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# Some Problems in the Asymptotic Analysis of Partial Differential Equations in Perforated Domains


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# Contents

# Introduction

The subject of this dissertation is the study of some aspects of the asymptotic behaviour of solutions of linear second order partial differential equations with Dirichlet boundary conditions in varying domains. More precisely, let $\Omega$ be an open subset of $\mathbf{R}^n$ and let $(\Omega_h)$ be an arbitrary sequence of open subsets of $\Omega$. Assume for simplicity that $\Omega$ is a bounded set with smooth boundary and that the partial differential operator is the Laplacian. We refer to the single chapters for the precise hypotheses. Let $f \in H^{-1}(\Omega)$. For each $h \in \mathrm{N}$ let $u_h$ be the solution of the Dirichlet problem

$$(0.0.1) \qquad \begin{cases} u_h \in H_0^1(\Omega_h) \\ -\Delta u_h = f \ \text{ in } \Omega_h \,, \end{cases}$$

extended by zero on $\Omega \backslash \Omega_h$. Using the variational method it can be easily proved that the sequence $(u_h)$ is bounded and so it has a subsequence that converges weakly in $H_0^1(\Omega)$ to some function $u$. We are interested in the equation satisfied by the limit function $u$.

One form of this equation can be obtained in the following way. Let $w_h$ be the solution of the Dirichlet problem

$$(0.0.2) \qquad \begin{cases} w_h \in H_0^1(\Omega_h) \\ -\Delta w_h = 1 \quad \text{in } \Omega_h \,, \end{cases}$$

extended by zero on $\Omega \backslash \Omega_h$. Like $(u_h)$, the sequence $(w_h)$ has a weak limit $w$ in $H_0^1(\Omega)$. Let now $\varphi \in C_0^\infty(\Omega)$ and take $w_h\varphi$ and $u_h\varphi$ as test functions in (0.0.1) and (0.0.2), respectively. Writing the equations in the weak form we obtain

$$\int_\Omega Du_h D(w_h\varphi)\,dx - \int_\Omega Dw_h D(u_h\varphi)\,dx + \langle 1, u_h\varphi\rangle \;=\; \langle f, w_h\varphi\rangle \,,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product. Note that here we can pass to the limit since the terms containing $Du_h Dw_h$ cancel out. Thus we get that $u$ is a solution of the problem

$$(0.0.3) \qquad \begin{cases} u \in H_0^1(\Omega) \\ \displaystyle\int_\Omega DuD(w\varphi)\,dx - \int_\Omega DwD(u\varphi)\,dx + \langle 1, u\varphi\rangle \;=\; \langle f, w\varphi\rangle \,, \ \forall \varphi \in C_0^\infty(\Omega) \,. \end{cases}$$

In Chapter 2 we provide a self-contained presentation of the properties of the solutions of an equation of the form (0.0.3), in the more general case of a linear elliptic operator of the second order with bounded measurable coefficients. The results mentioned therein were obtained in a joint work with G. Dal Maso (see [18]).

Remark that, if in (0.0.1) we consider an elliptic operator $A$ instead of the Laplacian, in (0.0.2) the adjoint operator $A^*$ will appear. Let us notice also that the solution $u_h$ of (0.0.1) satisfies an equation of the form (0.0.3) with $w$ replaced by the solution $w_h$ of (0.0.2).

Let $K = \{w \in H_0^1(\Omega) : -\Delta w \leq 1 \text{ in } \mathcal{D}'(\Omega),\ w \geq 0 \text{ a.e. in } \Omega\}$. Note that the solutions $w_h$ of (0.0.2) belong to $K$. We prove that for every $w \in K$ and $f \in H^{-1}(\Omega)$ there exists a unique solution $u$ of (0.0.3). Moreover the solution depends continuously on $w$ in the weak topology of $H_0^1(\Omega)$. We show also that the family of problems of type (0.0.3) with $w \in K$, is closed under the weak convergence of the solutions and that for each $w \in K$ there exists a sequence $(\Omega_h)$ of open subsets of $\Omega$ such that for every $f \in H^{-1}(\Omega)$ the solutions $u_h$ of (0.0.1) converge weakly in $H_0^1(\Omega)$ to the solution $u \in H_0^1(\Omega)$ of (0.0.3). This means that the family of problems of type (0.0.3) with $w \in K$ can be considered as the closure of the Dirichlet problems (0.0.1) with respect to the weak convergence in $H_0^1(\Omega)$.

Problem (0.0.3) was introduced by Dal Maso and Garroni in [13], where it is studied in an equivalent formulation. They proved that for every $w \in K$ there exists a positive Borel measure $\mu$ on $\Omega$, depending on $w$ and vanishing on the subsets of $\Omega$ of (harmonic) capacity zero, such that for each $f \in H^{-1}(\Omega)$ the solution $u$ of (0.0.3) coincides with the solution of the relaxed Dirichlet problem

$$(0.0.4) \qquad \begin{cases} u \in H_0^1(\Omega) \cap L_\mu^2(\Omega) \\ \langle -\Delta u, v\rangle + \displaystyle\int_\Omega uv\, d\mu = \langle f, v\rangle \\ \forall v \in H_0^1(\Omega) \cap L_\mu^2(\Omega). \end{cases}$$

Note that the relaxed Dirichlet problem (0.0.4) is not satisfied in the sense of distributions since we do not assume $\mu$ to be a Radon measure, and hence $C_0^\infty(\Omega)$ is, in general, not included in $H_0^1(\Omega) \cap L_\mu^2(\Omega)$. So one of the advantages of using problem (0.0.3) is that the space of admissible test functions is the usual space $C_0^\infty(\Omega)$. In Chapter 2, where we treat problem (0.0.3) independently of (0.0.4), we shall see that another advantage of (0.0.3) is that in the statements of the results (and most of the proofs) we do not have to use fine properties of Sobolev functions connected with capacity theory.

The following example shows that the limit function $u$ is not in general a solution of an equation of type (0.0.1), but of type (0.0.4). Consider the particular case in which

$\Omega = (0,1)^n$ and $(\Omega_h)$ is given by the following construction: for each $h \in \mathrm{N}$ divide $\Omega$ into $h^n$ cubes of edge $1/h$ and inside each such cube consider a closed ball of radius $r_h$, $0 < r_h < 1/(2h)$, concentric to the cube; let $E_h$ be the union of these balls and $\Omega_h = \Omega \setminus E_h$. Define, up to subsequences, a constant $k$ by

$$k = \begin{cases} \lim\limits_{h\to\infty} h^2/\ln(1/r_h)\,, & n = 2 \\ \lim\limits_{h\to\infty} h^n r_h^{n-2}\,, & n \geq 3\,. \end{cases} \tag{0.0.5}$$

Then it can be proved, using the results of [35] or [9], that the asymptotic behaviour of the sequence $(u_h)$ is determined by the constant $k$. If $k = +\infty$, then $u_h$ converge strongly in $H_0^1(\Omega)$ to zero; if $k = 0$, then $u_h$ converge strongly in $H_0^1(\Omega)$ to the solution $u$ of

$$\begin{cases} u \in H_0^1(\Omega) \\ -\Delta u = f \quad \text{in } \Omega\,. \end{cases}$$

If $0 < k < +\infty$, then the convergence is only weak and the limit function $u$ is the solution of

$$\begin{cases} u \in H_0^1(\Omega) \\ -\Delta u + c_n k u = f \quad \text{in } \Omega\,, \end{cases} \tag{0.0.6}$$

where $c_n$ is a strictly positive constant depending only on the dimension $n$ of the space. So, in the limit problem a new term appears, taking into account the asymptotic effect of the Dirichlet condition on the boundary of the balls. Roughly speaking, this is due to the fact that for $f \in L^2(\Omega)$, the equation satisfied by $u_h$ in $\Omega$ is

$$-\Delta u_h + \lambda_h = f \cdot 1_{\Omega_h}\,,$$

where $1_{\Omega_h}$ is the characteristic function of $\Omega_h$ and $\lambda_h$ is a measure concentrated on $\partial E_h$, whose physical meaning is that of charges induced on the boundary of the "conductor" $E_h$ kept at null potential.

Now we come back to the case of an arbitrary sequence $(\Omega_h)$ of open subsets of $\Omega$. It is known (see, for instance, [15]) that there exists a subsequence, which we still denote by $(\Omega_h)$, and a positive Borel measure $\mu$ on $\Omega$, absolutely continuous with respect to the (harmonic) capacity, such that the sequence $(u_h)$ weakly converges in $H_0^1(\Omega)$ to the solution $u$ of (0.0.4). If $\Omega$ is unbounded we replace equations (0.0.1) and (0.0.4) by

$$-\Delta u_h + \lambda u_h = f \quad \text{and} \quad \langle -\Delta u, v\rangle + \lambda \int_\Omega uv\,dx + \int_\Omega uv\,d\mu = f\,,$$

respectively, where $\lambda$ is some strictly positive real number.

This compactness result holds for a large class of linear elliptic equations and systems of equations. We treat in Chapter 3 the case of a vector valued function $u$ and a linear operator, studied together with G. Dal Maso in [19]. More precisely, let $\mathcal{A}\colon H^1_0(\Omega,\mathbf{R}^m)\to H^{-1}(\Omega,\mathbf{R}^m)$ be an elliptic operator of the form

$$\langle \mathcal{A}u, v\rangle = \int_\Omega (ADu, Dv)\,dx\,,$$

where $A(x)$ is a fourth order tensor and $(\cdot,\cdot)$ denotes the scalar product between matrices. Let $f\in H^{-1}(\Omega,\mathbf{R}^m)$ and let $u_h$ be, as above, the solution of the Dirichlet problem

$$\begin{cases} u_h \in H^1_0(\Omega_h,\mathbf{R}^m) \\ \mathcal{A}u_h = f \quad \text{in } \Omega_h\,, \end{cases} \tag{0.0.7}$$

extended by zero to $\Omega$. In the limit a relaxation phenomenon may occur. Namely, there exist an $m{\times}m$ matrix $T(x)$, with $|T(x)|=1$, and a measure $\mu$, not charging polar sets, such that the limit $u$ of the sequence $(u_h)$ is the solution of the relaxed Dirichlet problem

$$\begin{cases} u\in H^1_0(\Omega,\mathbf{R}^m)\cap L^2_\mu(\Omega,\mathbf{R}^m) \\ \displaystyle\int_\Omega (ADu,Dv)\,dx + \int_\Omega (Tu,v)\,d\mu = \langle f,v\rangle \\ \forall v\in H^1_0(\Omega,\mathbf{R}^m)\cap L^2_\mu(\Omega,\mathbf{R}^m)\,, \end{cases} \tag{0.0.8}$$

where, in the second integral, $(\cdot,\cdot)$ denotes the scalar product in $\mathbf{R}^m$. This result was established in [20] for symmetric $A$ and $T$, and in [7] in the general case.

The problem of finding the new term $\mu$ which appears in the limit was considered by many authors under different assumptions on the domains and on the operators. A wide bibliography on this subject can be found, for instance, in [12].

The capacity method allows us to determine it in a large variety of cases. We illustrate it here in the case of the Laplacian. For every $x\in\mathbf{R}^n$ let $D_\rho(x)$ be the closed ball of radius $\rho$ centered in $x$. Assume that the limit

$$\lim_{h\to\infty}\operatorname{cap}(D_\rho(x)\backslash\Omega_h,\Omega) = \alpha(D_\rho(x))$$

exists for every $x\in\Omega$ and for almost every $\rho>0$ such that $D_\rho(x)\subset\Omega$ and that for some bounded measure $\lambda$ on $\Omega$, absolutely continuous with respect to the capacity, we have $\alpha(D_\rho(x)) \le \lambda(D_\rho(x))$. The function $\alpha$ can be considered as an asymptotic capacity associated to the sequence $(\Omega_h)$. Then for $\lambda$-almost every $x \in \Omega$ the limit

$$\lim_{\rho\to 0} \frac{\alpha(D_\rho(x))}{\lambda(D_\rho(x))} = g(x)$$

exists and the measure $\mu$ is given by $\mu(E) = \int_E g(x)\, d\lambda$. The function $g$ can be seen as the asymptotic capacity density with respect to the measure $\lambda$.

In the case of a general elliptic operator (linear or not) a suitable notion of capacity associated to the operator has to be defined in order to repeat this construction. In Chapter 3 we use it to identify the pair $(T, \mu)$ which appears in the limit problem (0.0.8) and the notion of capacity we associate to the elliptic operator $\mathcal{A}$ is the following one. If $K$ is a compact subset of $\Omega$ and $\xi$, $\eta \in \mathbf{R}^m$, then the $\mathcal{A}$-capacity of $K$ in $\Omega$ relative to $\xi$ and $\eta$ is defined as

$$C_{\mathcal{A}}(K, \xi, \eta) = \int_{\Omega\setminus K} (ADu^\xi, Du^\eta)\, dx\,,$$

where, for every $\zeta \in \mathbf{R}^m$, $u^\zeta$ is the solution in $\Omega\setminus K$ of the Dirichlet problem

$$\begin{cases} u^\zeta \in H^1(\Omega\setminus K, \mathbf{R}^m)\,, \quad u^\zeta = \zeta \quad \text{on } \partial K\,, \quad u^\zeta = 0 \quad \text{on } \partial\Omega \\ \displaystyle\int_{\Omega\setminus K} (ADu^\zeta, Dv)\, dx = 0 \qquad \forall v \in H^1_0(\Omega\setminus K, \mathbf{R}^m)\,. \end{cases}$$

Note that we do not assume the system to be symmetric and this is the reason why we need two parameters $\xi$ and $\eta$ in the above definition of capacity (see the last section of Chapter 3 for the easier case of symmetric systems).

Assume that the limit

$$\lim_{h\to\infty} C_{\mathcal{A}}(D_\rho(x)\setminus\Omega_h, \xi, \eta) = \alpha(D_\rho(x), \xi, \eta)$$

exists for every $x \in \Omega$ and for almost every $\rho > 0$ such that $D_\rho(x) \subset \Omega$ (we shall see that this condition is always satisfied by a suitable subsequence). The main result of this chapter, Theorem 3.3.7, shows that, if the asymptotic capacity $\alpha$ can be majorized by a Kato measure $\lambda$, then for $\lambda$-almost every $x \in \Omega$ there exists an $m\times m$ matrix $G(x)$ such that

$$\operatorname*{ess\,lim}_{\rho\to 0} \frac{\alpha(D_\rho(x), \xi, \eta)}{\lambda(D_\rho(x))} = (G(x)\,\xi, \eta) \qquad \forall \xi, \eta \in \mathbf{R}^m\,.$$

Moreover, for every $f \in H^{-1}(\Omega, \mathbf{R}^m)$, the sequence $(u_h)$ of the solutions of (0.0.7) converges weakly in $H_0^1(\Omega, \mathbf{R}^m)$ to the solution $u$ of (0.0.8) with $T(x) = \frac{G(x)}{|G(x)|}$ and $\mu(E) = \int_E |G|\, d\lambda$. If $\mathcal{A}$ is symmetric, the same result (Theorem 3.4.3) holds under the weaker assumption that $\lambda$ is a bounded measure.

In the last two chapters we present some results on the asymptotic behaviour of the solutions of the wave equation on varying domains, which were originally published in [43] and [44]. We shall always assume that the solutions of problems (0.0.1) converge weakly in $H_0^1(\Omega)$ to the solution of problem (0.0.4). We show that a relaxation phenomenon occurs also for the wave equation, and that the measure $\mu$ appearing in the limit is the one that characterizes the asymptotic behaviour of the corresponding elliptic Dirichlet problems. In the fourth chapter we take up the case of finite time intervals.

Let $T > 0$. On the cylinders $Q_h = \Omega_h \times (0,T)$ we consider the problem

$$(0.0.9) \qquad \begin{cases} \dfrac{\partial^2 u_h}{\partial t^2} - \Delta u_h = f_h \text{ in } Q_h \\ u_h = 0 \quad \text{on } \partial\Omega_h \times (0,T) \\ u_h(0) = u_h^0 \quad \text{in } \Omega_h \\ \dot{u}_h(0) = u_h^1 \quad \text{in } \Omega_h. \end{cases}$$

Abstract results (see, e.g., [32]) show the existence and uniqueness of a solution $u_h$ of (0.0.9) satisfying $u_h \in C^0([0,T]; H_0^1(\Omega_h)) \cap C^1([0,T]; L^2(\Omega_h))$. We are interested in the asymptotic behaviour of the solutions $u_h$ of (0.0.9) as $h$ tends to infinity, without imposing any geometric restriction on $\Omega_h$. To characterize it, is enough to know the behaviour of the solutions of the corresponding elliptic Dirichlet problems.

Denoting by $H$ the closure of $H_0^1(\Omega) \cap L_\mu^2(\Omega)$ in the $L^2$-norm, we prove the following result. If the data of problem (0.0.9) satisfy

$$\begin{aligned} &f_h \rightharpoonup f \quad w\text{-}L^1(0,T; L^2(\Omega)), \\ &u_h^0 \rightharpoonup u^0 \quad w\text{-}H_0^1(\Omega), \\ &u_h^1 \rightharpoonup u^1 \quad w\text{-}L^2(\Omega) \quad \text{and } u^1 \in H, \end{aligned}$$

then the solutions $u_h$ of (0.0.9) converge to a function $u$ in the following sense

$$(0.0.10) \qquad u_h \rightharpoonup u \quad w^*\text{-}L^\infty(0,T; H_0^1(\Omega)),$$

$$(0.0.11) \qquad \dot{u}_h \rightharpoonup \dot{u} \quad w^*\text{-}L^\infty(0,T; L^2(\Omega)),$$

and the limit function $u$ is the solution of the relaxed evolution problem

$$(0.0.12)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \mu u = f \text{ in } Q = \Omega \times (0,T) \\ u = 0 \text{ on } \partial\Omega \times (0,T) \\ u(0) = u^0 \quad \text{in } \Omega \\ \dot{u}(0) = u^1 \quad \text{in } \Omega\,, \end{cases}$$

where the same measure $\mu$ that characterizes the limit of elliptic Dirichlet problems appears. Under stronger assumptions on the convergence of the data of (0.0.9) we obtain also the convergence of the energies. More precisely, if

$$\begin{aligned} & f_h \to f \quad s\text{-}L^1(0,T;L^2(\Omega))\,, \\ & u_h^0 \rightharpoonup u^0 \quad w\text{-}H_0^1(\Omega) \quad \text{and} \int_{\Omega_h} |Du_h^0|^2\,dx \to \int_\Omega |Du^0|^2 + \int_\Omega |u^0|^2\,d\mu\,, \\ & u_h^1 \to u^1 \quad s\text{-}L^2(\Omega) \quad \text{and } u^1 \in H\,, \end{aligned}$$

then (0.0.10) and (0.0.11) hold with $u$ solution of (0.0.12) and, in addition,

$$\int_{\Omega_h} |Du_h(\cdot)|^2\,dx \to \int_\Omega |Du(\cdot)|^2\,dx + \int_\Omega |u(\cdot)|^2\,d\mu \quad s\text{-}C^0([0,T])$$
$$\dot{u}_h(\cdot) \to \dot{u}(\cdot) \quad s\text{-}C^0([0,T];L^2(\Omega))\,.$$

Our results extend those of Cioranescu, Donato, Murat and Zuazua obtained in [10] under the assumption that the limit measure $\mu$ is a nonnegative Radon measure belonging to $H^{-1}(\Omega)$.

In the last chapter we study the behaviour, as the time goes to infinity, of the solutions of the wave equation on varying domains. We consider now the sets $\Omega_h$ of the following form. Let $(K_h)$ be a uniformly bounded sequence of compact subsets of $\mathbf{R}^n$ such that for every $h \in \mathbf{N}$, $\Omega_h = \mathbf{R}^n \setminus K_h$ is connected. The connectedness assumption implies, see, e.g., Theorem XI.91.5 in [38], that the Laplacian with Dirichlet boundary conditions on $\Omega_h$ has an absolutely continuous spectrum. We assume also that the limit operator $A = -\Delta + \mu$ has an absolutely continuous spectrum. Let us remark that the results in [38] imply that this assumption is satisfied for instance, by the measure arising from the sequences of domains considered in the example (0.0.5), (0.0.6), and more in general for those considered in [9], [41] and [35]. The behaviour of the solution $u_h$ of

$$
\begin{cases}
\dfrac{\partial^2 u_h}{\partial t^2} - \Delta u_h = 0 \quad \text{in } \Omega_h \times \mathbf{R} \\
u_h = 0 \quad \text{on } \partial\Omega_h \times \mathbf{R} \\
u_h(0) = u_h^0 \\
\dot{u}_h(0) = u_h^1
\end{cases}
\tag{0.0.13}
$$

as the time $t$ goes to infinity can be decribed using the wave operators $W_h$ which associate to the initial data $(u_h^0, u_h^1)$ of (0.0.13) an initial condition for the wave equation on the whole space $\mathbf{R}^n$ such that $u_h$ is asymptotically equal, as the time goes to infinity, to the solution of the free space equation, see Definition 5.2.1. The existence of $W_h$ was proved for instance, in [29] together with some fundamental properties of the wave operators. We are interested in the behaviour of $W_h$ as $h$ goes to infinity.

Following the lines of [29] we prove the existence and unitarity of the wave operator $W$ for the relaxed wave equation (see Theorem 5.2.6). Then we prove that the wave operators $W_h$ converge to $W$ in the following sense: if $\eta \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ is such that $\operatorname{supp} \eta \cap K_h = \emptyset$ for every $h$, then $W_h \eta$ converges to $W\eta$ in the energy norm (see Theorem 5.3.2). Our result generalizes for this relaxed formulation the one obtained by Rauch and Taylor (see [37]), which was confined to the case when the asymptotic elliptic Dirichlet problem (0.0.4) is not a relaxed one. The crucial step in the proof of this result is a uniform energy estimate (see Theorem 5.3.1) which shows that it is enough to know the behaviour of the solutions for finite time intervals and allows us to use the results of the previous chapter.

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# Chapter 1. Notation and preliminaries

## 1.1. Sobolev spaces and capacity

Let $\Omega$ be an open subset of $\mathbf{R}^n$, $n \geq 3$. If $x$, $y \in \mathbf{R}^n$, $x \cdot y$ denotes their scalar product; the Euclidean norm in $\mathbf{R}^n$ is denoted by $|\cdot|$. $\mathbf{M}^{n\times m}$ will denote the space of $n \times m$ matrices. Notice that, if $M \in \mathbf{M}^{n\times m}$ we shall write $|M|$ to denote its Euclidean norm as an element of $\mathbf{R}^{nm}$.

The space $\mathcal{D}'(\Omega)$ of distributions in $\Omega$ is the dual of $C_0^\infty(\Omega)$. Given two numbers $p$ and $q$, with $1 < p, q < +\infty$ and $1/p + 1/q = 1$, let $W^{1,p}(\Omega, \mathbf{R}^m)$ denote the usual Sobolev space, i.e. the space of all functions $u$ in $L^p(\Omega, \mathbf{R}^m)$ whose first order distribution derivatives $D_j u$ belong to $L^p(\Omega, \mathbf{R}^m)$, endowed with the norm

$$\|u\|^p_{W^{1,p}(\Omega,\mathbf{R}^m)} = \int_\Omega |Du|^p dx + \int_\Omega |u|^p dx\,,$$

where $Du = (D_j u^\alpha)$ is the Jacobian matrix of $u$. The space $W_0^{1,p}(\Omega, \mathbf{R}^m)$ is the closure of $C_0^\infty(\Omega, \mathbf{R}^m)$ in $W^{1,p}(\Omega, \mathbf{R}^m)$, and $W^{-1,q}(\Omega, \mathbf{R}^m)$ is the dual of $W_0^{1,p}(\Omega, \mathbf{R}^m)$. The symbol $\mathbf{R}^m$ will be omitted when $m = 1$. When $p = q = 2$ we shall use the notation $H^1(\Omega, \mathbf{R}^m)$, $H_0^1(\Omega, \mathbf{R}^m)$ and $H^{-1}(\Omega, \mathbf{R}^m)$, respectively.

For every subset $E$ of $\Omega$ the (harmonic) capacity of $E$ with respect to $\Omega$ is defined by $\mathrm{cap}(E) = \inf \int_\Omega |Du|^2 dx$, where the infimum is taken over all functions $u \in H_0^1(\Omega)$ such that $u \geq 1$ almost everywhere in a neighbourhood of $E$, with the usual convention $\inf \emptyset = +\infty$. We say that a property $\mathcal{P}(x)$ holds quasi everywhere (abbreviated as q.e.) in a set $E$ if it holds for all $x \in E$ except for a subset $N$ of $E$ with $\mathrm{cap}(N) = 0$. The expression almost everywhere (abbreviated as a.e.) refers, as usual, to the Lebesgue measure.

A function $u : \Omega \to \mathbf{R}^m$ is said to be quasicontinuous if for every $\varepsilon > 0$ there exists a set $E \subset \Omega$, with $\mathrm{cap}(E) \leq \varepsilon$, such that the restriction of $u$ to $\Omega \backslash E$ is continuous. We recall that for every $u \in H_0^1(\Omega, \mathbf{R}^m)$ there exists a quasicontinuous function $\tilde{u}$, unique up to sets of capacity zero, such that $u = \tilde{u}$ almost everywhere in $\Omega$. We shall always identify $u$ with its quasicontinuous representative $\tilde{u}$, so that the pointwise value of a function $u \in H_0^1(\Omega, \mathbf{R}^m)$ is defined quasi everywhere in $\Omega$.

For any $u \in H_0^1(\Omega)$ we shall denote by $u^+$ and $u^-$ the positive and the negative parts of $u$: $u^+ = u \vee 0$, $u^- = -(u \wedge 0)$. Then $u = u^+ - u^-$ and it can be easily proved that for any $u \in H_0^1(\Omega)$, $u^+$, $u^- \in H_0^1(\Omega)$.

If $U$ is an open subset of $\Omega$, each function $u \in H_0^1(U, \mathbf{R}^m)$ will always be extended to $\Omega$ by setting $u = 0$ in $\Omega \backslash U$. If $u \in H_0^1(\Omega, \mathbf{R}^m)$ and $u = 0$ q.e. on $\Omega \backslash U$, then $u \in H_0^1(U, \mathbf{R}^m)$. Let us recall that the weak convergence in $W_0^{1,p}(\Omega, \mathbf{R}^m)$ implies for a subsequence the strong convergence in $L^p$ on every compact set. Moreover, if a sequence $(u_h)$ converges to $u$ strongly in $W_0^{1,p}(\Omega, \mathbf{R}^m)$ then a subsequence of it converges to $u$ pointwise q.e. in $\Omega$. For fine properties of Sobolev functions and of their quasicontinuous representatives we refer to [48].

### 1.2. Measures

By a Borel measure on $\Omega$ we mean a nonnegative, countably additive set function with values in $[0, +\infty]$ defined on the $\sigma$-field $\mathcal{B}(\Omega)$ of all Borel subsets of $\Omega$; by a Radon measure on $\Omega$ we mean a Borel measure which is finite on every compact subset of $\Omega$. If $\mu$ is a Radon measure on $\Omega$, we shall always identify it with the corresponding continuous linear functional defined on the space of all continuous functions on $\Omega$ with compact support. For any Borel measure $\mu$, $L_\mu^r(\Omega)$, with $1 \leq r \leq +\infty$ denotes the usual Lebesgue space with respect to the measure $\mu$. When $\mu$ is the Lebesgue measure we use the standard notation $L^r(\Omega)$.

We denote by $\mathcal{M}_0(\Omega)$ the set of all positive Borel measures $\mu$ on $\Omega$ such that $\mu(E) = 0$ for every Borel set $E \subset \Omega$ with $\operatorname{cap}(E) = 0$. If $E$ is $\mu$-measurable in $\Omega$, we define the Borel measure $\mu \llcorner E$ by $(\mu \llcorner E)(B) = \mu(E \cap B)$ for every Borel set $B \subset \Omega$, while $\mu_{|_E}$ is the measure on $E$ given by $\mu_{|_E}(B) = \mu(B)$ for every Borel subset $B$ of $E$.

For every open subset $U \subset \Omega$ we define a Borel measure $\mu_U$ by

$$\mu_U(B) = \begin{cases} 0 & \text{if } \operatorname{cap}(B \backslash U) = 0\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{1.2.1}$$

for every $B \in \mathcal{B}(\Omega)$. As $U$ is open, it is easy to see that this measure belongs to $\mathcal{M}_0(\Omega)$.

For every $x \in \mathbf{R}^n$ and $\rho > 0$ we set $B_\rho(x) = \{y \in \mathbf{R}^n : |x - y| < \rho\}$, $D_\rho(x) = \overline{B}_\rho(x)$ and $B_\rho^c(x) = \mathbf{R}^n \backslash B_\rho(x)$. A special class of measures we shall frequently use is the Kato space.

**Definition 1.2.1.** The Kato space $K^+(\Omega)$ is the cone of all positive Radon measures $\mu$ on $\Omega$ such that

$$\lim_{\rho \to 0+} \sup_{x \in \Omega} \int_{\Omega \cap B_\rho(x)} |y - x|^{2-n} \, d\mu(y) = 0\,.$$

For every $\mu \in K^+(\Omega)$ and for every Borel subset $E$ of $\Omega$ we define

$$\|\mu\|_{K^+(E)} = \sup_{x\in E} \int_E |y-x|^{2-n}\, d\mu(y).$$

For every $\mu \in K^+(\Omega)$ it is easy to see that $\|\mu\|_{K^+(\Omega)} < +\infty$ and $\|\mu\|_{K^+(E)}$ tends to zero as $\operatorname{diam}(E)$ tends to zero. For more details about Kato measures we refer to [30] and [16].

If $X$ is a Banach space we shall denote by $\langle\cdot,\cdot\rangle$ the duality product between $X'$, the dual of $X$, and $X$ and it will be clear from the context which is the Banach space we refer to.

A positive Radon measure $\mu$ on $\Omega$ is said to belong to $W^{-1,q}(\Omega)$ if there exists $f \in W^{-1,q}(\Omega)$ such that

$$\langle f,\varphi\rangle = \int_\Omega \varphi d\mu \qquad \forall\varphi \in C_0^\infty(\Omega).$$

In this case the above equality holds for every $v \in W_0^{1,p}(\Omega)$ and we identify $f$ and $\mu$. Remark that if $\mu$ is a nonnegative Radon measure which belongs to $H^{-1}(\Omega)$, then $\mu \in \mathcal{M}_0(\Omega)$ and $H_0^1(\Omega) \subset L_\mu^1(\Omega)$.

Given an arbitrary subset $E \subset \mathbf{R}^n$ its characteristic function $1_E$ is defined by $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \in \mathbf{R}^n \setminus E$. For a function $v(x,t)$, $\dot v$ will denote the partial derivative with respect to $t$ and $Dv$ the gradient in $x$.

For $\mu \in \mathcal{M}_0(\Omega)$ we set $V = H_0^1(\Omega) \cap L_\mu^2(\Omega)$. Then $V$ is a Hilbert space with the scalar product given by

$$(u,v)_V = \int_\Omega DuDv\, dx + \int_\Omega uv\, dx + \int_\Omega uv\, d\mu\,.$$

We denote by $H$ the closure of $V$ with respect to the $L^2$-norm. Then $V \subset H \cong H' \subset V'$ with dense and continuous imbeddings (compact if $\Omega$ is bounded). Let us remark that functions in $V$ with compact support are dense in $V$ with respect to the norm induced by the above scalar product. Note that if $\mu = \mu_U$ for some open subset $U$ of $\Omega$ then $V = H_0^1(U)$.

**Definition 1.2.2.** For every measure $\mu \in \mathcal{M}_0(\Omega)$ we define $A_\mu$ as the union of all finely open subsets $A$ of $\Omega$ such that $\mu(A)$ is finite and by $S_\mu$ we denote the complementary of $A_\mu$. For the definition and properties of the fine topology we refer to [22].

In [4] it was proved that functions in $V$ are equal to zero q.e. on $S_\mu$. A similar characterization holds for $H$.

**Proposition 1.2.3.** *The space $H$ can be characterized by*

$$H = \{u \in L^2(\Omega) \,|\, u = 0 \ a.e. \ on \ S_\mu\}.$$

*Proof.* Let $Y = \{u \in L^2(\Omega) \,|\, u = 0$ a.e. on $S_\mu\}$. Let $u \in H$. Then there exists a sequence $u_n \in V$ such that $u_n \to u$ strongly in $L^2(\Omega)$, so there exists a subsequence converging pointwise a.e. to $u$. Since $u_n \in V$, $u_n = 0$ q.e. on $S_\mu$, hence $u = 0$ a.e. on $S_\mu$ and $H \subset Y$.

We have to prove now the opposite inclusion. Since there exists an increasing sequence $A_k$ of finely open sets such that $\mu(A_k) < \infty$ and $\mathrm{cap}(A_\mu \backslash \cup_k A_k) = 0$ (see [4]), it is enough to show that for a bounded finely open set $A \subset A_\mu$ with $\mu(A) < \infty$, we can approximate $1_A$ with functions in $V$. By Proposition 1.2 in [4] there exists an increasing sequence $u_n$ of functions in $H^1_0(A)$ such that $u_n \to 1_A$ pointwise q.e. (hence pointwise $\mu$-a.e.) and $0 \le u_n \le 1_A$. Since $\mu(A) < \infty$, $u_n \in L^2_\mu(\Omega)$, hence $u_n \in V$. □

### 1.3. $\gamma$-convergence and relaxed Dirichlet problems

With a measure $\mu \in \mathcal{M}_0(\Omega)$ we associate the functional $F_\mu$ defined by

$$F_\mu(u) = \begin{cases} \displaystyle\int_\Omega |Du|^2\,dx + \int_\Omega |u|^2\,d\mu & \text{if } u \in H^1_0(\Omega) \cap L^2_\mu(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \backslash (H^1_0(\Omega) \cap L^2_\mu(\Omega)). \end{cases}$$

On $\mathcal{M}_0(\Omega)$ we consider then the following notion of convergence introduced in [15] (see also Definition 2.7, Theorem 2.1 in [3]).

**Definition 1.3.1.** We say that a sequence $(\mu_h)$ of measures in $\mathcal{M}_0(\Omega)$ $\gamma$-converges to the measure $\mu \in \mathcal{M}_0(\Omega)$ if the functionals $F_{\mu_h}$ $\Gamma$-converge in $L^2(\Omega)$ to the functional $F_\mu$, that is if the following two conditions are satisfied:

(*i*) for every $u \in L^2(\Omega)$ and for every sequence $(u_h)$ converging to $u$ in $L^2(\Omega)$

$$F_\mu(u) \le \liminf_{h\to\infty} F_{\mu_h}(u_h);$$

(*ii*) for every $u \in L^2(\Omega)$ there exists a sequence $(u_h)$ converging to $u$ in $L^2(\Omega)$ such that

$$F(u) \ge \limsup_{h\to\infty} F_{\mu_h}(u_h).$$

We refer to [11] for the main properties of the $\Gamma$-convergence.

Let us recall here some results on the class $\mathcal{M}_0(\Omega)$. If $\mu_h \in \mathcal{M}_0(\Omega)$ $\gamma$-converge to a measure $\mu \in \mathcal{M}_0(\Omega)$ then for each open subset $\omega$ of $\Omega$ the sequence $(\mu_h^\omega)$ of the restrictions of the measures $\mu_h$ to $\omega$, $\gamma$-converges in $\omega$ to the restriction $\mu^\omega$ of $\mu$ to $\omega$ (see Theorem 4.10 of [13]). Moreover, for every $\mu \in \mathcal{M}_0(\Omega)$ there exists a sequence $S_h$ of compact subsets of $\Omega$ such that the sequence $(\infty_{S_h})$ $\gamma$-converges to $\mu$ (see Theorem 4.16 of [15]).

**Definition 1.3.2.** Given $\mu \in \mathcal{M}_0(\Omega)$ and $g \in H^{-1}(\Omega)$, a function $v$ is a solution of the relaxed Dirichlet problem

$$\begin{cases} v \in V \\ -\Delta v + v\mu = g \quad \text{in } \Omega \end{cases}$$

if $v \in V$ and for every $\varphi \in V$

$$\int_\Omega Dv D\varphi \, dx + \int_\Omega v\varphi \, d\mu = \langle g, \varphi \rangle .$$

If $\mu = \mu_U$ for some open subset $U$ of $\Omega$ then $v$ is a solution of the relaxed Dirichlet problem above if and only if $v \in H_0^1(U)$ and $-\Delta v = g$ in $U$. Note that for unbounded domains $\Omega$, a solution exists if and only if $\mu > \lambda m$ for some $\lambda > 0$, where $m$ denotes the Lebesgue measure on $\mathbf{R}^n$, see, for instance, [3]. Remark also that in general $v$ is not a solution in the sense of distributions and if we define $\beta = f + \Delta v$ then $\beta \in H^{-1}(\Omega)$ and $\operatorname{supp} \beta \subset \operatorname{supp} \mu$.

It can be proved that a sequence $(\mu_h)$ of measures in $\mathcal{M}_0(\mathbf{R}^n)$ $\gamma$-converges to the measure $\mu \in \mathcal{M}_0(\mathbf{R}^n)$ if and only if for any $g \in H^{-1}(\mathbf{R}^n)$ the solutions $v_h$ of the relaxed Dirichlet problems

$$\begin{cases} v_h \in H^1(\mathbf{R}^n) \cap L^2_{\mu_h}(\mathbf{R}^n) \\ -\Delta v_h + v_h + \mu_h v_h = g \quad \text{in } \mathbf{R}^n \end{cases} \tag{1.3.1}$$

weakly converge in $H^1(\mathbf{R}^n)$ to the solution $v$ of the relaxed Dirichlet problem

$$\begin{cases} v \in H^1(\mathbf{R}^n) \cap L^2_{\mu}(\mathbf{R}^n) \\ -\Delta v + v + \mu v = g \quad \text{in } \mathbf{R}^n. \end{cases} \tag{1.3.2}$$

For bounded domains $\Omega$ it is enough to consider $g = 1$. These particular solutions will play an important role in the sequel so let us fix a notation. If $A$ is the elliptic operator we study and $\Omega$ is bounded, let $w$ be the solution of the relaxed Dirichlet problem

$$\begin{cases} w \in H^1_0(\Omega) \cap L^2_\mu(\Omega) \\ Aw + w\mu = 1 \quad \text{in } \Omega\,. \end{cases} \tag{1.3.3}$$

If we consider a sequence $(\mu_h)$ of measures in $\mathcal{M}_0(\Omega)$ then $w_h$ will denote the solution of

$$\begin{cases} w_h \in H^1_0(\Omega) \cap L^2_{\mu_h}(\Omega) \\ Aw_h + w_h\mu_h = 1 \quad \text{in } \Omega\,. \end{cases} \tag{1.3.4}$$

It can also be proved (see [17]) that the set $\{w\varphi | \varphi \in C^\infty_0(\Omega)\}$ is dense in $V$. If $\Omega$ is unbounded we shall use in Chapter 4 test functions of the form $w\varphi$, where, for some $r > 0$, $w$ is the solution of problem (1.3.3) on $\Omega_r = \Omega \cap B_r$ and $\varphi \in C^\infty_0(\Omega_r)$. As $r > 0$ varies, the set of functions of this kind is dense in $V$.

# Chapter 2. Limits of Dirichlet problems on varying domains

The purpose of this chapter is to give an elementary description of the asymptotic behaviour of solutions of linear elliptic equations with Dirichlet boundary conditions in varying domains. When we say "elementary" we mean that we want to avoid (at least in the statement of the results) the use of fine properties of Sobolev functions connected with capacity theory.

Our problem can be described in the following way. Let $A$ be a linear elliptic operator of the second order with bounded measurable coefficients on a bounded open set $\Omega$ of $\mathbf{R}^n$. Let $f \in H^{-1}(\Omega)$ and let $(\Omega_h)$ be an arbitrary sequence of open subsets of $\Omega$. For each $h \in \mathrm{N}$ let us consider the solutions $u_h$ of the Dirichlet problem

$$(2.0.1) \qquad \begin{cases} u_h \in H_0^1(\Omega_h), \\ Au_h = f \ \text{ in } \Omega_h \end{cases}$$

and study the behaviour of $u_h$ when $h$ tends to infinity. Basic a priori estimates show that the sequence $(u_h)$ is bounded in $H_0^1(\Omega)$, hence it has a subsequence that converges weakly in $H_0^1(\Omega)$ to some function $u$. We want to find the equation satisfied by the limit function $u$. Let $w_h^*$ be the solution of the Dirichlet problem

$$(2.0.2) \qquad \begin{cases} w_h^* \in H_0^1(\Omega_h), \\ A^*w_h^* = 1 \ \text{ in } \Omega_h , \end{cases}$$

where $A^*$ is the adjoint operator of $A$. As we did for $u_h$, we can prove that $w_h^*$ has a weak limit $w^*$ in $H_0^1(\Omega)$. By taking now suitable test functions in (2.0.2) and (2.0.1) and passing to the limit we find that $u$ is a solution of the following problem:

$$(2.0.3) \qquad \begin{cases} u \in H_0^1(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \qquad \forall \varphi \in C_0^\infty(\Omega). \end{cases}$$

We provide here a self-contained presentation of the properties of the solutions of this equation and use it in the study of the asymptotic behaviour of the solutions of (2.0.1). Let us notice that the solution $u_h$ of (2.0.1) satisfies an equation of the form (2.0.3) with $w^*$ replaced by the solution $w_h^*$ of (2.0.2). We shall prove the existence and uniqueness of the solution of (2.0.3) and the continuous dependence of $u$ on $w^*$. More precisely, let us denote by $K^*$ the set of all functions $w^*$ such that

$$(2.0.4) \qquad w^* \in H_0^1(\Omega),\ A^*w^* \leq 1 \text{ in } \mathcal{D}'(\Omega), \text{ and } w^* \geq 0 \text{ a.e. in } \Omega.$$

Note that the solutions $w_h^*$ of (2.0.2) belong to $K^*$ (Proposition 2.1.1) and so does their limit. For any $w^* \in K^*$ and any $f \in H^{-1}(\Omega)$ we shall prove that there is one and only one solution of (2.0.3). Moreover, if $f \in L^\infty(\Omega)$, the solution $u$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. The estimates one can prove for this solution give its continuous dependence on $w^*$ in the weak topology of $H_0^1(\Omega)$. This shows, in particular, that the family of problems of type (2.0.3) with $w^* \in K^*$ is closed under the weak convergence of the solutions.

One of the advantages of studying limits of Dirichlet problems by using directly (2.0.3) is that some of the proofs can be made independently and in a rather elementary way. We do not have to use (in the statement of the results and most of the proofs) singular measures nor fine properties from capacity theory. Moreover the space of admissible test funtions is $C_0^\infty(\Omega)$. The degeneracy of the equation (2.0.3), that follows from the fact that $w^*$ can be zero on sets of positive measure, represents a difficulty of the problem but not a major one since it still allows us to prove the existence and uniqueness of the solution.

The third part of the chapter is devoted to the proof of the following density result. We shall show that for any $w^* \in K^*$ there exists a sequence $\Omega_h$ of open subsets of $\Omega$ such that for every $f \in H^{-1}(\Omega)$ the solutions $u_h$ of (2.0.1) converge weakly in $H_0^1(\Omega)$ to the solution $u$ of (2.0.3). This means that the family of problems of type (2.0.3) with $w^* \in K^*$ can be considered as the closure of the Dirichlet problems (2.0.1) with respect to the weak convergence in $H_0^1(\Omega)$. By the theorems proved in the previous sections it is enough to prove the existence of a sequence $\Omega_h$ such that the solutions $w_h^*$ of (2.0.2) converge weakly in $H_0^1(\Omega)$ to $w^*$. This will be done by using the method of Cioranescu and Murat [9] following a simplified version of [14].

## 2.1. Notation and preliminaries

Let us fix an $n\times n$ matrix $(a_{ij})$ of functions of $L^\infty(\mathbf{R}^n)$ satisfying, for a suitable constant $\alpha > 0$, the ellipticity condition

$$(2.1.1) \qquad \sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \geq \alpha|\xi|^2$$

for a.e. $x \in \mathbf{R}^n$ and for every $\xi \in \mathbf{R}^n$.

For every open set $U$ of $\mathbf{R}^n$ let $A : H^1(U) \to H^{-1}(U)$ and $A^* : H^1(U) \to H^{-1}(U)$ be the elliptic operators defined by

$$Au = -\sum_{i,j=1}^{n} D_i(a_{ij} D_j u) \qquad \text{and} \qquad A^* u = -\sum_{i,j=1}^{n} D_i(a_{ji} D_j u).$$

It is well known that, on $H_0^1(U)$, $A^*$ is the adjoint operator of $A$, that is: $\langle A^* u, v\rangle = \langle Av, u\rangle$ for every $u, v \in H_0^1(U)$.

Let $\Omega$ be a bounded open subset of $\mathbf{R}^n$. Let $K^*$ be the set of functions which satisfy (2.0.4); then it is easy to see that $K^*$ is a closed convex subset of $H_0^1(\Omega)$. Moreover, for every $w^* \in K^*$

$$\alpha \int_\Omega |Dw^*|^2\, dx \le \langle A^* w^*, w^*\rangle \le \int_\Omega w^*\, dx,$$

and this estimate shows that $K^*$ is bounded, and hence weakly compact in $H_0^1(\Omega)$.

Let $w_0^*$ be the solution of the Dirichlet problem

$$\begin{cases} w_0^* \in H_0^1(\Omega) \\ A^* w_0^* = 1 \quad \text{in } \Omega. \end{cases}$$

By the maximum principle we have $w^* \le w_0^*$ a.e. in $\Omega$ for every $w^* \in K^*$. As $w_0^* \in L^\infty(\Omega)$ (see [42]), the set $K^*$ is bounded in $L^\infty(\Omega)$.

Let us denote by $H^*$ the set of all functions $w^* \in H_0^1(\Omega)$ with the property that there exists an open subset $U$ of $\Omega$ such that $w^*$ is the solution of the Dirichlet problem

$$\begin{cases} w^* \in H_0^1(U) \\ A^* w^* = 1 \text{ in } U. \end{cases}$$

We shall show that the closure $\overline{H}^*$ of $H^*$ in the weak topology of $H_0^1(\Omega)$ coincides with $K^*$. We begin with the easier inclusion: $\overline{H}^* \subseteq K^*$. It is enough to prove that $H^* \subseteq K^*$.

**Proposition 2.1.1.** *Let $U$ be an open subset of $\Omega$ and let $w^*$ be the solution of the Dirichlet problem*

$$\text{(2.1.2)} \qquad \begin{cases} w^* \in H_0^1(U) \\ A^* w^* = 1 \textit{ in } U. \end{cases}$$

*Then $w^* \in K^*$.*

*Proof.* To prove that $w^* \in K^*$ we follow the argument of [5]. Let $z$ be the solution of the variational inequality

$$\begin{cases} z \in K_U \\ \langle A^* z - 1, v - z\rangle \geq 0 \qquad \forall v \in K_U\,, \end{cases}$$

where

$$K_U = \{v \in H_0^1(\Omega)\ :\ v \leq 0 \text{ a.e. on } \Omega \backslash U\}. \tag{2.1.3}$$

By the maximum principle we have $z \geq 0$ a.e. on $\Omega$ (see, e.g., [31], Chapter II, Theorem 6.4), so that $z = 0$ a.e. on $\Omega\backslash U$, hence $z \in H_0^1(U)$ (see, e.g., [2]). If $v \in H_0^1(U)$ and $v = 0$ a.e. on $\Omega\backslash U$, then $v \in K_U$. Therefore from the variational inequality we obtain easily that $z|_U$ is a solution of (2.1.2), hence $z = w^*$ a.e. in $\Omega$. Since all solutions of variational inequalities with an obstacle condition of the form (2.1.3) are subsolutions of the corresponding equation (see, e.g., [31], Chapter II, remark after Definition 6.3), we conclude that $A^* w^* \leq 1$ in $\Omega$ in the sense of distributions, hence $w^* \in K^*$. □

The inclusion $K^* \subseteq \overline{H}^*$ will be proved in the third section.

## 2.2. Some existence and uniqueness results for the limit problem

As mentioned in the introduction, we want to prove that for any $w^* \in K^*$ and any $f \in H^{-1}(\Omega)$ there exists one and only one solution of (2.0.3). In order to do this we need to prove first some lemmas. Let us begin with the case of $w^* \in W^{1,\infty}(\Omega)$.

**Lemma 2.2.1.** *Let $w^*$ be a function in $W^{1,\infty}(\Omega)$ such that $A^* w^* \leq 1$ in $\mathcal{D}'(\Omega)$ and $w^* \geq \varepsilon$ in $\Omega$, for some constant $\varepsilon > 0$. Then there exists a unique solution of the problem*

$$\begin{cases} u \in H_0^1(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^* w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \qquad \forall \varphi \in H_0^1(\Omega). \end{cases} \tag{2.2.1}$$

*Proof.* The existence of a unique solution of (2.2.1) is a consequence of the Lax-Milgram lemma. Indeed, let us consider the bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$ defined by:

$$a(u, \varphi) = \langle Au, w^*\varphi\rangle - \langle A^* w^*, u\varphi\rangle + \langle 1, u\varphi\rangle.$$

It can be easily seen that

$$a(u,\varphi) = \int_\Omega (\sum_{i,j=1}^n a_{ij} D_j u D_i \varphi) w^* \, dx - \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j w^* D_i \varphi) u \, dx + \int_\Omega u\varphi \, dx.$$

To show that $a$ is coercive we estimate

$$a(u,u) = \int_\Omega (\sum_{i,j=1}^n a_{ij} D_j u D_i u) w^* \, dx - \int_\Omega (\sum_{i,j=1}^n a_{ji} D_j w^* D_i u) u \, dx + \int_\Omega u^2 \, dx.$$

Since $w^* \in W^{1,\infty}(\Omega)$, the distribution $A^* w^*$ belongs to $H^{-1,\infty}(\Omega)$. Then the inequality $A^* w^* \le 1$ in $\mathcal{D}'(\Omega)$ implies that $\langle 1 - A^* w^*, v\rangle \ge 0$ for every $v \in H_0^{1,1}(\Omega)$, $v \ge 0$. As $u \in H_0^1(\Omega)$, we have $u^2 \in H_0^{1,1}(\Omega)$ and $u D_i u = \frac{1}{2} D_i(u^2)$. Then

$$-\int_\Omega (\sum_{i,j=1}^n a_{ji} D_j w^* D_i u) u \, dx + \frac{1}{2}\int_\Omega u^2 \, dx = \frac{1}{2}\langle 1 - A^* w^*, u^2\rangle \ge 0.$$

Since $w^* \ge \varepsilon$ a.e. in $\Omega$, the ellipticity condition (2.1.1) implies that

$$\int_\Omega (\sum_{i,j=1}^n a_{ij} D_j u D_i u) w^* \, dx \ge \varepsilon\alpha \|Du\|^2_{L^2(\Omega)},$$

which, together with the previous inequality gives

$$a(u,u) \ge \varepsilon\alpha\|Du\|^2_{L^2(\Omega)} + \langle 1 - A^* w^*, u^2\rangle + \frac{1}{2}\int_\Omega u^2 \, dx \ge$$

$$\ge \varepsilon\alpha\|Du\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)} \ge c_\varepsilon \|u\|^2_{H_0^1(\Omega)}$$

for some constant $c_\varepsilon > 0$, and this proves that $a$ is coercive. □

**Lemma 2.2.2.** *Under the hypotheses of the previous lemma the solution $u$ of (2.2.1) satisfies the estimate*

$$\|u\|_{H_0^1(\Omega)} \le c\|f\|_{H^{-1}(\Omega)},$$

*where the constant $c > 0$ depends only on $\Omega$ and on the ellipticity constant $\alpha$ and does not depend on $\varepsilon$.*

*Proof.* By taking $\varphi = \dfrac{u}{w^*}$ as test function in (2.2.1) we obtain

$$\langle Au, u\rangle - \langle A^* w^*, \frac{u^2}{w^*}\rangle + \langle 1, \frac{u^2}{w^*}\rangle = \langle f, u\rangle.$$

Since $\dfrac{u^2}{w^*} \in H_0^{1,1}(\Omega)$ and $1 - A^* w^* \ge 0$ we have $\langle 1 - A^* w^*, \dfrac{u^2}{w^*}\rangle \ge 0$. By the ellipticity condition we get

$$\alpha\|Du\|^2_{L^2(\Omega)} \le \langle f, u\rangle \le \|f\|_{H^{-1}(\Omega)}\|u\|_{H_0^1(\Omega)}$$

and the Poincaré Inequality implies the conclusion of the lemma. □

**Lemma 2.2.3.** *Under the hypotheses of Lemma* 2.2.1, if $f \geq 0$ in $\Omega$ then the solution $u$ of (2.2.1) is positive.

*Proof.* This can be easily seen by taking in (2.2.1) the test function $\varphi = \dfrac{u^-}{w^*}$. Indeed, we have

$$\langle Au, u^- \rangle + \langle 1 - A^*w^*, \frac{uu^-}{w^*} \rangle = \langle f, u^- \rangle.$$

Since $uu^- = -(u^-)^2$ and $1 - A^*w^* \geq 0$, we have $\langle 1 - A^*w^*, \dfrac{uu^-}{w^*} \rangle \leq 0$. As $f \geq 0$ in $\Omega$ and $u^- \geq 0$ a.e. in $\Omega$, we have $\langle f, u^- \rangle \geq 0$, hence $\langle Au, u^- \rangle \geq 0$. The definition of $u^-$ and the ellipticity condition (2.1.1) imply

$$0 \leq \langle Au, u^- \rangle = -\langle Au^-, u^- \rangle \leq -\alpha \|u^-\|^2_{H^1_0(\Omega)} \leq 0$$

so that $u^- = 0$ a.e. in $\Omega$. □

We shall use this lemma to compare the solution $u$ of (2.2.1) with the solutions of the problems

$$\text{(2.2.2)} \qquad \begin{cases} w \in H^1_0(\Omega) \cap L^\infty(\Omega) \\ \langle Aw, w^*\varphi \rangle - \langle A^*w^*, w\varphi \rangle + \langle 1, w\varphi \rangle = \langle 1, w^*\varphi \rangle \qquad \forall \varphi \in C_0^\infty(\Omega) \end{cases}$$

and

$$\text{(2.2.3)} \qquad \begin{cases} w_0 \in H^1_0(\Omega) \\ Aw_0 = 1 \qquad \text{in } \Omega. \end{cases}$$

**Lemma 2.2.4.** *Under the hypotheses of Lemma* 2.2.1 *problem (2.2.2) has a unique solution $w$ and $w \leq w_0$ a.e. in $\Omega$, where $w_0$ is the solution of (2.2.3).*

*Proof.* Lemma 2.2.1 gives the existence of a unique $w \in H^1_0(\Omega)$ that satisfies the equation in (2.2.2) for any $\varphi \in H^1_0(\Omega)$. By Lemma 2.2.3 we have $w \geq 0$. Then by taking the test functions $\dfrac{(w - w_0)^+}{w^*}$ in (2.2.2) and $(w - w_0)^+$ in (2.2.3) and taking the difference of the two equalities we obtain $(w - w_0)^+ = 0$ a.e. in $\Omega$, that is $w \leq w_0$ a.e. in $\Omega$. Since $w_0 \in L^\infty(\Omega)$ we get $w \in L^\infty(\Omega)$ and so $w$ is a solution of (2.2.2). The uniqueness follows by density arguments. □

**Lemma 2.2.5.** *Under the hypotheses of Lemma 2.2.1, if $f \in L^\infty(\Omega)$ then the solution $u$ of (2.2.1) satisfies the estimate $|u| \le \|f\|_{L^\infty(\Omega)} w$ a.e. in $\Omega$, where $w$ is the solution of (2.2.2).*

*Proof.* Let $c = \|f\|_{L^\infty(\Omega)}$. Multiplying the equation in (2.2.2) by $c$ and subtracting the equation (2.2.1) satisfied by $u$ we obtain that $cw - u$ is the solution of the equation in (2.2.1) with $f$ replaced by $c - f$. Applying now Lemma 2.2.3 we get $cw - u \ge 0$ a.e. in $\Omega$, hence $u \le cw$ a.e. in $\Omega$. The inequality $u \ge -cw$ is proved in a similar way. □

**Lemma 2.2.6.** *Let $w^* \in K^*$ and let $\Omega'$ be a regular bounded open subset of $\mathbf{R}^n$ such that $\Omega \subset\subset \Omega'$. If $w^*$ is extended by 0 on $\Omega' \setminus \Omega$, then $A^* w^* \le 1$ in $\mathcal{D}'(\Omega')$.*

*Proof.* This property was proved in [8], Lemma A, in the case of the Laplace operator $-\Delta$. For the sake of completeness, we repeat the proof here for a general linear elliptic operator $A$. Let us define the set $K^{w^*} = \{v \in H_0^1(\Omega') : v \le w^* \text{ a.e. in } \Omega'\}$ and let $z$ be the solution of

$$\begin{cases} z \in K^{w^*} \\ \langle A^* z - 1, v - z\rangle \ge 0 \qquad \forall v \in K^{w^*}. \end{cases} \tag{2.2.4}$$

Then $z \ge 0$ in $\Omega'$. Indeed, $z$ is the greatest subsolution of $A^* v = 1$ that belongs to $K^{w^*}$. (See, e.g., [31], Chapter II, Theorem 6.4.) As 0 is such a subsolution we have $z \ge 0$ in $\Omega'$.

We claim that $z = w^*$. If we take $v = w^*$ in (2.2.4), we obtain

$$\langle A^* z - 1, w^* - z\rangle \ge 0.$$

Since $0 \le z \le w^*$ in $\Omega'$, we have $w^* - z = 0$ on $\Omega' \setminus \Omega$ hence

$$\langle A^* z - 1, w^* - z\rangle \ge 0. \tag{2.2.5}$$

As $1 - A^* w^* \ge 0$ in $\Omega$, we get

$$\langle A^* w^* - 1, w^* - z\rangle \le 0,$$

and subtracting (2.2.5) we obtain

$$\langle A^*(w^* - z), w^* - z\rangle \le 0$$

and so, the ellipticity of $A^*$ implies $w^* - z = 0$ in $\Omega$. Since $0 \leq z \leq w^* = 0$ in $\Omega' \backslash \Omega$, we have shown that $w^* = z$ in $\Omega'$ and the conclusion follows from the inequality

$$1 - A^* z \geq 0 \text{ in } \Omega',$$

which holds for all solutions of variational inequalities with an obstacle of the form (2.2.4). (See, e.g., [31], Chapter II, remark after Definition 6.3.) □

**Theorem 2.2.7.** *For any $w^* \in K^*$ and any $f \in H^{-1}(\Omega)$ there exists a solution $u$ of the problem*

$$\text{(2.2.6)} \qquad \begin{cases} u \in H_0^1(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \qquad \forall \varphi \in C_0^\infty(\Omega), \end{cases}$$

*which satisfies the estimate $\|u\|_{H_0^1(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)}$, for a suitable constant $c$ that depends only on $\Omega$ and on the ellipticity constant $\alpha$ and does not depend on $w^*$.*

*Proof.* Let us consider a regular bounded open subset $\Omega'$ of $\mathbf{R}^n$ such that $\Omega \subset\subset \Omega'$ and let us extend $w^*$ by 0 on $\Omega' \backslash \Omega$.

By Lemma 2.2.6 we have that $\nu^* = 1 - A^*w^* \geq 0$ in $\mathcal{D}'(\Omega')$, hence $\nu^*$ is a positive Radon measure. As $A^*w^* \in H^{-1}(\Omega')$, we have also that $\nu^* \in H^{-1}(\Omega')$.

We can approximate $(a_{ij})$ by a sequence $(a_{ij}^\varepsilon)$ of matrices of class $C^\infty$ which converges a.e. to $(a_{ij})$ and satisfies the ellipticity and boundedness conditions with the same constants as $(a_{ij})$. We shall denote the corresponding operators by $A_\varepsilon$ and $A_\varepsilon^*$. Let $\nu_\varepsilon^* \in C^\infty(\Omega')$, $\nu_\varepsilon^* \geq 0$, approximate $\nu^*$ strongly in $H^{-1}(\Omega')$ and let $w_\varepsilon^*$ be the solution of the Dirichlet problem

$$\begin{cases} w_\varepsilon^* - \varepsilon \in H_0^1(\Omega'), \\ 1 - A_\varepsilon^* w_\varepsilon^* = \nu_\varepsilon^* \text{ in } H^{-1}(\Omega'). \end{cases}$$

From the regularity theory we deduce that $w_\varepsilon^* \in C^\infty(\Omega')$.

Let us prove that $w_\varepsilon^* - \varepsilon$ converges to $w^*$ weakly in $H_0^1(\Omega')$. Since $w_\varepsilon^* - \varepsilon$ is bounded in $H_0^1(\Omega')$ it has a weak limit $v \in H_0^1(\Omega')$. We write the weak form of the equation:

$$\int_{\Omega'} \Big(\sum_{i,j=1}^n a_{ji}^\varepsilon D_i\varphi D_j w_\varepsilon^*\Big)\, dx = \int_{\Omega'} \varphi\, dx - \langle \nu_\varepsilon^*, \varphi\rangle \qquad \forall \varphi \in H_0^1(\Omega').$$

As $(a_{ji}^\varepsilon)$ is bounded, we have $|a_{ji}^\varepsilon D_i\varphi| \leq M|D_i\varphi| \in L^2(\Omega')$ and the pointwise convergence a.e. of $a_{ji}^\varepsilon D_i\varphi$ to $a_{ji}D_i\varphi$ implies, by the Lebesgue Dominated Convergence

Theorem, the strong convergence in $L^2(\Omega')$. As $D_j w_\varepsilon^*$ converges weakly in $L^2(\Omega')$ to $D_j v$ we obtain that the left hand side of the equation converges to

$$\int_{\Omega'} \sum_{i,j=1}^{n} a_{ji} D_i \varphi D_j v \, dx .$$

Then, as $\nu_\varepsilon^* \to \nu^*$ in $H^{-1}(\Omega')$, $\langle \nu_\varepsilon^*, \varphi \rangle$ converges to $\langle \nu^*, \varphi \rangle$, so that $v$ satisfies the same equation as $w^*$, i.e.

$$\int_{\Omega'} \Big( \sum_{i,j=1}^{n} a_{ji} D_j v D_i \varphi \Big) \, dx = \int_{\Omega'} \varphi \, dx - \langle \nu^*, \varphi \rangle \qquad \forall \varphi \in H_0^1(\Omega'),$$

hence $w^* = v$ a.e. in $\Omega'$.

Let us prove now that $w_\varepsilon^* - \varepsilon \to w^*$ strongly in $H_0^1(\Omega')$. In the equations satisfied by $w^*$ and $w_\varepsilon^*$ we take as test functions $w^* - w_\varepsilon^* + \varepsilon$ and obtain

$$\langle A^* w^*, w^* - w_\varepsilon^* \rangle + \langle \nu^*, w^* - w_\varepsilon^* + \varepsilon \rangle = \langle 1, w^* - w_\varepsilon^* + \varepsilon \rangle = \langle A_\varepsilon^* w_\varepsilon^*, w^* - w_\varepsilon^* \rangle + \langle \nu_\varepsilon^*, w^* - w_\varepsilon^* + \varepsilon \rangle .$$

The ellipticity condition for $A_\varepsilon^*$ gives

$$\alpha \| D(w^* - w_\varepsilon^*) \|_{L^2(\Omega')}^2 \le \langle A_\varepsilon^* (w^* - w_\varepsilon^*), w^* - w_\varepsilon^* \rangle$$

and using the previous equality we substitute $\langle A_\varepsilon^* w_\varepsilon^*, w^* - w_\varepsilon^* \rangle$ and obtain

$$\alpha \| D(w^* - w_\varepsilon^*) \|_{L^2(\Omega')}^2 \le \langle A_\varepsilon^* w^* - A^* w^*, w^* - w_\varepsilon^* \rangle + \langle \nu_\varepsilon^* - \nu^*, w^* - w_\varepsilon^* + \varepsilon \rangle .$$

As $\nu_\varepsilon^* \to \nu^*$ strongly in $H^{-1}(\Omega')$ and $w_\varepsilon^* - \varepsilon \rightharpoonup w^*$ weakly in $H_0^1(\Omega')$, the second term in the right hand side converges to zero. Let us consider the first term

$$\langle A_\varepsilon^* w^* - A^* w^*, w^* - w_\varepsilon^* \rangle = \int_{\Omega} \Big( \sum_{i,j=1}^{n} (a_{ji}^\varepsilon - a_{ji}) D_j w^* D_i (w^* - w_\varepsilon^*) \Big) \, dx .$$

As $a_{ji}$ and $(a_{ji}^\varepsilon)$ are bounded, $|(a_{ji}^\varepsilon - a_{ji}) D_j w^*| \le M |D_j w^*| \in L^2(\Omega')$, $(a_{ji}^\varepsilon - a_{ji}) D_j w^*$ converges pointwise a.e. to zero, by the Lebesgue Dominated Convergence Theorem, the convergence is also in $L^2(\Omega')$. As $D_i(w^* - w_\varepsilon^*)$ converges weakly in $L^2(\Omega')$ to zero we get that the first term converges to zero and so $\| D(w^* - w_\varepsilon^*) \|_{L^2(\Omega')}^2 \to 0$, that is $w_\varepsilon^* - \varepsilon \to w^*$ strongly in $H_0^1(\Omega')$.

We shall continue now the proof of the existence of a solution of (2.2.6).

Let us consider the function $w_\varepsilon^* \vee \varepsilon$. As $A^* w_\varepsilon^* \leq 1$ and $A^*\varepsilon \leq 1$ we have that $1 - A^*(w_\varepsilon^* \vee \varepsilon) \geq 0$. (See, e.g., [31], Chapter II, Theorem 6.6.) Since $w_\varepsilon^* - \varepsilon \to w^*$ strongly in $H_0^1(\Omega')$ and $w^* \geq 0$, also $(w_\varepsilon^* \vee \varepsilon) - \varepsilon \to w^* \vee 0 = w^*$ strongly in $H_0^1(\Omega')$. As $(w_\varepsilon^* \vee \varepsilon) \in W^{1,\infty}(\Omega)$, by Lemma 2.2.1 there exists a function $u_\varepsilon \in H_0^1(\Omega)$ such that

$$(2.2.7)_\varepsilon \quad \langle A_\varepsilon u_\varepsilon, (w_\varepsilon^* \vee \varepsilon)\varphi\rangle - \langle A_\varepsilon^*(w_\varepsilon^* \vee \varepsilon), u_\varepsilon \varphi\rangle + \langle 1, u_\varepsilon \varphi\rangle = \langle f, (w_\varepsilon^* \vee \varepsilon)\varphi\rangle \qquad \forall \varphi \in H_0^1(\Omega).$$

By Lemma 2.2.2 we have $\|u_\varepsilon\|_{H_0^1(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)}$, so that, up to a subsequence, $u_\varepsilon$ converges weakly to a function $u \in H_0^1(\Omega)$. We shall consider now test functions $\varphi \in C_0^\infty(\Omega)$ and by passing to the limit in $(2.2.7)_\varepsilon$ we get that the limit function $u$ is a solution of (2.2.6). As $u_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$ and $\|u_\varepsilon\|_{H_0^1(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)}$, from the lower semicontinuity of the norm we obtain that $\|u\|_{H_0^1(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)}$. □

**Proposition 2.2.8.** *Let $w^* \in K^*$ and $f \in H^{-1}(\Omega)$. If $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution of (2.2.6) then $u$ satisfies the equation for any test function $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover, if $u_1$, $u_2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ are solutions of (2.2.6) then $u_1 = u_2$.*

*Proof.* Let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution of (2.2.6) and let $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Since $w^* \in K^*$ is also in $L^\infty(\Omega)$ the products $u\varphi$ and $w^*\varphi$ belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$, hence all terms of the equation are well defined. There exists a sequence $(\varphi_h)$ of functions in $C_0^\infty(\Omega)$, bounded in $L^\infty(\Omega)$, that converges strongly in $H_0^1(\Omega)$ to $\varphi$. We consider in (2.2.6) $\varphi_h$ as test function, pass to the limit in the equation

$$\langle Au, w^*\varphi_h\rangle - \langle A^* w^*, u\varphi_h\rangle + \langle 1, u\varphi_h\rangle = \langle f, w^*\varphi_h\rangle$$

and obtain that $u$ satisfies the equation with $\varphi$ as test function.

In order to prove the uniqueness let us denote by $u$ the difference $u_1 - u_2$. We have

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i \varphi\Big) w^* \, dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^* D_i \varphi\Big) u dx + \int_\Omega u\varphi \, dx = 0 \qquad \forall \varphi \in C_0^\infty(\Omega).$$

As $w^*$ and $u$ are bounded, this equality holds for any $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Then we can take $u$ as test function and obtain

$$(2.2.8) \qquad \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i u\Big) w^* \, dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^* D_i u\Big) u \, dx + \int_\Omega u^2 \, dx = 0 .$$

We have that

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a_{ji} D_j w^* D_i u\Big) u \, dx = \frac{1}{2}\langle A^* w^*, u^2\rangle$$

and since $\langle 1 - A^* w^*, u^2\rangle \geq 0$, we get

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij} D_j u D_i u\Big) w^* \, dx + \frac{1}{2}\int_\Omega u^2 \, dx \leq 0 \, .$$

As the first term is nonnegative, by the ellipticity of $(a_{ij})$ and the positivity of $w^*$, we obtain that $u = 0$ a.e. in $\Omega$ and so the uniqueness is proved. □

**Theorem 2.2.9.** *Let $w^* \in K^*$ and $f \in L^\infty(\Omega)$. Then there exists a unique solution of the problem*

$$(2.2.9) \qquad \begin{cases} u \in H_0^1(\Omega) \cap L^\infty(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^* w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \qquad \forall \varphi \in C_0^\infty(\Omega). \end{cases}$$

*Moreover, $u$ satisfies the equation for any test function $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and we have the following estimates $\|u\|_{H_0^1(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)}$ and $|u| \leq \|f\|_{L^\infty(\Omega)} w \leq \|f\|_{L^\infty(\Omega)} w_0$ a.e. in $\Omega$ where $w$ and $w_0$ are the solutions of (2.2.2) and (2.2.3), respectively, and $c$ is a constant depending only on $\Omega$ and on the ellipticity constant $\alpha$ and not on $w^*$.*

*Proof.* Let us consider the construction done in Theorem 2.2.7 for the proof of existence. If we denote by $w_0^\varepsilon$ the solution of the Dirichlet problem

$$\begin{cases} w_0^\varepsilon \in H_0^1(\Omega) \\ A_\varepsilon w_0^\varepsilon = 1 \qquad \text{in } \Omega \, , \end{cases}$$

and by $w_\varepsilon$ the solution of

$$\begin{cases} w_\varepsilon \in H_0^1(\Omega) \\ \langle A_\varepsilon w_\varepsilon, (w_\varepsilon^* \vee \varepsilon)\varphi\rangle - \langle A_\varepsilon^*(w_\varepsilon^* \vee \varepsilon), w_\varepsilon\varphi\rangle + \langle 1, w_\varepsilon\varphi\rangle = \langle 1, (w_\varepsilon^* \vee \varepsilon)\varphi\rangle \qquad \forall \varphi \in H_0^1(\Omega), \end{cases}$$

then by applying Lemma 2.2.5 and Lemma 2.2.4 to $A_\varepsilon$ and $(w_\varepsilon^* \vee \varepsilon)$, we obtain that $|u_\varepsilon| \leq \|f\|_{L^\infty(\Omega)} w_\varepsilon \leq \|f\|_{L^\infty(\Omega)} w_0^\varepsilon$. Since the weak convergence of $u_\varepsilon$ to $u$ and of $w_\varepsilon$ to $w$ proved in Theorem 2.2.7 implies the pointwise convergence of a subsequence, by passing to the limit we obtain $|u| \leq \|f\|_{L^\infty(\Omega)} w \leq \|f\|_{L^\infty(\Omega)} w_0$. Since $w_0 \in L^\infty(\Omega)$, also $u \in L^\infty(\Omega)$. Then the uniqueness and the fact that the equation is satisfied for any test function in $H_0^1(\Omega) \cap L^\infty(\Omega)$ follow from Proposition 2.2.8, and Theorem 2.2.7 gives the first estimate. □

Till now we did not use the capacity theory. To prove the uniqueness of the solution for (2.2.6) in $H_0^1(\Omega)$ for any $f \in H^{-1}(\Omega)$, we have to use the quasicontinuous representatives of Sobolev functions.

**Theorem 2.2.10.** *Let $w^* \in K^*$ and $f \in H^{-1}(\Omega)$. Then there exists a unique solution of the problem*

$$\begin{cases} u \in H_0^1(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \qquad \forall \varphi \in C_0^\infty(\Omega)\,. \end{cases}$$

*Proof.* The existence was proved in Theorem 2.2.7. By linearity, it suffices to prove the uniqueness for $f = 0$. We begin by showing that if $u \in H_0^1(\Omega)$ satisfies

$$\langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = 0 \qquad \forall \varphi \in C_0^\infty(\Omega) \tag{2.2.10}$$

then it also satisfies

$$\langle Au, w^*v\rangle - \langle A^*w^*, uv\rangle + \langle 1, uv\rangle = 0 \quad \forall v \in H_0^1(\Omega)\cap L^\infty(\Omega) \text{ such that } uDv \in L^2(\Omega, \mathbf{R}^n)\,. \tag{2.2.11}$$

Note that for this choice of $v$, the product $uv$ belongs to $H_0^1(\Omega)$. Indeed, assume that $u$ and $v$ are extended by zero to $\mathbf{R}^n$. Since $uv \in L^2(\Omega)$ and the derivatives in the sense of distributions $D_i(uv) = D_iuv + uD_iv$ belong to $L^2(\Omega)$, we have $uv \in H^1(\Omega)$. Now the quasicontinuous representative of $uv$ is zero q.e. on $\mathbf{R}^n\setminus\Omega$ and so, $uv \in H_0^1(\Omega)$. This shows that all terms in (2.2.11) are well defined.

Let $(\varphi_k)$ be a sequence of functions in $C_0^\infty(\Omega)$, bounded in $L^\infty(\Omega)$, which converges to $v$ strongly in $H_0^1(\Omega)$. Take $\varphi_k$ as test functions in (2.2.10). As $w^*\varphi_k$ converges to $w^*v$ strongly in $H_0^1(\Omega)$ (note that since $w^* \in L^\infty(\Omega)$, we may apply Lebesgue Dominated Convergence Theorem) we pass to the limit in the first term and get that $\langle Au, w^*\varphi_k\rangle \to \langle Au, w^*v\rangle$.

As $w^* \in K^*$, $\nu^* = 1 - A^*w^*$ is a nonnegative Radon measure belonging to $H^{-1}(\Omega)$ and

$$\langle 1, u\varphi_k\rangle - \langle A^*w^*, u\varphi_k\rangle = \int_\Omega u\varphi_k\, d\nu^*\,.$$

Since $\varphi_k$ converges to $v$ strongly in $H_0^1(\Omega)$, it converges, pointwise q.e. and hence $\nu^*$-a.e. Then also $u\varphi_k \to uv$ pointwise $\nu^*$-a.e. As $\int_\Omega |u\varphi_k|\, d\nu^* \leq M\int_\Omega |u|\, d\nu^*$ and

$u \in L^1_{\nu^*}(\Omega)$, we may apply Lebesgue Dominated Convergence Theorem to obtain that $u\varphi_k \to uv$ in $L^1_{\nu^*}(\Omega)$. So, passing to the limit in (2.2.10), we get

$$\langle Au, w^* v\rangle + \int_\Omega uv\, d\nu^* = 0$$

which is equivalent to

$$\langle Au, w^* v\rangle - \langle A^* w^*, uv\rangle + \langle 1, uv\rangle = 0\,.$$

To prove now the uniqueness, for each $k \in \mathbf{N}$ let $u_k$ be defined by truncation:

$$u_k(x) = \begin{cases} -k, \text{ if } u(x) < -k, \\ u(x) \text{ if } -k \le u(x) \le k, \\ k, \text{ if } u(x) > k, \end{cases}$$

Since $u_k \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and $uDu_k = u_k Du_k \in L^2(\Omega, \mathbf{R}^n)$, we may take $v = u_k$ as test function in (2.2.11) and obtain

$$\text{(2.2.12)} \quad \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u D_i u_k\Big) w^*\, dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^* D_i u_k\Big) u\, dx + \int_\Omega u u_k\, dx = 0\,.$$

As $w^* \ge 0$ and $D_j u D_i u_k = D_j u_k D_i u_k$ the first term is nonnegative by ellipticity. We have $uD_j u_k = \frac{1}{2} D_j(u_k^2)$, so that

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^* D_i u_k\Big) u\, dx + \int_\Omega u u_k\, dx =$$
$$= \frac{1}{2}\langle 1 - A^* w^*, u_k^2\rangle + \frac{1}{2}\int_\Omega (u u_k - u_k^2)\, dx + \frac{1}{2}\int_\Omega u u_k\, dx\,.$$

Since all the terms are nonnegative, from (2.2.12) we deduce that $u = 0$. □

**Proposition 2.2.11.** *Let $f \in H^{-1}(\Omega)$, let $U$ be an open subset of $\Omega$ and let $u$ and $w^*$ be the solutions of the problems*

$$\begin{cases} u \in H^1_0(U) \\ Au = f \text{ in } U \end{cases} \quad \textit{and} \quad \begin{cases} w^* \in H^1_0(U) \\ A^* w^* = 1 \text{ in } U\,. \end{cases}$$

*Then $u$ is the solution of*

$$
\text{(2.2.13)} \qquad \begin{cases} u \in H_0^1(\Omega) \\ \langle Au, w^*\varphi\rangle - \langle A^*w^*, u\varphi\rangle + \langle 1, u\varphi\rangle = \langle f, w^*\varphi\rangle \qquad \forall\varphi \in C_0^\infty(\Omega). \end{cases}
$$

*Proof.* Let $\varphi \in C_0^\infty(\Omega)$. Since $u$, $w^* \in H_0^1(U)$, the products $u\varphi$ and $w^*\varphi$ belong to $H_0^1(U)$ so that can be considered as test functions in the equations satisfied by $w^*$ and $u$, respectively. We obtain that

$$
\langle Au, w^*\varphi\rangle = \langle f, w^*\varphi\rangle, \qquad \langle A^*w^*, u\varphi\rangle = \langle 1, u\varphi\rangle, \qquad \forall\varphi \in C_0^\infty(\Omega)
$$

and by subtraction, we get that $u$ is the solution of (2.2.13). □

Let us study now the dependence on $w^*$ of the solutions of (2.2.13).

**Theorem 2.2.12.** *Let $f \in H^{-1}(\Omega)$, let $w_h^* \in K^*$ and let $u_h$ be the solution of the problem*

$$
\text{(2.2.14)} \qquad \begin{cases} u_h \in H_0^1(\Omega) \\ \langle Au_h, w_h^*\varphi\rangle - \langle A^*w_h^*, u_h\varphi\rangle + \langle 1, u_h\varphi\rangle = \langle f, w_h^*\varphi\rangle \qquad \forall\varphi \in C_0^\infty(\Omega). \end{cases}
$$

*Assume that $w_h^*$ converges weakly in $H_0^1(\Omega)$ to a function $w^* \in K^*$. Then $u_h$ converges weakly in $H_0^1(\Omega)$ to the solution $u$ of (2.2.13).*

*Proof.* The estimate $\|u_h\|_{H_0^1(\Omega)} \leq c\|f\|_{H^{-1}(\Omega)}$, proved in Theorem 2.2.9, gives the existence of a subsequence, still denoted by $u_h$, that converges weakly in $H_0^1(\Omega)$ to some function $u$. Then

$$
\langle Au_h, w_h^*\varphi\rangle - \langle A^*w_h^*, u_h\varphi\rangle + \langle 1, u_h\varphi\rangle = \langle f, w_h^*\varphi\rangle
$$

can be written as

$$
\int_\Omega \Big(\sum_{i,j=1}^n a_{ij} D_j u_h D_i\varphi\Big) w_h^*\, dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w_h^* D_i\varphi\Big) u_h dx + \int_\Omega u_h\varphi\, dx = \langle f, w_h^*\varphi\rangle,
$$

we may pass to the limit and obtain that $u$ is a solution of (2.2.13). The uniqueness of the solution implies that the whole sequence $(u_h)$ converges weakly in $H_0^1(\Omega)$ to the solution $u$ of (2.2.13). □

**Theorem 2.2.13.** *Let $\Omega_h$ be an arbitrary sequence of open subsets of $\Omega$. Then there exist a subsequence, still denoted by $\Omega_h$, and a function $w^* \in K^*$ such that for every $f \in H^{-1}(\Omega)$ the solution $u_h$ of (2.0.1) extended by 0 on $\Omega \setminus \Omega_h$ converges weakly in $H_0^1(\Omega)$ to the solution $u$ of (2.2.13).*

*Proof.* Let $w_h^*$ be the solution of (2.0.2). As $(w_h^*)$ is bounded in $H_0^1(\Omega)$ there exists a subsequence, still denoted by $(w_h^*)$, that converges weakly in $H_0^1(\Omega)$ to a function $w^*$. Since $w_h^* \in K^*$ (Proposition 2.1.1) and $K^*$ is weakly closed we obtain $w^* \in K^*$. Let $u_h$ be the solution of (2.0.1). Then, as $(u_h)$ is bounded in $H_0^1(\Omega)$, it has a subsequence that converges weakly in $H_0^1(\Omega)$ to some function $u$. By Proposition 2.2.11 $u_h$ is the solution of (2.2.14) and by applying now Theorem 2.2.12 we deduce that the limit $u$ is the solution of (2.2.13). □

## 2.3. A density result

Our purpose is now to complete the characterization given by Theorem 2.2.13, that is we want to show that if $w^* \in K^*$, $f \in H^{-1}(\Omega)$ and $u$ is the solution of (2.2.13) then there exists a sequence of domains $\Omega_h$ such that the corresponding solutions $u_h$ of (2.0.1) converge weakly in $H_0^1(\Omega)$ to $u$. From Theorem 2.2.12 it follows that we have only to prove that any $w^* \in K^*$ can be approximated by solutions of (2.0.2).

Let us return to the sets $K^*$ of all functions which satisfy (2.0.4) and $H^*$ defined in Section 1 (before Proposition 2.1.1). We shall prove that the weak closure in $H_0^1(\Omega)$ of $H^*$ is equal to $K^*$. As we have already remarked the closure of $H^*$ is contained in $K^*$ (Proposition 2.1.1). So we have only to prove that any function $w^* \in K^*$ can be approximated by functions in $H^*$. To this end let us define two auxiliary sets:

$K_1^*$ - the set of all functions in $K^*$ that are continuous and strictly positive on $\Omega$ and

$K_2^*$ - the set of all functions in $H_0^1(\Omega)$ that satisfy $A^*w^* + bw^* = 1$ in the sense of distributions on $\Omega$, for some continuous and positive function $b$.

**Remark 2.3.1.** $K_2^* \subseteq C^0(\Omega) \cap K^*$.

*Proof.* Let $w^* \in K_2^*$. By De Giorgi's Theorem (see, e.g., [25] Theorem 8.22) $w^* \in C^0(\Omega)$. Since $A^*w^* + bw^* = 1$ in the sense of distributions and $b \geq 0$ we get $w^* \geq 0$ and $A^*w^* \leq 1$. □

**Remark 2.3.2.** The closure of $K_2^*$ in the weak topology of $H_0^1(\Omega)$ contains $K_1^*$.

*Proof.* Indeed, let $w^* \in K_1^*$. Then $A^*w^* + \nu = 1$, where $\nu \in H^{-1}(\Omega)$ is a positive Radon measure. Since $w^* > 0$, we can define $\mu = \dfrac{\nu}{w^*}$. We can approximate $\mu$ strongly in $H^{-1}(\Omega)$ by continuous and positive functions $b_\varepsilon$.

Let $w_\varepsilon^* \in K_2^*$ be the solution of the following Dirichlet problem:

$$\begin{cases} w_\varepsilon^* \in H_0^1(\Omega), \\ A^*w_\varepsilon^* + b_\varepsilon w_\varepsilon^* = 1. \end{cases}$$

Then $w_\varepsilon^* \geq 0$ and, as $b_\varepsilon \geq 0$, $A^*w_\varepsilon^* \leq 1$ in the sense of distributions on $\Omega$.

By De Giorgi's Theorem $w_\varepsilon^* \in C^0(\Omega)$ hence $1 - A^*w_\varepsilon^* = b_\varepsilon w_\varepsilon^* \in C^0(\Omega)$. As $(w_\varepsilon^*)$ is bounded in $H_0^1(\Omega)$, there exists a function $\tilde{w} \in H_0^1(\Omega)$ such that $w_\varepsilon^*$ converges to $\tilde{w}$ weakly in $H_0^1(\Omega)$. Since $w_\varepsilon^* \leq w_0^*$, where $w_0^* \in H_0^1(\Omega)$ is the solution of $A^*w_0^* = 1$ in $\Omega$, and $w_0^* \in L^\infty(\Omega)$ we have that $(w_\varepsilon^*)$ is bounded in $L^\infty(\Omega)$, hence $\tilde{w} \in L^\infty(\Omega)$. We have

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w_\varepsilon^* D_i \varphi\Big)\, dx + \int_\Omega b_\varepsilon w_\varepsilon^* \varphi \, dx = \int_\Omega \varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Passing to the limit as $\varepsilon \to 0$, we get

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j \tilde{w} D_i \varphi\Big)\, dx + \int_\Omega \tilde{w} \varphi \, \frac{d\nu}{w^*} = \int_\Omega \varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega).$$

The above equation is satisfied for all $\varphi \in H_0^1(\Omega)$. This can be proved by density arguments using the fact that $\tilde{w} \in L^\infty(\Omega)$ and that $H_0^1(\Omega) \subseteq L_\mu^1(\Omega)$ for any positive Radon measure belonging to $H^{-1}(\Omega)$. As $w^*$ is a solution in $H_0^1(\Omega)$, we get that $w^* = \tilde{w}$ a.e. in $\Omega$. So, $w_\varepsilon^*$ converges to $w^*$ weakly in $H_0^1(\Omega)$. $\square$

**Remark 2.3.3.** The closure of $K_1^*$ in the weak topology of $H_0^1(\Omega)$ is equal to $K^*$.

*Proof.* Let us first remark that by definition, $K_1^* \subseteq K^*$. Let $w \in K^*$. Then $\nu = 1 - A^*w^*$ is a positive Radon measure that belongs to $H^{-1}(\Omega)$. We can approximate it strongly in $H^{-1}(\Omega)$ by a sequence of positive smooth functions $\nu_\varepsilon$. Let us consider the solution $v_\varepsilon$ of the Dirichlet problem

$$\begin{cases} v_\varepsilon \in H_0^1(\Omega) \\ A^*v_\varepsilon + \nu_\varepsilon v_\varepsilon = 1. \end{cases}$$

By the maximum principle $v_\varepsilon \geq 0$ and by De Giorgi's Theorem $v_\varepsilon \in C^0(\Omega)$. By the same arguments as before we obtain the weak convergence in $H_0^1(\Omega)$ of $v_\varepsilon$ to $w^*$. In order to obtain a sequence of functions in $K_1^*$ let us consider the solution $w_0^* \in H_0^1(\Omega)$ of $A^* w_0^* = 1$. By the strong maximum principle (see, for instance, [42]) we have that $w_0^* > 0$ and, by De Giorgi's Theorem, $w_0^* \in C^0(\Omega)$. We define then $w_\varepsilon^* = (1-\varepsilon)v_\varepsilon + \varepsilon w_0^*$. It is easy to see that $w_\varepsilon^* \in K_1^*$ and $w_\varepsilon^* \rightharpoonup w^*$ weakly in $H_0^1(\Omega)$. □

The conclusion of the Remarks 2.3.1-2.3.3 is that $K_2^*$ is dense in $K^*$ with respect to the weak topology of $H_0^1(\Omega)$, hence, in order to prove that $H^*$ is dense in $K^*$, it is enough to show that every element of $K_2^*$ can be approximated by elements of $H^*$.

**Theorem 2.3.4.** *The closure of $H^*$ with respect to the weak topology of $H_0^1(\Omega)$ contains $K^*$.*

*Proof.* As we have mentioned above it suffices to show that the closure of $H^*$ with respect to the weak topology of $H_0^1(\Omega)$ contains $K_2^*$. Let $w^* \in K_2^*$. This means that $w^* \in H_0^1(\Omega)$ and there exists a continuous, positive function $b$ on $\Omega$ such that $A^* w^* + b w^* = 1$ on $\Omega$ in the sense of distributions.

In order to get a sequence of functions in $H^*$ that converges weakly in $H_0^1(\Omega)$ to $w^*$ we shall use the method of Cioranescu and Murat [9] following the lines of [14]. There exist a sequence of open subsets $\Omega_h$ of $\Omega$, a sequence $z_h$ of functions in $H^1(\Omega)$ that converges weakly in $H^1(\Omega)$ to 1 and two sequences $\lambda_h$ and $\nu_h$ of measures in $H^{-1}(\Omega)$ such that $A^* z_h = \nu_h - \lambda_h$ in $\Omega$, $\lambda_h$ converges to $b$ weakly in $H^{-1}(\Omega)$, $\nu_h$ converges to $b$ strongly in $H^{-1}(\Omega)$ and $\langle \lambda_h, \varphi \rangle = 0$ for every function $\varphi \in H_0^1(\Omega_h)$. For the construction see [14]. We may assume that $0 \leq z_h \leq 1$ a.e. in $\Omega$.

Let us define $u_h = z_h w^*$. By construction $u_h \in H_0^1(\Omega_h)$. From the weak convergence of $z_h$ to 1 we deduce that $u_h$ converges to $w^*$ weakly in $H_0^1(\Omega)$.

Let $w_h^*$ be the solution of the Dirichlet problem

$$\begin{cases} w_h^* \in H_0^1(\Omega_h), \\ A^* w_h^* = 1 \text{ on } \Omega_h. \end{cases}$$

We extend $w_h^*$ by zero on $\Omega \setminus \Omega_h$. Then $(w_h^*)$ has a subsequence that converges to some function $v$ weakly in $H_0^1(\Omega)$. We want to prove that $w^* = v$. (As a consequence the whole sequence $(w_h^*)$ converges to $w^*$.) The properties of $z_h$ and $w^*$ imply that there exists $c_1 > 0$ such that $\|u_h\|_{L^\infty(\Omega)} \leq c_1$. There exists also $c_2 > 0$ such that $\|w_h^*\|_{L^\infty(\Omega)} \leq c_2$. We have

$$\langle A^* u_h, u_h - w_h^* \rangle = \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j u_h D_i (u_h - w_h^*)\Big) dx =$$

$$= \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j z_h w^* D_i (u_h - w_h^*)\Big) dx + \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^* z_h D_i (u_h - w_h^*)\Big) dx =$$

$$= \int_\Omega \sum_{i,j=1}^n a_{ji} D_j z_h D_i (w^*(u_h - w_h^*)) dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j z_h D_i w^*\Big)(u_h - w_h^*) dx +$$

$$+ \int_\Omega \sum_{i,j=1}^n a_{ji} D_j w^* D_i (z_h (u_h - w_h^*)) dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^* D_i z_h\Big)(u_h - w_h^*) dx =$$

$$= \int_\Omega w^*(u_h - w_h^*) d\nu_h + \int_\Omega (1 - bw^*) z_h (u_h - w_h^*) dx -$$

$$- \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j z_h D_i w^*\Big)(u_h - w_h^*) dx - \int_\Omega \Big(\sum_{i,j=1}^n a_{ji} D_j w^* D_i z_h\Big)(u_h - w_h^*) dx =$$

$$= I_1 + I_2 - I_3 - I_4,$$

where we have used the fact that $w^*(u_h - w_h^*) \in H_0^1(\Omega_h)$ so that $\langle \lambda_h, w^*(u_h - w_h^*)\rangle = 0$.

As $w^*, (u_h), (w_h^*)$ are bounded in $L^\infty(\Omega)$, the product $w^*(u_h - w_h^*)$ converges to $w^*(w^* - v)$ weakly in $H_0^1(\Omega)$. Then the strong convergence of $\nu_h$ to $b$ in $H^{-1}(\Omega)$ implies the convergence of $I_1$ to $\int_\Omega w^*(w^* - v) b dx$. Since $u_h - w_h^* \to w^* - v$ in $L^2(\Omega)$, the second term $I_2$ converges to $\int_\Omega (1 - bw^*)(w^* - v) dx$. From the weak convergence of $D_j z_h$ to 0 in $L^2(\Omega)$, the boundedness in $L^\infty(\Omega)$ of $(u_h - w_h^*)$ and its strong convergence to $w^* - v$ in $L^2(\Omega)$ we deduce that $I_3 \to 0$ and the same arguments hold for $I_4$. So that

$$\alpha \|u_h - w_h^*\|^2_{H_0^1(\Omega)} \leq \langle A^*(u_h - w_h^*), u_h - w_h^* \rangle =$$

$$= \langle A^* u_h, u_h - w_h^* \rangle - \langle A^* w_h^*, u_h - w_h^* \rangle = \langle A^* u_h, u_h - w_h^* \rangle - \langle 1, u_h - w_h^* \rangle = Z_h$$

Since $Z_h$ converges to $\int_\Omega w^*(w^* - v) b dx + \int_\Omega (1 - bw^*)(w^* - v) dx - \langle 1, w^* - v \rangle = 0$ we get $w^* = v$. So, for any $w^* \in K_2^*$ there exists a sequence of functions $w_h^*$ in $H^*$ such that $w_h^*$ converges to $w^*$ weakly in $H_0^1(\Omega)$, hence $H^*$ is dense in $K^*$. □

**Theorem 2.3.5.** *Let $w^* \in K^*$ and $f \in H^{-1}(\Omega)$. If $u$ is the solution of (2.2.13) then there exists a sequence $\Omega_h$ of open subsets of $\Omega$ such that the corresponding solutions $u_h$ of (2.0.1) extended by 0 on $\Omega \backslash \Omega_h$ converge to $u$ weakly in $H_0^1(\Omega)$.*

*Proof.* Theorem 2.3.4 gives the existence of a sequence $\Omega_h$ of open subsets of $\Omega$ such that the solution $w_h^*$ of (2.0.2) converges weakly in $H_0^1(\Omega)$ to $w^*$. Then the corresponding solutions $u_h$ of (2.0.1) converge weakly in $H_0^1(\Omega)$ to $u$. This can be seen for example by using Proposition 2.2.11 and Theorem 2.2.12. □

# Chapter 3. A capacity method for the study of limits of elliptic systems on varying domains

Let $\Omega$ be a bounded open subset of $\mathbf{R}^n$ and let $\mathcal{A}: H_0^1(\Omega, \mathbf{R}^m) \to H^{-1}(\Omega, \mathbf{R}^m)$ be an elliptic operator of the form

$$\langle \mathcal{A}u, v \rangle = \int_\Omega (ADu, Dv)\, dx\,,$$

where $A(x)$ is a fourth order tensor and $(\cdot,\cdot)$ denotes the scalar product between matrices. Given a sequence $(\Omega_h)$ of open subsets of $\Omega$, we consider for every $f \in H^{-1}(\Omega, \mathbf{R}^m)$ the sequence $(u_h)$ of the solutions of the Dirichlet problems

$$(3.0.1) \qquad \begin{cases} u_h \in H_0^1(\Omega_h, \mathbf{R}^m) \\ \mathcal{A}u_h = f \quad \text{in } \Omega_h\,, \end{cases}$$

extended to $\Omega$ by setting $u_h = 0$ on $\Omega \setminus \Omega_h$. We want to describe the asymptotic behaviour of $(u_h)$ as $j \to \infty$. As in the scalar case, a relaxation phenomenon may occur. Namely, if $(u_h)$ converges weakly in $H_0^1(\Omega, \mathbf{R}^m)$ to some function $u$, then there exist an $m \times m$ matrix $T(x)$, with $|T(x)| = 1$, and a measure $\mu$, not charging polar sets, such that $u$ is the solution of the relaxed Dirichlet problem

$$(3.0.2) \qquad \begin{cases} u \in H_0^1(\Omega, \mathbf{R}^m) \cap L^2_\mu(\Omega, \mathbf{R}^m) \\ \displaystyle\int_\Omega (ADu, Dv)\, dx + \int_\Omega (Tu, v)\, d\mu = \langle f, v \rangle \\ \forall v \in H_0^1(\Omega, \mathbf{R}^m) \cap L^2_\mu(\Omega, \mathbf{R}^m)\,, \end{cases}$$

where, in the second integral, $(\cdot,\cdot)$ denotes the scalar product in $\mathbf{R}^m$, while $\langle \cdot,\cdot \rangle$ is the duality pairing between $H^{-1}(\Omega, \mathbf{R}^m)$ and $H_0^1(\Omega, \mathbf{R}^m)$.

The problem we consider in this chapter is the identification of the pair $(T, \mu)$ which appears in the limit problem (3.0.2). To this aim we introduce a suitable notion of capacity. If $K$ is a compact subset of $\Omega$ and $\xi$, $\eta \in \mathbf{R}^m$, then the $\mathcal{A}$-capacity of $K$ in $\Omega$ relative to $\xi$ and $\eta$ is defined as

$$C_\mathcal{A}(K, \xi, \eta) = \int_{\Omega \setminus K} (ADu^\xi, Du^\eta)\, dx\,,$$

where, for every $\zeta \in \mathbf{R}^m$, $u^\zeta$ is the weak solution in $\Omega\backslash K$ of the Dirichlet problem

$$\begin{cases} u^\zeta \in H^1(\Omega\backslash K, \mathbf{R}^m), \quad u^\zeta = \zeta \quad \text{on } \partial K, \quad u^\zeta = 0 \quad \text{on } \partial\Omega \\ \displaystyle\int_{\Omega\backslash K} (ADu^\zeta, Dv)\, dx = 0 \qquad \forall v \in H_0^1(\Omega\backslash K, \mathbf{R}^m). \end{cases}$$

For every $x \in \mathbf{R}^n$ let $D_\rho(x)$ be the closed ball with centre $x$ and radius $\rho$. Assume that the limit

$$\lim_{j\to+\infty} C_{\mathcal{A}}(D_\rho(x)\backslash\Omega_h, \xi, \eta) = \alpha(D_\rho(x), \xi, \eta)$$

exists for every $x \in \Omega$ and for almost every $\rho > 0$ such that $D_\rho(x) \subset \Omega$. Our main result, Theorem 3.3.7, shows that, if $\alpha$ can be majorized by a Kato measure $\lambda$ (Definition 3.1.1), then for $\lambda$-almost every $x \in \Omega$ there exists an $m\times m$ matrix $G(x)$ such that

$$\operatorname*{ess\,lim}_{\rho\to 0} \frac{\alpha(D_\rho(x), \xi, \eta)}{\lambda(D_\rho(x))} = (G(x)\,\xi, \eta) \qquad \forall \xi, \eta \in \mathbf{R}^m.$$

Moreover, for every $f \in H^{-1}(\Omega, \mathbf{R}^m)$, the sequence $(u_h)$ of the solutions of (3.0.1) converges weakly in $H_0^1(\Omega, \mathbf{R}^m)$ to the solution $u$ of (3.0.2) with $T(x) = \frac{G(x)}{|G(x)|}$ and $\mu(E) = \int_E |G|\, d\lambda$. If $\mathcal{A}$ is symmetric, the same result (Theorem 3.4.3) holds whenever $\lambda$ is a bounded measure.

## 3.1. Notation and preliminaries

Let $\Omega$ be a bounded open subset of $\mathbf{R}^n$, $n \geq 3$. The case $n = 2$ can be treated in a similar way by using the logarithmic potentials. We assume that the boundary $\partial\Omega$ of $\Omega$ is of class $C^1$. Let $A(x) = (a_{\alpha\beta}^{ij}(x))$, with $1 \leq i, j \leq n$ and $1 \leq \alpha, \beta \leq m$, be a family of functions in $C(\bar{\Omega})$ satisfying the following conditions: there exist two constants $c_1 > 0$ and $c_2 > 0$ such that

$$\text{(3.1.1)} \qquad \begin{aligned} c_1|\xi|^2 &\leq \sum_{i,j}\sum_{\alpha,\beta} a_{\alpha\beta}^{ij}(x)\xi_j^\beta\xi_i^\alpha \qquad \forall x \in \Omega \quad \forall \xi \in \mathbf{M}^{m\times n}, \\ &\sum_{i,j}\sum_{\alpha,\beta} |a_{\alpha\beta}^{ij}(x)| \leq c_2 \qquad \forall x \in \Omega, \end{aligned}$$

and let $\mathcal{A}: H_0^1(\Omega, \mathbf{R}^m) \to H^{-1}(\Omega, \mathbf{R}^m)$ be the elliptic operator defined by

$$\langle \mathcal{A}u, v\rangle = \int_\Omega (ADu, Dv)\, dx,$$

where $ADu$ is the $m{\times}n$ matrix defined by $(ADu)_i^\alpha = \sum_j \sum_\beta a_{\alpha\beta}^{ij} D_j u^\beta$.

For fixed $x \in \Omega$ the Green's function $G(x,y) = G^x(y)$ is the solution of the problem

$$\begin{cases} \mathcal{A}^* G^x = \delta_x I & \text{in } \Omega \\ G^x \in W_0^{1,p}(\Omega, \mathbf{M}^{m\times m}), & 1 < p < \frac{n}{n-1}, \end{cases}$$

where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$, $\delta_x$ is the Dirac distribution at $x$, and $I$ is the $m{\times}m$ identity matrix. Since the coefficients are continuous the existence of the Green's function can be obtained by a classical duality argument. It is well-known that, as the boundary of $\Omega$ is of class $C^1$, there exists a constant $c_3 > 0$ such that

$$|G(x,y)| \le c_3 |x-y|^{2-n} \quad \text{for every } x, y \in \Omega. \tag{3.1.2}$$

This estimate can be proved by using classical regularity results, as in [1]. For any $\mathbf{R}^m$-valued bounded Radon measure $\mu$, the solution $u$ of the problem

$$\begin{cases} \mathcal{A}u = \mu & \text{in } \Omega \\ u \in W_0^{1,p}(\Omega, \mathbf{R}^m), & 1 < p < \frac{n}{n-1} \end{cases}$$

can be represented for almost every $x \in \Omega$ as

$$u(x) = \int_\Omega G(x,y)\, d\mu(y). \tag{3.1.3}$$

If, in addition, $\mu \in H^{-1}(\Omega, \mathbf{R}^m)$, then this formula provides the quasicontinuous representative of the solution $u$.

### 3.2. Definition and properties of the $\mu$-capacity

We introduce now two notions of capacity associated with the operator $\mathcal{A}$.

**Definition 3.2.1.** Let $\xi$, $\eta \in \mathbf{R}^m$ and let $K$ be a compact subset of $\Omega$. The capacity of $K$ in $\Omega$ relative to the operator $\mathcal{A}$ and to the vectors $\xi$ and $\eta$ is defined by

$$C_{\mathcal{A}}(K,\xi,\eta) = \int_{\Omega\setminus K} (ADu^\xi, Du^\eta)\, dx, \tag{3.2.1}$$

where, for every $\zeta \in \mathbf{R}^m$, $u^\zeta$ is the weak solution in $\Omega\backslash K$ of the Dirichlet problem

$$(3.2.2)\qquad \begin{cases} u^\zeta \in H^1(\Omega\backslash K, \mathbf{R}^m), \quad u^\zeta = \zeta \quad \text{on } \partial K, \quad u^\zeta = 0 \quad \text{on } \partial\Omega \\ \displaystyle\int_{\Omega\backslash K} (ADu^\zeta, Dv)\, dx = 0 \qquad \forall v \in H_0^1(\Omega\backslash K, \mathbf{R}^m). \end{cases}$$

We extend $u^\zeta$ to $\Omega$ by setting $u^\zeta = \zeta$ in $K$. In (3.2.2) the boundary conditions are understood in the following sense: for every $\varphi \in C_0^\infty(\Omega, \mathbf{R}^m)$ with $\varphi = \zeta$ on $K$ we have $u^\zeta - \varphi \in H_0^1(\Omega\backslash K, \mathbf{R}^m)$.

**Remark 3.2.2.** For every $\psi \in C_0^\infty(\Omega, \mathbf{R}^m)$ with $\psi = \eta$ on $K$ we have

$$C_{\mathcal{A}}(K, \xi, \eta) = \int_\Omega (ADu^\xi, D\psi)\, dx\,.$$

This can be easily seen by taking $u^\eta - \psi$, which belongs to $H_0^1(\Omega \backslash K, \mathbf{R}^m)$, as test function in the equation (3.2.2) satisfied by $u^\xi$.

**Remark 3.2.3.** The function $C_{\mathcal{A}}(K, \xi, \eta)$ is bilinear with respect to $\xi$ and $\eta$. Moreover there exist two constants $c_4 > 0$ and $c_5 > 0$, depending on $n$, $m$, and on the constants $c_1$ and $c_2$ which appear in (3.1.1), such that

$$C_{\mathcal{A}}(K, \xi, \xi) \geq c_4 \operatorname{cap}(K)|\xi|^2 \quad \text{and} \quad |C_{\mathcal{A}}(K, \xi, \eta)| \leq c_5 \operatorname{cap}(K)|\xi|\,|\eta|\,,$$

for every compact set $K \subset \Omega$ and for every $\xi$, $\eta \in \mathbf{R}^m$. For the proof see Proposition 3.2.7.

Let $\mu \in \mathcal{M}_0(\Omega)$ and let $T = (t_{\alpha\beta})$ be an $m \times m$ matrix of Borel functions satisfying the following conditions: there exist two constants $c_6 > 0$ and $c_7 > 0$ such that

$$(3.2.3)\qquad \begin{gathered} c_6|\xi|^2 \leq \sum_{\alpha,\beta} t_{\alpha\beta}(x)\xi^\alpha\xi^\beta\,, \\ \sum_{\alpha,\beta} |t_{\alpha\beta}(x)| \leq c_7 \end{gathered}$$

for $\mu$-almost every $x \in \Omega$ and every $\xi \in \mathbf{R}^m$.

**Definition 3.2.4.** Let $\xi$, $\eta \in \mathbf{R}^m$. For every Borel set $E \subset\subset \Omega$ the $(T,\mu)$-capacity of $E$ in $\Omega$ relative to $\mathcal{A}$, $\xi$, and $\eta$ is defined by

$$C_{\mathcal{A}}^{T,\mu}(E,\xi,\eta) = \int_\Omega (ADu^\xi, Du^\eta)\,dx + \int_E (T(u^\xi - \xi), (u^\eta - \eta))\,d\mu\,,$$

where, for every $\zeta \in \mathbf{R}^m$, $u^\zeta$ is the solution of

$$\text{(3.2.4)}\qquad \begin{cases} u^\zeta \in H_0^1(\Omega,\mathbf{R}^m), \quad u^\zeta - \zeta \in L_\mu^2(E,\mathbf{R}^m) \\ \displaystyle\int_\Omega (ADu^\zeta, Dv)\,dx + \int_E (T(u^\zeta - \zeta), v)\,d\mu = 0 \\ \forall v \in H_0^1(\Omega,\mathbf{R}^m) \cap L_\mu^2(E,\mathbf{R}^m)\,. \end{cases}$$

The existence and the uniqueness of the solution $u^\zeta$ of problem (3.2.4) follow from the Lax-Milgram Lemma.

**Remark 3.2.5.** For any $\psi \in H_0^1(\Omega,\mathbf{R}^m)$ with $\psi - \eta \in L_\mu^2(E,\mathbf{R}^m)$, we have

$$\text{(3.2.5)}\qquad C_{\mathcal{A}}^{T,\mu}(E,\xi,\eta) = \int_\Omega (ADu^\xi, D\psi)\,dx + \int_E (T(u^\xi - \xi), (\psi - \eta))\,d\mu\,.$$

To prove this fact it is enough to take $u^\eta - \psi$, which belongs to $H_0^1(\Omega,\mathbf{R}^m) \cap L_\mu^2(E,\mathbf{R}^m)$, as test function in the equation (3.2.4) satisfied by $u^\xi$. In particular (3.2.5) gives

$$C_{\mathcal{A}}^{T,\mu}(E,\xi,\eta) = \int_\Omega (ADu^\xi, D\psi)\,dx\,,$$

if $\psi = \eta$ $\mu$-almost everywhere on $E$.

**Remark 3.2.6.** If $\mu$ is bounded, then $u^\eta \in L_\mu^2(E,\mathbf{R}^m)$, thus we may take $u^\eta$ as test function in the equation satisfied by $u^\xi$ and we obtain

$$C_{\mathcal{A}}^{T,\mu}(E,\xi,\eta) = -\int_E (T(u^\xi - \xi), \eta)\,d\mu\,.$$

We shall compare now the capacity $C_{\mathcal{A}}^{T,\mu}$ with the $\mu$-capacity $C^\mu$ relative to the Laplacian, introduced in [15], Definition 5.1.

**Proposition 3.2.7.** *There exist two constants* $c_8 > 0$ *and* $c_9 > 0$, *depending on* $n$, $m$, *and on* $c_1$, $c_2$, $c_6$, $c_7$, *such that for every Borel set* $E \subset\subset \Omega$

$$c_8\, C^\mu(E)|\xi|^2 \le C^{T,\mu}_{\mathcal{A}}(E,\xi,\xi) \qquad \forall \xi \in \mathbf{R}^m, \tag{3.2.6}$$

$$|C^{T,\mu}_{\mathcal{A}}(E,\xi,\eta)| \le c_9\, C^\mu(E)|\xi|\,|\eta| \qquad \forall \xi,\eta \in \mathbf{R}^m. \tag{3.2.7}$$

*Proof.* To prove (3.2.6), let $v^\alpha = (u^\xi)^\alpha/\xi^\alpha$, if $\xi^\alpha \ne 0$, and $v^\alpha = 0$ otherwise. Then, using the ellipticity of $A$ and $T$, for every Borel subset $E \subset\subset \Omega$ and for every $\xi \in \mathbf{R}^m$ we obtain

$$\begin{aligned} C^{T,\mu}_{\mathcal{A}}(E,\xi,\xi) &= \int_\Omega (ADu^\xi, Du^\xi)\, dx + \int_E (T(u^\xi-\xi), u^\xi-\xi)\, d\mu \ge \\ &\ge k\Big(\int_\Omega |Du^\xi|^2 dx + \int_E |u^\xi-\xi|^2 d\mu\Big) \ge \\ &\ge k|\xi|^2 \sum_{\alpha=1}^m \Big(\int_\Omega |Dv^\alpha|^2 dx + \int_E |v^\alpha-1|^2 d\mu\Big) \ge m\,k\, C^\mu(E)|\xi|^2, \end{aligned}$$

where $k = \min\{c_1, c_6\}$.

Using Hölder Inequality it can be easily proved that

$$|C^{T,\mu}_{\mathcal{A}}(E,\xi,\eta)| \le c\,(C^{T,\mu}_{\mathcal{A}}(E,\xi,\xi))^{\frac12}(C^{T,\mu}_{\mathcal{A}}(E,\eta,\eta))^{\frac12}.$$

Hence it suffices to prove (3.2.7) for $\xi = \eta$. Let $v_E$ be the $C^\mu$-capacitary potential of $E$ in $\Omega$ (see [15], Definition 3.1). Define $\psi^\alpha = (1-v_E)\xi^\alpha$. By (3.2.5), using the boundedness of $A$ and $T$, Young Inequality, and then Poincaré Inequality we get

$$\begin{aligned} C^{T,\mu}_{\mathcal{A}}(E,\xi,\xi) &\le M\Big(\int_\Omega |Du^\xi|\,|D\psi|\, dx + \int_E |u^\xi-\xi|\,|\psi-\xi|\, d\mu\Big) \le \\ &\le \frac{M}{2}\Big(\varepsilon\int_\Omega |Du^\xi|^2 dx + \frac{1}{\varepsilon}\int_\Omega |D\psi|^2 dx + \varepsilon\int_E |u^\xi-\xi|^2 d\mu + \frac{1}{\varepsilon}\int_E |\psi-\xi|^2 d\mu\Big). \end{aligned}$$

For a suitable choice of $\varepsilon$ the sum of the terms containing $u^\xi$ can be majorized by $\frac{1}{M}C^{T,\mu}_{\mathcal{A}}(E,\xi,\xi)$, hence there exists a constant $K$ such that

$$\begin{aligned} C^{T,\mu}_{\mathcal{A}}(E,\xi,\xi) &\le K\Big(\int_\Omega |D\psi|^2 dx + \int_E |\psi-\xi|^2 d\mu\Big) \le \\ &\le K|\xi|^2\Big(\int_\Omega |Dv_E|^2 dx + \int_E |v_E|^2 d\mu\Big) = K|\xi|^2 C^\mu(E). \end{aligned}$$

□

**Theorem 3.2.8.** *For every Kato measure $\mu$ (see Definition 1.2.1) the solution $u^\zeta$ of (3.2.4) corresponding to a Borel subset $E$ of $\Omega$ of sufficiently small diameter belongs to $L^\infty(\Omega, \mathbf{R}^m)$ and tends to 0 in $L^\infty(\Omega, \mathbf{R}^m)$ as the diameter of $E$ tends to zero.*

*Proof.* Let $E$ be a Borel subset of $\Omega$ and let $u^\zeta$ be the solution of (3.2.4). If $u^\zeta \in L^\infty_\mu(\Omega, \mathbf{R}^m)$, then the representation formula (3.1.3) for the solution of a linear system of second order partial differential equations gives

$$u^\zeta(x) = -\int_E G(x,y)T(y)(u^\zeta(y) - \zeta)\, d\mu(y) \quad \text{for a.e. } x \in \Omega, \tag{3.2.8}$$

where $G(x,y)$ is the Green's function associated with the operator $\mathcal{A}$ and with the domain $\Omega$. In this case the measure $T(u^\zeta - \zeta)\mu \llcorner E$ belongs to $H^{-1}(\Omega, \mathbf{R}^m)$ and (3.2.8) provides the quasicontinuous representative of $u^\zeta$.

Let us consider the operator $U: L^\infty_\mu(\Omega, \mathbf{R}^m) \to L^\infty_\mu(\Omega, \mathbf{R}^m)$ defined by

$$Uf(x) = -\int_E G(x,y)T(y)(f(y) - \zeta)\, d\mu(y).$$

Since the functions $t_{\alpha\beta}$ are bounded, we may apply estimate (3.1.2) for the Green's function and we obtain

$$\|Uf_1 - Uf_2\|_{L^\infty_\mu(\Omega,\mathbf{R}^m)} \le c_3 c_7 \|f_1 - f_2\|_{L^\infty_\mu(\Omega,\mathbf{R}^m)} \sup_{x\in\Omega} \int_E |x-y|^{2-n}\, d\mu(y).$$

As $\mu \in K^+(\Omega)$, the integral in the above formula tends to zero as $\mathrm{diam}(E)$ tends to zero, so that for sets $E$ of sufficiently small diameter the operator $U$ is a contraction, hence it has a unique fixed point $w$ in $L^\infty_\mu(\Omega, \mathbf{R}^m)$. By (3.1.3), for $f \in L^\infty_\mu(\Omega, \mathbf{R}^m)$ the function $w_f = Uf$ is the solution of the Dirichlet problem

$$\begin{cases} w_f \in H^1_0(\Omega, \mathbf{R}^m) \\ Aw_f = -T(f - \zeta)\mu \llcorner E \quad \text{in } \Omega, \end{cases}$$

so that the fixed point $w$ belongs to $H^1_0(\Omega, \mathbf{R}^m)$ and is a solution in the sense of distributions of $Aw = -T(w - \zeta)\mu \llcorner E$, and hence a solution of (3.2.4). Therefore $u^\zeta = w$ and we conclude that for sets $E$ of sufficiently small diameter $u^\zeta \in L^\infty_\mu(\Omega, \mathbf{R}^m)$. Then, from (3.2.8), for the quasicontinuous representative of $u^\zeta$ we have

$$|u^\zeta(x)| = \left|\int_E G(x,y)T(y)(u^\zeta(y) - \zeta)\, d\mu(y)\right| \le \int_E |G(x,y)||T(y)||u^\zeta(y) - \zeta|\, d\mu(y) \le$$
$$\le c_3 c_7 \|u^\zeta - \zeta\|_{L^\infty_\mu(\Omega,\mathbf{R}^m)} \sup_{x\in\Omega} \int_E |x-y|^{2-n} d\mu(y),$$

which implies that $\|u^\zeta\|_{L^\infty(\Omega,\mathbf{R}^m)} \le c_E\|u^\zeta\|_{L^\infty_\mu(\Omega,\mathbf{R}^m)} + c_E|\zeta|$, where the coefficient $c_E$ is given by $c_3c_7\sup_{x\in\Omega}\int_E|x-y|^{2-n}d\mu$ and tends to zero as the diameter of $E$ tends to zero. As $u^\zeta \in H^1_0(\Omega,\mathbf{R}^m)$ and $\mu$ vanishes on sets of capacity zero, $\|u^\zeta\|_{L^\infty_\mu(\Omega,\mathbf{R}^m)} \le \|u^\zeta\|_{L^\infty(\Omega,\mathbf{R}^m)}$ and from the previous inequality we obtain that $\|u^\zeta\|_{L^\infty(\Omega,\mathbf{R}^m)}$ tends to zero as the diameter of $E$ tends to zero. □

**Proposition 3.2.9.** *If $\mu$ is a Kato measure then*

$$\lim_{\rho\to 0+}\frac{C^{T,\mu}_{\mathcal{A}}(D_\rho(x),\xi,\eta)}{\mu(D_\rho(x))} = (T(x)\,\xi,\eta)$$

*for $\mu$-almost every $x\in\Omega$ and for every $\xi,\ \eta\in\mathbf{R}^m$.*

*Proof.* Let $x\in\Omega$. Since every $\mu\in K^+(\Omega)$ is bounded, by Remark 3.2.6 we have

$$C^{T,\mu}_{\mathcal{A}}(D_\rho(x),\xi,\eta) = -\int_{D_\rho(x)}(T(y)(u^\xi(y)-\xi),\eta)\,d\mu(y)\,. \tag{3.2.9}$$

By the Besicovitch Differentiation Theorem (see, e.g., [9], 1.6.2),

$$\lim_{\rho\to 0+}\frac{1}{\mu(D_\rho(x))}\int_{D_\rho(x)}(T(y)\xi,\eta)\,d\mu(y) = (T(x)\,\xi,\eta) \tag{3.2.10}$$

for $\mu$-almost every $x\in\Omega$ and for every $\xi,\ \eta\in\mathbf{R}^m$. The conclusion follows now from (3.2.9), (3.2.10), and Theorem 3.2.8. □

### 3.3. $\gamma^{\mathcal{A}}$-convergence

In order to study the asymptotic behaviour of sequences of solutions of Dirichlet problems in varying domains we introduce the notion of $\gamma^{\mathcal{A}}$-convergence and show that under certain hypotheses the $\gamma^{\mathcal{A}}$-limit can be identified.

**Definition 3.3.1.** Let $(\Omega_h)$ be a sequence of open subsets of $\Omega$, let $\mu\in\mathcal{M}_0(\Omega)$, and let $T$ be an $m\times m$ matrix of Borel functions satisfying (3.2.3). We say that $(\Omega_h)$ $\gamma^{\mathcal{A}}_\Omega$-converges to $(T,\mu)$, and we use the notation $\Omega_h \xrightarrow{\gamma^{\mathcal{A}}_\Omega} (T,\mu)$, if for every $f\in H^{-1}(\Omega,\mathbf{R}^m)$ the sequence $(u_h)$ of the solutions of the problems

$$\begin{cases} u_h\in H^1_0(\Omega_h,\mathbf{R}^m) \\ \displaystyle\int_{\Omega_h}(ADu_h,Dv)\,dx = \langle f,v\rangle \qquad \forall v\in H^1_0(\Omega_h,\mathbf{R}^m), \end{cases}$$

extended by zero on $\Omega\backslash\Omega_h$, converges weakly in $H_0^1(\Omega,\mathbf{R}^m)$ to the solution of the relaxed Dirichlet problem

$$(3.3.1)\qquad \begin{cases} u \in H_0^1(\Omega,\mathbf{R}^m)\cap L^2_\mu(\Omega,\mathbf{R}^m) \\ \displaystyle\int_\Omega (ADu,Dv)\,dx + \int_\Omega (Tu,v)\,d\mu \;=\; \langle f,v\rangle \\ \forall v\in H_0^1(\Omega,\mathbf{R}^m)\cap L^2_\mu(\Omega,\mathbf{R}^m)\,. \end{cases}$$

**Remark 3.3.2.** Let $\mu\in\mathcal{M}_0(\Omega)$, let $T$ be an $m\times m$ matrix of Borel functions satisfying (3.2.3), and let $\nu$ and $S$ be defined by

$$\nu(E)=\int_E |T|\,d\mu\,,\qquad S(x)=\frac{T(x)}{|T(x)|}\,.$$

Then the measure $\nu$ belongs to $\mathcal{M}_0(\Omega)$ and the matrix $S$ satisfies (3.2.3). Moreover $\Omega_h \xrightarrow{\gamma^{\mathcal{A}}_\Omega} (T,\mu)$ if and only if $\Omega_h \xrightarrow{\gamma^{\mathcal{A}}_\Omega} (S,\nu)$. This shows that, in Definition 3.3.1, it is not restrictive to assume $|T(x)|=1$ for every $x\in\Omega$. However, it is sometimes useful to consider also matrices $T$ which do not satisfy this condition.

If $m=1$ and $\mathcal{A}=-\Delta$, we shall always assume that $T(x)=1$ for every $x\in\Omega$. In this case we use the notation $\Omega_h \xrightarrow{\gamma_\Omega} \mu$.

The following compactness result is proved in [7].

**Theorem 3.3.3.** *For every sequence $(\Omega_h)$ of open subsets of $\Omega$ there exist a subsequence $(\Omega_{j_k})$, a measure $\mu\in\mathcal{M}_0(\Omega)$, and an $m\times m$ matrix $T$ of Borel functions satisfying (2.3), such that $\Omega_{j_k} \xrightarrow{\gamma_\Omega} \mu$ and $\Omega_{j_k} \xrightarrow{\gamma^{\mathcal{A}}_\Omega} (T,\mu)$.*

The localization property of the $\gamma^{\mathcal{A}}$-convergence is also proved in [7].

**Theorem 3.3.4.** *If $\Omega_h \xrightarrow{\gamma^{\mathcal{A}}_\Omega} (T,\mu)$ then $\Omega_h\cap U \xrightarrow{\gamma^{\mathcal{A}}_U} (T_{|U},\mu_{|U})$ for every open subset $U$ of $\Omega$.*

**Proposition 3.3.5.** *Suppose that $\Omega_h \xrightarrow{\gamma^{\mathcal{A}}_\Omega} (T,\mu)$ and $\Omega_h \xrightarrow{\gamma^{\mathcal{A}}_\Omega} (\tilde{T},\tilde{\mu})$. If $\mu=\tilde{\mu}$ and $\mu(\Omega)<+\infty$, then $T(x)=\tilde{T}(x)$ for $\mu$-almost every $x\in\Omega$.*

*Proof.* Let $f\in H^{-1}(\Omega,\mathbf{R}^m)$ and let $u$ be the solution of the relaxed Dirichlet problem (3.3.1). Then we have

$$\int_\Omega ((T-\tilde{T})u,v)\,d\mu \;=\; 0 \qquad \forall v\in H_0^1(\Omega,\mathbf{R}^m)\cap L^2_\mu(\Omega,\mathbf{R}^m)\,.$$

In particular, since $\mu(\Omega) < +\infty$, this equality holds true for every $v \in C_0^\infty(\Omega, \mathbf{R}^m)$. So, varying $v$, we obtain that $(T - \tilde{T})u = 0$ $\mu$-almost everywhere in $\Omega$. Since $\mu(\Omega) < +\infty$, the set of all solutions $u$ of (3.3.1) corresponding to different data $f \in H^{-1}(\Omega, \mathbf{R}^m)$ is dense in $H_0^1(\Omega, \mathbf{R}^m)$. This implies that $T = \tilde{T}$ $\mu$-almost everywhere in $\Omega$. □

For every $x \in \Omega$ let $d_\Omega(x) = \mathrm{dist}(x, \partial\Omega)$.

**Theorem 3.3.6.** *If $\Omega_h \xrightarrow{\gamma_\Omega} \mu$, with $\mu(\Omega) < +\infty$, and $\Omega_h \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (T, \mu)$, then for every $x \in \Omega$ there exists a countable set $N(x) \subset \mathbf{R}$ such that*

$$C_{\mathcal{A}}(D_\rho(x) \backslash \Omega_h, \xi, \eta) \to C_{\mathcal{A}}^{T,\mu}(D_\rho(x), \xi, \eta)$$

*for every $\rho \in (0, d_\Omega(x)) \backslash N(x)$.*

*Proof.* Let us fix $x \in \Omega$. It is proved in [15] that there exists a countable set $N_1(x) \subset \mathbf{R}$ such that for all $\rho \in (0, d_\Omega(x)) \backslash N_1(x)$

$$\Omega \backslash (D_\rho(x) \backslash \Omega_h) \xrightarrow{\gamma_\Omega} \mu \llcorner D_\rho(x).$$

Then, applying Theorem 3.3.3 to the sequence $\tilde{\Omega}_h = \Omega \backslash (D_\rho(x) \backslash \Omega_h)$, we obtain that there exist a subsequence, still denoted by the same index $j$, and an $m \times m$ matrix $\tilde{T}$ of Borel functions satisfying (3.2.3) such that

$$\tilde{\Omega}_h \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (\tilde{T}, \mu \llcorner D_\rho(x)).$$

Now we apply the localization result (Theorem 3.3.4) to the sequence $(\Omega_h)$ and we obtain

$$\Omega_h \cap B_\rho(x) \xrightarrow{\gamma_{B_\rho(x)}} \mu_{|B_\rho(x)} \quad \text{and} \quad \Omega_h \cap B_\rho(x) \xrightarrow{\gamma_{B_\rho(x)}^{\mathcal{A}}} (T_{|B_\rho(x)}, \mu_{|B_\rho(x)}).$$

The same localization result applied to the sequence $\tilde{\Omega}_h$ gives

$$\Omega_h \cap B_\rho(x) = \tilde{\Omega}_h \cap B_\rho(x) \xrightarrow{\gamma_{B_\rho(x)}^{\mathcal{A}}} (\tilde{T}_{|B_\rho(x)}, \mu_{|B_\rho(x)}),$$

hence $T = \tilde{T}$ $\mu$-almost everywhere in $B_\rho(x)$ by Proposition 3.3.5. On the other hand, since $\mu(\Omega) < +\infty$, for every $x \in \Omega$ there exists a countable set $N_2(x) \subset \mathbf{R}$ such that $\mu(\partial D_\rho(x)) = 0$ for all $\rho \in (0, d_\Omega(x)) \backslash N_2(x)$. Together with the previous results this implies that

$$\tilde{\Omega}_h \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (T, \mu \llcorner D_\rho(x)) \text{ for every } \rho \in (0, d_\Omega(x)) \backslash (N_1(x) \cup N_2(x)).$$

Let $K_h = D_\rho(x)\backslash\Omega_h = \Omega\backslash\tilde\Omega_h$ and let $u_h$ be the weak solution in $\tilde\Omega_h$ of the problem

$$\begin{cases} u_h \in H^1(\tilde\Omega_h), \quad u_h = \xi \text{ on } \partial K_h, \quad u_h = 0 \text{ on } \partial\Omega \\ \displaystyle\int_{\tilde\Omega_h} (ADu_h, Dv)\,dx = 0 \qquad \forall v \in H^1_0(\tilde\Omega_h, \mathbf{R}^m). \end{cases}$$

As usual we extend $u_h$ to $\Omega$ by setting $u_h = \xi$ on $K_h$. Let $\varphi \in C_0^\infty(\Omega,\mathbf{R}^m)$ with $\varphi = \xi$ on $D_\rho(x)$, and let $z_h = u_h - \varphi$. Then $z_h$ is the solution of the problem

$$\begin{cases} z_h \in H^1_0(\tilde\Omega_h, \mathbf{R}^m) \\ \displaystyle\int_{\tilde\Omega_h} (ADz_h, Dv)\,dx = \langle f, v\rangle \qquad \forall v \in H^1_0(\tilde\Omega_h, \mathbf{R}^m), \end{cases}$$

where $f$ is the element of $H^{-1}(\Omega,\mathbf{R}^m)$ defined by $\langle f,v\rangle = -\int_\Omega (AD\varphi, Dv)\,dx$. By Definition 3.3.1 the sequence $(z_h)$ converges weakly in $H^1_0(\Omega,\mathbf{R}^m)$ to the solution $z$ of the problem

$$\begin{cases} z \in H^1_0(\Omega,\mathbf{R}^m) \cap L^2_\mu(D_\rho(x),\mathbf{R}^m) \\ \displaystyle\int_\Omega (ADz, Dv)\,dx + \int_{D_\rho(x)} (Tz, v)\,d\mu = \langle f, v\rangle \\ \forall v \in H^1_0(\Omega,\mathbf{R}^m) \cap L^2_\mu(D_\rho(x),\mathbf{R}^m). \end{cases}$$

This implies that $(u_h)$ converges weakly in $H^1_0(\Omega,\mathbf{R}^m)$ to the solution $u^\xi$ of (3.2.4) corresponding to $\zeta = \xi$ and $E = D_\rho(x)$. Consequently $(ADu_h)$ converges to $ADu^\xi$ weakly in $L^2(\Omega,\mathbf{M}^{m\times n})$. Let us fix now $\psi \in C_0^\infty(\Omega,\mathbf{R}^m)$ with $\psi = \eta$ on $D_\rho(x)$. Then, by Remarks 3.2.2 and 3.2.5,

$$C_{\mathcal{A}}(D_\rho(x)\backslash\Omega_h,\xi,\eta) = \int_\Omega (ADu_h, D\psi)\,dx \longrightarrow \int_\Omega (ADu^\xi, D\psi)\,dx = C_{\mathcal{A}}^{T,\mu}(D_\rho(x),\xi,\eta),$$

and the proof is concluded. □

Given a family $(f_\rho)_{\rho>0}$ of real numbers, we say that $\operatorname{ess\,lim}_{\rho\to 0} f_\rho = a$ if for every neighbourhood $V$ of $a$ there exists a neighbourhood $U$ of $0$ such that $f_\rho \in V$ for almost every $\rho \in U$.

Let $(\Omega_h)$ be a sequence of open subsets of $\Omega$. For every closed ball $D_\rho(x) \subset \Omega$ and for every $\xi$, $\eta \in \mathbf{R}^m$ we define

$$\begin{aligned} \alpha'(D_\rho(x),\xi,\eta) &= \liminf_{j\to\infty} C_{\mathcal{A}}(D_\rho(x)\backslash\Omega_h,\xi,\eta), \\ \alpha''(D_\rho(x),\xi,\eta) &= \limsup_{j\to\infty} C_{\mathcal{A}}(D_\rho(x)\backslash\Omega_h,\xi,\eta). \end{aligned} \tag{3.3.2}$$

We are now in a position to prove the main result of the chapter.

**Theorem 3.3.7.** *Assume that there exists a measure* $\lambda \in K^+(\Omega)$ *such that*

$$\alpha''(D_\rho(x),\xi,\xi) \le \lambda(D_\rho(x))|\xi|^2 \tag{3.3.3}$$

*for every closed ball* $D_\rho(x) \subset \Omega$ *and for every* $\xi \in \mathbf{R}^m$. *Assume, in addition, that for every* $x \in \Omega$

$$\alpha'(D_\rho(x),\xi,\eta) = \alpha''(D_\rho(x),\xi,\eta) \quad \textit{for a.e. } \rho \in (0, d_\Omega(x)). \tag{3.3.4}$$

*Then there exists an* $m{\times}m$ *matrix* $G(x)$ *of bounded Borel functions such that*

$$\operatorname*{ess\,lim}_{\rho\to 0} \frac{\alpha'(D_\rho(x),\xi,\eta)}{\lambda(D_\rho(x))} = \operatorname*{ess\,lim}_{\rho\to 0} \frac{\alpha''(D_\rho(x),\xi,\eta)}{\lambda(D_\rho(x))} = (G(x)\,\xi,\eta) \tag{3.3.5}$$

*for* $\lambda$*-almost every* $x \in \Omega$ *and for every* $\xi$, $\eta \in \mathbf{R}^m$. *Let* $T$ *and* $\mu$ *be defined by*

$$T(x) = \frac{G(x)}{|G(x)|} \qquad \textit{for every } x \in \Omega$$

$$\mu(E) = \int_E |G|\, d\lambda \qquad \textit{for every Borel set } E \subset \Omega\,,$$

*with the convention that 0/0 is the* $m{\times}m$ *identity matrix* $I$. *Then* $T$ *satisfies (3.2.3) and* $\Omega_h \xrightarrow{\gamma_\Omega^A} (T,\mu)$.

**Remark 3.3.8.** Theorems 3.3.3 and 3.3.6 imply that every sequence $(\Omega_h)$ has a subsequence which satisfies (3.3.4). Therefore condition (3.3.3) is the only non-trivial hypothesis of Theorem 3.3.7.

**Remark 3.3.9.** For every closed ball $D_\rho(x) \subset \Omega$ let

$$\beta''(D_\rho(x)) = \limsup_{j\to+\infty}\ \operatorname{cap}(D_\rho(x)\backslash\Omega_h)\,.$$

If there exists a measure $\lambda \in K^+(\Omega)$ such that $\beta''(D_\rho(x)) \le \lambda(D_\rho(x))$ then the estimates in Remark 3.2.3 imply that (3.3.3) is satisfied with $\lambda$ replaced by $c_5\lambda$. This condition is satisfied, for instance, in the periodic case with a critical size of the holes (see [9]) and for the sequences of domains considered in [35] and [42].

*Proof of Theorem 3.3.7.* Let us fix $x \in \Omega$. From the compactness result (Theorem 3.3.3) we obtain that there exist a subsequence, still denoted by $(\Omega_h)$, and a pair $(\tilde{T},\tilde{\mu})$, with $\tilde{T}$ satisfying (3.2.3) and $\tilde{\mu} \in \mathcal{M}_0(\Omega)$, such that $\Omega_h \xrightarrow{\gamma_\Omega} \tilde{\mu}$ and $\Omega_h \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (\tilde{B}, \tilde{\mu})$. By Theorem 5.15 in [15] for almost every $\rho \in (0, d_\Omega(x))$ we have $\mathrm{cap}(D_\rho(x) \setminus \Omega_h) \to C^{\tilde{\mu}}(D_\rho(x))$. The first estimate in Remark 3.2.3 gives

$$c_4 |\xi|^2 \mathrm{cap}(D_\rho(x) \setminus \Omega_h) \leq C_{\mathcal{A}}(D_\rho(x) \setminus \Omega_h, \xi, \xi),$$

and passing to the limit we get

$$c_4 C^{\tilde{\mu}}(D_\rho(x)) |\xi|^2 \leq \limsup_{\rho \to 0} C_{\mathcal{A}}(D_\rho(x) \setminus \Omega_h, \xi, \xi) = \alpha''(D_\rho(x), \xi, \xi) \leq \lambda(D_\rho(x)) |\xi|^2 .$$

Applying now Theorem 2.3 in [6] we get that $\tilde{\mu}$ is absolutely continuous with respect to $\lambda$. The above estimate implies that the density $\frac{d\tilde{\mu}}{d\lambda}(x)$ is bounded, so $\tilde{\mu} \in K^+(\Omega)$. Let

$$G(x) = \tilde{T}(x) \frac{d\tilde{\mu}}{d\lambda}(x), \quad T(x) = \frac{G(x)}{|G(x)|}, \quad \mu(E) = \int_E |G| \, d\lambda = \int_E |\tilde{T}| \, d\tilde{\mu},$$

with the convention that $0/0$ is the $m \times m$ identity matrix $I$. Then

$$T(x) = \frac{\tilde{T}(x)}{|\tilde{T}(x)|}, \text{ if } \frac{d\tilde{\mu}}{d\lambda}(x) > 0, \text{ and } T(x) = I, \text{ if } \frac{d\tilde{\mu}}{d\lambda}(x) = 0 .$$

As $\tilde{T}$ satisfies (3.2.3) $\tilde{\mu}$-almost everywhere, $T$ satisfies (3.2.3) $\mu$-almost everywhere. Since $\Omega_h \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (\tilde{T}, \tilde{\mu})$ and $T(x) = \frac{\tilde{T}(x)}{|\tilde{T}(x)|}$ $\tilde{\mu}$-almost everywhere in $\Omega$, by Remark 3.3.2 we have also $\Omega_h \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (T, \mu)$.

Let us prove now (3.3.1). Applying Theorem 3.3.6 we obtain that $C_{\mathcal{A}}(D_\rho(x) \setminus \Omega_h, \xi, \eta) \to C_{\mathcal{A}}^{\tilde{T}, \tilde{\mu}}(D_\rho(x), \xi, \eta)$ for almost every $\rho \in (0, d_\Omega(x))$. Thus

$$\alpha'(D_\rho(x), \xi, \eta) = \alpha''(D_\rho(x), \xi, \eta) = C_{\mathcal{A}}^{\tilde{T}, \tilde{\mu}}(D_\rho(x), \xi, \eta)$$

for almost every $\rho \in (0, d_\Omega(x))$ and for every $\xi, \eta \in \mathbf{R}^m$. We may now apply Proposition 3.2.9 and the Besicovitch Differentiation Theorem to obtain

$$\operatorname*{ess\,lim}_{\rho \to 0} \frac{\alpha'(D_\rho(x), \xi, \eta)}{\lambda(D_\rho(x))} = \operatorname*{ess\,lim}_{\rho \to 0} \frac{C_{\mathcal{A}}^{\tilde{T}, \tilde{\mu}}(D_\rho(x), \xi, \eta)}{\tilde{\mu}(D_\rho(x))} \operatorname*{ess\,lim}_{\rho \to 0} \frac{\tilde{\mu}(D_\rho(x))}{\lambda(D_\rho(x))} =$$
$$= (\tilde{T}(x) \xi, \eta) \frac{d\tilde{\mu}}{d\lambda}(x) = (G(x) \xi, \eta)$$

for every $\xi, \eta \in \mathbf{R}^m$ and for $\lambda$-almost every $x \in \Omega$ such that $\frac{d\tilde{\mu}}{d\lambda}(x) > 0$. Since $C_{\mathcal{A}}^{\tilde{T}, \tilde{\mu}}(D_\rho(x), \xi, \eta) \leq c_9 \, C^{\tilde{\mu}}(D_\rho(x)) |\xi| \, |\eta| \leq c_9 \tilde{\mu}(D_\rho(x)) |\xi| \, |\eta|$ by (3.2.7), we obtain that

$$\operatorname*{ess\,lim}_{\rho \to 0} \frac{\alpha'(D_\rho(x), \xi, \eta)}{\lambda(D_\rho(x))} = 0 = (G(x) \xi, \eta)$$

for $\lambda$-almost every $x \in \Omega$ such that $\frac{d\tilde{\mu}}{d\lambda}(x) = 0$. This concludes the proof of (3.3.5). □

### 3.4. The symmetric case

If the operator $\mathcal{A}$ is symmetric, then the $\mathcal{A}$-capacity can be obtained by solving a minimum problem. If $\Omega_h \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (T,\mu)$, with $\mu(\Omega) < +\infty$, then the matrix $T$ is symmetric (see [20], Corollary 5.4). In this case we have

$$C_{\mathcal{A}}^{T,\mu}(E,\xi,\xi) = \min_{u\in H_0^1(\Omega,\mathbf{R}^m)} \Big\{ \int_\Omega (ADu^\xi, Du^\xi)\,dx + \int_E (T(u^\xi-\xi),(u^\xi-\xi))\,d\mu \Big\}$$

for every measure $\mu \in \mathcal{M}_0(\Omega)$, for every $\xi \in \mathbf{R}^m$, and for every Borel set $E \subset\subset \Omega$.

**Remark 3.4.1.** Assume that $\mathcal{A}$ and $T$ are symmetric. If $\mu_1 \le \mu_2$, then $C_{\mathcal{A}}^{T,\mu_1}(E,\xi,\xi) \le C_{\mathcal{A}}^{T,\mu_2}(E,\xi,\xi)$ for every Borel subset $E$ of $\Omega$ and every $\xi \in \mathbf{R}^m$.

This monotonicity property of the capacity with respect to the measure allows us to extend the derivation theorem to any bounded measure in $\mathcal{M}_0(\Omega)$.

**Theorem 3.4.2.** *Assume that $\mathcal{A}$ is symmetric. Let $\mu$, $\nu \in \mathcal{M}_0(\Omega)$, with $\nu(\Omega) < +\infty$, and let $T$ be an $m\times m$ symmetric matrix of Borel functions satisfying (3.2.3). For every $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$ let*

$$(3.4.1) \qquad f(x,\xi) = \liminf_{\rho\to 0} \frac{C_{\mathcal{A}}^{T,\mu}(D_\rho(x),\xi,\xi)}{\nu(D_\rho(x))} \qquad \textit{(with the convention that 0/0=1).}$$

*Assume that there exists $\xi \in \mathbf{R}^m\setminus\{0\}$ such that*

$$(3.4.2) \qquad f(x,\xi) < +\infty \quad \forall x \in \Omega \qquad \textit{and} \qquad \int_\Omega f(x,\xi)\,d\nu < +\infty .$$

*Then $\mu(\Omega) < +\infty$, $\mu$ is absolutely continuous with respect to $\nu$, and*

$$f(x,\xi) = (T(x)\xi,\xi)\frac{d\mu}{d\nu}(x) \quad \textit{for } \nu\textit{-a.e. } x \in \Omega \textit{ and for every } \xi \in \mathbf{R}^m .$$

*Moreover, the* lim inf *in the definition of $f$ is a limit for $\nu$-almost every $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$.*

*Proof.* For every $x \in \Omega$ let

$$f_1(x) = \liminf_{\rho\to 0} \frac{C^\mu(D_\rho(x))}{\nu(D_\rho(x))} .$$

The estimates in Proposition 3.2.7 give

$$c_8|\xi|^2 f_1(x) \le f(x,\xi) \le c_9|\xi|^2 f_1(x) \qquad \forall x \in \Omega \quad \forall \xi \in \mathbf{R}^m, \tag{3.4.3}$$

thus $f_1 \in L^1_\nu(\Omega)$ and $f_1(x) < +\infty$ for every $x \in \Omega$. Then from Proposition 2.3 in [6] we deduce that $\mu(\Omega) < +\infty$ and that $\mu = f_1\nu$, i.e., $\mu(E) = \int_E f_1\,d\nu$ for every Borel set $E \subseteq \Omega$. By Proposition 2.5 of [3], there exist a measure $\lambda \in K^+(\Omega)$ and a Borel function $g\colon \Omega \to [0,+\infty]$ such that $\mu = g\lambda$. For every $k \in \mathbf{N}$ let $g_k(x) = \min\{g(x), k\}$. Since $g_k\lambda$ belongs to $K^+(\Omega)$, Proposition 3.2.9 implies the existence of a subset $E_1$ of $\Omega$ such that $\int_{E_1} g_k d\lambda = 0$ and

$$\lim_{\rho\to 0} \frac{C_{\mathcal{A}}^{T,g_k\lambda}(D_\rho(x),\xi,\xi)}{(g_k\lambda)(D_\rho(x))} = (T(x)\xi,\xi) \qquad \forall x \in \Omega\backslash E_1 \quad \forall \xi \in \mathbf{R}^m \quad \forall k \in \mathbf{N}.$$

Since $\lambda + \nu$ is a bounded measure on $\Omega$, by the Besicovitch Differentiation Theorem there exists a set $E_2 \subset \Omega$ such that $(\lambda+\nu)(E_2) = 0$ and

$$\lim_{\rho\to 0} \frac{(g_k\lambda)(D_\rho(x))}{(\lambda+\nu)(D_\rho(x))} = g_k(x)\frac{d\lambda}{d(\lambda+\nu)}(x) < +\infty \qquad \forall x \in \Omega\backslash E_2 \quad \forall k \in \mathbf{N},$$

$$\lim_{\rho\to 0} \frac{\nu(D_\rho(x))}{(\lambda+\nu)(D_\rho(x))} = \frac{d\nu}{d(\lambda+\nu)}(x) \le 1 \qquad \forall x \in \Omega\backslash E_2.$$

By (3.4.2) and (3.4.3) we have $f_1(x) < +\infty$ and $f(x,\xi) < +\infty$ for every $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$. Let $E = E_1 \cup E_2$. For $x \in \Omega\backslash E$ and $\xi \in \mathbf{R}^m$ we have

$$\begin{aligned} g_k(x)(T(x)\xi,\xi)\frac{d\lambda}{d(\lambda+\nu)}(x) &= \lim_{\rho\to 0}\frac{(g_k\lambda)(D_\rho(x))}{(\lambda+\nu)(D_\rho(x))}\lim_{\rho\to 0}\frac{C_{\mathcal{A}}^{T,g_k\lambda}(D_\rho(x),\xi,\xi)}{(g_k\lambda)(D_\rho(x))} = \\ = \lim_{\rho\to 0}\frac{C_{\mathcal{A}}^{T,g_k\lambda}(D_\rho(x),\xi,\xi)}{(\lambda+\nu)(D_\rho(x))} &\le \liminf_{\rho\to 0}\frac{C_{\mathcal{A}}^{T,g\lambda}(D_\rho(x),\xi,\xi)}{\nu(D_\rho(x))}\lim_{\rho\to 0}\frac{\nu(D_\rho(x))}{(\lambda+\nu)(D_\rho(x))} = \\ &= f(x,\xi)\frac{d\nu}{d(\lambda+\nu)}(x). \end{aligned}$$

So, for every Borel set $F \subset \Omega\backslash E$ and for every $\xi \in \mathbf{R}^m$ we have

$$\int_F \Big[g_k(x)(T(x)\xi,\xi)\frac{d\lambda}{d(\lambda+\nu)}(x)\Big]\,d(\lambda+\nu) \le \int_F \Big[f(x,\xi)\frac{d\nu}{d(\lambda+\nu)}(x)\Big]\,d(\lambda+\nu),$$

hence

$$\int_F g_k(x)(T(x)\xi,\xi)\,d\lambda \le \int_F f(x,\xi)\,d\nu$$

for every Borel set $F \subset \Omega$. Passing now to the limit as $k \to +\infty$, by the monotone convergence theorem we have

$$\int_F (T(x)\,\xi,\xi)\,d\mu \;=\; \int_F g(x)(T(x)\,\xi,\xi)\,d\lambda \;\le\; \int_F f(x,\xi)\,d\nu$$

for every Borel set $F \subset \Omega$ and every $\xi \in \mathbf{R}^m$. Thus, $f_1(x)(T(x)\,\xi,\xi) \le f(x,\xi)$ for $\nu$-almost every $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$. Since

$$C_{\mathcal{A}}^{T,\mu}(D_\rho(x),\xi,\xi) \;\le\; \int_{D_\rho(x)} (T(y)\xi,\xi) f_1(y)\,d\nu(y)\,,$$

by the Besicovitch Differentiation Theorem we obtain $f(x,\xi) \le f_1(x)(T(x)\,\xi,\xi)$ for $\nu$-almost every $x \in \Omega$ and for every every $\xi \in \mathbf{R}^m$. So we proved that $f(x,\xi) = f_1(x)(T(x)\,\xi,\xi)$ for every $\xi \in \mathbf{R}^m$ and $\nu$-almost every $x \in \Omega$. Moreover, by the Besicovitch Differentiation Theorem for $\nu$-almost every $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$ we have

$$\begin{aligned} f(x,\xi) \;&=\; \liminf_{\rho\to 0} \frac{C_{\mathcal{A}}^{T,\mu}(D_\rho(x),\xi,\xi)}{\nu(D_\rho(x))} \;\le\; \limsup_{\rho\to 0} \frac{C_{\mathcal{A}}^{T,\mu}(D_\rho(x),\xi,\xi)}{\nu(D_\rho(x))} \;\le \\ &\le\; \limsup_{\rho\to 0} \frac{1}{\nu(D_\rho(x))} \int_{D_\rho(x)} (T(y)\xi,\xi) f_1(y)\,d\nu(y) \;=\; f_1(x)(T(x)\,\xi,\xi)\,, \end{aligned}$$

and this completes the proof. □

The hypotheses in Theorem 3.3.7 can be weakened by using the monotonicity of the $\mathcal{A}$-capacity and the previous result.

**Theorem 3.4.3.** *Assume that $\mathcal{A}$ is symmetric and that there exists a bounded Radon measure $\lambda$ on $\Omega$ such that*

$$\alpha''(D_\rho(x),\xi,\xi) \le \lambda(D_\rho(x))|\xi|^2$$

*for every closed ball $D_\rho(x) \subset \Omega$ and for every $\xi \in \mathbf{R}^m$. Assume, in addition, that for every $x \in \Omega$ there exists a dense set $D \subset (0, d_\Omega(x))$ such that*

(3.4.4) $\qquad \alpha'(D_\rho(x),\xi,\xi) = \alpha''(D_\rho(x),\xi,\xi)$ *for every* $\rho \in D$ *and every* $\xi \in \mathbf{R}^m$.

*Then there exists an $m\times m$ symmetric matrix $G(x)$ of bounded Borel functions such that*

$$\operatorname*{ess\,lim}_{\rho\to 0} \frac{\alpha'(D_\rho(x),\xi,\xi)}{\lambda(D_\rho(x))} \;=\; \operatorname*{ess\,lim}_{\rho\to 0} \frac{\alpha''(D_\rho(x),\xi,\xi)}{\lambda(D_\rho(x))} \;=\; (G(x)\,\xi,\xi)$$

*for $\lambda$-almost every $x \in \Omega$ and for every $\xi \in \mathbf{R}^m$. Let $T$ and $\mu$ be defined by*

$$T(x) = \frac{G(x)}{|G(x)|} \qquad \text{for every } x \in \Omega\,,$$

$$\mu(E) = \int_E |G|\, d\lambda \qquad \text{for every Borel set } E \subset \Omega\,,$$

*with the convention that 0/0 is the $m \times m$ identity matrix $I$. Then $\mu \in \mathcal{M}_0(\Omega)$, $T$ satisfies (3.2.3), and $\Omega_h \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (B, \mu)$.*

*Proof.* Since $C_{\mathcal{A}}(\cdot, \xi, \xi)$ is an increasing set function, $\alpha'(D_\rho(x), \xi, \xi)$ and $\alpha''(D_\rho(x), \xi, \xi)$ are increasing functions of $\rho$, hence (3.4.4) implies that $\alpha'(D_\rho(x), \xi, \xi) = \alpha''(D_\rho(x), \xi, \xi)$ for almost every $\rho \in (0, d_\Omega(x))$. As in the proof of Theorem 3.3.7, we obtain that $\Omega_h \xrightarrow{\gamma_\Omega^{\mathcal{A}}} (\tilde{T}, \tilde{\mu})$, with $\tilde{\mu}$ absolutely continuous with respect to $\lambda$. Since $\frac{d\tilde{\mu}}{d\lambda}(x)$ is bounded, we have $\tilde{\mu}(\Omega) < +\infty$. Let $G(x) = \tilde{T}(x)\frac{d\tilde{\mu}}{d\lambda}(x)$. Since $\mu(E) = \int_E |G|\, d\lambda = \int_E |\tilde{T}|\, d\tilde{\mu}$, and $\tilde{\mu} \in \mathcal{M}_0(\Omega)$, we have $\mu \in \mathcal{M}_0(\Omega)$. The conclusion follows now by repeating the same arguments as in Theorem 3.3.7, the only difference being that now we apply Theorem 3.4.2 instead of Proposition 3.2.9. □

# Chapter 4. Wave equation on varying domains

In this chapter we study the wave equation on varying domains, with Dirichlet boundary conditions. Let $T > 0$, let $\Omega$ be an open subset of $\mathbf{R}^n$ $(n \geq 3)$ and let $(\Omega_h)$ be a sequence of open subsets of $\Omega$. On the cylinders $Q_h = \Omega_h \times (0,T)$ we consider the problem

$$(4.0.1) \qquad \begin{cases} \dfrac{\partial^2 u_h}{\partial t^2} - \Delta u_h = f_h \text{ in } Q_h \\ u_h = 0 \quad \text{on } \partial\Omega_h \times (0,T) \\ u_h(0) = u_h^0 \quad \text{in } \Omega_h \\ \dot{u}_h(0) = u_h^1 \quad \text{in } \Omega_h . \end{cases}$$

Abstract results (see, e.g., [32]) show the existence and uniqueness of a solution $u_h$ of (4.0.1) satisfying $u_h \in C^0([0,T]; H_0^1(\Omega_h)) \cap C^1([0,T]; L^2(\Omega_h))$. We are interested in the asymptotic behaviour of the solutions $u_h$ of (4.0.1) as $h$ tends to infinity.

Under the only hypothesis on the sequence $(\Omega_h)$ that there exists a measure $\mu$ absolutely continuous with respect to the Newtonian capacity such that for any $g \in H^{-1}(\Omega)$ the solutions $v_h$ of

$$(4.0.2) \qquad \begin{cases} v_h \in H_0^1(\Omega_h) \\ -\Delta v_h + v_h = g \quad \text{in } \Omega_h \end{cases}$$

converge to the solution $v$ of the relaxed Dirichlet problem

$$(4.0.3) \qquad \begin{cases} v \in H_0^1(\Omega) \cap L_\mu^2(\Omega) \\ -\Delta v + v + v\mu = g \quad \text{in } \Omega , \end{cases}$$

we prove a convergence result for the solutions of problem (4.0.1). More precisely, denoting by $H$ the closure of $H_0^1(\Omega) \cap L_\mu^2(\Omega)$ in the $L^2$-norm, we prove the following theorem.

**Theorem 4.0.1.** *Under the above assumption on the sequence $(\Omega_h)$, if the data of problem (4.0.1) satisfy*

$$\begin{aligned}
&f_h \rightharpoonup f \quad w\text{-}L^1(0,T;L^2(\Omega))\,,\\
&u_h^0 \rightharpoonup u^0 \quad w\text{-}H_0^1(\Omega)\,,\\
&u_h^1 \rightharpoonup u^1 \quad w\text{-}L^2(\Omega) \quad \text{and } u^1 \in H\,,
\end{aligned}$$

*then the solutions $u_h$ of (4.0.1) converge to a function $u$ in the following sense*

$$u_h \rightharpoonup u \quad w^*\text{-}L^\infty(0,T;H_0^1(\Omega))\,, \tag{4.0.4}$$

$$\dot{u}_h \rightharpoonup \dot{u} \quad w^*\text{-}L^\infty(0,T;L^2(\Omega))\,, \tag{4.0.5}$$

*and the limit function $u$ is the solution, in the sense of Definition 4.1.1, of the relaxed evolution problem*

$$\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} - \Delta u + \mu u = f \text{ in } Q = \Omega \times (0,T)\\
u = 0 \text{ on } \partial\Omega \times (0,T)\\
u(0) = u^0 \quad \text{in } \Omega\\
\dot{u}(0) = u^1 \quad \text{in } \Omega\,,
\end{cases} \tag{4.0.6}$$

*where the same measure $\mu$ that characterizes the limit of elliptic Dirichlet problems appears.* Under stronger assumptions on the convergence of the data of (4.0.1) we obtain also the convergence of the energies.

**Theorem 4.0.2.** *If in addition to the hypothesis on $(\Omega_h)$ considered in Theorem 4.0.1 we assume that*

$$\begin{aligned}
&f_h \to f \quad s\text{-}L^1(0,T;L^2(\Omega))\,,\\
&u_h^0 \rightharpoonup u^0 \quad w\text{-}H_0^1(\Omega) \quad \text{and} \quad \int_{\Omega_h} |Du_h^0|^2\,dx \to \int_\Omega |Du^0|^2 + \int_\Omega |u^0|^2\,d\mu\,,\\
&u_h^1 \to u^1 \quad s\text{-}L^2(\Omega) \quad \text{and } u^1 \in H\,,
\end{aligned}$$

*then the same conclusions of Theorem 4.0.1 hold and, in addition,*

$$\begin{aligned}
&\int_{\Omega_h} |Du_h(\cdot)|^2\,dx \to \int_\Omega |Du(\cdot)|^2\,dx + \int_\Omega |u(\cdot)|^2\,d\mu\,, \quad s\text{-}C^0([0,T])\\
&\dot{u}_h(\cdot) \to \dot{u}(\cdot) \quad s\text{-}C^0([0,T];L^2(\Omega))\,.
\end{aligned}$$

Our results extend those of Cioranescu, Donato, Murat and Zuazua obtained in [10] under the assumption that the limit measure $\mu$ is a nonnegative Radon measure belonging to $H^{-1}(\Omega)$.

Since choosing $\mu = \mu_{\Omega_h}$ we can consider problem (4.0.1) as a particular case of the relaxed evolution problem (4.0.6), in this chapter we study the general case of limits of relaxed problems corresponding to a $\gamma$-convergent sequence of measures, see Definition 1.3.1. Let us remark that the hypothesis we made on the sequence $(\Omega_h)$ corresponds to the $\gamma$-convergence of the measures $\mu_{\Omega_h}$. The convergence results proved in Theorems 4.2.1 and 4.2.5 are the analogues of Theorems 4.0.1 and 4.0.2 above.

## 4.1. Notation and preliminaries

Let $T > 0$, $\Omega$ be an open subset of $\mathbf{R}^n$, $n \geq 3$, $Q = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$. Let $\mu \in \mathcal{M}_0(\Omega)$ and $V$ and $H$ be defined as in Chapter 1. Given $f \in L^1(0,T;L^2(\Omega))$, $u^0 \in V$, $u^1 \in H$, the relaxed formulation of the wave equation we study is

$$(4.1.1)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + \mu u = f & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u^0 & \text{in } \Omega \\ \dot{u}(0) = u^1 & \text{in } \Omega. \end{cases}$$

For a function $v(x,t)$, $\dot{v}$ will denote the partial derivative with respect to $t$ and $Dv$ the gradient in $x$.

**Definition 4.1.1.** We say that a function $u$ is a solution of problem (4.1.1) if

$$\begin{cases} u \in C^0([0,T];V) \cap C^1([0,T];H) \\ u(0) = u^0, \quad \dot{u}(0) = u^1, \\ \dfrac{d^2}{dt^2}\displaystyle\int_\Omega u(t)v\,dx + \int_\Omega Du(t)Dv\,dx + \int_\Omega u(t)v\,d\mu = \int_\Omega f(t)v\,dx \text{ in } \mathcal{D}'(0,T),\ \forall v \in V. \end{cases}$$

The existence and uniqueness of the solution follow from the abstract theory developed in Chapter III §8 of [32] where the result is obtained first for $u \in L^\infty(0,T;V) \cap$

$W^{1,\infty}(0,T;H)$ and then it is proved that, after possibly a modification on a set of measure zero, $u \in C^0([0,T];V) \cap C^1([0,T];H)$. Then Lemma 8.3 [32] gives the following energy equality for the solution:

$$
\begin{aligned}
E(t) &= \int_\Omega |Du(t)|^2\,dx + \int_\Omega |u(t)|^2\,d\mu + \int_\Omega |\dot{u}(t)|^2\,dx = \\
&= \int_\Omega |Du^0|^2\,dx + \int_\Omega |u^0|^2\,d\mu + \int_\Omega |u^1|^2\,dx + \int_0^t \int_\Omega f(s)\dot{u}(s)\,dx\,ds\,.
\end{aligned}
\tag{4.1.2}
$$

Remark that on $(\Omega \backslash \operatorname{supp}\mu) \times (0,T)$ $u$ is a solution of the wave equation $\dfrac{\partial^2 u}{\partial t^2} - \Delta u = f$ in the sense of distributions.

Let $X$, $Y$ be reflexive Banach spaces such that $X \subset Y$ with continuous and dense imbedding. We shall use the following space introduced in [32], Chapter III, § 8.4:

$$
\begin{aligned}
C_s(0,T;Y) = \{f \in L^\infty(0,T;Y) \,:\, t \mapsto \langle v, f(t)\rangle_Y \text{ is continuous} \\
\text{from } [0,T] \text{ into } \mathbf{R} \text{ for every fixed } v \in Y'\}\,.
\end{aligned}
$$

According to Lemma 8.1 in Chapter III of [10] we have

$$
L^\infty(0,T;X) \cap C_s(0,T;Y) = C_s(0,T;X)\,.
$$

For a bounded domain $\Omega$ let $w$ be the solution of the relaxed Dirichlet problem

$$
\begin{cases}
w \in H_0^1(\Omega) \cap L_\mu^2(\Omega) \\
-\Delta w + w\mu = 1 \quad \text{in } \Omega\,.
\end{cases}
\tag{4.1.3}
$$

If we consider a sequence $(\mu_h)$ of measures in $\mathcal{M}_0(\Omega)$ then $w_h$ will denote the solution of

$$
\begin{cases}
w_h \in H_0^1(\Omega) \cap L_{\mu_h}^2(\Omega) \\
-\Delta w_h + w_h\mu_h = 1 \quad \text{in } \Omega\,.
\end{cases}
\tag{4.1.4}
$$

Recall that the set $\{w\varphi | \varphi \in C_0^\infty(\Omega)\}$ is dense in $V$. If $\Omega$ is unbounded we shall use test functions of the form $w\varphi$, where, for some $r > 0$, $w$ is the solution of problem (4.1.3) on $\Omega_r = \Omega \cap B_r$ and $\varphi \in C_0^\infty(\Omega_r)$. As $r > 0$ varies, the set of functions of this kind is dense in $V$.

## 4.2. Asymptotic behaviour of the solutions of the wave equation in varying domains

Cioranescu, Donato, Murat and Zuazua studied in [10] the asymptotic behaviour of the solutions of the wave equation on bounded perforated domains $\Omega_h$ with a critical size of the holes in the case when the measures $\mu_{\Omega_h}$ $\gamma$-converge to a Radon measure. We show in this section that their results hold also in the case of arbitrary measures in $\mathcal{M}_0(\Omega)$, with $\Omega$ a domain not necessarily bounded.

The main result of this section is

**Theorem 4.2.1.** *Let $\mu_h, \mu \in \mathcal{M}_0(\Omega)$, let $u_h^0 \in V_h$, $u_h^1 \in H_h$, $f_h \in L^1(0,T;L^2(\Omega))$. Assume that the following conditions are satisfied*

$$\begin{aligned}
&\mu_h \xrightarrow{\gamma} \mu\,,\\
&f_h \rightharpoonup f \quad w\text{-}L^1(0,T;L^2(\Omega))\,,\\
&u_h^0 \rightharpoonup u^0 \quad w\text{-}H_0^1(\Omega) \quad \text{and} \int_\Omega |u_h^0|^2\,d\mu_h \le c\,,\\
&u_h^1 \rightharpoonup u^1 \quad w\text{-}L^2(\Omega) \quad \text{and}\; u^1 \in H\,,
\end{aligned}$$

*with $c$ being a positive constant independent of $h$. Let $u_h \in C^0([0,T];V_h) \cap C^1([0,T];H_h)$ be the solution of the problem*

$$\text{(4.2.1)} \qquad \begin{cases}
\dfrac{\partial^2 u_h}{\partial t^2} - \Delta u_h + \mu_h u_h = f_h & \text{in } Q\\
u_h = 0 & \text{on } \Sigma\\
u_h(0) = u_h^0 & \text{in } \Omega\\
\dot{u}_h(0) = u_h^1 & \text{in } \Omega\,.
\end{cases}$$

*Then $u^0 \in V$ and*

$$\begin{aligned}
&u_h \rightharpoonup u \quad w^*\text{-}L^\infty(0,T;H_0^1(\Omega))\,,\\
&\dot{u}_h \rightharpoonup \dot{u} \quad w^*\text{-}L^\infty(0,T;L^2(\Omega))\,,\\
&\|u_h\|_{L^\infty(0,T;V_h)} \le C\,,
\end{aligned}$$

*where $u$ is the solution of problem (4.1.1) and $C$ is a constant independent of $h$. Moreover, for every $\theta \in H^{-1}(\Omega)$,*

$$\langle \theta, u_h(\cdot)\rangle \to \langle \theta, u(\cdot)\rangle \quad \text{strongly in } C^0([0,T])\,.$$

In order to prove this theorem we shall need some lemmas. In the case of a bounded domain the proof can be simplified using the compactness of the imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$.

**Lemma 4.2.2.** *Let $\Omega$ be an open set in $\mathbf{R}^n$, $\mu_h$, $\mu \in \mathcal{M}_0(\Omega)$ such that $\mu_h \xrightarrow{\gamma} \mu$, let $v_h \in L^\infty(0,T;V_h) \cap W^{1,\infty}(0,T;H_h)$ such that*

$$v_h \rightharpoonup v \quad w^*\text{-}L^\infty(0,T;H_0^1(\Omega)), \tag{1}$$

$$\dot{v}_h \rightharpoonup \dot{v} \quad w^*\text{-}L^\infty(0,T;L^2(\Omega)), \tag{2}$$

$$\|v_h\|_{L^\infty(0,T;V_h)} \leq C, \tag{3}$$

*where $C$ is a constant independent of $h$. Then $v \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H)$, for every $\theta \in H^{-1}(\Omega)$, $\langle \theta, v_h(\cdot)\rangle \to \langle \theta, v(\cdot)\rangle$ strongly in $C^0([0,T])$, $v(t) \in V$ for every $t \in [0,T]$, and $\|v\|_{L^\infty(0,T;V)} \leq C$.*

*Proof.* Remark that as $v_h \in L^\infty(0,T;V_h) \cap W^{1,\infty}(0,T;H_h)$, after possibly a modification on a set of measure zero, $v_h$ is a continuous mapping of $[0,T] \to H_h$, therefore, according to Lemma 8.1 in Chapter III of [32], $v_h \in C_s(0,T;V_h)$.

Let us prove now that for every $\theta \in L^2(\Omega)$ the function $g_h : [0,T] \to \mathbf{R}$ defined by $g_h(t) = \int_\Omega v_h(x,t)\theta(x)\,dx$ converges in $C^0([0,T])$ to the function $g : [0,T] \to \mathbf{R}$ given by $g(t) = \int_\Omega v(x,t)\theta(x)\,dx$. Using Ascoli-Arzelà Theorem, we show that $(g_h)$ is relatively compact in $C^0([0,T])$. Indeed, it is equibounded since

$$|g_h(t)| = \left|\int_\Omega v_h(x,t)\theta(x)\,dx\right| \leq \|v_h\|_{L^\infty(0,T;L^2(\Omega))}\|\theta\|_{L^2(\Omega)} \leq C$$

independently of $t$ and $h$. To see that it is also equicontinuous it is enough to show that $a_\varepsilon^h = \|v_h(\cdot,t+\varepsilon) - v_h(\cdot,t)\|_{L^\infty(0,T-\varepsilon;L^2(\Omega))}$ converges to zero uniformly in $h$. As

$$\|v_h(\cdot,t+\varepsilon) - v_h(\cdot,t)\|_{L^\infty(0,T-\varepsilon;L^2(\Omega))} \leq \sup_{t\in[0,T-\varepsilon]} \int_t^{t+\varepsilon} \|\dot{v}_h(s)\|_{L^2(\Omega)}ds$$

and due to the weak*-convergence in $L^\infty(0,T;L^2(\Omega))$, $\|\dot{v}_h(s)\|_{L^2(\Omega)} \leq C$, we get that $a_\varepsilon^h$ converges to zero as $\varepsilon \to 0$, uniformly in $h$. So, passing eventually to a subsequence we get that $g_h \to g$ in $C^0([0,T])$. Let us prove now that $\langle \theta, v_h(\cdot)\rangle \to \langle \theta, v(\cdot)\rangle$ strongly in $C^0([0,T])$ for every $\theta \in H^{-1}(\Omega)$. We show that it is a Cauchy sequence in $C^0([0,T])$. For $\hat{\theta} \in L^2(\Omega)$ let $\hat{g}_h(t) = \int_\Omega v_h(x,t)\hat{\theta}(x)\,dx$. Then

$$\begin{aligned} |g_h(t) - g_{h'}(t)| &\le |g_h(t) - \hat{g}_h(t)| + |\hat{g}_h(t) - \hat{g}_{h'}(t)| + |\hat{g}_{h'}(t) - g_{h'}(t)| = \\ &= |\langle \theta - \hat{\theta}, v_h\rangle| + |\langle \hat{\theta}, v_h - v_{h'}\rangle| + |\langle \theta - \hat{\theta}, v_{h'}\rangle| \le \\ &\le \big(\|v_h\|_{L^\infty(0,T;H_0^1(\Omega))} + \|v_{h'}\|_{L^\infty(0,T;H_0^1(\Omega))}\big)\|\theta - \hat{\theta}\|_{H^{-1}(\Omega)} + |\langle v_h - v_{h'}, \hat{\theta}\rangle| \,. \end{aligned}$$

Choosing now $\hat{\theta}$ in $L^2(\Omega)$ to approximate $\theta$ in $H^{-1}(\Omega)$ and using the previous step we have that $(g_h)$ is a Cauchy sequence, hence for every $\theta \in H^{-1}(\Omega)$

$$g_h(\cdot) = \langle \theta, v_h(\cdot)\rangle \to g(\cdot) = \langle \theta, v(\cdot)\rangle \quad \text{strongly in } C^0([0,T])\,.$$

In particular, for each $t \in [0,T]$, $v_h(t) \rightharpoonup v(t)$ weakly in $H_0^1(\Omega)$.

Using now the definition of the $\gamma$-convergence we get that $v(t) \in V$ for each $t \in [0,T]$. Indeed, in the case of bounded $\Omega$ this is obvious since the weak convergence in $H_0^1(\Omega)$ implies the strong convergence in $L^2(\Omega)$ and we can use the definition of the $\gamma$-convergence as it is given in Chapter 1. If $\Omega$ is unbounded we have $F_{\mu_h}(v_h(t)) \le C$, $v_h(t) \rightharpoonup v(t)$ weakly in $H_0^1(\Omega)$ for each fixed $t \in [0,T]$ and then using Lemma 2.2 of [3] we get $F_\mu(v(t)) \le \liminf_{h\to\infty} F_{\mu_h}(v_h(t)) < +\infty$ and hence $v(t) \in V$ for every $t$.

As, by (3), $\operatorname{ess\,sup}_{t\in[0,T]} \|v_h(t)\|_{V_h}^2 \le C$ for a constant $C$ independent of $h$, we also have $\operatorname{ess\,sup}_{t\in[0,T]} \|v(t)\|_V^2 \le C$. So, in order to prove that $v \in L^\infty(0,T;V)$ we have to prove the measurability of $v : [0,T] \to V$. Using Pettis' Theorem [21], since $V$ is separable, it is enough to prove that the map $t \to \langle \psi, v(t)\rangle$ is measurable for every $\psi \in V'$. Hence it suffices to prove the measurability of $t \to (\eta, v(t))_V$ for $\eta$ in a dense set in $V$. If $\Omega$ is bounded we take $\eta \in \{w\varphi \,|\, \varphi \in C_0^\infty(\Omega)\}$. If $\Omega$ is unbounded we also consider $\eta$ of the form $w\varphi$, now with $w$ being the solution of problem (4.1.3) in $\Omega_r$ for some $r > 0$ and $\varphi \in C_0^\infty(\Omega_r)$, both extended by zero on $\Omega\backslash\Omega_r$. Let $w_h$ be the solution of problem (4.1.4) on $\Omega$ or $\Omega_r$. Then $w_h\varphi \rightharpoonup w\varphi$ weakly in $H_0^1(\Omega)$. For each fixed $t \in [0,T]$ we have:

$$\begin{aligned} &\int_\Omega Dv_h(t)D(w_h\varphi)dx + \int_\Omega v_h(t)w_h\varphi d\mu_h = \int_\Omega Dv_h(t)\varphi Dw_h\,dx + \\ &+ \int_\Omega D\varphi v_h(t)Dw_h\,dx + \int_\Omega Dv_h(t)D\varphi w_h\,dx - \int_\Omega D\varphi v_h(t)Dw_h\,dx + \\ &+ \int_\Omega w_h v_h(t)\varphi d\mu_h = \int_\Omega v_h(t)\varphi\,dx + \int_\Omega Dv_h(t)D\varphi w_h\,dx - \int_\Omega D\varphi v_h(t)Dw_h\,dx\,. \end{aligned}$$

Since $v_h(t)$ converges to $v(t)$ weakly in $H_0^1(\Omega)$, the last expression converges to

$$\int_\Omega v(t)\varphi\,dx + \int_\Omega Dv(t)D\varphi w\,dx - \int_\Omega D\varphi v(t)Dw\,dx = \int_\Omega Dv(t)D(w\varphi)\,dx + \int_\Omega v(t)w\varphi d\mu\,.$$

Then since $\int_\Omega v_h(t)w_h\varphi dx \to \int_\Omega v(t)w\varphi dx$ and for every $h \in \mathbf{N}$

$$t \to \int_\Omega Dv_h(t)D(w_h\varphi)\,dx + \int_\Omega v_h(t)w_h\varphi d\mu_h + \int_\Omega v_h(t)w_h\varphi dx$$

is measurable, we obtain that also

$$t \to \int_\Omega Dv(t)D(w\varphi)\,dx + \int_\Omega v(t)w\varphi d\mu + \int_\Omega v(t)w\varphi dx$$

is measurable, and so $v \in L^\infty(0,T;V)$.

We have to show now that $\dot{v} \in L^\infty(0,T;H)$. As from (2) we deduce that $\dot{v} \in L^\infty(0,T;L^2(\Omega))$, it is enough to prove that $\dot{v}(t) \in H$ for a.e. $t \in (0,T)$. From the definition of $\dot{v}$ it follows that

$$\int_0^T \psi(t)\int_\Omega \dot{v}(t)\theta\,dx\,dt = -\int_0^T \dot{\psi}(t)\int_\Omega v(t)\theta\,dx\,dt\,,$$

for every $\psi \in C_0^\infty(0,T)$ and every $\theta \in L^2(\Omega)$. Choosing $\theta = 1_{S_\mu}\varphi$ with $\varphi \in C_0^\infty(\Omega)$, since $v(\cdot,t) = 0$ q.e. on $S_\mu$, we have

$$\int_0^T \psi(t)\int_\Omega \dot{v}(t)1_{S_\mu}\varphi\,dx\,dt = 0 \quad \forall\varphi \in C_0^\infty(\Omega) \quad \forall\psi \in C_0^\infty(0,T)$$

so $\dot{v}(\cdot,t) = 0$ a.e. on $S_\mu$ for a.e. $t \in (0,T)$, hence $\dot{v}(t) \in H$ for a.e. $t \in (0,T)$. □

*Proof of Theorem* 4.2.1. The hypothesis on $u_h^0$ and the definition of $\gamma$-convergence imply $u^0 \in V$. From the energy equality (4.1.2) and the convergence hypotheses on the data we get

$$\begin{aligned}
&(\|Du_h(t)\|^2_{L^2(\Omega)} + \|u_h(t)\|^2_{L^2_{\mu_h}(\Omega)} + \|\dot{u}_h(t)\|^2_{H_h})^{1/2} \le \\
&\le (\|Du_h^0\|^2_{L^2(\Omega)} + \|u_h^0\|^2_{L^2_{\mu_h}(\Omega)} + \|u_h^1\|^2_{H_h})^{1/2} + \frac{1}{\sqrt{2}}\int_0^t \|f_h(s)\|_{H_h}\,ds \le \\
&\le (\|Du_h^0\|^2_{L^2(\Omega)} + \|u_h^0\|^2_{L^2_{\mu_h}(\Omega)} + \|u_h^1\|^2_{H_h})^{1/2} + \frac{1}{\sqrt{2}}\int_0^t \|f_h(s)\|_{L^2(\Omega)}\,ds \le c
\end{aligned}$$

independently of $t$ and $h$, so $(\dot{u}_h)$ is bounded in $L^\infty(0,T;L^2(\Omega))$, and hence $(u_h)$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Since $(Du_h)$ is bounded in $L^\infty(0,T;L^2(\Omega,\mathbf{R}^n))$, it follows that $(u_h)$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$. (If $\Omega$ were bounded we could have used the compactness of the imbedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ to obtain the uniform boundedness of the $L^2$-norm of $u_h$.) Passing eventually to a subsequence we have,

$$\begin{aligned} u_h &\rightharpoonup u \quad w^*\text{-}L^\infty(0,T;H_0^1(\Omega)), \\ \dot{u}_h &\rightharpoonup \dot{u} \quad w^*\text{-}L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

Since the previous inequalities show also that $\|u_h\|_{L^\infty(0,T;L^2_{\mu_h}(\Omega))} \leq c$, we may apply Lemma 4.2.2 to obtain that $u \in L^\infty(0,T;V) \cap W^{1,\infty}(0,T;H)$.

We show now that $u$ is a solution of the relaxed wave equation. If $\Omega$ is bounded, in the equation satisfied by $u_h$ we take as test function $\psi(t)\varphi(x)w_h(x)$ with $\psi \in C_0^\infty(0,T)$, $\varphi \in C_0^\infty(\Omega)$ and $w_h$ the solution of problem (4.1.4). If $\Omega$ is unbounded, we fix $r > 0$ and consider a test function of the same kind as above, but with $w_h$ the solution of problem (4.1.4) on $\Omega_r$ and $\varphi \in C_0^\infty(\Omega_r)$. We obtain

$$\begin{aligned} \int_Q u_h\ddot{\psi}\varphi w_h\,dx\,dt + \int_Q Du_h Dw_h\psi\varphi\,dx\,dt + \int_Q Du_h D\varphi w_h\psi\,dx\,dt+ \\ + \int_Q u_h\psi\varphi w_h\,d\mu\,dt = \int_Q f_h\psi\varphi w_h\,dx\,dt. \end{aligned}$$

Applying Fubini's Theorem we get

$$\begin{aligned} \int_\Omega \varphi w_h\Big(\int_0^T \ddot{\psi}u_h\,dt\Big)dx + \int_\Omega Dw_h D\Big(\varphi\int_0^T u_h\psi\,dt\Big)dx- \\ - \int_\Omega Dw_h D\varphi\Big(\int_0^T u_h\psi\,dt\Big)dx + \int_\Omega w_h D\varphi D\Big(\int_0^T u_h\psi\,dt\Big)dx+ \\ + \int_\Omega w_h\varphi\Big(\int_0^T u_h\psi\,dt\Big)d\mu_h = \int_Q f_h\psi\varphi w_h\,dx\,dt. \end{aligned}$$

Note that, although the measure $\mu_h$ is not assumed to be $\sigma$-finite, Fubini's Theorem can be applied to the measure $w_h\mu_h$, which is always $\sigma$-finite. Now, using the equation satisfed by $w_h$

$$\begin{aligned} \int_\Omega Dw_h D\Big(\varphi\int_0^T u_h\psi\,dt\Big)dx + \int_\Omega w_h\varphi\Big(\int_0^T u_h\psi\,dt\Big)d\mu_h = \\ = \int_\Omega \varphi\Big(\int_0^T u_h\psi\,dt\Big)dx \end{aligned}$$

and the convergences

$$\int_0^T \varphi\ddot{\psi}(t)u_h(t)dt \to \int_0^T \varphi\ddot{\psi}(t)u(t)dt \quad s\text{-}L^2(\Omega)$$
$$\int_0^T \psi(t)u_h(t)dt \rightharpoonup \int_0^T \psi(t)u(t)dt \quad w\text{-}H_0^1(\Omega)$$

we pass to the limit and obtain

$$\int_\Omega \varphi w(\int_0^T \ddot{\psi}u\,dt)dx + \int_\Omega \varphi(\int_0^T u\psi dt)dx -$$
$$-\int_\Omega DwD\varphi(\int_0^T u\psi dt)dx + \int_\Omega wD\varphi D(\int_0^T u\psi dt)dx = \int_Q fw\psi\varphi dxdt\,.$$

Since $w$ is the solution of $-\Delta w + \mu w = 1$ and $\varphi \int_0^T \psi u\,dt$ is an admissible test function, the above equality becomes

$$\int_\Omega \varphi w(\int_0^T \ddot{\psi}u\,dt)dx + \int_\Omega Dw\varphi D(\int_0^T u\psi dt)dx +$$
$$+\int_\Omega wD\varphi D(\int_0^T u\psi dt)dx + \int_\Omega w\varphi(\int_0^T u\psi dt)d\mu = \int_Q f\psi\varphi w dxdt\,.$$

Using now Fubini's Theorem and the fact that the equality holds true for every $\psi \in C_0^\infty(0,T)$ we get

$$\frac{\partial^2}{\partial t^2}\int_\Omega u(t)w\varphi\,dx + \int_\Omega Du(t)D(w\varphi)\,dx + \int_\Omega u(t)w\varphi\,d\mu =$$
$$= \int_\Omega fw\varphi\,dx.$$

Now if $\Omega$ is bounded we use the fact that this holds true for every $\varphi \in C_0^\infty(\Omega)$ and $\{w\varphi\,|\,\varphi \in C_0^\infty(\Omega)\}$ is dense in $V$. If $\Omega$ is unbounded we take into account the fact that also $r > 0$ may vary. Then the set

$$\{w\varphi\,|\,r > 0,\ w \text{ is the solution of problem (4.1.3) on } \Omega_r\,,\ \varphi \in C_0^\infty(\Omega_r)\}$$

is dense in $V$. So, in both cases we obtain that the limit $u$ is a solution of

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u\mu = f.$$

In order to prove that the initial conditions are satisfied we need the following two lemmas (see Propositions 2.8 and 2.9 of [10]). If $X$ and $Y$ are Banach spaces we denote by $B(X,Y)$ the space of bounded linear operators from $X$ to $Y$.

**Lemma 4.2.3.** *Suppose $\Omega$ is bounded. Let $w_h$ and $w$ be the solutions of problem (4.1.4) and (4.1.3), respectively, and consider the operators $P_h : H_h \to L^2(\Omega)$ and $P : H \to L^2(\Omega)$ defined by $P_h\varphi = w_h\varphi$ and $P\varphi = w\varphi$. Then $P_h$ and $P$ can be extended as operators defined on $V_h'$ and $V'$, respectively, and, for every $q \in (1, \frac{n}{n-1})$, $P_h \in B(V_h', W^{-1,q}(\Omega))$ and $\|P_h\|_{B(V_h',W^{-1,q}(\Omega))} \leq c_q$, $P \in B(V', W^{-1,q}(\Omega))$ and $\|P\|_{B(V',W^{-1,q}(\Omega))} \leq c_q$.*

*Proof.* From the definition we have $P_h \in B(H_h, L^2(\Omega))$ and $\|P_h\|_{B(H_h,L^2(\Omega))} \leq c$. Fix $p > n$ and define $R_h\varphi = w_h\varphi$ for every $\varphi \in W_0^{1,p}(\Omega)$. Then

$$\int_\Omega |D(R_h\varphi)|^2\,dx \leq C_p\|\varphi\|_{W_0^{1,p}(\Omega)},$$

with $C_p$ independent of $h$, so $R_h \in B(W_0^{1,p}(\Omega), V_h)$. The previous estimate shows that $R_h \in B(W_0^{1,p}(\Omega), H_0^1(\Omega))$; if $\varphi \in W_0^{1,p}(\Omega)$ with $p > n$ then, by Sobolev Imbedding Theorem, $\varphi \in C^{0,\lambda}(\Omega)$ with $\lambda = 1 - \frac{n}{p}$ and

$$\int_\Omega |w_h\varphi|^2 d\mu_h \leq \sup_{\{w_h>0\}} |\varphi|^2 \int_\Omega |w_h|^2 d\mu_h \leq c\|\varphi\|^2_{W_0^{1,p}(\Omega)}.$$

Let us consider now the adjoint operator $R_h^* : V_h' \to W^{-1,q}(\Omega)$ given by

$$\langle R_h^*\psi, \varphi\rangle = \langle \psi, R_h\varphi\rangle = \langle \psi, w_h\varphi\rangle \quad \forall\psi \in V_h' \quad \forall\varphi \in W_0^{1,p}(\Omega).$$

If $\psi \in L^2(\Omega)$ then $\langle \psi, R_h\varphi\rangle = \int_\Omega \psi w_h\varphi\,dx = \int_\Omega \varphi P_h\psi\,dx$, hence $R_h^*$ is an extension of $P_h$. □

**Lemma 4.2.4.** *Assume $\Omega$ is bounded. If $v_h \in L^\infty(0,T;H_h) \cap W^{1,1}(0,T;V_h')$ and $v_h \rightharpoonup v$ $w^*$-$L^\infty(0,T;L^2(\Omega))$, $P_h\dot{v}_h \rightharpoonup P\dot{v}$ $w$-$L^1(0,T;W^{-1,q}(\Omega))$ for some $q \in (1, \frac{n}{n-1})$ then for every $\varphi \in L^2(\Omega)$*

$$\int_\Omega w_h v_h(\cdot)\varphi\,dx \to \int_\Omega wv(\cdot)\varphi\,dx \quad \textit{strongly in } C^0([0,T]).$$

*Proof.* We have $P_h v_h \rightharpoonup Pv$ $w^*$-$L^\infty(0,T;L^2(\Omega))$. It suffices to apply Corollary 2.6 in [10] to $P_h v_h$ with $X = L^2(\Omega)$ and $Y = W^{-1,q}(\Omega)$. □

*Proof of Theorem 4.2.1 (Continuation).* Let us prove now that the limit function $u$ satisfies also the initial conditions. From Lemma 4.2.2 we deduce that

$$\langle \theta, u_h(\cdot)\rangle \to \langle \theta, u(\cdot)\rangle \quad \text{strongly in } C^0([0,T]) \quad \forall \theta \in H^{-1}(\Omega).$$

Since $u_h(0) = u_h^0 \rightharpoonup u^0$ weakly in $H_0^1(\Omega)$, we get $u(0) = u^0$.

Suppose now that $\Omega$ is bounded. In order to prove that $\dot{u}(0) = u^1$ we want to apply Lemma 4.2.4 to $v_h = \dot{u}_h$. We have thus to show that $P_h\ddot{u}_h \rightharpoonup P\ddot{u}$ weakly in $L^1(0,T;W^{-1,q}(\Omega))$. We have $P_h\ddot{u}_h = P_h\Delta u_h + P_h f_h - P_h(u_h\mu_h)$. As $\|\Delta u_h - u_h\mu_h\|_{L^\infty(0,T;V_h')} \le c$, from Lemma 4.2.3 we get $\|P_h(\Delta u_h - u_h\mu_h)\|_{L^\infty(0,T;W^{-1,q})} \le C_q$ for every $q \in (1, \frac{n}{n-1})$.

Now $P_h f_h = w_h f_h$ is relatively compact in the weak topology of $L^1(0,T;L^2(\Omega))$. This can be proved using Dunford's Theorem (see Theorem 1 p.101 in [21]) since the hypotheses are satisfied; indeed,

1. $L^2(\Omega)$ has the Radon-Nikodym property,
2. $(w_h f_h)$ is bounded in $L^1(0,T;L^2(\Omega))$ (as $(f_h)$ is bounded in $L^1(0,T;L^2(\Omega))$ and $(w_h)$ is bounded in $L^\infty(\Omega)$),
3. $\displaystyle\int_E f_h w_h dt$ is bounded in $L^2(\Omega)$ for every measurable set $E \subset [0,T]$ and
4. $t \to \|w_h f_h\|_{L^2(\Omega)}$ is uniformly integrable on [0,T]. The last assertion follows from the fact that $(f_h)$ is relatively compact in $L^1(0,T;L^2(\Omega))$ that implies, see Theorem 4 p. 104 in [21], that $t \to \|f_h\|_{L^2(\Omega)}$ is uniformly integrable on [0,T].

Now $P_h\dot{u}_h = w_h\dot{u}_h \rightharpoonup w\dot{u}$ weakly in $L^\infty(0,T;L^2(\Omega))$, $\frac{\partial}{\partial t}(P_h\dot{u}_h) = P_h\ddot{u}_h$, so that $P_h\ddot{u}_h$ converges weakly to $P\ddot{u}$ and applying Lemma 4.2.4 we get

$$\text{(4.2.2)} \qquad \int_\Omega \dot{u}_h(\cdot)w_h\varphi\, dx \to \int_\Omega \dot{u}(\cdot)w\varphi\, dx \quad \text{strongly in } C^0([0,T]) \quad \forall\varphi \in L^2(\Omega).$$

In particular $\int_\Omega \dot{u}_h(0)w_h\varphi\, dx \to \int_\Omega \dot{u}(0)w\varphi\, dx$. Then, as $\dot{u}_h(0) = u_h^1 \rightharpoonup u^1 \in H$ weakly in $L^2(\Omega)$ and $\{w\varphi\}$ is dense in $H$, we get $\dot{u}(0) = u^1$.

If $\Omega$ is unbounded we fix some $r > 0$ and apply the above results constructing the operators using as functions $w_h$ and $w$ the solutions of problems (4.1.4) and (4.1.3), respectively, on $\Omega_r$, and taking the function $\varphi \in C_0^\infty(\Omega_r)$. Then since the set of $w\varphi$ constructed in this way is, as also $r > 0$ varies, dense in $H$ we get $\dot{u}(0) = u^1$ in this case, too.

Finally the uniqueness implies that the whole sequence converges and that $u \in C^0([0,T];V) \cap C^1([0,T];H)$. □

Under stronger assumptions on the data the convergence of the energies can be proved.

**Theorem 4.2.5.** *Let $\mu_h, \mu \in \mathcal{M}_0(\Omega)$, let $u_h^0 \in V_h$, $u_h^1 \in H_h$, $f_h \in L^1(0,T;L^2(\Omega))$. Assume that the following conditions are satisfied*

$$\mu_h \xrightarrow{\gamma} \mu ,$$

$$f_h \to f \quad s\text{-}L^1(0,T;L^2(\Omega)) ,$$

$$\begin{cases} u_h^0 \rightharpoonup u^0 \quad w\text{-}H_0^1(\Omega) \quad \text{and} \\ \displaystyle\int_\Omega |Du_h^0|^2\,dx + \int_\Omega |u_h^0|^2\,d\mu_h \to \int_\Omega |Du^0|^2\,dx + \int_\Omega |u^0|^2\,d\mu , \end{cases}$$

$$u_h^1 \to u^1 \quad s\text{-}L^2(\Omega) \quad \text{and } u^1 \in H .$$

*Let $u_h \in C^0([0,T];V_h) \cap C^1([0,T];H_h)$ be the solution of the problem (4.2.1). Then*

$$\int_\Omega |Du_h(\cdot)|^2\,dx + \int_\Omega |u_h(\cdot)|^2\,d\mu_h \to \int_\Omega |Du(\cdot)|^2\,dx + \int_\Omega |u(\cdot)|^2\,d\mu \quad s\text{-}C^0([0,T]) \tag{4.2.3}$$

$$\dot u_h(\cdot) \to \dot u(\cdot) \quad s\text{-}C^0([0,T];L^2(\Omega)) , \tag{4.2.4}$$

*where $u$ is the solution of problem (4.1.1).*

*Proof.* Fix $t \in [0,T]$. From Theorem 4.2.1 we get that $u_h(t)$ converges to $u(t)$ weakly in $H_0^1(\Omega)$ and also that $\|\dot u_h(t)\|_{L^2(\Omega)} \le C$. Hence there exist a subsequence $\dot u_{h_k}(t)$ and a function $v \in L^2(\Omega)$ such that $\dot u_{h_k}(t) \rightharpoonup v$ weakly in $L^2(\Omega)$. It follows then that

$$\int_\Omega \dot u_{h_k}(t) w_{h_k} \varphi\,dx \to \int_\Omega v w \varphi\,dx \qquad \text{for every } \varphi \in L^2(\Omega) ,$$

where, as in Theorem 4.2.1, $w_h$ and $w$ are the solutions of the corresponding relaxed elliptic Dirichlet problems (4.1.4) and (4.1.3) on $\Omega$, if $\Omega$ is bounded, or on some $\Omega_r$, if $\Omega$ is unbounded. In the proof of Theorem 4.2.1 we have seen that

$$\int_\Omega \dot u_{h_k}(t) w_{h_k} \varphi\,dx \to \int_\Omega \dot u(t) w \varphi\,dx \qquad \text{for every } \varphi \in L^2(\Omega) ;$$

this yields

$$\dot u(t) = P_H v , \tag{4.2.5}$$

where $P_H$ is the orthogonal projection from $L^2(\Omega)$ on $H$.

Since $u_h(t)$ converge to $u(t)$ weakly in $H_0^1(\Omega)$, by the definition of $\gamma$-convergence we have

$$\int_\Omega |Du(t)|^2\,dx + \int_\Omega |u(t)|^2\,d\mu \le \liminf_{h\to\infty}\Big(\int_\Omega |Du_h(t)|^2\,dx + \int_\Omega |u_h(t)|^2\,d\mu_h\Big)\,. \tag{4.2.6}$$

The weak convergence in $L^2(\Omega)$ of $\dot u_{h_k}(t)$ to $v$ implies

$$\int_\Omega |v|^2\,dx \le \liminf_{k\to\infty}\Big(\int_\Omega |\dot u_{h_k}(t)|^2\,dx\Big)\,. \tag{4.2.7}$$

From the hypothesis on $f_h$ and the convergence result for the solutions obtained in Theorem 4.2.1 it follows that

$$\int_0^T\int_\Omega f_h(s)\dot u_h(s)\,dx\,ds \to \int_0^T\int_\Omega f(s)\dot u(s)\,dx\,ds\,. \tag{4.2.8}$$

By the conservation of energy (4.1.2) we have

$$\int_\Omega |Du_{h_k}(t)|^2\,dx + \int_\Omega |u_{h_k}(t)|^2\,d\mu_{h_k} + \int_\Omega |\dot u_{h_k}(t)|^2\,dx =$$
$$= \int_\Omega |Du^0_{h_k}|^2\,dx + \int_\Omega |u^0_{h_k}|^2\,d\mu_{h_k} + \int_\Omega |u^1_{h_k}|^2\,dx + \int_0^T\int_\Omega f_{h_k}(s)\dot u_{h_k}(s)\,dx\,ds\,,$$

which by (4.2.8) and the hypotheses on the data, converges to

$$\int_\Omega |Du^0|^2\,dx + \int_\Omega |u^0|^2\,d\mu + \int_\Omega |u^1|^2\,dx + \int_0^T\int_\Omega f(s)\dot u(s)\,dx\,ds\,.$$

By applying again the conservation of energy (4.1.2), this expression is equal to

$$\int_\Omega |Du(t)|^2\,dx + \int_\Omega |u(t)|^2\,d\mu + \int_\Omega |\dot u(t)|^2\,dx,$$

which is less than or equal to

$$\int_\Omega |Du(t)|^2\,dx + \int_\Omega |u(t)|^2\,d\mu + \int_\Omega |v|^2\,dx\,.$$

From (4.2.6) and (4.2.7) we deduce that

$$\int_\Omega |v|^2\,dx = \int_\Omega |\dot u(t)|^2\,dx\,.$$

Then the above equality of the $L^2$-norms, together with (4.2.5), implies that $\dot{u}_h(t)$ converges to $\dot{u}(t)$ weakly in $L^2(\Omega)$, and moreover, that for every $t \in [0,T]$

$$\int_\Omega |Du_h(t)|^2\,dx + \int_\Omega |u_h(t)|^2\,d\mu_h \to \int_\Omega |Du(t)|^2\,dx + \int_\Omega |u(t)|^2\,d\mu\,,$$

and $\int_\Omega |\dot{u}_h(t)|^2\,dx \to \int_\Omega |\dot{u}(t)|^2\,dx$. Then it follows that for every $t \in [0,T]$, $\dot{u}_h(t) \to \dot{u}(t)$ strongly in $L^2(\Omega)$.

The convergence hypotheses we made on $f_h$, $u_h^0$ and $u_h^1$ and the fact that $\|\dot{u}_h\|_{L^\infty(0,T;L^2(\Omega))} \le C$, together with the energy equality (4.1.2) prove that $E_h(\cdot)$ is equicontinuous. The convergence of the energies

$$E_h(\cdot) \to E(\cdot) \quad s\text{-}C^0([0,T]) \tag{4.2.9}$$

follows then from the Ascoli-Arzela Theorem.

To prove (4.2.3) and (4.2.4) we need the following lemma (see also Proposition 4.3 in [10]).

**Lemma 4.2.6.** *Assume $\Omega$ is bounded and let*

$$e_h(v)(t) = \frac{1}{2}\|\dot{v}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|Dv(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v(t)\|^2_{L^2_{\mu_h}(\Omega)}$$

*for $v \in C^0([0,T];V_h) \cap C^1([0,T];H_h)$ and*

$$e(v)(t) = \frac{1}{2}\|\dot{v}(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|Dv(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v(t)\|^2_{L^2_{\mu}(\Omega)}$$

*for $v \in C^0([0,T];V) \cap C^1([0,T];H)$ and $w_h$ and $w$ be defined in (4.1.4) and (4.1.3). Then under the assumptions of Theorem 4.2.5 we have*

$$e_h(u_h - w_h\varphi)(\cdot) \to e(u - w\varphi)(\cdot) \quad s\text{-}C^0([0,T]),$$

*for every $\varphi \in C^\infty(Q)$ such that $\varphi(\cdot,t)$ has compact support in $\Omega$ for every $t \in (0,T)$.*

*Proof.* We have

$$\begin{aligned} e_h(u_h - w_h\varphi)(t) =& e_h(u_h)(t) + e_h(w_h\varphi)(t) - \int_\Omega |\dot{u}_h(t)w_h\dot{\varphi}(t)|^2\,dx - \\ & - \int_\Omega |Du_h(t)D(w_h\varphi(t))|^2\,dx - \int_\Omega |u_h(t)w_h\varphi(t)|^2\,d\mu_h. \end{aligned} \tag{4.2.10}$$

From (4.2.9) we obtain that $e_h(u_h)(\cdot) \to e(u)(\cdot)$ strongly in $C^0([0,T])$.

Consider now the second term in (4.2.10). We have

$$\|w_h\dot{\varphi}(\cdot)\|_{L^2(\Omega)} \to \|w\dot{\varphi}(\cdot)\|_{L^2(\Omega)} \quad s\text{-}C^0([0,T]).$$

Then

$$\int_\Omega |D(w_h\varphi(t))|^2\,dx + \int_\Omega |w_h\varphi(t)|^2\,d\mu_h = \int_\Omega Dw_h D(w_h\varphi(t)^2)\,dx + \int_\Omega |w_h|^2|D\varphi(t)|^2\,dx +$$
$$+\int_\Omega |w_h\varphi(t)|^2\,d\mu_h = \langle 1, w_h\varphi^2\rangle + \int_\Omega |w_h|^2|D\varphi(t)|^2\,dx$$

which converges in $C^0([0,T])$ to $\langle 1, w\varphi^2\rangle + \int_\Omega |w|^2|D\varphi|^2\,dx$. Hence $e_h(w_h\varphi)(\cdot) \to e(w\varphi)(\cdot)$ strongly in $C^0([0,T])$.

To show the convergence of the third term we approximate $\dot{\varphi}(x,t)$ in $C^0([0,T];L^2(\Omega))$ by functions of the form $\sum_{i=0}^k \eta_i(t)\psi_i(x)$, with $\eta_i$ continuous functions and $\psi_i \in L^2(\Omega)$. Applying now (4.2.2) to the functions $\psi_i$ we obtain

$$\int_\Omega \dot{u}_h(x,\cdot)w_h(x)\dot{\varphi}(x,\cdot)\,dx \to \int_\Omega \dot{u}(x,\cdot)w(x)\dot{\varphi}(x,\cdot)\,dx \quad s\text{-}C^0([0,T]).$$

Consider now the last two terms in (4.2.10). We have

$$\int_\Omega Du_h D(w_h\varphi)\,dx + \int_\Omega u_h w_h\varphi\,d\mu_h = \int_\Omega Dw_h D(u_h\varphi)\,dx + \int_\Omega Du_h w_h D\varphi\,dx -$$
$$-\int_\Omega Dw_h D\varphi u_h\,dx + \int_\Omega u_h w_h \varphi\,d\mu_h = \langle 1, u_h\varphi\rangle + \int_\Omega D(u_h w_h)D\varphi\,dx - 2\int_\Omega Dw_h D\varphi u_h\,dx,$$

and $\int_\Omega D(u_h w_h)D\varphi\,dx = \langle -\Delta\varphi, u_h w_h\rangle$. As $\Omega$ is bounded, from Theorem 4.2.1 we deduce that $u_h \to u$ strongly in $C^0([0,T],L^2(\Omega))$ and this implies that the above expression converges strongly in $C^0([0,T])$ to

$$\langle 1, u\varphi\rangle + \int_\Omega D(uw)D\varphi\,dx - 2\int_\Omega DwD\varphi u\,dx,$$

which completes the proof. □

*Proof of Theorem 4.2.5 (Continuation).* To use Lemma 4.2.6 we approximate $u$ by functions of the form $w\varphi$, with $\varphi$ smooth. First of all note that it is not restrictive to assume that $\Omega$ is bounded. Indeed, $u$ is the strong limit in $C^0([0,T];V)\cap C^1([0,T];H)$ of the sequence $\psi_k u$, where $\psi_k \in C_0^\infty(\mathbf{R}^n)$ are functions of the form $\psi_k(x)=\psi(\frac{|x|}{k})$ for a suitable $\psi\in C^\infty(\mathbf{R})$ such that $0\le\psi\le 1$, $\psi(s)=1$ for $0\le s\le 1$ and $\psi(s)=0$ for $s\ge 2$. Since $\psi_k u$ have compact support we may assume that $\Omega$ is bounded.

Let now $w$ be the solution of (4.1.3). We may assume that there exist positive $k$ and $\eta$ such that $0\le u(x,t)\le kw(x)$ for every $t\in[0,T]$ and q.e. $x\in\Omega$ and also that $\{u(\cdot,t)>0\}\subset\{w(\cdot)>\eta\}$. (Otherwise, let $v_\eta=\frac{(w\wedge 2\eta-\eta)^+}{\eta}$, for $\eta>0$ and $g_m$ be a function of class $C^\infty$ such that $g_m(s)=s$ for $s\le m-1$, $g_m=m$ for $s\ge m$ and $\dot g_m\le 2$. Then as $k\to\infty$ and $\eta\to 0$, with $2k\eta\to\infty$, $g_{2k\eta}(u)v_\eta\to u$ strongly in $C^0([0,T];V)\cap C^1([0,T];H)$ and $g_{2k\eta}(u)v_\eta\le kw$ and $\{g_{2k\eta}(u)v_\eta>0\}\subset\{w>\eta\}$.)

Then $u/w$ is bounded and belongs to $C^0([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))$. Hence there exists a sequence $\varphi_l\in C^\infty(Q)$ with $\varphi_l(\cdot,t)$ having compact support in $\Omega$ for every $t\in(0,T)$, and such that $w\varphi_l\to u$ strongly in $C^0([0,T];V)\cap C^1([0,T];H)$ as $l\to\infty$.

Now, let $w_h$ be the solution of problem (4.1.4). Then

$$\|\dot u_h-\dot u\|_{L^\infty(0,T,L^2(\Omega))}\le$$
$$\le\|\dot u_h-w_h\dot\varphi_l\|_{L^\infty(0,T,L^2(\Omega))}+\|(w_h-w)\dot\varphi_l\|_{L^\infty(0,T,L^2(\Omega))}+\|\dot u-w\dot\varphi_l\|_{L^\infty(0,T,L^2(\Omega))}.$$

From Lemma 4.2.6 we deduce that

$$\limsup_{h\to\infty}\|e_h(u_h-w_h\varphi_l)\|_{L^\infty(0,T)}\le 2\|e(u-w\varphi_l)\|_{L^\infty(0,T)}.$$

Since

$$\lim_{h\to\infty}\|(w_h-w)\dot\varphi_l\|_{L^\infty(0,T,L^2(\Omega))}=0\,,$$

we get that

$$\lim_{h\to\infty}\|\dot u_h-\dot u\|_{L^\infty(0,T,L^2(\Omega))}\le 2\|e(u-w\varphi_l)\|_{L^\infty(0,T)}+\|\dot u-w\dot\varphi_l\|_{L^\infty(0,T,L^2(\Omega))}$$

and passing now to the limit as $l\to\infty$ we obtain that $\dot u_h(\cdot)\to\dot u(\cdot)$ strongly in $C^0([0,T];L^2(\Omega))$. This convergence together with (4.2.9) implies that (4.2.3) also holds.

□

Let us consider now the case of a sequence of domains $\Omega_h \subset \Omega$ converging in the sense mentioned in the introduction. This corresponds to the existence of a measure $\mu \in \mathcal{M}_0(\Omega)$ such that the measures $\mu_{\Omega_h}$ defined by (1.2.1) $\gamma$-converge to the measure $\mu$. Then the results announced in the introduction hold.

*Proof of Theorems 4.0.1 and 4.0.2.* It suffices to apply Theorem 4.2.1 and 4.2.5, respectively, to the measures $\mu_h = \mu_{\Omega_h}$. □

**Remark 4.2.7.** If instead of the Laplacian we consider a symmetric linear elliptic operator, the results proved in this chapter also hold, with some obvious modifications.

# Chapter 5. Acoustic scattering in varying domains

In this chapter we study the behaviour, as the time goes to infinity, of the solutions of the wave equation on varying domains. Let $(K_h)$ be a uniformly bounded sequence of compact subsets of $\mathbf{R}^n$ such that for every $h \in \mathbf{N}$, $\Omega_h = \mathbf{R}^n \setminus K_h$ is connected. The behaviour of the solution $u_h$ of

$$(5.0.1) \qquad \begin{cases} \dfrac{d^2 u_h}{dt^2} - \Delta u_h = 0 \quad \text{in } \Omega_h \times \mathbf{R} \\ u_h = 0 \quad \text{on } \partial\Omega_h \times \mathbf{R} \\ u_h(0) = u_h^0 , \quad \dot{u}_h(0) = u_h^1 \end{cases}$$

as the time $t$ goes to infinity can be decribed using the wave operators $W_h$ which associate to the initial data $(u_h^0, u_h^1)$ of (5.0.1) an initial condition for the wave equation on the whole space $\mathbf{R}^n$ such that $u_h$ is asymptotically equal, as the time goes to infinity, to the solution of the free space equation, see Definition 5.2.1. We are interested in the behaviour of $W_h$ as $h$ goes to infinity.

On the sequence $(\Omega_h)$ we make the assumption that there exists a measure $\mu$, absolutely continuous with respect to the Newtonian capacity, such that for any $g \in H^{-1}(\mathbf{R}^n)$ the solutions $w_h$ of the elliptic Dirichlet problem

$$(5.0.2) \qquad \begin{cases} w_h \in H_0^1(\Omega_h) \\ -\Delta w_h + w_h = g \quad \text{in } \Omega_h \end{cases}$$

weakly converge in $H^1(\mathbf{R}^n)$ to the solution $w$ of the relaxed Dirichlet problem

$$(5.0.3) \qquad \begin{cases} w \in H^1(\mathbf{R}^n) \cap L^2_\mu(\mathbf{R}^n) \\ -\Delta w + w + \mu w = g \quad \text{in } \mathbf{R}^n. \end{cases}$$

As seen in the previous chapter, a relaxation phenomenon occurs also for (5.0.1), and the relaxed wave equation obtained in the limit contains the same measure $\mu$ which appears in (5.0.3).

Following the lines of [29] we prove in Theorem 5.2.6 that the wave operator $W$ for the relaxed wave equation

$$(5.0.4) \qquad \begin{cases} \dfrac{d^2 u}{dt^2} - \Delta u + \mu u = 0 \quad \text{in } \mathbf{R}^n \times \mathbf{R} \\ u(0) = u^0 \quad \dot{u}(0) = u^1 \end{cases}$$

exists and is unitary. Then we prove that the wave operators $W_h$ corresponding to (5.0.1) converge to $W$ in the following sense: if $\eta \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$ satisfies $\operatorname{supp} \eta \cap K_h = \emptyset$, then $W_h \eta$ converges to $W \eta$ in the energy norm (see Theorem 5.3.2).

Our result generalizes for this relaxed formulation the one obtained by Rauch and Taylor (see [37]), which was confined to the case when the asymptotic elliptic Dirichlet problem (5.0.3) is of the form (5.0.2). To our knowledge the study of the asymptotic behaviour of the wave operators corresponding to relaxed problems is new. In order to prove this convergence we show by a uniform energy estimate (see Theorem 5.3.1) that it is enough to know the behaviour of the solutions for finite time intervals and then use the results of the previous chapter.

Note that since $K_h$ are uniformly bounded, also the measure $\mu$ appearing in the limit has compact support. We assume also that the sets $\Omega_h$ are connected. This implies that the Laplacian with Dirichlet boundary conditions on $\Omega_h$ has an absolutely continuous spectrum (see, e.g., Theorem XI.91.5 in [38]). Our study of the wave operator $W$ is made under the hypothesis that the limit operator $A = -\Delta + \mu$ has also an absolutely continuous spectrum. This holds for instance if $\mu \in L^p(\mathbf{R}^n)$ with $p$ large enough and $\mu$ has compact support. A slightly more general example of measure $\mu$ for which $A$ has an absolutely continuous spectrum is discussed in Remark 5.1.1.

Since choosing $\mu = \mu_{\Omega_h}$ we can consider problem (5.0.1) as a particular case of the relaxed evolution problem (5.0.4), we study the convergence of the wave operators for the relaxed problems corresponding to a $\gamma$-convergent sequence of measures, see Definition 1.3.1. Let us recall that the hypothesis we made on the sequence $(\Omega_h)$ corresponds to the $\gamma$-convergence of the measures $\mu_{\Omega_h}$.

## 5.1. Notation and preliminaries

Throughout this chapter we assume that there exists $r > 0$ fixed such that the measures in $\mathcal{M}_0(\mathbf{R}^n)$ we shall consider have support included in $B_r$. Let $\mu \in \mathcal{M}_0(\mathbf{R}^n)$ be such a measure, let $V$ and $H$ be defined as in Chapter 1 and let $\mathcal{A} : V \to V'$ be the operator defined by $\mathcal{A} = -\Delta + \mu$, in the sense that

$$(5.1.1) \qquad \langle \mathcal{A}u, v \rangle_V = \int_{\mathbf{R}^n} DuDv \, dx + \int_{\mathbf{R}^n} uv \, d\mu \quad \text{for } u, v \in V.$$

Given a self-adjoint operator $T : H \to H$ we denote by $D(T)$, $R(T)$ and $N(T)$ its domain, image and null-space, respectively.

Let $A : D(A) \to H$ defined by $D(A) = \{u \in V \mid \mathcal{A}u \in H\}$ and $Au = \mathcal{A}u$ for every $u \in D(A)$ be its realization in $H$. Then, as $A$ is the operator associated to the lower semicontinuous quadratic form on $H$ given, for $u \in V$ by $\int_{\mathbf{R}^n} |Du|^2\, dx + \int_{\mathbf{R}^n} |u|^2\, d\mu$, we get that $A$ is self-adjoint and nonnegative (see, for instance, [11] Proposition 12.16 and Theorem 12.13). This implies that the spectrum $\sigma(A)$ of $A$ is included in $\overline{\mathbf{R}_+}$ and also that $A^{1/2}$ exists. The second representation theorem (Theorem VI.2.23 [28], p. 331) implies that $D(A^{1/2}) = V$.

Being self-adjoint, to $A : D(A) \to H$ we can associate a unique resolution of the identity $E$ such that

$$\langle Au, v\rangle_H = \int_{-\infty}^{+\infty} \lambda\, E_{u,v}(d\lambda) \quad \forall u \in D(A)\,,\ \forall v \in H\,.$$

In particular for each Borel subset $\omega$ of $\mathbf{R}$ $E(\omega) \in B(H,H)$ is a self-adjoint projection and the set-function $E_{u,v}(\cdot)$ defined for $u, v \in H$ by $E_{u,v}(\cdot) = \langle E(\cdot)u, v\rangle_H$ is a complex measure on $\mathcal{B}(\mathbf{R})$. Moreover $E$ is concentrated on the spectrum of $A$ in the sense that $E(\sigma(A))$ is equal to the identity operator $1_H : H \to H$. $E$ is also called the spectral decomposition of $A$ or the spectral measure of $A$. For the definition of a resolution of identity and other properties of it see, for instance, [39] and [28].

On $H_0 = L^2(\mathbf{R}^n)$, we consider the self-adjoint operator $A_0 : D(A_0) = H^2(\mathbf{R}^n) \to H_0$ given by $A_0 u = -\Delta u$. Then $A_0$ has an absolutely continuous spectrum, i.e., if we denote by $E_0(\cdot)$ the corresponding resolution of the identity, for every $u \in H_0$ the measure $\langle E_0(\cdot)u, u\rangle_{H_0}$ is absolutely continuous with respect to the Lebesgue measure (see, for instance, [28] Chapter X §1.1 and 1.2]).

We assume that also $A$ has an absolutely continuous spectrum.

**Remark 5.1.1.** Let $\Omega$ be a connected open subset of $\mathbf{R}^n$ such that $\mathbf{R}^n \backslash \Omega$ is compact, let $g \in L^p(\Omega)$ for $p > n/2$ be a function with compact support, and let $\mu = \mu_\Omega + g\, m$, $m$ being the Lebesgue measure on $\Omega$. Then the operator $A = -\Delta + \mu$ has an absolutely continuous spectrum. Indeed, in this case Majda showed in [34] that the spectrum $\sigma(A)$ of $A$ is absolutely continuous except for at most a countable set $\{k_j\}$ of isolated eigenvalues of finite multiplicities. He also pointed out that the essential property in proving the absolute continuity of the spectrum is the unique continuation property. It was shown then by Jerison and Kenig in [27] that the unique continuation property holds for $g \in L^p(\Omega)$, with $p > n/2$, hence in this case $A$ has no eigenvalues.

In most of the examples considered in [35], [9] and [41] the limit measure $\mu$ is of the type mentioned above and so, the limit operator has an absolutely continuous spectrum.

Let us introduce the unitary group $U(t)$ giving the generalized solution of the relaxed wave equation

$$\begin{cases} \dfrac{d^2u}{dt^2} - \Delta u + \mu u = 0 \quad \text{in } \mathbf{R}^n \times \mathbf{R} \\ u(0) = u^0 , \quad \dot{u}(0) = u^1 . \end{cases} \tag{5.1.2}$$

Note that the wave equation on an exterior domain $\Omega \subset \mathbf{R}^n$

$$\begin{cases} \dfrac{d^2u}{dt^2} - \Delta u = 0 \quad \text{in } \Omega \times \mathbf{R} \\ u = 0 \quad \text{on } \partial\Omega \times \mathbf{R} \\ u(0) = u^0 , \quad \dot{u}(0) = u^1 . \end{cases}$$

is a particular case of (5.1.2) corresponding to $\mu = \mu_\Omega$.

Let $B = A^{1/2}$. Then $B : D(B) = V \to H$ is a self-adjoint and nonnegative operator and

$$\|Bu\|_H^2 = \int_{\mathbf{R}^n} |Du|^2 \, dx + \int_{\mathbf{R}^n} |u|^2 \, d\mu . \tag{5.1.3}$$

It follows then that $B$ is injective since $Bu = 0$ implies $Du = 0$ and as $u \in V = H^1(\mathbf{R}^n) \cap L^2_\mu(\mathbf{R}^n)$, it follows that $u = 0$, and $\|u\|_B = \|Bu\|_H$ is a norm on $V$. We shall denote by $[D(B)] = \tilde{V}$ the completion of the domain of $B$ with respect to this norm. Remark that using Sobolev Imbedding Theorem it can be proved that $\tilde{V} = \{u \in L^{2^*}(\mathbf{R}^n) \mid Du \in L^2(\mathbf{R}^n, \mathbf{R}^n), u \in L^2_\mu(\mathbf{R}^n)\}$. We shall use in the sequel also the unitary extension $\tilde{B}$ of $B$ to $\tilde{V}$. As $R(B)^\perp = N(B)$, $\tilde{B} : \tilde{V} \to H$ is a bijection.

On $D(B) \times H$ we define $U(t)(u^0, u^1) = (u(t), v(t))$, for $-\infty < t < +\infty$, by

$$u(t) = \cos(tB)u^0 + B^{-1}\sin(tB)u^1$$
$$v(t) = -\sin(tB)Bu^0 + \cos(tB)u^1 ,$$

where $B^{-1}\sin(tB)$ is understood as the function of the operator $B$ corresponding to $f(z) = \frac{\sin(tz)}{z}$. Note that general facts of operational calculus (see, for instance, Lemma 13.23, Theorem 13.24 in [39]) imply that $u(t) \in D(B) = V$ for every $t$ and

$$\|Bu(t)\|_H^2 + \|v(t)\|_H^2 = \|Bu^0\|_H^2 + \|u^1\|_H^2 .$$

Then $U(t)$ can be extended uniquely to a unitary operator on $\mathcal{H} = \tilde{V} \times H$ with the norm given by

$$\|(u,v)\|_{\mathcal{H}}^2 = \|\tilde{B}u\|_H^2 + \|v\|_H^2 = \int_{\mathbf{R}^n} |Du|^2\,dx + \int_{\mathbf{R}^n} |u|^2\,d\mu + \int_{\mathbf{R}^n} |v|^2\,dx\,.$$

Moreover, $v(t)$ coincides with the derivative $\dot{u}(t)$ of $u(t)$ and $U(t)(u^0,u^1) = (u(t),\dot{u}(t))$ represents the unitary group giving the generalized solution of (5.1.2) at time $t$, with initial data $(u^0,u^1)$, in the sense considered, for instance, in [36] Chapter 4, p. 105.

Using Lemma 8.1 of [29] from the assumption that $A$ has an absolutely continuous spectrum we deduce that the self-adjoint operator generating $U$ has an absolutely continuous spectrum.

Note that the norm on $\mathcal{H}$ is the energy norm associated to the wave equation. In the same way $\mathcal{H}_0$ is defined by $[D(B_0)] \times H_0$, where $B_0 = A_0^{1/2}$ and $[D(B_0)]$ is the completion of its domain with respect to the norm $\|u\|_{B_0}^2 = \int_{\mathbf{R}^n} |Du|^2\,dx$. We use the following notation for the energy norms: for every Borel set $\omega \subset \mathbf{R}^n$

$$\|(u,v)\|_{E_0(\omega)}^2 = \|Du\|_{L^2(\omega)}^2 + \|v\|_{L^2(\omega)}^2\,; \qquad \|(u,v)\|_{E(\omega)}^2 = \|Du\|_{L^2(\omega)}^2 + \|u\|_{L^2_\mu(\omega)}^2 + \|v\|_{L^2(\omega)}^2\,.$$

Then $\|(u,v)\|_{E_0(\mathbf{R}^n)} = \|(u,v)\|_{\mathcal{H}_0}$ and $\|(u,v)\|_{E(\mathbf{R}^n)} = \|(u,v)\|_{\mathcal{H}}$.

We recall here some basic properties of the solutions of the wave equation that we shall use in the sequel.

**Remark 5.1.2.** Assume that the initial data $u^0$ and $u^1$ for the wave equation on $\mathbf{R}^n$ are smooth functions of compact support. Then the norm of the solution on compact sets tends to zero as the time tends to infinity. More precisely, let $r > 0$; we prove that for every $\varepsilon > 0$ there exists $T = T(\varepsilon, u^0, u^1) > 0$ such that

$$\begin{aligned} \|u(t)\|_{L^2(B_{r+1})} &\le \varepsilon \\ \|(u(t),\dot{u}(t))\|_{E_0(B_{r+1+t-T})} &\le \varepsilon \quad \text{for } t \ge T, \end{aligned}$$

where $u(t)$ denotes the solution at time $t$.

The solution of the wave equation on $\mathbf{R}^n$

$$\frac{d^2u}{dt^2} - \Delta u = 0, \quad u(0) = u^0, \quad \dot{u}(0) = u^1$$

is given by:

$$u(x,t) = c(n)\{\frac{\partial}{\partial t}\left(\frac{\partial}{t\partial t}\right)^{\frac{n-2}{2}} \int_0^t \left(\int_{|x-\xi|=\rho} u^0(\xi) d\sigma\right) \frac{1}{\sqrt{t^2-\rho^2}} d\rho +$$

$$+\left(\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \int_0^t \left(\int_{|x-\xi|=\rho} u^1(\xi) d\sigma\right) \frac{1}{\sqrt{t^2-\rho^2}} d\rho\}$$

(see [24]).

Choose $\delta > 0$ such that $\operatorname{supp} u^0$, $\operatorname{supp} u^1 \subset B_{\delta/2}$. Fix $c > 0$. If $T > c + r + 1 + \delta$ then for every $x \in B_{r+1+t-T}$ we have $B_{t-c}(x) \supset B_\delta \supset \operatorname{supp} u^0$, $\operatorname{supp} u^1$. So, for $\rho > t-c$ the integrals on the sphere $|x-\xi| = \rho$ are zero, hence in the formulas for $u$, $Du$, $u_t$ the integrals in $\rho$ are to be computed between zero and $t-c$. This implies that only $\frac{1}{\sqrt{t^2-\rho^2}}$ has to be derived with respect to $t$ and so, by standard estimates we get the decay of the norms as $t$ goes to infinity.

**Remark 5.1.3.** It can also be proved that the $L^2(\mathbf{R}^n)$-norm of the solution of the wave equation on $\mathbf{R}^n$ is bounded independently of $t$.

Indeed, if we compute the Fourier transform of the wave equation in the $x$ variable we get (see, e.g. [45])

$$\hat{u}(t,\xi) = \hat{u}^0(\xi)\cos(|\xi|t) + \hat{u}^1(\xi)\frac{\sin(|\xi|t)}{|\xi|}.$$

As $u^0$, $u^1$ belong to $C_0^\infty(\mathbf{R}^n)$ Paley-Wiener Theorem implies that $\hat{u}^0$, $\hat{u}^1$ are entire analytic functions on $\mathbf{C}^n$. Since $\hat{u}^1 \in \mathcal{S}$, $\hat{u}^1(\xi) \le (1+|\xi|)^{-k}$ for any natural $k$. As $n \ge 3$, $1/|\xi|^2$ is integrable near the origin and at infinity we use $\hat{u}^1 \in \mathcal{S}$, we have

$$\int_{\mathbf{R}^n} \frac{|\hat{u}^1(\xi)|^2}{|\xi|^2} d\xi < \infty,$$

and so,

$$\|\frac{\hat{u}^1(\xi)}{|\xi|}\sin(|\xi|t)\|_{L^2(\mathbf{R}^n)} \le c,$$

with $c$ dependent on $u^1$ but independent of $t$. We have also

$$\|\hat{u}^0(\xi)\cos(|\xi|t)\|_{L^2(\mathbf{R}^n)} \le c,$$

so, $\|u(t)\|_{L^2(\mathbf{R}^n)} \le c$, with $c$ dependent on the initial data but independent of $t$.

**Remark 5.1.4.** We shall need in the sequel also the following estimates for the resolvents of $A$ and $A_0$. Recall that $R_\lambda(A) = (A+\lambda)^{-1}$.

a) $\|R_\lambda(A)Au\|_{L^2(\mathbf{R}^n)} \leq 2\|u\|_{L^2(\mathbf{R}^n)}$, for $u \in D(A)$ (and obviously the same estimate holds for $A_0$);

b) let $\omega$ and $f$ be an open subset of $\mathbf{R}^n$ and, respectively, an element of $H^{-1}(\mathbf{R}^n)$ such that for some $r > 0$ and $s > 0$ either $\omega \subseteq B_r$ and $\operatorname{supp} f \subseteq B^c_{r+s}$, or $\operatorname{supp} f \subseteq B_r$ and $\omega \subseteq B^c_{r+s}$; then, for $s$ and $\lambda$ such that $s\sqrt{\lambda} > 2$,

$$\|(-\Delta+\lambda)^{-1} f\|^2_{L^2(\omega)} \leq c e^{-s\frac{\sqrt{\lambda}}{\sqrt{2}}} (\sqrt{\lambda})^{n-2} \|f\|^2_{H^{-1}(\mathbf{R}^n)} |B_r|,$$

where $c > 0$ is a constant independent of $\lambda$, $f$, $\omega$ and $s$, and $|B_r|$ is the Lebesgue measure of $B_r$.

Property (a) can be found, for instance, in [28] Chapter V, § 3.11, while (b) can be obtained by using the explicit representation of $R_\lambda(A_0)$ in terms of Green's function. Indeed, we have

$$(R_\lambda(A_0)f)(x) = \int_{\operatorname{supp} f} H_n(x-y,\sqrt{\lambda}) f(y)\,dy, \text{ where}$$
$$H_n(x,1) = (4\pi)^{-n/2} \int_0^\infty e^{-\delta} e^{-\frac{|x|^2}{4\delta}} \frac{d\delta}{\delta^{n/2}}$$

and

$$H_n(x,k) = k^{n-2} H_n(xk,1) \tag{5.1.4}$$

see [38] Problem 49, Chapter IX, Vol. II. By density we may assume $f = \operatorname{div} F$ with $F_i \in C_0^\infty(\mathbf{R}^n)$. Then

$$u(x) = (R_\lambda(A_0)f)(x) = \sum_{i=1}^n \int_{\operatorname{supp} f} D_i H_n(x-y,\sqrt{\lambda}) F_i(y)\,dy$$

and applying Hölder inequality and (5.1.4) we deduce that

$$\int_\omega |u(x)|^2\,dx \leq c\lambda^{n-2} \|f\|^2_{H^{-1}(\mathbf{R}^n)} \int_\omega \left( \sum_{i=1}^n \int_{\operatorname{supp} f} \left(D_i H_n((x-y)\sqrt{\lambda};1)\right)^2 dy \right) dx \leq$$

$$\leq c\lambda^{n-2} \|f\|^2_{H^{-1}(\mathbf{R}^n)} \sum_{i=1}^n \int_\omega \int_{\operatorname{supp} f} \left( \int_0^\infty e^{-\delta} \frac{(x_i-y_i)\lambda}{2\delta} e^{-\frac{|x-y|^2\lambda}{4\delta}} \frac{d\delta}{\delta^{n/2}} \right)^2 dy dx\,.$$

Applying Jensen Inequality for the measure $e^{-\delta}d\delta$ we get

$$\int_\omega |u(x)|^2\,dx \le c\lambda^{n-2}\|f\|^2_{H^{-1}(\mathbf{R}^n)} \int_\omega \int_{\operatorname{supp} f} \Big(\int_0^\infty \frac{|x-y|^2\lambda^2}{4\delta^2} e^{-\frac{|x-y|^2\lambda}{2\delta}} e^{-\delta}\frac{d\delta}{\delta^n}\Big) dydx\,.$$

so that

$$\|u\|^2_{L^2(\omega)} \le c\|f\|^2_{H^{-1}(\mathbf{R}^n)} \int_\omega \int_{\operatorname{supp} f} g(|x-y|,\lambda)dydx,$$

where

$$g(\rho,\lambda) = \lambda^n \int_0^\infty \frac{\rho^2}{4} e^{-\frac{\rho^2\lambda}{2\delta}} e^{-\delta}\frac{d\delta}{\delta^{n+2}}.$$

Since for every $\lambda > 0$ the function $g(\cdot,\lambda)$ is decreasing, using the assumptions made on $\omega$ and $\operatorname{supp} f$ we obtain

$$\int_\omega \int_{\operatorname{supp} f} g(|x-y|,\lambda)dydx \le |B_r|\omega_n \int_s^\infty g(\rho,\lambda)\rho^{n-1}d\rho\,.$$

Standard estimates show that for $s$ and $\lambda$ such that $s\sqrt{\lambda} > 2$,

$$\int_s^\infty g(\rho,\lambda)\rho^{n-1}d\rho \le c\epsilon^{-s\frac{\sqrt{\lambda}}{\sqrt{2}}}(\sqrt{\lambda})^{n-2}$$

and the proof is concluded.

## 5.2. Wave operators for the relaxed wave equation

This section is devoted to the proof of the existence of the wave operator for the relaxed wave equation and to the study of some of its properties. Let us remark that the case of an exterior domain $\Omega$, i.e., an open and connected subset of $\mathbf{R}^n$ whose complementary set is compact, corresponds to $\mu = \mu_\Omega$.

The behaviour of the solution of the wave equation as the time goes to infinity can be characterized by the following operators.

**Definition 5.2.1.** (cf. [29] Definition 3.1) Let $\mathcal{H}_1$, $\mathcal{H}_2$ be two separable Hilbert spaces, let $U_1$, $U_2$ be continuous unitary groups on $\mathcal{H}_1$ and $\mathcal{H}_2$, with self-adjoint generators $T_1$, $T_2$, i.e. $U_j(t) = e^{-itT_j}$ for $-\infty < t < +\infty$, $j = 1,2$, and assume that $T_1$ and $T_2$ have absolutely continuous spectrum. Let $J \in B(\mathcal{H}_1,\mathcal{H}_2)$. If

$$W_\pm = \mathrm{s} - \lim_{t\to\pm\infty} U_2(-t)JU_1(t) \in B(\mathcal{H}_1,\mathcal{H}_2)$$

# Chapter 1. Notation and preliminaries

## 1.1. Sobolev spaces and capacity

Let $\Omega$ be an open subset of $\mathbf{R}^n$, $n \geq 3$. If $x$, $y \in \mathbf{R}^n$, $x \cdot y$ denotes their scalar product; the Euclidean norm in $\mathbf{R}^n$ is denoted by $|\cdot|$. $\mathrm{M}^{n\times m}$ will denote the space of $n \times m$ matrices. Notice that, if $M \in \mathrm{M}^{n\times m}$ we shall write $|M|$ to denote its Euclidean norm as an element of $\mathbf{R}^{nm}$.

The space $\mathcal{D}'(\Omega)$ of distributions in $\Omega$ is the dual of $C_0^\infty(\Omega)$. Given two numbers $p$ and $q$, with $1 < p, q < +\infty$ and $1/p + 1/q = 1$, let $W^{1,p}(\Omega, \mathbf{R}^m)$ denote the usual Sobolev space, i.e. the space of all functions $u$ in $L^p(\Omega, \mathbf{R}^m)$ whose first order distribution derivatives $D_j u$ belong to $L^p(\Omega, \mathbf{R}^m)$, endowed with the norm

$$\|u\|^p_{W^{1,p}(\Omega,\mathbf{R}^m)} = \int_\Omega |Du|^p dx + \int_\Omega |u|^p dx ,$$

where $Du = (D_j u^\alpha)$ is the Jacobian matrix of $u$. The space $W_0^{1,p}(\Omega, \mathbf{R}^m)$ is the closure of $C_0^\infty(\Omega, \mathbf{R}^m)$ in $W^{1,p}(\Omega, \mathbf{R}^m)$, and $W^{-1,q}(\Omega, \mathbf{R}^m)$ is the dual of $W_0^{1,p}(\Omega, \mathbf{R}^m)$. The symbol $\mathbf{R}^m$ will be omitted when $m = 1$. When $p = q = 2$ we shall use the notation $H^1(\Omega, \mathbf{R}^m)$, $H_0^1(\Omega, \mathbf{R}^m)$ and $H^{-1}(\Omega, \mathbf{R}^m)$, respectively.

For every subset $E$ of $\Omega$ the (harmonic) capacity of $E$ with respect to $\Omega$ is defined by $\mathrm{cap}(E) = \inf \int_\Omega |Du|^2 dx$, where the infimum is taken over all functions $u \in H_0^1(\Omega)$ such that $u \geq 1$ almost everywhere in a neighbourhood of $E$, with the usual convention $\inf \emptyset = +\infty$. We say that a property $\mathcal{P}(x)$ holds quasi everywhere (abbreviated as q.e.) in a set $E$ if it holds for all $x \in E$ except for a subset $N$ of $E$ with $\mathrm{cap}(N) = 0$. The expression almost everywhere (abbreviated as a.e.) refers, as usual, to the Lebesgue measure.

A function $u : \Omega \to \mathbf{R}^m$ is said to be quasicontinuous if for every $\varepsilon > 0$ there exists a set $E \subset \Omega$, with $\mathrm{cap}(E) \leq \varepsilon$, such that the restriction of $u$ to $\Omega \backslash E$ is continuous. We recall that for every $u \in H_0^1(\Omega, \mathbf{R}^m)$ there exists a quasicontinuous function $\tilde{u}$, unique up to sets of capacity zero, such that $u = \tilde{u}$ almost everywhere in $\Omega$. We shall always identify $u$ with its quasicontinuous representative $\tilde{u}$, so that the pointwise value of a function $u \in H_0^1(\Omega, \mathbf{R}^m)$ is defined quasi everywhere in $\Omega$.

For any $u \in H_0^1(\Omega)$ we shall denote by $u^+$ and $u^-$ the positive and the negative parts of $u$: $u^+ = u \vee 0$, $u^- = -(u \wedge 0)$. Then $u = u^+ - u^-$ and it can be easily proved that for any $u \in H_0^1(\Omega)$, $u^+$, $u^- \in H_0^1(\Omega)$.

**Theorem 5.2.6.** *The wave operators $W_\pm(U, U_0, J)$ and $W_\pm(U_0, U, J^*)$ exist, are unitary and mutually adjoint.*

To prove this theorem we define

$$\tilde{J}(u,v) = (\tilde{B}^{-1} J \tilde{B}_0 u, Jv) \in \mathcal{H} \qquad \text{for every } (u,v) \in \mathcal{H}_0\,, \tag{5.2.1}$$

where $\tilde{B}$, $\tilde{B}_0$ are the unitary extensions of $B$ and $B_0 = A_0^{1/2}$ to $[D(B)] = \tilde{V}$ and $[D(B_0)]$, respectively, and we show that $W_\pm(U, U_0, \tilde{J})$ and $W_\pm(U_0, U, \tilde{J}^*)$ exist, are unitary and mutually adjoint. Then we prove that $J$ and $\tilde{J}$ are $(U_0, \pm)$-equivalent in the sense of Definition 5.2.2 and this will conclude the proof of Theorem 5.2.6.

In order to prove the existence of the wave operator for the groups $U(t)$ and $U_0(t)$ and identification operator $\tilde{J}$ we shall use a result of Kato, Theorem 9.3 in [29], where from the existence and completeness of the wave operators $C_\pm = W_\pm(B, B_0, J)$ it follows that the wave operators $W_\pm(U, U_0, \tilde{J})$ and $W_\pm(U_0, U, \tilde{J}^*)$ exist, are complete and are mutually adjoint partial isometries. (As we shall see, in our case they are unitary.) We begin by proving the existence of the wave operators $C_\pm$. This can be done applying the following theorem (see the proof of Theorem A.1 in [46]; the statement is modified according to [33]). If $X$, $Y$ are Banach spaces, $B_0(X,Y)$ will denote the space of compact operators from $X$ to $Y$, and $B_1(X,Y)$ that of the trace class operators (for definition and properties see, for instance, [28]).

**Theorem 5.2.7.** *Let $J_0 : H_0 \to H$ be a bounded linear operator such that $J_0 D(A_0) \subset D(A)$, $J_0^* D(A) \subset D(A_0)$ and assume that for every bounded interval $\omega \subset \mathbf{R}$*

(1) $$(AJ_0 - J_0 A_0) E_0(\omega) \in B_1(H_0, H)$$

(2) $$(J_0 J_0^* - 1) E(\omega) \in B_0(H, H)$$

(3) $$(J_0^* J_0 - 1) E_0(\omega) \in B_0(H_0, H_0)\,.$$

*Then the wave operators*

$$W_\pm = \mathrm{s} - \lim_{t \to \pm\infty} e^{itA_0} J_0^* e^{-itA}$$

$$W_\pm^0 = \mathrm{s} - \lim_{t \to \pm\infty} e^{itA} J_0 e^{-itA_0}$$

*exist and are unitary. In addition $W_\pm^* = W_\pm^0$ and the invariance principle holds, i.e., $W_\pm(\varphi(A_0), \varphi(A), J_0^*) = W_\pm$ for every continuous monotone increasing function $\varphi$. (It holds also for more general classes of functions $\varphi$.)*

**Proposition 5.2.8.** *Let $J : L^2(\mathbf{R}^n) \to H$ be the operator given by Definition 5.2.5. Then the wave operators $C_\pm = W_\pm(B, B_0, J)$ exist and are unitary.*

*Proof.* We apply the previous theorem for $J_0 = J$. Remark that since the support of $\mu$ is included in $B_r$, $jf \in H$, for every $f \in L^2(\mathbf{R}^n)$. We have $(J^*f)(x) = j(x)f(x)$. If $u \in D(A)$ then it is easy to see that $ju \in D(A_0)$ so that $J^*D(A) \subset D(A_0)$. From the definition of $j$ it follows also that $JD(A_0) \subset D(A)$.

Let us verify now (1): let $u \in D(A_0)$ and let $v = E_0(\omega)u$. Then, as $jv = 0$ on $B_r$ implies that $A(jv) = -\Delta(jv)$, it follows that

$$(AJ - JA_0)v = A(jv) - j(Av) = -\Delta(jv) - j(-\Delta v).$$

Since $v \in D(A_0)$ (see, for instance, [39] remark after Theorem 13.33) we may write everything in the classical form so that $-\Delta(jv) - j(-\Delta v) = -\Delta jv - 2DjDv$. This shows that only the spectral measure for $A_0$ is involved, so we get (1) using the same arguments as in [46].

(2): We have to prove that a bounded sequence in $H$ is transformed into a convergent one. Let $u_m \in H$ be such that $\|u_m\|_{L^2(\mathbf{R}^n)} \leq c$ and let $v_m = E(\omega)u_m$. We have $(JJ^* - 1)E(\omega)u_m(x) = (j^2(x) - 1)v_m(x)$. Since $j^2(x) - 1 = 0$ on $B^c_{r+1}$, $\operatorname{supp}((j^2 - 1)v_m) \subset B_{r+1}$. For $\omega_1 \cap \omega = \emptyset$ we have $(E(\omega_1)E(\omega)u_m, E(\omega)u_m) = 0$, hence

$$\int_{\mathbf{R}} \lambda^2(E(d\lambda)v_m, v_m) = \int_{\mathbf{R}} \lambda^2(E(d\lambda)E(\omega)u_m, E(\omega)u_m) =$$
$$= \int_{\omega} \lambda^2(E(d\lambda)E(\omega)u_m, E(\omega)u_m) \leq M\|u_m\|^2_{L^2(\mathbf{R}^n)}.$$

Then, using for instance Lemma 13.23 and Theorem 13.24 in [39] we get that $v_m \in D(A)$ and $\int_{\mathbf{R}} \lambda^2(E(d\lambda)v_m, v_m) = \|Av_m\|^2_H$. It follows that

$$(Av_m, v_m)_H = \int_{\mathbf{R}^n} |Dv_m|^2\, dx + \int_{\mathbf{R}^n} |v_m|^2\, d\mu \leq \|Av_m\|\,\|v_m\| \leq Mc^2$$

so that $\|Dv_m\|_{L^2(\mathbf{R}^n)} \leq c_1$. From Rellich's theorem we deduce that (2) is true. Then (3) can be proved in a similar way.

So the hypotheses of the theorem are satisfied and the existence of the wave operators $W_\pm(A, A_0, J)$ and $W_\pm(A_0, A, J^*)$ follows. Moreover, since the invariance principle holds, the wave operators $W_\pm(B_0, B, J^*)$ exist and are unitary, hence complete.

Let $U_1(t) = e^{-itB_0}$ and $U_2(t) = e^{-itB}$. According to Theorem 6.3 in [29] if $J$ is a $(U_2, \pm)$-asymptotic left inverse to $J^*$, the existence and completeness of $W_\pm(B_0, B, J^*)$ implies that the wave operators $C_\pm = W_\pm(B, B_0, J)$ exist, $J^*$ is a $(U_1, \pm)$-asymptotic left inverse to $J$ and $C_\pm^* = (C_\pm)^{-1} = W_\pm(B_0, B, J^*)$.

So it is enough to prove that $J$ is a $(U_2, \pm)$-asymptotic left inverse to $J^*$. According to Definition 5.1 in [29] this means that we have to prove that

$$\mathrm{s} - \lim_{t \to \pm\infty} (JJ^* - 1) U_2(t) = 0\,.$$

It suffices to prove that the limits are zero for $t \to \infty$ since we may then apply the result for $-B$. By the definition of $j$, $(j^2(x) - 1) = 0$ on $\mathbf{R}^n \setminus B_{r+1}$ so the following lemma will conclude the proof.

**Lemma 5.2.9.** *If* $u(t, \cdot) = e^{-itA^{1/2}} f$, $f \in H$, *then*

$$\lim_{t \to +\infty} \int_K |u(t,x)|^2 \, dx = 0 \quad \textit{for each compact set } K \subset \mathbf{R}^n.$$

*Proof.* Let $Q_K u(x) = 1_K(x) u(x)$, we have then to prove that

$$\lim_{t \to +\infty} \| Q_K e^{-itA^{1/2}} f \| = 0$$

for every compact set $K \subset \mathbf{R}^n$. Abstract theory (Theorem 2.3 in [47]) implies that if $Q_K$ is $A^{1/2}$-compact, this is true for every $f \in H$. Hence we have to show that $Q_K$ is $A^{1/2}$-compact, that is that for every $S \subset H$ bounded in the graph norm of $A^{1/2}$, which in our case is the norm of $V$, the set $Q_K S$ is precompact in $L^2(\mathbf{R}^n)$. Since $K$ is a compact set, this follows from Rellich's Theorem. □

**Remark 5.2.10.** Using the notion of equivalence of identification operators given in Definition 5.2.2, and the existence of the wave operators $C_\pm$ we can prove the existence of the wave operators with $J$ replaced by the projection $P$ on $H$. In order to do this is enough to verify that

$$\mathrm{s} - \lim_{t \to \pm\infty} (J - P) e^{-itA_0^{1/2}} = 0.$$

We have $u = u_1 + u_2$ with $u_1 = u$ on $B_{r+1}$ and $0$ outside and $u_2 = 0$ on $B_{r+1}$ and $u_2 = u$ on $B_{r+1}^c$. As $\operatorname{supp} \mu \subset B_r$, $u_2 \in H$ and by the definition of $J$ we have $Ju_2 = u_2 = Pu_2$. Then

$$\|(J - P)u\|_{L^2(\mathbf{R}^n)} = \|(J - P)u_1\|_{L^2(\mathbf{R}^n)} \le C \|u_1\|_{L^2(B_r)} = C \|u\|_{L^2(B_r)}.$$

Since now we consider $u$ of the form $e^{-itA_0^{1/2}}f$ with $f \in L^2(\mathbf{R}^n)$, the $L^2$-norm of $u$ on every fixed compact set tends to 0 as $t \to \pm\infty$, hence, $J$ and $P$ are equivalent.

The following result gives the existence of the wave operators with identification operator $\tilde{J}$.

**Theorem 5.2.11.** *Let $\tilde{J}$ be the operator defined by (5.2.1). Then the wave operators $W_\pm(U, U_0, \tilde{J})$ and $W_\pm(U_0, U, \tilde{J}^*)$ exist, are unitary and mutually adjoint.*

*Proof.* We shall use the wave operators $C_\pm = W_\pm(B, B_0, J)$. The intertwining property holds: $C_\pm B_0 \subset BC_\pm$ and $C_\pm^* B \subset B_0 C_\pm^*$.

Let $f \in D(B_0)$, $g \in R(B_0)$. Then $U_0(t)(f,g) = (u_0(t), \dot{u}_0(t))$ is given by

$$\begin{cases} u_0(t) = (\cos tB_0)f + B_0^{-1}\sin(tB_0)g \\ \dot{u}_0(t) = -(\sin tB_0)B_0 f + (\cos tB_0)g \end{cases}$$

so $B_0 u_0(t) = \frac{1}{2}e^{-itB_0}(B_0 f + ig) + \frac{1}{2}e^{itB_0}(B_0 f - ig)$ and, for $t \to \pm\infty$,

$$\begin{aligned} JB_0 u_0(t) &\sim \frac{1}{2}e^{-itB}C_\pm(B_0 f + ig) + \frac{1}{2}e^{itB}C_\mp(B_0 f - ig) = \\ &= \frac{1}{2}e^{-itB}(BC_\pm f + iC_\pm g) + \frac{1}{2}e^{itB}(BC_\mp f - iC_\mp g) \end{aligned}$$

where we used the intertwining relations. Similarly,

$$J\dot{u}_0(t) \sim \frac{-i}{2}e^{-itB}(BC_\pm f + iC_\pm g) + \frac{i}{2}e^{itB}(BC_\mp f - iC_\mp g).$$

Hence, as $t \to \pm\infty$ $JB_0 u_0(t) \sim Bu_\pm(t)$ and $J\dot{u}_0(t) \sim \dot{u}_\pm(t)$ in $H$, with $(u_\pm(t), \dot{u}_\pm(t)) = U(t)(f_\pm, g_\pm)$ and

$$\begin{aligned} f_\pm &= \frac{1}{2}(C_+ + C_-)f \pm \frac{i}{2}(C_+ - C_-)B_0^{-1}g \\ g_\pm &= \mp\frac{i}{2}(C_+ - C_-)B_0 f + \frac{1}{2}(C_+ + C_-)g . \end{aligned}$$

Note that $B_0^{-1}g$ is well defined since $B_0$ is injective and $g \in R(B_0)$. Set $(u(t), v(t)) = \tilde{J}(u_0(t), \dot{u}_0(t)) = \tilde{J}U_0(t)(f,g)$. From the definition of $\tilde{J}$ we have $\tilde{B}u(t) = J\tilde{B}_0 u_0(t)$ and $v(t) = J\dot{u}_0(t)$. Then

$$\begin{aligned} \|\tilde{J}U_0(t)(f,g) - U(t)(f_\pm, g_\pm)\|_{\mathcal{H}}^2 &= \|(u(t), v(t)) - (u_\pm(t), \dot{u}_\pm(t))\|_{\mathcal{H}}^2 = \\ &= \|\tilde{B}(u(t) - u_\pm(t))\|_H^2 + \|\dot{u}_\pm(t) - J\dot{u}_0(t)\|_H^2 \end{aligned}$$

which tends to zero as $t \to \pm\infty$.

Since $f \in D(B_0)$, $g \in R(B_0)$ form a dense set in $\mathcal{H}_0$, we conclude that there exist operators $W_\pm$ such that $W_\pm(f,g) = (f_\pm, g_\pm)$.

Let us prove now that $\tilde{J}^*$ is a $(U_0, \pm)$-asymptotic inverse of $\tilde{J}$, that is

$$\mathrm{s} - \lim_{t\to\pm\infty}(\tilde{J}^*\tilde{J} - 1)U_0(t) = 0 .$$

We have

$$\begin{aligned}\|(1-\tilde{J}^*\tilde{J})(u,v)\|_{\mathcal{H}_0} &= \|((1-\tilde{B}_0^{-1}J^*\tilde{B}\tilde{B}^{-1}J\tilde{B}_0)u,(1-J^*J)v)\|_{\mathcal{H}_0} = \\ &= \|B_0(1-\tilde{B}_0^{-1}J^*JB_0)u\|_{H_0} + \|(1-J^*J)v\|_{H_0} .\end{aligned}$$

In order to prove that it tends to zero as $t \to \pm\infty$ it is enough to show that

$$\|(1-J^*J)B_0u_0(t)\|_{H_0} \to 0 \quad \text{as } t \to \pm\infty$$
$$\|(1-J^*J)\dot{u}_0(t)\|_{H_0} \to 0 \quad \text{as } t \to \pm\infty$$

and the expressions for $B_0u_0(t)$ and $\dot{u}_0(t)$ we derived above imply that this follows from the convergence of $(1-J^*J)e^{-itB_0}\varphi$ to zero as $t \to \pm\infty$ for every $\varphi \in H_0$, which is a consequence of the completeness of $C_\pm$ (see [29]). Now we may apply Theorem 6.3 in [29] to obtain that $W_\pm(U, U_0, \tilde{J})$ is an isometry. Since by symmetry the same holds true for $W_\pm(U_0, U, \tilde{J}^*)$ we apply again Theorem 6.3 in [29] and get that $W_\pm(U_0, U, \tilde{J}^*)$ and $W_\pm(U, U_0, \tilde{J})$ are unitary and mutually adjoint. □

We prove now that the operators $J$ and $\tilde{J}$ are equivalent in the sense of Definition 5.2.2.

**Theorem 5.2.12.** *The identification operators $J$ and $\tilde{J}$ are $(U_0,\pm)$-equivalent, i.e.* $\mathrm{s} - \lim_{t\to\pm\infty}(J - \tilde{J})U_0(t) = 0$.

*Proof.* Let $\varepsilon > 0$ small and let $\varphi \in C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$. By Remark 5.1.2, we may choose $T > 0$ such that

$$\|U_0(t+T)\varphi\|_{E_0(B_{r+1+t})} < \varepsilon \quad \text{for } t \geq 0 . \tag{5.2.2}$$

It can be seen in the same way that the $L^2$-norm of the solution on a fixed compact subset of $\mathbf{R}^n$ converges to zero as the time $t$ tends to infinity. So, we may choose $T$ such that setting $(u_0(t), \dot{u}_0(t)) = U_0(t+T)\varphi$ we have also

$$\|u_0(t)\|_{L^2(B_{r+1})} < \varepsilon, \quad \text{for } t \geq 0. \tag{5.2.3}$$

Then we prove that for $t$ large enough

$$\|(\tilde{J}-J)U_0(t+T)\varphi\|_{E(\mathbf{R}^n)} \le c\varepsilon,$$

which, taking into account the definition of $\tilde{J}$, reduces to proving that for $t$ large enough

$$\|(JB_0 - BJ)u_0(t)\|_{L^2(\mathbf{R}^n)} \le c\varepsilon. \tag{5.2.4}$$

Now

$$\begin{aligned} &\|(JB_0 - BJ)u_0(t)\|_{L^2(\mathbf{R}^n)} = \\ &= \|(J-1)B_0u_0(t) + B_0((1-J)u_0(t)) + (B_0 - B)Ju_0(t)\|_{L^2(\mathbf{R}^n)}. \end{aligned} \tag{5.2.5}$$

First of all note that, by the choice of $T$,

$$\|B_0((1-J)u_0(t))\|_{L^2(\mathbf{R}^n)} = \|D((1-j)u_0(t))\|_{L^2(B_{r+1})} < \varepsilon.$$

To evaluate the other two terms we use the representation formula for the square root $A^{1/2}$ proved in [28] Chapter V, §3.11:

$$A^{1/2}u = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}R_\lambda(A)Au\,d\lambda \quad \text{for } u \in D(A), \tag{5.2.6}$$

where $R_\lambda(A) = (A+\lambda)^{-1}$ is the resolvent of $A$ and the integral is a Bochner integral in $H$, the analogous formula for $A_0$, and the estimates for the resolvent given in Remark 5.1.4. We have also $\|u_0(t)\|_{L^2(\mathbf{R}^n)} \le C$, with $C$ depending on $\varphi$ but independent of $t$, see Remark 5.1.3. Then we can choose $\lambda_0 > 0$ such that $\lambda_0^{1/2}C^2 \le \varepsilon$. The integral in the representation formula (5.2.6) will be written as the sum of two integrals on the intervals $(0,\lambda_0)$ and $(\lambda_0,+\infty)$. The two terms will be estimated using the inequalities (a) and (b) in Remark 5.1.4, respectively.

Let $s > 0$ be a parameter we shall choose later. Then there exists $t_\varepsilon(s)$ such that

$$\|u_0(t)\|_{L^2(B_{r+2s+1})} \le \varepsilon \qquad \text{for } t \ge t_\varepsilon(s). \tag{5.2.7}$$

Recall that by the choice of $T$, see (5.2.2) we also have $\|Du_0(t)\|_{L^2(B_{r+t+1})} \le \varepsilon$. Then there exists a function $\theta \in C_0^\infty(\mathbf{R}^n)$, with $0 \le \theta \le 1$, $\theta = 1$ on $B_{r+s}$ and $\operatorname{supp}\theta \subset B_{r+s+1}$ such that $\|B_0(\theta u_0(t))\|_{L^2(\mathbf{R}^n)} = \|D(\theta u_0(t))\|_{L^2(\mathbf{R}^n)} \le c\varepsilon$. We write $u_0(t) = \theta u_0(t) + (1-\theta)u_0(t)$ and

$$\|(J-1)B_0u_0(t)\|_{L^2(\mathbf{R}^n)} \le \|B_0(\theta u_0(t))\|_{L^2(\mathbf{R}^n)} + \|(J-1)B_0 w\|_{L^2(\mathbf{R}^n)},$$

with $w = (1-\theta)u_0(t)$.

As $w \in C_0^\infty(\mathbf{R}^n)$, to estimate $\|(J-1)B_0 w\|_{L^2(\mathbf{R}^n)}$ we can use the representation formula (5.2.6) and we obtain

$$\begin{aligned} &\|(J-1)B_0 w\|_{L^2(\mathbf{R}^n)} \leq \\ &\leq \frac{1}{\pi}\int_0^{\lambda_0} + \int_{\lambda_0}^\infty \lambda^{-1/2}\|R_\lambda(A_0)A_0 w\|_{L^2(B_{r+1})}\,d\lambda\,, \end{aligned} \tag{5.2.8}$$

where we used the fact that $\operatorname{supp}(j-1) \subset B_{r+1}$. Since $\|w\|_{L^2(\mathbf{R}^n)} \leq \|u_0(t)\|_{L^2(\mathbf{R}^n)} \leq C$, from estimate $(a)$ in Remark 5.1.4 and the choice of $\lambda_0$ we get that the first integral is less than $c\varepsilon$. Applying estimate $(b)$ in the same remark, and using the fact that $\operatorname{supp} w \subset B_{r+s}^c$ we can find $s$, independent of $t$ such that the second integral in (5.2.8) is less than $\varepsilon$.

It remains to estimate $\|(B-B_0)(Ju_0(t))\|_{L^2(\mathbf{R}^n)}$. We evaluate separately the norm on $B_{r+s}^c$ and on $B_{r+s}$. We have

$$\begin{aligned} &\|(B_0 - B)(Ju_0(t))\|_{L^2(B_{r+s}^c)} \leq \\ &\leq \frac{1}{\pi}\int_0^{\lambda_0} + \int_{\lambda_0}^\infty \lambda^{-1/2}\|R_\lambda(A_0)A_0(ju_0(t)) - R_\lambda(A)A(ju_0(t))\|_{L^2(B_{r+s}^c)}\,d\lambda\,. \end{aligned}$$

Now, for the first integral we use inequality $(a)$ in Remark 5.1.4 both for $R_\lambda(A_0)A_0(ju_0(t))$ and $R_\lambda(A)A(ju_0(t))$; by the choice of $\lambda_0$ we obtain that it can be estimated by $c\varepsilon$. For the second one, remark first of all that on $B_{r+s}^c$ $A(ju_0(t)) = A_0(ju_0(t))$. Let $f = A_0(ju_0(t))$ and $\beta_\lambda = (-\Delta+\lambda)(R_\lambda(A_0) - R_\lambda(A))f$. Then $\beta_\lambda \in H^{-1}(\mathbf{R}^n)$ and $(-\Delta+\lambda)R_\lambda(A)f = f - \beta_\lambda$. Using the relaxed Dirichlet problem satisfied by $R_\lambda(A)f$ we deduce that $\operatorname{supp}\beta_\lambda \subset \operatorname{supp}\mu \subset B_r$ and that $\|\beta_\lambda\|_{H^{-1}(\mathbf{R}^n)} \leq c$, with $c$ independent of $\lambda$ and $t$. With this notation the integral between $\lambda_0$ and infinity becomes $\int_{\lambda_0}^\infty \lambda^{-1/2}\|(-\Delta+\lambda)^{-1}\beta_\lambda\|_{L^2(B_{r+s}^c)}\,d\lambda$. Since $\operatorname{supp}\beta_\lambda \subset B_r$ estimate $(b)$ in Remark 5.1.4 shows that this integral can be made small by a suitable choice of $s$ independent of $t$.

We consider now the norm on $B_{r+s}$. As above, using (5.1.3) and (5.2.7), it can be proved that there exists a function $\theta \in C_0^\infty(\mathbf{R}^n)$, with $0 \leq \theta \leq 1$, $\theta = 1$ on $B_{r+2s}$ and $\operatorname{supp}\theta \subset B_{r+2s+1}$, such that $\|B_0(\theta j u_0(t))\|_{L^2(\mathbf{R}^n)} = \|D(\theta j u_0(t))\|_{L^2(\mathbf{R}^n)} < \varepsilon$ and $\|B(\theta j u_0(t))\|_{L^2(\mathbf{R}^n)} = \|D(\theta j u_0(t))\|_{L^2(\mathbf{R}^n)} < \varepsilon$, for every $t \geq t_\varepsilon(s)$ (note that the term containing $\mu$ in (5.1.3) disappears because $j = 0$ on $\operatorname{supp}\mu$). So we have now to estimate $\|(B_0 - B)((1-\theta)ju_0(t))\|_{L^2(B_{r+s})}$. Let $z = (1-\theta)ju_0(t)$. We shall use again the representation formula (5.2.6) and since we have to evaluate the norm on $B_{r+s}$ and $\operatorname{supp} z \subset B^c_{r+2s}$, for the term involving the resolvent of $A_0$ we apply the estimate (*b*) in Remark 5.1.4 and choose $s$ large enough (independent of $t$) to make its norm on $B_{r+s}$ less than $c\varepsilon$. So it remains to prove that $\|Bz\|^2_{L^2(B_{r+s})} \leq c\varepsilon$ for $s$ large enough. As $Az = A_0 z$, using (5.2.6) we have

$$\|Bz\|_{L^2(B_{r+s})} \leq \frac{1}{\pi}\int_0^{\lambda_0} + \int_{\lambda_0}^{\infty} \lambda^{-1/2}\|R_\lambda(A)A_0 z\|_{L^2(B_{r+s})} d\lambda\,.$$

The first integral can be made small using estimate (*a*) in Remark 5.1.4 and the choice of $\lambda_0$. To evaluate the second one, let $v = R_\lambda(A)A_0 z$. This means that

$$-\Delta v + \lambda v + \mu v = -\Delta z\,.$$

Taking $v$ as test function we get

$$\int_{\mathbf{R}^n} |Dv|^2 + \lambda\int_{\mathbf{R}^n} |v|^2 + \int_{\mathbf{R}^n} |v|^2\, d\mu = \int_{\mathbf{R}^n} DzDv \leq \frac{1}{2}\int_{\mathbf{R}^n} |Dz|^2 + \frac{1}{2}\int_{\mathbf{R}^n} |Dv|^2\,.$$

Let now $\chi$ be a cut-off function between $B_{r+s}$ and $B_{r+2s}$ and take $v\chi$ as test function. Then

$$\int_{\mathbf{R}^n} DvvD\chi + \int_{\mathbf{R}^n} |Dv|^2\chi + \lambda\int_{\mathbf{R}^n} |v|^2\chi + \int_{\mathbf{R}^n} |v|^2\, d\mu = 0$$

and we obtain

$$\lambda\int_{\mathbf{R}^n} |v|^2\chi \leq \|D\chi\|_{L^\infty(\mathbf{R}^n)}\frac{\|Dz\|^2_{L^2(\mathbf{R}^n)}}{\sqrt{\lambda}}\,.$$

Hence

$$\|v\|^2_{L^2(B_{r+s})} \leq \frac{C}{s\lambda^{3/2}}\,,$$

so that choosing $s$ large enough

$$\int_{\lambda_0}^{\infty} \lambda^{-1/2}\|v\|_{L^2(B_{r+s})} \leq \int_{\lambda_0}^{\infty} \lambda^{-1/2}\frac{C}{\sqrt{s}\lambda^{3/4}} \leq \varepsilon\,.$$

Chosen $s$ such that all conditions imposed above be satisfied, we can choose $t \geq t_\varepsilon(s)$ in order to obtain (5.2.4).

Let now $\psi \in \mathcal{H}_0$. Then there exists $\varphi \in C_0^\infty(\mathbf{R}^n)\times C_0^\infty(\mathbf{R}^n)$ such that $\|\psi-\varphi\|_{\mathcal{H}_0} \leq \varepsilon$. Then

$$\|(J-\tilde{J})U_0(t+T)\psi\|_{\mathcal{H}} \leq \|(J-\tilde{J})U_0(t+T)(\psi-\varphi)\|_{\mathcal{H}} + \|(J-\tilde{J})U_0(t+T)\varphi\|_{\mathcal{H}} \leq 2\varepsilon,$$

for $t$ large enough and the proof is concluded. □

*Proof of Theorem 5.2.6.* By Theorem 5.2.11 the wave operators $W_\pm(U, U_0, \tilde{J})$ and $W_\pm(U_0, U, \tilde{J}^*)$ exist, are unitary and mutually adjoint and by Theorem 5.2.12 the operators $J$ and $\tilde{J}$ are equivalent. It is easy to see that Definition 5.2.2 implies that $W_\pm(U, U_0, J) = W_\pm(U, U_0, \tilde{J})$ and $W_\pm(U_0, U, J^*) = W_\pm(U_0, U, \tilde{J}^*)$. □

## 5.3. A convergence result

Let us consider now a sequence of measures $\mu_h \in \mathcal{M}_0(\mathbf{R}^n)$, with $\operatorname{supp}\mu_h \subset B_r$ and assume that $\mu_h \xrightarrow{\gamma} \mu$ with $\mu \in \mathcal{M}_0(\mathbf{R}^n)$ and $\operatorname{supp}\mu \subset B_r$. Let $\mathcal{A}_h : V_h \to V_h'$ be defined by $\mathcal{A}_h = -\Delta + \mu_h$, see (5.1.1), where $V_h = H^1(\mathbf{R}^n) \cap L^2_{\mu_h}(\mathbf{R}^n)$, let $H_h = \overline{V_h}^{L^2(\mathbf{R}^n)}$ and let $A_h : D(A_h) \subset H_h \to H_h$ be the corresponding realization in $H_h$, defined by $D(A_h) = \{u \in V_h : \mathcal{A}_h u \in H_h\}$ and $A_h u = \mathcal{A}_h u$ for $u \in D(A_h)$. The operators $A_h$ are self-adjoint. Remark that since their domains of definition are not dense in $L^2(\mathbf{R}^n)$ we need to consider each operator defined on its own space. Let $B_h = A_h^{1/2}$, $\mathcal{H}_h = [D(B_h)] \times H_h$ and let $U_h : \mathcal{H}_h \to \mathcal{H}_h$ be the unitary group giving the solution of the relaxed wave equation defined by $A_h$. We assume that $A$ and $A_h$ have absolutely continuous spectrum.

Since the operator $J$ given in Definition 5.2.5 does not take into account the measure $\mu$, the same proofs for the existence and unitarity of the waves operators hold for every $A_h$. We are going to show now that the wave operators corresponding to $\mu_h$ converge. Let $W_h = W_+(U_h, U_0, J)$ and $W = W_+(U, U_0, J)$. With the notation for the energy norms used in Section 1, let $\|(u,v)\|_{E_h(\omega)}$ denote the $E$-norm on $\omega$ corresponding to $\mu = \mu_h$.

Let us show first that the behaviour of the solutions at a finite time $T$ characterizes the behaviour of the wave operator corresponding to initial data $\varphi \in C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$.

**Theorem 5.3.1.** *Given $\varphi \in C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$, for every $\varepsilon > 0$ there exist $T = T(\varepsilon, \varphi) > 0$ and $\psi \in C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ for which*

$$\|W\varphi - U(-T)\psi\|^2_{E(\mathbf{R}^n)} \le \varepsilon \qquad \textit{and}$$

$$\|W_h\varphi - U_h(-T)\psi\|^2_{E_h(\mathbf{R}^n)} \le \varepsilon$$

*uniformly with respect to $h$.*

*Proof.* Let $\varepsilon > 0$ small. We may choose $T > 0$ such that setting $(u_0, v_0) = U_0(T)\varphi$,

$$\begin{aligned} \|u_0\|_{L^2(B_{r+1})} &< \varepsilon \quad \text{and} \\ \|U_0(t+T)\varphi\|_{E_0(B_{r+1+t})} &< \varepsilon \quad \text{for } t \geq 0 \end{aligned} \tag{5.3.1}$$

(see Remark 5.1.2). Let $\psi = jU_0(T)\varphi$, where $j$ is the function considered in Definition 5.2.5. As for $|y| > r+1+t$, $B_t(y) \subset B^c_{r+1}$ and $\operatorname{supp}\mu \subset B_r$, on $B^c_{r+1+t}$, by the finite propagation property for the solutions of the wave equation and by the uniqueness of solution of the relaxed wave equation, we have $U(t)\psi = U_0(t)\psi$. Since on $B^c_{r+1}$ $\psi = U_0(T)\varphi$, it follows that on $B^c_{r+1+t}$ $U(t)\psi = U_0(t)\psi = U_0(t+T)\varphi$. As $U$ is unitary,

$$\|U(t)\psi\|^2_{E(\mathbf{R}^n)} = \|\psi\|^2_{E(\mathbf{R}^n)} = \|\psi\|^2_{E_0(\mathbf{R}^n)}, \ (\text{as } \psi = 0 \text{ on } B_r).$$

Now

$$\|\psi\|^2_{E_0(\mathbf{R}^n)} = \int_{\mathbf{R}^n} |Dj_0 u_0 + j_0 Du_0|^2\, dx + \int_{\mathbf{R}^n} |j_0 v_0|^2\, dx$$

and using the convexity of $x \mapsto |x|^2$, the conservation of energy $\|U_0(T)\varphi\|_{E_0(\mathbf{R}^n)} = \|\varphi\|_{E_0(\mathbf{R}^n)}$, and the estimate (5.3.1) we obtain that

$$\|U(t)\psi\|^2_{E(\mathbf{R}^n)} \leq c\varepsilon + \frac{1}{1-\varepsilon}\|\varphi\|^2_{E_0(\mathbf{R}^n)}.$$

Since

$$\begin{aligned} &\|U_0(t)\psi\|^2_{E_0(B^c_{r+1+t})} = \|U_0(t+T)\varphi\|^2_{E_0(B^c_{r+1+t})} = \\ &= \|U_0(t+T)\varphi\|^2_{E_0(\mathbf{R}^n)} - \|U_0(t+T)\varphi\|^2_{E_0(B_{r+1+t})} > \|\varphi\|^2_{E_0(\mathbf{R}^n)} - \varepsilon^2, \end{aligned}$$

for every $t \geq 0$, we have

$$\begin{aligned} \|U(t)\psi\|^2_{E(B_{r+1+t})} &= \|U(t)\psi\|^2_{E(\mathbf{R}^n)} - \|U_0(t)\psi\|^2_{E_0(B^c_{r+1+t})} \leq \\ &\leq c\varepsilon + \frac{1}{1-\varepsilon}\|\varphi\|^2_{E_0(\mathbf{R}^n)} - \|\varphi\|^2_{E_0(\mathbf{R}^n)} + \varepsilon^2 < c'\varepsilon, \end{aligned} \tag{5.3.2}$$

where $c'$ depends on $\varphi$ but not on $\varepsilon$. As on $B^c_{r+1+t}$ we have $U(t+T)U(-T)\psi = U_0(t+T)\varphi$ and on $B_{r+1+t}$ both terms have energy norm less than or equal to $C\varepsilon^{1/2}$ for some $C > 0$ ($\|\cdot\|^2_E \leq C\varepsilon$ implies $\|\cdot\|^2_{E_0} \leq C\varepsilon$), it follows that

$$\|U(t+T)U(-T)\psi - U_0(t+T)\varphi\|^2_{E_0(\mathbf{R}^n)} < c\varepsilon. \tag{5.3.3}$$

We shall show now that

$$\|W\varphi - U(-T)\psi\|^2_{E(\mathbf{R}^n)} \leq c\varepsilon.$$

Since $W\varphi = \text{s-}\lim_{t\to+\infty} U(-t)JU_0(t)\varphi$, it is enough to prove that for $t$ large enough

$$\|U(-(t+T))JU_0(t+T)\varphi - U(-T)\psi\|^2_{E(\mathbf{R}^n)} \leq c\varepsilon.$$

As $U$ is unitary and $\psi = jU_0(T)\varphi$, this is equivalent to

$$\|JU_0(t+T)\varphi - U(t)JU_0(T)\varphi\|^2_{E(\mathbf{R}^n)} \leq c\varepsilon.$$

By (5.3.1), $\|(J-1)U_0(t+T)\varphi\|_{E_0(\mathbf{R}^n)} \leq \varepsilon$, so, from (5.3.3) we obtain

$$\|JU_0(t+T)\varphi - U(t)JU_0(T)\varphi\|^2_{E_0(\mathbf{R}^n)} \leq c\varepsilon. \tag{5.3.4}$$

To pass from the $E_0$ energy to the $E$ energy we have to take into account the term containing the integral with respect to $\mu$. Since $\operatorname{supp}\mu \subset B_r$, this term is estimated by (5.3.2), which, together with (5.3.4) gives

$$\|JU_0(t+T)\varphi - U(t)JU_0(T)\varphi\|^2_{E(\mathbf{R}^n)} \leq c\varepsilon,$$

and the proof is concluded by remarking that the same arguments hold for each $h$ and the choice of $T$ and $\psi$ is independent of $h$, so that we have also

$$\|W_h\varphi - U_h(-T)\psi\|^2_{E_h(\mathbf{R}^n)} \leq c\varepsilon.$$

□

This theorem, together with the results obtained in the previous chapter for the solutions of the wave equation for finite time intervals, allows us to prove the convergence of the wave operators. Although the definitions used are different, the solutions considered in this chapter coincide with the solutions studied in the previous one.

**Theorem 5.3.2.** *Let $\mu_h, \mu \in \mathcal{M}_0(\mathbf{R}^n)$ with $\operatorname{supp}\mu_h$, $\operatorname{supp}\mu \subset B_r$. Assume that $\mu_h \xrightarrow{\gamma} \mu$ and that the corresponding operators $A_h$ and $A$ have absolutely continuous spectrum. Then the wave operators $W_+(U_h, U_0, J_h)$ converge to the wave operator $W_+(U, U_0, J)$ in the sense that for every $\varphi \in \mathcal{H}_0$, if we set $(w_h^1, w_h^2) = W_+(U_h, U_0, J_h)\varphi$ and $(w^1, w^2) = W_+(U, U_0, J)\varphi$, we have*

$$Dw_h^1 \rightharpoonup Dw^1 \quad w\text{-}L^2(\mathbf{R}^n) \quad \text{and} \quad w_h^2 \to w^2 \quad s\text{-}L^2(\mathbf{R}^n).$$

*Moreover,*

$$\|Dw_h^1\|^2_{L^2(\mathbf{R}^n)} + \|w_h^1\|^2_{L^2_{\mu_h}(\mathbf{R}^n)} \to \|Dw^1\|^2_{L^2(\mathbf{R}^n)} + \|w^1\|^2_{L^2_\mu(\mathbf{R}^n)}, \tag{5.3.5}$$

*and similar results hold for* $W_-(U_h, U_0, J_h)$. *For the wave operators* $W_+(U_0, U_h, J_h^*)$ *we have the following result:*

$$W_+(U_0, U_h, J_h^*)\varphi \to W_+(U_0, U, J^*)\varphi \qquad \textit{strongly in } \mathcal{H}_0$$

*for every* $\varphi \in \mathcal{H}_0$ *with* $\operatorname{supp}\varphi \cap B_r = \emptyset$, *and the same convergence holds for* $W_-(U_0, U_h, J_h^*)$.

*Proof.* Let $W_h = W_+(U_h, U_0, J_h)$ and $W = W_+(U, U_0, J)$. Let $\varphi \in \mathcal{H}_0$ and $\varepsilon > 0$. Then there exists $\tilde{\varphi} \in C_0^\infty(\mathbf{R}^n) \times C_0^\infty(\mathbf{R}^n)$ such that $\|\varphi - \tilde{\varphi}\|_{E_0(\mathbf{R}^n)} \le \varepsilon$. As $W_h$ is unitary, we have $\|W_h\varphi - W_h\tilde{\varphi}\|_{E_h(\mathbf{R}^n)} \le \varepsilon$.

We showed in Theorem 5.3.1 that there exist $T > 0$ and $\psi$ such that $\|W_h\tilde{\varphi} - U_h(-T)\psi\|_{E_h(\mathbf{R}^n)} \le \varepsilon$ for every $h$. Note that, by construction, $\psi = 0$ on $B_r$. We apply Theorem 4.2.5 and setting $(u_h, \dot{u}_h) = U_h(-T)\psi$ and $(u, \dot{u}) = U(-T)\psi$ we get that

$$\|Du_h\|^2_{L^2(\mathbf{R}^n)} + \|u_h\|^2_{L^2_{\mu_h}(\mathbf{R}^n)} \to \|Du\|^2_{L^2(\mathbf{R}^n)} + \|u\|^2_{L^2_\mu(\mathbf{R}^n)}$$

$$\dot{u}_h \to \dot{u} \quad \text{strongly in } L^2(\mathbf{R}^n).$$

Hence we deduce that (5.3.5) holds and also that $w_h^2 \to w^2$ strongly in $L^2(\mathbf{R}^n)$.

Let $W_{0,h} = W_+(U_0, U_h, J_h^*)$ and $W_0 = W_+(U_0, U, J^*)$. Let $\eta = (\eta^1, \eta^2) \in \mathcal{H}_0$ with $\operatorname{supp}\eta \cap B_r = \emptyset$. Then $\eta \in \mathcal{H}$ and $\eta \in \mathcal{H}_h$ for every $h$. As $W_{0,h} = (W_h)^*$ and $W_0 = W^*$, we have

$$\langle W_{0,h}\eta, \varphi\rangle_{\mathcal{H}_0} = \langle (W_h)^*\eta, \varphi\rangle_{\mathcal{H}_0} = \langle \eta, W_h\varphi\rangle_{\mathcal{H}_h}.$$

So, using the definition of the scalar product in $\mathcal{H}_h$, we have

$$\langle W_{0,h}\eta, \varphi\rangle = \int_\Omega D\eta^1 Dw_h^1\, dx + \int_\Omega \eta^1 w_h^1\, d\mu_h + \int_\Omega \eta^2 w_h^2\, dx$$

and the term containing $\mu_h$ disappears since $\operatorname{supp}\eta \cap B_r = \emptyset$. Now the convergence results we obtained for $w_h = (w_h^1, w_h^2)$ imply that

$$\langle W_{0,h}\eta, \varphi\rangle \to \langle W_0\eta, \varphi\rangle.$$

Since

$$\|W_{0,h}\eta\|_{E_0(\mathbf{R}^n)} = \|\eta\|_{E_h(\mathbf{R}^n)} = \|\eta\|_{E(\mathbf{R}^n)} = \|W_0\eta\|_{E_0(\mathbf{R}^n)},$$

from the weak convergence we deduce that

$$\|W_{0,h}\eta - W_0\eta\|_{E_0(\mathbf{R}^n)} \to 0.$$

□

In the particular case of exterior domains mentioned in the introduction we have the following result.

**Theorem 5.3.3.** *Let* $(\Omega_h)$ *be a sequence of open connected subsets of* $\mathbf{R}^n$ *such that* $K_h = \mathbf{R}^n \setminus \Omega_h$ *are contained in a ball* $B_r$*, independent of* $h$. *Assume that there exists a measure* $\mu$*, absolutely continuous with respect to the Newtonian capacity, such that the solutions* $w_h$ *of (5.0.2) weakly converge in* $H^1(\mathbf{R}^n)$ *to the solution* $w$ *of (5.0.3). Assume in addition that* $A = -\Delta + \mu$ *has an absolutely continuous spectrum. Let* $U_h$ *and* $U$ *be the unitary groups giving the solution of (5.0.1) and (5.1.2), respectively. Then the wave operators* $W_+(U_h, U_0, J_h)$ *converge to the wave operator* $W_+(U, U_0, J)$ *in the sense that for every* $\varphi \in \mathcal{H}_0$*, if we set* $(w_h^1, w_h^2) = W_+(U_h, U_0, J_h)\varphi$ *and* $(w^1, w^2) = W_+(U, U_0, J)\varphi$*, we have*

$$Dw_h^1 \rightharpoonup Dw^1 \quad w\text{-}L^2(\mathbf{R}^n) \quad \textit{and} \quad w_h^2 \to w^2 \quad s\text{-}L^2(\mathbf{R}^n).$$

*Moreover,*

$$\|Dw_h^1\|^2_{L^2(\mathbf{R}^n)} \to \|Dw^1\|^2_{L^2(\mathbf{R}^n)} + \|w^1\|^2_{L^2_\mu(\mathbf{R}^n)}, \tag{5.3.6}$$

*and similar results hold for* $W_-(U_h, U_0, J_h)$. *For the wave operators* $W_+(U_0, U_h, J_h^*)$ *we have the following result:*

$$W_+(U_0, U_h, J_h^*)\varphi \to W_+(U_0, U, J^*)\varphi \qquad \textit{strongly in } \mathcal{H}_0$$

*for every* $\varphi \in \mathcal{H}_0$ *with* $\operatorname{supp}\varphi \cap B_r = \emptyset$*, and the same convergence holds for* $W_-(U_0, U_h, J_h^*)$.

*Proof.* As remarked in the introduction the hypothesis made on the sequence $(\Omega_h)$ that the solutions of (5.0.2) converge weakly in $H^1(\mathbf{R}^n)$ to the solution of (5.0.3) is equivalent to the $\gamma$-convergence of the measures $\mu_h = \mu_{\Omega_h}$ to the measure $\mu$. Since the solution of the relaxed wave equation corresponding to $\mu_{\Omega_h}$ coincides with the solution of (5.0.1), also the wave operators for this problem coincide with the wave operators on the exterior domain $\Omega_h$ with the usual definition. Hence, in order to get the convergence of the wave operators, it suffices to apply the previous theorem to the measures $\mu_h = \mu_{\Omega_h}$. □

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