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Methods of Ordinary Differential Equations for Multi-Dimensional Problems in Calculus of Variations

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Introduction

The results presented here concern two problems that seem to interact enough to justify the title of this thesis. That is, a variational problem, (P_1), of the type

$$(P_1) \qquad \text{Minimize} \int_\Omega f(\nabla u(x)) + g(x, u(x))\, \mathrm{d}x$$

and a differential equation, (P_2), of the type

$$(P_2) \qquad x' = \nabla v(x).$$

Both problems are studied in $\mathbb{R}^n$ and v is, essentially, a convex function.

There are several papers concerned with problems of type (P_1), that arise from various applicative fields. When f is convex, the Direct Method of the Calculus of Variations can be applied and, so, the existence of a solution is assured under standard conditions. Besides the question of the existence of solutions, some further problems that are of interest even in the convex case are, for example, qualitative results on the solutions ([K.],[S.]), the equivalence of two problems ([B.S.]) and the validity of the Euler-Lagrange equations. When f is not necessarily convex, the Direct Method cannot be applied. Moreover, the existence of a solution is not guaranteed, even if the problem has a very simple formulation ([C.C.]). Still, for some particular non-convex problems arising from shape optimization, and apart from many papers concerned with numerical aspects, there are some existence results (e.g. [C.P.T.],[Ce.3],[Tr.]) and some results establishing the validity of the Euler-Lagrange equations ([C.P.]). One of the techniques used in dealing with the non-convex case is computing the integral on Ω by describing at first Ω as a collection of trajectories of the ordinary differential equation $x' = \nabla v(x)$, where v is the candidate solution, and then using the co-area formula. In this context, this is the link between the two problems stated above.

Most of the existence results cited here give solutions that are convex functions. This is why we begin to study (P_2) for the case of a convex v. If we only know that v is convex, the gradient of v does not necessarily exist everywhere. The classical way to deal with (P_2) in this case is to change it into the differential inclusion given by the subgradient, namely $x' \in \partial v$. However this approach does not seem to be helpful for the purpose of integrating on Ω, since we really need trajectories of $x' = \nabla v(x)$ in order to be able to perform the change of variables under the integral sign. That is why the approach presented here is different than the usual one.

The result presented in the first chapter is contained in [V.1]. It deals with the following problem :

$$\text{Minimize} \int_{\Omega} (h(|\nabla u(x)|) + u(x)) \, \mathrm{d}x$$

on $u \in W_0^{1,1}(\Omega)$, where Ω is an open, bounded and convex subset of $\mathbb{R}^n$ and h is not necessarily convex. There are several papers studying this type of problem ([Ce.3],[C.P.T.],[G.K.R.],[K.S.W.],[M.T.]). In particular, in [Ce.3], an existence result has been proved, when $n = 2$, under assumptions linking the properties of the function h to the properties of Ω. There the proof makes substantial use of the fact that Ω is a subset of $\mathbb{R}^2$. The result presented in this chapter holds for $\Omega \subset \mathbb{R}^n$ and it generalises the one in [Ce.3]. The main result is that, when a property connecting the geometry of $\partial\Omega$, the width of Ω and the subdifferential of h is satisfied, there exists a solution to the problem stated above. Also, this result is valid for a larger class of convex sets than the one considered in [Ce.3]. The solution turns out to be $-cd(x, \partial\Omega)$, where c is a constant depending on h. As stressed above, the solution is convex and the result is achieved by integrating on the trajectories of $x' = -c\nabla d(x, \partial\Omega)$. Notice that $d(x, \partial\Omega)$ in general will not be C^1. To use the co-area formula, one needs a lipschitzean mapping and has to compute jacobians. That is why, in order to obtain precise estimates, one must deal with principal curvatures of surfaces in $\mathbb{R}^n$ and n-th order determinants. The proof of the main result is rather technical, long and it uses some notions of differential geometry.

The results presented in the second chapter are based on [T.V.], a joint paper with Giulia Treu. In [B.S.] the following result is presented : set $I(u) \doteq \int_\Omega \frac{1}{2}|\nabla u|^2 + (\lambda u - f)u$, where $\lambda > 0$ and $f \in H^{1,\infty}(\Omega)$, and consider the following problems

$$\text{(1)} \qquad \text{Minimize } I(u) \text{ on } \{u \in H_0^1(\Omega) : |u(x)| \leq d(x, \partial\Omega)\}$$

and

$$\text{(2)} \qquad \text{Minimize } I(u) \text{ on } \{u \in H_0^{1,\infty}(\Omega) : |\nabla u(x)| \leq 1\}.$$

Under the assumption that $2\lambda \geq |\nabla f|_\infty$, (1) and (2) are equivalent. It appears interesting that a restriction on the gradient may be equivalent with a restriction on the function.

The problem considered in [B.S.] arises from the elasto-plastic torsion problem ([E.T.],[B.St.],[B.2]).

We consider the functional

$$I(u) = \int_\Omega f(\nabla u(x)) + g(x, u(x))\, \mathrm{d}x$$

defined on $u_0 + W_0^{1,p}(\Omega)$, where we ask only the convexity of f and no further regularity conditions on f, we ask g to be quite regular, C^1 in u and g'_u to be lipschitzean and we impose some lipschitz condition on u_0. We consider two problems : the first one is to minimize I on the set of those u such $\nabla u \in K$ a.e., where K is a compact and convex subset of $\mathbb{R}^n$ such that $0 \in \text{int}K$ and the second problem is to minimize I with suitable obstacles for u. Under an assumption involving the second derivatives of g, we prove an equivalence result that generalizes the one obtained in [B.S.]. Notice that we do not impose that $u_0 = 0$ and that, while the problem considered in [B.S.] admits a unique solution, in our case, the problem may lack uniqueness. Finally, when the problem is more regular, we give other equivalence results under weaker assumptions.

The results presented in the third chapter are contained in [C.V.], a joint paper with Arrigo Cellina. We study the trajectories of the differential equation $x' = \nabla v(x)$ where v is a convex function. As showed above, the motivation for this problem comes from the Calculus of Variations. In several papers ([C.P.T.],[Ce.3],[C.P.],[Tr.],[V.1]) the basic tool used to infer the truth of properties like the existence of minima or the validity of the Euler Lagrange equations was integration along the trajectories of $x' = \nabla v(x)$, where v is the candidate solution. Here the purpose is integrating over Ω by integrating along the trajectories of a differential equation; hence sets of measure zero do not count. This explains the different emphasis of this paper with respect to ways previously used in dealing with discontinuous right hand sides. In our approach we want $x'(t) = \nabla v(x(t))$ to hold for a.e. t and pay the price of losing a set of initial data having measure zero. The main results of this chapter are : given an open region Ω, for a.e. initial data (P_v) admits a unique (maximal) solution, which is defined on an open interval, and the trajectories of (P_v) fill Ω in a one to one way, on the complement of a null-measure set.

Besides convex mappings, we deal with mappings whose composition with a monotone function is convex and prove the same results stated above. The motivation for considering this class of mappings comes from the Calculus of Variations : in fact, it is known ([K.],[S.]) that the solution, v, to the basic problem

$$\min \int_{\Omega} (|\nabla u(x)|^2 + u(x))dx \ , \ u_{|\partial\Omega} = 0$$

on a convex domain Ω, is such that the mapping $x \rightarrow -(-v)^{\frac{1}{2}}$ is convex. Hence our results apply in particular to this solution v.

The results presented in the fourth chapter are contained in [V.2]. It treats the non-uniqueness set for the Cauchy Problem

$$(CP)(x_0) \begin{cases} x'(t) \in T(x(t)) \\ x(0) = x_0 \end{cases}$$

where T is a set-valued map. The main results are that, under some assumption on T, the set N, consisting of those x_0 for which $(CP)(x_0)$ admits at least two solutions has null measure, and the closure of N along the trajectories of (CP) has null measure, too. It was proved in [Ce.2] that, when T is maximal monotone, the set N has null measure. When T is the subgradient of some lipschitz function having convex sub-level sets, there is ([C.T.]) a finer result on the set of non-uniqueness. The hypothesis made on T in this chapter is weaker than the ones in [Ce.2] and [C.V.]. The class of those T satisfying the assumptions requested here contains both the class of lipschitzean mappings and the class of maximal monotone mappings. In particular, this result applies for the inclusion

$$(CP_-)(x_0) \begin{cases} x'(t) \in D^-(u(x(t))) \\ x(0) = x_0 \end{cases}$$

where u is a semiconvex function i.e. there is $c > 0$ such that $u(x) + \frac{1}{2}c|x|^2$ is convex. It appears that the set of semiconvex functions is particularly important when studying viscosity solutions ([E.]).

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Chapter 1. A Variational Problem on Subsets of $\mathbb{R}^n$

This chapter provides an existence result for a minimum problem in the scalar case of the Calculus of Variations. When one does not assume convexity in the variable gradient, the functional to be minimized is not lower semicontinuous. Hence, one cannot pass to the limit along minimizing sequences but one has to follow a constructive approach. The following problem, arising from shape optimization, has been considered by several authors ([Ce.3], [G.K.R.], [K.S.W.], [M.T.]) under different assumptions on the nonlinearity h :

$$\text{Minimize } \int_\Omega (h(|\nabla u(x)|) + u(x))\, dx$$

on $u \in W_0^{1,1}(\Omega)$, where Ω is an open, bounded and convex subset of $\mathbb{R}^2$ and h is not necessarily convex. As stressed in the introduction, an existence theorem has been proved in [Ce.3]. Here, the same problem is considered in the n-dimensional case and the results obtained are more general than those stated in [Ce.3]. The main result gives a condition under which the problem has a solution and the solution is given explicitly. The nature of the condition is geometric and, for given Ω and h, one may explicitly check if the condition is satisfied. However, the n-dimensional case is more complicated than the 2-dimensional one and the proofs are rather technical.

Notation and Preliminary Results

We denote by $|x|$ the euclidean norm of x. For a point x and a set U in some $\mathbb{R}^m$, by $d(x, U)$ we mean the distance from x to U i.e. $d(x,U) = \inf\{|x-y|, y \in U\}$. For a C^1 mapping $f : \mathbb{R}^m \to \mathbb{R}^k$, call $\mathbf{d}f$ the differential of f. For a matrix $A \in \mathbb{R}^{m\times k}$ the jacobian of A, $\mathbf{J}A$, satisfies $\mathbf{J}A = |\det A|$ when $m = k$, and, when

$m \geq k$,

$$\mathbf{J}A = \{(\sum A_i^2)^{1/2} : A_i \text{ is a } k\text{-order minor of } A\}.$$

For a C^1 function f, we set the jacobian of f, $\mathbf{J}f$, to be $\mathbf{J}(\nabla f)$ ([E.G.]).

In what follows we shall need some geometric considerations, in particular the notion of principal curvatures of a surface in $\mathbb{R}^n$. As we shall deal only with parametrized surfaces, we will present just this special case. A **parametrized $n-1$-surface** in $\mathbb{R}^n$ is a smooth map $\varphi : U \to \mathbb{R}^n$, $\varphi = (\varphi_1, \ldots, \varphi_n)$, where U is a connected open set in $\mathbb{R}^{n-1}$, which is **regular**, i.e. $\mathrm{d}\varphi(p)$ has rank $n-1$ for all $p \in U$ [Th., p.110]. By smoothness here we mean C^2 functions. Image $\mathrm{d}\varphi(p)$ is the **tangent hyperplane** to φ at the point $p \in U$. Consider, in the tangent hyperplane, the $n-1$ linearly independent vectors $\mathbf{v}_i$, which are $\mathbf{v}_i = \frac{\partial \varphi}{\partial \xi_i} = (\frac{\partial \varphi_1}{\partial \xi_i}, \ldots, \frac{\partial \varphi_n}{\partial \xi_i})$ [Th., p.114-115]. The normal at a point p, which we denote by $\mathbf{n}(p)$, is a unit vector, orthogonal to the tangent hyperplane, unique up to the sign, and we choose it to be such that

$$\det \begin{pmatrix} \mathbf{v}_1(p) \\ \vdots \\ \mathbf{v}_{n-1}(p) \\ \mathbf{n}(p) \end{pmatrix} > 0. \tag{1.1}$$

When the surface is the boundary of a convex set, this definition of the normal gives the **inward** normal. Observe that, as $\mathbf{n} \cdot \mathbf{n} = 1$, we have that $\mathbf{dn} \cdot \mathbf{n} = 0$. This means that $\mathbf{dn}$ is a linear mapping for which the tangent hyperplane is invariant. Call $T(p)$ the tangent hyperplane. The linear map

$$-\mathbf{dn} : T(p) \to T(p)$$

is the **Weingarten map** at $p \in U$. Its eigenvalues are called the **principal curvatures** of φ at p. In the case of a convex surface, one can prove that the principal curvatures are all (not necessarily strictly) positive. Denote by $\mathbf{V}$ the

$(n-1)\times n$ matrix where the i-th row is $\mathbf{v}_i$. We want to write $-\mathbf{dn}$ in the system of coordinates in T given by the matrix $\mathbf{V}$. Call Γ the $(n-1)\times(n-1)$ matrix of $-\mathbf{dn}$ in the new system of coordinates. We have

$$-\mathbf{dn} = \Gamma\mathbf{V}.$$

Setting $\mathbf{V}^t$ to be the transposed matrix of $\mathbf{V}$, we obtain

$$-\mathbf{dn}\cdot\mathbf{V}^t = \Gamma\cdot\mathbf{V}\mathbf{V}^t.$$

As $\mathbf{V}\cdot\mathbf{V}^t$ is invertible (the rank of $\mathbf{V}$ is maximal),

$$\Gamma = (-\mathbf{dn}\cdot\mathbf{V}^t)(\mathbf{V}\cdot\mathbf{V}^t)^{-1}.$$

Hence the principal curvatures are solutions of the equation

$$\det(-\mathbf{dn}\cdot\mathbf{V}^t - \lambda\mathbf{V}\cdot\mathbf{V}^t) = 0. \tag{1.2}$$

In the case the surface is given as the graph of a function, we may give an explicit equation whose solutions are principal curvatures. Let $\psi : U \to \mathbb{R}$ be a smooth function, and define $\varphi : U \to \mathbb{R}^n$ by $\varphi(p) = (p, \psi(p))$. In this case, the vectors $\mathbf{v}_i$ become $\mathbf{v}_i = (0,\ldots,1,0,\ldots,\frac{\partial}{\partial\xi_i}\psi)$ where 1 is in the i-th spot. The normal is

$$\mathbf{n} = \frac{1}{\omega}\left(-\frac{\partial\psi}{\partial\xi_1},\ldots,-\frac{\partial\psi}{\partial\xi_{n-1}},1\right)$$

where $\omega = \left(1+\left(\frac{\partial\psi}{\partial\xi_1}\right)^2+\ldots\left(\frac{\partial\psi}{\partial\xi_{n-1}}\right)^2\right)^{1/2}$. For simplicity, denote by H the hessian matrix of ψ and F the matrix which has the element on the i-th row and j-th column equal to $\frac{\partial\psi}{\partial\xi_i}\cdot\frac{\partial\psi}{\partial\xi_j}$. Then the element i,j of the matrix $\mathbf{V}\cdot\mathbf{V}^t$ equals $\mathbf{v}_i\cdot\mathbf{v}_j = \delta_i^j + \frac{\partial\psi}{\partial\xi_i}\frac{\partial\psi}{\partial\xi_j}$, where $\delta_i^j = 1$ if $i=j$ and $\delta_i^j = 0$ if $i\neq j$. So $\mathbf{V}\cdot\mathbf{V}^t = I+F$. The element i,j of the matrix $-\mathbf{dn}\mathbf{V}^t$ is

$$\gamma_{ij} = \frac{\partial}{\partial\xi_i}(-\mathbf{n})\cdot\mathbf{v}_j = \frac{\partial}{\partial\xi_i}\left(\frac{1}{\omega}\frac{\partial\psi}{\partial\xi_j}\right)\cdot 1 + \frac{\partial}{\partial\xi_i}\left(\frac{-1}{\omega}\right)\cdot\frac{\partial\psi}{\partial\xi_j}.$$

Computing this last quantity, one obtains

$$-\mathbf{dnV}^t = \frac{1}{\omega}H.$$

so formula (1.2) becomes

$$\det(\frac{1}{\omega}H - \lambda(I+F)) = 0. \tag{1.3}$$

Observe also that (1.3), in the case $\mathbf{d}\psi = 0$, becomes

$$\det(H - \lambda I) = 0. \tag{1.3'}$$

We shall also need the notion of focal point. Consider a parametrized surface $\varphi : U \to \mathbb{R}^n$ and the family of smooth maps $\varphi_s : U \to \mathbb{R}^n$ given by

$$\varphi_s(p) = g(p,s) = \varphi(p) + s\mathbf{n}(p) \text{ for } s \in \mathbb{R}.$$

Fix $p \in U$ and call $\beta : \mathbb{R} \to \mathbb{R}^n$ the map given by $\beta(s) = \varphi(p) + s\mathbf{n}(p)$. A point $z \in$ Image β is said to be a **focal point of** φ **along** β if $z = \beta(s_0)$, where s_0 is such that the map φ_{s_0} is not regular at p [Th., p.132]. The position of the focal points is well determined, in spite of the fact that the normal may be chosen in two different ways.

Lemma 1.1

Let $\varphi : U \to \mathbb{R}^n$ be a parametrized surface and call $g : U \times \mathbb{R} \to \mathbb{R}^n$ the function given by

$$g(p,s) = \varphi(p) + s\mathbf{n}(p).$$

Then $\mathbf{J}g$ may be written in the form

$$\mathbf{J}g(p,s) = \nu(p) \prod_{i=1,n-1} |1 - s\lambda_i| \tag{1.4}$$

where λ_i are the principal curvatures at p and

$$\nu(p) = \det \begin{pmatrix} \mathbf{v}_1(p) \\ \vdots \\ \mathbf{v}_{n-1}(p) \\ \mathbf{n}(p) \end{pmatrix} \tag{1.5}$$

■

Proof

$$\mathbf{J}g(p,s) = |\det \begin{pmatrix} \mathbf{d}\varphi + s\mathbf{dn} \\ \mathbf{n} \end{pmatrix}|$$

where $\mathbf{d}\varphi + s\mathbf{dn}$ is a $(n-1)\times n$ matrix and $\mathbf{n}$ is a row vector. So $\mathbf{J}g(p,s)$ is a polynomial in s. Moreover, $\mathbf{J}g(p,s) = 0$ if and only if there exists $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} \neq \mathbf{0}$ such that

$$\begin{pmatrix} \mathbf{d}\varphi + s\mathbf{dn} \\ \mathbf{n} \end{pmatrix} \cdot \mathbf{v} = \mathbf{0}.$$

That is $(\mathbf{d}\varphi + s\mathbf{dn}) \cdot \mathbf{v} = \mathbf{0}$ and $\mathbf{n} \cdot \mathbf{v} = 0$. This last equality is satisfied when $\mathbf{v} \in T(p)$, the tangent hyperplane at p. So $\mathbf{J}g(p,s) = 0$ if and only if there exists a $\mathbf{v} \in T(p)$ such that $(\mathbf{d}\varphi + s\mathbf{dn}) \cdot \mathbf{v} = \mathbf{0}$. But $\mathbf{d}\varphi + s\mathbf{dn} = \mathbf{d}\varphi_s$ and $\mathbf{d}\varphi_s \cdot \mathbf{v} = \mathbf{0}$ for some $\mathbf{v} \in T(p)$ if and only if φ_s is not regular at p. So $\mathbf{J}g(p,s) = 0$ if and only if $\varphi(p) + s\mathbf{n}(p)$ is a focal point along the normal line through p. From Theorem 1 in [Th.], pag. 132, we see that focal points have the form $\varphi(p) + \frac{1}{\lambda_i}\mathbf{n}(p)$ where λ_i are the non-zero principal curvatures at p. Since $\mathbf{J}g(p,s)$ is a polynomial in s, it has the form

$$\mathbf{J}g(p,s) = \overline{\nu}(p) \prod |s - \frac{1}{\lambda_i}|$$

where the product is made over the non-zero principal curvatures at p. One can easily see that we arrive to the form requested in the conclusion of our lemma. Moreover, the map ν appearing in (1.4) is $\mathbf{J}g(p,0)$ that is :

$$\nu(p) = \mathbf{J}g(p,0) = \det \begin{pmatrix} \mathbf{v}_1(p) \\ \vdots \\ \mathbf{v}_{n-1}(p) \\ \mathbf{n}(p) \end{pmatrix}$$

∎

Notice that the determinant appearing in **Lemma 1.1** is the same as in (1.1), it is strictly positive and depends only on p.

Notice also that in the product we have, in fact, only factors that correspond to non-zero principal curvatures.

We also need the following technical lemma :

Lemma 1.2

Let $\{x_i\}_{i=1,n}$ *be* n *strictly positive numbers and* $0 \leq T \leq \min_{i\in\{1,\ldots,n\}} x_i$. *Then the function* $u : [0,T] \to \mathbb{R}$ *given by*

$$u(t) = \frac{\displaystyle\int_t^T \prod_{i=1}^n (x_i - s)ds}{\displaystyle\prod_{i=1}^n (x_i - t)}$$

is decreasing. ∎

Proof Call $v(t) = \prod_{i=1}^n (x_i - t)$. Observe that u is well defined, that is it has a limit in T. If $T < \min x_i$ this is obvious and if v has T as a k-order root, then

$$\lim_{t\nearrow T} u(t) = \lim_{t\nearrow T} \frac{-v(t)}{v'(t)} = \lim_{t\nearrow T} \frac{-v(t)^{(k-1)}}{v(t)^{(k)}} = 0.$$

We have

$$u'(t) = -\frac{v^2(t) + v'(t)\displaystyle\int_t^T v(s)ds}{v^2(t)}.$$

We want to prove that

$$v^2(t) + v'(t)\int_t^T v(s)ds \geq 0, \text{ that is}$$

$$w(t) := \frac{v^2(t)}{v'(t)} + \int_t^T v(s)ds \leq 0, \text{ since } v \text{ is decreasing.}$$

Notice that $w(T) = \lim_{t \nearrow T} \frac{v^2(t)}{v'(t)}$ exists and it is ≤ 0. So if we prove that w is increasing, we are done.

$$w' = \frac{2vv'^2 - v''v^2}{v'^2} - v = \frac{v}{v'^2}(v'^2 - vv'').$$

We have to prove that $v'^2 \geq vv''$. Computing, one obtains

$$v'^2(t) - v(t)v''(t) = \sum_{i=1}^{n} \prod_{j \neq i} (x_j - t)^2 \geq 0.$$

■

We have the following

Proposition 1.3

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an invertible matrix and $B = (b_{ij}) \in \mathbb{R}^{(n-1) \times n}$ be such that

$$BA = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} = (I_{n-1}, \mathbf{0}). \tag{1.6}$$

Then

$$\mathbf{J}B = \frac{1}{\mathbf{J}A}\Big(\sum_{i=1,n} a_{in}^2\Big)^{1/2}.$$

■

Proof Call $A' = (a'_{ij})$ the inverse of A. We have that $(A')^{-1} = A$, so the element a_{ij} may be computed as follows :

Denote the $n-1$-order minor of A' obtained by removing the j-th row and the i-th column by A'_{ji}. Then

$$|a_{ij}| = |\frac{1}{\det A'} A'_{ji}|.$$

Multiply now the equality (1.6) by A' to obtain

$$B = (I_{n-1}, \mathbf{0}) \cdot A'$$

$$B = \begin{pmatrix} a'_{1,1} & \cdots & \cdots & a'_{1,n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ a'_{n-1,1} & \cdots & \cdots & a'_{n-1,n} \end{pmatrix}$$

So the minor obtained from B by removing the i-th column equals the minor of A' obtained by removing the n-th row and the i-th column, i.e. A'_{ni}. That is

$$\mathbf{J}B = \Big(\sum_{i=1,n} (A'_{ni})^2\Big)^{1/2} = |\det A'| \Big(\sum_{i=1,n} a_{in}^2\Big)^{1/2}$$

$$\mathbf{J}B = \frac{1}{\mathbf{J}A} \Big(\sum_{i=1,n} a_{in}^2\Big)^{1/2}.$$

■

Main Results

Let Ω be an open, bounded and convex subset of $\mathbb{R}^n$, $n \geq 2$. We shall assume that the map $h : [0, +\infty) \to [0, +\infty]$ satisfies the following

Assumption H

$$\sup\{t \geq 0 : h(t) = 0\} = \max\{t \geq 0 : h(t) = 0\} < +\infty.$$

Our purpose is to solve the following problem :

(P) Minimize $\int_\Omega (h(|\nabla u(x)|) + u(x))\, dx$ on $u \in W_0^{1,1}(\Omega)$.

We shall need the following definitions :

Definitions of ρ and Λ

Set $\rho = \max\{t \geq 0 : h(t) = 0\}$ and call A the set of supporting linear functions at ρ, $A = \{a : h(s) \geq a(s-\rho), \text{ for every } s \geq 0\}$. We have $0 \in A$. Call $\Lambda = \sup A$, $0 \leq \Lambda \leq \infty$.

Call, for $x \in \overline{\Omega}$, $\Pi(x) = \{y \in \partial\Omega : |x - y| = d(x, \partial\Omega)\}$ the projection of x in $\partial\Omega$. Notice that $\Pi : \overline{\Omega} \to \partial\Omega$ is an upper semicontinuous multivalued map. Denote also for $y \in \partial\Omega$, $\Pi^{-1}(y) = \{x \in \overline{\Omega} : y \in \Pi(x)\}$. $\Pi^{-1}(y)$ is closed and $y \in \Pi^{-1}(y)$. We have the following preliminary result :

Lemma 1.4

If there exists a function α in $L^\infty(\Omega)$ such that :

$$(i)\ 0 \leq \alpha(x) \leq \Lambda \ a.e.\ x \in \Omega$$

$$(ii)\ \int_\Omega (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta(x)\rangle + \eta(x))\, dx = 0 \ \textit{for every } \eta \textit{ in } C_0^\infty(\Omega)$$

then the function $u : \Omega \to \mathbb{R}$ given by $u(x) = -\rho d(x, \partial\Omega)$ is a solution to the problem (**P**). ■

Proof a) The map $x \to d(x, \partial\Omega)$ is differentiable a.e. and its gradient is (a.e.) $-\mathbf{n}(\Pi(x))$ [G.T., p.354]. In particular, $\Pi(x)$ is single valued for a.e. x in Ω. Then, a.e. on Ω, $\nabla u(x) = -\rho\mathbf{n}(y), y$ the unique point in $\Pi(x)$. In the case $\rho > 0, \dfrac{\nabla u(x)}{|\nabla u(x)|} = -\mathbf{n}(y)$.

Let α be a function in $L^\infty(\Omega)$ satisfying (i) and (ii). Then, for any vector v, when $\rho > 0$

$$h(|\nabla u(x) + v|) = h(|\nabla u(x)| + |\nabla u(x) + v| - |\nabla u(x)|) \geq$$

$$\geq h(|\nabla u(x)|) + \alpha(x)(|\nabla u(x) + v| - |\nabla u(x)|) \geq$$

$$\geq h(|\nabla u(x)|) + \alpha(x)\langle \frac{\nabla u(x)}{|\nabla u(x)|}, v\rangle = h(|\nabla u(x)|) + \alpha(x)\langle -\mathbf{n}(y), v\rangle$$

and, for $\rho = 0$,

$$h(|v|) \geq \alpha(x)|v| \geq \alpha(x)\langle -\mathbf{n}(y), v\rangle \text{ for a.e. } x \in \Omega.$$

Hence, for every ρ and for every function η in $W_0^{1,1}(\Omega)$, we have

$$\int_\Omega (h(|\nabla u + \nabla\eta|) + (u + \eta))\, dx \geq$$

$$\geq \int_\Omega (h(|\nabla u|) + u)\, dx + \int_\Omega (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta\rangle + \eta)\, dx.$$

By approximating a function η in $W_0^{1,1}(\Omega)$ by standard mollifiers, one sees that (ii) must be satisfied for every function η in $W_0^{1,1}(\Omega)$, so

$$\int_\Omega (h(|\nabla u + \nabla\eta|) + (u + \eta))\, dx \geq \int_\Omega (h(|\nabla u|) + u)\, dx$$

for all η in $W_0^{1,1}(\Omega)$, that is

$$\int_\Omega (h(|\nabla v|) + v)\, dx \geq \int_\Omega (h(|\nabla u|) + u)\, dx$$

for all $v \in W_0^{1,1}(\Omega)$. ■

So, in what follows, we shall find sufficient conditions in order to assure that u given above is a solution to our problem. We have to construct our α and to do this we shall need some assumptions on Ω, more precisely a smoothness assumption on the boundary.

Notice that, Ω being open and convex in $\mathbb{R}^n$, the Hausdorff measure on its boundary is the $n-1$-dimensional measure.

Definition 1.5

We say that Ω has **almost smooth boundary** if and only if $\partial\Omega$ may be written as

$$\partial\Omega = S \cup N$$

where

a) $\mu_{n-1}(N) = 0$

b) S is open in $\partial\Omega$ and it is of class C^2 i.e. for all $y \in S$, S is of class C^2 at y. ■

Lemma 1.6

Let Ω have almost smooth boundary. Then $\partial\Omega$ may be written as

$$\partial\Omega = (\bigcup_{i=1}^{\infty} S_i) \cup N$$

where $\mu_{n-1}(N) = 0$, S_i are open with respect to the topology of $\partial\Omega$ and two-by-two disjoint. Moreover, for each i there exists an open $U_i \subset \mathbb{R}^{n-1}$, a parametrized surface (i.e. a C^2 mapping) $\varphi_i : U_i \to \mathbb{R}^n$ such that $S_i = \varphi_i(U_i)$ and the determinant given by (1.1)*, computed for φ_i, is uniformly bounded and strictly positive on U_i.*

■

Proof $\partial\Omega = N' \cup S$ where $\mu_{n-1}(N') = 0$, S is open in $\partial\Omega$ and for all $y \in S$, S is of class C^2 at y i.e. for each $y \in S$ there exists an open neighbourhood $V_y \subset S$ and a C^2 one-to-one function $\varphi_y : U_y \to V_y$, for some open $U_y \subset \mathbb{R}^{n-1}$. Each open subset of V_y is, in turn, a parametrized surface, so we may choose V_y to be such that the determinant appearing in (1.1), computed for φ_y, is strictly positive on $\overline{U_y}$.

We want to apply the Vitali's covering theorem. To do this, notice that $\partial\Omega$ is homeomorphic with the unit sphere in $\mathbb{R}^n$, the semisphere is homeomorphic with the open unit ball in $\mathbb{R}^{n-1}$ and both homeomorphisms are lipschitz so preserve open sets and null-measure sets. Now, in $\mathbb{R}^{n-1}$ we may apply Vitali's covering theorem to obtain the covering we needed on S i.e. a countable family of open and disjoint sets, each subset of some V_y, whose union covers S up to a set of measure 0. ■

We have the following remark :

Proposition 1.7

Let $y \in \partial\Omega$ such that there exists $x \in \Omega \cap \Pi^{-1}(y)$. Then there exists a neighbourhood V_y of y in $\partial\Omega$ such that the surface V_y may be represented as the graph of a convex function, for a suitable choice of coordinates. ∎

Proof There exists an open ball around x, contained in Ω, such that each half-line issuing from a point of the ball in the direction of $y - x$ intersects $\partial\Omega$ exactly in one point. That it is at least one point is obvious and that there cannot be two points in the intersection follows from the convexity of Ω and the fact that x is an interior point. Choosing now a system of coordinates centered at y, with $n-1$ directions contained in the orthogonal hyperplane to $x - y$ and one direction as $x - y$, one may write, locally, $\partial\Omega$ as the graph of a function. The convexity of this function follows from the convexity of Ω. ∎

Lemma 1.8

We have, for all $y \in \partial\Omega$:

(i) When $\Pi^{-1}(y) \neq \{y\}$, $\partial\Omega$ is differentiable at y;

(ii) There exists a unique point $y* \in \overline{\Omega}$ such that

$$\Pi^{-1}(y) = \{\lambda y + (1-\lambda)y* : \lambda \in [0,1]\};$$

(iii) If $y* \neq y$, for all $\lambda \in (0,1)$, $\Pi(\lambda y + (1-\lambda)y*) = \{y\}$. ∎

Proof (*i*) Take $x \in \Pi^{-1}(y)$, $x \neq y$.

We may apply **Proposition 1.7** so, by moving the origin of the coordinates in y and taking one axis as $x - y$ and $n-1$ axes in the normal hyperplane to $x-y$ in order to obtain an orthonormal basis, there exists an open ball B in $\mathbb{R}^{n-1}$, containing 0, such that, locally, $\partial\Omega$ may be written as $z = \varphi(\xi)$ with φ convex on B.

Since $y \in \Pi(x)$, in the new system of coordinates we may write

$$|(0,\overline{x}) - (\xi, \varphi(\xi))|^2 \geq |(0,\overline{x}) - (0,0)|^2 \text{ , where } \overline{x} = d(y,x).$$

That is

$$\varphi(\xi)^2 - 2\overline{x}\varphi(\xi) + |\xi|^2 \geq 0 \text{ , for all } \xi \in B.$$

Since φ is convex, in order to prove that it is differentiable at 0, we have to prove that $\partial\varphi(0)$, the subgradient of φ, contains only one point. Suppose there is $v \neq 0$ such that $v \in \partial\varphi(0)$. Then

$$\varphi(tv)^2 - 2\overline{x}\varphi(tv) + t^2|v|^2 \geq 0 \text{ , for all } t \geq 0, \ t \text{ sufficiently small.}$$

$$\lim_{t \searrow 0} \frac{1}{t}\varphi(tv)^2 - 2\overline{x}\frac{1}{t}\varphi(tv) + t|v|^2 \geq 0. \text{ But}$$

$$\lim_{t \searrow 0} \frac{1}{t}\varphi(tv) = \varphi'(0;v) \geq |v|^2 > 0 \text{ and } \lim_{t \searrow 0} \varphi(tv) = 0.$$

So $0 \geq 2\overline{x}|v|^2 > 0$, a contradiction.

(ii) If $\Pi^{-1}(y) = \{y\}$, $y* = y$. If there is $x \in \Pi^{-1}(y)$, $x \neq y$, by (i) above, the tangent hyperplane to $\partial\Omega$ is unique and orthogonal to $x - y$, so $\Pi^{-1}(y)$ is included in a half-line. Moreover, Ω being bounded, $\Pi^{-1}(y)$ has to be bounded. So it has to be a closed segment line contained in $\overline{\Omega}$, having one of its extreme points y. The other one will be $y*$.

(iii) If, for some $\lambda \in (0,1)$, there exists $z \in \Pi(\lambda y + (1-\lambda)y*), z \neq y$, then

$$d(y*, z) < d(\lambda y + (1-\lambda)y*, y*) + d(\lambda y + (1-\lambda)y*, z) =$$

$$= d(\lambda y + (1-\lambda)y*, y*) + d(\lambda y + (1-\lambda)y*, y) = d(y*, y) \leq d(y*, z)$$

■

Notice that, whenever $\Pi^{-1}(y) \neq \{y\}$, it may be written as

$$\Pi^{-1}(y) = \{y + \lambda \mathrm{n}(y) : \lambda \in [0, d(y, y*)]\}$$

where $y*$ is the point given by (ii) above. Call

$$l(y) = d(y, y*). \tag{1.7}$$

Notice that, Ω being bounded, $l(y)$ has to be bounded. About the function $l : \partial\Omega \to \mathbb{R}$ we have the following

Lemma 1.9

Let $\partial\Omega$ be of class C^2 in a neighbourhood of y. Then :
(i) $l(y) \leq \frac{1}{\lambda}$ where λ is any principal curvature of $\partial\Omega$ at y;

(ii) When $l(y) < \frac{1}{\lambda}$ for all λ principal curvature of $\partial\Omega$ at y, then

$$\Pi(y + l(y)\mathrm{n}(y)) \neq \{y\};$$

(iii) l is continuous at y. ■

Proof To prove this result we shall recall Theorem 2 in [Th., p.134], that claims that, given a point y on a surface and $s \in \mathbb{R}$, the map that associates to each point of the surface the distance to $y + s\mathrm{n}(y)$ attains a local minimum at y if and only if there are no focal points between y and $y + s\mathrm{n}(y)$. If we denote by $r(y)$ the smallest radius of curvature, corresponding to the greatest principal curvature, we see [Th., p.132] that this may happen if and only if $s \leq r(y)$. Denote $x = y + l(y)\mathrm{n}(y)$. Point (i) is obvious from the considerations above, since $d(x, \partial\Omega) = l(y)$.

Notice that, if $\partial\Omega$ is C^2 at y, then $r(\cdot)$ is continuous at y.

(*ii*) If $l(y) < r(y)$, there exists a point $z_0 \in \Omega$ and t_0, $l(y) < t_0 < r(y)$, such that $z_0 = y + t_0\mathbf{n}(y)$. From [Th., p.134] we see that there exists a neighbourhood V around y in $\partial\Omega$ such that $t_0 = d(z_0, y) \leq d(z_0, \theta)$ for all $\theta \in V$. It is easy to prove that

$$d(z, y) \leq d(z, \theta)$$

for all $z \in \{y + t\mathbf{n}(y) : l(y) < t < t_0\}$ and $\theta \in V$. Consider $t_n \to l(y)$, $l(y) < t_n < t_0$. Then, as $y \notin \Pi(y + t_n\mathbf{n}(y))$ from the definition of l, we have

$$d(y + t_n\mathbf{n}(y), \partial\Omega) < d(y + t_n\mathbf{n}(y), y) \leq d(y + t_n\mathbf{n}(y), \theta) \text{ for all } \theta \in V.$$

So $\Pi(y + t_n\mathbf{n}(y)) \cap V = \emptyset$. If we suppose that $\Pi(x) = \{y\}$, then, because of the upper semicontinuity of Π, for all $z_n \to x$, $y_n \in \Pi(z_n)$, $y_n \to y$. In particular, $\Pi(y + t_n\mathbf{n}(y)) \cap V \neq \emptyset$, a contradiction.

An immediate consequence of (*ii*) is that, whenever $\partial\Omega$ is of class C^2 at y, $l(y) \neq 0$.

(*iii*) Consider $y_n \to y$ such that $l(y_n) \to \bar{l}$. As $y_n \in \Pi(y_n + l(y_n)\mathbf{n}(y_n))$, the upper semicontinuity of Π implies that $y \in \Pi(y + \bar{l}\mathbf{n}(y))$, so $\bar{l} \leq l(y)$. Suppose that $\bar{l} < l(y)$. If, on a subsequence, $l(y_n) = r(y_n)$, by continuity we obtain $\bar{l} = r(y) \geq l(y) > \bar{l}$. So we must have $l(y_n) < r(y_n)$. Point (*ii*) assures us that there exists $z_n \neq y_n$, $z_n \in \Pi(y_n + l(y_n)\mathbf{n}(y_n))$. On a subsequence, z_n will converge to some z_0 and, by the upper semicontinuity of Π, $z_0 \in \Pi(y + \bar{l}\mathbf{n}(y))$. From **Lemma 1.8 (iii)** we see that $z_0 = y$ (as $\bar{l} < l(y)$).

There exists a neighbourhood V' of y such that, locally, with a suitable choice of coordinates, $\partial\Omega$ may be represented as the graph of a C^2-function (see **Proposition 1.7**).

For n such that both y_n and z_n are in V', consider the two dimensional plane P that contains y_n, z_n and $y_n + l(y_n)\mathrm{n}(y_n)$. Its intersection with the surface $\partial\Omega$ is a 1-dimensional convex curve C. The distance from any point of C to $y_n + l(y_n)\mathrm{n}(y_n)$ is greater than or equal to $l(y_n)$. Moreover there are two points on C, y_n and z_n, that have the distance to $y_n + l(y_n)\mathrm{n}(y_n)$ equal to $l(y_n)$.

Claim 1.10 *There exists a point Q_n in $V' \cap C$ such that the radius of curvature of C at Q_n is smaller than or equal to $l(y_n)$. Moreover, $Q_n \to y$.* ■

Assume **Claim 1.10**. We have a point Q_n on $\partial\Omega$ and a curve through Q_n contained in $\partial\Omega$ such that the curvature of this curve is $\geq \frac{1}{l(y_n)}$. But [Th., p.135] the curvature of curves on a surface are bounded from below and from above by the minimal, respectively the maximal principal curvatures. So, in particular, using our notation, $r(Q_n) \leq l(y_n)$. As $Q_n \to y$, passing to the limit, we obtain $r(y) \leq \bar{l}$. But we assumed that $\bar{l} < l(y)$, so $r(y) < l(y)$ which contradicts point (i) above. ■

Proof of Claim 1.10 We shall prove the following :

Let $f : [0, a] \to \mathbb{R}$ be a convex function of class C^2, $a > 0$, such that :

a) $f(0) = 0, f'(0) = 0$

b) $|(t, f(t)) - (0, 1)| \geq 1$

c) $|(a, f(a)) - (0, 1)| = |(0, f(0)) - (0, 1)| = 1.$

Then there exists a point $(t_0, f(t_0))$ on the curve $(t, f(t))$ such that the radius of curvature at $(t_0, f(t_0))$ is smaller than or equal to 1.

It is easy to see that this implies the conclusion of **Claim 1.10**.

Suppose, on the contrary, that

$$\frac{f''(s)}{\left(1 + f'^2(s)\right)^{3/2}} < 1 \text{ for all } s \in [0, a].$$

Multiplying by $f'(s)$ and integrating from 0 to t, we obtain

$$1 - \frac{1}{\left(1 + f'^2(t)\right)^{1/2}} < f(t).$$

Since $f^2(t) - 2f(t) + t^2 \geq 0$ we have $f(t) \leq 1 - (1 - t^2)^{1/2}$. These two inequalities imply that

$$f'(t) < \frac{t}{(1 - t^2)^{1/2}}.$$

Integrating again from 0 to a, we obtain

$$f(a) < 1 - \left(1 - a^2\right)^{1/2}.$$

But $f(a) = 1 - \left(1 - a^2\right)^{1/2}$ and we obtain a contradiction. ∎

Definitions of α and β

Let $x \in \Omega$ be such that $\Pi(x) = \{y\}$ and $\partial\Omega$ is of class C^2 in a neighbourhood of y. Define $\alpha(x)$ by :

(1.8.1)
$$\alpha(x) = \frac{\displaystyle\int_{d(x,\partial\Omega)}^{l(y)} \prod_{i=1}^{n-1} (1 - t\lambda_i)\, dt}{\displaystyle\prod_{i=1}^{n-1} (1 - d(x, \partial\Omega)\lambda_i)}$$

where λ_i are the principal curvatures of $\partial\Omega$ at y. For all other $x \in \Omega$ define

(1.8.2)
$$\alpha(x) = 0.$$

For all $y \in \partial\Omega$ such that $\partial\Omega$ is of class C^2 in a neighbourhood of y set $\beta(y)$ to be

(1.9)
$$\beta(y) = \int_0^{l(y)} \prod_{i=1}^{n-1} (1 - t\lambda_i)\, dt$$

where λ_i are the principal curvatures of $\partial\Omega$ at y. ∎

We may state now our first theorem :

Theorem 1.11

When Ω has almost smooth boundary, the function α defined by (1.8) *satisfies*

:

$$\int_{\Omega}(\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta(x)\rangle + \eta(x))\, dx = 0, \textit{ for every } \eta \textit{ in } C_0^\infty(\Omega).$$

■

In order to prove the theorem, we shall need some preliminary considerations.

From **Lemma 1.6** we see that there exist open sets $U_i \subset \mathbb{R}^{n-1}$, C^2 mappings $\varphi_i : U_i \to \partial\Omega$ such that $\varphi_i(U_i)$ are disjoint and

$$\mu_{n-1}(\partial\Omega \setminus \bigcup_{i\in\mathbb{N}} \varphi_i(U_i)) = 0.$$

Call $V_i = \varphi_i(U_i)$ and

$$\Omega_i = \{x \in \Pi^{-1}(V_i) : d(x, \partial\Omega) < l(\Pi(x))\}.$$

Consider the map $\overline{g_i} : U_i \times \mathbb{R} \to \mathbb{R}^n$ given by

$$\overline{g_i}(\xi, s) = \varphi_i(\xi) + s\mathbf{n}(\varphi_i(\xi)).$$

Call g_i the restriction of $\overline{g_i}$ to the set $\{(\xi, s) : \xi \in U_i, 0 \le s \le l(\varphi_i(\xi))\}$. In order to simplify the notation, we shall use $l_i(\xi)$ for $l(\varphi_i(\xi))$ and $\mathbf{n}_i(\xi)$ for $\mathbf{n}(\varphi_i(\xi))$. Notice that l_i is continuous. Ω_i is the image of $\{(\xi, s) : \xi \in U_i, 0 \le s < l_i(\xi)\}$. From **Lemma 1.8 (iii)**, one sees that the projection of a point $g_i(\xi, s)$ in Ω_i is unique, so g_i is invertible on Ω_i. Set $U_i^\varepsilon = \{\xi \in U_i : l_i(\xi) > \varepsilon\}$. Call f_i the map from Ω_i to $\mathbb{R}^{n-1}$ given by

$$f_i(g_i(\xi, s)) = \xi.$$

We have :

Lemma 1.12

(i) $\mathbf{J}f_i = \dfrac{1}{\mathbf{J}g_i}$;

(ii) $\mathbf{J}g_i(\xi, s) = \nu_i(\xi)\prod_{k=1}^{n-1}(1 - s\lambda_k)$, *where* $\{\lambda_k\}$ *are the principal curvatures at* $\varphi_i(\xi)$ *and* ν_i *is the determinant given by* (1.1) *for the surface* φ_i*;*

(iii) g_i *is lipschitz on its domain ;*

(iv) Setting Ω_i^ε *to be the image of* $\{(\xi, s) : \xi \in U_i^\varepsilon, 0 \leq s < l_i(\xi) - \varepsilon\}$, f_i *is lipschitz on* Ω_i^ε. ■

Proof In order to prove (i) we shall apply **Proposition 1.3**. From the definition of f_i, we see that $\nabla f_i \cdot \nabla g_i = (I_{n-1}, 0)$. So we need to compute the derivatives of g_i with respect to the variable s. Fix ξ and set $\delta(s) := g_i(\xi, s)$. Then we see that $\nabla\delta(s) = \mathbf{n}_i(\xi)$, a vector of norm one, that is $|\nabla\delta| = 1$. Apply now **Proposition 1.3** to obtain

$$\mathbf{J}f_i = \frac{1}{\mathbf{J}g_i}|\nabla\delta| = \frac{1}{\mathbf{J}g_i}.$$

(ii) follows immediatly from **Lemma 1.1** and **Lemma 1.9 (i)**.

From (ii) we see that $\mathbf{J}g_i \leq \nu_i$, a bounded function, so g_i is lipschitz.

Since there exists $\zeta > 0$ such that $\nu_i(\xi) > \zeta$ for all $\xi \in U_i$ (see **Lemma 1.6**), appliyng (i) and (ii) we obtain that f_i is lipschitz on Ω_i^ε. ■

Lemma 1.13

Ω_i^ε *and* Ω_i *are open in* Ω *and satisfy*

$$\mu(\Omega \backslash \bigcup_i \Omega_i^\varepsilon) \to 0 \text{ as } \varepsilon \to 0;$$

$$\mu(\Omega_i \backslash \Omega_i^\varepsilon) \to 0 \ as \ \varepsilon \to 0;$$

$$\mu(\Omega \backslash \bigcup_i \Omega_i) = 0.$$

■

Proof We may write

$$\Omega \backslash \bigcup_i \Omega_i^\varepsilon = A_0 \cup B_0 \bigcup_i C_i^\varepsilon$$

where :

A_0 is the set of the $x \in \Omega$ such that $\Pi(x)$ is not single valued, a set of measure 0.

B_0 is the set of the $x \in \Omega$ such that $\Pi(x) \in N$, also a set of measure 0, as $\mu_{n-1}(N) = 0$.

$C_i^\varepsilon = (\Omega_i \backslash \Omega_i^\varepsilon) \cup \{x \in \Omega_i : d(x, \partial\Omega) = l(\Pi(x))\}$ is the image through g_i of the set contained in $\mathbb{R}^n$ given by

$$\{(\xi, s) : \xi \in U_i, \max(0, l_i(\xi) - \varepsilon) \leq s \leq l_i(\xi)\},$$

a set that has measure $\leq \varepsilon \mu_{n-1}(U_i)$.

g_i being lipschitz, $\mu(C_i^\varepsilon) \to 0$, as $\varepsilon \to 0$.

Notice also that l is continuous on each V_i, so Ω_i is open in Ω. ■

Lemma 1.14

$\alpha \in L^\infty(\Omega)$ and, for all $x \in \Omega$,

$$0 \leq \alpha(x) \leq \sup_y \beta(y) < +\infty.$$

■

Proof Applying **Lemma 1.2** we see that $0 \leq \alpha(x) \leq \beta(\Pi(x))$ when $\beta(\Pi(x))$ is defined and $\alpha(x) = 0$ when β is not defined. Moreover, $\beta(y) \leq \int_0^{l(y)} 1\, dt = l(y) \leq d(\Omega)$, the diameter of Ω. As l is continuous, α is continuous on Ω_i. **Lemma 1.13** proves that

$$\mu(\Omega \backslash \bigcup_i \Omega_i) = 0.$$

Since α is bounded, the conclusion follows. ∎

We may now prove our theorem.

Proof of Theorem 1.11

First, we prove that, for all $i \in \mathbb{N}$,

$$\int_{\Omega_i} (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta(x)\rangle + \eta(x))\, dx = 0, \text{ for every } \eta \text{ in } C_0^\infty(\Omega).$$

Let $i \in \mathbb{N}$ and η be any function in $C_0^\infty(\Omega)$. Call

$$I = \int_{\Omega_i} (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta(x)\rangle + \eta(x))\, dx$$

Since the integrand is in $L^\infty(\Omega)$, by **Lemma 1.13** we have also

$$I = \lim_{\varepsilon \to 0} \int_{\Omega_i^\varepsilon} (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta\rangle + \eta)\, dx =$$

$$= \lim_{\varepsilon \to 0} \int_{\Omega_i^\varepsilon} (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta\rangle + \eta)\frac{1}{\mathbf{J}f_i}\mathbf{J}f_i\, dx.$$

On Ω_i^ε, f_i is a Lipschitzean map with values in U_i^ε . By the co-area formula [E.G., p.117] we have

$$\int_{\Omega_i^\varepsilon} (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta\rangle + \eta)\frac{1}{\mathbf{J}f_i}\mathbf{J}f_i\, dx =$$

$$= \int_{U_i^\varepsilon} \left(\int_{\Omega_i^\varepsilon \cap f_i^{-1}(\xi)} (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta\rangle + \eta)\frac{1}{\mathbf{J}f_i}\, dH \right) d\xi$$

where H is the one-dimensional Hausdorff measure.

The set $f_i^{-1}(\xi)$ is the segment described by

$$\zeta = \varphi_i(\xi) + s\mathbf{n}_i(\xi)$$

for $0 \leq s \leq l_i(\xi)$. On it the Hausdorff measure coincides with the Lebesgue measure. We have :

$$\int_{\Omega_i^\varepsilon \cap f_i^{-1}(\xi)} \eta \frac{1}{\mathbf{J}f_i} \, dH = \int_{\Omega_i^\varepsilon \cap f_i^{-1}(\xi)} \eta \mathbf{J}g_i \, dH =$$

$$= \int_0^{l_i(\xi)-\varepsilon} \eta(\varphi_i(\xi) + s\mathbf{n}_i(\xi))\mathbf{J}g_i(\xi, s) \, ds.$$

Denote $G_i(\xi, s) = \int_s^{l_i(\xi)} \mathbf{J}g_i(\xi, t) \, dt$.

By integrating by parts, since $\mathbf{J}g_i(\xi, s) = -\frac{d}{ds}G_i(\xi, s)$ and $\frac{d}{ds}\eta(\varphi_i(\xi) + s\mathbf{n}_i(\xi)) = \langle \mathbf{n}, \nabla\eta \rangle$, we have

$$\int_{\Omega_i^\varepsilon \cap f_i^{-1}(\xi)} \eta \, \frac{1}{\mathbf{J}f_i} \, dH = -\eta G_i|_0^{l_i(\xi)-\varepsilon} + \int_0^{l_i(\xi)-\varepsilon} \langle \mathbf{n}, \nabla\eta \rangle G_i \, ds.$$

Since $\eta|_{\partial\Omega} = 0$ and $g_i(\xi, l_i(\xi) - \varepsilon) = \varphi_i(\xi) + (l_i(\xi) - \varepsilon)\mathbf{n}_i(\xi)$,

$$\int_{\Omega_i^\varepsilon \cap f_i^{-1}(\xi)} \eta \frac{1}{\mathbf{J}f_i} \, dH = -\eta(g_i(\xi, l_i(\xi) - \varepsilon))G_i(\xi, l_i(\xi) - \varepsilon) + \int_0^{l_i(\xi)-\varepsilon} \langle \mathbf{n}, \nabla\eta \rangle G_i \, ds.$$

Then

$$\int_{\Omega_i^\varepsilon \cap f_i^{-1}(\xi)} (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta \rangle + \eta) \frac{1}{\mathbf{J}f_i} \, dH =$$

$$= \int_0^{l_i(\xi)-\varepsilon} \langle \mathbf{n}, \nabla\eta \rangle \{-\alpha \mathbf{J}g_i + G_i\} \, ds - \eta(g_i(\xi, l(\xi) - \varepsilon))G_i(\xi, l(\xi) - \varepsilon)$$

where $\mathbf{J}g_i = \mathbf{J}g_i(\xi, s)$ and the functions $\nabla\eta$ and α appearing inside the integral are computed along $\{\varphi_i(\xi) + s\mathbf{n}_i(\xi) : 0 \leq s \leq l_i(\xi) - \varepsilon\}$.

Notice that α was chosen such that $\{-\alpha \mathbf{J}g_i + G_i\} \equiv 0$, so we obtain

$$\int_{\Omega_i^\varepsilon \cap f_i^{-1}(\xi)} (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta \rangle + \eta) \frac{1}{\mathbf{J}f_i} \, dH =$$

$$= \eta(g_i(\xi, l(\xi) - \varepsilon))\, G_i(\xi, l(\xi) - \varepsilon).$$

Hence

$$I = \lim_{\varepsilon \to 0} \int_{U_i^\varepsilon} \eta(g_i(\xi, l(\xi) - \varepsilon)) G_i(\xi, l(\xi) - \varepsilon)\, d\xi.$$

Since η is uniformly bounded and $G_i(\xi, l(\xi) - \varepsilon)$ converges uniformly to 0, the integral converges uniformly to 0 so that

$$\int_{\Omega_i} (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta\rangle + \eta)\, dx = 0. \tag{1.10}$$

Since $\mu(\Omega \backslash \bigcup_i \Omega_i) = 0$, by (1.10)

$$\int_{\Omega} (\alpha(x)\langle -\mathbf{n}(\Pi(x)), \nabla\eta(x)\rangle + \eta(x))\, dx = 0, \text{ for every } \eta \text{ in } C_0^\infty(\Omega).$$

■

We have the following

Theorem 1.15

Let Ω be an open, bounded, convex subset of $\mathbb{R}^n$ with almost smooth boundary. Assume that the map h satisfies **Assumption H** *and ρ and Λ are defined as above. Assume also that, for a.e. $y \in \partial\Omega$ such that $\partial\Omega$ is of class C^2 in a neighbourhood of y,*

$$\int_0^{l(y)} \prod_{k=1}^{n-1} (1 - s\lambda_k)\, ds \leq \Lambda$$

where λ_k are the principal curvatures of $\partial\Omega$ at y and $l(y)$ is defined by (1.7).

Then the function $u : \Omega \to \mathbb{R}$ given by $u(x) = -\rho d(x, \partial\Omega)$ is a solution to the problem (**P**). ■

Proof

The conclusion is an immediate consequence of **Theorem 1.11**, **Lemma 1.14** and **Lemma 1.4**. ■

We shall give now some estimates for $\int_0^{l(y)} \prod_{k=1}^{n-1}(1-s\lambda_k)\,ds$. We have

Corollary 1.16

Let Ω be an open, bounded, convex subset of $\mathbb{R}^n$ with almost smooth boundary. If one of the following conditions is satisfied, the function $u:\Omega \to \mathbb{R}$ given by $u(x) = -\rho d(x,\partial\Omega)$ is a solution to the problem **(P)** *:*

(a) $l(y) \leq \Lambda$ a.e. in $\partial\Omega$

(b) $W(\Omega) \leq \Lambda$ where $W(\Omega) = \sup\{d(x,\partial\Omega) : x \in \Omega\}$

(c) Ω is a ball of radius $\leq n\Lambda$

(d) Ω is a cube with side of lenght $\leq 2\Lambda$. ■

Proof *(a)* We have $0 \leq 1 - s\lambda_k \leq 1$ for all $s \in [0, l(y)]$ and λ_k principal curvature at y (see **Lemma 1.9 (i)**). So

$$\int_0^{l(y)} \prod_{k=1}^{n-1}(1-s\lambda_k)\,ds \leq \int_0^{l(y)} 1\,ds = l(y).$$

(b) For a.e. $y \in \partial\Omega$, $l(y) = d(y + l(y)\mathrm{n}(y), y) = d(y + l(y)\mathrm{n}(y), \partial\Omega) \leq$

$\leq \sup\{d(x,\partial\Omega) : x \in \Omega\} = W(\Omega)$. So we may apply *(a)*.

(c) If Ω is a ball of radius R, then $l(y) = R$ for all $y \in \partial\Omega$ and all curvatures equal $\frac{1}{R}$. So the condition in **Theorem 1.15** becomes

$$\int_0^R (1-\frac{s}{R})^{n-1}\,ds \leq \Lambda$$

The integral may be computed to obtain

$$\int_0^R (1-\frac{s}{R})^{n-1}\,ds = \frac{R}{n}.$$

(d) In the case of a cube with side a, the curvatures are all 0, $l(y) = \frac{a}{2}$ for almost all $y \in \partial\Omega$, so the condition becomes

$$\int_0^{\frac{a}{2}} 1\,ds = \frac{a}{2} \leq \Lambda.$$

■

Chapter 2. On the Equivalence of Two Variational Problems

The results of this chapter are based on [T.V.], a joint paper with Giulia Treu. We generalise here a result on the equivalence of two problems, proved in [B.S.]. As we remarked in the introduction, the following result has been proved in [B.S.] set $I(u) \doteq \int_\Omega \frac{1}{2}|\nabla u|^2 + (\lambda u - f)u$, where $\lambda > 0$ and $f \in H^{1,\infty}(\Omega)$, and consider the following problems

$$\text{Minimize } I(u) \text{ on } \{u \in H_0^1(\Omega) : |u(x)| \leq d(x, \partial\Omega)\} \tag{1}$$

and

$$\text{Minimize } I(u) \text{ on } \{u \in H_0^{1,\infty}(\Omega) : |\nabla u(x)| \leq 1\}. \tag{2}$$

Under the assumption that $2\lambda \geq |\nabla f|_\infty$, (1) and (2) are equivalent. This problem arises from the elasto-plastic torsion problem. There are several results regarding this problem, concerning the regularity of the solutions, the existence of the Lagrange multipliers and numerical aspects ([B.2],[H.S.],[E.T.],[B.St.]).

We are concerned in this chapter with a more general functional, namely

$$I(u) = \int_\Omega f(\nabla u(x)) + g(x, u(x))\, \mathrm{d}x,$$

where we ask f to be convex, we ask also some regularity conditions on g and we prove a more general equivalence result than the one proved in [B.S.]. The main improvements might be that we allow f not to be very regular, that we do not impose 0 as a boundary condition and that we control the directional derivatives, not only the Lipschitz constant. Also, our case may lack uniqueness of solutions. When g is C^2, we give some weaker conditions under which problems (1) and (2) are equivalent.

Notation and Preliminaries

Throughout this chapter, for $n \in \mathbb{N}$, we denote by :

Ω - an open and bounded subset of $\mathbb{R}^n$ with lipschitzean boundary $\partial\Omega$;

K - a compact and convex subset of $\mathbb{R}^n$ such that $0 \in \text{int}K$;

f - a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$;

g - a continuous function from $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}$.

Let $p > 1$ and $u_0 \in W^{1,p}(\Omega)$. On the set $u_0 + W_0^{1,p}(\Omega)$ we define the functional

$$I(u) \doteq \int_\Omega f(\nabla u(x)) + g(x, u(x))\ dx.$$

In this context we say that I is **coercive** with respect to the weak topology of $W^{1,p}$ if and only if the sub-level sets $\{u \in W^{1,p} : I(u) \leq c\}$ are weakly compact in $W^{1,p}$ for every $c \in \mathbb{R}$.

We denote by j_K the indicator function given by $j_K(x) = \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{if } x \in \mathbb{R}^n \setminus K \end{cases}$.

On $u_0 + W_0^{1,p}(\Omega)$ we also define the following functionals :

$$J^-(u) \doteq \int_\Omega j_K(\nabla u(x)) + u(x)\ dx$$

$$J^+(u) \doteq \int_\Omega j_K(\nabla u(x)) - u(x)\ dx\ .$$

Let us assume that there is at least a function $v \in u_0 + W_0^{1,p}(\Omega)$ such that $J^+(v) < +\infty$. Applying the direct method of the Calculus of Variations we see that both J^- and J^+ admit a unique minimizer. We will denote by u_0^-, u_0^+ the minimizers, respectively, of J^- and J^+. It is easy to check that $u_0^- \leq u_0^+$ a.e. on Ω.

We set

$K_{u_0} \doteq \{u \in u_0 + W_0^{1,p}(\Omega) : \nabla u \in K \text{ a.e. on } \Omega\}$

and, for $u_1, u_2 \in u_0 + W_0^{1,p}(\Omega)$, we denote

$W^p_{u_1,u_2} \doteq \{u \in u_0 + W^{1,p}_0(\Omega) : u_1 \le u \le u_2 \text{ a.e. on } \Omega\}$.

One can check that both $W^p_{u_1,u_2}$ and K_{u_0} are convex subsets of $u_0 + W^{1,p}_0(\Omega)$. It is also easy to see that $K_{u_0} \subset W^p_{u_0^-,u_0^+}$. We will consider the following two problems

$$(P_1) \qquad \text{Minimize} \int_\Omega f(\nabla u(x)) + g(x, u(x))\, dx \text{ on } W^p_{u_0^-,u_0^+}.$$

and

$$(P_2) \qquad \text{Minimize} \int_\Omega f(\nabla u(x)) + g(x, u(x))\, dx \text{ on } K_{u_0}.$$

We may already remark that, when $K_{u_0} \neq \emptyset$, then both u_0^- and u_0^+ are defined, and so, if f is convex and I is coercive, we can apply the direct method of the Calculus of Variations to prove that both (P_1) and (P_2) admit solutions. Our purpose is to find sufficient conditions in order to assure that the set of solutions to (P_1) coincides with the set of solutions to (P_2). To that purpose we need some preliminary results from convex analysis and some properties of lipschitzean functions.

We denote by K° the polar of K, $K^\circ \doteq \{x \in \mathbb{R}^n : (x, k) \le 1\ \forall k \in K\}$. When $G \subset \mathbb{R}^n$, γ_G denotes the Minkowski function $\gamma_G(x) \doteq \inf\{\lambda > 0 : x \in \lambda G\}$. We recall that the following property holds :

$$(*) \qquad \gamma_K(v)\gamma_{K^\circ}(h) \ge (v, h)$$

for every $v, h \in \mathbb{R}^n$ (see [R.]), where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$.

We have the following results on lipschitzean functions :

Lemma 2.1

Let $u : \mathbb{R}^n \to \mathbb{R}$. Then the following statements are equivalent :

$$(2.1) \qquad u \text{ is lipschitzean and } \nabla u(x) \in K \text{ for a.e. } x \in \mathbb{R}^n;$$

(2.2) $$u(y) - u(x) \leq \gamma_{K^\circ}(y - x) \textit{ for all } x, y \in \mathbb{R}^n;$$

(2.3) *there is* $\varepsilon > 0$ *such that* $u(y) - u(x) \leq \gamma_{K^\circ}(y-x) \forall x, y \in \mathbb{R}^n \ |x - y| < \varepsilon.$

■

Proof Using the fact that γ_{K° is positively homogeneous, it is very easy to see that (2.2) $\Leftrightarrow$ (2.3).

(2.1) $\Rightarrow$ (2.2) : Assume, by contradiction, that there are $x, y \in \mathbb{R}^n$ such that $u(y) - u(x) > \gamma_{K^\circ}(y - x)$ then there is $\delta > 0$ such that $u(y + v) - u(x + v) > \gamma_{K^\circ}(y - x)$ for every $v \in \mathbb{R}^n$, $|v| < \delta$. Then, we find v_0, of norm less than δ, such that, on the segment line between $x + v_0$ and $y + v_0$ we can integrate to obtain that $\gamma_{K^\circ}(y-x) \geq \int_0^1 \gamma_K(\nabla u(x + v_0 + t(y - x)))\gamma_{K^\circ}(y - x)\, dt$, since $\nabla u(x) \in K$ a.e. Now, using (*), we get $\gamma_{K^\circ}(y - x) \geq \int_0^1 (\nabla u(x + v_0 + t(y - x)), y - x)\, dt = u(y + v_0) - u(x + v_0) > \gamma_{K^\circ}(y - x)$, a contradiction.

(2.2) $\Rightarrow$ (2.1) : If (2.2) is satisfied, since $0 \in \text{int} K^\circ$, then u is lipschitzean and thus a.e. differentiable. Choose $x \in \mathbb{R}^n$ such that u is differentiable at x. Then, for every $h \in \mathbb{R}^n$ and $t > 0$, $\frac{1}{t}(u(x + th) - u(x)) \leq \frac{1}{t}\gamma_{K^\circ}(th)$. By letting $t \downarrow 0$, we obtain that $(\nabla u(x), h) \leq \gamma_{K^\circ}(h)$. In particular, for every $h \in K^\circ$, $\gamma_{K^\circ}(h) \leq 1$ and $(\nabla u(x), h) \leq 1$. This implies that $\nabla u(x) \in (K^\circ)^\circ = K$ (see [R.]). ■

Lemma 2.2

Let $G \subset \mathbb{R}^n$ *and let* $u : G \to \mathbb{R}$ *satisfy*

(2.2′) $$u(y) - u(x) \leq \gamma_{K^\circ}(y - x) \textit{ for all } x, y \in G.$$

Then there is $\overline{u} : \mathbb{R}^n \to \mathbb{R}$, *an extension of* u, *such that* $\overline{u}$ *satisfies (2.2).*

Moreover, if G *is closed,* $v : G \to \mathbb{R}$ *satisfies (2.2') and* $v_{|\partial G} = u_{|\partial G}$, *then the function* $\overline{v} : \mathbb{R}^n \to \mathbb{R}$, *given by*

$$\overline{v}(x) = \begin{cases} v(x) & \textit{if } x \in G \\ \overline{u}(x) & \textit{if } x \in \mathbb{R}^n \setminus G \end{cases},$$

also satisfies (2.2). ■

Proof We consider the family $\Gamma = \{(u^*, G^*) : G \subset G^* \subset \mathbb{R}^n, u^* : G^* \to \mathbb{R}, u^*_{|G} = u$, and u^* is such that $u^*(y) - u^*(x) \leq \gamma_{K^\circ}(y-x)$ for all $x, y \in G^*\}$. We order this family with the relation $(u_1^*, G_1^*) \leq (u_2^*, G_2^*)$ if and only if $G_1^* \subset G_2^*$ and $u^*_{2_{|G_1^*}} = u_1^*$. It is easy to see that $(\Gamma, \leq)$ is not empty and each chain has an upper bound, and so, by Zorn's Lemma, there is a maximal element $(u', G') \in \Gamma$. We will prove that $G' = \mathbb{R}^n$. Indeed, otherwise there is $x_0 \in \mathbb{R}^n \setminus G'$. We prove that u' may be extended to $G' \cup \{x_0\}$, which contradicts the fact that (u', G') is maximal. To do so, it is enough to find $\theta \in \mathbb{R}$ such that $-\gamma_{K^\circ}(x_0 - x) \leq u'(x) - \theta \leq \gamma_{K^\circ}(x - x_0)$ for all $x \in G'$. We can find such θ if and only if

$$\sup\{u'(x) - \gamma_{K^\circ}(x - x_0) : x \in G'\} \leq \inf\{u'(x) + \gamma_{K^\circ}(x_0 - x) : x \in G'\}$$

So we are left to prove that $u'(x) - \gamma_{K^\circ}(x - x_0) \leq u'(y) + \gamma_{K^\circ}(x_0 - y)$ for all $x, y \in G'$. Using the subadditivity of γ_{K° and the fact that $(u', G') \in \Gamma$ we find the result we needed.

Denote by $\overline{u}$ a maximal extension of u. To prove the second part of the lemma it is enough to check that $-\gamma_{K^\circ}(x - y) \leq v(y) - \overline{u}(x) \leq \gamma_{K^\circ}(y - x)$ for all $y \in G, x \in \mathbb{R}^n \setminus G$. For such x and y, there exists a point z on the segment line between x and y such that $z \in \partial G$. Since G is closed, $z \in G$ and then $v(y) - \overline{u}(x) = v(y) - v(z) + \overline{u}(z) - \overline{u}(x) \leq \gamma_{K^\circ}(y - z) + \gamma_{K^\circ}(z - x)$. Using now the fact that γ_{K° is positively homogeneous, we obtain $v(y) - \overline{u}(x) \leq \gamma_{K^\circ}(y - x)$. The other part of the inequality follows analogously. ■

Finally, for simplicity, we let functions defined on $\mathbb{R}^n$ to belong to Sobolev spaces on Ω, if their restriction to Ω does.

Main results

Our results concern mainly the equivalence (in the sense that the sets of solutions coincide) betweeen the problems (P_1) and (P_2) stated above.
We shall assume that f, g and u_0 satisfy the following

Assumption A

i) f *is convex;*

ii) *the restriction of* g *to* $\Omega \times \mathbb{R}$ *is continuously differentiable with respect to the last variable and* g'_u, *its differential, is lipschitzean;*

iii) $u_0(y) - u_0(x) \leq \gamma_{K^\circ}(y - x)$ *for all* $x, y \in \Omega$. ∎

We remark that we will not make further regularity restrictions on f.
Also, notice that, when u_0 satisfies iii) from the above assumption, due to **Lemma 2.2**, we may extend u_0 on $\mathbb{R}^n$ and then we may apply **Lemma 2.1** to obtain that $u_0 \in K_{u_0}$. In particular, $K_{u_0} \neq \emptyset$.
Since g'_u is lipschitzean it is differentiable a.e. We will impose on g further conditions. We remark that each of those conditions imply that $g(x, \cdot)$ is convex.

We have the following regularity result on the solutions to some variational problem :

Theorem 2.3

Let f, g *and* u_0 *satisfy* **Assumption A**. *Let* $u_1, u_2 \in K_{u_0}$. *Let* $\overline{u} \in W^p_{u_1,u_2}$ *be such that* $I(\overline{u}) \leq I(u)$ *for all* $u \in W^p_{u_1,u_2}$. *If*

$$\text{(2.4)} \quad \sup\{\gamma_K(-\nabla_x g'_u(x,u)) : (x,u) \in \Omega \times \mathbb{R}\} < \inf\{g''_{uu}(x,u) : (x,u) \in \Omega \times \mathbb{R}\}$$

then $\overline{u} \in K_{u_0}$. ∎

Proof First of all we apply **Lemma 2.2** to u_0 and we denote by u_0^* an extension of the function u_0 to $\mathbb{R}^n$, satisfying (2.2). We define also

$$u_1^*(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega \\ u_0^*(x) & \text{if } x \in \mathbb{R} \setminus \Omega \end{cases} \quad , \qquad u_2^*(x) = \begin{cases} u_2(x) & \text{if } x \in \Omega \\ u_0^*(x) & \text{if } x \in \mathbb{R} \setminus \Omega \end{cases} \quad \text{and}$$

$$\overline{u}^*(x) = \begin{cases} \overline{u}(x) & \text{if } x \in \Omega \\ u_0^*(x) & \text{if } x \in \mathbb{R} \setminus \Omega. \end{cases}$$

Since Ω is bounded, $u_1, u_2 \in K_{u_0}$ and u_0 is lipschitzean, then u_1 and u_2 are lipschitzean, too. In view of the second part of **Lemma 2.2** and of **Lemma 2.1**, u_1^*, $u_2^* \in K_{u_0}$ and both satisfy (2.2).

Let $h \in \mathbb{R}^n$, $h \neq 0$.

Using the same technique presented in [B.S.] we consider the following functions defined on $\mathbb{R}^n$

$$u_h^+(x) = \max\{\overline{u}^*(x+h) - \gamma_{K^\circ}(h), \overline{u}^*(x)\}$$

$$u_h^-(x) = \min\{\overline{u}^*(x-h) + \gamma_{K^\circ}(h), \overline{u}^*(x)\}$$

and the sets

$$E_h^+ = \{x \in \mathbb{R}^n : u_h^+(x) = \overline{u}^*(x+h) - \gamma_{K^\circ}(h) > \overline{u}^*(x)\}$$

$$E_h^- = \{x \in \mathbb{R}^n : u_h^-(x) = \overline{u}^*(x-h) + \gamma_{K^\circ}(h) < \overline{u}^*(x)\}.$$

The following properties hold true:

i) $u_1^* \leq u_h^- \leq u_h^+ \leq u_2^*$, a.e. in $\mathbb{R}^n$;

ii) $x \in \Omega$ for a.e. $x \in E_h^+ \cup E_h^-$;

iii) $E_h^+ = E_h^- - h$.

The inequality $u_h^- \leq u_h^+$ in i) is obviously satisfied. We prove only that $u_h^+ \leq u_2^*$, the remaining inequality can be proved in the same way. If x in $\mathbb{R}^n \setminus E_h^+$ the property is true by the choice of u_2^*. For a.e. $x \in E_h^+$

$$u_h^+(x) = \overline{u}^*(x+h) - \gamma_{K^\circ}(h) \leq u_2^*(x+h) - \gamma_{K^\circ}(h) \leq u_2^*(x).$$

ii) We prove that $x \in \Omega$ for a.e. $x \in E_h^+$. Indeed, we have, for a.e. $x \in E_h^+$, $\overline{u}^*(x+h) - \gamma_{K^\circ}(h) \leq u_2^*(x+h) - \gamma_{K^\circ}(h)$. Moreover, for every $x \in E_h^+ \setminus \Omega$, $u_2^*(x) = \overline{u}^*(x)$. Using the properties of u_2^*, we obtain the result we need.

The proof of property iii) follows immediately from the definition of the sets E_h^+ and E_h^-.

Then, by i), $u_h^-, u_h^+ \in W^p_{u_1,u_2}$. Now, by the assumption on $\overline{u}$ and by the definitions of u_h^+ and u_h^-, we have that, for $\lambda \in]0,1[$,

$$I(\overline{u} + \lambda(u_h^+ - \overline{u})) - I(\overline{u}) =$$

$$\int_{E_h^+} f(\nabla\overline{u}(x) + \lambda(\nabla u_h^+(x) - \nabla\overline{u}(x)) - f(\nabla\overline{u}(x))$$
$$+ g(x, \overline{u}(x) + \lambda(u_h^+(x) - \overline{u}(x)) - g(x, \overline{u}(x))\, dx =$$

$$\int_{E_h^+} f(\nabla\overline{u}(x) + \lambda(\nabla\overline{u}(x+h) - \nabla\overline{u}(x)) - f(\nabla\overline{u}(x))$$
$$+ g(x, \overline{u}(x) + \lambda(\overline{u}(x+h) - \gamma_{K^\circ}(h) - \overline{u}(x)) - g(x, \overline{u}(x))\, dx \geq 0 \tag{2.5}$$

and

$$I(\overline{u} + \lambda(u_h^- - \overline{u})) - I(\overline{u}) =$$

$$\int_{E_h^-} f(\nabla\overline{u}(x) + \lambda(\nabla u_h^-(x) - \nabla\overline{u}(x)) - f(\nabla\overline{u}(x))$$
$$+ g(x, \overline{u}(x) + \lambda(u_h^-(x) - \overline{u}(x)) - g(x, \overline{u}(x))\, dx =$$

$$\int_{E_h^-} f(\nabla\overline{u}(x) + \lambda(\nabla\overline{u}(x-h) - \nabla\overline{u}(x)) - f(\nabla\overline{u}(x))$$
$$+ g(x, \overline{u}(x) + \lambda(\overline{u}(x-h) + \gamma_{K^\circ}(h) - \overline{u}(x)) - g(x, \overline{u}(x))\, dx \geq 0. \tag{2.6}$$

In the last integral we make the change of variables that maps x in $x+h$ and we obtain

$$\int_{E_h^+} f(\nabla\overline{u}(x+h) + \lambda(\nabla\overline{u}(x) - \nabla\overline{u}(x+h)) - f(\nabla\overline{u}(x+h)) \tag{2.7}$$

$$+g(x+h,\overline{u}(x+h)+\lambda(\overline{u}(x)+\gamma_{K^\circ}(h)-\overline{u}(x+h))-g(x+h,\overline{u}(x+h))\ \mathrm{d}x \geq 0.$$

Adding term by term the inequalities (2.5) and (2.7) and using the convexity of f, we have that

$$\int_{E_h^+} g(x+h,\overline{u}(x+h)+\lambda(\overline{u}(x)+\gamma_{K^\circ}(h)-\overline{u}(x+h))-g(x+h,\overline{u}(x+h))$$

$$+g(x,\overline{u}(x)+\lambda(\overline{u}(x+h)-\gamma_{K^\circ}(h)-\overline{u}(x))-g(x,\overline{u}(x))\ \mathrm{d}x \geq 0.$$

We divide now the last inequality by $\lambda > 0$ and let $\lambda \downarrow 0$. Since g is continuously differentiable with respect to the last variable, we may apply the dominated convergence theorem to obtain

$$\text{(2.8)}\quad \int_{E_h^+} [g'_u(x+h,\overline{u}(x+h))-g'_u(x,\overline{u}(x))](\overline{u}(x)+\gamma_{K^\circ}(h)-\overline{u}(x+h))\ \mathrm{d}x \geq 0.$$

Now, we remark that, for every $x \in E_h^+$, $\overline{u}(x)+\gamma_{K^\circ}(h)-\overline{u}(x+h) < 0$. Let us consider the other factor. We have

$$\text{(2.9)}\qquad g'_u(x+h,\overline{u}(x+h))-g'_u(x,\overline{u}(x))$$

$$= g'_u(x+h,\overline{u}(x+h))-g'_u(x+h,\overline{u}(x))+g'_u(x+h,\overline{u}(x))-g'_u(x,\overline{u}(x)).$$

g'_u is lipschitzean and so it is differentiable a.e. Set then $i \doteq \inf\{g''_{uu}(x,u) : (x,u) \in \Omega\times\mathbb{R}\}$ and $s \doteq \sup\{\gamma_K(-\nabla_x g'_u(x,u)) : (x,u) \in \Omega\times\mathbb{R}\}$. Using the fact that g'_u is lipschitzean with respect to the first variable, by the proprerties of lipschitzean functions and recalling (*), we get, for a.e. $x \in E_h^+$, $g'_u(x+h,\overline{u}(x))-g'_u(x,\overline{u}(x)) \geq -s\gamma_{K^\circ}(h)$. By the same argument, we get also $g'_u(x+h,\overline{u}(x+h))-g'_u(x+h,\overline{u}(x)) \geq i(\overline{u}(x+h)-\overline{u}(x))$.

Thus, since for $x \in E_h^+$, $\overline{u}(x+h)-\overline{u}(x) > \gamma_{K^\circ}(h)$, we recall (2.4) to obtain that $g'_u(x+h,\overline{u}(x+h))-g'_u(x,\overline{u}(x)) > 0$ for a.e. $x \in E_h^+$.
Hence, we conclude that, for every $h \in \mathbb{R}^n$, $h \neq 0$, $\mu(E_h^+) = 0$, so
$\overline{u}^*(x+h)-\overline{u}^*(x) \leq \gamma_{K^\circ}(h)$ for a.e. $x \in \mathbb{R}^n$.

We apply this result letting h in a countable and dense subset of $\mathbb{R}^n$. By a standard argument, using the fact that $\nabla \overline{u}$ exists a.e. on Ω and recalling (*), we find that $\nabla \overline{u} \in K$ a.e. on Ω. ■

Notice that, since $K_{u_0} \subset W^p_{u_0^-, u_0^+}$, to prove that (P_1) and (P_2) are equivalent, it is enough to prove that (P_1) admits at least one solution and that each solution of (P_1) lies in K_{u_0}.

We state now our first equivalence result.

Theorem 2.4

Let f, g and u_0 satisfy **Assumption A** *and assume that I is coercive. Then both (P_1) and (P_2) admit solutions. Moerover, assume that one of the following statements holds true*

(2.10) $\sup\{\gamma_K(-\nabla_x g'_u(x,u)) : (x,u) \in \Omega \times \mathbb{R}\} < \inf\{g''_{uu}(x,u) : (x,u) \in \Omega \times \mathbb{R}\};$

(2.11) $\sup\{\gamma_K(-\nabla_x g'_u(x,u)) : (x,u) \in \Omega \times \mathbb{R}\} \leq \inf\{g''_{uu}(x,u) : (x,u) \in \Omega \times \mathbb{R}\}$

and for every $\lambda \in]0,1[$, $x \in K$, $y \in \mathbb{R}^n \backslash K$, $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$. Then u minimizes I on K_{u_0} if and only if u minimizes I on $W^p_{u_0^-, u_0^+}$. ■

Proof As we noticed above, under the conditions of **Theorem 2.4**, $K_{u_0} \neq \emptyset$ and both (P_1) and (P_2) admit solutions.

a) Assume that (2.10) holds true. From **Theorem 2.3**, any minimizer on $W^p_{u_0^-, u_0^+}$ belongs to K_{u_0}. Since $K_{u_0} \subset W^p_{u_0^-, u_0^+}$, we are done.

b) Notice that (2.11) implies that $g''_{uu} \geq 0$ and so $g(x, \cdot)$ is convex. Assume that (2.11) holds true and, for $l \in \mathbb{N}$, set $I_l(u) = I(u) + \frac{1}{l} \int_\Omega u^2(x)\, dx$. By setting $g_l(x,u) \doteq g(x,u) + \frac{1}{l} u^2$, we may apply the result of a) to I_l to find u_l, a minimizer for I_l on $W^p_{u_0^-,u_0^+}$, such that $u_l \in K_{u_0}$. Let now $l \to \infty$. We may pass to the limit on a subsequence of (u_l) and, by using the lower semicontinuity of I and the fact that u_l are uniformly bounded in $L^\infty(\Omega)$, we find $\overline{u} \in K_{u_0}$, a minimizer for I on $W^p_{u_0^-,u_0^+}$. Let now v be any minimizer for I on $W^p_{u_0^-,u_0^+}$. Then $I(v) = I(\overline{u})$. Set $v_K \doteq \{x \in \Omega : \nabla v(x) \in K\}$ and assume, by contradiction, that $\mu(\Omega \setminus v_K) > 0$. Then, for $\lambda \in]0,1[$,

$$\int_\Omega f(\lambda\nabla\overline{u}(x) + (1-\lambda)\nabla v(x))\,\mathrm{d}x < \int_{v_K} \lambda f(\nabla\overline{u}(x)) + (1-\lambda)f(\nabla v(x))\,\mathrm{d}x +$$

$$\int_{\Omega\setminus v_K} \lambda f(\nabla\overline{u}(x)) + (1-\lambda)f(\nabla v(x))\,\mathrm{d}x$$

and, since $g(x,\cdot)$ is convex, we find that

$$I(\lambda\overline{u} + (1-\lambda)v) < \lambda I(\overline{u}) + (1-\lambda)I(v) = I(\overline{u})$$

which contradicts the fact that $\overline{u}$ is a minimizer for I on $W^p_{u_0^-,u_0^+}$. ■

The following example shows that, when (2.10) is not true, the two minimum problems may not be equivalent :

Example 2.5 Let $f(\xi) = \frac{1}{4}\xi^2$ and $g(x,u) = u\cos x$, $\Omega =]-\pi,\pi[$ and $u_0(x) = -2$. Solving the Euler-Lagrange equation one obtains that the minimizer of $I(u)$ on $u_0 + W_0^{1,2}(\Omega)$ is $\overline{u}(x) = 2\cos x$. We can choose a positive a such that $\overline{u}$ is also the minimizer of the problem with the obstacles u_0^- and u_0^+ associated to the set $K = [-2+a, 2-a]$. Anyway, there exists a set A with positive measure such that $|u'(x)| > 2-a$, for every $x \in A$. ■

We give another example to show that we cannot drop the strict convexity required by (2.11) :

Example 2.6 Let $\Omega =]-1,1[$, $K = [-1,1]$, $u_0 = 0$, $g(x,u) = 0$ and

$$f(x) = \begin{cases} 0 & \text{if } |x| \leq 2 \\ x^2 - 4 & \text{if } |x| \geq 2. \end{cases}$$

Then $\sup\{\gamma_K(-\nabla_x g'_u(x,u)) : (x,u) \in \Omega \times \mathbb{R}\} = \inf\{g''_{uu}(x,u) : (x,u) \in \Omega \times \mathbb{R}\} = 0$ and

$$u(x) = \begin{cases} 0 & \text{if } |x| \geq \frac{1}{2} \\ 2|x| - 1 & \text{if } |x| \leq \frac{1}{2} \end{cases}$$

is a minimizer for I on $W^2_{u_0^-,u_0^+}(\Omega)$ such that $|u'| = 2$ on $]-\frac{1}{2},\frac{1}{2}[$. ∎

If the problem we deal with is more regular, we may weaken the restriction (2.10) from **Theorem 2.4.** We have

Theorem 2.7

Let f, g and u_0 satisfy **Assumption A**. *Assume that $g \in C^2(\mathbb{R}^n \times \mathbb{R})$ and that I is coercive and such that every minimizer of I on $W^p_{u_0^-,u_0^+}$ is continuous on $\overline{\Omega}$. If*

$$\text{(2.12)} \qquad \gamma_K(-\nabla_x g'_u(x,u)) < g''_{uu}(x,u) \textit{ for all } (x,u) \in \overline{\Omega} \times \mathbb{R}$$

then u minimizes I on K_{u_0} if and only if u minimizes I on $W^p_{u_0^-,u_0^+}$. ∎

Proof Again, our hypotheses assure the existence of solutions.
Let $\overline{u}$ be a minimizer for I on $W^p_{u_0^-,u_0^+}$. Since $\gamma_K(-\nabla_x g'_u)$ and g''_{uu} are uniformly continuous on compact sets and since $\overline{u}$ is uniformly continuous on $\overline{\Omega}$, in view of (2.12), there is $\delta > 0$ such that $\gamma_K(-\nabla_x g'_u(x_1, \overline{u}(y_1))) < g''_{uu}(x_2, \overline{u}(y_2))$ for every $x_1, x_2, y_1, y_2 \in \overline{\Omega}$ such that $|x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta$. Let now $h \in \mathbb{R}^n$, $h \neq 0$, $|h| < \delta$. The same construction that we made in the proof of **theorem 2.3** leads us to

$$\int_{E_h^+} [g'_u(x+h, \overline{u}(x+h)) - g'_u(x, \overline{u}(x))](\overline{u}(x) + \gamma_{K^\circ}(h) - \overline{u}(x+h))\, dx \geq 0.$$

We recall that $E_h^+ = \{x \in \Omega : \overline{u}(x+h) - \overline{u}(x) > \gamma_{K^\circ}(h)\}$. For $x \in E_h^+$ we may write $g_u'(x+h, \overline{u}(x+h)) - g_u'(x, \overline{u}(x)) = g_{uu}''(x+h, \overline{u}(\theta_1))(\overline{u}(x+h) - \overline{u}(x)) + (\nabla_x g_u'(\theta_2, \overline{u}(x)), h)$ where θ_1, θ_2 lie on the segment line between x and $x+h$. So $g_u'(x+h, \overline{u}(x+h)) - g_u'(x, \overline{u}(x)) > \gamma_K(-\nabla_x g_u'(\theta_2, \overline{u}(x)))\gamma_{K^\circ}(h) - (-\nabla_x g_u'(\theta_2, \overline{u}(x)), h) \geq 0$. As above, we have that $\mu(E_h^+) = 0$. In this case, $\overline{u}$ is continuous, so E_h^+ is open and thus we may apply directly **Lemma 2.1** to conclude that $\overline{u} \in K_{u_0}$. Again, since $K_{u_0} \subset W^p_{u_0^-, u_0^+}$, the two problems are equivalent. ∎

Our last result shows that, with more regularity, the condition (2.12) may replaced with

$$(2.13) \qquad \gamma_K(-\nabla_x g_u'(x,u)) \leq g_{uu}''(x,u) \text{ for all } (x,u) \in \Omega \times \mathbb{R} \text{ and}$$

for every $\lambda \in]0,1[$, $x \in K$, $y \in \mathbb{R}^n \setminus K$ $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda) f(y)$.

Corollary 2.8

Let f, g and u_0 satisfy **Assumption A**. *Assume that $p > n$, I is coercive and $g \in C^2(\mathbb{R}^n \times \mathbb{R})$. If (2.13) holds true then u minimizes I on K_{u_0} if and only if u minimizes I on $W^p_{u_0^-, u_0^+}$.* ∎

Proof To prove this corollary we use the same method as in b) from **theorem 2.4**. That is, we approximate I with the functionals $I_l(u) = I(u) + \frac{1}{l}\int_\Omega u^2(x)\,\mathrm{d}x$. The hypotheses of our corollary allow us to conclude that, for every $l \in \mathbb{N}$, there is a minimizer u_l for I_l on $W^p_{u_0^-, u_0^+}$ such that u_l is continuous on $\overline{\Omega}$ (since $p > n$). Thus, we may apply **Theorem 2.7** to obtain that $u_l \in K_{u_0}$. Passing to the limit, on a subsequence, we find $\overline{u}$, a minimizer for I on $W^p_{u_0^-, u_0^+}$, such that $\overline{u} \in K_{u_0}$.

By reasoning as in b) from **theorem 2.4**, we conclude that any other minimizer belongs to K_{u_0} and so, the two problems are equivalent. ∎

Chapter 3. On Gradient Flows

This chapter contains the results obtained in [C.V.], a joint paper with Arrigo Cellina. The purpose of these results is to contribute to the theory of existence of solutions to ordinary differential equations in $\mathbb{R}^n$, when the right hand side of the equation is not necessarily continuous. More precisely we are interested in the properties of a special family of right hand sides, namely the gradients of a class of functions that contains the family of convex functions. As shown in the introduction, the motivation for this problem comes to a large extent from the (multi-dimensional) Calculus of Variations. The technique of integrating along the trajectories of

$$(CP_\nabla)(x_0) \qquad\qquad x'(t) = \nabla u(x(t)) \ , \ x(0) = x_0,$$

where u is the candidate solution, is used in many papers. This technique raises the following questions : given an open region Ω, do solutions to $(CP_\nabla)(x_0)$ exist, at least a.e. with respect to x_0 in Ω ? do the trajectories of $(CP_\nabla)(x_0)$ fill Ω in a one to one way, at least on the complement of a null-measure set ?

Since here the purpose is to integrate over Ω, null measure sets do not count and so, the approach we use is not the usual one. Essentially, the previous emphasis in dealing with discontinuous right hand sides was in modifying the right hand side, by building an upper semicontinuous convex valued multifunction out of it, as in the definition of the subdifferential of a convex mapping or in defining the solutions in the sense of Filippov. In this way one can show the existence of a (generalized) solution for all initial data x_0. However, in this case, the derivative of the solution $x(t)$ does not, for a.e. t, equal the original right hand-side and this makes the trajectories unsuitable for integrating along. In our approach we want this equality to hold for a.e. t and pay the price of losing a set of initial data having measure zero. The results of this chapter are better clarified by examining the simple example where u is the convex mapping on $\mathbb{R}^2$ given by $x \to \sup\{|x_i|\}$ i.e. $u(x) = |x|_\infty$. Ω may be taken to be the unit square about the origin. It is obvious that if we regularise Ω by subtracting the set consisting of the two diagonal lines, then on the remaining set Ω_u the solutions to $(CP_\nabla)(x_0)$ exist on a maximal open interval of existence $(\omega_-(x_0), \omega_+(x_0))$ and are unique for all initial data in Ω_u. Moreover, for $t \to \omega_+$ solutions converge to the boundary of the original set Ω, while for $t \to \omega_-$ they converge to $\Omega \setminus \Omega_u$; finally, the collection of the trajectories is a partition of Ω_u.

The above result follows easily since in this case the set we have removed, the diagonals, is the set of non-uniqueness for the Cauchy problems

$$(CP_\partial)(x_0) \qquad\qquad x'(t) \in \partial u(x(t)) \ , \ x(0) = x_0$$

and it happens that this set coincides with the set of non-differentiability of u and is closed. However this property does not hold in general for a convex function : in fact, our **example 3.5** shows a convex mapping on $\mathbb{R}^2$ such that the corresponding set of initial points lacking uniqueness for (CP_∂) is dense in Ω and does not coincide with Ω. Still our theorems guarantee, for the class of mappings we consider, the validity of all the claims previously made for the special case discussed above.

As explained in the introduction, the results obtained in [S.] and [K.] induced us to deal here also with mappings whose composition with a monotone function is convex.

Notation and Preliminary Results

Let $n \geq 1$ be a natural number. For $S \subset \mathbb{R}^n$, ∂S denotes the boundary of S ; we use $(\cdot,\cdot)$ to denote the scalar product in $\mathbb{R}^n$, and μ to denote the n-dimensional Lebesgue measure ; μ_* and μ^* are, respectively, the interior and the exterior n-dimensional Lebesgue measure. For measurable $A \subset \mathbb{R}^n$ and $B \subset A$, $\mu(A) = \mu_*(B) + \mu^*(A \setminus B)$ ([Co.]pp.39,42). When $f : \mathbb{R}^n \to \mathbb{R}^n$ is a lipschitzean mapping of constant l, $\mu(f(A)) \leq l^n \mu(A)$ for every measurable $A \subset \mathbb{R}^n$ ([E.G.]). For the sake of simplicity, we deal with differential properties of functions only at points that are interior to the domain of definition. We denote by ∇u the gradient of a real valued function. When u is a convex function, ∂u denotes its subgradient ([R.]). When the real valued function u, defined on a subset of $\mathbb{R}^n$, is not necessarily convex, we consider the subdifferential of u at x to be the (possibly empty) set given by

$$(3.1) \quad D^-u(x) \doteq \{p \in \mathbb{R}^n : \liminf_{y \to x} \frac{u(y) - u(x) - (p, y - x)}{|y - x|} \geq 0\} \text{ ([B.D.],[E.]).}$$

When u is convex, $D^-u \equiv \partial u$. For $u : \text{Dom}u \to \mathbb{R}$, $S \subset \text{intDom}u$ and $x_0 \in S$ we will consider the following two types of Cauchy problems :

$$(CP_\nabla)(x_0) \begin{cases} x'(t) = \nabla u(x(t)) \\ x(0) = x_0 \end{cases} \text{ and } (CP_-)(x_0) \begin{cases} y'(t) \in D^-u(y(t)) \\ y(0) = x_0 \end{cases} .$$

We say that $x_{x_0} : I \to S$ is a solution to $(CP_\nabla)(x_0)$ if I is an interval containing 0 in its interior, $x_{x_0}(0) = x_0$, $x_{x_0}(\cdot)$ is absolutely continuous and, for a.e. $t \in I$, x_{x_0} is differentiable at t, u is differentiable at $x_{x_0}(t)$ and $x'_{x_0}(t) = \nabla u(x_{x_0}(t))$.

Also, we say that $x_{x_0} : I \to S$ is a solution to $(CP_-)(x_0)$ if I is an interval containing 0 in its interior, $x_{x_0}(0) = x_0$, $x_{x_0}(\cdot)$ is absolutely continuous and, for a.e. $t \in I$, x_{x_0} is differentiable at t, and $x'_{x_0}(t) \in D^-u(x_{x_0}(t))$.

Maximality of solutions is intended in the classical way and we shall deal only with maximal solutions. When u is convex, we will call (CP_∂) the Cauchy problem (CP_-). We recall ([A.C.],[B.1]) some properties of solutions to the differential inclusion (CP_∂), in the case S is open :

- for all $x_0 \in S$ there exists at least one (maximal) solution $x_{x_0} : (\omega_-, \omega_+) \to S$ to $(CP_\partial)(x_0)$;

- $(CP_\partial)(x_0)$ admits a unique solution in the past;

- the set $N = \{x_0 \in S : (CP_\partial)(x_0)$ admits at least two distinct solution in the future $\}$ has n-dimensional Lebesgue measure 0 ([C.T.],[Ce.2]);

- $u(x_{x_0}(\cdot))$ is non-decreasing;

- for a.e. $t \in (\omega_-, \omega_+)$, $x'_{x_0}(t) = v_m(t)$, where $|v_m(t)| = \min\{|v|, v \in \partial u(x_{x_0}(t))\}$;

- if we denote by m_u the minimum value level set of u then

$\lim_{t \downarrow \omega_-} |x_{x_0}(t)| = +\infty$ or $\lim_{t \downarrow \omega_-} x_{x_0}(t) \in \partial S \cup m_u$ and

$\lim_{t \uparrow \omega_+} |x_{x_0}(t)| = +\infty$ or $\lim_{t \uparrow \omega_+} x_{x_0}(t) \in \partial S$.

Finally, when $\Omega \subset \mathbb{R}^n$ we say that $A \subset \Omega$ is convex in Ω if A is the intersection of Ω and a convex set.

Main Results

It is our purpose to prove the following theorem on differential equations generated by gradients of convex mappings :

Theorem 3.1

Let u be a convex mapping with domain Dom$u \subset \mathbb{R}^n$*; let $\Omega \subset$* Dom$u$ *be open. Then there exists a set $\Omega_u \subset \Omega$ such that $\mu(\Omega \setminus \Omega_u) = 0$ and for every $x_0 \in \Omega_u$:*

(i) the Cauchy problem on Ω_u

$$(CP_\nabla)(x_0) \begin{cases} x'(t) = \nabla u(x(t)) \\ x(0) = x_0 \end{cases}$$

admits the unique (maximal) solution x_{x_0} on $(\omega_-^\nabla, \omega_+^\nabla)$;

(ii) the Cauchy problem on Ω

$$(CP_\partial)(x_0) \begin{cases} y'(t) \in \partial u(y(t)) \\ y(0) = x_0 \end{cases}$$

admits the unique (maximal) solution y_{x_0} on $(\omega_-^\partial, \omega_+^\partial)$;

(iii) $\omega_+^\partial = \omega_+^\nabla \doteq \omega_+$, $\omega_-^\partial \leq \omega_-^\nabla$, $y_{x_0} \equiv x_{x_0}$ on $(\omega_-^\nabla, \omega_+)$ and

$y_{x_0}((\omega_-^\partial, \omega_-^\nabla]) \subset \Omega \setminus \Omega_u$. ∎

The proof of the main result will need a technical lemma from measure theory.

Lemma 3.2

Let $C \subset R^n$ be compact and $f : C \to R^n$ be lipschitzean with lipschitz constant l. Set $Z = \{y \in f(C) :$ there are $x_1 \neq x_2 \in C, f(x_1) = f(x_2) = y\}$. Then, for all measurable $Y \subset f(C)$ such that $Y \cap Z = \emptyset$,

$$\mu(Y) \leq l^n \mu_*(f^{-1}(Y)) \ .$$

∎

Proof Let $Y \subset f(C) \setminus Z$ be measurable. For $k \in \mathbb{N}$ set

$$U_k \doteq \{x \in C : \forall y \in C, y \neq x \Longrightarrow \frac{|f(x) - f(y)|}{|x - y|} \geq \frac{1}{k}\}$$

$(U_k)_k$ is an increasing sequence of closed subsets of C. Set $V_k := f(U_k)$ and $f_k := f_{|U_k}$. It is easy to check that $V_k \cap Z = \emptyset$ and $f_k^{-1} : V_k \to U_k$ is lipschitz of constant k, so it maps measurable sets into measurable sets. Moreover, U_k and V_k are measurable (being closed), and so are $V_k \cap Y$ and $U_k \cap f^{-1}(Y) = f_k^{-1}(V_k \cap Y)$. So we may write

$$\mu(V_k \cap Y) = \mu(f_k(U_k \cap f^{-1}(Y))) \leq l^n \mu(U_k \cap f^{-1}(Y)) \leq l^n \mu_*(f^{-1}(Y))$$

Set $V \doteq \cup V_k$. Then $Y \cap V$ and $Y \setminus V$ are measurable. Moreover

$$\mu(Y \cap V) \leq l^n \mu_*(f^{-1}(Y))$$

We claim that $\mu(Y \setminus V) = 0$. In fact, we show that $Y \setminus V \subset f(B)$, where

$$B \doteq \{x \in C : \text{ there exist } x_k \in C, x_k \to x, \frac{|f(x_k) - f(x)|}{|x_k - x|} \to 0\}$$

Let $y \in Y \setminus V$ and let $x \in C$ be such that $f(x) = y$. Since for every k y is not in V_k then $x \notin U_k$, so there exists $x_k \in C, x_k \neq x$, such that $\frac{|f(x_k) - f(x)|}{|x_k - x|} < \frac{1}{k}$. C is compact, so a subsequence of $\{x_k\}$ converges to some $x^* \in C$. In particular, $f(x^*) = f(x) = y$. But, since $y \in Y$ and $Y \cap Z = \emptyset$, $x^* = x$. So $x \in B$ and $y \in f(B)$. Since B contains only points in C for which either f is not differentiable or the jacobian is 0, by the coarea formula ([11]) we see that $\mu(f(B)) = 0$. So

$$\mu(Y) = \mu(Y \cap V) \leq l^n \mu_*(f^{-1}(Y))$$

■

Proof of theorem 3.1 We wish to define the set Ω_u. For this purpose we consider solutions x_{x_0} of $(CP_\partial)(x_0)$. Obviously, $x_{x_0}(0) = x_0$. It is known that the set $N = \{x_0 \in \Omega : (CP_\partial)(x_0)$ admits at least two distinct solution in the future $\}$ has measure 0. We will consider points along x_{x_0} at negative times. These points are the initial data of new Cauchy problems. Hence we define the set $A = \{x_0 \in \Omega : \exists\ (t_i)_i, t_i \uparrow 0$ such that $(CP_\partial)(x_{x_0}(t_i))$ admits at least two solutions $\}$. Clearly $N \subset A$. Notice that for every solution x to (CP_∂), $x(t) \in A \Longrightarrow x(s) \in N$ for all $s < t$ such that $x(s)$ is defined.

a) It is our purpose to show that $\mu(A) = 0$.

It will be enough to prove that $\mu(A \cap B(z, R)) = 0$ for a generic closed ball $B(z, R)$ contained in Ω. Call, for $\lambda > 0$, $D_\lambda = B(z, R-\lambda)$. Set $M_\lambda = \sup\{|v| : v \in \partial u(x), x \in D_\lambda\}$. For $\varepsilon > 0$ define the mapping $f_\varepsilon : D_\lambda \to \mathbb{R}^n$ by $f_\varepsilon(x_0) = x_{x_0}(-\varepsilon)$. Notice that $D_\lambda = f_\varepsilon^{-1}(f_\varepsilon(D_\lambda))$. Since one has uniqueness in the past to (CP_∂) for all initial data, for $\varepsilon < \frac{\lambda}{2M_{\frac{\lambda}{2}}}$ f_ε is well defined. Moreover $D_{\lambda+\varepsilon M_\lambda} \subset f_\varepsilon(D_\lambda) \subset D_{\frac{\lambda}{2}}$. The monotonicity of ∂u implies that f_ε is lipschitz of constant 1. Since, for all $x_0 \in D_\lambda \cap A$, $x_{x_0}(t_0) \in N$ for some $t_0 > -\varepsilon$, a fortiori $f_\varepsilon(x_0) = x_{x_0}(-\varepsilon) \in N$. So $f_\varepsilon(D_\lambda \cap A) \subset f_\varepsilon(D_\lambda) \cap N$ and hence $D_\lambda \cap A \subset f_\varepsilon^{-1}(f_\varepsilon(D_\lambda \cap A)) \subset f_\varepsilon^{-1}(f_\varepsilon(D_\lambda) \cap N)$.

We want to apply **lemma 3.2** to f_ε and to $Y = f_\varepsilon(D_\lambda)\setminus N$. Since D_λ is compact, f_ε is lipschitz and $\mu(N) = 0$, $f_\varepsilon(D_\lambda)\setminus N$ is measurable and $\mu(f_\varepsilon(D_\lambda)\setminus N) = \mu(f_\varepsilon(D_\lambda))$. Moreover $Z \subset N$, where $Z = \{y \in f_\varepsilon(D_\lambda) : \exists x_1 \neq x_2$ such that $f_\varepsilon(x_1) = f_\varepsilon(x_2) = y\}$. In fact, let $y = x_{x_1}(-\varepsilon) = x_{x_2}(-\varepsilon)$, $x_1 \neq x_2 \in D_\lambda$. For $i = 1, 2$, call $y_i : [0,\varepsilon] \to D_\lambda$, $y_i(t) = x_{x_i}(t-\varepsilon)$. Then y_1 and y_2 solve $(CP_\partial)(y)$ on $[0,\varepsilon]$. Moreover $y_1(\varepsilon) = x_1 \neq x_2 = y_2(\varepsilon)$, so $y \in N$. Now, from **lemma 3.2** we have

$$\mu(f_\varepsilon(D_\lambda)) = \mu(f_\varepsilon(D_\lambda)\setminus N) \leq \mu_*(f_\varepsilon^{-1}(f_\varepsilon(D_\lambda)\setminus N))$$

We also have $D_\lambda = f_\varepsilon^{-1}(f_\varepsilon(D_\lambda)) = f_\varepsilon^{-1}(f_\varepsilon(D_\lambda)\setminus N) \cup f_\varepsilon^{-1}(f_\varepsilon(D_\lambda)\cap N)$ so that

$$\mu(D_\lambda) = \mu_*(f_\varepsilon^{-1}(f_\varepsilon(D_\lambda)\setminus N)) + \mu^*(f_\varepsilon^{-1}(f_\varepsilon(D_\lambda)\cap N))$$

hence $\mu(f_\varepsilon(D_\lambda)) \leq \mu(D_\lambda) - \mu^*(f_\varepsilon^{-1}(f_\varepsilon(D_\lambda)\cap N))$. Then

$$\mu^*(D_\lambda \cap A) \leq \mu^*(f_\varepsilon^{-1}(f_\varepsilon(D_\lambda)\cap N)) \leq \mu(D_\lambda) - \mu(f_\varepsilon(D_\lambda))$$

Since $D_{\lambda+\varepsilon M_\lambda} \subset f_\varepsilon(D_\lambda)$, we have $\mu^*(D_\lambda \cap A) \leq \mu(D_\lambda) - \mu(D_{\lambda+\varepsilon M_\lambda})$. By letting $\varepsilon \to 0$ one has that $\mu(D_\lambda \cap A) = \mu^*(D_\lambda \cap A) = 0$.

b) Set $\Omega_u = \Omega \setminus A$ and let $x_0 \in \Omega_u$.

To prove (ii) : by the definition of Ω_u, $(CP_\partial)(x_0)$ has a unique maximal solution, which we denote by $y_{x_0} : (\omega_-^\partial, \omega_+^\partial) \to \Omega$. Since Ω is open, the maximal interval of definition will be open too.

Set : $\omega_-^\nabla = \inf\{t \in (\omega_-^\partial, \omega_+^\partial) : y_{x_0}(t) \in \Omega_u\}$ and $\omega_+^\nabla = \omega_+^\partial$. Set also $x_{x_0}(t) \doteq y_{x_0}(t)$ for all $t \in (\omega_-^\nabla, \omega_+^\nabla)$. By the definition of ω_-^∇, $y_{x_0}(t) \in \Omega \setminus \Omega_u$ $\forall t \in (\omega_-^\partial, \omega_-^\nabla)$ and so, by the definition of Ω_u, $y_{x_0}(\omega_-^\nabla) \in \Omega \setminus \Omega_u$. Since $x_{x_0}(0) = x_0 \in \Omega_u$, $\omega_-^\nabla \leq 0$. We will prove that $\omega_-^\nabla < 0$ and that, for all $t \in (\omega_-^\nabla, \omega_+^\nabla)$, $x_{x_0}(t) \in \Omega_u$.

It cannot be that $\omega_-^\nabla = 0$. In fact, we would have for all $t \in (\omega_-^\partial, 0)$, $y_{x_0}(t) \in A$ which, according to a previous observation, implies that $y_{x_0}(t) \in N$ $\forall t \in (\omega_-^\partial, 0)$, hence $x_0 = y_{x_0}(0) \in A$, a contradiction. Assume now that for some $t \in (\omega_-^\nabla, \omega_+^\nabla)$ we have $x_{x_0}(t) = y_{x_0}(t) \in A$. Again, we see that $y_{x_0}(s) \in N \subset A$ for every $s \in (\omega_-^\nabla, t)$ which contradicts the choice of ω_-^∇. So $x_{x_0}(t) \in \Omega_u$ for all $t \in (\omega_-^\nabla, \omega_+^\nabla)$.

Since any solution to $(CP_\nabla)(x_0)$ is a solution to $(CP_\partial)(x_0)$ and $x_0 \notin N$, $(CP_\nabla)(x_0)$ admits at most one maximal solution which will coincide, on its interval of definition, with y_{x_0}. By the above, $y_{x_0}((\omega_-^\partial, \omega_-^\nabla]) \subset \Omega \setminus \Omega_u$, so the maximal interval of definition of the solution to $(CP_\nabla)(x_0)$ is contained in $(\omega_-^\nabla, \omega_+^\nabla)$.

So to prove the theorem, we are left to prove that for a.e. $t \in (\omega_-^\nabla, \omega_+^\nabla)$, $x'_{x_0}(t) = \nabla u(x_{x_0}(t))$.

For a.e. $t \in (\omega_-^\nabla, \omega_+^\nabla)$, x_{x_0} is differentiable at t and $x'_{x_0}(t) = v_m(t)$, where $|v_m(t)| = \min\{|v|, v \in \partial u(x(t))\}$. We show that for one such t, u is differentiable at $x_{x_0}(t)$. Assume this is not true. Then there is $v \in \partial u(x_{x_0}(t))$, $|v| > |v_m(t)|$. We claim that the following holds :

Claim 3.3

For every $x_0 \in \Omega$ and $v \in \partial u(x_0)$ there are $\varepsilon > 0$ and x, a solution to $(CP)_\partial(x_0)$ on $[0, \varepsilon]$, such that

$|x(s) - x_0| \geq s|v| \; \forall s \in [0, \varepsilon]$. ■

Proof of Claim 3.3 By choosing an appropriate $\varepsilon > 0$ one may construct, as in [A.C.] pp. 100, a sequence of piecewise linear mappings $x_n : [0, \varepsilon] \to \Omega$, that converge uniformly to $x : [0, \varepsilon] \to \Omega$, a solution to $(CP)_\partial(x_0)$.

At step n, set $x_n^1 = x_0 + \frac{\varepsilon}{n} v$. For $k \in \{1, ..., n-1\}$, choose $v_n^k \in \partial u(x_n^k)$ and set $x_n^{k+1} = x_n^k + \frac{\varepsilon}{n} v_n^k$. Denote also $v_n^0 = v$. $x_n : [0, \varepsilon] \to \Omega$ is given by $x_n(s) = x_n^k + (s - \frac{\varepsilon k}{n}) v_n^k$ when $s \in [\frac{\varepsilon k}{n}, \frac{\varepsilon(k+1)}{n}]$. The monotonicity of ∂u implies easily that $|v_n^i| \geq |v_n^{i-1}|$ for $i \geq 1$. We prove, inductively on k, that

$$(3.2_k) \qquad |x_n(s) - x_0|^2 \geq s^2|v|^2 \text{ on } (\frac{\varepsilon k}{n}, \frac{\varepsilon(k+1)}{n}]$$

When $k = 0$, $s \in (0, \frac{\varepsilon}{n}]$, $x_n(s) = x_0 + sv$ so (3.2_0) is obvious. When $n - 1 \geq k \geq 1$, assume (3.2_{k-1}) is true and let $s \in (\frac{\varepsilon k}{n}, \frac{\varepsilon(k+1)}{n}]$. Then

$$|x_n(s) - x_0|^2 = (\frac{\varepsilon}{n}(\sum_{0\leq i\leq k-1} v_n^i) + (s - \frac{\varepsilon k}{n})v_n^k, \frac{\varepsilon}{n}(\sum_{0\leq i\leq k-1} v_n^i) + (s - \frac{\varepsilon k}{n})v_n^k) =$$

$$= (s - \frac{\varepsilon k}{n})^2|v_n^k|^2 + 2(s - \frac{\varepsilon k}{n})\frac{\varepsilon}{n}(v_n^k, \sum_{0\leq i\leq k-1} v_n^i) + \frac{\varepsilon^2}{n^2}|\sum_{0\leq i\leq k-1} v_n^i|^2 =$$

$$= (s - \frac{\varepsilon k}{n})^2|v_n^k|^2 + 2(s - \frac{\varepsilon k}{n})\frac{\varepsilon}{n}(v_n^k, \sum_{0\leq i\leq k-1} v_n^i) + |x_n(\frac{\varepsilon k}{n}) - x_0|^2 \geq$$

$$\geq (s - \frac{\varepsilon k}{n})^2|v|^2 + 2(s - \frac{\varepsilon k}{n})\frac{\varepsilon}{n}(v_n^k, \sum_{0\leq i\leq k-1} v_n^i) + \frac{\varepsilon^2 k^2}{n^2}|v|^2.$$

The cyclical monotonicity of ∂u ([R.]) implies that

$(x_n^1 - x_0, v_n^0) + (x_n^2 - x_n^1, v_n^1) + \ldots + (x_n^k - x_n^{k-1}, v_n^{k-1}) \leq (x_n^k - x_0, v_n^k)$. So

$(v_n^k, \sum_{0\leq i\leq k-1} v_n^i) \geq \sum_{0\leq i\leq k-1} |v_n^i|^2 \geq k|v|^2$ and

$|x_n(s) - x_0|^2 \geq ((s - \frac{\varepsilon k}{n})^2 + 2(s - \frac{\varepsilon k}{n})\frac{\varepsilon k}{n} + \frac{\varepsilon^2 k^2}{n^2})|v|^2 = s^2|v|^2$.

Now, since x_n converge to x, $|x(s) - x_0| \geq s|v|$ $\forall s \in [0, \varepsilon]$. This proves the claim. ∎

Applying the previous claim to $x_{x_0}(t)$ and v we find $\varepsilon > 0$ and $x : [0, \varepsilon] \to \Omega$, a solution to $(CP_\partial)(x_{x_0}(t))$, such that $|x(s) - x_{x_0}(t)| \geq s|v|$ for all $s \in [0, \varepsilon]$. Since $x_{x_0}(t) \in \Omega_u = \Omega \backslash A$ and $x_{x_0}(= y_{x_0})$ is a solution to $(CP_\partial)(x_{x_0}(t))$, $x(s) = x_{x_0}(t+s)$ and $\frac{|x_{x_0}(t+s) - x_{x_0}(t)|}{s} \geq |v|$. However, $x_{x_0}(\cdot)$ is differentiable at t, so the above inequality implies that $|x'_{x_0}(t)| \geq |v| > |v_m(t)| = |x'_{x_0}(t)|$, a contradiction.

All the claims have been proved. ∎

Remark 3.4

The set $A = \Omega \setminus \Omega_u$, defined in the proof of the theorem, does not coincide, in general, with the set of points where u is not differentiable. For $\Omega = B[0,2]$ and $u(x) = \begin{cases} |x| \text{ for } |x| \le 1 \\ 2|x| - 1 \text{ for } 1 \le |x| \le 2 \end{cases}$, the set A consists of the point 0; the points $\{x : |x| = 1\}$ where u is not differentiable are "passed through" by the solutions, that remain absolutely continuous mappings satisfying a.e. (CP_∇).

Example 3.5 *A convex function u, defined on an open, bounded and convex set Ω, such that N is dense in Ω.*

Let $B \subset [0,1]$ be countable and dense and let $h : [0,1] \to [0,1]$ be an increasing mapping, discontinuous at every point of B and such that $h(0) = 0$ and $\int_0^1 h(s)\, ds = 1$. Then $f(x) = \int_0^x h(s)\, ds$ is a convex mapping which is not differentiable at any point of B and the set $\{(x_1, x_2) \in [0,1]^2 : f(x_1) < x_2$ and $x_1 < f(x_2)\}$ is convex, open, bounded and such that $\partial\Omega$ is not differentiable on a dense subset M.

Set $u : \Omega \to \mathbb{R}$ to be $u(x) = -d(x, \partial\Omega)$. Clearly u is convex and $\nabla u(x) = \mathbf{n}(\Pi(x))$ a.e. $x \in \Omega$, where $\Pi(\cdot)$ is the projection on $\partial\Omega$ and $\mathbf{n}(\cdot)$ is the outward normal to $\partial\Omega$ ([Ce.3]). For $x \in \Omega$, consider the differential inclusion

$$(CP_\partial)(x) \begin{cases} y'(t) \in \partial u(y(t)) \\ y(0) = x, y(t) \in \Omega \end{cases}$$

and set $N \doteq \{x \in \Omega : (CP_\partial)(x)$ admits at least two solutions $\}$. We will prove that N is dense in Ω.

Assume that there is an open ball $G \subset \Omega$ such that $G \cap N = \emptyset$. For $x \in G$ and $z \in \Pi(x)$ it is easy to check that the mapping $y : [0, |z - x|] \to \Omega$, $y(t) = x + t\dfrac{z - x}{|z - x|}$ solves $(CP_\partial)(x)$. Since $x \in G$, y has to be the unique solution to $(CP_\partial)(x)$, hence $\Pi(x)$ contains exactly one point. $\Pi(\cdot)$ is upper semicontinuous, so if we set $P \doteq \Pi_{|G}$, P is a continuous mapping from G to $\partial\Omega$. It was proved in [Ce.3] that $\Pi(\Omega)$,so also $P(G)$, contains only points of differentiability of $\partial\Omega$.

Moreover, G is open so $P(G)$ contains at least two points: let x_0 be the center of G; a point $x \in G$ on the normal through x_0 to $P(x_0) - x_0$ is such that $P(x) \neq P(x_0)$. Since G is arcwise connected, $P(G) \subset \partial\Omega$ is arcwise connected, so $P(G) \cap M \neq \emptyset$ which leads to a contradiction. ∎

The next theorem extends the results of **Theorem 3.1** to gradients of a larger class of mappings u :

Theorem 3.6

Let $\Omega \subset \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ be such that there exists a mapping $z : u(\Omega) \to \mathbb{R}$ for which :

(a) z is continuously differentiable on $\mathrm{int}u(\Omega)$ and z' is strictly positive.

(b) $v \doteq z \circ u$ is the restriction to Ω of a convex function.

Then there exists $\Omega_u \subset \Omega$ such that $\mu(\Omega \setminus \Omega_u) = 0$ and for all $x_0 \in \Omega_u$:

(i) the Cauchy problem on Ω_u

$$(CP_-)(x_0)\begin{cases} x'(t) \in D^-u(x(t)) \\ x(0) = x_0 \end{cases}$$

admits the unique (maximal) solution $x_{x_0} : I \to \Omega_u$ where I is an open interval containing 0;

(ii) for a.e. $t \in I$, x_{x_0} is differentiable at t, u is differentiable at $x_{x_0}(t)$ and $x'_{x_0}(t) = \nabla u(x_{x_0}(t))$. ∎

Proof Notice that, if u satisfies the conditions above, then its sub-level sets are convex in Ω, since they coincide with the sub-level sets of v. Denote by m_u the minimum value level set of u (which may be empty). Obviously, m_u is closed and convex in Ω. Moreover, $\mathrm{int}u(\Omega) = u(\Omega \setminus m_u)$ and so u is locally lipschitz on $\Omega \setminus m_u$. We first prove that, for all $x \in \Omega \setminus m_u$,

$$\partial v(x) = D^-v(x) = z'(u(x))D^-u(x) :$$

Indeed, for $p \in \mathbb{R}^n$,

$$\liminf_{\substack{y \to x \\ u(y) \neq u(x)}} \frac{u(y) - u(x)}{|y - x|} - \frac{(p, y - x)}{z'(u(x))|y - x|} =$$

$$\frac{1}{z'(u(x))} \liminf_{\substack{y \to x \\ u(y) \neq u(x)}} \frac{z(u(y)) - z(u(x)) - (p, y - x)}{|y - x|} +$$

$$+ \frac{1}{z'(u(x))} \lim_{\substack{y \to x \\ u(y) \neq u(x)}} \frac{u(y) - u(x)}{|y - x|} (z'(u(x)) - \frac{z(u(y)) - z(u(x))}{u(y) - u(x)}) =$$

$$\frac{1}{z'(u(x))} \liminf_{\substack{y \to x \\ u(y) \neq u(x)}} \frac{z(u(y)) - z(u(x)) - (p, y - x)}{|y - x|}$$

since u is locally lipschitz and z is differentiable. Also,

$$\liminf_{\substack{y \to x \\ u(y) = u(x)}} -\frac{(p, y - x)}{z'(u(x))|y - x|} = \frac{1}{z'(u(x))} \liminf_{\substack{y \to x \\ u(y) = u(x)}} -\frac{(p, y - x)}{|y - x|}.$$

So $p \in D^- v(x)$ if and only if $\frac{p}{z'(u(x))} \in D^- u(x)$. When $x \in m_u$, it is easy to see that $0 \in D^- u(x)$. Moreover, if $x \in \text{int} m_u$, then $D^- u(x) = \{\nabla u(x)\} = \{0\}$.

Since v is convex, there is a set $\Omega_v \subset \Omega$ such that $\mu(\Omega \setminus \Omega_v) = 0$ and that for all $x_0 \in \Omega_v$ the claims of **theorem 3.1** hold. Set $\Omega_u \doteq \Omega_v \setminus \partial m_u$. We shall prove that Ω_u satisfies the claims of our theorem. It is clear that $\mu(\Omega \setminus \Omega_u) \leq \mu(\Omega \setminus \Omega_v) + \mu(\partial m_u \cap \Omega) = 0$, since $\partial m_u \cap \Omega$ is contained in the boundary of a convex set. Fix $x_0 \in \Omega_u$.

Assume first that $x_0 \in \text{int} m_u$: then there exists a ball around x_0 such that u is constant on this ball. Then the unique maximal solution to $(CP_-)(x_0)$ is $x_{x_0}(t) = x_0 \forall t \in \mathbb{R}$ and all the claims of the theorem are satisfied.

Let now $x_0 \in \Omega_v \setminus m_u$. Then there are $\overline{\omega}_- < 0 < \overline{\omega}_+$ and $y_{x_0} : (\overline{\omega}_-, \overline{\omega}_+) \to \Omega$ such that y_{x_0} is the unique maximal solution on Ω to $(CP_\partial)_v(x_0) \begin{cases} y'(t) \in \partial v(x(t)) \\ y(0) = x_0 \end{cases}$.
Moreover, there is $\omega_-^* \in [\overline{\omega}_-, 0)$ such that $y_{x_0}((\overline{\omega}_-, \omega_-^*]) \subset \Omega \setminus \Omega_v$, $y_{x_0}((\omega_-^*, \overline{\omega}_+)) \in \Omega_v$, and for a.e. $t \in (\omega_-^*, \overline{\omega}_+)$, y_{x_0} is differentiable at t, v is differentiable at $y_{x_0}(t)$

and $y'_{x_0}(t) = \overset{*}{\nabla} v(y_{x_0}(t))$. From the classical theory of monotone differential inclusions we know that $u(y_{x_0}(\cdot))$ is increasing and :

$\lim_{t \downarrow \overline{\omega}_-} |y_{x_0}(t)| = +\infty$ or $\lim_{t \downarrow \overline{\omega}_-} y_{x_0}(t) \in \partial\Omega \cup \partial m_u$;

$\lim_{t \uparrow \overline{\omega}_+} |y_{x_0}(t)| = +\infty$ or $\lim_{t \uparrow \overline{\omega}_+} y_{x_0}(t) \in \partial\Omega$.

Notice that, for all $t > \omega^*_-$, $y_{x_0}(t) \in \Omega_u$: indeed, otherwise $y_{x_0}(t) \in \Omega_v \cap m_u$ and the Cauchy problem $(CP_\partial)_v(y_{x_0}(t))$ admits two distinct solutions (one constant and the other one a reparametrization of y_{x_0}) in contradiction with the choice of Ω_v.

Set $h : (\omega^*_-, \overline{\omega}_+) \to (0, +\infty)$ to be $h(t) \doteq \dfrac{1}{z'(u(y_{x_0}(t)))}$ and consider the Cauchy problem (3.3) $\begin{cases} \lambda'(t) = h(\lambda(t)) \\ \lambda(0) = 0 \end{cases}$. Since h is continuous and strictly positive, (3.3) admits a unique maximal solution $\lambda : (\omega_-, \omega_+) \to (\omega^*_-, \overline{\omega}_+)$ which is continuously differentiable, strictly increasing and $\lim_{t \uparrow \omega_+} \lambda(t) = \overline{\omega}_+$, $\lim_{t \downarrow \omega_-} \lambda(t) = \omega^*_-$.

Define now $x_{x_0}(t) : (\omega_-, \omega_+) \to \Omega_u$ by $x_{x_0}(t) \doteq y_{x_0}(\lambda(t))$.

We will prove that x_{x_0} satisfies the claims of our theorem.

(i) λ is continuously differentiable, y_{x_0} is absolutely continuous so x_{x_0} is absolutely continuous and for a.e. $t \in (\omega_-, \omega_+)$

$$x'_{x_0}(t) = \lambda'(t) y'_{x_0}(\lambda(t)) \in \lambda'(t) D^- v(y_{x_0}(\lambda(t))) =$$

$$= \lambda'(t) z'(u(y_{x_0}(\lambda(t)))) D^- u(y_{x_0}(\lambda(t))) = D^- u(x_{x_0}(t)).$$

Since $x_{x_0}(0) = y_{x_0}(\lambda(0)) = x_0$, x_{x_0} is a solution to $(CP_-)(x_0)$.

Moreover, if $\lim_{t \uparrow \omega_+} |x_{x_0}(t)| \neq +\infty$ then $\lim_{t \uparrow \omega_+} x_{x_0}(t) = \lim_{t \uparrow \omega_+} y_{x_0}(\lambda(t)) = \lim_{s \uparrow \overline{\omega}_+} y_{x_0}(s) \in \partial\Omega$. Also, either $\lim_{t \downarrow \omega_-} |x_{x_0}(t)| = +\infty$ or $\lim_{t \downarrow \omega_-} x_{x_0}(t) = \lim_{t \downarrow \omega_-} y_{x_0}(\lambda(t)) = \lim_{s \downarrow \omega^*_-} y_{x_0}(s) \in \partial\Omega \cup (\Omega \setminus \Omega_v)$, from the properties of ω^*_-. In both situations we find that x_{x_0} is maximal. So to prove (i) we are left to prove that x_{x_0} is the unique solution to $(CP_-)(x_0)$. We will prove only the uniqueness in the past since the uniqueness in the future follows analogously. Assume, by contradiction, that there is $\overline{x} : I_1 \to \Omega_u$, a maximal solution to $(CP_-)(x_0)$, and $t_0 \in I_1 \cap (\omega_-, 0]$ such that $\overline{x}(t_0) \neq x_{x_0}(t_0)$. Since $x_{x_0}(t_0) \notin m_u$ and $\overline{x}(t_0) \notin m_u$, there is $\varepsilon > 0$ such that $\overline{x}(t) \notin m_u$, $x_{x_0}(t) \notin m_u$ for all $t \in (t_0 - \varepsilon, 0]$.

Let $\overline{h} : (t_0 - \varepsilon, 0] \to (0, +\infty)$ be given by $\overline{h}(t) \doteq z'(u(\overline{x}(t)))$. Consider the Cauchy problem (3.4) $\begin{cases} \mu'(t) = \overline{h}(\mu(t)) \\ \mu(0) = 0 \end{cases}$. As above, (3.4) admits a unique maximal solution $\mu : (\alpha, 0] \to (t_0 - \varepsilon, 0]$ which is continuously differentiable, strictly increasing and $\lim_{t \downarrow \alpha} \mu(t) = t_0 - \varepsilon$. Let $\overline{y} : (\alpha, 0] \to \Omega$ be given by $\overline{y}(t) = \overline{x}(\mu(t))$. Then $\overline{y}$ solves $(CP_\partial)_v(x_0)$ so it has to coincide with y_{x_0} on $(\alpha, 0]$. So, for all $t \in (\alpha, 0]$, $\mu'(t) = z'(u(\overline{x}(\mu(t)))) = z'(u(\overline{y}(t))) = z'(u(y_{x_0}(t)))$ and

$$\mu'(\lambda(s)) = z'(u(y_{x_0}(\lambda(s)))) = \frac{1}{\lambda'(s)} \; \forall s \in (\lambda^{-1}(\alpha), 0].$$

Since $\mu(\lambda(0)) = 0$, $\mu(\lambda(s)) = s$ for all $s \in (\lambda^{-1}(\alpha), 0]$ and $\lambda^{-1}(\alpha) = t_0 - \varepsilon$. Then

$$x_{x_0}(s) = y_{x_0}(\lambda(s)) = \overline{y}(\lambda(s)) = \overline{x}(\mu(\lambda(s))) = \overline{x}(s) \text{ on } (t_0 - \varepsilon, 0]$$

a contradiction.

(ii) For all $t \in (\omega_-, \omega_+)$, $u(y_{x_0}(\lambda(t))) \in \operatorname{int} u(\Omega)$ and for a.e. $t \in (\omega_-, \omega_+)$ y_{x_0} is differentiable at $\lambda(t)$, v is differentiable at $y_{x_0}(\lambda(t))$ and $y'_{x_0}(\lambda(t)) = \nabla v(y_{x_0}(\lambda(t)))$. For such t, x_{x_0} is differentiable at t, $u = z^{-1} \circ v$ is differentiable at $x_{x_0}(t) = y_{x_0}(\lambda(t))$ and

$$x'_{x_0}(t) = \lambda'(t) y'_{x_0}(\lambda(t)) = \frac{1}{z'(u(y_{x_0}(\lambda(t))))} z'(u(y_{x_0}(\lambda(t)))) \nabla u(x_{x_0}(t)) = \nabla u(x_{x_0}(t)).$$

∎

Chapter 4. A Qualitative Result for a Class of Differential Inclusions

In the previous chapter, we have considered the Cauchy Problem

$$(CP_\partial)(x_0)\begin{cases} x'(t) \in \partial u(x(t)) \\ x(0) = x_0 \end{cases}$$

where u is a convex function with domain in $\mathbb{R}^n$ and we have proved that the set N, consisting of those x_0 for which $(CP_\partial)(x_0)$ admits at least two solutions has null measure, and that the closure of N along the trajectories of (CP_∂) has null measure, too. In this chapter we prove that these results are true for a larger class of differential inclusions.

We shall assume that T satisfies the following

Assumption S

$T : DomT \subset \mathbb{R}^n \to \mathbb{R}^n$ is a locally bounded upper semicontinuous multifunction with convex and compact values and with open domain DomT; Moreover, there exists $c > 0$ such that $(u - v, x - y) \geq -c|x - y|^2$ for all $x, y \in \Omega, u \in Tx, v \in Ty$.

■

We may state now the first theorem of this chapter :

Theorem 4.1

Let T satisfy **Assumption S**. *For $x_0 \in DomT$, consider the Cauchy Problem*
$$(CP)(x_0)\begin{cases} x'(t) \in T(x(t)) \\ x(0) = x_0 \end{cases}$$
Set $N \doteq \{x_0 \in DomT : (CP)(x_0)$ admits at least two distinct (maximal) solutions $\}$. Then $\mu(N) = 0$. ■

Proof Since T is upper semicontinuous and it has convex and compact values, $(CP)(x_0)$ admits at least one solution for every $x_0 \in$ Dom$T$. It is easy to see that if $x(\cdot)$ and $y(\cdot)$ are, respectively, solutions to $(CP)(x_0)$ and $(CP)(y_0)$ then, for every $t > 0$ such that both $x(-t)$ and $y(-t)$ are defined,

$$|x(-t) - y(-t)| \leq |x_0 - y_0|e^{ct}.$$

This implies that $(CP)(x_0)$ admits a unique solution in the past. We use the notation $x_{x_0}(\cdot)$ to denote the solution to $(CP)(x_0)$ at negative times.

For $x \in \mathrm{Dom}T$ and $t \in \mathbb{R}$, set $\mathcal{A}_t(x) \doteq \{y \in \mathrm{Dom}T : \exists \overline{x}$, a solution to $(CP)(x)$, such that $\overline{x}(t) = y\}$. Since T is locally bounded, there exists $\varepsilon > 0$ such that, for all $t \in (0, \varepsilon)$, $\mathcal{A}_t(x)$ is compact and connected ([A.C.]).

For $t > 0$, set $D_t \doteq \{x_0 \in \mathrm{Dom}T$ such that $x_{x_0}(-t)$ is defined$\}$. Define $f_t : D_t \to \mathrm{Dom}T$ by $f_t(x_0) = x_{x_0}(-t)$. f_t is lipschitzean with constant e^{ct}. We say that x is a *critical value* for f_t if and only if there is $y \in f_t^{-1}(x)$ such that either the jacobian of f_t at y is 0 or f_t is not differentiable at y. Denote by V_{f_t} the set of critical values of f_t. The co-area formula ([E.G.]) assures that $\mu(V_{f_t}) = 0$.

Claim 4.2 *For every $x_0 \in N$, there is an open interval $I_{x_0} \subset (0, +\infty)$ such that x_0 is a critical value for f_t, $\forall t \in I_{x_0}$.* ■

Proof of Claim 4.2 Since $x_0 \in N$, there is $\bar{t} > 0$ such that $\mathcal{A}_{\bar{t}}(x_0)$ contains at least two points. Therefore, there is x, a solution to $(CP)(x_0)$, and there are $t_0 > 0$ and $\varepsilon > 0$ such that, for every $t \in (0, \varepsilon)$, $\mathcal{A}_t(x(t_0))$ contains at least two points. Moreover, there is $\delta > 0$, $\delta < \varepsilon$, such that, for all $t \in (0, \delta)$, $\mathcal{A}_t(x(t_0))$ is compact and connected. Fix $t \in (0, \delta)$. Since $\mathcal{A}_t(x(t_0))$ is connected and contains two distinct points, it contains an infinity of points and so, being compact, it contains at least an acumulation point y_0. So there are $y_i \to y_0$, $y_i \neq y_0$ such that $f_{t+t_0}(y_i) = f_{t+t_0}(y_0) = x_0$. If f_{t+t_0} is differentiable at y_0, denote by v a vector of norm 1 in $\mathbb{R}^n$ such that a subsequence of $\dfrac{y_i - y_0}{|y_i - y_0|}$ converges to v. Then $df_{t+t_0}(y_0)v = 0$ and thus the jacobian of f_{t+t_0} at y_0 is 0. We may then set

$I_{x_0} \doteq (t_0, t_0 + \delta)$. ■

Return to the proof of the theorem. Set $G \doteq \bigcup_{x_0 \in N} I_{x_0}$. G is an open subset of $\mathbb{R}$, so we may write $G = \bigcup_{k \in \mathbb{N}} J_k$, where J_k are open and two-by-two disjoint intervals. For $k \in \mathbb{N}$, choose $t_k \in J_k$. Then $N \subset \bigcup_{k \in \mathbb{N}} V_{f_{t_k}}$ and so $\mu(N) = 0$. ■

Theorem 4.3

Let T satisfy **Assumption S**. *Let $\Omega \subset DomT$ be open. Then there exists $\Omega_T \subset \Omega$ such that $\mu(\Omega \setminus \Omega_T) = 0$ and, for every $x_0 \in \Omega_T$, the Cauchy Problem*

$$(CP)(x_0) \begin{cases} x'(t) \in T(x(t)) \\ x(0) = x_0 \ , \ x(t) \in \Omega \end{cases}$$

admits the unique (maximal) solution x_{x_0}, which is defined on some open interval (ω_-, ω_+). Moreover, there is ω_-^*, $\omega_- \leq \omega_-^* < 0$, such that $x_{x_0}((\omega_-, \omega_-^*]) \subset \Omega \setminus \Omega_T$ and $x_{x_0}((\omega_-^*, \omega_+)) \subset \Omega_T$. ■

Proof The proof is analogous to the proof of **Theorem 3.1**. Define the set $A = \{x_0 \in \Omega : \exists \ (t_i)_i, t_i \downarrow 0 \text{ such that } x_{x_0}(-t_i) \in N\}$. We show that $\mu(A) = 0$.

It will be enough to prove that $\mu(A \cap B(z, R)) = 0$ for a generic closed ball $B(z, R)$ contained in Ω. As in **theorem 3.1**, for $\lambda < R$, define $D_\lambda = B(z, R - \lambda)$, $M_\lambda = \sup\{|v| : v \in T(x), x \in D_\lambda\}$. For $\varepsilon > 0$ define the mapping $f_\varepsilon : D_\lambda \to \mathbb{R}^n$ by $f_\varepsilon(x_0) = x_{x_0}(-\varepsilon)$. Since one has uniqueness in the past to (CP) for all initial data, for $\varepsilon < \dfrac{\lambda}{2M_{\frac{\lambda}{2}}}$, f_ε is well defined. Moreover, $D_{\lambda + \varepsilon M_\lambda} \subset f_\varepsilon(D_\lambda) \subset D_{\frac{\lambda}{2}}$. f_ε is lipschitz with constant $e^{c\varepsilon}$. We conclude, analogously to the proof of the theorem from the previous chapter, that $D_\lambda \cap A \subset f_\varepsilon^{-1}(f_\varepsilon(D_\lambda \cap A)) \subset f_\varepsilon^{-1}(f_\varepsilon(D_\lambda) \cap N)$. Since D_λ is compact, f_ε is lipschitz and $\mu(N) = 0$, $f_\varepsilon(D_\lambda) \setminus N$ is measurable and $\mu(f_\varepsilon(D_\lambda) \setminus N) = \mu(f_\varepsilon(D_\lambda))$. Moreover $Z \subset N$, where $Z = \{y \in f_\varepsilon(D_\lambda) : \exists x_1 \neq x_2 \text{ such that } f_\varepsilon(x_1) = f_\varepsilon(x_2) = y\}$. Apply now **lemma 3.2** to f_ε and to $Y = f_\varepsilon(D_\lambda) \setminus N$. Then

$$\mu(f_\varepsilon(D_\lambda)) = \mu(f_\varepsilon(D_\lambda) \setminus N) \leq e^{nc\varepsilon} \mu_*(f_\varepsilon^{-1}(f_\varepsilon(D_\lambda) \setminus N))$$

We conclude that

$\mu^*(D_\lambda \cap A) \leq \mu(D_\lambda) - e^{-nc\varepsilon}\mu(D_{\lambda+\varepsilon M_\lambda})$. By letting $\varepsilon \to 0$ one has that $\mu(D_\lambda \cap A) = \mu^*(D_\lambda \cap A) = 0$.

Set now $\Omega_T = \Omega \setminus A$ and let $x_0 \in \Omega_u$. By the definition of Ω_T, $(CP)(x_0)$ has a unique maximal solution, $x_{x_0} : (\omega_-, \omega_+) \to \Omega$. Since Ω is open, the maximal interval of definition will be open too. Set $\omega_-^* \doteq \inf\{t \in (\omega_-, \omega_+) : x_{x_0}(t) \in \Omega_T\}$. By reasoning as in the proof of **theorem 3.1**, it is easy to see that all the claims of the theorem are satisfied. ∎

We denote by $D^-u(x)$ the subdifferential of u at x (see (3.1)).

Corollary 4.4 *Let $\Omega \subset \mathbb{R}^n$ be open and $u : \Omega \to \mathbb{R}$ be semiconvex. Then there exists $\Omega_u \subset \Omega$ such that $\mu(\Omega \setminus \Omega_u) = 0$ and, for every $x_0 \in \Omega_u$, the Cauchy Problem*

$(CP_-)(x_0)\begin{cases} x'(t) \in D^-(u(x(t))) \\ x(0) = x_0 \ , \ x(t) \in \Omega \end{cases}$ *admits a unique (maximal) solution x_{x_0}, which is defined on some open interval (ω_-, ω_+). Moreover, there is ω_-^*, $\omega_- \leq \omega_-^* < 0$, such that $x_{x_0}((\omega_-, \omega_-^*]) \subset \Omega \setminus \Omega_u$ and $x_{x_0}((\omega_-^*, \omega_+)) \subset \Omega_u$.* ∎

Proof Since u is semiconvex, there is $c > 0$ such that $g(x) \doteq u(x) + \frac{1}{2}c|x|^2$ is convex ([E.]). From the definition of D^- it is easy to see that $p \in D^-u(x)$ if and only if $p + cx \in \partial g(x)$. The monotony of ∂g assures that we can apply the results of the previous theorems to (CP_-). ∎

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