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q-Euclidean Covariant Qunatum Mechanics on $\mathbf{R}_q^N$: Isotropic Harmonic Oscillator and Free Particle

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Contents

Chapter 1

Introduction

The development of physics has been often characterized by the introduction of some more general and accurate theories as sort of deformations of already known and accepted ones. A well-known example is special relativity, which can be viewed as a deformation of Galileo's relativity; the velocity of light plays the role of deformation parameter. Another example is quantum mechanics, which can be seen as a deformation of classical mechanics, Planck constant being the deformation parameter.

In recent years the mathematical development of so-called quantum groups [7] and related objects (quantum spaces [35], braid groups [26]) has disclosed a bunch of new opportunities [53, 49, 36] for implementing in quantum physics deformations of both space(time) geometry and related fundamental symmetries. These fascinating mathematical objects are highly nontrivial examples of noncommutative geometries [5].

One of the essential features of quantum mechanics is that it invalidats the classical geometrical description of the state of a physical system as a point in the corresponding phase space; the naive formulation of this fact are Heisenberg's indetermination relations. The notion of a " point " in continuous phase space is replaced by the more general notion of a ($\mathbf{C}^*$-)algebra of operators, acting on a Hilbert space. Geometry may be recovered as a useful structure for the spectral decomposition of operators. For instance, the spectral decomposition of the vector components of the position operator of one quantum particle typically is realized on a manifold coinciding with classical configuration space. This is due to the axiom that these components commute. In a noncommutative-geometric approach to quantum mechanics one essentially releases the latter axiom.

As known, formulation of ordinary quantum field theory is based on locality, i.e. on fields and interactions which are functions of points of a classical (i.e. continuous) spacetime manifold. It is a common hope, and indeed there are some encouraging indications, that a list of long-standing problems in local quantum field theory (ultraviolet divergencies, regularization [32], quantization of gravity [30],...) could find a natural solution in the framework of a formulation of QFT on " noncommutative-geometric manifolds ", thought as deformations of ordinary spacetime ones; of course the effects of such a deformation should be practically undetectable at least in the domain of observability of already well-explained physical phenomena. A first step in this direction would be to reach the same target for quantum mechanics of a finite number of degrees of freedom, starting from a single first-quantized particle.

This is the common rationale behind several recent endeavours of developing and applying the formal machinery of noncommutative geometry to some simple quantum systems, and is also the main motivation underlying the work of this thesis.

1.1 A simple quantum system

For the sake of concreteness, and in order to introduce some important concepts which will be heavily used in next chapters it is useful to consider a simple realistic example of a physical system, a nonrelativistic particle in ordinary $\mathbb{R}^3$ space, say. Let r,π be the position and the momentum of the particle; (r,π) define the state of the system. We introduce a reference frame S and the position and momentum (vector) functions $\vec{x},\vec{p}$:

$$\vec{x} : r \to \vec{x}(r) \in \mathbb{R}^3 \qquad \vec{p} : \pi \to \vec{p}(\pi) \in \mathbb{R}^3; \tag{1.1}$$

$x^i(r), p^i(\pi)$ are respectively the position and momentum vector components of the particle in the frame S. The functions x^i, p^i commute:

$$[x^i, x^j] = 0 = [p^i, p^j] = [x^i, p^j]. \tag{1.2}$$

All other observables are functions (power series) of x^i, p^j and make up a (commutative) $\mathbf{C}^*$-algebra $Fun(\mathbb{R}^3_{\bar{x}} \times \mathbb{R}^3_{\bar{p}})$. To switch to ordinary quantum mechanics one replaces commutation relations (1.2) by

$$[X^i, X^j] = 0 = [P^i, P^j] \qquad [X^i, P^j] = i\hbar\delta^{ij} \neq 0; \tag{1.3}$$

X^i, P^j generate a $\mathbf{C}^*$-algebra of operators including the observables of the system. A " noncommutative phase space " takes the place of the classical one. The further step towards a " noncommutative configuration space " would be implemented by replacing commutation relations $(1.3)_a$ with some more general ones where

$$[X^i, X^j] \neq 0. \tag{1.4}$$

Of course, there are infinitely too many ways to do such a generalization, if one has no other requirement but the fact that the new relations should be " tiny " deformations of the classical ones (i.e. undetectable in the domain of already well explained physical phenomena).

Here one makes symmetries enter the game. It is hard to overestimate the importance of symmetries for the progress of physics in the past century; the latter would have been unconceivable without physicists' guiding principle that all " good " fundamental physical laws should be covariant w.r.t. some " well chosen " symmetry transformation. The same attitude is taken over here. We stick to our simple model and ask what kind of symmetry transformation and what kind of deformation (1.4) could be compatible with each other. The physical description of the " undeformed " model is covariant w.r.t. the Galileo group, in particular we stick to its covariance w.r.t. its $G := SO(3, \mathbb{R})$ rotation subgroup. As a first step, it is convenient to recall how one can formulate this covariance using the dual languages of " coaction + corepresentations ".

Let $Fun(G)$ be the commutative algebra of functions on the group $G := SO(3, \mathbb{R})$. The functions can be expressed as power series in the tautological functions $T^i_j \in Fun(G)$, $i, j = 1, 2, 3$, which are defined by $T^i_j(g) = g^i_j$ ($g \in G$ and $\|g^i_j\| \in adj(G)$, where $adj(G)$ denotes the adjoint representation of G). Any vector $\vec{V} \equiv (V^i)$ provides a fundamental corepresentation of the (left) coaction $\phi_L : Fun(\mathbb{R}^3) \to Fun(G) \otimes Fun(\mathbb{R}^3)$ of $Fun(G)$:

$$\phi_L(V^i) := T^i_j \otimes V^j. \tag{1.5}$$

If $\vec{V} = \vec{X}$ ($\vec{X} \equiv$the position operator of the quantum particle in a reference frame S), then the position operator $\vec{X}'$ of the particle in the frame S' obtained from S by a rotation g will be given by

$$[\phi_L(X^i)](g, \cdot) := [T^i_j \otimes X^j](g, \cdot) = T^i_j(g) X^i = g^i_j X^j = X'^i. \tag{1.6}$$

The same works if $V^i = x^i \equiv$ *classical coordinates.* Only, in the quantum case we require in addition that the transformation (1.6) of X^i is unitarily implemented. ϕ_L is extended as an algebra homomorphism to functions of V^i. of $Fun(G)$, in the sense that

$$\phi_L(ab) := \phi_L(a)\phi_L(b), \qquad a, b \in Fun(\mathbb{R}^3_{\vec{V}} \times \mathbb{R}^3_{\vec{p}}), \tag{1.7}$$

and one says that $Fun(\mathbb{R}^3_V)$ is a " comodule " of $Fun(G)$. For instance,

$$\phi_L(X^iX^j) = T^i_hT^j_k \otimes X^hX^k. \tag{1.8}$$

This formula is consistent with the commutation relations

$$[X^i, X^j] = 0, \qquad\qquad [T^i_h, T^j_k] = 0 \tag{1.9}$$

One can guess from formula (1.8), and indeed we are going to check now, that deformation of commutation relations $(1.9)_a$ (while keeping the definition (1.5),(1.7) of ϕ_L) is strictly coupled to deformation of commutation relations $(1.9)_b$, and conversely. In fact (1.5),(1.7) imply

$$\phi_L([X^i, X^j]) = T^i_hT^j_k \otimes X^hX^k - T^j_kT^i_h \otimes X^kX^h. \tag{1.10}$$

Then

$$\{(1.10)\ \&\ [X^i, X^j] = 0\} \Rightarrow 0 = \phi_L([X^i, X^j]) = (T^i_hT^j_k - T^j_kT^i_h) \otimes X^hX^k \Rightarrow \tag{1.11}$$

$$\Rightarrow [T^i_h, T^j_k] = 0. \tag{1.12}$$

To prove the converse implication we need to specify the kind of deformed commutation relations we wish for the coordinates. We assume that they are homogeneous:

$$\mathcal{P}_A{}^{ij}_{hk}X^hX^k = 0, \tag{1.13}$$

with a nontrivial matrix $\mathcal{P}_A$. Then

$$\{(1.13)\ \&\ [T^i_h, T^j_k] = 0\} \Rightarrow 0 = \phi_L(\mathcal{P}_A{}^{ij}_{hk}X^hX^k) = \mathcal{P}_A{}^{ij}_{hk}T^h_lT^k_m \otimes X^lX^m \Rightarrow$$

$$\Rightarrow \mathcal{P}_A{}^{ij}_{hk}T^h_lT^k_m = T^i_hT^j_k\mathcal{P}_A{}^{hk}_{lm} \Rightarrow \mathcal{P}_A{}^{ij}_{hk} \propto (\delta^i_h\delta^j_k - \delta^i_k\delta^j_h)$$

$$\Rightarrow \{(1.13)\ reads\ [X^i, X^j] = 0\}; \tag{1.14}$$

the second implication in the second line is due to the fact that the relations on its left must be equivalent to the commutation relations $[T^i_h, T^j_k] = 0$.

We conclude that relations $(1.9)_a, (1.9)_b$ have to be deformed in a related way. In other words, also the manifold G has to be replaced by a " noncommutative-geometric manifold ", and one should look for a different implementation of symmetries at the quantum level than the standard one based on unitary representations of Lie groups [52]. This should not be so unwelcome, since it is after all contrasting with the spirit of quantum mechanics that one can conceive an " infinitely precise change of frame " corresponding to the point $g \in G$, although one cannot determine in a frame the position of a particle with infinite precision. Rather, it would be quite natural that both X and T should be thought as operators.

In our simple model the introduction of the quantum space $\mathbb{R}_q^3$ and of the quantum group $SO_q(3,\mathbb{R})$ (more precisely, of the Hopf algebra $Fun(SO_q(3,\mathbb{R}))$ is conceived to perform a consistent q-deformation of both commutation relations $(1.9)_a$ and $(1.9)_b$.

Generally speaking, for any classical Lie group G [9] the Hopf algebra $Fun(G_q)$ of " function-on-the-group " type consists of an algebra structure & a compatible Hopf (coalgebra & antipode) structure; when $q = 1$ the former is trivial (i.e. abelian), the latter is equivalent to the Lie group structure of G.

1.2 Subject and plan of the thesis

Keeping the previous example in mind, we come now to the actual subject of this thesis. In this work we deal with a q-Euclidean covariant formulation of quantum mechanics of one particle on the Euclidan quantum space $\mathbb{R}_q^N$ $(N \geq 3)$. $\mathbb{R}_q^N$ is a comodule of the quantum group $SO_q(N)$, which is the q-deformation of the Lie group $SO(N)$. [1] The q-Euclidean quantum group E_q^N which has been considered is obtained as the (braided) semidirect product $\mathbb{R}_q^N \rtimes SO_q(N)$, in manifest analogy with the most obvious definition of the classical Euclidean group E^N. $q \in \mathbb{C}$ is the parameter of deformation; in all physical applications we will take $q \in \mathbb{R}^+$ to guarantee the existence of the " complex conjugation " $*$. Precise definitions and related references will be given at the right time. The main motivation for considering the quantum Euclidean objects (group, space,...) is that they should be the right ones to be used in a future q-deformed Euclidean formulation of quantum field theory.

We collect all our contributions to the subject together with the necessary preliminaries. The original material is enriched here and there with some further elaboration and is presented in a rather self-contained way.

Chapter 2 is a review chapter introducing to the Euclidean Hopf algebras (in fact there

[1] The $\hat{R}$ braid matrix of $SO_q(N)$ is not of Hecke, but of BWM type. Compared with the more known $SL_q(N)$ case, this implies some complications in all computations . Generally speaking, only a $\hat{R}$ of the second type admits a " singlet projection operator ", implying the existence of a q-covariant metric on the corresponding quantum space; a metric is necessary to deal with a realistic q-version of space(time).

are *two* independent nonreal Hopf structures) $Fun(E_q^N), Fun(\bar{E}_q^N)$ of " functions-on-the-group " type and to their comodules, in particular $Fun(\mathbb{R}_q^N)$. As anticipated, $E_q^N, \bar{E}_q^N$ are essentially semidirect products of the simpler objects $SO_q(N), \mathbb{R}_q^N$. Essential bibliography are Ref. [9], [43], [25].

Chapter 3 is also of preliminary character and deals with the implementation of differential calculus on $\mathbb{R}_q^N$ and its formal developments. From the introduction of one new notion (the exterior derivative) on $\mathbb{R}_q^N$ one derives consistent notions of both differential forms and partial derivatives. Then we concentrate on the study of the " differential algebra $Diff(\mathbb{R}_q^N)$ " generated by coordinates x and derivatives ∂. Finally we derive some useful results (ϵ_q-tensor, volume form, Hodge duality,...) from properties of the exterior algebra. Essential references are [50, 2, 39, 40, 12] plus some parts of [11, 14].

Chapter 4 deals with the dual objects of those of Chapter 2. From the (co)representation point of view, the essential point to keep in mind is that a comodule (e. g. $\mathbb{R}_q^N$) in the approach of Chapter 2 is a module in the dual one, and viceversa. As known, the dual of the quantum space $\mathbb{R}_{q,\vec{x}}^N$ of x-coordinates is the quantum space $\mathbb{R}_{q,\vec{\partial}}^N$ of partial q-derivatives, the dual of $Fun(SO_q(N))$ is $U_q(so(N))$ [9, 7]. We realize the dual Hopf algebras $U_q(e^N), U_q(\bar{e}^N)$ of $Fun(E_q^N), Fun(\bar{E}_q^N)$ as subalgebras of the differential algebra $Diff(\mathbb{R}_q^N)$. These results were obtained by us essentially in Ref. [14, 15], and have a sufficiently simple formulation in order to tackle the construction of the corresponding representations, which is the subject of Chapter 7.

Chapter 5 is devoted to the construction of a q-integration on $\mathbb{R}_q^N$ starting from the requirement of its q-Euclidean covariance, and is based on our work in Ref. [11], Sects. 3,4, & Appendices. Actually, invariance of q-integration w.r.t. to translations (in infinitesimal form, Stoke's theorem) alone is sufficient to guarantee this larger covariance, and is used as a constructive tool. We compare our original results with some subsequent works dealing with the same subject, and we find essentially equivalent results. This concludes the algebraic part of the thesis.

Part two is devoted to the implementation of the physical ideas described at the beginning of this introduction. We consider only two physical one-particle quantum models, the isotropic harmonic oscillator (Chapter 6) and the free particle on $\mathbb{R}_q^N$ (Chapter 7), but all the necessary ingredients are available for considering with the same approach one-particle quantum systems with arbitrary potentials $V(x)$ on $\mathbb{R}_q^N$. Observables are found within $Diff(\mathbb{R}_q^N)$. We propose a *nonstandard* approach based on a *pair* of conjugated configuration-space realizations (the unbarred and the barred) for implementing hermiticity of observables depending on q-derivatives; it is based on an idea that we first introduced in Ref. [11]. In either case one finds a nice and consistent quantum mechanical model for any $q \in \mathbb{R}^+$, namely one finds a positive definite scalar product, a complete set of commuting observables, energy spectrum bounded from below, etc. In the limit $q \to 1$ we find the corresponding classical models (correspondence principle).

A mathematically-oriented synthesis of Chapter 7 is that it is devoted to $*$-representations of the q-Euclidean algebra $u_q(e^N)$ (of u.e.a. type). From this viepoint the q-representations can be considered as lattice-regularizations of the classical ones, in the sense that momentum operators have discrete (rather than continuous) spectra and their eigenvectors are square integrable functions (rather than generalized ones). We can already give at this preliminary stage of the exposition a flavour of how discrete spectra come about, sticking for simplicity to the case $N = 3$. The momentum components are p^{-1}, p^0, p^1 with q-deformed commutation relations

$$p^{-1}p^0 = qp^0p^{-1}, \qquad\qquad p^0p^1 = qp^1p^0. \tag{1.15}$$

p^0 is hermitean. If $|\psi>$ is an eigenvector of p^0 with eigenvalue M, then also $(p^{\pm 1})^m|\psi>$ ($m \in \mathbb{N}$) is, with eigenvalue $Mq^{\pm m}$: the p^0-spectrum is an " exponential " lattice.

We conclude by observing that already at the present stage of investigations q-deformation appears as a promising lattice-regularization device in field theory, at least for Euclidean QFT. We justify this statement just with a sketchy comparison with two popular and widely used regularization methods in Euclidean QFT. First, representations of $u_q(e^N)$, thought as lattice-regularized versions of representations of the (undeformed) Euclidean algebra $U(e^N)$, by construction are closed under the (discrete) transformation implemented through " a lattice-regularized " version of $U(e^N)$ itself, $u_q(e^N)$. On the contrary, standard lattice-regularized momentum space [6] is not invariant w.r.t. the action of a lattice-regularized (i.e. discrete) version of the whole Euclidean group. Second, the existence of a q-deformed completely antisymmetric tensor for $Fun(SO_q(N))$ (see section 3.4) should in principle allow the application of a regularization scheme based on q-Euclidean symmetry also to field theories with chiral coupling. In dimensional regularization, on the contrary, there exists no completely antisymmetric tensor analogue in $N \neq 4$ dimensions.

Part I

Algebraic Part

Chapter 2

The quantum Euclidean Hopf algebras of f.o.g. type and their comodules

In this chapter we review how one can introduce (two) q-deformed N-dimensional Euclidean Hopf algebras $Fun(E_q^N), Fun(\bar{E}_q^N)$ of " f.o.g. " (" functions-on-the-group ") type and their comodules, in such a way to mimic the classical construction of the Euclidean group E^N, namely the construction of E^N as the semidirect product of $\mathbb{R}^N$ with the group of rotations $SO(N)$. Here $N \geq 3$. In part II of this work (the physical part) we will be interested only in their real sections which can be obtained when $q \in \mathbb{R}^+$.

The basic ingredients of the construction are:

- the Hopf algebra $Fun(SO_q(N))$ [9] (which is reviewed in section 1);
- its comodules [9], in particular the algebra $Fun(\mathbb{R}_q^N)$ of functions on the quantum Euclidean space $\mathbb{R}_q^N$ (which is reviewed in section 2);
- the braided group structure [24, 25, 26] of the latter (which is reviewed in section 3);
- the braided semidirect product [24, 26] $Fun(\mathbb{R}_q^N) \rtimes Fun(SO_q(N))$, giving the Euclidean Hopf algebras of Ref. [43] (this is reviewed in section 4).

The " semidirect " approach to inhomogeneous quantum groups, which we apply to the case of the Euclidean Lie group/algebra, is not the only possible one; a different (and chronologically preceding) approach is based on contractions of homogeous quantum groups of higher rank, see for instance Ref. [4, 20, 21]; another one based on projections of bicovariant differential calculi on homogeous quantum groups of higher rank has been recently proposed in Ref. [3]

2.1 The quantum group $SO_q(N)$ and its real section $SO_q(N,\mathbb{R})$

In Ref. [9] one parameter deformations of the classical simple Lie groups and Lie algebras are presented. For each Lie group G a family $Fun(G_q)$ of Hopf algebras parametrized by $q \in \mathbb{C}$ ($q \equiv$ the parameter of deformation) is given, and for $q = 1$ (which corresponds to the socalled classical limit) $Fun(G_q)$ reduces to the Hopf algebra $Fun(G)$ of functions on G. With a suggestive expression, $Fun(G_q)$ is said to be the Hopf algebra of functions on the " quantum group " G_q. If $q \neq 1$ the expression " quantum group G_q " has no intrinsic meaning: all " geometrical " properties of G_q are to be translated and understood in terms of properties of $Fun(G_q)$. We will be concerned with the deformation $SO_q(N)$ ($N \geq 3$) of $SO(N)$, more precisely with its real section $SO_q(N,\mathbb{R})$.

The elements of the Hopf algebra $Fun(SO_q(N))$ are formal ordered power series in the generating elements $\{T^i_j\}$, with $i,j = -n, 1-n, -1, 0, 1, 2, ..., n$ if $N = 2n+1$ and $i,j = -n, 1-n, -1, 1, 2, ..., n$ if $N = 2n$. Throughout all this work n will denote the integral part of $\frac{N}{2}$. The elements T^i_j satisfy the relations

$$\hat{R}(T \otimes' T) = (T \otimes' T)\hat{R} \tag{2.1}$$

and

$$T^i_j C^{jl} T^k_l = \mathbf{1}_{SO_q(N)} C^{ik}, \qquad T^j_i C_{jl} T^l_k = \mathbf{1}_{SO_q(N)} C_{ik}. \tag{2.2}$$

Here $C := ||C_{ij}||$ denotes the (q-deformed) metric matrix, $\mathbf{1}_{SO_q(N)}$ denotes the unit of the algebra, and the tensor product $\otimes'$ appearing in eq. (2.1) just means that we are tensoring indices. For instance, equation (2.1) explicitly reads

$$\hat{R}^{ij}_{hk} T^h_l T^k_m = T^i_h T^j_k \hat{R}^{hk}_{lm}. \tag{2.3}$$

When $q = 1$ $\hat{R}$ reduces to the permutation matrix P, $P^{ij}_{hk} := \delta^i_k \delta^j_h$. The R matrix defined by $R := P\hat{R}$ is lower triangular, i.e. $R^{ij}_{hk} = 0$ if either $i < h$, or $i = h$ and $j < k$. When $N > 2$ and $(1+q^2)(1+\epsilon q^{\epsilon-N+1})(1-\epsilon q^{\epsilon-N-1}) \neq 0$ the matrix $\hat{R}$ has the spectral decomposition

$\hat{R} := ||\hat{R}^{ij}_{hk}||$ is the braid matrix and satisfies the Yang-Baxter equation (in the braid version)

$$(\hat{R} \otimes' \mathbf{1}_1)(\mathbf{1}_1 \otimes' \hat{R})(\hat{R} \otimes' \mathbf{1}_1) = (\mathbf{1}_1 \otimes' \hat{R})(\hat{R} \otimes' \mathbf{1}_1)(\mathbf{1}_1 \otimes' \hat{R}), \tag{2.4}$$

where $\mathbf{1}_1$ is the unit matrix acting on $\mathbb{C}^N$. For $q = 1$ $\hat{R}$ reduces to the permutation matrix P, $P^{ij}_{hk} := \delta^i_k \delta^j_h$. The sides of eq. (2.4) are matrices acting on $\mathbb{C}^N \otimes' \mathbb{C}^N \otimes' \mathbb{C}^N$. Eq. (2.4) itself is most commonly written in the form

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \tag{2.5}$$

with self-explaining notation. Eq.'s (2.1), (2.5) imply

$$f(\hat{R})(T \otimes' T) = (T \otimes' T)f(\hat{R}) \tag{2.6}$$

$$f(\hat{R}_{12})\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}f(\hat{R}_{23}) \tag{2.7}$$

for any polynomial $f(\hat{R})$ in the variable $\hat{R}$. $C, \hat{R}$ are explicitly given by

$$C_{ij} := q^{-\rho_i}\delta_{i,-j}, \tag{2.8}$$

$$\hat{R} = q\sum_{i\neq -i} e^i_i \otimes' e^i_i + \sum_{i\neq j,-j,\ or\ i=j=0} e^j_i \otimes' e^i_j + q^{-1}\sum_{i\neq -i} e^{-i}_i \otimes' e^i_{-i} + \tag{2.9}$$

$$+ (q - q^{-1})[\sum_{i<j} e^i_i \otimes' e^j_j - \sum_{i<j} q^{-\rho_i+\rho_j} e^{j'}_i \otimes' e^j_{i'}], \tag{2.10}$$

where $(e^i_j)^h_k := \delta^{ih}\delta_{jk}$ and

$$(\rho_i) := \begin{cases} (\frac{N}{2}-1, \frac{N}{2}-2, ..., 0, 0, ..., 1-\frac{N}{2}) & if\ N\ even \\ (\frac{N}{2}-1, \frac{N}{2}-2, ..., \frac{1}{2}, 0, -\frac{1}{2}..., 1-\frac{N}{2}) & if\ N\ odd \end{cases}. \tag{2.11}$$

Notice that $N = 2 - 2\rho_n$ both for even and odd N. For instance for $N = 3$

$$C = \left\| \begin{matrix} & & q^{\frac{-1}{2}} \\ & 1 & \\ q^{\frac{1}{2}} & & \end{matrix} \right\| \tag{2.12}$$

C is not symmetric and coincides with its inverse: $C^{-1} = C$, so that $C^{ij} = C_{ij}$. The fact that the matrix C is not diagonal for q=1 is due to the choice of non real coordinates for the fundamental representation of $SO(N)$. The $\hat{R}$-matrix can be decomposed using the three orthogonal projectors corresponding to its three eigenvalues $q, -q^{-1}, q^{1-N}$:

$$\hat{R} = q\mathcal{P}_S - q^{-1}\mathcal{P}_A + q^{1-N}\mathcal{P}_1, \tag{2.13}$$

and

$$\mathbf{1}_2 = \mathcal{P}_S + \mathcal{P}_A + \mathcal{P}_1 \tag{2.14}$$

Here now $\mathbf{1}_2$ denotes the unit matrix acting on $\mathbf{C}^N \otimes' \mathbb{C}^N$. Therefore any polynomial in the $\hat{R}$ variable reduces to a combination of these three projectors. It can be also expressed as a combination of three other linearly independent functions of $\hat{R}$. For instance the choice of the variables $\hat{R}, \mathbf{1}, \mathcal{P}_1$ will be convenient for many calculations, so we solve here equations (2.13), (2.14) for $\mathcal{P}_A, \mathcal{P}_S$:

$$\begin{aligned} \mathcal{P}_A &= \tfrac{1}{q+q^{-1}}[-\hat{R} + q\mathbf{1} - (q - q^{1-N})\mathcal{P}_1]; \\ \mathcal{P}_S &= \tfrac{1}{q+q^{-1}}[\hat{R} + q^{-}\mathbf{1} - (q^{-1} + q^{1-N})\mathcal{P}_1]. \end{aligned} \tag{2.15}$$

The projectors $\mathcal{P}_A, \mathcal{P}_1, \mathcal{P}_S$, have respectively dimension $\frac{N(N-1)}{2}, 1, \frac{N(N+1)}{2} - 1$. If q=1 they reduce to the projectors over the irreducible corepresentations (antisymmetric, singlet and symmetric respectively) of the tensor product $x \otimes' x$ of the fundamental corepresentation (x) of $SO(N)$. The projector $\mathcal{P}_1$ is related to the metric matrix C by

$$\mathcal{P}_1{}^{ij}_{hk} = \frac{C^{ij}C_{hk}}{Q_N}, \qquad Q_N := C^{ij}C_{ij} \tag{2.16}$$

The matrix $\hat{R}$ is symmetric

$$\hat{R}^T = \hat{R}; \tag{2.17}$$

$\hat{R}$ and its inverse satisfy the relations

$$C_{mi}\hat{R}^{\pm 1}{}^{ij}_{hk} = \hat{R}^{\mp 1}{}^{jn}_{ml}C_{nk}. \tag{2.18}$$

As a direct consequence, they also satisfy the following one

$$[\hat{R}^{\pm 1}, P \cdot (C \otimes' C)] = 0, \qquad [\hat{R}^{\pm 1}, P \cdot (C^t \otimes' C^t)] = 0, \tag{2.19}$$

and so does any polynomial function $f(\hat{R})$ (in particular each one of the three projectors):

$$[f(\hat{R}), P \cdot (C \otimes' C)] = 0, \qquad [f(\hat{R}), P \cdot (C^t \otimes' C^t)] = 0, \tag{2.20}$$

$$f(\hat{R})^T = f(\hat{R}) \tag{2.21}$$

The coproduct ϕ

$$\phi : Fun(SO_q(N)) \to Fun(SO_q(N)) \otimes Fun(SO_q(N)) \tag{2.22}$$

and counity ε

$$\varepsilon : Fun(SO_q(N)) \to \mathbb{C} \tag{2.23}$$

are defined on the basic variables by

$$\phi(T^i_j) := T^i_k \otimes T^k_j, \qquad \varepsilon(T^i_j) := \delta^i_j \tag{2.24}$$

and extended to all $Fun(SO_q(N))$ as algebra homomorphisms:

$$\phi(ab) = \phi(a)\phi(b), \qquad \varepsilon(ab) = \varepsilon(a)\varepsilon(b) \qquad \forall a, b \in Fun(SO_q(N)). \tag{2.25}$$

The antipode S is defined by

$$S(T^i_j) = C^{il} T^m_l C_{mj} \tag{2.26}$$

on the basic variables and is extended as a linear antihomomorphism to all $Fun(SO_q(N))$:

$$S(ab) = S(b)S(a). \tag{2.27}$$

It is easy to verify that $Fun(SO_q(N))$ equipped with ϕ, ε, S is a Hopf algebra, namely that ϕ, ε, S, m satisfy the properties

$$(\phi \otimes id) \circ \phi = (id \otimes \phi) \circ \phi \qquad (id \otimes \varepsilon) \circ \phi = id = (\varepsilon \otimes id) \circ \phi \tag{2.28}$$

$$m \circ (id \otimes S) \circ \phi = i \circ \varepsilon = m \circ (S \otimes id) \circ \phi, \tag{2.29}$$

$$(\varepsilon \otimes id) \circ \phi = id = (id \otimes \varepsilon) \circ \phi \tag{2.30}$$

$$\varepsilon \circ S = \varepsilon \tag{2.31}$$

$$\phi \circ S = \tau \circ (S \otimes S) \circ, \qquad \tau(a \otimes b) := (b \otimes a). \tag{2.32}$$

Here id is the identity operator on $Fun(SO_q(N))$,

$$m : Fun(SO_q(N)) \otimes Fun(SO_q(N)) \rightarrow Fun(SO_q(N)) \tag{2.33}$$

is the multiplication operator ($m(a \otimes b) := ab$) and $i : \mathbb{C} \rightarrow Fun(SO_q(N))$ is the injection operator defined by $i(c) := c\mathbf{1}_{SO_q(N)}$.

There exists an alternative Hopf structure, $\phi^{op}, S^{op}, \varepsilon, m$ which can be introduced by

$$\phi^{op} = \tau \circ \phi \qquad S^{op} T = C^t T^t C^t, \tag{2.34}$$

where τ is the flip operator $\tau(a \otimes b) = (b \otimes a)$ and the upper index t denotes transposition of matrix indices.

$Fun(SO_q(N))$ is said to be a Hopf-algebra of (dual)-quasitriangular type, since it is endowed with a (dual)-quasitriangular structure $\mathcal{R} : Fun(SO_q(N)) \otimes Fun(SO_q(N)) \rightarrow \mathbb{C}$

in the sense of Ref. [31] (see for instance, this reference for more details). On the basic generators the latter is defined by

$$\mathcal{R}(T^i_j \otimes T^h_k) = qR^{ih}_{jk} \qquad R := P\hat{R} \tag{2.35}$$

(or, alternatively, with $\hat{R}^{-1}$), and it is extended as a skew bialgebra-bicharacter:

$$\mathcal{R}(T_1T_2...T_{s-1} \otimes T_s) = R_{1s}R_{2s}...R_{s-1,s}q^{s-1} \tag{2.36}$$

$$\mathcal{R}(T_1 \otimes T_2...T_s) = R_{s1}R_{s-1,1}...\hat{R}_{21}q^{s-1}. \tag{2.37}$$

There is an alternative (dual) quasitriangular structure, the one which is obtained by replacing $q\hat{R} \to q^{-1}\bar{\hat{R}}$ in the preceding formulae.

If $q \in \mathbb{R}$ there exists an antilinear involution $*$ on $Fun(SO_q(N))$. It is called complex conjugation, since it reduces to the ordinary complex conjugation for q=1. On the basic variables it is defined by

$$(T^i_j)^* := S(T^j_i) = C^{li}T^l_m C_{jm} \tag{2.38}$$

and extended as an antilinear antihomomorphism to all $Fun(SO_q(N))$:

$$(ab)^* := b^*a^*. \tag{2.39}$$

It is easy to show that $*$ is compatible with the relations (2.1),(2.2) defining $Fun(SO_q(N))$, namely that the relations obtained by taking the complex conjugated of (2.1),(2.2) are satisfied. We explicitly check the compatibility with the second relation, which we rewrite here by showing indices

$$T^i_j C^{jk} T^l_k = C^{il}\mathbf{1}_{SO_q(N)} \tag{2.40}$$

Since $\mathbf{1}^*_{SO_q(N)} = \mathbf{1}_{SO_q(N)}$ and the matrix elements of C are real, complex conjugation on the RHS gives

$$(RHS(2.40))^* = C^{il}\mathbf{1}_{SO_q(N)}; \tag{2.41}$$

as for the LHS, using formula (2.2) itself and the property $C^{-1} = C$ we find

$$(LHS(2.40))^* = (T^l_k)^* C^{jk} (T^i_j)^* = C^{ml}T^m_n C_{kn} C^{jk} C^{pi} T^p_q C_{jq} = \tag{2.42}$$

$$= C^{ml}T^m_j C_{jq} T^p_q C^{pi} = C^{ml}C_{mp}C^{pi}\mathbf{1}_{SO_q(N)} = C^{il}\mathbf{1}_{SO_q(N)}; \tag{2.43}$$

they are equal, as announced. Similarly one shows that (2.1) is transformed by $*$ into an identity. By multiplying relations (2.1),(2.2) by powers of T^i_j one generates new relations involving higher order polynomials in T^i_j; the latter are compatible with $*$ too, since $*$ is an antihomomorphism.

The algebra $Fun(SO_q(N))$ equipped with $*$ is the compact real section of this algebra and will be denoted by $Fun(SO_q(N,\mathbb{R}))$. It is straightforward to show that when $q \in \mathbb{R}$ both $[Fun(SO_q(N,\mathbb{R})), \phi, S, \varepsilon, m, *]$ and $[Fun(SO_q(N,,\mathbb{R})), \phi^{op}, S^{op}, \varepsilon, m, *]$ are $*$-Hopf algebras, namely that the complex conjugation " commutes " with the coproducts and the counit

$$\phi \circ * = (* \otimes *) \circ \phi \qquad \phi^{op} \circ * = (* \otimes *) \circ \phi^{op}; \qquad * \circ \varepsilon = \varepsilon \circ *, \tag{2.44}$$

and that it satisfies the properties

$$(S \circ *)^2 = id \qquad (S^{op} \circ *)^2 = id. \tag{2.45}$$

If $q = 1$ they reduce to the Hopf algebra of functions on the compact group $SO(N,\mathbb{R})$.

2.2 $Fun(SO_q(N))$-comodules; the quantum Euclidean space $\mathbb{R}_q^N$

A left (respectively right) comodule [9, 35] $\mathcal{M}$ of a Hopf algebra $\mathcal{H}$ is by definition an algebra endowed with an algebra homorphism $\phi_L : \mathcal{M} \to \mathcal{H} \otimes \mathcal{M}$ (resp. $\phi_R : \mathcal{M} \to \mathcal{M} \otimes \mathcal{H}$) such that

$$(\Delta \otimes id_{\mathcal{M}}) \circ \phi_L = (id_{\mathcal{H}} \otimes \phi_L) \circ \phi_L \tag{2.46}$$

$$(id_{\mathcal{M}} \otimes \Delta) \circ \phi_R = (\phi_R \otimes id_{\mathcal{H}}) \circ \phi_R \tag{2.47}$$

$$(\varepsilon \otimes id_{\mathcal{M}}) \circ \phi_L = id_{\mathcal{M}} \qquad (resp. \quad (id_{\mathcal{M}} \otimes \varepsilon) \circ \phi_R = id_{\mathcal{M}}), \tag{2.48}$$

wher Δ is the coproduct in $\mathcal{H}$.

According to Ref. [9], given a polynomial function $f(t)$ in one variable, a left (resp. right) $Fun(SO_q(N))$-comodule C_f^N can be defined as the Poincare' algebra of power series in variables $\{y^i\}$, $(i = -n, 1-n, ..., n)$ satisfying relations

$$[f(\hat{R})]^{ij}_{hk} y^h y^k = 0. \tag{2.49}$$

Actually one can easily realize that for any choice of f the previous relation amounts to imposing $\mathcal{P}^{ij}_{hk} y^h y^k = 0$ for one or more projector $\mathcal{P} = \mathcal{P}_S, \mathcal{P}_A, \mathcal{P}_1$. On the generators ϕ_L coacts in the following way

$$\phi_L(\mathbf{1}_{C_f^N}) = \mathbf{1}_{Fun(G_q)} \otimes \mathbf{1}_{C_f^N} \qquad \phi_L(y^i) = T^i_k \otimes y^k; \tag{2.50}$$

and it is extended as a linear homomorphism to all C_f^N:

$$\phi_L(ab) = \phi_L(a)\phi_L(b), \qquad \phi_L(a+b) = \phi_L(a) + \phi_L(b) \qquad \forall\ a,b \in C^N_{f,G_q}. \quad (2.51)$$

To see that ϕ_L is well defined we note that the conditions (2.49) are covariant w.r.t. the quantum group $SO_q(N)$ (namely they are compatible with the coaction) because of relation (2.6): (with $f(\hat{R}) = \mathcal{P}_A$):

$$\phi_L(f(\hat{R})^{ij}_{hk} y^h y^k) = f(\hat{R})^{ij}_{hk} T^h_{h'} T^k_{k'} \otimes y^{h'} y^{k'} = T^i_{i'} T^j_{j'} \otimes \mathcal{P}_A{}^{i'j'}_{hk} y^h y^k. \quad (2.52)$$

Imposing condition (2.49) puts both sides equal to zero. ϕ_R is defined by $\tau \circ \phi_L$.

The axioms (2.46) (2.47) is satisfied if we set $\Delta = \phi$ in the case of ϕ_L, $\Delta = \phi^{op}$ in the case of ϕ_R:

$$(\phi \otimes id_{C^N_f}) \circ \phi_L = (id_{Fun(SO_q(N))} \otimes \phi_L) \circ \phi_L \quad (2.53)$$

$$(id_{C^N_f} \otimes \phi^{op}) \circ \phi_R = (\phi_R \otimes id_{Fun(SO_q(N))}) \circ \phi_R. \quad (2.54)$$

As the first example of comodule we take the algebra $Fun(\mathbb{R}^N_q)$ (O^N_q in the notation of Ref. [9]) of functions on the N-dimensional quantum euclidean space $\mathbb{R}^N_q$, which is introduced as the algebra of formal ordered power series in the generating elements $\{x^i\}$, $i = 1, 2, ..., N$ modulo the the $\frac{N(N-1)}{2}$ independent conditions

$$\mathcal{P}_A{}^{ij}_{hk} x^h x^k = 0 \quad (2.55)$$

(in other words $f(\hat{R}) \equiv \mathcal{P}_A$). x^i are vector type components; one can also introduce covector components x_i. In general, indices are raised and lowered through the metric matrix C, for instance

$$a_i = C_{ij} a^j, \qquad a^i = C^{ij} a_j. \quad (2.56)$$

For any " vectors " $a := (a^i), b := (b^i)$ let us define

$$(a \circ b)_j := \sum_{l=1}^{j} a^{-l} b_{-l} + \begin{cases} \frac{q}{q+1} a^0 b_0 & if\ N\ odd \\ 0 & if\ N\ even \end{cases}, \qquad 0 \le j \le n \quad (2.57)$$

$$\mathcal{A}^j(a,b) := a^j b^{-j} - a^{-j} b^j - (q^2-1) q^{-\rho_j - 2} (a \circ b)_{j-1}, \qquad j \ge 1 \quad (2.58)$$

$$(1 + q^{-2\rho_j})(a \cdot b)_j := \sum_{l=-j}^{j} a^l b_l \qquad 0 < j \le n \quad (2.59)$$

(when this causes no confusion we will also use the notation $a \cdot b := (a \cdot b)_n$). ρ_i were defined in formula (2.11). Then it is easy to verify that

$$(a \cdot b)_j = (a \circ b)_j + \frac{\sum_{l=1}^{j} \mathcal{A}^l(a,b) q^{-\rho_l}}{1 + q^{-2\rho_j}} \quad (2.60)$$

Note that the preceding four formulae make sense for any $n \geq j$ and do not formally depend on n.

Relations (2.55) amount respectively to

$$x^i x^j = q x^j x^i, \qquad i < j, \tag{2.61}$$

$$\mathcal{A}^i(x, x) = 0, \qquad i = 1, 2, ..., n \tag{2.62}$$

For instance for $N = 3$ eq.'s (2.55) amount to the three independent relations

$$x^1 x^2 - q x^2 x^1 = 0, \qquad x^2 x^3 - q x^3 x^2 = 0, \qquad x^1 x^3 - x^3 x^1 + (q^{\frac{1}{2}} - q^{\frac{-1}{2}})(x^2)^2 = 0. \tag{2.63}$$

For $q = 1$ and any N $\mathcal{P}_A{}^{ij}_{hk} = \frac{1}{2}(\delta^i_h \delta^j_k - \delta^i_k \delta^j_h)$, so that the x^i " coordinates " become commuting variables and their order in each monomial doesn't matter any more. In other terms the underlying geometry is no more noncommutative, but classical (i.e. commutative).

The left coaction $\phi_L : Fun(\mathbb{R}^N_q) \rightarrow Fun(SO_q(N)) \otimes Fun(\mathbb{R}^N_q)$ of the quantum group $SO_q(N)$ is defined on the basic variables by

$$\phi_L(x^i) = T^i_j \otimes x^j \tag{2.64}$$

and is extended as an algebra homomorphism:

$$\phi_L(ab) = \phi_L(a)\phi_L(b) \qquad a, b \in Fun\mathbb{R}^N_q). \tag{2.65}$$

It is easy to check that the square length $x \cdot x = \frac{x^i C_{ij} x^j}{1+q^{N-2}}$ is central in $Fun(\mathbb{R}^N_q)$ and is a scalar under the coaction of the quantum group:

$$\phi_L(x \cdot x) = 1 \otimes (x \cdot x). \tag{2.66}$$

The existence of scalars of this form within comodules is a consequence of the peculiar structure of $Fun(SO_q(N))$, and can be considered as one of the basic motivations for its definition. As for the first point, note that relation (2.13),(2.14),(2.55) imply

$$q^{-1} \hat{R}^{ij}_{hk} x^h x^k = x^i x^j + \frac{1 - q^2}{\mu} q^{-N} C^{ij}(xCx) \tag{2.67}$$

$$q \hat{R}^{-1}{}^{ij}_{hk} x^h x^k = x^i x^j + \frac{q^2 - 1}{\mu} C^{ij}(xCx); \tag{2.68}$$

hence,

$$\begin{aligned}
x^i(xCx) &= x^i x^l C_{lm} x^m \stackrel{(2.67)}{=} [q^{-1}\hat{R}^{il}_{hk} x^h x^k - \tfrac{1-q^2}{\mu} q^{-N} C^{il}(xCx)] C_{lm} x^m \\
&\stackrel{(2.18)}{=} q^{-1} C_{hl} \hat{R}^{-1}{}^{li}_{km} x^h x^k x^m - \tfrac{1-q^2}{\mu} q^{-N}(xCx)x^i \\
&\stackrel{(2.68)}{=} q^{-1}\hat{C}_{hl} x^h [q^{-1} x^l x^i + q^{-1}\tfrac{q^2-1}{\mu} C^{li} xCx] - \tfrac{1-q^2}{\mu} q^{-N}(xCx)x^i \\
&= [q^{-2} + q^{-2}\tfrac{q^2-1}{\mu} + q^{-N}\tfrac{q^2-1}{\mu}](xCx)x^i = (xCx)x^i,
\end{aligned} \tag{2.69}$$

as claimed. As for the second point, relation (2.66), it is a straightforward consequence of the definition of ϕ_L and property (2.2).

As before if $q \in \mathbb{R}$ one can define an antilinear involution $*$, the complex conjugation (we will use the same symbol used for $Fun(SO_q(N))$) on $Fun(\mathbb{R}_q^N)$ by setting

$$(x^i)^* := x^j C_{ji} \tag{2.70}$$

on the basic variables $\{x^i\}$ and extending it as an antilinear antihomomorphism. The basic conditions (2.55) defining $Fun\mathbb{R}_q^N)$ are compatible with complex conjugation. In fact, using properties (2.19), (2.20), we find

$$[\mathcal{P}_A{}^{ij}_{hk} x^h x^k]^* = \mathcal{P}_A{}^{ij}_{hk} x^{k'} x^{h'} C_{h'h} C_{k'k} = (\mathcal{P}_A{}^{j'i'}_{k'h'} x^{k'} x^{h'}) C^{j'j} C^{i'i}; \tag{2.71}$$

Imposing eq.'s (2.55) sets both handsides of this relation equal to zero. The coaction and the complex conjugation commute:

$$\phi_L \circ * = (* \otimes *) \circ \phi_L. \tag{2.72}$$

It is enough to check this property on the basic variables x^i, since $\phi_L, *$ are respectively an homomorphism and an antihomomorphism. Indeed

$$\phi_L((x^i)^*) = C_{ji}\phi_L(x^j) = C_{ji} T^j_h \otimes x^h \tag{2.73}$$

and

$$(T^i_k)^* \otimes (x^k)^* = C^{ji} T^j_h C_{kh} \otimes x^l C_{lk} = C_{ji} T^j_h \otimes x^h, \tag{2.74}$$

upon use of (2.38), (2.70) and of the property $C^{-1} = C$.

It is easy to check that $(x \cdot x)^* = x \cdot x$, i.e. that the square lenght is real (for $q \in \mathbb{R}$).

The algebra $Fun(\mathbb{R}_q^N)$ equipped with the involution $*$ will be suggestively called " the algebra of functions on the N-dimensional *real* quantum Euclidean space $\mathbb{R}_q^N$ ", and will be denoted by the same symbol $Fun(\mathbb{R}_q^N)$ (again the expression $\mathbb{R}_q^N$ has a well-defined meaning only in relation with $Fun(\mathbb{R}_q^N)$). In the sequel it will be clear from the context when we are dealing with this real section or the algebra without $*$-structure. From the above contruction it should be clear that the structure of the real quantum space $\mathbb{R}_q^N$ is strictly dependent from the symmetry we endow it with, namely the symmetry w.r.t. the quantum group $SO_q(N,\mathbb{R})$. Then the construction of $\mathbb{R}_q^N$ from $Fun(SO_q(N,\mathbb{R})$ can be seen as a noncommutative realization of Felix Klein's program for geometry: the geometry of spaces should be determined by the symmetries we want them to satisfy. In this approach the quantum group is defined before the quantum space. On the contrary, in Manin's approach [35] the definition of quantum spaces (as deformations of the commutative spaces) is given first, and the quantum groups are defined so as to be their symmetries.

2.3 Braided group structure for $\mathbb{R}_q^N$

An additional structure can be added to any $Fun(SO_q(N))$-comodule $\mathcal{M}$ (in particular $Fun(\mathbb{R}_q^N)$) to convert it into a " braided group " (or " braided Hopf algebra ") [27], because of the fact that $Fun(SO_q(N))$ is equipped with a (dual)-quasitriangular structure $\mathcal{R}$: $Fun(SO_q(N))\otimes Fun(SO_q(N)) \rightarrow \mathbb{C}$ [31]. A braided Hopf algebra is a " braided algebra " equipped with a " braided Hopf structure ", consisting of suitably matched " braided coproduct, counit and antipode ". We briefly recall here the meaning of these notions; a proper introduction to these concepts can be found in Ref. [26].

One says that an algebra $\mathcal{A}$ is a braided algebra essentially when the tensor product of many copies of $\mathcal{A}$ is not performed according to the ordinary notion of tensor product but as a " braided tensor product " $\underline{\otimes}$. The ordinary multiplication rule of tensor products is modified in the following sense:

$$(a\underline{\otimes}b)(c\underline{\otimes}d) = (a\underline{\otimes}1)\Psi(b\underline{\otimes}c)(1\underline{\otimes}d) \qquad a,b,c,d \in \mathcal{A}. \tag{2.75}$$

The " braiding " Ψ is a linear map $\Psi : (\underline{\otimes}^k\mathcal{A})\underline{\otimes}(\underline{\otimes}^l\mathcal{A}) \rightarrow (\underline{\otimes}^{k-l}\mathcal{A})$ satisfying the rules of braiding; in the simplest case the latter read

$$\Psi(a\underline{\otimes}(b\underline{\otimes}c)) = (id\underline{\otimes}\Psi)[\Psi(a\underline{\otimes}b)\underline{\otimes}c]. \tag{2.76}$$

In the cases of major interest Ψ is a q-deformation of the flip operator τ: $\tau(a\otimes b) := b\otimes a$.

If $\mathcal{A} = \mathcal{M}$ the braiding is determined by the requirement that the braided tensor products of $\mathcal{M}$ are comodules of the bialgebra $A(R)$ generated by matrices satisfying relation (2.1) only (the Hopf algebra $Fun(SO_q(N))$ is obtained by quotienting $A(R)$ by relations (2.2)):

$$\Psi(a\underline{\otimes}b) := \sum_{IJ} \mathcal{R}(S(a'^J)\otimes S(b'^I))(b^I\otimes a^J) \qquad (resp.\Psi(a\underline{\otimes}b) := \sum_{IJ}(b^I\otimes a^J)\mathcal{R}(a'^J\otimes b'^I) \tag{2.77}$$

where

$$\phi_L(c) = \sum_I c'^I \otimes c^I, \qquad \phi_R(d) = \sum_I d^I \otimes d'^I, \tag{2.78}$$

and ϕ_L (resp. ϕ_R) is defined as in formulae (2.64). Correspondingly, $\mathcal{M}$ will be said to be a braided algebra of vectors (resp. covectors) of $A(R)$.

From relation (2.77) it follows that the braiding is defined on the basic generators x^i of vector type of $Fun(\mathbb{R}_q^N)$ in one of the two following ways (corresponding to the two possible choices of the dual-quasitriangular structure in (2.35)):

$$\Psi(x^i\underline{\otimes}x^j) = [(q\hat{R})^{\pm 1}]^{ij}_{kh} x^h\underline{\otimes}x^k. \tag{2.79}$$

Then Ψ is extended to products functorially:

$$\Psi(a\underline{\otimes}b\cdot c) = (\cdot\underline{\otimes}id)\Psi[a\underline{\otimes}(b\underline{\otimes}c)]; \tag{2.80}$$

the above definitions are compatible with the quantum space relations (2.55). It is immediate to verify that in terms of the components $x_i := C_{ij}x^j$ $Fun(\mathbb{R}_q^N)$ is a algebra of covectors:

$$\Psi(x_i\underline{\otimes}x_j) = [(q\hat{R})^{\pm 1}]^{hk}_{ij} x_h\underline{\otimes}x_k, \tag{2.81}$$

When $q=1$ $\Psi=\tau$.

One can slightly enlarge $Fun(SO_q(N))$ to a new Hopf algebra $Fun(\tilde{SO}_q(N))$ by adding central generators $g^{\pm 1}$ $(gg^{-1}=1)$ to the generators of $Fun(SO_q(N))$ and defining

$$\phi(g) = g\otimes g \qquad \varepsilon(g) = 1 \qquad S(g) = g^{-1}. \tag{2.82}$$

The braided tensor products of $\mathcal{M}$ are not $Fun(SO_q(N))$-comodules, but $Fun(\tilde{SO}_q(N))$-comodules, w.r.t. the modified coactions ϕ'_L, ϕ'_R which on the generators x^i are defined by

$$\phi'_L(x^i) = gT^i_j\otimes x^j \qquad \phi'_R(x^i) = x^j\otimes gT^i_j. \tag{2.83}$$

The braided Hopf structure of $Fun(\mathbb{R}_q^N)$ is introduced in the following way. We first introduce a braided comultiplication $\underline{\Delta}$, i.e. an algebra homomorphism w.r.t. the braided tensor product, called " coaddition ", starting from

$$\underline{\Delta}(x^i) = x^i\underline{\otimes}\mathbf{1} + \mathbf{1}\underline{\otimes}x^i; \tag{2.84}$$

$\underline{\Delta}(x^i)$ gives in the classical limit the (vectorial) sum of two vectors. Secondly, the (braided) counit $\underline{\varepsilon} : Fun(\mathbb{R}_q^N) \to \mathbb{C}$ and the braided antipode $\underline{\sigma} : Fun(\mathbb{R}_q^N) \to Fun(\mathbb{R}_q^N)$ are defined on the generators by

$$\underline{\varepsilon}(1) = 1 \qquad\qquad \underline{\varepsilon}(x^i) = 0 \tag{2.85}$$

$$\underline{\sigma}(p^i) = -p^i, \tag{2.86}$$

and are extended respectively as multiplicative and braided antimultiplicative maps:

$$\underline{\varepsilon}(ab) = \underline{\varepsilon}(a)\underline{\varepsilon}(b) \qquad\qquad \underline{\sigma}(ab) = \cdot\Psi(\underline{\sigma}(a)\underline{\otimes}\underline{\sigma}(b)). \tag{2.87}$$

$\underline{\Delta}, \underline{\varepsilon}, \underline{\sigma}, m$ satisfy the axioms of an Hopf algebra (see section 1), provided we replace $\otimes$'s by $\underline{\otimes}$'s. For instance

$$(id\underline{\otimes}\underline{\sigma})\underline{\Delta} = (\underline{\sigma}\underline{\otimes}id)\underline{\Delta} = 0, \tag{2.88}$$

which essentially says that $\underline{\sigma}(a)$ is the " opposite " of a.

2.4 The quantum groups Euclidean E_q^N, $\bar{E}_q^N$

The modified coaction ϕ'_L (2.83) (or equivalently ϕ'_R) introduced in the previous section is compatible with the coaddition (2.84) of $\mathbb{R}_q^N$; in other words we may either first do a translation $\vec{x} \to \vec{x}' := \vec{x} + \vec{a}$ and then a rotation of $\vec{x}'$, or do first a rotation of $\vec{x}$ and then translate it by the rotated $\vec{a}$.

This enables the semidirect product $\mathbb{R}_q^N \rtimes Fun(\tilde{SO}_q(N))$ of $\mathbb{R}_q^N$ with $Fun(\tilde{SO}_q(N))$ [29]; in this way the braided Hopf algebra $\mathbb{R}_q^N$ is embedded into an ordinary one, a process which was called " bosonization " [29]. The result is the one which was found in Ref [43] and which we are reporting here.

The Euclidean Hopf algebra $Fun(E_q^N) := \mathbb{R}_q^N \rtimes Fun(\tilde{SO}_q(N))$ (corresponding to the first choice of the braiding in formula (2.79)) as an algebra is unital and is generated by elements $g^{\pm 1}, T^i_j, y^i$ satisfying relations (2.1)(2.2)(2.55) and cross relations

$$gy^i = q^{-1}y^ig, \qquad\qquad [g, T^i_j] = 0, \qquad\qquad y^iT^j_h = \hat{R}^{ij}_{lm}T^l_hy^m; \tag{2.89}$$

as for the Hopf structure, its coproduct, counit and antipode $\phi^E, \varepsilon^E, S^E$ coincide with ϕ, ε, S on the generators $g^{\pm 1}, T^i_j$ (see formulae (2.24),(2.82)) and are defined on the y^i by

$$\phi^E(y^i) = gT^i_j\otimes y^j + y^i\otimes 1 \qquad \varepsilon^E(y^i) = 0 \qquad S^E(y^i) = -g^{-1}S(T^i_j)y^j = -g^{-1}(CT^tC)^i_jy^j. \tag{2.90}$$

$\phi^E, \varepsilon^E, S^E$ are extended as algebra (anti)homomorphisms as usual. One can easily verify on the generators that the axioms (2.24)-(2.32) are verified, i.e. that $Fun(E_q^N)$ is really an Hopf algebra.

Similarly we define the Euclidean Hopf algebra $Fun(\bar{E}_q^N) := \mathbb{R}_q^N \rtimes' Fun(\bar{SO}_q(N))$ (corresponding to the second choice of the braiding in formula (2.79)) by replacing $q, \hat{R}, g \to q^{-1}, \hat{R}^{-1}, g^{-1}$ in formulae (2.89)(2.90); the corresponding generators of translations will be denoted by $\bar{y}$ (instead of y).

When $q \in \mathbb{R}$ we can define the complex conjugation $*$ as an antilinear involutive antihomomorphism mapping $Fun(E_q^N) \leftrightarrow Fun(\bar{E}_q^N)$ by defining it on the generators through the formulae (2.38) plus the new ones

$$\bar{y}^i = (y^j)^* C^{ji} \qquad\qquad g^* = g^{-1}. \tag{2.91}$$

One can immediately verify through formulae (2.38)(2.20) that we cannot impose the reality condition $y^i = (y^j)^* C^{ji}$ which we adopted for the quantum space $\mathbb{R}_q^N$ (formula (2.70)), otherwise we would spoil the algebra relations (2.89). One could define a larger Hopf algebra $Fun(E_q^N)_{ext}$ containing both $Fun(E_q^N)$ and $Fun(\bar{E}_q^N)$, namely an algebra generated by $T^i_j, g^{\pm 1}, y^i, \bar{y}^i$ satisfying the previous relations plus the new ones

$$y^i \bar{y}^j = q^{-1} \hat{R}^{ij}_{lm} \bar{y}^l y^m, \tag{2.92}$$

and with coproduct, counit, antipode defined as in formulae (2.90) (& and their barred versions). Then one can easily verify that $Fun(E_q^N)_{ext}$ would be a $*$-Hopf algebra, i.e. would satisfy axioms (2.44).

From the previous construction it is clear that $Fun(SO_q(N))$ is a Hopf subalgebra of $Fun(E_q^N), Fun(\bar{E}_q^N), Fun(E_q^N)_{ext}$ and that it can be obtained from the latter by applying the " projection operator " $\Pi : Fun(E_q^N)_{ext} \to Fun(SO_q(N))$ defined as a $*$-algebra homomorphism which on the generators reduces to

$$\Pi(T^i_j) = T^i{}_j \qquad \Pi(g^{\pm 1}) = 1, \qquad \Pi(y) = 0 = \Pi(\bar{y}). \tag{2.93}$$

Finally $Fun(\mathbb{R}_q^N)$ can be equipped with a left (or right, as well) E_q^N- and $\bar{E}_q^N$-comodule structure introducing respectively the left coactions defined on the basic generators by

$$\phi_L^E(x^i) := gT^i_j \otimes x^j + y^j \otimes 1 \qquad or \qquad \phi_L^E(x^i) := g^{-1}T^i_j \otimes x^j + \bar{y}^j \otimes 1; \tag{2.94}$$

indeed, one can easily check that they are compatible with the $Fun(E_q^N)$ and $Fun(\bar{E}_q^N)$ coproduct ϕ^E, i.e. satisfy axiom (2.46). It is immediate to check that $(\Pi \otimes id) \circ \phi_L^E = \phi_L$, namely that Π projects the inhomogenous coaction to the homogeneous one.

Chapter 3

The E_q^N-covariant differential calculus on $\mathbf{R}_q^N$

The differential calculus on $\mathbb{R}_q^N$ is the omnipresent ingredient used in the q-deformed constructions of all the next chapters of this thesis. The reason is that we try to mimic the undeformed ones (where it plays a fundamental role) wherever it is possible. We prefer to introduce it using the differential-geometric approach of Ref. [50, 42] (Section 1); now also a more familiar interpretation of partial q-derivatives as sort of q-incremental ratios [28] is available, and we just mention it in Section 2. Section 3 is devoted to exploring the properties of the algebra generated by coordinates and derivatives, Section 4 those of the exterior algebra.

3.1 The basics: exterior derivative, basic 1-forms and partial derivatives

The differential calculus on the quantum space $\mathbb{R}_q^N$ (see Ref. [2] is defined by a few essential requirements: its $SO_q(N)$-covariance; Leibniz rule and nilpotency for the exterior derivative d whatever q; homogeneous commutation relations between 1-forms and 0-forms (i.e. functions). The approach is the same which was first suggested in Ref. [50] and [42] to define the differential calculus on the quantum hyperplane.

E_q^N-covariance of the differential calculus then follows from the braided semiproduct structure $E_q^N = \mathbb{R}_q^N \rtimes SO_q(N)$ (see the preceding Chapter). It turns out that the differential calculus reduces to the classical one on $\mathbb{R}^N$ for $q = 1$. The steps are the following. One first introduces the basic 1-forms ξ^i by applying the exterior derivative to x^i's. Then one looks at sensible homogeneous commutation relations between x^i's and ξ^j. Finally one introduces " derivatives " ∂_i's " w.r.t. the coordinates " by setting $d := \xi^i \partial_i$. We will see that two linearly independent sets of such objects will be available - what we call the unbarred and barred ones. When $q \in \mathbb{R}$, they are mapped one into the other by the complex conjugation; moreover, one can find a nonlinear map which allows to express unbarred objects as functions of barred ones, or viceversa.

The exterior derivative of the differential calculus D is denoted by d. It is nilpotent, namely

$$d^2 = 0. \tag{3.1}$$

Let $\bigwedge_q^p$ be the space of differential p-forms over $\mathbb{R}_q^N$, and let $\alpha_p \in \bigwedge_q^p$. We define

$$d\alpha_p| := d\alpha_p - (-1)^p \alpha_p d. \tag{3.2}$$

Leibniz rule can be expressed by the statement

$$\alpha_p \in \bigwedge_q^p \quad \Rightarrow \qquad d\alpha_p| \in \bigwedge_q^{p+1}. \tag{3.3}$$

In fact it implies

$$d\alpha_p \beta| = d\alpha_p|\beta + (-1)^p \alpha_p d\beta| \tag{3.4}$$

for any form β. In the common notation $d\alpha_p|$ would be denoted by $(d\alpha_p)$, so that Leibniz rule (3.4) would take the usual form $d(\alpha_p\beta_q) = (d\alpha_p)\beta_q + (-1)^p\alpha_p(d\beta_q)$. We prefer to use the new symbol $d\alpha_p|$ to keep in mind that this form can be written as the difference (3.2); this will enable us to define the complex conjugation $*$ in an explicit antihomomorphic form on all arguments (also on d, see below) and to avoid many notation ambiguities.

In particular if $f \in Fun(\mathbb{R}_q^N) \equiv \bigwedge_q^0$, $df|$ is a 1-form. We denote by $\xi^i := dx^i|$ the exterior derivatives of the basic coordinates x^i; $\{\xi^i\}$ is a " basis " of $\bigwedge_q^1$, the space of 1-forms. The latter can be obtained by arbitary combinations of formal products $f(x)\xi^i g(x)$, $f, g \in Fun(\mathbb{R}_q^N)(\mathbb{R})$. One should be able to reduce all such combinations to combinations of terms either of the type $f\xi^i$ or of the type $\xi^i g$, and to this end one needs to prescribe commutations relations between the x^i's and the ξ^j's. In the classical case these relations are homogeneous; therefore, as already anticipated, we look for homogeneous ones for any q:

$$x^i \xi^j = M^{ij}_{hk} \xi^h x^k, \qquad M \ invertible. \tag{3.5}$$

M is fixed by the requirement of covariance, consistency with the q-space relations (2.55) for x^i's and consistency with Leibniz rule. By covariance we mean that the coaction should be naturally extended as an homomorphism to all $\bigwedge_q^p$ by the fundamental requirement that the exterior derivative " commutes " with the coaction:

$$\phi_L^E \circ d = (id_{E_q^N} \otimes d) \circ \phi_L^E, \tag{3.6}$$

or equivalently that d is E_q^N-invariant: $\phi_L^E(d) = id_{E_q^N} \otimes d$. Projecting this relation on the Hopf subalgebra $Fun(SO_q(N))$ and using the definition $\xi^i := dx^i|$ we get for instance

$$\phi_L(\xi^i) = \phi_L(\xi^i) = T^i_j \otimes \xi^j. \tag{3.7}$$

Applying ϕ_L to both sides of (3.5) we find that M must satisfy the condition

$$T^i_{i'}T^j_{j'} \otimes x^{i'}x^{j'} = M^{ij}_{hk}T^h_{h'}T^k_{k'} \otimes \xi^{h'}x^{k'} = M^{ij}_{hk}T^h_{h'}T^k_{k'}M^{-1}{}_{i'j'}^{h'k'} \otimes x^{i'}\xi^{j'}, \tag{3.8}$$

in other words

$$M(T \otimes' T) = (T \otimes' T)M. \tag{3.9}$$

Therefore M must be a function $f(\hat{R})$ of $\hat{R}$ (see formula (2.6)), hence a combination of three linearly independent polynomial functions of $\hat{R}$, let us take $\hat{R}, \hat{R}^{-1}, \mathcal{P}_1$:

$$M = a\hat{R} + b\hat{R}^{-1} + c\mathcal{P}_1. \tag{3.10}$$

Let us now consider the matrix $B(a,b,c)$ appearing in the commutation relations

$$x^i x^j \xi^k = B^{ijk}_{lmn}\xi^l x^m x^n \tag{3.11}$$

and obtained by applying two times relations (3.5) with M as given in (3.10). Consistency with the relations (2.55) means that contracting both sides of (3.11) with $\mathcal{P}_A{}^{hk}_{ij}$ we should get zero, namely

$$(\mathcal{P}_A)_{12}B(\mathcal{P}_\Omega)_{23} = 0, \qquad \Omega = S, 1 \tag{3.12}$$

A little lenghty calculation shows that this gives two equations in the unknowns a, b, c; they have two solutions:

$$either \qquad b = c = 0, \qquad\qquad or \qquad a = c = 0 \tag{3.13}$$

The remaining constant is fixed by the requirement of consistency of Leibniz rule with the relations (2.55). Upon use of the (2.13),(3.10) wefind for the two solutions (3.13) respectively the identities

$$d\mathcal{P}_A{}^{ij}_{hk}x^h x^k| = \mathcal{P}_A{}^{ij}_{hk}\xi^h x^k + \mathcal{P}_A{}^{ij}_{hk}x^h\xi^k = \mathcal{P}_A{}^{ij}_{hk}\xi^h x^k \cdot \begin{cases} (1 - aq^{-1}) \\ (1 - bq) \end{cases}; \tag{3.14}$$

the LHS vanishes because of (2.55), (2,2) so we conclude that it must be $a = q$ and $b = q^{-1}$ respectively. We denote the 1-forms and the exterior derivative corresponding to the first and second solution by ξ^i, d and $\bar{\xi}^i, \bar{d}$ respectively. Summing up, $\xi^i =: dx^i|$, $\bar{\xi}^i := \bar{d}x^i|$ and

$$x^i\xi^j = q\hat{R}^{ij}_{hk}\xi^h x^k \tag{3.15}$$

$$x^i\bar{\xi}^j = q^{-1}\hat{R}^{-1}{}^{ij}_{hk}\bar{\xi}^h x^k. \tag{3.16}$$

The wedge product of two 1-forms is essentially determined by the requirement of nilpotency of the exterior derivative. In fact, using relation (3.16) and the decomposition (2.13) of $\hat{R}$ we find

$$d^2x^ix^j| = d(\xi^ix^j + x^i\xi^j)| = d[(1+q\hat{R})^{ij}_{hk}\xi^hx^k]| = \tag{3.17}$$

$$= -[(1+q^2)\mathcal{P}_S + (1+q^{2-N})\mathcal{P}_1]^{ij}_{hk}\xi^h\xi^k; \tag{3.18}$$

consistency of this relation with $d^2 = 0$ requires that

$$\mathcal{P}_S(\xi \otimes' \xi) = 0 = \mathcal{P}_1(\xi \otimes' \xi). \tag{3.19}$$

Therefore the wedge product $\wedge$ of the 1-forms ξ^i is to be defined as the tensor product $\otimes'$ modulo the relations (3.19). In other words:

$$\mathcal{P}_S(\xi \wedge \xi) = 0 = \mathcal{P}_1(\xi \wedge \xi). \tag{3.20}$$

In this way one defines $\bigwedge^2_q$. Higher degree forms are to be defined in a similar way, namely as tensor products of 1-forms modulo relations (3.19) for all neighbouring tensor factors. In a similar way one shows that $\bar{d}^2 = 0$ implies

$$\mathcal{P}_S(\bar{\xi} \wedge \bar{\xi}) = 0 = \mathcal{P}_1(\bar{\xi} \wedge \bar{\xi}) \tag{3.21}$$

for the barred 1-forms. This enables us to define the space $\bar{\bigwedge}^p_q$ of barred p-forms. As a direct consequence of eq.'s (3.20), (3.21)

$$\mathcal{P}_A(\xi \wedge \xi) = (\xi \wedge \xi), \qquad \mathcal{P}_A(\bar{\xi} \wedge \bar{\xi}) = (\bar{\xi} \wedge \bar{\xi}). \tag{3.22}$$

In the sequel we will drop the symbol $\wedge$.

The decompositions $d = \xi^i\partial_i$, $\bar{d} = \bar{\xi}^i\bar{\partial}_i$ define the derivatives ∂_i, $\bar{\partial}_i$ corresponding to each coordinate x^i. The coaction should be extended in a natural way as an homomorphism to the larger algebras generated by x^i, ξ^i, ∂^i and $x^i, \bar{\xi}^i, \bar{\partial}^i$ respectively. The invariance (3.6) of the exterior derivative under E_q^N implies that the coaction must act on the derivatives in the following way:

$$\phi_L^E(\partial^i) := g^{-1}T_j^i \otimes \partial^j \qquad \phi_L^E(\bar{\partial}^i) := g^{-1}T_j^i \otimes \bar{\partial}^j. \tag{3.23}$$

(we have used the orthogonality relations (2.2) of $SO_q(N)$).

The commutation relations involving the derivatives are already fixed by the previous constraints, let us determine them. First, notice that dx^i can be written in two ways

$$dx^i = \xi^j \partial_j x^i, \qquad dx^i = \xi^i + x^i d = \xi^i + x^i \xi^h \partial_h = \xi^i + q\hat{R}^{ih}_{jk}\xi^j x^k \partial_h, \tag{3.24}$$

whence, by comparison,

$$\partial_j x^i = \delta^i_j + q\hat{R}^{ih}_{jk} x^k \partial_h, \tag{3.25}$$

or, equivalently,

$$\partial^i x^j = C^{ij} + q\hat{R}^{-1}{}^{ij}_{hk} x^h \partial^k. \tag{3.26}$$

These are the " commutation relations " of the derivatives with the coordinates; notice that the Leibniz rule for the derivatives ∂ holds only for $q = 1$ (in fact for $q = 1$ $\hat{R}^{ij}_{hk} = \delta^i_k \delta^j_h$). Similarly one can show that

$$\bar{\partial}^i x^j = C^{ij} + q^{-1}\hat{R}^{ij}_{hk} x^h \bar{\partial}^k. \tag{3.27}$$

To find the commutation relations between two derivatives note that

$$d^2 = d\xi^i \partial_i = -\xi^i d\partial_i = -\xi^i \xi^j \partial_j \partial_i. \tag{3.28}$$

Using properties (3.20) (3.22) of the wedge product and formula (3.1) we derive that the commutation relations between the derivatives must be of the form

$$(\mathcal{P}_A + a\mathcal{P}_S + b\mathcal{P}_1)^{ij}_{hk}\partial^h \partial^k = 0. \tag{3.29}$$

Apply both sides of the preceding relation to x^m and use twice the derivation rule (3.25). It is easy to check that the constants a, b have to vanish to get again zero at both sides. An analogous argument applies to the $\bar{\partial}^i$ derivatives. Summing up:

$$\mathcal{P}_A{}^{ij}_{hk}\partial^h \partial^k = 0 \tag{3.30}$$

$$\mathcal{P}_A{}^{ij}_{hk}\bar{\partial}^h \bar{\partial}^k = 0 \tag{3.31}$$

In other words both the unbarred the barred derivatives generate Euclidean quantum spaces.

It remains to find out the commutation relations between the derivatives and the 1-forms. They can be determined by the requirement of covariance and consistency with the commutation relations (3.15), (3.25). The calculations are straightforward and one finds:

$$\partial^i \xi^j = q^{-1} \hat{R}^{ij}_{hk} \xi^h \partial^k \tag{3.32}$$

$$\bar{\partial}^i \bar{\xi}^j = q \hat{R}^{-1}{}^{ij}_{hk} \bar{\xi}^h \bar{\partial}^k . \tag{3.33}$$

By the above discussion we have shown that the forementioned consistency requirements are satisfied for any product $\eta^i\eta^j\eta^k$ of three basic elements η ($\eta^l = x^l, \xi^l, \partial^l$). Then consistency will be satisfied for any product of any arbitrary number n of elements. One can easily show that this statement is a consequence of two facts: first, ϕ (resp. $*$) is a homomorphism (resp. a antihomomorphism); second, the matrices $\hat{R}_{i,i+1}, \quad i = 1, 2, ..., n$ provide a representation of the braid group (with generating elements $\sigma_{i,i+1}, \sigma^{-1}_{i,i+1}$).

One can easily show that no combination of the 1-forms $\{\xi^i\}$ can reproduce the $\{\bar{\xi}^j\}$ satisfying the above relations, in other terms barred and non barred forms are linearly independent objects. This is also the case for the $\{\partial^i\}$ and $\{\bar{\partial}^j\}$ derivatives.

The above relations define the differential calculi $D = \{x^i, d, \ \xi^i, \ \partial^i\}$ and $\bar{D} = \{x^i, \bar{d}, \ \bar{\xi}^i, \bar{\partial}^i\}$. Note that all relations concerning barred objects can be obtained from the corresponding relations for unbarred ones by replacing $d, \xi^i, \partial^i, \Delta, \bigwedge_q, q, \hat{R}_q, ...$ by $\bar{d}, \bar{\xi}^i, \bar{\partial}^i, \bar{\Delta}, \bar{\bigwedge}_q, q^{-1}, \hat{R}_q^{-1}, ...$ This statement can be rephrased as follows.

Remark. The set of valid formulae involving unbarred/barred is invariant under the above replacements. In the rest of this work it will be often convenient to extend this " invariance " also to the new results, so that results concerning barred objects can be obtained from the ones concerning unbarred objects by performing these replacements. However, to preserve the invariance one has to keep in mind the following two points:

- Invariance holds only for the formulae which are written in an explicitly $Fun(SO_q(N))$-covariant ($Fun(SO_q(N))$-scalars, -vectors, -tensors) form, i.e. where space indices are free or saturated with covariant matrices (e.g. $f(\hat{R}_{12})$ etc.).

- All new definitions of unbarred/barred ($Fun(SO_q(N))$-covariant) objects $\mathcal{O}, \bar{\mathcal{O}}$ have to be given in such a way that $\bar{\mathcal{O}}$ can be obtained from $\mathcal{O}$ by performing the above-mentioned replacements.

We don't give the commutation relations between objects in D and objects in $\bar{D}$, since we won't need them.

Since the unbarred and the barred derivatives form two Euclidean quantum spaces, $\partial \cdot \partial$, $\bar{\partial} \cdot \bar{\partial}$ are central elements respectively in the algebra of the ∂ and $\bar{\partial}$ derivatives and are both scalars under the coaction of the quantum group:

$$\partial^i(\partial \cdot \partial) = (\partial \cdot \partial)\partial^i \qquad \bar{\partial}^i(\bar{\partial} \cdot \bar{\partial}) = (\bar{\partial} \cdot \bar{\partial})\bar{\partial}^i, \tag{3.34}$$

$$\phi_L((\partial \cdot \partial)) = 1 \otimes (\partial \cdot \partial), \qquad \phi_L(\bar{\partial} \cdot \bar{\partial}) = 1 \otimes (\bar{\partial} \cdot \bar{\partial}). \tag{3.35}$$

therefore $\partial \cdot \partial$, $\bar{\partial} \cdot \bar{\partial}$ will be called the unbarred and barred laplacians respectively, since their classical limit is the classical laplacian.

Now let us consider the effect of the application of the complex conjugation on both handsides of all the commutation relations of this section. For $q \in \mathbb{R}$ we define the complex conjugation $*$ as an involutive antilinear antihomomorphism acting on the algebras generated by D and $\bar{D}$ respectively:

$$(AB)^* = B^* A^* \tag{3.36}$$

One can show that no new relations are introduced by the application of $*$ to the upper relations; more precisely, the relations involving the D calculus are transformed into relations equivalent to those involving the $\bar{D}$ calculus, and viceversa. Let us consider for instance relation (3.15). It can be rewritten in the form

$$\xi^l x^m = q^{-1} \hat{R}^{-1}{}_{ij}^{lm} x^i \xi^j. \tag{3.37}$$

By taking the complex conjugation of both handsides and using the definition (1.45) we get

$$x^{m'} C_{m'm} (\xi^l)^* = q^{-1} \hat{R}^{-1}{}_{ij}^{lm} x^{i'} (\xi^j)^* C_{i'i} \tag{3.38}$$

It is straightforward to check that, performing the replacement $(\xi^i)^* \rightarrow \bar{\xi}^j C_{ji}$ and using property (2.18) of the $\hat{R}$-matrix, the previous relation becomes exactly the commutation relation (3.16) between the coordinates x^i's and the barred 1-forms $\bar{\xi}^j$, so one can identify $(\xi^i)^*$ and $\bar{\xi}^j C_{ji}$. Looking at the explicit definition $\xi^i = dx^i - x^i d$, $\bar{\xi}^i = \bar{d}x^i - x^i \bar{d}$ we see that d^* behaves as $-\bar{d}$, so they can be identified,too. Finally, by means of the commutation relations (3.32) between derivatives and 1-forms and the decompositions $d = \xi^i \partial_i$, $\bar{d} = \bar{\xi}^i \bar{\partial}_i$ of the exterior derivatives, we are led to the identification of $\bar{\partial}^i$ with $-q^N (\partial^j)^* C_{ji}$; in fact one can check that the latter is consistent with all other relations (3.27) etc. Summarizing, one can say that $*$ maps $x^i, \xi^i, \partial^i, d$ into a combination of $\bar{x}^i, \bar{\xi}^i, \bar{\partial}^i, \bar{d}$ respectively, in the following way

$$(x^i)^* = x^j C_{ji}, \quad (\xi^i)^* = \bar{\xi}^j C_{ji}, \quad (\partial^i)^* = -q^{-N} \bar{\partial}^j C_{ji}, \quad d^* = -\bar{d}. \tag{3.39}$$

For q=1 the two calculi D, $\bar{D}$ are the same and these relations become the usual ones characterizing the classical calculus.

A direct consequence of (3.39) is the relation

$$(\partial \cdot \partial)^* = q^{-2N}(\bar{\partial} \cdot \bar{\partial}). \tag{3.40}$$

Definition. We define the differential algebra $Diff(\mathbb{R}_q^N)$ on $\mathbb{R}_q^N$ as the left and right $SO_q(N)$-comodule algebra generated by x^i, ∂^i, more precisely the ring of power series in the generators x^i, ∂^i satisfying relations (2.55)(3.30)(3.25), with coefficients given by rational functions in q.

One can prove [39] that $Diff(\mathbb{R}_q^N)$ is essentially an universal object independent of q, in particular it coincides with the differential algebra of classical coordinates and derivatives. Actually one can find a formal invertible map from classical coordinates and derivatives to q-ones; the map is not of algebraic, but of trancendent type (it involves exponentials of products $x\partial$). Of course, only the q-objects will be $SO_q(N, \mathbb{R})$-covariant in the sense of formula (2.50). We will see in next subsection that $\bar{\partial}$ derivatives can be expressed as elemnts of $Diff(\mathbb{R}_q^N)$.

For any function $f(x) \in Fun(\mathbb{R}_q^N)$ $\partial_i f$ can be expressed in the form

$$\partial_i f = f_i + f_i^j \partial_j, \qquad f_i, f_i^j \in Fun(\mathbb{R}_q^N) \tag{3.41}$$

(with f_i, f_j^i uniquely determined) upon using the derivation relations (9) to move step by step the derivatives to the right of each x^l variable of each term of the power expansion of f, as far as the extreme right. We denote f_i by $\partial_i f|$. This defines the action of ∂_i as a differential operator $\partial_i : f \in Fun(\mathbb{R}_q^N) \to \partial_i f| \in Fun(\mathbb{R}_q^N)$: we will say that $\partial_i f|$ is the " evaluation " of ∂_i on f. For instance:

$$\partial_i 1| = 0, \qquad \partial^i x^j| = C^{ij}, \qquad \partial^i x^j x^k| = C^{ij} x^k + q \hat{R}^{-1}{}^{ij}_{hl} x^h C^{lk} \tag{3.42}$$

By its very definition, ∂_i satisfies the generalized Leibnitz rule:

$$\partial_i(fg)| = \partial_i f|g + O_i^j f|\partial_j g|, \qquad f, g \in Fun(\mathbf{R_q^N}), \quad O_i^j \in Diff(\mathbb{R}_q^N) \tag{3.43}$$

$(O_i^j f| = f_i^j)$. Any $D \in Diff(\mathbb{R}_q^N)$ can be considered as a differential operator on $Fun(\mathbb{R}_q^N)$ by defining its evaluation in a similar way; a corresponding Leibnitz rule will be associated to it. In chapter 4 we will consider as differential operators the q-deformed angular momentum components (i.e. the generators of q-rotations).

In the rest of the work we will deal with some example of q-differential equations

$$\mathcal{D}(\partial,x)\phi(x)| = 0, \qquad\qquad \mathcal{D}\in Diff(Fun(\mathbb{R}_q^N)). \tag{3.44}$$

In section 3.3 we will se that there exists a dilatation operator $\Lambda \in Diff(Fun(\mathbb{R}_q^N))$ satisfying the properties

$$[\Lambda,x]_q = 0 = [\Lambda,\partial]_{q^{-1}}, \qquad\qquad \Lambda\mathbf{1}| = \mathbf{1} \tag{3.45}$$

For any given $f = f(x,\Lambda) \in Diff(Fun(\mathbb{R}_q^N))$ we can order all Λ-powers to the right of the x-powers; let $f(x|\Lambda)$ denote the corresponding " normal ordered " series expansion.

Proposition 1 *If ϕ satisfies the equation (3.44), then it is straightforward to find a solution $\tilde{\phi}$ of the equation*

$$\tilde{\mathcal{D}}(\partial,x)\tilde{\phi}(x)| = 0, \qquad\qquad \tilde{\mathcal{D}}(\partial,x) := \mathcal{D}(s(\Lambda)\partial, xs^{-1}(\Lambda)). \tag{3.46}$$

Proof. The key observation is that the derivation formulae (3.25) are invariant under the replacements $\partial, x \rightarrow s\partial, xs^{-1}$, what can be immediately checked. The hypothesis amounts to

$$\mathcal{D}(\partial,x)\phi(x) = \phi^i(x,\partial)\partial_i; \tag{3.47}$$

performing the just mentioned replacements and " normal ordering " in ϕ we find

$$\tilde{\mathcal{D}}(\partial,x)\phi_s(x|\Lambda) = \phi^i(x,\partial)\partial_i, \tag{3.48}$$

where $\phi_s(x,\Lambda) := \phi(xs^{-1}(\Lambda))$. Evaluating both sides on $\mathbf{1}_{Fun(\mathbb{R}_q^N)}|$ and using the relation $\Lambda\mathbf{1}_{Fun(\mathbb{R}_q^N)}| = \mathbf{1}_{Fun(\mathbb{R}_q^N)}$ we find the claim for $\tilde{\phi} := \phi_s(x|\Lambda = 1)$. $\diamondsuit$

3.2 Partial q-derivatives as incremental ratios for " infinitesimal q-translations "

There is an alternative and suggestive way to define the partial derivatives $\partial, \bar{\partial}$, due to S. Majid [28], which is closer to the classical way of defining derivatives. It is based on the quantum analogue of finite translations on the Euclidean space, i.e. with the coadditive braided structure of $Fun(\mathbb{R}_q^N)$ recalled in Chapter 1. For notational convenience let us write the action of the coaddition $\underline{\Delta}$ in the following way:

$$\underline{\Delta}(x^i) = x^i + a^i, \tag{3.49}$$

where we have performed in the RHS the replacements $1 \otimes x^i \to x^i$, $x^i \otimes 1 \to a^i$. Let us apply $\underline{\Delta}$ to the monomial $x^{i_1}...x^{i_m}$; we get

$$\underline{\Delta}(x^{i_1}...x^{i_m}) = (a^{i_1} + x^{i_1})...(a^{i_m} + x^{i_m}). \tag{3.50}$$

Let us now expand the sum in the RHS and reorder the a's to the left of the x's according to the two braidings (2.79). One can easily check that

$$coeff_{a^i}\underline{\Delta}(x^{i_1}...x^{i_m}) = \begin{cases} \partial_i x^{i_1}...x^{i_m}| & \textit{with the braiding}(2.79)_+ \\ \bar{\partial}_i x^{i_1}...x^{i_m}| & \textit{with the braiding}, (2.79)_- \end{cases} \tag{3.51}$$

where $coeff_{a^i}$ denotes the coefficient (belonging to $Fun(\mathbb{R}_q^N)$) of the first degree monomial in a a^i. More suggestively, we can rewrite the previous relation for an arbitrary $f \in \mathbb{R}_q^N$ as

$$\begin{cases} \partial_i \\ \bar{\partial}_i \end{cases} f(x)| = \frac{f(a+x) - f(x)}{a^i}|_{a\equiv 0} := coeff_{a^i} f(a+x). \tag{3.52}$$

An equivalent way of defining the partial derivatives would be as (braided) duals of the coordinates. One can easily show that the quantum space relations (3.30)(3.31) for the derivatives can be derived from the properties of braiding [28].

3.3 The differential algebra $Diff(\mathbb{R}_q^N)$: some explicit formulae

In this section we give some explicit formulae regarding $Diff(\mathbb{R}_q^N)$. They are based on results obtained in Refs. [2, 39],[40, 14]

The derivation relations (3.30) explicitly read

$$\begin{cases} \partial_k x^j = q x^j \partial_k - (q^2-1) q^{-\rho_j - \rho_k} x^{-k} \partial_{-j}, & j < -k, j \neq k \\ \partial_k x^j = q x^j \partial_k & j > -k, j \neq k \\ \partial_{-k} x^k = x^k \partial_{-k}, & k \neq 0 \\ \partial_i x^i = 1 + q^2 x^i \partial_i + (q^2-1) \sum_{j>i} x^j \partial_j, & i > 0 \\ \partial_i x^i = 1 + q^2 x^i \partial_i + (q^2-1) \sum_{j>i} x^j \partial_j - q^{-2\rho_i}(q^2-1) x^{-i} \partial_{-i}, & i < 0 \\ \partial_0 x^0 = 1 + q x^0 \partial_0 + (q^2-1) \sum_{j>0} x^j \partial_j, & (\textit{only for } N \textit{ odd}). \end{cases} \tag{3.53}$$

Here are some useful formulae which can be easily proved directly from equations (3.30) (sum over l is understood):

$$\partial^i (x \cdot x)_n = q^{2\rho_n} x^i + q^2 (x \cdot x)_n \partial^i \qquad (\partial \cdot \partial)_n x^i = q^{2\rho_n} \partial^i + q^2 x^i (\partial \cdot \partial)_n \tag{3.54}$$

$$(x^l\partial_l)x^i = x^i + q^2 x^i x^l \partial_l + (1-q^2)(x\cdot x)_n \partial^i \qquad \partial^i(x^l\partial_l) = \partial^i + q^2(x^l\partial_l)\partial^i + (1-q^2)x^i(\partial\cdot\partial)_n. \tag{3.55}$$

In Ref. [39],[40] the dilatation operator Λ_n

$$\Lambda_n(x,\partial)^2 := 1 + (q^2-1)x^i\partial_i + q^{N-2}(q^2-1)^2(x\cdot x)(\partial\cdot\partial) \tag{3.56}$$

was introduced; it fulfils the relations

$$\Lambda_n^2 x^i = q^2 x^i \Lambda_n^2, \qquad \Lambda_n^2 \partial^i = q^{-2}\partial^i \Lambda_n^2, \tag{3.57}$$

Then one can prove [40] that

$$\bar{\partial}^k = \Lambda_n^{-2}[\partial^k + q^{N-2}(q^2-1)x^k(\partial\cdot\partial)], \tag{3.58}$$

$$(\bar{\partial}\cdot\bar{\partial}) = q^{N-2}\Lambda^{-2}(\partial\cdot\partial). \tag{3.59}$$

In the sequel we will also need the operator

$$\mathcal{B}_n := 1 + q^{N-2}(q^2-1)(x\cdot\partial) \tag{3.60}$$

it is easy to show that it is the only operator of degree one in $x^i\partial^j$ satisfying the relations

$$\mathcal{B}_n(x\cdot x) = q^2(x\cdot x)\mathcal{B}_n, \qquad \mathcal{B}_n(\partial\cdot\partial) = q^{-2}(\partial\cdot\partial)\mathcal{B}_n; \tag{3.61}$$

Under complex conjugation

$$\mathcal{B}_n^* = q^{-N}\mathcal{B}\Lambda_n^{-2} \qquad \Lambda_n^* = q^{-N}\Lambda_n^{-1} \qquad if \;\; q\in\mathbb{R}^+. \tag{3.62}$$

In the sequel we will drop the index n in $\mathcal{B}_n, \Lambda_n$ when this causes no confusion.

We write two formulae which will be often used in the sequel

$$[\partial\cdot\partial, x\cdot x]_{q^2} = \frac{q^{2+2\rho_N}}{q^2-1}(\Lambda^2 - q^{2\rho_n-2}), \tag{3.63}$$

$$(\partial\cdot\partial)(x\cdot x)^h = \frac{(q^{2-N}+1)q^{-N+2+2h}}{q^2-1}h_{q^2}(x\cdot x)^{h-1}\mathcal{B} - \frac{q^{4-N}(q^2+1)i}{q^2-1}h_{q^4}(x\cdot x)^{h-1} + q^{4h}(x\cdot x)^h(\partial\cdot\partial). \tag{3.64}$$

By definition a scalar $I\in Diff(\mathbb{R}_q^N)$ transforms trivially under the coaction associated to the quantum group of symmetry $SO_q(N,\mathbb{R})$ [2]:

$$\phi_L(I) = \mathbf{1}_{Fun(SO_q(N))}\otimes I. \tag{3.65}$$

Any scalar polynomial $I(x,\partial)\in Diff(\mathbb{R}_q^N)$ of degree $2p$ in x,∂ is a polynomial in scalar variables of the form

$$\tilde{I} = (\eta_{\varepsilon_1})^{i_1}(\eta_{\varepsilon_2})^{i_2}...(\eta_{\varepsilon_p})^{i_p}(\eta_{\varepsilon'_p})_{i_p}...(\eta_{\varepsilon'_2})_{i_2}(\eta_{\varepsilon'_1})_{i_1}, \tag{3.66}$$

where $\varepsilon_i, \varepsilon'_j = +, -$, $\eta_+ := x$ and $\eta_- := \partial$. From here we see that no polynomial of odd degree in η^i_ε can be a scalar. One can show that any scalar I can be expressed as an ordered polynomial in two particular scalar variables:

Proposition 2 *Any scalar $I \in Diff(\mathbb{R}^N_q)$ can be expressed as a power series in the (ordered) variables $(x \cdot x), (\partial \cdot \partial)$.*

Proof. First we prove that any scalar polynomial $I(x, \partial)$ in x^i, ∂^j can be expressed as a polynomial in the (ordered) variables $x \cdot x, \partial \cdot \partial$ alone. The thesis for a generic scalar element of $Diff(\mathbb{R}^N_q)$ will follow from the fact that it admits a power expansion of the form (3.66) in x^i, ∂^j.

Let I_{2m} be a scalar polynomial of degree $2m$ (consequently it will contain only even powers $\leq 2m$ of η^i_ε). The only four independent I_2 are $1, x \cdot x, \partial \cdot \partial, x^i \partial_i$, and they all can be expressed as polynomials in $x \cdot x, \partial \cdot \partial$ because of formula (3.64).

Our claim amounts to showing that for any I_{2m} $(m \geq 0)$ there exist an ordered polynomial $P_I(x\cdot, \partial \cdot \partial)$ in $x\cdot, \partial \cdot \partial$ such that

$$I_{2m} = P_I(x \cdot x, \partial \cdot \partial) \tag{3.67}$$

The claim is obviously true for $m = 0$. The general proof is by induction: assume that it is true for $m = k$. Since any $I_{2(k+1)}$ can be written as a polynomial in some scalar variables $\tilde{I}_{2n}$ of the form (3.66) with $n \leq k + 1$, it is sufficient to prove the claim for a $\tilde{I}_{2(k+1)}$ whatsoever. By the induction hypothesis and the very definition of the $\tilde{I}$ variables $\tilde{I}_{2(k+1)}$ can be written in the form

$$\tilde{I}_{2(k+1)} = (\eta_\varepsilon)^i \tilde{P}(x \cdot x, \partial \cdot \partial)(\eta_{\varepsilon'})_i \tag{3.68}$$

with some polynomial $\tilde{P}$. Decomposing the latter in a sum of monomials and using formulae (3.34)(3.54) to move the η^i's step by step through all the factors $x \cdot x, \partial \cdot \partial$ as far as the extreme right we will be able to write the RHS of (3.68) as a combination of terms of the type $\tilde{P}'(x \cdot x, \partial \cdot \partial) \cdot (\eta_{\varepsilon''})^i(\eta_{\varepsilon'})_i$; but $(\eta_{\varepsilon''})^i(\eta_{\varepsilon'})_i$ is a polynomial of the type I_2 for which the claim (3.67) holds, hence it holds also for $\tilde{I}_{2(k+1)}$ and the statement (3.67) is completely proved. $\diamondsuit$

Remark As an immediate consequence of the previous proposition and of formulae (3.63) (3.64), if I is a scalar with definite natural dimension $d = 0$, then it can be expressed as a power series $I(B, \Lambda)$ in the commuting variables B, Λ.

Throughout this work we will use two types of q-deformed integers:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} \qquad (n)_q := \frac{q^n - 1}{q - 1}; \tag{3.69}$$

both $[n]_q$ and $(n)_q$ go to n when $n \to 1$.

The q-exponential function is introduced by

$$e_q[Z] := \sum_{n=0}^{\infty} \frac{Z^n}{(n)_q!}, \tag{3.70}$$

and satisfies the q derivative property

$$\frac{e_q[qZ] - e_q[Z]}{(q-1)Z} = e_q[Z]. \tag{3.71}$$

From (3.54) one can easily prove the relations

$$\partial^i exp_{q^2}[\alpha(x\cdot x)] = \alpha q^{2-N} x^i exp_{q^2}[\alpha(x\cdot x)] + exp_{q^2}[q^2\alpha(x\cdot x)]\partial^i \tag{3.72}$$

$$\bar{\partial}^i e_{q^{-2}}[\alpha(x\cdot x)] = \alpha x^i e_{q^{-2}}[\alpha(x\cdot x)] + e_{q^{-2}}[q^{-2}\alpha(x\cdot x)]\bar{\partial}^i \tag{3.73}$$

which imply $\partial^i exp_{q^2}[\alpha(x\cdot x)] = q^{2-N}\alpha x^i exp_{q^2}[\alpha(x\cdot x)]$, $\bar{\partial}^i e_{q^{-2}}[\alpha(x\cdot x)] = \alpha x^i e_{q^{-2}}[\alpha(x\cdot x)]$. These simple derivation formulae will make these exponentials particularly useful in the sequel.

3.3.1 Decoupled generators of $Diff(\mathbb{R}_q^N)$

In Ref. [39] it is shown that there exists a natural embedding of $Diff(\mathbb{R}_q^{N-2})$ into $Diff(\mathbb{R}_q^N)$. Incidentally, in next chapter we will see that it naturally induces an embedding of $U_q(so(N-2))$ into $U_q(so(N))$, as well. We just need to do the change of generators of $Diff(\mathbb{R}_q^N)$ $(x^i, \partial^j) \to (X^i, D^j)$ $(|i|, |j| \leq n)$, with

$$\begin{cases} x^i = \mu_n^{\frac{1}{2}} X^i, & \partial_i = \mu_n^{\frac{1}{2}} D_i, \qquad |i| < n \\ x^n = X^n & \partial_n = D_n \\ x^{-n} = \Lambda_n \mu_n^{\frac{-1}{2}} X^{-n} - q^{-2-\rho_n}(q^2-1)(X\cdot X)_{n-1} D_n & \\ \partial_{-n} = q^{-1}\Lambda_n \mu_n^{\frac{-1}{2}} D_{-n} - q^{-2-\rho_n}(q^2-1)X^n(D\cdot D)_{n-1} & \end{cases} \tag{3.74}$$

and

$$\mu_n := \mu(X^n, D_n) := D_n X^n - X^n D_n = 1 + (q^2-1)X^n D_n. \tag{3.75}$$

Then the variables X^i, D^j, $(|i|, |j| \leq n-1)$ satisfy the commutation and derivation relations (2.55),(3.30),(3.25) for $Diff(\mathbb{R}_q^{N-2})$, whereas

$$[X^{\pm n}, X^i] = 0 \qquad [X^{\pm n}, D_i] = 0 \qquad [D_{\pm n}, X^i] = 0 \qquad [D_{\pm n}, D_i] = 0,$$

$$[D_{\pm n}, X^{\mp n}] = 0 \ , \qquad [D_n, D_{-n}] = 0 \qquad [X^n, X^{-n}] = 0 \tag{3.76}$$

and

$$D_n X^n = 1 + q^2 X^n D_n \qquad\qquad D_{-n} X^{-n} = 1 + q^{-2} X^{-n} D_{-n}. \tag{3.77}$$

As a direct consequence of the previous relations, μ_n commutes with all the X, D variables, except X^n, D_n themselves:

$$\mu_n X^n = q^2 X^n \mu_n, \qquad\qquad \mu_n D_n = q^{-2} D_n \mu_n. \tag{3.78}$$

The dilatation operator Λ_n in terms of X^i, D^j variables reads $\Lambda_n^2(x,\partial) = \Lambda_{n-1}^2(X,D)\mu_n\mu_{-n}$, where $\mu_{-n} := (D_{-n}X^{-n} - X^{-n}D_{-n})^{-1}$ and $\Lambda_{n-1}^2(X,D)$ depends only on X^i, D_j ($|i|, |j| \leq n-1$) as dictated by formula (3.56) (after the replacement $n \to n-1$).

For odd N it is convenient to start the chain of embeddings from the " differential algebra of the quantum line " $Diff(\mathbb{R}_q^1)$ generated by x^0, ∂_0 satisfying the relation

$$\partial_0 x^0 = 1 + q x^0 \partial_0. \tag{3.79}$$

For N even, it is convenient to start the chain from the differential algebra $Diff(\mathbb{R}_q^2)$ of two commuting quantum lines; it is generated by the four variables $x^{\pm 1}, \partial_{\pm 1}$ all commuting with each-other, except for the two relations

$$\partial_{\pm 1} x^{\pm 1} = 1 + q^2 x^{\pm 1} \partial_{\pm 1}. \tag{3.80}$$

Summing up, we have the two chains of embeddings

$$Diff(\mathbb{R}_q^h) \hookrightarrow Diff(\mathbb{R}_q^{h+2}) \hookrightarrow Diff(\mathbb{R}_q^{h+4}) \hookrightarrow \ldots\ldots\ldots \tag{3.81}$$

where $h = \begin{cases} 1 & for\ odd\ N's \\ 2 & for\ even\ N's \end{cases}$.

From the abovementioned embeddings it trivially follows the important

Proposition 3

$$F(x^i, \partial_j) = 0 \quad in\ Diff(\mathbb{R}_q^{N-2}) \quad \Rightarrow \quad F(X^i(x^h, \partial_k), D_j(x^h, \partial_k)) = 0 \quad in\ Diff(\mathbb{R}_q^N), \tag{3.82}$$

with $|i|, |j| \leq n-1, \qquad |h|, |k| \leq n$. *In the LHS* x^i, ∂_j *are the* (x,∂)*-type generators for* $Diff(\mathbb{R}_q^{N-2})$*, in the RHS* X^i, D_j *and* x^h, ∂_k *are respectively* X, D*- and* (x,∂)*-type generators for* $Diff(\mathbb{R}_q^N)$*, and* F *for our purposes will be some polynomial function in the variables* $x, \partial, \mu_{n-1}^{\pm\frac{1}{2}}\Lambda_{n-1}$.

Let us introduce variables $\chi^i, \mathcal{D}_i$, $i \in \mathbb{C}$, such that

$$\mathcal{D}_i\chi^i = 1 + a\chi^i\mathcal{D}_i \qquad a = \begin{cases} q^2 & if\ i>0 \quad or\ N\ even\ and\ i=-1 \\ q & if\ i=0 \\ q^{-2} & otherwise \end{cases}, \tag{3.83}$$

$$[\eta, \xi] = 0 \qquad if \quad \eta = \chi^i, \mathcal{D}_i, \quad \xi = \chi^j, \mathcal{D}_j \quad with\ \ i \neq j. \tag{3.84}$$

By iterating the transformation (3.74) one arrives precisely at generators of $Diff(\mathbb{R}_q^N)$ of the type $\chi^i, \mathcal{D}_i$ with $|i| \leq n$ (and $i \neq 0$ when N is even), by identifying

$$X^{\pm n} = \chi^{\pm n}, \quad D_{\pm n} = \mathcal{D}_{\pm n}, \qquad X^{n-1} = \chi^{n-1}, \quad D_{n-1} = \mathcal{D}_{n-1} \quad .... \tag{3.85}$$

We generalize the definition (3.78) in the following way

$$\begin{cases} (\mu_{\pm i})^{\pm 1} := \mathcal{D}_{\pm i}\chi^{\pm i} - \chi^{\pm i}\mathcal{D}_{\pm i} = 1 + (q^{\pm 2} - 1)\chi^{\pm i}\mathcal{D}_{\pm i} & i>0, \quad except\ N\ even\ \&\ i=1; \\ \mu_{\pm 1} := \mathcal{D}_{\pm 1}\chi^{\pm 1} - \chi^{\pm 1}\mathcal{D}_{\pm 1} = 1 + (q^2-1)\chi^{\pm 1}\mathcal{D}_{\pm 1} & when\ N\ even; \\ (\mu_0)^{\frac{1}{2}} := \mathcal{D}_0\chi^0 - \chi^0\mathcal{D}_0 = 1 + (q-1)\chi^0\mathcal{D}_0 = \mathcal{B}_0 & when\ N\ odd. \end{cases} \tag{3.86}$$

Consequently,

$$[\mu_i, \mu_j] = 0 \qquad \mu_i\chi^j = \chi^j\mu_i \cdot \begin{cases} q^2 & if\ \ i=j \\ 1 & if\ \ i\neq j \end{cases}, \qquad \mu_i\mathcal{D}_j = \mathcal{D}_j\mu_i \cdot \begin{cases} q^{-2} & if\ \ i=j \\ 1 & if\ \ i\neq j \end{cases}. \tag{3.87}$$

and $\Lambda_n^2 = \prod_{i=-n}^{n} \mu_i$. In terms of X, D and $\chi, \mathcal{D}$ variables the square lenght $x \cdot x$ and the laplacian $\partial \cdot \partial$ take respectively the forms

$$\begin{aligned} x \cdot x &= \Lambda_n \mu_n^{-\frac{1}{2}} X^n X^{-n} q^{\rho_n} + q^{-2}(X \cdot X)_{n-1} \\ &= \sum_{i=1}^{n} \Lambda_i \mu_i^{-\frac{1}{2}} \chi^i \chi^{-i} q^{\rho_i - 2(n-i)} + \begin{cases} 0 & if\ \ N=2n \\ \frac{q^{-2n+1}}{q+1}\chi^0\chi^0 & if\ \ N=2n+1 \end{cases} \end{aligned} \tag{3.88}$$

$$\begin{aligned} \partial \cdot \partial = \Lambda_n \mu_n^{-\frac{1}{2}} D^n D^{-n} q^{\rho_n - 1} + q^{-2}(D \cdot D)_{n-1} = \sum_{i=1}^{n} \Lambda_i \mu_i^{-\frac{1}{2}} \mathcal{D}_i \mathcal{D}_{-i} q^{\rho_i - 1 - 2(n-i)} \\ + \begin{cases} 0 & if\ \ N=2n \\ \frac{q^{-2n+1}}{q+1}\mathcal{D}_0\mathcal{D}_0 & if\ \ N=2n+1 \end{cases} \end{aligned} \tag{3.89}$$

3.4 Exterior algebra on $\mathbb{R}_q^N$: volume form, completely antisymmetric tensor and q-determinant of $SO_q(N)$), Hodge duality

In this section we develop the study of the exterior algebra defined in section 3.1 and draw some useful consequences out of it. The main results were obtained by us in Ref. [12].

We are going to show that the pair of conditions (3.20) is enough to order the variables ξ^i's belonging to $\bigwedge_q(\mathbb{R}_q^N)$ in any prescribed way, so that we can uniquely define the volume N-form and therefore the q-deformed analogue $\varepsilon_q^{i_1 i_2 \dots i_N}$ of the completely antisymmetric tensor. This will in turn allow to define an Hodge duality operation and the q-determinant. Finally we show that we can set the determinants equal to one, since their squares are equal to one, so that d_qV is invariant under the E_q^N coaction ϕ_L^E.

We write the pair of conditions (3.20) in a more manageable form. On account of the decomposition $(2.15)_b$, note that the pair (3.20) is equivalent to

$$F^{ij}_{hk}\xi^h\xi^k = 0, \qquad F = \hat{R} + q^{-1}\mathbf{1}. \tag{3.90}$$

A complete set of independent relations can be obtained from (3.90) by choosing either $i \leq j, \quad i \neq -j$ or $-i = j \geq 0$. By combining them linearly we can put them in the following more explicit form. One can easily show that the Equations defining the quantum Euclidean exterior algebra amount to the system

$$\begin{cases} q\xi^i\xi^j + \xi^j\xi^i = 0 \qquad i < j, \quad i \neq -j, \\ \xi^i\xi^i = 0 \qquad i \neq 0, \\ \xi^l\xi^{-l}q^{-1} + \xi^{-l}\xi^l q = \xi^{l-1}\xi^{1-l} + \xi^{1-l}\xi^{l-1}, \qquad n \geq l > 1, \\ q\xi^{-1}\xi^1 + q^{-1}\xi^1\xi^{-1} = (q - q^{-1})\sum_{i\leq -1} q^{-\rho_i}\xi^i\xi^{-i} \qquad\qquad if\ G_q = SO_q(2n), \\ \begin{cases} q^{\frac{-3}{2}}(\xi^{-1}\xi^1 + \xi^1\xi^{-1}) - (q-q^{-1})\sum_{i<-1} q^{-\rho_i}\xi^i\xi^{-i} = 0 \\ (q^{\frac{1}{2}} + q^{\frac{-1}{2}})\xi^0\xi^0 = q\xi^{-1}\xi^1 + q^{-1}\xi^1\xi^{-1}, \end{cases} \qquad if\ G_q = SO_q(2n+1). \end{cases} \tag{3.91}$$

From these relations we realize that any product $\xi^l\xi^m$ can be written as a combination of terms of the type $\xi^i\xi^j$, where i, j satisfy a prescribed order relation. In particular, if we take the usual order relation, i.e. we require $i < j$ for all i, j, we see that this statement is true since:

- Eq.'s $(3.91)_a, (3.91)_b$ let us order $\xi^l\xi^m$ if $l \neq -m$.
- If l=-m (say $l \geq 0$), then
  - a) if $l > 1$ repeated application of eq. $(3.91)_c$ lets us write $\xi^l\xi^{-l}$ as a combination of $\xi^j\xi^{-j}$, $-n \leq j \leq 1$: it remains to order $\xi^1\xi^{-1}$, $\xi^0\xi^0$;
  - b) if $l = 1$ and $G_q = SO_q(2n)$ (respectively $G_q = SO_q(2n+1)$) we use eq. $(3.91)_d$ (respectively $(3.91)_e$) to write $\xi^1\xi^{-1}$ as a combination of $\xi^j\xi^{-j}$, $-n \leq j \leq -1$.

– c) if $l=0$ (and $G_q = SO_q(2n+1)$) eq. $(3.91)_g$ and the remarks of point b) let us reduce $\xi^0\xi^0$ to a combination of $\xi^j\xi^{-j}$ with $-n \leq j \leq -1$.

By repeated application of the ordering procedure we can rewrite any monomial $M := \xi^{i_1}\xi^{i_2}...\xi^{i_k}$ as a combination of monomials $\xi^{j_1}\xi^{j_2}...\xi^{j_k}$, where the indices $j_1, j_2, ...j_k$ satisfy a prescribed order relation, in particular the usual one $j_1 < j_2 < ... < j_k$. It is immediate to realize that if $k = N$, just as when $q = 1$, any monomial M is either zero or proportional to

$$d_qV := \begin{cases} \xi^{-n}\xi^{-n+1}...\xi^{-1}\xi^{1}...\xi^{n-1}\xi^{n} & if\ N\ is\ even \\ \xi^{-n}\xi^{-n+1}...\xi^{-1}\xi^{0}\xi^{1}...\xi^{n-1}\xi^{n} & if\ N\ is\ odd, \end{cases} \tag{3.92}$$

which therefore will be called the " volume form "; whereas if $k \geq N+1$ $M = 0$. We define the q-deformed antisymmetric tensor $\varepsilon_q^{i_1 i_2 ... i_N}$ by

$$\xi^{i_1}\xi^{i_2}...\xi^{i_N} := \varepsilon_q^{i_1 i_2 ... i_N} d_qV. \tag{3.93}$$

We can define $\varepsilon_{q,i_1 i_2 ... i_N}$ by lowering indices by means of the metric matrix:

$$\varepsilon_{q,i_1 i_2 ... i_N} := C_{i_1 j_1} C_{i_2 j_2} ... C_{i_N j_N} \varepsilon_q^{j_1 j_2 ... j_N} \tag{3.94}$$

The ε_q-tensor satisfies the relations

$$\mathcal{P}^{S\ lm}_{\ \ i_j i_{j+1}} \varepsilon_q^{i_1 i_2 ... i_N} = 0 = \mathcal{P}^{(0)lm}_{\ \ i_j i_{j+1}} \varepsilon_q^{i_1 i_2 ... i_N}, \qquad \mathcal{P}_{S\ lm}^{\ \ i_{j+1} i_j} \varepsilon_{q,i_1 i_2 ... i_N} = 0 = \mathcal{P}_{(0)lm}^{\ \ i_{j+1} i_j} \varepsilon_{q,i_1 i_2 ... i_N}, \tag{3.95}$$

with $j = 1, 2, ..., N-1$. Therefore the completely antisymmetric projectors $\mathcal{P}_{A,m}$ acting on $\otimes'^m \mathbb{C}^N$, which are defined by the conditions

$$\mathcal{P}_{i,i+1}\mathcal{P}_{A,m} = 0 = \mathcal{P}_{A,m}\mathcal{P}_{i,i+1} \qquad \mathcal{P} = \mathcal{P}_S, \mathcal{P}_1, \qquad i = 1, 2, ..., m-1, \tag{3.96}$$

can be expressed in terms of ε_q-tensors by the formula

$$\mathcal{P}^{A\ i_1 ... i_m}_{\ \ j_1 ... j_m} = \alpha_m \varepsilon_q^{i_1 ... i_m i_{m+1} ... i_N} \varepsilon_{q, i_N i_{N-1} ... i_{m+1} j_m ... j_1}. \tag{3.97}$$

where α_m is the normalization constant such that $(\mathcal{P}^A)^2 = \mathcal{P}^A$.

If N is even, then one easily realizes that $\varepsilon_q^{i_1 i_2 ... i_N}$ will be different from zero only if $(i_1, i_2, ..., i_N)$ is a permutation of $(-n, -n+1, ..., -1, 1, ..., n-1, n)$. If N is odd, $\varepsilon_q^{i_1 i_2 ... i_N}$ may be different from zero also if $n_0 > 1$ (n_0 odd) elements of the row $(i_1, i_2, ..., i_N)$ are equal to 0, and the remaining ones are all different from each other. A glance at the relation

$$(\xi^0)^{2h} = [\sum_{j<0} \xi^j \xi^{-j} q^{-\rho_j} \frac{(q - q^{-1})}{1 + q^{-1}}]^h \tag{3.98}$$

$$= \left(\frac{q-q^{-1}}{1+q^{-1}}\right)^h \sum_{j_1<j_2<\ldots j_h<0} q^{h(h-1)-(\rho_{j_1}+\rho_{j_2}\ldots+\rho_{j_h})} h! \xi^{j_1}\ldots\xi^{j_h}\xi^{-j_h}\ldots\xi^{-j_1} \tag{3.99}$$

$(0 < h \leq n)$ will convince the reader that this condition is not sufficient to guarantee that $\varepsilon_q^{i_1 i_2 \ldots i_N} \neq 0$. Consider for instance a monomial such as $\xi^{-n}\xi^{1-n}\ldots\xi^{-1}(\xi^0)^{2h+1}$; it vanishes, as in any term obtained upon use of relation (3.99) at least one ξ^{-j} with a $j > 0$ will appear two times. In fact, if ξ^j appears in a term in the RHS of (3.99), then in the same term there appears also ξ^{-j}. Therefore $\varepsilon_q^{-n,1-n,\ldots,-1,0,0,\ldots} = 0$. A little reasoning will convince the reader that for odd N the vanishing condition can be summarized as follows. Assume that $n_0 \leq n$ indices i_j are equal to zero and let $X := \{|i_j| \ : \ |i_j| \neq 0\}$; then $\varepsilon^{i_1 i_2 \ldots i_N} = 0$ iff $\#(X) > n - [\frac{n_0}{2}]$, where $[a]$ denotes the integral part of a.

From this result we deduce that whenever $\varepsilon_q^{j_1 j_2 \ldots j_N} \neq 0$, then for each value of the index $i = -n, \ldots, n$ there is a i' such that $j_i = -j_{i'}$. Together with the definition (3.94), this immediately leads to

$$\varepsilon_{q, i_1 i_2 \ldots i_N} = \varepsilon_q^{-i_1, -i_2, \ldots, -i_N}. \tag{3.100}$$

The $\bar{\xi}$ introduced in Section 1 satisfy the same commutation relations (3.91) of the ξ, therefore

$$\bar{\xi}^{i_1}\bar{\xi}^{i_2}\ldots\bar{\xi}^{i_N} := \varepsilon_q^{i_1 i_2 \ldots i_N}\overline{d_q V} \qquad where \qquad \overline{d_q V} := \bar{\xi}^{-n}\bar{\xi}^{-n+1}\ldots\bar{\xi}^{n-1}\bar{\xi}^{n}. \tag{3.101}$$

When $q \in \mathbb{R}$, it is easy to show that if we define the complex conjugation on the 1-forms ξ by $(\xi^i)^* = \bar{\xi}^j C_{ji}$, we get just the 1-forms $\bar{\xi}$. From the definitions we immediately find

$$(d_q V)^* = \overline{d_q V} \tag{3.102}$$

Finally, taking the complex conjugate of (3.93), recalling that ε is real and comparing with (3.100)(3.101) we find

$$\varepsilon_{q, i_1 i_2 \ldots i_N} = \varepsilon_q^{i_N, i_{N-1}, \ldots, i_1}. \tag{3.103}$$

To build the Hopf algebra $Fun(SL_q(N))$ one chooses a R-matrix which allows the definition of the q-determinant $det_q T$ of the generators T^i_j of the bialgebra $Fun(GL_q(N))$ as the only nontrivial central element in this algebra; then one can set $det_q T = 1$ as the characterizing condition for $Fun(SL_q(N))$. On the contrary, $Fun(SO_q(N))$ (as well as $Fun(Sp_q(n))$), is characterized by the quadratic relations (2.1)(2.2) in the generators T^i_j, which guarantee that the corresponding transformations leave the " distance " in the underlying quantum spaces unchanged. When q=1 (i.e. in the classical limit) from these relations it follows $det(T)^2 = 1$.

We now investigate whether a determinant can be defined and whether its square is automatically 1 when $q \neq 1$. We will see that using the transformation properties of the volume form under (left) coaction one can define a q-determinant $det_q T$ which has the same properties as the one of $Fun(SL_q(N))$. From the commutation relation between the coordinates and the volume form it immediately follows that $det_q T$ is central in $Fun(SO_q(N))$. Then we prove that $(det_q T)^2 = 1$. This is compatible with setting $det_q(T) = 1$. Hence the mentioned volume form transforms as a scalar under the quantum group coaction, namely it has the expected transformation law which we would require to an integration measure; in fact in Chapter 5 we will define an integration where the volume form plays such a role.

Now we are in the condition to define the q-deformed determinant of $T = ||T^i_j||$. By definition

$$det_q T := T^{-n}_{i_1} T^{-n+1}_{i_2} ... T^{n}_{i_N} \varepsilon_q^{i_1 i_2 ... i_N}, \tag{3.104}$$

or, equivalently,

$$\phi_L(d_q V) := det_q T \otimes d_q V \qquad \qquad \phi_R(d_q V) := d_q V \otimes det_q T \tag{3.105}$$

Note that the definition (3.93) implies

$$T^{j_1}_{i_1} T^{j_2}_{i_2} ... T^{j_N}_{i_N} \varepsilon_q^{i_1 i_2 ... i_N} = \varepsilon_q^{j_1 j_2 ... j_N} det^L_q T. \tag{3.106}$$

To find the result of the coproducts ϕ, ϕ^{op} of $SO_q(N)$ on $det_q T$ we use a standard argument: applying both sides of the identities (2.46), (2.47) to $d_q V$ we find

$$\phi(det_q T) = det_q T \otimes det_q T, \qquad \qquad \phi^{op}(det_q T) = det_q T \otimes det_q T, \tag{3.107}$$

i.e. $det_q T$ is grouplike under the comultiplication ϕ, ϕ^{op}.

We give the explicit expression for the tensor ε_q in the case $N = 3, 4$ and the q-determinant in the case $N = 3$:

$\varepsilon_q^{-101} = 1$	$\varepsilon_q^{-110} = -q$	$\varepsilon_q^{0-11} = -q$	$\varepsilon_q^{01-1} = q$
$\varepsilon_q^{10-1} = -q^2$	$\varepsilon_q^{1-10} = q$	$\varepsilon_q^{000} = -q(q^{\frac{1}{2}} - q^{\frac{-1}{2}})$	$\varepsilon_q^{ijk} = 0 \quad otherwise$

(3.108)

$\varepsilon_q^{-2-112} = 1$	$\varepsilon_q^{-21-12} = -q^2$	$\varepsilon_q^{-2-121} = -q$	$\varepsilon_q^{-212-1} = q^3$
$\varepsilon_q^{-22-11} = q^2$	$\varepsilon_q^{-221-1} = -q^4$	$\varepsilon_q^{-1-212} = -q$	$\varepsilon_q^{-11-22} = q^2$
$\varepsilon_q^{-1-221} = q^2$	$\varepsilon_q^{-12-21} = -q^2$	$\varepsilon_q^{-121-2} = q^3$	$\varepsilon_q^{-112-2} = -q^2$
$\varepsilon_q^{1-1-22} = -q^4$	$\varepsilon_q^{1-2-12} = q^3$	$\varepsilon_q^{1-12-2} = q^4$	$\varepsilon_q^{12-1-2} = -q^5$
$\varepsilon_q^{12-2-1} = q^4$	$\varepsilon_q^{1-22-1} = -q^4$	$\varepsilon_q^{2-2-11} = -q^2$	$\varepsilon_q^{2-1-21} = q^3$
$\varepsilon_q^{21-2-1} = -q^5$	$\varepsilon_q^{2-21-1} = q^4$	$\varepsilon_q^{2-11-2} = -q^4$	$\varepsilon_q^{21-1-2} = q^6$
$\varepsilon_q^{ijkl} = 0 \quad otherwise$			

(3.109)

The determinant $det_q T$ (when $N = 3$) is given by

$$det_q T = T_{-1}^{-1} T_0^0 T_1^1 - q T_{-1}^{-1} T_1^0 T_0^1 - q T_0^{-1} T_{-1}^0 T_1^1 + q T_0^{-1} T_1^0 T_{-1}^1 \tag{3.110}$$

$$- q^2 T_1^{-1} T_0^0 T_{-1}^1 + q T_1^{-1} T_{-1}^0 T_0^1 - q(q^{\frac{1}{2}} - q^{\frac{-1}{2}}) T_0^{-1} T_0^0 T_0^1. \tag{3.111}$$

We see that $det_q T$ is the sum of seven terms, one more (the one proportional to $T_0^{-1} T_0^0 T_0^1$) than in the classical case.

Next, we show that there is a simple commutation relation between the volume form $d_q V$ and the coordinates x^i, and that consequently $det_q T$ is central in $Fun(SO_q(N))$. The proof will be based on the fact that the $R = P\hat{R}$ matrix is triangular. By repeated application of formula (3.37) we find the following commutation rule of x^i with $d_q V$:

$$x^i d_q V = q^N R_{l_1\ v_1}^{-n\ i} R_{l_2\ v_2}^{-n+1\ v_1} \ldots R_{l_{N-1}\ v_{N-1}}^{n-1\ v_{N-2}} R_{l_N\ v_N}^{n\ v_{N-1}} \varepsilon_q^{l_1 l_2 \ldots l_N} d_q V x^{v_N} \tag{3.112}$$

(here $R = P\hat{R}$). Because of the lower-triangularity of R, the matrix element $R_{l_1 v_1}^{-n\ i}$ is different from zero only when $l_1 = -n$, and a glance at the explicit expression for R shows that $R_{-n\ v_1}^{-n\ i} = \delta_{v_1}^i \cdot \sigma_{-n,i}$, where

$$\sigma_{i,j} := \begin{cases} q & if \quad i = j \\ q^{-1} & if \quad i = -j \\ 1 & otherwise. \end{cases} \tag{3.113}$$

Therefore in formula (3.112) we can replace l_1 by $-n$ and $R_{-n\ v_1}^{-n\ i}$ by $\delta_{v_1}^i \cdot \sigma_{-n,i}$, Then the ε_q-tensor forces l_2 to run over values $\geq -n+1$. Next, we use the triangularity of R to reduce $R_{l_2\ v_2}^{-n+1\ i}$ to $\delta_{v_2}^i \cdot \sigma_{1-n,i}$. Using the same kind of argument again and again we see that after n (resp. $n+1$) steps if N is even (resp. odd) relation (3.112) has become

$$x^i d_q V = (\prod_{n \geq j \geq 0} \sigma_{-j,i}) R_{l_{N-n+1}\ v_{N-n+1}}^{1\ i} R_{l_{N-n+2}\ v_{N-n+2}}^{2\ v_{N-n+1}} \ldots R_{l_N\ v_N}^{n\ v_{N-1}} \varepsilon_q^{-n,1-n,\ldots,l_{N-n+1}\ldots l_N} d_q V x^{v_N}. \tag{3.114}$$

Because of the remarks following Eq. (3.100), both for even and odd N $\varepsilon_q^{-n,1-n,\ldots,l_{N-n+1}\ldots l_N}$ will be different from zero only if $(l_{N-n+1}, l_{N-n+2}, ..., l_N)$ is a permutation of $(1, 2, ...n)$. Then the enforcement of the same kind of argument (based on the lower triangularity of R) as before will reduce relation (3.114) after n steps to

$$x^i d_q V = (\prod_{j=-n}^{n} \sigma_{j,i}) d_q V x^i. \tag{3.115}$$

A glance at the definition (3.113) will now convince the reader that we have proved the

Proposition 4

$$x^i d_q V = d_q V x^i. \tag{3.116}$$

Now it is straightforard to show that $det_q T$ is central in $Fun(SO_q(N))$. Applying the left coaction ϕ_L to both sides of (3.116) we obtain

$$T^i_j det_q T^L \otimes x^j d_q V = (det^L_q T) T^i_j \otimes d_q V x^j = (det^L_q T) T^i_j \otimes x^j d_q V, \tag{3.117}$$

whence

$$T^i_j (det_q T) = (det_q T) T^i_j. \tag{3.118}$$

Now we can show that $(det_q T)^2 = 1$. This is the exact analogue of the classical situation. Let us consider the algebra $\bigwedge_q^N \underline{\otimes} \bigwedge_q^{',N}$ obtained as the braided product of two independent copies $\bigwedge_q^N$, $\bigwedge_q^{',N}$ of the exterior algebra on $\mathbb{R}_q^N$, generated respectively by the sets of 1-forms $\{\xi^i\}$, $\{\xi'^j\}$ both satisfying conditions (3.20). The braiding will be given by commuation relations of the form

$$\xi^i \xi'^j = M^{ij}_{hk} \xi'^h \xi^k \tag{3.119}$$

with M proportional to $\hat{R}$ or $\hat{R}^{-1}$, for instance $M = -\hat{R}$. This algebra is therefore a left and right $Fun(SO_q(N))$-comodule, with coactions ϕ_L and ϕ_R respectively. Let us apply the coactions to the form $(\xi^i C_{ij} \xi'^j)^N \in \bigwedge_q^N \underline{\otimes} \bigwedge_q^{',N}$. Using relations (2.2) we obtain

$$\phi_L((\xi^i C_{ij} \xi'^j)^N) = \mathbf{1}_{Fun(SO_q(N))} \otimes (\xi^i C_{ij} \xi'^j)^N. \tag{3.120}$$

On the other hand, using relations (3.119) to move all ξ' to the right of all the ξ we find that $(\xi^i C_{ij} \xi'^j)^N = \beta d_q V \cdot d_q V'$ with $\beta \neq 0$, and therefore

$$\phi_L((\xi^i C_{ij} \xi'^j)^N) = (det_q T)^2 \otimes (\xi^i C_{ij} \xi'^j)^N. \tag{3.121}$$

Comparing (3.105) and (3.122) we derive the claimed equality $(det_qT)^2 = 1$. By applying ϕ_R to $(\xi^i C_{ij} \xi'^j)^N$ we would find the same result.

If we set

$$det_qT = \pm 1, \tag{3.122}$$

when $q \in \mathbb{R}$, $N = 2n + 1$ the coactions perform respectively only a pure rotation of the coordinates, or a rotation together with an inversion. When $N = 2n$ we will consider only the plus sign, which classically corresponds to the most general (real) rotation.

Formulae (3.105)(3.122) show that d_qV transforms as a (pseudo)-scalar under the coactions ϕ_L, ϕ_R; this is the right transformation property we would require an integration measure. Actually in chapter 5 we show that it is possible to introduce an integration on the quantum euclidean space (based on the Stoke's theorem) where d_qV plays the role of volume form.

To define a Hodge duality operation $\star$ in $\bigwedge_q^N$ we basically require that ${}^\star\alpha_p$, α_p have inverse transformation properties w.r.t. the $Fun(SO_q(N))$ coaction, for any p-form α_p, and that $({}^\star)^2 = \pm 1$; in other words we require that the N-form $\alpha_p \wedge^\star \alpha_p$ (which is proportional to d_qV) transforms as a scalar. In fact this is what occurs in the classical case.

Let us define the Hodge duality operation ${}^\star : \bigwedge_q^m(\mathbb{R}_q^N) \to \bigwedge_q^{N-m}(\mathbb{R}_q^N)$ by

$${}^\star\xi^{i_1}...\xi^{i_m} \propto \varepsilon_{q,i_N i_{N-1}...i_1} \xi^{i_{m+1}}...\xi^{i_N}. \tag{3.123}$$

In particular we find

$${}^\star d_qV = 1 \qquad\qquad {}^\star 1 = d_qV. \tag{3.124}$$

Relation (3.123) can be inverted as

$$\xi^{i_1}...\xi^{i_m} \propto \varepsilon_q^{i_{m+1}...i_N j_1...j_m}\ {}^\star(\xi^{j_1}...\xi^{j_m}) \tag{3.125}$$

Then we can easily show that

$$\phi_L({}^\star\xi^{i_1}...\xi^{i_m}) \propto \varepsilon_{q,i_N,...i_1} \varepsilon_q^{j_{m+1}...j_N h_1...h_m} T^{i_{m+1}}_{j_{m+1}}...T^{i_N}_{j_N} \otimes\ {}^\star\xi^{h_1}...\xi^{h_m}. \tag{3.126}$$

These are just the right transformation properties we were mentioning, since they imply

$$\phi_L(\xi^{i_1}...\xi^{i_m} \wedge^\star \xi^{i_1}...\xi^{i_m}) = 1_{Fun(SO_q(N))} \otimes \xi^{i_1}...\xi^{i_m} \wedge^\star \xi^{h_1}...\xi^{h_m}. \tag{3.127}$$

By fixing suitably the normalization constants in formula (3.123) one could show that $({}^\star)^2 = (-1)^{m(N-m)}$.

Chapter 4

The Euclidean Hopf-algebras of u.e.a. type

The structure of a quantum group and of the corresponding quantum space on which it coacts are intimely interrelated [9]. As we have seen in Chapter 3, the differential calculus on the quantum Euclidean space [2] is built so as to extend the covariant coaction of the quantum group to derivatives. In absence of deformations, a function of the space coordinates is mapped under an infinitesimal $SO(N)$ transformation or an infinitesimal translation of the coordinates to a new one which can be obtained through the action of some differential operators, the angular momentum components and the pure derivatives respectively. In other words the algebra $Fun(\mathbb{R}^N)$ of functions on $\mathbb{R}^N$ is the base space of a reducible representation of $so(N)$ (which we can call the regular (vector) representation of $so(N)$), and a reducible representation of the abelian group $\mathbb{R}^N$ of translations. It is interesting to ask whether an analogue of these facts occur in the q-deformed case; in proper language, whether $Fun(\mathbb{R}_q^N)$ can be considered as a left (or right) module of the universal enveloping algebra $U_q(so(N))$, the latter being realized as some subalgebra U_q^N of the algebra of differential operators $Diff(\mathbb{R}_q^N)$ on $\mathbb{R}_q^N$, and of some suitable q-deformed version of the abelian algebra of infinitesimal translation; altogether, whether $Fun(\mathbb{R}_q^N)$ can be considered as a left (or right) module of some suitable q-deformed version of the euclidean algebra of u.e.a. type.

In this chapter we give positive answers to these questions.

The result mimics the classical (i.e. q=1) one: starting from the only $2N$ objects $\{x^i, \partial_j\}$ (the coordinates and derivatives, i.e. the generators of $Diff(\mathbb{R}_q^N)$) with already fixed commutation and derivation relations, we end up with a very economic way of realizing $U_{q^{-1}}(so(N))$ and its regular vector representation (the fact that in this way we realize $U_{q^{-1}}(so(N))$ rather than $U_q(SO(N))$ is due to the choice that our differential operators act from the left as usual, rather than from the right). In this framework, the real structure of $Diff(\mathbb{R}_q^N)$ induces the real structure of $U_{q^{-1}}(so(N))$. This we do in the first three sections, and is based on the work in Ref. [14, 13].

What is more, this approach makes inhomogeneous extensions of $U_{q^{-1}}(so(N))$ and the study of the corresponding representation spaces immediately feasible, without introducing any new generator: it essentially suffices to add derivatives to the generators of $U_{q^{-1}}(so(N))$ to find a realization of the q-deformed universal enveloping algebra of the Euclidean algebra in N dimensions [15] containing $U_{q^{-1}}(so(N))$ as a subalgebra. In fact this method was used in Ref. [51] to find the q-deformed Poincare' Hopf algebra. In both cases the inhomogeneous Hopf algebra contains the homogeneous one as a Hopf subalgebra. We will see that our q-deformed Euclidean Hopf algebras (in fact there are two independent Hopf structures) can be considered the duals of the Hopf algebras $Fun(E_q^N), Fun(\bar{E}_q^N)$ of chapter 2. This is done in the fourth section, and is based on the work in Ref. [15]. In section 5 we define a new set of generators of the q-Euclidean algebra, in view of the study of representations, in section 6 we determine the casimirs, in section 7 we propose a nonstandard approach to the " real section " of these Hopf algebras.

In Chapter 7 we will study Hilbert space representations of these Hopf algebras.

As anticipated in Chapter 2, there are some alternative approaches to inhomogeneous quantum groups. A different (and chronologically preceding) approach is based on contractions of homogeous quantum groups of higher rank, see for instance Ref. [4, 20, 21]; another one based on projections of bicovariant differential calculi on homogeous quantum groups of higher rank has been recently proposed in Ref. [3]

The plan of the chapter is as follows. In section 1 we define a subalgebra $U_q^N \subset Diff(\mathbb{R}_q^N)$ by requiring that its elements commute with scalars, introduce two different sets of generators for it and study the commutation relations of the second set. In section 2 we find the commutation relations of these generators with the coordinates and derivatives and derive the natural Hopf algebra structure associated to U_q^N (thought as algebra of differential operators on $Fun(\mathbb{R}_q^N)$); the Hopf algebra U_q^N is then identified with $U_{q^{-1}}(so(N))$. In Section 3 we find that the q-deformed homogeneous symmetric spaces are the base spaces of the irreducible representations of U_q^N in $Fun(\mathbb{R}_q^N)$, and we show that they can be explicitly constructed as highest weight representations. When $q \in \mathbb{R}^+$ the representations are unitary and the hermitean conjugation coincides with the complex conjugation in $Diff(\mathbb{R}_q^N)$.

We will treat by a unified notation odd and even N's whenever it is possible, and n will be related to N by the formulae $N = 2n+1$ and $N = 2n$ respectively. To this end we define $h = \begin{cases} 0 & if \;\; N = 2n+1 \\ 1 & if \;\; N = 2n \end{cases}$. When $N = 2n$ there is a complete symmetry of $\mathbb{R}_q^N$ under the exchange of coordinates $x^{-1} \leftrightarrow x^1$, and this implies a complete invariance of the differential calculus, and all the results of this chapter and and of the following ones under the exchange of indices $-1 \leftrightarrow 1$. We will assume that q is generic. Finally, we will often use the shorthand notation $[A,B]_a := AB - aBA$ ($\Rightarrow [\cdot,\cdot]_1 = [\cdot,\cdot]$).

4.1 The u.e.a. U_q^N of the angular momentum on $\mathbb{R}_q^N$

Inspired by the classical (i.e. q=1) case, we give the following

Definition: the universal enveloping algebra U_q^N of the angular momentum on $\mathbb{R}_q^N$ is the subalgebra of $Diff(\mathbb{R}_q^N)$ whose elements commute with any scalar $I(x,\partial) \in Diff(\mathbb{R}_q^N)$. Since any such I can be expressed as a function of the laplacian and of the square lenght $x \cdot x, \partial \cdot \partial$, our definition amounts to

$$U_q^N := \{u \in Diff(\mathbb{R}_q^N): \quad [u, x \cdot x] = 0 = [u, \partial \cdot \partial]\} \tag{4.1}$$

In the next two subsections we consider two sets of generators of U_q^N (actually we will prove in Appendix B that any $u \in U_q^N$ can be expressed as a function of them). The generators of the first set transform in the same way as the products $x^i x^j$ under the coaction, since (up to a scalar) they are q-antisymmetrized products of x, ∂ variables, but have rather complicated commutation relations; nevertheless Casimirs have a very compact expressions in terms of them. The generators of the second set have a quite simple form in terms of $\chi, \mathcal{D}$ variables and are much more useful for practical purposes, since they have simple commutation relations and are directly connected with the Cartan-Weyl generators of $U_{q^{-1}}(so(N))$

4.1.1 The set of generators $\{l^{ij}, B\}$

Keeping the classical case in mind, where the angular momentum components are antisymmetrized products $x^i \partial^j - x^j \partial^i$ of coordinates and derivatives, we try with the q-deformed antisymmetrized products

$$\mathcal{L}^{ij} := \mathcal{P}_A{}^{ij}_{hk} x^h \partial^k = -q^{-2} \mathcal{P}_A{}^{ij}_{hk} \partial^h x^k. \tag{4.2}$$

From relations (25),(8) it follows that

$$\mathcal{L}^{ij}x\cdot x = q^2 x\cdot x\mathcal{L}^{ij} \qquad \mathcal{L}^{ij}\partial\cdot\partial = q^{-2}\partial\cdot\partial\mathcal{L}^{ij}. \tag{4.3}$$

This implies that $\mathcal{L}^{ij}$ commutes only with scalars having natural dimension $d=0$. This shortcoming can be cured by introducing a scalar $S \in Diff(\mathbb{R}_q^N)$ with natural dimension $d=0$ and such that $Sx\cdot x = q^{-2}x\cdot xS$, $S\partial\cdot\partial = q^2\partial\cdot\partial S$; then by defining $l^{ij} := \mathcal{L}^{ij}S$ we get

$$[l^{ij}, I] = 0; \tag{4.4}$$

l^{ij} are therefore candidates to the role of angular momentum components. The simplest choice is to take $S=\Lambda^{-1}$, as we did in Ref. [12], and will be adopted in the sequel.

Starting from commutation relations for the $\mathcal{L}^{ij}$'s we get corresponding relations for the l^{ij}'s by multiplying them by a suitable power of Λ^{-1}. In fact, it is clear that the former must be homogeneous in $\mathcal{L}$'s to be consistent with relation (4.4). Nevertheless, commutation relations including factors such as $\mathcal{L}^{ij}\mathcal{L}^{-j,l}$ cannot be of this form. In fact, performing the derivations $\partial^{-j}x^j$ according to rules (3.53) one lowers by 1 the degree in $x\partial$ of some terms; this can be taken into account only by considering homogeneous relations both in $\mathcal{L}^{ij}$'s and $\mathcal{B}$ ($\mathcal{B}$ was defined in (3.60)), since $\mathcal{B}$ is the only other 1^{st} degree polynomial in $x^i\partial_j$ with the same scaling law (3.61) as $\mathcal{L}^{hk}$. Summing up, we expect homogeneous commutation relations in the l^{ij}'s and $B := \mathcal{B}\Lambda^{-1}$. B is not really an independent generator, as we will see below. Therefore, the alternative choice $S := \mathcal{B}^{-1}$ (as considered in ref. [13]) would yield the same algebra.

Remark 1: When $q=1$ $\mathcal{B}=1=\Lambda$ and l^{ij} reduce to the classical " angular momentum " components, i.e. to generators of $U(so(N))$ (note that they are expressed as functions of the non-real coordinates x^i of $\mathbb{R}^N$ and of the corresponding derivatives). In this limit one can take as generators of the Cartan subalgebra the $l^{i,-i}$'s, as ladder operators corresponding to positive (resp. negative) roots the l^{jk}'s with $|j|<|k|$ and $k>0$ (resp. $k<0$), as ladder operators corresponding to simple roots the $l^{1-i,i}$'s together with $l^{j,2}$ ($i=2,...,n$, and $j = \begin{cases} 0 & if\ N=2n+1 \\ 1 & if\ N=2n \end{cases}$). A Chevalley basis is formed by the set of triples $\{(l^{1-i,i}, l^{-i,i-1}, l^{i,-i}-l^{i-1,1-i}),\quad i=1,...,n\}$ if $N=2n+1$ (here $l^{0,0}=0$) and $\{(l^{1,2}, l^{-2,-1}, l^{2,-2}+l^{1,-1}), (l^{1-i,i}, l^{-i,i-1}, l^{i,-i}-l^{i-1,1-i}),\quad i=2,...,n\}$ if $N=2n$. The correspondence with spots in the Dynkin diagrams of the classical series $\mathbf{B}_n, \mathbf{D}_n$ is shown in figure 1.

Remark 2: One could work with $\bar{\partial}$ instead of ∂ derivatives and define $\bar{l}^{ij} := \mathcal{P}_A{}^{ij}_{hk} x^h \bar{\partial}^k \Lambda$ $\bar{B}_n := \bar{\mathcal{B}}_n \Lambda$, where $\bar{\mathcal{B}}_n := 1 + (q^{-2} - 1)(x \cdot \bar{\partial})$. But using formulae (3.58) one shows that $q^{-1}\bar{l}^{ij} = q l^{ij}$.

The scalar $(l \cdot l)_n := l^{ij} L_{ji}$ commutes with any l^{ij} and reduces (up to a factor) to the classical square angular momentum when q=1. We will call this casimir the (q-deformed) square angular momentum. Higher order Casimirs can be obtained by forming nontrivial independent scalars out of $j-th$ powers ($j > 2$) of the L's,

$$(\underbrace{l \cdot l \cdot ... l}_{j\ times})_n := l^{i_1 i_2} l_{i_2}^{\ i_3} l_{i_3}^{\ i_4} ... l_{i_j, i_1}, \tag{4.5}$$

for the same values of j as in the classical case.

Proposition 5 *The following important relation connects $B, (l \cdot l)$:*

$$1 = (B)^2 - \frac{(q^2-1)(q^2-q^{-2})}{(1+q^{2\rho_n})(1+q^{-2\rho_n-2})}(l \cdot l)_n \tag{4.6}$$

Proof. Using formulae (2.19),(2.15), (2.35) one can easily show [11] that

$$(\mathcal{L} \cdot \mathcal{L})_n = \alpha_N(q) x^i \partial_i + \beta_N(q) x^i x^j \partial_j \partial_i + \gamma_N(q)(x^i x_i)(\partial^i \partial_i), \tag{4.7}$$

where

$$\begin{cases} \alpha_N(q) := \frac{(q^{2-\frac{N}{2}} + q^{\frac{N}{2}-2})(q^{1-N} - q^{N-1})}{(q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1})(q^{-2} - q^2)} \\ \beta_N := \frac{q^3 + q^{N-1}}{(1+q^{2-N})(q+q^{-1})} \qquad \gamma_N := -\frac{(q^{5-N}+q)(1+q^{-N})}{(1+q^{2-N})^2(q+q^{-1})}. \end{cases} \tag{4.8}$$

Performing derivations in $\mathcal{B}^2$ according to formula (3.55) we realize that

$$\Lambda_n^2 = (\mathcal{B}_n)^2 - \frac{(q^2-1)(q^2-q^{-2})}{(1+q^{2\rho_n})(1+q^{-2\rho_n-2})}(\mathcal{L} \cdot \mathcal{L})_n \tag{4.9}$$

which is equivalent to the claim. $\diamondsuit$

As a consequence, B^2 is not an independent generator, as anticipated, but depends on $l \cdot l$.

When $q \in \mathbb{R}$ from formulae (2.20), (3.39), (3.55), (3.58) it follows that under complex conjugation

$$(l^{i,j})^* = q^{\rho_i + \rho_j} l^{-j,-i}; \tag{4.10}$$

this implies in particular that $l \cdot l$, the other casimirs and the l^i's are real. Moreover, it is easy to show that all the l^i's commute with each other, as the $\mathcal{L}^i$'s do.

The basic commutation relations between l^{ij}, B are quadratic in these variables but rather complicated and we won't give them here.

Rather, we show that there exist closed commutation relations between the B, l^{ij}'s and the components v^h of any vector $\vec{v} \in Diff(\mathbb{R}_q^N)$ (e.g. $v^h = x^h, \partial^h, \bar{\partial}^h$)

Proposition 6

$$v^i B = q \left[1 + \frac{q^2 - 1}{1 + q^{2-N}}\right] B v^i + \frac{(q^{-1} - q)(q^4 - 1)}{(1 + q^{N-2})(1 + q^{4-N})} l^{ib} v_b \tag{4.11}$$

$$v^i l^{jk} = \left[\frac{q^3 - q^{-1}}{1 + q^{4-N}} C^{ih} \mathcal{P}_A{}^{jk}_{ha} C_{bc} + q(\hat{R}_{12}\hat{R}_{23})^{ijk}{}_{abc}\right] l^{ab} v^c \tag{4.12}$$

$$+ q^{-1} C^{ih} \mathcal{P}{}^{jk}{}_A{}_{hl} B v^l + \alpha \mathcal{P}_{3,A}{}^{ijk}_{abc} l^{ab} v^c \tag{4.13}$$

where the last term (with an arbitrary $\alpha \in \mathbb{C}$) is identically zero in this framework (because of formulae (2.55)(2.30)), but can be taken different from zero when l^{ij} are not the purely orbital part of the angular momentum components, but include also some intrisic angular momentum (spin) part (see section 6.4)

Proof. As an example we prove the first relation. First, using formula (2.15) the definitions (2.59),(4.2), and equations (3.60) it is easy to prove the following relation:

$$x^i (\partial \cdot \partial) = \frac{x^l \partial_l}{1 + q^{N-2}} \partial^i + \frac{(q^2 + 1) q^{2-N}}{1 + q^{4-N}} \mathcal{L}^{ij} \partial_j. \tag{4.14}$$

Then,

$$\mathcal{B} \partial^i \stackrel{(3.55)}{=} q^{-2} \partial^i \mathcal{B} + \frac{1 - q^{-2}}{1 + q^{2-N}} \left[q^{2-N} \partial^i + (q^2 - 1) x^i (\partial \cdot \partial)\right]$$

$$\stackrel{(4.14)}{=} q^{-2} \partial^i \mathcal{B} + \frac{1 - q^{-2}}{1 + q^{2-N}} \left[q^{2-N} \mathcal{B} \partial^i + \frac{(1 - q^{-2})(q^4 - 1)}{(1 + q^{N-2})(1 + q^{4-N})} \mathcal{L}^{ij} \partial_j\right]. \tag{4.15}$$

We find relation (4.11) if we collect terms in $\mathcal{B}\partial^i$ together, multiply both sides of this identity by $q^2 \Lambda^{-1}$, use relation (3.57) and the definition $B = \mathcal{B}\Lambda^{-1}$. $\diamondsuit$

4.1.2 The set of generators $\{\mathbf{L}^{ij}, (\mathbf{k}^i)^{\pm 1}\}_{i \neq j}$

On the contrary, the new generators defined below admit very simple commutation relations, allowing a straightforward proof of the isomorphism $U_q^N \approx U_{q^{-1}}(so(N))$. It is convenient to use $\chi, \mathcal{D}$ variables to define and study them. The definitions of $\mathbf{L}^{ij}, \mathbf{k}^i$ involve only $\chi^l, \mathcal{D}_m$ variables with $|l|, |m| \leq J := max\{|i|, |j|\}$, so that in terms of these variables it makes sense for any $n \geq J$. Hence, it trivially follows the embedding $U_q^N \hookrightarrow U_q^{N+2}$, since, as we will show in Appendix B, $\mathbf{L}^{ij}, (\mathbf{k}^i)^{\pm 1}$ generate U_q^N.

Proposition 7 *The elements*

$$\mathbf{k}^i := \mu_i\mu_{-i}^{-1} \in Diff(\mathbb{R}_q^N) \qquad 0 < i \leq n \tag{4.16}$$

belong to U_q^N *and commute with each other.*

Proof. The thesis is a trivial consequence of formulae (3.86),(3.88), (3.89). ◇

We will call the subalgebra generated by $\mathbf{k}^i$ the " Cartan subalgebra " $H_q^N \subset U_q^N$.

In Appendix A we show that the elements $\mathbf{k}_i \in U_q^N$ can be expressed as functions of B, l^{ij}.

Now we define the generators $\mathbf{L}^{ij} \in U_q^N$, which correspond to roots. Since the generators of U_q^{N-2} belong also to U_q^N (in the sense of the abovementioned embedding), we can stick to the definition of the new generators, i.e. the ones belonging to $(U_q^N - U_q^{N-2})$. For this purpose it is convenient to use the X, D variables of $Diff(\mathbb{R}_q^N)$.

Definition:

$$\begin{cases} \mathbf{L}^{ln} := q^{-2}\Lambda_n^{-1}\mu_{-n}[D^l,(X\cdot X)_{n-1}]D^n - \mu_n^{-\frac{1}{2}}X^nD^l, & (pos.\ roots) \\ \mathbf{L}^{-nl} := q^{-1}\Lambda_n^{-1}\mu_{-n}X^{-n}[(D\cdot D)_{n-1},X^l] - \mu_n^{-\frac{1}{2}}X^lD^{-n}, & (neg.\ roots) \end{cases} \quad |l| < n. \tag{4.17}$$

In particular, it is easy to show that the complete list of generators corresponding to simple roots of U_q^N (i.e. the ones with indices as prescribed in Remark 1) in terms of $\chi, \mathcal{D}$ variables reads

$$\begin{cases} \mathbf{L}^{1-k,k} := \mu_k^{-\frac{1}{2}}\left[q^{2\rho_k}(\mu_{-k}\mu_{k-1})^{\frac{1}{2}}\chi^{1-k}\mathcal{D}^k - \chi^k\mathcal{D}^{1-k}\right] & n \geq k \geq j = \begin{cases} 2 & if\ \ N = 2n+1 \\ 3 & if\ \ N = 2n \end{cases} \\ \mathbf{L}^{01} := (\mu_1)^{-\frac{1}{2}}\left[q^{-2}(\mu_{-1})^{\frac{1}{2}}\chi^0\mathcal{D}^1 - \chi^1\mathcal{D}^0\right] & if\ \ N = 2n+1, \\ \mathbf{L}^{\pm 1,2} := \mu_2^{-\frac{1}{2}}\left[q^{-2}(\mu_{-2}\mu_{\mp 1})^{\frac{1}{2}}(\mu_{\pm 1})^{-\frac{1}{2}}\chi^{\pm 1}\mathcal{D}^2 - \chi^2\mathcal{D}^{\pm 1}\right] & if\ \ N = 2n; \end{cases} \tag{4.18}$$

the list of corresponding negative Chevalley partners is given by

$$\begin{cases} \mathbf{L}^{-k,k-1} := \mu_k^{-\frac{1}{2}}\left[q^{2\rho_k-1}(\mu_{-k}\mu_{k-1})^{\frac{1}{2}}\chi^{-k}\mathcal{D}^{k-1} - \chi^{k-1}\mathcal{D}^{-k}\right] & 2 \leq k \leq n, \\ \mathbf{L}^{-1,0} = (\mu_1)^{-\frac{1}{2}}\left[\mu_{-1}^{\frac{1}{2}}\chi^{-1}\mathcal{D}^0 - \chi^0\mathcal{D}^{-1}\right] & if\ \ N = 2n+1, \\ \mathbf{L}^{-2,\pm 1} := \mu_2^{-\frac{1}{2}}\left[q^{-1}(\mu_{-2}\mu_{\pm 1})^{\frac{1}{2}}(\mu_{\mp 1})^{-\frac{1}{2}}\chi^{-2}\mathcal{D}^{\pm 1} - \chi^{\pm 1}\mathcal{D}^{-2}\right] & if\ \ N = 2n. \end{cases} \tag{4.19}$$

Note that when $N = 2n+1$ $l^{0\pm 1} = (\mathbf{k}^1)^{\frac{1}{2}}\mathbf{L}^{0\pm 1}$, when $N = 2n$ $l^{\pm 1,2} = (\mathbf{k}^2)^{\frac{1}{2}}(\mathbf{k}^1)^{\pm\frac{1}{2}}\mathbf{L}^{\pm 1,2}$, $l^{-2,\pm 1} = q(\mathbf{k}^2)^{\frac{1}{2}}(\mathbf{k}^1)^{\mp\frac{1}{2}}\mathbf{L}^{-2,\pm 1}$. In appendix A we show that the simple roots and their Chevalley partners are functions of l^{ij}, B.

Proposition 8 *:* $\mathbf{L}^{ln}, \mathbf{L}^{-n,l} \in U_q^N$.

Proof. In terms of X, D variables, formulae (3.88), (3.84), (3.87) yield

$$[\mathbf{L}^{ln},(x{\cdot}x)_n] = \left[q^{-2}\Lambda_n^{-1}\mu_{-n}[D^l,(X\cdot X)_{n-1}]D^n, \Lambda_n\mu_{-n}^{-\frac{1}{2}}X^nX^{-n}q^{\rho_n}\right] - [\mu_n^{-\frac{1}{2}}X^nD^l, q^{-2}(X{\cdot}X)_{n-1}]$$

$$= q^{\rho_n-2}\mu_{-n}\mu_n^{-\frac{1}{2}}[D^l,(X\cdot X)_{n-1}][D^n,X^{-n}]X^n - q^{-2}\mu_n^{-\frac{1}{2}}X^n[D^l,(X\cdot X)_{n-1}] = 0, \quad (4.20)$$

and formulae (3.86),(3.89),(3.84) yield

$$[\mathbf{L}^{ln},(\partial{\cdot}\partial)_n] = -q^{\rho_n}\mu_n^{-1}\Lambda_n[X^n,D^{-n}]_{q^{-2}}D^lD^n + q^{-4}\Lambda_n^{-1}\mu_{-n}D^n\left[[D^l,(X\cdot X)_{n-1}],(D\cdot D)_{n-1}\right]_{q^{-2}}$$

$$= q^{2\rho_n-2}\mu_n^{-1}\Lambda_nD^lD^n - q^{-6}\Lambda_n^{-1}\mu_{-n}D^n\left[D^l,\Lambda_{n-1}^2\frac{q^{4+2\rho_n}}{q^2-1}\right] = 0 \qquad (4.21)$$

(here we have used the identity (3.63) $[\partial\cdot\partial, x\cdot x]_{q^2} = \frac{q^{2+2\rho_n}}{q^2-1}(\Lambda_n^2-1)$); namely $\mathbf{L}^{ln}\in U_q^N$. Similarly one proves that $\mathbf{L}^{-nl}\in U_q^N$. $\diamondsuit$.

Lemma 1

$$[\mathbf{L}^{hn},\partial_n]_q = \partial^h, \qquad [\mathbf{L}^{-n,h},x^n]_{q^{-1}} = -q^{\rho_n}x^h \qquad |h|<n, \quad (4.22)$$

$$[\partial^{n-1},\mathbf{L}^{1-n,n}]_{q^{-1}} = q^{\rho_n-1}\partial^n, \qquad [\mathbf{L}^{-n,n-1},x^{1-n}]_{q^{-1}} = q^{\rho_l}x^{-n} \qquad n>1, \quad (4.23)$$

$$[\partial^0,\mathbf{L}^{01}] = q^{-1}\partial^1 \qquad [\mathbf{L}^{-10},x^0] = x^{-1} \qquad if\ \ N=3. \quad (4.24)$$

Proof. For the proof see Proposition 15 of next section and the remark following it. $\diamondsuit$

The following proposition allows to construct all the roots starting from the Chevalley ones.

Proposition 9 *The following relations hold in* $Diff(\mathbb{R}_q^N)$:

$$\begin{cases}[\mathbf{L}^{-jl},\mathbf{L}^{-lk}]_q = q^{\rho_l}\mathbf{L}^{-j,k},\\ [\mathbf{L}^{-kl},\mathbf{L}^{-l,j}]_q = q^{\rho_l+1}\mathbf{L}^{-k,j},\end{cases} \qquad n\ge k>l>j\ge\begin{cases}0 & if\ \ N=2n+1\\ -1 & if\ \ N=2n\end{cases} \qquad (4.25)$$

$$[\mathbf{L}^{l-1,k},\mathbf{L}^{1-l,l}]_{q^{-1}} = q^{\rho_l-1}\mathbf{L}^{lk} \qquad [\mathbf{L}^{-l,l-1},\mathbf{L}^{-k,1-l}]_{q^{-1}} = q^{\rho_l}\mathbf{L}^{-k,-l} \qquad 2\le l<k\le n \quad (4.26)$$

$$[\mathbf{L}^{0k},\mathbf{L}^{01}] = q^{-1}\mathbf{L}^{1k} \qquad [\mathbf{L}^{-10},\mathbf{L}^{-k0}] = \mathbf{L}^{-k,-1} \qquad 1<k\le n\ \ if\ \ N=2n{+}1. \quad (4.27)$$

Proof. As an example we prove equation $(4.25)_1$. First consider the case $n=k$. We note that

$$[\mathbf{L}^{-jl},\mathbf{L}^{-lk}]_q = q^{-2}\Lambda_k^{-1}\mu_{-k}\left[[\mathbf{L}^{-j,l},D^{-l}]_q,(X\cdot X)_{k-1}\right]D^k - \mu_k^{-\frac{1}{2}}X^k[\mathbf{L}^{-j,l},D^{-l}]_q, \quad (4.28)$$

as $[(X\cdot X)_{k-1},\mathbf{L}^{-j,l}]=0$. But

$$[\mathbf{L}^{-j,l},D^{-l}]_q=D^{-j}q^{\rho_l} \tag{4.29}$$

as a consequence of the preceding Lemma and Proposition 1, therefore the RHS of equation $(4.25)_1$ gives $q^{\rho_l}\mathbf{L}^{-jk}$. Applying Proposition 1 $(n-k)$ times we prove formula $(4.25)_1$ in the general case. The proofs of the other equations are similar. ◇

Proposition 10 *When* $q\in\mathbb{R}$

$$(\mathbf{k}^i)^*=\mathbf{k}^i,\qquad (\mathbf{L}^{1-k,k})^*=q^{-2}\mathbf{L}^{-k,k-1}\qquad k\geq 2,$$
$$\begin{cases}(\mathbf{L}^{01})^*=q^{-\frac{3}{2}}\mathbf{L}^{-10} & if\ \ N=2n+1\\ (\mathbf{L}^{12})^*=q^{-2}\mathbf{L}^{-2,-1} & if\ \ N=2n\end{cases} \tag{4.30}$$

Proof. The thesis can be proved by writing these $\mathbf{k},\mathbf{L}$ generators in terms of the B,L ones as shown in Appendix A and by using the conjugation relations (4.10). ◇

The following three propositions give the basic commutation relations among the Chevalley generators. More relations for the other roots can be obtained from these ones using the relations of Proposition 9 In the following two propositions we assume that $k\geq\begin{cases}1 & if\ \ N=2n+1\\ 2 & if\ \ N=2n\end{cases}$

Proposition 11

$$[\mathbf{k}^i,\mathbf{L}^{\pm(1-k),\pm k}]_a=0\qquad a=\begin{cases}q^{\pm 2} & if\ i=k\leq n\\ q^{\mp 2} & if\ \ i=k-1\\ 1 & otherwise\end{cases}$$
$$[\mathbf{k}^i,\mathbf{L}^{\pm 1,\pm 2}]_a=0\qquad a=\begin{cases}q^{\pm 2} & if\ i=1,2\\ 1 & otherwise\end{cases} \tag{4.31}$$

Proof: a trivial consequence of formulae (3.87) and of the definition of $\mathbf{L},k$'s. ◇

Proposition 12 *(commutation relations between positive and negative simple roots)*

$$[\mathbf{L}^{1-m,m},\mathbf{L}^{-k,k-1}]_a=0\qquad a=\begin{cases}q^{-1} & m-1=k\\ 1 & if\ \ m-1>k\end{cases} \tag{4.32}$$

$$[\mathbf{L}^{-m,m-1},\mathbf{L}^{1-k,k}]_a=0\qquad a=\begin{cases}q & if\ \ m-1=k\\ 1 & if\ \ m-1>k\end{cases} \tag{4.33}$$

$$[\mathbf{L}^{12},\mathbf{L}^{-2,1}]=0\qquad [\mathbf{L}^{-1,2},\mathbf{L}^{-2,-1}]=0\qquad if\quad N=2n, \tag{4.34}$$

$$\begin{cases}[\mathbf{L}^{1-m,m},\mathbf{L}^{-m,m-1}]_{q^2}=q^{1+2\rho_m}\frac{1-\mathbf{k}^{m-1}(\mathbf{k}^m)^{-1}}{q-q^{-1}} & 2\leq m\leq n\\ [\mathbf{L}^{01},\mathbf{L}^{-1,0}]_q=q^{-\frac{1}{2}}\frac{1-(\mathbf{k}^1)^{-1}}{q-q^{-1}} & if\ \ N=2n+1\\ [\mathbf{L}^{12},\mathbf{L}^{-2,-1}]_{q^2}=q^{-1}\frac{1-(\mathbf{k}^2\mathbf{k}^1)^{-1}}{q-q^{-1}} & if\ \ N=2n.\end{cases} \tag{4.35}$$

Proof. Use equations (3.84),(3.87) and perform explicit computations. ◇

Proposition 13 *(Serre relations)*

$$[\mathbf{L}^{1-m,m},\mathbf{L}^{1-k,k}]=0 \qquad [\mathbf{L}^{-m,m-1},\mathbf{L}^{-k,k-1}]=0 \qquad m,k>0,\quad |m-k|>1 \quad (4.36)$$

$$[\mathbf{L}^{12},\mathbf{L}^{1-j,j}]=0 \qquad [\mathbf{L}^{-2,-1},\mathbf{L}^{-j,j-1}]=0 \qquad j=2,4,5,...,n, \qquad N=2n, \quad (4.37)$$

$$[\mathbf{L}^{1+j-m,m-j},\mathbf{L}^{2-m,m}]_a=0=[\mathbf{L}^{-m,m-2},\mathbf{L}^{j-m,m-j-1}]_a \qquad a=\begin{cases}q & if\ j=0\\ q^{-1} & if\ j=1\end{cases} \qquad m\geq 3 \quad (4.38)$$

$$\begin{cases}[\mathbf{L}^{01},\mathbf{L}^{12}]_{q^{-1}}=0\\ [\mathbf{L}^{-1,2},\mathbf{L}^{02}]_q=0\end{cases} \qquad \begin{cases}[\mathbf{L}^{-2,-1},\mathbf{L}^{-1,0}]_{q^{-1}}=0\\ [\mathbf{L}^{-2,0},\mathbf{L}^{-2,1}]_q=0\end{cases} \qquad if\ \ N=2n+1, \quad (4.39)$$

$$\begin{cases}[\mathbf{L}^{12},\mathbf{L}^{13}]_{q^{-1}}=0\\ [\mathbf{L}^{-23},\mathbf{L}^{13}]_q=0,\end{cases} \qquad \begin{cases}[\mathbf{L}^{-3,-1},\mathbf{L}^{-2,-1}]_{q^{-1}}=0\\ [\mathbf{L}^{-3,-1},\mathbf{L}^{-3,2}]_q=0,\end{cases} \qquad if\ \ N=2n. \quad (4.40)$$

Proof. Use the definitions (4.17), commutation and derivation relations for the X, D variables, equation (3.87) and perform explicit computations. ◇

We collect below all the basic commutation relations characterizing U_q^3, U_q^4. Their algebras read respectively

$$\begin{cases}[(\mathbf{k}^1)^{\frac{1}{2}},\mathbf{L}^{01}]_q=0\\ [(\mathbf{k}^1)^{\frac{1}{2}},\mathbf{L}^{-10}]_{q^{-1}}=0\\ [\mathbf{L}^{01},\mathbf{L}^{-10}]_q=q^{-\frac{1}{2}}\frac{1-(\mathbf{k}^1)^{-1}}{q-q^{-1}}\end{cases} \quad (4.41)$$

and

$$\begin{cases}[(\mathbf{k}^1\mathbf{k}^2)^{\frac{1}{2}},\mathbf{L}^{12}]_{q^2}=0\\ [(\mathbf{k}^1\mathbf{k}^2)^{\frac{1}{2}},\mathbf{L}^{-2,-1}]_{q^{-2}}=0\\ [\mathbf{L}^{12},\mathbf{L}^{-2,-1}]_{q^2}=q^{-1}\frac{1-\mathbf{k}^1\mathbf{k}^2}{q-q^{-1}}\end{cases} \qquad \begin{cases}[((\mathbf{k}^1)^{-1}\mathbf{k}^2)^{\frac{1}{2}},\mathbf{L}^{-1,2}]_{q^2}=0\\ [((\mathbf{k}^1)^{-1}\mathbf{k}^2)^{\frac{1}{2}},\mathbf{L}^{-2,1}]_{q^{-2}}=0\\ [\mathbf{L}^{-12},\mathbf{L}^{-2,1}]_{q^2}=q^{-1}\frac{1-(\mathbf{k}^1)^{-1}\mathbf{k}^2}{q-q^{-1}}\end{cases} \quad (4.42)$$

$$[L,L']=0 \qquad L=\mathbf{L}^{12},\mathbf{L}^{-2,-1},(\mathbf{k}^1\mathbf{k}^2) \qquad L'=\mathbf{L}^{-12},\mathbf{L}^{-21},(\mathbf{k}^1)^{-1}\mathbf{k}^2; \quad (4.43)$$

We see that U_q^4 is the direct sum of two (commuting) identical algebras, (the ones in the L and L' generators respectively). This is no surprise, since it preludes to the relation $U_q^4 \approx U_q(so(4)) \approx U_q(su(2)) \otimes U_q(su(2))$, which we will prove in next section.

4.2 The Hopf algebra structure of U_q^N and its identification as $U_{q^{-1}}(so(N))$

In this section we show that U_q^N is an Hopf algebra, more precisely that it is isomorphic to $U_{q^{-1}}(so(N))$.

A natural bialgebra structure can be associated to U_q^N for the reason that its elements satisfy some Leibnitz rule when acting as differential operators on $Fun(\mathbb{R}_q^N)$. A matched antipode can be found in a straightforward way, so that U_q^N acquires a Hopf algebra structure. As for the mentioned isomorphism, we will prove it by constructing an invertible transformation from the generators of U_q^N to those of $U_{q^{-1}}(so(N))$, in such a way that the commutation relations,coproduct, counit, antipode of U_q^N are mapped into the ones of $U_{q^{-1}}(so(N))$.

This means that the Hopf algebra $U_{q^{-1}}(so(N)$ admits a representation on all of $Fun(\mathbb{R}_q^N)$.

The Hopf algebra $U_q(so(N))$ [7, 9] is generated by X_i^+, X_i^-, H_i $(i = 1, ..., n)$ satisfying the commutation relations

$$\begin{cases} [H_i, H_j] = 0, & [H_i, X_j^{\pm}] = \pm(\alpha_i, \alpha_j)X_i^{\pm}, \\ [X_i^+, X_j^-] = \delta_{ij}\frac{q^{H_i}-q^{-H_i}}{q-q^{-1}} & \\ \sum_{t=1}^{m_{ij}}(-1)^t \begin{bmatrix} m_{ij} \\ t \end{bmatrix}_{q^i} (X_i^{\pm})^t X_j^{\pm}(X_i^{\pm})^{m_{ij}-t} = 0 & i \neq j, \end{cases} \tag{4.44}$$

where

$$q^i = q^{(\alpha_i, \alpha_i)}, \qquad m_{ij} = 1 - \frac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \qquad \begin{bmatrix} m \\ t \end{bmatrix}_q := \frac{[m]_q}{[t]_q[m-t]_q} \tag{4.45}$$

and the $(n \times n)$ matrix of scalar products between the simple roots α_i is given by

$$||b_{ij}|| := ||(\alpha_i, \alpha_j)|| = \left\| \begin{matrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & & & & \\ & & & & \cdot & \cdot & \cdot \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{matrix} \right\| \tag{4.46}$$

if $N = 2n+1$ and by

$$\|b_{ij}\| := \|(\alpha_i, \alpha_j)\| = \left\| \begin{matrix} 2 & & -1 & & & & & & \\ & 2 & -1 & & & & & & \\ -1 & -1 & 2 & -1 & & & & & \\ & & -1 & 2 & -1 & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & & \cdot & \cdot & \cdot & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{matrix} \right\| \tag{4.47}$$

if $N = 2n$. Moreover, when $q \in \mathbb{R}$ they also satisfy the adjointness relations

$$H_i^\dagger = H_i \qquad (X_i^+)^\dagger = X_i^-. \tag{4.48}$$

The coproduct, counit, antipode $\Phi_q, \epsilon, \sigma_q$ are defined respectively by

$$\Phi_q(H_i) = \mathbf{1} \otimes H_i + H_i \otimes \mathbf{1} \qquad \Phi_q(X_i^\pm) = X_i^\pm \otimes q^{-\frac{H_i}{2}} + q^{\frac{H_i}{2}} \otimes X_i^\pm \tag{4.49}$$

$$\epsilon(X_i^\pm) = 0, \qquad \epsilon(H_i) = 0 \tag{4.50}$$

$$\sigma_q(H_i) = -H_i \qquad \sigma_q(X_i^\pm) = -q^{-\frac{(\alpha_i, \alpha_i)}{2}} X_i^\pm \tag{4.51}$$

on the generators and extended as algebra homomorphisms/antihomomorphisms.

Now we show that there exist closed commutation relations between the generators of U_q^N and the coordinates x^i.

Proposition 14

$$[\mathbf{k}^i, x^h]_a = 0, \qquad a = \begin{cases} q^2 & if\ h = i > 0 \\ q^{-2} & if\ h = -i < 0 \\ 1 & otherwise \end{cases} \qquad in\ Diff(\mathbb{R}_q^N) \tag{4.52}$$

Proof. One has just to write x^h as functions of $\chi^j, \mathcal{D}_j$ and use relations (3.87). ◇

As for the commutation relations between roots **L**'s and x^i, we write down only the ones involving simple roots and their opposite (the other ones can be obtained in the same way or using Proposition 9).

Proposition 15 *Let* $m \geq \begin{cases} 1 & if\ N = 2n+1 \\ 2 & if\ N = 2n \end{cases}$. *Then*

$$[\mathbf{L}^{1-m,m}, x^i]_a = 0 \qquad a = \begin{cases} 1 & if\ |i| < m-1\ or\ |i| > m \\ q^{-1} & if\ i = 1-m, m \end{cases}, \tag{4.53}$$

$$[\mathbf{L}^{-m,m-1}, x^i]_a = 0 \qquad a = \begin{cases} 1 & if \quad |i| < m-1 \quad or \quad |i| > m \\ q & if \quad i = -m, m-1 \end{cases}, \tag{4.54}$$

$$[\mathbf{L}^{1-m,m}, x^{m-1}]_q = -q^{\rho_m} x^m \qquad [\mathbf{L}^{1-m,m}, x^{-m}]_q = q^{\rho_m} x^{1-m}, \tag{4.55}$$

$$[\mathbf{L}^{-m,m-1}, x^m]_{q^{-1}} = -q^{\rho_m} x^{m-1} \qquad [\mathbf{L}^{-m,m-1}, x^{1-m}]_{q^{-1}} = q^{\rho_m} x^{-m}, \tag{4.56}$$

$$[\mathbf{L}^{01}, x^0] = -q^{-1} x^1 \qquad [\mathbf{L}^{-1,0}, x^0] = x^{-1} \qquad if \quad N = 2n+1 \tag{4.57}$$

$$\begin{cases} [\mathbf{L}^{12}, x^1]_{q^{-1}} = 0 & [\mathbf{L}^{-2-1}, x^{-1}]_q = 0 \\ [\mathbf{L}^{12}, x^2]_{q^{-1}} = 0 & [\mathbf{L}^{-2-1}, x^{-2}]_q = 0 \\ [\mathbf{L}^{12}, x^{-1}]_q = -q^{-1} x^2 & [\mathbf{L}^{-2-1}, x^1]_{q^{-1}} = q^{-1} x^{-2} \\ [\mathbf{L}^{12}, x^{-2}]_q = q^{-1} x^1 & [\mathbf{L}^{-2-1}, x^2]_{q^{-1}} = -q^{-1} x^{-1} \end{cases} \qquad if \quad N = 2n. \tag{4.58}$$

Proof. One has just to write x^h as functions of $\chi^j, \mathcal{D}_j$ and use relations (3.84),(3.87). $\diamondsuit$

The commutation relations of $\mathbf{k}^i, \mathbf{L}^{ij}$'s with $\partial_i, \bar{\partial}_i$ are the same, since $\partial_i \propto [\partial \cdot \partial, x_i]_{q^2}$, $\bar{\partial}_i \propto [\bar{\partial} \cdot \bar{\partial}, x_i]_{q^{-2}}$ and the $\mathbf{k}, \mathbf{L}$'s commute with scalars. The knowledge of the latter commutation relations will allow us in section 6.4 to construct the inhomogeneous extension of $U_q^N \approx U_{q^{-1}}(so(N))$, i.e. the universal enveloping algebra of the quantum Euclidean group, adding derivatives as new generators [15] (see section 4.4)

We can consider $\mathbf{L}^{ij}$'s, $\mathbf{k}^i$'s as differential operators on $Fun(\mathbb{R}_q^N)$ in the same way as we did in section 2 with ∂_i. The commutation relations (4.52)-(4.58) allow us to define iteratively their evaluations and Leibnitz rules starting from

$$\mathbf{k}^i \mathbf{1}| = 1 \qquad \mathbf{L}^{ij} \mathbf{1}| = 0 \tag{4.59}$$

($\mathbf{1}$ denotes the unit of $Fun(\mathbb{R}_q^N)$). For instance, by applying $\mathbf{k}^i$ to x^h, using eq. (4.52) and the previous relation we find

$$\mathbf{k}^i x^h| = x^h \cdot \begin{cases} q^{\pm 2} & if \quad \pm h = i > 0 \\ 1 & otherwise; \end{cases} \tag{4.60}$$

by applying $\mathbf{k}^i$ to $x^h \cdot g$ ($g \in Fun(\mathbb{R}_q^N)$) and using again eq. (4.52) we find

$$\mathbf{k}^i (f \cdot g)| = \mathbf{k}^i f | \mathbf{k}^i g| \tag{4.61}$$

for $f = x^h$ first, and then by recurrence for any $f \in Fun(\mathbb{R}_q^N)$. The latter relation is the Leibnitz rule for $\mathbf{k}^i$. (4.59),(4.60),(4.61) are equivalent to (4.52),(4.59) and determine the evaluation of $\mathbf{k}^i$ on all of $Fun(\mathbb{R}_q^N)$. Similarly the Leibnitz rule for the simple roots is determined to be

$$\mathbf{L}^{1-m,m}(f \cdot g)| = \mathbf{L}^{1-m,m} f | g + (k^{m-1}(k^m)^{-1})^{\frac{1}{2}} f | \mathbf{L}^{1-m,m} g| \qquad m \geq \begin{cases} 1 & if \quad N = 2n+1 \\ 2 & if \quad N = 2n \end{cases} \tag{4.62}$$

$$\mathbf{L}^{12}(f\cdot g)| = \mathbf{L}^{12}f|g + (\mathbf{k}^1\mathbf{k}^2)^{-\frac{1}{2}}f|\mathbf{L}^{12}g| \qquad if \;\; N=2n \tag{4.63}$$

$(f,g \in Fun(\mathbb{R}_q^N))$, and the same formulae hold by replacing each simple root by its negative partner.

More abstractly, the above formulae define: 1) a counit $\epsilon : U_q^N \to \mathbb{C}$, by setting $\epsilon(u) := \pi(u\mathbf{1}|)$, $\quad u \in U_q^N$ and $\pi(\alpha\mathbf{1}) := \alpha \quad \forall\alpha \in \mathbb{C}$, implying that ϵ is an homomorphism which on the generators $\mathbf{k}^i, \mathbf{L}^{ij}$ takes the form

$$\epsilon(\mathbf{L}^{ij}) = 0 \qquad\qquad \epsilon(\mathbf{k}^i) = 1; \tag{4.64}$$

2) a coassociative coproduct $\phi : U_q^N \to U_q^N \otimes U_q^N$ which on the generators $\mathbf{k}^i, \mathbf{L}^{ij}$ takes the form

$$\phi(\mathbf{k}^i) = \mathbf{k}^i \otimes \mathbf{k}^i \tag{4.65}$$

$$\begin{cases} \phi(\mathbf{L}^{1-m,m}) = \mathbf{L}^{1-m,m} \otimes \mathbf{1}' + (\mathbf{k}^{m-1}(k^m)^{-1})^{\frac{1}{2}} \otimes \mathbf{L}^{1-m,m} \\ \phi(\mathbf{L}^{-m,m-1}) = \mathbf{L}^{-m,m-1} \otimes \mathbf{1}' + (k^{m-1}(k^m)^{-1})^{\frac{1}{2}} \otimes \mathbf{L}^{-m,m-1} \end{cases} \qquad m \geq \begin{cases} 1 & if \;\; N=2n+1 \\ 2 & if \;\; N=2n \end{cases} \tag{4.66}$$

$$\begin{cases} \phi(\mathbf{L}^{12}) = \mathbf{L}^{12} \otimes \mathbf{1}' + (\mathbf{k}^1\mathbf{k}^2)^{-\frac{1}{2}} \otimes \mathbf{L}^{12} \\ \phi(\mathbf{L}^{-2,-1}) = \mathbf{L}^{-2,-1} \otimes \mathbf{1}' + (\mathbf{k}^1\mathbf{k}^2)^{-\frac{1}{2}} \otimes \mathbf{L}^{-2,-1} \end{cases} \qquad if \;\; N=2n \tag{4.67}$$

($\mathbf{1}'$ here denotes the unit of $Diff(\mathbb{R}_q^N)$, which acts as the identity when considered as an operator on $Fun(\mathbb{R}_q^N)$, and $\mathbf{k}^0 \equiv \mathbf{1}'$), and is extended to all of U_q^N as an homomorphism. ϵ, ϕ are matched so as to form a bialgebra; in particular the coassociativity of ϕ follows from the associativity of the Leibnitz rule, which in turn is a consequence of the associativity of $Diff(\mathbb{R}_q^N)$.

An antipode σ which is matched with ϕ, ϵ, (i.e. satisfies all the required axioms) is found by first imposing the two basic axioms

$$m \circ (\sigma \otimes id) \circ \phi = m \circ (id \otimes \sigma) \circ \phi = i \circ \epsilon \tag{4.68}$$

on the generators of U_q^N, and then by extending it as an antihomomorphism; here m denotes the multiplication in U_q^N and i is the canonical injection $i : \mathbb{C} \to U_q^N$. Computations are straightforward:

$$\sigma(\mathbf{k}^i) = (\mathbf{k}^i)^{-1}, \tag{4.69}$$

$$\begin{cases} \sigma(\mathbf{L}^{1-m,m}) = -(\mathbf{k}^m(\mathbf{k}^{m-1})^{-1})^{\frac{1}{2}}\mathbf{L}^{1-m,m} \\ \sigma(\mathbf{L}^{-m,m-1}) = -(\mathbf{k}^m(\mathbf{k}^{m-1})^{-1})^{\frac{1}{2}}\mathbf{L}^{-m,m-1} \end{cases} \qquad m \geq \begin{cases} 1 & if \;\; N=2n+1 \\ 2 & if \;\; N=2n \end{cases} \tag{4.70}$$

$$\begin{cases} \sigma(\mathbf{L}^{12}) = -(\mathbf{k}^1\mathbf{k}^2)^{-\frac{1}{2}}\mathbf{L}^{12} \\ \sigma(\mathbf{L}^{-2,-1}) = -(\mathbf{k}^1\mathbf{k}^2)^{-\frac{1}{2}}\mathbf{L}^{-2,-1} \end{cases} \qquad if \;\; N=2n. \tag{4.71}$$

Finally, when $q \in \mathbb{R}$ it is straightforward to check that the complex conjugation $*$ (the antilinear involutive antihomomorphism defined in section 3, which acts on the basic generators as shown in formula (4.30)) is compatible with the Hopf algebra structure of U_q^N, so that U_q^N gets a $*$-Hopf algebra.

Now it is easy to identify the Hopf algebra U_q^N.

Proposition 16 *All the relations characterizing the $(*)$-Hopf algebra $U_{q^{-1}}(so(N))$ are mapped into the ones characterizing the $(*)$-Hopf algebra U_q^N through the transformation of generators*

$$[\mathbf{k}^i(\mathbf{k}^{i-1})^{-1}]^{\frac{1}{2}} = q^{H_i} \qquad \mathbb{L}^{1-i,i} = q^{\rho_i-\frac{3}{2}}X_i^+ q^{-\frac{H_i}{2}} \qquad \mathbb{L}^{-i,i-1} = q^{\rho_i+\frac{3}{2}}X_i^- q^{-\frac{H_i}{2}},$$

$$(i \geq \begin{cases} 1 & if \;\; N = 2n+1 \\ 2 & if \;\; N = 2n \end{cases}, \; \mathbf{k}^0 \equiv 1), \; and$$

$$[\mathbf{k}^2(\mathbf{k}^1)]^{\frac{1}{2}} = q^{H_1} \qquad \mathbb{L}^{1,2} = q^{-\frac{5}{2}}X_1^+ q^{-\frac{H_1}{2}} \qquad \mathbb{L}^{-2,-1} = q^{\frac{1}{2}}X_1^- q^{-\frac{H_1}{2}}, \qquad N = 2n. \tag{4.72}$$

after setting $\Phi_{q^{-1}} = \phi$, $\sigma_{q^{-1}} = \sigma$ (or, alternatively, $\tau \circ \Phi_q = \phi$, $\sigma_q = \sigma$, τ being the permutation operator). In other words $U_q^N \approx U_{q^{-1}}(so(N))$.

Proof. Straightforward computations. $\diamondsuit$

Note that if we had defined the elements of U_q^N as differential operators acting on $Fun(\mathbb{R}_q^N)$ from the right (instead of from the left), we would have got the isomorphism $U_q^N \approx U_{q^{-1}}(so(N))$.

As a concluding remark, the final lesson we learn is that the product in $Fun(\mathbb{R}_q^N)$ realizes the tensor product of representations of $U_{q^{-1}}(so(N))$, the Leibnitz rule satisfied by the differential operators of U_q^N realizes the corresponding coproduct, and the real structure of $Diff(\mathbb{R}_q^N)$ realizes the real structure of $U_{q^{-1}}(so(N))$.

4.3 Vector representations of U_q^N on $Fun(\mathbb{R}_q^N)$

Let us now look at U_q^N as an operator algebra over $Fun(\mathbb{R}_q^N)$. In other words we consider " evaluations " of its elements on $Fun(\mathbb{R}_q^N)$ as defined in the previous section. We look for its irreducible representations. Since $l \cdot l$ commutes with any l^{ij}, it is proportional to the identity matrix on the base space W of each of them.

As a first remark, we note that any W must consist of polynomials of fixed degree in x, as any $u \in U_q^N$ is a power series in the products $x^i\partial_j$. Of course, the degree of these polynomials must be the same, say k, also after factoring out all powers of $x \cdot x$, since $[u, x \cdot x] = 0$. One can easily realize (see Ref. [11]) that the subspace of $Fun(\mathbb{R}_q^N)$ satisfying these two requirements is

$$W_k := Span_{\mathbb{C}}[\mathcal{P}_{k,S}{}^{j_1 \ldots j_k}_{i_1 \ldots i_k} x^{i_1} \ldots x^{i_k}], \qquad k \in \mathbb{N}, \tag{4.73}$$

and that W_k is an eigenspace of $l \cdot l$. Here $\mathcal{P}_{k,S}$ denotes the (q-deformed) k-symmetric (modulo trace) projector, which can be defined through

$$\mathcal{P}_{k,S}\mathcal{P}_{Ai,(i+1)} = 0 = \mathcal{P}_{k,S}\mathcal{P}_{1i,(i+1)}, \qquad \mathcal{P}_{Ai,(i+1)}\mathcal{P}_{k,S} = 0 = \mathcal{P}_{1i,(i+1)}\mathcal{P}_{k,S}, \qquad (\mathcal{P}_{k,S})^2 = \mathcal{P}_{k,S}, \tag{4.74}$$

$1 \leq i \leq k-1$, and

$$(\mathcal{P}_{k,S})^2 = \mathcal{P}_{k,S}, \tag{4.75}$$

where $\mathcal{P}_{Ai,(i+1)} = (\otimes 1)^{i-1} \otimes \mathcal{P}_A \otimes (\otimes 1)^{n-i-1}$, etc. (the second relation only fixes a normalization constant). Hence $W \subset W_k$.

In particular the fundamental (vector) representation W_1 is spanned by the N independent vectors x^i.

Below we are going to see that the representations of U_q^N in W_k's are irreducible and of highest weight type. When $q = 1$ they reduce to the vector representations of $so(N)$.

As " ladder operators " corresponding to positive, negative, simple roots we take the ones indicated in Remark 1 for the case $q = 1$. Correspondingly,

Proposition 17 *The highest (respectively lowest) weight eigenvector is the vector $u_k^n := (x^n)^k$ (respectively $(x^{-n})^k$). W_k is generated by iterated application of negative (resp. positive) ladder operators and is an eigenspace of $l \cdot l$ with eigenvalue*

$$l_{k,N}^2 = [k]_q[k+N-2]_q \frac{(q^{\rho_n+1} + q^{-\rho_n-1})}{(q+q^{-1})(q^{\rho_n} + q^{-\rho_n})} \tag{4.76}$$

Proof. Using the derivation rules (3.83) it is straightforward to show that all positive ladder operators $\mathbf{L}^{jk}$ annihilate $(x^n)^k$. Moreover, it is easy to show that this vector is an eigenvector of $l \cdot l$ (with eigenvalue l_k^2) and therefore belongs to W_k. This follows from the fact that it is an eigenvector of $x^l \partial_l$ (with eigenvalue $(k)_{q^2}$) and from formulae (4.9),(3.60). As already noted, the application of negative ladder operators then yields a space $W \subset W_k$. As known, $W = W_k$ when $q = 1$; but $dim(W), dim(W_k)$ are constant with q, therefore $W = W_k \; \forall q$. Similarly one proves that $(x^{-n})^k$ is the lowest weight eigenvector. $\diamondsuit$.

Let us consider the space of homogeneous polynomials of degree k

$$M_k := Span_{\mathbb{C}}[x^{i_1} \ldots x^{i_k}]. \tag{4.77}$$

As a consequence of the definition of W_l, we are able to decompose M_k into irreducible representations of U_q^N (see [12]), just as in the case $q = 1$:

$$M_k = \bigoplus_{0 \leq m \leq \frac{k}{2}} W_{k-2m}(x \cdot x)^m. \tag{4.78}$$

Recall that $dim(M_k) = \binom{N+k-1}{N-1}$, therefore this formula allows to recursively find $dim(W_k)$: $dim(W_k) = dim(M_k) - dim(M_{k-2})$. The formula

$$Fun(\mathbb{R}_q^N) = \bigoplus_{l=0}^{\infty} M_l = \bigoplus_{l=0}^{\infty} \bigoplus_{0 \leq m \leq \frac{l}{2}} W_{l-2m}(x \cdot x)^m, \tag{4.79}$$

gives the formal decomposition of $Fun(\mathbb{R}_q^N)$ into irreducible vector representations of $U_{q^{-1}}(so(N))$. All of them are involved (infinitely many times), and therefore $Fun(\mathbb{R}_q^N)$ can be called the base space of the " regular " representation $U_{q^{-1}}(so(N))$, in analogy with the classical case.

When $q \in \mathbb{R}$, starting from the prescriptions $(u_k^n, u_k^n) := 1$, $u^\dagger := u^* \quad \forall u \in U_q^N$, and using the commutation relations of U_q^N one can define an inner product $(\cdot , \cdot)$ in all of W_k.

Proposition 18 *When $q \in \mathbb{R}^+$ the inner product $(\cdot , \cdot)$ is positive definite, i.e. the representations W_k of the $*$-Hopf algebra U_q^N become Hilbert space representations.*

Proof. For the proof we use the notion of integration $\int d_qV$ on $\mathbb{R}_q^N$ which will be introduced in next chapter; the latter satisfies a q-deformed Stoke's theorem. According to it

$$(f,g) = c_k \int dV \ f^* g \ \rho(x \cdot x), \qquad f, g \in W_k. \tag{4.80}$$

Indeed, integrating by parts " border terms " vanish, and therefore taking the adjoint $u^\dagger$ of u w.r.t this inner product amounts to taking its complex conjugate. In this formula $\rho(x \cdot x)$ denotes a " rapidly decreasing function " function of the square lenght such as the q-deformed gaussian $exp_q(-ax \cdot x)$ and the normalization factor c_k is chosen so that $(u_k^n, u_k^n) = 1$. But we will prove in next Chapter, Theorem 3, that $(\cdot , \cdot)$ is positive definite. $\diamondsuit$

According to the theory of representations of $U_{q^{-1}}(so(N))$, when $q \in \mathbb{R}$ the Cartan subalgebra generators H_i make up a complete set of commuting observables in W_k, $\forall k \geq 0$. The highest weight associated to W_k is the n-ple $(0, 0, ..., 0, k)$ of eigenvalues of the n-ple of operators $(H_1, H_2, ..., H_n)$ on u_k^n.

According to the commutation relations (4.31), a basis E_k of W_k consisting of eigenvectors of $(H_1, H_2, ..., H_n)$ is obtained by considering all the independent vectors obtained by applying negative root operators to u_k^n.

For instance, in the case $N = 3$ the dimension of W_k is $2k+1$ and

$$E_k := \{u_{k,h} := (\mathbf{L}^{-10})^{-k-h} u_k^1, \quad h = -k, -k+1, ..., k\}. \tag{4.81}$$

For any monomial $M(k,\{i\}) := x^{i_1} x^{i_2} ... x^{i_k}$ define $t(M) := i_1 + i_2 + ... i_k$. Looking at formulae (3.83) we realize that the effect of the action of $\mathbf{L}^{0\pm1}$ on any monomial $M_{\{i\}}$ is to give a combination of monomials M' with $t(M') = t(M) \pm 1$. Therefore $u_{k,h}$ is a combination of monomials M with $t(M) = h$.

The functions $(x \cdot x)^{\frac{-k}{2}} u$ $(u \in E_k)$ will be said q-deformed spherical functions of degree k, since they reduce to the classical ones in the limit $q = 1$, when we express $x^i (x \cdot x)^{\frac{-1}{2}}$ in terms of angular coordinates.

4.4 Euclidean Hopf algebra of u.e.a. type

In this section we are going to introduce (two) q-deformed Euclidean Hopf algebras $U_q(e^N)$,$U_q(\bar{e}^N)$ of u.e.a. type in N dimensions as " extensions " of $U_q^N = U_{q^{-1}}(so(N))$, more precisely by adding to its generators " infinitesimal " generators p^i of translations and the generator of dilatation. $U_q(e^N)$, $U_q(\bar{e}^N)$ will be the duals of the Hopf algebras $Fun(E_q^N)$,$Fun(\bar{E}_q^N)$ (see Chapter 2).

Two notational remarks are necessary before the beginning. In the sequel $\mathbb{N} := \{0, 1, 2, ...\}$. Moreover for brevity we will omit to write some relations in the case $N = 2n$; they can be obtained by performing the index replacement $1 \leftrightarrow -1$ in all formulae where a space index i takes the values $i = \pm 1$. This can be done because of the complete $1 \leftrightarrow -1$ symmetry of the formulae of the quantum Euclidean space and differential calculus w.r.t. a space index i in the case $N = 2n$.

We first introduce the algebraic structure of the Hopf algebras $U_q(e^N)$, $U_q(\bar{e}^N)$.

As we have mentioned in section 2.3, the dual of a copy $\mathbb{R}_{q,\bar{x}}^N$ of the Euclidean quantum space (thought as the braid group of finite translations) is another copy $\mathbb{R}_{q,\bar{p}}^N$ of this quantum space, with generators p^i such that $< p_i, x^j >= \delta_j^i$, and $Fun(SO_q(N))$ coacts on it in the way shown in formula (2.50), provided we use the controvariant components $p^i = p_j C^{ji}$. The condition that the p^i generate $Fun(\mathbb{R}_{q,\bar{p}}^N)$ of course means that they satisfy the commutation relations

$$\mathcal{P}_{A}{}^{ij}{}_{hk} p^h p^k = 0. \tag{4.82}$$

The dual version of the statement that $Fun(SO_q(N))$ coacts on $Fun(\mathbb{R}^N_{q,\vec{p}})$ in the same way as on $Fun(\mathbb{R}^N_{q,\vec{x}})$ is that the commutation relations between the generators of U_q^N and p^i must be the same as those with x^i (this is no surprise since we already saw that also the derivatives $\partial^i, \bar{\partial}^i$ satisfy commutation relations (4.52)- (4.58)). We rewrite them here for convenience. Let $m \geq \begin{cases} 1 & if \quad N = 2n+1 \\ 2 & if \quad N = 2n \end{cases}$. Then:

$$[\mathbf{k}^i, p^h]_a = 0, \qquad a = \begin{cases} q^2 & if\ h = i > 0 \\ q^{-2} & if \quad h = -i < 0 \\ 1 & otherwise \end{cases}$$

$$[\mathbf{L}^{1-m,m}, p^i]_a = 0 \qquad a = \begin{cases} 1 & if \quad |i| < m-1 \quad or \quad |i| > m \\ q^{-1} & if \quad i = 1-m, m \end{cases}, \tag{4.83}$$

$$[\mathbf{L}^{-m,m-1}, p^i]_a = 0 \qquad a = \begin{cases} 1 & if \quad |i| < m-1 \quad or \quad |i| > m \\ q & if \quad i = -m, m-1 \end{cases},$$

$$[\mathbf{L}^{1-m,m}, p^{m-1}]_q = -q^{\rho_m} p^m \qquad [\mathbf{L}^{1-m,m}, p^{-m}]_q = q^{\rho_m} p^{1-m},$$

$$[\mathbf{L}^{-m,m-1}, p^m]_{q^{-1}} = -q^{\rho_m} p^{m-1} \qquad [\mathbf{L}^{-m,m-1}, p^{1-m}]_{q^{-1}} = q^{\rho_m} p^{-m}, \tag{4.84}$$

$$[\mathbf{L}^{01}, p^0] = -q^{-1} p^1 \qquad [\mathbf{L}^{-1,0}, p^0] = p^{-1} \qquad if \quad N = 2n+1.$$

We see that the algebra generated by $\mathbf{L}, \mathbf{k}, p$ is closed.

To find a coproduct we will need to introduce one more generator, the generator Λ of dilatation, such that:

$$[\Lambda, \vec{p}]_{q^{-1}} = 0 \qquad [\Lambda, \mathbf{k}] = 0 \qquad [\Lambda, \mathbf{L}] = 0; \tag{4.85}$$

we will see that it is the dual of g.

Definition In the sequel $\hat{u}_q(e^N)$ will denote the algebra generated by $\mathbf{L}, \mathbf{k}, p$, $u_q(e^N)$ the algebra generated by $\mathbf{L}, \mathbf{k}, p, \Lambda^{\pm 1}$; these generators satisfy relations (4.31)-(4.40).

$U_q^N \subset \hat{u}_q(e^N) \subset u_q(e^N)$ as subalgebras; more precisely these subalgebras (4.82)-(4.85) can be projected into each other by setting $\Lambda = 1$ and $p = 0$ respectively.

Remark Note that there exists a natural embedding $\hat{u}_q(e^N) \hookrightarrow \hat{u}_q(e^{N+2})$ obtained by setting equal to zero all the generators of $p^i, \mathbf{L}^{ij}, \mathbf{k}^i$ of $\hat{u}_q(e^{N+2})$ having $i = \pm(n+1)$ for some space index i.

There exists a trivial embedding $\hat{u}_q(e^N) \hookrightarrow \hat{u}_q(e^{N+2})$, which is obtained by setting equal to zero all the generators of $\hat{u}_q(e^{N+2})$ having $i = \pm(n+1)$ for some space index i.

Actually as a set of algebraically independent generators of $u_q(e^N)$ one should take the Chevalley generators of U_q^N, $\Lambda^{\pm1}$ and *one* particular momentum component (e.g. p^n), since equations (4.84) can be used to *define* the others; then relations (4.83) will be linear in p and of degree $d \geq 1$ in the Chevalley generators and will play a role analogous to Serre relations (4.36)-(4.40).

$\mathbf{L}, \mathbf{k}, p, \Lambda$ should now be thought as abstract generators, not as elements of $Diff(\mathbb{R}_q^N)$, even though we found the relations of their algebra realizing them as differential operators on $\mathbb{R}_q^N$; at the representation-theoretic level this will be necessary since we are interested in finding all representations of $u_q(e^N)$ and not only those with zero spin.

We try now to endow $u_q(e^N)$ with some Hopf structures compatible with its algebraic structure.

We first look for the coalgebra structure. Again, as we did in section 4.2 with U_q^N, our postulates will be suggested by the explicit realization of the generators p^i as differential operators on $Fun(\mathbb{R}_q^N)$.

A first possibility is to realize p^i by $p^i \equiv \partial^i$, a second, by $p^i \equiv \bar{\partial}^i$. Using relations (3.58)(3.54),(4.2) one can easily show that more generally $p^i \equiv f(\Lambda)(\alpha\partial^i + \beta\bar{\partial}^i)$ is a realization of the p^i's as differential operators on $\mathbb{R}_q^N$. The follwing theorem shows that there are no more alternatives.

Theorem 1 *The only way to realize the generators p^i satisfying the algebra (4.83)-(4.85) as elements of $Diff(\mathbb{R}_q^N)$ is by a combination*

$$p^i \equiv f(\Lambda)(\alpha\partial^i + \beta\bar{\partial}^i), \qquad \alpha, \beta \in \mathbb{C}, \quad f(t) \in \mathbb{C}[t] \tag{4.86}$$

Proof. The commutation relations (4.83),(4.83) impose for p^i the general form $p^i = S_1\partial^i + S_2x^i$, where $S_1, S_2 \in Diff(\mathbb{R}_q^N)$ are scalars. The commutation relation (4.85) implies that the natural dimensions d of S_1, S_2 are given by $d(S_1) = 0$, $d(S_2) = 2$. Hence, using the results mentioned in Proposition 2 in section 3.3 and the remark following it, we are led to an improved ansatz $p^i = S_1'(B,\Lambda)\partial^i + S_2'(B,\Lambda)x^i\partial\cdot\partial$ (recall that $B \in U_q^N$). Now we replace this general form for p^i into the commutation relations (4.82). We move $\partial\cdot\partial$ to the right of $S_1', S_2', x^i, \partial^i$ by using relations (3.57). The three coefficients of the powers 0,1,2 of $\partial\cdot\partial$ must vanish independently:

$$\begin{cases} \mathcal{P}_A{}^{ij}_{hk} S_1'(B,\Lambda)\partial^h S_1'(B,\Lambda)\partial^k = 0 \\ \mathcal{P}_A{}^{ij}_{hk}[S_1'(B,\Lambda)\partial^h S_2'(B,\Lambda)x^k + S_2'(B,\Lambda)x^h S_1(B,\Lambda q^2)\partial^k + q^{2-N}S_2x^h S_2(B,\Lambda q^2)\partial^k] = 0 \\ \mathcal{P}_A{}^{ij}_{hk} S_2'(B,\Lambda)x^k S_2'(B,\Lambda q^2)x^k = 0. \end{cases} \tag{4.87}$$

We expand S_i' in a power series $S_i' = \sum_{n=0}^{\infty} B^n S_{i,n}'(\Lambda)$ and use the commutation relations (4.11) to move the ∂ (resp. the x) to the right of all the B's in the first (resp. third) equation; we get an expansion in powers of the independent generators B, l^{ij} of U_q^N. Setting all their coefficients equal to zero implies $S_{i,n}' \equiv 0$ $n \geq 1$ (only the coefficient of the constant term vanishes automatically because of the relation (2.55) (resp. (3.30)). In other words $S_i' = S_i'(\Lambda)$ only. This allows to rewrite the second equation as

$$[-q^2 S_1' S_2'(\Lambda q) + S_2' S_1'(\Lambda q) + S_2' S_2'(\Lambda q) q^{2-N}] l^{ij} \partial \cdot \partial = 0; \tag{4.88}$$

This implies that the term in square brackets must vanish. A nontrivial solution of the equations (4.88) is therefore $S_2' = 0$, $S_1' = S_1'(\Lambda)$, yielding the solution with $\beta = 0$ given in the claim. If $S_2' \neq 0$ we can set $s(\Lambda) := \frac{S_1'(\Lambda)}{S_2'(\Lambda)}$, and the preceding equation is equivalent to the new one

$$q^{2-N} - q^2 s(\Lambda) + s(q\Lambda) = 0. \tag{4.89}$$

Expanding $s(\Lambda)$ in power series one immeditely finds that the general solution of the latter equation is $s = \frac{q^{2-N}}{q^2-1} + a\Lambda^2$, $a \in \mathbb{C}$, which yields the remaining solutions of the claim (after use of eq. (3.58)). $\diamondsuit$

To each of the above realizations of p^i there corresponds a coalgebra structure; below we find the latter in two particular cases.

So far we have not considered the $*$ structure. Of course, we are finally interested in a $*$-Hopf algebra. At the algebra level (i.e. within $u_q(e^N)$) we can impose the equations

$$(p^i)^* = p^j C_{ji} \tag{4.90}$$

and the $*$-relations (3.62)(4.30) and get in this way a $*$-algebra (in fact, these are the $*$ relations which are satisfied by the subalgebra of $Diff(\mathbb{R}_q^N)$ generated by $U_q^N, \Lambda^{\pm 1}, x^i!$). However, at the Hopf level one should check the compatibility of the $*$ structure with the Hopf axioms. To this aim we ask whether there exists a differential realization (4.86) compatible with relation $*$ (all other compatibilities are already satisfied). Using relations (3.2) one immediately shows the following

Corollary 1 *The only way to realize the generators p^i satisfying the algebra (4.82) (4.85) and the $*$ relations (4,90) as elements of $Diff(\mathbb{R}_q^N)$ is (up to a global factor) through*

$$p^i \equiv -i(\partial^i q^N + \bar{\partial}^i). \tag{4.91}$$

One could easily prove that there is a one-to-one correspondence between differential realizations and bialgebra structures. Therefore we could rephrase the two previous propositions by saying that the only Hopf-algebra structures compatible with the algebra described are those corresponding to the realizations (4.86), and the only one compatible also with the $*$ structure is the one corresponding to the realization (4.91).

Now we derive these coalgebra structures (and then complete as Hopf ones) in two particular cases. We follow the same approach used in section 4.2 with U_q^N, i.e. the counit derives from evaluation of differential operators on the unit $1 \in Fun(\mathbb{R}_q^N)$, the coproduct from their Leibnits rules, and the antipode from consistency of the coproduct with the axioms of a Hopf algebra.

The two cases are respectively those with $\alpha = 0$, $\beta = 0$ in formula (4.86). These will turn out to be much simpler than the one corresponding to (4.91). Concrete computations in the latter case are extremely hard; in fact, the derivation relations of $p^i = q^N \partial^i + \bar{\partial}^i$ with x have no simple expression such as (3.25) in terms of some $\hat{R}$ matrix. For this reason we have searched and found [10, 11, 16] a nonstandard formulation of a couple of quantum mechanical physical systems on $\mathbb{R}_q^N$ (the harmonic oscillator and the free particle), which we will present in Chapters 6.,7. This formulation is based on a non-standard way of realizing the momentum operators p^i in " configuration space representation " as the *pair* of q-derivatives $\partial, \bar{\partial}$ (see for instance section 7.2). On an abstract level, this formulation suggests a non-standard way to construct tensor product representations (see section 4.6).

As already noticed, we only need to consider one momentum component, since the other ones can be iteratively obtained from it through commutators (4.84) with the Chevalley generators of U_q^N, and the coproduct, counit, antipode (ϕ, ϵ, σ will be constructed as (anti)homomorphisms.

In the first case $p^i \equiv \partial^i$ it is convenient to consider the component $p_n \equiv \partial_n$. From

$$[\Lambda, \vec{x}]_q = 0 \tag{4.92}$$

and

$$\partial_n x^n = 1 + q^2 x^n \partial_n, \qquad \partial_n x^{-n} = x^{-n} \partial_n, \qquad \partial_n x^l = q x^l \partial_n, \quad |l| < n \tag{4.93}$$

we infer

$$\phi(\Lambda) := \Lambda \otimes \Lambda \qquad \epsilon(\Lambda) = 1 \qquad \sigma(\Lambda) = \Lambda^{-1} \tag{4.94}$$

and

$$\phi(\partial_n) := \partial_n \otimes 1 + \Lambda(\mathrm{k}^n)^{\frac{1}{2}} \otimes \partial_n \qquad \epsilon(\partial_n) = 0 \qquad \sigma(\partial_n) = -\Lambda^{-1}(\mathrm{k}^n)^{-\frac{1}{2}}\partial_n. \quad (4.95)$$

As already anticipated, we extend ϕ, ϵ as algebra homomorphisms, σ as an algebra anti-homomorphism, to the rest of $u_q(e^N)$. Then $u_q(e^N)$ is turned into a Hopf algebra, since the basic axioms are satisfied on the generators. In particular we readily find when $i \geq h$, for instance,

$$\phi(\partial_i) = \partial_i \otimes 1' + \Lambda(\mathrm{k}^i)^{\frac{1}{2}} \otimes \partial_i + (1-q^2)\sum_{j>i} q^{-\rho_i}\Lambda \mathrm{L}^{-i,j}(\mathrm{k}^j)^{\frac{1}{2}} \otimes \partial_j \qquad \epsilon(\partial_i) = 0 \quad (4.96)$$

$$\sigma(\partial_i) = -\Lambda^{-1}(\mathrm{k}^i)^{-\frac{1}{2}}\partial_i + (q^2-1)\sum_{j>i} q^{-\rho_i}\Lambda^{-1}(\mathrm{k}^j)^{-\frac{1}{2}}\sigma(\mathrm{L}^{-i,j})\partial_j. \qquad (4.97)$$

We call this realization the Hopf algebra $U_q(e^N)$.

In the second case $p^i \equiv \bar{\partial}^i$ it is convenient to consider the component $p_{-n} \equiv \partial_{-n}$. From

$$\bar{\partial}_{-n}x^{-n} = 1 + q^{-2}x^{-n}\bar{\partial}_{-n}, \qquad \bar{\partial}_{-n}x^n = x^n\bar{\partial}_{-n}, \qquad q\bar{\partial}_{-n}x^l = x^l\bar{\partial}_{-n}, \quad |l| < n \quad (4.98)$$

we find

$$\bar{\phi}(\bar{\partial}_{-n}) := \bar{\partial}_{-n} \otimes 1 + \Lambda^{-1}(\mathrm{k}^n)^{\frac{1}{2}} \otimes \bar{\partial}_{-n} \qquad \bar{\epsilon}(\bar{\partial}_n) = 0 = \bar{\epsilon}(\bar{\partial}_i) \qquad \bar{\sigma}(\bar{\partial}_{-n}) = -\Lambda(\mathrm{k}^n)^{-\frac{1}{2}}\bar{\partial}_{-n}. \quad (4.99)$$

We call this second realization the Hopf algebra $U_q(\bar{e}^N)$.

Whenever $q \in \mathbb{R}^+$ we can define a $*$ such that $U_q(e^N)$, $U_q(\bar{e}^N)$ are mapped into each other under complex conjugation $*$, more precisely $\partial \overset{*}{\leftrightarrow} \bar{\partial}$, $\Lambda \overset{*}{\leftrightarrow} \Lambda^{-1}$, $U_q^N \overset{*}{\leftrightarrow} U_q^N$, as given in formulae (3.39), (3.62),(4.30). One easily checks that $\phi, \bar{\phi}, \sigma, \bar{\sigma}$ satisfy the equations

$$(* \otimes *) \circ \phi = \bar{\phi} \circ *, \qquad \sigma \circ * \bar{\sigma} \circ * = id = \bar{\sigma} \circ * \sigma \circ *? \qquad (4.100)$$

If we define a larger algebra $U_{q,ext}(e^N)$ with generators $\mathrm{L}, \mathrm{k}, \partial, \bar{\partial}\Lambda^{\pm 1}$, $U_{q,ext}(e^N)$ will contain both $U_q(e^N)$ and $U_q(\bar{e}^N)$ as sub-Hopf algebras and will be a $*$-Hopf algebra, in the sense that the axioms

$$(* \otimes *) \circ \phi = \phi \circ * \qquad (\sigma \circ *)^2 = id. \qquad (4.101)$$

But to achieve this goal we pay the price of doubling the number of p^i.

As already noticed, however, if we forget the Hopf structures, i.e. we just consider the algebra $u_q(e^N)$, then we can impose the equations $(p^i)^* = p^j C_{ji}$ (4.90) instead of equations

(3.39), and $u_q(e^N)$ gets a $*$-algebra. Actually the $*$-algebra structure is all what we need in order to find the irreducible (one-particle) representations of $u_q(e^N)$ (see Chapter 7).

Summing up, our definition of the quantum Euclidean Hopf algebras (of u.e.a. type) $U_q(e^N)$, $U_q(\bar{e}^N)$ is

Definition: $U_q(e^N)$, $U_q(\bar{e}^N)$ are the (Poincaré) algebras generated by elements $\Lambda^{\pm 1}\mathbf{k}^i, \mathbf{L}^{-i,i+1}, \mathbf{L}^{-i-1,i}$ $(i = h, h+1, ..., n)$ and $p^{\pm n}$ (in the $N = 2n$ case one should also add $\mathbf{L}^{1,2}, \mathbf{L}^{-2,-1}$). The Hopf structure are given respectively by (4.64)-(4.71),(4.94),(4.95) or (4.64)-(4.71),(4.94),(4.99)

We can easily recognize that $U_q(e^N)$, $U_q(\bar{e}^N)$ can be considered as the duals of the two possible versions of the Euclidean Hopf algebra of functions-on-the-group type introduced in Ref. [43], $Fun(E_q^N), Fun(\bar{E}_q^N)$.

The statement that $U_q(e^N)$, $U_q(\bar{e}^N)$ are the duals of $Fun(E_q^N)$, $Fun(\bar{E}_q^N)$ respectively is clarified first by the specification of the pairing $< \cdot , \cdot >$ between the elements of a basis of $U_q(e^N)$ (resp. $U_q(\bar{e}^N)$) and the elements of a basis of $Fun(E_q^N)$ (resp. $Fun(\bar{E}_q^N)$). Unfortunately, since $Fun(E_q^N), Fun(\bar{E}_q^N)$ are not dual quasitriangular Hopf algebras (implying that the basic commutation relations between all its generators cannot be put in the form (2.1), with a suitable $\hat{R}$ matrix), the pairing cannot be introduced through an $\hat{R}$ matrix.

Let us start from the unbarred hopf algebras. As a first step, the pairing with the units is clearly given by $< \mathbf{1}', A >:= \varepsilon(A)$, $< a, \mathbf{1} >:= \epsilon(a)$, $a, \mathbf{1}' \in U_q(e^N)$, $A, \mathbf{1} \in Fun(E_q^N)$ and ε is the counit in $Fun(E_q^N)$. As pairing between the sub-Hopf-algebras U_q^N, $Fun(SO_q(N))$ we obviously take the canonical one of Ref. [9], which on the generators $L^{\pm}$ takes the form

$$< L^{\pm,i}_{\ \ h}, M^j_k >= R^{\pm\ ij}_{\ \ hk} \qquad R^+ = \hat{R}P, \qquad R^- = \hat{R}^{-1}P; \tag{4.102}$$

note that there you use the basis $L^{\pm,i}_j$ of the subalgebra $Fun(SO_q(N))^*_{reg} \subset U_q^N$ (with the notation of Ref [9]), not the Drinfeld-Jimbo one [7] (which is essentially that with generators $\mathbf{L}, \mathbf{k}$, apart unessential rescaling of the roots by elements of the Cartan subalgebra). The pairing between the ∂_i's (resp. $\bar{\partial}^i$'s) and the x^j (resp $\bar{x}^j$) is postulated in the standard way

$$< \partial_i, x^j >= \delta_i^j =< \bar{\partial}_i, \bar{x}^j >, \tag{4.103}$$

and as usual we postulate a trivial pairing between U_q^N and the subalgebras of momenta, $< P, U_q^N >= 0$. The pairing is extended from the generators to the rest of the algebras through one of the two (equivalent) postulates

$$< ab, A > = < a \otimes b, \phi_I(A) > \qquad < a, AB > = < \phi(a), A \otimes B > \tag{4.104}$$

$a, b \in U_q(e^N)$, $\quad A, B \in Fun(E_q^N)$ $\;(resp. \;\in\; Fun(\bar{E}_q^N))$ where

$$< a \otimes b, A \otimes B > := < a, A > < b, B > . \tag{4.105}$$

In particular we find that

$$< \partial_{i_1}\partial_{i_2}...\partial_{i_l}, x^{j_1}x^{j_2}...x^{j_m} > = \begin{cases} 0 & if \;\; m \neq l \\ ([1;\hat{R}^{\pm 1}][2;\hat{R}^{\pm 1}]...[m;\hat{R}^{\pm 1}])^{j_1 j_2 ... j_m}_{i_m i_{m-1}... i_1} & if \;\; m = l, \end{cases} \tag{4.106}$$

where (with a notation suggested by, but slightly different from that of Ref. [28])

$$[h, M] := 1 + M_{h-1,h} + M_{h,h+1}M_{h-1,h}... + M_{m-1,m}M_{m-2,m-1}...M_{h-1,h}. \tag{4.107}$$

$\hat{R}^{\pm 1}$ correspond to $Fun(E_q^N), Fun(\bar{E}_q^N)$-duality respectively. Note that formulae (4.106) alone can be obtained also by using the fact that the braid group $\mathbb{R}^N_{q,\vec{\partial}}$ (resp. $\mathbb{R}^N_{q,\bar{\vec{\partial}}}$) of momenta and the braid group $\mathbb{R}^N_{q,\vec{x}}$ of finite translations x are " braided " paired, in the sense that their pairing is generated from the basic formulae (4.103),(4.104), provided ϕ is replaced by the coaddition $\underline{\Delta}$ (3.49) and tensor products are understood as braided ones (see Section 2.3).

From the viewpoint of duality the difficulty one meets in finding $*$-structures compatible the Hopf ones in $Fun(E_q^N), U_q(e^N), \mathbb{R}^N_{q,\vec{\partial}}, \mathbb{R}^N_{q,\vec{x}}$ (resp. $Fun(\bar{E}_q^N), U_q(\bar{e}^N), \mathbb{R}^N_{q,\bar{\vec{\partial}}}, \mathbb{R}^N_{q,\bar{\vec{x}}}$) are all related and are a general feature of inhomogeneous quantum groups of BWM type [1].

In the usual approach to representation theory, these difficulties would open many serious problems to physical applications of inhomogeneous quantum groups to quantum mechanical systems. Indeed, since one is essentially interested in Hilbert space representations, at the representation-theoretic level we need a notion of hermitean conjugation of operators, (equivalently, a notion of complex conjugation at the abstract level) which is compatible with the coalgebra structure; only in this case we could get representations out of tensor products of irreps, what is essential to allow a description of complex (" many-particle ") physical systems in terms of simpler (" one-particle ") ones.

We think that maybe a new approach and some new notion is needed in order to face in full generality the difficulties just mentioned, and we hope to address this problem on a more abstract level elsewhere. The idea is somehow to combine the two conjugated Hopf algebras $U_q(e^N)$ $U_q(\bar{e}^N)$ at the representation level. Here we would like to work in a rather concrete manner by sticking to the quantum Euclidean case and exploiting the knowledge we have about the differential algebra on $\mathbb{R}_q^N$. Developing an idea which was first introduced in [11], we would like to show how the structure of $Diff(\mathbb{R}_q^N)$ naturally leads to a combined use of each pair of objects (the unbarred and the barred one) in trying to overcome the mentioned difficulties. This will be sketched in section 4.7. In the following two sections we work only at the algebra level, the only one which is involved when studying Irreps.

4.5 New L generators of the Euclidean algebra $u_q(e^N)$

In Chapter 7 we will construct the Hilbert space irreducible representations of the Euclidean algebra. Since the element $(p\cdot p)_n$ of the algebra commutes with all the generators of the algebra but Λ, with which it q-commutes, on a given Irrep either is identically zero (so that the Irrep is actually an Irrep of U_q^N) or it never vanishes. In the former case it follows that $p^{\pm n}$ cannot vanish as well; the same holds with all other components p^i, which can be obtained from p^n by iterated q-commutation with the L's. Therefore all $(p\cdot p)_i$ never vanish, and we can define their inverse.

Let us consider the relations

$$[\mathbf{L}^{-m,m+1},(p\cdot p)_l]=0 \qquad if \quad l\neq m; \tag{4.108}$$

$$[\mathbf{L}^{-m,m+1},(p{\cdot}p)_m]=-q^{2\rho_{m+1}}p^{-m}p^{m+1}, \qquad (\Rightarrow \quad [\mathbf{L}^{-m,m+1},\frac{1}{(p\cdot p)_m}]=\frac{q^{2+2\rho_{m+1}}}{[(p\cdot p)_m]^2}p^{-m}p^{m+1}), \tag{4.109}$$

which can be easily drawn from equations (4.83),(4.84). As a consequence of the second formula, $\mathbf{L}$ would not map eigenvectors of $(p\cdot p)_m$ into each-other. However, one can define improved generators L which actually do. From (4.84) we find

$$\begin{cases} [\mathbf{L}^{-m,m+1},\frac{1}{(p\cdot p)_m}]=\frac{q^{2\rho_{m+1}+2}}{[(p\cdot p)_m]^2}p^{-m}p^{m+1}, \\ [\mathbf{L}^{1,2},\frac{1}{(p\cdot p)_1}]=\frac{1}{[(p\cdot p)_1]^2}p^1p^2, \qquad if \quad N=2n; \end{cases} \tag{4.110}$$

therefore we define

$$\begin{cases} L^{-m,m+1}:=\mathbf{L}^{-m,m+1}+\frac{q^{2\rho_{m+1}+2}}{(1-q^2)(p\cdot p)_m}p^{-m}p^{m+1} \\ L^{-m-1,m}:=\mathbf{L}^{-m-1,m}+\frac{q^{2\rho_{m+1}+1}}{(1-q^2)(p\cdot p)_m}p^{-m-1}p^{m}. \end{cases} \tag{4.111}$$

(similarly for L^{12}). Note that this redefinition is possible only when $q\neq 1$. The basic property of the new generators is the fact that if $i > 0$

$$[L^{-m,m+1}, p^i]_a = 0 \qquad [L^{-m-1,m}, p^i]_a = 0 \qquad a = \begin{cases} q & if \quad i = -m-1, m \\ q^{-1} & if \quad i = -m, m+1 \\ 1 & otherwise, \end{cases} \tag{4.112}$$

if $N = 2n+1$ and $i = 0$

$$[L^{01}, p^0] = 0 = [L^{-10}, p^0], \tag{4.113}$$

implying

$$[L^{-m,m+1}, (p \cdot p)_i] = 0 = [L^{-m-1,m}, (p \cdot p)_i] \qquad \forall i, m; \tag{4.114}$$

moreover, it is easy to see that the L's satisfy the same *-conjugation relations as the $\mathbf{L}$'s.

Let us find out now the commutation relations satisfied by the L's. We can define other roots L starting from simple ones, just in the same way as we did with the $\mathbf{L}$'s, using relations (4.25)-(4.27) (with the replacement $\mathbf{L} \to L$). L roots can be divided into positive and negative ones according to the same convention used for the $\mathbf{L}$'s.

Proposition 19 *Let $k \geq h+1$. The commutation relations between positive and negative simple roots are*

$$[L^{1-m,m}, L^{-k,k-1}]_a = 0 \qquad a = \begin{cases} q^{-1} & m-1 = k \\ 1 & if \quad m-1 > k \end{cases} \tag{4.115}$$

$$[L^{-m,m-1}, L^{1-k,k}]_a = 0 \qquad a = \begin{cases} q & if \quad m-1 = k \\ 1 & if \quad m-1 > k \end{cases} \tag{4.116}$$

$$[L^{1,2}, L^{-2,1}] = \frac{(p \cdot p)_2 p^1 p^1}{(1-q^2)[(p \cdot p)_1]^2}, \qquad [L^{-1,2}, L^{-2,-1}] = \frac{(p \cdot p)_2 p^{-1} p^{-1}}{(1-q^2)[(p \cdot p)_1]^2}, \qquad if \quad N = 2n, \tag{4.117}$$

$$\begin{cases} [L^{1-m,m}, L^{-m,m-1}]_{q^2} = q^{1+2\rho_m} \frac{1-\mathbf{k}^{m-1}(\mathbf{k}^m)^{-1}}{q-q^{-1}} + C_m & 2 \leq m \leq n \\ [L^{01}, L^{-1,0}]_q = q^{-\frac{1}{2}} \frac{1-(\mathbf{k}^1)^{-1}}{q-q^{-1}} + C_1 & if \quad N = 2n+1 \end{cases} \tag{4.118}$$

where

$$C_1 := \frac{q^{\frac{1}{2}}}{1-q^2}\left[1 + q\frac{(p \cdot p)_1}{(p \cdot p)_0}\right] \qquad if \quad N = 2n+1 \tag{4.119}$$

$$C_{m+1} := \frac{q^{2\rho_m}}{1-q^2}\left[1 - \frac{(p \cdot p)_{m-1}(p \cdot p)_{m+1}}{[(p \cdot p)_m]^2}\right], \qquad m \geq 1. \tag{4.120}$$

Proof. Use equations (4.32)-(4.35),(4.110),(4.114) perform explicit computations. ◇

The list of commutation relations involving the L's is completed by the following proposition:

Proposition 20 *The* $[\mathbf{k}, L]$ *relations and the Serre relations for the* L *generators are the same as those of the* $\mathbf{L}$ *generators*

Summing up, the commutations relations of the L's are the same as those of the $\mathbf{L}$'s, if we add some " central charges ". **Remark** Note that the embedding $\hat{u}_q(e^N) \hookrightarrow \hat{u}_q(e^{N+2})$ mentioned after formula (4.85) is obtained now by setting equal to zero all the generators of $p^i, L^{ij}, \mathbf{k}^i$ of $\hat{u}_q(e^{N+2})$ having $i = \pm(n+1)$ for some space index i.

Definition We denote by $u_q^{\pm,N}$ the subalgebra of $u_q(e^N)$ generated by the positive roots L's (resp. negative roots L's and $p^{\pm 1}$).

4.6 Casimirs of $\hat{u}_q(e^N)$

As in the classical case, $\hat{u}_q(e^N)$ has $n+1-h$ casimirs; their definition mimics the classical one. They can be built in the most compact way as follows. Define the " Pauli-Lubanski " $(2l+1)$ - form

$$\omega_{2l+1} := (\omega)^l p \qquad \omega := l^{ij}\xi_j\xi_i, \qquad p := p^i\xi_i \qquad l = 0, 1, ..., n-1. \tag{4.121}$$

Theorem 2 *The* $n+1-h$ *casimirs* Ω^l *of* $\hat{u}_q(e^N)$ *are defined by*

$$\Omega^l \; dV := \omega_{2l+1} \wedge^\star \omega_{2l+1}, \tag{4.122}$$

and

$$\Omega^n \; dV := o_{2n+1} \qquad\qquad if \quad N = 2n+1, \tag{4.123}$$

where dV *denotes the volume* $N-form$*, and* $\star$ *denotes the Hodge duality operation introduced in section 3.4*

Proof. The general proof will be given in a separate paper [16]; it involves the use of the general $[l^{ij}, p^h]$ commutation relations and of specific properties of the q-deformed completely antisymmetric projector with $m \leq N$ indices. The theorem will be verified in the cases $N = 3, 4$ using the L generators in section 3.4. ◇

The irreps of $\hat{u}_q(e^N)$ are characterized by the values of the casimirs and as we will see, by the sign of p_0 in the case $N = 2n+1$.

In particular, when $l = 0$ we find the square momentum casimir

$$\Omega^0 \equiv (p \cdot p)_n; \tag{4.124}$$

when $N=4$, Ω^1 is the q-deformed analogue of the Wick-rotated casimir which is constructed from the Pauli-Lubanski vector and gives the intrinsic spin of each irrep. Of course we expect that the casimirs Ω^l, $l\geq 1$, are zero in the singlet representation, i.e. that characterized by the trivial weight, since then both p^l and l^{ij} can be expressed as differential operators on $\mathbb{R}_q^N$, namely $l^{ij}=\mathcal{P}_{A\ hk}^{\ ij}x^h\partial^k\Lambda^{-1}$, $p^l=\partial^l\Lambda^a$, and $\mathcal{P}_{A\ hk}^{ij}\partial^h\partial^k=0$.

4.6.1 The casimirs of $\hat{u}_q(e^N)$ in the cases $N=3,4$ in terms of the L,k,p generators

Proposition 21 *When $N=3,4$, the Casimirs Ω_1 (4.123), (4.122) in terms of p,L,k generators take respectively the form*

$$\Omega_1=p^0(\mathrm{k}^1)^{-\frac{1}{2}}-q(q+1)\frac{(p\cdot p)_1}{p^0}(\mathrm{k}^1)^{\frac{1}{2}}+q^{\frac{1}{2}}(1-q)(1-q^2)L^{-1,0}L^{0,1}(\mathrm{k}^1)^{\frac{1}{2}} \tag{4.125}$$

and

$$\Omega_1=(L^{-2,1}L^{-1,2})(L^{-2,-1}L^{1,2})\mathrm{k}^2+\frac{q^{-2}}{(q^2-1)^2}(p\cdot p)_1\{\mathrm{k}^1(L^{-2-1}L^{12})+(\mathrm{k}^1)^{-1}(L^{-21}L^{-12})\}$$
$$-\frac{q^{-4}(p\cdot p)_1}{(q^2-1)^2}\left[L^{-2,-1}L^{1,2}\mathrm{k}^1+L^{-2,1}L^{-1,2}(\mathrm{k}^1)^{-1}\right]+\frac{1}{(q^2-1)^4}\left[q^{-4}(p\cdot p)_1(\mathrm{k}^2)^{-1}-\mathrm{k}^2\frac{[(p\cdot p)_2]^2}{(p\cdot p)_1}\right].$$
$$-\frac{q^{-2}(p\cdot p)_2}{(1-q^2)^2(p\cdot p)_1}[p^{-1}p^{-1}L^{-21}L^{12}+p^1p^1L^{-2-1}L^{-12}]\mathrm{k}^2 \tag{4.126}$$

Proof. We prove that Ω_1 as defined by equation (4.122), (4.123) take the forms (4.125),(4.126); then it is straightforward to verify that they are casimirs of $u_q(e^N)$ using relations (4.112),(4.120).

Case $N=3$. Using the definition of $\mathcal{P}_A$ one verifies that there are only three independent l^{ij}, $l^{-1,0},l^{01},l^{1,-1}$, say. After formula (4.19) we mentioned that $l^{0,1}=(\mathrm{k}^1)^{\frac{1}{2}}\mathrm{L}^{0,1}$, $l^{-1,0}=(\mathrm{k}^1)^{\frac{1}{2}}\mathrm{L}^{-1,0}$; in appendix 4.8 we show thT $l^{1,-1}\frac{(\mathrm{k}^1)^{\frac{1}{2}}-B}{q-1}$ (B was defined in equation (); we are using the normalization of appendix 4.8). B can be easily expressed as a linear funtion of the quadratic casimir $C_{U_q^3}$ of U_q^3, which in terms of L,k generators reads

$$C_{U_q^3}=q^{\frac{3}{2}}\mathrm{L}^{-1,0}\mathrm{L}^{01}(\mathrm{k}^1)^{\frac{1}{2}}+\frac{[(\mathrm{k}^1q)^{\frac{1}{4}}-(\mathrm{k}^1q)^{-\frac{1}{4}}]^2}{(q-q^{-1})(q^{\frac{1}{2}}-q^{-\frac{1}{2}})}; \tag{4.127}$$

one finds

$$B_1(1+q)=(\mathrm{k}^1)^{\frac{1}{2}}+\mathrm{k}^{-\frac{1}{2}}+q^2(q-q^{-1})(q^{\frac{1}{2}}-q^{-\frac{1}{2}})\mathrm{L}^{-1,0}\mathrm{L}^{01}(\mathrm{k}^1)^{\frac{1}{2}}. \tag{4.128}$$

This allows an expression for Ω_1 involving only L,k generators. Finally, one replaces L's as functions of $\mathrm{k}^1,L^{0,1},L^{-1,0}$ and finds the expression (4.125).

The case $N = 4$ is proved in a similar way. Only, one has to remember that $U_q^4 \approx U_{q^{-1}}(su(2)) \otimes U_{q^{-1}}(su(2))$ has two independent quadratic casimirs, each corresponding to one of the two $U_{q^{-1}}(su(2))$ factors; the symmetric combination $\mathcal{C}_{U_q^4}$ of the two is

$$\mathcal{C}_{U_q^4} = q^3(\mathbf{L}^{-2,1}\mathbf{L}^{-1,2} + \mathbf{L}^{-2,-1}\mathbf{L}^{1,2}) + \left[\frac{(\mathbf{k}^2(\mathbf{k}^1)^{-1}q^2)^{\frac{1}{4}} - (\mathbf{k}^2(\mathbf{k}^1)^{-1}q^2)^{-\frac{1}{4}}}{q-q^{-1}}\right]^2 + \left[\frac{(\mathbf{k}^2\mathbf{k}^1q^2)^{\frac{1}{4}} - (\mathbf{k}^2\mathbf{k}^1q^2)^{-\frac{1}{4}}}{q-q^{-1}}\right]^2. \tag{4.129}$$

B can be expressed as a linear function of this casimir, as follows

$$(q+q^{-1})B = 2 + (q-q^{-1})^2\mathcal{C}_{U_q^4}. \tag{4.130}$$

Using the formulae $l^{\pm1,2} = (\mathbf{k}^2)^{\frac{1}{2}}(\mathbf{k}^1)^{\pm\frac{1}{2}}$, $l^{-2,\mp1} = q(\mathbf{k}^2)^{\frac{1}{2}}(\mathbf{k}^1)^{\mp\frac{1}{2}}$ etc. (see formula (4.19) and appendix 4.8) and formula (3.109), one can express Ω_1 as a function of $\mathbf{k}^i, \mathbf{L}$ and finds formula (4.126). $\diamondsuit$

4.7 $*$-Hopf algebra $\tilde{U}_q(e^N)$ as the pair $\{U_q(e^N), U_q(\bar{e}^N)\}$

In section 4.4 we have anticipated that the technical difficulties arising from the search and the manipulation of a (proper) $*$-Hopf structure for $u_q(e^N)$ can be bypassed by a combined use of the two $*$-conjugated Hopf algebras $U_q(e^N), U_q(\bar{e}^N)$ for representation theory purposes.

Assume that $\mathcal{S}_1$,$\mathcal{S}_2$ are two physical quantum systems with Hilbert spaces of states $\mathcal{H}_1$,$\mathcal{H}_2$, and that the latter are the base spaces of two $*$ irreducible representations Γ^1, Γ^2 of the algebra $u_q(e^N)$. We want to build a $*$-representation of $u_q(e^N)$ as a tensor product of the two. We have the conjugated coproduct, antipode, counit ϕ, σ, ϵ and $\bar{\phi}, \bar{\sigma}, \bar{\epsilon}$ at our disposal. Hinted by the " configuration space realization " of the algebra $u_q(e^N)$, which we will present in section 7.2, we introduce abstract coproduct, antipode, counit $\Phi, \sum, E$ and a pair of morphisms $\rho, \bar{\rho}$ such that (by definition)

$$\begin{array}{ccc} & & \phi, \sigma, \epsilon \\ & \overset{\rho}{\nearrow} & \\ \Phi, \sum, E & & \\ & \overset{\bar{\rho}}{\searrow} & \\ & & \bar{\phi}, \bar{\sigma}, \bar{\epsilon} \end{array} \tag{4.131}$$

We will say that $\Phi, \sum, E, m$ (m is the algebra product) satisfy the axioms of a Hopf-algebra, meaning that their $\rho, \bar{\rho}$ images do.

We define the tensor product representation Γ by

$$\Gamma := (\Gamma^1 \otimes \Gamma^2) \circ (\rho + \bar{\rho}) \circ \Phi = (\Gamma^1 \otimes \Gamma^2) \circ (\phi + \bar{\phi}) \tag{4.132}$$

Given vectors $|\varphi_1>, |\psi_1> \in \mathcal{H}_1$ $|\varphi_2>, |\psi_2> \in \mathcal{H}_2$, and defined in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ the scalar product as usual,

$$(\psi, \varphi) := (\psi_1, \varphi_1)_1 (\psi_2, \varphi_2)_2, \qquad if \qquad \begin{cases} |\varphi> := |\varphi_1> \otimes |\varphi_2> \\ |\psi> := |\psi_1> \otimes |\psi_2>, \end{cases} \tag{4.133}$$

then it is easy to check that Γ is a $*$-representation of $u_q(e^N)$, using property (4.100) of $\phi, \bar{\phi}, *$:

$$(\psi, \Gamma(u)\varphi) := (\psi, [\Gamma \circ (\phi + \bar{\phi})](u)\varphi) = (\{[\Gamma \circ (\phi + \bar{\phi})](u)\}^{*\otimes *}\psi, \varphi)$$
$$([\Gamma \circ (\phi + \bar{\phi})](u^*)\psi, \varphi) = (\Gamma(u^*)\psi, \varphi) \tag{4.134}$$

More generally, given Hilbert spaces representations Γ^i, $i = 1, 2, ..., I$, we find that

$$\Gamma := (\bigotimes_{i=1}^{I})\Gamma^i) \circ (\rho + \bar{\rho}) \circ [\underbrace{id \otimes ... \otimes id}_{(i-2)\ times} \otimes \Phi] \circ [\underbrace{id \otimes ... \otimes id}_{(i-3)\ times} \otimes \Phi] \circ ... \circ \Phi \tag{4.135}$$

is a Hilbert space representation itself. Since this holds for any set $\{\Gamma^i\}$, the result is that on an abstract level we can proceed as if

$$(* \otimes *) \circ \Phi = \Phi \circ *. \tag{4.136}$$

We can summarize the situation as follows. When $q \in \mathbb{R}$ there exists two $*$-conjugate Hopf algebras $U_q(e^N)$, $U_q(\bar{e}^N)$. but we can think of them as a unique $*$-Hopf algebra (with coproduct Φ, antipode $\sum$, counit E, product m, and complex conjugation $*$), $U_q(e^N)$, provided we always apply $\rho + \bar{\rho}$ to its objects before enforcing the formal machinery that we would normally use to compute their tensor product representations (applying first Φ etc, then the Γ's).

We call $\tilde{U}_q(e^N)$ the algebra $u_q(e^N)$ equipped with the abstract $\Phi, \sum, E, m, *$, and the homomorphisms $\rho, \bar{r}$. Because of the formal relations (4.36), we can consider $\tilde{U}_q(e^N)$ as an example of some new non-standard notion of $*$-Hopf algebra.

4.8 Appendix

In this appendix we show how to express the generators $k_i, \mathbf{L}^{ij}$ as functions of l^{ij}, B_n. One can easily check that this map is invertible.

Instead of the N linearly dependent operators $\mathcal{L}^{-i,i}$ one can use their n linearly independent combinations

$$\mathcal{L}^i := \mathcal{A}^i(x,\partial), \qquad i = 1,2,...,n. \tag{4.137}$$

As for the operators $\mathcal{L}^{ij}$, $i \neq -j$, for simplicity we will renormalize them as follows

$$\mathcal{L}^{ij} := (1+q^2)\mathcal{P}_A{}^{ij}{}_{hk}x^i\partial^j = (x^i\partial^j - qx^j\partial^i), \qquad i<j, \qquad \mathcal{L}^{ij} = -q\mathcal{L}^{ji}, \qquad i>j. \tag{4.138}$$

We first introduce some useful combinations F of the l^i, B variables introduced in subsection 3.1.

Let us iteratively define objects $\mathcal{F}_n^l \in Diff(\mathbb{R}_q^N)$, ($N = \begin{cases} 2n+1 & for\ odd\ N \\ 2n & for\ even\ N \end{cases}$ as usual) by

$$\mathcal{F}_l^{l+1} := \mathcal{B}_l, \qquad l \geq 0 \quad \forall N \geq 2; \qquad \mathcal{F}_1^{-1} = \mu_{-1} \quad if \ \ N=2 \tag{4.139}$$

($\mathcal{B}_0 = 1$ when $N=2$),

$$\mathcal{F}_n^{l+1}(x,\partial) := \mu_n\mathcal{F}_{n-1}^{l+1}(X,D), \quad n > l \geq 0; \qquad \mathcal{F}_n^{-1}(x,\partial) := \mu_n\mathcal{F}_{n-1}^{-1}(X,D) \qquad if\ N=2n. \tag{4.140}$$

Let $F_n^l := \mathcal{F}_n^l\Lambda^{-1}$. One easily checks that $F_n^l \in U_q^N$, more precisely

$$F_n^{l+1} = B_n + \frac{q^2-1}{1+q^{-2\rho_n}}[\sum_{j=l+1}^{n} l^j q^{-\rho_j} - (q^2-1)\frac{(n-l)_{q^2}}{1+q^{2\rho_l}}\sum_{j=1}^{l} l^j q^{-\rho_j}] \tag{4.141}$$

$$F_n^{-1} = B_n + \frac{q^2-1}{1+q^{-2\rho_n}}\sum_{j=1}^{n} l^j q^{-\rho_j} + (1-q^2)l^1 = F_n^1 + (1-q^2)l^1 = F_n^2 - \frac{q^2-1}{2}l^1. \tag{4.142}$$

Next we define

$$K_n^{i+1} := (\Lambda_n)^{-2}\Lambda_i(\mu_{i+1})^2...(\mu_n)^2 \in Diff(\mathbb{R}_q^N), \qquad 0 \leq i \leq n-1 \tag{4.143}$$

and observe that

Proposition 22 *K_n^i's belong to U_q^N and*

$$K_n^{i+1} = (F_n^{i+1})^2 - \frac{q^2-1}{q^{2\rho_i}+1}\frac{q^2-q^{-2}}{1+q^{-2\rho_i-2}}(l\cdot l)_i; \tag{4.144}$$

Proof. From the definitions of Λ_l, μ_l it immediately follows the first part of the proposition. Relation (4.144) is a consequence of formula (4.9) and of the definitions of the F's.◇

Formulae (4.141),(4.142),(4.144) allow to express $\mathbf{k}_i$ as functions of l^{ij}, B after noting that

$$\mathbf{k}_i = K_n^i (K_n^{i+1})^{-1} \qquad (K_n^{n+1} \equiv 1) \tag{4.145}$$

As for the L's, we find the

Proposition 23

$$\begin{cases} \mathbf{L}^{1-k,k} = (K_n^k)^{-1} \left[F_n^{k-1} l^{1-k,k} - \frac{q^2-1}{q^2+q^{-2-2\rho_k}} \sum_{l=2-k}^{l=k-2} l^{1-k,l} l_l{}^k \right] \\ \mathbf{L}^{-k,k-1} = q^{-1}(K_n^k)^{-1} \left[l^{-k,k-1} F_n^{k-1} - \frac{q^2-1}{q^2+q^{-2-2\rho_k}} \sum_{l=2-k}^{l=k-2} l^{-k,l} l_l{}^{k-1} \right] \end{cases} \quad 2 \le k \le n \tag{4.146}$$

and

$$\begin{cases} \mathbf{L}^{0\pm1} = (F_n^1)^{-1} l^{0\pm1} & if \quad N = 2n+1 \\ \mathbf{L}^{\pm(1,2)} = (F_n^1)^{-1} l^{\pm(1,2)} & if \quad N = 2n \end{cases}. \tag{4.147}$$

Proof. As an example we prove equation $(4.146)_1$. As usual, it is sufficient to prove the claim when $k = n$, and then use Proposition 1 to extend it to $n > k$. Inverting relation $(3.74)_4$ we get $D^n = q\Lambda^{-1}\mu_n^{\frac{1}{2}}[\partial^n + q^{-2-2\rho_n}(q^2-1)X^n(D\cdot D)_{n-1}]$. Replacing this expression in the definition (4.17) of $\mathbf{L}^{1-n,n}$ and using the definition (3.56) for Λ_{n-1} we easily find

$$\begin{aligned} \Lambda_n^2 \mathbf{L}^{1-n,n} &= \mu_{-n}\mu_n^{\frac{1}{2}} \left\{ [D^{1-n}, (X\cdot X)_{n-1}]\partial^n - [D^{1-n}, (1+q^{-2-2\rho_n})(X\cdot D)_{n-1}]X^n \right\} \\ &= \mu_{-n}\mu_n^{\frac{1}{2}} \{ \mu_{n-1}(q^{2\rho_n+2}X^{1-n}\partial^n - X^n D^{1-n}) + (1-q^{-2})[(X\cdot X)_{n-2}D^{1-n}\partial^n \\ &+ X^{1-n}(D\cdot D)_{n-2}X^n] - (q^2-1)(1+q^{-2\rho_n-4})(X\cdot D)_{n-2}]X^n D^{1-n} \}; \end{aligned} \tag{4.148}$$

on the other hand, using the normalization (4.138) for $\mathcal{L}^{ij}$,

$$\begin{aligned} \frac{\sum_{l=2-n}^{n-2} \mathcal{L}^{1-n,l}\mathcal{L}_l{}^n}{1+q^{-2\rho_n-4}} &= (\partial\cdot x)_{n-2}x^{1-n}\partial^n - (x\cdot x)_{n-2}\partial^{1-n}\partial^n - x^{1-n}(\partial\cdot\partial)_{n-2}x^n + q^2(x\cdot\partial)_{n-2}\partial^{1-n}x^n \\ &= \mu_n^{\frac{3}{2}} \{ (X\cdot D)_{n-2}(q^2 D^{1-n}X^n + q^{-2-2\rho_n}X^{1-n}\partial^n) + \frac{q^2 - q^{2\rho_n+4}}{q^2-1}\mu_{n-1}X^{1-n}\partial^n \\ &\qquad - (X\cdot X)_{n-2}D^{1-n}\partial^n - X^{1-n}(D\cdot D)_{n-2}X^n \} \end{aligned} \tag{4.149}$$

and

$$\mathcal{F}_n^{n-1}\mathcal{L}^{1-n,n} = \mu_n^{\frac{3}{2}} \left[1 + (q^2-1)(X^{n-1}D_{n-1} + q^{-2\rho_n-4}(X\cdot D)_{n-2}) \right] (X^{1-n}\partial^n - X^n D^{1-n}). \tag{4.150}$$

From the preceding three formulae we find that

$$\Lambda_n^2 \mathbf{L}^{1-n,n} = \mu_{-n}(\mu_n)^{-1}[\mathcal{F}_n^{n-1}\mathcal{L}^{1-n,n} - \frac{q^2-1}{q^2+q^{-2-2\rho_n}} \sum_{l=2-n}^{l=n-2} \mathcal{L}^{1-n,l}\mathcal{L}_l{}^{n}] \tag{4.151}$$

which is equivalent to the claim upon use of formula (4.145). $\diamondsuit$

Note that $K_n^1 = (F_n^1)^2$ both for odd and even N, and $(K_n^2)^2 = K_n^1(F^{-1})^2$ when $N = 2n$. All K_n^i go to 1 in the limit $q \to 1$. Moreover, for $N = 3$ $F_1^1 = (\mathrm{k}^1)^{\frac{1}{2}}$ and for $N = 4$ $F_2^1 = (\mathrm{k}^1\mathrm{k}^2)^{\frac{1}{2}}$ $F_2^{-1} = (\mathrm{k}^1)^{-\frac{1}{2}}(\mathrm{k}^2)^{\frac{1}{2}}$.

4.9 Appendix

Define

$$\begin{cases} \hat{\mathbf{L}}^{in} := X^i D^n - q^{-2-2\rho_n}\mu_n^{\frac{1}{2}}\Lambda_n^{-1}[X^i,(D\cdot D)_{n-1}]X^n \\ \hat{\mathbf{L}}^{-n,i} := X^{-n}D^i - q^{-3-2\rho_n}\Lambda_n^{-1}\mu_n^{-\frac{1}{2}}\mu_{-n}[D^i,(X\cdot X)_{n-1}]D^{-n} \end{cases} \qquad |i| < n. \tag{4.152}$$

Lemma 2 $\hat{\mathbf{L}}^{in}, \hat{\mathbf{L}}^{-n,i} \in U_q^N$ *and can be easily expressed as simple functions of the* $\mathbf{L}, \mathbf{k}$*'s.*

Since $[\mathrm{k}^n, \chi^j]_a = 0 = [\mathrm{k}^n, \mathcal{D}_j]_b$ with some a, b, we can introduce a grading $p \in \mathbb{C}$ in $Diff(\mathbb{R}_q^N)$ and decompose the latter as follows

$$Diff(\mathbb{R}_q^N) = \bigoplus_{p\in\mathbb{C}} Diff^p \qquad \textit{where} \qquad \mathrm{k}^n Diff^p := q^{2p} Diff^p \mathrm{k}^n; \tag{4.153}$$

note that for each monomial $M(\chi,\mathcal{D}) := (\chi^n)^l(\chi^{-n})^m(\mathcal{D}_n)^s(\mathcal{D}_{-n})^r$

$$p(M) = l + r - m - s. \tag{4.154}$$

Decomposition (4.153) induces the decomposition $U_q^N = \bigoplus_{p\in\mathbb{C}} U_q^N \bigcap Diff^p$.

Now we can sketch the proof of the main theorem of this appendix.

Proposition 24

$$u \in U_q^N \qquad \Rightarrow \qquad u = u(\mathrm{k}^i, \mathbf{L}^{jk}), \quad i = 1, ..., n. \qquad |j|,|k| \leq n. \tag{4.155}$$

Moreover

$$f \in Diff(\mathbb{R}_q^N): \quad \left[f, \begin{Bmatrix} x\cdot x \\ \partial\cdot\partial \end{Bmatrix}\right] = 0 \qquad \Rightarrow \qquad f = \sum_l u_l \begin{cases} f_l(x) \\ f_l(\partial) \end{cases}, \quad u_l \in U_q^N. \tag{4.156}$$

Sketch of the Proof. As a preliminary remark, let us recall that $[\Lambda_n, u] = 0$, namely u has natural dimension zero. Our proof will be by induction in n. It is easy to prove that $U_q^1 = \mathbf{1} \cdot \mathbb{C}$, and that U_q^2 is generated by k^1. Now assume that the thesis is true for U_q^{N-2}.

The most general $u \in Diff(\mathbb{R}_q^N)$ can be written in the form

$$u = \sum_{l,m=0}^{\infty} \{(\chi^{-n})^l(\mu_n^{-\frac{1}{2}}\chi^n)^m v_{l,m}(\mu_n, \mu_{-n}, \chi^j, \mathcal{D}_j) + (\mu_n^{-\frac{1}{2}}\mathcal{D}_n)^l(\mathcal{D}_{-n})^m v_{-l,-m}(\mu_n, \mu_{-n}, \chi^j, \mathcal{D}_j) +$$

$$(\mathcal{D}_{-n})^l(\mu_n^{-\frac{1}{2}}\chi^n)^m v_{-l,m}(\mu_n, \mu_{-n}, \chi^j, \mathcal{D}_j) + (\chi^{-n})^l(\mu_n^{-\frac{1}{2}}\mathcal{D}_n)^m v_{l,-m}(\mu_n, \mu_{-n}, \chi^j, \mathcal{D}_j) + \tag{4.157}$$

$|j| < n$. In fact the dependence on powers of $\chi^{\pm n}\mathcal{D}_{\pm n}$ can be reabsorbed into the dependence on $\mu_{\pm n}$.

It is easy to realize that, if we impose the constraint that the natural dimension $d(u)$ of u is zero, formula (4.157) can be rewritten in the form

$$u = \sum_{\{l_i, l_i'\}} (\mathbf{L}^{-n,1-n})^{l_{1-n}} \ldots (\mathbf{L}^{-n,n-1})^{l_{n-1}} (\mathbf{L}^{1-n,n})^{l'_{1-n}} \ldots (\mathbf{L}^{n-1,n})^{l'_{n-1}} \cdot$$

$$\sum_{p=0}^{\infty}\left[\sum_{h=0}^{p}[(\mu_n^{-\frac{1}{2}}\chi^n)^h(\mathcal{D}_{-n})^{p-h} v^{p,h}_{\{l_i,l_i'\}}(\mu_n, \mu_{-n}, \chi^j, \mathcal{D}_j) + (\mu_n^{-\frac{1}{2}}\mathcal{D}_n)^h(\chi^{-n})^{p-h} v^{-p,-h}_{\{l_i,l_i'\}}(\mu_n, \mu_{-n}, \chi^j, \mathcal{D}_j)]\right], \tag{4.158}$$

where $i = 1-n, ..., n-1$. We sketch the procedure which leads to this result. For each $\mu_n^{-\frac{1}{2}}\chi^n$ or χ^{-n} (respectively $\mathcal{D}^{-n}$ or $\mu_n^{-\frac{1}{2}}\mathcal{D}^n$) we can extract out of the corresponding coefficient function v a D^i (respectively a X^i) variable (since $d(u) = 0$) and replace the LHS's of the following identities by the RHS's (see the definitions (4.17)):

$$\mu_n^{-\frac{1}{2}}\chi^n D^i = q^{-2}\Lambda_{n-1}^2 \mathrm{k}_n^{-\frac{1}{2}}[D^i, (X \cdot X)_{n-1}]\mathcal{D}^n - \mathbf{L}^{in},$$

$$\chi^{-n} D^i = q^{-2}\Lambda_{n-1}^2 \mathrm{k}_n^{-\frac{1}{2}}[D^i, (X \cdot X)_{n-1}]\mathcal{D}^n - \hat{\mathbf{L}}^{in} \tag{4.159}$$

$$\mu_n^{-\frac{1}{2}}\mathcal{D}^{-n} X^i = q^{-1}\Lambda_{n-1}^2 \mathrm{k}_n^{-\frac{1}{2}}[(D \cdot D)_{n-1}, X^i]\chi^{-n} - \mathbf{L}^{-n,i},$$

$$\chi^{-n} D^i = q^{-2}\Lambda_{n-1}^2 \mathrm{k}_n^{-\frac{1}{2}}[D^i, (X \cdot X)_{n-1}]\mathcal{D}^n - \hat{\mathbf{L}}^{in}. \tag{4.160}$$

Then each factor $\chi^n\mathcal{D}_n$, $\chi^{-n}\mathcal{D}_{-n}$ can be reabsorbed into the μ_n, μ_{-n}-dependence of the coefficient functions v's. Finally, we arrive at (4.158) using the result of Lemma 2 and the commutation relations of section 3, which allow us to reorder all $\mathbf{L}, \mathrm{k}$'s according to the ordering shown in that formula.

Now we impose the conditions $[u, x\cdot x] = 0 = [u, \partial\cdot\partial]$ explicitly. They reduce to

$$\begin{cases} \left[\sum_{h=0}^{p}(\mu_n^{-\frac{1}{2}}\chi^n)^h(\mathcal{D}_{-n})^{p-h}v^{p,h}_{\{l_i,l_i'\}}(\mu_n,\mu_{-n},\chi^j,\mathcal{D}_j), \begin{Bmatrix} x\cdot x \\ \partial\cdot\partial \end{Bmatrix}\right] = 0 \\ \left[\sum_{h=0}^{p}(\mu_n^{-\frac{1}{2}}\mathcal{D}_n)^h(\chi^{-n})^{p-h}v^{-p,-h}_{\{l_i,l_i'\}}(\mu_n,\mu_{-n},\chi^j,\mathcal{D}_j), \begin{Bmatrix} x\cdot x \\ \partial\cdot\partial \end{Bmatrix}\right] = 0. \end{cases} \tag{4.161}$$

In fact the powers of L's appearing in formula (4.158) belong to a Poincare' basis of U_q^N, therefore are independent, and their coefficient functions can be split into components belonging to different subspaces $Diff^p$ (4.153). Using a procedure which, for the sake of brevity, we describe only in the case $p=1$, it is easy to show that from the latter equations it follows decompositions of the type

$$\begin{cases} \sum_{h=0}^{p}(\mu_n^{-\frac{1}{2}}\chi^n)^h(\mathcal{D}_{-n})^{p-h}v^{p,h}_{\{l_i,l_i'\}}(\mu_n,\mu_{-n},\chi^j,\mathcal{D}_j) = \sum_{\{i_1,\ldots i_p\}} f_{\{i_1,\ldots i_p\}}(\mathbf{k}^n)u_{\{i_1,\ldots i_p\}}\mathbf{L}^{i_1,n}\ldots\mathbf{L}^{i_p,n}, \\ \sum_{h=0}^{p}(\mu_n^{-\frac{1}{2}}\mathcal{D}_n)^h(\chi^{-n})^{p-h}v^{-p,-h}_{\{l_i,l_i'\}}(\mu_n,\mu_{-n},\chi^j,\mathcal{D}_j) = \sum_{\{i_1,\ldots i_p\}} f_{\{i_1,\ldots i_p\}}(\mathbf{k}^n)u_{\{i_1,\ldots i_p\}}\mathbf{L}^{-n,i_1}\ldots\mathbf{L}^{-n,i_p}, \end{cases} \tag{4.162}$$

$u_{\{i_1,\ldots i_p\}} \in U_q^{N-2}$, which completes the proof of formula (4.155). When $p=1$, upon use of formulae (3.83),(3.87), it is easy to verify that the LHS's of equations (4.161) are combination of $(\mu_n^{-\frac{1}{2}}\chi^n)^2\chi^{-n}, \mathcal{D}_{-n}, \mu_n^{-\frac{1}{2}}\chi^n$ and $\mu_n^{-\frac{1}{2}}\mathcal{D}_n(\mathcal{D}_{-n})^2, \mathcal{D}_{-n}, \chi^n$ respectively, and that setting their coefficients equal to zero amounts to

$$v^m = v^m(\mathbf{k}^n,\chi^j,\mathcal{D}_j) \qquad m=0,1, \quad |j|<n \qquad\qquad [v^0,(X{\cdot}X)_{n-1}] = 0 = [v^1,(D{\cdot}D)_{n-1}] \tag{4.163}$$

$$v^0 = -q^{-2-2\rho_n}\Lambda_{n-1}^{-1}(\mathbf{k}^n)^{-\frac{1}{2}}[v^1,(X\cdot X)_{n-1}]. \tag{4.164}$$

Hence

$$0 = \Big[[v^1,(X\cdot X)_{n-1}],(X\cdot X)_{n-1}\Big]_{q^2} = \Big[[v^1,(X\cdot X)_{n-1}]_{q^2},(X\cdot X)_{n-1}\Big] \tag{4.165}$$

implying upon use of the recursion hypothesis (4.156), formula (3.54) and of relations $d(v^1)=1$,

$$[(D\cdot D)_{n-1},(X\cdot X)_{n-1}]_{q^2}(q^2-1) = q^{4+2\rho_n}(\Lambda_{n-1}^2 - q^{2\rho_n}), \tag{4.166}$$

the equation

$$[v^1,(X{\cdot}X)_{n-1}]_{q^2} = u_iX^i, \quad u_i\in U_q^{N-2} \qquad \Rightarrow \qquad v^1 \propto \left[v^1, \frac{q^{4+2\rho_n}(\Lambda_{n-1}^2 - q^{2\rho_n})}{q^2-1}\right]_{q^2} \propto u_iD^i. \tag{4.167}$$

This yields $v^1(\mu_n^{-\frac{1}{2}}\chi^n) + v^0\mathcal{D}_{-n} \propto u_i\mathbf{L}^{in}$, as claimed.

The proof of (4.153) can be given recursively by constraining the general expansion (4.157) in a similar way. $\diamondsuit$.

Chapter 5

Integration over $\mathbf{R}_q^N$

Riemann integration over the Euclidean space $\mathbb{R}^N$ is covariant under the action of the Euclidean group, and in particular is invariant under finite translations. In infinitesimal form, the latter invariance implies the validity of Stoke's theorem for all integrable functions which are differentiable. We ask whether in the q-deformed case it is possible to define an integration which is covariant w.r.t. the quantum Euclidean group E_q^N (see Chapter 2). The reason why we want a q-covariant integration is ultimately related to our wish of doing q-covariant Physics.

The answer is positive, and, more importantly, one can construct this q-integration by essentially imposing only its invariance w.r.t. to finite translations (or equivalently the validity of Stoke's theorem, in infinitesimal form), together with the obvious requirement of linearity. Its $SO_q(N,\mathbf{R})$-covariance will follow from the $SO_q(N,\mathbf{R})$-covariance of the braided coaddition (2.84) (or, equivalently, of the differential calculus on $\mathbb{R}_q^N$).

Riemann integration also satisfies some other important properties, namely the reality condition (the complex conjugate of an integral is the integral of the complex conjugate of the function to be integrated) and the positivity condition (the integrand of a positive definite function is positive). We ask whether the q-deformed integration satisfies also these properties, and the answer is again affirmative.

In the classical case, if $f = P_n(x)exp[-a|x|^2]$ (P_n denotes a polynomial of degree n in x and $|x|^2$ the square lenght), then

$$\partial^i P_n(x)exp[-a|x|^2]| = P_{n-1}(x)exp[-a|x|^2] + P_{n+1}exp[-a|x|^2]; \tag{5.1}$$

Stoke's theorem then implies

$$\int dV\ P_{n-1}(x)exp[-a|x|^2] + \int dV\ P_{n+1}exp[-a|x|^2] = 0. \tag{5.2}$$

This relation allows to recursively define the integral $\int dV\ f$ (for any function f of the same kind) in terms of $\int dV\ exp[-a|x|^2]$ (which fixes the normalization of the integration). This result suggests to investigate whether it is possible to define a q-deformed integration in a similar way, namely by reducing the evaluation of all integrals to the integral of one particular function; we call the latter " reference function ", and as a particular candidate we expect some suitably defined q-deformed gaussian.

This approach has been thoroughly developed for the first time by us in Ref. [11]. The idea that a sensible definition of q-deformed integration should imply a q-deformed version of Stoke's theorem goes back to Ref. [50] (where the authors explicitly considered the case of the quantum hyperplane); however, it has exploited only recently by the second author et al. in Ref. [54] for a consistent definition of the integration on the quantum hyperplane.

At some stage of our construction of the integration over $\mathbb{R}_q^N$, namely when we have performed integration over the " angular coordinates " and are left only with the integration over the " radial " one, we realize that the latter satisfies the properties of Jackson q-integral [18]. Therefore an alternative way to define q-integration is to introduce at this stage Jackson q-integral over the radial coordinate; this has been pursued in Ref. [48]. The two approaches are essentially equivalent, except that some differences may arise in determining the domains of integrable functions, and are often complementary for practical purposes.

Finally, a very recent work on algebraic q-integration [19] deals with integration over generic braided space, including the socalled braided line, the quantum hyperplane, the quantum Euclidean and Minkowski space. Only in the case of the braided line and the quantum hyperplane the authors succeed in introducing a notion of " indefinite " q-integration through an algebraic formulation of the Jackson q-integral as the inverse of q-derivation. Definite integrals (and in particular the integral over the whole line/space) are numbers obtained by taking traces of the indefinite integrals - considered as operators - over their representation spaces. Only by a " gaussian " approach - which essentially coincides with the " reference function " approach that we introduced in Ref. [11] and that we are going to present here - they can define a definite q-integral over the whole spaces also in the remaining cases. The " reference function " approach seems therefore the most general for this purpose, up to now.

We are introducing the latter in the concrete case of the quantum Euclidean case, as in our original paper [11], but formulae are essentially valid whenever the derivatives and coordinates satisfy derivation rules (3.25) with some braid matrix $\hat{R}$ admitting a decomposition into projectors including a singlet one (i.e. there exists an covariant metric on the quantum space). Particular attention will be given to the role of q-covariance, which in our opinion has not been stressed enough elsewhere. Moreover, the reality and positivity of q-integration when $q \in \mathbb{R}$ will be thoroughly explored; Ref. [11] and the present work are, up to our knowledge, the only ones where this point has been fully taken into account.

5.1 Formal requirements

As in the classical case we can equivalently formulate properties of integration over $\mathbb{R}_q^N$ in terms of integrals of " functions " $f \in \mathbb{R}_q^N$ or of differential forms $\omega \in \bigwedge_q^N$. The obvious relation between the two formulations is that any N-form ω_N can be written in the form $\omega_N = d_qV\ f$ (d_qV is the volume form introduced in section, see the discussion there) with some $f \in \mathbb{R}_q^N$, so that

$$\int \omega_N = \int d_qV\ f \tag{5.3}$$

trivially. In the sequel statements regarding integrals will be written typically in only one of the two versions. For instance, Stoke's theorem takes respectively the two forms

$$\int d_qV \partial^i f| = 0 \quad i = 1, 2, ..., n, \qquad\qquad \int d\omega_{n-1}| = 0; \tag{5.4}$$

$\partial^i f|$ denotes the (total derivative) function which was introduced in formula (3.41).

A complication (which is absent e.g. in the quantum hyperplane case) seems to arise, because as we know there exist two sets of linearly independent derivatives, $\partial, \bar{\partial}$ (see Chapter 2), hence potentially two kinds of integrations $\int d_qV$, $\overline{\int d_qV}$ and another version of Stoke's theorem:

$$\overline{\int d_qV}\ \bar{\partial}^i f| = 0 \quad i = 1, 2, ..., N; \qquad\qquad \overline{\int} \bar{d}\omega_{N-1}| = 0. \tag{5.5}$$

At first sight the reality (for $q \in \mathbb{R}^+$) condition for each of the two integrations $\int d_qV$, $\overline{\int d_qV}$ seems no more guaranteed by Stoke's theorems (5.4) because $*$ maps derivatives $\partial \in D$ into derivatives $\bar{\partial} \in \bar{D}$ (and viceversa). Quite surprisingly, in next section we will see that the two integrations coincide, if we impose that they coincide on the reference function; therefore the reality condition holds.

For the moment we keep $\int d_qV$ and $\overline{\int d_qV}$ distinct. We list the requirements that these integrations should satisfy and show that they are compatible with each other. In next section one of these requirements, Stoke's theorem, will be used to recursively define the integrations.

We would like an integration $\int d_qV$ to be defined on a not too poor subspace $\mathcal{V}$ of $Fun(\mathbb{R}_q^N)$ and to satisfy:

- 1) linearity;
- 2) covariance;
- 3) continuity in q and correspondence principle for $q \to 1$;
- 4) reality (when $q \in \mathbb{R}^+$);
- 5) positivity (when $q \in \mathbb{R}^+$).

Of course linearity means

$$\int d_qV\ (\alpha f + \beta)g = \alpha \int d_qV\ f + \beta \int d_qV\ g, \qquad \alpha, \beta \in \mathbb{C} \quad f, g \in \mathcal{V} \tag{5.6}$$

and one has to check that if f vanishes because of relations (2.9), then so does $\int d_qV\ f$, in other terms

$$f(x) = A_{ij}\mathcal{P}_{A}{}^{ij}{}_{hk}x^hx^k \cdot g(x) \quad \Rightarrow \quad \int d_qV\ f = 0. \tag{5.7}$$

By covariance we mean

$$\mathbf{1}_{E_q^N} \int d_qV\ f = (id_{E_q^N} \otimes \int d_qV\ ) \circ \phi_L^E(f), \tag{5.8}$$

where $\mathbf{1}_{E_q^N}$ and $id_{E_q^N}$ denote respectively the unit element and the identity operator on the quantum Euclidean group $Fun(E_q^N)$, and ϕ_L^E is the left coaction of E_q^N (2.94) on $Fun(\mathbb{R}_q^N)$. As we have seen in Chapter 2, E_q^N can be considered as the semidirect product of $SO_q(N)$ times $\mathbb{R}_q^N$ thought as braided group of translations. If we apply the projection Π defined in formula (2.93) equation, so that we coact only with ϕ_L, and consider a function $f^{i_1i_2...i_k} := x^{i_1}x^{i_2}...x^{i_k}g(x \cdot x)$, then covariance more explicitly reads

$$\mathbf{1}_{SO_q(N)} \int d_qV\ f^{i_1i_2...i_k} = T^{i_1}_{j_1}T^{i_2}_{j_2}...T^{i_k}_{j_k} \int d_qV\ f^{j_1j_2...j_k}, \tag{5.9}$$

in other words the numbers $\int d_qV\ f^{i_1i_2...i_k}$, $i_j = 1, 2, ..., N$, are the components of an " isotropic " tensor; in the classical case relation (5.9) corresponds to the well-known property of tensors such as

$$\int d^Nx\ g(|x|^2)x^i = 0, \qquad \int d^Nx\ g(|x|^2)x^ix^j \propto \delta^{ij}, \tag{5.10}$$

$$\int d^N x \; g(|x|^2) x^i x^j x^k x^l \propto (\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}), ... \tag{5.11}$$

namely the property that the latter are invariant under an orthogonal transformation of the coordinates $x^i \rightarrow x'^i := g^i_j x^j$. The simplest nontrivial example of a tensor satisfying (5.9) is for $k = 2$, $\int d_qV \; f^{ij} \propto C^{ij}$. In general tensors satisfying (5.9) involve matrix products among $\hat{R}$-matrices (or, equivalently, $\hat{R}^{-1}$-matrices) and contractions with metric matrices C: the former reorder indices by means of the RTT relations (2.1), whereas the latter transform a couple of neighbouring T-matrices into a commuting number (see relation (2.2)). Therefore an integral $\int d_qV \; x^{i_1}...x^{i_k} g(x \cdot x)$ should be factorizable as a product

$$\int d_qV \; x^{i_1}...x^{i_k} g(x \cdot x) = S^{i_1 \cdots i_k} \alpha_g, \tag{5.12}$$

and $S^{i_1 \cdots i_k} = 0$ for k odd; the g-dependence of the RHS of (5.12) is concentrated in the constant α_g, which essentially is a (yet unspecified) integral along the " radial " direction. Explicit solutions $S^{i_1 \cdots i_k}, \bar{S}^{i_1 \cdots i_k}$ satisfying (5.9) will be found in next section. The translation part invariance involved in equation (5.8) is best singled out when formulated in terms of the braided coaddition (2.84):

$$\int d_qV \; f(a+x) := \int d_qV \; \underline{\Delta} f(x) = \underline{\Delta}(\int d_qV \; f(x)) = \int d_qV \; f(x). \tag{5.13}$$

By expanding powers $(a+x)$ in the LHS, reordering the a's to the left of the x's and putting equal to zero the coefficients of the a powers, we find Stoke's theorem (5.4) (because of formula (3.52)), which can be therefore considered as the infinitesimal version of translation invariance (5.13). On the other hand, the infinitesimal version of $Fun(SO_q(N))$-covariance will amount to the possibility of integrating by parts (i.e. of applying Stoke's theorem) also with the infinitesimal generators of rotations $L, \mathbf{L}$ (see the preceding Chapter) in the place of the derivatives, and it will follow from the fact that the former can be expressed as differential operators themselves. We will see that, as in the classical case, Stoke's theorem involves a definite choice of the " radial " part of the integration (whereas the latter is left unspecified by requirement 2) alone). As already noticed, Stoke's theorem is a formidable tool to define (up to a normalization factor) the corresponding integration.

Point 3) means that we require the q-deformed integral to reduce to the classical Riemann one when q=1. Maybe it is timely to recall the fact that the x^i coordinates are not real (even for q=1), but are complex combinations of the usual real cartesian coordinates; the latter can be used to perform the integral when q=1.

The reality condition 4) reads

$$(\int d_qV\ f)^* = \int d_qV\ f^* \tag{5.14}$$

for any q ($\in \mathbb{R}^+$), and similarly for the $\bar{\int}$. Actually we will first prove an apparently weaker form reality condition, namely $\int d_qV\ f^* = \bar{\int}\overline{d_qV}\ f^*$, involving both $\int$ and $\bar{\int}$.

The positivity condition 5) in its usual form reads

$$\int d_qV\ f^*f \geq 0, \qquad \int d_qV\ f^*f = 0 \Leftrightarrow f = 0; \tag{5.15}$$

We will prove that it is valid $\forall q \in \mathbb{R}^+$, and the proof is in two steps, namely by integrating over " angular " and the " radial " variables separatly. Then the usual definition of scalar product

$$(f,g) := \int d_qV\ f^*g, \qquad f,g \in \mathbb{R}_q^N \tag{5.16}$$

is sufficient to single out a domain of square integrable functions within $Fun(\mathbb{R}_q^N)$ which could be considered as a Hilbert space of a q-defomed physical system. However we will see that definition (5.16) is not suitable to make operators as the momentum components hermitean, and in fact we will see in two cases of physical interest (Chapters 6,7) that a more sophisticated definition is required. Nevertheless, also in the latter cases the proof of the relations

$$(f,g)^* = (g,f); \qquad (f,f) \geq 0, \ \ (f,f) = 0 \Leftrightarrow f = 0 \qquad f,g \in \mathcal{V}, \tag{5.17}$$

will be based on the properties of the q-integration (like the positivity of integration over " angular " variables, and some other property).

Now we briefly discuss compatibility of requirements 1) - 5).

It is straightforward to check that linearity is compatible with covariance because of property (2.6) (where we take $f(\hat{R}) = \mathcal{P}_A$). Requirement 3) is obviously compatible with 1),2) since Riemann integration is linear and has an E^N-invariant integration measure. It is straightforward to prove that reality (5.14) is compatible with linearity (because of relation (2.6), where again we take $f(\hat{R}) = \mathcal{P}_A$), with covariance (apply $*$ to eq. (5.9) and use definitions (2.38),(2.70)) and with the correspondence principle (Riemann real integration satisfies the reality condition). Positivity in the form (5.15) is clearly compatible with requirements 1),3) and with reality (5.14). At this stage is not easy to understand if it is compatible with covariance. In section 3 we will prove that the integration *constructed* by imposing covariance 2) (i.e. essentially Stoke's theorem) *is* positive definite.

Now we show the

Proposition 25 *When $q \in \mathbb{R}^+$ the integrations $\int d_qV$, $\overline{\int d_qV}$ satisfying Stoke's theorems* (5.4) *and (5.5) are compatible with reality in the form*

$$\int d_qV\ f^* = \overline{\int \overline{d_qV}}\ f^*, \tag{5.18}$$

Then compatibility with reality in the form (5.14) will follow from the equality $\int d_qV \ = \overline{\int \overline{d_qV}}$, which we will prove in next section.

Proof: Let us consider the spaces of formal relations

$$\mathcal{F} = Span_{\mathbb{C}}\{\partial^i f - \partial^i f| - f^i_j\partial^j = 0, \qquad i,j = 1,..,N \quad f \in \mathcal{V}\} \tag{5.19}$$

$$\bar{\mathcal{F}} := Span_{\mathbb{C}}\{\bar{\partial}^i f - \bar{\partial}^i f| - \bar{f}^i_j\bar{\partial}^j = 0, \qquad i,j = 1,..,N \quad f \in \mathcal{V}\}, \tag{5.20}$$

where: 1) $\partial^i f|, f^i_j, \bar{\partial}^i f|, \bar{f}^i_j$ are the functions introduced in formula (3.41); 2) $\mathcal{V}$ is some subspace of $Fun(\mathbb{R}^N_q)$ closed under complex conjugation and derivation, in the sense $f \in \mathcal{V}$ implies $\partial^i f|, f^i_j, \bar{\partial}^i f|, \bar{f}^i_j$. In the classical case the space V_{cl} of functions of the type $P(x)exp[-a|x|^2]$ (P being a polynomial) is an example of such a subspace $\mathcal{V}$, and we will see that analogous examples will be available in the q-deformed case, too. Under these assumptions it is immediate to recognize that the two spaces (5.19),(5.20) are mapped into each other by $*$, since $* : D \rightarrow \bar{D}$ and $* : \bar{D} \rightarrow D$. In other terms $\mathcal{F}^* = \bar{\mathcal{F}}$. If we define subspaces $\mathcal{A}, \bar{\mathcal{A}} \subset \mathcal{V}$ as the linear spans of functions of the form $\partial^i f|$ and $\bar{\partial}^i f|$ respectively, the previous remark implies

$$\mathcal{A}^* = \bar{\mathcal{A}} \tag{5.21}$$

(it suffices to recall the definition (3.41) to write $(\partial^i f|)^* = (\partial^i f - f^i_j\partial^j)^*$). For each $a \in \mathcal{A}$ let $\bar{a} \in \bar{\mathcal{A}}$ be the function such that $a^* = \bar{a}$. Stoke's theorems respectively imply

$$(5.4) \quad \Rightarrow \quad \int d_qV\ a = 0 = \int d_qV\ a^* \quad \forall a \in \mathcal{A} \tag{5.22}$$

$$(5.5) \quad \Rightarrow \quad \overline{\int \overline{d_qV}}\ \bar{a} = 0 = \overline{\int \overline{d_qV}}\ \bar{a}^* \quad \forall \bar{a} \in \bar{\mathcal{A}}, \tag{5.23}$$

hence reality in both the form (5.14) and (5.18) is trivially satisfied for the integrals $\int d_qV\ a$, $\overline{\int \overline{d_qV}}\ \bar{a}$. If q=1 and we take $\mathcal{V} = V_{cl}$ one easily realizes that any $f \in \mathcal{V}$ can be expressed in the form

$$f = a + c_f f_0, \quad a \in \mathcal{A}, \ c_f \in \mathbb{C} \tag{5.24}$$

(as anticipated at the beginning of this section), where f_0 is defined by $f_0 := exp[-a|x|^2]$. Consequently

$$\int dV\ f = c_f \int dV\ f_0. \tag{5.25}$$

For self-evident reasons we call f_0 the reference function of the integral. In next sections we will see that a similar situation occurs also in the q-deformed case, for instance by taking $\mathcal{V} = \{f = P(x)e_{q^2}[-a(x \cdot x)] \ \mid\ P \equiv \ a\ polynomial\}$ and $f_0 := e_{q^2}[-a(x \cdot x)]$. In any case f_0 should be a real function not belonging to $\mathcal{A}$ and should go to a smooth rapidly decreasing classical function in the limit $q \to 1$. Taking the complex conjugate of eq. (5.24) we get

$$f^* = \bar{a} + c_f^* f_0, \quad \bar{a} \in \bar{\mathcal{A}},\ c_f \in \mathbb{C}, \tag{5.26}$$

which implies

$$\bar{\int} \overline{d_qV}\ f^* = c_f^* \bar{\int} \overline{d_qV}\ f_0 \tag{5.27}$$

We are still free to fix $\int d_qV f_0, \bar{\int}\overline{d_qV} f_0$ as we like. If we impose the reality condition in the form (5.18) on the reference function we see that it is transfered to all functions belonging to $\mathcal{V}$, as claimed $\diamondsuit$.

Since $f_0^* = f_0$, the reality condition (5.18) on the reference function reads $\int d_qV\ f_0 = \bar{\int}\overline{d_qV}\ f_0^*$. In the sequel we will take $\int d_qV\ f_0 \in \mathbb{R}^+$.

5.2 Construction of the integration

In this section we use Stoke's theorem (in its two versions (5.4),(5.5)) as a tool for constructing the integrations. The systematic enforcement of Stoke's theorems generates a set of formal relations between integrals of different functions. In this section we assume for simplicity that the reference function is scalar. Consequently, we determine these relations in two steps. First, we find out the isotropic tensors $S^{i_1\ldots i_k}, \bar{S}^{i_1\ldots i_k}$: hence, according to (5.12), the integrals $\int d_qV\ f$, $\bar{\int}\overline{d_qV}\ f$ of a non scalar function f will be expressed in terms of integrals of a scalar one. Due to the fact that $S^{i_1\ldots i_k}, \bar{S}^{i_1\ldots i_k}$ turn out to be proportional, the two integrations coincide. Second, we determine the equations relating integrals of different scalar functions; in this way we will be able to express integrals of scalar functions in terms of the integral $\int d_qV\ f_0$ of a particular one, what we call the reference function f_0. $\int d_qV\ f_0$ is a normalization constant and can be fixed quite arbitrarily (see the end of the preceding section). So to say, the second step amounts to integration over the radial coordinate. In Chapter 7 we will consider a physical system (free particle) such that the scalar product of vectors can be expressed in terms of the integral of a non-scalar reference function (giving the norm of the ground state). As an example we will explicitly consider in this section the reference function $f_0 = e_{q^2}[-\alpha x \cdot x q^{N-2}]$; in Chapter 6 we will take an other reference function which is conceived for defining the scalar products of states of the harmonic oscillator. In this way one can define the integrals for infinitely many independent functions $\{f_i\}_{i \in N}$ and therefore for finite combinations of them. This is enough for the scopes of the present work, since it will enable us to define a positive definite scalar product inside some subspace of $Fun(\mathbb{R}_q^N)$; then the completion of this pre-Hilbert space will be done w.r.t. the corresponding norm. Nevertheless, to further enlarge the domain of definition of the integrals one could consider functions admitting series expansions in the $\{f_i\}$, and we will briefly address this problem at the end of this section.

The preliminary discussion of the previous section has shown that the two basic integrations $\int d_qV$, $\overline{\int d_qV}$ are linear, covariant and coincide with the classical Riemann integration for q=1. Therefore the explicit recursive application of the two Stoke's theorems will determine (up to a factor) isotropic tensors $S^{i_1...i_k}$, $\bar{S}^{i_1...i_k}$ (see (5.9)). As we are going to see, up to a factor these tensors coincide and do not depend on the choice of the function $g(x \cdot x)$ in formula (5.12) The relevant results of this section are summarized in Propositions 25,26,27.

The choice $g = e_{q^2}[-\alpha x \cdot x q^{N-2}]$ (or, alternatively, we could take $g = e_{q^{-2}}[-\alpha x \cdot x]$) is particularly convenient for this goal. Using relation (2.29) we find

$$\partial^{i_1} x^{i_2}...x^{i_k} e_{q^2}[-\alpha x \cdot x q^{N-2}]| = \tag{5.28}$$

$$= -\alpha x^{i_1} x^{i_2}...x^{i_k} e_{q^2}[-\alpha x \cdot x q^{N-2}] + e_{q^2}[-q^2 \alpha x \cdot x q^{N-2}] \partial^{i_1} x^{i_2}...x^{i_k}| = \tag{5.29}$$

$$= -\alpha x^{i_1} x^{i_2}...x^{i_k} e_{q^2}[-\alpha x \cdot x q^{N-2}] + e_{q^2}[-q^2 \alpha x \cdot x q^{N-2}] M_{k,j_3...j_k}^{\ i_1...i_k} x^{j_3}...x^{j_k} \tag{5.30}$$

where the tensors $M_{k,j_3...j_k}^{\ i_1...i_k}$, $N_{k,j_1...j_k}^{\ i_1...i_k}$ are introduced by the defining relation

$$\partial^{i_1} x^{i_2}...x^{i_k} =: M_{k,j_3...j_k}^{\ i_1...i_k} x^{j_3}...x^{j_k} + N_{k,j_1...j_k}^{\ i_1...i_k} x^{j_1}...x^{j_{k-1}} \partial^{j_k} \tag{5.31}$$

Taking the integral $\int d_qV$ of (5.28) and applying Stoke's theorem we find

$$\int d_qV \ x^{i_1}...x^{i_k} e_{q^2}[-\alpha x \cdot x q^{N-2}] = \frac{1}{\alpha} M_{k,j_3...j_k}^{\ i_1...i_k} \int d_qV \ x^{j_3}...x^{j_k} e_{q^2}[-q^2 \alpha x \cdot x q^{N-2}] \tag{5.32}$$

Starting from $k = 0, 1$ and noting that Stoke's theorem (or, equivalently, covariance) imply $\int d_qV \ x^i e_{q^2}[-\alpha x \cdot x q^{N-2}] = 0$, we see that the recursive application of relation (5.32) determines tensors $S_k^{i_1 \dots i_k}$. The result is summarized in the

Proposition 26 *The tensors*

$$S_k := \begin{cases} 0 & if \ k \ is \ odd \\ S_{2m} := M_{2m} \cdot M_{2(m-1)} \cdot \dots M_2 & if \ k = 2m \end{cases}, \qquad k, m \in \mathbb{N} \tag{5.33}$$

satisfy the covariance condition (5.9).

Here we have used the shorthand notation

$$(M_{2m} \cdot M_{2(m-1)})^{i_1 i_2 \dots i_{2m}}_{j_5 j_6 \dots j_{2m}} = M_{2m, l_3 l_4 \dots l_{2m}}^{\ \ i_1 i_2 \dots i_{2m}} M_{2(m-1), j_5 j_6 \dots j_{2m}}^{\ \ l_3 l_4 \dots l_{2m}}. \tag{5.34}$$

As a direct consequence of the proposition and of relation (5.32), the integral (5.32) will vanish when k is odd and

$$\int d_qV \ x^{i_1} x^{i_2} \dots x^{i_{2m}} e_{q^2}[-q^2 \alpha x \cdot x q^{N-2}] \propto S_{2m}^{i_1 i_2 \dots i_{2m}}. \tag{5.35}$$

Similarly one can determine tensors $\bar{S}_k^{i_1 \dots i_k}$ satisfying the analog of relation (3.12) for the integration $\overline{\int d_qV}$; to this end we only need to replace $\partial, M, N, S, \hat{R}^{\pm 1}, q$ with $\bar{\partial}, \bar{M}, \bar{N}, \bar{S}, \hat{R}^{\mp 1}, q^{-1}$ in the preceding formulae.

Proposition 27

$$\bar{S}_{2m}^{i_1 \dots i_{2m}} \propto S_{2m}^{i_1 \dots i_{2m}}. \tag{5.36}$$

Proof: this immediately follows from the very useful formulae

$$(\partial \cdot \partial)^n x^{i_1} x^{i_2} \dots x^{i_{2m}} | = (q^{2-N})^n n_{q^2}! S_{2m}^{i_1 i_2 \dots i_{2m}} \tag{5.37}$$

$$(\bar{\partial} \cdot \bar{\partial}) x^{i_1} x^{i_2} \dots x^{i_{2m}} | = n_{q^{-2}}! \bar{S}_{2m}^{i_1 i_2 \dots i_{2m}} \tag{5.38}$$

and from equations (3.57),(3.59). The proof of relations (5.37) is by induction and will be given in appendix 5.4 $\diamondsuit$.

Next, it is easy to realize that equation (5.35) can be generalized to

$$\int d_qV \ x^{i_1} x^{i_2} \dots x^{i_{2m}} g(x \cdot x) \propto S_{2m}^{i_1 i_2 \dots i_{2m}}. \tag{5.39}$$

with any function $g(x \cdot x)$. In fact, looking at the power series defining g one immediately finds that $\partial^i g(x \cdot x)| = \tilde{g}(x \cdot x)x^i$, with some functions $\tilde{g}, \in Fun(\mathbf{R}_q^N)$. Then, applying both sides of (5.31) to g and taking the integral $\int d_qV$ we find

$$0 = \int d_qV\ \partial^{i_1} x^{i_2} ... x^{i_{2m}} g| = M_{2m,j_3...j_{2m}}^{\ \ i_1 i_2 ... i_{2m}} \int d_qV\ x^{j_3} ... x^{j_{2m}} g + N_{2m,j_1...j_{2m}}^{\ \ i_1 ... i_{2m}} \int d_qV\ x^{j_1} ... x^{j_{2m}} \tilde{g}. \tag{5.40}$$

This result holds for any function g, in particular for the previous choice $g = e_{q^2}[-\alpha x \cdot x q^{N-2}]$; by comparison with (5.32),(5.33) we infer the invertibility of the matrices N_{2m}, the relations

$$N_{2m}^{-1} \cdot S_{2m} \propto S_{2m} \tag{5.41}$$

and hence the relations

$$\int d_qV\ x^{i_1} ... x^{i_{2m}} \tilde{g} = c_{n,g} S_{2m}^{i_1 ... i_{2m}}, \tag{5.42}$$

for any function $\tilde{g}(x \cdot x)$. By contracting the free indices $i_1, i_2, ..., i_{2m}$ with $C_{i_1 i_2}, ..., C_{i_{2m-1} i_{2m}}$ we reduce the determination of the constant $c_{n,\tilde{g}}$ to the evaluation of the integral of a purely scalar function. The same arguments can be applied to the integration $\int \overline{d_qV}$. Thus we are led to the

Proposition 28

$$\int d_qV\ x^{i_1} ... x^{i_{2m}} \tilde{g} = S_{2m}^{i_1 ... i_{2m}} \frac{\int d_qV\ (xCx)^n \tilde{g}}{\mathcal{S}_{2m}}, \tag{5.43}$$

$$\int \overline{d_qV}\ x^{i_1} ... x^{i_{2m}} \tilde{g} = S_{2m}^{i_1 ... i_{2m}} \frac{\int \overline{d_qV}\ (xCx)^n \tilde{g}}{\mathcal{S}_{2m}}; \tag{5.44}$$

here

$$\mathcal{S}_{2m} := C_{i_1 i_2} ... C_{i_{2m-1} i_{2m}} S_{2m}^{\ \ i_1, i_2, ..., i_{2m}}. \tag{5.45}$$

The constant $\mathcal{S}_{2m}$ is positive for any $q \in \mathbf{R}^+$.

(The positivity of $\mathcal{S}_{2m}$ can be easily proved using formulae (3.63),(5.37)).

Let us analyze the " radial " dependence of the two integrals $\int d_qV$, $\int \overline{d_qV}$. We introduce recall the definition (3.60) of $\mathcal{B}$

$$\mathcal{B} := 1 + \frac{q^2 - 1}{\mu} x^i \partial_i = q^{-N}(1 + \frac{q^2 - 1}{\mu} \partial^i x_i) \tag{5.46}$$

and introduce an anlogous barred differential operator by

$$\bar{\mathcal{B}} := 1 + \frac{q^{-2} - 1}{\bar{\mu}} x^i \bar{\partial}_i = q^N(1 + \frac{q^{-2} - 1}{\bar{\mu}} \bar{\partial}^i x_i); \tag{5.47}$$

it is straightforward to check that $B(x\cdot x) = q^2(x\cdot x)B$, $\bar{B}(x\cdot x) = q^{-2}(x\cdot x)\bar{B}$ and therefore

$$Bf(x\cdot x) = f(q^2 x\cdot x)B, \qquad \bar{B}f(x\cdot x) = f(q^{-2}x\cdot x)\bar{B}, \tag{5.48}$$

for any $f \in Fun(\mathbb{R}_q^N)$ depending only on $(x\cdot x)$; hence

$$q^{-N}(f+\frac{q^2-1}{1+q^{2-N}}\partial^i x_i f|) = f(q^2 x\cdot x), \qquad q^N(f+\frac{q^{-2}-1}{1+q^{N-2}}\bar{\partial}^i x_i f|) = f(q^{-2}x\cdot x). \tag{5.49}$$

By taking the integrals $\int d_qV$, $\bar{\int}\overline{d_qV}$ respectively of $(5.49)_a$, $(5.49)_b$ and by applying Stoke's theorems (5.4),(5.5) we find the formal relations

$$\int d_qV\ f(q^2 x\cdot x)q^N = \int d_qV\ f(x\cdot x), \qquad \bar{\int}\overline{d_qV}\ f(q^2 x\cdot x)q^N = \bar{\int}\overline{d_qV}\ f(x\cdot x) \tag{5.50}$$

for both integrations. These relations remind the characterizing property of the socalled Jackson integral, and in fact one can define now the integral of scalr functions using the latter; this will be shown below. For the moment we go on with our abstract discussion.

As we will see in a moment, equation (5.50) will determine the integrals of scalar functions belonging to some domain in terms of that of the reference function f_0; if we set $\int d_qV\ f_0 = \bar{\int}\overline{d_qV}\ f_0 (\in \mathbf{R}^+)$, this implies the formal relation

$$\int d_qV\ f = \bar{\int}\overline{d_qV}\ f, \tag{5.51}$$

at least for $f = f(x\cdot x)$. But looking back at relations (5.43),(5.44) we realize that previous equation holds for any f. This concludes the proof of the

Proposition 29 *the two integrations* $\int d_qV$, $\bar{\int}\overline{d_qV}$ *(formally) coincide.*

Since the integral $\int d_qV\ f$ of any $f \in Fun(\mathbb{R}_q^N)$, if it exists, is reduced to a combinations of integrals of radial functions by means of relation (5.43), then property (5.50) is generalized by the

Proposition 30

$$\int d_qV\ f(qx)q^N = \int d_qV\ f(x). \tag{5.52}$$

This fundamental relation characterizes the integration defined by means of Stoke's theorem and will be called " scaling property " for reasons which will become clear at the end of this section.

So far we have not specified the domain of functions $f \in Fun(\mathbf{R}_\mathbf{q}^\mathbf{N})$ for which the integral $\int d_qV\ f$ can be defined. Therefore all the previous relations were purely formal. Now we pick up a particular reference function $f_0 = f_0(x \cdot x)$. We ask what are the functions f such that the corresponding integral $\int d_qV\ f$ can be reduced to the one $\int d_qV\ f_0$ by means of iterated application of Stoke's theorem and of linearity, and turn out to be finite. Of course we wish to include in this space of " integrable " functions as many $f \in Fun(\mathbf{R}_\mathbf{q}^\mathbf{N})$ as possible.

As an example we take $f_0 = e_{q^2}[-\alpha x \cdot x q^{N-2}]$, $\alpha > 0$, which for q=1 reduces to a well known smooth rapidly decreasing classical function, the gaussian. First we consider functions f of the type $f(x \cdot x) = P(x \cdot x)e_{q^2}[-\frac{\alpha x C x}{\mu}]$, P being an arbitrary polynomial. Using property (5.52) and the q-derivative property (3.71) of the exponential we show the

Proposition 31

$$\int d_qV\ e_{q^2}[-\alpha x \cdot x](x \cdot x)^h = \left(\frac{1}{\alpha}\right)^h \left(h - 1 + \frac{N}{2}\right)_{q^2} \cdots \left(\frac{N}{2}\right)_{q^2} \cdot q^{-h(N+h-1)}c, \qquad (5.53)$$

where $c := \int d_qV\ e_{q^2}[-\alpha x \cdot x] \in \mathbf{R}^+$ plays the role of normalization factor.

Proof :

$$\int d_qV\ e_{q^2}[-\alpha x \cdot x](x \cdot x)^{k-1} \overset{(5.52)}{=} q^{N+2(k-1)} \int d_qV\ e_{q^2}[-q^2\alpha x \cdot x](x \cdot x)^{k-1}$$

$$\overset{(3.71)}{=} q^{N+2(k-1)} \int d_qV\ e_{q^2}[-\alpha x \cdot x](x \cdot x)^{k-1} - \alpha(q^2-1)q^{N+2(k-1)} \int d_qV\ e_{q^2}[-\alpha x \cdot x](x \cdot x)^k, \qquad (5.54)$$

whence

$$\int d_qV\ e_{q^2}[-\alpha x \cdot x](x \cdot x)^k = (\frac{1}{\alpha})(k - 1 + \frac{N}{2})_{q^2} q^{-N-2(k-1)} \int d_qV\ e_{q^2}[-\alpha x \cdot x](x \cdot x)^{k-1}; \qquad (5.55)$$

applying h times formula (5.55) for $k = h, h-1, ..., 1$ we find (5.553) $\diamondsuit$.

Relations (5.43),(5.53) allow to define the integration $\int d_qV$, on all functions of the type $f = P(x)f_0$, where $P(x)$ is an arbitary polynomial in x and $f_0 := e_{q^2}[-\alpha x \cdot x]$. We could enlarge the domain of definition of the integrations by admitting functions $P(x)$ in the form of power series $P(x) = \sum_{n=0}^{\infty} \sum_{i_1,i_2,...,i_n} A_{i_1 i_2 ... i_n} x^{i_1} x^{i_2} ... x^{i_n}$ such that the series

$$\sum_{n=0}^{\infty} \sum_{i_1,i_2,...,i_n} A_{i_1 i_2 ... i_n} \int d_qV\ x^{i_1} x^{i_2} ... x^{i_n} f_0 \qquad (5.56)$$

converges; the integral $\int d_qV\ f$ would then be defined as the limit (5.56) A further step towards the enlargement of the domain of definition of the integrations could be done defining a new reference function by letting $f_0 := P(x)e_{q^2}[-\alpha x \cdot x]$ for some $p \in$

$Fun(\mathbb{R}_q^N)$: by means of formulae (5.43),(5.52) we should be able to evaluate $\int d_qV\ \tilde{P}(x)f_0$ in terms of c' for all polynomials $\tilde{P}(x)$. Thus one could include in the domain of integrable functions also functions f susceptible of a decomposition $f = \tilde{P}(x)f_0$, $\tilde{P}(x) = \sum_{n=0}^{\infty}\sum_{i_1,i_2,...,i_n} A_{i_1 i_2 ... i_n} x^{i_1}x^{i_2}...x^{i_n}$ such that the series (5.56) with this new f_0 converges. For instance, in Chapter we will consider as reference function (for odd N) a function of the form $f_0 := e_q^2[-m^2 x\cdot x]e_{q^{-1}}[iMx^0]e_{q^{-1}}[-iq^{-1}iMx^0]$. It is natural to figure that to the new choice of the reference function there should correspond an actual enlargement of the domain of integrable functions.

This operation could be iterated in a sort of continuation of the functional $\int d_qV$ so as to enlarge to the maximum possible size the space of integrable functions. It is out of the scope of this work to face this problem by analyzing which conditions the coefficients $\{A_{i_1 i_2 ... i_n}\}$ of an expansion of the type $f = f_0 \sum_{n=0}^{\infty}\sum_{i_1,i_2,...,i_n} A_{i_1 i_2 ... i_n} x^{i_1}x^{i_2}...x^{i_n}$ should satisfy in order that f be integrable.

Incidentally, we just briefly note that, having defined the integrations $\int d_qV$ using Stoke's theorems (3.7), one could define a new integration $\int \rho(d_qV)$satisfying (at least) requirements 1) - 3) of the preceding section, by setting

$$\int \rho(d_qV)\ f := \int d_qV\ f\cdot\rho; \tag{5.57}$$

the " weight " ρ should be a real rapidly decrasing function.

If we replace $d_qV, \overline{d_qV}$ by the formal expression $d_qr\ r^{N-1}$ ($r^2 := x\cdot x$), we realize that a natural way to satisfy the formal relations (5.52) is to consider r as a real variable and to *define* as in Ref. [48] $\int d_qVf(x\cdot x)$, $\int\overline{d_qV}\ f(x\cdot x)$ by means of the Jackson integral [18] $\int d_qr$:

$$\int d_qVf(x\cdot x) := \bar{\int}\overline{d_qV}f(x\cdot x) := \int_0^{\infty} d_qr\ r^{N-1}f(r^2). \tag{5.58}$$

we recall that the latter is defined by

$$\int_0^{\infty} d_qr\ g(r) := |q-1|\sum_{l=-\infty}^{\infty} q^l g(q^l), \qquad q\in\mathbb{R}^+ - 1 \tag{5.59}$$

and is formally zero if g is the q-derivative $g(r) = \frac{h(qr)-h(r)}{(q-1)r}$ of another funtion $h(r)$. This integral is finite and goes to the Riemann integral $\int_0^{\infty} dr\ g(r)$ in the limit $q\to 1$ whenever $g(r)$ is continuous and Riemann integrable in the interval $(0,\infty)$. As natural, one says that g is Jackson-integrable if the series (5.59) converges. Of course, a very important question is to investigate what is the domain of Jackson-integrable functions, since it will determine the domain of q-integrable functions on $\mathbb{R}_q^N$, if we adopt the definition (5.58). It turns out that there are functions (typically the q-special functions) which are badly non-integrable w.r.t. to Riemann integration but integrable w.r.t. Jackson one. An important theorem due to the authors of Ref. [48] helps to identify the latter. Formula (5.58) is not the only manner to implement the radial integral (5.58) by a series in such a way that the scaling property (5.52) is satisfied. For instance, one would obtain another implementation by a " shift " of the integration variable r,

$$\int d_q^a r \ g(r) := |q-1| \sum_{l > log_q(a)} q^l g(q^l - a), \qquad a, \in \mathbb{R}^+, \ q, \in \mathbb{R}^-\{1\} \tag{5.60}$$

also this expression would go to the classical Riemann integral of g in the limit $q \to 1$, under the same regularity hypotheses as before. More generally, as suggested by the authors of Ref. [19], the integral (5.58) could be implemented as the trace of $(x \cdot x)^{\frac{N}{2}} f(x \cdot x)$ represented as an operator acting on the Hilbert space of some q-deformed quantum systems [19], and the definitions (5.58),(5.59) would be too strict to embrace this case. At this level we prefer therefore for the integration on the whole $\mathbb{R}_q^N$ the reference function approach, since this is based only on algebraic requirements and therefore will give results independent of the above choice of the representation.

Now let us come back to property (5.52). By its iterative application we find

$$\int d_qV \ f(q^n x) q^{nN} = \int d_qV \ f(x), \qquad n \in \mathbb{Z}. \tag{5.61}$$

Relation (4.30) states that under the change of integration variables $x \to ax$ with $a = q^n$ the integral $\int$ is invariant if we let d_qV transform according to $d_qV \to a^N d_qV$, namely according to the law of transformation of $d^N x$. This justifies the name " scaling property " for relations (5.50),(5.52),(5.61). In the classical and the q-deformed case this property holds $\forall a \in \mathbb{R}^+$ and characterizes Riemann integral, which has a " homogeneous " (i.e. translation invariant) measure.

One can now ask if the scaling property holds even if the dilatation parameter $a \notin Q := \{q^n, \ n \in \mathbb{C}\}$. One can easily check that this is not the case. In other terms, the function

$$F(a) := \int d_qV \ f(ax) a^N \tag{5.62}$$

is periodic in the variable $b = ln(a)$ with period $ln(q)$, but is not constant and generally speaking may be even undefined when $a \neq q^m$ (consider fro instance q-special functions and apply theorem of ref. [48].

In the limit $q \to 1$ we recover the classical scaling property only in the sense that $\forall a, \epsilon > 0$ there exist a neighbourhood of $q = 1$ such the latter holds for a scaling factor a' with $|a - a'| < \epsilon$: take $a' = q^m$ with a suitable $m \in \mathbb{C}$.

Let us consider now eq. (5.61) from the dimensional viewpoint. The series expansion of the function f makes sense only if $f(x)$ is of the form $f(x) = g(c_0 x)$, where c_0 is some constant with dimension of inverse lenght, $[c_0] = L^{-1}$ (in the case of the harmonic oscillator and of the free particle to be considered in Chapters 6,7 $c_0 = \sqrt{\omega}, M$ respectively). For the sake of brevity assume that $[g] = 1$. Since the volume form has dimension L^N, then $[\int d_qV\ f(x)] = [\int d_qV\ g(c_0x)] = L^N$; this implies

$$\int d_qV\ f(x) = \int d_qV\ g(c_0x) \propto c_0^{-N} \tag{5.63}$$

We cannot require property (5.63) to be valid for a scaling $c_0 \to c_0' := \alpha c_0$ of the fundamental constant with an arbitrary $\alpha \in \mathbb{R}^+$, otherwise the scaling property of the integral would hold for an arbitrary scaling parameter as well. The scaling is not incompatible with the "quantized" scaling property (5.61) only if $c_0' \in \mathcal{I}_{c_0} := \{c = q^m c_0 : \quad m \in \mathbb{C}\}$. In other words choosing an integration implies a restriction of the admitted values for the fundamental constants characterizing the system.

5.3 The positivity of q-integration and the Hilbert space $\mathcal{L}^2(\mathbb{R}_q^N)$

In this section we prove the positivity (5.15) of the q-integration defined in the two previous sections. This enables the introduction of a Hilbert space $\mathcal{L}^2(\mathbb{R}_q^N)$ of " square integrable functions on $\mathbb{R}_q^N$ " as a subspace of $Fun(\mathbb{R}_q^N)$.

Let us rewrite the formal decomposition (4.79) of $Fun(\mathbb{R}_q^N)$ by factorizing the subspaces with fixed square angular momentum $(l \cdot l)$:

$$Fun(\mathbb{R}_q^N) = \bigoplus_{k=0}^{\infty} W_k \otimes \mathbb{C}[x \cdot x]. \tag{5.64}$$

Here $\mathbb{C}(t)$ denotes the algebra of formal power series with complex coefficients in one variable t. Decomposition (5.64) induces the decomposition of any subspace $\mathcal{V} \subset Fun(\mathbb{R}_q^N)$ in its components with fixed square angular momentum:

$$\mathcal{V} = \bigoplus_{k=0}^{\infty} \mathcal{V}_k \qquad\qquad \mathcal{V}_k := \mathcal{V} \cap (W_k \otimes \mathbb{C}[x \cdot x]). \tag{5.65}$$

Since the angular momentum components l^{ij} defined in subsection 4.1.1 can also be written as $l^{ij} = -q^{-2}\mathcal{P}_A{}^{ij}_{hk}\partial^h x^k \Lambda^{-1}$, it is clear that $l^{ij}f|$ is a total derivative $\forall f \in Fun(\mathbb{R}_q^N)$, so that formally $\int d_qV\ l^{ij}f| = 0$, because of Stoke's theorem. The same holds if we apply functions of l^{ij} to f, e.g. $l \cdot l$ or the L angular momentum components introduced in subsection 4.1.2. Moreover, we have shown in sections 4.2 (Propositions 6,14,15) that there exist closed commutation relations between the generators l^{ij}, B (or $\mathbf{L}, \mathbf{k}$) and the x's. The expected consequence of this is the content of Proposition 18, namely that when representing U_q^N as an algebra of differential operators on $\mathbb{R}_q^N$ (for real q) hermitean conjugation of these operators will amount to complex conjugation.

By introducing in $\mathcal{V}$ the (formal) scalar product

$$(f,g) := \int d_qV\ f^* g, \qquad\qquad f,g \in \mathcal{V}, \tag{5.66}$$

and by applying the previous observation to the differential operator $l \cdot l$ we find that

$$\mathcal{V}_k \perp \mathcal{V}_h \qquad\qquad if \quad h \neq k. \tag{5.67}$$

The immediate consequence of this is that the question of the positivity of the inner product (5.66) (equivalently of the q-integration) is reduced to the question of its positivity within each subspace $\mathcal{V}_k$. The answer to this question is affirmative and can be given in two steps. The first step concerns positivity of the scalar product within each W_k, equivalently positivity of " q-integration over q-angular variables " (only). The second step concerns positivity of the scalar product when varying (only) the radial dependence of the functions belonging to $\mathcal{V}_k$, equivalently positivity of " q-integration over the radial direction ".

According to the results of Section 4.3, any $f_k \in \mathcal{V}_k$ can be written in the form

$$f_k = D_{l_1 l_2 \ldots l_k}\mathcal{P}_{k,S}{}^{l_1 l_2 \ldots l_k}_{i_1 i_2 \ldots i_k} x^{i_1} \ldots x^{i_k} f_{k,r}(x \cdot x), \qquad\qquad f_{k,r}(t) \in \mathbb{C}[t]. \tag{5.68}$$

Then the norm of f_k according to the definition (5.66) and formula (5.43) will be given by

$$(f_k, f_k) = \frac{||D_{k.S}||^2}{S_{2k}} \int d_qV\ (x \cdot x)^k |f_{k,r}(x \cdot x)|^2, \tag{5.69}$$

where

$$||D_{k,S}||^2 := D^*_{p_1 p_2 \ldots p_k}\mathcal{P}_{k,S}{}^{p_1 p_2 \ldots p_k}_{j_1 j_2 \ldots j_k} D_{l_1 l_2 \ldots l_k}\mathcal{P}_{k,S}{}^{l_1 l_2 \ldots l_k}_{i_1 i_2 \ldots i_k} C^{h_1 j_1} C^{h_2 j_2} \ldots C^{h_k j_k} S^{h_k \ldots h_1 i_1 \ldots i_k}. \tag{5.70}$$

The following Lemma is the first step, i.e it amounts to the the positivity of the " angular integration "; it implies that the inner product introduced in section 4.3 within the vector representation space of U_q^N is positive definite.

Lemma 3

$$||D_{k,S}||^2 \geq 0 \qquad \forall\{D_{l_1 l_2 \dots l_k}\} \subset \mathbb{C} \qquad\qquad ||D_{k,S}||^2 = 0 \quad \Rightarrow \quad f = 0. \qquad (5.71)$$

Proof. First, using property (5.84) of the symmetric projectors we can rewrite $||D_{k,S}||^2$ in the following way:

$$||D_{k,S}||^2 = [(\otimes^k C) \cdot D]^T_{j_1 \dots j_k} \mathcal{P}_{k,S}{}^{j_k \dots j_1}_{h_k \dots h_1} \mathcal{P}_{k,S}{}^{l_1 \dots l_k}_{i_1 \dots i_k} S^{h_k \dots h_1 i_1 \dots i_k} D_{l_1 l_2 \dots l_k}. \qquad (5.72)$$

In appendix 5.6, formula (5.92) we prove the following relation:

$$\mathcal{P}_{k,S}{}^{j_k \dots j_1}_{h_k \dots h_1} S^{h_k \dots h_1 i_1 \dots i_k} \mathcal{P}_{k,S}{}^{l_1 l_2 \dots l_k}_{i_1 i_2 \dots i_k} = \sigma_k [\mathcal{P}_{k,S} \cdot (\otimes^k C) P_k \mathcal{P}_{k,S}]^{l_1 \dots l_k}_{j_1 \dots j_k}, \qquad \sigma_k > 0; \quad (5.73)$$

P_k is the permutator on $\otimes^k \mathbb{C}^N$ defined by

$$P_k(v_1 \otimes' v_2 \otimes' \dots \otimes' v_k) = v_k \otimes' \dots \otimes' v_2 \otimes' v_1, \qquad v_i \in \mathbb{C}^N. \qquad (5.74)$$

Replacing into (5.72) and using property (5.84) again, we can rewrite $||D_S||^2$ as a sum of positive terms

$$\sum_{m_1, m_2, \dots, m_k} |[(\otimes^k C) \cdot \mathcal{P}_{k,S} \cdot D]^{m_1 m_2 \dots m_k}|^2; \qquad (5.75)$$

this expression is always ≥ 0 and is zero if and only if $\mathcal{P}_{k,S} \cdot D = 0$ (in fact $(\otimes^k C)$ is a nondegenerate matrix) $\diamondsuit$.

The second step is the positivity of the integral $\int d_q V \; (x \cdot x)^k |f_{k,r}(x \cdot x)|^2 \; \forall f_{k,r} \in \mathbb{C}[t]$, $q \in \mathbb{R}_q^+$. This is trivial if we evaluate this integral as a Jackson integral (see formula (5.58)) or in any case as some trace of the positive definite operator $(x \cdot x)^{k+\frac{N}{2}} |f_{k,r}(x \cdot x)|^2$. In these cases we therefore arrive at the

Theorem 3 *q-integration satisfies the positivity (5.15), or equivalently the inner product (5.66) is positive definite*

As in Ref. [48] we can give now the following

Definition We define the pre-Hilbert space $\tilde{\mathcal{L}}^2(\mathbb{R}_q^N)$ by

$$\tilde{\mathcal{L}}^2(\mathbb{R}_q^N) := \{f \in Fun(\mathbb{R}_q^N) \quad | \quad |f|^2 := (f, f) < \infty\}; \qquad (5.76)$$

Its completion $\mathcal{L}^2(\mathbb{R}_q^N)$ w.r.t. the norm $|f|$ will be denoted by called the " Hilbert space of square integrable functions on $\mathbb{R}_q^N$ ".

5.4 Appendix

We prove formulae (5.37),(5.38) by induction. For $n = 1$ (5.37),(5.38) are true, since $(\partial \cdot \partial)x^{i_1}x^{i_2}| = q^{2-N}\partial^{i_1}x^{i_2} = q^{2-N}C^{i_1 i_2}$, $(\bar{\partial} \cdot \bar{\partial})x^{i_1}x^{i_2}| = \bar{\partial}^{i_1}x^{i_2} = C^{i_1 i_2}$. Now assume that they are true for $n = m - 1$. Then

$$(\partial \cdot \partial)^m x^{i_1}x^{i_2}...x^{i_{2m}}| = q^{2-N}(\partial \cdot \partial)^{m-1}\partial^{i_1}x^{i_2}...x^{i_{2m}}| + q^2(\partial \cdot \partial)^{m-1}x^{i_1}(\partial \cdot \partial)x^{i_2}...x^{i_{2m}}| = \tag{5.77}$$

$$= q^{2-N}(1+q^2)(\partial \cdot \partial)^{m-1}\partial^{i_1}x^{i_2}...x^{i_{2m}}| + q^4(\partial \cdot \partial)^{m-2}x^{i_1}(\partial \cdot \partial)^2 x^{i_2}...x^{i_{2m}}| = \tag{5.78}$$

$$= .... = \tag{5.79}$$

$$= q^{2-N}m_{q^2}(\partial \cdot \partial)^{m-1}\partial^{i_1}x^{i_2}...x^{i_{2m}}| + q^{2m}x^{i_1}(\partial \cdot \partial)^m x^{i_2}...x^{i_{2m}}|. \tag{5.80}$$

The second term in the last expression is zero, since the $2m$ derivatives contained in $(\partial \cdot \partial)^m$ act on $(2m-1)$ coordinates x; using the definition (5.31) of the tensor M_{2m}, the induction hypothesis and the definition (5.33) of S_{2m} we are able to rewrite the first term as

$$q^{2-N}m_{q^2}M_{2m,j_3...j_{2m}}^{i_1 i_2...i_{2m}}(\partial \cdot \partial)^{m-1}x^{j_3...j_{2m}}| = (q^{2-N})^m m_{q^2}!M_{2m,j_3...j_{2m}}^{\;\;i_1 i_2...i_{2m}}S_{2(m-1)}^{j_3...j_{2m}} = \tag{5.81}$$

$$= (q^{2-N})^m m_{q^2}!S_{2m}^{i_1 i_2...i_{2m}}, \tag{5.82}$$

which shows that (5.37) is true also for $n = m$. In a similar way one proves (5.38).

5.5 Appendix

The relations (4.74),(4.75) defining the projectors $\mathcal{P}_{k,S}$ admit one and only one solution $\forall q \in \mathbb{R}^+$, since they do when $q = 1$, and the number of independent relations which they amount to is q-independent (in fact the dimension of the projectors $\mathcal{P}_A, \mathcal{P}_1, \mathcal{P}_S$ is). In this appendix we give two properties of the projectors $\mathcal{P}_{k,S}$.

First, $\mathcal{P}_{k,S}$ is symmetric

$$\mathcal{P}_{k,S}^T = \mathcal{P}_{k,S}; \tag{5.83}$$

this follows directly from the defining relations (5.74), after taking the total transpose and recalling that $\mathcal{P}_A, \mathcal{P}_1$ are symmetric.

Second,

Lemma 4

$$[P_k \cdot (\otimes^k C), \mathcal{P}_{k,S}] = 0, \tag{5.84}$$

where P_k is the permutator introduced in formula (4.74),(4.75).

Proof. Let

$$\mathcal{P}'_{k,S} := P_k \cdot (\otimes^k C)\mathcal{P}_{k,S} P_k \cdot (\otimes^k C) \tag{5.85}$$

By sandwiching the set of defining relations (2.20) between two copies of the matrix $P_k \cdot (\otimes^k C)$ it is staightforward to verify that this set is invariant under the replacement $\mathcal{P}_{k,S} \to \mathcal{P}'_{k,S}$. In fact: 1) from formula (2.20) it follows

$$P_k \cdot (\otimes^k C)\mathcal{P}_{i,i+1} = \mathcal{P}_{k-i,k+1-i} P_k \cdot (\otimes^k C) \qquad \mathcal{P} = \mathcal{P}_A, \mathcal{P}_1, \qquad i = 1, ..., k-1; \tag{5.86}$$

2) from $C^2 = 1$ we find that $[P_k \cdot (\otimes^k C)]^2 = 1$ as well. But the solution of the equations (4.74),(4.75) is unique, as already noticed, implying the equality $\mathcal{P}'_{k,S} = \mathcal{P}_{k,S}$, which is equivalent to the claim. $\diamondsuit$

One could show (we will deal with this point elsewhere) that $\mathcal{P}_{k,S}$ can be expressed in the form

$$\mathcal{P}_{k,S} = \pi_k(\hat{R}_{i,i+1}, \mathcal{P}^1_{i,i+1}), \qquad i = 1, 2, ..., k-1, \tag{5.87}$$

where π_k is a polynomial function in $\hat{R}_{i,i+1}, \mathcal{P}^1_{i,i+1}$. Here we give, as an example, the explicit form of $\mathcal{P}_{3,S}$ in terms of $\hat{R}_{i,i+1}, (\mathcal{P}_1)_{i,i+1}$:

$$\mathcal{P}_{3,S} = \frac{1}{3_{q^2}!}\{1 + q(\hat{R}_{12} + \hat{R}_{23}) + q^2(\hat{R}_{12}\hat{R}_{23} + \hat{R}_{23}\hat{R}_{12}) + q^3\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} + \tag{5.88}$$

$$-\frac{(q^N-1)\mu}{2(q^{N+2}-1)}[(2q^2+1-q^4)((\mathcal{P}_1)_{12}+(\mathcal{P}_1)_{23})+2q^4 Q_N((\mathcal{P}_1)_{12}(\mathcal{P}_1)_{23}+(\mathcal{P}_1)_{23}(\mathcal{P}_1)_{12}) \tag{5.89}$$

$$q^2(q+q^{-1})(\hat{R}_{12}(\mathcal{P}_1)_{23} + (\mathcal{P}_1)_{23}\hat{R}_{12} + \hat{R}_{23}(\mathcal{P}_1)_{12} + (\mathcal{P}_1)_{12}\hat{R}_{23}) + \tag{5.90}$$

$$+ (\hat{R}_{12}(\mathcal{P}_1)_{23}\hat{R}_{12} + \hat{R}_{23}(\mathcal{P}_1)_{12}\hat{R}_{23})]\}. \tag{5.91}$$

5.6 Appendix

Proposition 32

$$\mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1} S^{h_k...h_1 i_1...i_k} \mathcal{P}_{k,S}{}^{l_1 l_2...l_k}_{i_1 i_2...i_k} = \sigma_k[\mathcal{P}_{k,S} \cdot (\otimes^k C) P_k \mathcal{P}_{k,S}]^{l_1...l_k}_{j_1...j_k}, \qquad \sigma_k > 0; \tag{5.92}$$

Proof. From relation (5.37) we infer that

$$\mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1} \mathcal{P}_{k,S}{}^{l_1 l_2...l_k}_{i_1 i_2...i_k} S^{h_k...h_1 i_1...i_k}$$

$$= \sigma'_k \mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1} \mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k} (\partial \cdot \partial)^k x^{h_k}...x^{h_1} x^{i_1}...x^{i_k}|, \qquad \sigma'_k > 0. \tag{5.93}$$

Using relations (5.34) and the one $\mathcal{P}_S x\partial = \mathcal{P}_S \partial x$ we can rewrite the RHS of the latter equation in the following way:

$$\sigma'_k \mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1} \mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k} (\partial\cdot\partial)^{k-1}(q^{2-N}\partial^{h_k} + q^2 x^{h_k}(\partial\cdot\partial))x^{h_{k-1}}...x^{h_1}x^{i_1}...x^{i_k}| = \tag{5.94}$$

$$= \sigma'_k \mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1} \mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k} (\partial\cdot\partial)^{k-1}[q^{2-N}k_{q^2}x^{h_k}...x^{h_2}\partial^{h_1}x^{i_1}...x^{i_k}|+ \tag{5.95}$$

$$q^{2k}x^{h_k}...x^{h_1}(\partial\cdot\partial)x^{i_1}...x^{i_k}|]. \tag{5.96}$$

The second term in the square brackets will yield a vanishing contribution. In fact, the operator $(\partial\cdot\partial)^{k-1}$ can transform at most $(k-1)$ of the k x^{h_i} into ∂^{h_i}, and the remaining x^{h_i} can be moved to the left of all derivatives using property $\mathcal{P}_S x\partial = \mathcal{P}_S\partial x$; such an expression is zero, since it contains a number $l > k$ of derivatives acting on $x^{i_1}...x^{i_k}$. Using k times the same kind of argument we end up with

$$\mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1} \mathcal{P}_{k,S}{}^{l_1 l_2...l_k}_{i_1 i_2...i_k} S^{h_k...h_1 i_1...i_k} = \tag{5.97}$$

$$\sigma''_k \mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1} \mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k}[\partial^{h_k}...\partial^{h_2}\partial^{h_1}x^{i_1}...x^{i_k}|, \qquad \sigma''_k > 0. \tag{5.98}$$

Now let us perform the remaining derivations in the RHS of (5.99). Using relation (2.18) it becomes

$$RHS(5.99) = \sigma''_k \mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1} \mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k}[\partial^{h_k}...\partial^{h_2}(C^{h_1 i_1}x^{i_2}...x^{i_k} + qC^{h_1 l}\hat{R}^{i_1 i_2}_{lp}x^p x^{i_3}...x^{i_k} + ...)|, \tag{5.99}$$

and using relations (2.13),(4.74) it can be written in the form

$$RHS() = \sigma''_k \mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1} \mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k}[\partial^{h_k}...\partial^{h_2}k_{q^2}C^{h_1 i_1}x^{i_2}...x^{i_k}| = \tag{5.100}$$

$$= .... = \tag{5.101}$$

$$= \sigma_k \mathcal{P}_{k,S}{}^{j_k...j_1}_{h_k...h_1} \mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k} C^{h_1 i_1}C^{h_2 i_2}...C^{h_l i_k} = \tag{5.102}$$

$$= \sigma_k[\mathcal{P}_{k.S}P_k(\otimes^k C)\mathcal{P}_{k,S}]^{j_k...j_1}_{l_1...l_k}, \qquad \sigma_k > 0; \tag{5.103}$$

for the last equality we have used properties (5.83),(5.84). Relation (5.92) is thus proved. ♢

Part II

Physical applications

Chapter 6

The isotropic harmonic oscillator on $\mathbf{R}_q^N$

In this chapter we build $\forall q \in \mathbb{R}^+$ a sensible quantum mechanical model on the quantum Euclidean space $\mathbb{R}_q^N$ $(N \geq 3)$ as the simplest q-deformation of the (time-independent) classical isotropic harmonic oscillator on ordinary $\mathbb{R}^N$. Correspondingly, the symmetry group $SO(N,\mathbb{R})$ of rotations of the hamiltonian is deformed into the quantum group symmetry $SO_q(N,\mathbb{R})$. We will call this model the $Fun(SO_q(N,\mathbb{R}))$-isotropic harmonic oscillator on $\mathbb{R}_q^N$. The chapter is essentially based on our work in Refs. [10, 11, 14]; there are some improvements w.r.t. to the work of Refs. [10, 11], regarding mainly the first two sections.

We will prove that such a model satisfies the fundamental axioms of quantum mechanics. We will show that it has a lower bounded energy spectrum and that the scalar product is strictly positive for any $q \in \mathbb{R}^+$. Generalizing the classical algebraic construction, the Hilbert space of physical states will be built applying creation operators to the (unique) ground state. Observables will be defined as hermitean operators, as usual. In particular we will construct the observables hamiltonian, square angular momentum, angular momentum components, position operators, momentum operators. As in the classical case, the first two and any angular momentum component will commute with each other; when $N = 3,4$ they make up a complete set of commuting observables.

The plan of the chapter is as follows.

In section 6.1 we start from the configuration-space realization of the classical hamiltonian and select our q-deformed (unbarred) hamiltonian $h_\omega \in Diff(Fun(\mathbb{R}_q^N))$ on the basis of both physical and technical requirements; h_ω is a combination of the q-deformed laplacian and square lenght (3.34),(2.66). In section 6.2 we find (in q-configuration space) a sequence of solutions of the corresponding Schroedinger equation that has bounded (from below) energy spectrum; the sequence is generated by applying suitably defined creation/annihilation operators to the (unique) ground state. The eigenfunctions are the "q-deformed" Hermite functions. They will span the physical Hilbert space of the model(in the socalled " unbarred realization"). The physical results of this section are then formulated in abstractly. In section 6.3 we introduce the " barred realization " of these abstract results.

In the remaining sections we construct the Hilbert space structure and observables of the harmonic oscillator. The very interesting and peculiar point is now how the two realizations (the unbarred and the barred) are used together to define the scalar product. First a pre-Hilbert space structure is introduced so as to construct in the simplest possible way hermitean operators as functions of coordinates and derivatives (section 6.4). It is shown that the position/momentum operators and the hamiltonian of the harmonic oscillator are observables, i.e. hermitean operators. In Sect. 6.5 we consider the set of commuting observables consisiting of the hamiltonian, the square angular momentum and a Cartan subalgebra of $U_q^N = U_{q^{-1}}(so(N))$; the latter operators can be considered as the q-deformed version of the classical angular momentum components and square angular momentum, and the results of sections 4.1-4.3 can be applied in this case to find corresponding the spectra and eigenfuntions. With the help of these results we prove (Sect. 6.6) the positivity of the scalar product introduced in Sect. 4. This allows the completion of $\mathcal{H}$ into a Hilbert space $[\mathcal{H}]$.

q-deformed harmonic oscillators have already been treated by other authors [22] starting from a purely algebraic approach, in the sense that creation/annihilation operators with some prescribed commutations relation are postulated from the very beginning without any reference to a geometrical framework. The deformation considered there concerned the well-known hidden $su(n)$ symmetries of the harmonic oscillator hamitonians. Here and in Refs. [10, 11], on the contrary, the deformation concerns the rotation symmetry of the space itself: in other words, a geometrical framework is the starting point and creation/destruction operators are constructed out of the deformed " coordinates " and " derivatives ".

In this chapter we sistematically do a change of conventions. We use rescaled square lenght and laplacians xCx,Δ, $\bar{\Delta}$ instead of $x \cdot x, \partial \cdot \partial, \bar{\partial} \cdot \bar{\partial}$ used in the rest of the work (see formulae (2.59) for their normalization):

$$xCx := x^i x_i = (1+q^{N-2}) x{\cdot}x, \qquad \Delta := \partial^i \partial_i = (1+q^{N-2})\partial{\cdot}\partial, \qquad \bar{\Delta} := \bar{\partial}^i \bar{\partial}_i = (1+q^{N-2})\bar{\partial}{\cdot}\bar{\partial}. \tag{6.1}$$

Moreover, we introduce the following abbreviations

$$\mu := 1 + q^{2-N} \qquad\qquad \bar{\mu} := 1 + q^{N-2}. \tag{6.2}$$

In this way the correspondence between barred and unbarred formulae - which was mentioned in the remark after fromula (3.33) - is implemented by the replacements listed there (and other ones, that we are going to introduce), and we can usually avoid to write formulae in the barred realization.

6.1 Choice of the hamiltonian

We want to q-deform the ordinary isotropic harmonic oscillator on the Euclidean space $\mathbb{R}^N$; therefore, as a first task we have to fix within the differential algebra $Diff(\mathbb{R}_q^N)$ a suitable q-analogue h_ω of the classical hamiltonian $h_\omega^{cl} := \frac{1}{2}(-\Delta + \omega^2 x^i x_i)$ (x, ∂ being classical coordinates and derivatives) with characteristic constant ω.

Minimal requirements on $h_\omega \in Diff(\mathbb{R}_q^N)$ are of course that:

- 1) it should be a $Fun(SO_q(N))$-scalar (this is the meaning of the word " isotropic " in the q-deformed setting);
- 2) it should have a homogenous natural dimension $d(h_\omega) = d(\partial^2) = 2$ and should reduce to h_ω^{cl} in the limit $q \to 1$;
- 3) it should be the configuration-space realization of a hermitean operator H_ω on some Hilbert space $\mathcal{H}$ (to be defined);
- 4) the spectrum of H_ω in $\mathcal{H}$ should be bounded from below, in order that H_ω can be considered as the hamiltonian of a sensible (i.e. stable) quantum mechanical system.

According with the results of Proposition 1 in section 3.1 & remark, requirements 1),2) impose for h_ω the general form

$$h_\omega = F(B,\Lambda)\Delta + \omega^2 G(B,\Lambda) xCx, \qquad \lim_{q\to 1} F = -\frac{1}{2} = -\lim_{q\to 1} G. \tag{6.3}$$

h_ω being a scalar, it commutes with U_q^N and in particular with the square angular momentum (defined before in formula (4.5)) $l \cdot l$; this means that when realized as operators, h_ω,$l \cdot l$ can be diagonalized simultaneously. There is still a great freedom in defining h_ω; because of the yet unspecified B-dependence of h_ω and of formula (4.6), the eigenvalues of h_ω will in general depend on the ones of $l \cdot l$, contrary to what happens in the classical case; in fact in the latter case we know that the existence of an accidental $U(N)$-symmetry guarantees the $(l \cdot l)$-independence of the energy levels. Here we want to consider the simplest q-deformation of the classical model and therefore we add the requirement

- 5) F, G satisfy the condition

$$F = F(\Lambda), \quad G = G(\Lambda) \qquad \Leftrightarrow \qquad (l \cdot l) - independent\ energy\ levels. \tag{6.4}$$

Due to the remark following proposition 1 in section 3.1, we can assume without essential loss of generality $G \equiv 1$. Finally,

- 6) we would like to algebraically solve the Schroedinger equation through the introduction of $Fun(SO_q(N))$-vectors of energy creators $\vec{a}^+ := (a^{i+}) \in Diff(\mathbb{R}_q^N)$ and annihilators $\vec{a} := (a^i) \in Diff(\mathbb{R}_q^N)$, as in the classical case. More precisely, we require commutations relations of the type

$$h_\omega a^{i\pm} = a^{i\pm} f^{i\pm}(h_\omega), \qquad f^{i\pm}(t) \in \mathbb{C}[t]. \tag{6.5}$$

In fact, if a relation of the form (6.5) were satisfied, given an eigenvector ψ of h_ω, $h_\omega \psi = E\psi$, then $a^{i\pm}\psi$ would be an eigenvector of h_ω with eigenvalue $f^{i\pm}(E)$.

According to the first part of the previous requirement and Proposition 2 in section 3.2, the most general $Diff(\mathbb{R}_q^N)$-realization of a^i (or a^{i+}) will be of the form

$$a^i = g(xCx, \Delta, \omega)x^i + f(xCx, \Delta, \omega)\partial^i, \qquad d(g) = 1 = -d(f); \tag{6.6}$$

when $q = 1$ f, g should reduce to constants.

For the sake of simplicity we choose for $a^{i\pm}$ the following ansatz

$$a^{i\pm} = x^i b^{i\pm}(\Lambda)\beta^{i\pm}(h_\omega) + c^{i\pm}(\Lambda)\gamma^{i\pm}(h_\omega); \tag{6.7}$$

then, $a^{i\pm}\psi$ would have a quite manageable form

$$a^{i\pm}\psi = [x^i b^{i\pm}(\Lambda)\beta^{i\pm}(E) + c^{i\pm}(\Lambda)\gamma^{i\pm}(E)]\psi. \tag{6.8}$$

Proposition 33 *In order that the ansatz (6.7) and the commutation relations (6.5) are fulfilled it must be $F = const.$ and $f^{i\pm}$ must coincide with a solution $f^{\pm}(h_\omega)$ of the equation*

$$(qh_\omega) - f)(q^{-1}h_\omega - f) = \mu^2 q^N \omega^2; \tag{6.9}$$

consequently the annihilators/creators are of the form

$$a^{i\pm} = \Lambda^{-\frac{1}{2}}[x^i \beta^{\pm}(h_\omega) + \partial^i \gamma^{\pm}(h_\omega)] \tag{6.10}$$

where $\beta^{\pm}, \gamma^{\pm}$ satisfy the condition

$$\frac{\beta^{\pm}}{\gamma^{\pm}} = \frac{q^{-N}}{\mu}(q^{-1}h_\omega - f^{\pm}). \tag{6.11}$$

We omit the straightforward but slightly lenghty proof. Indeed, we will use the proposition just to definitively choose hamiltonian and find ansätze for $a^{i\pm}$. In next section we will do detailed calculations according to these choices.

Fixing the constant F, our final candidate for h_ω corresponding to creators/annihilators of the form (6.7) is

$$h_\omega := \frac{1}{2}(-q^N \Delta + \omega^2 x C x). \tag{6.12}$$

It will be called " the hamiltonian in the unbarred realization ", for reasons that will become clear below. The above hamiltonian is not real, so that the fulfilment of requirement 3) seems endangered. Using formula (3.39), (3.59), (3.62) one can immediately check that the unique real hamiltonian of the form (6.3) with $G = 1$ is $h'_\omega := (-a\Delta\Lambda^{-1} + \omega^2 x C x)$. Then, the search for corresponding annihilators/creators could only be possible by choosing a more general ansatz than (6.7), what would create many technical complications for the solution of the Schroedinger equation.

As a matter of fact, requirement 3) is not satisfied only if we adopt the standard approach to configuration-space realization of quantum mechanics. However, we are going to propose a *non − standard* one based on a *pair* of " conjugated " configuration-space realizations (the " unbarred " and the " barred "); the very interesting thing is that these two realizations are put together in the configuration-space realization of the scalar product in a very peculiar way. We will come back to this point in section 6.4. For the moment we pick up definition (6.12) and investigate its consequences. It remains to fulfil requirements 3), 4). We will impose them at the representation-theoretic level in next sections.

6.2 The Schroedinger equation in the unbarred realization

6.2.1 Solving the equation

From formulae (3.54) we get for any $\alpha \in \mathbb{C}$

$$2h_\omega(x^i + \alpha\partial^i)\Lambda^{-\frac{1}{2}} =$$

$$= \{x^i(xCx)\omega^2 - \alpha q^N\partial^i\Delta - q^N(\mu\partial^i + q^2x^i\Delta) + \omega^2\alpha q^{-2}[\partial^i(xCx) - \mu x^i]\}\Lambda^{-\frac{1}{2}} =$$

$$= x^i\Lambda^{-\frac{1}{2}}[q2h_\omega - \mu\alpha\omega^2q^{-2}] + \partial^i[q^{-1}\alpha 2h_\omega - \mu q^N]. \tag{6.13}$$

Now assume that $\psi = \psi(x\sqrt{\omega})$ is an eigenvector of h_ω

$$h_\omega\psi| = E\psi. \tag{6.14}$$

We look for some other eigenvector ψ' of h_ω in the form $\psi' = a^{i\pm}\psi$, with $a^{i\pm}$ as in formula (6.6); let E' be the corresponding eigenvalue. Apply the LHS of equation (6.13) to ψ, and rescale the result by $\beta^\pm(E)$. We find the equation

$$2E'(x^i+\alpha\partial^i)\Lambda^{-\frac{1}{2}}\psi| = 2h_\omega(x^i+\alpha\partial^i)\Lambda^{-\frac{1}{2}}\psi| = x^i\Lambda^{-\frac{1}{2}}[q2E-\mu\alpha\omega^2q^{-2}]\psi+\partial^i\Lambda^{-\frac{1}{2}}[\alpha q^{-1}2E-\mu q^N]\psi| \tag{6.15}$$

where now $\alpha = \alpha(E) = \frac{\gamma(E)}{\beta(E)}$; namely

$$\begin{cases} 2E'\alpha = \alpha 2Eq^{-1} - \mu q^N \\ 2E' = qE - \mu\alpha\omega^2q^{-2} \end{cases} \tag{6.16}$$

This system can be solved now for E' and $\alpha(E)$, yielding

$$\begin{cases} \alpha = \frac{E(q-q^{-1})\mp\sqrt{E^2(q-q^{-1})^2+\mu\bar{\mu}\omega^2}}{\mu\omega^2q^{-2}} =: \alpha_\pm \\ E' = E_\pm := Eq^{-1} - \frac{\mu}{2\alpha_\pm}q^N = \frac{1}{2}\{E(q+q^{-1}) \pm \sqrt{E^2(q-q^{-1})^2 + \mu\bar{\mu}\omega^2}\}. \end{cases} \tag{6.17}$$

Formulae (6.10),(6.17) can be used to find by induction spectra and eigenfunctions of h_ω, starting from a known eigenfunction ψ.

Let us analyze now the resulting spectrum.

First we recall what happens in the classical case. When $q = 1$ the above formulae reduce to

$$\begin{cases} \alpha = \alpha_\pm = \mp\frac{1}{\omega} \\ E' = E_\pm = E \pm \omega; \end{cases} \tag{6.18}$$

$\alpha_\pm$ therefore correspond to energy annihilators/creators respectively. Applying first an annihilator and then a creator (or viceversa) we get the same energy. If we don't impose any other condition we have a $\mathbb{Z}$-labeled spectrum not bounded from below, what is physically unacceptable. As we know, the requirement that the space of solutions is a Hilbert space, i.e. that it is endowed with a positive definite inner product, prevents the occurrence of negative energy states, in that it requires the introduction of a ground state (the latter is defined as a state annihilated by the $x^i + \frac{\partial^i}{\omega}$). In fact, a normalizable wave-function must decrease in a sufficiently rapid way at spatial infinity; this allows integration by parts and makes the laplacian $-\Delta$ a positive definite operator on the Hilbert space (as well as the square lenght), and therefore the energy positive definite.

When $q \neq 1$ it is difficult at this stage of the analysis to impose the existence of a Hilbert structure on the space of the eigenvectors; on the contrary, it is much simpler to impose requirement 4) that the energy spectrum is bounded from below. If $\frac{E}{\omega}$ is generic we can easily check that:

- Applying first a would-be annihilator and then a would-be creator (respectively $\pm$ sign in formula ()) - or viceversa - we *wouldn't* get the same energy, therefore we would virtually find a uncountable spectrum (implying that the Hilbert space is not separable); the names " annihilator/creator " themselves would seem unappropriate compared with their ordinary meaning.

- The spectrum would not be bounded from below.

To verify the second point it suffices to apply the would-be annihilators a^i m times to ψ; let us denote by E^m the energy of the state $(a^i)^m\psi$, then

$$2(\delta E)_m := 2(E_{m+1} - E_m) = E_m(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 - \sqrt{E_m^2(q - q^{-1})^2 + \mu\bar{\mu}\omega^2} < 0 \qquad (6.19)$$

is twice the m-th energy variation. It is straightforward to see that

$$\begin{cases} E_m > 0 \quad \Rightarrow \quad 2[(\delta E)_m - (\delta E)_{m-1}] < (\delta E)_{m-1}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2 \quad \Rightarrow \quad |(\delta E)_m| > \frac{q+q^{-1}}{2}|(\delta E)_{m-1}| \\ E_m \leq 0 \quad \Rightarrow \quad |(\delta E)_m| > \omega\sqrt{\mu\bar{\mu}}. \end{cases} \qquad (6.20)$$

Therefore $\lim_{m\to\infty} E_m = -\infty$ $\forall E \in \mathbb{R}$, unless E is such that after $m < \infty$ steps we find an eigenstate ψ_0 which is annihilated by the would-be annihilator:

$$(x^i + \alpha_-(E_0)q^{-1}\partial^i)\psi_0 = 0, \qquad (6.21)$$

where

$$h_\omega\psi_0 =: E_0\psi_0. \qquad (6.22)$$

The unique solution in $Fun(\mathbb{R}_q^N)$ of the first equation is $\psi_0 = e_{q^2}[-\frac{[illegible]}{\alpha_-\mu}]$ (see equation (3.72)); replacing this expression into the second equation and performing the derivations by means of formulae (3.25),(3.72), we find $\alpha_- = \alpha_{-,0} := \frac{[illegible]}{[illegible]}$ and

$$\psi_0 = e_{q^2}[-\frac{\omega q^{-N} xCx}{\mu}], \qquad E_0 = \frac{[illegible]}{[illegible]} \tag{6.23}$$

(Q_N was defined in formula (2.16))

Now we devote our attention to the sequence of solutions that are determined from ψ_0 ($\bar{\psi}_0$) by a recursive application of the would-be creators [illegible]. The coefficient and eigenvalue obtained after n steps will be called [illegible].

Lemma 5

$$\alpha_{+,n} = -\frac{q^{2-n}}{\omega}, \quad n \geq 1; \qquad E_n = \frac{1}{2}\omega(q^{\frac{N}{2}-1} - [illegible] \qquad n \geq 0; \tag{6.24}$$

(the q-deformed integers $[n]_q$ were defined in formula [illegible]

Proof. The proof of this proposition is straightforward: [illegible] yield for E_0 the values of formulae (6.23), and it is trivial algebra to show that if the proposition is true for $n = m$ then formulae (6.17) imply that it is true also for $n = m - 1$ [illegible]

Let us consider now the action of the annihilators [illegible] on one of the states with energy $E = E_{n-1}$. Equation (6.17) yields

$$\alpha_-(E_{n-1}) = \frac{q^{n+N}}{\omega} =: \alpha_{-,n} \qquad E' = E_-(E_{n-1}) = E_{n-2}. \tag{6.25}$$

We see that no new eigenvalue is introduced in this way. [illegible] in next subsection we are going to show that *eigenfunctions* of energy E_{n-1} are mapped into eigenfunctions of energy E_{n-2}, i.e. *no new eigenfunction* is introduced by applying the a^i's. In other words, the names " annihilators/creators " for a^i, a^{i-} are appropriate.

We note that the energies are invariant under the replacement $q \to q^{-1}$. This is due to the particular choice (6.12) of the coefficients of Δ, xCx. However, a different choice of these coefficients

$$\tilde{h}_\omega := -a^2 q^N \Delta + b^2 [illegible] xCx \tag{6.26}$$

would only modify the energy levels by a global factor: $\tilde{E}_n = \frac{b}{a} E_n$. In any case the spectrum is bounded from below (by construction) and increasing with n for any $q \in \mathbb{R}^+$; energy levels are not equidistant as in the classical case ($q = 1$) and the difference between neighbouring energy levels diverges with n, so it would yield to a macroscopic energy gap for great n bothe for $q > 1$ and $0 < q < 1$. This is a difference w.r.t. other q-deformed oscillators (see e.g. [22]), where the energies are proportional to the q-deformed integers $(l)_{q^{\pm 2}}$ (see definitions (3.69)), which have a finite limit for $0 < q < 1$ (resp. $q > 1$), instead of being proportional to the q-deformed integers $[l]_q$. In next subsection we will see that the energy levels have the same degeneracy as in the classical case.

As a consequence of formulae (6.24) and the very construction of the eigenfunctions, at level n the latter are given by

$$\psi_n^{i_n i_{n-1}\dots i_1} =$$

$$= (x^{i_n} + \alpha_n \partial^{i_n})(x^{i_{n-1}} + q\alpha_{n-1}\partial^{i_{n-1}})....(x^{i_1} + q^{n-1}\alpha_1\partial^{i_1})e_{q^2}[-\frac{\omega q^{-n-N} xCx}{\mu}]| =$$

$$= (x^{i_n} - \frac{q^{2-n}}{\omega}\partial^{i_n})(x^{i_{n-1}} - \frac{q^{4-n}}{\omega}\partial^{i_{n-1}})....(x^{i_1} - \frac{q^n}{\omega}\partial^{i_1})e_{q^2}[-\frac{\omega q^{-n-N} xCx}{\mu}]| \qquad (6.27)$$

The eigenfunctions (6.27), (3.21) are the "q-deformed Hermite functions ", since they reduce to the classical Hermite functions for $q = 1$.

On the linear span of these eigenfunctions it is sensible to express the energy-level dependence of the coefficients $\alpha_{\pm,n}$ as an analytical dependence of α_ω on h_ω, as we wished (and suggested in Ref. [47]). Inverting relation (6.24) w.r.t q we get

$$q^{n+\frac{N}{2}} = KE_n + [1 + (KE_n)^2]^{\frac{1}{2}}, \qquad K := \frac{q - q^{-1}}{\omega(q^{1-\frac{N}{2}} + q^{\frac{N}{2}-1})} \qquad (6.28)$$

whence, replacing in formulae $(6.24)_a, (6.25)_b$

$$\alpha_\pm(h_\omega) = \mp q^{\mp\frac{N}{2}}(Kh_\omega + [1 + (Kh_\omega)^2])^{\pm\frac{1}{2}}) \begin{cases} q^{-2} \\ 1 \end{cases} \qquad (6.29)$$

and the commutation relations (6.5) take the rather heavy form

$$h_\omega a^{\pm i} = a^{\pm i}(2K)^{-1}\left\{q^{\pm 1}(Kh_\omega + [1 + (Kh_\omega)^2])^{\frac{1}{2}}) - q^{\mp 1}(Kh_\omega + [1 + (Kh_\omega)^2])^{-\frac{1}{2}})\right\}. \qquad (6.30)$$

6.2.2 The linear span of the q-deformed Hermite functions

In the classical case any function of the type $P(x)exp[-\frac{\omega(xCx)}{2}]$ ($P(x)$ being a polynomial) can be expressed as a combination of particular functions of this type, the well-known Hermite functions with characteristic constant ω. Moreover, any eigenfunction of the corresponding harmonic oscillator hamiltonian is a combination of Hermite functions of one (an the same) level. In this subsection we want to understand whether analogous statements hold in the q-deformed case. The answer will be essentially affirmative, but the formulation of these statements is rather unusual and slightly technical; the reader who is not interested in details can skip this subsection without relevant consequences for the general understanding.

What seems to complicate the analysis in the q-deformed case is the fact that the exponents in the exponentials of formula (6.27) have a n-dependent q-power; consequently an enormous proliferation of functions of the type *polynomial* $\cdot$ *exponential* for the same characteristic constant ω seems to take place. Nevertheless, upon iterative use of relation the q-derivative property (2.71) of the exponential we easily realize that one function of this type can be expressed in infinitely many equivalent ways,

$$P_n(x)e_{q^2}[-\frac{\omega q^{-N-m}(xCx)}{\mu}] =$$
$$= P_n(x)[1-\omega q^{-N-m-2}\frac{q^2-1}{\mu}(xCx)]e_{q^2}[-\frac{\omega q^{-N-m-2}(xCx)}{\mu}] =$$
$$\equiv P_{n+2}(x)e_{q^2}[-\frac{\omega q^{-N-m-2}(xCx)}{\mu}] = \ldots$$
$$\equiv P_{n+2h}(x)e_{q^2}[-\frac{\omega q^{-N-m-2h}(xCx)}{\mu}], \qquad h \geq 0, \tag{6.31}$$

where by $P_n(x)$ we mean a polynomial of degree n in x; so this proliferation is (to a great extent) only apparent.

To simplify the analysis we can consider polynomials containing only either even or odd powers of x, since both the enforcement of relation (3.71) and the application of the operators Δ, xCx appearing in h_ω to a function of the type *monomial* $\cdot$ *exponential* change the degree of the monomial only by ± 2. Therefore from now on

$$P_n(x) := a\ polynomial\ in\ x\ containing\ only\ powers\ of\ degree\ p = n(mod\ 2), \tag{6.32}$$

and formula (6.31) holds also with this restricted acceptation of the symbol P_n.

We introduce the following notation:

$$\Psi_n := Span_{\mathbb{C}}\{\psi_n\text{'s of formula (2.37)}\} \tag{6.33}$$

$$V_n := Span_{\mathbb{C}}\{f(x) \in Fun(\mathbb{R}_q^N) \mid \exists n \in \mathbb{N}\ s.t.\ f = P_n(x)e_{q^2}[-\frac{\omega q^{-n-N}(xCx)}{\mu}]\}. \tag{6.34}$$

$$V := \sum_{n=0}^{\infty} V_n \tag{6.35}$$

and

$$S := Span_{\mathbb{C}}\{\psi \in Fun(\mathbb{R}_q^N) \mid \psi = P(x)e_{q^2}[-\frac{\alpha(xCx)}{\mu}]\ and\ eigenfunctions\ of\ h_\omega\}, \tag{6.36}$$

then one can prove

Proposition 34

$$V_n = \tilde{\Psi}_n := \bigoplus_{0\le m\le \frac{n}{2}} \Psi_{n-2m}; \qquad dim(\Psi_n) = dim(M_n) = \binom{N+n-1}{N-1}, \qquad n \in \mathbb{N} \tag{6.37}$$

(here M_n denotes the space of homogenous polynomials of degree n, see formula (4.78)).

Corollary 2

$$V = S = \bigoplus_{n=0}^{\infty} \Psi_n. \tag{6.38}$$

Proof of the corollary The following chain of inclusion relations holds:

$$V \supset \ S \supset \bigoplus_{n=0}^{\infty} \Psi_n = \sum_{n=0}^{\infty} V_n =: V. \tag{6.39}$$

The first equality holds by proposition 23; the second inclusion relation is trivial since the eigenfunctions (6.27) are eigenfunctions of the form *polynomial* $\cdot$ *gaussian.* As for the first inclusion, note that if n is the degree of the polynomial $P(x)$ for a given element $P(x)e_{q^2}[-\frac{\alpha(xCx)}{\mu}] \in S$, then α must be given by $\alpha = \omega q^{-n-N}$ in order that ψ is an eigenvector of h_ω; in fact this is the condition which must be satisfied to annihilate the coefficient of the term of degree $n+2$ in the LHS of the eigenvalue equation $h_\omega\psi| = E\psi$ (as one can easily check from formulae (3.23) (3.72)). Since the two extrema of the chain of inclusions (6.39) coincide, the claim is proved. $\diamondsuit$

Now we come to the

Proof of the Proposition. The Proposition is trivially true for $n = 0, 1$ (see (formulae (6.23),(3.72)). The general proof is by induction. Assume that the proposition is true for $n = m-2, m-1$.

We first prove the statement

$$dim(\Psi_n) = dim(M_n) \qquad n = m, m+1 \tag{6.40}$$

which is a direct consequence of the

Lemma 6

$$\begin{pmatrix} A_{i_1...i_n} \text{ are such that} \\ A_{i_1...i_n}x^{i_1}...x^{i_n} = 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} A_{i_1...i_n} \text{ are such that} \\ A_{i_1...i_n}\psi_n^{i_1...i_n} = 0 \end{pmatrix}. \tag{6.41}$$

In other words, when evaluated as operators on $\mathcal{H}$ the creators a^{+i} satisfy the relations

$$\mathcal{P}_{A\ hk}^{\ ij} a^{+h}a^{+k} = 0 \tag{6.42}$$

i.e. generate a Euclidean quantum space. A similar relation holds for the annihilators.

To prove the $\Rightarrow$ implication in (6.41) we note that

$$A_{i_1...i_n} x^{i_1}...x^{i_n} = 0 \tag{6.43}$$

with nontrivial coefficients $A_{i_1...i_n}$ can only occurr if at least one of the relations

$$A_{i_1...i_n} = \mathcal{P}_A{}^{r_j r_{j+1}}_{i_j i_{j+1}} A_{i_1...i_{j-1} r_j r_{j+1} i_{j+2}...i_n} \qquad 1 \leq j \leq n-1 \tag{6.44}$$

is satisfied (recall that $\mathcal{P}_A{}^{ij}_{hk} x^h x^k = 0$). But if this is the case then also the expression $A_{i_1...i_n} \psi_n^{i_1...i_n}$ vanishes because of the relation

$$\mathcal{P}_A{}^{ij}_{hk} (x^h + \beta \partial^h)(x^k + \beta q^2 \partial^k) = 0 \qquad \forall \beta \in \mathbb{C} \tag{6.45}$$

(which can be easily checked on the basis of relations (2.55),(2.27),(2.31),(1.12)).

To prove the $\Leftarrow$ implication note from equations (3.72) it follows that $\forall \beta \gamma \in \mathbb{C}$

$$(x^i + \beta \partial^i)(x^j + \gamma \partial^j) e_{q^2}[\frac{\alpha x C x}{\mu}]| = a x^i x^j e_{q^2}[\frac{\alpha x C x}{\mu}] + b e_{q^2}[\frac{\alpha q^2 x C x}{\mu}] \partial^i x^j| \tag{6.46}$$

($a = cost \neq 0$) and therefore

$$\psi_n^{i_n...i_1} = (x^{i_n} - \frac{q^{2-n}}{\omega} \partial^{i_n})...(x^{i_3} - \frac{q^{n-4}}{\omega} \partial^{i_3}) \cdot$$

$$\cdot \{a x^{i_2} x^{i_1} e_{q^2}[-\frac{\omega q^{-n-N} x C x}{\mu}] + b e_{q^2}[-\frac{\omega q^{2-n-N} x C x}{\mu}] \partial^{i_2} x^{i_1}\}|; \tag{6.47}$$

the second term in the braces can only give a term of the type $P_{n-2}^{i_n...i_1} e_{q^2}[-\frac{\omega q^{2-n-N} x C x}{\mu}]$ when the operator standing to its left acts on it, therefore

$$\psi_n^{i_n...i_1} = a(x^{i_n} - \frac{q^{2-n}}{\omega} \partial^{i_n})...(x^{i_3} - \frac{q^{n-4}}{\omega} \partial^{i_3}) e_{q^2}[-\frac{\omega q^{-n-N} x C x}{\mu}] x^{i_2} x^{i_1}| + \tag{6.48}$$

$$+ P_{n-2}^{i_n...i_1}(x) e_{q^2}[-\frac{\omega q^{2-n-N} x C x}{\mu}]. \tag{6.49}$$

By applying the same argument to the first term in the RHS, and then again and again, we end up with

$$\psi_n^{i_n...i_1} = \tilde{a} x^{i_n}...x^{i_1} e_{q^2}[-\frac{\omega q^{-n-N} x C x}{\mu}] + \tag{6.50}$$

$$+ \tilde{P}_{n-2}^{i_n...i_1}(x) e_{q^2}[-\frac{\omega q^{2-n-N} x C x}{\mu}], \qquad \tilde{a} \neq 0. \tag{6.51}$$

Let us consider

$$A_{i_n...i_1} \psi_n^{i_n...i_1} = \tilde{a} A_{i_n...i_1} x^{i_n}...x^{i_1} e_{q^2}[-\frac{\omega q^{-n-N} x C x}{\mu}]| + \tag{6.52}$$

$$+ A_{i_n...i_1} \tilde{P}^{i_n...i_1}_{n-2}(x) e_{q^2}[-\frac{\omega q^{2-n-N} xCx}{\mu}]. \tag{6.53}$$

According to the induction hypothesis the second term in the RHS belongs to $\tilde{\Psi}_{n-2}$ and the first term doesn't (unless it vanishes). Consequently the vanishing of the LHS implies the vanishing of the both terms in the RHS, i.e. the $\Leftarrow$ implication of (6.41). Similarly one proves the analogous results for the anninilators. So the proof of the lemma is completed. $\diamond$

Now the proof of the claim $V_n = \Psi_n$ for $n = m, m+1$ is straightforward, since clearly $V_n \supset \tilde{\Psi}_n$ (to check this inclusion relation use formula (6.31) to reduce the exponents to $[-\frac{\omega q^{-n-N} xCx}{\mu}]$ when necessary), and

$$dim V_n = dim(space\ of\ polyn.\ P_n\ 's) = dim M_n + dim(space\ of\ polyn.\ P_{n-2}\ 's) = \tag{6.54}$$

$$= dim\Psi_n + dim V_{n-2} = dim\Psi_n + dim\tilde{\Psi}_{n-2} = dim\tilde{\Psi}_n, \qquad n = m, m+1. \tag{6.55}$$

Here the first two equalities are trivial; statement (6.40) has been used to justify the third equality, whereas the fourth and the fifth hold because of the induction hypothesis and the defintion of $\tilde{\Psi}_n$, respectively.

It remains to show that $dim M_n = \binom{N+n-1}{N-1}$ as in the case $q = 1$. In the classical case (i.e. for $q = 1$) $\binom{N+n-1}{N-1}$ is the number of sets $\{r_1, r_2, ..., r_N\}$ satisfying the condition $\sum_{i=1}^{N} r_i = n$, or, equivalently, the number of independent ordered monomials $x^{i_1}...x^{i_n}$ modulo the relations (2.55)

$$\mathcal{P}_A{}^{ij}_{hk} x^h x^k = 0 \tag{6.56}$$

where $\mathcal{P}_A{}^{ij}_{hk} = \frac{1}{2}(\delta^i_h \delta^j_k - \delta^i_k \delta^j_k)$ (for $q = 1$). The antisymmetric projector $\mathcal{P}_A$ is deformed for $q \neq 1$, but the number of relations (2.55) remains the same; consequently also the number of independent monomials. The proof of Proposition 2 is so completed $\diamondsuit$.

$\forall q \in \mathbb{R}^+$ relations (2.55) are sufficient to order any monomial according with a prescribed order relation for the indices, for instance according to increasing order; hence a basis in M_n is

$$\{x^{i_1} x^{i_2} ... x^{i_n},\ i_1 \leq i_2 \leq ... \leq i_n\} \tag{6.57}$$

and a basis in Ψ_n (because of lemma 6) is provided by

$$\{\psi^{i_1 i_2 ... i_n},\ i_1 \leq i_2 \leq ... \leq i_n\}. \tag{6.58}$$

Finally we note that V can be split into subspaces V^+, V^- of opposite parity:

$$V^+ := \bigoplus_{h=0}^{\infty} \Psi_{2h} \qquad V^- := \bigoplus_{h=0}^{\infty} \Psi_{2h+1}, \qquad \Rightarrow \qquad V = V^+ \oplus V^- \tag{6.59}$$

The main results of this section can be summarized into the abstract relations (6.30),

$$h_\omega a^{\pm i} = a^{\pm i}(2K)^{-1}\left\{q^{\pm 1}(Kh_\omega + [1+(Kh_\omega)^2])^{\frac{1}{2}}) - q^{\mp 1}(Kh_\omega + [1+(Kh_\omega)^2])^{-\frac{1}{2}})\right\}. \tag{6.60}$$

the ones

$$\mathcal{P}_{A}{}^{ij}_{hk}a^h a^k = 0 = \mathcal{P}_{A}{}^{ij}_{hk}a^{+h}a^{+k} \tag{6.61}$$

and the ground state condition

$$a^i\psi_0 = 0 \qquad\qquad h_\omega\psi_0 = E_0\psi_0 \tag{6.62}$$

(note that the eigenvalue E_0 can be easily derived imposing that ψ_0 is an eigenvector of h_ω which is annihilated by the function of h_ω which is contained in the braces at the RHS of formula ()).

6.3 Barred realization

As we noticed in section 6.1, the unbarred hamiltonian h_ω (6.12) is not real. More precisely,

$$h_\omega^* = \bar{h}_\omega, \tag{6.63}$$

where

$$\bar{h}_\omega := \frac{1}{2}(-q^{-N}\bar{\Delta} + \omega^2(xCx)); \tag{6.64}$$

$\bar{h}_\omega$ will be called the " barred " hamiltonian. The choice of the coefficients of Δ, xCx in definition (6.12) was deliberately done in such a way that $\bar{h}_\omega$ can be obtained from h_ω by the replacements $q, \partial \to q^{-1}, \bar{\partial}$ (so as to preserve the symmetry mentioned at the beginning of the chapter also in the future developments of this chapter; see also remark following equation (3.33)).

Because of this symmetry, all computations and arguments of the preceding two sections and of the rest of this chapter are invariant under the simultaneous replacements

$$\begin{array}{ccc} \mathcal{P} & & \mathcal{P} \\ q & & q^{-1} \\ \hat{R}, & & \hat{R}^{-1} \\ \partial & & \bar{\partial} \\ h_\omega & & \bar{h}_\omega \\ \psi_n & & \bar{\psi}_n \\ \alpha_{+,n} & \longleftrightarrow & \bar{\alpha}_{+,n} \\ \alpha_{-,n} & & \bar{\alpha}_{-,n} \\ a^{\pm i} & & \bar{a}^{\pm i} \\ \ldots & & \ldots \\ \mathcal{O} & & \bar{\mathcal{O}} \end{array} \tag{6.65}$$

(where $\mathcal{P} = \mathcal{P}_A, \mathcal{P}_S, \mathcal{P}_1$, $\mathcal{O}$ denotes any new composite object that we are going to introduce in the sequel, and $\bar{\mathcal{O}}$ the barred partner obtained by the above replacements, see remark following equation (3.33)) in all formulae.

In particular, the main results of the previous section listed at its end hold and take the same form (in fact, note that they were invariant under $q \to q^{-1}$). Since the energy levels (6.24) are invariant under the replacement $q \to q^{-1}$, they are also the energy levels in the unbarred realization.

Definition We will call " unbarred/barred " configuration-space realization of the harmonic oscillator the configuration-space realization in which we use only unbarred/barred objects respectively.

In the case $q = 1$ both schemes reduce to the classical ones.

As an example, we report here a few formulae in both realizations:

$$h_\omega \psi_n^{i_n i_{n-1} \ldots i_1} = E_n \psi_n^{i_n i_{n-1} \ldots i_1} \qquad \psi_n^{i_n i_{n-1} \ldots i_1} := a_n^{i_n +} a_{n-1}^{i_{n-1} +} \ldots a_1^{i_1 +} \psi_0 \tag{6.66}$$

$$\bar{h}_\omega \bar{\psi}_n^{i_n i_{n-1} \ldots i_1} = E_n \bar{\psi}_n^{i_n i_{n-1} \ldots i_1} \qquad \bar{\psi}_n^{i_n i_{n-1} \ldots i_1} := \bar{a}_n^{i_n +} \bar{a}_{n-1}^{i_{n-1} +} \ldots \bar{a}_1^{i_1 +} \bar{\psi}_0. \tag{6.67}$$

where the indices i_j, $j = 1, \ldots, n$, are the space indices. $\psi_0, \bar{\psi}_0$ denote the ground state eigenfunctions

$$\psi_0 := e_{q^2}[-\frac{q^{-N} \omega x C x}{\mu}] \qquad \bar{\psi}_0 := e_{q^{-2}}[-\frac{q^{N} \omega x C x}{\bar{\mu}}], \tag{6.68}$$

and

$$a_h^{i+} := b_h(q)(x^i - \frac{q^{2-h}}{\omega}\partial^i)\Lambda^{-\frac{1}{2}} \qquad \bar{a}_h^{i+} := b_h(q^{-1})(x^i - \frac{q^{h-2}}{\omega}\bar{\partial}^i)\Lambda^{\frac{1}{2}} \qquad i = 1, 2, \ldots, N \tag{6.69}$$

are the " creation " operators at level h in the unbarred and barred scheme respectively. In the limit $q = 1$ the eigenfunctions (6.65),(6.66) become the classical Hermite functions. The operators

$$a_h^i := d_h(q)(x^i + \frac{q^{h+N}}{\omega}\partial^i)\Lambda^{-\frac{1}{2}} \qquad \bar{a}_h^i := d_h(q^{-1})(x^i + \frac{q^{-h-N}}{\omega}\partial^i)\Lambda^{\frac{1}{2}} \tag{6.70}$$

are destruction operators (at level $h-2$), since $a_h^i\psi_{h-1}, \bar{a}_h^i\psi_{h-1}$ are eigenvectors of level $(h-2)$. At this stage we are free to fix the coefficients d_n as we wish. In section 6.4 we will fix b_n, d_n so as to build in the simplest way well-defined position/momentum observables.

In next section we will show that these two schemes can be seen as two different " realizations " of the same Hilbert space.

6.4 The Hilbert space of the harmonic oscillator and the observables R^i, P_j, H_ω

In this section we define the pre-Hilbert space of states of the harmonic oscillator on $\mathbb{R}_q^N$ and define the observables Hamiltonian, position and momentum. We first generate the space through the application of creation operators to the ground state, then endow it with a scalar product which mixes the barred and unbarred representation; the scalar product is conceived to make differential operators (such as the hamiltonian) hermitean in a straightforward way. The second part is technical and may be skipped by the reader without serious consequences for the general understanding. It starts just after formula (6.90) and deals with the determination of some coefficients which appear in the definition of the creation/destruction operators.

We introduce the pre-Hilbert space $\mathcal{H}$ of the $SO_q(N)$-isotropic harmonic oscillator with characteristic constant ω in the following way. Let $|0>$ be the ground state with the energy E_0 given in formula (6.23). We introduce a direct (ρ, V) and a barred $(\bar{\rho}, \bar{V})$ representation $(V, \bar{V})$ were defined in Sect. 2) by first assuming

$$\begin{array}{ccc} & & e_{q^2}[-\frac{\omega q^{-N}(xCx)}{\mu}] \in V \\ & \overset{\rho}{\nearrow} & \\ |0> \in \mathcal{H} & & \\ & \overset{\bar{\rho}}{\searrow} & \\ & & e_{q^{-2}}[-\frac{\omega q^{N}(xCx)}{\bar{\mu}}] \in \bar{V}. \end{array} \tag{6.71}$$

Creation and destruction operators A^{+i}, A^i are to be represented abstractly by

$$A^{\pm i} \begin{array}{l} \stackrel{\rho}{\nearrow} \quad a^{\pm i} \\[2ex] \stackrel{\bar{\rho}}{\searrow} \quad \bar{a}^{\pm i} \end{array} \tag{6.72}$$

or respectively by

$$A^{+i} \begin{array}{l} \stackrel{\rho}{\nearrow} \quad b_n(q)(x^i - \frac{q^{2-n}}{\omega}\partial^i)\Lambda^{-\frac{1}{2}} \equiv a_n^{+i} \\[2ex] \stackrel{\bar{\rho}}{\searrow} \quad b_n(q^{-1})(x^i - \frac{q^{n-2}}{\omega}\bar{\partial}^i)\Lambda^{\frac{1}{2}} \equiv \bar{a}_n^{+i} \end{array} \tag{6.73}$$

when acting on states of level $(n-1)$ (to give states of level n), and by

$$A^{i} \begin{array}{l} \stackrel{\rho}{\nearrow} \quad d_n(q)(x^i + \frac{q^{n+N}}{\omega}\partial^i)\Lambda^{-\frac{1}{2}} \equiv a_n^{i} \\[2ex] \stackrel{\bar{\rho}}{\searrow} \quad d_n(q^{-1})(x^i + \frac{q^{-n-N}}{\omega}\bar{\partial}^i)\Lambda^{\frac{1}{2}} \equiv \bar{a}_n^{i} \end{array} \tag{6.74}$$

when acting on states of level $(n-1)$ (to give states of level $(n-2)$); the coefficients b_n, d_n will be fixed below. The space $\mathcal{H}_n$ of states of level n will be introduced as linear span of the vectors

$$|i_n, i_{n-1}, ...i_1 > := A^{+i_n}A^{+i_{n-1}}...A^{+i_1}|0 > \begin{array}{l} \stackrel{\rho}{\nearrow} \quad \psi_n^{i_n i_{n-1}...i_1} \\[2ex] \stackrel{\bar{\rho}}{\searrow} \quad \bar{\psi}_n^{i_n i_{n-1}...i_1} \end{array} . \tag{6.75}$$

The vector $|i_n, ...i_1 >$ can be assigned the $SO_q(N, \mathbb{R})$ transformation law

$$\phi_l(|i_n, ...i_1 >) = T^{i_n}_{j_n}...T^{i_1}_{j_1} \otimes |j_n, ...j_1 > \tag{6.76}$$

since both $\psi_n^{i_n...i_1}$ and $\bar{\psi}_n^{i_n...i_1}$ have transformation laws of this kind. Any $|u_n > \in \mathcal{H}_n$ is an eigenvector with eigenvalue

$$E_n = \omega\frac{1}{2}(q^{\frac{N}{2}-1} + q^{1-\frac{N}{2}})[\frac{N}{2} + n]_q, \qquad n \geq 0 \tag{6.77}$$

of the hamiltonian H_ω, which is represented by

$$h_\omega = \frac{1}{2}(-q^N\Delta + \omega^2(xCx))$$

$$H_\omega \quad \overset{\rho}{\nearrow} \quad \overset{\bar{\rho}}{\searrow} \tag{6.78}$$

$$h_\omega = \frac{1}{2}(-q^{-N}\bar{\Delta} + \omega^2(xCx));$$

$\mathcal{H}$ itself is defined as

$$\mathcal{H} := \bigoplus_{n=1}^{\infty} \mathcal{H}_n. \tag{6.79}$$

By the above construction any vector $|u> \in \mathcal{H}$ will be represented both by a vector $\psi_u \in V$ and by a vector $\bar{\psi}_u \in \bar{V}$:

$$\psi_u$$

$$|u> \quad \overset{\rho}{\nearrow} \quad \overset{\bar{\rho}}{\searrow} \tag{6.80}$$

$$\bar{\psi}_u.$$

With reference to the notation of Sect. 6.2.2, we know that any function of the type $\psi_u = P_n(x)e_{q^2}[-\frac{\omega q^{-n-N-2m}xCx}{\mu}]$ belongs to V. From the above construction the corresponding $\bar{\psi}_u := \bar{\rho}\rho^{-1}\psi_u \in \bar{V}$ will be of the form $\bar{\psi}_u = \bar{P}_n(x)e_{q^{-2}}[-\frac{\omega q^{+n+N+2m}xCx}{\bar{\mu}}]$ where the polynomial $\bar{P}_n(x)$ is obtained from $P_n(x)$ by the following steps: 1) writing $P_n(x)e_{q^2}[-\frac{\omega q^{-n-N-2m}xCx}{\mu}]$ as a combinations of the ψ_m's of formula (6.27); 2) replacing ψ_m's by $\bar{\psi}_m$'s. If we consider the explicit form of $\psi_m, \bar{\psi}_m$ involving only the coordinates (without derivatives) the second step amounts to the substitutions $q \leftrightarrow q^{-1}$, $\hat{R} \leftrightarrow \hat{R}^{-1}$; in particular if the $\hat{R}, \hat{R}^{-1}$ matrices are written in terms of the projectors $\mathcal{P}_S, \mathcal{P}_A, \mathcal{P}_1$ alone, then we only need to interchange q with q^{-1}.

We define the scalar product of two vectors $|v>, |u> \in \mathcal{H}$ by the sum of two " conjugate " terms:

$$(u,v) := \int d_qV \bar{\psi}_u^* \psi_v + \int d_qV \psi_u^* \bar{\psi}_v. \tag{6.81}$$

Indeed (,) is manifestly sesquilinear and (using property (5.14))

$$(v,u)^* = (\int d_qV \bar{\psi}_v^* \psi_u)^* + \int d_qV (\psi_v^* \bar{\psi}_u)^* = \int d_qV \psi_u^* \bar{\psi}_v + \int d_qV \bar{\psi}_u^* \psi_v = (u,v) \tag{6.82}$$

as required (see relation (5.17)). Relation (6.80) implies that $(u,u) \in \mathbb{R}$; its positivity (i.e. $(u,u) \geq 0$ and $(u,u) = 0 \Leftrightarrow u = 0$) $\forall q \in \mathbb{R}^+$ will be proved in Sect. 7. Here we just note that it must hold at least in a ($|u>$-dependent) neighbourhood of q=1, as it holds for q=1 and (u,u) is a continuous function of q.

The abstract definition of the hermitean conjugate $T^\dagger$ of an operator T is the usual one

$$(u, Tv) = (T^\dagger u, v). \tag{6.83}$$

We have chosen for the scalar product the (apparently cumbersome) form (6.80) to make the operator H_ω hermitean. Let us check that this is the case. Using the notation introduced in formula (3.41),(3.42)

$$\Delta f = f'(x) + f_j(x,\partial)\partial^j =: \Delta f| + f_j(x,\partial)\partial^j, \qquad f \in Fun(\mathbb{R}_q^N) \tag{6.84}$$

and the relation $\bar{\Delta} = q^{2N}\Delta^*$ it is straightforward to show that

$$(\Delta f|)^* g := f'^* g = f^* \cdot (q^{-2N}\bar{\Delta} g|) - \partial^{j*} f_j^* g|. \tag{6.85}$$

Hence

$$\int d_qV \psi_u^* \bar{h}_\omega \bar{\psi}_v = \int d_qV \psi_u^* (\omega^2 xCx - q^{-N}\bar{\Delta})\bar{\psi}_v| \tag{6.86}$$

and

$$\int d_qV (h_\omega \psi_u)^* \bar{\psi}_v = -q^N \int d_qV (\Delta\psi_u|)^* \bar{\psi}_v + \int d_qV \omega^2 xCx \psi_u^* \bar{\psi}_v \tag{6.87}$$

$$= -q^{-N} \int d_qV \psi_u^* \bar{\Delta}\bar{\psi}_v| + \int d_qV \omega^2 xCx \psi_u^* \bar{\psi}_v + q^N \int d_qV \partial^{j*} \psi_{uj}^* \bar{\psi}_v|; \tag{6.88}$$

The last term vanishes because of Stoke's theorem (5.4),(5.5) (in fact ∂^{j*} are derivatives of $\bar{\partial}$ type), therefore $\int d_qV \psi_u^* \bar{h}_\omega \bar{\psi}_v = \int d_qV (h_\omega \psi_u)^* \bar{\psi}_v$. Similarly one proves that $\int d_qV \bar{\psi}_u^* h_\omega \psi_v = \int d_qV (\bar{h}_\omega \bar{\psi}_u)^* \psi_v$. Hence we find the

Proposition 35 *the Hamiltonian H_ω is hermitean:*

$$(u, H_\omega v) = (H_\omega u, v). \tag{6.89}$$

As an immediate consequence of the hermiticity of the hamiltonian, if $|u>, |v>$ are two eigenvectors of H_ω with different eigenvalues, then

$$(u, v) = 0. \tag{6.90}$$

Looking back at the previous proof we see that in fact a stronger property holds:

$$n \neq m \quad \Rightarrow \quad \int d_qV \bar{\psi}_n^* \psi_m = 0, \quad \int d_qV \psi_n^* \bar{\psi}_m = 0 \quad \psi_p \in \Psi_p, \;\; \bar{\psi}_p \in \bar{\psi}_p \tag{6.91}$$

(Ψ_p was defined in formula (6.33), $\bar{\Psi}_p$ is its barred partner). For the evaluation of the scalar products (,) it is only necessary to find out integrals of the type $\int d_qV(xCx)^k f(xCx)$ with

$$f = e_{q^{-2}}[-\frac{\omega q^{N+k+2m}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-N-k-2m}xCx}{\mu}], \tag{6.92}$$

since their tensor structure is already determined by the general knowledge of the tensors $S^{i_1\ldots i_{2n}}$ of Section 4.; these integrals will be determined in Appendix 6.8.

We still have to fix the coefficients b_n, d_n to complete the definitions (6.72),(6.73) of A^i, A^{+i}. We determine them imposing two requirements: 1) that creation/destruction operators are hermitean conjugate of each-other, equation (6.93); 2) that the position operators do not contain derivatives, equation (6.100). The final result is shown in equations (6.108),(6.110). The reader not interested in these computations might simply give a glance to these formulae. Before starting, we mention two relations which we will use in doing this job:

$$\int d_qV(\bar{a}^{+i}_{n+1}\bar{a}^{+j}_n\bar{\psi})^*\psi = q^{-1-N}\frac{b_{n+1}(q^{-1})b_n(q^{-1})}{d_n(q)d_{n-1}(q)}\int d_qV\bar{\psi}^*a^{j'}_{n-1}a^{i'}_n\psi C_{j'j}C_{i'i} \tag{6.93}$$

$$\int d_qV(a^{+i}_{n+1}a^{+j}_n\psi)^*\bar{\psi} = q^{1+N}\frac{b_{n+1}(q)b_n(q)}{d_n(q^{-1})d_{n-1}(q^{-1})}\int d_qV\psi^*\bar{a}^{j'}_{n-1}\bar{a}^{i'}_n\bar{\psi}C_{j'j}C_{i'i}; \tag{6.94}$$

they can be derived using Stoke's theorem (5.4) and the scaling property (5.52).

First we require that

$$(A^{+i})^\dagger = A^lC_{li} \qquad (A^i)^\dagger = A^{+l}C_{li}; \tag{6.95}$$

using the orthogonality relations (6.87) this requirement reduces to

$$(A^{+i}u_n, u_{n+1}) = (u_n, A^lu_{n+1})C_{li} \tag{6.96}$$

$$(A^iu_{n+1}, u_n) = (u_{n+1}, A^{+l}u_n)C_{li}, \qquad \forall n \geq 0, \quad \forall u_n \in \mathcal{H}_n, \quad u_{n+1} \in \mathcal{H}_{n+1}. \tag{6.97}$$

Conditions (6.94),(6.95) is equivalent to

$$\begin{cases} (A^{+i}A^{+j}u_m, u_{m+2}) = (u_m, A^kA^lu_{m+2})C_{kj}C_{li} \quad \forall m \geq 0, \\ (A^iA^ju_{m+2}, u_m) = (u_{m+2}, A^{+k}A^{+l}u_m)C_{kj}C_{li}, \qquad \forall u_m \in \mathcal{H}_m, u_{m+2} \in \mathcal{H}_{m+2} \end{cases} \tag{6.98}$$

$$(A^{+i}u_0, u_1) = (u_0, A^lu_1)C_{li} \qquad (A^iu_1, u_0) = (u_1, A^{+l}u_0)C_{li} \tag{6.99}$$

The implication [(6.94) + (6.95)] $\Rightarrow$ (6.96) is trivial. To prove the converse one we just need to express the vectors $|u_n>$ as combinations of the vectors (6.74). Writing down conditions $(6.96)_a$ explicitly in terms of b_n, d_n and using relations (6.91),(6.92), we can check that these conditions are satisfied if

$$\frac{b_{m+2}(q^{-1})b_{m+1}(q^{-1})}{q^{N+1}d_{m+1}(q)d_m(q)}\int d_qV\bar{\psi}^*_{u_m}a^{j'}_m a^{i'}_{m+1}\psi_{u_{m+2}} = \int d_qV\bar{\psi}^*_{u_m}a^{j'}_{m+2}a^{i'}_{m+3}\psi_{u_{m+2}} \qquad \forall q\in\mathbb{R}^+, \tag{6.100}$$

and similarly for the conjugate term. According to formula (6.89) we can replace in the RHS of (6.98) the function $a^{j'}_m a^{i'}_{m+1}\psi_{u_{m+2}}$ by the one $P_{\Psi_m}[a^{j'}_m a^{i'}_{m+1}\psi_{u_{m+2}}]$ (P_{Ψ_m} denotes the projector onto the space Ψ_m). Using formula (6.144) in the appendix we can check that

$$P_{\Psi_m}[a^{j'}_m a^{i'}_{m+1}\psi_{u_{m+2}}] =$$
$$= \frac{q^{m+1+N}+q^{-m-1}}{q^{m+3+N}+q^{-1-m}}\frac{q^{m+N}+q^{-m}}{q^{m+2+N}+q^{-m}}\frac{d_{m+1}(q)d_m(q)}{d_{m+3}(q)d_{m+2}(q)}a^{j'}_{m+2}a^{i'}_{m+3}\psi_{u_{m+2}} \tag{6.101}$$

Collecting this information we find that (6.98) and hence $(6.96)_a$ are satisfied if

$$\frac{b_{m+2}(q^{-1})}{d_{m+3}(q)} = q^{N+1}\frac{1+q^{2m+4+N}}{1+q^{2m+N}}\frac{d_{m+2}(q)}{b_{m+1}(q^{-1})} \qquad \forall m\geq 0, \quad \forall q\in\mathbb{R}^+. \tag{6.102}$$

As for conditions $(6.96)_b$, an explicit computation shows that they are satisfied if

$$d_2(q) = b_1(q^{-1})q^{-N-\frac{1}{2}}\frac{1+q^N}{1+q^2}\phi(q) \qquad \forall q\in\mathbb{R}^+, \tag{6.103}$$

where the constant $\phi(q)$ is defined in formula (6.167). Solving the recursive equation (6.100) by taking relation (6.101) as the initial condition, we find

$$\frac{d_{m+2}(q)}{b_{m+1}(q^{-1})} = \left[\frac{(1+q^2)q^{\frac{N}{2}}}{\phi(1+q^{2+N})}\right]^{(-1)^{m+1}} q^{\frac{-N-1}{2}}\frac{1+q^{2m+N}}{1+q^{2m+2+N}} \qquad \forall q\in\mathbb{R}^+, \quad m\geq 0. \tag{6.104}$$

Summing up, relation (6.101) guarantees that equation (6.93) is satisfied.

As a direct consequence of equation (6.93), if $f_l\in\mathbb{C}$ are numbers such that

$$C_{ml}f_l^* = f_m, \tag{6.105}$$

then the operators $f_l(A^l+A^{+l}), \frac{f^l}{i}(A^l-A^{+l})$ are hermitean operators. There exist N independent solutions f_l^i, $i=1,2,...,N$ of equations (6.103). For instance if $N=3$ we can take

$$||f_l^i|| = \frac{1}{2}\left\|\begin{matrix} 1 & 0 & q^{\frac{1}{2}} \\ 0 & 1 & 0 \\ \frac{1}{i} & 0 & iq^{\frac{1}{2}} \end{matrix}\right\|. \tag{6.106}$$

In general, ,given N solutions f_l^i of (6.103) we define

$$R^i := \frac{1}{\sqrt{\omega}}f_l^i(A^l+A^{+l}) \qquad P^i := \frac{1}{i\sqrt{\omega}}f_l^i(A^l-A^{+l}). \tag{6.107}$$

Additional conditions on the coefficients b_n, d_n arise if we impose the requirement that the operators $\rho(R^i), \bar{\rho}(R^i)$ (acting respectively in $V, \bar{V}$) contain no derivative:

$$R^i \begin{array}{l} \overset{\rho}{\nearrow} \quad g_n(q) f^i_l x^l \Lambda^{-\frac{1}{2}} \;\; on \;\; \Psi_n \\ \\ \underset{\bar{\rho}}{\searrow} \quad g_n(q^{-1}) f^i_l x^l \Lambda^{-\frac{1}{2}} \;\; on \;\; \bar{\psi}_n \end{array} \quad . \tag{6.108}$$

Equation (6.106) and the definitions (6.72),(6.73) of A^l, A^{+l} imply

$$d_{m+1}(q) = q^{-2m-N} b_{m+1}(q), \qquad g_m(q) = \frac{1+q^{-2m-N}}{\sqrt{\omega}} b_{m+1}(q). \tag{6.109}$$

Relations (6.102),(6.107) determine all the coefficients b_n, d_n, $n \geq 1$, in terms of b_1. The solutions are

$$d_n(q) = q^{2-2n-N} b_n(q), \qquad b_n(q) = \frac{(1+q^N)q^{2n-2}}{1+q^{2n-2+N}} \cdot \begin{cases} b_1(q) & if \; n \; is \; odd \\ \phi \frac{1+q^{N+2}}{1+q^2} q^{\frac{-1}{2}} b_1(q^{-1}) & if \; n \; is \; even \end{cases} . \tag{6.110}$$

Consequently

$$g_n = \begin{cases} \frac{1+q^{-N}}{\sqrt{\omega}} b_1(q) \equiv g_+ & if \; n \; is \; even \\ \frac{(1+q^N)(1+q^{-2-N})}{(1+q^2)\sqrt{\omega}} q^{\frac{3}{2}} \phi b_1(q^{-1}) \equiv g_- & if \; n \; is \; odd \end{cases} \tag{6.111}$$

We are still free to fix b_1. In the sequel we will take

$$b_1(q) = \frac{\sqrt{2\omega}}{1+q^{-N}} \qquad \Rightarrow \qquad g_+ = \sqrt{2}, \quad g_- = \phi q^{\frac{3}{2}} \frac{1+q^{-2-N}}{1+q^2} \sqrt{2}. \tag{6.112}$$

The above choice is such that $b_n(1) = \sqrt{\frac{\omega}{2}} = d_n(1)$, so that in the classical limit $q = 1$ A^{+i}, A^i give the classical creation/ annihilation operators. Let

$$\mathcal{H}^+ := \sum_{h=0}^{\infty} \mathcal{H}_{2h} \qquad \mathcal{H}^- := \sum_{h=0}^{\infty} \mathcal{H}_{2h+1} \tag{6.113}$$

We can summarize formulae (6.106),(6.109) by

$$R^i|u> \begin{array}{l} \overset{\rho}{\nearrow} \quad g_\pm(q) f^i_l x^l \Lambda^{-\frac{1}{2}} \psi_u \\ \\ \underset{\bar{\rho}}{\searrow} \quad g_\pm(q^{-1}) f^i_l x^l \Lambda^{\frac{1}{2}} \bar{\psi}_u \end{array} \qquad if \; |u> \in \mathcal{H}_\pm \quad , \tag{6.114}$$

where $\psi_u = \rho(|u>) \in V_\pm$, $\bar{\psi}_u = \bar{\rho}(|u>) \in \bar{V}_\pm$. It is natural to call the observables R^i the position operators of the system, since they reduce to the ordinary position operators when

$q=1$. Because of our requirement (6.106) $\rho(R^i)\Lambda^{\frac{1}{2}}$, $\bar{\rho}(R^i)\Lambda^{-\frac{1}{2}}$ act as pure multiplication by a combination of coordinates x^i. The classical commutation relations $[R^i,R^j]=0$ are replaced by the new ones

$$\tilde{\mathcal{P}}_{A}{}^{ij}_{hk}R^hR^k=0, \qquad \tilde{\mathcal{P}}_A:=(f\otimes' f)\mathcal{P}_A(f^{-1}\otimes' f^{-1}) \tag{6.115}$$

Up to a factor there exixts only one quadratic function of the R^i's which is a scalar and an observable, $R^2:=\frac{1}{2}R^i(f^{-1T}Cf^{-1})_{ij}R^j$ (notice that the matrix $(f^{-1T}Cf^{-1})$ is hermitean), therefore we will call it the square lenght. Since the action of R^i flips the parity of a vector, R^2 is represented in the same way on all of $\mathcal{H}$:

$$R^2 \begin{array}{l} \overset{\rho}{\nearrow}\ \phi_{\frac{q^{1+\frac{N}{2}}+q^{-1-\frac{N}{2}}}{q+q^{-1}}}\,q^{-1-\frac{N}{2}}xCx\Lambda \\ \underset{\bar{\rho}}{\searrow}\ \phi_{\frac{q^{1+\frac{N}{2}}+q^{-1-\frac{N}{2}}}{q+q^{-1}}}\,q^{1+\frac{N}{2}}xCx\Lambda^{-1} \end{array} . \tag{6.116}$$

According to the definition (6.105) the observable P^i will be represented by

$$P^i|u_n> \begin{array}{l} \overset{\rho}{\nearrow}\ \frac{1}{i\sqrt{\omega}}b_{n+1}(q)f^i_l[(q^{-2n-N}-1)x^l+2\frac{q^{1-n}}{\omega}\partial^l]\Lambda^{-\frac{1}{2}}\psi_{u_n} \\ \underset{\bar{\rho}}{\searrow}\ \frac{1}{i\sqrt{\omega}}b_{n+1}(q^{-1})f^i_l[(q^{2n+N}-1)x^l+2\frac{q^{n-1}}{\omega}\bar{\partial}^l]\Lambda^{\frac{1}{2}}\bar{\psi}_{u_n}. \end{array} \qquad if\ |u_n>\in\mathcal{H}_n \tag{6.117}$$

They will be called momentum observables,since they reduce to the ordinary momentum operators in $\mathbb{R}^N$ when $q=1$. Contrary to the classical case, from formula (6.115) we recognize that $\rho(P^i)\Lambda^{\frac{1}{2}},\bar{\rho}(P^i)\Lambda^{-\frac{1}{2}}$ are not pure derivatives. A straightforward computation shows that the classical commutation relations $[P^i,P^j]=0$ are replaced by the new ones

$$\tilde{\mathcal{P}}_{A}{}^{ij}_{hk}P^hP^k=0. \tag{6.118}$$

The reader might ask why we have not defined the position/momentum operators so that the square lenght and the square momentum be represented in V (resp. in $\bar{V}$) by $xCx, q^N\Delta$ (resp. $xCx, q^{-N}\bar{\Delta}$). The reason is that the operators $xCx, q^N\Delta$ (resp. $xCx, q^{-N}\bar{\Delta}$) do not map all of V (resp. $\bar{V}$) into itself (for instance, if ψ_0 denotes the ground state in V, then it is easy to check that $(xCx)\psi_0\notin V$). Therefore the hamiltonian (6.77) cannot be written as a combination of the square lenght and square momentum.

6.5 The angular momentum observables

To proceed in the study of the model we look for other hermitean operators such that they commute with the hamiltonian H_ω and with each other (for $N = 3, 4$ we will actually find complete sets of commuting observables).

Since h_ω is a scalar, then $[U_q^N, h_\omega] = 0$. Recalling the results of Chapter, we can add as new commuting observables $l{\cdot}l$ and the basis $\mathbf{k}^i$ $i = 1, 2, ..., n$ of the Cartan subalgebra $H \subset U_q^N$. This is the q-deformed analogue of adding the angular momentum components to a classical hamiltonian with central potential. Note that since the commutation relations of the generators of U_q^N with $\partial^i, \bar{\partial}^i$ are the same (see section 4.2), angular momentum operators will be realized in the same way in the barred and unbarred scheme. The hermiticity of the above observables trivially follows from their reality.

Using property $\mathcal{P}_S x\partial = \mathcal{P}_S \partial x$, it is straightforward to realize that

$$[(\mathcal{P}_1 \otimes' ... \otimes' \mathcal{P}_1 \otimes' \mathcal{P}_{n-2m,S})\psi_n]^{l_1...l_n} \propto P_S^{l_{2m+1}...l_n} p_{n,m}(xCx) e_{q^2}[-\frac{\omega q^{-n-N} xCx}{\mu}], \quad (6.119)$$

where $p_{n,m}$, $0 \leq m \leq \frac{n}{2}$ are polynomials and

$$P_S^{l_{2m+1}...l_n} := \mathcal{P}_{n-2m,S,j_{2m+1}...j_n}^{\ \ \ \ \ \ \ \ \ \ \ l_{2m+1}...l_n} x^{j_{2m+1}}...x^{j_n} \in W_{n-2m}, \quad (6.120)$$

with the notation of section 4.3; hence these functions are eigenvectors of $l{\cdot}l$ with eigenvalue l^2_{n-2m}. Using the property (6.60) ($\mathcal{P}_{A\ i,i+1}\psi_n = 0$) we conclude that $\mathbf{1}_{M_n}$ is the identity operator in $\Psi_n, \bar{\Psi}_n$ as well. Therefore

$$\Psi_n = \bigoplus_{0 \leq m \leq \frac{n}{2}} \Psi_{n,n-2m} \quad (6.121)$$

where

$$\Psi_{n,n-2m} := (\mathcal{P}_1 \otimes' ... \otimes' \mathcal{P}_1 \otimes' \mathcal{P}_{n-2m.S})\Psi_n, \quad (6.122)$$

is the eigenspace of $h_\omega, l \cdot l$ with eigenvalues E_n, l^2_{n-2m}. The same holds for the analogous combinations of $\bar{\psi}$'s.

We introduce the subspaces $\mathcal{H}_{n,n-2m} \subset \mathcal{H}$ by

$$\begin{array}{ccc} & & \Psi_{n,n-2m} \\ & \overset{\rho}{\nearrow} & \\ \mathcal{H}_{n,n-2m} & & \\ & \overset{\bar{\rho}}{\searrow} & \\ & & \bar{\psi}_{n,n-2m} \end{array} \quad (6.123)$$

We summarize the preceding results in the

Proposition 36 *$\mathcal{H}_{n,n-2m}$ ($n \geq 0$, $0 \leq m \leq \frac{n}{2}$) is an eigenspace of the operators $H_\omega, l \cdot l$ defined by (6.77),(4.5) with eigenvalues E_n, l^2_{n-2m} (see (6.76),(4.76)) respectively. Moreover*

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \bigoplus_{0 \leq m \leq \frac{n}{2}} \mathcal{H}_{n,n-2m}. \tag{6.124}$$

$\bigoplus$ is to be understood in the sense of direct sum of mutually orthogonal subspaces w.r.t. the scalar product (,).

Actually, one can easily check that a slightly stronger property holds:

$$\int d_q V \bar{\psi}^*_{n,k} \psi_{n',k'} = 0 = \int d_q V \psi^*_{n,k} \bar{\psi}_{n',k'} \qquad if \;\; (n,k) \neq (n',k'), \tag{6.125}$$

where $\psi_{p,h} \in \Psi_{p,h}$, $\bar{\psi}_{p,h} \in \bar{\psi}_{p,h}$.

When $N = 3,4$ a complete set of commuting observables of the isotropic harmonic oscillator is respectively given by $H_\omega, l \cdot l, \mathbf{k}^1$ and $H_\omega, l \cdot l, \mathbf{k}^1, \mathbf{k}^2$. Because of the decomposition (6.122), the spectrum and the eigenfunctions of the operators $l \cdot l, \mathbf{k}^1, \mathbf{k}^2$ are those of the corresponding representations analyzed in section 4.3.

6.6 Positivity of the scalar product

In this section we prove the positivity of the scalar product (,); in this way the proof that $\mathcal{H}$ is a pre-Hilbert space is concluded. Then completion of $\mathcal{H}$ can be performed in the standard way. The section is not essential for a conceptual understanding of the work and may be skipped by the reader if he/she is not interested in computations.

Proposition 37 *$\forall q \in \mathbf{R}^+$ the scalar product introduced in section 6.4 is positive definite:*

$$(u,u) \geq 0 \qquad (u,u) = 0 \;\Leftrightarrow\; u = 0, \qquad u \in \mathcal{H}. \tag{6.126}$$

Proof : The results of the preceding section imply that it is sufficient to prove positivity inside each subspace $\mathcal{H}_{n,n-2m}$. The most general $|u> \in \mathcal{H}_{n,k}$, $k = n-2m$, $0 \leq m \leq \frac{n}{2}$ is of the form

$$u \quad \begin{matrix} \overset{\rho}{\nearrow} \; D_{l_1 l_2 \ldots l_k} \psi^{l_1 l_2 \ldots l_k}_{n,(k,S)} \\ \\ \underset{\bar{\rho}}{\searrow} \; D_{l_1 l_2 \ldots l_k} \bar{\psi}^{l_1 l_2 \ldots l_k}_{n,(k,S)} \end{matrix} \qquad D_{l_1 \ldots l_k} \in \mathbf{C} \tag{6.127}$$

(see (6.125)) where

$$\psi^{l_1 l_2 \dots l_k}_{n,(k,S)} := (a^+_n C a^+_{n-1})\dots(a^+_{k+2} C a^+_{k+1}) \psi^{l_1 l_2 \dots l_k}_{k,S} \tag{6.128}$$

$$\bar{\psi}^{l_1 l_2 \dots l_k}_{n,(k,S)} := (\bar{a}^+_n C \bar{a}^+_{n-1})\dots(\bar{a}^+_{k+2} C \bar{a}^+_{k+1}) \bar{\psi}^{l_1 l_2 \dots l_k}_{k,S} \tag{6.129}$$

and

$$\psi^{l_1 l_2 \dots l_k}_{k,S} := \mathcal{P}_{k,S}\,{}^{l_1 \dots l_k}_{i_1 \dots i_k} \psi^{i_1 \dots i_k}_k = t_k(q) \mathcal{P}_{k,S}\,{}^{l_1 \dots l_k}_{i_1 \dots i_k} x^{i_1} \dots x^{i_k} e_{q^2}[-\frac{\omega q^{-k-N} x C x}{\mu}] \tag{6.130}$$

$$\bar{\psi}^{l_1 l_2 \dots l_k}_{k,S} := \mathcal{P}_{k,S}\,{}^{l_1 \dots l_k}_{i_1 \dots i_k} \bar{\psi}^{i_1 \dots i_k}_k = t_k(q^{-1}) \mathcal{P}_{k,S}\,{}^{l_1 \dots l_k}_{i_1 \dots i_k} x^{i_1} \dots x^{i_k} e_{q^{-2}}[-\frac{\omega q^{k+N} x C x}{\bar{\mu}}]. \tag{6.131}$$

Here $a_m C a_{m+1} := a^i_m C_{ij} a^j_{m+1}$ and a^{+i}_m, a^i_m are the creation/destruction operators introduced in (6.72), (6.73). A glance at formula $(6.108)_b$ and an easy calculation show that the coefficients $t_k(q)\ t_k(q^{-1})$ are positive $\forall q \in \mathbb{R}^+$. Actually, performing derivartions the sign $-$ before of the derivatives cancel against the one of the exponent of e_{q^2}, and we recall that $\mathcal{P}_S x\partial = \mathcal{P}_S \partial x$. In the rest of this section $a \propto b$ will mean $a = \sigma b,\ \ \sigma > 0$. The square norm of u is given by

$$(u, u) = D^*_{p_1 \dots p_k} D_{l_1 \dots l_k} [\int d_q V (\bar{\psi}^{p_1 \dots p_k}_{n,(k,S)})^* \psi^{l_1 \dots l_k}_{n,(k,S)} + \int d_q V (\psi^{p_1 \dots p_k}_{n,(k,S)})^* \bar{\psi}^{l_1 \dots l_k}_{n,(k,S)}]. \tag{6.132}$$

We are going to show our claim by proving that each one of the conjugate terms in the RHS of relation (6.130) is positive.

Lemma 7

$$\int d_q V (\bar{\psi}^{p_1 \dots p_k}_{n,(k,S)})^* \psi^{l_1 \dots l_k}_{n,(k,S)}, \int d_q V (\psi^{p_1 \dots p_k}_{n,(k,S)})^* \bar{\psi}^{l_1 \dots l_k}_{n,(k,S)} \propto$$

$$\mathcal{P}_{k,S}\,{}^{l_1 \dots l_k}_{i_1 \dots i_k} \mathcal{P}_{k,S}\,{}^{p_1 \dots p_k}_{j_1 \dots j_k} C^{h_1 j_1} \dots C^{h_k j_k} \cdot \frac{S^{h_k \dots h_1 i_1 \dots i_k}}{S_{2k}} \cdot \rho_k \tag{6.133}$$

where

$$\rho_k := \int d_q V e_{q^{-2}}[-\frac{\omega q^{k+N} x C x}{\bar{\mu}}] (x C x)^k e_{q^2}[-\frac{\omega q^{-k-N} x C x}{\mu}], \tag{6.134}$$

and S, S_{2k} were given in formulae (5.33),(5.45).

Proof of the Lemma. From the definition $(6.126)_a$ and formulae (6.91), (6.92),(6.108) it follows

$$\int d_q V (\bar{\psi}^{p_1 \dots p_k}_{n,(k,S)})^* \psi^{l_1 \dots l_k}_{n,(k,S)} \propto \int d_q V (\bar{\psi}^{p_1 \dots p_k}_{k,S})^* f^{l_1 \dots l_k}_{k,S} \tag{6.135}$$

where

$$f^{l_1 \dots l_k}_{k,S} := (a_k C a_{k+1}) \dots (a_{n-4} C a_{n-3})(a_{n-2} C a_{n-1}) \psi^{l_1 l_2 \dots l_k}_{n,(k,S)}. \tag{6.136}$$

Because of formula (6.89), only the component of $f_{k,S}^{l_1...l_k}$ belonging to Ψ_k contributes to the integral LHS(6.132). Looking at formula (6.145), we can decompose the operator $(a_{n-2}Ca_{n-1})$ in the following way

$$(a_{n-2}Ca_{n-1}) = \alpha_{n-1,2}(a_{n+2}^{+}Ca_{n+1}^{+}) + \beta_{n-1,2}(a_nCa_{n+1}) + \gamma_{n-1,2}(a_n^{+}Ca_{n+1}) +$$

$$+ \delta_{n-1,2}(a_{n+2}Ca_{n+1}^{+}), \tag{6.137}$$

which is appropriate to clearly display the result of its action on Ψ_n: we see that it maps $\psi_{n,(k,S)}$ into a combination of functions $\psi'_{n+2}, \psi'_n, \psi'_{n-2}$ belonging respectively to $\Psi_{n+2}, \Psi_n, \Psi_{n-2}$. Next, the operator $(a_{n-4}Ca_{n-3})$ acts on $\psi'_{n+2}, \psi'_n, \psi'_{n-2}$. For each of these three functions we choose the appropriate decomposition of $(a_{n-4}Ca_{n-3})$. Doing the same job again and again, we end up with a combination of functions belonging to $\Psi_{2n-k}, \Psi_{2n-2-k}, ...\Psi_k$. Only the component belonging to Ψ_k will contribute to the integral. It is not difficult to realize that the latter is given by

$$\mathcal{P}_{\Psi_k}(f_{k,S}^{l_1...l_k}) = \prod_{h=1}^{m} \beta_{n-2h+1,2}(a_{k+2}Ca_{k+3})...(a_nCa_{n+1})\psi_{n,(k,S)}^{l_1l_2...l_k}, \tag{6.138}$$

where $\mathcal{P}_{\Psi_k}$ denotes the projector on Ψ_k. Since all coefficients $\beta_{l,m}$ are positive for $q \in \mathbb{R}^+$, by picking the explicit definition (6.126) of $\psi_{n,(k,S)}$ we find

$$\mathcal{P}_{\Psi_k}(f_{k,S}^{l_1...l_k}) \propto (a_{k+2}Ca_{k+3})...(a_nCa_{n+1})(a_n^{+}Ca_{n-1}^{+})\psi_{n-2,(k,S)}^{l_1l_2...l_k}. \tag{6.139}$$

In the appendix 6.7it is proved that

$$(a_nCa_{n+1})(a_n^{+}Ca_{n-1}^{+})\psi_{n-2,(k,S)}^{l_1l_2...l_k} \propto \psi_{n-2,(k,S)}^{l_1l_2...l_k} \tag{6.140}$$

(see Proposition 37); hence

$$\mathcal{P}_{\Psi_k}(f_{k,S}^{l_1...l_k}) \propto (a_{k+2}Ca_{k+3})...(a_{n-2}Ca_{n-1})\psi_{n-2,(k,S)}^{l_1l_2...l_k} \tag{6.141}$$

using $m = \frac{n-k}{2}$ times the same kind of argument we conclude that

$$\mathcal{P}_{\Psi_k}(f_{k,S}^{l_1...l_k}) \propto \psi_{k,S}^{l_1l_2...l_k}. \tag{6.142}$$

From eq.'s (6.132), (6.89), (6.140), (6.126), (6.127),(5,43) it follows that

$$\int d_qV(\bar{\psi}_{n,(k,S)}^{p_1...p_k})^{*}\psi_{n,(k,S)}^{l_1...l_k} \propto \int d_qV(\bar{\psi}_{k,S}^{p_1...p_k})^{*}\psi_{k,S}^{l_1...l_k} \propto$$

$$\propto \mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k}\mathcal{P}_{k,S}{}^{p_1...p_k}_{j_1...j_k}C^{h_1j_1}...C^{h_kj_k}. \tag{6.143}$$

$$\int d_qVx^{h_k}...x^{h_1}x^{i_1}...x^{i_k}e_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}]e_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}] = RHS(7.6) \qquad (6.144)$$

Similarly one can show the claim for $\int d_qV(\psi^{p_1...p_k}_{n,(k,S)})^*\bar{\psi}^{l_1...l_k}_{n,(k,S)}$. $\diamond$

Now we can complete the proof of Proposition 36. From the preceding lemma we find that $(u,u) = \|D_{k,S}\|^2\frac{\rho_k}{S_{2k}}$; since $\rho_k > 0$, $\|D_{k,s}\|^2 > 0$, $S_{2k} > 0$ (see formulae (5.45)(6.189)), we immediately derive the thesis $\diamondsuit$.

Now we can introduce a norm $\|\ \|$ in $\mathcal{H}$ by setting

$$\|u\|^2 = (u,u). \qquad (6.145)$$

The completion $[\mathcal{H}]$ of $\mathcal{H}$ w.r.t. this norm can be performed in the standard way. It induces completions $[V]$, $[\bar{V}] \subset Fun(\mathbb{R}^N_q)$ of the spaces $V, \bar{V}$ introduced in subsection 6.4.2. It would be interesting to investigate if the latter can be characterized in an intrinsic way, e.g. by characterizing their (formal) power expansion in x^i's. This is left as a possible subject for some future work.

6.7 Appendix

In this section we give three results concerning annihilation/creation operators.

We start with a useful decomposition of $a_{n-1}Ca_n$. From the definition (6.72), (6.73) of the creation/destruction operators it immediately follows

$$\frac{a^i_n}{d_n(q)} = \frac{q^{n+N}+q^{2-n-m}}{q^{n+N+m}+q^{2-n-m}}\frac{a^i_{n+m}}{d_{n+m}(q)} + \frac{q^{n+N+m}-q^{n+N}}{q^{n+N+m}+q^{2-n-m}}\frac{a^{+i}_{n+m}}{b_{n+m}(q)} \qquad m\in\mathbb{Z}, \quad (6.146)$$

whence

$$a_{n-1}Ca_n := a^i_{n-1}C_{ij}a^j_n = \alpha_{n.m}(q)(a^+_{n+m+1}Ca^+_{n+m}) + \beta_{n.m}(q)(a_{n+m-1}Ca_{n+m}) + \quad (6.147)$$

$$+\gamma_{n.m}(q)(a^+_{n+m-1}Ca_{n+m}) + \delta_{n.m}(q)(a_{n+m+1}Ca^+_{n+m}), \qquad (6.148)$$

where $\beta_{n.m}(q)$ is positive $\forall q \in \mathbf{R}^+$.

Second, using the explicit definition of the creation/destruction operators and the Schroedinger equation for $\psi_n \in \Psi_n$ it is straightforward to show that

$$a^+_nCa_{n+1}\psi_n = \sigma B\psi_n \qquad a_{n+2}Ca^+_{n+1}\psi_n = \sigma' B\psi_n, \qquad \sigma,\sigma' > 0, \qquad (6.149)$$

where B was deined in formula (); if ψ_n is the scalar eigenfunction of level $n = 2m$, then B acts as the identity operator and therefore

$$a_n^+ C a_{n+1} \psi_n = \sigma \psi_n \qquad a_{n+2} C a_{n+1}^+ \psi_n = \sigma' \psi_n, \qquad \sigma, \sigma' > 0. \tag{6.150}$$

Third, we prove the

Proposition 38

$$(a_n C a_{n+1})(a_n^+ C a_{n-1}^+) \psi_{n-2,(k,S)}^{l_1 \ldots l_k} = v_{n-2,k} \psi_{n-2,(k,S)}^{l_1 \ldots l_k} \qquad \textit{with} \quad v_{n-2,k} > 0 \quad \forall q \in \mathbb{R}^+ \tag{6.151}$$

(the function $\psi_{n-2,(k,S)}$ $(k = n - 2m)$ *was defined in (6.126)).*

Proof. We know that equation (6.148) holds with some constant $v_{n-2,k} \in \mathbb{C}$, since both sides are eigentuntions of $h_\omega, l \cdot l$ with the same eigenvalues and have the same transformation properties under the coaction of the quantum group $SO_q(N, \mathbb{R})$. We now show that the constant $v_{n-2,k}$ is positive $\forall q \in \mathbf{R}^+$. In the sequel by $a \propto b$ we will mean $a = \sigma b$ with $\sigma > 0$. Note that $\psi_{n-2,(k,S)}^{l_1 \ldots l_k}$ can be written in the form

$$\psi_{n-2,(k,S)}^{l_1 \ldots l_k} = [c(xCx)^{m-1} + \ldots] e_{q^2}[-\frac{q^{-n-N+2} \omega x C x}{\mu}] \mathcal{P}_{k,S}{}^{l_1 \ldots l_k}_{i_1 \ldots i_k} x^{i_1} \ldots x^{i_k}, \tag{6.152}$$

where (as in the sequel) the dots in the square bracket denote lower degree powers of (xCx). The strategy will be to find out $v_{n-2,k}$ by only looking at the term of highest degree in xCx at each step of the derivation. From the definition (6.72),(6.73) of the creation/destruction operators and the definition (3.60) of the $\mathcal{B}$ operator we get

$$(a_n C a_{n+1})(a_n^+ C a_{n-1}^+) = q^{-3}[xCx + \frac{q^{2(n+1+N)}}{\omega^2} \Delta + \frac{q^{n+2N} \mu^2}{\omega(q^2-1)} \mathcal{B} + const.] \cdot \tag{6.153}$$

$$[xCx + \frac{q^{10-2n}}{\omega^2} \Delta - \frac{q^{2-n+N} \mu^2}{\omega(q^2-1)} \mathcal{B} + const.] \Lambda^{-2} d_n(q) d_{n+1}(q) b_n(q) b_{n-1}(q) \tag{6.154}$$

The Δ's in the first and second square bracket have to act respectively on functions belonging to $\Lambda^{-1}\Psi_n$ and to $\Lambda^{-2}\Psi_{n-2}$, therefore they can be respectively replaced by $(q^{-N-4}\omega^2 xCx - q^{-N-2} E_n)$ and $(q^{-N-8}\omega^2 xCx - q^{-N-4} E_{n-2})$. Hence

$$(a_n C a_{n+1})(a_n^+ C a_{n-1}^+) \propto E \cdot F, \tag{6.155}$$

where

$$E := [xCx(1 + q^{N+2(n-1)}) + \frac{q^{n+2N} \mu^2}{\omega(q^2-1)} \mathcal{B} + const.] \tag{6.156}$$

$$F := [xCx(1+q^{2(1-n)-N}) - \frac{q^{2-n+N}\mu^2}{\omega(q^2-1)}\mathcal{B} + const.]\Lambda^{-2}. \qquad (6.157)$$

From property $\mathcal{P}_S x\partial = \mathcal{P}_s \partial x$ one easily derives the identity

$$\mathcal{B}\mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k} x^{i_1}...x^{i_k}| = \frac{q^{2k}+q^{2-N}}{\mu}\mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k} x^{i_1}...x^{i_k} \qquad (6.158)$$

Using the fundamental property (3.61) of $\mathcal{B}$, formulae (6.150) and (6.154) we find

$$F\psi^{l_1...l_k}_{n-2,(k,S)} = [c(xCx)^m q^{4-4m}(1+q^{2(1-n)-N})e_{q^2}[-\frac{q^{-n-N-2}\omega xCx}{\mu}]+$$

$$-c\frac{\mu q^{4-n-2m+N}(q^{2k}+q^{2-N})}{\omega(q^2-1)}(xCx)^{m-1}e_{q^2}[-\frac{q^{-n-N}\omega xCx}{\mu}]+...]\cdot$$

$$\mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k} x^{i_1}...x^{i_k}; \qquad (6.159)$$

applying the q-derivative property (3.71) to the exponential $e_{q^2}[-\frac{q^{-n-N}\omega xCx}{\mu}]$ we find

$$F\psi^{l_1...l_k}_{n-2,(k,S)} \propto [c(xCx)^m + ...]e_{q^2}[-\frac{q^{-n-N-2}\omega xCx}{\mu}]\mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k} x^{i_1}...x^{i_k}. \qquad (6.160)$$

After similar steps one can see that the result of the action of E on $\psi_{n-2,(k,S)}$ is

$$E\cdot F\psi^{l_1...l_k}_{n-2,(k,S)} \propto e[c(xCx)^{m+1} + ...]e_{q^2}[-\frac{q^{-n-N-2}\omega xCx}{\mu}]\mathcal{P}_{k,S}{}^{l_1...l_k}_{i_1...i_k} x^{i_1}...x^{i_k} \qquad (6.161)$$

where

$$e := (1-q^{2m})(1-q^{2(n-m-1)+N}). \qquad (6.162)$$

Using again the q-derivative property 23.71), we increase by 4 the degree of the q-power in the exponent and we lower by 2 the degree of the polynomial in (xCx) contained in the square bracket, with the result that eq. (6.148) holds with $v_{n-2,k}$ given by

$$v_{n-2,k} \propto e(\frac{-\mu}{\omega(q^2-1)q^{-n-N-2}})(\frac{-\mu}{\omega(q^2-1)q^{-n-N}}). \qquad (6.163)$$

We see that $v_{n-2,k} > 0 \quad \forall q \in \mathbb{R}^+$. $\diamondsuit$

6.8 Appendix

In this appendix we first show how to evaluate integrals of the type

$$\int d_qV(xCx)^m e_{q^{-2}}[-\frac{\omega q^{N+k}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-N-k}xCx}{\mu}] \qquad (6.164)$$

taking $f_0 := e_{q^{-2}}[-\frac{\omega q^N xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-N}xCx}{\mu}]$ as reference function. The outcoming results, together with formulae (5.43), will allow the determination of all integrals involved in the scalar products of vectors of $\mathcal{H}$. As we have seen before, the evaluation of these scalar products can be performed also in a purely algebraic way using A, A^+ operators. Second, we give some results concerning the action of creation/destruction operators on functions $\psi \in V$.

Proposition 39

$$\int d_qV(xCx)^m e_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}] =$$
$$= (\frac{q^{1-\frac{N}{2}}+q^{\frac{N}{2}-1}}{\omega})^m[\frac{N}{2}+m-1]_q[\frac{N}{2}+m-2]_q...[\frac{N}{2}]_q\frac{[m-k]_q!}{[2(m-k)]_q!!}\cdot$$
$$\int d_qV e_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}]. \tag{6.165}$$

Proof. We start from

$$\int d_qV(xCx)^m e_{q^{-2}}[-\frac{\omega q^{k+N-2}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}] =$$
$$= q^{N+2m}\int d_qV(xCx)^m e_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{2-k-N}xCx}{\mu}] \tag{6.166}$$

which is a direct consequence of the scaling property (5.52) of the integrals. Using the q-derivative properties (3.71) of the exponentials to expand the functions $e_{q^{-2}}[-\frac{\omega q^{N-2+k}xCx}{\bar{\mu}}]$ and $e_{q^2}[-\frac{\omega q^{2-N-k}xCx}{\mu}]$ in the LHS/RHS respectively, we find

$$\int d_qV(xCx)^{m+1}e_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}] = (\frac{q^{1-\frac{N}{2}}+q^{\frac{N}{2}-1}}{\omega})\cdot$$
$$\cdot[\frac{N}{2}+m]_q\frac{[m-k]_q}{[2(m-k)]_q}\int d_qV(xCx)^m e_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}], \tag{6.167}$$

whnce the claim follows directly. $\diamondsuit$

Now consider the integral $\int d_qVf_0$. If $k=2l$, upon use of of the q-derivative properties (3.71) one finds

$$\int d_qVf_0 = \int d_qVe_{q^{-2}}[-\frac{\omega q^{2l+N}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-2l-N}xCx}{\mu}]\cdot$$
$$\cdot[\prod_{h=0}^{l-1}(1-q^{2(h-l)-N}\omega\frac{q^2-1}{\mu}xCx)][\prod_{h=0}^{l-1}(1-q^{2(l-h)+N}\omega\frac{q^{-2}-1}{\bar{\mu}}xCx)]. \tag{6.168}$$

Expanding the products contained in the square brackets and using formula (6.161) to evaluate all the integrals one finds for any even k

$$\int d_qVf_0 = z_k\int d_qVe_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}] \tag{6.169}$$

with a suitable constant z_k. If k is even, this formula, together with (6.161), allows to evaluate any integral (6.160) in terms of $\int d_qVf_0$ (which is taken as the normalization factor of the integral). If k is odd, by repeating the previous steps we obtain

$$\int d_qVf_0' = z_k' \int d_qV e_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}], \tag{6.170}$$

where $f_0' := e_{q^{-2}}[-\frac{\omega q^{N+1}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-N-1}xCx}{\mu}]$. Following the line suggested at the end of sect. 4, it is possible to find the constant $\phi(q)$ such that

$$\int d_qV e_{q^{-2}}[-\frac{\omega q^{1+N}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-1-N}xCx}{\mu}] =$$

$$\phi(q)\int d_qV e_{q^{-2}}[-\frac{\omega q^{N}xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-N}xCx}{\mu}] \tag{6.171}$$

and to show that it is positive $\forall q \in \mathbb{R}^+$. We don't perform here this computation, but just notice that by continuity the positivity of ϕ must hold at least in a neighbourhood of $q = 1$, since $\phi(1) = 1$. In this way all the integrals (6.160) are evaluated in terms of the normalization constant $\int d_qVf_0$. Note that from the definition (6.167) it follows $\phi(q^{-1}) = \phi(q)$.

Proposition 40

$$\rho_k := \int d_qV e_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}](xCx)^k e_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}] \; > \; 0 \qquad \forall q \in \mathbb{R}^+ \tag{6.172}$$

Proof. First we consider the case $k = 2h$. Using the scaling property (5.52) of the integral we find

$$\rho_k = a\int d_qV\bar{\psi}_0(xCx\Lambda^{-1})^k\psi_0, \qquad a > 0. \tag{6.173}$$

Looking back at formulae $(6.105)_a$, (6.114) it is easy to prove that $(xCx)\Lambda^{-1}$ can be decomposed in the following way

$$(xCx)\Lambda^{-1} = \frac{q+q^{-1}}{\omega(1+q^{-2-N})}[(a_nCa_{n+1})+(a_n^+Ca_{n+1})+(a_{n+2}Ca_{n+1}^+)+(a_{n+2}^+Ca_{n+1}^+)]\forall n \geq 0. \tag{6.174}$$

Only the component $\mathcal{P}_{\Psi_0}((xCx\Lambda^{-1})^k\psi_0)$ belonging to Ψ_0 of the function $(xCx\Lambda^{-1})^k\psi_0$ gives a nonvanishing contribution to the integral (6.169), because of property (6.89). Using the decomposition (B.26) with $n = 0, 2, ..., 2(k-1)$, properties (6.148),(6.147) we see that

$$\mathcal{P}_{\Psi_0}((xCx\Lambda^{-1})^k\psi_0) = \tau_k\psi_0, \tag{6.175}$$

where τ_k is given by a sum of positive constants ($\forall q \in \mathbb{R}^+$). This proves (6.168) in the case $k = 2l$.

If $k = 2l+1$ an analogous reduction shows that

$$\int d_qV e_{q^{-2}}[-\frac{\omega q^{k+N}xCx}{\bar{\mu}}](xCx)^k e_{q^2}[-\frac{\omega q^{-k-N}xCx}{\mu}] =$$

$$= \tau'_k \int d_qV e_{q^{-2}}[-\frac{\omega q^{1+N}xCx}{\bar{\mu}}](xCx)e_{q^2}[-\frac{\omega q^{-1-N}xCx}{\mu}], \tag{6.176}$$

where $\tau'_k 0 \quad \forall q \in \mathbb{R}^+$. formulae (6.163), (6.167) imply

$$\int d_qV e_{q^{-2}}[-\frac{\omega q^{1+N}xCx}{\bar{\mu}}](xCx)e_{q^2}[-\frac{\omega q^{-1-N}xCx}{\mu}] = \tag{6.177}$$

$$= \frac{q^{\frac{2-N}{2}} + q^{\frac{N-2}{2}}}{\omega}[\frac{N}{2}]_q\frac{1}{[2]_q}\phi(q)\int d_qV e_{q^{-2}}[-\frac{\omega q^N xCx}{\bar{\mu}}]e_{q^2}[-\frac{\omega q^{-N}xCx}{\mu}]. \tag{6.178}$$

Since $\phi(q)$ is positive $\forall q \in \mathbb{R}^+$, (6.168) is proved for any k. $\diamondsuit$

Chapter 7

The free particle on $\mathbf{R}_q^N$

In this chapter we study physically relevant representations of the q-deformed Euclidean Hopf algebra introduced in Chapter 4. To understand the underlying motivations for that, let us first recall what happens the classical (in the sense of undeformed) context.

The interest in the study of representations of the Euclidean Lie group E^N (and of its Lie algebra e^N) is based on many strong motivations of both physical and mathematical kind.

For instance, in nonrelativistic mechanics on $\mathbb{R}^N$ (the universal covering of) E^N is the symmetry group of the hamiltonian of systems of free quantum particles of arbitrary " spin ", therefore this study is relevant for describing the Hilbert spaces of states of such systems. Note that in these models the whole Hilbert spaces states could be obtained as the carrier spaces of *-representations of the group $E^N \otimes \mathbb{R}^+$, where $\mathbb{R}^+$ is the group of dilatations of the space around a fixed origin.

Another physical motivation is due for instance to the fact that E^N is also the Wick rotated section of the Poincare' group in $(N-1)+1$ dimensions, and in fact under certain restrictions the fundamental Hilbert space representations with positive mass of the latter (describing quantum relativistic particles) can be Wick-rotated into Hilbert space representations of E^N satisfying some subsidiary condition, which takes the place of the positive energy condition [8].

Having the q-deformed versions of these Lie groups/algebras at hand, it is natural to investigate their representations. Of course, in the quantum symmetry setting we should speak of representations only for the Hopf algebras of u.e.a. type, and of corepresentations for the Hopf algebras of " functions-on-the-group type "; the former should be related to the latter essentially by duality.

Incidentally, we note that also a q-deformed version of Wick-rotation in 3+1 dimensions Minkowski space is now available (it was recently proposed in Ref. [33]); Our subalgebra $\hat{u}_q(e^4)$ can be considered as the Wick rotated version of the q-deformed Poincare' algebra of Ref. [51, ?], as proved in Ref. [?]. In fact the 4-dimensional Euclidean quantum space proposed by S. Majid in Ref. [33] (having $SO_q(4) \approx SU_q(2) \otimes SU_q(2)$ as quantum group symmetry), which enters the semidirect product defining his Euclidean Hopf algebra, is nothing but the 4-dimensional case of the quantum Euclidean space introduced by [9], which we are treating in all this work. This can be easily realized since there exists a linear map from the generators of Ref. [33] to the ones x^i of the $*$-algebra $Fun(\mathbb{R}_q^N)$. This increases the interest in looking for the representations of $\hat{u}_q(e^N)$, in particular $u_q(e^4)$. As already said, however, Wick-rotated versions of the physically relevant irreps of the Poincare' algebra require some subsidiary condition, namely the analogue of the positivity of the energy component of p^i; we don't consider them here, but hope to address to them elsewhere.

Since we are interested in potential applications to quantum physics, we look only for Hilbert space $*$-representations of the q-Euclidean Hopf algebra of u.e.a. type presented in Chapter 4. One of the major motivations is the fact that we expect a discrete spectrum for all observables, including those which are functions of the momenta, and corresponding normalizable eigenvectors, because the momentum component don't commute, but q-commute. In other words we expect a sort of q-regularization of the representations. This should allow for instance to find a q-deformed version of Fourier transform which *really is* a (unitary) basis transformation from (normalizable) eigenvectors of some observables depending on the " position " to (normalizable) eigenvectors of some observables depending on the " momentum ". Here we stick in particular to fundamental (i.e. one-particle and irreducible) $*$-representations, which will be called irreps in the sequel; then only the algebraic structure of $U_q(e^N)$ (or $U_q(\bar{e}^N)$) is involved (the Hopf structure is irrelevant), in other words we actually deal with irreducible Hilbert space representations of the $*$-algebra $u_q(e^N)$. The content of the Chapter is based on our work in Ref. [15].

From the physical viewpoint, the subalgebra $\hat{u}_q(e^N) = u_q(e^N)/\{\Lambda = 1\}$ can be considered as the quantum group symmetry of the hamiltonian

$$H := \frac{(p \cdot p)}{2M}, \tag{7.1}$$

describing the (time-independent) dynamics of a free nonrelativistic particle with arbitrary

" generalized " $U_q^N = U_{q^{-1}}(so(N))$-spin on $\mathbb{R}_q^N$. Therefore we expect that all states with a given energy are obtained from each other by the action of $\hat{u}_q(e^N)$, and that, as in the classical case, different eigenspaces of the energy can be obtained from each other by the action of the dilatation operators $\Lambda^{\pm 1}$. The results of the chapter will disclose the structure of the Hilbert space of such a q-deformed physical system.

In section 1 we do an abstract study, with an approach similar to the one used in Ref. [41] for the q-Poincare' algebra. One important fact is that the irreps turn out to be of highest weight type, and they can be obtained from tensor products of a single one (the singlet, i.e. the one describing a particle with zero spin) with some representation of $U_q^N = U_{q^{-1}}(so(N))$. The spectra of all observables are discrete, in particular the spectra of squared momentum components (subsection 1.1), as expected. The corresponding eigenvectors are normalizable and make up an orthogonal basis of the Hilbert space of each irrep. In other words we have " lattice-regularized " the irrep, in the sense mentioned before. We could use this result as a regularizing device in the sense of QFT in the Euclidean formulation, and we hope to report on this subject elsewhere [17].

We then concentrate on the structure of the singlet representation. The transparency of the analysis is strongly increased by the use of the L-generators introduced in section 4.5, instead of **L** ones. A cumbersome " kinematical parity asymmetry " in the structure of the spectra of the angular momentum observables, which is an anusual feature for a " lattice " theory; of course, it disappears in the limit $q \to 1$.

In section 2 we find configuration-space realizations of the abstract singlet representation, which is essentially a new result for inhomogeneous quantum groups, as far as we know. A suggestive final comment to sections 1,2 singles out that ultimately the x^i-coordinate description of the space of physical states (heuristically, the Euclidean space itself) arises naturally from the existence of its Hilbert structure.

In section 3 we just briefly look at the limit $q \to 1$ of the representations.

We will assume that $0 < q \leq 1$; the case $q > 1$ can be treated in an analogous way, and in doing so one essentially will interchange the roles of annihilation and creation operators. Moreover, we set $h = \begin{cases} 0 & if \;\; N = 2n+1 \\ 1 & if \;\; N = 2n \end{cases}$ to allow a compact way of writing relations valid both for even and odd N. Finally, in our conventions $\mathbb{N} := \{0, 1, 2, ...\}$.

7.1 Fundamental representations of $u_q(e^N)$

As noticed before, When $q \in \mathbb{R}$, $u_q(e^N)$ is a $*$ algebra, with $*$-relations (4.30),(4.90,)(3.62). We remind that a $*$-representation Γ [44] of $*$-algebra A on a Hilbert $\mathcal{H}$ space is essentially a representation of A such that $\Gamma(a^*) = \Gamma(a)^\dagger$ ($T^\dagger$ is the adjoint of T) at least on a dense subset of the Hilbert space. The positivity of the scalar product

$$(u,u) \geq 0 \qquad\qquad (u,u) = 0 \quad \Leftrightarrow u = 0, \quad \forall u \in \mathcal{H}, \tag{7.2}$$

will be imposed *apriori* at each step of our construction, and of course will be essential in determining the structure of the representations.

As a first observation, the generator Λ commutes with U_q^N and q-commutes with p; therefore we immediately realize that a representation space of $u_q(e^N)$ is " foliated " by $\Lambda^{\pm 1}$ into representation spaces of the subalgebra $\hat{u}_q(e^N) := u_q(e^N)/\{\Lambda = 1\}$ (the algebra where we have moded out q-dilatations, leaving only q-rotations) which coincide with each other, provided we rescale the casimirs of $\hat{u}_q(e^N)$. Therefore the study is reduced to the study of the irreps of $\hat{u}_q(e^N)$.

7.1.1 Spectra and eigenspaces of the squared momentum observables

Contrary to the classical case, the momenta p^i don't commute with each-other, therefore cannot be chosen as (part of) a set of commuting observables to study the Hilbert spaces of the irreps. Among the commuting observables of a complete set characterizing an irrep we can always take (see sections 4.4, 4.5)

$$p_0, (p\cdot p)_1, ..., (p\cdot p)_{n-1}, (p\cdot p)_n; \mathbf{k}^1, ..., \mathbf{k}^n \qquad\qquad (p_0 \equiv 0 \quad if \quad N = 2n) \tag{7.3}$$

(in fact we will see that they actually make up a complete set for the " singlet " irrep). Notice that if we had taken $(p\cdot p)_0$ instead of p_0 we could not distinguish between positive and negative eigenvalues of p_0. It is easy to realize from the commutation relations of $u_q(e^N)$ that the sign of p_0 will be the same within each irrep. If we choose an irrep of $\hat{u}_q(e^N)$ with $p^0 > 0$, we can interpret it as the q-deformed " Wick-rotated " version of a massive irrep of the Poincaré algebra in $N = 2n+1$ dimensions with Wick rotated energy p^0.

We make an ansatz, assuming existence of eigenspaces of the first $n + h$ observables consisting only of normalizable eigenvectors; then we find that all other eigenspaces consist of normalizable eigenvectors, too.

Let $\mathcal{H}$ be the Hilbert base space of the considered irrep of $u_q(e^N)$. Given an eigenspace $\hat{\mathcal{H}} \in \mathcal{H}$ of $(p \cdot p)_n$ with eigenvalue M^2, $\hat{\mathcal{H}}_{\pi_n} := \Lambda^{2\pi_n+2}\hat{\mathcal{H}}$ ($\pi_n \in \mathbb{Z}$) will be an eigenspace as well, with eigenvalue $M^2 q^{2\pi_n+2}$. Here M^2 is a nonnegative constant characterizing the irrep; it has the dimension of a mass squared and is determined up to integer powers of q^2. Each $\hat{\mathcal{H}}_{\pi_n}$ will be an irrep of $\hat{u}_q(e^N)$. Finally, the irrep of $u_q(e^N)$ characterized by $M = 0$, i.e. $(p \cdot p)_n \equiv 0$, is actually an irrep of U_q^N.

Therefore, as anticipated, our problem is thus reduced to the study of the irreps of $\hat{u}_q(e^N)$.

From (2.59) we find

$$(p \cdot p)_l = p^l p^{-l} q^{\rho_l} + q^{-2}(p \cdot p)_{l-1} = p^{-l} p^l q^{\rho_l} + (p \cdot p)_{l-1} \tag{7.4}$$

Each $p^l p^{-l}$ is positive definite ($\Rightarrow (p \cdot p)_l \geq (p \cdot p)_{l-1} \geq 0 \quad \forall l$).

Assume that $|\psi > \in \mathcal{H}$ is an eigenvector of all $(p \cdot p)_i$. According to eq. (2.61), $|\psi_{l,\pm r} > := (p^{\pm l})^r |\psi >$ ($r \in \mathbb{N}, \quad l \leq n$), will also be an eigenvector of all of them; the eigenvalues of $|\psi >, |\psi_{l,\pm r} >$ will differ by an integer power of q. Let a_i be the eigenvalues of $(p \cdot p)_i$ on $|\psi >$. The norm of $|\psi_{l,-r-1} >$ will be given by

$$< \psi_{l,-r-1} | \psi_{l,-r-1} > = < \psi_{l,-r} | \psi_{l,-r} > (a_l - q^{-2r-2} a_{l-1}) \tag{7.5}$$

If $a_{l-1} \neq 0$, there must exist a r such that $(p^{-l})^{r+1} |\psi > = 0$, otherwise the above norm would get negative for large r. In other words there must exist a state which is annihilated by p^{-l}.

Let us start by setting $l = n$ in the previous argument; then we iterate it by setting $l = n-1, ..., h+1$. We infer the existence of a nonempty subspace $\mathcal{H}_{\pi_n,0} \subset \hat{\mathcal{H}}_{\pi_n}$ such that $(p \cdot p)_n = M^2 q^{2\pi_n+2}$, $p^{-l}\mathcal{H}_{\pi_n,0} = 0 \quad \forall l$. Let $\vec{\pi} := (\pi_h, \pi_{h+1}, ..., \pi_n) \in \mathbb{N}^{n-h} \times \mathbb{Z}$. We define $\mathcal{H}_{\vec{\pi}} := (p^n)^{\pi_{n-1}} ... (p^{h+1})^{\pi_h} \mathcal{H}_{\pi_n,0}$. Clearly the maps $p^{\pm l} : \mathcal{H}_{\vec{\pi}} \to \mathcal{H}_{\vec{\pi} \pm e_{l-1}}$ are invertible, the inverse being $[p^l]^{-1} = \frac{q^{\rho_l}}{(p \cdot p)_l - (p \cdot p)_{l-1}} p^{-l}$ and $[p^{-l}]^{-1} = \frac{q^{\rho_l}}{(p \cdot p)_l - q^{-2}(p \cdot p)_{l-1}} p^l$ respectively (note that the denominator is always different from zero by eq. (7.4)), as one can easily check using equations (7.4). Therefore we arrive at the proposition

Proposition 41 *$\mathcal{H}$ can be decomposed into the direct sum*

$$\mathcal{H} = \bigoplus_{\vec{\pi} \in \mathbb{N}^{n-h} \times \mathbb{Z}} \mathcal{H}_{\vec{\pi}}, \qquad \mathcal{H}_{\vec{\pi}} := \Lambda^{\pi_n} (p^n)^{\pi_{n-1}} ... (p^{h+1})^{\pi_h} \mathcal{H}_{\vec{0}} \tag{7.6}$$

of orthogonal eigenspaces $\mathcal{H}_{\vec{\pi}}$ of the observables $(p \cdot p)_i$,

$$(p \cdot p)_l \mathcal{H}_{\vec{\pi}} = M^2 q^{\sum_{k=l}^{n} 2(1+\pi_k)} \mathcal{H}_{\vec{\pi}} \qquad l = h, h+1, ..., n. \tag{7.7}$$

Remark. As expected the spectra of $(p\cdot p)_i$ are discrete; they are particularly simple, since they consist only of q-powers. Note that none of them contains the zero eigenvalue (but the latter is an accumulation point of the spectra); in particular $(p\cdot p)_n > 0$ always, i.e. " there is no state in which the nonrelativistic quantum particle is at rest ".

Assume that $|\phi>\in\mathcal{H}$ is an eigenvector of $\mathbf{k}^i$: $\mathbf{k}^i|\phi>=\lambda_i|\phi>\ \forall i=1,...,n$ It is easy to realize from eq.s (4.31),(4.83) that application of all generators of $u_q(e^N)$ to $|\phi>$ will yield eigenvectors of $\mathbf{k}^i$ with eigenvalues $\lambda_i q^{j_i}$, with fixed λ_i and $\vec{j}:=(j_1,...,j_n)\in\mathbb{Z}^n$. We can assume without loss of generality that $1\geq\lambda_i>q^2$. We will see in the sequel which further restrictions there are on the values of λ_i, j_i. Summing up, our Hilbert space will be spanned by orthonormal vectors $|\vec{\pi};\vec{j};\alpha>$ such that

$$(p\cdot p)_l|\vec{\pi};\vec{j};\alpha>=M^2 q^{\sum\limits_{k=l}^{n}2(1+\pi_k)}|\vec{\pi};\vec{j};\alpha>, \qquad \mathbf{k}^i|\vec{\pi};\vec{j};\alpha>=\lambda_i q^{j_i}|\vec{\pi};\vec{j};\alpha>\ . \tag{7.8}$$

α stands for possible further labels necessary to completely identify the vectors of a basis of $\mathcal{H}$. They occur if the set of commuting observables (7.3) is not complete on $\mathcal{H}$; they label the eigenvalues of the commuting observables which are to be added to the ones reported in formula (7.3), to get a complete set.

In the case $N=2n+1$ we will attach superscripts $\pm$ when we want to specify that we are dealing with irreps characterized by positive (resp. negative) eigenvalues of p_0. Then:

$$p_0\mathcal{H}^{\pm}_{\vec{\pi}}=\pm M[1+q^{-1}]^{\frac{1}{2}}q^{\sum\limits_{k=0}^{n}(1+\pi_k)}\mathcal{H}^{\pm}_{\vec{\pi}}. \tag{7.9}$$

In the sequel we look for a definition of the action of all the generators of $u_q(e^N)$ on these vectors which is consistent with all the algebra relations; maximal extension of the domain of these operators (in particular of the symmetric ones, in order to get self-adjoint operators in $\mathcal{H}$) is out of the scope of this work.

Proposition 42

$$p^l|\vec{\pi};\vec{j},\alpha>=M[1-q^{2(\pi_{l-1}+1)}]^{\frac{1}{2}}q^{\sum\limits_{k=l}^{n}(1+\pi_k)}|\vec{\pi}+\vec{e}_{l-1};\vec{j}+\vec{y}_l,\alpha> \tag{7.10}$$

$$p^{-l}|\vec{\pi};\vec{j},\alpha>=M[1-q^{2\pi_{l-1}}]^{\frac{1}{2}}q^{-\rho_l+\sum\limits_{k=l}^{n}(1+\pi_k)}|\vec{\pi}-\vec{e}_{l-1};\vec{j}-\vec{y}_l,\alpha>\ . \tag{7.11}$$

Here $l > h, \vec{e}_l \in \mathbb{N}^{n-h}, \vec{y}_i \in \mathbb{Z}^n$ *with* $(\vec{e}_l)^j = \delta^j_l$, $(\vec{y}_i)^j = \delta^j_i$; *whereas*

$$\begin{cases} p_0|\vec{\pi};\vec{j},\alpha> = \pm M[1+q^{-1}]^{\frac{1}{2}} q^{\sum_{k=0}^{n}(1+\pi_k)} |\vec{\pi};\vec{j},\alpha>, & if \quad N=2n+1, \quad |\vec{\pi};\vec{j},\alpha>\in\mathcal{H}^{\pm}; \\ p^{\pm 1}|\vec{\pi};\vec{j},\alpha> = M q^{\sum_{k=1}^{n}(1+\pi_k)} |\vec{\pi};\vec{j}\pm 1,\alpha>, & if \quad N=2n \end{cases} \tag{7.12}$$

(we have set all the arbitrary phase factors equal to 1).

Proof. Let us consider for instance the proof of relation (7.10). One starts from the ansatz $p^l|\vec{\pi};\vec{j},\alpha> = A|\vec{\pi}+\vec{e}_{l-1};\vec{j}+\vec{y}_l,\alpha>$, takes the norm of this vector, uses hermitean conjugation and knowledge of the eigenvalues of formula (7.7) to find $|A|^2$; the arbitrary phase factor of A is taken equal to one for convenience. ◇

Formula $(7.12)_b$ implies that the range of j_1 in the case $N=2n$ is $\mathbb{Z}$.

Let us consider now the action of the other generators of $\hat{u}_q(e^N)$. As already noticed, it is convenient to separate the action of changing $\vec{\pi}$ from that of changing $\vec{j}$ by using the operators L instead of the operators $\mathbf{L}$.

In fact, relation (4.113) implies $L : \mathcal{H}_{\vec{\pi}} \to \mathcal{H}_{\vec{\pi}}$; due to equation (7.6), if we knew the structure of any subspace $\mathcal{H}_{\vec{\pi}}$ and the way the L's operators act on it, we would be able to extend this knowledge in a straightforward way to all the other subspaces, through application to $\mathcal{H}_{\vec{\pi}}$ of powers in the momenta.

7.1.2 Structure of $\mathcal{H}_{\vec{0}}$

As a particular subspace we take $\mathcal{H}_{\vec{0}}$; next, we are going to investigate its structure. Note that the " central charges " of formula (4.117) reduce to

$$C_m|_{\mathcal{H}_{\vec{0}}} = \begin{cases} 0 & if \quad m > h+1 \\ \frac{q^{-\frac{3}{2}}}{q^{-1}-1} & if \quad N=2n+1 \;\; and \;\; m=1 \\ \frac{1}{1-q^2} & if \quad N=2n \;\; and \;\; m=2 \end{cases} \tag{7.13}$$

Correspondingly the triples $\mathcal{T}_m := L^{1-m,m}, L^{-m,m-1}, \mathrm{k}^m(\mathrm{k}^{m-1})^{-1}$ generate $U_q(su(2))$ subalgebras (since they satisfy the same commutation relations of $M^{1-m,m}, M^{-m,m-1}, \mathrm{k}^m(\mathrm{k}^{m-1})^{-1}$) when $m > h+1$, whereas in the remaining cases they generate subalgebras characterized by the following modified relations:

$$[L^{01}, L^{-1,0}]_q = q^{\frac{1}{2}}\frac{q^{-1}+(\mathrm{k}^1)^{-1}}{1-q^2} \tag{7.14}$$

if $N = 2n+1$ and

$$\begin{cases} [L^{1,2}, L^{-2,1}] = \frac{q^{-2}p^1p^1}{(1-q^2)(p\cdot p)_1}, & [L^{-1,2}, L^{-2,-1}] = \frac{q^{-2}p^{-1}p^{-1}}{(1-q^2)(p\cdot p)_1}, \\ [L^{-1,2}, L^{1,2}] = 0 & [L^{-2,1}, L^{-2,-1}] = 0, \end{cases} \tag{7.15}$$

$$[L^{\pm 1,2}, L^{-2,\mp 1}]_{q^2} = \frac{(\mathbf{k}^2)^{-1}(\mathbf{k}^1)^{\mp 1}}{1-q^2}, \tag{7.16}$$

if $N = 2n$.

As a first task we want to determine the constants λ_i involved in the definition (7.8) of the eigenvalues of the $\mathbf{k}^i$.

To each triad $\mathcal{T}_m$, $m > h+1$, we can apply the representation theory of $U_q(su(2))$. $\mathcal{H}_{\vec{0}}$ can be completely decomposed into the direct sum of the representation spaces of the irreps of each $U_q(su(2))$ triple $\mathcal{T}_m$, separatly. Therefore well-known results concerning the irreps of $U_q(su(2))$ imply that there exists $l \in \mathbb{N}$ characterizing each irrep of $\mathcal{T}_m$ such that the eigenvalues of $log_q[(\mathbf{k}^m)(\mathbf{k}^{m-1})^{-1}]$ are $-l, 1-l, ..., l$, implying $log_q(\frac{\lambda_m}{\lambda_{m-1}}) \in \mathbb{Z}$ in all $\mathcal{H}_{\vec{0}}$. We recall that these well-known results follow from the existence of both an highest and a lowest weight within each irrep of $U_q(su(2))$.

It remains to evaluate λ_1. We are going to show that this constant is not constrained by the representation theory of the algebrae (7.14),[(7.15),(7.16)].

$\mathcal{H}_{\vec{0}}$ can be completely decomposed into the direct sum of the representation spaces $\mathcal{H}^s_{\vec{0}}$ of the corresponding $*$-irreps. Let us study them.

We start from the first algebra (i.e. with odd N). It is immediate to realize that, because of the embedding mentioned in section 4.5 after Proposition 20, the casimir Ω of the algebra (7.14) is formally given by the casimir $\Omega = \Omega^1|_{\mathcal{H}_{\vec{0}}}$ of $\hat{u}_q(e^3)$, namely

$$\Omega = L^{-10}L^{01}(\mathbf{k}^1)^{\frac{1}{2}} + q^{\frac{1}{2}}\frac{(\mathbf{k}^1)^{-\frac{1}{2}} - (\mathbf{k})^{\frac{1}{2}}}{(q^2-1)(q-1)}. \tag{7.17}$$

Let $\Omega\mathcal{H}^s_{\vec{0}} = \omega^s\mathcal{H}^s_{\vec{0}}$, $\omega^s \in \mathbb{R}$, let $|\psi> \in \mathcal{H}^s_{\vec{0}}$ be an eigenvector of $\mathbf{k}^1$, $\mathbf{k}^1|\psi> = \mu^2|\psi>$, and define $|\psi_m> := (L^{01})^m|\psi>$. Then

$$<\psi_{m+1}|\psi_{m+1}> = q^{-\frac{3}{2}} <\psi_m|L^{-10}L^{01}|\psi_m> = <\psi_m|\psi_m> \{\omega^s q^{-m}\mu^{-1} + q^{\frac{1}{2}}\frac{1-\mu^{-2}q^{-2m}}{(q^2-1)(q-1)}\}. \tag{7.18}$$

Assuming $\mu > 0$, we realize that $<\psi_{m+1}|\psi_{m+1}>$ would get negative for large m unless it vanishes for some m. The latter condition means the existence of a highest weight vector $|\tau_s> \in \mathcal{H}^s_{\vec{0}}$,

$$L^{01}|\tau_s> = 0, \qquad \mathbf{k}^1|\tau_s> = \tau_s^2|\tau_s>, \tag{7.19}$$

and $\omega^s = q^{\frac{1}{2}}\frac{\tau_s^{-1}-\tau_s}{(1-q^2)(1-q)}$. If we repeat the same argument with $|\psi_{-m}> := (L^{-10})^m|\psi>$, we see that its norm keeps positive for large m, hence there exists no lowest weight vector and no restriction on the value of τ_s. The initial vector $|\psi>$ can be reconstructed from $|\tau_s>$ through $|\psi> = \alpha(L^{-10})^m|\tau_s>$ (with some $\alpha \in \mathbb{R}$); in fact each power of $L{-}10L^{01}$ can be expressed as a diagonal operator combination of $\omega^s(\mathbf{k}^1)^{-\frac{1}{2}}$ and a function of $\mathbf{k}^1$, therefore $(L^{-10})^m(L^{01})^m$ is diagonal. If it were $\mu < 0$ we would find a lowest weight vector and no highest weight, on the contrary; for the sake of brevity, in the sequel we will assume that $\mu > 0$.Since, if $N = 5, 7, ...,$ different subspaces $\mathcal{H}^s_{\vec{0}} \subset \mathcal{H}_{\vec{0}}$ are mapped into each other by the remaining L operators - as well as by the p^i - in such a way that the $\mathbf{k}^1$ eigenvalues are only rescaled by even powers of q, we infer that the constant λ_1 $(1 \geq \lambda_1 > q^2)$ involved in the $\mathbf{k}^1$ eigenvalues is characteristic of the $\hat{u}_q(e^N)$ irrep and unconstrained. One can show in general that its value is a function of the casimirs of the irrep. Note that no classical analogue of such representations with $1 > \lambda_1 > q^2$ is available

Now we consider the $N = 2n$ case. Since the expression for the casimir of the algebra (7.15)+(7.16) is not so simple, we prefer to use the Lemma of the appendix to prove the existence of highest weight vectors, in the sense that they are annihilated by $L^{\pm 1,2}$. In fact there are an infinite series, for, if $|\phi>$ is one, then $(\frac{p^{\pm 1}}{M})^l|\phi>$ is another $\forall l \in \mathbb{Z}$. For proving that the whole representation space can be reconstructed from the highest weight vectors one needs using the explicit expression for the casimir, and we omit the computations here. Reasoning as in the $N = 2n+1$ case one concludes as before that there is a unique constant λ_1 (involved in the $\mathbf{k}^1$ eigenvalue) characterizing the irrep.

Let $P^{+,N}$ be the subalgebra of $u_q(e^N)$ generated by p^l, $l > h$, and let

$$u_q^{c,N} := \begin{cases} U_q^{-,N} \otimes P^{+,N} & if \quad N = 2n+1 \\ U_q^{-,N} \otimes P^{+,N} \otimes (\mathbb{C}[p^1]) \otimes (\mathbb{C}[p^{-1}]) & if \quad N = 2n \end{cases}. \tag{7.20}$$

Theorem 4 *The subspace $\mathcal{H}_G$ of " highest weight vectors ", i.e.*

$$\mathcal{H}_G := \{|\phi>\in \mathcal{H} \ \mid \ L|\phi>= 0, \qquad p^{-l}|\phi>= 0 \qquad \forall L \in U_q^{+,N}, \ l > h\} \tag{7.21}$$

is nonempty and $\mathcal{H}_{\vec{0}} = u_q^{-,N}\mathcal{H}_G$. It is one-dimensional in the case $N = 2n+1$ and infinite dimensional in the case $N = 2n$. In the latter case a basis of $\mathcal{H}_G$ is provided by the vectors $\{(p^{\pm 1})^r|\phi>, \ \ r \in \mathbb{N}\}$, where $|\phi>$ is any nontrivial vector of $\mathcal{H}_G$. Using the results of Proposition 41 we can say that $|\phi>$ is cyclic in $\mathcal{H}$ w.r.t. the subalgebra $u_q^{c,N}$. (In the sequel by " the highest weight vector " we will mean one of these vectors, in the case $N = 2n$). The eigenvalues k^i of the operators $\mathbf{k}^i$ are of the type $k^i = q^{2l}\lambda_1$, $l \in \mathbb{Z}$, and the constant λ_1, $1 \geq \lambda_1 > q^2$, is a function of the casimirs characterizing the irrep.

Proof. We note that it is sufficient to prove the theorem in $\mathcal{H}_{\vec{0}}$, by proposition 41. The proof for $N = 3, 4$ amounts to the discussion preceding the claim. The general proof will be given in the appendix.

Remark: As already noted, there is no lowest weight vector in $\mathcal{H}_{\vec{0}}$, due to the presence of non-vanishing C_1 in the commutation relation (4.117), when N is odd (resp. C_2, when N is even). For this reason the application of $U_q^{-,N}$ to a highest weight vector generates an infinite dimensional space (the whole $\mathcal{H}_{\vec{0}}$ when $N = 2n+1$, its subspace characterized by odd/even j_1 if $N = 2n$).

In the sequel we will stick to irreps having classical analogue, namely those characterized by $\lambda = 1, q$.

For this class of irreps we can introduce a vector $\vec{w}$ such that $\mathbf{k}^i|\phi >= q^{w_i}|\phi >$. The vector $\vec{w}$ depends on the casimirs and together with the value of M completely characterizes an irrep. We will therefore attach it as a superscript to the symbol $\mathcal{H}$ and write $\mathcal{H}^{\vec{w}}$ when we want to specify that we are considering the Hilbert space base with highest weight $\vec{w}$; correspondingly, we will attach superscripts to the ket symbols: $|\vec{0}, \vec{j}, ... >^{\vec{w}}$; the highest weight vector itself will be denoted by $|\vec{0}, \vec{w} >^{\vec{w}}$ So far we have avoided an heavy notation distinguishing the abstract generators $p, L, \mathbf{k}, \mathbf{L}$ of the Euclidean algebra from their realizations as operators belonging to a particular representation of the algebra. From now on we will denote by $\Gamma^{\vec{w}}$ the Irrep with highest weight $\vec{w}$, and we will write $\Gamma^{\vec{w}}(p), \Gamma^{\vec{w}}(L), ...$ instead of $p, L, ...$ when we want to stress that we are dealing with operators on $\mathcal{H}^{\vec{w}}$.

Definition We define the singlet Irrep as the one characterized by the highest weight $\vec{w} = 0$. It will play a crucial role in the sequel, as one expects from comparison with the classical case.

It is immediate to verify that on the singlet Irreps of $\hat{u}_q(e^3)$, $\hat{u}_q(e^4)$ the casimirs Ω_1 take zero values (see formulae (4.125),(4.126)), since they annihilate the corresponding highest weight states. This is no surprise, since they vanish in the classical case as well.

Now we are going to determine possible highest weights $\vec{w}$ and construct generic (not necessarily singlet) Irrep of $\hat{u}_q(e^N)$ from tensor products. It is easy to verify that by making the tensor product of the singlet Irrep $(\Gamma^{\vec{0}}, \mathcal{H}^{\vec{0}})$ of $\hat{u}_q(e^N)$ and an Irrep $(\Gamma_o^{\vec{u}}, \mathcal{H}_o^{\vec{u}})$ of $U_q^N \equiv U_{q^{-1}}(so(N))$ with highest weight $\vec{u}$ we find a reducible Hilbert space representation of $\hat{u}_q(e^N)$ characterized by the same mass, as it occurs in the classical case. In fact, let $\tilde{\mathcal{H}}^{\vec{u}} := \mathcal{H}^{\vec{0}} \otimes \mathcal{H}_o^{\vec{u}}$ and define $\tilde{\Gamma}^{\vec{u}}$ on $\tilde{\mathcal{H}}^{\vec{u}}$ by

$$\tilde{\Gamma}^{\vec{u}} := (\Gamma^{\vec{0}} \otimes \Gamma_o^{\vec{u}}) \circ \phi, \qquad \Gamma_o^{\vec{u}}(p^i) := 0 =: \Gamma_o^{\vec{u}}((p \cdot p)_i), \tag{7.22}$$

where ϕ is the coproduct of $U_q(e^N)$. Then it is immediate to verify that $\tilde{\Gamma}^{\vec{u}}(p^i), \tilde{\Gamma}^{\vec{u}}(L^{ij}), \tilde{\Gamma}^{\vec{u}}(\mathbf{k}^i)$ satisfy the commutations of $U_q(e^N)$ and the spectra of the operators $(p \cdot p)_i$ are the same as in the singlet Irrep (i.e. the " mass " scale M characterizing the Irrep is the same); in fact, for instance, one can immediately verify that

$$[\tilde{\Gamma}^{\vec{u}}(L^{-m,m+1}), \tilde{\Gamma}^{\vec{u}}(L^{-m-1,m})]_{q^2} = q^{1+2\rho_m} \frac{1 \otimes 1 - \mathbf{k}^(\mathbf{k}^{m+1})^{-1} \otimes \mathbf{k}^m(\mathbf{k}^{m+1})^{-1}}{q - q^{-1}} + c_m \otimes 1$$

$$= \tilde{\Gamma}^{\vec{u}} \left[\frac{1 - \mathbf{k}^m(\mathbf{k}^{m+1})^{-1}}{q^2 - 1} q^{2\rho_m} + c_m \right] \qquad m \geq 1 \tag{7.23}$$

where the central charges C_m were defined in formulae (4.118),(4.119).

We would like now to single out the Irreps contained in $\tilde{\Gamma}^{\vec{u}}$. In the classical theory this can be done imposing the " wave-equations " on the tensor product space; each kind of wave equation selects the subspace corresponding to an Irrep, which consists of the tensor product of a one-dimensional representation (characterized by a vector $\vec{p}$) of the translation subalgebra and an Irrep of the little subgroup $SO(N-1) \subset SO(N)$ of the direction of $\vec{p}$ in the momentum space. An equivalent approach is to tensor this one-dimensional representation directly to the little group of $\vec{p}$. It doesn't seem either approach can be applied in the quantum case, since we don't know natural embeddings $U_q(so(N-1)) \hookrightarrow U_q(so(N))$, except when $N = 3, 4$. When $N = 3$ we have in fact a natural embedding $U(so(2)) \approx U(1) \hookrightarrow U_q(so(3))$: $U(so(2))$ is the classical subalgebra generated by $\mathbf{k}^1$. In the case $N = 4$ we have also a natural embedding, since $U_q(so(3)) \approx U_q(su(2)) \hookrightarrow U_q(su(2)) \otimes U_q(su(2)) \approx U_q(so(4))$

However, there is a natural way to find the Irreps contained in $\tilde{\Gamma}^{\vec{u}}$ for any N, due to the fact that the representation is of highest weight type; it is the usual procedure that e.g. one uses to determine the irrep decomposition of the tensor product of two irreps of the same (classical) Lie algebra. It arises from the highest weight and Hilbert space structures of the tensor product irrep.

For the sake of being explicit let us consider the case $N = 2n + 1$. The vector

$$|\vec{0}, \vec{u}>^{\vec{u}} := |\vec{0}, \vec{0}>^{\vec{0}} \otimes ||\vec{u}> \tag{7.24}$$

is clearly an highest weight vector of $u_q(e^N)$ (in the sense of theorem 1) and when we apply $U_q^{-,N}$ to it we generate the Irrep $(\Gamma^{\vec{u}}, \mathcal{H}^{\vec{u}})$; in particular $\Gamma^{\vec{u}}(L^{-1,0})|\vec{0}, \vec{u}>^{\vec{u}} \in \mathcal{H}^{\vec{u}} \subset \tilde{\mathcal{H}}^{\vec{u}}$.

Since the subspace of $\tilde{\mathcal{H}}^{\vec{u}}$ with $\vec{j} = \vec{u} - \vec{y}_1$ is two-dimensional (it is spanned by the vectors $|\vec{0}, \vec{0}>^{\vec{0}} \otimes ||\vec{u} - \vec{y}_1>$, $|\vec{0}, -\vec{y}_1>^{\vec{0}} \otimes ||\vec{u}>$), its orthogonal complement is one-dimensional and we can easily verify that it is spanned by

$$|\vec{0}, \vec{u}-\vec{y}_1>^{\vec{u}-\vec{y}_1} := [q^{-\frac{3}{2}}(u_1)_{q^{-2}}]^{\frac{1}{2}}|\vec{0}, -\vec{y}_1>^{\vec{0}} \otimes ||\vec{u}> - \left[\frac{q^{\frac{5}{2}}(1+\lambda_1 q)}{q^{-2}-1}\right]^{\frac{1}{2}} |\vec{0}, \vec{0}>^{\vec{0}} \otimes ||\vec{u}-\vec{y}_1> \tag{7.25}$$

(as usual, the tensor product scalar product is defined on the vector of a basis as the product of the scalar product of the tensor factors and is extended by linearity).

$|\vec{0}, \vec{u}-\vec{y}_1>^{\vec{u}-\vec{y}_1}$ clearly is a highest weight vector itself, because $\tilde{\Gamma}^{\vec{u}}$ is a $*$-representation, and when we apply $U_q^{-,N}$ to it we generate a different Irrep, $(\Gamma^{\vec{u}-\vec{y}_1}, \mathcal{H}^{\vec{u}-\vec{y}_1})$. It is evident that if we reiterate this procedure $2u_1$ times we find $2u_1+1$ Irreps.

The same argument can be applied in the case $N=2n$ to the highest weight vector of $(\tilde{\Gamma}^{\vec{u}}, \tilde{\mathcal{H}}^{\vec{u}})$. We conjecture that all highest weights (for the class of representations with $\lambda_1 = 1, q$) can be obtained in this way. The final result is summarized in the

Proposition 43 *The irreps of $u_q(e^N)$ (characterized by integer λ_1) are highest weight irreps. Possible highest weights are of the form $\vec{\pi} \equiv 0$, $\vec{w} \equiv \vec{u} - l\vec{y}_1$, $0 \leq l \leq 2u_1$, if $N = 2n+1$, $\vec{w} \equiv \vec{w}(l,l') := \vec{u} - l \cdot sign(u_2-u_1)(\vec{y}_2 - \vec{y}_1) - l'(\vec{y}_2+\vec{y}_1)$, $0 \leq l \leq |u_2-u_1|$, $0 \leq l' \leq u_1+u_2$, if $N=2n$; $\vec{u}$ denote weights of U_q^N . In particular, when $N=3,4$ the sets $\{\vec{w}\}$ of weight satisfy the inclusion $\{w_1\} \supset \mathbb{Z}$, $\{\vec{w}\} \supset \mathbb{Z} \otimes \mathbb{Z}$ respectively. We have the following tensor product decomposition*

$$\tilde{\Gamma}^{\vec{u}} = \begin{cases} \bigoplus_{l=0}^{2u_1} \Gamma^{\vec{u}-l\vec{y}_1} & if \quad N=2n+1 \\ \bigoplus_{\substack{0\leq l \leq |u_2-u_1|;\\ 0 \leq l' \leq u_1+u_2}} \Gamma^{\vec{w}(l,l')} & if \quad N=2n+1 \end{cases} \tag{7.26}$$

Highest weight vectors can be easily determined from the above tensor product construction procedure.

7.1.3 Moding out singular vectors in the singlet irrep

According to theorem 1, $\mathcal{H}_{\vec{0}}$ is generated by application of the Borel subalgebra $U_q^{-,N}$ to its highest weight vector. A Poincaré-Birkhoff-Witt basis for $U_q^{-,N}$ is the set of monomials $(L_{\alpha_1})^{m_1} ... (L_{\alpha_s})^{m_s} (p^{sign(m) \cdot 1})^{|m|}$, where $s = \begin{cases} n^2 & if \quad N=2n+1 \\ n(n-1) & if \quad N=2n \end{cases}$; by L_{α_i} we denote the $L^{j,k}$ corresponding to the negative root α_i $(i=1,2,...,s)$, and the roots have been ordered according to an admissible total order ($\alpha_i > \alpha_l$ if $i > l$). Then, to each such monomial there corresponds a vector of $\mathcal{H}_{\vec{0}}$; not all of these vectors, however, can be considerated as independent.

In fact, as in the classical case, the requirement that the representation is of Hilbert space type makes many combinations of the previous vectors singular. We remind that a vector $|\chi>$ is said to be singular if $<\psi|\chi>=0\ \forall|\psi>\in\mathcal{H}_{\vec{0}}$. The simplest examples of singular vectors can be found in the singlet representation when $N>4$. They are $L^{-i,i-1}|\vec{0},\vec{0}>$, where $i>h+1$. Actually they are orthogonal to all vectors with different weights, by the hermicity of the $\mathbf{k}^l$, and have zero norm because of eq (4.117),(7.13) and the definition of the singlet irrep. Therefore we have to set

$$L^{-i,i-1}|\vec{0},\vec{0}>=0 \tag{7.27}$$

if we want the basic requirement $(7.2)_b$ of a Hilbert space to be satisfied. A slightly more complicated example is provided in the same representation and for $N\geq 5$ odd by the vectors $|1>:=L^{-1,0}L^{-2,1}L^{-1,0}|\vec{0},\vec{0}>$, $|2>:=L^{-2,1}(L^{-1,0})^2|\vec{0},\vec{0}>$, since we find, after straightforward use of formula (4.117), Proposition 20,

$$|2'>:=|1>-\frac{1}{q+q^{-1}} \qquad \Rightarrow \qquad <1|2'>=0=<2|2'> \qquad \Rightarrow \qquad <\psi|2'>=0\quad \forall\psi\in\mathcal{H}_{\vec{0}}. \tag{7.28}$$

When $N=2n$ another example of singular vector is

$$|\tau>:=(L^{-2,-1}-q^2L^{-2,1}\frac{p^{-1}p^{-1}(\mathbf{k}^1)^{-1}}{(p\cdot p)_1})|\vec{0},\vec{0}>, \tag{7.29}$$

as it can be easily verified by applying commutation relations (4.116)(4.117).

An Irrep of our algebra is obtained by moding out all singular vectors which are generated in the previous construction. Now we want to identify them.

Let us stick for the moment to the singlet representation of $\hat{u}_q^N$.

Theorem 5 *In the singlet Irrep the commutation relations*

$$\begin{cases} g_1:=L^{-2,-1}-q^2L^{-2,1}\frac{p^{-1}p^{-1}(\mathbf{k}^1)^{-1}}{(p\cdot p)_1}=0 \\ \tilde{g}_1:=L^{-2,1}-q^2L^{-2,-1}\frac{p^1p^1\mathbf{k}^1}{(p\cdot p)_1}=0 \end{cases} \qquad if \quad N=2n, \tag{7.30}$$

$$g_i:=L^{-i,i-1}L^{-i-1,i}-\frac{(\mathbf{k}^i)^{-\frac{1}{2}}-(\mathbf{k}^i)^{\frac{1}{2}}}{q(\mathbf{k}^i)^{-\frac{1}{2}}-q^{-1}(\mathbf{k}^i)^{\frac{1}{2}}}L^{-i-1,i}L^{-i,i-1}=0 \qquad i>h \tag{7.31}$$

and their hermitean conjugates hold. All singular vectors of the singlet representation can be obtained by application either of any element of $\hat{u}_q^{-,N}$ to a singular vector, or of some g_i to a nonsingular vector. Therefore if we mode out the singular vectors, an orthogonal basis $\mathcal{B}_q$ of $\mathcal{H}_{\vec{0}}$ will consist of the vectors

$$|\vec{0},-\vec{j}>:=(L^{-n,n-1})^{J_n}(L^{1-n,n-2})^{J_{n-1}}...(L^{-2,1})^{J_2}.\begin{cases}(L^{-1,0})^{J_1}|\vec{0},\vec{0}> & if \quad N=2n+1\\ (p^{-sign(J_1)\cdot 1})^{|J_1|}|\vec{0},\vec{0}> & if \quad N=2n\end{cases} \tag{7.32}$$

where the integers J_i *satisfy the conditions* $J_{i+1} \leq J_i \quad if \;\; i \geq h+1$. *The integers* $\vec{j} \in \mathbb{N}^n$ *specify the eigenvalues of* $\mathbf{k}^i$ *and are given by*

$$j_i = J_i - J_{i+1} \geq \qquad\qquad (J_{n+1} \equiv 0) \tag{7.33}$$

Remarks

- It is easy to show that the above relations are compatible with the Serre relations of Proposition 20, and the commutation relations (4.112), namely they yield identities $0 = 0$ when plugged into the latter. The validity of commutations relations (7.30), (7.31) depends crucially on the fact that $C_1 \neq 0, \; C_2 \neq 0$ in relations (4.117).
- As a consequence of formula (7.32), each weight in the singlet irrep has multiplicity 1. This is a strong indication that it is convenient to introduce a " configuration space realization " of the singlet irrep; actually we will see in section 7.2 how to represent its vectors as functions on the quantum Euclidean space.

By inverting relations (7.33) one gets the equivalent relation:

$$J_i = \sum_{l=i}^{n} j_i \tag{7.34}$$

Proof of the theorem. It is immediate to check that commutation relations (7.30), (7.31) hold when applied to $|\vec{0},\vec{0}>$. Let $\mathcal{H}_{\vec{0},(\vec{j}),\leq} := Span_{\mathbb{C}}\{|\vec{0},\vec{l},...> \;|\; l^i \geq j^i\}$. The proof of these equations is by induction in the weights $\vec{j}$, i.e. assuming that they hold within $\mathcal{H}_{\vec{0},(\vec{j}),\leq}$ we show that they hold when applied on $\mathcal{H}_{\vec{0},\vec{j}-\vec{e}_l+e_{l-1},\leq}$.

As a consequence of formulae (7.30),(7.31) one can easily show that within $\mathcal{H}_{\vec{0},(\vec{j}),\leq}$

$$L^{-i,i-1}L^{1-i,i} = \begin{cases} q^{-\frac{3}{2}}\frac{(\mathbf{k}^1)^{-1}-1}{(1-q)(q^{-1}-q)} & if \quad N=2n+1 \;\; and \;\; i=1 \\ \frac{q^{-2}\mathbf{k}^1(1-(\mathbf{k}^2)^{-1})}{(1-q^2)^2} & if \quad N=2n \;\; and \;\; i=2 \\ q^{2\rho_i}\frac{[(\mathbf{k}^i)^{-1}-1][1-q^{-2}\mathbf{k}^{i-1}]}{(q-q^{-1})^2} & otherwise. \end{cases} \tag{7.35}$$

First part of the thesis: if $|\chi>$ is singular, then for any $u \in \hat{u}_q^{-,N}$ $u|\chi>$ is. It is sufficient to consider only $u = p^{\pm 1}$ or $u = L^{-l,l-1}$, and vectors $|\chi>$ of the form $|\chi>= vg_iv'|0>$, where v, v' are two ordered polynomials in the generators of $\hat{u}_q^{-,N}$; in fact, by the induction hypothesis the latter are singular. Any other singular vector can be expressed as a combination of singular vectors of this form.

We consider first relations (7.30) in the case $N = 2n$. Since $[g_1, U_q^{-,N}] = 0$, $|\chi>$ is of the form $|\chi>= vg_1|\vec{0},\vec{0}> \; v \in \hat{u}_q^{-,N}$. The vector $|\chi'>:= L^{-2,1}|\chi>$ is trivially orthogonal to any vector corresponding to a different weight. In addition it is orthogonal to the only two independent vectors $|1>, |2>$ with the same weight, which are (according to formula (7.32) of the induction hypothesis)

$$|1'>:=L^{-2,1}|1>:=L^{-2,1}vL^{-2,-1}|\vec{0},\vec{0}> \qquad |2'>:=L^{-2,1}|2>:=\frac{L^{-2,1}vL^{-2,1}q^{-2}p^{-1}p^{-1}}{(1-q^2)(p\cdot p)_1}|\vec{0},\vec{0}>. \tag{7.36}$$

In fact, one can easily check that for any v as considered above,

$(v^*L^{-1,2})L^{-2,1} = a\,L^{-2,1}(v^*L^{-1,2})+\mathrm{k}v^*$ (k being an element of the Cartan subalgebra), $[v^*, L^{-12}]=0$, implying

$$<1'|\chi'>\propto<\vec{0},\vec{0}|L^{1,2}(v^*L^{-1,2})L^{-2,1}|\chi>\propto<\vec{0},\vec{0}|L^{1,2}L^{-2,1}(v^*L^{-1,2})|\chi>+b<\vec{0},\vec{0}|L^{1,2}v^*|\chi>$$

$$=<\vec{0},\vec{0}|L^{-2,1}L^{1,2}|\chi>+c<2||\chi>+b'<1|\chi>=0; \tag{7.37}$$

in the last identity we have used the singularity of $|\chi>$ and the fact that $<\vec{0},\vec{0}|L^{-2,1}=0$. Similarly one shows that $<2'|\chi'>=0$. Summing up, $|\chi'>$ is singular. The corresponding result for $\tilde{g}_1$ is a direct consequence of the above.

Now we consider the remaing commutation relations (7.31) (for even and odd N at the same time). One can easily check that they imply the relations

$$L^{-m,m-1}g_i = g_i f_l L^{-m,m-1} \qquad f_m := \begin{cases} q\frac{1-q^{-2}\mathrm{k}^i}{1-\mathrm{k}^i} & if \;\; m=i,i+1 \\ q\frac{1-q^{-2}\mathrm{k}^{i+1}}{1-\mathrm{k}^{i+1}} & if \;\; m=i+2 \\ q^{-1}\frac{1-\mathrm{k}^{i-1}}{1-q^{-2}\mathrm{k}^{i-1}} & if \;\; m=i-1 \\ 1 \;\; otherwise; \end{cases} \tag{7.38}$$

since they hold in $\mathcal{H}_{\vec{0},(\vec{j}),<}$, we can reduce any singular vector to one of the form $|\chi>= g_i u''|0>$. Using commutation relations (7.31) and relations (7.32) only in $\mathcal{H}_{\vec{0},(\vec{j}),<}$ one can easily show that

$$L^{-l,l-1}L^{1-l,l}g_i = s_{i,l}(\mathrm{k})g_i; \qquad or\ equivalently: \qquad L^{1-l,l}L^{-l,l-1}g_i = s'_{i,l}(\mathrm{k})g_i. \tag{7.39}$$

The vector $|\chi'>:=L^{-l,l-1}|\chi>$ is trivially orthogonal to any vector corresponding to a different weight; in addition it has zero norm

$$<\chi'|\chi'>\propto<\chi|s'_{i,l}|\chi>=0 \tag{7.40}$$

because of the preceding relation and the fact that $|\chi>$ is singular; therefore $|\chi'>$ is singular itself.

Second part of the thesis: if $|\phi> \in \mathcal{H}_{\vec{0},(\vec{j}),\leq}$ is nonsingular, then $|\phi'>:= g_i|\phi>$ is singular.

Again, we consider first relations (7.30) in the case $N = 2n$. As already noticed, it is easy to write this vector in the form $|\phi'>= vg_1'|\vec{0},\vec{0}>$; then the proof that $|\phi'>$ is singular goes as for $|\chi'>$.

As for the remaining relations, using formulae (4.117),(7.13) we find, when $i > h+1$

$$[L^{-i,i+1}L^{1-i,i}, g_i]_{q^2} = q^{2\rho_{i+1}}\mathbf{k}^i \frac{1-q^2(\mathbf{k}^{i+1})^{-1}}{1-q^{-2}\mathbf{k}^i} L^{-i,i-1}L^{1-i,i}+$$

$$q^{2\rho_i}\frac{1-q^{-2}(\mathbf{k}^{i-1})^{-1}}{1-q^{-2}\mathbf{k}^i}\left(q^2L^{-i-1,i}L^{-i,i+1}+q^{2\rho_{i+1}}\frac{1-\mathbf{k}^i(\mathbf{k}^{i+1})^{-1}}{1-q^{-2}}\right); \qquad (7.41)$$

within $\mathcal{H}_{\vec{0},(\vec{j}),\leq}$ we can replace the operators $L^{-i,i-1}L^{1-i,i}$, $L^{-i-1,i}L^{-i,i+1}$ in the RHS by their expressions (7.39), and we find that the RHS vanishes. A similar argument can be used when $i = h+1$, and also when the order of the two positive roots appearing in the LHS is reversed. We find that on $\mathcal{H}_{\vec{0},(\vec{j}),\leq}$

$$[L^{-i,i+1}L^{1-i,i}, g_i]_a = 0 \qquad [L^{1-i,i}L^{-i,i+1}, g_i]_a = 0 \qquad a = \begin{cases} q & if\ \ N = 2n+1,\ \ i=1 \\ q^2 & otherwise. \end{cases} \qquad (7.42)$$

Now consider the vector $g_i|\vec{0},-\vec{j}>$. On one hand, it is trivially orthogonal to any other vector of $\mathcal{H}_{\vec{0}}$ of different weight; on the other, it is orthogonal both to $|1>:= L^{-i-1,i}L^{-i,i-1}|\vec{0},-\vec{j}>$ and $|2>:= L^{-i,i-1}L^{-i-1,i}|\vec{0},-\vec{j}>$

$$<2|g_i|\vec{0},-\vec{j}> \propto <\vec{0},-\vec{j}|g_iL^{-i,i+1}L^{1-i,i}|\vec{0},-\vec{j}>= 0$$

$$<1|g_i|\vec{0},-\vec{j}> \propto <\vec{0},-\vec{j}|g_iL^{1-i,i}L^{-i,i+1}|\vec{0},-\vec{j}>= 0, \qquad (7.43)$$

where we have used equations (7.41), the fact that the vectors $g_iL^{-i,i+1}L^{1-i,i}|\vec{0},-\vec{j}>$, $g_iL^{1-i,i}L^{-i,i+1}|\vec{0},-\vec{j}>$ belong to $\mathcal{H}_{\vec{0},\vec{j},\leq}$ and the induction hypothesis. Therefore the vector $g_i|\vec{0},-\vec{j}>$ is singular in $\mathcal{H}_{\vec{0}}$ and must be set equal to zero. $\diamondsuit$

Let us consider now a generic representation $(\Gamma^{\vec{w}},\mathcal{H}^{\vec{w}})$. Its singular vectors can be determined using relations (7.30),(7.31) in $\mathcal{H}^{\vec{0}}$ and the tensor product construction of $\Gamma^{\vec{w}},\mathcal{H}^{\vec{w}}$, described in Proposition 3. Of course, in building $\mathcal{H}^{\vec{w}}$ through application of $u_q^{c,N}$ to the highest weight vector $|\vec{0},\vec{w}>^{\vec{w}}$, one has to impose relations (7.30),(7.31) only on the generators L acting on the first tensor factor (the singlet representation) of the product

(7.22). The existence of singular vectors in $\mathcal{H}^{\vec{w}}$ is therefore only due to the existence of singular vectors in the representation $\Gamma^{\vec{w}}$ of U_q^N.

Finally, let us comment a little on the structure of the pre-Hilbert spaces $\mathcal{H}^{\vec{w}}$. First, we note that the domain of the labels $\vec{j}$ in $\mathcal{H}^{\vec{w}}_{\vec{\pi}}$ is:

$$\mathcal{J} := \{\vec{j} \in \mathbb{Z}^n \mid j_i \leq \pi_{i-1} + w_i, \quad i = h+1, h+2, ..., n; \qquad j_1 \in \mathbb{Z} \quad if \ N = 2n.\} \tag{7.44}$$

This follows from inequalities (7.33), the tensor product construction of Proposition 43 and formulae $(4.83)_a$. We give an intuitive picture of the physical content of the spectra of the observables (7.3) in the singlet representation. The subspace $\mathcal{H}_i^{\vec{0}} := \bigoplus_{\{\vec{\pi}, \mid \pi_{i-1}=0\}} \mathcal{H}^{\vec{0}}_{\vec{\pi}}$ is the eigenspace of the observable $p^{-i}p_{-i} = (p \cdot p)_i - (p \cdot p)_{i-1}$ with the minimum eigenvalue compatible with a given eigenvalue of $(p \cdot p)_i$, namely $p^{-i}p_{-i} = M^2 q^{\sum_{k=i}^{n} 2(1+\pi_k)}(q^2 - 1)$; it never vanishes when $q \neq 1$. This means that the there is always a " point zero " momentum component available in the plane of the coordinates $i, -i$. Now let us ask in which " directions " of this plane this point zero momentum component can be pointed.

The admitted eigenvalues of $ln_q(\mathrm{k}^i)$, i.e. of the angular momentum component in the plane, are $j_i \leq 0$ (see (7.33)) and show that (except when $N = 2n$, $i = 1$) only a " clockwise " or " radial " orientation are possible The anticlockwise is excluded! If $N = 3$, for instance, minimum p^1p_1 means that the momentum is almost pointed in the p^0 direction; j_1 represents the p^0-direction component of the (orbital) angular momentum; therefore the " orbital helicity " j_1p_0 has a definite sign. We find a sort of a purely " kinematical " parity asymmetry, which is a surprising feature for a lattice theory; in fact, at least usual equispatiated lattice theories, which are commonly used nowadays for regularization purposes, cannot have a parity asimmetry by a well-known no-go-theorem [38]. In next section we will see in which sense in the classical limit $q \to 1$, however, parity symmetry is recovered.

Both for odd and even N value of j_i is not bounded from below; larger and larger absolute values of the angular momentum with a fixed amount of momentum available in the plane $i, -i$ (in particular, with the minimum amount $p^{-i}p_{-i} = (p \cdot p)_i - (p \cdot p)_{i-1}$) can be intuitively described only if the corresponding states have larger and larger values of the mean distance from the origin of the plane. Thus we can visualize the states of $\mathcal{H}_i^{\vec{0}}$ with larger and larger $|j_i|$ as states with larger and larger mean distance from the origin in the x^i, x^{-i} plane.

7.2 Configuration space realization of the singlet irrep

As known, the classical (i.e. $q = 1$) Euclidean algebra e^N can be realized in the singlet (i.e. spin zero) representation as the algebra of differential operators acting on suitable subspaces of the algebra of " functions " on $\mathbb{R}^N$ configuration space (e.g. the subspace of smooth functions, or square integrable, or distributions...). Working in configuration-space realization rather than at an abstract level is very useful in the classical context for many purposes; for instance, questions regarding in concrete cases the domain of definition of some elements of $U(e^N)$, considered as operators in the singlet representation, - e.g. the essential adjointness domain of would-be observables, etc. -, are best treated in configuration (or, depending on cases, momentum) space realization.

On the other hand, compared, say, with the representation theory of compact Lie groups, the representation theory of $U(e^N)$ is complicated by the fact that the eigenvectors of translation generators are not normalizable functions, but distributions; nevertheless in a generalized sense they are used to form a " basis " of the Hilbert space $\mathcal{L}^2(\mathbb{R})$ carrying the representation, namely they can be " combined " into Fourier transforms to yield elements of $\mathcal{L}^2(\mathbb{R})$. The occurence of distributions is (ultimately) related to the the fact that the spectrum of these operators is continuous.

In the case $q \neq 1$ the spectrum is discrete, as we have seen, and the corresponding eigenvectors are normalizable. In the preceding sections we have investigated in an abstract way the singlet irrep of $u_q(e^N)$, we would now be happy to find its corresponding q-deformed configuration-space counterpart, if any, in view of further developments of the theory (concerning, for instance, domain questions). The existence of such a counterpart is expected (and desired) also because theorem 2shows that the notation with L generators applied to $|\vec{0},\vec{0}>$ is too heavy to identify a basis of $\mathcal{H}^{\vec{0}}$ once one has moded out the singular vectors. The vectors of the basis $\mathcal{B}_q$ of formula (7.32) are labeled just by the corresponding powers π_m, J_l of the generators of $u_q^{c,N}$ applied on $|\vec{0},\vec{0}>$; in other words these vectors don't depend (apart from a factor) on the order in which these generators are applied. This result can be obtained in a natural way in a configuration-space realization.

We are going to show that such a target can be reached by constructing and mixing a *pair* of configuration-space realizations (the barred and the unbarred), just in the same way as we did in chapter 6 for the harmonic oscillator.

As we noted in section 4.4, all the replacements $p \to F(\Lambda)\partial, \bar{F}(\Lambda)\bar{\partial}$ allow to represent the subalgebra of momenta as differential operators on $Fun(\mathbb{R}_q^N)$, whatever $F(t), \bar{F}(t) \in$

$\mathbb{C}[t]$. These differential operators are never real, nevertheless we hope to realize hermitean conjugation by the same mechanism as the one described in the previous chapter. On the other hand, the generators $\mathbf{L}, \mathbf{k}$ of the rotation part of $U_q(e^N)$ were realized as differential operators satisfying the $*$-relations (4.30) since their very construction in [14] (as we have seen in Chapter 4), and therefore their hermitean conjugation will amount to $*$-conjugation in $Diff(\mathbb{R}_q^N)$.

For instance, we can introduce a *pair* of realizations (what we call the unbarred and the barred) of the algebra of $U_q(e^N)$, i.e. a pair of algebra homomorphisms $\rho, \bar{\rho}$

$$\begin{array}{ccc} & & Diff(\mathbb{R}_q^N),\ \ Fun(\mathbb{R}_q^N) \\ & \overset{\rho}{\nearrow} & \\ U_q(e^N),\ \ \mathcal{H} & & \\ & \underset{\bar{\rho}}{\searrow} & \\ & & Diff(\mathbb{R}_q^N),\ \ Fun(\mathbb{R}_q^N), \end{array} \tag{7.45}$$

through

$$\begin{cases} \rho(p^i) := -i\Lambda^a\partial^i, \\ \bar{\rho}(p^i) := -i\bar{\partial}^i\Lambda^{-a}q^{-N(1+a)} \end{cases}, \qquad \rho(u) := u =: \bar{\rho}(u) \ \ \forall u \in U_q^N \qquad \rho(\Lambda) := \Lambda := \bar{\rho}(\Lambda) \tag{7.46}$$

($a \in \mathbb{Z}$). It is easy to check that the complex conjugation in $u_q(e^N)$, defined in formulae (4.30),(4.90) (we call it $\dagger$ here, to avoid confusion) satisfies the relation

$$\rho(u^\dagger) = [\bar{\rho}(u)]^* \qquad \bar{\rho}(u^\dagger) = [\rho(u)]^*, \tag{7.47}$$

where $*$ denotes the complex conjugation in $Diff(\mathbb{R}_q^N)$. To be precise we define the homomorphisms $\rho, \bar{\rho}$ first on the vectors of the basis $\mathcal{B}_q$ (see theorem 5), and then we extend them linearly to all of $\mathcal{H}$.

The $\rho, \bar{\rho}$-images of the vectors of the basis $\mathcal{B}_q$ are determined by the requirement that they satisfy q-differential equations which are respectively the ρ- and $\bar{\rho}$-image of the equations satisfied by $|\phi>$. Actually we can limit ourselves to the search of ρ- and $\bar{\rho}$-images of the highest weight vector(s), since they are cyclic in the representations spaces. If we find solutions to these differential equations we obtain (two) representations of $u_q(e^N)$; then we have to see whether they are $*$ representations and, if so, identify them with some irreps studied in the previous section.

This is the program before us. We will see that it can carried through, and that we obtain a *unique* configuration-space realization of the singlet irrep based on the use of a *pair* of non-$*$-representations (the unbarred and the barred).

As shown by Proposition 1 in section 3.1, all choices of a in equation (7.46) are essentially equivalent. The choice $a=-1$ will be particularly convenient for representing scalar products in $\mathcal{H}^{\vec{0}}$ as integrals of functions on $\mathbb{R}_q^N$, and we adopt it here.

Then we can easily prove the important

Lemma 8 *If the vector $|\phi>$ satisfies the equations*

$$(a+a^ip_i+u)|\phi>=0, \qquad a\in\mathbb{C},\ u\in U_q^N, \qquad\qquad (p\cdot p)_n|\phi>=M^2|\phi> \quad (7.48)$$

and $\varphi:=\rho(|\phi>)\in Fun(\mathbb{R}_q^N)$ is the function that represents $|\phi>$ in the unbarred realization with $a=-1$, then its barred partner $\bar{\varphi}:=\bar{\rho}(|\phi>)$ is given by

$$\bar{\varphi}(x)=e_{q^2}[-m^2x\cdot x]\varphi(q^{-1}x) \qquad\qquad m^2:=(q^{-2}-1)q^{-2\rho_n-1}M^2 \qquad (7.49)$$

Proof. After a shift $x\to q^{-1}x$, $\partial\to q\partial$ the hypothesis reads

$$(a-ia^i\Lambda^{-1}q\partial_i+u)\varphi(q^{-1}x)=0, \qquad a\in\mathbb{C},\ u\in U_q^N, \qquad\qquad -q\Lambda^{-2}(\partial\cdot\partial)_n\varphi(q^{-1}x)=M^2\varphi(q^{-1}x); \quad (7.50)$$

then the thesis can be verified by a straightforward calculation, using the expression (3.58) of $\bar{\partial}$ in terms of ∂, x and formulae (3.54)(3.55). $\diamond$

Now comes the

Proposition 44 *One can solve the highest weight vector conditions (7.21) in the $\rho,\bar{\rho}$ realization. The resulting representations of $\hat{u}_q(e^N)$ on $Fun(\mathbb{R}_q^N)$ coincide with the singlet representation $(\Gamma^{\vec{0}},\mathcal{H}^{\vec{0}})$. The cyclic vector $|\vec{0},\vec{0}>^{\vec{0}}$ is represented in the unbarred realization by*

$$\rho(|\vec{0},\vec{0}>^{\vec{0}})=:\varphi_0=\begin{cases} e_{q^{-1}}[iM_0x^0], & M_0=\pm(1+q^{-1})^{\frac{1}{2}}q^{n+1} \qquad if \quad N=2n+1\\ \sum\limits_{n=0}^{\infty}\frac{[-(x\cdot x)_1M_0^2]^n}{[(n)_{q^{-2}}!]^2} & M_0=q^{n+\frac{1}{2}}M \qquad if \quad N=2n.\end{cases} \quad (7.51)$$

The subspace $\rho(\mathcal{H}_{\vec{0}}^{\vec{0}})\subset\rho(\mathcal{H}^{\vec{0}})$ is spanned in the unbarred representation by $\rho(\mathcal{B}_q)$, whose elements are the functions

$$\rho(|\vec{0},-\vec{j}>^{\vec{0}})\propto(x^{-n})^{j_n}...(x^{-2})^{j_2}\cdot\begin{cases}(x^{-1})^{j_1}e_{q^{-1}}[iM_0x^0] & if \quad N=2n+1\\ (x^{-sign(j_1)\cdot 1})^{|j_1|}\varphi^{J_1} & if \quad N=2n\end{cases} \quad (7.52)$$

where $\varphi^{J_1}:=\sum\limits_{n=0}^{\infty}\frac{[-(x\cdot x)_1M_0^2]^n}{(n)_{q^{-2}}!(n+J_1)_{q^{-2}}!}$ and J_1 was defined in (7.34).

Proof. The most general expression for the function representing the cyclic vector is

$$\varphi_0 = \sum_{l=-n}^{n} \sum_{i_l=0}^{\infty} A_{i_{-n},...,i_n} (x^{-n})^{i_{-n}} ... (x^n)^{i_n}. \tag{7.53}$$

The requirement that it is annihilated by $\rho(p^{-i})$, $i > h$, implies that there can be no dependence on x^i (take in the order $i = n, n-1, ...$ and perform the derivations). Similarly, since on $\mathcal{H}_{\vec{0}}$ $L^{1-i,i} = \mathrm{L}^{1-i,i} + cost \cdot p^i p^{1-i}$, the requirement that it is annihilated by $L^{1-i,i}$ implies that there can be no dependence on x^{-i}. This is straightforward to check when $i > h+1$, and a little more lenghty when $i = h+1$. In the case $N = 2n+1$, for instance, it is easy to check that

$$L^{0,1}\varphi_0 = 0 \qquad \Rightarrow \qquad \begin{cases} A_{i_{-1},i_0+1} \propto \frac{A_{i_{-1},i_0}}{(i_0)_{q^{-1}}} & when \;\; i_{-1} \geq 0 \\ A_{i_{-1},i_0+1} \propto \frac{A_{i_{-1},i_0}}{(i_0+1)_{q^{-1}}} & when \;\; i_{-1} \geq 1; \end{cases} \tag{7.54}$$

the first (resp. second) condition comes from setting the coefficient of $(x^{-1})^{i_{-1}}(x^0)^{i_0-1}x^1$ (resp. $(x^{-1})^{i_{-1}-1}(x^0)^{i_0+1}$) equal to zero. They are incompatible, therefore $A_{i_{-1}i_0} = 0$ if $i_{-1} > 0$. Finally the requirement that φ_0 is an eigenvector of $\rho(p^0)$ in the case $N = 2n+1$ or $\rho((p \cdot p)_1), \mathbf{k}^1$ in the case $N = 2n$ with the prescribed eigenvalues (following from the conditions $(p \cdot p)_n = M^2$, $p^{-l}|\phi_0 >= 0)$ yields the expression $(7.51)_a$.

A direct application of the commutation relations of Proposition 15 yields (7.52) as the basis of $\rho(\mathcal{H}^{\vec{0}}_{\vec{0}})$; the commutation relations of section 4.5 can be easily checked applying them to the elements of this basis.

Thus we have obtained two representations of $u_q(e^N)$. They coincide with the singlet representation of the previous section since the eigenvalues of $\mathbf{k}^i$ on the highest weight vectors coincide with those of the singlet representation both in the unbarred and in the barred case. ◇

It remains to realize the scalar product yielding the hermiticity structure described in the abstract context in section 7.1.

The relevance of the double realization manifests itself in all its (so far hidden) importance in the

Theorem 6 *The scalar product in $\mathcal{H}^{\vec{0}}$ can be realized in configuration-space by*

$$< \phi_1|\phi_2 >= \int d_qV \;\; [\bar{\varphi}_1]^* \varphi_2, \tag{7.55}$$

where $\int d_qV$ is the integration first defined in [11] and briefly reviewed in Chapter 5, with a suitable normalization.

Proof. Let $<\phi_1|\phi_2>' := RHS(7.55)$. Because of the lemma,

$$<\phi_1|\phi_2>' = \int d_qV \;\; e_{q^2}[-m^2 x\cdot x][\varphi_1(q^{-1}x)]^*\varphi_2(x) \qquad m^2 := (1-q^2)q^{-2\rho_n-3}M^2; \quad (7.56)$$

$<\;,\;>'$ is a well defined inner product in $\mathcal{H}_{\vec{0}}$: in fact, the integral (7.56) is well defined according to the definition given in Chapter 5, due to the presence of the the q^2-gaussian damping factor $e_{q^2}[-m^2 x\cdot x]$, which can be taken as " reference function of the integration" : in the frame of that definition it suffices to expand $[\varphi_1(q^{-1}x)]^*\varphi_2(x)$ in powers of x, perform the integrations and resum the integrals to obtain the above integral. From the practical point of view it is convenient, however, to choose directly $[\bar{\varphi}_0]^*\varphi_0$, where $|\phi_0> = |\vec{0},\vec{0}>$ stands for the cyclic vector, as (non-scalar) reference function of the integration (in the sense mentioned in Chapter 5), since, as we will see below, all integrals of the form (7.56) can be evaluated from this basic integral.

Because the integration satisfies Stoke's theorems (5.4),(5.5), i.e. " boundary terms " vanish, and formula (7.47) holds, the complex conjugation (3.39) $*$ followed by an exchange of the two realizations acts as hermitean conjugation of all differential operators of $u_q(e^N)$ with respect to the scalar product of $\mathcal{H}^{\vec{0}}$. This ensures that the inner product $<\;,\;>'$ preserves the orthogonality relations between different vectors of the basis $\mathcal{B}_q$ of $\mathcal{H}^{\vec{0}}$ since they have different eigenvalues of the observables (7.3).

If $|\phi_i> = \mathcal{D}_i|\phi_0>$, $\mathcal{D}_i \in u_q^{c,N}$ ($u_q^{c,N}$ was defined in formula (7.56)), $i = 1,2$, then because of formula ()

$$\int d_qV \;\; [\bar{\varphi}_1]^*\varphi_2 = \int d_qV \;\; [\bar{\varphi}_0]^*\rho(\mathcal{D}_1^*\mathcal{D}_2|\phi_0>). \qquad (7.57)$$

Only the ρ image of the nonzero $|\phi_0>$-component of $\mathcal{D}_1^*\mathcal{D}_2|\phi_0>$, if any, will contribute to the above integral. This explains why the evaluation of all integrals of the form (7.56) is reduced to the basic integral $\int d_qV \; [\bar{\varphi}_0]^*\varphi_0$, as claimed. Finally, we normalize the integration so that $\int d_qV[\bar{\varphi}_0]^*\varphi_0 = 1$. This concludes the proof that $<\;,\;>' = <\;,\;>$. $\diamondsuit$

Remark 1 To recognize that the inner product $<\;,\;>'$ is sesquilinear it is convenient to consider its more symmetric but equivalent form

$$<\phi_1|\phi_2>' := \int d_qV \;\; ([\bar{\varphi}_1]^*\varphi_2 + [\varphi_1]^*\bar{\varphi}_2) \qquad (7.58)$$

(alternatively one could include in the RHS only the second term). In fact, the sesquilinearity of the scalar product is immediate in this form. That this form is equivalent to the former can be esaily understood. Actually each one of the two terms separatly allows to formally realize the †-structure of equations (4.90) in terms of differential operators acting on spaces of functions on $\mathbb{R}_q^N$, and the whole Hilbert space $\mathcal{H}^{\vec{0}}$ is built up from a single cyclic vector through application of elements of $u_q(e^N)$; as we have seen all inner products are therefore completely determined in terms respectively of the integrals $\int d_qV \quad [\bar{\varphi}_0]^*\varphi_0$ and (7.58) involving the cyclic state $|\phi_0>$. It sufficient to normalize both equal to 1 to make them coincide.

Important remark 2. Formula (7.56) allows to explain from the point of view of configuration space the regularizing effect of taking the q-deformed version of the Euclidean Hopf-algebra instead of the classical one. Only when $q \neq 1$ the damping factor $e_{q^2}[-m^2x \cdot x] \neq 1$ in the integral (7.56) makes the norms of eigevectors of the observables (7.3) finite.

Remark 3 The final lesson we learn from interpreting the results of sections 7.1,7.2 is the following. If we stick for simplicity to the highest weight singlet representation of $u_q(e^N)$, then the fact that power series in the coordinates x^i (the ones belonging to subspaces $\rho(\mathcal{H}^{\vec{0}}), \bar{\rho}(\mathcal{H}^{\vec{0}}) \subset Fun(\mathbb{R}_q^N)$ which we have constructed) arise as a natural basis to identify elements of the singlet irrep of $u_q(e^N)$ (instead of the vectors obtained by applying combinations of the elements of a Poincare'-Birkhoff-Bott basis of $u_q^{c,N}$ to the highest weight state) ultimately is traced back to the requirement that the representation is of Hilbert-space type. It is this requirement, in fact, which makes a huge amount of singular vectors to appear within the space of such combinations; having moded the latter out, we have shown that the abovementioned power series in $Fun(\mathbb{R}_q^N)$ are sufficient to identify the remaining vectors.

7.3 Classical limit of the singlet irrep

In this section we just briefly sketch what we are allowed to mean by " $\lim_{q\to 1}\Gamma_q = \Gamma_{q=1}$ ", i.e. by saying that an irrep of $u_q(e^N)$ goes to an Irrep $U(e^N)$ in the limit $q \to 1$. If we compare the behaviour of the representations Γ_q of $u_q(e^N)$ under this limit with that of the representations of U_q^N (which is the " compact " subalgebra of $u_q(e^N)$), we find important differences. For simplicity we stick to the singlet Irrep of $u_q(e^N)$; the other Irreps of $u_q(e^N)$ can be obtained by the tensor product(7.22), and the properties of the Irreps of U_q^N are essentially known.

The commuting observables

$$p_0, (p\cdot p)_1, ..., (p\cdot p)_{n-1}, (p\cdot p)_n; h_1, ..., h_n \qquad (p_0 \equiv 0 \quad if \quad N = 2n+1). \quad (7.59)$$

($h_i := ln_{q^2}(\mathbf{k}^i)$) make up a complete set both when $q \neq 1$ and $q = 1$. We have chosen h_i

instead of $\mathbf{k}^i$ because it is the set of generators $\{L^{i,j}, h_i, p^i\}$ which has classical commutation relations in the limit $q \to 1$. The eigenvalues of the observables (7.3) label the vectors of an orthonormal basis $\mathcal{B}_q$ (7.32) (of eigenvectors) of the singlet Irrep for all $q \in \mathbb{R}^+$; when $q = 1$ the vectors of this basis are distributions (see the configuration space realization of the preceding section), i.e. $\mathcal{H}_{q=1}$ is no more a Hilbert space but the space of functionals on some space of smooth functions on $\mathbb{R}^n$, e.g. $\mathcal{S}(\mathbb{R}^N)$.

For each fixed eigenvector $|\vec{\pi}, \vec{j}>$ the eigenvalues j_i of $h_i := log_{q^2}(\mathbf{k}^i)$ don't depend on q and are integers; whereas the eigenvalues of $(p \cdot p)_i$ (non-uniformly) " collapse " to M^2:

$$\lim_{q \to 1} c_i(q, \vec{\pi}) = M^2 \qquad with \qquad (p \cdot p)_i |\vec{\pi}, \vec{j}> =: c_i(q, \vec{\pi}) |\vec{\pi}, \vec{j}> . \tag{7.60}$$

Therefore all vectors $|\vec{\pi}, \vec{j}>$ of $\mathcal{B}_q$ with fixed $\vec{j}$ would have the same limit when $q \to 1$, and the latter would coincide with a vector of $\mathcal{B}_{q=1}$; consequently, $\{\lim_{q \to 1} |\vec{\pi}, \vec{j}> \ , \ |\vec{\pi}, \vec{j}> \in \mathcal{B}_q\} \neq \mathcal{B}_{q=1}$. Therefore the limit $\lim_{q \to 1} \Gamma_q = \Gamma_{q=1}$ cannot be given a literal sense.

However, we can give a weaker sense to the above limit, as we are going to explain. Assume that $q = 1$ and the vector $\vec{r} := (r_h, ..., r_{n-1}, r_n)$ consists of components $r_n \in \mathbb{R}^+$, $0 \leq r_i \leq 1 \ h \leq i \leq n-1$, $\vec{j} \in \mathbb{Z}^n$; let $||\vec{r}, -\vec{j}> \in \mathcal{B}_{q=1}$ be the vector with eigenvalues $h_i = -j_i$, $(p \cdot p)_i = (p \cdot p)_{i+1} r_i$, $(p \cdot p)_n = M^2 r_n$. Define functions $\tilde{\pi}_i(r_i, q) := \left[ln_{q^2}(r_i) \right]$ ($[a]$ denotes the integral part of the number $a \in \mathbb{R}$), and set $\tilde{\Pi}_i := \sum_{l=i}^{n-1} \tilde{\pi}_l$. If $q < 1$ (as we have assumed in all this chapter) and $(1-q)$ is sufficiently small then $-j_i \leq \tilde{\pi}_{i-1}$, and we can define

$$|\psi_{\vec{r}, -\vec{j}}> := |\vec{\pi} = \tilde{\vec{\pi}}(\vec{r}, q), -\vec{j}> \propto \Lambda^{\tilde{\pi}_n} (p^n)^{\tilde{\pi}_{n-1}} ... (p^{h+1})^{\tilde{\pi}_h}$$

$$\cdot (L^{-n,n-1})^{J_n + \tilde{\Pi}_{n-1}} ... (L^{-2,1})^{J_2 + \tilde{\Pi}_1} \begin{cases} (L^{-1,0})^{J_1 + \tilde{\Pi}_0} |\vec{0}, \vec{0}> & if \ \ N = 2n+1 \\ (p^{-sign(J_1 + \tilde{\Pi}_1) \cdot 1})^{|J_1| + \tilde{\Pi}_1} |\vec{0}, \vec{0}> & if \ \ N = 2n, \end{cases} \tag{7.61}$$

where J_i are related to j_i by formula (7.34). The weak sense which can be given to the limit $\lim_{q \to 1} \Gamma_q = \Gamma_{q=1}$ is at least that for any $||\vec{r}, -\vec{j}> \in \mathcal{B}_{q=1}$ and small $\varepsilon > 0$ we can find a $q < 1$ and a vector $|\psi_{\vec{r}, -\vec{j}}> \in \mathcal{B}_q \subset \mathcal{H}_q$ such that the corresponding eigenvalues of the observables $(p \cdot p)_i$ differ by less than ε: indeed, we only need to set $q \equiv 1 - \varepsilon q^{-r_n}$, solve for q and define $|\psi_{\vec{r}, -\vec{j}}>$ as in formula (7.61). Note that the convergence of the q-deformed eigenvalues (selected in this way) to the classical ones is uniform only " on the states localized on the sphere $(p \cdot p)_n = c$ ", i.e. within each subspace characterized by a fixed value $(p \cdot p)_n = c$.

In the limit $q \to 1$ the " parity asymmetry " in the spectrum of the observables (7.3) noticed at the end of section 7.1 disappears, in the sense that the range of each j_i (as a function of the square momenta) becomes the whole set $\mathbb{Z}$, whenever $r_{i-1} < 1$, i.e. $(p\cdot p)_{i-1} < (p\cdot p)_i$, i.e. " almost everywhere " in momentum space. (In fact, the condition $(p\cdot p)_{i-1} = (p\cdot p)_i$ fixes a cylinder in the classical momentum space $\mathbb{R}^N_{\vec{p}}$, this is a subset of $\mathbb{R}^N_{\vec{p}}$ of zero measure). The same is true also in the other (not necessarily singlet) Irreps.

7.4 Appendix

We will denote by $\mathcal{H}_{\vec{0},\vec{\mu}} \subset \mathcal{H}_{\vec{0}}$ the eigenspace of all $\mathbf{k}^i$'s with eigenvalues, say, μ_i. We can easily show that $\mathcal{H}_{\vec{0},\vec{\mu}}$ is finite dimensional. In fact, if $|\phi> \in \mathcal{H}_{\vec{0},\vec{\mu}}$ the whole $\mathcal{H}_{\vec{0},\vec{\mu}}$ has to be obtained by applying zero-graded (w.r.t. the grading of $\mathbf{k}^i$) operators obtained as products of elements L_{+a}, L_{-a} (a are the roots of U_q^N) of the Poincare'-Birkhoff-Witt bases of the subalgebras $u_q^{+,N}, u_q^{-,N}$ introduced in section 4.5. But there is only a finite number of linearly independent products, since the existence of the casimirs (which are indeed among these operators) essentially reduces the products of high powers in $L_{\pm a}$ to products of lower ones.

Assume for brevity that all eigenvalues μ_i are positive. Let L be one of the positive roots, L^- its Cartan-Weyl partner; then their commutation relation is of the form

$$[L, L^-]_a = c\frac{(1+d) - \mathbf{k}}{1-a^{-1}}, \qquad 0 < a < 1, c > 0 \qquad [\mathbf{k}, L]_{a^{-2}} = 0 = [\mathbf{k}, L^-]_{a^2}, \quad (7.62)$$

where k is an element of the Cartan subalgebra, and in the sequel

$$a = \begin{cases} q & if \quad N = 2n+1 \quad and \quad L = L^{0l} \\ q^2 & otherwise \end{cases}.$$

Lemma 9 *Let L, L^-, k satisfy (7.62) and let L^-L be hermitean positive definite (both conditions are satisfied if L is a simple roots, for instance). For any $\vec{\mu}$ there $\exists m \in \mathbb{N}$ such that $(L)^m\mathcal{H}_{\vec{0},\vec{\mu}} = \mathbf{0}$; if $1 + d < 0$, then $\exists m' \in \mathbb{N}$ such that $(L^-)^{m'}\mathcal{H}_{\vec{0},\vec{\mu}} = \mathbf{0}$*

Proof. Let $\{|f_i>\}_{i\in I}$ be a basis of $\mathcal{H}_{\vec{0},\vec{\mu}}$ consisting of eigenvectors of L^-L and let l_i be the corresponding eigenvalues. By recursively applying commutation relation (7.62) we find that $L^m|f_i>$, $m \in \mathbb{N}$, is an eigenvector of the positive definite operator L^-L:

$$(L^-L)(L^m)|f_i> = l(l_i, m)(L^m)|f_i> \qquad l(l_i, m) := \left[l_i a^{-m-1} - (m+1)_{a^{-1}} c a^{-1}\frac{(1+d) - \lambda a^{-m}}{1-a^{-1}}\right]. \quad (7.63)$$

where λ denotes the eigenvalue of $\mathbf{k}$ in $\mathcal{H}_{\vec{0},\vec{\mu}}$. We see that $l(l_i, m) \leq l(b, m)$ ($b \geq l_i \ \forall i$), and that $l(b, m)$ would get negative for large m, unless there exists a $m \in \mathbb{N}$ such that $L^m|f_i> = 0 \ \forall i \in I$. Similarly one proves the second part of the Lemma. $\diamondsuit$.

Applying repeatedly Serre relations (proposition 13) one can prove the following

Lemma 10 *Let* $|j| < k,\ k \geq i \geq h+1$.

$$[\mathbf{L}^{1-i,i}, \mathbf{L}^{j,k}]_a = 0 \qquad a = \begin{cases} q & if \quad i = k, \quad j \neq 1-i, i-1 \\ q^{-1} & if \quad i < k, \quad j = i, 1-i \\ 1 & if \quad i < k, \quad j \neq \pm i, \pm(i-1); \end{cases} \tag{7.64}$$

as a consequence, if $k > j > i \geq h$

$$[\mathbf{L}^{j,k}, \mathbf{L}^{i,k}]_{q^{-1}} = 0 \qquad\qquad [\mathbf{L}^{-j,k}, \mathbf{L}^{-i,k}]_q = 0. \tag{7.65}$$

The same holds if we replace the $\mathbf{L}$ *by the* L *roots.*

Proof of theorem4. Because of proposition 41, it is sufficient to prove the theorem within $\mathcal{H}_{\vec{0}}$. We prove only the first part of the thesis, stating the existence of a highest weight vector. The rest of the proof will be given elsewhere [17], since we need some explicit knowledge about the casimirs of $\hat{u}_q(e^N)$.

Given any $|\psi_0>\in \mathcal{H}_{\vec{0}}$, let $k^i|\psi_0>=\lambda_i|\psi_0>$ and apply lemma 9 to $\mathcal{H}_{\vec{0},\vec{\lambda}}$ by setting $L \equiv L^{01}$ if $N = 2n+1$ and $L \equiv L^{\pm 1,2}$ if $N = 2n$; we will respectively determine integers $p, p_{\pm} \in \mathbb{N}$ such that

$$(L^{01})^{p+1}|\psi_0>=0 \qquad (L^{01})^p|\psi_0>\neq 0 \qquad if \quad N = 2n+1$$

$$\begin{cases} (L^{1,2})^{p_+ +1}(L^{-1,2})^{p_-}|\psi_0>=0=(L^{1,2})^{p_+}(L^{-1,2})^{p_- +1}|\psi_0>, \\ (L^{1,2})^{p_+}(L^{-1,2})^{p_-}|\psi_0>\neq 0 \end{cases} \quad if \quad N = 2n. \tag{7.66}$$

If we define

$$|\psi_1>:=\begin{cases} (L^{01})^p|\psi_0> & if \quad N = 2n+1 \\ (L^{1,2})^{p_+}(L^{-1,2})^{p_-}|\psi_0> & if \quad N = 2n, \end{cases} \tag{7.67}$$

this means that $|\psi_1>$ is annihilated by $U_q^{+,3}$ (resp. $U_q^{+,4}$).

Now the proof goes on by induction. Assume that we have determined a nontrivial vector $|\psi_{j-1}>\in \mathcal{H}_{\vec{0}}$ such that $U_q^{+,2j-1+h}|\psi_{j-1}>=\{0\}$, $j = 2, ..., n$. Moreover, assume that we have determined integers $p_l \in \mathbb{N}$, $l = j-1, j-2, ..., i$, $j-1 \geq i \geq 2-j$, such that

$$\begin{cases} |\psi_{i,j}>:=(L^{i,j})^{p_i}(L^{i+1,j})^{p_{i+1}}...(L^{j-1,j})^{p_{j-1}}|\psi_{j-1}>\neq 0, \\ L^{i,j}|\psi_{i,j}>=0 \qquad U_q^{+,2j-1+h}|\psi_{i,j}>=0 \end{cases} \tag{7.68}$$

Then we can determine an integer $p_{i-1} \in \mathbb{N}$ such that

$$\begin{cases} |\psi_{i-1,j}>:=(L^{i-1,j})^{p_{i-1}}(L^{i,j})^{p_i}...(L^{j-1,j})^{p_{j-1}}|\psi_{j-1}>\neq 0, \\ L^{i-1,j}|\psi_{i-1,j}>=0 \qquad U_q^{+,2j-1+h}|\psi_{i-1,j}>=0 \end{cases} \tag{7.69}$$

In fact, on one hand we set $L \equiv L^{i-1,j}$ and try to apply lemma 9. It is well known that when L is not simple the Cartan-Weyl partner of L, L^-, differs from $(L^{i-1,j})^\dagger \propto L^{-j,1-i}$;

its polynomial expression in terms of simple negative roots can be obtained from the one of $L^{-j,-i}$ by the replacement $q \to q^{-1}$. Viceversa, the polynomial expression in terms of simple positive roots of $L' := (L^-)^\dagger$ can be obtained from the one of $L^{i-1,j}$ by the same replacement. For instance,

$$L := L^{-1,3} \propto [L^{-1,2}, L^{-2,3}]_q \qquad (L)^\dagger = L^{-3,1} \propto [L^{-3,2}, L^{-2,1}]_q; \tag{7.70}$$

$$L^- \propto [L^{-3,2}, L^{-2,1}]_{q^{-1}} \qquad L' \propto [L^{-1,2}, L^{-2,3}]_{q^{-1}}. \tag{7.71}$$

One can easily verify that in any case we can find $u' \in U_q^{+,N-2}$, $\quad u \in U_q^{+,N}$ $\quad a > 0$ such that

$$L' = aL + uu' \qquad \Rightarrow \qquad L^-L' = aL^-L + L^-uu'. \tag{7.72}$$

Now the second term in the RHS can be neglected because it always gives zero when applied to $(L^{i-1,j})^m|\psi_{i,j}>$, because of the induction hypothesis (7.68). Therefore we are in the conditions to apply lemma 9 to the subspace $\mathcal{H}_{\vec{0},\lambda}$ containing $|\psi_{i,j}>$, since L^-L' is positive definite and L, L^- belong to a Cartan-Weyl triple.

On the other hand, one can easily show that the generators of $U_q^{+,2j-1+h}$ $L^{1-k,k}$, with $k = h+1, h+2, ..., j-1$, annihilate $|\psi_{i-1,j}>$ as well, by application of the commutation relations (7.64),(7.65), their consequences

$$L^{1-k,k}(L^{-k,j})^p = q^p(L^{-k,j})^pL^{1-k,k} + aq^p(L^{-k,j})^{p-1}L^{1-k,j} \qquad a > 0 \tag{7.73}$$

$$L^{1-k,k}(L^{k-1,j})^p = q^p(L^{-k,j})^pL^{1-k,k} + a'q^p(L^{-k,j})^{p-1}L^{k,j} \qquad a' > 0 \tag{7.74}$$

and the induction hypothesis. In the least simple case, $i-1 \leq -k$, for instance,

$$\begin{aligned}
L^{1-k,k}|\psi_{i-1,j}> &\overset{(7.64)_c)}{\propto} ...L^{1-k,k}(L^{-k,j})^{p_{-k}}...(L^{j-1,j})^{p_{j-1}}|\psi_{j-1}> \\
&\overset{(7.73)}{\propto} ...[(L^{-k,j})^{p_{-k}}L^{1-k,k} + a(L^{-k,j})^{p-1}L^{1-k,j}](L^{1-k,j})^{p_{1-k}}...(L^{j-1,j})^{p_{j-1}}|\psi_{j-1}> \\
&\overset{(7.68)_b}{=} ...L^{1-k,k}(L^{k-1,j})^{p_{k-1}}...(L^{j-1,j})^{p_{j-1}}|\psi_{j-1}> \\
&\overset{(7.74)}{\propto} ...[(L^{k-1,j})^{p_{k-1}}L^{1-k,k} + a'(L^{k-1,j})^{p_{k-1}-1}L^{k,j}](L^{k,j})^{p_k}...(L^{j-1,j})^{p_{j-1}}|\psi_{j-1}> \\
&\overset{(7.68)_b}{\propto} ......(L^{j-1,j})^{p_{j-1}}L^{1-k,k}|\psi_{j-1}> \overset{(7.68)_c}{=} 0.
\end{aligned} \tag{7.75}$$

The dots stand for the same powers of the roots L which appear in the definition $(7.68)_a$ of $|\psi_{i-1,j}>$.

Finally define

$$|\psi_j> := (L^{1-j,j})^{p_{1-j}}(L^{2-j,j})^{p_{2-j}}...(L^{j-1,j})^{p_{j-1}}|\psi_{j-1}>; \tag{7.76}$$

it is easy to show the relation $U_q^{+,2j+1+h}|\psi_j>=\{\mathbf{0}\}$, i.e. the induction hypothesis for the subsequent step.

In fact, on one hand $L^{1-j,j}|\psi_j>=0$ trivially, because of equation (7.69) and the definition (7.76); on the other, $U_q^{+,2j-1+h}|\psi_j>=\{\mathbf{0}\}$ because $U_q^{+,2j+1+h}$ annihilates by construction each of the vectors (7.68)

The first part of the thesis is proved if we set $|\phi>=|\psi_{1-n,n}>$. $\diamondsuit$.

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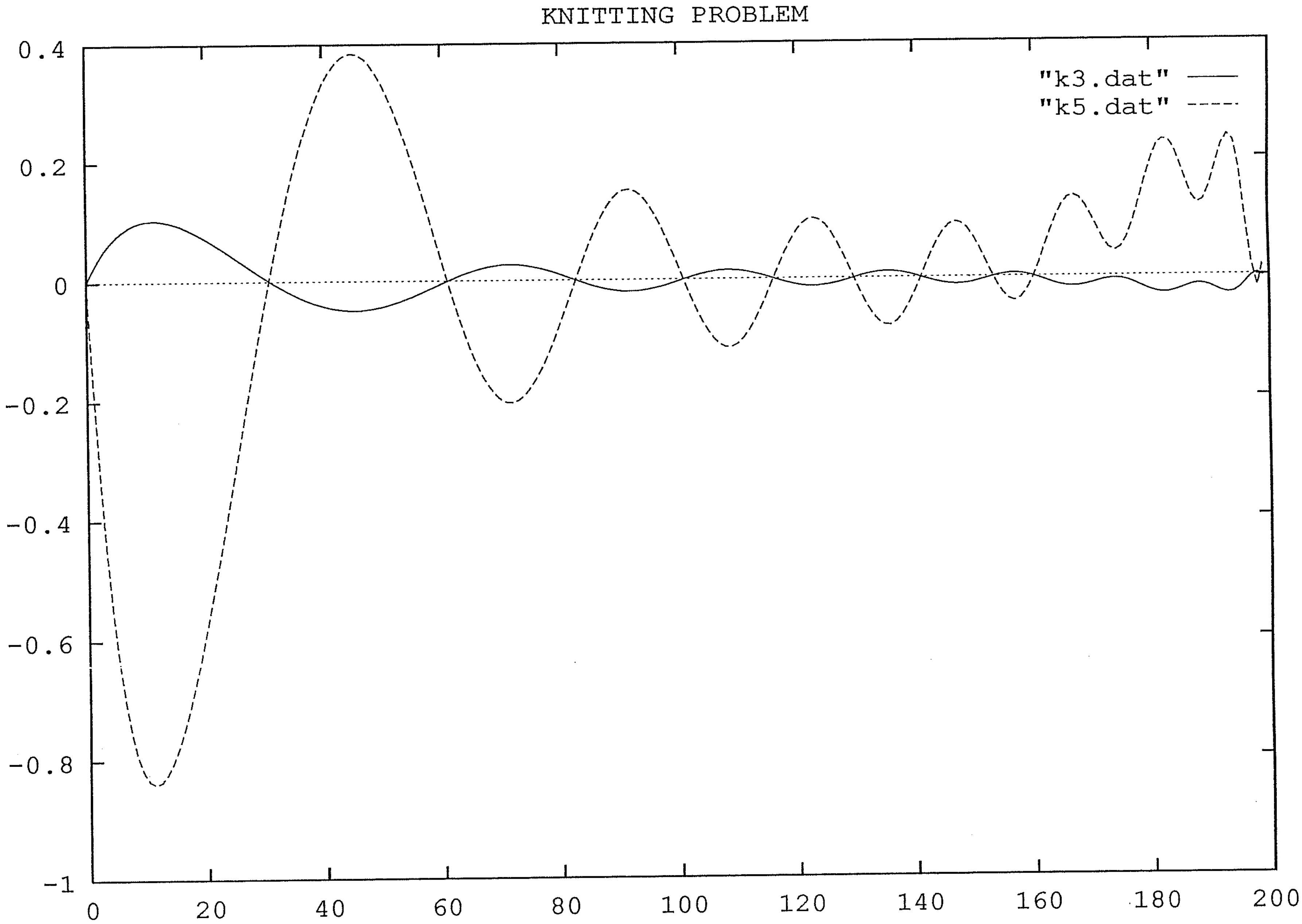
```
/* THIS CODE GENERATES THE POSSIBLE CONFIGURATIONS FOR  TWO SAW's ON     */
/* A STRIP OF WIDTH n                                                    */



#include stdio.h

typedef char column[4];         /* define strip columns as char vectors */

main()
{
        column config[10];      /* a vector of column type elements contains */
                                /* all the configurations                    */

        char hole=0;            /* the hole variable tells if there are       */
                                /* holes to be filled in the present config.  */
                                /* the value 0 means that there are no holes  */

        char tail=0;            /* the same as the hole variable, but for    */
                                /* the tail of the configuration             */

        char label;             /* the label variable tells which site       */
                                /* is connected to which by assigning them   */
                                /* the same number                           */


        int n=4;
        int m;

        int h,i;                /* i spans the vector of configurations,      */

        int j,k;                /* j,k span the single column-configuration  */

        int l;                  /* significative length of the configuration */

        int subconfig[20]       /* the vector subconfig contains the final   */
                                /* index for the configuration of length l   */

        int p,q;                /* indices for the subconfig vector          */

        m = n-1;

        for (j=0 ; j<n ; j++)   config[0][j] = 0;
                                /* puts to zero (empty) the first config     */


        for (l=2, i=1 ; l<=n ; l++)
        /* l spans all the possible significative configuration lengths */
        {
            for (j=0 ; j<(n-1) ; j++) config[i][j] = 0;
            j = n-1;
            label = 1;
            config[i][j] = label;
            for (k=j+1 ; k<n ; k++)
            {
                config[i][k] = label;
                if ((k-j-1)>=2)
                {
                   hole = 1;
```

```
           p = k-j-1;
        }
        if ((n-k-1)>=2)
        {
           tail = 1;
           q = n-k-1;
        }


        subconfig[l] = i;
        if (hole)
             for (h=1 ; h<=subconfig[p] ; h++)
             {
                  i=i+1;
                  config[i][j] = label;
                  config[i][k] = label;
                  for (z=1 ; z<=p ; z++)
                       config[i][j+z] = config[h][n-p+z-1]+label;
                  subconfig[l] = i;
             }
        if (tail)
             for (h=1 ; h<=subconfig[q] ; h++)
             {
                  i=i+1;
                  config[i][j] = label;
                  config[i][k] = label;
                  for (z=1 ; z<=q ; z++)
                       config[i][k+z] = config[h][n-q+z-1]+label;
                  subconfig[l] = i;
             }
        if (tail && hole)
             for (h=1 ; h<=subconfig[p] ; h++)
             {
                  i=i+1;
                  config[i][j] = label;
                  config[i][k] = label;
                  label1 = label;
                  for (z=1 ; z<=p ; z++)
                  {
                       temp = config[h][n-p+z-1]+label;
                       config[i][j+z] = temp;
                       if (label1 < temp) label1 = temp;
                  }
                  for (f=1 ; f<subconfig[q] ; f++)
                       for (w=1 ; w<=q ; w++)
                            config[i][k+w] = config[f][n-q+w-1]+label1;
```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140
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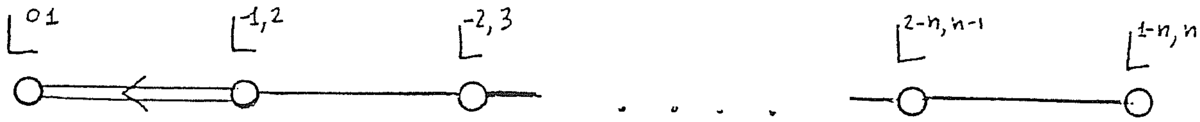
"k5.dat"

Fig. 1

Series B_n :



Series D_n :