the theorem. In the last section we will apply theorem 0 to a vanishing for the cohomology of a stable rank two vector bundle on $\mathbf{P}^3$ (see [E],[F] and [G]).

I Some preliminaries

Let us start with the following:

Lemma I.1: Let $\mathcal{E}$ be a stable rank two vector bundle on P^2 with $c_1 = 0$, $c_2 \geq 9$. Set $t = min\{n \in \mathbf{N} : h^0(\mathcal{E}(n)) \neq 0\}$. Assume $h^1(\mathcal{E}(c_2 - k)) \neq 0$, with $k \leq \frac{c_2+3}{2}$, and consider a section of $\mathcal{E}(t)$: $0 \to O \to \mathcal{E}(t) \to I_Z(2t) \to 0$, then one of the following is true:

i) Z contains a subscheme Z', of length $z' \geq c_2 + t - k + 2$, which is contained in a line;

ii) Z contains a subscheme $Z"$, of length $z" \geq 2(c_2 + t - k + 1)$, which is contained in a conic.

Proof: We would like to apply Corollary 2 of [EP] with $s = 3$. Set $\tau = max\{n \in \mathbf{N} : h^1(I_Z(n)) \neq 0\}$, then from the hypothesis we have $\tau \geq c_2 - k + t$. We have to check a) $d := degZ \geq 9$, b) $\tau \geq \frac{d}{3}$. a) follows directly from $c_2 \geq 9$. b) follows if we prove that $t + c_2 - k > \frac{d}{3} = \frac{c_2+t^2}{3}$. The hypothesis $k \leq \frac{c_2+3}{2}$ implies that it suffices to show that $3t + \frac{c_2}{2} > t^2 + \frac{9}{2}(+)$. If $t = 1$, this follows from $c_2 \geq 9$. If $t \geq 2$, we start by recalling that the proposition 7.4 of [H1] implies $c_2 - k < c_2 - t^2 - 1$, thus $2 \leq t \leq (k-1)^{\frac{1}{2}}$. To check $(+)$ it is then sufficient to show that $3t + \frac{c_2}{2} \geq k - 1 + \frac{9}{2}$ which is always true because $k - 1 + \frac{9}{2} \leq \frac{c_2}{2} + 5$ by hypothesis and $\frac{c_2}{2} + 5 \leq \frac{c_2}{2} + 3t$ if $t \geq 2$. ◇

Corollary I.2: We assume the same hypothesis of lemma I.1. One of the following is true:

a) there is an unique line L containing a subscheme $Z' \subset Z$, of length $z' \geq c_2 + t - k + 1$;

b) there is a reduced conic C containing a subscheme $Z" \subset Z$, of length $z" \geq 2(c_2+t-k+1)$. Moreover, if C is reducible, we can suppose each of the components of C to contain at least $c_2 + t - k + 1$ points of Z".

Proof: from lemma I.1 we know that Z contains either a subscheme Z' of length $\geq c_2 + t - k + 1$ contained in a line L, or a subscheme Z" of length $\geq 2(c_2 + t - k + 1)$ contained in a conic C. What we have to show here is that, if L is not the unique line containing a subscheme of Z of length $\geq c_2 + t - k + 1$, then there is a reducible conic containing a subscheme of Z of length $\geq 2(c_2+t-k+1)$ (s.t. every component does contain at least $c_2 + t - k + 1$ points), and that if the conic C cannot supposed to be reduced, then there is a unique line containing a subscheme of Z of length $\geq c_2 + t - k + 1$. But this is very simple to prove, indeed if the line of the point i) of the lemma is not unique, then, simply by taking the union of two such lines, we get a reducible conic of the type we need. Moreover, if the conic C cannot supposed to be reduced, its support, which is a line, does contain a subscheme of Z of length $\geq c_2 + t - k + 1$. Finally this line is necessarily unique otherwise C could be supposed to be reduced. $\diamondsuit$

Proposition I.3: If the statement b) of the corollary I.2 happens, then $t = 1$, $k = \frac{c_2+3}{2}$ and Z is contained in the conic C.

The proof of the proposition rests on the following:

Lemma I.3': The conic C of the statement b) of the corollary I.2 is unstable of order $r \geq c_2 - k + 1$.

Proof of the lemma: we begin by supposing C to be smooth. The exact sequence $0 \to O(-t) \to \mathcal{E} \to I_Z(t) \to 0$ $(+)$ tensored by O_C yields a surjection $\mathcal{E}_C \to O_C(t) \otimes O_C(-Z") \to 0$, whose kernel is $O_C(-t) \otimes O_C(Z")$.

The sequence splits because $degO_C(-t) \otimes O_C(Z") > 0$, thus the statement follows by the hypothesis on the length of Z". Let us suppose now the conic C to be reducible: $C = L_1 \cup L_2$. Again, tensoring the sequence $(+)$ by O_C we get a surjection as above. Let us call K the kernel of such a surjection. Tensoring by O_{L_i} the sequence $0 \to K \to \mathcal{E} \to O_C(t) \otimes O_C(-Z") \to 0$ we see that $K\mid_{L_i} \simeq O_{L_i}(degL_i \cap Z" - t)$. As $degL_i \cap Z" - t \geq c_2 - k + 1$ and because of the fact that $h^0(K) = h^0(K\mid_{L_1}) + h^0(K\mid_{L_2}) - 1$, we see that $h^0(\mathcal{E}_C(-c_2 + k - 1)) \neq 0$ and the lemma follows. $\diamondsuit$

Remark The same proof of lemma I.3' shows that a line coming from the statement a) of corollary I.2 is unstable of order $\geq c_2 - k + 1$.

Proof of the proposition I.3: let us perform a reduction step with respect to C:

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow I_C(-r) \longrightarrow 0 \qquad (++)$$

with $r \geq c_2 - k + 1$. From proposition 1.1 of [H3], we deduce $c_1(\mathcal{E}') = -2$, $c_2(\mathcal{E}') \leq c_2 - 2(c_2 - k + 1) \leq 1$, thus $c_1(\mathcal{E}'(1)) = 0$, $c_2(\mathcal{E}'(1)) \leq c_2 - 2(c_2 - k + 1) - 1 \leq 0$, and the equality in the last formula implies $k = \frac{c_2+3}{2}$. On the other hand, (++) and the stability of $\mathcal{E}$ imply that $\mathcal{E}'$ is semistable and $c_2(\mathcal{E}'(1)) \geq 0$. Putting all together we conclude that $k = \frac{c_2+3}{2}$, $\mathcal{E}' \simeq O(-1) \oplus O(-1)$ (indeed $c_1(\mathcal{E}'(1)) = c_2(\mathcal{E}'(1)) = 0$ implies $\mathcal{E}'(1) \simeq O \oplus O$), and $t = 1$ (indeed $h^0(\mathcal{E}'(1)) = 2$). Finally from the statement b) of corollary I.2 we conclude that $degZ" = c_2 + 1 = degZ$ and the zero locus of a least-degree section of $\mathcal{E}$ is contained in C. $\diamondsuit$

Corollary I.4: If the conic C is smooth (resp. if $C = L_1 \cup L_2$), then $\mathcal{E}\mid_C \simeq O_C(-\frac{c_2-1}{2}) \oplus O_C(\frac{c_2-1}{2})$ ($\mathcal{E}\mid_{L_i} \simeq O_{L_i}(-\frac{c_2-1}{2}) \oplus O_{L_i}(\frac{c_2-1}{2})$).

Proof. It follows from the fact that the order of instability of $\mathcal{E}\mid_C$ is $\frac{degZ"-2}{2}$ (see the proof of lemma I.3'). $\diamondsuit$

II A characterization of t'Hooft bundles

Suppose now E to be a stable rank two vector bundle on $\mathbf{P}^3$ with $c_1 = 0$, $c_2 \geq 20$ and s.t. $h^1(E_H(c_2 - k)) \neq 0$ if H is general and if k is constrained as in lemma I.1. Then, from corollary I.2 and proposition I.3, we know that there are only two possibilities for the restriction of E to H:

a) there exists an unique line $L_H \subset H$ which is unstable of order $r \geq c_2 - k + 1$ (see the remark following lemma I.3');

b) the second Chern class is odd, $k = \frac{c_2+3}{2}$ and there exists a section $s \in H^0(E_H(1))$ s.t. the zero locus Z_H of s is contained in a reduced conic C_H.

What we are going to do now is to analyze what kind of implications the statement b) above has. Let us assume, then, c_2 to be odd, $h^1(E_H(\frac{c_2-3}{2})) \neq 0$, $h^0(E_H(1)) = 2$ for H general plane. The goal of this section is to prove the following:

Theorem II.1: let us assume that the statement of point b) above holds. Then E is a t'Hooft bundle associated to c_2+1 skew lines on a quadric Q.

The proof of this theorem will follow after two propositions and a lemma (see proposition II.4).

Let us start with the following:

Proposition II.2: let E be as above. Then we have $h^0(E(1)) \neq 0$. Moreover, if Y is the zero locus of a section $s \in H^0(E(1))$, then Y is contained in a quadric Q.

Proof: set $l := min\{n \in \mathbf{N} : \ h^0(E(n)) \neq 0\}$. Let us consider the usual sequence associated to a section $s \in h^0(E(l))$

$$0 \longrightarrow O \longrightarrow E(l) \longrightarrow I_Y(2l) \longrightarrow 0$$

Twisting and tensoring by O_{C_H} we find a surjection $f : E_{C_H} \to O_{C_H}(l) \otimes O_{C_H}(-\Delta) \to 0$ where $\Delta := Y \cap C_H$. Set $K := kerf$. If C_H is smooth, then $K \simeq O_{C_H}(\Delta) \otimes O_{C_H}(-l)$. If $C_H = L_1 \cup L_2$, then $K\ |_{L_i} \simeq O_{L_i}(\delta_i - l)$ where $\delta_i := degY \cap L_i$. On the other hand corollary I.4 says that, in the first case, $E\ |_{C_H} \simeq O_{C_H}(-\frac{c_2-1}{2}) \oplus O_{C_H}(\frac{c_2-1}{2})$, and, in the second, $E\ |_{L_i} \simeq O_{L_i}(-\frac{c_2-1}{2}) \oplus O_{L_i}(\frac{c_2-1}{2})$. Recalling that $-l > -(3c_2+1)^{\frac{1}{2}}$ (see [H3]), and that $(3c_2+1)^{\frac{1}{2}} < \frac{c_2-1}{2}$ because $c_2 \geq 20$, we conclude $-l > -\frac{(c_2-1)}{2}$. Putting all together we see that the sequence $0 \to K \to E_{C_H} \to O_{C_H}(\Delta) \otimes O_{C_H}(-l) \to 0$ ($0 \to K\ |_{L_i} \to E_{L_i} \to O_{L_i}(\delta_i - l) \to 0$) splits when C_H is smooth (when $C_H = L_1 \cup L_2$), thus $deg\Delta = (c_2 - 1 + 2l) \geq 20$ ($deg\delta_i = \frac{c_2-1}{2} + l \geq 10$). In any case every quartic containing Δ must contain C_H, thus there cannot be syzygyes in degrees 3 and 4, hence, by a theorem of R. Strano (see [S]), the conic C_H lifts to a quadric Q containing a curve Y' s.t. $Y' \cap H = \Delta$. The conclusion comes at once by lemma II.3 bellow. $\diamondsuit$

Lemma II.3: let F be a stable rank two reflexive sheaf on $\mathbf{P}^3$ with $c_1 = 0$. Set $l := min\{n \in \mathbf{N} : \ h^0(F(n)) \neq 0\}$ and call Y the zero locus of a section $s \in H^0(F(l))$. Suppose there exists a curve Y' and a quadric Q s.t. $Y' \subset Y$, $Y' \subset Q$ and $degY' \geq c_2 - 1 + 2l$. Then $l = 1$ and $Y' = Y$.

Proof: let us call $W := Res_Q Y$. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & O(-l) & \to & O(-l) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F' & \to & F & \xrightarrow{f} & I_{Y\cap Q,Q} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & I_W(l-2) & \to & I_Y(l) & \to & I_{Y\cap Q,Q} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $F' := kerf$ and the bottom exact sequence is the usual one. From corollary 1.5 of [H2], we conclude that F' is a reflexive sheaf. Furthermore, an easy computation show that $c_1(F'(1)) = 1$ and $c_2(F'(1)) \leq 2 - 2l$ (see [H3] prop. 1.1). On the other hand the diagram above shows that $F'(1)$ is semistable hence $c_2(F'(1)) \geq 0$. Putting all together we have $l = 1$ and deg $Y' = degY = c_2 + 1$. $\diamondsuit$

Proposition II.4: the quadric Q of the last proposition is smooth and E is a t'Hooft bundle associated to $c_2 + 1$ skew lines on Q.

Proof: firstly Q cannot be reducible. Indeed if it were $Q = H_1 \cup H_2$, $H_1 \neq H_2$, then, by corollary I.4, both H_1 and H_2 would be unstable of order $\geq \frac{c_2-1}{2}$. From the restriction sequence $0 \to E(-1) \to E \to E_H \to 0$ we see that $h^1(E(-\frac{c_2+1}{2})) \neq 0$, hence $spE = (-m, -m+1, ..., -1, 0, 1, ..., m-1, m)$ where $m := \frac{c_2-1}{2}$ and spE is the spectrum of E (see theorems 7.1-5 of [H2]). Furthermore lemma 9.11 of [H2], implies that there is an unique unstable plane of order $\frac{c_2-1}{2}$ for E and this contradicts the hypothesis $H_1 \neq H_2$. Hence Q is irreducible. As in the last proposition, call Y the zero locus of a section of $E(1)$. We have $degY = c_2 + 1 \geq 21$, $\omega_Y \simeq O_Y(-2)$ and $deg\omega_Y = -2(c_2+1)$. A curve with arithmetic genus ≤ 0 lying in an irreducible quadric must fall in one of the following classes: a) a line L in a quadric cone with $deg\omega_L = -2$; b) a conic C in a quadric cone with $deg\omega_C = -4$; c) a rational normal cubic C in a quadric cone with $deg\omega_C = -6$; d) a curve W of type $(0, r)$ in a smooth quadric with $deg\omega_W = -2r$; e) a curve W' of type $(1, r)$ in a smooth quadric with $deg\omega_{W'} = -2$; f) a quartic X on a smooth quadric with $deg\omega_X = 0$. Only case d) is compatible with the condition on the degree. $\diamondsuit$

III Conclusion of the proof of Theorem 0

In this section we are going to consider the case a) at the beginning of section II. Therefore we suppose that, for the general plane H, there exists an unique line L_H which is unstable for E of order $\geq c_2 - k + 1$, with $0 \leq k \leq \frac{c_2+3}{2}$. What we are going to do is to prove the following

Proposition III.1: Let E be as above. Then c_2 is odd, $k = \frac{c_2+3}{2}$ and the spectrum of E is maximal (see [H.2] section 9 for a description of such bundles).

The proof of the proposition rests on the following

Lemma III.2: there exists an unstable plane K for E of order $r \geq c_2 - k + 1$ s.t., if H is general, then $L_H = K \cap H$.

We would like to give two different proves of this lemma. The first is a simple consequence of a theorem of R. Strano. The second is more geometric and improves a method used by the author in the paper [F].

First proof : set $l := min\{n \in \mathbf{N} : h^0(E(n)) \neq 0\}$ and call X the zero locus of a section $s \in H^0(E(l))$. Tensoring by O_{L_H} the exact sequence associated to s: $0 \to O(-l) \to E \to I_X(l) \to 0$ we get a surjection $E_{L_H} \to O_{L_H}(l - \Gamma) \to 0$ ($\Gamma := L_H \cap X$), whose kernel is $O_{L_H}(\Gamma - l)$. As $l < (3c_2+1)^{\frac{1}{2}}$ and $E_{L_H} \simeq O_{L_H}(r) \oplus O_{L_H}(-r)$ with $r \geq c_2 - k + 1 \geq \frac{c_2-1}{2} > (3c_2+1)^{\frac{1}{2}}$ we have that $-l > -r$ and the sequence $0 \to O_{L_H}(\Gamma - l) \to E_{L_H} \to O_{L_H}(l - \Gamma) \to 0$ splits thus $deg\Gamma = r + l \geq c_2 - k + 1 + l \geq \frac{c_2-1}{2} + l > 20$. It is clear that every cubic containing Γ must contains L_H, thus there cannot exist syzygyes in degrees ≤ 3. By a theorem of R. Strano (see [S]), the line L_H lifts to a plane K containing a curve X' s.t. $X' \cap K = \Gamma$. Obviously K is unstable of the same order of its general line.

Second proof : by letting the general plane H to vary, we obtain a two dimensional family of lines $U := \{L_H : h^0(E_H) = 0\} \subset \mathbf{G}(1, \mathbf{P}^3)$ in the grassmannian. Following [E] we will call super jumping line (s.j.l) a line which is in the closure of U into the grassmannian. Proposition I.3 of [F] shows that there are planes containing infinitely many s.j.l. Arguing as in lemma I.5 of [F] it is easy to see that a plane containing infinitely many s.j.l is unstable of order $\geq \frac{c_2}{4} - \frac{1}{2}$ and the same proof of lemma I.6 of [F] shows that there can be only finitely planes containing infinitely many s.j.l and in fact only one by the uniqueness of L_H for H general. Call K such a plane it is clear that K is unstable for E of the same order of its general line. ♢

Proof of proposition III.1: from the restriction sequence $0 \to E(-1) \to E \to E_K \to 0$ we see that, if K is unstable of order r, then $h^1(E(-r-1)) \neq 0$. The argument at the beginning of the proof of proposition II.4 shows that $r \leq \frac{c_1-1}{2}(+)$ and, if the equality in the last formula holds, the spectrum of E is maximal (in this case the description of E is given in lemma 9.15 of [H2]). Combining (+) with the hypothesis on r (see lemma III.2) we get $k \geq \frac{c_2+3}{2}$ and equality implies maximality of the spectrum. The conclusion comes at once by the hypothesis on k. ♢

IV A vanishing for the cohomology of stable rank 2 vector bundles on P^3

In this section we would remark how theorem 0 allows us to give a vanishing for the cohomology of a stable rank two vector bundle on $\mathbf{P}^3$. (see for more details [G], [E] section II and [F] section II). The vanishing to which we allude is the following

Theorem III.1: Let E be a stable rank two vector bundle on $\mathbf{P}^3$ with $c_1 = 0$. Then $h^1(E(k)) = 0$ if $k \geq \frac{c_2^2}{2} + \frac{c_2}{2} + \frac{m(1-m)}{2} - c_1(c_2 - m) + \frac{c_1 c_2}{2}$, where $m = [\frac{c_2-1}{2}]$ ♢.

The proof of the theorem rests on the following

Lemma : Let F be a stable rank-two reflexive sheaf on $\mathbf{P}^3$ s.t. $c_2 > 35$. Suppose there exists an integer $x > 0$ s.t. $h^1(F_H(k)) = 0$ if $k \geq x$ and H is a general plane. Then $h^1(F(x)) \leq x\{c_2 - \frac{x+3}{2} - c_1\} + c_2(1 + \frac{c_1}{2}) - \frac{c_3}{2}$ and $h^1(F(t)) = 0$ if $t \geq x\{c_2 - \frac{x+1}{2} - c_1\} + c_2(1 + \frac{c_1}{2}) - \frac{c_3}{2}$.

Proof of the lemma: (see [F] Lemma II.4). ♢

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