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Form Factors in Two-Dimensional Quantum Field Theory

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CANDIDATE
Gesualdo Delfino

SUPERVISOR
Dr. Giuseppe Mussardo

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TRIESTE

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Introduction

Two-dimensional models have been during last years an extraordinary laboratory for testing fundamental ideas of Quantum Field Theory (QFT). This is due to two basic reasons. The first one is the existence in two-dimensions of theories possessing an infinite number of integrals of motion, the so-called integrable models. It was realized at the end of the seventies that this circumstance has important consequences on the scattering theory since it completely prevents particle production and permits the factorization of multiparticle amplitudes in a product of two-body amplitudes. The additional dynamical requirement that stable asymptotic particles must be associated to the poles of the scattering amplitudes (the bootstrap principle) permitted the determination of the exact S-matrices of many lagrangian integrable theories, as the Sine-Gordon model and the O(n) non-linear sigma model [1].

The second reason relies in the infinite dimensional character of conformal symmetry in two dimensions which allowed the complete solution (i.e. the determination of all correlation functions) of an infinite number of QFTs corresponding to fixed points of the renormalization group [2, 3]. Many of these theories have been identified as describing the universality classes of widely studied two-dimensional statistical systems.

An important progress in the understanding of the general structure of the space of two-dimensional QFTs was made with the discover that some particular perturbations of conformal field theories preserve an infinite number of conserved currents and lead to new not-scale invariant integrable models whose exact S-matrices can be determined by the bootstrap method [4]. A particularly relevant example of a theory solved in this way is provided by the Ising model in a magnetic field.

It is commonly believed that the S-matrix encodes all the informations necessary to determine the off-shell physics of a theory, and in particular the correlation functions. In the domain of integrable theories, an important step in this direction is represented by the so-called form factor bootstrap [5, 6]. Form factors are matrix elements of local operators between asymptotic states. A series of requirements on their analytic structure (strongly simplified by the integrability condition) gives rise to functional equations which have been solved in a large number of cases. Once the form factors are known, correlation functions can be expressed as infinite sums over intermediate multiparticle states. This thesis is mainly devoted to the description of this approach to the computation of correlation functions and to his extension to massless and not-translational invariant integrable theories.

In chapter 1, after reviewing the basic lines of the S-matrix and form factor bootstrap for massive integrable models, we discuss the $\Phi_{1,3}$ deformation of the minimal non-unitary model $\mathcal{M}_{3,5}$. Such model appears as particularly suitable for detailed analysis since it exhibits both simple short distance structure and scattering theory. We compute in closed form the exact matrix elements of the trace of the energy-momentum tensor $\Theta(x)$ which enter the spectral representation of the two-point correlation function. Comparison with conformal perturbation theory (see appendix A) shows that the first few terms in the expansion over intermediate massive states are sufficient to reproduce with remarkable precision the correlation function up to scales of the order $10^{-2}\xi$, where ξ is the correlation length of the theory. This extremely fast convergence of the form factors series is a general feature of massive integrable models and is the basis of the numerical computation of correlation functions in physically relevant problems, such as dilute polymers and percolation.

The functional equations for form factors are obtained without reference to any particular operator, so that finding their most general solution amounts to classify the operator content of the quantum theory under investigation. This observation shows that the form factor approach provides a well founded theoretical framework for studying the mapping and reduction phenomena among different integrable models. A particularly suggestive example of this kind is considered in chapter 2, where the form factor approach is used to investigate the so-called "staircase model". This model, that can be formulated as a scattering theory for a single scalar particle of mass m, is characterized by a remarkable off-shell pattern [37]: the thermodynamic Bethe ansatz method (see appendix B) reveals that the renormalization group trajectory associated to this theory crosses an infinite series of fixed points before developing a finite correlation length at very large distances. In particular, the last fixed point visited by the trajectory before the crossover to the massive region corresponds to the Ising model. The scattering amplitude of the Staircase model depends on a real parameter θ_0 and can be considered as an analytic continuation to complex (non-perturbative) values of the coupling constant of the S-matrix of the Sinh-Gordon model. Using this analytic continuation, we study the behaviour of the form factors of Sinh-Gordon model varying

θ_0 and we show that, for bounded values of the momenta of the particles ($p << me^{\theta_0}$), the Sinh-Gordon model actually collapses into the thermally perturbed Ising model. In particular, we establish that the elementary field $\phi(x)$ and the trace of the energy-momentum tensor $\Theta(x)$ of the Sinh-Gordon model are mapped into the magnetization operator $\sigma(x)$ and the energy operator $\varepsilon(x)$ of the Ising model, respectively. It is worth to stress that the reduction just described is essentially different from the usual quantum group reduction. Indeed, in the latter scheme the reduced models are obtained as projection of some original theory at specific values of the coupling constant. On the contrary, the staircase model provides an explicit example of a theory able to show a rich variety of different behaviours depending on the energy scale at which it is observed.

For higher energy ranges the analytically continued Sinh-Gordon model is expected to reduce to a massless theory describing the crossover among two successive fixed points. In spite of some subtleties in the definition of two-dimensional massless scattering, many important progresses have been made in recent years in the description of massless integrable theories in terms of factorized S-matrices [7, 8, 9]. We show in chapter 3 that also the form factor bootstrap can be extended to the massless case once the differences in the analytic structure with respect to massive models have been properly identified. Although the knowledge of form factors is itself a relevant theoretical result (they can be used, for instance, to prove the locality of the theory [6]), their utility for the computation of correlation functions in massless theories is a priori doubtful due to the absence of any infrared cutoff. We investigate this important issue taking as an example the flow from tricritical to critical Ising model; among other interesting features, this model provides the simplest example of spontaneous supersymmetry breaking. A factorized S-matrix for this integrable flow can be constructed in terms of left and right-mover neutral fermions [7]. We compute the form factors of the trace of the energy-momentum tensor $\Theta(x)$ and of the magnetization operator $\sigma(x)$ and show that, when inserted in the spectral representation for correlation functions, they give rise to two different situations. The expansion of the two-point correlator of $\sigma(x)$ turns out to be infrared divergent; nevertheless, we can show that the form factor series, after regularization and resummation, gives the expected scale dimension in the large distance limit. The need for resummation can be traced back to the fact that the magnetization operator lacks a local representation in terms of the fermions. On the contrary, $\Theta(x)$ has a simple infrared fermionic expansion and its two-point correlator exhibits fast convergence properties analogous to those observed in the massive case. Although further investigations are certainly needed, we expect that a similar pattern generalizes to other massless integrable models.

The main purpose of chapter 4 is to show that the bootstrap approach can be successfully extended to integrable models with linear inhomogeneities and that the computation of the correlation functions for those systems can be achieved by means of a suitable generalization of the form factor techniques. We analyze the general situation in which translation invariance is broken by the presence of defect lines allowing reflection and transmission processes. The particular case of vanishing transmission corresponds to the boundary field theories which have recently received a lot of attention in view of their potential application to a wide class of physical situations. In the S-matrix approach, the problem is reduced to the determination of the reflection and transmission amplitudes of the bulk particles with the "impurities". We show that the requirement of factorization of scattering amplitudes severely constrains the possible bulk S-matrices if both reflection and transmission are different from zero. Correlation functions can be computed passing to the quantization scheme in which the spatial dimension is parallel to the defect line, so that the hilbert space and, consequently, the form factors of the theory are the same as in the bulk case. Then, a spectral representation can be introduced in which the defect (or boundary, in the purely reflecting case) appears as an additional operator localized in time whose form factors are expressed in terms of the reflection and transmission amplitudes. We use this technique to study, directly in the continuum limit, the Ising model with a defect line and to exhibit its ultraviolet non-universal behaviour.

Chapter 1

Bootstrap methods in massive integrable theories

1.1 Factorized S-matrices

Two-dimensional Quantum Field Theory (QFT) is nowadays the most advanced subject in the domain of relativistic field theory. This fact relies on a fundamental peculiarity of two dimensional models, i.e. the existence of theories admitting an infinite number of integrals of motion, the so-called integrable models. Among those, an important role is played by the models corresponding to the fixed points of the renormalization group, namely the Conformal Field Theories (CFTs) [2, 3]. Here integrability emerges in the form of a very rigid algebraic structure based on the infinite dimensional Virasoro algebra which permits the complete solution of the theory, namely the computation of all correlation functions. As shown in ref.[4], an infinite number of integrals of motion can survive when perturbing a CFT by particular relevant operators which lead to not scale-invariant (massive or massless) integrable models. While the algebraic language characteristic of CFTs is no longer useful for the study of such models, the most effective approach turns out to be the one based on the direct computation of the exact S-matrix of the theory. The origin of the effectiveness of this method can be understood analyzing the main consequences induced on the scattering theory by the presence of infinite integrals of motion [1].

Consider a relativistic scattering theory containing N types of particles with masses m_a, $a = 1, \ldots, N$. We will denote $A_a(p^\mu)$ the particle a with two-momentum p_μ satisfying the mass-shell constraint $p_\mu p^\mu = m_a^2$. We also define the light-cone components of p^μ

$$p = p^0 + p^1 \; ; \qquad \bar{p} = p^0 - p^1 \quad . \tag{1.1}$$

The states

$$|A_{a_1}(p_1^\mu)A_{a_2}(p_2^\mu)\dots A_{a_n}(p_n^\mu)>_{in(out)} \tag{1.2}$$

form the basis of the asymptotic in (out)-states which is assumed to be complete in a local field theory. The S-matrix is the operator connecting these two sets of asymptotic states. Let's assume that the theory possesses an infinite number of integrals of motion P_s with different spin s, whose action on the asymptotic states (1.2) is given by

$$P_s|A_{a_1}(p_1^\mu)\dots A_{a_n}(p_n^\mu)>_{in(out)}=\sum_{i=1}^{n}\omega_s^{a_i}(p_i^\mu)|A_{a_1}(p_1^\mu)\dots A_{a_n}(p_n^\mu)>_{in(out)} \quad . \tag{1.3}$$

The additivity of the contributions of the largely separated particles contained in asymptotic states follows from the fact that P_s is assumed to be the integral of a local density. Moreover the spin structure dictates the following form for the eigenvalues $\omega_s^a(p)$

$$\omega_s^a(p^\mu) = k_s^a p^s \ , \tag{1.4}$$

where p is defined in eq.(1.1) and k_s^a are constants.

If we now consider a generic scattering process in which the state $|A_{a_1}(p_1^\mu)\dots A_{a_n}(p_n^\mu)>_{in}$ evolves into the final state $|A_{b_1}(q_1^\mu)\dots A_{b_m}(q_m^\mu)>_{out}$, conservation of charges P_s implies

$$\sum_{i=1}^{n}\omega_s^{a_i}(p_i^\mu) = \sum_{j=1}^{m}\omega_s^{b_j}(q_j^\mu) \quad . \tag{1.5}$$

Since we required the existence of infinite charges with different spin, eq.(1.5) is a system of infinite equations for a finite number of unknowns which will be satisfied in general only if $n = m$ and the set of initial two-momenta $\{p_1^\mu,\dots,p_n^\mu\}$ equals the set of final two-momenta $\{q_1^\mu,\dots,q_m^\mu\}$. In other words, we conclude that

a) only elastic scattering processes are admitted (i.e. there is no particle production) in which each initial two-momentum is individually conserved (this automatically implies the conservation of the number of particles with a given mass).

All this does not mean that the scattering is trivial since, for instance, if the initial state contains particles with the same mass, they can exchange momenta in the final state, or be replaced by other particles with the same mass.

The second fundamental simplification induced by integrability on the scattering theory can be intuitively understood by the following argument [10]. Consider the collision process of three particles with spatial momenta k_1, k_2, k_3; the three possible space-time diagrams are depicted in fig. 1. Let's now apply to the initial state the operator $\exp(iaP_s)$, with $s > 1$ and a some real parameter. Since the charge is locally conserved and the particles are initially widely separated, the exponential operator will act separately on the wave packet of the i-th particle which we write in the form

$$|\psi_i(x)>=|\int_{-\infty}^{+\infty} dk \, \exp[ik(x-x_i)]\, f(k)> \ , \tag{1.6}$$

where $f(k)$ is any reasonable function peaked at $k = k_i$. We have

$$\begin{aligned} |\psi'_i(x)> &\equiv \exp(iaP_s)|\psi_i(x)> \\ &= |\int_{-\infty}^{+\infty} dk \, \exp[ia\omega_s(k)] \exp[ik(x-x_i)]\, f(k)> \quad . \end{aligned} \tag{1.7}$$

By stationary phase approximation, we see that $\psi_i(x)$ is peaked at x_i, while ψ'_i is peaked around the value

$$x = x_i - a\left[\frac{d}{dk}\omega_s(k)\right]_{k=k_i} \quad . \tag{1.8}$$

Then, recalling eq.(1.4), we conclude that the exponential operator with $s > 1$ displaces the "centre of mass" of the i-th wave packet by an amount depending on the mean momentum k_i, so that taking the parameter a sufficiently large we can alter the relative position between any two particles with different momenta. As a consequence, we can pass in fig. 1 from a diagram to another applying $\exp[iaP_s]$ to the initial state. Since this operator commutes with the S-matrix, we conclude that the amplitudes for the three possible sequences of collisions in fig. 1 are equal and factorize in the product of three two-body S-matrices. With obvious notations

$$S(123) = S(23)S(13)S(12) = S(12)S(13)S(23) \quad . \tag{1.9}$$

These cubic relations, also known as *star-triangle* or *Yang-Baxter* equations, are trivially satisfied if the $S(ij)$ are ordinary functions, but give rise on the contrary to severe constraints to the two-body S-matrix in the case the spectrum of the theory contains particles with the same mass which differ for quantum numbers related to some internal symmetry.

Generalizing the displacement argument just illustrated one concludes that

b) any n-particle scattering amplitude can be factorized in $n(n-1)/2$ two-particle amplitudes as if the n-particle process consists in a sequence of pair collisions [11].

Hence, we see that the problem of obtaining the complete S-matrix for a two-dimensional scattering theory possessing non-trivial integrals of motion reduces to the determination of the two-body S-matrix. We will now show that severe constraints on the two-particle scattering amplitudes are imposed by the general requirements of analyticity, unitarity and crossing symmetry.

According to the fundamental properties a) and b) above, we define the n-particle scattering amplitude for a relativistic integrable model by the relation

$$|A_{a_1}(p_1^\mu)A_{a_2}(p_2^\mu)\dots A_{a_n}(p_n^\mu)>_{in}= S_{a_1a_2\dots a_n}^{b_1b_2\dots b_n}(p_1^\mu,p_2^\mu,\dots,p_n^\mu)|A_{b_1}(p_1^\mu)A_{b_2}(p_2^\mu)\dots A_{b_n}(p_n^\mu)>_{out} \tag{1.10}$$

(here and in the following summation over repeated indices is understood). Due to factorization, we immediately restrict our attention to the two-particle process

$$|A_a(p_1^\mu)A_b(p_2^\mu)>_{in}= S_{ab}^{cd}(p_1^\mu,p_2^\mu)|A_c(p_1^\mu)A_d(p_2^\mu)>_{out} \quad . \tag{1.11}$$

The two-body S-matrix actually depends only on the Mandelstam variable

$$s=(p_1^\mu+p_2^\mu)^2 \ , \tag{1.12}$$

which is the only independent relativistic invariant which can be constructed out of the momenta entering the process. It is easy to show that as an analytic function of the (formally) complex variable s, the matrix S_{ab}^{cd} must exhibit two square root branch cuts along the real axis for $s\leq(m_a-m_b)^2$ and $s\geq(m_a+m_b)^2$ (see fig. 2a). Indeed, above the two-particle threshold $s=(m_a+m_b)^2$, the unitarity condition for the S-matrix reads

$$S_{ab}^{ef}(s)\ \left[S_{ef}^{cd}(s)\right]^*=\delta_a^c\delta_b^d \ , \tag{1.13}$$

where the star denotes complex conjugation. Using the standard real analyticity property

$$\left[S_{ab}^{cd}(s)\right]^*=S_{ab}^{cd}(s^*) \ , \tag{1.14}$$

we see that eq.(1.13) requires the presence of a cut for real values of s above the threshold. Denoting by $s+i0$ and $s-i0$ the values of the upper and lower edges of the cut respectively, we have

$$S_{ab}^{ef}(s+i0)S_{ef}^{cd}(s-i0)=\delta_a^c\delta_b^d \ . \tag{1.15}$$

This equation precisely implies that the matrix $S_{ab}^{cd}(s)$ takes on again its initial value when analytically continued twice around the branch point $s=(m_a+m_b)^2$.

On the other hand, in a relativistic theory crossing symmetry implies the coincidence between the amplitudes for the process $A_a(p_1^\mu)A_b(p_2^\mu)\to A_c(p_1^\mu)A_d(p_2^\mu)$ and the crossed one $A_a(p_1^\mu)A_{\bar d}(-p_2^\mu)\to A_c(p_1^\mu)A_{\bar b}(-p_2^\mu)$ (the bar denotes charge conjugation) [12, 13]. Then we get the equation

$$S_{ab}^{cd}(s+i0)=S_{a\bar d}^{c\bar b}(2m_a^2+2m_b^2-s-i0) \ , \tag{1.16}$$

which maps the unitarity cut into an analogous one originating at $s = (m_a - m_b)^2$. The presence of such cuts gives to the s-plane the structure of a Riemann surface with several sheets; the first of them is usually referred to as the "physical" sheet.

Besides the two cuts, the only other singularities of the two-particle S-matrix are supposed to be simple poles located in the interval $(m_a - m_b)^2 < s < (m_a + m_b)^2$ of the real axis and corresponding to stable bound states.

The analytic structure of scattering amplitudes is substantially simplified if we introduce the parameterization in terms of the rapidity variables θ which are related to the energy and momentum of a particle of mass m by the relations

$$\begin{aligned} p^0 &= m\cosh\theta \\ p^1 &= m\sinh\theta \ . \end{aligned} \tag{1.17}$$

In terms of rapidities relation (1.12) becomes

$$s = \left(p_1^\mu(\theta_1) + p_2^\mu(\theta_2)\right)^2 = m_a^2 + m_b^2 + 2m_a m_b \cosh\theta \ , \tag{1.18}$$

where we defined $\theta \equiv \theta_1 - \theta_2$. This relation shows that, modulo $2\pi i$-translations, two values of the rapidity difference (θ and $-\theta$) correspond to the same value of s. Thus the physical sheet of the s-plane can be mapped into the strip $0 \leq Im\theta \leq \pi$ of the θ-plane as shown in fig. 2 in such a way that the two-particle S-matrix becomes a meromorphic function of the rapidity difference (the other sheets of the Riemann surface are mapped into successive strips of width $i\pi$ in the θ-plane). According to this mapping, eqs.(1.15) and (1.16) are immediately translated into

$$S_{ab}^{ef}(\theta) S_{ef}^{cd}(-\theta) = \delta_a^c \delta_b^d \tag{1.19}$$

and

$$S_{ab}^{cd}(\theta) = S_{a\bar{d}}^{c\bar{b}}(i\pi - \theta) \tag{1.20}$$

respectively.

The rapidity formalism permits also a very elegant and suggestive algebraic description for the scattering processes in integrable theories. In order to illustrate this point, let's rewrite eq.(1.11) in the form

$$|A_a(\theta_1)A_b(\theta_2) >_{in} = S_{ab}^{cd}(\theta_1 - \theta_2) |A_c(\theta_1)A_d(\theta_2) >_{out} \ . \tag{1.21}$$

In doing this we have to fix $\theta_1 > \theta_2$ in order to specify that the S-matrix is computed in this case on the upper edge of the unitarity cut in the s-plane. Using eq.(1.19) we also obtain

$$|A_a(\theta_1)A_b(\theta_2) >_{out} = S_{ab}^{cd}(\theta_2 - \theta_1) |A_c(\theta_1)A_d(\theta_2) >_{in} \ , \tag{1.22}$$

where S_{ab}^{cd} is now taken on the lower edge of the cut since $\theta_2 - \theta_1 < 0$. Hence, we see that eqs. (1.21) and (1.22) are both encoded in the equation

$$|A_a(\theta_1)A_b(\theta_2)>= S_{ab}^{cd}(\theta_1-\theta_2)|A_d(\theta_2)A_c(\theta_1)> , \qquad (1.23)$$

provided we define the state $|A_a(\theta_i)A_b(\theta_j)>$ to be an in-state for $\theta_i > \theta_j$ and an out-state for $\theta_i < \theta_j$. This ordering prescription generalizes in natural way to the n-particle states since, due to factorization, the analytic structure of the n-particle amplitude is reduced to that of each two-particle sub-channel. Then we define

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)\dots A_{a_n}(\theta_n)>= \begin{cases} |A_{a_1}(\theta_1)A_{a_2}(\theta_2)\dots A_{a_n}(\theta_n)>_{in} & \text{if } \theta_1>\theta_2>\dots>\theta_n \\ |A_{a_1}(\theta_1)A_{a_2}(\theta_2)\dots A_{a_n}(\theta_n)>_{out} & \text{if } \theta_1<\theta_2<\dots<\theta_n \end{cases} . \qquad (1.24)$$

Moreover, if we introduce the particle creation operators $A_a(\theta)$ through the definition

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)\dots A_{a_n}(\theta_n)>= A_{a_1}(\theta_1)A_{a_2}(\theta_2)\dots A_{a_n}(\theta_n)|0> \qquad (1.25)$$

where $|0>$ is the vacuum state, eq.(1.23) immediately leads to the so-called Faddeev-Zamolodchikov algebra

$$A_a(\theta_1)A_b(\theta_2) = S_{ab}^{cd}(\theta_1-\theta_2)A_d(\theta_2)A_c(\theta_1) . \qquad (1.26)$$

This algebra encodes the main properties of scattering theory for integrable models (apart from crossing symmetry which is an additional constraint peculiar to the relativistic case). Indeed, the unitarity relation (1.19) appears as the consistency condition of the algebra (1.26) when it is applied twice. On the other hand, the Yang-Baxter equation (1.9) can be obtained requiring the associativity of the Faddeev-Zamolodchikov algebra: starting with the product $A_a(\theta_1)A_b(\theta_2)A_c(\theta_3)$, there are two ways (differing for the sequence in which the pair commutations are performed) to get the product $A_i(\theta_3)A_h(\theta_2)A_f(\theta_1)$ in which the order of the rapidities is reversed; if the algebra (1.26) is associative the two procedures must give the same result and we get the equation

$$S_{ab}^{de}(\theta_{12})S_{dc}^{fg}(\theta_{13})S_{eg}^{hi}(\theta_{23}) = S_{bc}^{de}(\theta_{23})S_{ae}^{gi}(\theta_{13})S_{gd}^{fh}(\theta_{12}) , \qquad (1.27)$$

where $\theta_{ij} \equiv \theta_i - \theta_j$. This equation can be graphically represented as in fig. 3.

The basic dynamical principle entering the determination of the S-matrix is the so-called bootstrap condition. The poles located on the real axis of the s-plane between the two branch points are mapped into the interval $(0, i\pi)$ of the imaginary axis of the θ-plane. If the amplitude $S_{ab}^{de}(\theta)$ has a simple pole with positive residue at $\theta = iu_{ab}^c$ (due to real analyticity $S_{ab}^{de}(\theta)$ is real for purely imaginary rapidities), then the particle A_c with mass

$$m_c^2 = s(iu_{ab}^c) = m_a^2 + m_b^2 + 2m_am_b\cos u_{ab}^c \qquad (1.28)$$

is a direct channel "bound state" of A_a and A_b. According to the last equation the quantity $\bar{u}_{ab}^c \equiv \pi - u_{ab}^c$ can be interpreted as the internal angle of the triangle with sides m_a, m_b, m_c so that we get the relation (see fig. 5)

$$u_{ab}^c + u_{b\bar{c}}^{\bar{a}} + u_{\bar{c}a}^{\bar{b}} = 2\pi \ . \tag{1.29}$$

We require that the particle A_c belongs to the set of "fundamental" particles A_{a_i} $(i = 1, 2, \ldots, N)$ used to construct the asymptotic states. The pole term

$$S_{ab}^{de}(\theta) \simeq \frac{i g_{ab}^c g_c^{de}}{\theta - i u_{ab}^c} \tag{1.30}$$

corresponds to the diagram in fig. 4, where the vertices represent the three-particle couplings g. If the residue turns out to be negative, then, according to (1.20), the pole corresponds to a bound state in the crossed channel. A displacement argument analogous to that used to obtain the Yang-Baxter equation leads to the bootstrap equation

$$g_{ab}^c S_{dc}^{ef}(\theta) = g_{hj}^f S_{da}^{ih}(\theta - \bar{u}_{\bar{c}a}^{\bar{b}}) S_{ib}^{ej}(\theta + \bar{u}_{b\bar{c}}^{\bar{a}}) \ , \tag{1.31}$$

which is represented in fig. 6.

The combined use of eqs.(1.19), (1.20), (1.27) and (1.31) has allowed the determination of the factorized scattering theories associated to a large number of integrable models. Among those we simply recall the Sine-Gordon theory [1], the Ising model in a magnetic field [4], the $O(n)$ non linear sigma model (related to the problem of self-avoiding polymers in the limit $n \to 0$) [14] and the q-state Potts model (related to percolation in the limit $q \to 1$) [15]. For details and a complete bibliography on the subject we refer the reader to the review article ref.[16].

1.2 Form factors

The knowledge of the exact S-matrix amounts to a complete reconstruction of the on-shell physics of a theory. Hence, it is natural to start from this result in order to approach the problem of computing also off-shell quantities as the correlation functions. This would allow us, in particular, to recover through the S-matrix approach the conformal data, as the anomalous dimensions, characteristic of the ultraviolet regime. In the sequel we will prove that a relevant part of such ambitious program can be actually realized in the context of integrable models.

A fundamental achievement has been obtained through the so-called *form factor bootstrap* proposed in ref.[6]. Form factors are the matrix elements of a local operator

$O(x)$ between asymptotic states

$$F^{a'_1\ldots a'_m}_{a_1\ldots a_n}(\theta'_1,\ldots,\theta'_m|\theta_1,\ldots,\theta_n) = {}_{out} < A_{a'_1}(\theta'_1)\ldots A_{a'_m}(\theta'_m)|O(0)|A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n) >_{in} \quad . \tag{1.32}$$

In the following we will suppress the subscripts "in" and "out" and we will consider the form factors as analytic functions of the rapidities which assume their physical values when the rapidities fulfill the ordering prescriptions (1.24). In particular, we restrict our attention to the matrix elements

$$F_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n) =< 0|O(0)|A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n) > \ , \tag{1.33}$$

which are pictorially represented in fig. 7. There is no loss of generality in doing this, since crossing symmetry provides the relation

$$F^{a'_1\ldots a'_m}_{a_1\ldots a_n}(\theta'_1,\ldots,\theta'_m|\theta_1,\ldots,\theta_n) = F_{\bar{a}'_1\ldots\bar{a}'_m a_1\ldots a_n}(\theta'_1+i\pi,\ldots,\theta'_m+i\pi,\theta_1,\ldots,\theta_n) \tag{1.34}$$

(the shift of $i\pi$ in the rapidities of the crossed particles corresponds to change the sign of the two-momentum). Let's now analyze the fundamental properties of the form factors.

1. Two-dimensional Lorentz covariance implies, for a generic operator of spin s, the relation

$$F_{a_1\ldots a_n}(\theta_1+\Lambda,\ldots,\theta_n+\Lambda) = e^{s\Lambda}F_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n) \ , \tag{1.35}$$

showing in particular that the form factors of a scalar operator depend only on the rapidity differences $\theta_{ij} \equiv \theta_i - \theta_j$.

2. In order to deduce the monodromy properties of form factors, let's consider the two-particle case for a scalar operator $O(x)$ (the spin structure is trivial under monodromy) [5]. In momentum notation we have

$$F_{ab}(s+i0) =< 0|O(0)|A_a(p_1^\mu)A_b(p_2^\mu) >_{in} \ , \tag{1.36}$$

while through analytic continuation to the lower edge of the unitarity cut we obtain

$$F_{ab}(s-i0) =< 0|O(0)|A_a(p_1^\mu)A_b(p_2^\mu) >_{out} \quad . \tag{1.37}$$

Inserting a complete set of out-states between $O(0)$ and the two-particle state in (1.36) and using property a) of section 1, we immediately determine the discontinuity of $F_{ab}(s)$ across the unitarity cut

$$\begin{aligned} F_{ab}(s+i0) &= < 0|O(0)|A_c(p_1^\mu)A_d(p_2^\mu) >_{out\ out} < A_c(p_1^\mu)A_d(p_2^\mu)|A_a(p_1^\mu)A_b(p_2^\mu) >_{in} = \\ &= F_{cd}(s-i0)S^{cd}_{ab}(s+i0) \ . \end{aligned} \tag{1.38}$$

This is the only branch cut for the two-particle form factor in the s-plane. Indeed, crossing symmetry (1.34) together with eqs. (1.36) and (1.37) implies

$$ {}_{out} < A_{\bar{b}}(p_2^\mu)|O(0)|A_a(p_1^\mu) >_{in} = < 0|O(0)|A_a(p_1^\mu)A_b(-p_2^\mu) >_{in} = F_{ab}(2m_a^2+2m_b^2-s-i0) \ , \tag{1.39} $$

$$ {}_{in} < A_{\bar{b}}(p_2^\mu)|O(0)|A_a(p_1^\mu) >_{out} = < 0|O(0)|A_a(p_1^\mu)A_b(-p_2^\mu) >_{out} = F_{ab}(2m_a^2+2m_b^2-s+i0) \ . \tag{1.40} $$

On the other hand, the two matrix elements on the left-hand side of previous equations should coincide (this is immediately seen using completeness of "in" and "out" bases and the general result ${}_{out} < A_b(p_2^\mu)|A_a(p_1^\mu) >_{in} = \delta_{ab}\delta(p_1^\mu - p_2^\mu)$) so that

$$ F_{ab}(2m_a^2 + 2m_b^2 - s - i0) = F_{ab}(2m_a^2 + 2m_b^2 - s + i0) \ . \tag{1.41} $$

In terms of rapidities, eqs.(1.38) and (1.41) can be recast in the form (valid for an operator with generic spin)

$$ F_{ab}(\theta_1, \theta_2) = F_{cd}(\theta_2, \theta_1) S_{ab}^{cd}(\theta_1 - \theta_2) \ , \tag{1.42} $$

$$ F_{ab}(\theta_2, \theta_1 - i\pi) = F_{ab}(\theta_1 + i\pi, \theta_2) \tag{1.43} $$

(to be precise, these equations are obtained for $\theta_1 - \theta_2 > 0$ but the result holds true on the whole strip $0 \leq Im(\theta_1 - \theta_2) \leq 2\pi$ by analytic continuation). Sending θ_1 into $\theta_1 + i\pi$, eq.(1.43) leads to the relation

$$ F_{ab}(\theta_1 + 2i\pi, \theta_2) = F_{ab}(\theta_2, \theta_1) \ . \tag{1.44} $$

The generalization to the n-particle case is straightforward once one realizes that, due to factorization, the properties (1.42) and (1.44) should hold for each two-particle sub-channel. Then we have [5, 6]

$$ F_{a_1 \ldots a_i a_{i+1} \ldots a_n}(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots, \theta_n) = S_{a_i a_{i+1}}^{b_i b_{i+1}}(\theta_i - \theta_{i+1}) F_{a_1 \ldots b_{i+1} b_i \ldots a_n}(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots, \theta_n) \tag{1.45} $$

$$ F_{a_1 a_2 \ldots a_n}(\theta_1 + 2i\pi, \theta_2, \ldots, \theta_n) = F_{a_2 \ldots a_n a_1}(\theta_2, \ldots, \theta_n, \theta_1) \ . \tag{1.46} $$

Notice that eq.(1.45) can also be obtained simply using the Faddeev-Zamolodchikov algebra (1.26).

3. The form factors are meromorphic functions of the rapidity differences θ_{ij} in the strip $0 \leq Im\theta_{ij} \leq 2\pi$ with simple poles of two different kinds.

i) *Bound state poles.* If the scattering amplitude $S_{ab}^{de}(\theta)$ has a pole at $\theta = iu_{ab}^c$ in the physical strip corresponding to a bound state A_c, then the form factor $F_{a_1 \ldots a_n ab}(\theta_1, \ldots, \theta_{n+2})$ has a pole at $\theta_{n+1} - \theta_{n+2} = iu_{ab}^c$ with residue (see fig. 8)

$$ i \, res_{\theta_{n+1} - \theta_{n+2} = iu_{ab}^c} F_{a_1 \ldots a_n ab}(\theta_1, \ldots, \theta_n, \theta_{n+1}, \theta_{n+2}) = g_{ab}^c F_{a_1 \ldots a_n c}(\theta_1, \ldots, \theta_n, \theta'_{n+1}) \ , \tag{1.47} $$

where g_{ab}^c is the three-particle coupling constant appearing in (1.30) and $\theta'_{n+1} = \theta_{n+1} - i \arctan \frac{\mu \sin u_{ab}^c}{1+\mu \cos u_{ab}^c}$, $\mu \equiv \frac{m_b}{m_a} = \frac{\sin u_{\bar{c}a}^{\bar{b}}}{\sin u_{b\bar{c}}^{\bar{a}}}$ (the expression for θ'_{n+1} is obtained requiring conservation of energy and momentum at the three-particle vertex). Equation (1.47) provides a recursive relation among form factors with n and $n+1$ particles.
ii) *Annihilation poles.* In order to understand the origin of the second kind of poles let's consider the particular case of (1.32) in which $m = n = 1$ and $O(x)$ coincides with the identity operator. We have

$$_{out} < A_{a'}(\theta')|A_a(\theta) >_{in} = 2\pi\delta_{aa'}\delta(\theta - \theta') \ . \tag{1.48}$$

This simple example shows that in the general case a singular contribution is present in the right hand side of eq.(1.34) when a particle A_a appears both in the initial and the final state with the same rapidity θ. Intuitively, this circumstance can be understood thinking that, when crossed from the final to the initial state, the particle $A_a(\theta)$ becomes $A_{\bar{a}}(\theta + i\pi)$ and annihilates the particle $A_a(\theta)$ already present in the initial state. A more formal argument based on LSZ reduction shows that, if $\theta' \geq \theta > \theta_1 > \ldots > \theta_n$, we have to write [6]

$$\begin{aligned} F_{aa_1\ldots a_n}^{a'}(\theta'|\theta,\theta_1,\ldots,\theta_n) &= F_{\bar{a}'aa_1\ldots a_n}(\theta'+i\pi,\theta,\theta_1,\ldots,\theta_n) \\ &+ 2\pi\delta_{aa'}\delta(\theta-\theta')F_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n) \ , \end{aligned} \tag{1.49}$$

$$\begin{aligned} F_{a_n\ldots a_1 a}^{a'}(\theta'|\theta_n,\ldots,\theta_1,\theta) &= F_{a_n\ldots a_1 a\bar{a}'}(\theta_n,\ldots,\theta_1,\theta,\theta'-i\pi) \\ &+ 2\pi\delta_{aa'}\delta(\theta-\theta')F_{a_n\ldots a_1}(\theta_n,\ldots,\theta_1) \ . \end{aligned} \tag{1.50}$$

Using eqs.(1.45) and (1.19), the last equation can be transformed into

$$\begin{aligned} F_{aa_1\ldots a_n}^{a'}(\theta'|\theta,\theta_1,\ldots,\theta_n) &= F_{aa_1\ldots a_n\bar{a}'}(\theta,\theta_1,\ldots,\theta_n,\theta'-i\pi) \\ &+ 2\pi\delta_{aa'}\delta(\theta-\theta')S_{a_1\ldots a_n}^{b_1\ldots b_n}(\theta_1,\ldots,\theta_n|\theta)F_{b_1\ldots b_n}(\theta_1,\ldots,\theta_n) \ , \end{aligned} \tag{1.51}$$

where

$$S_{a_1\ldots a_n}^{b_1\ldots b_n}(\theta_1,\ldots,\theta_n|\theta) \equiv S_{a_1\alpha_n}^{b_1\alpha_1}(\theta_1-\theta)S_{a_2\alpha_1}^{b_2\alpha_3}(\theta_2-\theta)\ldots S_{a_n\alpha_{n-1}}^{b_n\alpha_n}(\theta_n-\theta) \ . \tag{1.52}$$

Subtracting eq.(1.49) from eq.(1.51) we find

$$\begin{aligned} F_{\bar{a}'aa_1\ldots a_n}(\theta'+i\pi,\theta,\theta_1,\ldots,\theta_n) &= F_{aa_1\ldots a_n\bar{a}'}(\theta,\theta_1,\ldots,\theta_n,\theta'-i\pi) \\ &+ 2\pi\delta_{aa'}\delta(\theta-\theta')\left(S_{a_1\ldots a_n}^{b_1\ldots b_n}(\theta_1,\ldots,\theta_n|\theta) - \delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_n}^{b_n}\right)F_{b_1\ldots b_n}(\theta_1,\ldots,\theta_n) \ . \end{aligned} \tag{1.53}$$

While for $a \neq a'$ or $\theta \neq \theta'$ this equation is equivalent to (1.46), it also implies that the form factor $F_{\bar{a}aa_1\ldots a_n}(\theta',\theta,\theta_1,\ldots,\theta_n)$ has an "annihilation" pole at $\theta' = \theta + i\pi$ whose residue is given by

$$-i\, res_{\theta'=\theta+i\pi} F_{\bar{a}' a a_1\ldots a_n}(\theta',\theta,\theta_1,\ldots,\theta_n) = \tag{1.54}$$
$$\left[\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_n}^{b_n} - S_{a_1\ldots a_n}^{b_1\ldots b_n}(\theta_1,\ldots,\theta_n|\theta)\right] F_{b_1\ldots b_n}(\theta_1,\ldots,\theta_n) \ .$$

This equation provides a recursive relation among form factors with n and n+2 particles.

The set of equations (1.35), (1.45), (1.46), (1.47), (1.54) proved to be powerful enough to permit in many cases the complete reconstruction of the form factors [6, 17, 18, 19, 20]. In particular, Smirnov was able to use the form factor bootstrap to prove the local commutativity of the fields. Others interesting applications can be found in refs.[21, 22, 23, 24, 25, 26, 27].

1.3 Massive deformation of the model $\mathcal{M}_{3,5}$

The aim of this section is to show the efficacy of the form factor bootstrap through the application to a specific model, namely the $\Phi_{1,3}$ massive deformation of the minimal non-unitary model $\mathcal{M}_{3,5}$ [20]. As we will see, such model exhibits a relatively simple scattering theory which allows to minimize the technical complications and to expose concisely the fundamental steps of the method. As usually happens when dealing with non-unitary theories, there is however a price to be paid for this simplification, i.e. properties taken for granted in ordinary QFT, such as the positivity norm of the states in the Hilbert space, are no longer guaranteed for non-unitary models and may require a generalization to handle apparently paradoxical situations [28, 29, 30, 31].

The model we will discuss in the following can be formally described by the action

$$\mathcal{A} = \mathcal{A}_{\rm CFT} - i\lambda \int \varphi(x)\, d^2x \ , \tag{1.55}$$

where we defined $\varphi \equiv \Phi_{1,3}$ and $\mathcal{A}_{\rm CFT}$ stays for the action of the minimal model $\mathcal{M}_{3,5}$. The important data of this CFT are collected in tables 1 and 2. The presence of i in the action reflects the non-unitary features of the model. With respect to the Kac-table and the fusion rules of the conformal point, the field φ is an operator even under a Z_2 symmetry, with conformal weight $\Delta_\varphi = 1/5$. With regard to the massive phase, the corresponding theory is integrable [30, 4] and the spectrum is given by one Z_2 odd particle A in bootstrap interaction [1]. The mass m of the particle A may be expressed

[1] As discussed in ref. [30] and confirmed in ref. [31], the model presents a Z_2 spontaneous symmetry breaking and it is very similar in many respects to the Ising model in the low temperature phase. The actual spectrum consists in two kink excitations above a doubly degenerate vacuum. Since the

in terms of the coupling constant appearing in the action (1.55) as [32]

$$\lambda = (0.333121)\, m^{8/5} \ . \tag{1.56}$$

The two-particle elastic S-matrix was determined in ref.[30] and is given by

$$S(\beta) = -i \tanh \frac{1}{2}\left(\beta - i\frac{\pi}{2}\right) \ . \tag{1.57}$$

This scattering amplitude satisfy the unitarity equation (1.19) and presents no poles in the physical sheet so that no additional bound states are created. The aforementioned unusual features of the model appear in the crossing symmetry relation

$$S(\beta) = -S(i\pi - \beta) \tag{1.58}$$

which differs for a minus sign from the standard equation (1.20). This circumstance suggests that a factor i should be associated to each crossed particle. As a consequence, the charge conjugation C of the particle A obeys the relation $C^2 = -1$.

We now turn to the computation of form factors. In the present case the notation can be simplified denoting by $F_n(\beta_1, \ldots, \beta_n)$ the matrix elements in eq.(1.33). Notice that the absence of bound states in the model under consideration leaves us with only the kinematical recursive equation connecting the form factors with n and $n+2$ particles; there is no way to relate form factors with a different parity in the number of particles. This situation is typical of the models which exhibit a Z_2-symmetry and whose physically relevant operators have nonvanishing matrix elements only on states containing a number of particles with fixed parity. To be specific, in the following we will be particularly interested in the trace of the energy-momentum tensor $\Theta(x) = T^{\mu}_{\mu}(x)$ which is expressed in terms of the perturbing field by the relation

$$\Theta(x) = -2\pi i \lambda (2 - 2\Delta_{\varphi}) \varphi(x) \ . \tag{1.59}$$

Since the fields σ, which is supposed to create the particles, and φ are respectively odd and even with respect to the Z_2 symmetry of the fusion rules (see table 2), we conclude that $\Theta(x)$ has nonvanishing matrix elements only on an even number of external particles. For this reason we will restrict our attention to the form factors $F_{2n}(\beta_1, \ldots, \beta_{2n})$ in the remaining part of this section.

Of course, the anomalous crossing property (1.58) induces slight modifications (in the form of insertion of powers of i) into the equations for form factors related to crossing symmetry, namely eqs. (1.46) and (1.54), which now read

$$F_{2n}(\beta_1 + 2\pi i, \ldots, \beta_{2n-1}, \beta_{2n}) = (-1)^{n+1}\, F_{2n}(\beta_2, \ldots, \beta_{2n}, \beta_1) \tag{1.60}$$

internal degree of freedom of the kink excitations are frozen, they behave as a single particle A but with the unusual features discussed in the text.

and

$$-i\, res_{\beta'=\beta+i\pi} F_{2n+2}(\beta',\beta,\beta_1,\beta_2,\ldots,\beta_{2n}) = \left(1-(-1)^n \prod_{i=1}^{2n} S(\beta_i-\beta)\right) F_{2n}(\beta_1,\ldots,\beta_{2n}) \ , \tag{1.61}$$

respectively. On the contrary (eq.1.45), which relies just on unitarity, remains unchanged and in the present notation can be written as

$$F_{2n}(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_{2n}) = F_{2n}(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_{2n})\, S(\beta_i-\beta_{i+1}) \ . \tag{1.62}$$

To find the solution of the above functional and recursive equations, the first step is to solve the simplest system

$$\begin{aligned} F_2(\beta) &= S(\beta)\, F_2(-\beta) \ , \\ F_2(\beta+2i\pi) &= F_2(-\beta) \ . \end{aligned} \tag{1.63}$$

Let $F_{\rm min}(\beta)$ be the solution of (1.63) with no poles and zeros in the physical sheet. Explicitly,

$$F_{\rm min}(\beta) = \mathcal{N} \sinh\frac{\beta}{2} \prod_{k=0}^{\infty} \left| \frac{\Gamma\left(k+\frac{3}{4}+i\frac{\hat\beta}{2\pi}\right)}{\Gamma\left(k+\frac{5}{4}+i\frac{\hat\beta}{2\pi}\right)} \right|^2 \tag{1.64}$$

($\hat\beta = i\pi-\beta$), where we choose the normalization constant $\mathcal{N}$ such that $F_{\rm min}(i\pi)=1$, i.e.

$$\mathcal{N} = -i(\sqrt{2}\pi)^{-1/2} \exp\left(\mathcal{G}/\pi\right) \ , \tag{1.65}$$

where $\mathcal{G}$ is Catalan's constant. $F_{\rm min}(\beta)$ satisfies the functional equation

$$F_{\rm min}(\beta+i\pi)\, F_{\rm min}(\beta) = -\frac{\pi}{2}\mathcal{N}^2 \frac{\sinh\beta}{\sinh\frac{1}{2}\left(\beta+i\frac{\pi}{2}\right)} \ , \tag{1.66}$$

and for large values of β behaves as

$$F_{\rm min}(\beta) \simeq \Xi \exp\left[\frac{\beta-i\pi}{4}\right] \ , \tag{1.67}$$

where $\Xi = i\frac{\mathcal{N}}{2}\sqrt{\pi}$.

The general parameterization of the form factors which takes into account the kinematical poles and the monodromy structure is given by

$$F_{2n}(\beta_1,\ldots,\beta_{2n}) = H_{2n}\, Q_{2n}(x_1,\ldots,x_{2n}) \prod_{i<j} \frac{F_{\rm min}(\beta_{ij})}{x_i+x_j} \ , \tag{1.68}$$

where H_{2n} are normalization constants, $Q_{2n}(x_1,\ldots,x_{2n})$ symmetric functions of $x_1,\ldots,x_{2n}$ and $x_i \equiv e^{\beta_i}$. By using the residue condition (1.61) and the functional equation (1.66), we obtain for the Q_{2n}'s the recursive equations

$$Q_{2n+2}(-x,x,x_1,\ldots,x_{2n}) = x^{n+1}\sqrt{\sigma_{2n}(x_1,\ldots,x_{2n})}\,\mathcal{D}_{2n}(x|x_1,\ldots,x_{2n})\,Q_{2n}(x_1,\ldots,x_{2n}) \ , \tag{1.69}$$

where

$$\mathcal{D}_{2n}(x|x_1,\ldots,x_{2n}) = \sum_{k=0}^{2n} \sin\frac{\pi k}{2}\, x^{2n-k}\sigma_k(x_1,\ldots,x_{2n}) \tag{1.70}$$

and $\sigma_k(x_1,\ldots,x_m)$ $(k=0,1,\ldots m)$ are the elementary symmetric polynomials in m-variables generated by [33]

$$\prod_{i=1}^{m}(x+x_i) = \sum_{k=0}^{m} x^{m-k}\sigma_k(x_1,\ldots,x_m) \ . \tag{1.71}$$

In writing eq. (1.69), we chose the normalization constants such that

$$H_{2n+2} = 2\left(\pi\mathcal{N}^2/2\right)^{-2n} H_{2n} \ . \tag{1.72}$$

We want to stress that equation (1.69) was obtained without reference to any particular operator, so that its general solution would provide the form factors with even number of particles of all the operators of the theory local with respect to the field which creates the particles [2]. Rather than facing this general problem, in the following we will restrict our attention to the trace of the energy-momentum tensor $\Theta(x)$ since it has a particular physical interest and will be important in the applications of the next section. In order to select the form factors of $\Theta(x)$ [17, 25], we exploit the fact that the the energy-momentum tensor is a conserved current satisfying the continuity equation $\partial_\mu T^{\mu\nu}(x) = 0$. This allows us to write

$$T^{\mu\nu}(x) = (\partial^\mu\partial^\nu - g^{\mu\nu}\partial^\lambda\partial_\lambda)A(x) \ , \tag{1.73}$$

where $A(x)$ is a scalar operator. Introducing the light-cone coordinates $x^\pm = x^0 \pm x^1$ and the energy-momentum components

$$T_{++}(x) = \partial_+^2 A(x) \tag{1.74}$$

$$T_{--}(x) = \partial_-^2 A(x) \tag{1.75}$$

one easily gets the following relations among form factors (in obvious notations)

$$\begin{aligned} \sigma_1 F_m^{T_{++}} &= -\frac{1}{4}\frac{\sigma_{m-1}}{\sigma_m}F_m^{\Theta} \\ \frac{\sigma_{m-1}}{\sigma_m}F_m^{T_{--}} &= -\frac{1}{4}\sigma_1 F_m^{\Theta} \ . \end{aligned} \tag{1.76}$$

[2] See ref.[19] for an analysis of this point in the context of the Sinh-gordon model

Since all the components of the energy-momentum tensor are supposed to have the same singularity structure, the last equations implies that for $m > 2$ the form factors $F_m^\Theta(\beta_1, \ldots, \beta_m)$ contain an overall factor $\sigma_1 \sigma_{m-1}$. Then we will look for those solutions of eq.(1.69) in the form

$$Q_{2n}(x_1, \ldots, x_n) = \sigma_1 \sigma_{2n-1} R_{2n}(x_1, \ldots, x_{2n}) \ , \qquad n > 1 \ . \tag{1.77}$$

Since $\Theta(x)$ is a scalar operator, equation (1.35) implies that the total order of $Q_{2n}(x_1, \ldots, x_{2n})$ is equal to $n(2n-1)$. Moreover, from the relations

$$E = \frac{1}{2\pi} \int dx^1 T^{00}(x) \ , \tag{1.78}$$

$$< A(\beta)|E|A(\beta') >= 2\pi m \delta(\beta - \beta') \cosh\beta \ , \tag{1.79}$$

where E is the energy, one gets the result

$$F_2^\Theta(i\pi) =< A(\beta)|\Theta(0)|A(\beta') >= 2\pi m^2 \tag{1.80}$$

which provides the initial conditions for the recursive equations (1.69) and (1.72)

$$Q_2(x_1, x_2) = \sigma_1 \ , \tag{1.81}$$

$$H_2 = 2\pi m^2 \ . \tag{1.82}$$

The following property of the symmetric polynomials σ_k is extremely useful in the solution of eq.(1.69)

$$\sigma_k(-x, x, x_1, \ldots, x_n) = \sigma_k(x_1, \ldots, x_n) - x^2 \sigma_{k-2}(x_1, \ldots, x_n) \ . \tag{1.83}$$

One finds

$$R_{2n} = \begin{cases} (\sigma_1)^{-1} & \text{if } n = 1 \ , \\ (-1)^n (\sigma^{\frac{n-1}{2}})_{2n} \ \det A_{kj} & n \geq 2 \ . \end{cases} \tag{1.84}$$

In the previous expression A_{kj} is a $(n-2) \times (n-2)$ matrix with entries

$$A_{kj} = \sigma_{4j-2k+1} \ . \tag{1.85}$$

For $n = 0$ one also recovers the vacuum expectation value

$$F_0^\Theta =< 0|\Theta(0)|0 >= \pi m^2 \tag{1.86}$$

obtained from the thermodynamic Bethe ansatz [32].

1.4 Correlation functions

The computation of correlation functions is the central problem in QFT. Unfortunately, no general method is known to face it, apart from perturbation theory which is affected by the usual problems of convergence and renormalization. Even in two dimensions, although the situation appears very satisfactory for the CFTs, the only exactly computed off-critical correlator remains the two-point function of the magnetization operator in the scaling Ising model which is known to be a solution of Painleve' equation [34]. The purpose of this section is to show how the possibility to compute exactly the form factors in integrable models leads to a very effective way to evaluate correlation function in such theories.

The basic idea is very simple [23]. Consider for instance the two-point function of a local operator $O(x)$; inserting a complete set of asymptotic states we find

$$
\begin{aligned}
< O(x)O(0) >= & \sum_{n=0}^{\infty} \int_{\beta_1>\beta_2>\ldots>\beta_n} \frac{d\beta_1 \ldots d\beta_n}{(2\pi)^n} < 0|O(x)|A_{a_1}(\beta_1)\ldots A_{a_n}(\beta_n) > \\
\times & < A_{a_1}(\beta_1 \ldots A_{a_n}(\beta_n)|O(0)|0 > \\
= & \sum_{n=0}^{\infty} \int_{\beta_1>\beta_2>\ldots>\beta_n} \frac{d\beta_1 \ldots d\beta_n}{(2\pi)^n} F_{a_1\ldots a_n}(\beta_1,\ldots,\beta_n) F_{a_n\ldots a_1}(\beta_n,\ldots,\beta_1) e^{-r\sum_{i=1}^{n} m_i \cosh\beta_i} ,
\end{aligned}
\tag{1.87}
$$

where the restriction on the integration ranges ensures that the sum is performed over physical states and we used euclidean invariance to choose $x^\mu = (ir, 0)$. Since the form factors $F_{a_1\ldots a_n}(\beta_1,\ldots,\beta_n)$ are supposed to be known for the integrable theory under consideration and since all the integrals above are nonsingular and convergent, expression (1.87) provides a representation of the correlator in the form of an infinite sum over multiparticle intermediate states. Of course, from the practical point of view, the utility of such representation relies on the hope that the convergence of the series is fast enough to provide good numerical approximations also at intermediate scales using partial sums (clearly, the series (1.87) is dominated at large distances by the contributions coming from the states with lowest number of particles, so that it can be seen as an infrared expansion). We will now show, using as an example the model discussed in the previous section, that this hope is in fact completely confirmed.

Let us consider the correlation function $G(x) = \langle\Theta(x)\Theta(0)\rangle$ where $\Theta(x)$ is the trace of the energy-momentum tensor of the model $\mathcal{M}_{3,5}$ perturbed by the relevant operator $\Phi_{1,3}$[20]. The form factors $F^{\Theta}_{2k}(\beta_1,\ldots,\beta_k)$ for this model have been explicitly computed in previous section. In the present case, the representation (1.87) specializes to

$$G(x) = \sum_{k=0}^{\infty}(-1)^k \int \frac{d\beta_1 \ldots d\beta_{2k}}{(2k)!(2\pi)^{2k}} F_{2k}^{\Theta}(\beta_1, \ldots, \beta_{2k}) F_{2k}^{\Theta}(\beta_{2k}, \ldots, \beta_1) \exp\left(-mr\sum_{i=1}^{2k}\cosh\beta_i\right) , \tag{1.88}$$

where the integration now ranges over all the real values of rapidities and the over-counting of the states is prevented by the factorial in the denominator. The factor $(-1)^k = i^{2k}$ is peculiar to the model under consideration and is present because we need to cross n particles in order to pass from the first to the second line of eq.(1.87). A first important check of the validity of our solution (1.84) and of the fast rate of convergence of the series (1.88) can be performed by extracting CFT data directly from the massive phase of the model. A relevant quantity is the central charge of the original $\mathcal{M}_{3,5}$ model, $c = -\frac{3}{5}$, which can be obtained from the correlator $< \Theta(x)\Theta(0) >$ in terms of the c-theorem sum rule [35]

$$c = \frac{3}{4\pi} \int |x|^2 < \Theta(x)\Theta(0) > d^2x \ . \tag{1.89}$$

Using the spectral representation (1.88) and keeping only the two-particle contribution, we get for the previous quantity the approximate value $c = -0.600316$. Including also the four-particle contribution, the resulting estimate $c = -0.600006$ becomes very close to the exact value. As clarified in ref.[22], such fast convergence is due to the softening of the multi-particle branch cuts. The remarkable convergence of the series also extends to the very short distance region of the correlation function and allows us to probe its ultraviolet behaviour. This can be directly seen by comparing the short-distance values of $G(r)$ with the perturbative expansion discussed in appendix A and based on the operator product

$$\varphi(r)\varphi(0) = C_{\varphi\varphi}^{I}(r)\, I + C_{\varphi\varphi}^{\varphi}(r)\, \varphi(0) + \cdots \tag{1.90}$$

where the off-critical structure constants have the following regular expansion in λ

$$\begin{aligned} C_{\varphi\varphi}^{I}(r) &= r^{-4/5}\,(1 - i\lambda\, r^{8/5}\, Q_1 + \cdots) \\ C_{\varphi\varphi}^{\varphi}(r) &= r^{-2/5}\,(C_{\varphi\varphi}^{\varphi} - i\lambda\, r^{8/5}\, Q_2 + \cdots) \ . \end{aligned} \tag{1.91}$$

The conformal structure constant $C_{\varphi\varphi}^{\varphi}$ can be found in table 2 and the first coefficients Q_1 and Q_2 are computed as explained in appendix A. From the general formula (A.25) we read

$$\begin{aligned} Q_1 &= -\int' < \varphi(y)\varphi(1)\varphi(0) >_{\text{CFT}} d^2y \ , \\ Q_2 &= -\int' < \varphi(\infty)\varphi(y)\varphi(1)\varphi(0) >_{\text{CFT}} d^2y \ , \end{aligned} \tag{1.92}$$

where the prime on the integrals denotes the regularization with respect to the infrared divergencies. Q_1 can be computed exactly

$$Q_1 = C_{\varphi\varphi}^{\varphi} \frac{\tan\left(\frac{\pi}{5}\right)}{2} \left| \frac{\Gamma^2\left(\frac{4}{5}\right)}{\Gamma\left(\frac{8}{5}\right)} \right|^2 . \tag{1.93}$$

On the other hand, Q_2 has been computed numerically. Combining the null-vector conditions at levels 3 and 4 satisfied by the operator φ [2, 36], the 4-point conformal correlation function

$$< \varphi(\infty)\varphi(x,\overline{x})\varphi(1)\varphi(0) >_{\rm CFT} = |x(1-x)|^{12/5} F(x,\overline{x}) \tag{1.94}$$

satisfies the second-order differential equation

$$\begin{aligned} x^2(1-x)^2(x^2-x+1)F''(x) - x(1-x)(6x^3-9x^2+11x-4)F'(x) + \\ +\frac{4}{25}(39x^4-78x^3+117x^2-78x+14)F(x) = 0 \end{aligned} \tag{1.95}$$

(analogously for $\overline{x}$). The solutions of this differential equation can be expressed in power series in the annuluses around the singular points $x=0$ and $x=\infty$ and in each of such domain combined into a monodromy invariant combination. The numerical integration of (1.92) gives for the finite part of the integral the value $Q_2 = -1.58 \pm 0.01$. Using the relationship (1.59) and the vacuum expectation value (1.86), one obtains the following short-distance perturbative expansion for the two-point correlator

$$\begin{aligned} G(r) &= -\left(\frac{16}{5}\pi\lambda\right)^2 \left[1 - i\lambda\, Q_1\, r^{8/5}\right] r^{-4/5} + \\ &\quad -\frac{16}{5} i\lambda\pi^2 m^2 \left[C_{\varphi\varphi}^{\varphi} - i\lambda\, Q_2\, r^{8/5}\right] r^{-2/5} + \mathcal{O}\left((mr)^{12/5}\right) \end{aligned} \tag{1.96}$$

In fig. 9 the zero-order (dashed line) and first-order corrected (full line) short distance expansion and the large distance expansion (1.88) with up to four-particle contribution (dotted line) are compared at intermediate distances. A very impressive agreement is observed which extends also to the ultraviolet region, as shown in fig. 10 where the infrared expansion with up to two-particle (dashed line) and four-particle (dotted line) contributions is compared with the short distance expansion (full line). Thus, it is evident that the first form factors are able to reproduce with high accuracy the behaviour of the function, following very closely its power law singularities at short-distance scales. Concluding this section, we want to stress that this remarkable fact is not peculiar to the model under consideration [3] but emerges as a general feature of massive integrable theories.

[3] See for instance refs.[17, 23] for other detailed examples.

Chapter 2

The staircase model

2.1 Roaming trajectories

Typically, when a fixed point of the renormalization group is perturbed by a relevant operator, two qualitatively different behaviours are expected: a) the perturbed theory develops a finite correlation length corresponding to massive excitations and the correlation functions decay exponentially at large distances; b) the correlation length remains infinite also at large distances and the theory, massless although not scale-invariant, interpolates among two different critical regimes. In this chapter we will discuss in some detail a model which provides a remarkable example of intermediate behaviour with respect to the two typical situations outlined above. This model was proposed by Al. Zamolodchikov in ref. [37] and consists in a relatively simple purely elastic scattering theory which under TBA analysis reveals a very unusual off-shell pattern. The theory contains a single particle, which is chosen to be a boson of mass m, and is defined by the two-particle amplitude

$$S(\theta) = \frac{\sinh\theta - i\cosh\theta_0}{\sinh\theta + i\cosh\theta_0} \ , \tag{2.1}$$

where θ_0 is a real parameter. This amplitude satisfies the usual requirements of unitarity and crossing symmetry which for a single particle theory read simply

$$S(\theta)S(-\theta) = 1 \ , \tag{2.2}$$

$$S(\theta) = S(i\pi - \theta) \ . \tag{2.3}$$

$S(\theta)$ exhibits two simple zeroes in the physical strip at positions $\theta = \frac{i\pi}{2} \pm \theta_0$, paired via the unitarity relation to two simple poles in the unphysical strip at positions $\theta = -\frac{i\pi}{2} \pm \theta_0$.

The TBA analysis of this model goes along the standard lines described in the appendix B for the "fermionic case" since from eq.(2.1) we get $S(0) = -1$ (note that the same off-shell pattern discussed below can be obtained supposing that the particle of the theory is a fermion and changing the sign of the amplitude (2.1)). Since only one particle is present, one has to deal with the single TBA equation

$$Rm\cosh\theta = \varepsilon(\theta) + \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi}\,\phi(\theta-\theta')L(\theta')\;, \tag{2.4}$$

where

$$L(\theta) \equiv \ln\left(1 + e^{-\varepsilon(\theta)}\right) \tag{2.5}$$

and the kernel $\phi(\theta)$ defined in eq.(B.13) in this specific case is given by

$$\phi(\theta) = \frac{1}{\cosh(\theta+\theta_0)} + \frac{1}{\cosh(\theta-\theta_0)} \quad . \tag{2.6}$$

The ultraviolet effective central charge $\tilde{c}_{UV} = \tilde{C}(mR=0)$ is easily obtained using the relations (B.32), (B.27) and (B.26); it turns out to be

$$\tilde{c}_{UV} = 1 \quad . \tag{2.7}$$

The interesting features of the model under consideration appear when equation (2.4) is solved numerically and the effective central charge at intermediate distances is computed through eq.(B.24) [37]. The results of this analysis for various values of the parameter θ_0 are shown in figs. 11a-d where the effective central charge $\tilde{C}$ is plotted as a function of the logarithmic scale

$$x = \ln\frac{mR}{2} \quad . \tag{2.8}$$

For $\theta_0 = 0$, $\tilde{C}(x)$ shows the usual behaviour smoothly interpolating between the ultraviolet limit $\tilde{C}(x=-\infty) = \tilde{c}_{UV}$ and the value $\tilde{C}(x=+\infty) = 0$ characteristic of massive theories; but for $\theta_0 \neq 0$ the situation becomes highly non-trivial and $\tilde{C}(x)$ develops a "staircase" pattern which becomes more and more visible as θ_0 increases. More precisely, for θ_0 sufficiently large (say $\theta_0 > 20$) $\tilde{C}(x)$ clearly exhibits a series of plateaux at values coinciding with the central charges of unitary minimal models $\mathcal{M}_p$; the plateau at $\tilde{C} = 1 - 6/p(p+1)$ lies inside the interval $-(p-2)\theta_0/2 < x < -(p-3)\theta_0/2$ with $p = 3, 4, \ldots$. Since the difference in the heights of the neighbouring steps becomes small as $x \to -\infty$, the numerical resolution becomes insufficient in the deep ultraviolet limit and the picture is slurred. Nevertheless, at $\theta_0 = 50$ one can clearly distinguish 8 steps, the highest being of height $21/22$ and corresponding to $\mathcal{M}_{11}$ central charge.

These results unavoidably lead to an interpretation of the model defined by eq.(2.1) closely related to the massless RG flows between the theories $\mathcal{M}_p$ and $\mathcal{M}_{p-1}$ induced by the perturbing field $\Phi^p_{(1,3)}$. Indeed the characteristic pattern of figs. 11a-d are suggestive for a one-parameter family of *roaming trajectories* interpolating between all the fixed points $\mathcal{M}_p$: according to eq.(2.7), each trajectory starts from the limiting fixed point $\mathcal{M}_\infty$ and then, for θ_0 large enough, flows very close to the fixed points $\mathcal{M}_p$ spending approximately the same fraction $\theta_0/2$ of the RG time x near each one. In ref. [37] it was shown that for $\theta_0 \gg 1$ and $x \sim -(p-2)\theta_0/2$ (this is the value corresponding to the switching from c_{p+1} to c_p) the function $\tilde{C}(x)$ reproduces with high accuracy the values obtained by the TBA system proposed in ref.[7] as describing the flow from $\mathcal{M}_{p+1}$ to $\mathcal{M}_p$. Thus we can imagine that the limiting $\theta_0 \to \infty$ trajectory starts at $\mathcal{M}_\infty$ and then proceed on the critical surface following the massless trajectories $\mathcal{M}_p \to \mathcal{M}_{p-1}$ until at $\mathcal{M}_3$ it develops a finite correlation length and gives rise to a massive infrared behaviour.

One can use the numerical data for $\tilde{C}(x)$ to compute the beta-functions along the RG trajectories [37]. Indeed, let's call Φ the operator which draws the field theory along the trajectory and g the conjugated coupling constant. Then the expectation value of Φ on the TBA geometry can be written as

$$< \Phi >_R = \frac{\pi}{6R}\frac{\partial \tilde{C}}{\partial g} \quad . \tag{2.9}$$

We now eliminate the dependence on R fixing $R = 1$ and normalize the field Φ through

$$< \Phi >_{R=1} = -\frac{\pi}{6} \ , \tag{2.10}$$

where the minus sign is due to the fact that $\tilde{C}$ monotonically decreases along the trajectory. Requiring g to be zero at the ultraviolet fixed point, eqs.(2.9) and (2.10) give

$$g(x) = 1 - \tilde{C}(x) \quad . \tag{2.11}$$

On the other hand the beta-function is simply defined as the derivative of the coupling constant with respect to the scale parameter so that

$$\theta(g) = -\frac{\partial \tilde{C}}{\partial x} \quad . \tag{2.12}$$

The last two relations give a parametric representation of $\theta(g)$. figs. 12 show the behaviour of the beta-function for different values of the parameter θ_0: $\theta(g)$ develops deep minima in correspondence of the values $g = 6/p(p+1)$, $p = 3, 4, \ldots$ which become progressively indistinguishable from zeroes when θ_0 increases. Note that, while the higher minima turn subsequently to zeroes when θ_0 grows, the beta-function in between to zeroes is stabilized at the corresponding interpolating shape.

An interpretation of the results presented above from the conformal perturbation theory point of view was proposed in ref. [39] where the following hamiltonian density was argued to describe the staircase model:

$$\mathcal{H} = \mathcal{H}_p + \lambda\Phi^p_{(1,3)} - \tilde{\lambda}\Phi^p_{(3,1)} \quad . \tag{2.13}$$

Here $\mathcal{H}_p$ stays for the hamiltonian density of the minimal conformal model $\mathcal{M}_p$ and the suffix p for the fields denotes that they belong to $\mathcal{M}_p$. For $\lambda > 0$ and $\tilde{\lambda} = 0$ eq.(2.13) simply corresponds to the deformation studied in sec. 1.6 and interpolating between $\mathcal{M}_p$ and $\mathcal{M}_{p-1}$. The aim of the irrelevant perturbation coming from $\Phi^p_{(3,1)}$ is then to deform this interpolating trajectory in such a way to avoid that it stops at $\mathcal{M}_p$ ($\mathcal{M}_{p-1}$) in the ultraviolet (infrared) limit. This would make possible a multiple crossover of the type described above. A detailed perturbative study of the RG equations corresponding to the deformation (2.13) was carried out in ref. [39] in the limit of large values of p. To leading order in $1/p$ it was shown that, for λ and $\tilde{\lambda}$ both positive, there exists a unique one parameter family of solutions exhibiting the characteristic behaviour of roaming trajectories. Indeed, if we denote by x the RG time and by θ_0 the parameter labelling the solutions, it turns out that each trajectory come close to each fixed point $\mathcal{M}_{p'}$ in the time interval $(p-p'-1/2)\theta_0 < \theta < (p-p'+1/2)\theta_0$. In particular, for the RG flow of the function $C(x)$ defined in sec. 1.5 one obtains

$$\begin{aligned} C(k\theta_0) &= c_{p-k} + O(p^{-4}) \ , & (2.14) \\ C((k+1/2)\theta_0) &= c_{p-k} - \frac{1}{2}(c_{p-k} - c_{p-k-1}) + O(p^{-4}) & (2.15) \end{aligned}$$

for integer k.

In conclusion we make some remark about the particular deformation of the fixed point $\mathcal{M}_p$ defined by eq.(2.13). Both the perturbations $\Phi_{(1,3)}$ and $\Phi_{(3,1)}$ are separately integrable (see sec. 2.1 and ref. [4]). But, while in the general case a linear combination of two integrable perturbation does not generate an integrable field theory off criticality (this is the case, for example, of Ising model under simultaneous thermal ($\Phi_{(1,3)}$) and magnetic ($\Phi_{(1,2)}$) perturbations), the combination of $\Phi_{(1,3)}$ and $\Phi_{(3,1)}$ was argued to be integrable in ref. [38] so that one can expect (2.13) to correspond to a factorized scattering theory.

The multiple crossover exhibited by the theory (2.13) for λ and $\tilde{\lambda}$ positive gives rise to an interesting critical behaviour as a function of the relevant "temperature-like" parameter λ. Indeed for $\lambda = 0$ the theory is in the universality class of $\mathcal{M}_p$

but the thermodynamic singularities as $\lambda \to 0$ are determined not by $\mathcal{M}_p$ alone, but simultaneously by all the fixed points $\mathcal{M}_p, \mathcal{M}_{p-1}, \ldots, \mathcal{M}_3$. Some exact exponents are obtained in ref. [40].

2.2 Sinh-Gordon theory

The scattering amplitude (2.1) can be considered as an analytical continuation of the S-matrix of the Sinh-Gordon theory, namely the theory of a two-dimensional scalar field $\phi(x)$ with the action

$$A = \int d^2x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{g^2}\cosh g\phi(x)\right] \quad . \tag{2.16}$$

It can be regarded as a perturbation of the free massless conformal action by means of the relevant operator $\cosh g\phi(x)$ of anomalous dimension $\Delta = -\theta^2/8\pi$ or, alternatively, as a deformation of the conformal Liouville action

$$A = \int d^2x \left[\frac{1}{2}(\partial_\mu \phi)^2 - \lambda e^{g\phi}\right] \tag{2.17}$$

by the relevant operator $e^{-g\phi}$.

Action (2.16) possesses a $\mathbf{Z}_2$-symmetry under the substitution $\phi \to -\phi$ and can be mapped into the action of the Sine-Gordon model by an analytic continuation in g, namely $g \to ig$. In a perturbative approach, ultraviolet divergencies come only from tadpole diagrams and can be removed by a normal ordering prescription. This gives rise to finite wave function and mass renormalization, while the coupling constant g does not renormalize.

The Sinh-Gordon model (SGM) is the simplest example of a large class of integrable theories, the affine Toda field theories. Integrability allows the determination of the exact S-matrix which is given by [41]

$$S(\theta, B) = \frac{\sinh\theta - i\sin\frac{\pi B}{2}}{\sinh\theta + i\sin\frac{\pi B}{2}} \,, \tag{2.18}$$

where B is the following function of the coupling constant g:

$$B(g) = \frac{2g^2}{8\pi + g^2} \quad . \tag{2.19}$$

It is evident from this relation that in the Sinh-Gordon theory $B(g)$ takes values in the range $[0, 2)$. The two-particle amplitude (2.18) has no poles in the physical strip (then there are not bound states) and exhibits two zeroes at the crossing symmetric positions $i\pi B/2$ and $i\pi(2-B)/2$.

An interesting feature of (2.18) is its invariance under the substitution

$$B \to 2 - B \ , \tag{2.20}$$

corresponding through eq.(2.19) to the strong-weak coupling constant duality

$$g \to \frac{8\pi}{g} \ . \tag{2.21}$$

At the *self-dual point* $B(\sqrt{8\pi}) = 1$ the two zeroes of the scattering amplitude collide at $\theta = i\pi/2$. If we now analytically continue the parameter B to the complex values

$$B = 1 \pm \frac{2i}{\pi}\theta_0 \tag{2.22}$$

the zeroes split again but along a direction parallel to the real θ-axis and (2.18)exactly coincides with the scattering amplitude (2.1).

The form factors for the Sinh-Gordon model were computed in ref. [18]. They can be parameterized as

$$F_n^k(\theta_1,\ldots,\theta_n) \;=\; H_n^k\, Q_n^k(x_1,\ldots,x_n) \prod_{i<j} \frac{F_{\text{min}}(\theta_{ij})}{(x_i+x_j)} \ , \tag{2.23}$$

where $x_i \equiv e^{\theta_i}$ and $\theta_{ij} = \theta_i - \theta_j$. $F_{\text{min}}(\theta)$ is an analytic function given by

$$\begin{aligned} F_{\text{min}}(\theta,B) \;&=\; \mathcal{N}(B)\ \Xi(\theta,B) \\ \Xi(\theta,B) \;&=\; \exp\left[8\int_0^\infty \frac{dx}{x}\frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}(1-\frac{B}{2})\right)\sinh\frac{x}{2}}{\sinh^2 x}\sin^2\left(\frac{x\hat{\theta}}{2\pi}\right)\right] \\ \mathcal{N}(B) \;&=\; \exp\left[-4\int_0^\infty \frac{dx}{x}\frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}(1-\frac{B}{2})\right)\sinh\frac{x}{2}}{\sinh^2 x}\right] \end{aligned} \tag{2.24}$$

($\hat{\theta} = i\pi - \theta$). $F_{\text{min}}(\theta,B)$ has a simple zero at the threshold $\theta = 0$ and no poles in the physical strip $0 \leq \text{Im}\,\theta \leq \pi$, with an asymptotic behaviour $\lim_{\theta\to\infty} F_{\text{min}}(\theta,B) = 1$. In eq. (2.23) H_n^k are normalization constants which depend on the operator one is considering. The functions $Q_n^k(x_1,\ldots,x_n)$ are symmetric polynomials in the variables x_i, solutions of the recursion equations which link the n-particle and the $(n+2)$-particle form factors

$$-i\lim_{\tilde{\theta}\to\theta}(\tilde{\theta}-\theta)F_{n+2}^k(\tilde{\theta}+i\pi,\theta,\theta_1,\theta_2,\ldots,\theta_n) = \left(1-\prod_{i=1}^n S(\theta-\theta_i,B)\right)\, F_n^k(\theta_1,\ldots,\theta_n). \tag{2.25}$$

For form factors of spinless operators, the total degree of Q_n^k is equal to $n(n-1)/2$ whereas their partial degree in each variable x_i depends on the operator $\mathcal{O}_k$ which is considered. It was shown in ref. [19] that a general solution for the Q_n^k can be written in terms of the so-called *elementary solutions* $\mathcal{Q}_n(p)$ given by

$$\mathcal{Q}_n(p) = \det M_{ij}(p) \ , \tag{2.26}$$

where $M_{ij}(p)$ is an $(n-1)\times(n-1)$ matrix with entries $M_{ij}(p) = \sigma_{2i-j}\,[i-j+p]$ (σ_l are the elementary symmetric polynomials, p an arbitrary integer and $[n] \equiv \frac{\sin(n\pi B/2)}{\sin \pi B/2}$).

Form Factors of $\phi(x)$ and $\Theta(x)$

Important operators of the SGM are the elementary field $\phi(x)$ and the trace of the stress-energy tensor $\Theta(x)$. They are odd and even operators respectively under the Z_2 symmetry of the model with normalizations given by $< 0 \mid \phi(0) \mid \theta >= 1$ and $< \theta \mid \Theta(0) \mid \theta >= 2\pi M^2$, where M is the physical mass. The whole set of form factors of the elementary field $\phi(x)$ is given by

$$F_n^{\phi}(\theta_1,\ldots,\theta_n) = \left(\frac{4\sin(\pi B/2)}{\mathcal{N}(B)}\right)^{(n-1)/2} \mathcal{Q}_n(0) \prod_{i<j} \frac{F_{\rm min}(\theta_{ij})}{x_i + x_j} \ . \tag{2.27}$$

They are automatically zero for even n (in agreement with the Z_2 parity of the model) whereas for odd n they vanish asymptotically when $\theta_i \to \infty$, as follows from the LSZ reduction formula. Concerning the form factors of $\Theta(x)$, $F_{2n+1}^{\Theta} = 0$ whereas F_{2n}^{Θ} are given by

$$F_{2n}^{\Theta}(\theta_1,\ldots,\theta_{2n}) = \frac{2\pi M^2}{\mathcal{N}(B)} \left(\frac{4\sin(\pi B/2)}{\mathcal{N}(B)}\right)^{n-1} \mathcal{Q}_{2n}(1) \prod_{i<j} \frac{F_{\rm min}(\theta_{ij})}{x_i + x_j} \ , \tag{2.28}$$

and they go to a constant when $\theta_i \to \infty$

Kernel Solutions

The general structure of the form factors of the SGM is that of a sequence of finite linear spaces whose dimensions grow linearly as n increasing the number $2n-1$ or $2n$ of external particles. In fact, at each level of the recursive process the space of the form factors is enlarged by including the kernel solutions of the recursive equation (2.25), i.e. $Q_n(-x, x, x_1, \ldots, x_{n-2}) = 0$. With the constraint that the total order of the polynomials is $\frac{n(n-1)}{2}$, the kernel is unique and given by $\Sigma_n(x_1,\ldots,x_n) = \det \sigma_{2i-j}$. This solution gives rise to the half-infinite chain under the recursive pinching $x_1 = -x_2 = x$

$$\ldots \to Q_{n+4}^{(n)} \to Q_{n+2}^{(n)} \to Q_n^{(n)} = \Sigma_n \to 0 \tag{2.29}$$

and therefore the whole space of form factors presents the foliation structure[1]

$$\begin{array}{ccccccccccccccc}
\ldots & \rightarrow & Q^{(1)}_{n+4} & \rightarrow & Q^{(1)}_{n+2} & \rightarrow & Q^{(1)}_{n} & \rightarrow & Q^{(1)}_{n-2} & \rightarrow & \ldots & \rightarrow & Q^{(1)}_{3} & \rightarrow & 1 \\
\ldots & \rightarrow & Q^{(3)}_{n+4} & \rightarrow & Q^{(3)}_{n+2} & \rightarrow & Q^{(3)}_{n} & \rightarrow & Q^{(3)}_{n-2} & \rightarrow & \ldots & \rightarrow & \Sigma_3 & & \\
 & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & & & \\
 & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & & & & \\
\ldots & \rightarrow & Q^{(n-2)}_{n+4} & \rightarrow & Q^{(n-2)}_{n+2} & \rightarrow & Q^{(n-2)}_{n} & \rightarrow & \Sigma_{n-2} & & & & & & \\
\ldots & \rightarrow & Q^{(n)}_{n+4} & \rightarrow & Q^{(n)}_{n+2} & \rightarrow & \Sigma_{n} & & & & & & & & \\
\ldots & \rightarrow & Q^{(n+2)}_{n+4} & \rightarrow & \Sigma_{n+2} & & & & & & & & & &
\end{array} \qquad (2.30)$$

The explicit expressions of such solutions can be found by determining the linear combination of $Q_n(k)$ which reduces to Σ_n at the level n.

2.3 C-theorem in the staircase model

Since the staircase model may be seen as the SGM at $B = 1 \pm \frac{2i}{\pi}\theta_0$, it is natural to study the behaviour of the form factors of the latter model under this analytic continuation [24]. As we show, the presence of a scale θ_0 in the rapidity axes may induce a non-uniform convergence in series expansions obtained in the original Sinh-Gordon model. Consider for instance the total variation of the central charge $\Delta c = c_{uv} - c_{ir}$ going from the short to the large distances. For both the SGM and the staircase model, $\Delta c = 1$. Let us try to express it as a sum-rule fulfilled by the two-point function of the trace $\Theta(x)$

$$\Delta c = \frac{3}{4\pi} \int r^2 < \Theta(r)\Theta(0) > d^2 r = \sum_{n=1}^{\infty} \Delta c^{(2n)} \, , \qquad (2.31)$$

where $\Delta c^{(2n)}$ is the contribution to the variation of the central charge coming from the $2n$-intermediate states. In the original SGM with real coupling constant, the convergence of the series to the value $\Delta c = 1$ is extremely fast and almost saturated by the two-particle contribution $\Delta c^{(2)}$ [18]. This has to be expected, given the massive behaviour of the model and the threshold suppression phenomena analyzed in [22]. Similar behaviour has been also observed in supersymmetric models [42]. However, in the staircase model the situation is drastically different. Consider initially the two-particle contribution to the c-theorem sum rule

$$\Delta c^{(2)}(\theta_0) = \frac{3}{2} \int_0^{\infty} d\theta \, \frac{|\Xi(2\theta, \theta_0)|^2}{\cosh^4 \theta} \, . \qquad (2.32)$$

[1]This is the structure for form factors of odd operators. Analogous structure arises for the form factors of even operators.

The plot of such a quantity (fig. 13) shows that $\Delta c^{(2)}(\theta_0)$ monotonically decreases from the value very close to 1 at $\theta_0 = 0$ (corresponding to the Sinh-Gordon self-dual point) to $1/2$ for $\theta_0 \to \infty$. The asymptotic value $1/2$ can be easily obtained analytically by noticing that

$$\Xi(\theta,\theta_0) = \sinh\frac{\theta}{2}\, h(\theta,\theta_0)\,, \tag{2.33}$$

$$h(\theta,\theta_0) \simeq \begin{cases} \exp\left(-\frac{\theta-\theta_0}{2}\right) & \theta > \theta_0 \\ -i & \theta < \theta_0 \end{cases}\,,$$

and therefore for $\theta_0 \to \infty$ the integral (2.32) simply reduces to

$$\Delta c^{(2)}(|\theta_0| \to \infty) = \frac{3}{2}\int_0^\infty d\theta\, \frac{\sinh^2\theta}{\cosh^4\theta} = \frac{1}{2}\,. \tag{2.34}$$

Concerning the higher particles contributions $\Delta c^{(2n)}$, all of them vanish in the limit $\theta_0 \to \infty$. In fact, the $2n$-particle form factor entering the formula (2.31) for $\Delta c^{(2n)}$ is given by eq.(2.28) and after the analytic continuation they may be written as

$$F_{2n}(\theta_1,\ldots,\theta_n) = 2\pi m^2 g_{2n}(\theta_0)\, \mathcal{Q}_{2n}(1) \prod_{i<j} \frac{\sinh\frac{\theta_{ij}}{2}\, h(\theta_{ij},\theta_0)}{x_i + x_j}\,, \tag{2.35}$$

where $g_{2n}(\theta_0) = (4\cosh\theta_0)^{n-1}\mathcal{N}^{2n(n-1)}(\theta_0)$. Analogously to the two-particle case, the θ_0-dependence coming from $h(\theta_{ij})$ is strongly suppressed in the integration over rapidities and the asymptotic behaviour in θ_0 of $\Delta c^{(2n)}$ is only determined by the exponential factors contained in g_{2n} and $\mathcal{Q}_{2n}(1)$. In the large θ_0 limit, $\mathcal{N}(\theta_0) \sim \exp\left(-\frac{|\theta_0|}{2}\right)$ and then $g_{2n}(\theta_0) \sim \exp\left\{-(n-1)^2|\theta_0|\right\}$. On the other hand, for $\theta_0 \to \infty$ $\mathcal{Q}_{2n}(1) \sim \exp\{(n-1)(n-2)|\theta_0|\}\,\mathcal{P}(x_i)$ where $\mathcal{P}(x_i)$ is a symmetric polynomial. So, for $n > 1$ $\Delta c^{(2n)}(|\theta_0| \to \infty) \to 0$ as $\exp(-(n-1)\theta_0)$. Therefore the result of the series (2.31) is $\Delta c = 1/2$ instead of $\Delta c = 1$, i.e. a violation of the c-theorem sum rule.

Although striking, the non-uniform convergence of the series has a natural interpretation once the nontrivial interplay between the two scales θ and θ_0 of the problem is correctly taken into account. In fact, since the n-particle contribution in (2.31) behaves as $e^{-n(Mr)}$, given any length scale r there is always an integer N_r such that the states with a number of particles $n \geq N_r$ give a negligible contribution to the series (2.31). This means that any partial sum $\Delta c_N \equiv \sum_{m=1}^{N} \Delta c^{(2m)}$ only reproduces the variation of the c-function from the infrared limit $r = \infty$ up to a certain scale $r^{(N)}$. In usual situations, when $c(r)$ is a smooth function in the deep ultraviolet region, the first few $\Delta c^{(2n)}$ are sufficient to give the correct value of Δc, with high level of precision. But for the staircase model this is not the case. Consider a scale r_1 such that $c(r_1, \theta_0 = 0) > 1/2$

(fig. 14). According to the results of the TBA analysis, after the first jump from 0 to 1/2, the function $c(r, \theta_0)$ stays constant at 1/2 until a value r_2 proportional to $e^{-|\theta_0|/2}$ is reached and, only at this point the second jump takes place. The other jumps occur at $r_n \sim e^{-|\theta_0|(n-1)/2}$ and for $\theta_0 \to \infty$, they accumulate to the origin. Truncating the series (2.31) to any N, there is *always* a value θ_0^* such that $c(r_1^{(N)}, |\theta_0| > |\theta_0^*|) = 1/2$, i.e. the point of the second jump is always ahead of the corresponding length scale $r_1^{(N)}$, however small $r_1^{(N)}$ may be, and therefore

$$\lim_{N\to\infty} \lim_{|\theta_0|\to\infty} \Delta c_N(\theta_0) = \frac{1}{2} \ . \tag{2.36}$$

2.4 Collapse of the Sinh-Gordon model to the Ising model

Taking the limit $\theta_0 \to \infty$ (keeping θ fixed), the S-matrix of the SGM goes to $S = -1$, i.e. to the S-matrix of the thermal perturbed Ising model. Together with (2.36), these results naturally suggest that for $\theta_0 \to \infty$ the Hilbert space of the original SGM collapses to that of the Ising model, spanned in the local sector only by three independent families of fields, those of identity $\{1\}$, magnetization $\{\sigma\}$ and energy $\{\epsilon\}$ operators. It is therefore interesting to find the mapping between the operator content of the two models.

It is easy to see that the elementary field $\phi(x)$ of the SGM is mapped onto the magnetization operator $\sigma(x)$ of the Ising model. In fact, analytically continuing the form factors (2.27) and taking the limit $\theta_0 \to \infty$, the θ_0 dependences coming from different terms of the original expression compensate each other and we obtain the following finite result

$$F_{2n+1}^{\phi}(\theta_1, \ldots, \theta_{2n+1}) \to (i)^n \prod_{i<j}^{2n+1} \tanh \frac{\theta_{ij}}{2} \ . \tag{2.37}$$

These are precisely the form factors of the magnetization operator $\sigma(x)$ of the thermal perturbed Ising model [5, 23]. This field belongs to the interacting sector of the theory and its correlation functions satisfy non-trivial differential equations [34, 43]. Notice that in this limit the boundary conditions of the field ϕ have been modified: in the original SGM its form factors vanish for large values of θ_i whereas in the resulting expression (2.37) they go to a constant.

On the other hand, taking the limit $\theta_0 \to \infty$ for the analytic continuation of the form factors of Θ (2.28), all of them vanish but $F_2 = 2\pi m^2 \sinh \theta/2$. Hence the operator $\Theta(x)$ of the original SGM is mapped onto the energy operator $\epsilon(x)$ of the Ising model.

This is a free field (a result which is manifest by the absence of higher form factors) and its correlators can be easily expressed in terms of Bessel functions. Also in this case the boundary condition of the field Θ has been changed, since originally F_2^{Θ} goes to a constant for large values of θ_i whereas after taking the limit $\theta_0 \to \infty$ it diverges at infinity.

It is also interesting to analyze the behaviour for $\theta_0 \to \infty$ of the kernel solutions. In this limit the recursive equations (2.25) become

$$\begin{aligned} Q_{n+2}(-x, x, x_1, \ldots, x_n) &= -x^{n+1} \sigma_n \, Q_n(x_1, \ldots, x_n) \quad n = \text{odd} \\ Q_{n+2}(-x, x, x_1, \ldots, x_n) &= 0 \qquad\qquad\qquad\qquad\quad\; n = \text{even} \end{aligned} \tag{2.38}$$

The kernel solutions of the Z_2 even operators of the original SGM are therefore mapped onto the free sectors of the Ising model, i.e. those given by the identity and energy operators. Indeed, their form factors are different from zero only at a given level n in the number of external particles (where they coincide with Σ_n defined in sec. 2.2.2) and, due to the second equation in (2.38), they decouple from the rest of the recursive chain. Correlators of the operators defined by such form factors can be also expressed in terms of Bessel functions.

Such a decoupling in the recursive chain does not occur, on the contrary, for the kernel solutions of the odd operators of the original SGM. Their explicit expressions may be written as determinants of minors of the matrix Σ_n. In fact, consider the half-infinite chain of form factors $Q_{n+2m}^{(n)}$ (n odd and $m = 1, 2, \ldots$) satisfying the first equation in (2.38), with the initial condition

$$Q_{n+2}^{(n)} = -x^{n+1} \sigma_n \, \Sigma_n \; . \tag{2.39}$$

It is easy to see that[2] $Q_{n+2}^{(n)} = [\Sigma_{n+2}]_{(\frac{n+1}{2}, n-1)}$ and in general

$$Q_{n+2m}^{(n)} = \left[\left[\ldots [\Sigma_{n+2m}]_{\left(\frac{n+2m-1}{2}, n+2m-1\right)} \cdots \right]_{\left(\frac{n+3}{2}, n+3\right)} \right]_{\left(\frac{n+1}{2}, n+1\right)} \; . \tag{2.40}$$

Such form factors define matrix elements of operators belonging to the magnetization sector. For instance $Q_n^{(1)}$ defines the form factors of the magnetization operator itself whereas $Q_n^{(3)}$ those of the operator $\mathcal{O}^{(3)} = (\sigma(x) + 1/M^2 \partial^2 \sigma(x))$ etc. In general such operators have the distinguishing property that their two-point correlation function $< \mathcal{O}^{(n)}(r) \mathcal{O}^{(n)}(0) >$ decreases at infinity as $\exp[-nMr]$.

[2] We denote by $[A]_{(a,b)}$ the determinant of the matrix obtained by A eliminating its a row and b column.

Chapter 3

Massless integrable models

3.1 Massless scattering

We have seen in the previous chapter that, in presence of integrability, the bootstrap method based on factorized scattering is the most powerful approach to the study of massive models. Hence, it is natural to investigate if and to which extent similar techniques can be extended to the analysis of massless integrable models. Recently, many important progresses have been made in this direction in spite of some subtleties arising when trying to define the notion of massless scattering in two-dimensional space-time [7, 8, 9]. These results, which we will briefly review in this section, rely on the basic assumption that the fundamental properties a) and b) of section 1 characterize the dynamics of scattering processes also in the case of massless integrable theories.

The best way to introduce massless scattering is probably to consider it as the limiting case of some massive relativistic scattering theory. In order to avoid inessential notational complications, we will consider a model whose spectrum consists of a single self-conjugated particle $A(\theta)$ of mass m; the extension to the general case is straightforward. Taking the limit $m \to 0$ of a two-particle process, two different situation can occur in two-dimensions: a) one of the incoming particle becomes a right-mover and the other a left-mover; b) both the particles become right-movers or left-movers.

Let's first analyze case a). The Mandelstam variable s is the only independent relativistic invariant quantity. In the case of diagonal scattering as the one we are considering, the s-plane has only two sheets; we will call "physical" the first one and "unphysical" the second. As m goes to zero, the unitarity and crossing cuts of the s-plane (see fig. 2a) join in the origin so that the Riemann surface splits into two distinct sheets: the "upper" ("lower") one contains the half of the physical (unphysical) sheet with $Im\, s > 0$ and the half of the unphysical (physical) sheet with $Im\, s < 0$. The unitarity and crossing relations (1.15) and (1.16) are immediately specialized to the present case and read

$$S_{RL}(s+i0)S_{RL}(s-i0) = 1 \ , \tag{3.1}$$

$$S_{RL}(s+i0) = S_{RL}(-s-i0) \ , \tag{3.2}$$

where we denoted by S_{RL} the scattering amplitude for the right-left scattering (obviously the massless scattering processes are diagonal in the indices R and L). Some care is required to pass to a parameterization in terms of rapidities. We want to send m to zero in eq.(1.17) while keeping energy and momentum finite; then it is sufficient to replace θ by $\beta \pm \beta_0/2$ and to take the limit $m \to 0, \beta_0 \to +\infty$ in such a way that the massive parameter $M \equiv m e^{\beta_0/2}$ remains finite. According to the sign in front of β_0 we get

$$\begin{aligned} p^0 = p^1 &= \frac{M}{2} e^{\beta} \qquad \text{for right-movers} \ , \\ p^0 = -p^1 &= \frac{M}{2} e^{-\beta} \qquad \text{for left-movers} \ . \end{aligned} \tag{3.3}$$

For practical manipulations, one can think that a right-mover $A_R(\beta)$ or a left-mover $A_L(\beta)$ are obtained from $A(\theta)$ in the limits

$$\begin{aligned} A_R(\beta) &= \lim_{\beta_0 \to +\infty} A(\beta + \beta_0/2) \ , \\ A_L(\beta) &= \lim_{\beta_0 \to +\infty} A(\beta - \beta_0/2) \ . \end{aligned} \tag{3.4}$$

From eqs.(3.3) we obtain the relation

$$s = (p_R^{\mu}(\beta_1) + p_L^{\mu}(\beta_2))^2 = M^2 e^{\beta_1 - \beta_2} \ , \tag{3.5}$$

showing that, contrary to the massive case, each value of $\beta \equiv \beta_1 - \beta_2$ in the strip $0 < Im\beta < 2\pi$ corresponds now to a different value of s. This implies that in the massless case we need two $2\pi i$-periodic, meromorphic functions of the rapidity difference in order to represent a function defined on the two sheets of the s-plane. We denote by $S_{RL}(\beta)$ ($\tilde{S}_{RL}(\beta)$) the values of the scattering amplitude on the "upper" ("lower") sheet defined above. Then, by definition, $S_{RL}(\beta)$ ($\tilde{S}_{RL}(\beta)$) must be obtained from the value $S(\theta)$ ($S(-\theta)$) of the massive amplitude in the limit $A(\theta_1) \to A_R(\beta_1)$ and $A(\theta_2) \to A_L(\beta_2)$

$$\begin{aligned} S_{RL}(\beta) &= \lim_{\beta_0 \to +\infty} S(\beta + \beta_0) \ , \\ \tilde{S}_{RL}(\beta) &= \lim_{\beta_0 \to +\infty} S(-\beta - \beta_0) \ . \end{aligned} \tag{3.6}$$

Now it is straightforward to take the limit of eqs.(1.19) and (1.20) to get the massless unitarity and crossing relations [7]

$$S_{RL}(\beta)\tilde{S}_{RL}(\beta) = 1 \ , \tag{3.7}$$

$$S_{RL}(\beta) = \tilde{S}_{RL}(\beta + i\pi) \ . \tag{3.8}$$

Of course, they are nothing else than the translation in the rapidity language of eqs.(3.1) and (3.2) respectively. From their combination we obtain the cross-unitarity equation

$$S_{RL}(\beta) S_{RL}(\beta - i\pi) = 1 \ . \tag{3.9}$$

Turning to right-right and left-left scattering, we notice that the variable s identically vanishes so that all the analyticity arguments characteristic of the S-matrix formalism cannot be applied to this case. This circumstance clearly reflects the loss of intuitive understanding with respect to the scattering process of massless particles moving in the same direction in one spatial dimension. Nevertheless, the amplitudes S_{RR} and S_{LL} can be formally introduced starting from the massive theory and taking the massless limit prescribed by eqs.(3.4). Since in this case the rapidity shifts cancel, we simply find

$$S_{RR}(\beta) = S_{LL}(\beta) = S(\beta) \ , \tag{3.10}$$

so that, in the formalism of rapidities, RR and LL scattering follow the same formal rules of massive scattering. In particular, the following "unitarity" and "crossing" relations hold

$$\begin{aligned} S_{RR}(\beta) S_{RR}(-\beta) &= 1 \ , \\ S_{RR}(\beta) &= S_{RR}(i\pi - \beta) \ , \end{aligned} \tag{3.11}$$

(the same for S_{LL}). Notice that, since in the present case the only relativistic invariants are ratios of momenta, according to eqs.(3.3), the amplitudes S_{RR} and S_{LL} correctly depends on the difference of rapidities. Due to this fact, they are completely independent on the mass scale M. Thus we conclude that this parameter plays a role (namely, scale invariance is broken) only in presence of a rapidity dependent RL scattering.

We have seen in the previous chapter that an ordering prescription over rapidities can be introduced in order to identify the physical asymptotic states and that such prescription arises from the analytic properties of the two-particle scattering amplitude. In the massless case the amplitude $S_{RR}(\beta)$ ($S_{LL}(\beta)$) behaves as the massive amplitude $S(\theta)$ so that it is natural to assume the prescription (1.24) for the ordering of right (left) particles in "in" and "out" states. Hence, what we need is to assign a prescription for the ordering of right-movers with respect to left-movers. Obviously, a two-particle state containing a right-mover and a left-mover is (for all possible values of the rapidities) an asymptotic in-state if the right-mover is at $x = -\infty$ while the left-mover is at $x = +\infty$; it is an out-state in the opposite case. This observation suggests the definitions

$$|A_R(\beta_1) A_L(\beta_2) >_{in} \ = \ |A_R(\beta_1) A_L(\beta_2) > \ ,$$

$$|A_R(\beta_1)A_L(\beta_2)>_{out} \quad = \quad |A_L(\beta_2)A_R(\beta_1)> \quad . \tag{3.12}$$

The two asymptotic states are related by the S-matrix in the following way

$$|A_R(\beta_1)A_L(\beta_2)>= S_{RL}(\beta_1-\beta_2)|A_L(\beta_2)A_R(\beta_1)> \quad . \tag{3.13}$$

Using eq.(3.7) we also obtain

$$|A_L(\beta_2)A_R(\beta_1)>= \tilde{S}_{RL}(\beta_1-\beta_2)|A_R(\beta_1)A_L(\beta_2)> \quad . \tag{3.14}$$

We are now in the position to introduce a Faddeev-Zamolodchikov algebra for massless scattering in the form

$$A_{\alpha_1}(\beta_1)A_{\alpha_2}(\beta_2) = S_{\alpha_1\alpha_2}(\beta_1-\beta_2)A_{\alpha_2}(\beta_2)A_{\alpha_1}(\beta_1) \ , \tag{3.15}$$

where $\alpha_i = R, L$ and $A_R(\beta)$, $A_L(\beta)$ are the creation operators for right and left movers respectively. Clearly, the last equation agrees with eq.(3.14) provided that

$$S_{LR}(\beta_1-\beta_2) = \tilde{S}_{RL}(\beta_2-\beta_1) \quad . \tag{3.16}$$

This equation can also be obtained directly in the limit from the massive case (see eq.(3.6)). Equation (3.15) suggests the best interpretation for the scattering amplitude of two massless particles moving in the same direction: it is the phase which results when commuting two particle creation operators. The appearance of this phase is explicitly seen in the lattice Bethe ansatz approach [46, 47] or in the classical limit [48] where the two-particle wavefunction and the two-soliton solution respectively are observed to exhibit non-trivial monodromy when one particle is moved through the other.

Extending the previous considerations to a generic state containing r right-movers and $l = n - r$ left-movers we conclude that it is an in-state if it is written as

$$\begin{aligned} &|A_R(\beta_1)A_R(\beta_2)\dots A_R(\beta_r)A_L(\beta_{r+1})A_L(\beta r+2)\dots A_L(\beta_n)> \\ &\beta_1>\beta_2>\dots>\beta_r \ , \qquad \beta_{r+1}>\beta_{r+2}>\dots\beta_n \ ; \end{aligned} \tag{3.17}$$

the corresponding out-state is

$$|A_L(\beta_n)A_L(\beta_{n-1})\dots A_L(\beta_{r+1})A_R(\beta_r)A_R(\beta r-1)\dots A_R(\beta_1)> \ , \tag{3.18}$$

with the rapidities satisfying the same inequalities as before.

3.2 Massless form factors

Massless form factors are defined to be, as in the massive case, the matrix elements of local operators between asymptotic states. In order to determine their basic analytic properties we will follow the same strategy used for the scattering amplitudes in the previous section. Let's start with the two-particle massive form factor of a scalar operator $O(x)$

$$F(s) = < 0|O(0)|A(p_1^\mu)A(p_2^\mu) > , \tag{3.19}$$

satisfying eqs.(1.38) and (1.41). In the massless limit in which the two particles become a right-mover and a left-mover, these equations reduce to

$$\begin{aligned} F_{RL}(s+i0) &= S_{RL}(s+i0)F_{RL}(s-i0) , \\ F_{RL}(-s+i0) &= F_{RL}(-s-i0) . \end{aligned} \tag{3.20}$$

As explained for the scattering amplitudes, we need two functions of rapidities to cover the two sheets of the s-plane. We denote by $F_{RL}(\beta_1,\beta_2)$ ($\tilde{F}_{RL}(\beta_1,\beta_2)$) the values on the "upper" ("lower") sheet defined in the previous section. According to eqs.(3.4), these functions are obtained from the massive form factor $F(\theta_1,\theta_2)$ as the limits

$$\begin{aligned} F_{RL}(\beta_1,\beta_2) &= \lim_{\beta_0\to+\infty} F\left(\beta_1+\frac{\beta_0}{2},\beta_2-\frac{\beta_0}{2}\right) , \\ \tilde{F}_{RL}(\beta_1,\beta_2) &= \lim_{\beta_0\to+\infty} F\left(\beta_2-\frac{\beta_0}{2},\beta_1+\frac{\beta_0}{2}\right) . \end{aligned} \tag{3.21}$$

Then we can take the limit of eqs.(1.42) and (1.43) to get the equations

$$\begin{aligned} F_{RL}(\beta_1,\beta_2) &= S_{RL}(\beta_1-\beta_2)\tilde{F}_{RL}(\beta_1,\beta_2) , \\ F_{RL}(\beta_1+i\pi,\beta_2) &= \tilde{F}_{RL}(\beta_1-i\pi,\beta_2) , \end{aligned} \tag{3.22}$$

which translate eqs.(3.20) and hold for operators of generic spin. From their combination we get

$$F_{RL}(\beta_1,\beta_2) = S_{RL}(\beta_1-\beta_2)F_{RL}(\beta_1+2\pi i,\beta_2) . \tag{3.23}$$

Since

$$F_{LR}(\beta_2,\beta_1) = \lim_{\beta_0\to+\infty} F(\beta_2-\frac{\beta_0}{2},\beta_1+\frac{\beta_0}{2}) = \tilde{F}_{RL}(\beta_1,\beta_2) \tag{3.24}$$

and since $F_{RR}(\beta_1,\beta_2)$ and $F_{LL}(\beta_1,\beta_2)$ behaves as in the massive case, the monodromy equations for the massless two-particle form factors can be written in the compact form

$$\begin{aligned} F_{\alpha_1\alpha_2}(\beta_1,\beta_2) &= S_{\alpha_1\alpha_2}(\beta_1-\beta_2)F_{\alpha_2\alpha_1}(\beta_2,\beta_1) , \\ F_{\alpha_1\alpha_2}(\beta_1+2\pi i,\beta_2) &= F_{\alpha_2\alpha_1}(\beta_2,\beta_1) , \end{aligned} \tag{3.25}$$

where $\alpha_i = R, L$. The generalization to the n-particle form factors

$$F_{\alpha_1\ldots\alpha_n}(\beta_1,\ldots,\beta_n) = <0|O(0)|A_{\alpha_1}(\beta_1),\ldots,A_{\alpha_n}(\beta_n)> \tag{3.26}$$

reads

$$\begin{aligned} &F_{\alpha_1\ldots\alpha_i\alpha_{i+1}\ldots\alpha_n}(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_n) = \\ &\quad S_{\alpha_i\alpha_{i+1}}(\beta_i-\beta_{i+1})F_{\alpha_1\ldots\alpha_{i+1}\alpha_i\ldots\alpha_n}(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_n)\ , \end{aligned} \tag{3.27}$$

$$F_{\alpha_1\alpha_2\ldots\alpha_n}(\beta_1+2\pi i,\beta_2,\ldots,\beta_n) = F_{\alpha_2\ldots\alpha_n\alpha_1}(\beta_2,\ldots,\beta_n,\beta_1)\ . \tag{3.28}$$

The matrix elements (3.26) are meromorphic functions of rapidities defined in the strips $0 \leq Im\beta_i < 2\pi$. We have seen that in the massive case the form factors admit simple poles associated to bound states in the scattering amplitudes and to particle-antiparticle annihilation. Stable bound states are usually forbidden in massless theories due to the absence of thresholds. This leads to exclude the presence of the first kind of singularities in the right-left sub-channels, the only ones to which standard S-matrix theory can be applied. On the other hand, we have seen that right-right and left-left scattering formally behave as in the massive case. Hence, whenever the amplitude $S_{\alpha\alpha}(\beta)$ has a simple pole for $\beta = iu$ ($u \in (0,\pi)$) with residue g, we expect the following equation to hold

$$i\, res_{\beta_{n+1}-\beta_{n+2}=iu} F_{\alpha_1\ldots\alpha_n\alpha\alpha}(\beta_1,\ldots,\beta_n,\beta_{n+1},\beta_{n+2}) = g F_{\alpha_1\ldots\alpha_n\alpha}(\beta_1,\ldots,\beta_n,\beta_{n+1}-iu)\ . \tag{3.29}$$

Concerning the annihilation poles, it is clear that again they can only occur in right-right or left-left sub-channels since the two-momenta of a right-mover and a left-mover cannot be made opposite in sign through analytic continuation in the rapidities. The residue equation in this case reads

$$-i\, res_{\beta'=\beta+i\pi} F_{\alpha\alpha_1\ldots\alpha_n}(\beta',\beta,\beta_1,\ldots,\beta_n) = \left(1 - \prod_{i=1}^{n} S_{\alpha_i\alpha}(\beta_i-\beta)\right) F_{\alpha_1\ldots\alpha_n}(\beta_1,\ldots,\beta_n)\ . \tag{3.30}$$

We conclude this section noting that symmetry under spatial inversion provides the relation

$$F_{\alpha_1\ldots\alpha_n}(\beta_1,\ldots,\beta_n) = F_{\mathcal{P}[\alpha_n]\ldots\mathcal{P}[\alpha_1]}(-\beta_n,\ldots,-\beta_1)\ , \tag{3.31}$$

where $\mathcal{P}[R] = L$, $\mathcal{P}[L] = R$.

3.3 The flow from tricritical to critical Ising model

3.3.1 General features

It is well known that the perturbation of the minimal unitary model $\mathcal{M}_p$ with central charge

$$c = 1 - \frac{6}{p(p+1)} \tag{3.32}$$

by the relevant operator $\phi_{1,3}$ leads (for positive values of the coupling constant) to a massless theory flowing to the model $\mathcal{M}_{p-1}$ in the infrared limit [44, 45]. It was shown in ref.[4] that the QFTs describing these flows admit an infinite number of conserved currents and therefore are integrable; the corresponding factorized scattering theories were determined in refs.[7, 9]. In the following we will consider in detail the simplest of these theories, namely the flow from the tricritical Ising model ($p = 4$) to the critical Ising model ($p = 3$) with the purpose to apply to it the massless form factors bootstrap introduced in the previous section.

The minimal unitary model $\mathcal{M}_4$ ($c = 7/10$) describes the tricritical point related to the spontaneous breakdown of Z_2-symmetry in two-dimensional systems [49, 50]. The conformal dimensions of the model are collected in table 3; a set of primary mutually local scalar fields whose conformal families form a closed operator algebra is listed in table 4.

The tricritical behaviour can be observed, for instance, in an Ising ferromagnet with vacancies described by the lattice hamiltonian

$$H = -\beta \sum_{(ij)} \sigma_i \sigma_j t_i t_j + k \sum_i t_i \ , \tag{3.33}$$

where the sum (ij) runs over nearest neighbors only; the Ising spin σ takes values ± 1 while t takes values $0, 1$ so that the parameter k is a chemical potential controlling the vacancy density. The phase diagram of this system is schematically represented in fig. 15. A line of phase transition points separates region I with spontaneous magnetization from the disordered phase II. The tricritical point T divides the second order phase transition line (continuous part) from the first order one (dashed part); the point C, located on the transition line at zero vacancy density, corresponds to the critical Ising model.

A field theoretical description of tricritical behaviour can be given in terms of a Landau-Ginzburg lagrangian for a single scalar order parameter (the "spin" field). The possible coexistence of three different phases, implying three degenerate minima for the potential, leads to the choice

$$\mathcal{L}_{LG} = \frac{1}{2}(\partial_\mu \phi)^2 + \sum_{k=1}^{6} g_k \phi^6 \ . \tag{3.34}$$

The tricritical point corresponds to $g_i = 0$ for $i = 1, \ldots, 5$. The fields appearing in the lagrangian (3.34) can be matched with the primary operators of table 4 using a prescription suggested by A. Zamolodchikov [51]. We start with the natural identification $\phi = \sigma$ and define recurrently the composite operator $: \phi^k :$ as the most singular term remaining as $x \to 0$ in the operator expansion $\phi(x) : \phi^{k-1}(0) :$ after the subtraction of the contributions with dimension lower than the dimension of $: \phi^{k-1} :$. Using the fusion rules of $\mathcal{M}_4$ (see table 5) we find $: \phi^2 := \varepsilon$, $: \phi^3 := \sigma'$, $: \phi^4 := \varepsilon'$ and $: \phi^5 :\sim \partial_\mu \partial^\mu \sigma = \partial_\mu \partial^\mu \phi$. Last relation expresses the equation of motion for the field theory (3.34) at the tricritical point; it also shows that $: \phi^5 :$ is an irrelevant operator. Also $: \phi^6 :$ is irrelevant and is usually identified with ε''.

The tricritical Ising model (TIM) possesses two kinds of discrete symmetry. The first one is the Z_2-symmetry under spin reversal $\sigma \to -\sigma$ which, according to the fusion rules in table 5, divides the theory in an even sector containing the conformal families $\{I\}, \{\varepsilon\}, \{\varepsilon'\}, \{\varepsilon''\}$ and an odd sector containing $\{\sigma\}, \{\sigma'\}$. The second symmetry concerns the "Kramers-Wannier" or "duality" transformation which, in analogy to what happens in Ising model, maps the spin fields σ, σ' into the dual "disorder" fields μ, μ' with the same conformal dimensions $(3/80, 3/80)$ and $(7/16, 7/16)$, and changes the sign of the energy ε; the fusion rules imply that the operators ε' and ε'' are respectively even and odd under this transformation. The fields σ, σ' and their dual μ, μ' are not mutually local since the fermion fields $\psi \equiv \phi_{(6/10,1/10)}$ and $\bar{\psi} \equiv \phi_{(1/10,6/10)}$ are generated in their OPE. For instance, we have

$$\sigma(z,\bar{z})\,\mu(0,0) \sim (z\bar{z})^{1/40}\left[z^{1/2}\psi(z,\bar{z}) + \bar{z}^{1/2}\bar{\psi}(z,\bar{z})\right] + \ldots \ . \tag{3.35}$$

The operator algebra generated by the primary fields $\{I, \sigma, \varepsilon, \sigma', \varepsilon', \varepsilon''\}$ and its dual generated by $\{I, \mu, -\varepsilon, \mu', \varepsilon', -\varepsilon''\}$ are isomorphic and represent two different "local sections" of the conformal theory $\mathcal{M}_4$. In this sense we say that the TIM is self-dual.

Self-duality as well as invariance under spin reversal are expected to characterize the whole phase transition line in fig. 15. Hence, the subleading energy ε', being the only relevant operator in $\mathcal{M}_4$ invariant under both the discrete symmetries, is uniquely identified as the field which drives the flow along the line. The action for this flow can formally be written as

$$\mathcal{A} = \mathcal{A}_{\mathcal{M}_4} + \tilde{\lambda} \int d^2x\, \varepsilon'(x) \ , \tag{3.36}$$

$\tilde{\lambda}$ being a coupling constant with scale dimension $\tilde{\lambda} \sim (mass)^{4/5}$ whose sign determines the direction of the flow.

Further insight into the physical properties of the model can be obtained using a lagrangian description alternative to (3.34). This possibility arises from the fact that TIM is the only model of the minimal unitary series appearing also in the discrete series of superconformal theories discovered by Friedan, Qiu and Shenker [49, 50]. Then we can describe the tricritical point by the super Landau-Ginzburg action proposed by A. Zamolodchikov in ref.[51]

$$\mathcal{A}_{\mathcal{M}_4} = \int d^2z d^2\theta \, (D\Phi\bar{D}\Phi + \frac{g}{3}\Phi^3) \; , \tag{3.37}$$

where $D = \partial_\theta + \theta\partial_z$, $\bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}$ and the superfield $\Phi(z,\theta;\bar{z},\bar{\theta}) = \varepsilon(z,\bar{z}) + \theta\psi(z,\bar{z}) + \bar{\theta}\bar{\psi}(z,\bar{z}) + \theta\bar{\theta}\varepsilon'(z,\bar{z})$ coincides with the superprimary $\Phi_{(1/10,1/10)}$ of the Neveu-Schwartz sector. In this formalism the spin fields appear as primary fields in the Ramond sector (corresponding to periodic boundary condition on the cylinder) and the order-disorder degeneracy is a consequence of the commutativity of the zero mode Ramond generator G_0 with the dilatation generator L_0: $\mu = G_0\sigma$, $\mu' = G_0\sigma'$. The subenergy $\varepsilon' = \int d^2\theta\Phi$ preserves global supersymmetry and the perturbed action (3.36) can be written as [52, 53]

$$\mathcal{A} = \int d^2z d^2\theta \, (D\Phi\bar{D}\Phi + \frac{g}{3}\Phi^3 + \lambda\Phi) \; . \tag{3.38}$$

Self-duality manifests itself as the invariance of the above action under the substitutions $\Phi \to -\Phi, \theta \to -\theta$ or, equivalently, $\varepsilon \to -\varepsilon, \psi \to \psi, \bar{\psi} \to -\bar{\psi}, \varepsilon' \to \varepsilon'$. The integration in θ and $\bar{\theta}$ and the exclusion of ε' lead to the following expression for the action

$$\mathcal{A} = \int d^2z \left[\partial\varepsilon\bar{\partial}\varepsilon - \psi\bar{\partial}\psi - \bar{\psi}\partial\bar{\psi} - \frac{1}{4}(g\varepsilon^2 + \lambda)^2 - 2g\psi\bar{\psi}\varepsilon\right] \; . \tag{3.39}$$

For $\lambda/g < 0$ the bosonic potential $V(\varepsilon) = (g\varepsilon^2 + \lambda)^2/4$ has two degenerate minima at $\varepsilon_\pm = \pm\sqrt{-\lambda/g}$ and the fermion and the boson acquire the same mass $m = 2\sqrt{-g\lambda}$; since $V(\varepsilon_\pm) = 0$, global supersymmetry is preserved. The factorized S-matrix of this massive supersymmetric integrable theory describing the scale region of TIM along the dashed line in fig. 15 was determined by A. Zamolodchikov in ref.[52].

For $\lambda/g > 0$ the bosonic potential has a single minimum at $\varepsilon = 0$; since $V(0) > 0$, supersymmetry is spontaneously broken. The boson becomes massive ($m_\varepsilon = \sqrt{2\lambda g}$) while the fermion stays massless and plays the role of goldstino. A low energy effective action for the goldstino can be obtained integrating out the scalar field in the Landau-Ginzburg action (3.39). At first order one finds [53]

$$\mathcal{A}_{\text{eff}} = -\int d^2z \left[\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} - \frac{4}{\lambda^2}(\psi\partial\psi)(\bar{\psi}\bar{\partial}\bar{\psi}) + \ldots\right] \; . \tag{3.40}$$

As could be expected, this action corresponds to the Ising model (free massless majorana fermion) perturbed by the irrelevant operator $(\psi\partial\psi)(\bar{\psi}\bar{\partial}\bar{\psi}) \sim T\bar{T}$, which is the lowest dimension nonderivative field invariant under Z_2 and duality transformations. The two discrete symmetries of the theory appear in this context as the invariance property of the action (3.40) under the substitutions $\psi \to \psi, \bar{\psi} \to -\bar{\psi}$ and $\psi \to -\psi, \bar{\psi} \to -\bar{\psi}$ which we identify as corresponding to the duality transformation and to spin reversal, respectively.

3.3.2 Scattering theory and form factors

The factorized scattering theory for the flow from tricritical to critical Ising was determined by Al. Zamolodchikov in ref.[7]. The basic assumption is that the massless neutral fermion appearing in the action (3.39) is the only stable particle of the theory while the massive boson may decay and does not enter the asymptotic states. Thus, the spectrum of particles is the same as in the infrared fixed point. Since the right-right and left-left amplitudes cannot vary along the flow, they will preserve the infrared value

$$S_{RR}(\beta) = S_{LL}(\beta) = -1 \ . \tag{3.41}$$

Concerning right-left scattering, TBA analysis shows that the minimal solution of the cross-unitarity equation (3.9) tending to -1 in the infrared limit, namely

$$S_{RL}(\beta) = \tanh\left(\frac{\beta}{2} - \frac{i\pi}{4}\right) , \tag{3.42}$$

is in fact the correct choice. The values of this amplitude on the "upper" sheet of the s-plane read

$$S(s) = \frac{s - iM^2}{s + iM^2} , \tag{3.43}$$

where M is the mass scale entering the parameterization (3.3). Using eq.(3.1) to get the values on the "lower" sheet we see that, while the physical sheet is free of poles, the unphysical one contains two poles at $s = \pm iM^2$ which can be interpreted as a reminiscence of the unstable boson. Using the effective action (3.40) to compute perturbatively the next to leading term in the infrared expansion of $S_{RL}(s)$ and comparing with the exact expression (3.43) we get the relation $M^2 = \lambda^2/2$.

We now turn to the computation of form factors using the general properties exposed in the previous section. In order to simplify the notation, we consider the following subset of form factors:

$$F_{r,l}(\beta_1, \beta_2, \ldots, \beta_r; \beta'_1, \beta'_2, \ldots, \beta'_l) = \\ < 0|O(0)|A_R(\beta_1)A_R(\beta_2)\ldots A_R(\beta_r)A_L(\beta'_1)A_L(\beta'_2)\ldots A_L(\beta'_l) > \tag{3.44}$$

Any other form factor of the local operator $O(x)$ can be obtained from these using eq.(3.27). We parameterize the functions (3.44) as

$$F_{r,l}(\beta_1,\beta_2,\ldots,\beta_r;\beta'_1,\beta'_2,\ldots,\beta'_l) = H_{r,l}Q_{r,l}(x_1,x_2,\ldots,x_r;y_1,y_2,\ldots,y_l)\times \prod_{1\le i<j\le r}\frac{f_{RR}(\beta_i-\beta_j)}{x_i+x_j}\prod_{i=1}^{r}\prod_{j=1}^{l}f_{RL}(\beta_i-\beta'_j)\prod_{1\le i<j\le l}\frac{f_{LL}(\beta'_i-\beta'_j)}{y_i+y_j}\,, \quad (3.45)$$

where $x_i \equiv e^{\beta_i}$, $y_i \equiv e^{-\beta'_i}$ and $H_{r,l}$ are normalization constants introduced for later convenience. The auxiliary functions f_{RR}, f_{LL} and f_{RL} satisfy the following equations obtained combining eqs.(3.25) for a scalar operator

$$f_{\alpha_1\alpha_2}(\beta) = S_{\alpha_1\alpha_2}(\beta)f_{\alpha_1\alpha_2}(\beta+2\pi i) \quad (3.46)$$

Given the scattering amplitudes (3.41), (3.42) we choose the following solutions with neither poles nor zeroes in the strip $0 <\text{Im}\beta < 2\pi$

$$f_{RR}(\beta) = f_{LL}(\beta) = \sinh\frac{\beta}{2}\,, \quad (3.47)$$

$$f_{RL}(\beta) = \exp\left(\frac{\beta}{4}-\int_0^\infty\frac{dt}{t}\frac{\sin^2\left(\frac{(i\pi-\beta)t}{2\pi}\right)}{\sinh t\cosh\frac{t}{2}}\right)\,. \quad (3.48)$$

The function $f_{RL}(\beta)$ satisfies the equation

$$f_{RL}(\beta\pm i\pi)f_{RL}(\beta) = \frac{i\gamma}{1\pm ie^{-\beta}}\,, \quad (3.49)$$

where $\gamma = \sqrt{2}e^{2G/\pi}$, G being the Catalan costant. The functions (3.47), (3.48) completely take into account the monodromy properties in eq.(3.45). Since the scattering amplitudes S_{RR} and S_{LL} in this theory are free of poles, only kinematical poles appear in the form factors and are explicitly inserted in the parameterization (3.45) through the factors x_i+x_j and y_i+y_j in the denominator. As a consequence, after requiring the form factors to be power bounded in the momenta, $Q_{r,l}$ have to be rational functions separately symmetric in the $\{x_i\}$ and $\{y_i\}$ with at most poles located at $x_i = 0$ or $y_i = 0$.

Inserting the parameterization (3.45) into the residue equations (3.30) we find the recursive relations satisfied by $Q_{r,l}$

$$Q_{r+2,l}(-x,x,x_1,\ldots,x_r;y_1,\ldots,y_l) = x^{r-l+1}\frac{\rho_r(\{x_i\})}{\lambda_l(\{y_i\})}\times\sum_{k=0}^{l}{}'(-ix)^k\lambda_k(\{y_i\})Q_{r,l}(x_1,\ldots,x_r;y_1,\ldots,y_l)\,,$$

$$Q_{r,l+2}(x_1,\ldots,x_r;y_1,\ldots,y_l,y,-y) \;=\; y^{l-r+1}\frac{\lambda_l(\{y_i\})}{\rho_r(\{x_i\})}$$
$$\times \sum_{k=0}^{r}{}' (-iy)^k \rho_k(\{x_i\}) Q_{r,l}(x_1,\ldots,x_r;y_1,\ldots,y_l) \;, \qquad (3.50)$$

where the primed sums run over odd indices if $(r+l)$ is even and vice versa, and $\rho_k(\{x_i\})$ is the basis of symmetric polynomials in the variables $\{x_i\}$ generated by

$$\prod_{j=1}^{r}(x+x_j) = \sum_{k=0}^{r} x^{r-k}\rho_k(\{x_i\}) \qquad (3.51)$$

(analogously for $\lambda_k(\{y_i\})$). In writing eqs. (3.50) we have chosen

$$H_{r,l} = -\frac{\gamma^l}{i^r 2^{2r+1}} H_{r+2,l} = -\frac{\gamma^r}{i^l 2^{2l+1}} H_{r,l+2} \;. \qquad (3.52)$$

Equations (3.50) are quite general since they were obtained without reference to any particular operator. In the following we will restrict our attention to some operators of particular physical relevance, namely the trace of the stress-energy tensor $\Theta(x) = T^\mu_\mu(x)$, the magnetization (order) operator $\sigma(x)$ and the disorder operator $\mu(x)$. Since all these fields are spinless, eq.(1.35) implies that under a lorentz transformation the functions $Q_{r,l}(\{x_i\};\{y_i\})$ behave as

$$Q_{r,l}(\{e^\Lambda x_i\};\{e^{-\Lambda}y_i\}) \;=\; e^{\left(\frac{r(r-1)}{2}-\frac{l(l-1)}{2}\right)\Lambda}\, Q_{r,l}(\{x_i\};\{y_i\}) \;. \qquad (3.53)$$

Selection rules for the form factors are obtained assigning the transformation properties of the massless particles under the discrete symmetries of the theory. According to the above discussed invariance properties of the fermionic action (3.40), we assign even Z_2-parity to both right and left-movers, and even (odd) parity to right (left) movers under duality transformation.

Trace of the energy-momentum tensor

The trace of the energy-momentum tensor is expressed in terms of the ultraviolet perturbing field by the relation

$$\Theta(x) = 2\pi\tilde{\lambda}(2-2\Delta_{\varepsilon'})\varepsilon'(x) \;. \qquad (3.54)$$

Since the subenergy ε' is even under both spin reversal and duality transformation, Θ will have nonvanishing form factors $F_{r,l}$ only for even r and l, starting from $r=l=2$ (the vacuum expectation value $F_{0,0}$ must be identically zero since it is known to vanish in the infrared limit). Moreover, adapting the argument of section 1.3, one easily realizes that the conservation of the energy-momentum tensor implies the factorization

$$Q_{r,l}(\{x_i\};\{y_i\}) = \rho_1\lambda_1 T_{r,l}(\{x_i\};\{y_i\}) \,. \tag{3.55}$$

Inserting this expression into (3.50) we get the recursive equations for $T_{r,l}$

$$\begin{aligned} T_{r+2,l}(-x,x,\{x_i\};\{y_i\}) &= x^{r-l+1}\frac{\rho_r}{\lambda_l}\sum_{k=0}^{l}{}' (-ix)^k\lambda_k\, T_{r,l}(\{x_i\};\{y_i\}) \,, \\ T_{r,l+2}(\{x_i\};\{y_i\},y,-y) &= y^{l-r+1}\frac{\lambda_l}{\rho_r}\sum_{k=0}^{r}{}' (-iy)^k\rho_k\, T_{r,l}(\{x_i\};\{y_i\}) \,. \end{aligned} \tag{3.56}$$

The leading infrared contribution to $F_{2,2}$ is easily computed using the action (3.40)

$$F_{2,2}(\beta_1,\beta_2;\beta_1',\beta_2') \to -4\pi M^2 \sinh\frac{\beta_1-\beta_2}{2}\,\sinh\frac{\beta_1'-\beta_2'}{2}\, e^{\beta_1+\beta_2-\beta_1'-\beta_2'} \,. \tag{3.57}$$

With this information we fix the exact $F_{2,2}$ to be

$$F_{2,2}(\beta_1,\beta_2;\beta_1',\beta_2') = \frac{4\pi M^2}{\gamma^2}\sinh\frac{\beta_1-\beta_2}{2}\prod_{i,j=1,2} f_{RL}(\beta_i-\beta_j')\sinh\frac{\beta_1'-\beta_2'}{2} \,. \tag{3.58}$$

The recursive equations (3.56), (3.52) are then iteratively solved by using (3.58) as initial condition. With the normalization constant

$$H_{2n,2m} = \pi M^2 i^{n(n+1)+m(m+1)} 2^{2(n^2+m^2)-n-m}\gamma^{-2nm} \tag{3.59}$$

the first right chains are determined to be

$$\begin{aligned} T_{2n,2}(\{x_i\};\{y_i\}) &= (i)^{n^2-1}\left(\rho_{2n}\frac{\lambda_1}{\lambda_2}\right)^{n-1} \,, \\ T_{2n,4}(\{x_i\};\{y_i\}) &= (i)^{n^2-2n}\left(\frac{\rho_{2n}}{\lambda_4}\right)^{n-2}\sum_{k=0}^{n-1}\rho_{2k+1}\lambda_1^{n-1-k}\lambda_3^k \,, \\ &\vdots \end{aligned} \tag{3.60}$$

Using the right-left symmetry relation $T_{r,l}(\{x_i\};\{y_i\}) = T_{l,r}(\{y_i\};\{x_i\})$ one can immediately obtain the solution for the corresponding left chains.

Order and disorder operators

The invariance of the theory under spin reversal implies that the magnetization (order) operator σ has nonvanishing form factors only on an odd number of particles. Taking

$F_{1,0} = F_{0,1} = 1$ as initial conditions for the recursive equations, we obtain

$$
\begin{aligned}
Q_{r,0} &= \rho_r^{(r-1)/2} \, , \\
Q_{r,1} &= \frac{\rho_r^{r/2-1}}{\lambda_1^{r/2}} \, , \\
Q_{r,2} &= \rho_r^{(r-3)/2} \sum_{k=0}^{r}{}' \rho_k \lambda_2^{(k-r+1)/2} \, , \\
Q_{r,3} &= \frac{\rho_r^{(r/2-2)}}{\lambda_3^{r/2-1}} \sum_{k=0}^{r}{}' \rho_k \lambda_2^{k/2} \, , \\
&\vdots
\end{aligned} \tag{3.61}
$$

where the prime denotes sum over even indices only and r should be chosen such that $r + l$ is odd.

The disorder operator μ is nonlocal with respect to the magnetization. This circumstance induces a slight modification in eq. (3.50) [6, 23, 20, 54] (the minus sign in front of the product becomes plus). One can easily check that the functions $Q_{r,l}$ for μ derived from the modified recursive equation are still given by the formulas (3.61) with $r + l$ even (μ is Z_2-even).

3.3.3 C-theorem and correlation functions

We have seen in the first chapter that the knowledge of form factors in massive integrable theories allows us to express the correlation functions as infinite sums over intermediate multiparticle states and that very precise numerical estimates are provided by the first few terms of the series. An analogous expansion can be formally written also in the massless case but obviously serious doubts on its practical utility are raised by the absence of any natural infrared cutoff. To be concrete, let's consider the two-point function of a local operator $O(x)$ in the massless flow we are considering. According to the prescriptions introduced in section 1.1 to select the physical asymptotic states, the spectral representation of this correlator reads

$$
\begin{aligned}
< O(x)O(0) >= &\sum_{r,l=0}^{\infty} \frac{1}{r!\,l!} \int_{-\infty}^{+\infty} \frac{d\beta_1 \ldots d\beta_r d\beta_1' \ldots d\beta_l'}{(2\pi)^{r+l}} |F_{r,l}(\beta_1, \ldots, \beta_r; \beta_1' \ldots \beta_l')|^2 \\
&\times \exp\left[-\frac{Mr}{2} \left(\sum_{j=1}^{r} e^{\beta_j} + \sum_{j=1}^{l} e^{-\beta_j'} \right) \right] \, ,
\end{aligned} \tag{3.62}
$$

where we used euclidean invariance to set $x = (ir, 0)$. This expression clearly shows that, contrary to the massive case, the convergence in the infrared limit $\beta_i \to -\infty, \beta_i' \to$

$+\infty$ is no longer guaranteed by the exponential factor inside the integral and completely relies on the behaviour of the form factors $F_{r,l}$ in this limit.

Let's first analyze the case of the trace of the energy-momentum tensor. As explicitly shown by the relation (3.57), $F_{2,2}$ goes exponentially to zero in the infrared limit so that the 4-particle contribution to the correlation function $G_\Theta(r) =< \Theta(r)\Theta(0) >$ is convergent; it can be easily checked that also the form factors with higher number of particles are well behaved and that their contributions to the infrared behaviour of the correlator are subleading with respect to the 4-particle contribution. Then, plugging expression (3.57) into the integral (3.62), we find

$$G_\Theta(r) \simeq \frac{16}{\pi^2 M^4 r^8} , \quad r \to \infty . \tag{3.63}$$

A logarithmic plot of $G_\Theta(r)$ obtained including in the spectral representation (3.62) the first two contributions is shown in fig. 16; the dashed line represents the expected ultraviolet asymptotic. Using eq.(3.54) and the conformal OPE one finds

$$G_\Theta(r) \simeq M^4 \left(\frac{8}{5}\pi\alpha\right)^2 (Mr)^{-12/5} , \quad r \to 0 \tag{3.64}$$

where the constant $\alpha = \tilde{\lambda}/M^{4/5} = 0.148695516$ was determined in ref.[38]. The figure clearly shows a very fast ultraviolet convergence analogous to that observed in the massive case. Such convergence is confirmed by the computation of the difference between the central charges of the ultraviolet and infrared fixed points. Using the formula [35]

$$\Delta c = \frac{3}{2}\int dr\, r^3 < \Theta(r)\Theta(0) > , \tag{3.65}$$

we obtain $\Delta c^{(4)} = 0.19600 \pm .00007$ from the 4-particle contribution and $\Delta c^{(6)} = 0.1995 \pm .0004$ adding the 6-particle one. We recall that the expected value is $\Delta c = 7/10 - 1/2 = 0.2$.

A very different situation must be faced when trying to compute the correlation functions of the order and disorder operators. The leading infrared behaviour of the form factors of σ and μ can be determined in general specializing the form factor bootstrap to the Ising critical point where $S_{RL} = -1$ and $f_{RL}(\beta - \beta') = e^{(\beta-\beta')/2}$. Inserting the initial conditions $F_{1,0} = F_{0,1} = F_{0,0} = F_{1,1} = 1$ in the recursive equations obtained in this way (a simplified form of eqs.(3.50)), we find

$$F^{IR}_{r,l}(\beta_1,\ldots,\beta_r;\beta'_1,\ldots,\beta'_l) = \prod_{i<j} \tanh\frac{\beta_i-\beta_j}{2}\tanh\frac{\beta'_i-\beta'_j}{2} , \tag{3.66}$$

where the superscript IR denotes the infrared solution and $r+l$ must be taken odd for σ and even for μ. Expression (3.66) clearly shows that the integrals in the spectral representation of the correlators $G_\sigma(r) =< \sigma(r)\sigma(0) >$ and $G_\mu(r) =< \mu(r)\mu(0) >$ are infrared divergent. This circumstance can be ascribed to the fact that the fields $\sigma(x)$ and $\mu(x)$, contrary to $\Theta(x)$, do not enjoy a simple infrared expansion in terms of free fermions. As a consequence, the computation of $G_\sigma(r)$ and $G_\mu(r)$ requires the resummation of the whole (suitably regularized) form factor series. We now show how this kind of resummation can be performed in the low energy limit to extract the infrared conformal dimensions $\Delta_\sigma = \Delta_\mu = 1/16$. Using the form factors (3.66) and exploiting their complete factorization in a left and a right part we can write

$$\tilde{G}^{IR}(t) \equiv G_\sigma^{IR}(t) + G_\mu^{IR}(t) = \left[\sum_{r=1}^{\infty} \frac{1}{r!} \int_{-\Lambda}^{\infty} \prod_{i=1}^{r} \frac{d\beta_i}{2\pi} \left(\prod_{i<j} \tanh\frac{\beta_i - \beta_j}{2}\right)^2 e^{-t\sum_{i=1}^{r} e^{\beta_i}}\right]^2 , \tag{3.67}$$

where $t \equiv Mr/2$ and Λ is an infrared cutoff. Shifting the rapidities we obtain the expression

$$\tilde{G}^{IR}(t) = \left[\sum_{r=1}^{\infty} \frac{1}{r!} \int_{0}^{\infty} \prod_{i=1}^{r} \frac{d\beta_i}{2\pi} \left(\prod_{i<j} \tanh\frac{\beta_i - \beta_j}{2}\right)^2 e^{-te^{-\Lambda}\sum_{i=1}^{r} e^{\beta_i}}\right]^2 , \tag{3.68}$$

which can be considered as the square of the partition function of a classical gas of particles living on a semi-infinite line and subject to the pairwise interaction

$$V(\beta_i - \beta_j) = -\ln\left(\tanh\frac{\beta_i - \beta_j}{2}\right)^2 ; \tag{3.69}$$

the activity of the gas depends on the coordinates and is given by

$$U(\beta) = \frac{1}{2\pi} e^{-te^{\beta-\Lambda}} . \tag{3.70}$$

This function keeps a constant value $U_0 = 1/2\pi$ inside a box of length $L \sim \ln\frac{e^\Lambda}{t}$. Thus we see that removing the cutoff Λ amounts in this context to take the thermodynamic limit $L \to \infty$ of a gas with constant activity U_0. In this limit, the standard relation between the partition function and the bulk-free energy per unit length $f(U_0)$ allows us to write

$$\tilde{G}^{IR}(t) \sim e^{-2f(\frac{1}{2\pi})L} \sim \left(\frac{e^\Lambda}{t}\right)^{-2f(\frac{1}{2\pi})} . \tag{3.71}$$

The bulk-free energy $f(U_0)$ for the classical gas with interaction (3.69) was computed in ref.[55, 23] and reads

$$f(U_0) = \frac{1}{2\pi^2}\arcsin^2(2\pi U_0) - \frac{1}{2\pi}\arcsin(2\pi U_0) . \tag{3.72}$$

Thus we have $f(1/2\pi) = -1/8$ and the relation (3.71) gives the expected infrared behaviour for the two point functions of the fields $\sigma(x)$ and $\mu(x)$.

Chapter 4

Applications to non-homogeneous systems

4.1 Factorized scattering in presence of a defect line

We have seen in the previous chapters that the bootstrap techniques based on the factorizable S-matrix are extremely effective in the description of homogeneous integrable systems. Such systems, however, are in many cases a mathematical idealization of the real physical samples which may present instead boundary effects and various types of inhomogeneity or defects. It is an interesting problem in statistical mechanics to estimate the influence of the inhomogeneities on the results obtained in pure cases and to develop the corresponding theory. With reference to systems with boundaries, they have been the subject of a wide investigation which has employed a large variety of techniques [56-62]. An important progress toward the understanding of QFT with boundary has been recently provided by the bootstrap approach developed by Goshal and Zamolodchikov for the integrable models defined on half plane [63]. The aim of the present chapter is to describe this approach starting from the slightly more general situation in which an infinite "defect line" breaks the translation invariance of a theory defined on the whole plane [64]; as we will see, the boundary case is recovered in the particular situation of an impenetrable defect. Before developing the bootstrap theory, it is worth to briefly discuss general aspects of statistical models with lines of defect in order to gain some insight to their properties [65-71].

One of the main reasons for considering extended lines of inhomogeneities is that only such kind of defects may affect the critical properties of the pure systems. Indeed, in the opposite case where there are only a finite number of localized inhomogeneities in the lattice, they would be eventually neutralized by iterating the Renormalization Group transformations so that the regime of the pure model will definitely take over.

Scaling considerations are also useful to understand in simple terms the continuum version of the models with an infinite line of defect and to show that they may interpolate between a bulk or a boundary statistical behaviour. For sake of clarity, let us consider the simplest physical realization given by a system at temperature T in the bulk but heated at a different temperature $\tilde{T}$ along a line placed at the y axis. This system may be equally regarded as two semi-infinite copies of the model at temperature T coupled together through the energy density at the defect line. Its continuum properties are described by the euclidean action

$$\mathcal{A} = \mathcal{A}_B + g \int d^2r \, \delta(x) \, \epsilon(r) \ , \tag{4.1}$$

where $\mathcal{A}_B$ is the action relative to the bulk and $\epsilon(r)$ is the energy density with scaling dimension ν. The scaling dimension of the coupling constant $g = (\tilde{T} - T)$ is then given by $y_g = 1 - \nu$. Consequently, all those systems with an irrelevant energy operator of scaling dimension $\nu > 1$ will exhibit the bulk critical behaviour near a defect line. On the contrary, those models which have a relevant energy operator with $\nu < 1$ will present a surface critical behaviour. The reason is that, in the former case the effective coupling constant may become arbitrary small and then the action reduces to that of the bulk theory, whereas, in the latter case it may take arbitrary large values suppressing all the fluctuations across the defect line between the two semi-infinite copies which will eventually decouple.

An exception to the above pictures is given by the purely marginal case, i.e. $\nu = 1$ which is realized in the Ising model. The interesting result obtained in the past by Bariev [66] and McCoy and Perk [67] is that the model presents a non universal critical behaviour, with the critical indices of the magnetization operators continuously dependent on the parameter g of the action (A.1). The energy operator on the contrary remains a purely marginal operator for all values of the coupling constant g since its critical exponent ν is fixed at the Ising value of 1 [66, 67, 68, 69].

Let us now turn our attention to the boostrap theory of the integrable statistical models with an extended line of defect placed along the y-axis in the two-dimensional plane. The general action of the system can be written as

$$\mathcal{A} = \mathcal{A}_B + \int d^2r \, \delta(x) \mathcal{L}_D \left(\phi_i, \frac{d\phi_i}{dy} \right) \ , \tag{4.2}$$

where $\mathcal{A}_B$ stays for the action of the integrable bulk theory whose factorized S-matrix is supposed to be known. The additional interaction, responsible for scattering processes which take place on the defect line, will generally spoil the original integrability of the theory: particles which hit the defect with sufficient energy may excite internal degrees of freedom of the defect (being eventually absorbed by it), or may give rise to production processes with multiparticle states propagating through the two semi-infinite systems placed on the two sides of the defect line. However, assuming that the additional interaction along the defect line is still compatible with the existence of an infinite number of conserved charges in involution, the dynamics drastically simplifies and consequently is suitable for an exact analysis, as we show in the sequel.

By translation invariance along the y-direction (which we here identify with the time axis in the Minkowski space), for the theory described by the action (4.2) we still have the conservation of the energy but not of the momentum. Therefore we may have scattering processes with an exchange of momentum on the defect line, compatible though with the conservation of the energy. If in addition to the energy other higher charges are also conserved, the scattering processes at the defect line must be completely elastic. In particular, this means that a particle which hits the defect line with rapidity β can only proceed forward with the same rapidity or reverses its motion acquiring a rapidity of $-\beta$. A further effect of the interaction with the defect line may be a change of the label of the particle inside its multiplet of degeneracy. The interactions of the particles $\mid \beta; i >$ with the defect line will then be described in terms of the transmission and reflection amplitudes, denoted respectively by $T_{ij}(\beta)$ and $R_{ij}(\beta)$ (fig. 17).

The interaction of the particles at the line of inhomogeneity may be encoded in a set of algebraic relations analogous to those which describe the scattering processes in the bulk. In order to illustrate this, an additional operator D associated to the defect line should be introduced in the theory[1]. This operator may be considered in relation to an additional particle state with zero rapidity in the entire time evolution of the system. Its commutation relations with the creation operators $A_i^\dagger(\beta)$ associated to the asymptotic particles in the bulk are given by

$$\begin{aligned} A_i^\dagger(\beta)\, D &= R_{ij}(\beta)\, A_j^\dagger(-\beta)\, D + T_{ij}(\beta)\, D\, A_j^\dagger(\beta) \;; \\ D\, A_i^\dagger(\beta) &= R_{ij}(-\beta)\, D\, A_j^\dagger(-\beta) + T_{ij}(-\beta)\, A_j^\dagger(\beta)\, D \;. \end{aligned} \tag{4.3}$$

The first of these equations expresses the scattering of a particle that hits the defect coming from the semi-infinite system on the left hand side with rapidity β. The second of (4.3) is obtained by an analytic continuation $\beta \to -\beta$ of the scattering amplitudes of a particle that approaches the defect coming from the semi-infinite system on the right

[1]For simplicity, we discuss the case of a defect without internal degrees of freedom and therefore carrying no additional indices. The formulation of the more generale case is straightforward.

hand side. The consistency condition of this algebra requires the unitarity equations

$$\begin{aligned} R_{ij}(\beta)\, R_{jk}(-\beta) + T_{ij}(\beta)\, T_{jk}(-\beta) &= \delta_{ik} \ ; \\ R_i^j(\beta)\, T_j^k(-\beta) + T_i^j(\beta)\, R_j^k(-\beta) &= 0 \ . \end{aligned} \tag{4.4}$$

Additional constraints emerge from the crossing relations

$$\begin{aligned} R_{ij}(\beta) &= S_{\bar{j}l}^{k,\bar{i}}(2\beta)\, R_{kl}(i\pi - \beta) \ ; \\ T_{i\bar{j}}(\beta) &= T_{ij}(i\pi - \beta) \ . \end{aligned} \tag{4.5}$$

The first equation in (4.5) is obtained according to the argument proposed in [63] which exploits the quantization of the theory in the scheme where the time axis is placed along the x-axis. With reference to the second equation in (4.5), the transmission channel of the process shares the same properties of ordinary scattering in the bulk, the only difference being the occurrence of the particle D with zero rapidity. Thus, it is natural to assume that in the transmission channel the crossing symmetry is implemented in the usual way. We will assume the validity of eqs. (4.5) and will check that they are actually satisfied each time we will provide explicit solutions of the scattering theories with a line of defect.

As we discussed in section 1.1, the presence of an infinite number of integrals of motion usually implies not only the elasticity of all scattering processes but also their complete factorization, i.e. an n-particle scattering amplitude can be entirely expressed in terms of the elementary two-body interactions. A crucial step for proving the factorization property of the total S-matrix is to impose the associativity condition of the algebra (4.3). In terms of physical process, this means that we prepare initially an asymptotic two-particle state consisting of $\mid A_i^\dagger(\beta_1)\, A_j^\dagger(\beta_2) >$ with $\beta_1 > \beta_2$, and we let it scatter with the defect particle D with zero rapidity. The final output of the process should be independent from the temporal sequence of the elementary two-body interactions. Although what we have just described looks like an ordinary three-body process of the type that occurs in the bulk, there is however one distinguishing feature. In fact, in the three-body processes which take place in the bulk, given an initial state $\mid A_i^\dagger(\beta_1) A_j^\dagger(\beta_2) A_k^\dagger(\beta_3) >$ identified by a set of three ordered rapidities $\beta_1 > \beta_2 > \beta_3$, there is an *unique* final state given by the reverse ordering of the rapidities and possible exchange of the internal indices among the particles. On the contrary, for the scattering processes on the defect line we may have four possible final states, namely: (a) the state with both particles reflected by the defect line; (b) the state with both particles transmitted; (c and d) the states with one particle reflected whereas the other one transmitted. The final states may also differ from the initial one for the exchange of the internal indices and the above four possibilities give rise to a set of reflection-transmission (RT) equations shown in fig. 18a-d.

The first of these (fig. 18a) coincides with the well-known boundary equations already analysed in [61, 62, 63],

$$S_{ac}^{ef}(\beta_1-\beta_2)R_{fg}(\beta_2)S_{ge}^{dh}(\beta_1+\beta_2)R_{gb}(\beta_1) = R_{ah}(\beta_2)S_{ch}^{fe}(\beta_2+\beta_1)R_{fg}(\beta_2)S_{eg}^{bd}(\beta_1-\beta_2) \ . \tag{4.6}$$

The RT equations associated with the configurations of figs. 18b-d are given respectively by

$$\begin{aligned} S_{ac}^{lm}(\beta_1-\beta_2)\,T_{lb}(\beta_1)\,T_{md}(\beta_2) &= S_{ml}^{bd}(\beta_1-\beta_2)\,T_{cm}(\beta_2)\,T_{dl}(\beta_1) \ ; \\ S_{ac}^{fe}(\beta_1-\beta_2)\,T_{fb}(\beta_1)\,R_{ed}(\beta_2) &= R_{ce}(\beta_2)\,S_{ae}^{fd}(\beta_1+\beta_2)\,T_{fb}(\beta_1) \ ; \\ S_{ac}^{fe}(\beta_1-\beta_2)\,R_{fg}(\beta_2)\,S_{ge}^{dh}(\beta_1+\beta_2)\,T_{hb}(\beta_1) &= T_{ab}(\beta_1)\,R_{cd}(\beta_2) \ . \end{aligned} \tag{4.7}$$

Although a general solution of these equations is still lacking, it is easy to see that they become extremely restrictive once applied to QFT with a non-degenerate spectrum, i.e. those which have a diagonal S-matrix in the bulk. In fact, whereas eq. (4.6) and the first in (4.7) are identically satisfied, the last two equations in (4.7) become in this case

$$\begin{aligned} S_{ab}(\beta_a+\beta_b) &= S_{ab}(\beta_b-\beta_a) \ , \\ S_{ab}(\beta_a+\beta_b)\,S_{ab}(\beta_a-\beta_b) &= 1 \ , \end{aligned} \tag{4.8}$$

Hence, from the first equation in (4.8) we see that the S-matrix in the bulk has to be a constant and from the second equation (or equivalently from the unitarity condition) this constant is fixed to be ± 1. Thus we conclude that the only integrable QFT with diagonal S-matrix in the bulk and factorizable scattering in the presence of the defect line are those associated to generalized-free theories.

Obviously this restriction on the bulk S-matrix does not apply when one considers purely reflective theories because they are ruled only by equations (4.6). Non-trivial solutions of these equations have been analysed for several models and they provide explicit examples of QFT with boundary [72], some of them of relevant importance in statistical mechanics.

4.2 Ising model with a defect line

As we have seen at the end of the previous section, the validity of the Transmission-Reflection Equations in the case of non-degenerate mass spectrum selects $S=-1$ as a possible scattering matrix in the bulk. This solution can be identified as the scattering amplitude of the particle excitations of the Ising model, given by the massive Majorana fermions. The Lagrangian density of the continuum theory in the bulk is given by

$$\mathcal{L}_B = \overline{\Psi}(x,t)\,(i\gamma^\mu\partial_\mu - m)\,\Psi(x,t) \ . \tag{4.9}$$

In the Majorana representation, given by $\gamma^0 = \sigma_2$, $\gamma^1 = -i\sigma_1$, the fermionic field $\Psi(x,t)$ is real, i.e. $\Psi^\dagger(x,t) = \Psi(x,t)$. The physical content of the model, as defined by the Lagrangian (4.9), does not depend on the sign in front of the mass term since it can be altered by the transformation $\Psi \to \Psi$; $\overline{\Psi} \to -\overline{\Psi}$ of the fermionic field. As it is well known, the mass m is a linear measurement of the deviation of the temperature with respect to the critical one

$$m = 2\pi(T - T_c) \,, \tag{4.10}$$

and the symmetry $m \to -m$ simply expresses the self-duality of the model. In the high temperature phase given by $m > 0$, the vacuum expectation value of the magnetization operator σ vanishes, whereas, the corresponding quantity of the disorder operator μ is different from zero. Under the duality transformation, the role of order and disorder operators is reversed whereas for the energy operator ϵ, given by $\epsilon = i\overline{\Psi}\Psi$, we simply have a change in its sign.

On a square lattice, the Ising model with a line of defect can be realized in two different ways (fig. 19). The first is the chain geometry with bulk coupling constants J and modified couplings $\tilde{J}$ parallel to the defect line. The second one is the ladder geometry, with the modified set of couplings placed in the perpendicular direction. Since the two geometric realizations are related by Kramer-Wannier duality symmetry, from now on we can restrict our attention to one of them, say the chain geometry. In the continuum formulation, the defect line introduces the additional term[2]

$$\mathcal{L}_D = -g\,\delta(x)\overline{\Psi}(x,t)\Psi(x,t) \tag{4.11}$$

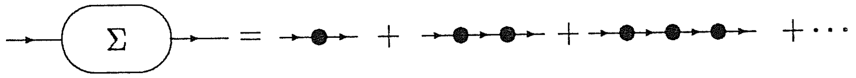
to the Lagrangian (4.9). The new interaction is purely marginal and therefore the beta-function associated to the coupling constant g is identically zero. The marginality of the interaction has as a consequence that the theory presents a non-universal ultraviolet behaviour in the magnetization sector, with the critical exponent of the magnetization operators which depend continuously on the parameter g, whereas the energy operator always keeps its original value of 1 of the Ising model [66, 67, 68, 69].

In this section we are interested in determining the reflection $R(\beta)$ and transmission $T(\beta)$ amplitudes for the scattering of the fermion with the defect line, i.e. the S-matrix elements between initial and final states $u(p_i)$ and $\overline{u}(p_f)$ with $p_f = \pm p_i$. To this aim, let us consider the perturbative series of the Green function of the fermion field Ψ based on the Feynman rules

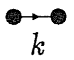
[2]The exact relationship between g and the lattice coupling constants will be established in section 4.5 by comparing correlation functions computed in the lattice and in the continuum formulation.

$= i(2\pi)^2\, \delta^2(p-p')\, \frac{\not p + m}{p^2 - m^2 + i\epsilon}$  $= -ig\, 2\pi\, \delta(p_0 - p_0')$

For the self-energy entering the exact propagator we have the following series of diagrams



where we have to integrate on the spatial component of the momentum running in the internal lines. The integral on the intermediate state is given by



$$= (-ig)^2\, i\, \delta(k_0 - p_0) \int \frac{dk^1}{2\pi} \frac{\not k + m}{k^2 - m^2 + i\epsilon} = -g^2\, \delta(k_0 - p_0)\, \frac{p_0 \gamma^0 + m}{2\omega} \quad .$$

In the above quantity we have discarded by parity the infinity related to the linear term in k. With this prescription, the geometric series for Σ is finite and can be expressed in a closed form as

$$\Sigma(p_0) = 2\pi\, i\, \delta(p_0 - p_0')\, \sin\chi\, \frac{\omega - i\frac{g}{2}(p_0\gamma^0 - m)}{\omega - im\, \sin\chi} \,, \tag{4.12}$$

where

$$\omega = \sqrt{p_0^2 - m^2} \quad , \quad \sin\chi = -\frac{g}{1 + \frac{g^2}{4}} \quad .$$

We can now apply the usual LSZ reduction formulae, and for the transmission and reflection amplitudes defined by

$${}_{out}< \beta' \mid \beta >_{in} = 2\pi\delta(\beta - \beta')\, T(\beta, g) + 2\pi\delta(\beta + \beta')\, R(\beta, g) \ ,$$

we have

$$T(\beta, g) = \frac{\cos\chi\, \sinh\beta}{\sinh\beta - i\sin\chi} \,,$$
$$R(\beta, g) = i\, \frac{\sin\chi\, \cosh\beta}{\sinh\beta - i\sin\chi} \quad . \tag{4.13}$$

The transmission amplitude also contains the disconnected part relative to the free motion.

Before commenting on the properties of these amplitudes, it is interesting to present an alternative derivation of (4.13). This is obtained by implementing the algebra (4.3) on the creation operators of the fermion field. Let $\Psi_\pm(x,t)$ be the solutions of the free Dirac equation in the two intervals $x > 0$ and $x < 0$, i.e.

$$\Psi(x,t) = \theta(x)\, \Psi_+(x,t) + \theta(-x)\, \Psi_-(x,t) \,, \tag{4.14}$$

with the value at the origin given by $\Psi(0,t) = \frac{1}{2}(\Psi_+(0,t) + \Psi_-(0,t))$. The mode expansion of the two components of the fields $\Psi_\pm(x,t)$ is expressed as

$$\begin{aligned}
\psi^{(1)}_{(\pm)}(x,t) &= \int \frac{d\beta}{2\pi} \left[\omega e^{\frac{\beta}{2}} A_{(\pm)}(\beta)\, e^{-im(t\cosh\beta - x\sinh\beta)} + \overline{\omega} e^{\frac{\beta}{2}} A^{\dagger}_{(\pm)}(\beta)\, e^{im(t\cosh\beta - x\sinh\beta)}\right] \\
\psi^{(2)}_{(\pm)}(x,t) &= -\int \frac{d\beta}{2\pi} \left[\overline{\omega} e^{-\frac{\beta}{2}} A_{(\pm)}(\beta)\, e^{-im(t\cosh\beta - x\sinh\beta)} + \omega e^{-\frac{\beta}{2}} A^{\dagger}_{(\pm)}(\beta)\, e^{im(t\cosh\beta - x\sinh\beta)}\right] ,
\end{aligned} \tag{4.15}$$

with $\omega = \exp(i\pi/4)$, $\overline{\omega} = \exp(-i\pi/4)$. The operators $A_\pm(\beta)$ and $A^\dagger_\pm(\beta)$ satisfy the usual anti-commutation relations of a free fermion

$$\left\{A_\pm(\beta), A^\dagger_\pm(\theta)\right\} = 2\pi\, \delta(\beta - \theta) \ , \tag{4.16}$$

although they are not all independent. They are related to each other by the conditions at $x = 0$ which arise from applying the eqs. of motion to (4.14), i.e.

$$\begin{aligned}
(\psi^{(2)}_+ - \psi^{(2)}_-)(0,t) &= \tfrac{g}{2}(\psi^{(1)}_+ + \psi^{(1)}_-)(0,t) \ ; \\
(\psi^{(1)}_+ - \psi^{(1)}_-)(0,t) &= \tfrac{g}{2}(\psi^{(2)}_+ + \psi^{(2)}_-)(0,t) \ .
\end{aligned} \tag{4.17}$$

These equations are equivalent to the relationship between the modes

$$M \begin{pmatrix} A^\dagger_-(\beta) \\ A^\dagger_+(-\beta) \end{pmatrix} = N \begin{pmatrix} A^\dagger_-(-\beta) \\ A^\dagger_+(\beta) \end{pmatrix} , \tag{4.18}$$

where

$$M = \begin{pmatrix} \omega e^{-\frac{\beta}{2}} + \frac{g}{2}\overline{\omega} e^{\frac{\beta}{2}} & -\omega e^{\frac{\beta}{2}} + \frac{g}{2}\overline{\omega} e^{-\frac{\beta}{2}} \\ \omega e^{-\frac{\beta}{2}} + \frac{2}{g}\overline{\omega} e^{\frac{\beta}{2}} & \omega e^{\frac{\beta}{2}} - \frac{2}{g}\overline{\omega} e^{-\frac{\beta}{2}} \end{pmatrix} ;$$

$$N = \begin{pmatrix} -\omega e^{\frac{\beta}{2}} - \frac{g}{2}\overline{\omega} e^{-\frac{\beta}{2}} & \omega e^{-\frac{\beta}{2}} - \frac{g}{2}\overline{\omega} e^{\frac{\beta}{2}} \\ -\omega e^{\frac{\beta}{2}} - \frac{2}{g}\overline{\omega} e^{-\frac{\beta}{2}} & -\omega e^{-\frac{\beta}{2}} + \frac{2}{g}\overline{\omega} e^{\frac{\beta}{2}} \end{pmatrix} .$$

Hence,

$$\begin{pmatrix} A^\dagger_-(\beta) \\ A^\dagger_+(-\beta) \end{pmatrix} = M^{-1} N \begin{pmatrix} A^\dagger_-(-\beta) \\ A^\dagger_+(\beta) \end{pmatrix} = \begin{pmatrix} R(\beta,g) & T(\beta,g) \\ T(\beta,g) & R(\beta,g) \end{pmatrix} \begin{pmatrix} A^\dagger_-(-\beta) \\ A^\dagger_+(\beta) \end{pmatrix} \tag{4.19}$$

with $R(\beta,g)$ and $T(\beta,g)$ given in (4.13). Note that, although the boundary conditions (4.17) are both linear in g, there is however a feedback between the two components of the fermionic field. The final dependence from the coupling constant is then expressed in terms of trigonometric functions of the auxiliary angle χ.

Given the explicit expressions of the amplitudes (4.13), it is easy to check that they satisfy the unitarity and crossing equations (4.4) and (4.5). They present several interesting features. Firstly, by taking their sum and difference we obtain

$$e^{2i\delta_0} \equiv T(\beta,g) + R(\beta,g) = \frac{\sinh\frac{1}{2}(\beta + i\chi)}{\sinh\frac{1}{2}(\beta - i\chi)} \ ;$$

$$e^{2i\delta_1} \equiv T(\beta,g) - R(\beta,g) = \frac{\cosh\frac{1}{2}(\beta - i\chi)}{\cosh\frac{1}{2}(\beta + i\chi)} ,$$

which can be considered as partial-wave phase shifts, with δ_0 and δ_1 crossed functions of each other. Secondly, notice that as functions of the coupling constant g, they satisfy a strong-weak duality given by

$$T\left(\beta, \frac{4}{g}\right) = -T(\beta,g) \quad , \quad R\left(\beta, \frac{4}{g}\right) = R(\beta,g) \ . \tag{4.20}$$

At the self-dual points $g = \pm 2$ the transmission amplitude vanishes and therefore the defect line behaves as a pure reflecting surface. From the unitarity equations (4.4), the corresponding reflection amplitudes $R(\beta, \pm 2)$ become pure phases, as can be explicitly seen by their equivalent expressions

$$R(\beta, \pm 2) = -\frac{\cosh\frac{\beta}{2} \pm i\sinh\frac{\beta}{2}}{\cosh\frac{\beta}{2} \mp i\sinh\frac{\beta}{2}} \ . \tag{4.21}$$

They coincide with the reflection amplitudes of the Ising model with fixed and free boundary conditions respectively, as determined in [63]. To establish directly the pure reflecting properties of the defect line at the self-dual points, let us analyse more closely the decoupling which occurs in the boundary conditions when $g = \pm 2$. For $g = 2$, the boundary conditions (4.17) become

$$\begin{aligned} (\psi_+^{(2)} - \psi_-^{(2)})(0,t) &= (\psi_+^{(1)} + \psi_-^{(1)})(0,t) \ ; \\ (\psi_+^{(1)} - \psi_-^{(1)})(0,t) &= (\psi_+^{(2)} + \psi_-^{(2)})(0,t) \ , \end{aligned} \tag{4.22}$$

and taking their sum and difference, they can be written as

$$\begin{aligned} (\psi_+^{(2)} - \psi_+^{(1)})(0,t) &= 0 \ ; \\ (\psi_-^{(2)} + \psi_-^{(1)})(0,t) &= 0 \ . \end{aligned} \tag{4.23}$$

For $g = -2$, the original boundary conditions (4.17) are reduced instead to

$$\begin{aligned} (\psi_+^{(2)} + \psi_+^{(1)})(0,t) &= 0 \ ; \\ (\psi_-^{(2)} - \psi_-^{(1)})(0,t) &= 0 \ . \end{aligned} \tag{4.24}$$

Equations (4.23) and (4.24) explicitly show that the two semi-infinite systems across the defect line are completely decoupled, and each of them can be treated as a QFT in the presence of pure reflecting surface whose role is to supply the appropriate boundary conditions [63]. At first sight, though, one may be surprised by the asymmetric form assumed by the eqs. (4.23) and (4.24) which treat differently the two fermionic fields $\Psi_\pm(x,t)$. However, this asymmetry has a physical origin. By means of the mode expansion (4.15), the first equation in (4.23) and (4.24) can be used to determine directly the reflection amplitudes $R(\beta, \pm 2)$. By the same token, using the second equation in (4.23) and (4.24) we find $R(-\beta, \pm 2)$, instead. But, this is physically correct, the reason being that, in order to have a reflection of a particle described by $\Psi_+(x,t)$ with the defect (boundary) line, this particle must approach the origin with positive rapidity β. On the contrary, a reflection of a particle described by $\Psi_-(x,t)$ with the defect (boundary) line is only realized for negative values of its rapidity.

Further support of the identification of $R(\beta, \pm 2)$ with the reflection amplitudes of the Ising model with fixed and free boundary conditions comes from the analysis of the relationship between the lattice and the continuum formulations of the chain geometry, which will be established in section 4.5. Anticipating the result, this is provided by the formula

$$\sin \chi = \tanh 2(J - \tilde{J}) \ . \tag{4.25}$$

Hence, the condition $\sin \chi = -1$ corresponds to a coupling constant $\tilde{J}$ along the defect line infinitely larger (and positive) than the coupling constant J of the bulk. As a consequence, the spins along the defect line are frozen into a fixed boundary condition. On the other hand, the condition $\sin \chi = 1$ is obtained in the anti-ferromagnetic limit $\tilde{J} \longrightarrow -\infty$ where the spins along the defect line are aligned in antiparallel configurations. Since the nearby spins couple to a surface with vanishing magnetization, this situation corresponds to the free boundary conditions [65].

Let us now turn our attention to the analytic structure of the reflection and transmission amplitudes. For negative values of g, the interaction with the defect line is attractive and consequently the theory presents a bound state with binding energy $e_b = m \cos \chi$. It is quite instructive to calculate the transmission and reflection amplitudes $T_b(\beta)$ and $R_b(\beta)$ relative to the scattering of the fermion with the excited state present on the defect line. The first thing to observe is that both amplitudes $R(\beta)$ and $T(\beta)$ present a pole singularity at $\beta = i\chi$ and $\beta = i(\pi - \chi)$. The reflection amplitude $R(\beta)$ has positive residue at both locations, given by $i \sin \chi$. On the other hand, $T(\beta)$ presents a positive residue with the same value as $R(\beta)$ at $\beta = i\chi$ and a negative residue $-i \sin \chi$ at the other pole $\beta = i(\pi - \chi)$. The problem of identifying which one of the two poles corresponds to the bound state is solved by selecting the singularity with positive residue in both amplitudes. This is the pole at $\beta = i\chi$. The relative binding energy is positive, as it should be. To recover the transmission and reflection amplitudes relative to the excited state, we have to impose the commutativity of the graphs shown in figs. 20a-d. Since the S-matrix in the bulk is -1, the reflection amplitude $R_b(\beta)$ coincides with the original one i.e. $R_b(\beta) = R(\beta)$ whereas the transmission amplitude is given by $T_b(\beta) = -T(\beta)$. If we again identify the singularity associated to a bound state as that pole with a positive residue in both channels, we see that for the defect bound state amplitudes the role of the two poles has been reversed! Namely, the pole which corresponds to the bound state in $R_b(\beta)$ and $T_b(\beta)$ is now located at $\beta = i(\pi - \chi)$ and is relative to the original ground state of the defect line[3].

As a final remark of this section, the marginal nature of the defect interaction in the Ising model can be also inferred by looking at the high-energy limit of the amplitudes. For large values of β we have

$$T(\beta) \sim \cos\chi \ , \quad R(\beta) \sim i \sin\chi \ . \tag{4.26}$$

Hence, except for the special values of the coupling constant g where one of the two quantities vanish, both amplitudes are always simultaneously present. Since this limit probes the short distance scales of the model, we see that its critical properties of bulk and surface behaviour are simultaneously present[4].

4.3 Bosonic theory with a defect line

Another solution of the reflection-transmission equations is provided by the massive free bosonic theory with the S-matrix in the bulk given by $S = 1$. As an example of a bosonic theory with a line of defect, we consider the model described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\varphi)^2 - m^2\varphi^2 - g\,\delta(x)\varphi^2\right] \ . \tag{4.27}$$

For the equation of motion we have

$$\left[\Box + m^2 + g\,\delta(x)\right]\varphi = 0 \ . \tag{4.28}$$

As for the Ising model, one can obtain the reflection and transmission amplitudes by an exact resummation of the perturbative series in the coupling constant g. The calculations are analogous to the fermionic case, and rather than repeating them here,

[3]Note that the presence of the transmission amplitude has been quite crucial in order to discriminate which one of the two poles with positive residue in the reflection channel corresponds to the bound state. In the pure reflecting situation, as for instance may be the case of the Ising model with a boundary magnetic field considered in [63], the occurrence of positive residue at both poles in the reflection amplitude and a misinterpretation of their role could in fact lead to a paradoxical hierarchy of bound states obtained by applying iteratively the boundary bootstrap equations.

[4]For an irrelevant interaction which leads to a bulk critical behaviour near the defect line, we expect in fact a vanishing of the reflection amplitude in the high-energy limit. On the contrary, for a relevant interaction, the system should show a purely surface critical behaviour characterized by the vanishing of the transmission amplitude.

we prefer to exploit the algebraic approach directly. The solution of the equation of motion may be written as

$$\varphi(x,t) = \theta(x)\varphi_+(x,t) + \theta(-x)\varphi_-(x,t) \ , \tag{4.29}$$

where the mode decomposition of the two fields $\varphi_\pm(x,t)$ is given by

$$\varphi_{(\pm)}(x,t) = \int \frac{d\beta}{2\pi} \left[A_{(\pm)}(\beta)\, e^{-im(t\cosh\beta - x\sinh\beta)} \ + \ A^\dagger_{(\pm)}(\beta)\, e^{im(t\cosh\beta - x\sinh\beta)}\right] \ . \tag{4.30}$$

The operators $A_\pm(\beta)$ and $A^\dagger_\pm(\beta)$ satisfy the usual commutation relations of a free massive boson

$$\left[A_\pm(\beta), A^\dagger_\pm(\theta)\right] \ = \ 2\pi\,\delta(\beta - \theta) \ . \tag{4.31}$$

The interaction along the defect however makes them not linearly independent. In fact, substituting eq. (4.29) into the equation of motion, the latter is equivalent to the boundary conditions

$$\begin{aligned} \varphi_+(0,t) - \varphi_-(0,t) \ &= \ 0 \ ; \\ \frac{\partial}{\partial x}(\varphi_+(0,t) - \varphi_-(0,t)) \ &= \ \frac{g}{4m}(\varphi_+(0,t) + \varphi_-(0,t)) \ , \end{aligned} \tag{4.32}$$

which, in terms of the mode, can be written as

$$\begin{pmatrix} A^\dagger_-(\beta) \\ A^\dagger_+(-\beta) \end{pmatrix} = \begin{pmatrix} R(\beta,g) & T(\beta,g) \\ T(\beta,g) & R(\beta,g) \end{pmatrix} \begin{pmatrix} A^\dagger_-(-\beta) \\ A^\dagger_+(\beta) \end{pmatrix} \ . \tag{4.33}$$

The transmission and reflection amplitudes in the above formula are given by

$$\begin{aligned} T(\beta,g) \ &= \ \frac{\sinh\beta}{\sinh\beta + ig/4m} \ , \\ R(\beta,g) \ &= \ -\frac{ig/4m}{\sinh\beta + ig/4m} \ . \end{aligned} \tag{4.34}$$

These amplitudes satisfy the unitarity and crossing equations (4.4) and (4.5). It is easy to see that by substituting $\sinh\beta$ in (4.34) with the linear momentum k, the two resulting amplitudes are the same as those obtained in one-dimensional quantum mechanics for the scattering in a δ-function potential (see, for instance, [73]). However, due to the relativistic nature of the QFT, there is an important difference between the two cases, as shown by the analysis which follows on the pole structure of the amplitudes (4.34).

For the $2\pi i$ periodicity of the amplitudes, we can restrict our attention to the strip $-i\pi \leq \beta \leq i\pi$. Let us consider initially the case when g is a positive quantity. As long as g satisfies the condition $0 < g < 4m$, there are two poles on the negative imaginary axis relative to the unphysical sheet. By increasing the value of g they approach each other, and there is a critical value $g_{c_1} = 4m$ where they collide at position $\beta = -i\pi/2$. Additional increment of the coupling constant causes the poles to move in the complex strip keeping their imaginary part equal to $-i\pi/2$ but acquiring a real part (fig. 21). In terms of QFT, this means that the bosonic theory with a coupling constant of the defect line larger than $4m$ presents two resonance states. As g grows, these poles move to infinity, and in the limit $g \to \infty$, the defect line acts as a pure reflecting surface. Indeed, the transmission amplitude vanishes, whereas the reflection amplitude expresses the fixed boundary condition $\varphi(0,t) = 0$.

Let us now analyse the case when g is a negative quantity. In the range $-4m < g < 0$, the amplitudes present two poles placed on the positive imaginary axis relative to the physical sheet. The closest one to the origin can be interpreted as a defect bound state. By decreasing g, these two poles approach each other until they finally collide at $\beta = i\pi/2$ for the critical value $g_{c_2} = -4m$. Further decrement of the coupling constant makes them move in the complex strip with an imaginary part equals to $i\pi/2$ and with a real component which increases by decreasing g. However, these poles are now located in the physical strip and therefore the theory presents instability properties. The easiest way to explicitly illustrate this instability is to consider the analytic continuation $\beta \to \left(i\frac{\pi}{2} - \beta\right)$ in $R(\beta)$. As discussed in section 4.5, the resulting quantity $\hat{R}(\beta)$, given by

$$\hat{R}(\beta, g) = -\frac{g/4m}{\cosh\beta + g/4m}\,, \tag{4.35}$$

can be interpreted as the amplitude relative to the emission of a pair of particles with momentum β and $-\beta$ from the defect line placed along the x-axis [63]. Then, for $g < -4m$, $\hat{R}(\beta)$ presents a pole for real values of β that induces a spontaneous emission of pairs of particles. The occurrence of such processes obviously spoils the stability of the theory.

In light of the above results, we can summarize the discussion by saying that the QFT associated to the Lagrangian (4.27) makes sense only for values of g in the range $-4m < g \leq \infty$. In a path integral approach to the problem, it is easy to see that there may be a competition in the Lagrangian (4.27) between the genuine mass term and the defect interaction. Adopting the interpretation of the δ-function interaction as a suitable limit of a constant potential in the strip $(-\epsilon, \epsilon)$ around the origin, when g is sufficiently positive in this interval, we may have an effective mass of the field φ in this strip higher than the threshold mass m in the bulk. This produces the resonance poles in the transmission and reflection amplitudes. Viceversa, for negative values of g, the effective mass of the field φ in the tiny interval around the origin is smaller than the mass gap in the bulk and it decreases until it vanishes at $g = -4m$. After this value it becomes imaginary, giving rise to the instability property previously discussed.

It is likewise interesting to understand the different behaviour of the bosonic and the fermionic theories in terms of the coupling constant. The reason is that the physical content of the fermionic model does not depend on the sign of the mass term, which enters linearly in the action. Therefore, by varying the coupling constant g, there is no a real competition with the genuine mass term in the action, so that the fermionic model cannot present instabilities or resonance states. In fact, crossing the critical values $g = \pm 2$, the poles simply interchange their positions, i.e. the weak coupling regime swaps with the strong coupling one.

As a last comment on the bosonic theory analysed in this section, the defect interaction is associated to an irrelevant operator and therefore the defect line should be completely transparent in the ultraviolet limit. Indeed, taking the the high-energy limit $\beta \to \infty$ of the amplitudes (4.34), the reflection amplitude vanishes whereas the transmission amplitude is identically equal to 1.

4.4 Models with multi-defect lines

The solutions so far determined for the fermionic and bosonic theories in the presence of a single line of defect can be generalized and geometrical situations with a richer structure of defect lines can be also included. In this section, we analyse the case of two parallel lines of defect, and then the quantization conditions induced by a periodic array of defects. Due to the different behaviour of the fermionic and bosonic theories, it is convenient to discuss them separately.

4.4.1 Fermionic theory

Let us initially consider the Ising model with two parallel lines of defect, one placed at the origin along the y-axis with strength g_1, the other shifted by a and with strength g_2. In the fermionic formulation of the model, the field $\Psi(x,t)$ has a free motion in each of the three intervals $I_- \equiv (-\infty, 0)$, $I_0 \equiv (0, a)$ and $I_+ \equiv (a, +\infty)$ separated by the two defect lines. Therefore in each of the three intervals the field $\Psi(x,t)$ admits the usual decomposition in modes and the role of the defect lines is to provide the boundary conditions at the edges of the intervals. The first of them is at $x = 0$ and is given by

$$\begin{aligned}(\psi_0^{(2)} - \psi_-^{(2)})(0,t) &= \tfrac{g_1}{2}(\psi_0^{(1)} + \psi_-^{(1)})(0,t) \ ; \\ (\psi_0^{(1)} - \psi_-^{(1)})(0,t) &= \tfrac{g_1}{2}(\psi_0^{(2)} + \psi_-^{(2)})(0,t) \ ,\end{aligned} \tag{4.36}$$

whereas for the second boundary condition at $x = a$ we have

$$\begin{aligned}(\psi_+^{(2)} - \psi_0^{(2)})(a,t) &= \tfrac{g_2}{2}(\psi_+^{(1)} + \psi_0^{(1)})(a,t) \ ; \\ (\psi_+^{(1)} - \psi_0^{(1)})(a,t) &= \tfrac{g_2}{2}(\psi_+^{(2)} + \psi_0^{(2)})(a,t) \ .\end{aligned} \tag{4.37}$$

In these equations the intervals are labelled by the subscript of the fields while their components by the upper indices. By using the notation R_i and T_i ($i = 1,2$) for the reflection and the transmission amplitudes relative to the defect line with strength g_i, it is easy to see that eliminating the intermediate modes relative to the interval I_0, there is a linear relationship between the modes of the fields in the intervals I_- and I_+ given by

$$\begin{pmatrix} A_-^{\dagger}(\beta) \\ A_+^{\dagger}(-\beta) \end{pmatrix} = \begin{pmatrix} R(\beta,g_1,g_2,a) & T(\beta,g_1,g_2,a) \\ T(\beta,g_1,g_2,a) & R(\beta,g_1,g_2,a) \end{pmatrix} \begin{pmatrix} A_-^{\dagger}(-\beta) \\ A_+^{\dagger}(\beta) \end{pmatrix} , \tag{4.38}$$

where

$$\begin{aligned} T(\beta,g_1,g_2,a) &= \frac{T_1(\beta)T_2(\beta)}{1-\eta(\beta,a)R_1(\beta)R_2(\beta)} \ , \\ R(\beta,g_1,g_2,a) &= \frac{R_1(\beta)+\eta(\beta,a)R_2(\beta)[T_1^2(\beta)-R_1^2(\beta)]}{1-\eta(\beta,a)R_1(\beta)R_2(\beta)} \ . \end{aligned} \tag{4.39}$$

In the above expressions $\eta(\beta,a)$ is a pure phase given by $\eta(\beta,a) = \exp[-2ima\sinh\beta]$.

The above amplitudes satisfy the unitarity and crossing equations (4.4) and (4.5). They describe the physical situation of a particle coming from the interval I_- with rapidity β which hits the first defect line and, as result of this interaction, it can be either reflected or transmitted. When it is reflected, it appears as an asymptotic particle with rapidity $-\beta$ whereas when it is transmitted it approaches the next defect and can again be reflected or transmitted. As shown in fig. 22, these two types of process may be repeated an arbitrary number of times at the two defect lines.

Due to the existence of the fixed points $g = \pm 2$ of a single defect line, it is interesting to analyse some special limits of the expressions (4.39). To begin with, note that, at the values $g_1 = \pm 2$ where $T_1 = 0$, the total transmission amplitude $T(\beta,g_1,g_2,a)$ vanishes as well, whereas the reflection amplitude reduces to a pure phase given by $R(\beta,\pm 2,g_2,a) = R_1(\beta,\pm 2)$. In this case, the first defect acts as a pure reflecting surface which therefore completely screens the presence of the second defect. The total transmission amplitude also vanishes when $g_2 = \pm 2$. Concerning the reflection amplitude, it becomes a pure phase given by

$$R(\beta, g_1, \pm 2, a) = \eta(\beta, a)\, R_2(\beta, \pm 2) \frac{\sinh \beta (1 + \eta^{-1}(\beta, a) \sin \chi_1) + i \sin \chi (1 - \eta^{-1}(\beta, a))}{\sinh \beta (1 + \eta(\beta, a) \sin \chi_1) - i \sin \chi (1 - \eta(\beta, a))} . \tag{4.40}$$

The total reflection process is now the result of an infinite sequence of elementary transmission and reflection scatterings at the first defect line combined with pure reflecting processes at the second defect line. Hence it is not surprising that the final expression depends on both $R_2(\beta, \pm 2)$ and the separation distance a.

Except for the values of g when the defects behave as mirror surfaces, the possibility for the fermion to go back and forth between the two defect lines produces typical resonance phenomena which are illustrated for instance by plotting the absolute value of $T(\beta, g_1, g_2, a)$. An example is shown in fig. 23.

Finally, by taking the limit $a \to 0$, the two defect lines behave as a single one but with an effective strength g given by

$$g = \frac{g_1 + g_2}{1 + g_1 g_2 / 4} . \tag{4.41}$$

This composition law of the defect strengths is similar to the addition of velocities in relativistic dynamics. The effective coupling constant g has as critical values $g = \pm 2$ and reaches these limits when either g_1 or g_2 are equal to ± 2. This can be also seen by analysing the fixed points of the composition law defined by the iterative map

$$g_{n+1} = \frac{g_n + g}{1 + g_n g / 4} , \tag{4.42}$$

for some initial value g.

The natural generalization of the situation with two defect lines is to consider a periodic array of defects all with equal strength g and separated by a distance a. The fermionic field satisfies in this case the equation

$$\left[i\gamma^\mu \partial_\mu - m - g \sum_{n=-\infty}^{\infty} \delta(x + na) \right] \Psi(x,t) = 0 , \tag{4.43}$$

and admits the decomposition

$$\Psi(x,t) = \sum_{n=-\infty}^{\infty} \theta(x - na)\theta(-x + (n+1)a)\Psi_n(x,t) , \tag{4.44}$$

with $\Psi_n(x,t)$ solutions of the free Dirac equation. The dynamics of the model is entirely encoded into an infinite set of linear equations relative to the boundary conditions between the interval na and $(n-1)a$, i.e.

$$\begin{aligned}(\psi_{n-1}^{(2)} - \psi_n^{(2)})(na,t) &= \tfrac{g}{2}(\psi_{n-1}^{(1)} + \psi_n^{(1)})(na,t) \; ; \\ (\psi_{n-1}^{(1)} - \psi_n^{(1)})(na,t) &= \tfrac{g}{2}(\psi_{n-1}^{(2)} + \psi_n^{(2)})(na,t) \; .\end{aligned} \tag{4.45}$$

The simplest way to solve these equations is to employ a relativistic generalization of the Bloch theorem [74], i.e to associate a wave vector k to the spinor field Ψ such that

$$\Psi(x+a,t) = e^{ika}\,\Psi(x,t) \; . \tag{4.46}$$

Equivalently,

$$\Psi_n(na,t) = e^{ika}\,\Psi_{n-1}((n-1)a,t) \; . \tag{4.47}$$

The wave vector k can always be confined to the first Brillouin zone $-\pi/a \leq k \leq \pi/a$. Plugging (4.47) into eqs. (4.45), the resulting system is compatible provided that the equation

$$\cos ka = \frac{1}{\cos\chi}\left[\cos(ma\sinh\beta) - \sin\chi\frac{\sin(ma\sinh\beta)}{\sinh\beta}\right] \tag{4.48}$$

is valid. This equation gives rise to a band structure in the energy levels of the Majorana fermion of the Ising model, completely analogous to the periodic potentials considered in condensed matter physics. In fact, eq. (4.48) can be satisfied for real k if and only if the right hand side of the equation is less than unity. Consequently, there will be allowed and forbidden regions of β and the corresponding spectrum of the energy, given by $E = m\cosh\beta$, consists of a family of energy bands. A characteristic form of the spectrum is plotted in fig. 24. For the pure reflecting values $g = \pm 2$, the above equation reduces to the quantization condition of the rapidity variable β

$$\sinh\beta = \pm\tan(ma\sinh\beta) \; , \tag{4.49}$$

which arises by considering the fermionic field defined in a strip of width a with fixed $(+)$ or free $(-)$ boundary conditions at the edge of the interval.

4.4.2 Bosonic theory

The discussion of the bosonic theory largely follows the previous one and eqs. (4.39) is valid as it stands on the condition that we insert the bosonic amplitudes instead. Also in this case there are typical resonance phenomena produced by the trapping of the bosonic particle between the two defect lines. There is however a significant difference with respect to the fermionic case and this concerns the composition law relative to two defect lines with a separation $a \to 0$. In this limit, the two defect lines behave as a single one with an effective strength g given by

$$g = g_1 + g_2 \ . \tag{4.50}$$

Due to the peculiar properties of the bosonic system discussed in section 4.3, this composition law implies that a system with two defect lines in the limit $a \to 0$ may become unstable although each of the defect lines taken individually does not present any instability property. Viceversa, one can obtain a well-defined bosonic system as a result of the limit $a \to 0$ of a system which presents instability properties at one defect line and resonance states at the other.

Taking the limit $g_1 \to +\infty$, the first defect line becomes a pure reflecting surface and the total transmission amplitude vanishes. In this case the reflection amplitude reduces to $R(\beta, +\infty, g_2, a) = -1$. The total transmission amplitude also vanishes when the second defect line acts as a pure reflecting surface. The corresponding reflection amplitude is a pure phase given by

$$R(\beta, g_1, +\infty, a) = -\eta(\beta, a) \frac{\sinh\beta - i\frac{g}{4m}(1 - \eta^{-1}(\beta, a))}{\sinh\beta + i\frac{g}{4m}(1 - \eta(\beta, a))} \ . \tag{4.51}$$

As in the fermionic case, the presence of an infinite periodic array of defect lines of strength g and separation a gives rise to a band structure described by a Kronig-Penney type equation

$$\cos ka = \cos(ma \sinh\beta) + \frac{g}{m} \frac{\sin(ma\sinh\beta)}{\sinh\beta} \ . \tag{4.52}$$

The pure reflective case $g \to +\infty$ gives rise to the quantization condition

$$ma \sinh\beta = \pi n \ , \quad (n = 0, \pm 1, \ldots) \tag{4.53}$$

relative to the bosonic field in a strip of width a with fixed boundary conditions $\varphi(0,t) = \varphi(a,t) = 0$ at the end points of the interval.

4.5 Correlation functions

The purpose of this section is to show that the spectral methods described in the previous chapters dealing with bulk theories are also suitable for computing the correlation functions of models with linear inhomogeneities.

The easiest way to approach the problem is to use a formalism which takes full advantage of the solution of the theory in the bulk. To this aim, it is convenient to interchange the original role of the x and the t axes by the transformation $x \to -it$,

$t \to ix$. The new space has a Minkowski structure with the defect line placed now at $t = 0$. In this new geometry, the space of the states is the same as in the bulk, and therefore, even in the presence of the defect line, the local operators ϕ_i can be completely characterized by their known form factors. The presence of the defect line can be taken into account by defining an operator $\mathcal{D}$ placed at $t = 0$, acting on the bulk states. This operator plays the role of the S-matrix of the problem, and therefore, standard formulas of QFT allow the correlation functions to be expressed as

$$< \Phi_1(x_1,t_1)\ldots\Phi_n(x_n,t_n) > = \frac{< 0 \mid T[\phi_1(x_1,t_1)\ldots\mathcal{D}\ldots\phi_n(x_n,t_n)] \mid 0 >}{< 0 \mid \mathcal{D} \mid 0 >} \quad . \tag{4.54}$$

In the above formula, $\Phi_i(x_i,t_i)$ are the fields in the Heisenberg representation, i.e. the representation where the time evolution is ruled by the exact Hamiltonian of the problem, including the defect interaction. On the other hand, $\phi_i(x_i,t_i)$ are the field operators of the bulk theory and, as such, their time evolution operator is the bulk Hamiltonian[5]. The main advantage of eq. (4.54) is that, using the completeness relation of the bulk states, its right hand side can be entirely expressed in terms of the Form Factors of the bulk fields and the matrix elements of the operator $\mathcal{D}$ which are determined as follows.

The defect operator $\mathcal{D}$ encodes all information relative to the physical processes which take place at the defect line. To examine them, we have to initially realize that the first effect of the interchange of the x and the t axes consists in an analytic continuation of the original rapidity $\beta \to \left(i\frac{\pi}{2} - \beta\right)$, the reason being that, to preserve the Minkowski structure in the new set of axes, we have to interchange correspondingly the momentum and the energy role. For convenience, it is useful to introduce the new transmission and reflection amplitudes, given by

$$\hat{T}(\beta) = T\left(i\frac{\pi}{2} - \beta\right) \quad , \quad \hat{R}(\beta) = R\left(i\frac{\pi}{2} - \beta\right) \quad . \tag{4.55}$$

They enter the expression of the simplest matrix elements of the operator $\mathcal{D}$, given by $\mathcal{D}_{1,1} =< \beta \mid \mathcal{D} \mid \theta >$, $\mathcal{D}_{2,0} =< \beta_1,\beta_2 \mid \mathcal{D} \mid 0 >$ and $\mathcal{D}_{0,2} =< 0 \mid \mathcal{D} \mid \beta_1,\beta_2 >$. For the fermionic and the bosonic theory analysed in the previous sections, the first matrix element is easily computed by resumming the perturbative series with the defect interaction now localized at $t = 0$ and the result is

$$< \beta \mid \mathcal{D} \mid \theta > = 2\pi\,\hat{T}(\beta)\,\delta(\beta - \theta) \quad . \tag{4.56}$$

[5]An equivalent way to look at eq. (4.54) is to consider a transfer matrix approach in the euclidean space. The transfer matrix may be written as $\mathcal{T} = \exp[-H_B t]$ for all t but $t = 0$, where it is placed the defect line. Hence $\mathcal{D}$ in (4.54) can be interpreted as the continuum limit of the transfer matrix operator which connects the states below and above the defect line.

By the same means, for the other two matrix elements, we have respectively

$$< \beta_1, \beta_2 \mid \mathcal{D} \mid 0 >= 2\pi\, \hat{R}(\beta_1)\, \delta(\beta_1 + \beta_2) \;, \tag{4.57}$$

and

$$< 0 \mid \mathcal{D} \mid \theta_1, \theta_2 >= 2\pi\, \hat{R}(\theta_1)\, \delta(\theta_1 + \theta_2) \;. \tag{4.58}$$

Hence, $\hat{T}(\beta)$ describes the process where a particle with rapidity β hits the defect line and is transmitted through it, keeping the same value of the rapidity. On the contrary, $\hat{R}(\beta)$ may be interpreted as the amplitude for the creation or the annihilation of a pair of particles with equal and opposite rapidity β. These three processes are compatible with the dynamics of the model because in a situation where the defect line is placed at $t = 0$, the processes are constrained by the conservation of the momentum but not of the energy.

For the general matrix elements of the operator $\mathcal{D}$, we can exploit the factorization property of the scattering theory and write down a set of recursive equations which involve the elementary two-body interactions considered above. For the bosonic case, the recursive equations are expressed by

$$\begin{aligned} &< \beta_1, \ldots, \beta_i, \ldots, \beta_m, \beta \mid \mathcal{D} \mid \theta_1, \ldots \theta_n >= \\ &= 2\pi \sum_{i=1}^{m} \hat{R}(\beta)\delta(\beta + \beta_i) \;< \beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots \beta_m \mid \mathcal{D} \mid \theta_1, \ldots \theta_n > + \\ &+ 2\pi \sum_{j=1}^{n} \hat{T}(\beta)\, \delta(\beta - \theta_j) \;< \beta_1, \ldots \beta_m \mid \mathcal{D} \mid \theta_1, \ldots \theta_{j-1}, \theta_{j+1}, \ldots, \theta_n > \;; \end{aligned} \tag{4.59}$$

$$\begin{aligned} &< \beta_1, \ldots, \beta_m, \mid \mathcal{D} \mid \theta_1, \ldots \theta_n, \theta >= \\ &= 2\pi \sum_{i=1}^{n} \hat{R}(\theta)\delta(\theta + \theta_i) \;< \beta_1, \ldots, \beta_m \mid \mathcal{D} \mid \theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n > + \\ &+ 2\pi \sum_{j=1}^{m} \hat{T}(\theta)\delta(\theta - \beta_j) \;< \beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_m \mid \mathcal{D} \mid \theta_1, \ldots, \theta_n > \;. \end{aligned} \tag{4.60}$$

For the fermionic case, taking into account the anti-commutation relations of the fields, they can be written as

$$\begin{aligned} &< \beta_1, \ldots, \beta_i, \ldots, \beta_m, \beta \mid \mathcal{D} \mid \theta_1, \ldots \theta_n >= \\ &= 2\pi \sum_{i=1}^{m} (-1)^{m+1-i}\, \hat{R}(\beta)\delta(\beta + \beta_i) \;< \beta_1, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots \beta_m \mid \mathcal{D} \mid \theta_1, \ldots \theta_n > + \\ &+ 2\pi \sum_{j=1}^{n} (-1)^j\, \hat{T}(\beta)\, \delta(\beta - \theta_j) \;< \beta_1, \ldots \beta_m \mid \mathcal{D} \mid \theta_1, \ldots \theta_{j-1}, \theta_{j+1}, \ldots, \theta_n > \;; \end{aligned} \tag{4.61}$$

$$< \beta_1, \ldots, \beta_m, | \mathcal{D} | \theta_1, \ldots \theta_n, \theta > =$$
$$= 2\pi \sum_{i=1}^{n} (-1)^i \hat{R}(\theta) \delta(\theta + \theta_i) \; < \beta_1, \ldots, \beta_m | \mathcal{D} | \theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n > \tag{4.62}$$
$$+ 2\pi \sum_{j=1}^{m} (-1)^{m-j} \hat{T}(\theta) \delta(\theta - \beta_j) \; < \beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_m | \mathcal{D} | \theta_1, \ldots, \theta_n > ,$$

These recursive equations express the exact resummation of the perturbative series associated to the scattering matrix elements $< m \mid \mathcal{D} \mid n >$. Since the particles are created or destroyed in couples, the non-vanishing matrix element $< m \mid \mathcal{D} \mid n >$ are only those with $m - n$ even. They are proportional to $< 0 \mid \mathcal{D} \mid 0 >$ (which, for convenience, is set equal to 1) and the recursive equations permit to express all of them in terms of the elementary matrix elements $\mathcal{D}_{1,1}, \mathcal{D}_{2,0}$ and $\mathcal{D}_{0,2}$ as previously determined.

A useful method for solving the recursive equations is to introduce a generating functional of the matrix elements of $\mathcal{D}$ by the formula

$$\mathcal{G}(\eta, \gamma) = \exp \left\{ \int d\beta \left(\frac{\hat{R}(\beta)}{2} [\eta(-\beta)\eta(\beta) + \gamma(\beta)\gamma(-\beta)] + \hat{T}(\beta)\eta(\beta)\gamma(\beta) \right) \right\} . \tag{4.63}$$

$\mathcal{G}$ depends on the two currents $\eta(\beta)$ and $\gamma(\beta)$, which commute or anti-commute, depending on whether we are considering the bosonic theory or the fermionic one. The matrix elements of $\mathcal{D}$ are then given by

$$< \beta_1, \ldots, \beta_m, | \mathcal{D} | \theta_1, \ldots \theta_n > = (2\pi)^{\frac{m+n}{2}} \frac{\partial}{\partial \gamma(\theta_n)} \cdots \frac{\partial}{\partial \gamma(\theta_1)} \frac{\partial}{\partial \eta(\beta_1)} \cdots \frac{\partial}{\partial \eta(\beta_m)} \mathcal{G} \Big|_{\eta=\gamma=0} . \tag{4.64}$$

We are now in the position to compute correlation functions of local operators of the Ising model and the bosonic theory with a line of defect. Note that in computing the left hand side of eq. (4.54) we should consider two different cases, namely: (a) the case where some of the operators Φ_i are in the upper half-plane and the others are in the lower one, or (b) the case where the operators Φ_i are all in one semi-plane, for example the upper one. In the former case, one has to use the general matrix elements $< i \mid \mathcal{D} \mid j >$, and consequently both transmission and reflection amplitudes will enter the final expression of the correlation functions. In the latter case, on the contrary, the correlation functions will depend only on the reflection amplitudes $\hat{R}(\beta)$ because, in virtue of the time ordering in eq. (4.54), the defect operator $\mathcal{D}$ is in this case the last in the row and so, it acts directly on the vacuum state $\mid 0 >$. Hence, the only matrix elements which enter the computation are $\mathcal{D}_{i,0} =< i \mid \mathcal{D} \mid 0 >$. Those describe the creation of the particle pairs and therefore only depend on $\hat{R}(\beta)$.

In the remaining part of this section, using the form factors of the Ising model determined in [5, 23], and the matrix elements of the defect operator we compute some correlation functions of this model in the presence of the defect line[6]. The simplest one is the one-point function of the energy operator $\epsilon(x,t)$ which can be computed through the formula

$$\epsilon_0(t) \,=\, \sum_{n=0}^{\infty} < 0 \mid \epsilon(x,t) \mid n >< n \mid \mathcal{D} \mid 0 > \quad . \tag{4.65}$$

The energy operator couples the vacuum only to the two particle state, as can be easily checked by the fermionic representation of this operator, and for its matrix element we have

$$\begin{aligned} < 0 \mid \epsilon(x,t) \mid \beta_1, \beta_2 >= \; & 2\pi i\, m \, \sinh \frac{\beta_1 - \beta_2}{2} \, \times \\ \times \, \exp\left[-mt\left(\cosh\beta_1 + \cosh\beta_2\right) + imx\left(\sinh\beta_1 + \sinh\beta_2\right)\right] \; , \end{aligned} \tag{4.66}$$

Hence the above sum (4.65) consists of only one term and using eq. (4.57), it can be expressed as

$$\epsilon_0(t) \;=\; m\, \sin\chi \int_0^\infty d\beta \frac{\sinh^2\beta}{\cosh\beta - \sin\chi}\, e^{-2mt\cosh\beta} \quad . \tag{4.67}$$

The one-point function does not depend on x, as it can be equivalently argued by translation invariance along this axis. The above integral reduces to closed expressions in terms of Bessel functions when the defect line acts as pure reflecting surface. In the case of fixed boundary conditions, we have

$$\epsilon(t) \;=\; -m\, \left[K_1(2mt) - K_0(2mt)\right] \; , \tag{4.68}$$

whereas for free boundary conditions

$$\epsilon(t) \;=\; m\, \left[K_1(2mt) + K_0(2mt)\right] \; , \tag{4.69}$$

In the general case, the one-point function interpolates between the two curves. The critical exponent of the energy operator in the presence of the defect line can be extracted by looking at the ultraviolet limit $t \to 0$ of its one-point function. For this limit we have

$$\epsilon_0(t) \sim \frac{\sin\chi}{2t} \quad . \tag{4.70}$$

From this expression, we see that the defect line does not influence the critical exponent of the energy operator, which is the same as in the bulk, but rather enters the universal amplitude of the one-point function. For the pure reflecting case, the universal amplitudes coincide with those calculated in [60].

[6] All correlation functions will be computed in the euclidean space obtained by the analytic continuation $t \to it$.

The relationship between the coupling constant in the continuum theory and in the discrete formulation can be extracted by comparing eq. (4.70) with the analogous lattice computation, which reads [75]

$$\epsilon_0(t) \sim \frac{\tanh 2(J-\tilde{J})}{2t} \quad . \tag{4.71}$$

Hence, we have the following identification

$$\sin\chi = \tanh 2(J-\tilde{J}) \quad . \tag{4.72}$$

In addition to the one-point function of the energy operator, it is also interesting to compute its two-point function. To simplify calculations, it is convenient to define the function

$$F(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} d\beta \, \frac{\exp[-t\cosh\beta + ix\sinh\beta]}{\cosh\beta - \sin\chi} \quad . \tag{4.73}$$

Let us initially consider the situation where the energy density operators are on opposite sides of the defect line, i.e. $t_2 > 0$ and $t_1 < 0$. The relevant expression in this case is given by[7]

$$G_1(\rho_1,\rho_2) = \sum_{i,j} < 0 \mid \epsilon(\rho_2) \mid i >< i \mid \mathcal{D} \mid j >< j \mid \epsilon(\rho_1) \mid 0 > \quad . \tag{4.74}$$

As before, the above series terminate. To explicitly evaluate it, in addition to the matrix elements $\mathcal{D}_{2,0}$ and $\mathcal{D}_{0,2}$, we also need the matrix element $\mathcal{D}_{2,2}$ given by

$$\begin{aligned} < \beta_1,\beta_2 \mid \mathcal{D} \mid \theta_1,\theta_2 > \; = \; & (2\pi)^2 \left[\hat{R}(\beta_1)\,\hat{R}(\theta_1)\delta(\beta_1+\beta_2)\delta(\theta_1+\theta_2) + \right. \\ & + \hat{T}(\beta_1)\,\hat{T}(\beta_2)\delta(\beta_1-\theta_1)\,\delta(\beta_2-\theta_2) + \\ & \left. - \; \hat{T}(\beta_1)\,\hat{T}(\beta_2)\,\delta(\beta_1-\theta_2)\,\delta(\beta_2-\theta_1)\right] \quad . \end{aligned} \tag{4.75}$$

With the notation $t \equiv t_2 - t_1$ and $x \equiv x_2 - x_1$, eq. (4.74) can be expressed as

$$\begin{aligned} G_1(\rho_1,\rho_2) \; = \; & \cos^2\chi \left[\left(\frac{\partial^2}{\partial x \partial t}F(x,t)\right)^2 + \left(\frac{\partial^2}{\partial t^2}F(x,t)\right)^2 - \left(\frac{\partial}{\partial t}F(x,t)\right)^2\right] + \\ & + \epsilon_0(t_1)\,\epsilon_0(t_2) \quad . \end{aligned} \tag{4.76}$$

When the defect line acts as a pure reflecting surface, all fluctuations across it are suppressed and this formula correctly reduces to the vacuum expectation values of the energy densities.

Let us consider now the situation where the two energy operators are on the same side of the defect line, with $t_2 \geq t_1 > 0$. For convenience, let us introduce the notation

[7]To simplify the notation, in the sequel we denote the couple of coordinate (x_i, t_i) simply by ρ_i.

$t \equiv t_2 - t_1$, $\bar{t} \equiv t_2 + t_1$, $x \equiv x_2 - x_1$ and $r \equiv \sqrt{x^2 + t^2}$. The two point function can be written in this case as

$$G_2(\rho_1, \rho_2) = \sum_{i,j} < 0 \mid \epsilon(\rho_2) \mid i >< i \mid \epsilon(\rho_1) \mid j >< j \mid \mathcal{D} \mid 0 > \quad . \tag{4.77}$$

There are only a finite number of non-vanishing matrix elements of the energy density and therefore the series truncates. It can be written as

$$G_2(\rho_1, \rho_2) = I_1 + I_2 + I_3 \ , \tag{4.78}$$

where

$$\begin{aligned}
I_1 &= \frac{1}{2!}\int_{-\infty}^{+\infty} \frac{d\beta_1}{2\pi}\frac{d\beta_2}{2\pi} < 0 \mid \epsilon(\rho_2) \mid \beta_1, \beta_2 >< \beta_1, \beta_2 \mid \epsilon(\rho_1) \mid 0 >< 0 \mid \mathcal{D} \mid 0 > \\
I_2 &= \frac{1}{2!\,2!}\int_{-\infty}^{+\infty} \frac{d\beta_1}{2\pi}\cdots\frac{d\beta_4}{2\pi} < 0 \mid \epsilon(\rho_2) \mid \beta_1, \beta_2 >< \beta_1, \beta_2 \mid \epsilon(\rho_1) \mid \beta_3, \beta_4 >< \beta_3, \beta_4 \mid \mathcal{D} \mid 0 > \\
I_3 &= \frac{1}{2!\,4!}\int_{-\infty}^{+\infty} \frac{d\beta_1}{2\pi}\cdots\frac{d\beta_6}{2\pi} < 0 \mid \epsilon(\rho_2) \mid \beta_1, \beta_2 >< \beta_1, \beta_2 \mid \epsilon(\rho_1) \mid \beta_3, .., \beta_6 >< \beta_3, .., \beta_6 \mid \mathcal{D} \mid 0 >
\end{aligned}$$

I_1 coincides with the two-point function of the energy operator in the bulk,

$$I_1 = m^2 \left[\left(\frac{\partial}{\partial x} K_0(mr) \right)^2 + \left(\frac{\partial}{\partial t} K_0(mr) \right)^2 - (K_0(mr))^2 \right] \quad .$$

The quantities which appear in I_2 and I_3 are the higher matrix elements of the energy density (which may be directly computed by the fermionic representation of this operator, $\epsilon = i\overline{\Psi}\Psi$) and the matrix elements of the defect operator $\mathcal{D}$, given by (4.64). Considering that the computation of these quantities is lengthy but straightforward, we shall only present the final result

$$\begin{aligned}
I_2 &= 2m^2 \sin\chi \left[\left(\frac{\partial}{\partial x} K_0(r) \right) \left(\frac{\partial}{\partial x} F(x, \bar{t}) \right) - K_0(r) \left(\frac{\partial^2}{\partial x^2} F(x, \bar{t}) \right) \right] \ , \\
I_3 &= m^2 \sin^2\chi \left[\left(\frac{\partial}{\partial x} F(x, \bar{t}) \right)^2 - \left(\frac{\partial^2}{\partial x^2} F(x, \bar{t}) \right)^2 - \left(\frac{\partial^2}{\partial x \partial \bar{t}} F(x, \bar{t}) \right)^2 \right] + \\
&\quad + \epsilon_0(t_1)\epsilon_0(t_2) \quad .
\end{aligned}$$

Returning to eq. (4.77), the two-point function can be cast in the form

$$\begin{aligned}
G_2(\rho_1, \rho_2) \quad &= \epsilon_0(t_1)\epsilon_0(t_2) + \left[\frac{\partial}{\partial x} K_0(r) + \sin\chi \frac{\partial}{\partial x} F(x, \bar{t}) \right]^2 + \left[\frac{\partial}{\partial t} K_0(r) \right]^2 \\
&- \left[\sin\chi \frac{\partial}{\partial x \partial \bar{t}} F(x, \bar{t}) \right]^2 - \left[K_0(r) + \sin\chi \frac{\partial^2}{\partial x^2} F(x, \bar{t}) \right]^2 \quad .
\end{aligned} \tag{4.79}$$

It is now easy to verify that the expressions (4.76) and (4.79) coincide with those obtained in the lattice calculation [75].

As our last example, we discuss the one-point function of the magnetization operator $\sigma(\rho)$ in the low temperature phase in the presence of the defect line. It can be calculated through the formula

$$\sigma_0(t) = \sum_{n=0}^{\infty} < 0 \mid \sigma(\rho) \mid n >< n \mid \mathcal{D} \mid 0 > \quad . \tag{4.80}$$

The magnetization operator couples the vacuum to all states with an even number of particles and its form factors are given by [5, 23]

$$< 0 \mid \sigma(0,0) \mid \beta_1, \ldots \beta_{2n} >= (-i)^n \prod_{i<j} \tanh \frac{\beta_i - \beta_j}{2} \quad . \tag{4.81}$$

Since the matrix elements of $\mathcal{D}$ in (4.80) are different from zero only for pairs of particles of opposite momentum, we are lead to consider the matrix elements of the magnetization operator given by $< 0 \mid \sigma(0) \mid -\beta_1, \beta_1, \ldots, -\beta_n, \beta_n >$. They can be conveniently written as

$$< 0 \mid \sigma(0,0) \mid -\beta_1, \beta_1, \ldots - \beta_n, \beta_n >= i^n \left(\prod_{i=1}^{n} \tanh \beta_i \right) \times \det W(\beta_i, \beta_j) \ , \tag{4.82}$$

where $W(\beta_i, \beta_j)$ is the $n \times n$ matrix given by

$$W(\beta_i, \beta_j) = \left(\frac{2\sqrt{\cosh \beta_i \cosh \beta_j}}{\cosh \beta_i + \cosh \beta_j} \right) \quad . \tag{4.83}$$

Hence, the one-point function is the sum of an infinite number of terms and can be expressed as a Fredholm determinant

$$\begin{aligned} \sigma_0(t) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\beta_1 \ldots d\beta_n \left(\prod_{k=0}^{n} i \tanh \beta_k \, \hat{R}(\beta_k) \, e^{-2mt \cosh \beta_k} \right) \det W(\beta_i, \beta_j) = \\ &= \mathrm{Det} \, (1 + z \, \mathcal{W}) \ . \end{aligned} \tag{4.84}$$

The explicit form of the kernel is given by

$$\mathcal{W}(\beta_i, \beta_j, \chi) = \frac{E(\beta_i, mt, \chi) E(\beta_j, mt, \chi)}{\cosh \beta_i + \cosh \beta_j} \ , \tag{4.85}$$

where

$$E(\beta, mt, \chi) = \sinh \beta \, e^{-mt \cosh \beta} (\cosh \beta - \sin \chi)^{-1/2} \quad , \quad z = \frac{\sin \chi}{2\pi} \ . \tag{4.86}$$

In terms of the eigenvalues of the integral operator and their multiplicity, $\sigma_0(t)$ can be also expressed as

$$\sigma_0(t) = \prod_{i=1}^{\infty} (1 + z\,\lambda_i)^{a_i} \tag{4.87}$$

As far as mt is finite, the kernel is square integrable and therefore all results valid for bounded symmetric integral operators apply (see, for instance [76]). In particular, for $mt \to \infty$, $\sigma_0(t)$ falls off exponentially to the bulk vacuum expectation value. However, when $mt \to 0$, the integral operator becomes unbounded. The multiplicity of the eigenvalues grows logarithmically as $a \sim \frac{1}{\pi} \ln \frac{1}{mt}$ whereas the eigenvalues become dense in the interval $(0, \infty)$ according to the distribution

$$\lambda(p) = \frac{2\pi}{\cosh \pi p} \ . \tag{4.88}$$

Hence, for the critical exponent of the magnetization operator, defined by

$$\sigma_0(t) \sim \frac{C}{(2t)^{x_\sigma}} \ , \ t \to 0 \ , \tag{4.89}$$

we have

$$x_\sigma(\chi) = -\frac{1}{\pi} \int_0^\infty dp \, \ln \left(1 + \frac{2\pi z}{\cosh p}\right) = -\frac{1}{8} + \frac{1}{2\pi^2} \arccos^2(-\sin \chi) \ . \tag{4.90}$$

This expression agrees with the lattice calculations [66, 67] and since it depends on the coupling constant, it explicitly shows the non-universality of the model.

Appendix A

Conformal perturbation theory

Consider an off-critical theory obtained perturbing a CFT by a relevant operator $\varphi(x)$ of conformal dimension $\Delta \equiv \Delta_\varphi \leq 1$; the action can be represented as

$$\mathcal{A} = \mathcal{A}_{CFT} + g \int d^2x \; \varphi(x) \; , \tag{A.1}$$

where the coupling constant g has scale dimension $g \sim (mass)^{2-2\Delta}$. The problem we want to face is the perturbative computation of the correlation functions of the local fields $A_i(x)$ of the theory. In the following we assume a one to one correspondence between the off-critical (renormalized) fields A_i and the conformal fields of the ultraviolet fixed point, denoted by $\tilde{A}_i$. Let's consider for instance the two-point correlator of a field $\Phi(x)$; formally, from the action (A.1) we get

$$< \Phi(x)\Phi(0) >= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int < \Phi(x)\Phi(0)\varphi(y_1)\ldots\varphi(y_n) >_{CFT} d^2y_1 \ldots d^2y_n \; . \tag{A.2}$$

Although all the conformal correlators appearing in the right hand side are in principle exactly known, the above expression cannot be used in practice since the integrals are in general both ultraviolet and infrared divergent. While one can think to eliminate the ultraviolet divergences through a renormalization procedure, the infrared divergences cannot be absorbed in any local quantity of the theory and lead to non-analyticity in the coupling constant g. An alternative approach for the computation of correlation functions was proposed by Al.B. Zamolodchikov in ref.[17]. The starting point is the operator product expansion (OPE) of the perturbed theory

$$\Phi(x)\Phi(0) = \sum_i C^i_{\Phi\Phi}(x) A_i(0) \; . \tag{A.3}$$

The basic observation is that, due to the local character of the OPE, a regular expansion in g should be expected for the off-critical structure functions $C^i_{\Phi\Phi}(x)$

$$C^i_{\Phi\Phi}(x) = x^{\Delta_i - 2\Delta_\Phi} \bar{x}^{\bar{\Delta}_i - 2\bar{\Delta}_\Phi} (C^{i\,(0)}_{\Phi\Phi} + C^{i\,(1)}_{\Phi\Phi} t + C^{i\,(2)}_{\Phi\Phi} t^2 + \ldots) \; , \tag{A.4}$$

where $t = gr^{2-2\Delta}$ is the dimensionless expansion parameter and $C_{\Phi\Phi}^{i\,(0)}$ is the CFT structure constant. As a consequence, non-locality (and then non-analyticity) in the computation of $< \Phi(x)\Phi(0) >$ is entirely encoded in possible non-zero vacuum expectation values (VEVs) of the fields A_i. We recall that, while only $< I > \neq 0$ is CFT, non-zero VEVs can be developed in the off-critical theory by the fields which are "neutral" with respect to all the symmetries of the theory. Dimensional analysis gives

$$< A_i >= g^{\frac{\Delta_i}{1-\Delta}} Q_i \ , \tag{A.5}$$

where Q_i are pure numbers. Some of these VEVs are available by non-perturbative techniques like the thermodynamic bethe ansatz. Then one can think to compute perturbatively the correlator in the form

$$< \Phi(x)\Phi(0) >= \sum_i C_{\Phi\Phi}^{i}(x) < A_i > \ , \tag{A.6}$$

the problem being reduced to the determination of the coefficients $C_{\Phi\Phi}^{i\,(n)}$ $(n > 0)$ in eq.(A.4). We will now show, following ref.[17], how to develop the infrared finite perturbation theory for the structure functions. For simplicity, we will assume that both Δ and Δ_Φ are lesser than 1/2 so that the fields φ and Φ do not need renormalization.

Consider the matrix

$$\begin{aligned} \tilde{I}_l^k(g,R,\varepsilon) &= < \tilde{A}^k(\infty)\tilde{A}_l(0) >_g^{(R,\varepsilon)} \\ &= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int_{\varepsilon<|y_i|<R} < \tilde{A}^k(\infty)\varphi(y_1)\ldots\varphi(y_n)\tilde{A}_l(0) >_{CFT} d^2y_1 \ldots d^2y_n \ , \end{aligned} \tag{A.7}$$

where ε and R are ultraviolet and infrared cut-off respectively. Using the standard CFT normalization we can write

$$\tilde{I}_k^l(g,R,\varepsilon) = \delta_k^l + \mathcal{O}(g) \ . \tag{A.8}$$

The matrix elements $\tilde{I}_k^l$ are in general divergent in the limit $\varepsilon \to 0$; the singular dependence can be factorized in the form

$$\tilde{I}_k^l(g,R,\varepsilon) = U_l^{l'}(g,\varepsilon) I_{l'}^k(g,R) \ . \tag{A.9}$$

From dimensional arguments and analyticity in g we have

$$U_l^{l'}(g,\varepsilon) = \sum_{n=0}^{\infty} U_l^{l'(n)} \frac{(g\varepsilon^{2-2\Delta})^n}{\varepsilon^{2(\Delta-\Delta_{l'})}} \ . \tag{A.10}$$

Since only the terms with negative powers of ε survive in the limit $\varepsilon \to 0$ and since $\Delta < 1$, we conclude that only a finite number of terms is present in the right hand side of eq.(A.10). It is also clear that

$$U_l^{l'}(g,\varepsilon) = 0 \qquad \text{for } \Delta_{l'} > \Delta_l \ , \tag{A.11}$$

so that, if we order the field with increasing dimensions ($\Delta_0 \leq \Delta_1 \leq \Delta_2 \leq \ldots$), then $U_l^{l'}$ has triangular form. Defining the renormalized fields as

$$A_k = (U^{-1})_k^l \tilde{A}_l \tag{A.12}$$

and fixing the normalization of the matrix $U_l^{l'}$ as

$$U_l^{l'}(g,\varepsilon) = \delta_l^{l'} + \mathcal{O}(g) \ , \tag{A.13}$$

we obtain $A_k = \tilde{A}_k + \ldots$ with only a finite number of fields of dimension lower than Δ_k omitted. The renormalized matrix elements

$$I_l^k(g,R) = < \tilde{A}_k(\infty) A_l(0) >_g^{(R)} \tag{A.14}$$

are independent on the ultraviolet regularization and have the following dimensional structure [1]

$$I_l^k(g,R) = \sum_{n=0}^{\infty} I_l^{k(n)} \frac{(gR^{2-2\Delta})^n}{R^{2(\Delta_l-\Delta_k)}} \ . \tag{A.15}$$

Since now we should take the terms with positive powers of R, the series eq.(A.15) is infinite and its resummation gives rise to a non-trivial function of R. In particular the limits

$$\lim_{R\to\infty} \frac{I_l^k(g,R)}{I_0^k(g,R)} = \lim_{R\to\infty} \frac{< \tilde{A}_k(\infty) A_l(0) >_g^{(R)}}{< \tilde{A}_k(\infty) >} = g^{\Delta_l/(1-\Delta)} Q_l^{(k)} \tag{A.16}$$

are the VEVs (A.5). Define the quantities

$$\begin{aligned} G_{\Phi\Phi}^k(g,x,R) &= < \tilde{A}^k(\infty)\Phi(x)\Phi(0) >_g^{(R)} = \\ &= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int_{|y_i|<R} < \tilde{A}^k(\infty)\varphi(y_1)\ldots\varphi(y_n)\Phi(x)\Phi(0) >_{CFT} d^2y_1 \ldots d^2y_n \ . \end{aligned} \tag{A.17}$$

In the previous expression we do not need the cut-off ε since the ultraviolet singularities of the above correlators are integrable due to our assumption $\Delta, \Delta_\Phi < 1/2$. Substituting eq.(A.3) for the product $\Phi(x)\Phi(0)$ in eq.(A.17) we get

$$G_{\Phi\Phi}^k(g,x,R) = \sum_l C_{\Phi\Phi}^l(x) I_l^k(g,R) \tag{A.18}$$

so that

$$C_{\Phi\Phi}^k(x) = \sum_l G_{\Phi\Phi}^l(g,x,R)(I^{-1})_l^k(g,R) \ . \tag{A.19}$$

[1] We recall that $< \tilde{A}_k(\infty) \equiv \lim_{z,\bar{z}\to\infty} z^{2\Delta_k}\bar{z}^{2\bar{\Delta}_k} < \tilde{A}_k(z,\bar{z})$ in such a way that $< \tilde{A}_k(\infty)\tilde{A}_l(0) >_{CFT} = \delta_{kl}$.

Since the structure functions $C^k_{\Phi\Phi}(x)$ appear in the local OPE (A.3) they cannot depend on what happens at spatial infinity and are expected to be finite as $R \to \infty$. To first order in g we have

$$\begin{aligned}
\tilde{I}^k_l(g,R,\varepsilon) &= <\tilde{A}^k(\infty)\tilde{A}_l(0)>_{CFT} \\
&\quad -g\int_{\varepsilon<|y|<R} <\tilde{A}^k(\infty)\varphi(y)\tilde{A}_l(0)>_{CFT} d^2y \qquad (A.20)\\
&= \delta^k_l - 2\pi g C^{k\,(0)}_{\varphi l}\int_{\varepsilon<r<R} dr\, r^{-2\Delta-2\Delta_l+2\Delta_k+1} \\
&= \delta^k_l - \pi g C^{k\,(0)}_{\varphi l}\,\frac{R^{2(\Delta_k-\Delta_l-\Delta+1)} - \varepsilon^{2(\Delta_k-\Delta_l-\Delta+1)}}{\Delta_k-\Delta_l-\Delta+1} \quad .
\end{aligned}$$

Therefore to first order we find

$$I^k_l(g,R) = \delta^k_l - \pi g C^{k\,(0)}_{\varphi l}\frac{R^{2(\Delta_k-\Delta_l-\Delta+1)}}{\Delta_k-\Delta_l-\Delta+1}\ , \qquad (A.21)$$

$$U^k_l(g,\varepsilon) = \delta^k_l + \pi g C^{k\,(0)}_{\varphi l}\frac{\varepsilon^{2(\Delta_k-\Delta_l-\Delta+1)}}{\Delta_k-\Delta_l-\Delta+1}\ . \qquad (A.22)$$

Substituting eq.(A.21) into eq.(A.19) we get [2]

$$\begin{aligned}
C^k_{\Phi\Phi}(x) &= \sum_l \left(<\tilde{A}^l(\infty)\Phi(x)\Phi(0)>_{CFT} - g\int_{|y|<R} <\tilde{A}^l(\infty)\varphi(y)\Phi(x)\Phi(0)>_{CFT} d^2y \right) \\
&\quad \times\left(\delta^k_l + \pi g C^{k\,(0)}_{\varphi l}\frac{R^{2(\Delta_k-\Delta_l-\Delta+1)}}{\Delta_k-\Delta_l-\Delta+1}|x|^{2(\Delta_l-2\Delta_\Phi)}\right) + \mathcal{O}(g^2) \qquad (A.23)\\
&= C^{k\,(0)}_{\Phi\Phi}|x|^{2(\Delta_k-2\Delta_\Phi)} - g\int_{|y|<R} <\tilde{A}^k(\infty)\varphi(y)\Phi(x)\Phi(0)>_{CFT} d^2y \\
&\quad +\pi g\sum_l C^{k\,(0)}_{\varphi l}C^{l\,(0)}_{\Phi\Phi}\frac{R^{2(\Delta_k-\Delta_l-\Delta+1)}}{\Delta_k-\Delta_l-\Delta+1}|x|^{2(\Delta_l-2\Delta_\Phi)} + \mathcal{O}(g^2)\ .
\end{aligned}$$

If $|y| > |x|$ we can use the OPE (A.3) to evaluate the integral in eq.(A.23)

$$\begin{aligned}
&\int_{|y|<R} <\tilde{A}^k(\infty)\varphi(y)\Phi(x)\Phi(0)>_{CFT} d^2y \qquad (A.24)\\
&= 2\pi\sum_l C^{l\,(0)}_{\Phi\Phi}C^{k\,(0)}_{\varphi l}\int_{r<R} r dr\, r^{2(\Delta_k-\Delta_l-\Delta)}\ |x|^{2(\Delta_l-2\Delta_\Phi)} \\
&= \pi\sum_l C^{k\,(0)}_{\varphi l}C^{l\,(0)}_{\Phi\Phi}\frac{R^{2(\Delta_k-\Delta_l-\Delta+1)}}{\Delta_k-\Delta_l-\Delta+1}|x|^{2(\Delta_l-2\Delta_\Phi)}\ .
\end{aligned}$$

Thus we see that the last term in eq.(A.23) exactly cancels all the possible infrared divergences. Moreover, since only positive powers of R are non-zero in the limit $R \to$

[2]This formula is valid for $\Delta_k - \Delta_l \neq 1 - \Delta$; if such "resonances" occur logarithmic divergences are present in the perturbative expansion which require a particular treatment.

∞, only a finite number of terms is present in the sum. For practical application it is useful to exploit the scaling properties of conformal fields to recast eq.(A.23) in the form

$$C^k_{\Phi\Phi}(x) = |x|^{2(\Delta_k - 2\Delta_\Phi)} \Big[C^{k\,(0)}_{\Phi\Phi} - g|x|^{2(1-\Delta)} \times \tag{A.25}$$
$$\Big(\int_{|y|<R} < \tilde{A}^k(\infty)\varphi(y)\Phi(1)\Phi(0) >_{CFT} d^2y - \pi \sum_l C^{k\,(0)}_{\varphi l} C^{l\,(0)}_{\Phi\Phi} \frac{R^{2(\Delta_k - \Delta_l - \Delta + 1)}}{\Delta_k - \Delta_l - \Delta + 1} \Big) \Big] + \mathcal{O}(g^2),$$

in which the conformal correlator appears in the standard form.

Appendix B

Thermodynamic Bethe ansatz

In this appendix we will describe the basic lines of a powerful method, known as *thermodynamic Bethe ansatz* (TBA) [77, 78, 79, 80, 81, 82], which gives the possibility to recover the ultraviolet data of an integrable model using only the on-shell informations contained in the S-matrix. This allows us, in particular, to test if the *minimal* S-matrix obtained using the bootstrap procedure is the correct one to describe the massive theory arising from the relevant perturbation of a CFT.

Let's start by considering a relativistic field theory living on a torus generated by two orthogonal circles B and C of circumference L and R respectively. One can develop an hamiltonian approach to this situation choosing the time direction along the circle B or, alternatively, along the circle C. This leads to the definition of two different Hamiltonian, H_C and H_B respectively. Therefore the partition function of the theory can be written in two different ways:

$$Z(R,L) = \mathrm{tr}_{\mathcal{C}} e^{-LH_C} = \mathrm{tr}_{\mathcal{B}} e^{-RH_B} \ , \qquad \text{(B.1)}$$

where $\mathcal{B}$ ($\mathcal{C}$) is the space of states on B (C).

Consider now the limit $L \to \infty$, $L \gg R$. This corresponds to the thermodynamic limit for the system living on B and we can write

$$ln Z(R,L) \simeq -RLf(R) \ , \qquad \text{(B.2)}$$

where $f(R)$ is the free energy for unit length at temperature $T = 1/R$. On the other hand, in the limit we are considering, the second expression for $Z(R,L)$ in eq.(B.1) is dominated by the contribution of the ground state of H_C whose energy $E_0(R)$ depends on the size R of the system:

$$Z(R,L) \simeq e^{-E_0(R)L} \ . \qquad \text{(B.3)}$$

In the ultraviolet limit $R \to 0$, where the system is described by a CFT with central charge c, the ground state energy behaves as $E_0(R) \simeq -\pi\tilde{c}/6R$, where

$$\tilde{c} = c - 12 d_{min} \ , \tag{B.4}$$

d_{min} being the lowest anomalous scale dimension in the CFT. In a unitary theory $d_{min} = 0$ and $\tilde{c} = c$. In light of this result we introduce the scaling function $\tilde{C}(mR)$ (m is the mass scale of the theory) such that $\tilde{C}(0) = \tilde{c}$ and write

$$E_0(R) = -\frac{\pi}{6R}\tilde{C}(mR) \ . \tag{B.5}$$

Putting together eqs.(B.2), (B.3) and (B.5) one sees that the determination of $\tilde{C}(mR)$ reduces to the computation of the free energy of a system of relativistic particles living on a line of length $L \to \infty$ at temperature $1/R$. We will discuss this problem for the case in which the system of particles is described by a diagonal scattering theory containing n species of particles with masses m_a, $a = 1, \dots, n$.

Generally speaking, the wave function formalism is inappropriate to describe a system of relativistic particles due to the virtual and real particle creation. But for a system of N particles on a line of length L much larger than the correlation length ξ there exist regions of the configuration space in which the particles are widely separated from each other: $|x_i - x_j| \gg \xi$, $\forall i \neq j$ ($\xi \sim 1/m_1$ if m_1 stays for the mass of the lightest particle in the theory). In these *free regions* the off-shell effects can be neglected and it is sensible to describe the system by a *Bethe wave function* $\psi(x_1, \dots, x_n)$ proportional to the free wave function $\prod_{i=1}^{N} e^{ip_i x_i}$. Now the important step: the passage from a free region to a different one in which, for example, particles i and j exchanged their positions can be described simply by multiplying the original wave function by the scattering amplitude $S_{ij}(\theta_{ij})$. In particular, if we impose periodic boundary conditions, this criterion leads to the relation

$$e^{ip_i L} \prod_{j:j\neq i} S_{ij}(\theta_{ij}) = 1 \ , \qquad i = 1, \dots, N \ . \tag{B.6}$$

Defining

$$S_{ij}(\theta) = e^{i\delta_{ij}(\theta)} \ , \tag{B.7}$$

eq.(B.6) can be written as

$$m_i L \sinh\theta_i + \sum_{j:j\neq i} \delta_{ij}(\theta_{ij}) = 2\pi n_i \ , \tag{B.8}$$

with N integers numbers n_i. This system of transcendental equations selects admissible sets of rapidities in free regions. Note that in the non-interacting case, where $\delta_{ij} = 0$, eq.(B.8) reduces to the usual quantization condition for the momenta of a particle in a box: $p_i = 2\pi n_i/L$.

If identical particles are present, statistical constraints on the wave functions must be taken into account. The unitarity condition allows two different cases:

$$\text{(a)} \qquad S_{ij}(0) = -1 \ . \tag{B.9}$$

Then the wave function is antisymmetric in the coordinates of two identical particles with the same rapidity. This is allowed for fermions but implies that a system of bosons cannot contains identical particles with the same rapidities. Under this respect bosons behaves like fermions since each value of rapidity can be occupied by at most one particle. As a consequence, all the integers n_i in eq.(B.8) must be different and we refer to this case as "fermionic". On the other hand, if the identical particles are fermions, the states with coinciding rapidity are allowed and there are no constraints on the numbers n_i; we will refer to this case as "bosonic".

$$\text{(b)} \qquad S_{ij}(0) = 1 \ . \tag{B.10}$$

In this case the situation in just inverted, bosons giving rise to the bosonic case and fermions to the fermionic case.

Since each set of N integers n_i for which the system (B.8) admits a solution selects a set of rapidities, we can imagine a system of levels in the rapidity space. In the thermodynamic limit $L \to \infty$ the distance between two adjacent levels behaves as $\theta_i - \theta_{i+1} \sim 1/m_1 L$ (think to the free case) and it is useful to introduce continuous level densities $\rho_a(\theta)$, with the index a referring to different species of particles. We also define rapidity densities of particles $\tilde{\rho}_a(\theta)$ as

$$\tilde{\rho}_a(\theta) = \frac{n_a(\theta)}{\Delta\theta} \ , \tag{B.11}$$

where $n_a(\theta)$ is the number of particles of species a contained in an interval $\Delta\theta$ around θ such that $1/m_1 L \ll \Delta\theta \ll 1$. Using these definitions equation (B.8) can be rewritten in the integral form

$$m_a L \cosh\theta + \sum_{b=1}^{n} \int d\theta' \, \phi_{ab}(\theta - \theta')\tilde{\rho}_b(\theta') = 2\pi\rho_a(\theta) \ , \tag{B.12}$$

where

$$\phi_{ab}(\theta) \equiv \frac{\partial}{\partial\theta}\delta_{ab}(\theta) = -i\frac{\partial}{\partial\theta} ln S_{ab}(\theta) \ . \tag{B.13}$$

The free energy Lf can be computed by the usual thermodynamic relation

$$Lf(\rho,\tilde{\rho}) = H_B(\rho) - \frac{1}{R}\mathcal{S}(\rho,\tilde{\rho}) \ , \tag{B.14}$$

where the total energy $H_B(\rho)$ is given by

$$H_B(\rho) = \sum_{a=1}^{n} \int d\theta \, m_a \tilde{\rho}_a(\theta) \cosh\theta \ , \tag{B.15}$$

and $\mathcal{S}(\rho, \tilde{\rho})$ is the entropy of the system. The number of particles in a rapidity interval $\Delta\theta$ is $\tilde{\rho}_a(\theta)\Delta\theta$ and $\rho_a(\theta)\Delta\theta$ is the number of levels in the same interval for a given species of particles with mass a. Therefore the number of possible distributions of such particles among these levels is

$$\frac{[\rho_a(\theta)\Delta\theta]!}{[\tilde{\rho}_a(\theta)\Delta\theta]! \, [(\rho_a(\theta) - \tilde{\rho}_a(th))\Delta\theta]!} \tag{B.16}$$

in the fermionic case, and

$$\frac{[(\rho_a(\theta) + \tilde{\rho}_a(\theta)\Delta\theta]!}{[\rho_a(\theta)\Delta\theta]! \, [\tilde{\rho}(\theta)\Delta\theta]!} \tag{B.17}$$

in the bosonic case. Since the entropy is the logarithm of the number of possible distributions for given densities ρ and $\tilde{\rho}$ in the limit $L \to \infty$, we have

$$\mathcal{S}_{Fermi}(\rho, \tilde{\rho}) = \sum_{a=1}^{n} \int d\theta \, [\rho_a ln\rho_a - \tilde{\rho}_a ln\tilde{\rho}_a - (\rho_a - \tilde{\rho}_a) ln(\rho_a - \tilde{\rho}_a)] \tag{B.18}$$

and

$$\mathcal{S}_{Bose}(\rho, \tilde{\rho}) = \sum_{a=1}^{n} \int d\theta \, [(\rho_a + \tilde{\rho}_a) ln(\rho_a + \tilde{\rho}_a) - \rho_a ln\rho_a - \tilde{\rho}_a ln\tilde{\rho}_a] \ . \tag{B.19}$$

At this point we have to minimize the free energy in order to determine the densities ρ and $\tilde{\rho}$ at equilibrium. They are related by the dynamical equation (B.12) from which we get

$$\frac{\delta\rho_a(\theta')}{\delta\tilde{\rho}_b(\theta)} = \frac{1}{2\pi}\phi_{ab}(\theta' - \theta) \ . \tag{B.20}$$

Define the *pseudoenergies* $\varepsilon_a(\theta)$ by

$$\frac{\tilde{\rho}_a(\theta)}{\rho_a(\theta)} = \frac{e^{-\varepsilon_a(\theta)}}{1 \pm e^{-\varepsilon_a(\theta)}} \tag{B.21}$$

and introduce also

$$L_a(\theta) = \pm ln\left(1 \pm e^{-\varepsilon_a(\theta)}\right) \tag{B.22}$$

(here and in the following upper and lower signs refer to the particle a being of fermionic or bosonic type, respectively). Using eq.(B.20) to compute the variation of H_B and $\mathcal{S}$ with respect to $\tilde{\rho}_a$, the extremum condition for f take the form

$$Rm_a \cosh\theta = \varepsilon_a(\theta) + \sum_{b=1}^{n} \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \phi_{ab}(\theta - \theta') L_b(\theta') \ . \tag{B.23}$$

These are the TBA equations written in unified form for fermionic and bosonic case. The numerical solution of these integral equations provides the values of the pseudoenergies $\varepsilon_a(\theta)$ which are the necessary ingredients for the determination of the function $\tilde{C}(r)$ $(r \equiv mR)$ through the formula

$$\begin{aligned} \tilde{C}(r) &= -\frac{6R^2}{\pi} f(R) \\ &= \frac{3}{\pi^2} \sum_{a=1}^{n} \int_{-\infty}^{+\infty} d\theta \, L_a(\theta) \frac{m_a}{m_1} r \cosh\theta \quad . \end{aligned} \tag{B.24}$$

This expression can be explicitly evaluated in the limit $r \to 0$. In this limit θ has to be taken very large in order to give a non-negligible contribution to the left-hand side of equation (B.23). One has

$$r\frac{m_a}{m_1}\cosh\theta \sim \frac{r}{2}\frac{m_a}{m_1}e^{\theta} \sim \frac{m_a}{m_1} exp\left(\theta - ln\frac{2}{r}\right) \quad . \tag{B.25}$$

Numerical work shows that the pseudoenergies assume constant values ε_a in the range $-ln(2/r) \ll \theta \ll ln(2/r)$, where the left-hand side of eq.(B.23) can be neglected, and grow exponentially at very large values of $|\theta|$. In the interval in which they are constant eq.(B.23) reduces to

$$\varepsilon_a = \pm \sum_{b=1}^{n} N_{ab} \, ln(1 \pm e^{-\varepsilon_b}) \ , \tag{B.26}$$

with

$$N_{ab} = -\int_{-\infty}^{+\infty} \frac{d\theta}{2\pi} \, \phi_{ab}(\theta) = -\frac{1}{2\pi}(\delta_{ab}(\infty) - \delta_{ab}(-\infty)) \quad . \tag{B.27}$$

On the other hand this region does not contribute to the integral in eq.(B.24) which is sensible only to the large $|\theta|$ limit for $r \to 0$. In this limit the pseudoenergies $\hat{\varepsilon}_a(\theta)$ are determined by eq.(B.23) in the form

$$\frac{r}{2}\frac{m_a}{m_1}e^{\theta} = \hat{\varepsilon}_a(\theta) + \sum_{b=1}^{n} \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \, \phi_{ab}(\theta - \theta')\hat{L}_b(\theta') \ , \tag{B.28}$$

where

$$\hat{L}_a(\theta) = \pm ln(1 \pm e^{-\hat{\varepsilon}_a(\theta)}) \ , \tag{B.29}$$

and we have

$$\tilde{C}(0) = \frac{6}{\pi^2} \sum_{a=1}^{n} \lim_{r \to 0} \int_0^{\infty} d\theta \, \hat{L}_a(\theta) \frac{r}{2}\frac{m_a}{m_1} e^{\theta} \tag{B.30}$$

(in last line the parity of $\varepsilon_a(\theta)$ was used to change the lower boundary of integration). Taking a derivative of eq.(B.28) with respect to θ one obtains another expression for

$\frac{r}{2}\frac{m_a}{m_1}e^{\theta}$ which can be substituted in eq.(B.30) and, after several integration by parts, allows to arrive at the final result

$$\tilde{c} = \sum_{a=1}^{n} \tilde{c}_a^{\pm}(\varepsilon_a) \ , \tag{B.31}$$

where

$$\tilde{c}_a^{\pm}(\varepsilon_a) = \frac{6}{\pi^2} \times \begin{cases} L\left(\frac{1}{1+e^{\varepsilon_a}}\right) \\ L(e^{-\varepsilon_a}) \end{cases} = \frac{6}{\pi^2}\int_0^{\infty} dx \, \frac{x+\varepsilon_a/2}{e^{x+\varepsilon_a}\pm 1} \ . \tag{B.32}$$

Here the ε_a are determined by equation (B.26) and $L(x)$ is Rogers' dilogarithm function [83]

$$L(x) = -\frac{1}{2}\int_0^x dy \left[\frac{ln\, y}{1-y} + \frac{ln(1-y)}{y}\right] \ . \tag{B.33}$$

In conclusion we give an expression for the quantities N_{ab} defined in eq.(B.27) and entering eq.(B.26). Since the S-matrix element $S_{ab}(\theta)$ for a purely elastic scattering theory can be written as product of the building blocks

$$f_\alpha(\theta) = \frac{\sinh\frac{1}{2}(\theta + i\alpha\pi)}{\sinh\frac{1}{2}(\theta - i\alpha\pi)} \ , \tag{B.34}$$

then $\phi_{ab}(\theta) = \sum_i \phi[f_{\alpha_i}](\theta)$ and $N_{ab} = \sum_i N[f_{\alpha_i}]$, in an obvious notation. Direct computation gives

$$\phi[f_\alpha](\theta) = -i\frac{d}{d\theta}ln f_\alpha(\theta) = -\frac{\sin\alpha\pi}{\cosh\theta - \cos\alpha\pi} \tag{B.35}$$

and

$$N[f_\alpha] = (1-|\alpha|)\text{sgn}\alpha \qquad \text{for } -1 < \alpha < 1 \ , \tag{B.36}$$

where sgnα is the sign of α (sgn$0 = 0$).

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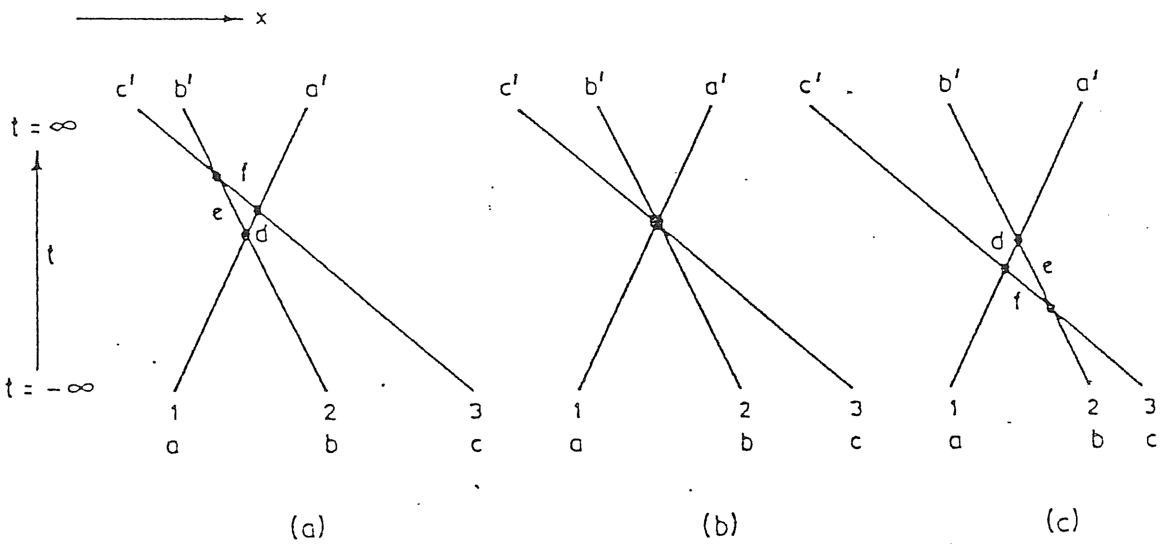
Figures and tables



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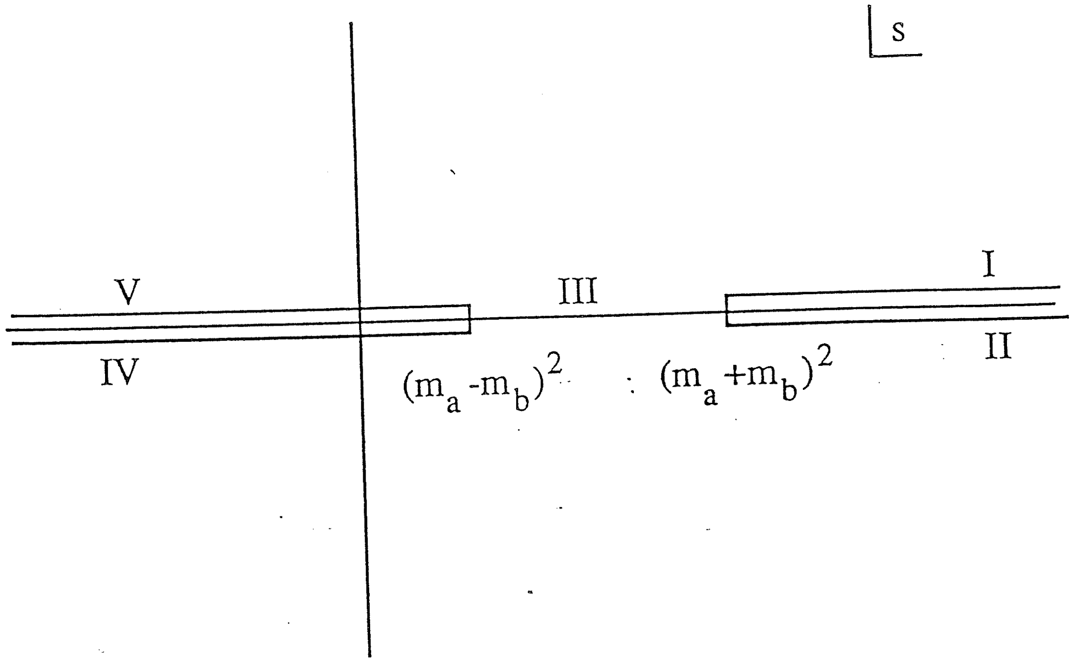

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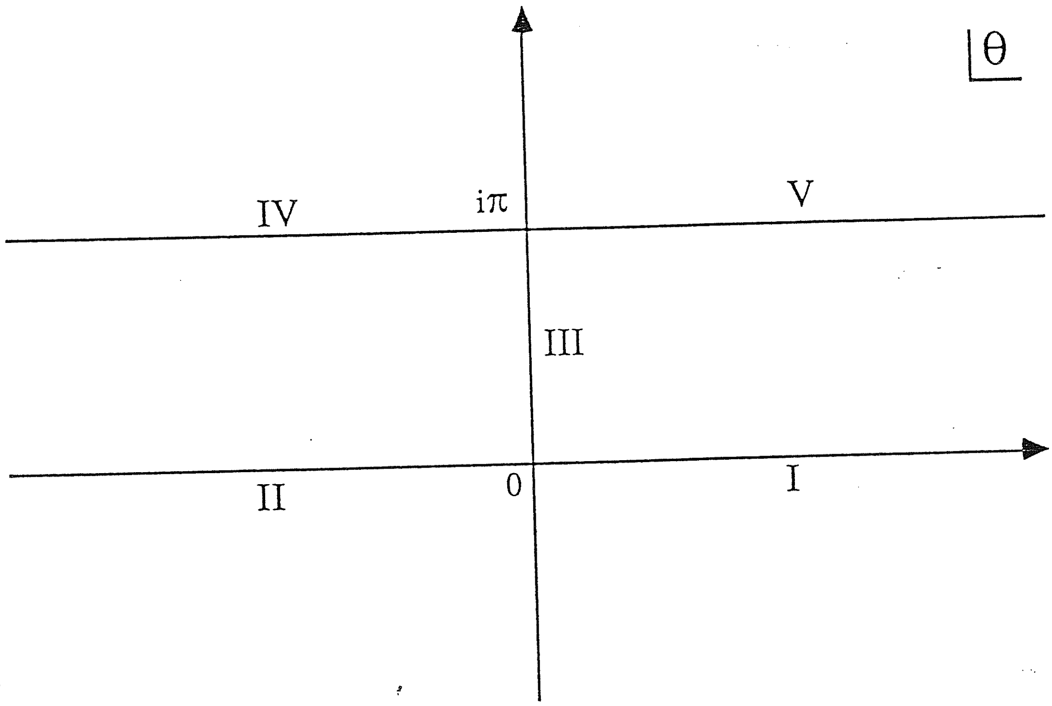

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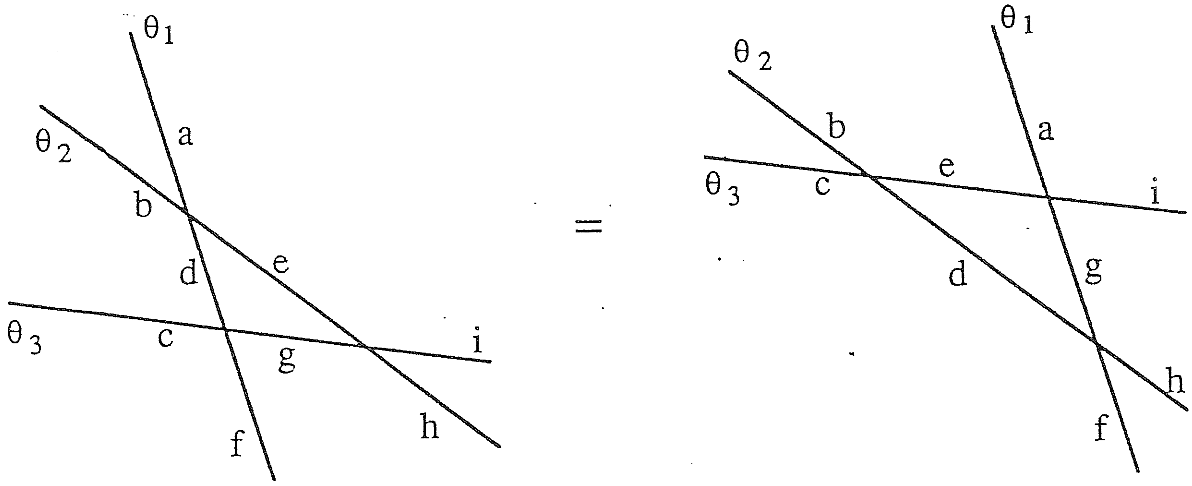

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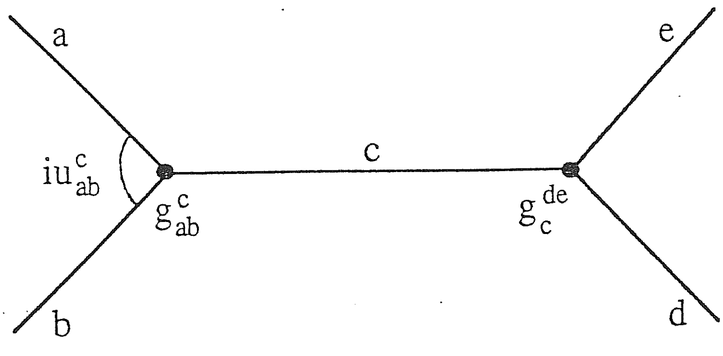

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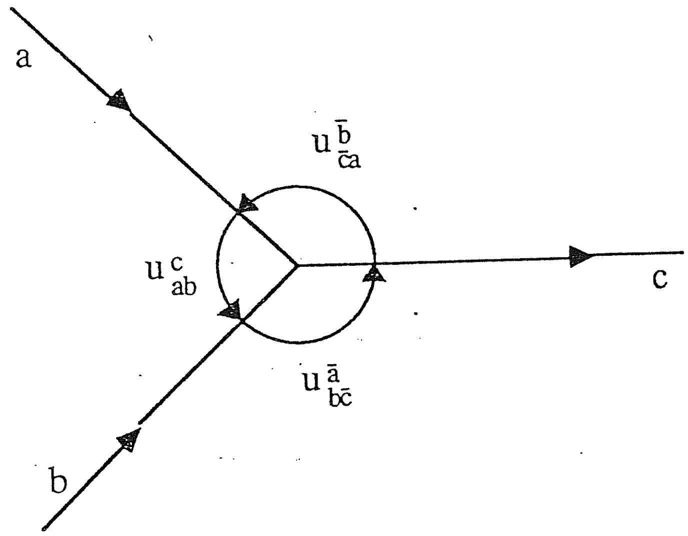

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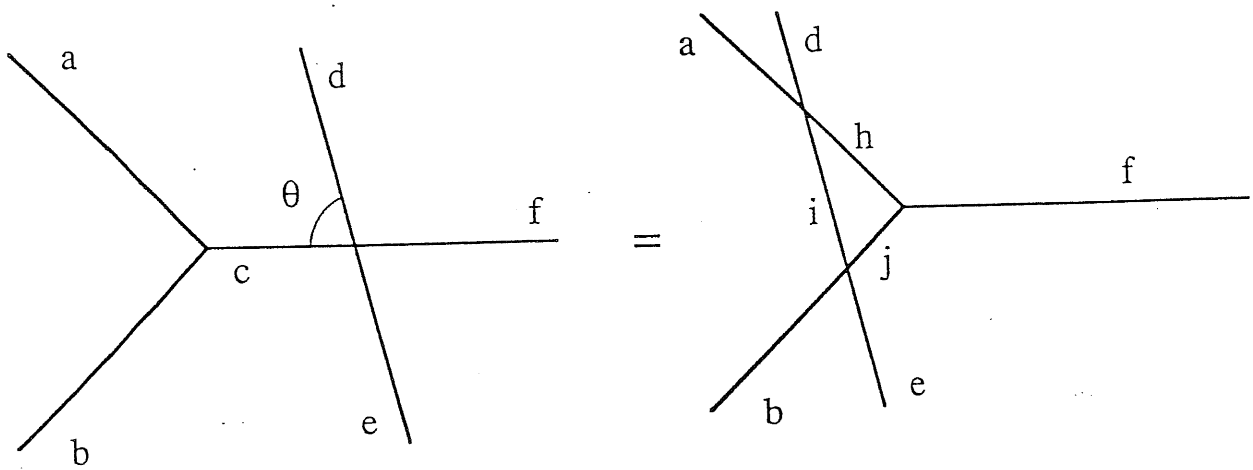

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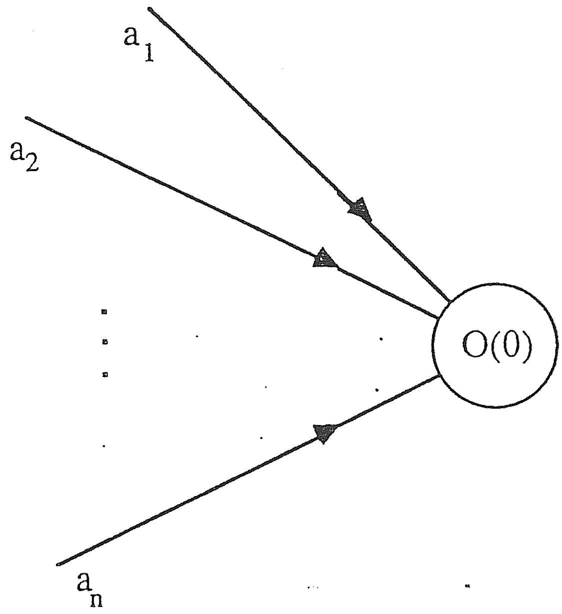

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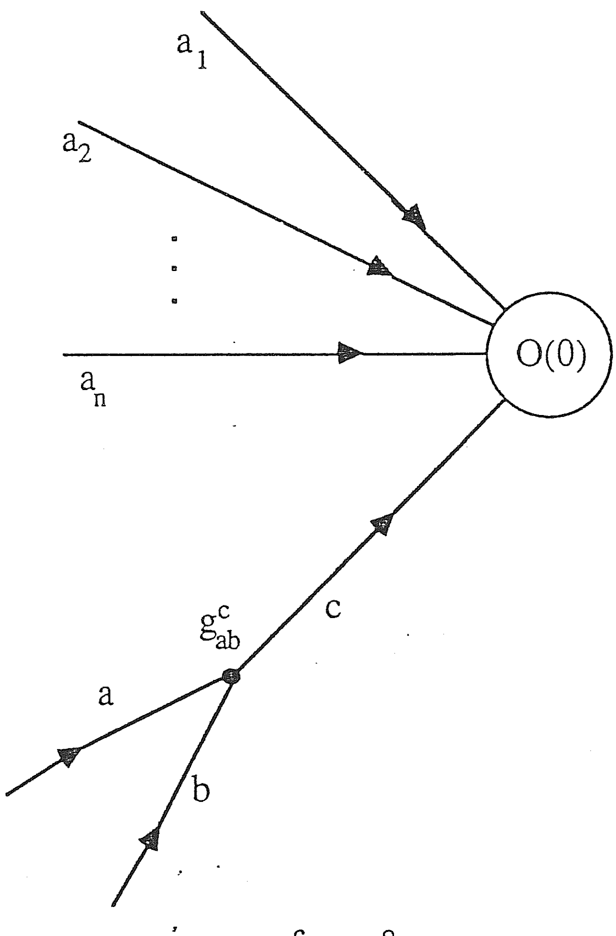

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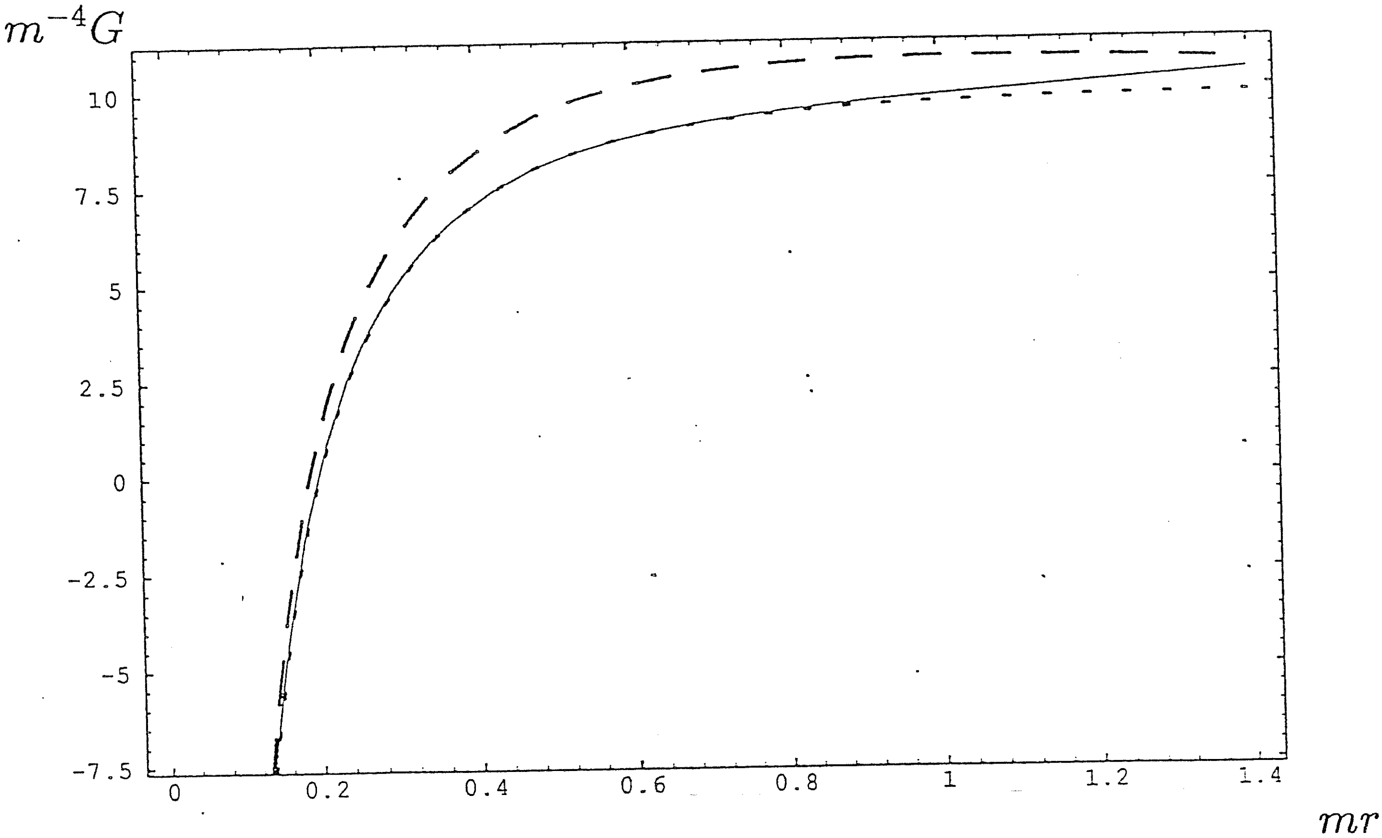

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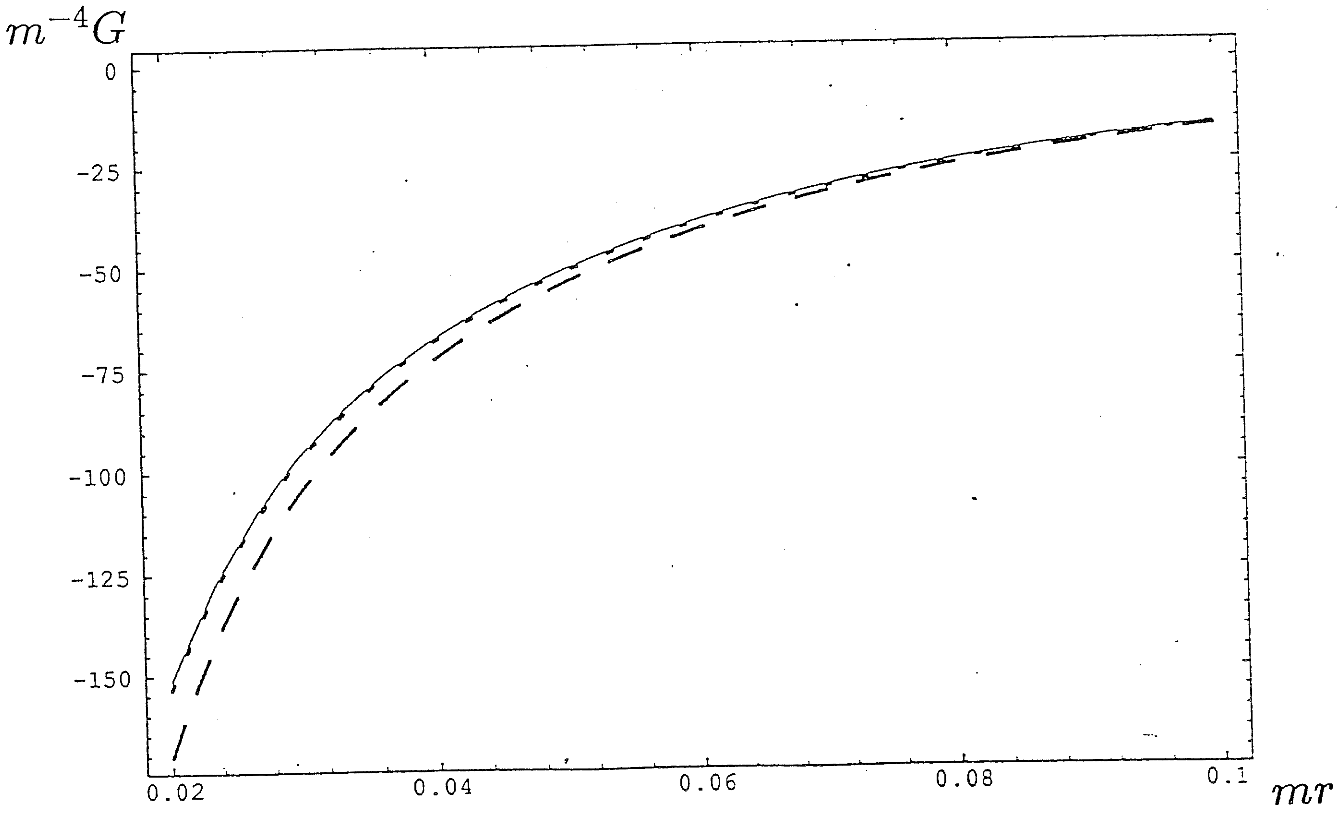

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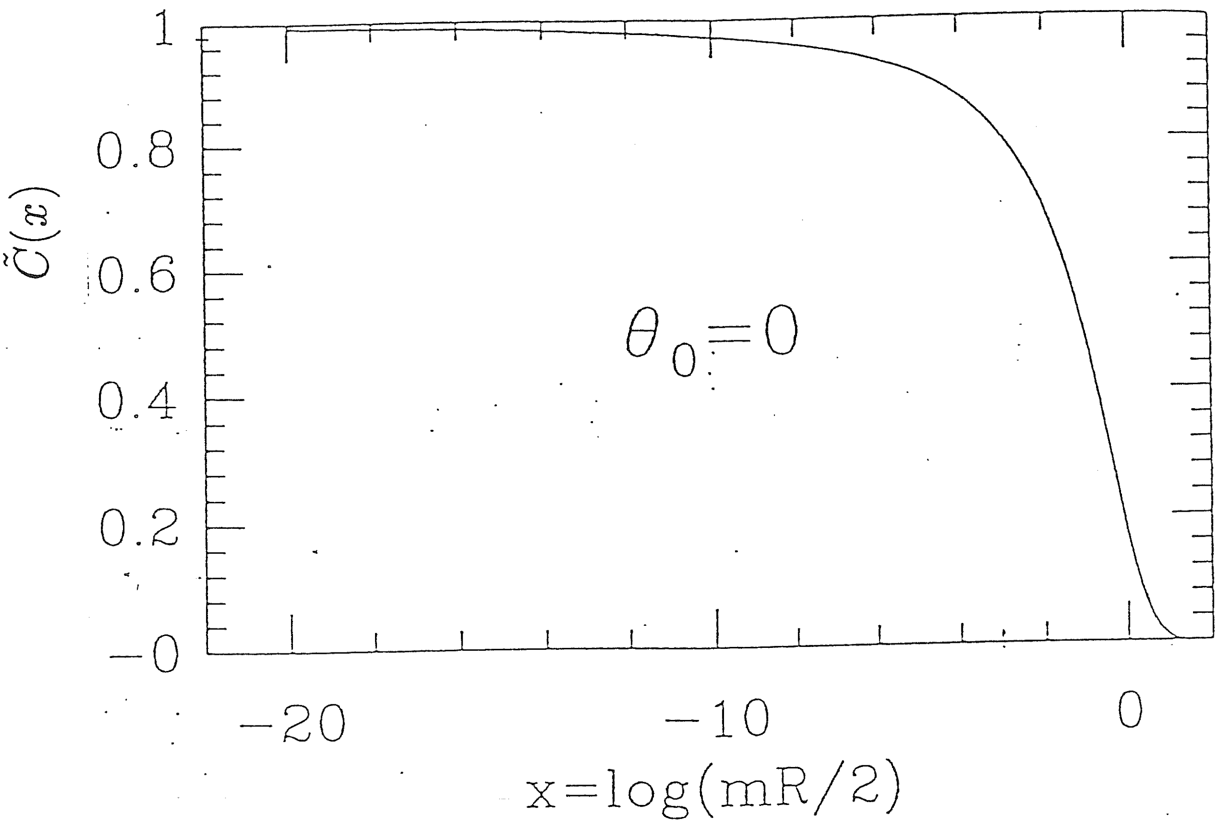

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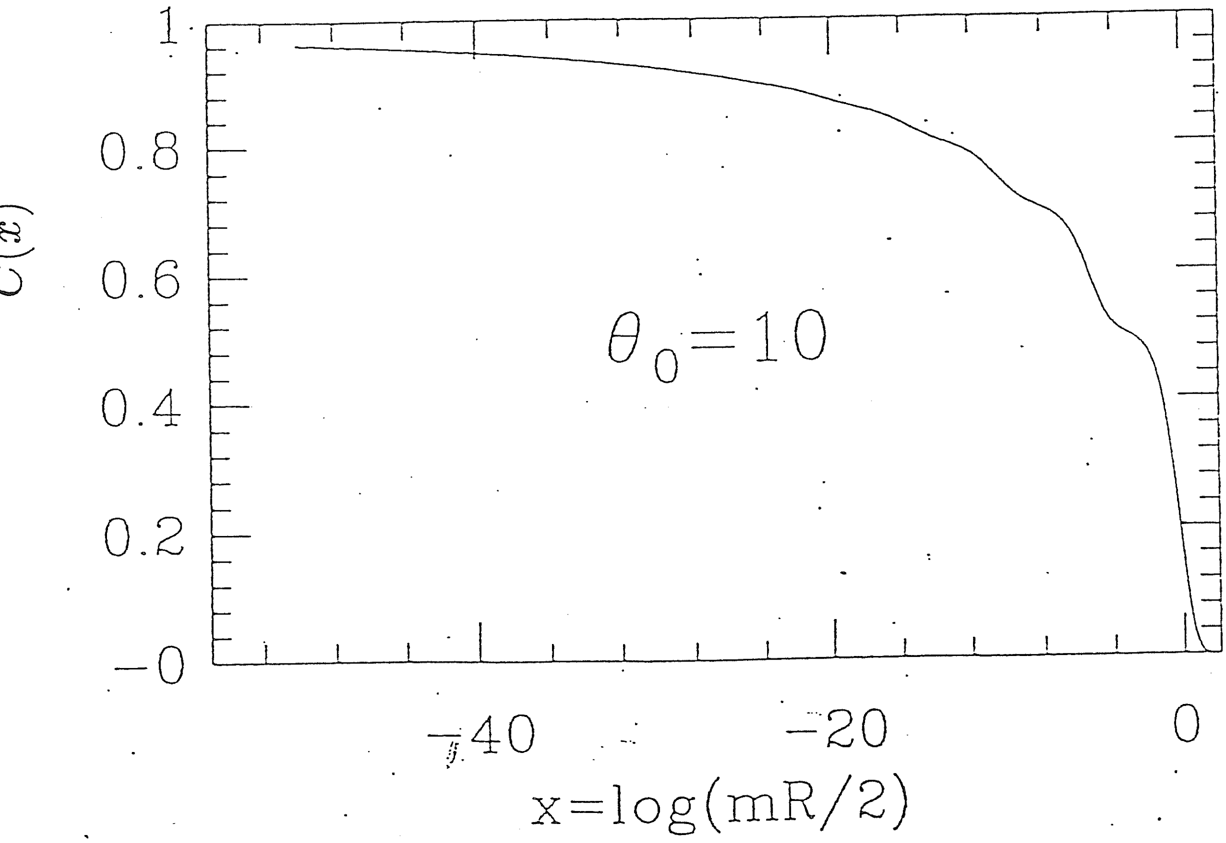

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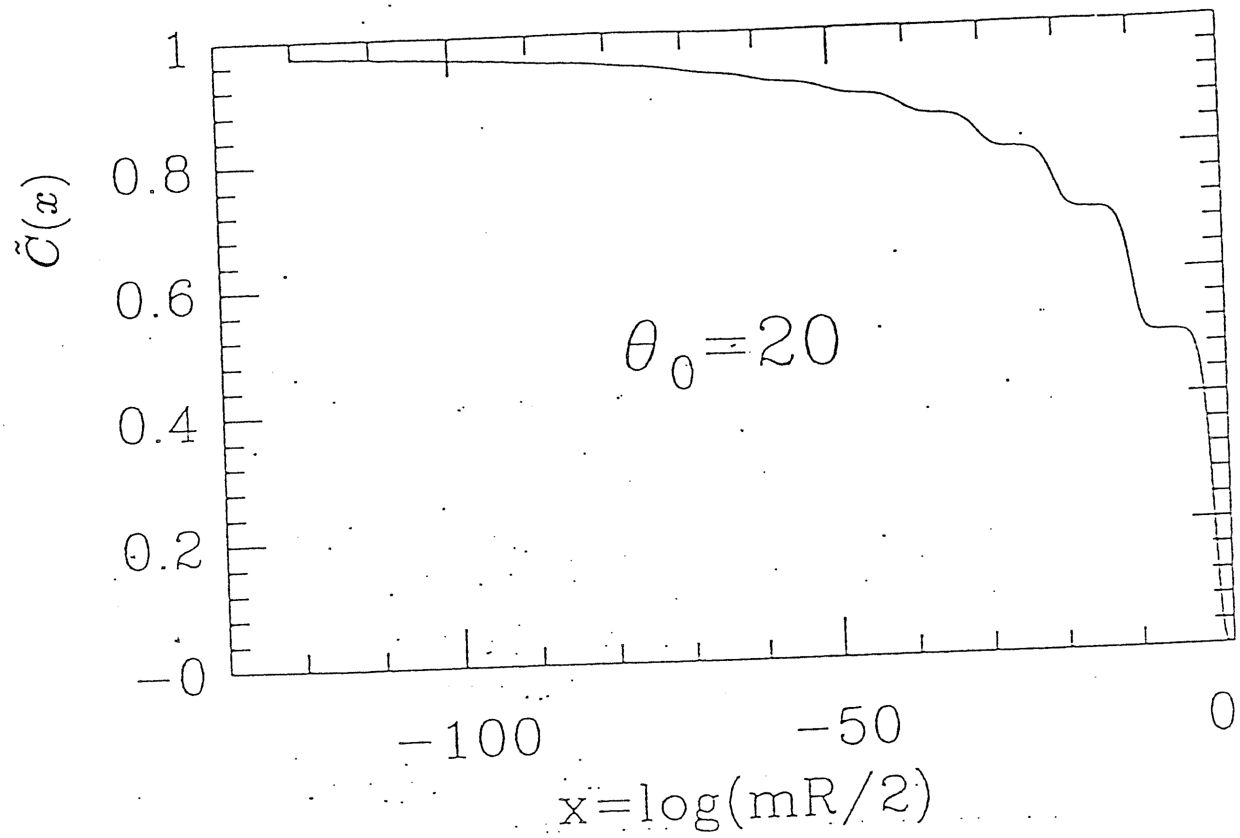

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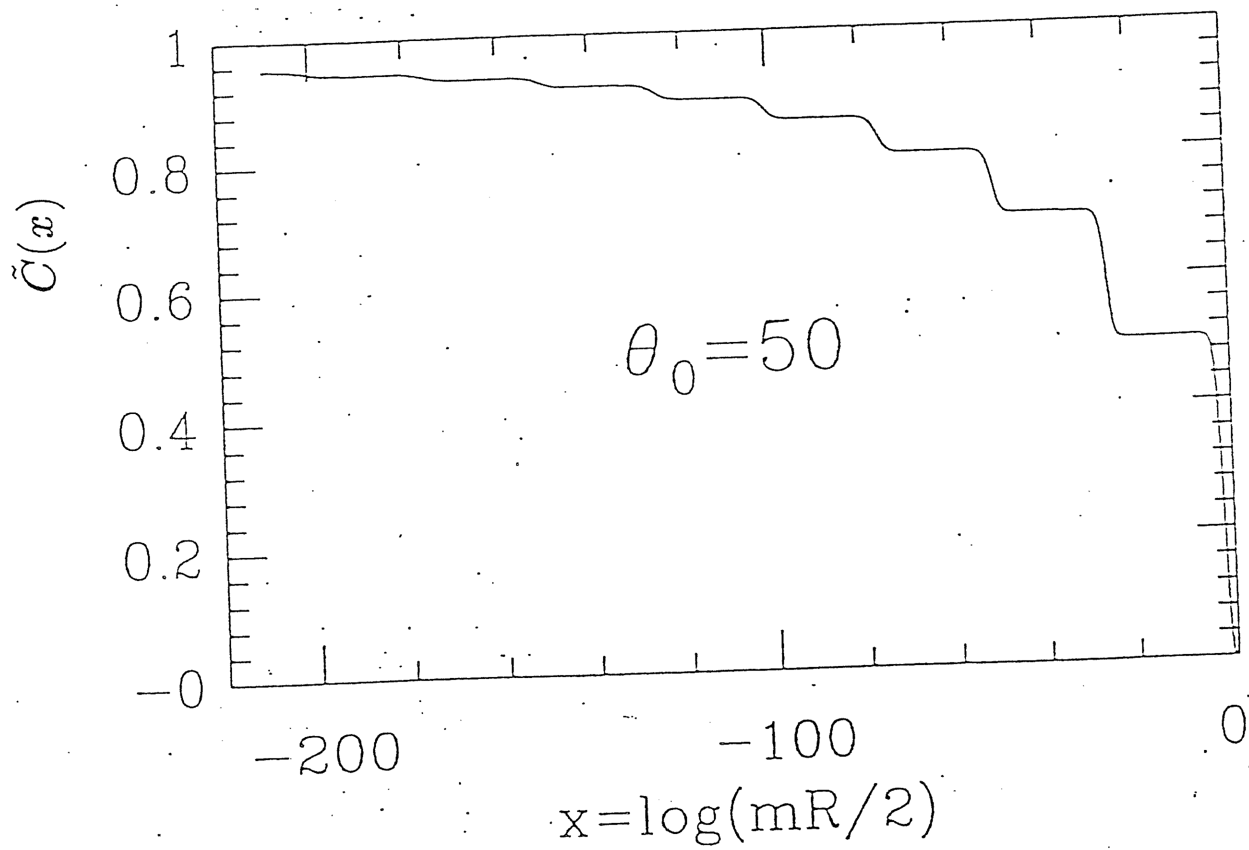

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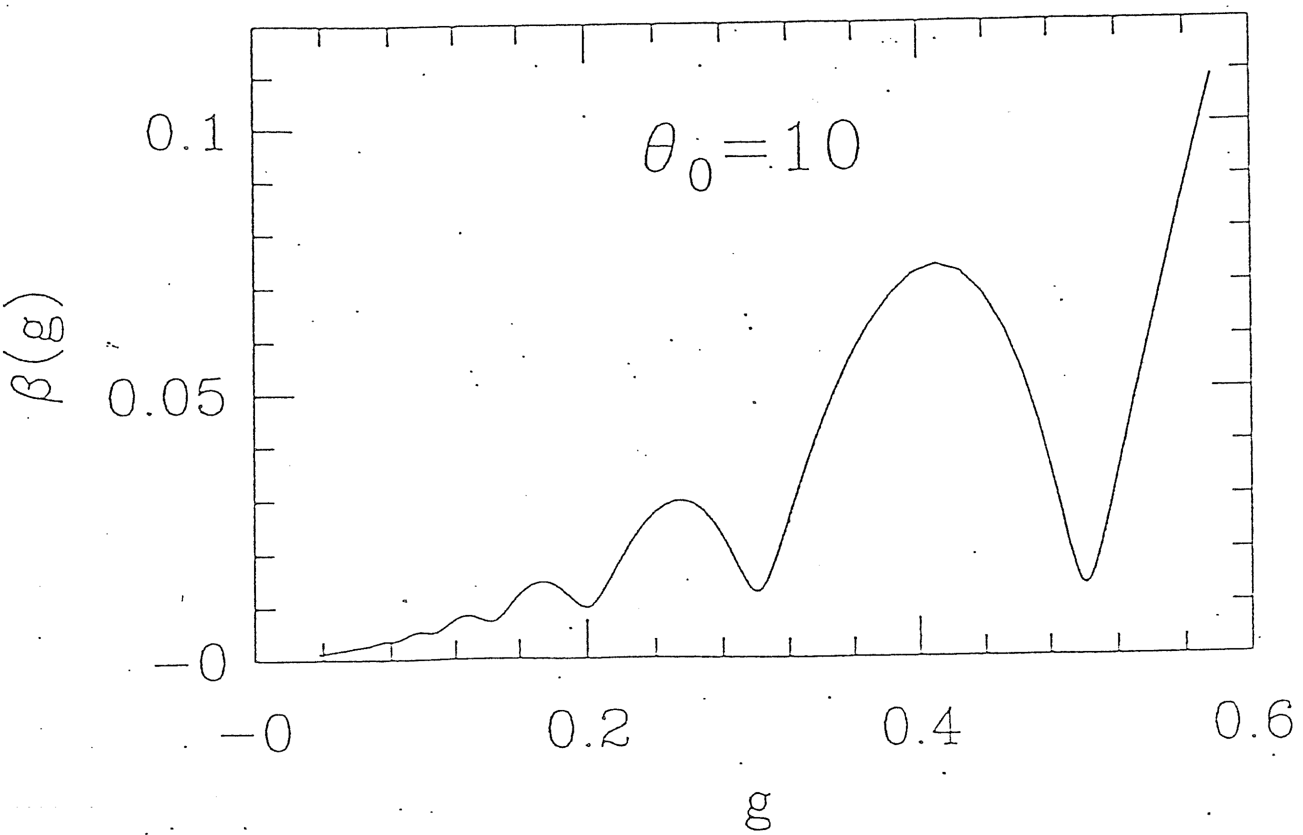

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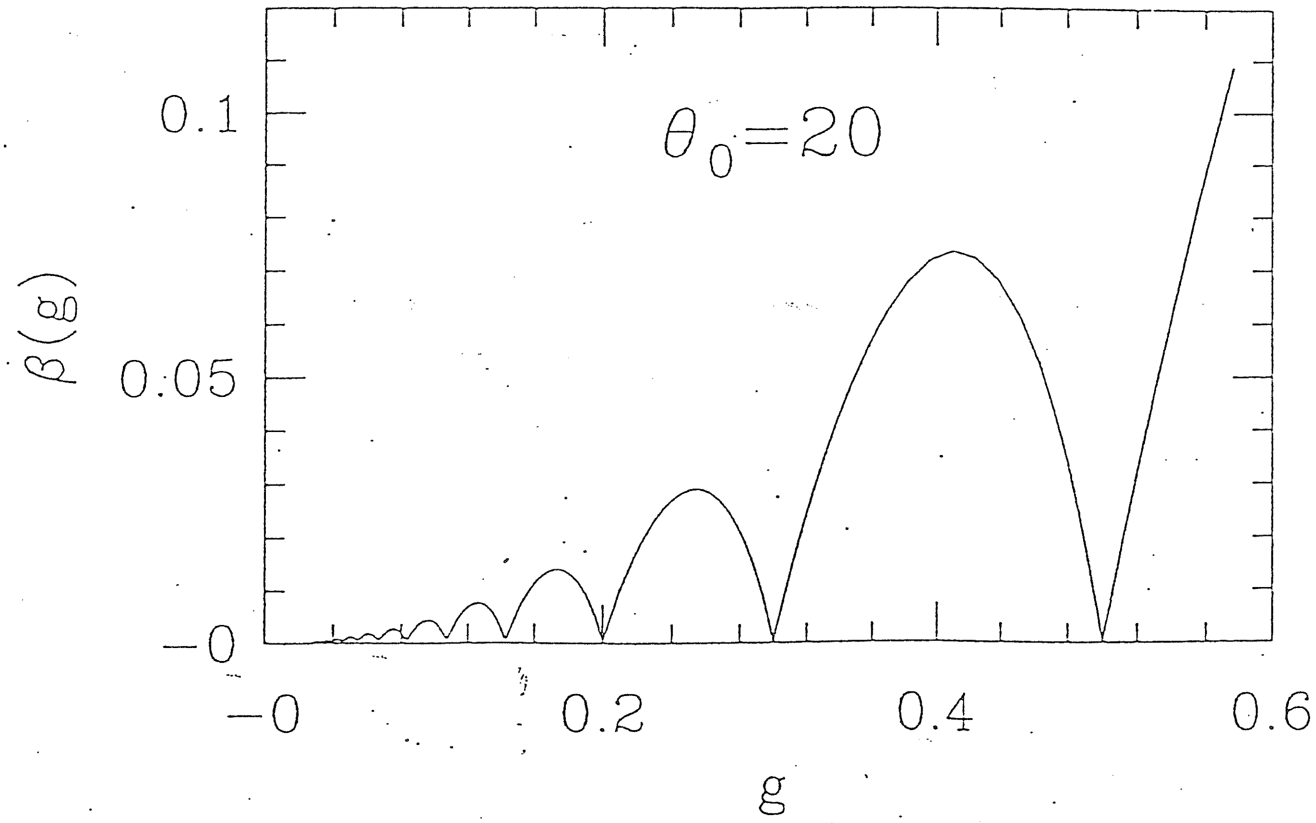

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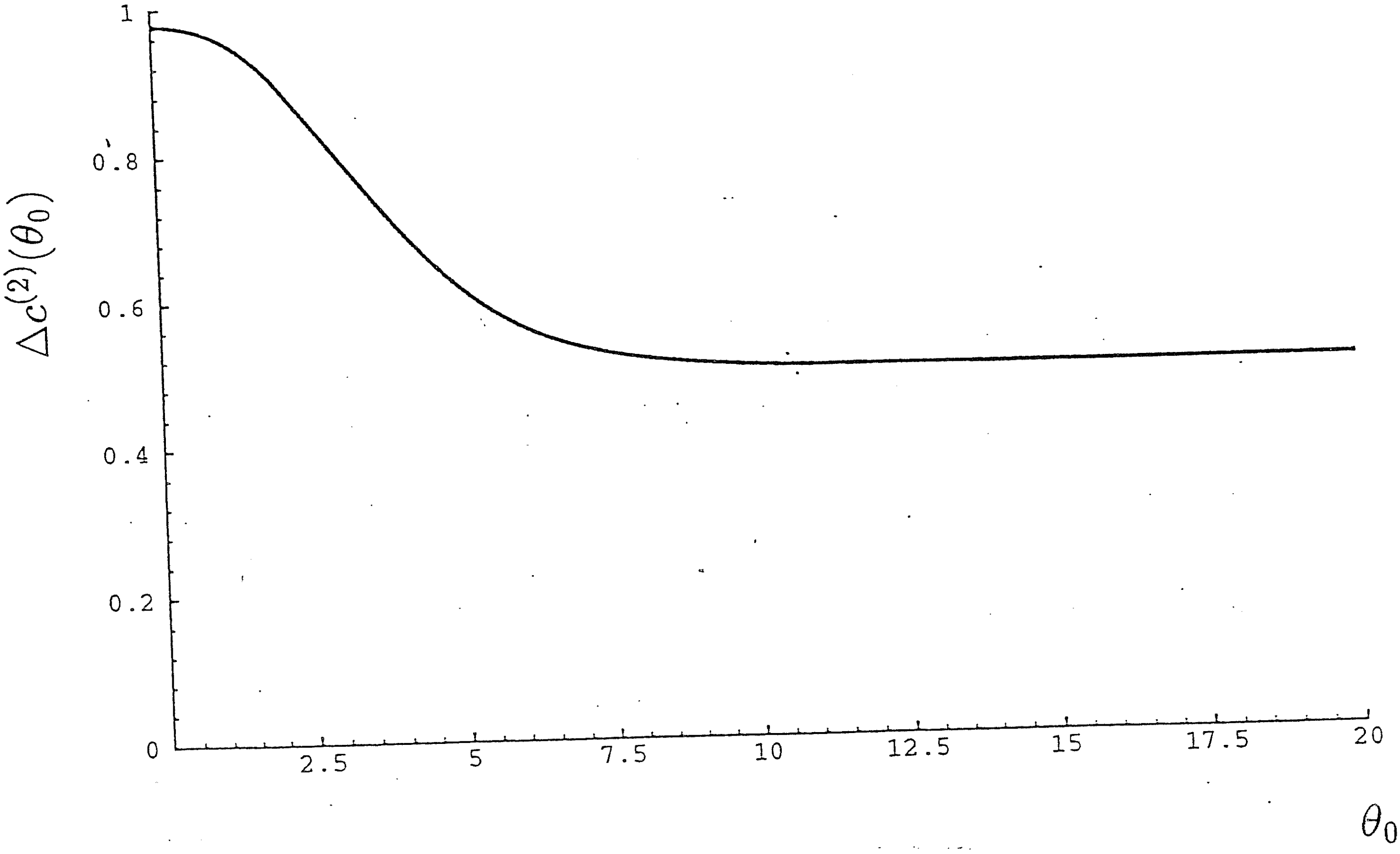

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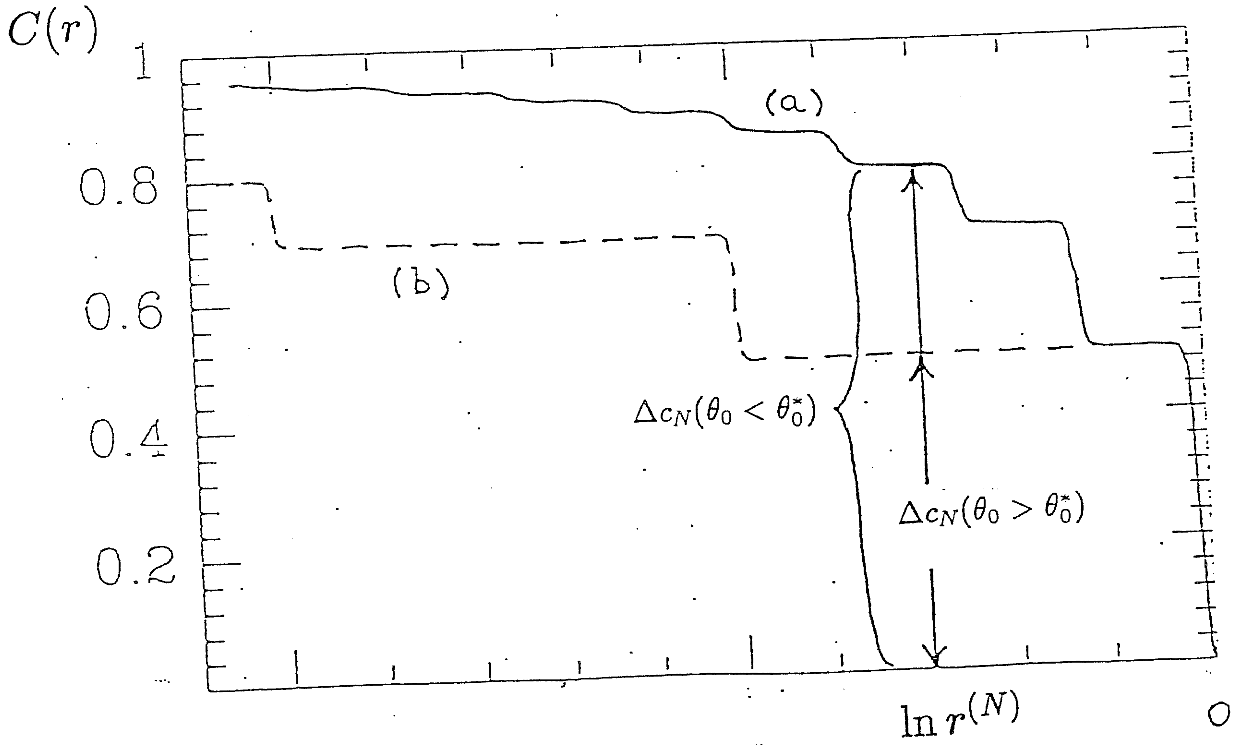

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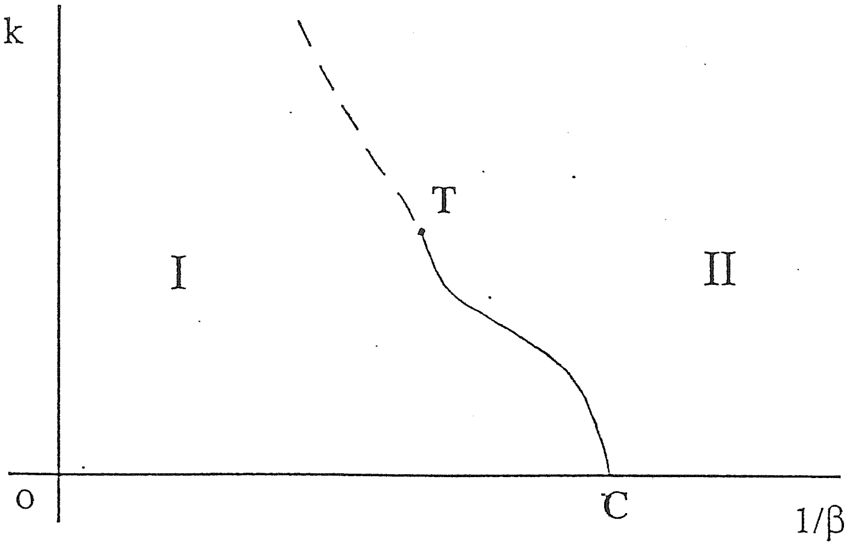

figure 15

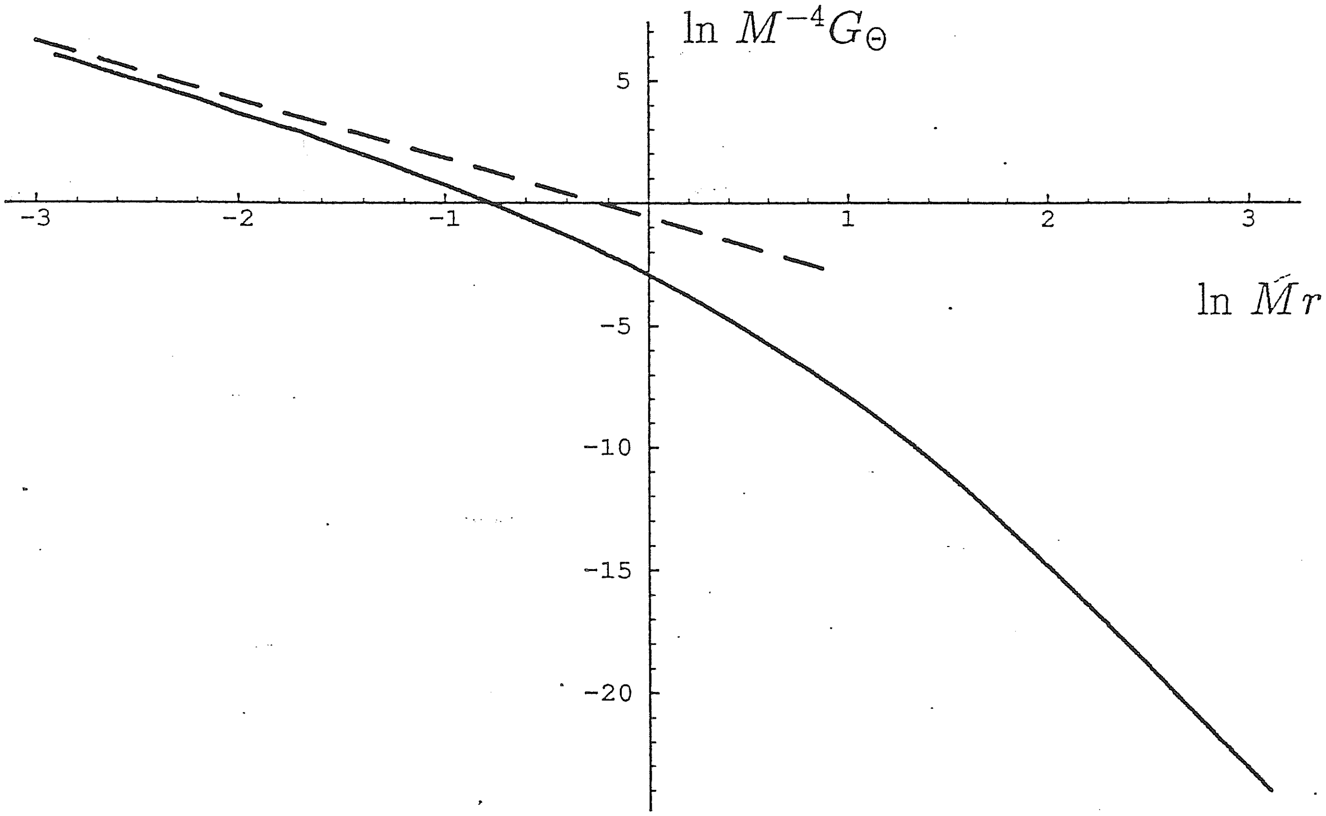

figure 16

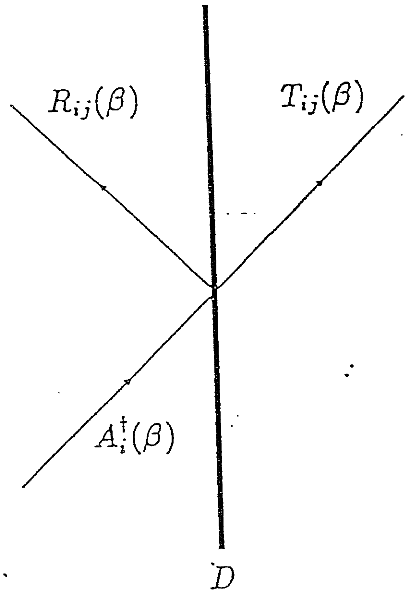

figure 17

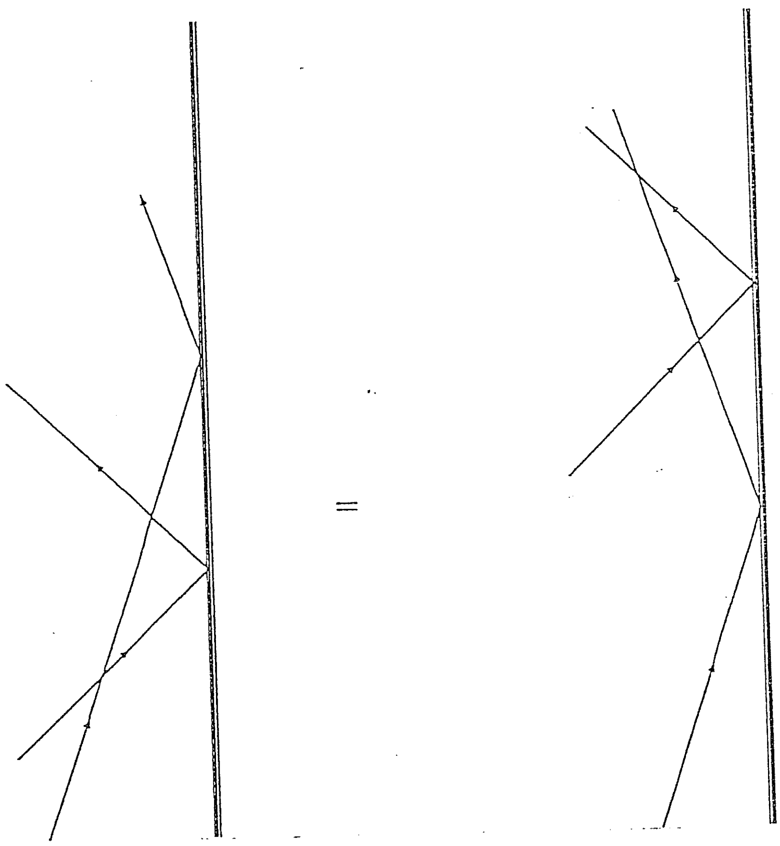

figure 18a

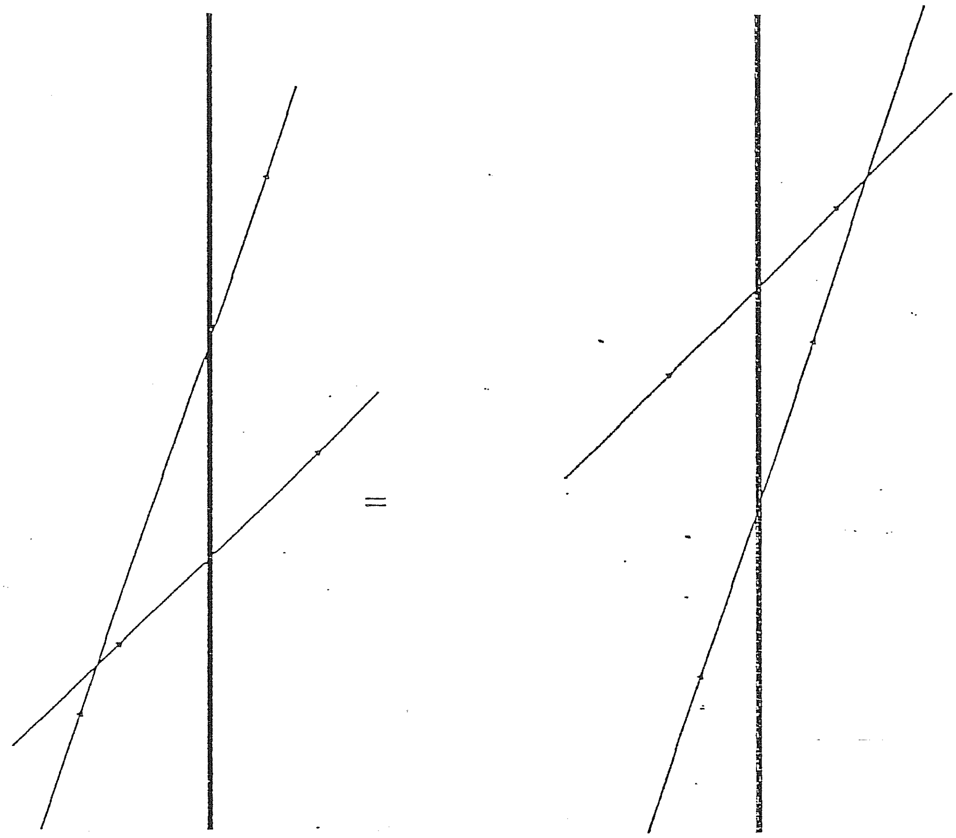

figure 18b

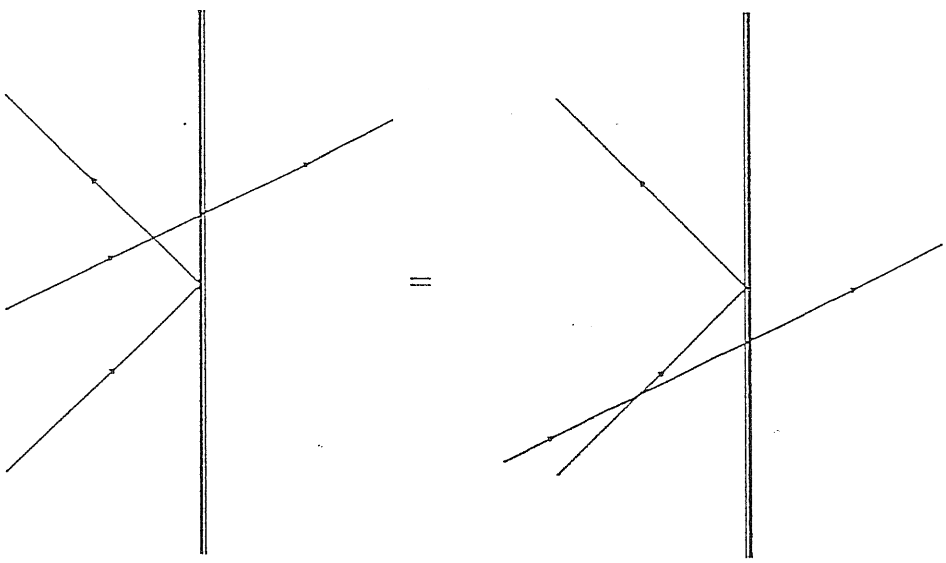

figure 18c

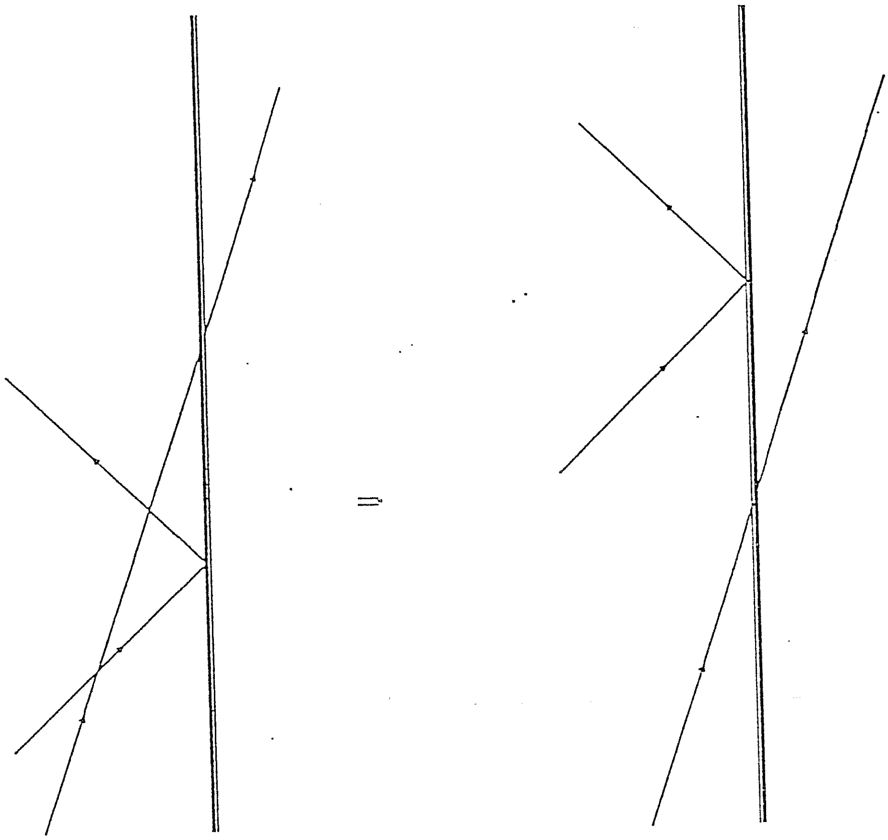

figure 18d

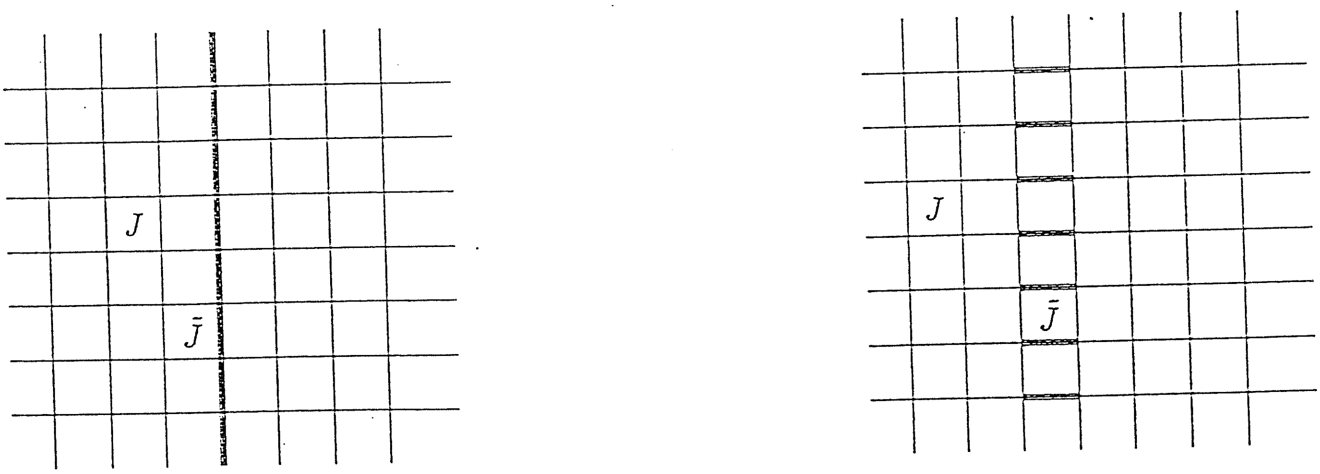

figure 19

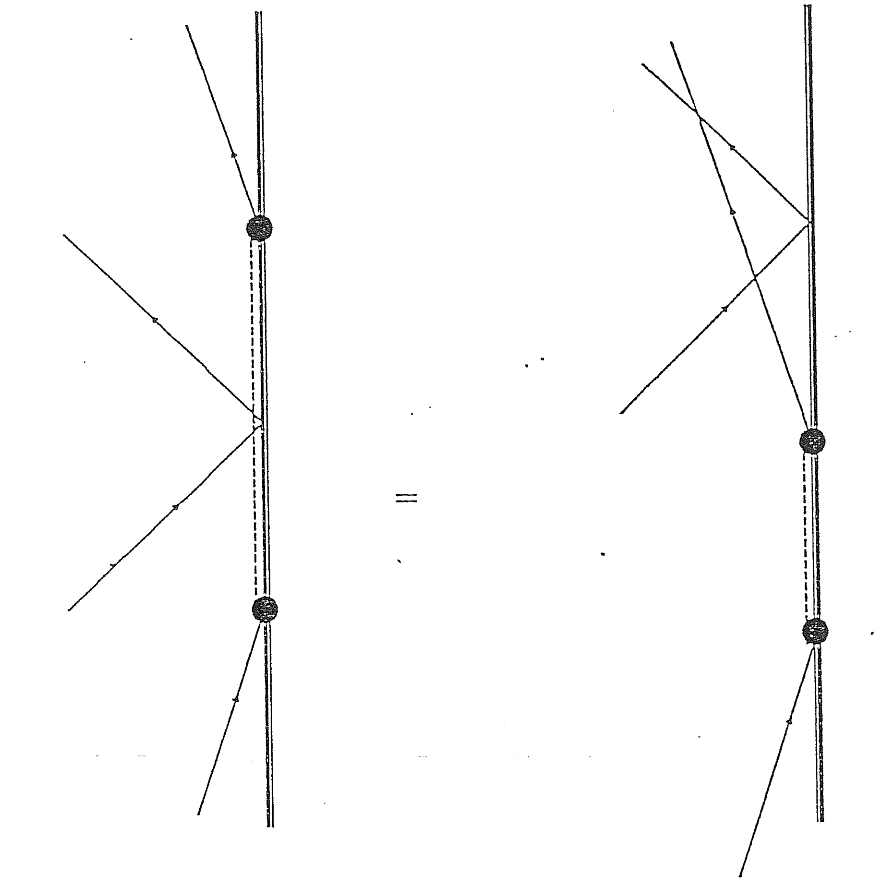

figure 20a

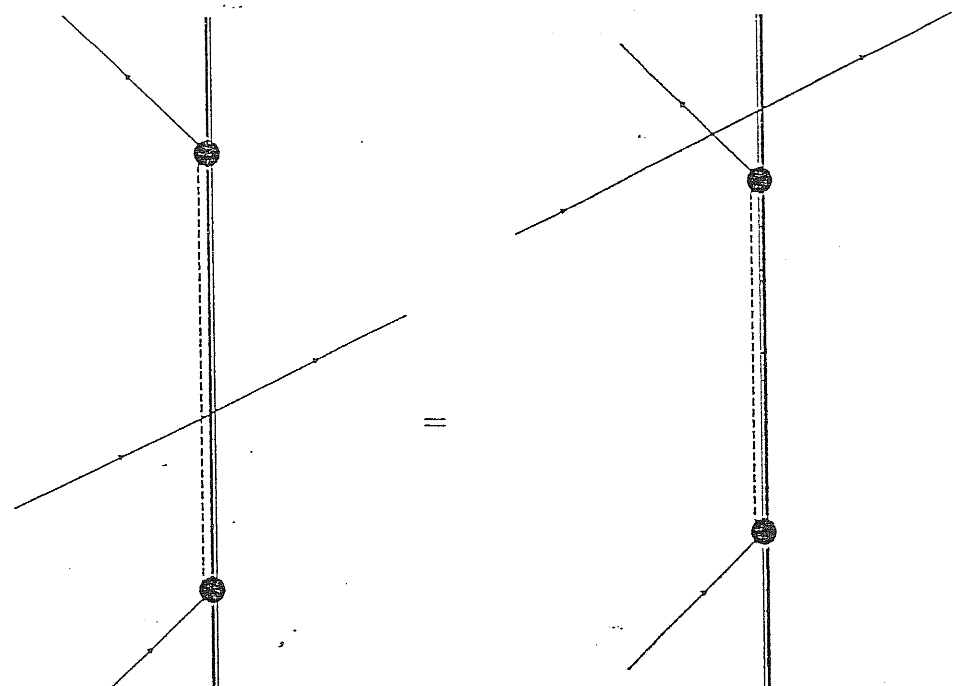

figure 20b

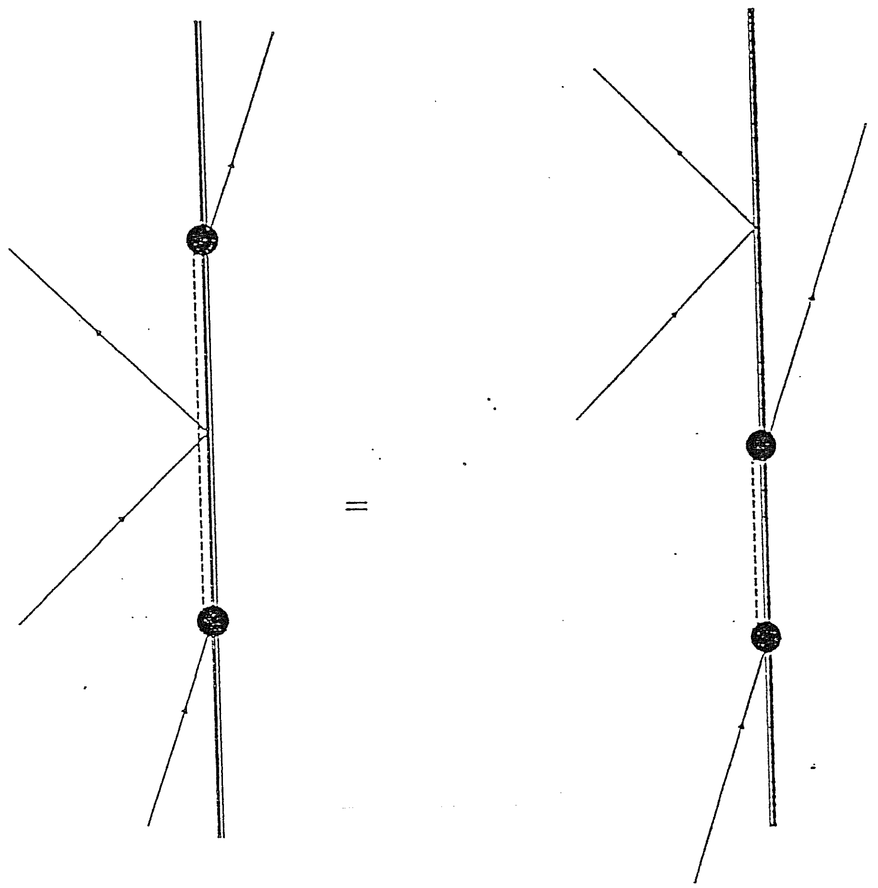

figure 20c

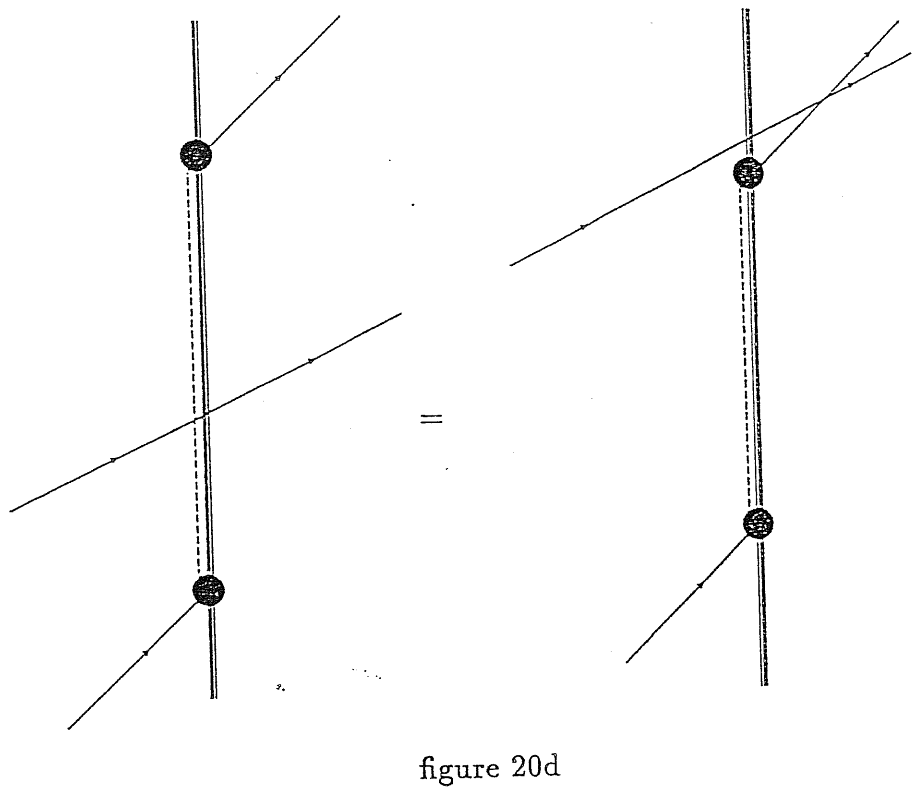

figure 20d

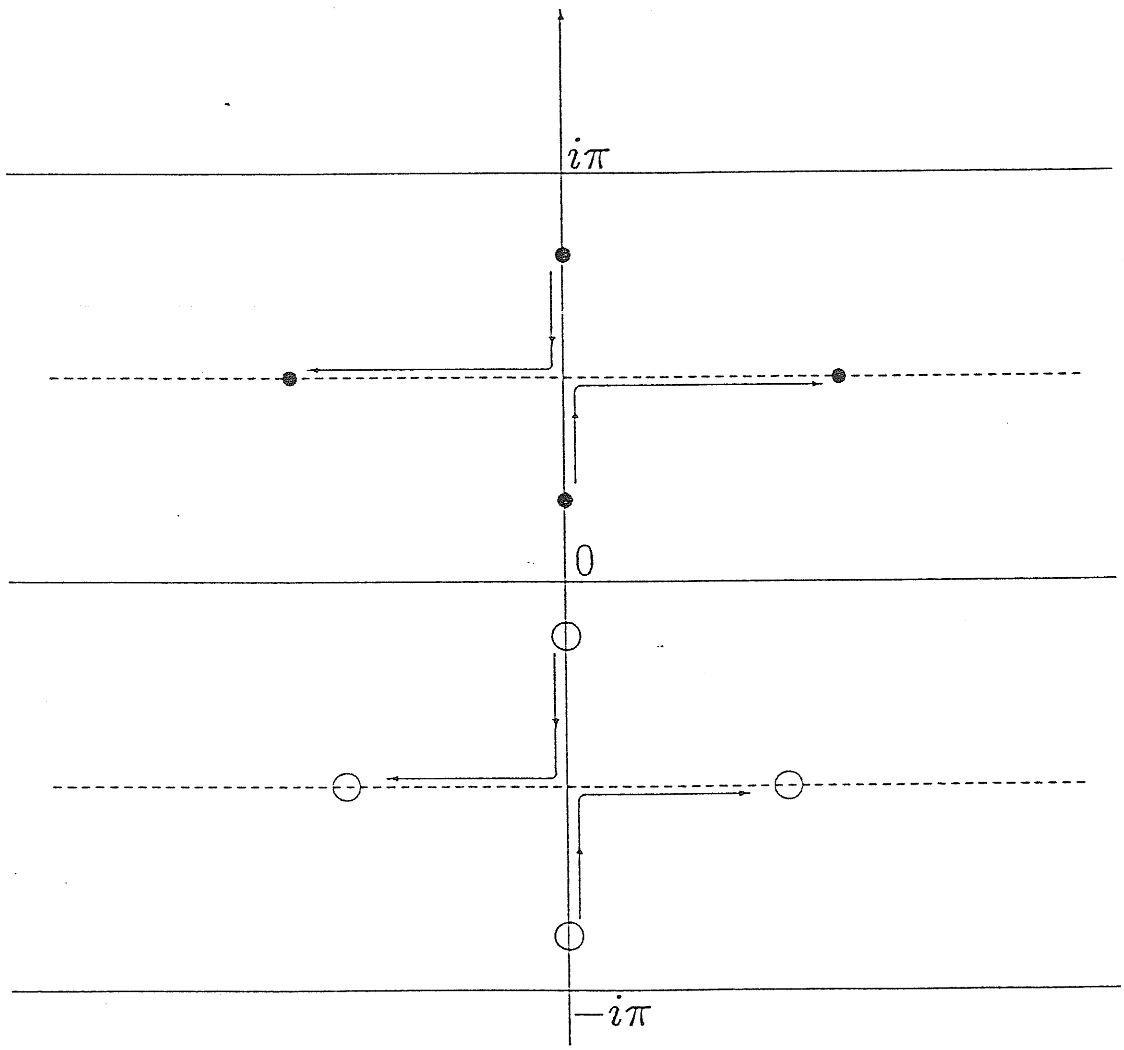

figure 21

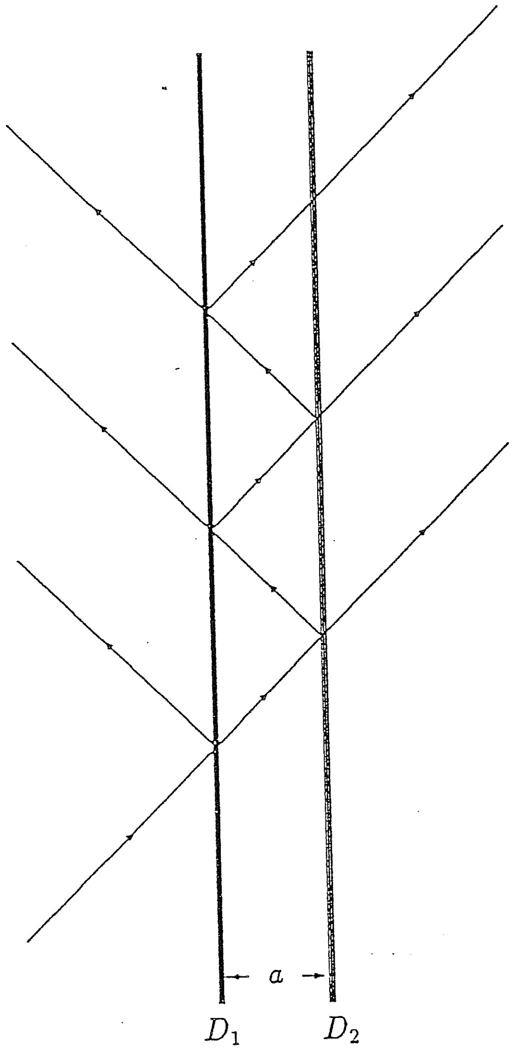

figure 22

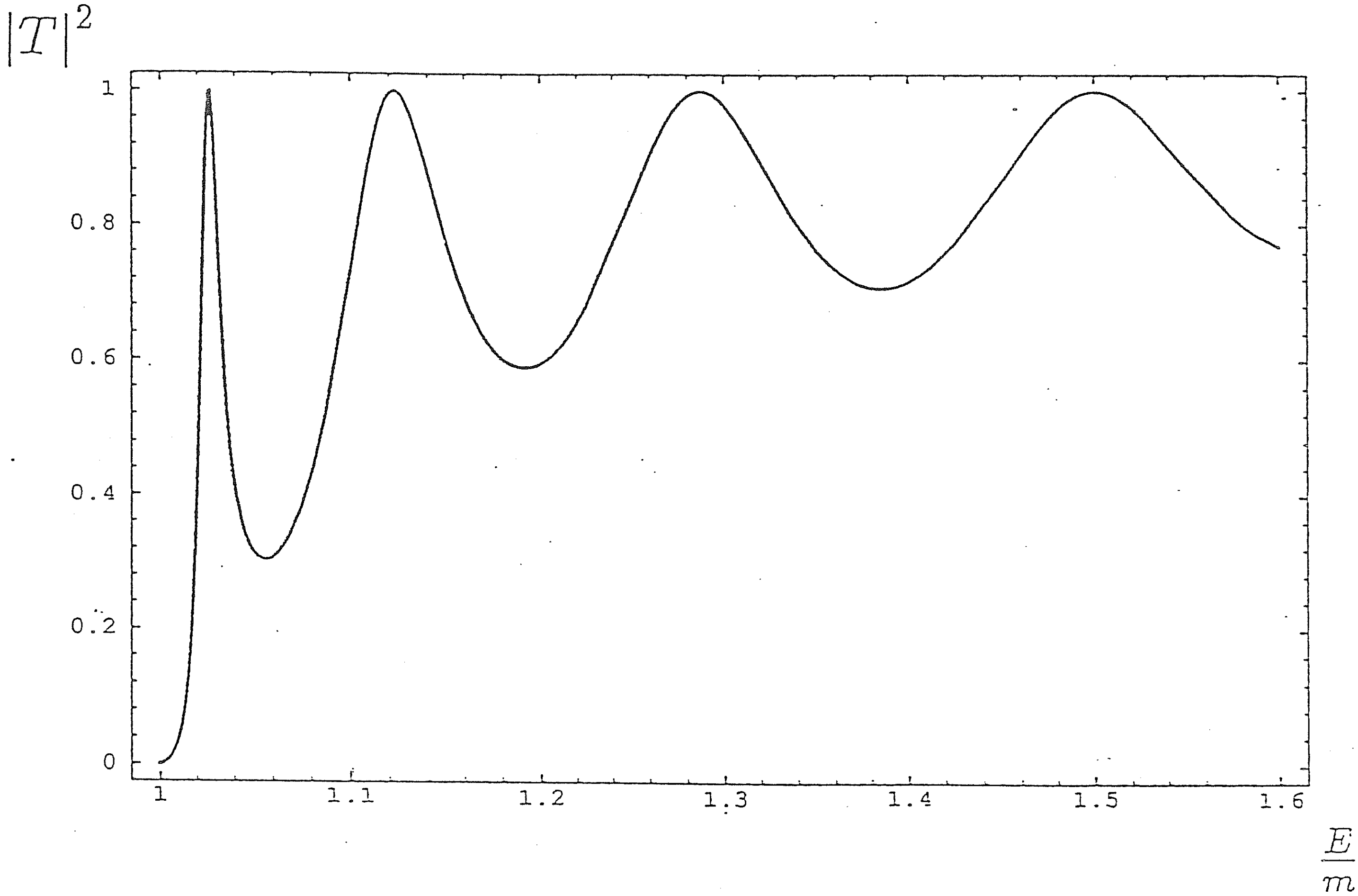

figure 23

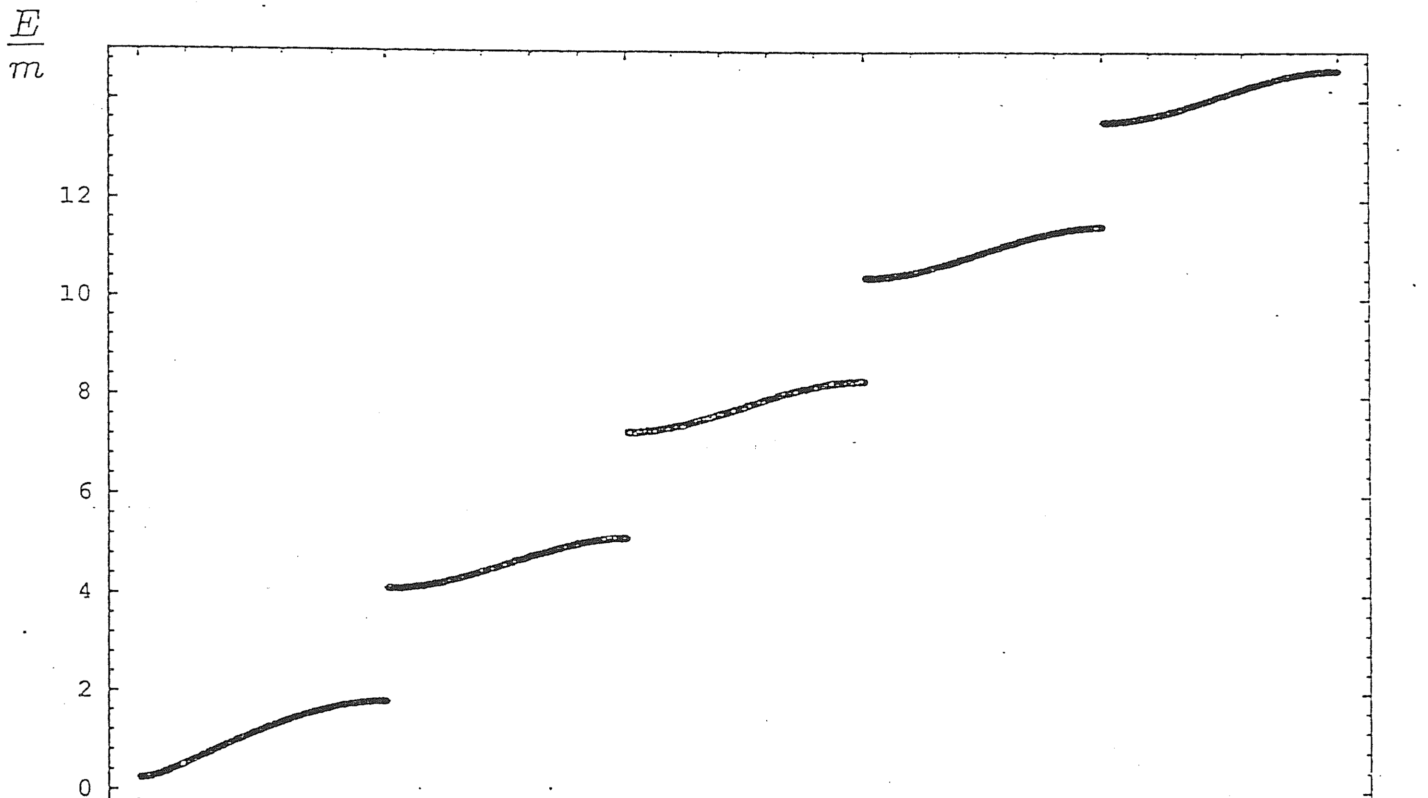

figure 24

$\frac{3}{4} \quad 0$ $\frac{1}{5} \quad -\frac{1}{20}$ $-\frac{1}{20} \quad \frac{1}{5}$ $0 \quad \frac{3}{4}$	$I = \Phi_{0,0}$ $\sigma = \Phi_{-\frac{1}{20},-\frac{1}{20}}$ $\varphi = \Phi_{\frac{1}{5},\frac{1}{5}}$ $\psi = \Phi_{\frac{3}{4},\frac{3}{4}}$

table 1

$\sigma * \sigma = [I] + C_{\sigma\sigma}^{\varphi}\,[\varphi]$ $\sigma * \varphi = C_{\sigma\sigma}^{\varphi}\,[\sigma] + C_{\sigma\varphi}^{\psi}\,[\psi]$ $\varphi * \varphi = [I] + C_{\varphi\varphi}^{\varphi}\,[\varphi]$ $\psi * \psi = [I]$	$C_{\sigma\sigma}^{\varphi} = i\,\frac{\Gamma^2\left(\frac{4}{5}\right)}{\Gamma\left(\frac{6}{5}\right)\Gamma\left(\frac{2}{5}\right)}\sqrt{\frac{\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{2\pi}{5}\right)}}$ $C_{\varphi\varphi}^{\varphi} = i\,\left(\frac{2}{5}\right)^2\frac{\Gamma^2\left(-\frac{1}{5}\right)}{\Gamma\left(\frac{6}{5}\right)\Gamma\left(\frac{2}{5}\right)}\sqrt{\frac{\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{2\pi}{5}\right)}}$ $C_{\sigma\varphi}^{\psi} = \frac{1}{2}$

table 2

$\frac{3}{2}$	$\frac{7}{16}$	0
$\frac{6}{10}$	$\frac{3}{80}$	$\frac{1}{10}$
$\frac{1}{10}$	$\frac{3}{80}$	$\frac{6}{10}$
0	$\frac{7}{16}$	$\frac{3}{2}$

table 3

$I = \Phi_{0,0}$	identity
$\sigma = \Phi_{\frac{3}{80},\frac{3}{80}}$	magnetization
$\varepsilon = \Phi_{\frac{1}{10},\frac{1}{10}}$	energy
$\sigma' = \Phi_{\frac{7}{16},\frac{7}{16}}$	sub-magnetization
$\varepsilon' = \Phi_{\frac{6}{10},\frac{6}{10}}$	vacancy density
$\varepsilon'' = \Phi_{\frac{3}{2},\frac{3}{2}}$	(irrelevant)

table 4

even * even	
$\varepsilon * \varepsilon = [1] + c_1[\varepsilon']$	
$\varepsilon' * \varepsilon' = [1] + c_2[\varepsilon']$	$c_1 = \frac{2}{3}\sqrt{\frac{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}}$
$\varepsilon * \varepsilon' = c_1[\varepsilon] + c_3[\varepsilon'']$	
$\varepsilon'' * \varepsilon'' = [1]$	$c_2 = c_1$
even * odd	$c_3 = \frac{3}{7}$
$\varepsilon * \sigma' = c_4[\sigma]$	$c_4 = \frac{1}{2}$
$\varepsilon * \sigma = c_4[\sigma'] + c_5[\sigma]$	$c_5 = \frac{3}{2}c_1$
$\varepsilon' * \sigma' = c_6[\sigma]$	$c_6 = \frac{3}{4}$
$\varepsilon' * \sigma = c_6[\sigma'] + c_7[\sigma]$	$c_7 = \frac{1}{4}c_1$
odd * odd	$c_8 = \frac{7}{8}c_1$
$\sigma' * \sigma' = [1] + c_8[\varepsilon'']$	$c_9 = \frac{1}{56}$
$\sigma' * \sigma = c_4[\varepsilon] + c_6[\varepsilon']$	
$\sigma * \sigma = [1] + c_5[\varepsilon] + c_7[\varepsilon'] + c_9[\varepsilon'']$	

table 5

Ringraziamenti

Desidero innanzitutto esprimere la mia gratitudine al mio relatore Giuseppe Mussardo per le cose che mi ha insegnato e per il legame di schietta amicizia che ha voluto stabilire con me.

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